Algebraic topology I - VI

Stefan Friedl

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Introduction

These are the lecture notes for a sequence of courses on algebraic and geometric topology at the University of Regensburg that started in the winter term of 2016/17 and which is still going on.

There are some issues with the lecture notes which are known to me:

(1) Some references point to future, so far unwritten material.
(2) The part on differential topology needs to be reorganized and cleaned up.
(3) The definition of CW-complexes is suboptimal and will be revised.

If you have any comments, especially if you find mistakes or typos, please let me know.

sfriedl@gmail.com

Part I

General Topology

1. Preliminaries

In this preliminary chapter we introduce some concepts and results from set theory which we will use throughout these notes, but which otherwise have little to do with topology itself. The reader might want to skip this chapter and directly move to the next chapter.

1.1. **Basic set theory.** Throughout these notes we work with "naive set theory". Like most working mathematicians this author does not want to get drawn into a discussion about what a set really is. Furthermore, when we describe sets we sometimes use creative notation and language. The hope is that the language is chosen in such a way that any good-willed reader will understand what is meant.

In this short section we recall a few basic definitions and results from set theory which will frequently use throughout these notes. Usually we will not refer to the statements explicitly, unless it helps to clarify an argument.

First we recall the following standard notation.

Notation. We denote by $\varnothing$ the empty set.

We continue with two of the most important definitions of set theory.

Definition. Let $\{U_i\}_{i\in I}$ be a family of subsets of a given set X. We write

$$\begin{aligned} \bigcup_{i\in I} U_i &= \{x \in X \mid \text{ there exists an } i \in I \text{ with } x \in X_i\} \\ \bigcap_{i\in I} U_i &= \{x \in X \mid \text{ for every } i \in I \text{ we have } x \in X_i\}. \end{aligned}$$

In particular, if $I = \varnothing$, then $\bigcup_{i\in I} U_i = \varnothing$ and $\bigcap_{i\in I} U_i = X$.[1]

Next we state, without proof, three elementary lemmas from set theory. Even though they are elementary it is helpful to have them on record.

Without further apologies, let us state the first lemma.

Lemma 1.1. *Let X be a set and let $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be two families of subsets of X. Then*

$$\Big(\bigcup_{i\in I} A_i\Big) \cap \Big(\bigcup_{j\in J} B_j\Big) = \bigcup_{i\in I, j\in J} (A_i \cap B_j).$$

We continue with de Morgan's laws.

Lemma 1.2. (De Morgan's laws) *Let X be a set and let $\{A_i\}_{i\in I}$ be a family of subsets of X. If given a subset $Y \subset X$ we denote its complement by Y^c, then*

$$\Big(\bigcap_{i\in I} A_i\Big)^c = \bigcup_{i\in I} A_i^c \quad \textit{and} \quad \Big(\bigcup_{i\in I} A_i\Big)^c = \bigcap_{i\in I} A_i^c.$$

Or equivalently, in a more cumbersome notation, we have

$$X \setminus \Big(\bigcap_{i\in I} A_i\Big) = \bigcup_{i\in I} (X \setminus A_i) \quad \textit{and} \quad X \setminus \Big(\bigcup_{i\in I} A_i\Big) = \bigcap_{i\in I} (X \setminus A_i).$$

[1] This simple observation will be useful at times.

The following lemma concerns the interaction between operations on sets and maps between sets.

Lemma 1.3. *Let $f\colon X \to Y$ be a map between two sets X and Y. Then*

(1) *for every $U \subset X$ we have* $U \subset f^{-1}(f(U))$
furthermore, if f is injective[2]*, then we have* $U = f^{-1}(f(U))$

(2) *for every $V \subset Y$ we have* $V \supset f(f^{-1}(V))$
furthermore, if f is surjective, then we have $V = f(f^{-1}(V))$

(3) *for every family $\{U_i\}_{i\in I}$ of subsets of X we have* $f\Big(\bigcup\limits_{i\in I} U_i\Big) = \bigcup\limits_{i\in I} f(U_i)$

(4) *for every family $\{U_i\}_{i\in I}$ of subsets of X we have* $f\Big(\bigcap\limits_{i\in I} U_i\Big) \subset \bigcap\limits_{i\in I} f(U_i)$

(5) *for every family $\{V_i\}_{i\in I}$ of subsets of Y we have* $f^{-1}\Big(\bigcup\limits_{i\in I} V_i\Big) = \bigcup\limits_{i\in I} f^{-1}(V_i)$

(6) *for every family $\{V_i\}_{i\in I}$ of subsets of Y we have* $f^{-1}\Big(\bigcap\limits_{i\in I} V_i\Big) = \bigcap\limits_{i\in I} f^{-1}(V_i)$

(7) *for every $V \subset Y$ we have* $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$.

We conclude this short section with the following notation.

Notation. Given a set X we denote by $\mathcal{P}(X)$ its *power set*, i.e. $\mathcal{P}(X)$ is the set of all subsets of X.

1.2. **Relations and orders on sets.** We start out with recalling the notion of a relation.

Definition. A *relation* on a set S is a subset of $S \times S$.

The definition of a relation in its full generality is pretty useless. It becomes more interesting once we introduce several adjectives.

Definition. Let S be a set and let $R \subset S \times S$ be a relation on S.

(1) The relation is called *reflexive*, if for every $s \in S$ we have $(s,s) \in R$.
(2) The relation is called *transitive*, if for every (s,t) and (t,u) in R we also have that $(s,u) \in R$.
(3) The relation is called *symmetric*, if for every (s,t) in R we also have that $(t,s) \in R$.

We adopt the following notation.

Notation. Let S be a set and let $R \subset S \times S$ be a relation on S. Given $x, y \in S$ we write $x\,R\,y$ if and only if $(x,y) \in R$.

Example. We consider the set $\mathbb{R}$ together with the relation

$$\leq \;:=\; \{(x,y) \in \mathbb{R}^2 \,|\, y - x \text{ is not negative}\}.$$

One can easily verify that this relation $\leq$ is reflexive and transitive, but that it is not symmetric. Note that given $x, y \in \mathbb{R}$ the above expression "$x \leq y$" has the usual meaning that we are all used to.

[2] As a totally irrelevant side remark we would like to point out that the fundamental terms "surjective", "injective" and "bijective" are actually fairly recent additions to the mathematical language. They were first introduced on page 80 of Nicolas Bourbaki's "Théorie des ensembles" first published in 1954, see [**Bou54**]. The reader who has need read the biography of the great mathematician Nicolas Bourbaki is strongly encouraged to do so.

Most of the relations that we will consider are going to be reflexive and transitive. There is a big difference though in the roles of symmetric and non-symmetric relations. Let us consider the symmetric ones first.

Definition. An *equivalence relation* on a set S is a relation that is reflexive, transitive and symmetric.

Convention. Equivalence relations on a set S are almost invariably denoted by "$\sim$" or some variation thereof.

To practice our conventions and notations, the three properties of an equivalence relation "$\sim$" on a set S are precisely the following three statements:

(1) for any $s \in S$ we have $s \sim s$ (reflexivity),
(2) for any $s, t, u \in S$ with $s \sim t$ and $t \sim u$ we have $s \sim u$ (transitivity),
(3) for any $s, t \in S$ with $s \sim t$ we also have $t \sim s$ (symmetry).

We continue with the following definition.

Definition. Let S be a set together with an equivalence relation "$\sim$". An *equivalence class* of $(S, \sim)$ is a subset C of S with the following three properties:

(1) the subset is non-empty,
(2) given any two $s, t \in C$ we have $s \sim t$, and
(3) given $s \in C$ and $t \in S$ with $s \sim t$ we have $t \in C$.

We refer to [**Halm60**, Section 7] for a confirmation of the above definitions.

The following lemma summarizes the key properties of equivalence classes.

Lemma 1.4. *Let S be a set together with an equivalence relation "$\sim$".*

(1) *Every element of S is contained in a unique equivalence class, namely*
$$[s] := \{t \in S \,|\, t \sim s\}.$$
(2) *Two equivalence classes are either disjoint or the same.*

Proof. Surely the reader will have seen the proof of the lemma before. Nonetheless we give the proof, in particular to highlight the fact that we do indeed need the symmetry condition.

(1) Let $s \in S$. Since $\sim$ is reflexive we see that $s \in [s]$. Thus it remains to show $[s] := \{t \in S \,|\, t \sim s\}$ is indeed an equivalence class.
  (a) We already showed that $s \in [s]$, in particular we see that $[s]$ is non-empty.
  (b) Let $t, t' \in [s]$. By definition this means that $t \sim s$ and $t' \sim s$. By symmetry we $s \sim t'$. By transitivity we deduce from $t \sim s$ and $s \sim t'$ that $t \sim t'$.
  (c) Let $t \in [s]$ and suppose that $t \sim u$. Since by definition we have $s \sim t$ we obtain from transitivity that $s \sim u$, i.e. we see that $u \in [s]$.
(2) Let C and D be two equivalence classes. We suppose that $C \cap D \neq \varnothing$. Thus there exists an $x \in C \cap D$. We need to show that $C = D$. Thus let $d \in D$. By Condition (2) of an equivalence class applied to D we know that $x \sim d$. By condition (1) of an equivalence class, applied to C, we see that $d \in C$. Thus we have shown that $D \subset C$. Evidently the same argument shows that $C \subset D$. ■

Throughout these notes we adopt the following standard notation.

Notation. Let S be a set together with an equivalence relation "$\sim$".

(1) We denote by $S/\sim$ the set of equivalence classes.
(2) Given $s \in S$ we denote by $[s]$ the unique equivalence class that contains $[s]$.

Example. Let $n \in \mathbb{N}$. We consider the set $\mathbb{Z}$ together with the equivalence relation that is given by

$$x \sim y \quad :\Longleftrightarrow \quad (x - y) \text{ is divisible by } n.$$

There are exactly n equivalence classes, namely

$$\begin{array}{rcl} [0] & = & \{\dots, -2n, -n, 0, n, 2n, 3n, \dots\} \\ [1] & = & \{\dots, 1-2n, 1-n, 1, 1+n, 1+2n, 1+3n, \dots\} \\ & \vdots & \\ [n-1] & = & \{\dots, -2n-1, -n, -1, n-1, 2n-1, 3n-1, \dots\}. \end{array}$$

To the despair of many of my cultured friends we write $\mathbb{Z}_n := \mathbb{Z}/\sim$.

Next we turn our gaze towards relations which are not required to be symmetric. We have the following harmless definition.

Definition.

(1) A relation "$\leq$" on a set S is called *antisymmetric* if $a \leq b$ and $b \leq a$ implies $a = b$.
(2) A relation that is reflexive, transitive and antisymmetric is called a *partial order*.
(3) A set together with a partial order is called a *partially ordered set*.
(4) Let "$\leq$" be a partial order on a set A. Given $a, b \in A$ we write $a < b$ if $a \leq b$ and if $a \neq b$.
(5) Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. A map $f\colon A \to B$ is called *order-preserving*, if for every two elements $a_1, a_2 \in A$ with $a_1 \leq a_2$ we also have $f(a_1) \leq f(a_2)$.[3]
(6) We say that two partially ordered sets $(A, \leq)$ and $(B, \leq)$ have the same *order type* if there exists a bijection $f\colon A \to B$ which is order-preserving and such that the inverse $f^{-1}\colon B \to A$ is also order-preserving.

Now we give a few examples of partial orders.

Examples.

(1) Let X be a subset of $\mathbb{R}$ and denote by $\leq$ the usual "less or equal" relation on the real numbers. Then $(X, \leq)$ is a partially ordered set.
(2) Let M be a set. We denote by $\mathcal{P}(M)$ the power set of M, i.e. the set of all subsets of M. Then "being a subset", i.e. the relation given by "$\subset$", is a partial order on $\mathcal{P}(M)$.
(3) We consider the three reflexive transitive relations that are illustrated in Figure 1. Here it is understood that $P \leq Q$ if there exists a "directed path" from Q to P.
   (a) The example on the left has a loop. Given any two points P, Q on the loop we have $P \leq Q$ and $Q \leq P$, hence this example is not a partial order.

[3]Note that "order-preserving" does not mean that $a_1 < a_2$ implies $f(a_1) < f(a_2)$.

(b) This example is a partial order.
(c) This example is also a partial order.

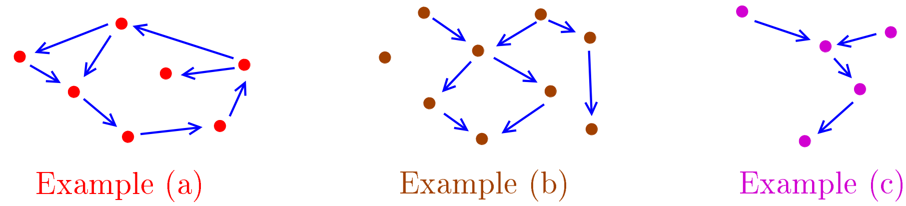


FIGURE 1

The following construction lets us build new examples of partially ordered sets out of given ones.

Definition. Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. Given (a, b) and (a', b') in $A \times B$ we define[4]

$$(a, b) < (a', b') \quad \Longleftrightarrow \quad \text{either } a < a' \text{ or } a = a' \text{ and } b < b'$$

and we define $(a, b) \leq (a', b')$ if either $(a, b) = (a', b')$ or $(a, b) < (a', b')$. One can easily verify that "$\leq$" is a partial order on $A \times B$. We refer to to this partial order on $A \times B$ as the *lexicographic order on* $A \times B$.

We need a few more definitions before we can state the first non-trivial result.

Definition. Let $(P, \leq)$ be a partially ordered set.

(1) A *chain* in P is a subset Q such that for any two elements $q, q' \in Q$ we have $q \leq q'$ or $q' \leq q$.
(2) An *upper bound* of a subset Q is an element $p \in P$ such that $q \leq p$ for all $q \in Q$.
(3) A *maximal element* of a subset Q is an element $q \in Q$ such that there is no $r \in Q$ with $q < r$.

Examples.

(1) In Figure 2 we once again consider the partially ordered set in the middle of Figure 1. We show a chain, we show upper bounds for the chain and we show maximal elements for P.

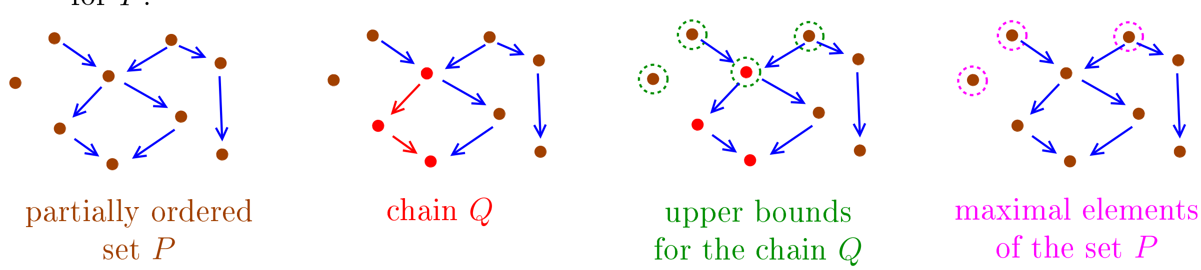


FIGURE 2

(2) We consider $P = [0, \infty)$ with the usual partial ordering. Every subset is a chain, but not every subset has an upper bound. For example P does not have an upper bound.

[4]Note that if $a < a'$, then we do not demand that $b \leq b'$.

(3) We consider $P = (-\infty, 0]$ with the usual partial ordering. Every subset is a chain and 0 is an upper bound for every subset furthermore it is a maximal element of P.

Next we formulate Zorn's Lemma. Its modest name conceals its true importance in mathematics.

Lemma 1.5. (Zorn's Lemma) *Suppose a partially ordered set $(P, \leq)$ has the property that every chain has an upper bound in P. Then the set P contains at least one maximal element.*

Remark.

(1) Zorn's Lemma even made it into "The Simpsons", see

`http://www.simonsingh.net/Simpsons_Mathematics/zorns-lemma/`

(2) The lemma is named after Max Zorn [**Zo35**] who published a proof in 1935. In fact the same statement was already proved by Casimir Kuratowski [**Kura22**] in 1922.

Proof. A nice discussion of the proof and its historical context is given in [**Sti13**, Section 7.2]. Alternatively see also [**Cie97**, Theorem 4.3.4] and [**Je73**, p. 9] for a proof. In fact, as is shown in [**Sti13**, p. 154] or alternatively [**Je73**, Theorem 2.1], Zorn's Lemma is equivalent to the Axiom of Choice. ■

We continue with the next adjective that we can associate to a relation.

Definition.

(1) A relation "$\leq$" on a set S is called *connex* if for every $a, b \in S$ we have $a \leq b$ or $b \leq a$.
(2) A partial order that is connex is called a *total order.*
(3) A set together with a total order is called a *totally ordered set.*

Examples.

(1) Evidently the usual "$\leq$"-relation on $\mathbb{R}$ is connex.
(2) None of the partial orders considered in Figure 1 is connex.
(3) Given two totally ordered sets $(A, \leq)$ and $(B, \leq)$ one can show easily that the lexicographic order on $A \times B$ is again a total order.

Finally we introduce one last type of partial orders.

Definition.

(1) A total order $\leq$ on a set S is called a *well-order* if every non-empty subset has a minimal for all $u \in T$.
(2) A set together with a well-order is called a *well-ordered set.*

Examples.

(1) The usual order on $\mathbb{Q}$ is a total order but it is not a well-order, for example the subset $T := \{x \in \mathbb{Q} \,|\, x^2 > 2\}$ does not have a minimal element. But since $\mathbb{Q}$ is in bijection to $\mathbb{N}$ we see that $\mathbb{Q}$ admits a well-order.
(2) The usual order on $\mathbb{R}$ is a total order but it is not a well-order. It is a priori not so clear how one can define a well-order on $\mathbb{R}$.

(3) Given two well-ordered sets $(A, \leq)$ and $(B, \leq)$ one can again show easily that the lexicographic order on $A \times B$ is a well-order.

Perhaps somewhat shockingly the following theorem, due to Ernst Zermelo, says that every set admits a well-ordering.

Theorem 1.6. Well-ordering Theorem *Every set admits a well-order.*

Proof. We refer to [**Cie97**, Theorem 4.3.3] or alternatively to [**Je73**, p. 9] for a proof. In fact, as for Zorn's Lemma it is shown in [**Je73**, Theorem 2.1] that the Well-ordering Theorem is equivalent to the Axiom of Choice. ■

1.3. **The cardinality of sets.** In this section we introduce the notion of the cardinality of a set.

Definition.

(1) We say that two given sets S and T are *equinumerous* if there exists a bijection $\varphi\colon S \to T$.[5]
(2) Given a set S we define the *cardinality $\#S$ of S* as the equinumerosity equivalence class of S in the class of all sets. Put differently we have

$$\#S \;:=\; \text{class of all sets } T \text{ that are equinumerous with } S.$$

(3) If S is a finite set, then we write $\#S = n$ for the unique $n \in \mathbb{N}_0$ which satisfies $\#S = \#\{1, \ldots, n\}$. If S infinite, then sometimes we write $\#S = \infty$. At times we also write $|S| \in \mathbb{N}_0 \cup \{\infty\}$ instead of $\#S \in \mathbb{N}_0 \cup \{\infty\}$.
(4) Let S be a set. If $\#S = n$ for some $n \in \mathbb{N}_0$ or $\#S = \#\mathbb{N}$, then we say that S is *countable*, otherwise we say that S is *uncountable*.

Example. We have $\#(\mathbb{R} \setminus \{0\}) = \#\mathbb{R}$. Indeed a bijection is given by

$$\begin{aligned} \mathbb{R} \setminus \{0\} &\to \mathbb{R} \\ x &\mapsto \begin{cases} x, & \text{if } x \in \mathbb{R} \setminus \mathbb{N}_0, \\ x-1, & \text{if } x \in \mathbb{N}. \end{cases} \end{aligned}$$

Remark.

(1) Another way of putting the definition of cardinality is that given two sets S and T we have $\#S = \#T$ if and only if there exists a bijection from S to T.
(2) In almost any proper course on real analysis it is shown that $\mathbb{Q}$ is countable and that $\mathbb{R}$ is not countable. Thus we $\#\mathbb{Q} \neq \#\mathbb{R}$. This example also shows that the above notation $\#S = \infty$ for an infinite set S is rather dangerous, since we have $\#\mathbb{Q} = \infty$ and $\#\mathbb{R} = \infty$ even though $\#\mathbb{Q} \neq \#\mathbb{R}$.

On several occasions we will use the following lemma.

Lemma 1.7.

(1) *Every subset of a countable set is again countable.*

[5]It follows immediately from the definitions that being equinumerous defines an equivalence relation on the class of all sets.

(2) *Let S be a countable set and let $f\colon S \to T$ be a map. The image $f(S)$ is again countable.*
(3) *The union of countably many countable sets is again countable.*
(4) *Let T be a set and let S be a subset such that $T \setminus S$ is countable. Then S is countable if and only if T is countable.*
(5) *The product of finitely many countable sets is again countable.*
(6) *If S is a finite set and if T is a countable set, then the set of maps from S to T is also countable.*

Sketch of proof.

(1) After excluding the case that S is finite and the case that T is finite one sees that one has to prove the following: given any infinite subset T of $\mathbb{N}$ there exists a bijection $\varphi\colon \mathbb{N} \to T$. Such a bijection is given by defining $\varphi(i)$ as the i-th smallest element of T.
(2) Let S be a countable set and let $f\colon S \to T$ be a map. We only need to consider the case that $S = \mathbb{N}$ and that $f(S)$ is infinite. In this case the bijection $\varphi\colon \mathbb{N} \to f(S)$ is given by defining $\varphi(1) = f(1)$ and by defining iteratively
$$\varphi(i+1) \;:=\; \left\{ f(k) \;\middle|\; \begin{array}{c} k \in \mathbb{N} \text{ is the smallest number such} \\ \text{that } f(k) \notin \{\varphi(1), \dots, \varphi(i)\} \end{array} \right\}.$$
(3) It is fairly straightforward, albeit notationally slightly messy to give a direct proof. For completeness we refer to [**Cie97**, Corollary 5.2.6] for a proof.
(4) This statement follows immediately from (2) and (3).
(5) By induction and by ignoring the rather dull case of finite sets it remains to show that $\mathbb{N}_0 \times \mathbb{N}_0$ is countable. This can be shown by the well-known argument sketched in Figure 3.
(6) Let $S = \{v_1, \dots, v_n\}$ be a finite set and let T be a countable set. We have an obvious bijection between T^n and the set of maps from S to T. Thus it follows from (4) that the set of maps from S to T is countable. ■

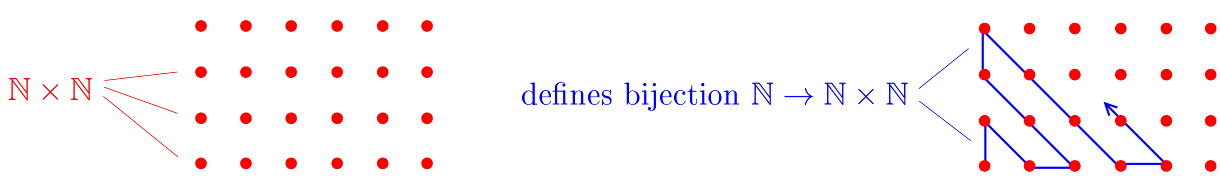


Figure 3. Illustration for the proof of Lemma 1.7.

We move on to the next definition.

Definition. Let S and T be two sets.
(1) We write $\#S \le \#T$ if there exists a monomorphism $\varphi\colon S \to T$.
(2) We write $S < T$ if $\#S \le \#T$ but $\#S \neq \#T$.

Examples.
(1) We have $\#\mathbb{Q} \le \#\mathbb{R}$ but we had just pointed out that $\#\mathbb{Q} \neq \#\mathbb{R}$ since $\#\mathbb{Q} = \#\mathbb{N}$ but $\#\mathbb{R} \neq \#\mathbb{N}$. Therefore $\#\mathbb{Q} < \#\mathbb{R}$.
(2) It follows from the previous remark that a set S is countable if and only if $\#S \le \#\mathbb{N}$.

(3) Given a set X we denote by $P(X)$ its power set, i.e. the set of all subsets of X. For every X we have $\#X < \#P(X)$.[6] Indeed, the map

$$\begin{array}{rcl} X & \to & P(X) \\ x & \mapsto & \{x\} \end{array}$$

is an injection, hence $\#X \leq \#P(X)$. Now let $f\colon X \to P(X)$ be a map. We have to show that it cannot be surjective. Indeed, the set

$$Y \;:=\; \{x \in X \mid x \notin f(x)\}$$

is not in the range of f: if there did exist a $z \in X$ with $f(z) = Y$, then $z \in Y$ if and only if $z \notin Y$, which is a contradiction.

Now we can formulate the final result of this chapter.

Theorem 1.8. (Bernstein-Schröder)[7] *Let S and T be two sets. Then*

$$\#S \leq \#T \text{ and } \#T \leq \#S \quad \Longrightarrow \quad \#S = \#T.$$

Example. Let U be a non-empty subset of $\mathbb{R}$ which contains an open interval (a,b) with $a < b$. Then $\#U = \#\mathbb{R}$. Indeed, the inclusion $U \to \mathbb{R}$ is of course injective, thus we see that $\#U \leq \#\mathbb{R}$. Furthermore the map $\varphi\colon (a,b) \to \mathbb{R}$ given by $x \mapsto \frac{x}{(x-a)(x-b)}$ defines a bijection between (a,b) and $\mathbb{R}$. Thus $\mathbb{R} \xrightarrow{\varphi^{-1}} (a,b) \to U$ is also an injection. This means that $\#\mathbb{R} \leq \#U$. Therefore it follows from the Bernstein-Schröder-Theorem that there exists a bijection from U to $\mathbb{R}$.

Sketch of proof $(*)$. Let $f\colon A \to B$ and $g\colon B \to A$ be two injective maps between two sets A and B. Without loss of generality we can assume that A and B are disjoint.[8] We denote by $\sim$ the equivalence relation on $A \sqcup B$ that is generated by $a \sim f(a)$ for $a \in A$ and by $g \sim g(b)$ for $b \in B$. Note that if $a \in A$ lies in the image of g, then by the injectivity of f there exists a unique element $g^{-1}(a) \in B$ and we have $a \sim g^{-1}(a)$. The same remark works with the roles of f and g reversed.

Since $A \sqcup B$ is the disjoint union of all equivalence classes it suffices to prove the following claim.[9]

Claim. For each equivalence class I there exists a bijection $A \cap I \to B \cap I$.

---

[6] This statement is known as Cantor's Theorem.

[7] The theorem was initially stated by Cantor without a proof and it was first proved by Felix Bernstein (1878-1956) and Ernst Schröder (1841-1902) in 1896. Sometimes it is also called the Cantor-Bernstein-Schröder theorem.

[8] If A and B are not disjoint, then we replace A by $A' = A \times \{0\}$ and B by $B' = B \times \{1\}$, then A' and B' are disjoint, since any two elements in A' and B' now have different second coordinate.

[9] Indeed, recall that equivalence classes always give a disjoint decomposition of a set. In our case, if I_j, $j \in J$ denote the set of equivalence classes of $\sim$ on $A \sqcup B$, then we have

$$A \sqcup B \;=\; \bigsqcup_{j \in J} I_j \quad \text{and therefore also} \quad A \;=\; \bigsqcup_{j \in J} (A \cap I_j) \text{ and } B \;=\; \bigsqcup_{j \in J} (B \cap I_j).$$

Therefore it suffices to give a bijection $A \cap I_j \to B \cap I_j$ for every $j \in J$.

So let I be an equivalence class and let $a \in A \cap I$.[10] The subset $I \subset A \sqcup B$ consists of all elements equivalent to a. The equivalence class I is then of one of the following three types:[11]

$$\begin{array}{llllllllll}
 & & & A & B & A & B & A & B & \\
(1) & a' & \ldots & f^{-1}(g^{-1}(a)), & g^{-1}(a), & a, & f(a), & g(f(a)), & f(g(f(a))), & \ldots \text{ with } a' \in A \setminus g(B) \\
(2) & b' & \ldots & f^{-1}(g^{-1}(a)), & g^{-1}(a), & a, & f(a), & g(f(a)), & f(g(f(a))), & \ldots \text{ with } b' \in B \setminus f(A) \\
(3) & \ldots & \ldots & f^{-1}(g^{-1}(a)), & g^{-1}(a), & a, & f(a), & g(f(a)), & f(g(f(a))), & \ldots
\end{array}$$

In case (1) the map f defines a bijection $A \cap I \to B \cap I$, in case (2) the map g^{-1} defines a bijection $A \cap I \to B \cap I$ and in case (3) both f and g^{-1} define a bijection $A \cap I \to B \cap I$. ■

Remark. The *continuum hypothesis*, first formulated by Cantor in 1878, states that there exists no set S with $\#\mathbb{Q} < \#S < \#\mathbb{R}$. Any attempt at proving this statement relies on first of all making precise what "sets" are supposed to be or what axioms they are supposed to satisfy. As wikipedia writes "the answer to this problem is independent of ZFC set theory (that is, Zermelo-Fraenkel set theory with the axiom of choice included), so that either the continuum hypothesis or its negation can be added as an axiom to ZFC set theory, with the resulting theory being consistent if and only if ZFC is consistent." This independence was proved in 1963 by Paul Cohen for which he obtained a Fields medal in 1966. More useful information can as always be found on wikipedia:

`https://en.wikipedia.org/wiki/Continuum_hypothesis`

1.4. **Conventions for (linear) algebra.** Next we want to introduce the category of rings. Since the definition of a "ring" in the literature can differ considerably let us state precisely what we mean by a ring in these notes.

Definition.

(1) A *ring* is a triple $(R, +: R \times R \to R,\ \cdot : R \times R \to R)$ where $(R,+)$ is an abelian group and such that the following conditions are satisfied:
  (a) the multiplication "$\cdot$" is associative,
  (b) the multiplication "$\cdot$" is distributive,
  (c) there exists a multiplicatively neutral element $1 = 1_R$.
(2) A *ring homomorphism* between two rings $(R, +, \cdot)$ and $(S, +, \cdot)$ is a map $R \to S$ that preserves the additive and multiplicative structures and that sends 1_R to 1_S.
(3) A *field* is a commutative ring such that every non-zero element has a multiplicative inverse.

Remark. Recall that "associative" means that for any $a, b, c \in R$ we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. It is well-known that this means that we can unambiguously define the product of finitely many elements using any choice of parentheses. Even though this statement is completely

[10] There always exists an $a \in A \cap I$. Indeed, I is by definition non-empty. If it contains an element in A we are done. Otherwise it contains an element in $b \in B$, but then it also contains $g(b) \in A$, since $b \sim g(b)$.

[11] Note that we do not claim that the elements in the given sequences are distinct, for example $f: A \to B$ could be a bijection and $g = f^{-1}$ its inverse, then the equivalence class consists only of the two elements a and $g(a)$.

familiar, if one thinks about it, one realizes that this requires a proof. Such a proof is for example provided in [**DF04**, p. 19].

We move on to the next definition.

Definition. Let R be a commutative ring, let V and W be two free R-modules with finite bases $\mathcal{B} = \{b_1, \ldots, b_m\}$ and $\mathcal{C} = \{c_1, \ldots, c_n\}$ and let $\varphi \colon V \to W$ be an R-homomorphism. We refer to the unique $(n \times m)$-matrix $P = (p_{ij})$ with $\varphi(v_i) = \sum_{j=1}^{n} p_{ji} \cdot w_j$ as the *matrix representing φ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.*

The previous definition is the standard definition well-known from earlier courses in linear algebra. Towards the end of the notes we will work with module homomorphisms over non-commutative rings. At that point we will see that at times it is convenient to switch to a different convention.

We conclude this chapter with the following definition.

Definition. Let R be a commutative domain. Given an R-module M we define

$$\operatorname{rank}(M) \quad := \quad \dim_{K(R)}(M \underset{\substack{\uparrow \\ \text{tensor product over } R \text{ as defined in Section ??}}}{\otimes_R} K(R))$$

where $K(R)$ denotes the quotient field of R.

Exercises for Chapter 1.

Exercise 1.1. Let $(V, \leq)$ be a totally ordered set of cardinality $n \in \mathbb{N}$. We equip the set $\{1, \ldots, n\}$ with the usual total order $\leq$. Show that there exists a *unique* order-preserving bijection $\{1, \ldots, n\} \to V$.

Exercise 1.2. Let $f \colon X \to Y$ be a map between two sets. Show that there exist total orders "$\leq_X$" on X and "$\leq_Y$" on Y such that the map f is order-preserving.

Exercise 1.3. Show that there exists an uncountable well-ordered set $(X, \leq)$ such that for every $y \in X$ the set $\{x \in X \mid x \leq y\}$ is countable.

Exercise 1.4. We consider $\mathbb{N}$ and $\mathbb{Z}$ with the usual partial order $\leq$.

(a) Does the lexicographically ordered set $\mathbb{N} \times \mathbb{N}$ have the same order type as $\mathbb{N}$?
(b) Does the lexicographically ordered set $\mathbb{Z} \times \mathbb{Z}$ have the same order type as $\mathbb{Z}$?

Exercise 1.5. We consider the half-open interval $[0, 1)$, the set $\mathbb{R}_{\geq 0}$ and the set $\mathbb{N}_0$, each equipped with the usual partial order $\leq$.

(a) Show that the lexicographically ordered set $[0, 1) \times \mathbb{N}_0$ has the same order type as $\mathbb{R}_{\geq 0}$.
(b) Show that the lexicographically ordered set $\mathbb{N}_0 \times [0, 1)$ does not have the same order type as $\mathbb{R}_{\geq 0}$.

Exercise 1.6.

(a) Let $n \in \mathbb{N}$. Show that $\mathbb{R}$ and $\mathbb{R}^n$ have the same cardinality.
(b) Show that $\mathbb{R}$ and $\mathbb{R}^{\mathbb{N}}$ have the same cardinality.

2. Topological spaces: basic definitions and properties

In this chapter we give the definition of topological spaces, we provide many examples and we discuss a long list of basic properties. Many of these results might well be familiar to the reader. Nonetheless we attempt to give a fairly complete list of basic statements so that later on we have a firm common foundation to work with.

2.1. **The definition of a topological space.** In this section we give the definition of a topological space and we discuss some very basic definitions and statements. Presumably most readers will have seen this material. Throughout these lecture notes, in the proofs, we strive for giving precise references to earlier results. The main exceptions are the lemmas from this Section 2.1. Usually we use them without referring to them explicitly.

The following definition will keep us occupied for the rest of these notes.

Definition.

(1) Let X be a set. A *topology on* X is a subset $\mathcal{T}$ of $\mathcal{P}(X)$[12] with the following properties:
   (a) the empty set and the entire set X lie in $\mathcal{T}$,
   (b) the intersection of *finitely many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$,
   (c) the union of *arbitrarily many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$.

   The sets in $\mathcal{T}$ are called *open with respect to* $\mathcal{T}$. If $\mathcal{T}$ is understood from the context, then we just say *open*.
(2) A *topological space* is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a *topology on* X.

Remark.

(1) Usually we suppress the topology from the notation, e.g. usually we just write "let X be a topological space".
(2) If X is a topological space, then the properties of a topology imply immediately the following three statements:
   (a) the empty set and the entire set X are open,
   (b) the intersection of *finitely many* open sets is again open,
   (c) the union of *arbitrarily many* open sets is again open.
(3) It took mathematicians several decades to settle on the above definition, the long and tortuous history of the definition of a topological space is presented in [**Moo08**].

We start out with some very basic examples of topological spaces.

Examples.

(1) Let X be a set, then $\mathcal{T} = \{\varnothing, X\}$ is a topology on X. This topology is sometimes called the *trivial topology on* X. Other names that are used are the *indiscrete topology* and the *chaotic topology*.
(2) Let X be a set. Then $\mathcal{T} = \mathcal{P}(X)$ is also a topology on X. In other words, $\mathcal{T}$ is the topology such that every subset of X is open. This topology is usually referred to as the *discrete topology on* X. Sometimes we say that a topological space is discrete, if the topology is indeed the discrete topology.

[12]In more humble words, $\mathcal{T}$ is a set of subsets of X.

(3) Let X be a set.
   (a) We define $\mathcal{T} \subset \mathcal{P}(X)$ as follows:

$$U \in \mathcal{T} \;:\Longleftrightarrow\; \text{either } U = \varnothing \text{ or } U \text{ is the complement of a finite subset of } X.$$

   It follows easily from de Morgan's laws, stated in Lemma 1.2, that $\mathcal{T}$ is a topology on X. We refer to it as the *cofinite topology*. Sometimes this topology is also called the *finite-complement topology*.
   (b) Similarly we can define the *cocountable topology* $\mathcal{T} \subset \mathcal{P}(X)$ as follows:

$$U \in \mathcal{T} \;:\Longleftrightarrow\; \text{either } U = \varnothing \text{ or } U \text{ is the complement of a countable subset of } X.$$

   It follows from de Morgan's laws together with Lemma 1.7 that this is a topology on X.

(4) Let X be the empty set. Together with $\mathcal{T} = \{\varnothing\}$ this is a topological space, called the *empty topological space*. It is a never ending source of counterexamples to carelessly formulated statements.

(5) (a) Let $X = \{*\}$ be a set with a single element $*$. It follows immediately from the definition of a topology there is a unique topology on $X = \{*\}$, namely the topology given by $\mathcal{T} = \{\varnothing, \{*\}\}$. Together with the empty set this is the only case where the topology is already determined by the set.
   (b) Given a set X with 2 elements it is straightforward to see that there exist four distinct topologies on X.

   Not surprisingly the number of topologies on a finite set grows rapidly with the cardinality of the set, we refer to **[KR70, BMc02]** for details.

Surely some of the most interesting examples of topological spaces arise from metric spaces. For completeness' sake we recall the definition of metric spaces.

Definition. A *metric space* is a pair (X, d) consisting of a set X and a *metric* d on X, i.e. a map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \,|\, x \geq 0\} \\ (x, y) & \mapsto & d(x, y), \end{array}$$

with the following properties:

(1) $d(x, y) = 0$ if and only if $x = y$,
(2) for all $x, y \in X$ we have

$$d(x, y) \;=\; d(y, x) \qquad \text{(symmetry)}$$

(3) for all $x, y, z \in X$ we have

$$d(x, z) \;\leq\; d(x, y) + d(y, z) \qquad \text{(triangle inequality)}.$$

Sometimes we refer to $d(x, y)$ as the *distance between x and y*.

Example. In the following we consider $\mathbb{R}^n$ as a metric space with the *Euclidean metric* given by

$$d(x, y) \;:=\; \|x - y\| \quad \text{where} \quad \|z\| \;=\; \|(z_1, \ldots, z_n)\| \;:=\; \sqrt{z_1^2 + \cdots + z_n^2}\,.$$

In Exercise 2.4 we will equip $\mathbb{R}^n$ with two other natural metrics, namely the Manhattan metric and the SNCF metric.

The following lemma shows that metric spaces give rise to topological spaces.

Lemma 2.1. *Let (X, d) be a metric space. A subset U of X is called* open *if for every $x \in U$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) := \{y \in X \mid d(x, y) < \epsilon\}$ is contained in U. These open sets define a topology on X.*

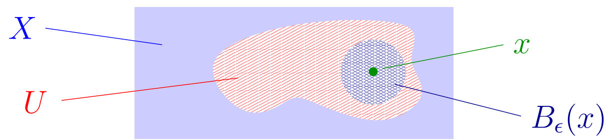


FIGURE 4. Illustration of Lemma 2.1.

PROOF.

(1) It is clear that the empty set and X itself are open.
(2) Suppose $U_1, \dots, U_k$ are open subsets of X. We need to show that the intersection $U_1 \cap U_2 \cap \dots \cap U_k$ is also open. Thus let $x \in U_1 \cap U_2 \cap \dots \cap U_k$. For each $i \in \{1, \dots, k\}$ there exists an $\epsilon_i > 0$ with $B_{\epsilon_i}(x) \subset U_i$. We set $\epsilon := \min\{\epsilon_1, \dots, \epsilon_k\}$. Then
$$B_\epsilon(x) \underset{\substack{\uparrow \\ \text{since } \epsilon \le \epsilon_i}}{\subset} B_{\epsilon_1}(x) \cap \dots \cap B_{\epsilon_k}(x) \underset{\substack{\uparrow \\ \text{since } B_{\epsilon_i}(x) \subset U_i}}{\subset} U_1 \cap U_2 \cap \dots \cap U_k.$$
(3) Let $\{U_i\}_{i \in I}$ be a family of open sets. We want to show that $\bigcup_{i \in I} U_i$ is also open. Thus let $x \in \bigcup_{i \in I} U_i$. This means that there exists an $i \in I$ with $x \in U_i$. Since U_i is open there exists an $\epsilon > 0$ with $B_\epsilon(x) \subset U_i$. Thus we have
$$B_\epsilon(x) \subset U_i \subset \bigcup_{i \in I} U_i. \qquad \blacksquare$$

Convention. Given a metric space (X, d) we always view X as a topological space with the topology defined in Lemma 2.1. In particular, unless we say explicitly otherwise, we view $\mathbb{R}^n$ with the topology arising from the Euclidean metric.

We proceed with the next definition.

Definition. Let X be a topological space. We say a subset A of X is *closed* if the complement $X \setminus A$ is open.

The following lemma is an immediate consequence of the definition of a topology and de Morgan's laws, see Lemma 1.2. The lemma mirrors the remark on page 83.

Lemma 2.2. *Let X be a topological space.*

(1) *the empty set and the entire set X are closed,*
(2) *the intersection of* arbitrarily many *closed sets is again closed,*
(3) *the union of* finitely many *closed sets is again closed.*

Remark. Let X be a set. If we specify what we mean by closed sets such that the three properties of Lemma 2.2 are satisfied, then it follows once again from de Morgan's laws that the complements of the closed sets define a topology.

The following proposition gives us important examples of closed and open subsets.

Proposition 2.3. *Let (X,d) be a metric space, let $y \in X$ and let $r \in \mathbb{R}$.*

(1) *the set $B_r(y) = \{x \in X \mid d(x,y) < r\}$ is open in X,*
(2) *the set $\overline{B}_r(y) = \{x \in X \mid d(x,y) \le r\}$ is closed in X,*
(3) *the set $S_r(y) = \{x \in X \mid d(x,y) = r\}$ is closed in X.*

Proof. Let (X,d) be a metric space, let $y \in X$ and let $r \in \mathbb{R}$.

(1) Let $x \in B_r(y)$. We set $s := d(x,y)$. By definition of $B_r(y)$ we have $s < r$. We set $t := r - s$. Since $t > 0$ it suffices to show that $B_t(x) \subset B_r(y)$. Thus let $z \in B_t(x)$. We see that

$$d(y,z) \underset{\substack{\uparrow \\ \text{triangle inequality}}}{\le} d(x,y) + d(x,z) < s + t = r.$$

Thus we see that $z \in B_r(y)$, which is what we needed to show. We refer to Figure 5 for an illustration.
(2) Almost the same argument as in (1) shows that $X \setminus \overline{B}_r(y)$ is open, which implies that $\overline{B}_r(y)$ is closed.
(3) We have $X \setminus S_r(y) = B_r(y) \cup (X \setminus \overline{B}_r(y))$. By (1) and (2) we see that $X \setminus S_r(y)$ is the union of two open sets, thus by the third property of a topological space we see that $X \setminus S_r(y)$ is open, thus $S_r(y)$ is indeed closed. ■

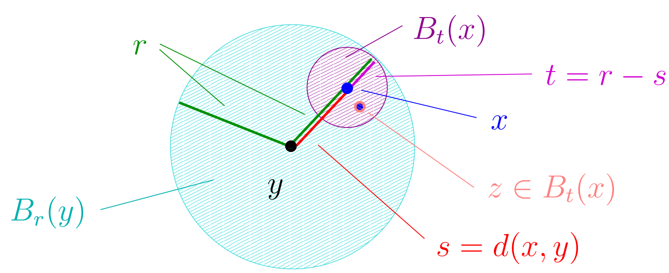


Figure 5

The most frequent application of Proposition 2.3 is to the metric space $\mathbb{R}^n$. Thus we introduce the following notation.

Notation. Let $n \in \mathbb{N}_0$. Given $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$ we write[13]

$$\begin{array}{rcll} B^n_r(y) & := & \{x \in \mathbb{R}^n \mid \|x-y\| < r\} & \text{called } \textit{open } n\textit{-ball} \\ \overline{B}^n_r(y) & := & \{x \in \mathbb{R}^n \mid \|x-y\| \le r\} & \text{called } \textit{closed } n\textit{-ball} \\ S^{n-1}_r(y) & := & \{x \in \mathbb{R}^n \mid \|x-y\| = r\} & \text{called } (n-1)\textit{-sphere.} \end{array}$$

We adopt the following conventions:

(1) Not surprisingly a 2-ball is often called a *disk* and a 1-sphere is usually referred to as a circle.
(2) If the parameter n is clear from the context we sometimes drop it from the notation.
(3) If $y = 0$, then we drop y from the notation and if $r = 1$, then we drop r from the notation.
(4) Sometimes we write rB^n instead of $B_r^n(0)$ and sometimes we write $r\overline{B}^n$ instead of $\overline{B}_r^n(0)$.
(5) We refer to the point $(0, \dots, 0, 1) \in S^n = \{x \in \mathbb{R}^{n+1} \,|\, \|x\| = 1\}$ as the *North Pole* of S^n and we refer to the point $(0, \dots, 0, -1) \in S^n = \{x \in \mathbb{R}^{n+1} \,|\, \|x\| = 1\}$ as the *South Pole* of S^n.

We sketch B^1, B^2, S^1 and S^2 in Figure 6. Of course one cannot easily sketch the 3-sphere S^3. But it is worth pointing out that in some form it has been around for a while. In [**Lip14**, Chapter 2] and [**Petn79**] the argument is made that the universe in Dante's Divina Commedia is precisely the 3-sphere.

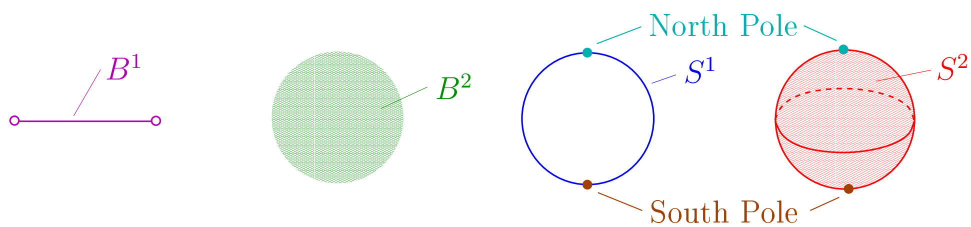


FIGURE 6

The next definition gives us a rich source of new examples of topological spaces.

Definition. Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$ be a subset. It follows almost immediately from Lemma 1.1 that

$$\mathcal{S} \;:=\; \{Y \cap U \,|\, U \in \mathcal{T}\}$$

is a topology on Y. We refer to $\mathcal{S}$ as the *subspace topology* on Y.

Convention.

(1) Unless we say something else, we consider each subset Y of $\mathbb{R}^n$ always as a topological space with respect to the subspace topology.
(2) Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$ be a subset. If Y is indeed equipped with the subspace topology, then sometimes we refer to Y as a *subspace* of X.

We continue with the following useful lemma.

Lemma 2.4. *Let X be a topological space and let $Y \subset X$ be a subset. We equip Y with the subspace topology.*

(1) *A set $V \subset Y$ is open in Y if and only if there exists an open subset $U \subset X$ with $V = U \cap Y$.*

[13] These definitions also make sense for $r < 0$, in that case all of the sets are the empty set. Also note that for $n = 0$ we obtain that $S_r^{n-1}(y) = S_r^{-1}(y) = \varnothing$.

(2) *A set $B \subset Y$ is closed in Y if and only if there exists a closed subset $A \subset X$ with $B = A \cap Y$.*

Proof (∗). [14] The first statement is just a reformulation of the definition of the subspace topology. We turn to the proof of the second statement. Now let $B \subset Y$. Elementary set theory shows that for any subset U of X we have

$$(*) \qquad U \cap Y = Y \setminus B \quad \Longleftrightarrow \quad (X \setminus U) \cap Y = B.$$

We refer to Figure 7 for an illustration. Now the second statement is basically obvious. Indeed, we have

$$\begin{aligned} B \text{ is closed in } Y \quad &\Longleftrightarrow \quad Y \setminus B \text{ is open in } Y \\ &\Longleftrightarrow \quad \text{there exists } U \subset Y \text{ open with } U \cap Y = Y \setminus B \\ &\underset{\text{by } (*)}{\Longleftrightarrow} \quad \text{there exists } U \subset Y \text{ open with } (X \setminus U) \cap Y = B \\ &\Longleftrightarrow \quad \text{there exists } A \subset Y \text{ closed with } A \cap Y = B. \end{aligned}$$

We have thus proved the desired statement. ■

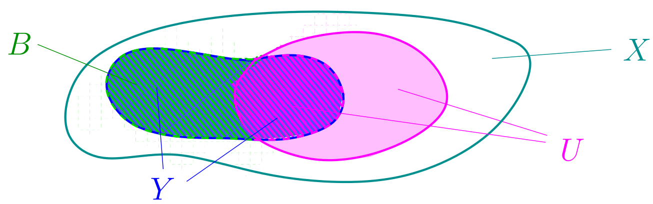


Figure 7. Illustration for the proof of Lemma 2.4.

Example. We consider $Y = (-1, 2]$ equipped with the subspace topology coming from $\mathbb{R}$.

(1) It follows from Lemma 2.4 that $V = (0, 2]$ is an open subset of Y, since we can write $V = (0, 4) \cap Y$ and $(0, 4)$ is an open subset of $\mathbb{R}$.

(2) It follows from Lemma 2.4 that $B = (-1, 0]$ is a closed subset of Y, since we can write $B = [-3, 0] \cap Y$ and $[-3, 0]$ is a closed subset of $\mathbb{R}$.

These two examples are illustrated in Figure 8.

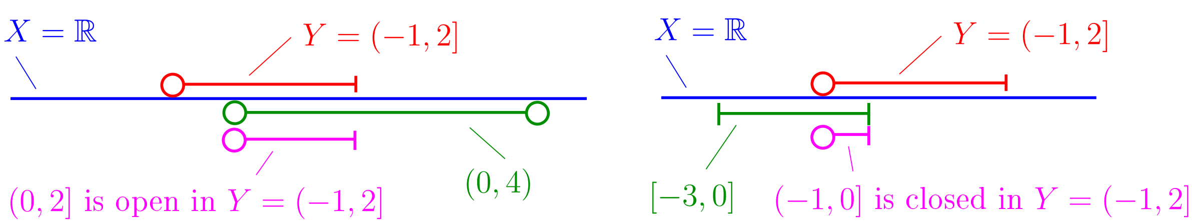


Figure 8

We continue with the next basic definition.

[14] We indicate proofs and statements that can safely be skipped at a first reading of these notes by the symbol (∗).

Definition. Let X be a topological space.

(1) Let $A \subset X$ be a subset. We say $U \subset X$ is a *neighborhood of A* if there exists an open set V such that $A \subset V \subset U$. We say U is an *open neighborhood of A*, if U is furthermore open. This definition is illustrated in Figure 9.
(2) An (open) neighborhood of a point $x \in X$ is an (open) neighborhood of $\{x\}$.

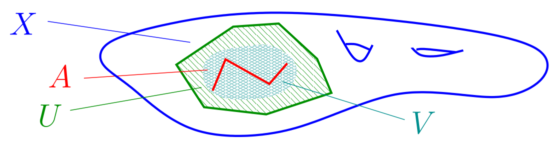


FIGURE 9

Remark. Some books, see e.g. [**Mun75**, p. 96], demand that a neighborhood is open. Our definition of a neighborhood agrees with the definition in most books, e.g. [**Bre93, Hat02, Jä05**].

Example. If $X = \mathbb{R}$ and $A = [0, 2)$, then $U = (-1, 3]$ and $V = (-2, \infty)$ are neighborhoods of A in X.

The following lemma gives a useful criterion for showing that a given subset is open.

Lemma 2.5. *Let X be a topological space and let $U \subset X$ be a subset. If given any $x \in U$ there exists a neighborhood V of x that is contained in U, then U is open.*

PROOF $(*)$. Let X be a topological space and let $U \subset X$ be a subset. Suppose that given any $x \in U$ there exists a neighborhood V_x of x that is contained in U. By definition of a neighborhood there exists for each $x \in U$ an open subset W_x with $x \in W_x \subset V_x$. We see that

$$U \;=\; \bigcup_{x \in U} \{x\} \;\subset\; \bigcup_{x \in U} W_x \;\underset{\substack{\uparrow \\ \text{since each } W_x \text{ is contained in } U}}{\subset}\; U.$$

It follows that $U = \bigcup\limits_{x \in U} W_x$, i.e. U is the union of open subsets of X, thus by the third property of a topology, U itself is open. ■

On many occasions it will be useful to "break" a topological space into smaller pieces. This leads us to the following definition.

Definition. Let X be a topological space.

(1) A *cover* of a subset $A \subset X$ is a family $\{U_i\}_{i \in I}$ of subsets of X with $A \subset \bigcup\limits_{i \in I} U_i$.
(2) We say a cover $\{U_i\}_{i \in I}$ is *open* if each U_i is open. Similarly we define a *closed cover*.
(3) We say a cover $\{U_i\}_{i \in I}$ is *finite* if I is finite. Otherwise we say that the cover is *infinite*.
(4) We say a cover $\{U_i\}_{i \in I}$ is *locally finite* if each $x \in X$ admits an open neighborhood W such that $\{i \in I \,|\, U_i \cap W \neq \varnothing\}$ is finite.

Remark. What we call a "cover" is often also called a "covering". We will reserve the name "covering" for a completely different concept.

Example. In Figure 10 to the right we sketch a closed cover $\{U_i\}_{i\in I}$ of $X=\mathbb{R}^2$ which has the property that each $x\in X$ is contained in only finitely many U_i, but which nonetheless fails to be locally finite.

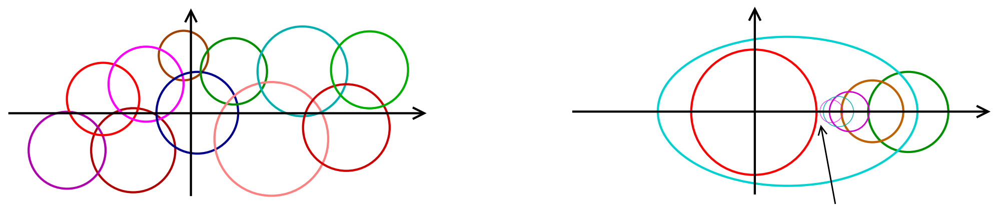


FIGURE 10

The following lemma is crucial for many arguments.

Lemma 2.6. *Let X be a topological space and let M be a subset of X.*

(1) *Let $\{U_i\}_{i\in I}$ be an open cover of X. The following two statements hold:*
   (a) *M is open if and only if each intersection $M\cap U_i$ is open in U_i,*
   (b) *M is closed if and only if each intersection $M\cap U_i$ is closed in U_i.*
(2) *Let $\{A_i\}_{i\in I}$ be a locally finite closed cover of X. (In most applications we will deal with a finite closed cover.) The following two statements hold:*
   (a) *M is open if and only if each intersection $M\cap A_i$ is open in A_i,*
   (b) *M is closed if and only if each intersection $M\cap A_i$ is closed in A_i.*

Remark. Note that in Lemma 2.6 we allow an arbitrary family of open subsets but we only allow a finite family of closed subsets. One can easily see that this restriction is necessary. For example, consider $X=\mathbb{R}$ and the infinite family $\{A_i\}_{i\in I}$ of closed subsets of $X=\mathbb{R}$ given by all subsets consisting of a single point. Let $M\subset\mathbb{R}$ be a subset. For each $i\in I$ the intersection $M\cap A_i$ is either the empty set or all of A_i, in particular the intersection $M\cap A_i$ is closed in A_i. But of course there is no reason why M should be a closed subset of $\mathbb{R}$.

PROOF. Let X be a topological space and let M be a subset of X.

(1) Let $\{U_i\}_{i\in I}$ be an open cover of X. The "only if"-directions of (a) and (b) are an immediate consequence of Lemma 2.4. It remains to prove the "if"-directions.
   (a) Suppose that for every $i\in I$ the intersection $M\cap U_i$ is an open subset of U_i. By definition of the subspace topology this means that for each $i\in I$ there exists an open subset Z_i of X with $Z_i\cap U_i=M\cap U_i$. We have
$$M \underset{\substack{\uparrow\\ \text{since } \bigcup_{i\in I} U_i = X}}{=} \bigcup_{i\in I}(M\cap U_i) = \bigcup_{i\in I}\overbrace{(Z_i\cap U_i)}^{\text{intersection of open subsets of } X\text{, hence open in } X} \underset{\substack{\uparrow\\ \text{union of open subsets, hence open}}}{=} \text{open subset of } X.$$
   (b) Now suppose that each intersection $M\cap U_i$ is closed in U_i. In other words, each set $U_i\setminus(M\cap U_i)$ is open in U_i. Since $(X\setminus M)\cap U_i=U_i\setminus(M\cap U_i)$ we obtain from (a) that $X\setminus M$ is open in X. In other words, M is closed in X.

(2) First note that, as in (1) it remains to prove the "if"-directions.

(b) First we deal with the case that we are given a finite closed cover $\{A_i\}_{i=1,\dots,m}$ of X such that for each i the intersection $M \cap A_i$ is a closed subset of A_i. It follows from Lemma 2.4 that for each $i \in \{1,\dots,m\}$ there exists a closed subset Z_i of X with $Z_i \cap A_i = M \cap A_i$. We have

$$M \underset{\substack{\uparrow \\ \text{since } A_1 \cup \dots \cup A_m = X}}{=} \bigcup_{i=1}^{m} (M \cap A_i) = \bigcup_{i=1}^{m} \overbrace{(Z_i \cap A_i)}^{\text{intersection of closed subsets of } X\text{, hence closed in } X} \underset{\substack{\uparrow \\ \text{union of finitely many closed subsets,} \\ \text{hence closed}}}{=} \text{closed subset of } X$$

Now suppose that we are given a locally finite closed cover $\{B_i\}_{i\in I}$ of X such that for each $i \in I$ the intersection $M \cap B_i$ is a closed subset of B_i. Since the cover is locally finite there exists for each $x \in X$ an open neighborhood W_x such that $I_x := \{i \in I \,|\, W_x \cap B_i \neq \varnothing\}$ is finite. Since $\{W_x \cap B_i\}_{i\in I_x}$ is a finite closed cover of W_x we obtain from the above that $M \cap W_x$ is closed. But since $\{W_x\}_{x\in X}$ is an open cover of X we obtain from (1b) that M is a closed subset of X.

(a) As in (1) we can reduce the proof of (a) to the proof of (b). ■

We continue with the following definition which will show up every now and then.

Definition. Let X be a topological space. We say a subset A of X is *dense* if every non-empty open subset of X has a non-empty intersection with A.

Examples.

(1) Let $n \in \mathbb{N}$. It follows from basic facts in real analysis that $A = \mathbb{Q}^n$ is a dense subset of $X = \mathbb{R}^n$.

(2) It is very tempting to think that dense open subsets of say $X = [0,1]$ must be "almost the whole set". But the following example might dent one's optimism. First we pick a bijection $f\colon \mathbb{N} \to \mathbb{Q} \cap [0,1]$. For $i \in \mathbb{N}$ we write $x_i := f(i)$. We consider the set

$$U := \bigcup_{i\in\mathbb{N}} \underbrace{(x_i - \tfrac{1}{3^i}, x_i + \tfrac{1}{3^i}) \cap [0,1]}_{\text{open in } [0,1]} \subset [0,1].$$

The same argument as in (1) shows that U is a dense subset of $[0,1]$. Since U is the union of open subsets we see that U is also an open subset of $[0,1]$. On the other hand its Lebesgue measure satisfies

$$\text{vol}(U) = (\nu)\Big(\bigcup_{i\in\mathbb{N}} \big(x_i - \tfrac{1}{3^i}, x_i + \tfrac{1}{3^i}\big) \cap [0,1]\Big) \le \sum_{i\in\mathbb{N}} (\nu)\big((x_i - \tfrac{1}{3^i}, x_i + \tfrac{1}{3^i})\big) \le \sum_{i\in\mathbb{N}} \frac{2}{3^i} = \frac{2}{3}.$$

Thus we have shown that the Lebesgue measure of U is less than the Lebesgue measure of $[0,1]$. So by no means can one reasonably claim that U is "almost all of $[0,1]$".

(3) Here is an immediate, perhaps not entirely serious application of topology to the eternal problem for how to capture a lion in the desert [**Barr81**]: We give the desert the *leonine topology*, in which a subset is closed if it is the whole desert or if it contains

no lions.[15] The set of lions is now a dense subset.[16] Now put an open cage in the desert. By density it contains a lion. Once we shut the cage we have caught the lion![17]

After this joke of dubious taste we return to serious mathematics.

Lemma 2.7.

(1) *Every open subset of $\mathbb{R}^n$ is the union of open balls of the form $B_\epsilon(x)$ where ϵ is rational and $x \in \mathbb{Q}^n$.*
(2) *Every open subset of $\mathbb{R}^n$ is the union of countably many open balls.*

Proof (∗).

(1) Let U be an open subset of $\mathbb{R}^n$. For $x \in U$ we set $\mu_x := \sup\{\mu \in [0,1] \,|\, B_\mu(x) \subset U\}$. The hypothesis that U is open implies that $\mu_x > 0$. We pick a rational number $\epsilon_x \in (\frac{1}{2}\mu_x, \mu_x)$. Note that $B_{\epsilon_x}(x) \subset U$. It remains to prove the following claim.

Claim. The subset U is the union of all $B_{\epsilon_x}(x)$ with $x \in U \cap \mathbb{Q}^n$.

Let $y \in U$. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ we can pick $x \in U \cap \mathbb{Q}^n$ with $\|x - y\| < \frac{1}{4}\epsilon_y$. Note that this is equivalent to saying that $y \in B_{\frac{1}{4}\epsilon_y}(x)$. Thus it remains to prove that $B_{\frac{1}{4}\epsilon_y}(x) \subset B_{\epsilon_x}(x)$. Recall that we pointed out above that $B_{\epsilon_y}(y) \subset U$. Since $\|x - y\| < \frac{1}{4}\epsilon_y$ we have by the triangle inequality that $B_{\frac{3}{4}\epsilon_y}(x) \subset B_{\epsilon_y}(y)$. Thus $B_{\frac{3}{4}\epsilon_y}(x) \subset U$. By definition of μ_x this implies that $\frac{3}{4}\epsilon_y \leq \mu_x$. Hence we have $\frac{1}{4}\epsilon_y \leq \frac{3}{8}\epsilon_y \leq \frac{1}{2}\mu_x \leq \epsilon_x$. But then we also have $B_{\frac{1}{4}\epsilon_y}(x) \subset B_{\epsilon_x}(x)$, as desired. ⊞

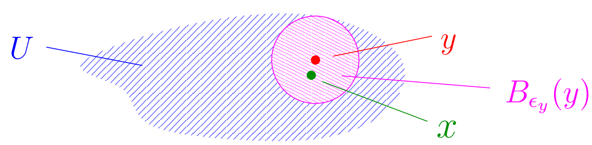


Figure 11. Illustration for the proof of Lemma 2.7.

(2) This statement follows immediately from (1). ■

We move on to the next definition.

Definition. Let X be a topological space and let A be a subset of X. We make the following two definitions:

interior $\mathring{A} :=$ union of all open sets of X that are contained in A *closure* $\overline{A} :=$ intersection of all closed sets of X that contain A

Furthermore we define the *boundary of A in X* as $\partial A := \overline{A} \setminus \mathring{A}$.

We will give examples in a second, but first we record a few observations that follow almost immediately from the definitions.

[15] Why is this a topology?
[16] Why is that?
[17] This and many other mathematical approaches to hunting lions can be found in [**Boas95**].

Lemma 2.8. *Let X be a topological space and let A be a subset of X.*

(1) *The interior $\mathring{A}$ is an open subset of X.*
(2) *The closure $\overline{A}$ of A is a closed subset of X.*
(3) *We have $\mathring{A} \subset A \subset \overline{A}$.*
(4) *If A is open, then $\mathring{A} = A$.*
(5) *If A is closed, then $\overline{A} = A$.*
(6) *We have $\mathring{A} = A \setminus \partial A$.*

Proof.

(1) The union of arbitrarily many open sets is, by definition of a topology, again an open set. Thus the interior $\mathring{A}$ is an open subset of X.
(2) In Lemma 2.2 we saw that the intersection of arbitrarily many closed sets is again a closed set. It follows immediately that the closure $\overline{A}$ is a closed subset of X.

The remaining four statements follow almost immediately from the definitions. ■

The above definition is succinct and precise, but using the definition it is perhaps difficult to find the interior and the closure of a given subset of a topological space. In many situations the following lemma takes care of that difficulty.

Lemma 2.9. *Let X be a topological space and let A be a subset of X. We have the following three equalities:*

$$\begin{array}{lrcl} (1) & \mathring{A} & = & \{x \in A \,|\, \textit{there exists a neighborhood } U \textit{ of } x \textit{ that is contained in } A\} \\ (2) & \overline{A} & = & \{x \in X \,|\, \textit{every neighborhood of } x \textit{ contains at least one point in } A\} \\ (3) & \partial A & = & \{x \in X \,|\, \textit{every neighborhood of } x \textit{ contains at least one point in } A \\ & & & \hfill \textit{and one point that does not lie in } A\} \end{array}$$

Proof. The proof of the lemma is Exercise 2.3. ■

Using Lemma 2.9 we can give examples of interiors, closures and boundaries.

Examples.

(1) We consider the topological space $X = \mathbb{R}$ and the subset $A = [-1, 2)$. Using Lemma 2.9 one can easily see that the interior of A is the open interval $(-1, 2)$ and that the closure of A is the closed interval $[-1, 2]$. Furthermore $\partial A = \{-1, 2\}$.
(2) Let $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$. We view $B^n_r(x)$, $\overline{B}^n_r(x)$ and $S^{n-1}_r(x)$ as subsets of $\mathbb{R}^n$. Using Lemma 2.9 one can show fairly easily that
$$\overline{B^n_r(x)} \;=\; \overline{B}^n_r(x) \quad \text{and} \quad \partial\overline{B}^n_r(x) \;=\; \partial B^n_r(x) \;=\; \partial S^{n-1}_r(x) \;=\; S^{n-1}_r(x).$$
(3) In Figure 12 we show a subset of $\mathbb{R}^2$, and we also show its interior $\mathring{A}$, its closure $\overline{A}$ and its boundary ∂A.
(4) It is rather unfortunate that the notations $\mathring{A}$, $\overline{A}$ and ∂A hide the role of X. For example let us consider $A = [0, 2]$. If we view it as a subset of $X = \mathbb{R}$, then $\partial A = \partial[0, 2] = \{0, 2\}$. On the other hand, if we view $A = [0, 2]$ as a subset of $X = \mathbb{R}_{\geq 0}$, then $\partial A = \partial[0, 2] = \{2\}$. This ambiguity in the notation can sometimes

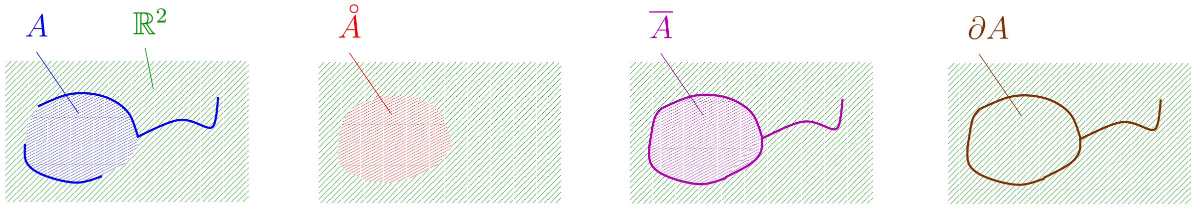


FIGURE 12

be a source of confusion. Thus it would be more appropriate to specify the role of X in the above notation. But mathematicians, as all humans, tend to be lazy and thus tend to simplify the notation.

We conclude this section with the following technical lemma.

Lemma 2.10. $(*)$ *Let X be a topological space and let $\{U_i\}_{i\in I}$ be a family of subsets. The following inclusions hold:*

$$(1)\qquad \textit{interior of } \bigcup_{i\in I} U_i \;\supset\; \bigcup_{i\in I} \mathring{U}_i \qquad \textit{and} \qquad (2)\qquad \textit{closure of } \bigcap_{i\in I} U_i \;\subset\; \bigcap_{i\in I} \overline{U_i}.$$

Furthermore, if we are given finitely many subsets $U_1,\dots,U_n$ of X, then the following equalities hold:

$$(3)\qquad \textit{interior of } \bigcap_{i=1}^{n} U_i \;=\; \bigcap_{i=1}^{n} \mathring{U}_i \qquad \textit{and} \qquad (4)\qquad \textit{closure of } \bigcup_{i=1}^{n} U_i \;=\; \bigcup_{i=1}^{n} \overline{U_i}.$$

Example. It is easy to come up with examples of two subsets U_1, U_2 in $\mathbb{R}^n$ such that inclusions in Lemma 2.10 (1) and (2) are actually proper inclusions. Some examples are illustrated in Figure 13.

FIGURE 13

PROOF. The statements follow fairly easily from the definitions. We leave it to the reader to verify the statements. ■

2.2. **The Hausdorff property and convergence of sequences.** Before we introduce the Hausdorff property we introduce two more examples of topological spaces. Both are of the type "$\mathbb{R}^n$ together with an extra point".

Examples.

(1) Let $n \in \mathbb{N}$. We consider the set

$$X \;:=\; \mathbb{R}^n \cup \{\infty\},$$

i.e. X consists of $\mathbb{R}^n$ and an extra point ∞. We say $U \subset X$ is open, if both of the following two conditions are satisfied:
(a) for each point $x \in U \cap \mathbb{R}^n$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset U$,
(b) if $\infty \in U$, then there exists a $C > 0$ such that $\{x \in \mathbb{R}^n \,|\, \|x\| > C\} \subset U$.

It is straightforward to see that this defines indeed a topology on X. We refer to $\mathbb{R}^n \cup \{\infty\}$ as "$\mathbb{R}^n$ with a point at infinity". For $n = 1$ we refer to $\mathbb{R} \cup \{\infty\}$ also as the "line with a point at infinity".

(2) We consider the set
$$X \;:=\; \mathbb{R} \cup \{*\},$$
i.e. X consists of $\mathbb{R}$ and an extra point $*$. We say $U \subset X$ is open, if the following two conditions are satisfied:
(a) for each point $x \in U \cap \mathbb{R}$ there exists an $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset U$,
(b) if $* \in U$, then there exists an $\epsilon > 0$ such that $(-\epsilon, 0) \cup (0, \epsilon) \subset U$.
One can easily verify that this is indeed a topology on X. We refer to this topological space as the "line with two zeros".

Even though the "line with two zeros" looks at first glance similar to the "line with a point at infinity" we will soon see that they are strikingly different topological spaces.

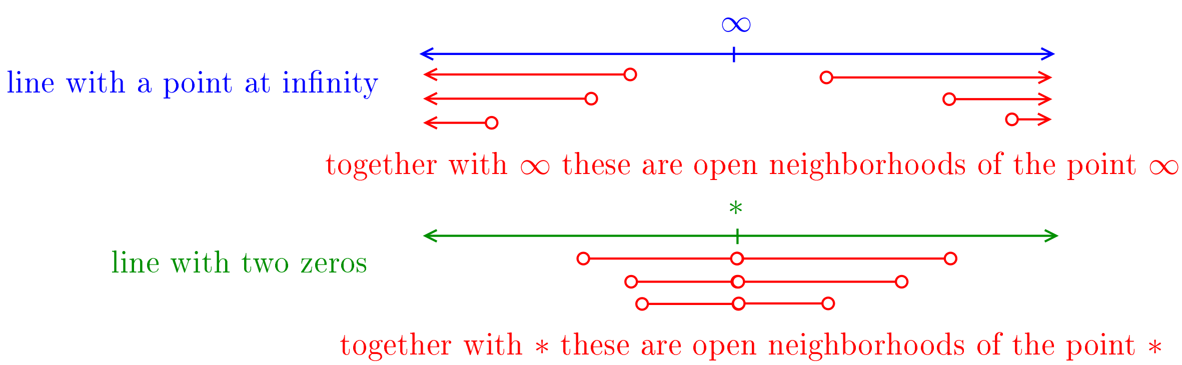


FIGURE 14

Now we provide the first definition of this section.

Definition.
(1) We say a topological space X is *Hausdorff*, if given any two distinct points $x \neq y$ there exist disjoint open neighborhoods U of x and V of y.
(2) A *Hausdorff space* is a topological space that is Hausdorff.

The following proposition, which very likely is well-known to most readers, says that topological spaces arising from metric spaces are Hausdorff.

Proposition 2.11. *Let (X, d) be a metric space. The corresponding topological space is Hausdorff.*

PROOF. Let $x, y \in X$ be two different points. We set $\epsilon := \frac{1}{2}d(x, y)$. It follows from Proposition 2.3 that $U := B_\epsilon(x)$ and $V := B_\epsilon(y)$ are neighborhoods of x respectively y. An elementary argument using the triangle inequality shows that U and V are disjoint. ■

Example. In Exercise 2.20 we will see that "$\mathbb{R}^n$ with a point at infinity" is Hausdorff and that on the other hand "the line with two zeros" is not Hausdorff.

The following lemma is basically trivial, but since it gets used on many occasions it is still worth spelling it out.

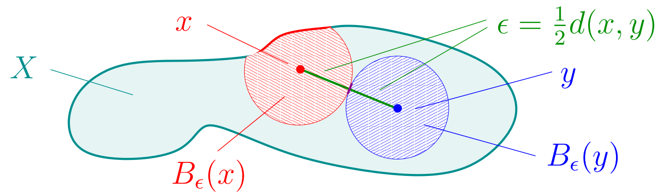


FIGURE 15

Lemma 2.12. *Let X be a topological space and let A be a subset of X. If X is Hausdorff, then A is also Hausdorff.*

The following gives us a modest but still somewhat useful consequence of being Hausdorff.

Lemma 2.13. *Let X be a topological space. If X is Hausdorff, then for every $x \in X$ the corresponding subset $\{x\}$ is closed.*

PROOF. Let $x \in X$. Since X is Hausdorff we know that for every $y \neq x$ there exist open disjoint neighborhoods U_y of x and V_y of y. We have

$$X \setminus \{x\} \;=\; \bigcup_{y \in X \setminus \{x\}} \{y\} \;\underset{\substack{\uparrow \\ \text{since } y \in V_y}}{\subset}\; \bigcup_{y \in X \setminus \{x\}} V_y \;\underset{\substack{\uparrow \\ \text{since } x \notin V_y}}{\subset}\; X \setminus \{x\}.$$

Thus we see that the inclusions are equalities. Since each V_y is open we see that $X \setminus \{x\}$ is the union of open sets, in particular $X \setminus \{x\}$ is open. In other words, $\{x\}$ is a closed subset of X. ■

Next we introduce the notion of a convergent sequence.

Definition. Let X be a topological space. We say that a sequence $\{a_n\}_{n \in \mathbb{N}}$ *converges* if there exists an $a \in X$ such that for any neighborhood U of a there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n \in U$. We refer to a as a *limit* of the sequence.

Example. We consider the line with two zeros, i.e. the topological space $X = \mathbb{R} \cup \{*\}$ defined above. The sequence $a_n = \frac{1}{n}$ converges, more or less evidently with limit 0. But strikingly enough, it follows easily from the definitions that the sequence also converges to $*$. Thus we see that the limit is not unique.

The following lemma is most likely already familiar to the reader from real analysis.

Lemma 2.14. *Let X be a topological space. If X is Hausdorff, then every convergent sequence in X has a unique limit.*

PROOF. Assume there exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ in X that converges to two different points x and y. Since X is Hausdorff there exist two disjoint neighborhoods U and V of x and y. But by our hypothesis, for large enough n the a_n have to lie in U and V at the same time. But that is not possible and we have thus obtained a contradiction. ■

2.3. **Compact topological spaces.** In this section we introduce the notion of compactness. Usually it takes a while till one appreciates the brilliance of the definition.

Definition. We say a topological space X is *compact* if for each open cover $\{U_i\}_{i\in I}$ of X there exist finitely many indices $i_1,\dots,i_k \in I$ such that

$$X \;=\; U_{i_1} \cup \dots \cup U_{i_k}.$$

Remark. The introduction of the notion of compact topological space by Pavel Alexandrov and Pavel Urysohn [**AU29**] in 1929 shows that often in mathematics the really clever thing is not necessarily the proofs, but coming up with just the right definition. It is difficult to exaggerate the role of compactness in topology.

Example. Every finite topological space (regardless of the topology) is compact, since there exist only finitely many distinct open subsets. This simple example can sometimes be quite useful.

In the following we will often use the following convention.

Convention. Let P be a property of a topological space (e.g. being discrete, Hausdorff, compact). We say that a subset Y of a topological space X has the property P if Y, equipped with the subspace topology, has the property P.

Examples.

(1) The set $\mathbb{Z} \subset \mathbb{R}$ is easily seen to be a discrete subset.
(2) Any finite set of a topological space is, by the above, a compact subset.

Frequently and often subconsciously the following elementary lemma is employed.

Lemma 2.15. *Let X be a topological space, let $A \subset X$ be a subset and let $\{U_i\}_{i\in I}$ be an open cover of A. If A is compact, then there exist finitely many indices $i_1,\dots,i_k \in I$ such that*

$$A \;\subset\; U_{i_1} \cup \dots \cup U_{i_k}.$$

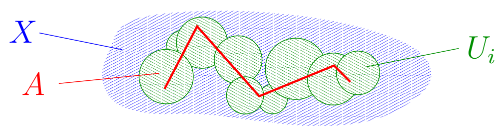


FIGURE 16. Illustration of Lemma 2.15.

PROOF (∗). Given $i \in I$ we set $V_i := U_i \cap A$. By definition of the subspace topology each V_i is an open subset of A. Thus $\{V_i\}_{i\in I}$ is an open cover of A. Since A is compact there exist $i_1,\dots,i_k \in I$ with $A = V_{i_1} \cup \dots \cup V_{i_k}$. But this implies that $A \subset U_{i_1} \cup \dots \cup U_{i_k}$. ■

We continue with the following frequently employed lemma.

Lemma 2.16. *Let X be a topological space.*

(1) *The union of finitely many compact subsets of X is again a compact subset.*
(2) *The intersection of arbitrarily many compact subsets of X is again a compact subset.*

PROOF. The statement follows easily from the definitions, we leave it to the reader to fill in the details. ■

Before we give some more interesting examples of compact topological spaces we want to discuss a few properties of compact topological spaces and compact subsets. The following lemma is frequently used, often subconsciously, in topology.

Lemma 2.17.

(1) *Let X be a topological space. If X is compact, then any closed subset of X is also compact.*
(2) *Let X be a topological space and let $A \subset X$ be a compact subset. If X is Hausdorff, then A is a closed subset of X.*

Examples.

(1) Let X be a topological space. As we pointed out on page 97, any finite subset of X is compact. Thus if X is Hausdorff, then it follows from Lemma 2.17 (2) that any finite subset of X, in particular any point in X, is a closed subset of X. Thus we have found a new proof for Lemma 2.13.
(2) Let X be a topological space, let A be a closed subset and let K be a compact subset of X. It follows from Lemma 2.17 (1) that $A \cap K$ is also compact.

Proof.

(1) Let X be a compact topological space and let A be a closed subset. Let $\{U_i\}_{i \in I}$ be an open cover of A. By definition of the subset topology there exists for every $i \in I$ an open subset V_i of X such that $U_i = V_i \cap A$. Since A is closed the complement $X \setminus A$ is open. The open subsets $\{V_i\}_{i \in I}$ together with $X \setminus A$ are an open cover of X. Since X is compact we can cover X by finitely many of these open sets. But then we can also cover A by finitely many of the U_i.
(2) Let X be a Hausdorff space and let $A \subset X$ be a compact subset. We want to show that $X \setminus A$ is open. By Lemma 2.5 it suffices to prove the following claim.

   Claim. Let $x \in X \setminus A$. Then there exists an open neighborhood V of x that is contained in $X \setminus A$.

   We apply the Hausdorff property to x and every $y \in A$. For every $y \in A$ we obtain disjoint open neighborhoods U_y of y and V_y of x. Evidently we have
$$A = \bigcup_{y \in A} \{y\} \subset \bigcup_{y \in A} (U_y \cap A) \subset A.$$
   Thus we see that $\{U_y \cap A\}_{y \in A}$ is an open cover of A. Since A is compact there exist $y_1, \dots, y_k \in A$ such that
$$A = \bigcup_{i=1}^{k} (U_{y_i} \cap A).$$
   Now we consider
$$V := \bigcap_{i=1}^{k} V_{y_i}.$$
   Since V is the intersection of finitely many open sets, it is open itself. Furthermore V does not intersect any of the U_{y_i}, $i = 1, \dots, k$. This implies that V is disjoint from $A \subset U_{y_1} \cup \dots \cup U_{y_k}$. ■

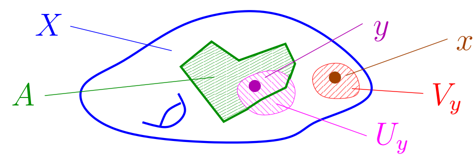


FIGURE 17. Illustration for the proof of Lemma 2.17 (2).

Before we move on to giving more interesting examples of compact topological spaces we state and prove the following elementary lemma.

Lemma 2.18.

(1) *Every discrete topological space that is compact is finite.*
(2) *Let X be a compact topological space and let A be a closed subset. If A is discrete, then A is finite.*

Remark. Note that the conclusion of Lemma 2.18 (2) does not necessarily hold, if we drop the hypothesis that A is closed. For example, consider the topological space $X = [0, 1]$. The subset $A := \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ is a discrete subset, but evidently it is not finite.

PROOF (∗).

(1) Let X be a discrete topological space. Each point thus defines an open set. Evidently X is covered by all the open sets given by the points in X. Since X is compact we can cover X by finitely many of those open sets. But this means that X contains only finitely many points.
(2) Let X be a compact topological space and let A be a closed subset. It follows from Lemma 2.17 (1) that A is also compact. So if A is discrete, then it follows from (1) that A is finite. ■

We continue with the following self-explanatory definition.

Definition. A subset A of $\mathbb{R}^n$ is called *bounded* if there exists a $C \geq 0$ such that $A \subset \overline{B}^n_C(0)$.

Example. Let $y \in \mathbb{R}^n$ and $r \geq 0$. It follows easily from the triangle inequality that the balls $B^n_r(y)$, $\overline{B}^n_r(y)$ and the sphere $S^{n-1}_r(y)$ are bounded.

Lemma 2.19. *Let $A \subset \mathbb{R}^n$ be a subset. If A is compact, then A is closed and it is bounded.*

PROOF. Let $A \subset \mathbb{R}^n$ be a compact subset.

(1) We want to show that $\mathbb{R}^n \setminus A$ is open. Let $x \in \mathbb{R}^n \setminus A$. We need to show that there exists an $\epsilon > 0$ with $B^n_\epsilon(x) \subset \mathbb{R}^n \setminus A$. Given $j \in \mathbb{N}$ we consider the open subset $U_j = \mathbb{R}^n \setminus \overline{B}^n_{\frac{1}{j}}(x)$. Evidently these sets cover A. Since A is compact we obtain from Lemma 2.15 that there exist $j_1, \dots, j_k$ with $A \subset U_{j_1} \cup \dots \cup U_{j_k}$. We set $\epsilon = \min\{\frac{1}{j_1}, \dots, \frac{1}{j_k}\}$. We obtain that $A \subset \mathbb{R}^n \setminus \overline{B}^n_\epsilon(x)$. Thus we have found the desired ϵ.
(2) We need to show that A is bounded, i.e. we need to show that there exists a $C \geq 0$ with $A \subset \overline{B}^n_C(0)$. We consider the open subsets $U_j = B^n_j(0)$ with $j \in \mathbb{N}$. Evidently these sets cover A. Since A is compact we obtain again from Lemma 2.15 that there

exist $j_1, \dots, j_k$ such that $A \subset U_{j_1} \cup \dots \cup U_{j_k}$. We set $C = \max\{j_1, \dots, j_k\}$. We obtain that $A \subset B^n_C(0) \subset \overline{B}^n_C(0)$. We have thus found the desired C. ■

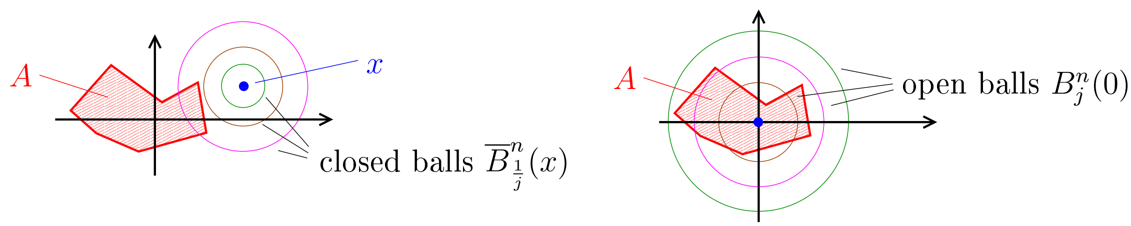


FIGURE 18. Illustration for the proof of Lemma 2.19.

The ever-popular Heine-Borel Theorem says that mercifully the converse to Lemma 2.19 holds.

Theorem 2.20. (Heine-Borel) *A subset of $\mathbb{R}^n$ is compact if and only if it is bounded and closed.*

Examples.

(1) Let $y \in \mathbb{R}^n$ and $r \geq 0$. It follows from the Heine-Borel Theorem 2.20 and Proposition 2.3 that the closed ball $\overline{B}^n_r(y)$ and the sphere $S^{n-1}_r(y)$ are compact.

(2) It is an amusing exercise, see Exercise 2.20, to show, using the Heine-Borel Theorem 2.20, that "$\mathbb{R}^n$ with a point at infinity" is compact.

(3) It is an equally entertaining exercise, see Exercise 2.20, to show that the "line with two zeros" is not compact.

(4) We set $C_0 = [0,1]$. We define sets C_i, $i \in \mathbb{N}$ iteratively by setting

$$C_i \;:=\; \tfrac{1}{3}C_{i-1} \cup \left(\tfrac{2}{3} + \tfrac{1}{3}C_{i-1}\right)$$

i.e. we obtain C_i from C_{i-1} by removing the "open middle third" of each interval. See Figure 19. The *Cantor set* is defined as

$$\text{Cantor set } C \;:=\; \bigcap_{i \in \mathbb{N}_0} C_i.$$

Each C_i is closed, hence it follows from Lemma 2.2 that C itself is a closed subset of $\mathbb{R}$. It is evidently bounded, hence the Heine-Borel Theorem 2.20 implies that the Cantor set is compact.

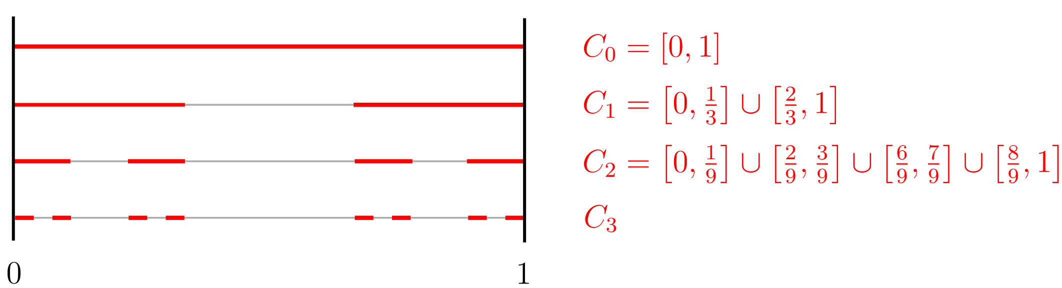


FIGURE 19

(5) Since the concepts of being bounded and closed are initially much more intuitive than being compact it is not uncommon that people like to replace "compact" by

"bounded and closed". But this equality only holds for subsets of $\mathbb{R}^n$, it does not hold for other metric spaces. For example, the set $[0,1]\cap\mathbb{Q}$ is a bounded and closed subset of the metric space $(\mathbb{Q}, d(x,y) = |x-y|)$, but as we will see in Exercise 2.44, it is not compact. Furthermore in Exercise 2.45 we will see that in general the conclusion of the Heine-Borel Theorem 2.20 does not hold for infinite-dimensional real vector spaces.

The following proposition is the key ingredient in the proof of the Heine-Borel Theorem 2.20.

Proposition 2.21. *Let $n \in \mathbb{N}$ and let $a_i < b_i$, $i = 1, \dots, n$ be real numbers. The hyper-rectangle*

$$\underbrace{[a_1, b_1] \times \ldots \times [a_n, b_n]}_{:=\{(x_1,\dots,x_n)\in\mathbb{R}^n \,|\, x_i \in [a_i,b_i] \text{ for } i=1,\dots,n\}} \subset \mathbb{R}^n$$

is compact.

PROOF. To simplify the notation we only treat the case $n = 1$. The general case is treated in a very similar fashion.[18] To simplify the notation we assume that $a = 0$ and $b = 1$. Thus we will now show that $I := [0,1]$ is compact. In an attempt at a proof by contradiction we suppose that I is not compact. This means that there exists an open overing $\{U_j\}_{j\in J}$ such that I cannot be covered by finitely many of the U_j.

Claim. There exists a sequence of closed intervals

$$I = I_0 \supset I_1 \supset I_2 \supset \ldots$$

such that for any $m \in \mathbb{N}_0$ the following two conditions hold:

(1) I_m cannot be covered by finitely many of the U_j,
(2) the length of I_m equals 2^{-m}.

We prove the claim by induction. We set $I_0 := I$. Suppose we have already found $I_0, I_1, \dots, I_m$. We divide the interval I_m into two closed intervals I' and I'' of half the length. Since I_m cannot be covered by finitely many U_j it follows that at least one of the intervals I' and I'' cannot be covered by finitely many U_j either. We declare this interval to be I_{m+1}. ⊞

Claim. There exists an $x \in I$ with $x \in \bigcap\limits_{m\in\mathbb{N}} I_m$.

For every $m \in \mathbb{N}$ we pick an $a_m \in I_m$. For any $m \geq n$ we know that a_m and a_n both lie in I_n. In particular

$$|a_m - a_n| \leq \text{length of } I_n = \frac{1}{2^n}.$$

It follows that $(a_m)_{m\in\mathbb{N}}$ is a Cauchy sequence. We denote by x the limit of this Cauchy sequence. We want to show that $x \in \bigcap\limits_{m\in\mathbb{N}} I_m$. Put differently, we want to show that for every $m \in \mathbb{N}$ we have $x \in I_m$. Thus let $m \in \mathbb{N}$. For all $n \geq m$ we have $a_n \in I_n \subset I_m$.

[18]The full details are for example provided in [**RudiW76**, Theorem 2.4.1].

In particular, $(a_n)_{n\geq m}$ is a sequence that converges in I_m. Since I_m is closed we see[19] that $\lim_{n\to\infty} a_n \in I_m$. ⊞

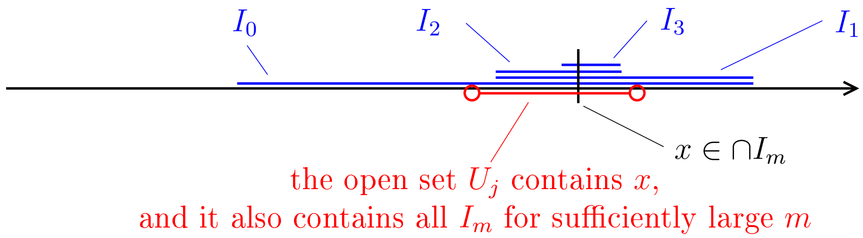


FIGURE 20. Illustration for the proof of Proposition 2.21.

Since $\{U_j\}_{j\in J}$ is an open cover of I there exists a $j \in J$ with $x \in U_j$. Since U_j is open there exists an $\epsilon > 0$ with $(x-\epsilon, x+\epsilon)\cap I \subset U_j$. We pick $m \in \mathbb{N}$ with $2^{-m} < \epsilon$. Now note that

$$I_m \underset{\substack{\uparrow\\ \text{since } x\in I_m \text{ and since}\\ \text{the length of } I_m \text{ equals } 2^{-m}}}{\subset} [x-2^{-m}, x+2^{-m}]\cap I \underset{\substack{\uparrow\\ \text{choice of } m}}{\subset} (x-\epsilon, x+\epsilon)\cap I \underset{\substack{\uparrow\\ \text{choice of } U_j}}{\subset} U_j$$

i.e. I_m is contained in a unique U_j. But this contradicts property (1) of the intervals $I_m, m \in \mathbb{N}$. ■

Now we can finally give the proof of the Heine-Borel Theorem 2.20.

PROOF OF THE HEINE-BOREL THEOREM 2.20. In light of Lemma 2.19 we only need to show that if A is a bounded and closed subset of $\mathbb{R}^n$, then it is compact. Since A is bounded there exists a $C \geq 0$ such that $A \subset [-C, C]^n$. By Proposition 2.21 we know that $A \subset [-C, C]^n$ is compact. Since A is closed in $\mathbb{R}^n$ it follows from Lemma 2.4 that is also a closed subset of $[-C, C]^n$. It follows from Lemma 2.17 that A itself is compact. ■

2.4. **Bases of a topology.** We start out with the following definition.

Definition. Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$ be a subset of the power set of X. We say $\mathcal{B}$ has the *basis property* if the following two conditions are satisfied:

(B1) For any $x \in X$ there exists a $B \in \mathcal{B}$ with $x \in B$.
(B2) Given any $B_1, B_2 \in \mathcal{B}$ and given any $x \in B_1 \cap B_2$ there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$. (See Figure 21 for an illustration.)

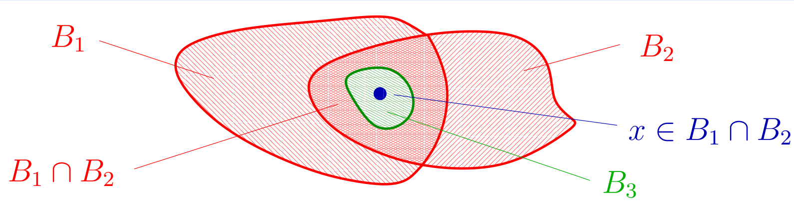


FIGURE 21

The following lemma justifies our sudden interest in this new definition.

[19]Why do we see this?

Lemma 2.22. *Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. We define*

$$\mathcal{T}(\mathcal{B}) \;:=\; \{V \subset X \,|\, \textit{for every } x \in V \textit{ there exists a } B \in \mathcal{B} \textit{ with } x \in B \subset V\}.$$

If $\mathcal{B}$ has the basis property, then $\mathcal{T}(\mathcal{B})$ is a topology on X.

Definition. Let X be a set.

(1) If $\mathcal{B} \subset \mathcal{P}(X)$ has the basis property, then we refer to $\mathcal{T}(\mathcal{B})$ as the *topology on X generated by $\mathcal{B}$*.
(2) Conversely, if $\mathcal{T}$ is a given topology on X and if $\mathcal{B} \subset \mathcal{P}(X)$ has the basis property and if $\mathcal{T} = \mathcal{T}(\mathcal{B})$, then we say that $\mathcal{B}$ is a *basis of the topology $\mathcal{T}$ of X*.

Examples.

(1) If (X, d) is a metric space, then
$$\mathcal{B} \;=\; \{\text{all balls } B_r(x) \text{ with } r > 0 \text{ and } x \in X\} \;\subset\; \mathcal{P}(X)$$
satisfies (B1) and (B2).[20] The topology generated by this family of subsets is, by definition, precisely the topology we already introduced on page 85.
(2) Let X be a set and let $\mathcal{T}$ be a topology on X. It follows easily from the definitions that $\mathcal{T}$ has the basis property. Furthermore it follows from Lemma 2.5 that the topology generated by $\mathcal{T}$ is in fact $\mathcal{T}$.
(3) Let $n \in \mathbb{N}$ and let $\mathbb{R}^n \cup \{\infty\}$ be the topological space defined on page 95. It follows immediately from the definition of the topology that a basis for the topological space $\mathbb{R}^n \cup \{\infty\}$ is given by
$$\mathcal{B} \;=\; \big\{B^n_r(x) \,\big|\, r > 0 \text{ and } x \in \mathbb{R}^n\big\} \;\cup\; \big\{\{\infty\} \cup (\mathbb{R}^n \setminus \overline{B}^n_C) \,\big|\, C \in \mathbb{R}\big\}.$$

For completeness we provide the elementary proof of Lemma 2.22.

Proof ($*$). Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. We write $\mathcal{T} := \mathcal{T}(\mathcal{B})$. We need to show that $\mathcal{T}$ satisfies the three properties (1), (2) and (3) in the definition of a topological space, see page 83. Given any $x \in X$ there exists by (B1) a $B \in \mathcal{B}$ with $x \in B$. By definition we see that $X \in \mathcal{T}$. Furthermore it is clear that the empty set $\varnothing$ lies in $\mathcal{T}$.
We need to show that the intersection of finitely many sets in $\mathcal{T}$ is again a set in $\mathcal{T}$.

First we consider the case of two sets in $\mathcal{T}$. Thus let $U, V \in \mathcal{T}$. We need to show that $U \cap V \in \mathcal{T}$. Let $x \in U \cap V$. Since U and V lie in $\mathcal{T}$ there exist $B, C \in \mathcal{B}$ with $x \in B \subset U$ and $x \in C \subset V$. By (B2) there exists a $D \in \mathcal{B}$ with $x \in D$ and $D \subset B \cap C$. In particular we have $x \in D \subset U \cap V$. This shows that $U \cap V \in \mathcal{T}$.

Now let $U_1, \dots, U_k$ be sets in $\mathcal{T}$. Using the above statement and an elementary induction argument we see that $U_1 \cap \dots \cap U_k$ lies in $\mathcal{T}$ as well.
Let $U_j, j \in J$ be a family of sets in $\mathcal{T}$. We need to show that the union $\bigcup\limits_{j \in J} U_j$ lies in $\mathcal{T}$. Thus let $x \in \bigcup\limits_{j \in J} U_j$. This means that there exists a $k \in J$ with $x \in U_k$. Since $U_k \in \mathcal{T}$ there exists a $B_x \in \mathcal{B}$ with $x \in B_x \subset U_k$. But then we also have
$$x \;\in\; B_x \;\subset\; U_k \;\subset\; \bigcup_{j \in J} U_j. \qquad \blacksquare$$

[20]Why is (B2) satisfied?

The following lemma can be used to show that two topologies on a given set agree.

Lemma 2.23. *Let X be a set and let $\mathcal{S}$ and $\mathcal{T}$ be topologies for X. Suppose that $\mathcal{C}$ is a basis for $\mathcal{T}$. (As discussed above we could take $\mathcal{C} = \mathcal{T}$.) If given any $x \in X$ and given any $V \in \mathcal{C}$ there exists a $U \in \mathcal{S}$ with $x \in U \subset V$, then $\mathcal{T} \subset \mathcal{S}$.*

Example. Let $n \in \mathbb{N}$. We consider the two metrics

$$\begin{array}{rcl} \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (v,w) & \mapsto & \|v-w\| \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (v,w) & \mapsto & \max\{|v_i - w_i| \,|\, i \in \{1,\dots,n\}\}. \end{array}$$

The former is the Euclidean metric that we introduced on page 84. We refer to the second metric as the *maximum's metric*. We leave it as a task to the dedicated reader to show, using Lemma 2.23, that the two topologies on $\mathbb{R}^n$ agree.

Proof. It follows immediately from Lemma 2.5 that any set in $\mathcal{C}$ is open in $\mathcal{S}$. But then, once again using Lemma 2.5 we see that any set in $\mathcal{T}$ is open in $\mathcal{S}$, i.e. $\mathcal{T} \subset \mathcal{S}$. ■

Sometimes the following two lemmas can be useful.

Lemma 2.24. *Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. If $\mathcal{B}$ has the basis property, then*

$$V \subset X \text{ is open with respect to } \mathcal{T}(\mathcal{B}) \quad \Longleftrightarrow \quad V \text{ is the union of sets in } \mathcal{B}.$$

Proof of Lemma 2.24 $(*)$. We first prove the "$\Leftarrow$"-direction. Thus let V be the union of sets in $\mathcal{B}$. Since sets in $\mathcal{B}$ are open with respect to $\mathcal{T}(\mathcal{B})$ and since $\mathcal{T}(\mathcal{B})$ is a topology we see that $V \in \mathcal{T}(\mathcal{B})$.

Now we prove the "$\Rightarrow$"-direction. Thus we suppose that $V \subset X$ is open with respect to $\mathcal{T}(\mathcal{B})$. For each $x \in V$ there exists, by definition of $\mathcal{T}(\mathcal{B})$, a set $B_x \in \mathcal{B}$ with $x \in B_x \subset V$. We obtain that

$$V = \bigcup_{x \in V} \{x\} \subset \bigcup_{x \in V} B_x \subset V, \quad \text{thus it follows that } V = \bigcup_{x \in V} B_x. \qquad ■$$

Lemma 2.25. *Let $\mathcal{B}$ be a basis for the topology of a topological space X. Furthermore let $\{U_j\}_{j \in J}$ be an open cover of X, i.e. $\{U_j\}_{j \in J}$ is a family of open sets with $\bigcup_{j \in J} U_j = X$. Then*

$$\{B \in \mathcal{B} \,|\, \text{ there exists some } j \in J \text{ with } B \subset U_j\}$$

is also a basis for the topology of X.

Proof $(*)$. We write

$$\mathcal{C} := \{B \in \mathcal{B} \,|\, \text{ there exists some } j \in J \text{ with } B \subset U_j\}.$$

Since $\mathcal{C} \subset \mathcal{B}$ we have $\mathcal{T}(\mathcal{C}) \subset \mathcal{T}(\mathcal{B})$. It remains to show the reverse inclusion $\mathcal{T}(\mathcal{B}) \subset \mathcal{T}(\mathcal{C})$. It follows easily from Lemma 2.5 that it suffices to show that $\mathcal{B} \subset \mathcal{T}(\mathcal{C})$. In other words, it suffices to prove the following claim.

Claim. Given any $x \in X$ and any $B \in \mathcal{B}$ with $x \in B$ there exists a $C \in \mathcal{C}$ such that $x \in C \subset B$.

So let $x \in X$ and let $B \in \mathcal{B}$ with $x \in B$. By hypothesis there exists a j such that $x \in U_j$. Since U_j is open and since $\mathcal{B}$ is a basis for the topology of X there exists a $B' \in \mathcal{B}$ with $x \in B' \subset U_j$. By (B2) there exists a $B'' \subset B$ with $x \in B''$ and $B'' \subset B \cap B' \subset U_j$. By definition we have $B'' \in \mathcal{C}$. This concludes the proof of the claim and thus of the lemma. (We refer to Figure 22 for an illustration of the argument.) ■

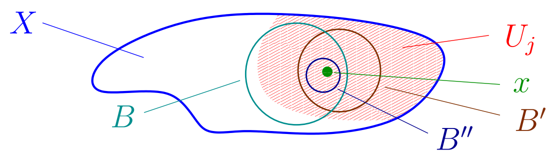


FIGURE 22. Illustration for the proof of Lemma 2.25.

Example. As we pointed out above, $\mathcal{B} = \{B^n_r(x) \,|\, r > 0 \text{ and } x \in \mathbb{R}^n\}$ is a basis for the usual topology of $\mathbb{R}^n$. We denote by $\mathcal{U}$ the family of all open sets that are contained in an open ball of radius 1. Then the open sets in $\mathcal{U}$ cover $\mathbb{R}^n$. It follows from Lemma 2.25 that

$$\{B^n_r(x) \,|\, r \in (0,1) \text{ and } x \in \mathbb{R}^n\}$$

is also a basis for $\mathbb{R}^n$.

The following lemma gives us in particular another source of topologies.

Lemma 2.26. *Let X be a set. For any $\mathcal{C} \subset \mathcal{P}(X)$ the set*

$$\mathcal{B}(\mathcal{C}) \;:=\; \begin{array}{c}\textit{all subsets of } X \textit{ that can be written as}\\ \textit{intersections of a finite family of sets in } \mathcal{C}\end{array}$$

has the basis property.

PROOF (∗).

(1) On page 72 we pointed out that if we take the empty family of subsets of X, then the intersection is all of X. In other words, we see that $X \in \mathcal{B}(\mathcal{C})$. It follows that $\mathcal{B}(\mathcal{C})$ satisfies (B1).
(2) It is easy to verify that $\mathcal{B}(\mathcal{C})$ satisfies (B2). ■

The previous lemma leads us to the following definition.

Definition. Let X be a set and let $\mathcal{C} \subset \mathcal{P}(X)$.

(1) We define $\mathcal{B} = \mathcal{B}(\mathcal{C})$ as in Lemma 2.26 and we refer to $\mathcal{T}(\mathcal{B})$ as the *topology generated by $\mathcal{C}$*.
(2) Conversely, if $\mathcal{T}$ is a topology on X and if the topology generated by $\mathcal{C}$ agrees with $\mathcal{T}$, then we say that $\mathcal{C}$ is a *subbasis for $\mathcal{T}$*.

Example.

(1) Let $\mathcal{C}$ be the set of all intervals in $\mathbb{R}$ of length one. One can easily verify that $\mathcal{C}$ generates the usual topology on $\mathbb{R}$.
(2) It follows easily from the definitions that a basis of a given topology is also a subbasis of the topology.

The following lemma gives us a convenient criterion for showing that a set of open subsets is in fact a (sub-) basis for a given topology.

Lemma 2.27. *Let X be a topological space and let $\mathcal{C}$ be a family of open sets.*

(1) *Suppose $\mathcal{C}$ has the following property:*
   (∗) *Given any open set U and given any $x \in U$ there exists a $C \in \mathcal{C}$ with $x \in C \subset U$.*
   Then $\mathcal{C}$ is a basis for the topology of X.

(2) *Suppose $\mathcal{B}$ is a basis for the topology of X. If given any $x \in X$ and given any $B \in \mathcal{B}$ there exist finitely many $C_1, \ldots, C_m \in \mathcal{C}$ with $x \in C_1 \cap \cdots \cap C_m \subset B$, then $\mathcal{C}$ is a subbasis for the topology of X.*

Example. Using Lemma 2.27 (1) and using the density of $\mathbb{Q} \subset \mathbb{R}$ one can easily show that the set of *rational* intervals forms a basis for the usual topology on $\mathbb{R}$. In particular $\mathbb{R}$ has a countable basis for its topology.

Proof (∗).

(1) We denote by $\mathcal{T}$ the topology of X. First we show that $\mathcal{C}$ has the basis property:
   (B1) If we apply the property of $\mathcal{C}$ to $U = X$ then we obtain immediately that given any $x \in X$ there exists a $C \in \mathcal{C}$ with $x \in C$.
   (B2) Let $x \in C_1 \cap C_2$ with $C_1, C_2 \in \mathcal{C}$. Since C_1 and C_2 are open the intersection $U = C_1 \cap C_2$ is also open. It follows from the hypothesis on $\mathcal{C}$, applied to U, that there exists a $C \in \mathcal{C}$ with $x \in C \subset C_1 \cap C_2$.

   We denote by $\mathcal{S}$ the topology on X that is generated by $\mathcal{C}$. We need to show that $\mathcal{S} = \mathcal{T}$.

   First we show the inclusion $\mathcal{S} \subset \mathcal{T}$. Since all sets in $\mathcal{C}$ are open with respect to $\mathcal{T}$ it follows from Lemma 2.24 that $\mathcal{S} \subset \mathcal{T}$.

   Finally we show the reverse inclusion $\mathcal{T} \subset \mathcal{S}$. Thus let $U \in \mathcal{T}$. We need to show that $U \in \mathcal{S}$. Thus let $x \in U$. By our hypothesis there exists a $C \in \mathcal{C}$ with $x \in C \subset U$. Therefore by definition we have $U \in \mathcal{S}$. We have thus shown that $\mathcal{T} \subset \mathcal{S}$.

(2) We leave the verification of this statement as a rather uninspiring exercise to the reader. ■.

We conclude this section with the following not terribly inspiring lemma.

Lemma 2.28. (∗) *Let X be a topological space and let $A \subset X$ be a subset. If $\mathcal{B}$ is a basis for the topology of X, then $\{B \cap A \,|\, B \in \mathcal{B}\}$ is a basis for the topology of A, equipped with the subspace topology.*

Proof (∗). We prove the lemma using Lemma 2.27 (1). Thus let $U \subset A$ be an open subset in A and let $x \in U$. By definition of the subspace topology there exists an open subset $V \subset X$ of X with $U = V \cap A$. Since $\mathcal{B}$ is a basis for the topology of X there exists a $B \in \mathcal{B}$ with $x \in B \subset V$. This implies that $x \in B \cap A \subset V \cap A = U$. It follows from Lemma 2.27 (1) that $\{B \cap A \,|\, B \in \mathcal{B}\}$ is indeed a basis for the topology of A. ■

Example. We consider

$$S^1 \;=\; \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\} \;\subset\; \mathbb{R}^2.$$

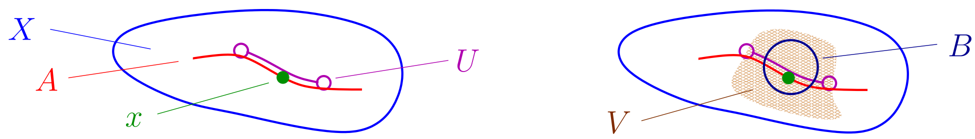


FIGURE 23. Illustration for the proof of Lemma 2.28.

As we saw on page 103, a basis for the topology on $\mathbb{R}^2$ is given by

$$\mathcal{B} \;=\; \{B_r(x,y)\,|\,(x,y)\in\mathbb{R}^2 \text{ and } r>0\}.$$

The intersection of such an open ball with S^1 is an "open interval on S^1", i.e. it is a subset of the form $\{e^{\mathrm{i}\varphi}\,|\,\varphi\in(a,b)\}$. It follows from Lemma 2.28 that these "open intervals" form a basis for the topology on S^1. We refer to Figure 24 for an illustration.

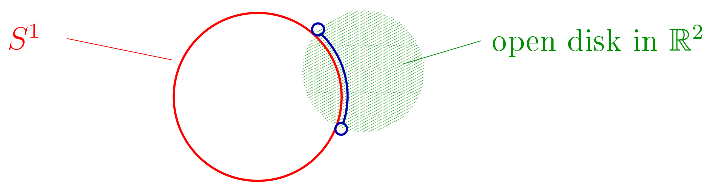


FIGURE 24

2.5. **Continuous maps.** The following is the key definition in the context of topological spaces.

Definition. We say a map $f\colon X\to Y$ between two topological spaces X and Y is *continuous*, if for each open set U in Y the preimage $f^{-1}(U)$ is open in X.

Examples.

(1) Let X and Y be two topological spaces. If Y is equipped with the trivial topology or if X is equipped with the discrete topology, then any map $f\colon X\to Y$ is continuous.
(2) We consider the set $X=\{A,B,C,D\}$ where the topology is given by the set

$$\mathcal{T} \;:=\; \{\varnothing,\{A\},\{C\},\{A,C\},\{A,B,C\},\{A,C,D\},X\}.$$

We refer to this to topological spaces as the *pseudocircle*. It follows easily from the definitions that the map

$$\begin{array}{rcl} f\colon S^1 & \to & X \\ e^{\mathrm{i}t} & \mapsto & \begin{cases} A, & \text{if } t\in(-\frac{\pi}{4},\frac{\pi}{4}),\\ B, & \text{if } t=\frac{\pi}{4},\\ C & \text{if } t\in(\frac{\pi}{4},\frac{3\pi}{4}),\\ D & \text{if } t=\frac{3\pi}{4}.\end{cases}\end{array}$$

is continuous. This map is illustrated in Figure 25.
(3) Let X and Y be topological spaces and let $y\in Y$. The constant map

$$\begin{array}{rcl} c_y\colon X & \to & Y \\ x & \mapsto & y \end{array}$$

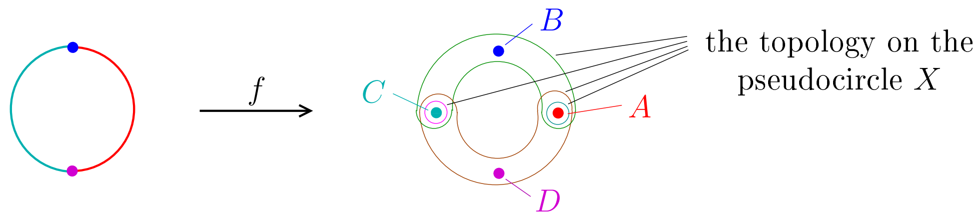


FIGURE 25

is continuous since the preimage of any set in Y is either the $\varnothing$ or all of X. This simple example will come in handy on many occasions.

The following lemma gives a few more examples of continuous maps.

Lemma 2.29. *Let $n \in \mathbb{N}$.*

(1) *For each $i \in \{1, \ldots, n\}$ the projection map*
$$\begin{array}{rcl} \mathbb{R}^n & \to & \mathbb{R} \\ (x_1, \ldots, x_n) & \mapsto & x_i \end{array}$$
is continuous.

(2) *Let $f = (f_1, \ldots, f_n)\colon X \to \mathbb{R}^n$ be a map from a topological space X to $\mathbb{R}^n$. Then f is continuous if and only if each $f_i\colon X \to \mathbb{R}$ is continuous.*

PROOF. We leave the slightly messy proof of the lemma to the reader. ■

Lemma 2.30.

(1) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are two continuous maps between topological spaces, then the composition $g \circ f\colon X \to Z$ is also continuous.*
(2) *Let X be a topological space and let A be a subset. If we equip A with the subspace topology, then the inclusion map $i\colon A \to X$ is continuous.*
(3) *Let $f\colon X \to Y$ be a continuous map between topological spaces and let A be a subset. If we equip A with the subspace topology, then the restriction of f to the map $f|_A\colon A \to Y$ is also continuous.*
(4) *Let $f\colon X \to Y$ be a continuous map. If $Z \subset Y$ is a subset with $f(X) \subset Z$, then the map $f\colon X \to Z$ is also continuous.*

PROOF.

(1) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two continuous maps between topological spaces. We need to show that the composition $g \circ f\colon X \to Z$ is also continuous. Let $U \subset Z$ be an open subset. Then
$$(g \circ f)^{-1}(U) \;=\; f^{-1}(g^{-1}(U)) \underset{\substack{\uparrow \\ \text{since } g \text{ is continuous and } U \text{ is open}}}{=} f^{-1}(\text{the open set } g^{-1}(U)) \underset{\substack{\uparrow \\ \text{since } f \text{ is continuous}}}{=} \text{open}.$$
(2) This statement follows immediately from the definitions.
(3) This statement is an immediate consequence of (1) and (2) since the restriction of f to A is precisely the same as $f \circ i$ where $i\colon A \to X$ denotes the inclusion.

(4) For clarity we denote by $g\colon X \to Z$ the map given by $x \mapsto f(x)$. Let $U \subset Z$ be an open subset. We need to show that $g^{-1}(U)$ is an open subset of X. By definition of the subspace topology there exists an open subset $V \subset X$ with $U = Z \cap V$. Thus we see that

$$g^{-1}(U) \;=\; g^{-1}(Z\cap V) \underset{\substack{\uparrow\\ \text{definition of } g}}{=} f^{-1}(Z\cap V) \underset{\substack{\uparrow\\ \text{since } f(X)\subset Z}}{=} f^{-1}(V) \underset{\substack{\uparrow\\ \text{since } f \text{ is continuous}}}{=} \text{open subset of } X.$$

■

Now we can give many more examples of continuous maps.

Lemma 2.31. *Let X be a topological space. If $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are continuous maps, then the maps*

$$\begin{array}{rcl} X & \to & \mathbb{R} \\ x & \mapsto & f(x)+g(x) \end{array} \qquad \begin{array}{rcl} X & \to & \mathbb{R} \\ x & \mapsto & f(x)\cdot g(x) \end{array} \qquad \begin{array}{rcl} X & \to & \mathbb{R} \\ x & \mapsto & \max\{f(x),g(x)\} \end{array} \qquad \begin{array}{rcl} X & \to & \mathbb{R} \\ x & \mapsto & \min\{f(x),g(x)\} \end{array}$$

are also continuous.

Example. Let A be a real $(m \times n)$–matrix. It follows easily from Lemma 2.29 (1), Lemma 2.30 (1) and Lemma 2.31 that the map

$$\begin{array}{rcl} f\colon \mathbb{R}^n & \to & \mathbb{R}^m \\ v & \mapsto & Av \end{array}$$

is continuous.

Proof. In Exercise 2.27 we will show that the maps

$$\begin{array}{rcl} \alpha\colon \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x+y \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \beta\colon \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x\cdot y \end{array}$$

are continuous. The map $x \mapsto f(x)+g(x)$ equals the map

$$X \xrightarrow{x\mapsto (f(x),g(x))} \mathbb{R}^2 \xrightarrow{\alpha} \mathbb{R}.$$

By Lemma 2.29 (2) the first map is continuous. By the above we know that α is continuous. From Lemma 2.30 (1) we deduce that the map $f+g$ is continuous. The same argument also takes care of the map $x \mapsto f(x)\cdot g(x)$. Finally note that for any $a,b \in \mathbb{R}$ we have

$$\max\{a,b\} \;=\; \tfrac{1}{2}(a+b+|a-b|) \qquad \text{and} \qquad \min\{a,b\} \;=\; \tfrac{1}{2}(a+b-|a-b|).$$

Using this observation one can now easily show, by a slight modification of the previous arguments, that the maps $x \mapsto \max\{f(x),g(x)\}$ and $x \mapsto \min\{f(x),g(x)\}$ are continuous. ■

At times it is also useful to have a characterization of continuous maps in terms of closed sets.

Lemma 2.32. *A map $f\colon X \to Y$ between two topological spaces X and Y is continuous if and only if for each closed set A in Y the preimage $f^{-1}(A)$ is closed in X.*

Proof. This lemma is a straightforward consequence of Lemma 1.3 (7) and the definitions. We leave it to the reader to fill in the details. ■

For completeness' sake we also state the following lemma which we will prove in Exercise 2.11.

Lemma 2.33. $(*)$ *Let $f\colon X \to Y$ be a map between topological spaces. For any subset $B \subset Y$ we have $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$ and the analogous statement holds with the closures replaced by the interiors.*

By Lemma 2.30 (3) we know that the restriction of a continuous map to any subset is still continuous. The following lemma can be viewed as a converse: if "enough" restrictions of a given map are continuous, then the original map is also continuous.

Lemma 2.34. *Let X be a topological space and let $\{X_i\}_{i\in I}$ be a family of subsets. The following two statements are equivalent:*

(1) *A subset $U \subset X$ is open if each $U \cap X_i$ is open in X_i.*
(2) *A subset $U \subset X$ is closed if each $U \cap X_i$ is closed in X_i.*

Furthermore, if (1) or equivalently (2) holds, then also the following statement holds:

(3) *A map $f\colon X \to Y$ to some topological space Y is continuous if each restriction $f|_{X_i}\colon X_i \to Y$ is continuous.*

PROOF.

(1)$\Rightarrow$(2) We assume that (1) holds. Now let $U \subset X$ be a subset. We have

$$\begin{array}{c}\text{for every } i \in I \text{ the set}\\ X_i \cap U \text{ is closed in } X_i.\end{array} \overset{\substack{\text{since } (X\setminus U)\cap X_i = X_i \setminus (X_i \cap U)\\ \downarrow}}{\Longrightarrow} \begin{array}{c}\text{for every } i \in I \text{ the set}\\ (X\setminus U)\cap X_i \text{ is open in } X_i\end{array}$$
$$\underset{\substack{\uparrow\\ \text{since (1) holds}}}{\Longrightarrow} X \setminus U \text{ is open in } X \Longrightarrow U \text{ is closed in } X.$$

(2)$\Rightarrow$(1) This proof is basically the same as the above argument.

(1)$\Rightarrow$(3) Let $f\colon X \to Y$ be a map to some topological space Y. We suppose that each $f|_{X_i}$ is continuous. Let $U \subset Y$ be an open subset. We need to show that $f^{-1}(U)$ is an open subset of X. By (1) it suffices to show that each intersection $f^{-1}(U) \cap X_i$ is an open subset of X_i. But this follows from the observation that $f^{-1}(U) \cap X_i = (f|_{X_i})^{-1}(U)$ and the hypothesis that $f|_{X_i}$ is continuous. ■

Initially the main application of Lemma 2.34 is the following lemma which we will use abundantly.

Lemma 2.35. (Pasting Lemma) *Let $f\colon X \to Y$ be a map between topological spaces.*

(1) *If there exists an open subsets $\{W_i\}_{i\in I}$ of X such that each restriction $f|_{W_i}$ is continuous, then f itself is continuous.*
(2) *If there exists a* locally finite *closed cover $\{A_i\}_{i\in I}$ of X such that each restriction $f|_{A_i}$ is continuous, then f itself is continuous.*

In practice often the following variation on (2) gets used:

(2') *If X is Hausdorff and if there exists a locally finite cover $\{A_i\}_{i\in I}$ of X such that each A_i is compact and such that each restriction $f|_{A_i}$ is continuous, then f itself is continuous.*

Remark.

(1) In practice Lemma 2.35 (2) often gets used in the following way: Suppose we are given two topological spaces X and Y and closed subsets $A_1, \dots, A_m$ of X such that $X = A_1 \cup \dots \cup A_m$. Furthermore we are given continuous maps $f_i \colon A_i \to Y$, $i = 1, \dots, m$. If the f_i agree on the overlaps, then the map

$$\begin{array}{rcll} X & \to & Y & \\ P & \mapsto & f_i(P) & \text{if } P \in A_i \end{array}$$

is well-defined and it follows from Lemma 2.35 (2) that it is continuous.

(2) Note that in Lemma 2.35 (1) we are allowed to deal with arbitrarily many open subsets whereas in Lemma 2.35 (2) we are restricted to finitely many closed subsets. It is clear that Lemma 2.35 (2) cannot be generalized to arbitrarily many closed subsets, after all, the restriction of a map $f \colon \mathbb{R} \to \mathbb{R}$ to any one-point subset $\{x\}$ is continuous, but clearly not every map $f \colon \mathbb{R} \to \mathbb{R}$ is continuous.

Proof. Let $f \colon X \to Y$ be a map between topological spaces.

(1) This statement follows immediately from the combination of Lemma 2.6 (1a) and Lemma 2.34.

(2) Similar to (1) this statement follows immediately from the combination of Lemma 2.6 (2a) and Lemma 2.34.

(2') This statement follows immediately from (2) and Lemma 2.17 (2). ■

The notion of continuity is of course already familiar from earlier analysis courses. The following proposition shows that for maps between metric spaces the earlier "ϵ–δ"–definition of continuity agrees with the above definition of continuity.

Proposition 2.36. *Let $f \colon X \to Y$ be a map between metric spaces. Then the following holds:*

$$f \text{ is continuous} \quad \iff \quad \underset{x\in X}{\forall}\ \underset{\epsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x'\in B_\delta(x)}{\forall}\ d(f(x), f(x')) < \epsilon.$$

Remark. We use Proposition 2.36 to import from real analysis the various continuous functions on $\mathbb{R}$ and $\mathbb{C}$, e.g. polynomials, trigonometric functions and the exponential function. The replacement of the classic "$\epsilon - \delta$"–definition of continuity by the definition in terms of open sets can even be helpful in the context of real analysis. For example, let us consider the function

$$\begin{array}{rcl} f\colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} \frac{1}{2}x^2, & \text{if } x \geq 0, \\ -x, & \text{if } x \leq 0. \end{cases} \end{array}$$

With the "$\epsilon - \delta$"–definition it is slightly annoying to prove continuity at 0. But applying Lemma 2.35 (2) to the closed subsets $A_1 = (-\infty, 0]$ and $A_2 = [0, \infty)$ it is totally evident that this function is continuous. We refer to Figure 26 for an illustration.

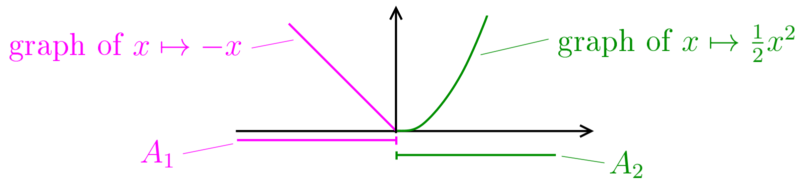


FIGURE 26

The previous remark leads us straight to the following convention.

Remark. The purist will correctly complain that the definition of the map in the previous remark is irregular, since the two cases treated in the definition of the map f are not disjoint. Nonetheless, on numerous occasions we will use such irregular definitions, usually to highlight the fact that we can use Lemma 2.35 to show that the map is continuous.

The proof of Proposition 2.36 requires some preparations. In particular it builds on the following proposition that frequently makes it much easier to show that a given map is continuous.

Proposition 2.37. *Let $f\colon X \to Y$ be a map between topological spaces. Let $\mathcal{C}$ be a subbasis for the topology of Y (e.g. $\mathcal{C}$ could be a basis for the topology). Then the following holds*

$$f \text{ is continuous} \iff \text{for each } C \in \mathcal{C} \text{ the preimage } f^{-1}(C) \text{ is open in } X.$$

PROOF (∗). Let $f\colon X \to Y$ be a map between topological spaces and let $\mathcal{C}$ be a subbasis for the topology of Y. As in Lemma 2.26 we denote by $\mathcal{B} := \mathcal{B}(\mathcal{C})$ all subsets of Y that can be obtained by intersecting finitely many sets of $\mathcal{C}$.

The "⇒"-direction of the proposition follows from the observation that elements of a subbasis are in fact open. Thus let us turn to the "⇐"-direction. Now we assume that for each $C \in \mathcal{C}$ the preimage $f^{-1}(C)$ is open in X. We start out with a little observation. Suppose we are given $C_1, \dots, C_k \in \mathcal{C}$, then

$$f^{-1}(C_1 \cap \dots \cap C_k) \overset{\substack{\text{Lemma 1.3}\\ \downarrow}}{=} \underbrace{f^{-1}(C_1)}_{\text{open in } X} \cap \dots \cap \underbrace{f^{-1}(C_k)}_{\text{open in } X}$$

is also open in X. This shows, by definition, that for each $B \in \mathcal{B} = \mathcal{B}(\mathcal{C})$ the preimage $f^{-1}(B)$ is open in X.

Now we turn to the actual proof that f is continuous. Thus let $U \subset Y$ be an open subset. By Lemma 2.24 there exists a family $\{B_j\}_{j\in J}$ of sets in $\mathcal{B}$ with

$$U = \bigcup_{j\in J} B_j.$$

Then we see that

$$f^{-1}(U) = f^{-1}\Big(\bigcup_{j\in J} B_j\Big) \overset{\substack{\text{Lemma 1.3}\\ \downarrow}}{=} \underbrace{\bigcup_{j\in J} \underbrace{f^{-1}(B_j)}_{\text{open in } X}}_{\substack{\text{union of open}\\ \text{sets, thus open in } X}}.$$

■

Now that we have Proposition 2.37 under the belt we can easily provide a proof of Proposition 2.36.

PROOF OF PROPOSITION 2.36 (∗). Let $f\colon X \to Y$ be a map between metric spaces. We need to show that

$$f \text{ is continuous} \quad \Longleftrightarrow \quad \underset{x\in X}{\forall}\ \underset{\epsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{y\in B_\delta(x)}{\forall}\ d(f(x), f(y)) < \epsilon.$$

We prove the two directions separately:

"⇒" Let $x \in X$ and let $\epsilon > 0$. We consider $V := f^{-1}(B_\epsilon(f(x))) \subset X$. Since f is continuous we know that V is open. By the definition of the topology on a metric space there exists a $\delta > 0$ such that $B_\delta(x) \subset V$. This delta has the desired property.

"⇐" On page 103 we saw that the open balls form a basis for the topology of metric spaces. Thus by Proposition 2.37 it suffices to show that if $y \in Y$ and $\mu > 0$, then $f^{-1}(B_\mu(y))$ is an open subset of X. By the definition of the topology on X it suffices to prove the following claim.

Claim. Given any $x \in f^{-1}(B_\mu(y))$ there exists an $\epsilon > 0$ with $B_\epsilon(x) \subset f^{-1}(B_\mu(y))$.

We set $\delta = \mu - d(f(x), y)$. Note that $\delta > 0$. By the triangle inequality we have $B_\delta(f(x)) \subset B_\mu(y)$. Furthermore by our hypothesis on f there exists an $\epsilon > 0$ such that $f(B_\epsilon(x)) \subset B_\delta(f(x))$. Thus we see that $B_\epsilon(x) \subset f^{-1}(B_\mu(y))$. We refer to Figure 27 for an illustration of this argument. ■

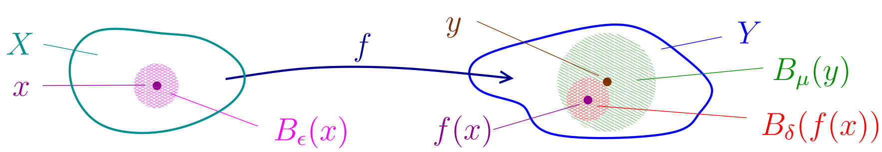


FIGURE 27. Illustration of the proof of Proposition 2.36.

We continue with the following straightforward definition.

Definition. Let $f\colon X \to Y$ be a map between topological spaces.
(1) We say f is *open* if the image of every open subset of X is open in Y.
(2) We say f is *closed* if the image of every closed subset of X is closed in Y.

Examples.
(1) The projection $\mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x$ is easily seen to be an open map but it is not a closed map.[21]
(2) On the other hand the inclusion $i\colon \mathbb{R} \to \mathbb{R}^2$, $i(x) = (x, 0)$ is evidently not an open map but it is a closed map.

Lemma 2.38. *Let X be a topological space.*
(1) *If $U \subset X$ is an open subset, then the inclusion map $i\colon U \to X$ is open.*
(2) *If $A \subset X$ is a closed subset, then the inclusion map $i\colon A \to X$ is closed.*

[21]Why is not a closed map?

Proof.

(1) Suppose that V is an open subset of U. By definition of the subspace topology there exists an open subset $W \subset X$ with $V = U \cap W$. But then $i(V) = V = U \cap W$ is the intersection of two open subsets of X, thus it is an open subset of X.
(2) The proof of this statement is analogous to the proof of Statement (1), we just not to appeal to Lemma 2.4 (2) to make the necessary adjustments. ■

We have the following analogue to Proposition 2.37.

Lemma 2.39. *Let $f\colon X \to Y$ be a map between topological spaces. Let $\mathcal{C}$ be a subbasis for the topology of X (e.g. $\mathcal{C}$ could be a basis for the topology). Then the following holds*

$$f \text{ is open} \iff \text{for each } C \in \mathcal{C} \text{ the image } f(C) \text{ is open in } Y.$$

Proof. The logic of the proof is very similar to the proof of Proposition 2.37. Eventually the proof boils down to Lemmas 2.24 and 1.3. ■

The first statement of the following lemma is perhaps the most important feature of compact sets.

Lemma 2.40.

(1) *Let $f\colon X \to Y$ be a continuous map. If X is compact, then $f(X)$ is also compact.*
(2) *Let $f\colon X \to \mathbb{R}$ be a continuous map. If X is compact and non-empty, then f assumes its maximum and its minimum, i.e. there exist $x_0, x_1 \in X$ such that*

$$f(x_0) \;\le\; f(x) \;\le\; f(x_1) \quad \text{for all } x \in X.$$

(3) *Let $f\colon X \to Y$ be a continuous map. If X is compact and if Y is Hausdorff, then $f(X)$ is a closed subset of Y.*

Example. Let X be a topological space, let Y be a subset and let $K \subset Y$ be another subset. If K is a compact subset of Y, i.e. then K is also a compact subset of X. This elementary observation follows immediately from Lemma 2.40 and the fact, observed above, that inclusion maps are continuous.

Remark. The statement of Lemma 2.40 (1) is really quite astounding. More precisely, we already saw that the image of an open set under a continuous map is not necessarily open. Similarly, the image of a closed set under a continuous map is not necessarily closed. For example consider the inclusion map $i\colon X = (0,1] \to Y = \mathbb{R}$. Evidently X is a closed subset of itself, but $i(X)$ is not a closed subset of $\mathbb{R}$. We will discuss such examples in more detail in Exercise 2.26.

Proof.

(1) Let $\{U_i\}_{i\in I}$ be an open cover of $f(X)$. We need to show that $f(X)$ is contained in the union of finitely many U_i's. We have

$$X \underset{\substack{\uparrow \\ \text{Lemma 1.3 (1)}}}{=} f^{-1}(f(X)) \;\subset\; f^{-1}\Big(\bigcup_{i\in I} U_i\Big) \underset{\substack{\uparrow \\ \text{Lemma 1.3 (5)}}}{=} \bigcup_{i\in I} f^{-1}(U_i).$$

By the continuity of f we obtain that $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of X. Since X is compact there exist $i_1,\dots,i_k \in I$ such that

$$X \;=\; f^{-1}(U_{i_1}) \cup \dots \cup f^{-1}(U_{i_k}).$$

We see that

$$\begin{aligned} f(X) &= f\left(f^{-1}(U_{i_1}) \cup \dots \cup f^{-1}(U_{i_k})\right) \\ &\underset{\substack{\uparrow \\ \text{Lemma 1.3 (3)}}}{=} f\left(f^{-1}(U_{i_1})\right) \cup \dots \cup f\left(f^{-1}(U_{i_k})\right) \underset{\substack{\uparrow \\ \text{Lemma 1.3 (2)}}}{\subset} U_{i_1} \cup \dots \cup U_{i_k}. \end{aligned}$$

(2) It follows from (1) and the Heine-Borel Theorem 2.20 that $f(X) \subset \mathbb{R}$ is bounded and closed. Thus $y_0 := \inf(f(X))$ and $y_1 := \sup(f(X))$ exist and they lie in $f(X)$. Any choice of points $x_0 \in f^{-1}(\{y_0\})$ and $x_1 \in f^{-1}(\{y_1\})$ have the desired properties.

(3) This statement is an immediate consequence of Lemma 2.17 (1) and the first statement. ■

On many occasions we will use the following elementary lemma.

Lemma 2.41. (∗) *Let $f\colon K \to X$ be a continuous map from a compact topological space K to a topological space X.*[22] *Suppose there exists a sequence $U_1, U_2, \dots$ of subsets of X with the following properties:*

(1) *each U_i is open,*
(2) *the sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$,*
(3) *we have $X = \bigcup\limits_{i\in\mathbb{N}} U_i$.*

Then there exists a $k \in \mathbb{N}$ with $f(K) \subset U_k$.

Proof (∗). For $i \in \mathbb{N}$ we write $V_i = f^{-1}(U_i)$. Since f is continuous it follows from (1) that each V_i is an open subset of K. By (3) the V_i cover all of K. Since K is compact there exist $i_1,\dots,i_n$ such that $K = V_{i_1} \cup \dots \cup V_{i_n}$. We set $k := \max\{i_1,\dots,i_n\}$. Since the U_i are nested the V_i are also nested. It follows that K is already contained in V_k, put differently, $f(K)$ is contained in U_k. ■

In most instances we are dealing with continuous maps. Since constantly writing and reading "continuous maps" is tiresome for the author and the reader we adopt throughout the remainder of these lecture notes the following convention:

Convention. Henceforth any map between two topological spaces is assumed to be continuous, unless we say explicitly otherwise.

2.6. **Homeomorphisms.** We introduce the main objects of interest of this section.

Definition.

[22] A common application is that K is a compact subset of X and f is the inclusion map.

(1) We say a map $f\colon X \to Y$ between two topological spaces X and Y is a *homeomorphism* if the following three properties are satisfied:
   (a) f is continuous,
   (b) f is bijective,
   (c) the inverse map $f^{-1}\colon Y \to X$ is also continuous.

   If there exists a homeomorphism between X and Y we say that X and Y are *homeomorphic* and sometimes we write $X \cong Y$.
(2) We say a map $f\colon X \to Y$ between two topological spaces X and Y is an *embedding* if f is an embedding, i.e. if $f\colon X \to f(X)$, where $f(X)$ is equipped with the subspace topology, is a homeomorphism. We say the embedding is *closed* if $f(X)$ is a closed subset of Y and we say the embedding is *open* if $f(X)$ is an open subset of Y.

Remark.

(1) If two topological spaces are homeomorphic, then they have the same topological properties, i.e. they share all properties that are defined purely in terms of the topology. For example, if X and Y are homeomorphic, then X is Hausdorff if and only if Y is Hausdorff, X is compact if and only if Y is compact and so on.
(2) It follows from Lemma 2.38 that an open (respectively closed) embedding is also an open (respectively closed) map.

Convention. At times we refer to a topological space that is homeomorphic to a ball or to a sphere also as a ball or a sphere.

Examples.

(1) Given $n \in \mathbb{N}$ we denote by
$$S^n_{\geq 0} \;:=\; \{(x_1,\dots,x_{n+1}) \in S^n \mid x_{n+1} \geq 0\}$$
the upper hemisphere and we define similarly the lower hemisphere $S^n_{\leq 0}$ to be
$$S^n_{\leq 0} \;:=\; \{(x_1,\dots,x_{n+1}) \in S^n \mid x_{n+1} \leq 0\}.$$
The maps
$$\begin{array}{rcl} \varphi\colon \overline{B}^n & \mapsto & S^n_{\geq 0} \\ x & \mapsto & \big(x, \sqrt{1-\|x\|^2}\big) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} p\colon S^n_{\geq 0} & \to & \overline{B}^n \\ (x_1,\dots,x_n,x_{n+1}) & \mapsto & (x_1,\dots,x_n) \end{array}$$
are easily seen to be continuous and inverses to one another. Thus both maps are homeomorphisms. The same way we see that the lower hemisphere is also homeomorphic to the closed ball $\overline{B}^n$.
(2) Let $A \in \mathrm{GL}(n,\mathbb{R})$ be an invertible real matrix. It follows from the discussion on page 109 that the map $\mathbb{R}^n \to \mathbb{R}^n$ given by multiplication by A is a homeomorphism. This simple observation has a useful consequence: Let V be an n-dimensional real vector space. We pick an isomorphism $\varphi\colon \mathbb{R}^n \to V$ and we equip V with the unique topology that turns φ into a homeomorphism. By the above discussion this definition of a topology does not depend on the choice of φ. Unless we say something else any finite-dimensional real vector space will be equipped with this topology. Note that with these topologies any isomorphism between real vector spaces of the same finite dimension is also a homeomorphism.

(3) We consider the map

$$\begin{array}{rcl} f\colon [0,1) & \to & S^1 \\ t & \mapsto & e^{2\pi i t}. \end{array}$$

The map f is a continuous bijection but it is not a homeomorphism. One can see this by either directly showing that f^{-1} is not continuous or by noting that $[0,1)$ is not compact whereas S^1 is compact. We illustrate this example in Figure 28. The importance of this example cannot be overestimated. It is an unfortunate fact of life that all too often one writes down a continuous map $f\colon X \to Y$ between topological spaces, checks that it is a bijection, and walks away under the warm, but mistaken, impression that one now has found a homeomorphism.

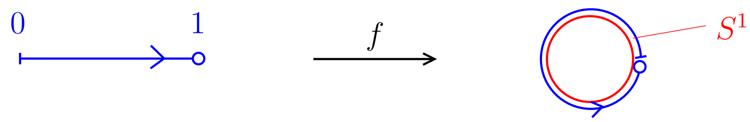


Figure 28

(4) To drive home the point we made in (3) we show in Figure 29 an injective continuous map $f\colon [-1,1) \to \mathbb{R}^2$ which is *not* an embedding. Especially once we move to more abstract topological spaces it is easy to loose track of the fact that an injective homeomorphism is not necessarily an embedding.

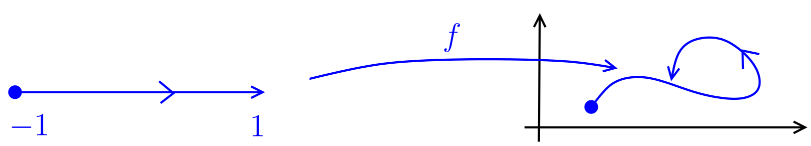


Figure 29

In many situations one is given an explicit continuous bijection $f\colon X \to Y$ between two topological spaces and one would like to show that f is in fact a homeomorphism. Unfortunately usually it is very painful to write down the explicit inverse map $f^{-1}\colon Y \to X$, let alone to verify that it is continuous. Fortunately we have several results which often help us in circumventing this difficulty.

Lemma 2.42. $(*)$ *Let $f\colon X \to Y$ be a map between topological spaces that is continuous. We assume that f is open or that f is closed.*

(1) *If f is an injection, then f is an open (respectively closed) embedding.*
(2) *If f is a bijection, then f is a homeomorphism.*

Proof $(*)$. Let $f\colon X \to Y$ be a map between topological spaces that is continuous

(2) We assume that f is a bijection. Furthermore we now assume that f is open. We want to show that f is a homeomorphism. It remains to show that $g := f^{-1}\colon Y \to X$ is continuous. Thus let $U \subset X$ be an open set. We need to show that $g^{-1}(U)$ is an open subset of Y. But this is obvious, since $g^{-1}(U) = f(U)$ is an open subset since we assume that f is an open map.

Now suppose that the map f is closed. The proof is almost identical to the above argument, except that now we have to use the continuity criterion provided by Lemma 2.32.

(1) We assume that f is an injection and we assume that f is open. We write $Z := f(X)$. Since $f\colon X \to Y$ is open it follows basically immediately from the definition of the subspace topology that $f\colon X \to Z$ is also open. Thus we obtain from (2) that f is an embedding. Furthermore, since f is open we see that f is an open embedding. The proof in the case that f is closed is basically identical. ■

Example. Let U be an open subset of $\mathbb{R}$ and let $f\colon U \to \mathbb{R}$ be a strictly monotonously increasing map, i.e. for any $x < y$ in U we have $f(x) < f(y)$. Note that open intervals form a basis for the topology of U. By the Intermediate Value Theorem the image of an open interval under f is again an open interval. Thus it follows from Lemma 2.39 that $f\colon U \to \mathbb{R}$ is an open map. Lemma 2.42 implies that $f\colon U \to f(U)$ is a homeomorphism.

The next proposition is certainly one of the most frequently used results in topology.

Proposition 2.43. *Let $f\colon X \to Y$ be a continuous map between topological spaces. If X is compact and if Y is Hausdorff, then the following statements hold:*

(1) *The map $f\colon X \to Y$ is closed.*
(2) *If f is an injection, then f is a closed embedding.*
(3) *If f is a bijection, then f is a homeomorphism.*

Remark. In the above example we saw that certainly we cannot drop the condition that X is compact from the proposition. But what about the "Hausdorff condition"? In Exercise 2.41 we will come up with an example of a bijective continuous map $f\colon X \to Y$ between two topological spaces where X is compact but such that f is nonetheless not a homeomorphism.

Proof. Let $f\colon X \to Y$ be a continuous map between topological spaces. We assume that X is compact and that Y is Hausdorff.

(1) Let A be a closed subset of X. We need to show that $f(A)$ is a closed subset of Y. In fact we see that

$$A \subset X \text{ closed} \underset{\substack{\uparrow\\ \text{by Lemma 2.17 since } X \text{ is compact}}}{\Longrightarrow} A \text{ is compact} \underset{\substack{\uparrow\\ \text{Lemma 2.40 (1)}}}{\Longrightarrow} f(A) \text{ is compact} \underset{\substack{\uparrow\\ \text{by Lemma 2.17 (2) since } Y \text{ is Hausdorff}}}{\Longrightarrow} f(A) \text{ is closed.}$$

(2) If f is an injection, then it follows immediately from (1) together with Lemma 2.42 (1) that f is a closed embedding.
(3) This follows immediately from (2). ■

We formulate our first application of Proposition 2.43 (3) as a lemma.

Lemma 2.44. *Let $n \in \mathbb{N}$. We consider the map*

$$\begin{array}{rcl} \Phi\colon S^n & \to & \mathbb{R}^n \cup \{\infty\} \\ (x_1,\dots,x_{n+1}) & \mapsto & \begin{cases} \left(\frac{x_1}{1-x_{n+1}},\dots,\frac{x_n}{1-x_{n+1}}\right), & \text{if } x_{n+1} < 1, \\ \infty, & \text{if } x_{n+1} = 1. \end{cases} \end{array}$$

where we equip $\mathbb{R}^n \cup \{\infty\}$ with the topology that we introduced on page 95. We refer to Φ as the stereographic projection. *This map has the following properties:*

1. *The map Φ sends the North Pole $N = (0,\dots,0,1) \in S^n$ to ∞.*
2. *For any $P \in S^n$ that does not equal the North Pole N the point $\Phi(P) \in \mathbb{R}^n$ is the unique point such that the ray emanating from N and that goes through P intersects the plane $\mathbb{R}^n \times \{0\}$ in $(\Phi(P), 0)$. We refer to Figure 30 for an illustration.*
3. *For any $(v_1,\dots,v_n,0) \in S^n \cap (\mathbb{R}^n \times \{0\})$ we have $\Phi(v_1,\dots,v_n,0) = (v_1,\dots,v_n)$.*
4. *The map Φ is continuous.*
5. *The map Φ is a homeomorphism.*
6. *The restriction of Φ to a map $S^n \setminus \{N\} \to \mathbb{R}^n$ is also a homeomorphism.*

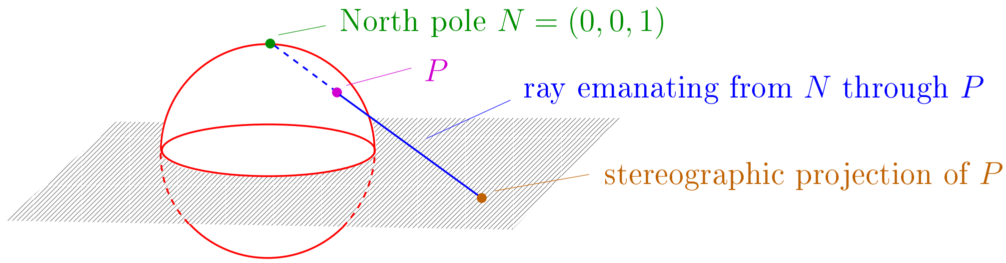


FIGURE 30. Stereographic projection from $S^2 \setminus \{(0,0,1)\}$ onto $\mathbb{R}^2$.

Convention. Let $n \in \mathbb{N}$. On many occasions we will use the homeomorphism from Lemma 2.44 to make the identification $S^n = \mathbb{R}^n \cup \{\infty\}$. In particular we have $S^2 = \mathbb{C} \cup \{\infty\}$.

PROOF.

1. This statement follows immediately from the definition of Φ.
2. This statement follows from an elementary calculation which we leave to the reader.
3. This statement follows immediately from the definition of Φ.
4. We will prove this statement in Exercise 2.51.
5. We leave it to the reader to verify that Φ is a bijection. Note that S^n is compact by the Heine-Borel Theorem 2.20. Furthermore note that on page 95 we pointed out that $\mathbb{R}^n \cup \{\infty\}$ is Hausdorff. Hence it follows from Proposition 2.43 (3) that Φ is a homeomorphism.
6. This statement follows immediately from (5). ■

If we are not in the setting of Proposition 2.43 (3), then it is often a major nuisance to try to show that a given map $f\colon X \to Y$ is in fact a homeomorphism. The following proposition relaxes the condition on X being compact.

Proposition 2.45. *Let X be a topological space, let $n \in \mathbb{N}$ and let $f\colon X \to \mathbb{R}^n$ be an injective continuous map. Furthermore suppose that X is the union of compact subsets $\{K_i\}_{i\in I}$ such that the following condition is satisfied:*

($*$) *Given any $r > 0$ there exist only finitely many $i \in I$ with $f(K_i) \cap B^n_r \neq \varnothing$.*

Then f is an embedding.

Remark. The statement and the proof of Proposition 2.45 also holds if we replace $\mathbb{R}^n$ by an arbitrary metric space (Y, d) and if we replace the open balls B^n_r by open balls in (Y, d) centered around a fixed point.

Example. Let $X = \mathbb{R}_{\geq 0}$. We consider the "outward spiraling" map $f\colon X \to \mathbb{R}^2$ that is sketched in Figure 31. Given $i \in \mathbb{N}_0$ we set $K_i := [i, i+1]$. These compact subsets cover all of $X = \mathbb{R}_{\geq 0}$ and they satisfy the condition ($*$) of the proposition. Thus we obtain from Proposition 2.45 that the map f is an embedding.

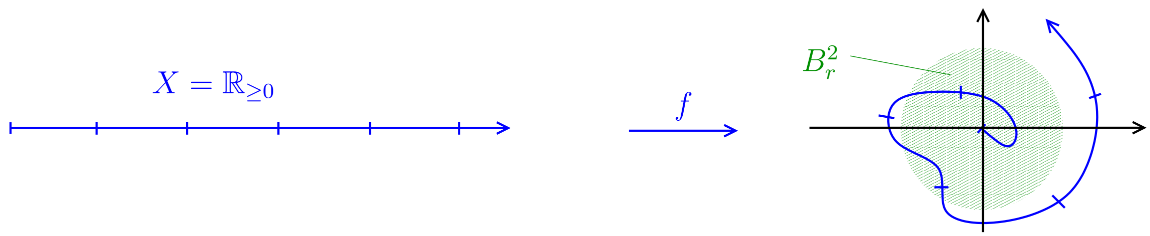


FIGURE 31

PROOF. By Lemma 2.42 (1) it remains to show that the map $f\colon X \to f(X)$ is closed. In fact, by definition of the subspace topology we only need to show that the map $f\colon X \to \mathbb{R}^n$ is closed.

Thus let A be a closed subset of X. We need to show that $f(A)$ is a closed subset of $\mathbb{R}^n$. Given $s \in \mathbb{N}_0$ we consider $C_s := \overline{B}^n_{s+1} \setminus B^n_s$. Clearly $\{C_s\}_{s\in\mathbb{N}_0}$ is a locally finite closed cover of $\mathbb{R}^n$. By Lemma 2.6 (2b) it suffices to show that each $C_s \cap f(A)$ is closed. Thus let $s \in \mathbb{N}_0$. By hypothesis we know that $J := \{i \in I \mid f(K_i) \cap \overline{B}^n_s \neq \varnothing\}$ is a finite set. Now we see that

$$\begin{aligned} C_s \cap f(A) &= C_s \cap f\Big(\bigcup_{i\in I} K_i\Big) \overset{\substack{\text{definition of } J\\ \downarrow}}{=} C_s \cap f\Big(\bigcup_{j\in J} K_j\Big) \overset{\substack{\text{by Lemma 2.16 (1), since each } K_j \text{ is compact}\\ \text{and since } J \text{ is finite}\\ \downarrow}}{=} C_s \cap f(\text{compact subset}) \\ &\underset{\substack{\uparrow\\ \text{Lemma 2.40}}}{=} C_s \cap \text{compact subset of } \mathbb{R}^n \underset{\substack{\uparrow\\ \text{Lemma 2.17}}}{=} C_s \cap \text{closed subset of } \mathbb{R}^n \underset{\substack{\uparrow\\ \text{definition of subspace topology}}}{=} \text{closed subset of } C_s. \end{aligned}$$

■

2.7. **Normal topological spaces ($*$).** There are many other "separation axioms" that are similar to the Hausdorff condition. We refer to [**Mun75**, Chapter 4.2] for more details. The only property we will need at some point is the following.

Definition. A topological space X is called *normal* if any two disjoint closed subsets A and B of X are separated by open neighborhoods, i.e. if there exist disjoint open neighborhoods U of A and V of B.

The following lemma nicely combines two key notions that we have introduced so far.

Lemma 2.46. ($*$) *Let X be a Hausdorff space.*

(1) *Let A be a subset of X and let B be a compact subset of X. There exists an open subset U of X with $A \subset U$ and $U \cap B = \varnothing$.*
(2) *If X is compact, then it is also normal.*

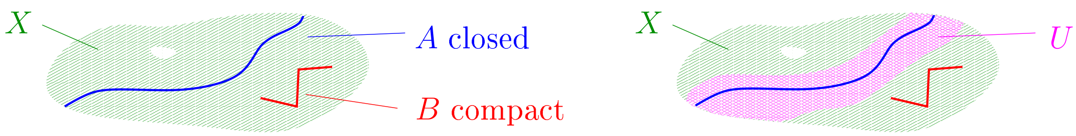


FIGURE 32. Illustration of Lemma 2.46 (1).

PROOF ($*$). Let X be a Hausdorff space. We start out with the following claim.

Claim. Let A be a subset of X and let B be a compact subset of X. Given any $a \in A$ there exists an open neighborhood U_a of a and an open neighborhood V_a of B that are disjoint.

Let $a \in A$. Since X is Hausdorff we can find for any $b \in B$ disjoint open neighborhoods R_b and S_b of a and b. Furthermore, since B is compact there exist $b_1, \dots, b_n \in B$ such that $B \subset S_{b_1} \cup \dots \cup S_{b_n}$. The open sets

$$U_a := R_{b_1} \cap \dots \cap R_{b_n} \quad \text{and} \quad V_a := S_{b_1} \cup \dots \cup S_{b_n}$$

have the desired properties. ⊞

Now we turn to the proofs of the two statements.

(1) Let A be a subset of X and let B be a compact subset of X. Given any $a \in A$ we pick U_a and V_a as in the claim. The set $U := \bigcup_{a \in A} U_a$ is evidently an open neighborhood of A. Furthermore, since for each $a \in A$ we have $U_a \cap B = \varnothing$ we see also that $U \cap B = \varnothing$. In other words, U has all the desired properties.
(2) Now assume that X is also compact. We need to show that X is normal. Thus let A and B be two disjoint closed subsets of X. Note that Lemma 2.17 (1) implies that A and B are also compact.

    Since B is compact we can make use of the above claim. In the following we use the notation of the above claim. Since A is compact we can find $a_1, \dots, a_m \in A$ such that $A \subset U_{a_1} \cup \dots \cup U_{a_m}$. The open sets

$$U := U_{a_1} \cup \dots \cup U_{a_m} \quad \text{and} \quad V := V_{a_1} \cap \dots \cap V_{a_m}$$

    have the desired properties. ■

Remark. Mathematical psychology tells us that Hausdorff does not imply normal. Indeed, an example of a topological space that is Hausdorff but not normal is given in [**Mun75**, Chapter 4.2].

The following proposition gives us an often convenient approach to showing that a topological space is normal.

Proposition 2.47. *Let (X, d) be a metric space. If we equip X with the corresponding topology, then X is normal.*

Proof. We leave the proof of this proposition as a challenging exercise to the reader. Alternatively we refer to [**Wil70**, Theorems 20.9 and 20.10] for a proof. ■

The following lemma is basically just a convenient reformulation of being normal.

Lemma 2.48. *Let X be a topological space and let $A \subset X$ be a closed subset and $U \subset X$ be an open subset with $A \subset U$. If X is normal, then there exists an open subset V with $A \subset V \subset \overline{V} \subset U$.*

Figure 33. Illustration of Lemma 2.48.

Proof. Let X be a topological space and let $A \subset X$ be a closed subset and $U \subset X$ be an open subset with $A \subset U$. We set $B := X \setminus U$. This is a closed subset with $A \cap B = \varnothing$. Since X is normal there exist disjoint neighborhoods V of A and W of B. It remains to show that $\overline{V} \subset U$. In fact we have $V \subset X \setminus W \subset U$. But since W is open we know that $X \setminus W$ is closed. By definition of $\overline{V}$ we have $\overline{V} \subset X \setminus W$. ■

Lemma 2.49. *Let X be a topological space and let $U_1, \dots, U_m$ be open subsets such that $X = \bigcup_{i=1}^{m} U_i$. If X is normal, then there exist open subsets $V_i \subset U_i$ such that $\overline{V}_i \subset U_i$ and such that $X = \bigcup_{i=1}^{m} V_i$.*

Remark. A much stronger version of this lemma can be found in [**Dug66**, Theorem 6.1] or alternatively in [**Grote69**, Satz 43].

Proof (∗). To simplify the notation we only discuss the case $m = 2$. We leave it to the reader to prove the general case.

Thus let X be a topological space that is normal. Furthermore let U_1 and U_2 be two open subsets with $U_1 \cup U_2 = X$. This data is equivalent to the statement that $X \setminus U_1$ and $X \setminus U_2$ are disjoint closed subsets of X. Since X is normal there exists, by Lemma 2.48, a subset $W_1 \subset X$ with $X \setminus U_1 \subset W_1 \subset \overline{W_1} \subset U_2$. In particular we have $\overline{W_1} \cap (X \setminus U_2) = \varnothing$. Again, because of Lemma 2.48, there exists a subset $W_2 \subset X$ with $X \setminus U_2 \subset W_2 \subset \overline{W_2} \subset X \setminus \overline{W_1}$. In particular we have $\overline{W_2} \cap \overline{W_1} = \varnothing$. We claim that $V_1 = X \setminus \overline{W_1}$ and $V_2 = X \setminus \overline{W_2}$ have the desired properties.

First note that for $i = 1, 2$ we have

$$V_i \;=\; X \setminus \overline{W_i} \;\subset\; X \setminus W_i \;\subset\; U_i.$$

Finally note that

$$V_1 \cup V_2 \;=\; (X \setminus \overline{W_1}) \cup (X \setminus \overline{W_1}) \;=\; X \setminus (\overline{W_1} \cap \overline{W_2}) \;=\; X.$$

We refer to Figure 34 for an illustration. ■

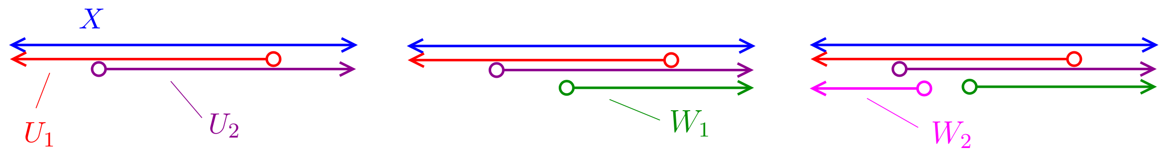


FIGURE 34

We conclude this discussion of normal topological spaces with two significantly deeper results, namely Urysohn's Lemma and the Tietze Extension Theorem. We will not really make use of these results, thus we will not provide the (non-trivial!) proofs.

We start out with what is known as Urysohn's Lemma. Despite its modest name it is actually highly non-trivial.

Lemma 2.50. (Urysohn's Lemma) *Let X be a topological space. If X is normal, then for any two disjoint closed subsets A and B there exists a continuous function $f\colon X \to [0,1]$ with $f|_A \equiv 0$ and $f|_B \equiv 1$.*

PROOF. Proofs of Urysohn's Lemma are provided in many textbooks on general topology, see e.g. [**Kel75**, Lemma 4.4], [**Mun75**, Theorem 4.3.1] or [**Jä05**, p. 109]. ■

Theorem 2.51. (Tietze Extension Theorem) *Let X be a topological space, let $A \subset X$ be a closed subset and let $f\colon A \to \mathbb{R}$ be a continuous map. If X is normal, then there exists a continuous map $g\colon X \to \mathbb{R}$ with $g|_A = f$.*

Remark. Let X be a topological space and let A and B be two disjoint closed subsets. We consider the map on $A \cup B$ that is given by $f|_A \equiv 0$ and $f|_B \equiv 1$. If X is normal, then we can extend f by the Tietze Extension Theorem 2.51 to a continuous map on all of X. This shows that Urysohn's Lemma is an immediate consequence of the Tietze Extension Theorem 2.51.

PROOF. We refer to [**Kel75**, p. 242], [**Mun75**, Theorem 4.3.2] or [**Jä05**, p. 114] for proofs. ■

2.8. **Convex subsets.** We start out with the definition of a convex subset.

Definition.

(1) We say that a subset $A \subset \mathbb{R}^n$ is *convex*, if for every two points P and Q in A the segment $\overline{PQ} := \{t \cdot P + (1-t) \cdot Q \,|\, t \in [0,1]\}$ also lies in A.
(2) Given a subset S of $\mathbb{R}^k$ the *convex hull* of S is defined as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. Since the intersection of convex sets is again convex we see that the convex hull of S is a convex subset of $\mathbb{R}^n$.

Examples.

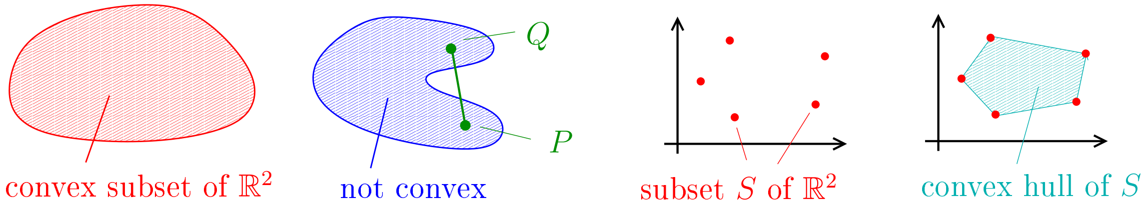


FIGURE 35

(1) For any point $x \in \mathbb{R}^n$ and any $\epsilon > 0$ the open ball $B^n_\epsilon(x)$ is convex. Indeed, let $P, Q \in B^n_\epsilon(x)$. Then for any $t \in [0,1]$ we have

$$\begin{aligned} \|t\cdot P + (1-t)\cdot Q - x\| &= \|t\cdot(P-x) + (1-t)\cdot(Q-x)\| \\ &\underset{\substack{\uparrow \\ \text{triangle inequality and homogeneity of } \|-\|}}{\leq} t\cdot\|P-x\| + (1-t)\cdot\|Q-x\| \underset{\substack{\uparrow \\ \text{since } P,Q\in B^n_\epsilon(x)}}{<} t\cdot\epsilon + (1-t)\cdot\epsilon = \epsilon. \end{aligned}$$

We have thus shown that $t\cdot P + (1-t)\cdot Q \in B^n_\epsilon(x)$. Basically the same way one shows that the closed ball $\overline{B}^n_\epsilon(x)$ is also convex.

(2) It is straightforward to verify that any hyperrectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is convex.

(3) In Exercise 2.36 we will show that the convex hull of points $P_1, \ldots, P_n \in \mathbb{R}^k$ is given by the set

$$\Big\{ \sum_{i=1}^n t_i P_i \,\Big|\, t_1, \ldots, t_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^n t_i = 1 \Big\}.$$

(4) One of the most useful properties of convex subsets is that the intersection of arbitrarily many convex subsets is, basically by definition, again convex.

The following proposition gives a useful criterion for showing that subsets of $\mathbb{R}^n$ are homeomorphic to an open or to a closed ball.

Proposition 2.52.

(1) *If A is a bounded open convex subset of $\mathbb{R}^n$, then given any $Q \in A$ there exists a canonical*[23] *homeomorphism*

$$f_Q \colon A \to B^n := \{x \in \mathbb{R}^n \mid \|x\| < 1\}$$

with $f_Q(Q) = 0$.

(2) *If A is a bounded closed*[24] *convex subset of $\mathbb{R}^n$, then given any point Q in the interior of A*[25] *there exists a canonical homeomorphism*[26]

$$f_Q \colon A \to \overline{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

with the following properties:

(a) *the homeomorphism f_Q sends Q to the origin,*

(b) *the homeomorphism f_Q restricts to a homeomorphism $\partial A \to S^{n-1}$,*

Examples.

(1) The open square $(0,1) \times (0,1)$ is clearly bounded and convex, thus it follows from Proposition 2.52 (1) that it is homeomorphic to the open disk B^2.
(2) It follows from Proposition 2.52 (2) that any triangle, i.e. any subset of $\mathbb{R}^2$ of the form $A = \{P + sv + tw \,|\, s, t \in [0,1] \text{ and } s + t \leq 1\}$ where $P \in \mathbb{R}^2$ and v, w are two linearly independent vectors, is homeomorphic to $\overline{B}^2$.

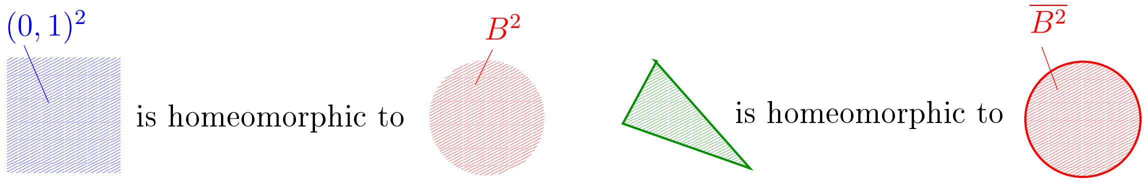


FIGURE 36

For readers who like gory details we present the more technical sibling of Proposition 2.52.

Proposition 2.53.

(1) *Let A is a bounded* open *convex subset of $\mathbb{R}^n$ and let $Q \in A$.*[27] *We consider the map*

$$\begin{array}{rcl} \rho_A \colon A \setminus \{Q\} & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \sup \big\{ \|rx\| \,\big|\, r \in \mathbb{R}_{>0} \text{ and } r(x-Q)+Q \in A \big\}. \end{array}$$

This map is well-defined in the sense that the supremum exists for each $x \in A \setminus \{Q\}$. Furthermore the map ρ_A is continuous and the map

$$\begin{array}{rcl} f_Q \colon A & \to & B^n \\ x & \mapsto & \begin{cases} (x-Q) \cdot \frac{1}{\rho_A(x)}, & \text{if } x \neq Q, \\ 0, & \text{if } x = Q. \end{cases} \end{array}$$

is a homeomorphism with $f_Q(Q) = 0$.

(2) *Let A be a bounded* closed *convex subset of $\mathbb{R}^n$. If Q is a point in the interior of A, then the same expression as in (1) defines a homeomorphism $f_Q \colon A \to \overline{B}^n$ with $f_Q(Q) = 0$. Furthermore f_Q restricts to a homeomorphism $\partial A \to S^{n-1}$.*

Examples.

[23] Throughout the lecture notes we will on occasions write that "some objects exists canonically". This means that in the proof we show the existence of the object by giving an explicit construction that does not rely on any choices.

[24] Of course "bounded and closed" is in this instance, by the Heine-Borel Theorem 2.20 equivalent to "compact". In this instance we prefer to talk of "bounded closed" to stress the analogy to statement (1) of the lemma.

[25] Note that the interior might well be empty, i.e. such a point Q might not exist.

[26] This says in particular that if A is a bounded closed convex subset of $\mathbb{R}^n$ with *non-empty interior*, then A is homeomorphic to $\overline{B}^n$.

[27] The statement becomes slightly more readable if one sets $Q = 0$.

(1) For $i = 1, \ldots, n$ let $a_i < b_i$ be real numbers. We consider the "hyperrectangle" $[a_1, b_1] \times \cdots \times [a_n, b_n]$. It follows from the discussion on page 124 and from Proposition 2.21 that the hyperrectangle is convex and compact. Proposition 2.53 (2) gives us an explicit homeomorphism

$$f_Q \colon [a_1, b_1] \times \cdots \times [a_n, b_n] \ \to \ \overline{B}^n$$

that sends the boundary of the hyperrectangle to the sphere S^{n-1} and that sends the "midpoint" $Q := (\frac{a_1+b_1}{2}, \ldots, \frac{a_n+b_n}{2})$ to the origin. Throughout this course we will use this homeomorphism to identify the left-hand side with the right-hand side. It follows easily from the explicit description of f_Q that the restriction of f_Q to each open cone on a side is smooth.[28] The homeomorphism from Proposition 2.53 (2) is illustrated in Figure 37.

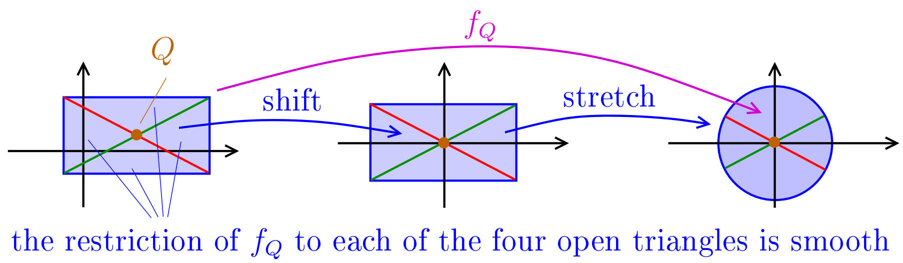


FIGURE 37

(2) Let A be a bounded convex subset of $\mathbb{R}^n$ such that the origin 0 lies in the interior of A. Proposition 2.53, applied to $Q = 0$, implies in particular that the map

$$\begin{aligned} f \colon \partial A &\to S^{n-1} \\ x &\mapsto \tfrac{x}{\|x\|} \end{aligned}$$

is a homeomorphism. This statement is illustrated in Figure 38.

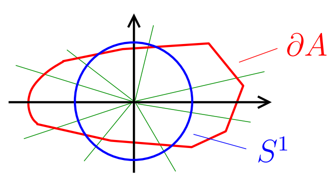


FIGURE 38

The proof of Proposition 2.53 relies on the following lemma.

Lemma 2.54. $(*)$ *Let A be a bounded closed convex subset of $\mathbb{R}^n$ such that the origin 0 lies in the interior of A.*

(1) *The map $\rho_A \colon A \setminus \{0\} \to \mathbb{R}_{\geq 0}$ defined in Proposition 2.53 is continuous.*

[28] We leave it to the reader to make it precise what "open cone on a side" means. We will make use of this observation only once, and then it should be clear what we mean.

(2) *The map*

$$\begin{aligned} f_0\colon A &\to \overline{B}^n \\ x &\mapsto \begin{cases} x \cdot \frac{1}{\rho_A(x)}, & \textit{if } x \neq 0, \\ 0, & \textit{if } x = 0 \end{cases} \end{aligned}$$

is continuous and it satisfies $f_0^{-1}(S^{n-1}) = \partial A$.

PROOF OF LEMMA 2.54 (∗). We start out with the following claim.

Claim. The map $\rho_A\colon A \setminus \{0\} \to \mathbb{R}$ is defined, it is continuous, it is bounded from above and it is bounded from below by 1.

Since A is bounded it follows that ρ_A is defined and that it is bounded from above. It follows from the definition of ρ_A that ρ_A is bounded from below by 1. Therefore it remains to show that ρ_A is continuous. Now let $x \in A \setminus \{0\}$ and let $\epsilon > 0$. It suffices to show the following two statements:

(1) there exists an open neighborhood U of x in $A \setminus \{0\}$ such that $\rho_A(y) > \rho_A(x) - \epsilon$ for all $y \in U$,
(2) there exists an open neighborhood V of x in $A \setminus \{0\}$ such that $\rho_A(y) < \rho_A(x) + \epsilon$ for all $y \in V$.

We first show the existence of U. After possibly replacing ϵ by $\min\{\frac{1}{2}\rho_A(x), \epsilon\}$ we can suppose that $\epsilon \in (0, \rho_A(x))$.

Since A is closed we have $f_0(x) \in A$. Since 0 lies in the interior of A there exists an $\eta > 0$ such that $B^n_\eta(0) \subset A$. Since A is convex, the convex hull C of $B^n_\eta(0) \cup \{f_0(x)\}$ is still contained[29] in A. Furthermore, since $B^n_\eta(0)$ is open it is straightforward to see that $C' := C \setminus \{f_0(x)\}$ is an open subset of $\mathbb{R}^n$. We denote by $S^{n-1}_{\rho_A(x)-\epsilon}$ the sphere of radius $\rho_A(x) - \epsilon$ around 0. The point $x \cdot \frac{\rho_A(x)-\epsilon}{\|x\|}$ lies on $S^{n-1}_{\rho_A(x)-\epsilon}$ and it lies in C'. Since C' is open there exists an open neighborhood U' on $S^{n-1}_{\rho_A(x)-\epsilon}$ that is contained in C'. We set

$$U := \{rz \mid z \in U' \text{ and } r \in (0,1)\}.$$

This is an open neighborhood of x and for any $y \in U$ we have $(\rho_A(x) - \epsilon) \cdot y \in A$, i.e. for any $y \in A$ we have $\rho_A(y) \geq \rho_A(x) - \epsilon$. Thus we have shown that the desired neighborhood U exists. The existence of V is proved in a very similar way. We refer to [**Berger09**, Chapter 11.3] for full details. ⊞

From the continuity of ρ_A it follows that f is continuous on $A \setminus \{0\}$. From the fact that ρ_A is bounded it follows easily that f_Q is continuous in 0. Finally one can easily verify, using the convexity of A, that $f_0^{-1}(S^{n-1}) = \partial A$. We leave it to the reader to fill in the details. ■

PROOF OF PROPOSITION 2.53 (∗).

(2) Let A be a bounded closed convex subset of $\mathbb{R}^n$ and let Q be a point in the interior of A. After a translation we can assume that $Q = 0$ is the origin, in particular we

[29] Why is that the case?

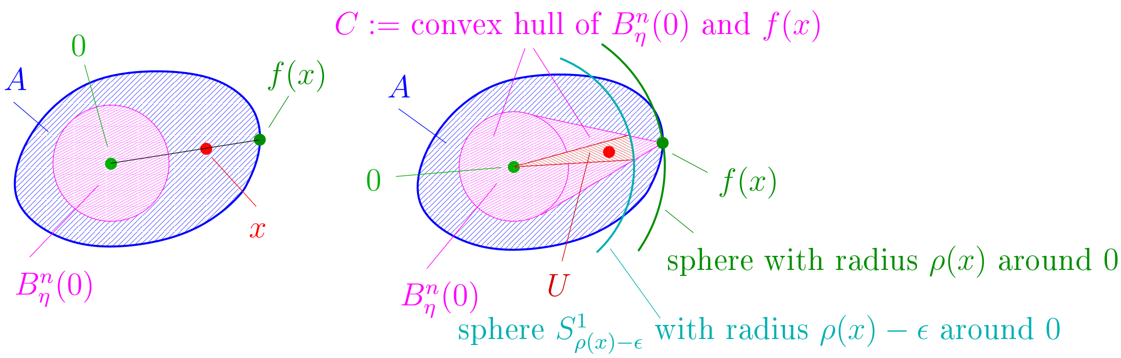


FIGURE 39. Illustration for the proof of Lemma 2.54.

can assume that the origin 0 lies in the interior of A. By Lemma 2.54 we know that the map $\rho_A\colon A \setminus \{0\} \to \mathbb{R}_{\geq 0}$ is defined and continuous. Next we consider the map

$$\begin{array}{rcl} f_0\colon A & \to & \overline{B}^n \\ x & \mapsto & \begin{cases} x \cdot \frac{1}{\rho_A(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{array}$$

It is straightforward to verify that this map is bijective[30] and by Lemma 2.54 we know that f_0 is continuous and that $f_0^{-1}(S^{n-1}) = \partial A$. Note that our hypothesis on A and Heine-Borel Theorem 2.20 imply that A is compact. Thus we obtain from Proposition 2.43 (3) that f_0 is in fact a homeomorphism. Finally note that it follows from $f_0^{-1}(S^{n-1}) = \partial A$ that f_0 restricts to a homeomorphism $\partial A \to S^{n-1}$.

(1) We denote by $C = \overline{A}$ the closure of A. Since A is open and convex we know by Exercise 2.9 that $A = \mathring{C}$. One verifies fairly easily that C is convex. It is clear that C is bounded and it follows from Lemma 2.8 that C is closed. One verifies easily that the map f_0 defined for $C = \overline{A}$ restricts to the map f_0 defined for A. Thus it follows from (2) that the map $f_0\colon C = \overline{A} \to \overline{B}^n$ is a homeomorphism and it restricts to a homeomorphism $f_0\colon \mathring{C} \to B^n$. Since $\mathring{C} = A$ we get the desired homeomorphism. ■

Remark. One again let A be a bounded closed convex subset A of $\mathbb{R}^n$ and let Q be a point in the interior of A. Proposition 2.53 provides us with a homeomorphism $f\colon A \to \overline{B}^n$. Since ∂A is not necessarily "smooth" (here we use smooth in a naive sense) one cannot hope to get such a map f that is smooth. But it does make sense to ask whether one can find such a map f such that the restriction to the interior $\mathring{A}$ of A is smooth, or even better, a diffeomorphism. In general the homeomorphism that we provide in Proposition 2.53 will not have this property. But we can modify the above construction to obtain a homeomorphism $f\colon A \to \overline{B}^n$ such that the restriction to $\mathring{A} \to B^n$ is a diffeomorphism. This can be achieved by replacing the linear stretching on the rays by an appropriately chosen different map,

[30] It follows easily from the convexity of A, the hypothesis that $\mathring{A} \neq \varnothing$ and the definition of f_0 that for each $v \in S^{n-1}$ the map f_0 restricts to a bijection on the "ray defined by v", i.e. it restricts to a bijection

$$A \cap \{rv \,|\, r \in \mathbb{R}_{>0}\} \;\to\; \overline{B}^n \cap \{rv \,|\, r \in \mathbb{R}_{>0}\}.$$

e.g. we should stretch by 1 close to the origin to obtain differentiability at 0. It takes some effort to carry out this program rigorously, details can be found in [**Feru07**, Satz 237].[31] Other approaches to finding homeomorphisms $f\colon A \to \overline{B}^n$ that restrict to a diffeomorphism $\mathring{A} \to B^n$ are sketched in [**BJ82**, p. 86] and [**GT98**, p. 60].

2.9. **The two notions of connected topological spaces.** In this section we will see that there are two notions of connectedness of a topological space. As we will see, for most "reasonable" topological spaces the two notions mercifully coincide.

Definition.

(1) We say that a topological space X is *path-connected* if given any two points x and y there exists a path from x to y, i.e. there exists a map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.
(2) We say that a topological space X is *connected* if the following holds[32]

$$\text{if } X = U \sqcup V \text{ is the disjoint union of open sets } U, V \quad \Longrightarrow \quad X = U \text{ or } X = V.$$

Examples.

(1) It follows basically from the definitions that any convex subset of some $\mathbb{R}^n$ is path-connected. But it is much less clear whether say $\mathbb{R}^n$ is connected in the above sense.
(2) Next let $G \subset \mathbb{R}^2$ be the graph of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$. As in Figure 40 we consider the complement $X = \mathbb{R}^2 \setminus G$. It is pretty clear that X is not path-connected, surely there is no path from a point P below G to a point above G. But how can one show the non-existence of such a path?[33] But it is clear that X is not connected, since X is the disjoint union of the two open non-empty sets $U := \{(x,y) \,|\, y < x^2\}$ and $V := \{(x,y) \,|\, y > x^2\}$.

So we see that both notions of connectedness have their advantages and disadvantages. It is often easy to show that a given topological space is path-connected but it can be a pain to show that a given topological space is not path-connected. With connectedness it is precisely the other way round.

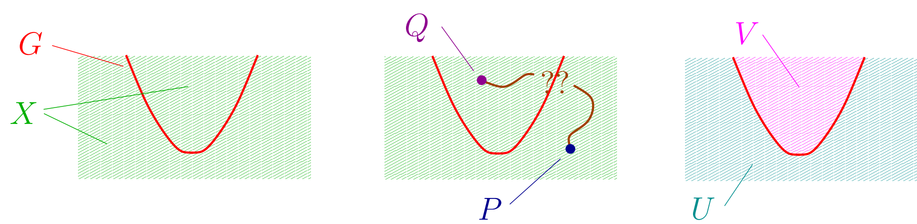


FIGURE 40

[31] It is worth noting that this diffeomorphism is orientation-preserving. (See page 299 for the definition of "orientation-preserving".)

[32] Recall that we write $X = U \sqcup V$ if X is the disjoint union of U and V, i.e. if $X = U \cup V$ and if $U \cap V = \varnothing$.

[33] The fact that such a path does not exist can be shown, with some dexterity, using real analysis. How?

It is basically obvious that any interval is path-connected. But as we will see, it takes some effort to show that the interval $[0,1]$ is connected.

Proposition 2.55. *The interval* $[0,1]$ *is connected.*

PROOF. Let U and V be two disjoint open subsets of $[0,1]$ with $U \cup V = [0,1]$. Without loss of generality we can assume that $0 \in U$. We want to show that $U = [0,1]$. We consider

$$A \ := \ \big\{ t \in [0,1] \,\big|\, [0,t] \subset U \big\}.^{34}$$

The set A contains 0, hence it is non-empty. Evidently the set is bounded thus we can consider $T := \sup A$. We need to show that $T = 1$ and that $T \in U$.

First let us suppose that $T \notin U$. This implies that $T \in V$. Since V is open it follows that there exists an $\epsilon > 0$ such that $(T-\epsilon, T+\epsilon) \cap [0,1]$ is also contained in V. In particular we have $(T-\epsilon, T+\epsilon) \cap A = \varnothing$ which implies that $\sup A \leq T - \epsilon$. Thus we have obtained a contradiction.

Now we know that $T \in U$. We still need to show that $T = 1$. Since U is open it follows that there exists an $\epsilon > 0$ such that $(T-\epsilon, T+\epsilon) \cap [0,1] \subset U$. Thus $[0, T+\epsilon) \cap [0,1] \subset U$. In light of the definition of T that is only possible if $T = 1$. ■

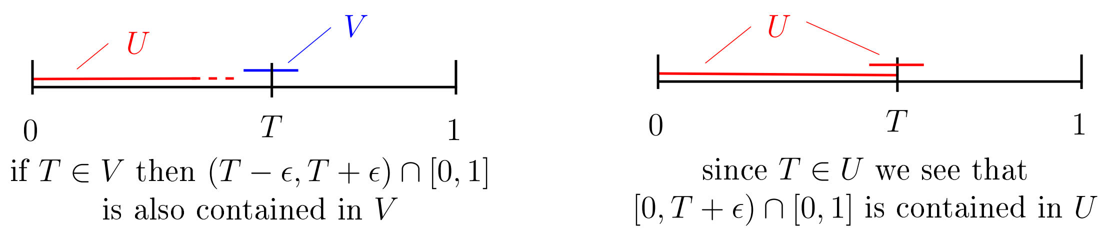


FIGURE 41. Illustration of the proof of Proposition 2.55.

Corollary 2.56. *Every path-connected topological space is also connected.*

PROOF. Let X be a topological space that is path-connected. Suppose that X is not connected. This means that there exist disjoint open non-empty subsets U and V with $X = U \cup V$. Since U and V are non-empty we can find points $x \in U$ and $y \in V$. Since X is path-connected there exists a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\gamma^{-1}(U) \cup \gamma^{-1}(V)$ is a decomposition of $[0,1]$ into two disjoint non-empty open subsets. But that is a contradiction to Proposition 2.55. ■

The following lemma comes in handy on several occasions.

Lemma 2.57. *Let* $f\colon X \to Y$ *be a map between topological spaces.*

(1) *If* X *is path-connected, then* $f(X)$ *is also path-connected.*
(2) *If* X *is connected, then* $f(X)$ *is also connected.*

PROOF. We leave the elementary proof to the reader. ■

[34]Put differently,

$$A \ = \ \text{the set of all } t \in [0,1] \text{ such that the interval } [0,t] \text{ is contained in } U.$$

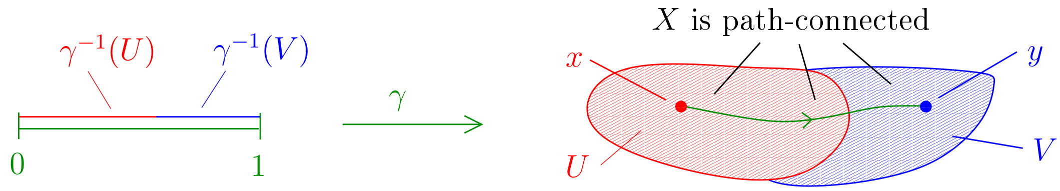


FIGURE 42. Illustration of the proof of Corollary 2.56.

In the following lemma we show that most of the topological spaces that we had encountered so far are path-connected and connected. We will use this lemma on numerous occasions without explicitly referring to it.

Lemma 2.58.

(1) *Let $n \geq 1$. Every open ball $B^n_r(y)$, every closed ball $\overline{B}^n_r(y)$ and all of $\mathbb{R}^n$ are path-connected and connected.*
(2) *Every sphere $S^n_r(y)$ with $n \geq 1$ is path-connected and connected.*

PROOF. By Corollary 2.56 it suffices to show that all these topological spaces are path-connected. It is clear that convex subsets of $\mathbb{R}^n$ are path-connected.

(1) It follows from the example on page 124 that all balls (open or closed) are path-connected.
(2) We will show in Exercise 2.58 that spheres of dimension ≥ 1 are path-connected. ■

For better or worse, the converse to Corollary 2.56 is unfortunately not always correct.

Example. Consider the topological space

$$X := \left\{(0,y) \,\middle|\, y \in [-1,1]\right\} \cup \left\{(x, \sin(\tfrac{1}{x})) \,\middle|\, x \in (0,\pi]\right\} \subset \mathbb{R}^2.$$

The topological space X is the union of the graph of $x \mapsto \sin(\frac{1}{x})$ for $x \in (0,\pi]$ and an interval on the y-axis. We sketch the topological space X in Figure 43.[35] It is a very instructive exercise, see Exercise 2.52 to show that X is connected and it is an equally instructive exercise to show that X is *not* path-connected.

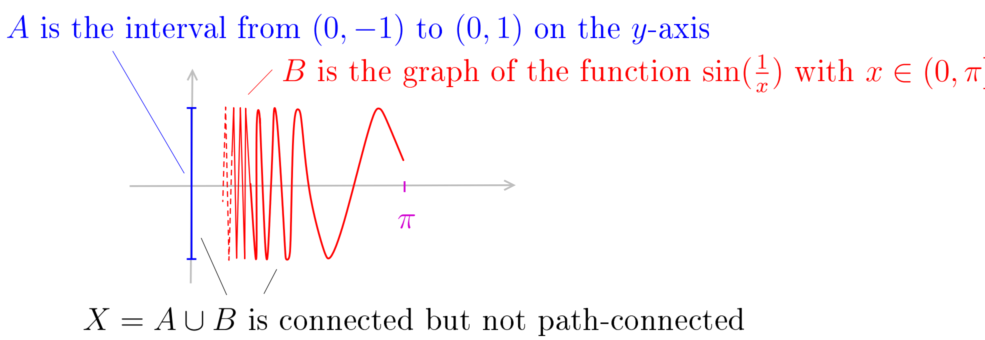


FIGURE 43

[35] This topological space is sometimes referred to as the *topologist's sine curve.*

Remark. The fact that an example of a topological space that is connected but not path-connected is so "wild" indicates that perhaps for most "reasonable" spaces the two notions of connectedness and path-connectedness agree. In fact in basically all examples of topological spaces that we encounter throughout these notes the two notions of connectedness will agree.

By a clever use of the notion of connectedness one can sometimes prove statements which at first glance have nothing to do with connectedness. For example we can now prove the following lemma.

Lemma 2.59. *Let $n \in \mathbb{N}_{\geq 2}$.*

(1) *The topological spaces $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic.*
(2) *The topological spaces $[0,1]$ and $[0,1]^n$ are not homeomorphic.*

Remark. Later on, in Proposition 43.6, we will study the question whether for $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ can be homeomorphic.

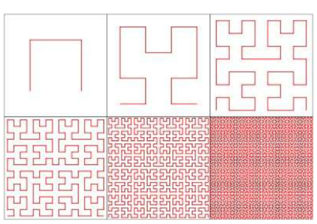
PROOF.

(1) Let us suppose that there exists a homeomorphism $f\colon \mathbb{R} \to \mathbb{R}^n$. Let $P \in \mathbb{R}$ be a point. Then f restricts to a homeomorphism from $\mathbb{R} \setminus \{P\}$ to $\mathbb{R}^2 \setminus \{f(P)\}$. The topological space $\mathbb{R}\setminus\{P\} = (-\infty, P) \cup (P, \infty)$ is not path-connected[36], while one can easily verify that $\mathbb{R}^n \setminus \{f(P)\}$ is path-connected. But as we remarked on page 116, homeomorphic spaces have the same topological properties. In particular two homeomorphic spaces are either both path-connected or they are both not path-connected. Thus we have obtained a contradiction.
(2) The proof of this statement is basically identical to the proof of (1). ∎

By Lemma 2.59 (2) we know that for $n \geq 2$ the interval $[0,1]$ is not homeomorphic to the cube $[0,1]^n$. One might think that this should surely be obvious since $[0,1]^n$ is "clearly bigger" than $[0,1]$. But the following proposition will disabuse the reader of this idea.

Proposition 2.60. *Let $n \in \mathbb{N}$.*

(1) *There exists a map*[37] *$f\colon [0,1] \to [0,1]^n$ that is surjective.*
(2) *There exists an* injective *map $f\colon S^1 \to [0,1]^2$ such that $f(S^1)$ has non-zero 2-dimensional Lebesgue measure.*



first six stages in the construction
of the Hilbert space-filling curve

FIGURE 44. Illustration of Proposition 2.60.

[36]Why is this topological space not path-connected? How can one show this?
[37]Just to make things clear, here we demand of course that f is continuous.

Remark.

(1) A surjective map from $[0,1]$ to $[0,1]^n$, or more generally from a one-dimensional smooth manifold to some smooth manifold of dimension ≥ 2, is often called a *space-filling curve*.
(2) An injective map $f\colon S^1 \to [0,1]^2$ such that $f([0,1])$ has non-zero 2-dimensional Lebesgue measure is called an *Osgood curve*.
(3) By Proposition 2.43 (3) and Lemma 2.59 (2) we know that for $n \geq 2$ an *injective* map $f\colon [0,1] \to [0,1]^n$ cannot be possible be surjective. In other words, the maps promised by Proposition 2.60 (1) are necessarily non-injective.

PROOF.

(1) Many examples of such maps are given in [**Sag94**], see e.g. [**Sag94**, Theorem 2.1].
(2) In 1903 William Osgood [**Osg03**] and Henri Lebesgue [**Leb1903**] independently showed that there exist injective maps $f\colon [0,1] \to [0,1]^2$ such that $f([0,1])$ has non-zero 2-dimensional Lebesgue measure. Many more examples of such maps are given in [**Sag94**, Chapter 8.2]. In Exercise 2.57 we will see how this fact can be used to find injective maps $g\colon S^1 \to [0,1]^2$ such that $g(S^1)$ has non-zero 2-dimensional Lebesgue measure. ■

We continue with the following elementary but useful lemma.

Lemma 2.61. *Every map $f\colon X \to Y$ from a connected topological space X to a discrete topological space is constant.*

Remark. One could argue that Lemma 2.61 is the lemma in topology that is most relevant to real life. More precisely, an immediate consequence of Lemma 2.61 is that any *non-constant* map $(a,b) \to A$ from an interval to a finite set is not continuous. This is of huge significance since humanity likes to squeeze a real-parameter (e.g. time, intelligence, performance) into a finite set of values (e.g. historical stages, grades in school and exams, hired-or-fired). Any such function is non-continuous and hence very problematic.

PROOF. We need to show that for any $x \in X$ we have $f(x) = f(X)$. Since Y is a discrete topological space we know that $\{f(x)\}$ and $Y \setminus \{f(x)\}$ are open subsets of Y. The preimages $U := f^{-1}(f(x))$ and $V := f^{-1}(Y \setminus \{f(x)\})$ are also open and they are evidently disjoint. Since U is non-empty and since X is connected we see that $V = \varnothing$. But this means that f is constant. ■

In the remainder of this section we will discuss a few lemmas which will be useful for showing that certain topological spaces are (path-) connected. For example the next lemma gives a criterion for the union of (path-) connected subsets to be (path-) connected.

Lemma 2.62. $(*)$ *Let X be a topological space and let $\{B_i\}_{i\in I}$ be a family of subsets such that the intersection $\bigcap\limits_{i\in I} B_i$ is non-empty.*

(1) *If each B_i, $i \in I$ is path-connected, then $\bigcup\limits_{i\in I} B_i$ is path-connected.*
(2) *If each B_i, $i \in I$ is connected, then $\bigcup\limits_{i\in I} B_i$ is connected.*

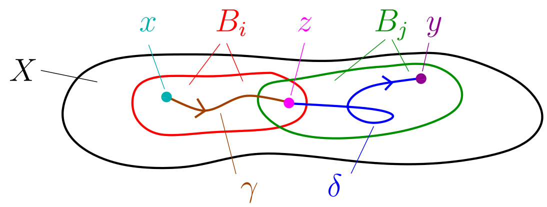


FIGURE 45. Illustration for the proof of Lemma 2.62 (1).

Remark. In Exercise 2.55 we will provide a partial converse to Lemma 2.62.

PROOF (∗). Let X be a topological space and let $\{B_i\}_{i\in I}$ be a family of subsets such that the intersection $\bigcap\limits_{i\in I} B_i$ contains a point z. We write $Z := \bigcup\limits_{i\in I} B_i$.

(1) We suppose that each B_i, $i \in I$ is path-connected. So let $x, y \in Z$. There exists an $i \in I$ with $x \in B_i$ and there exists a $j \in I$ with $y \in B_j$. Since B_i and B_j are path-connected we can pick a path $\gamma\colon [0,1] \to B_i$ from x to z and we can pick a path $\delta\colon [0,1] \to B_j$ from z to y. It is clear that

$$\begin{array}{rcl} [0,1] & \to & Z \\ t & \mapsto & \begin{cases} \gamma(2t), & \text{if } t \in [0,\frac{1}{2}] \\ \delta(2t-1), & \text{if } t \in [\frac{1}{2},1] \end{cases} \end{array}$$

is a path[38] from x to y.

(2) We suppose that each B_i, $i \in I$ is connected. We write $Z = U \sqcup V$ where U and V are open subsets. Without loss of generality we can assume that $z \in U$. For each $i \in I$ we have a decomposition $B_i = (U \cap B_i) \sqcup (V \cap B_i)$ into disjoint open subsets (here we use the definition of the subspace topology). Since B_i is connected and since $U \cap B_i \neq \emptyset$ we see that $B_i = U \cap B_i$. Thus we have shown that

$$U \;=\; \bigcup_{i\in I}(B_i \cap U) \;=\; \bigcup_{i\in I} B_i \;=\; Z. \qquad \blacksquare$$

Sometimes it is also useful to have criteria which show that removing a subset does not change connectedness.

Lemma 2.63. (∗) *Let X be a topological space and let $A \subset X$ be a subset.*

(1) *If X is path-connected and if ∂A is path-connected, then $X \setminus \mathring{A}$ is also path-connected.*
(2) *If X is connected and if ∂A is connected, then $X \setminus \mathring{A}$ is also connected.*

We will prove Lemma 2.63 in Exercise 2.54.

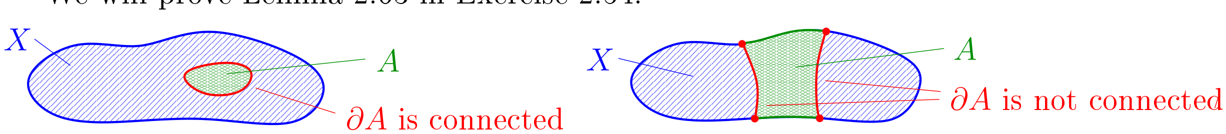


FIGURE 46. Illustration of Lemma 2.63.

[38] It follows from Lemma 2.35 (2) that this map is indeed continuous.

2.10. **The (path-) components of a topological space.** Let us start with the following simple-minded definition.

Definition. Let x and y be two points in a topological space X. We say that the points x and y are *path-equivalent*, if there exists a map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Lemma 2.64. *For every topological space X the notion of path-equivalence is indeed an equivalence relation.*

Proof. All properties of an equivalence relation are trivial except possibly for transitivity. The fact that path-equivalence is transitive follows immediately from the argument in Lemma 2.62 (1). ■

Definition. Let X be a topological space.

(1) We call the path-equivalence classes of points the *path-components* of X.
(2) We denote by $\pi_0(X)$ the set of path-components of X.[39] Furthermore we write $\pi_0(X) = 0$ if X is non-empty and if it consists of a single path-component.

Example. It is a straightforward consequence of the Intermediate Value Theorem that each point in $\mathbb{Q}$ is in fact already a path-component of $\mathbb{Q}$.

Later on it will at times be useful to know the path-components of certain matrix groups. The relevant results are summarized in the following lemma.

Lemma 2.65. *Let $n \in \mathbb{N}$.*

(1) *The real general linear group*
$$\mathrm{GL}(n,\mathbb{R}) \;=\; \{A \in \mathrm{M}(n \times n, \mathbb{R}) \,|\, \det(A) \neq 0\} \;\subset\; \mathrm{M}(n \times n, \mathbb{R}) \;=\; \mathbb{R}^{n^2}$$
has precisely two path-components, namely
$$\mathrm{GL}_+(n,\mathbb{R}) \;:=\; \{A \in \mathrm{GL}(n,\mathbb{R}) \,|\, \det(A) > 0\} \quad \textit{and} \quad \{A \in \mathrm{GL}(n,\mathbb{R}) \,|\, \det(A) < 0\}.$$
Put differently, two invertible matrices lie in the same path-component if and only if the signs of their determinants agree. The same statement also holds for $\mathrm{O}(n)$, *more precisely* $\mathrm{O}(n)$ *has precisely two path-components, namely*
$$\mathrm{SO}(n,\mathbb{R}) \;=\; \{A \in \mathrm{O}(n,\mathbb{R}) \,|\, \det(A) > 0\} \quad \textit{and} \quad \{A \in \mathrm{O}(n,\mathbb{R}) \,|\, \det(A) < 0\}.$$
(2) *The group* $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{M}(n \times n, \mathbb{R}) \,|\, \det(A) = 1\}$ *is path-connected.*
(3) *The unitary group* $\mathrm{U}(n)$ *and the special unitary group*
$$\mathrm{SU}(n) \;=\; \{A \in U(n) \,|\, \det(A) = 1\}$$
are both path-connected, thus they consist of a single path-component.
(4) *The complex general linear group*
$$\mathrm{GL}(n,\mathbb{C}) \;=\; \{A \in \mathrm{M}(n \times n, \mathbb{C}) \,|\, \det(A) \neq 0\} \;\subset\; \mathrm{M}(n \times n, \mathbb{C}) \;=\; \mathbb{C}^{n^2}$$
is path-connected.

[39] The notation "$\pi_0(X)$" might look rather odd at the moment, but it will become clearer later on, when we introduce the fundamental groups and the higher homotopy groups, why this is a perfectly sensible notation.

Remark. In Lemma 8.8 we collect more information on the relationships between the topological spaces that are given by the various matrix groups.

Proof. It is an instructive exercise in linear algebra to prove this lemma, see Exercise 2.60. Alternatively, the proof for these statements is given in [**Bak02**, Chapter 9.2]. ■

The following lemma follows immediately from the definitions and basic properties of equivalence relations.

Lemma 2.66. *Let X be a topological space.*

(1) *As a set, X is the disjoint union of its path-components.*
(2) *Each path-component of X is path-connected.*
(3) *Each path-component C of X is a maximal path-connected subset, i.e. if $C \subsetneq D$, then D is not path-connected.*

Our goal now is to define the "components" of a topological space that should play the same role as the path-components, i.e. we want to introduce the notion of "components" such that the statements of Lemma 2.66 hold, but with "path-connected" replaced by "connected" and "path-component" replaced by "component".

We start out with the following definition.

Definition. Let x and y be two points in a topological space X. We say that the points x and y are *equivalent*, if there exists a connected subset of X that contains both x and y.

Lemma 2.67. *For every topological space X the above notion of equivalence is indeed an equivalence relation.*

Proof. All properties of an equivalence relation are trivial except for transitivity. So suppose that x and y are equivalent and that y and z are equivalent. By definition there exists a connected subset A that contains x and y and there exists a connected subset B that contains y and z. By Lemma 2.62 (2) the union $A \cup B$ is connected and it thus follows that x and z are equivalent. ■

Definition. Let X be a topological space. We call the equivalence classes of points on X the *components* of X.

Examples.

(1) The topologist's sine curve that we illustrated in Figure 43 has two path-components but only a single component.
(2) We consider $X = \mathbb{Q}$ with the subspace topology coming from $\mathbb{R}$. In Exercise 2.17 we show that the only connected subsets of $\mathbb{Q}$ are the subsets consisting of at most a point. It follows that the components of $\mathbb{Q}$ are precisely the one-point subsets.

The following is in particular the promised analogue of Lemma 2.66.

Lemma 2.68. *Let X be a topological space.*

(1) *As a set X is the disjoint union of its components.*
(2) *Each component of X is connected.*

(3) *Each component C of X is a maximal connected subset of X, i.e. if $C \subsetneq D$, then D is not connected.*
(4) *If A is a subset of X that is closed and open, then A is the union of components of X.*
(5) *Each component of X is the union of path-components of X.*

PROOF (∗). Let X be a topological space.

(1) This statement is clear.
(2) Now let C be a component of X. We pick $c \in C$. For each $d \in C$ there exists by definition a connected subset Y_d of X that contains c and d. We have

$$C = \bigcup_{c \in C} \{c\} \subset \bigcup_{c \in C} Y_c \underset{\substack{\uparrow \\ \text{by definition all points in } Y_d \text{ are equivalent to } c, \text{ thus we have } Y_d \subset C}}{\subset} C \quad \text{which implies that} \quad C = \bigcup_{c \in C} Y_c.$$

It follows from Lemma 2.62 (2) that C is connected.
(3) This statement follows immediately from the definitions.
(4) Let A be a subset of X that is closed and open. Let C be a component of X. Note that $A \cap C$ is a subset of C that is closed and open. By (2) we know that C is connected, hence we see that either $A \cap C = \varnothing$ or $A \cap C = C$. This shows that A is precisely the union of all the components of X that A intersects.
(5) Let C be a component of X. It follows from Lemma 2.66 that it suffices to show that if P is a path-component of X, then either $P \subset C$ or $P \cap C = \varnothing$. Thus let P be a path-component of X. Note that $P \cap C$ is Corollary 2.56 ■

We wrap up the discussion of the components of a topological space with the following lemma.

Lemma 2.69. *Let X be a topological space.*

(1) *Every component of X is a closed subset of X.*
(2) *If X has only finitely many components, then each component is also an open subset of X.*

Example. We consider $X = \mathbb{Q}$. In this case every point is already a component. We see that every component is closed, but in this case no component is an open subset. Therefore we cannot drop the condition on the number of components in the second statement of the lemma.

PROOF (∗). Let X be a topological space.

(1) Let C be a component of X. We need to show that $X \setminus C$ is open. By Lemma 2.5 it suffices to show that any $P \in X \setminus C$ admits a neighborhood that does not intersect C. Thus let $P \in X \setminus C$. By Lemma 2.68 we know that $C \cup \{P\}$ is not connected. Therefore there exists a decomposition $C \cup \{P\} = U \sqcup V$ with U and V open subsets such that both U and V are non-empty. Without loss of generality $C \cap U \neq \varnothing$. Since C is connected and since $C = (U \cap C) \sqcup (V \cap C)$ we see that $C = U \cap C$. Since V is also non-empty we see that $V = \{P\}$. By the definition of the subspace topology that means that there exists an open subset W of X such that $W \cap (C \cup \{P\}) = V = \{P\}$.

But this shows that W is an open neighborhood of P that does not intersect C. We have thus shown that the complement of C is open. We refer to Figure 47 for an illustration of the above argument.

(2) Now suppose that X has only finitely many components. Let C be a component of X. We already know from (1) that C is closed. By definition the complement of C is the union of all other components. Since there are only finitely many and since they are closed by (1) it follows that the complement of C is closed, i.e. C is open. ■

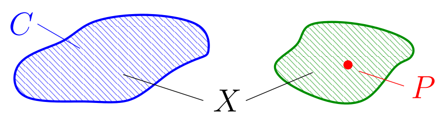


FIGURE 47. Illustration for the proof of Lemma 2.69 (1).

Remark. Let X be a topological space. We say that the points x and y are *quasi-equivalent*, if the following holds

$$\text{if } X = U \sqcup V \text{ is the disjoint union with } U, V \text{ open} \implies \text{either } x, y \in U \text{ or } x, y \in V.$$

It is straightforward to see that this indeed an equivalence relation on X. The equivalence classes of this equivalence relation are called *quasi-components*. It follows immediately from the definitions that components are contained in quasi-components. But the notions are different, for example it is shown in [**SS78**, p. 137] that there exist topological spaces such that quasi-components are not connected. We will not make use of the concept of quasi-equivalence and quasi-components.

We conclude this section with the following lemma that can be viewed as a generalization of Lemma 2.57 and Lemmas 2.61.

Lemma 2.70. (∗) *Let X and Y be topological space. If X is (path-) connected and if Y consists of (path-) components $Y_i, i \in I$, then given any map $f\colon X \to Y$ there exists an $i \in I$ with $f(X) \subset Y_i$.*

PROOF (∗). Again we leave the proof as a straightforward exercise to the reader. ■

2.11. **Local properties.** As so often, let us start out with a definition.

Definition. Let P be a property of topological spaces and let X be a topological space.

(1) We say X is *regionally* P if given any $Q \in X$ and any neighborhood U of Q there exists a neighborhood V of Q that is contained in U that has the property P.

(2) We say X is *locally* P if given any $Q \in X$ and any neighborhood U of Q there exists an *open* neighborhood V of Q that is contained in U that has the property P.

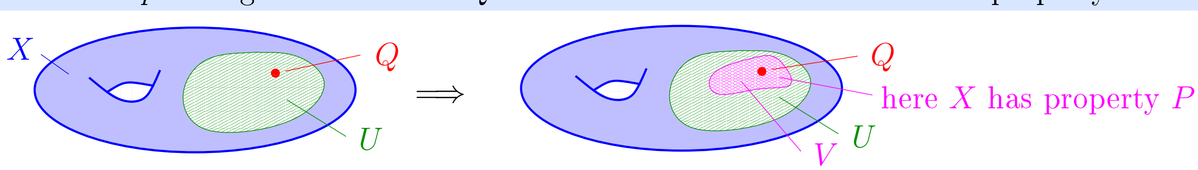


FIGURE 48

Warning. Our usage of the expression "locally P" is non-standard. In fact most textbooks, see e.g. [**Hat02, Bre93, Rot88**], define "locally P" as what we call "regionally P". (Note though that, as is apparent from [**Mun75**, p. 161], in [**Mun75**] our notion of "locally P" is being used.) The property that we now call "locally P" does not seem to have an established name. For our purpose it is useful to give names to both concepts since both have their uses and justifications. Later on we will also introduce notion of a local property of a map, in this context most books implicitly or explicitly seem to work with open neighborhoods. Thus it seems reasonable to us to use the word "local" whenever we deal with open neighborhoods and to rename the concept when any neighborhood is allowed.

Examples.

(1) Every open subset of $\mathbb{R}^n$ is locally path-connected. Indeed, suppose A is an open subset of $\mathbb{R}^n$. Let $Q \in A$ and let U be a neighborhood of Q in A. Then there exists an $\epsilon > 0$ such that $B^n_\epsilon(x) \subset U$. The ϵ-ball $B^n_\epsilon(x)$ is of course path-connected.
(2) The space $X = (-2, 1) \cup [3, 6] \subset \mathbb{R}$ with the usual topology is not path-connected but it is locally path-connected.[40] On the other hand $X = \mathbb{Q}$ is not even locally path-connected.
(3) The line with two zeros, i.e. the topological space $\mathbb{R} \cup \{*\}$ with the topology from page 95, is not Hausdorff, but one can easily show that $\mathbb{R} \cup \{*\}$ is locally Hausdorff.

The following lemma can at times be useful.

Lemma 2.71. *Let P be a property of topological spaces and let X be a topological space. If X is locally P, then the open subsets of X that have P form a basis for the topology of X.*

PROOF. This lemma follows immediately from Lemma 2.27 (1). ■

The following lemma is the key to showing that for "reasonable" topological spaces the notions of being connected and being path-connected are actually the same.

Lemma 2.72. *Let X be a topological space that is locally path-connected. Then*

$$X \text{ is connected} \quad \Longleftrightarrow \quad X \text{ is path-connected.}$$

PROOF. The "$\Leftarrow$"-direction is an immediate consequence of Corollary 2.56. In Exercise 2.63 we will show that if X is locally path-connected, then the converse also holds. ■

In the following we discuss the notion of being regionally compact.

Examples.

(1) Every open subset of $\mathbb{R}^n$ is regionally compact. Indeed, suppose A is an open subset of $\mathbb{R}^n$. Let $Q \in A$ and let U be a neighborhood of Q in A. Then there exists an $\epsilon > 0$ such that $B^n_\epsilon(x) \subset U$. The closed ϵ-ball $\overline{B}^n_{\frac{\epsilon}{2}}(x)$ is a neighborhood of x and it is compact by the Heine-Borel Theorem 2.20.
(2) Similar to (1) one can easily show that every interval is regionally compact. In fact later on we will mostly need the fact that the interval $[0, 1]$ is regionally compact.

[40] Why is that?

In Exercise 2.65 we will see that there exists a topological space that is compact but not regionally compact. The following lemma shows that this slightly unintuitive phenomenon cannot occur if we restrict ourselves to topological spaces that are Hausdorff.

Lemma 2.73. (∗) *Let X be a Hausdorff space.*

(1) *If X is compact, then X is also regionally compact.*
(2) *If given any point $P \in X$ there exists a compact neighborhood of P, then X is regionally compact.*

Proof.

(1) We will use Lemma 2.73 only once, in a rather inessential way. Therefore we leave the proof of this lemma as a not entirely trivial exercise to the reader. Alternatively we refer to [**Wil70**, Theorem 18.2] for a proof.
(2) This statement follows almost immediately from (1). ■

We conclude the discussion of regionally compact topological spaces with another health warning.

Warning. Recall that what we call "regionally P" is often called "locally P". Even more confusing, in the literature frequently "locally compact" is used with different definitions. For example in [**HY88**, p. 71] and [**Dug66**, p. 237] a topological space is called "locally compact" if every point admits an open neighborhood such that its closure is compact. Furthermore in [**SS78**, p. 20], [**Bre93**, p. 31] and [**Jos83**, p. 266] a topological space is called "locally compact" if every point is contained in a compact neighborhood. The latter two notions of "locally compact" are in general different, see e.g. [**SS78**, Example 52]. Furthermore note that for the above two concepts of "locally compact" every compact topological space is already locally compact. But as we mentioned above, in Exercise 2.65 we will see that there exists a compact topological space that is not regionally compact topological. Nonetheless, it is a "metatheorem" that all concepts of "locally compact" in the literature agree for Hausdorff spaces.

In the following definition we turn from properties of topological spaces to properties of maps.

Definition. Let P be a property of maps between topological spaces and let $f\colon X \to Y$ be a map between topological spaces.

(1) We say that $f\colon X \to Y$ is *locally P at $x \in X$* if there exists an open neighborhood U of x and an open neighborhood V of $f(x)$ such that $f\colon U \to V$ has the property P.
(2) We say that f is *locally P* if it is locally P at any $x \in X$.

Examples.

(1) One easily verifies that the map $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is a local homeomorphism at all points $x \neq 0$.
(2) The map

$$\begin{array}{rcl} p\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & e^{2\pi \mathrm{i} t} \end{array}$$

is a local homeomorphism. Indeed, let $t \in \mathbb{R}$. We set $U = (t-\frac{1}{2}, t+\frac{1}{2})$ and $V = p(U)$. The restriction of p to $p\colon U \to V$ is continuous and a bijection. Furthermore it is not difficult to show that p is an open map, see also Exercise 2.34. It follows from Lemma 2.42 (2) that $p\colon U \to V$ is a homeomorphism.

We will often use the statement of the following lemma without making reference to the lemma.

Lemma 2.74. *Let $f\colon X \to Y$ be a map between two topological spaces (for once we do not assume that f is continuous). If f is locally continuous, then f itself is continuous.*

Proof. Let $f\colon X \to Y$ be a locally continuous map between two topological spaces. It follows from Lemma 2.30 (2) that given any $x \in X$ that there exists an open neighborhood U_x of x such that the restriction of f to U_x is continuous. It is an immediate consequence of Lemma 2.35 (1) that f itself is continuous. ■

2.12. **The Lebesgue Lemma and uniformly continuous maps (∗).** In this section we want to prove two statements that concern specifically maps whose domain is actually a metric space. It is perhaps initially difficult to get excited about these results, but they will come in handy later on.

First we intend to formulate and prove the Lebesgue Lemma. Despite its humble name it does actually play an important role in topology. The formulation of the Lebesgue Lemma requires the following definition.

Definition. Let X be a metric space. The *diameter* of a non-empty subset A of X is defined as

$$\operatorname{diam}(A) \;:=\; \sup\{d(a,b)\,|\,a,b \in A\} \;\in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

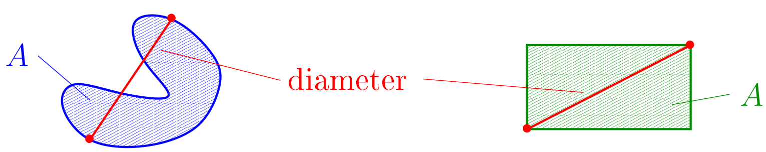


Figure 49

Example. One can easily show that

$$\text{diameter of the hyperrectangle } [a_1,b_1] \times \cdots \times [a_n,b_n] \;=\; \sqrt{(b_1-a_1)^2 + \cdots + (b_n-a_n)^2}.$$

Now we can state the Lebesgue Lemma.

Lemma 2.75. (Lebesgue Lemma) *Let K be a compact metric space and let $\{U_i\}_{i\in I}$ be an open cover of K. Then there exists a $\delta > 0$ such that for every subset A with $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ with $A \subset U_i$.*

Remark. Any $\delta > 0$ that has the property stated in the Lebesgue Lemma is usually called a *Lebesgue number.*

Proof. Since K is compact we can cover K with finitely many of the U_i's. Put differently, without loss of generality we can assume that $I = \{1,\ldots,n\}$ is a finite set.

If there exists an $i \in \{1,\ldots,n\}$ with $K = U_i$, then any $\delta > 0$ has the desired property. Now suppose that this is not the case, i.e. we suppose that for every i we have $U_i \subsetneq K$.[41] We first prove the following claim.

Claim. The function

$$\begin{array}{rcl} f\colon K & \to & \mathbb{R} \\ x & \mapsto & f(x) := \frac{1}{n}\sum\limits_{i=1}^{n} d(x, K\setminus U_i) \end{array}$$

is continuous and positive.

It follows easily from the definitions and the triangle inequality that f is continuous. Now let $x \in K$. We want to show that $f(x) > 0$. Since $K = U_1 \cup \cdots \cup U_n$ there exists an i with $x \in U_i$. Since U_i is open there exists an $r > 0$ with $B_r(x) \subset U_i$. It follows that $d(x, K\setminus U_i) \geq r$, hence $f(x) \geq \frac{r}{n}$. ⊞

Since f is continuous and since K is compact it follows from Lemma 2.40 that the function f has a global minimum δ on K. It follows from the claim that this minimum δ is greater than 0. Now we want to show that this δ has the desired property. More precisely, we want to prove the following claim.

Claim. Let A be a subset of K with $\operatorname{diam}(A) < \delta$. Then there exists an $i \in \{1,\ldots,n\}$ with $A \subset U_i$.

Let $x \in A$. We choose an $m \in \{1,\ldots,n\}$ such that $d(x, K\setminus U_m)$ is maximal. We want to show that $A \subset U_m$. Since the diameter of A is less than δ we have $A \subset B_\delta(x)$. Thus it suffices to show that $B_\delta(x) \subset U_m$. Put differently, we need to show that $d(x, K\setminus U_m) \geq \delta$. Indeed we have

$$d(x, K\setminus U_m) \underset{\substack{\uparrow \\ \text{since } d(x,K\setminus U_m) \geq d(x,K\setminus U_i)}}{\geq} \frac{1}{n}\sum_{i=1}^{n} d(x, K\setminus U_i) = f(x) \underset{\substack{\uparrow \\ \text{by the choice of } \delta}}{\geq} \delta.$$

■

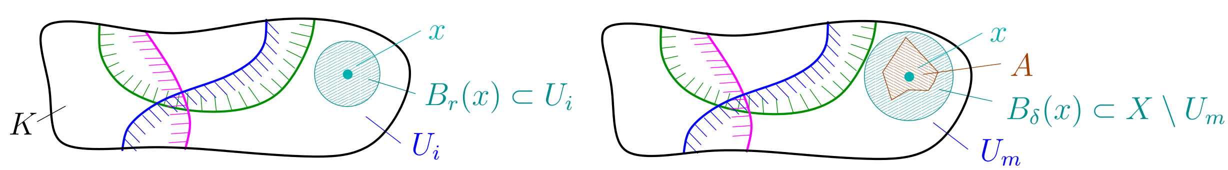


FIGURE 50. Illustration of the proof of the Lebesgue Lemma.

In many cases the following corollary is even more useful.

Corollary 2.76. *Let $f\colon [0,1]^n \to X$ be a map from the cube $[0,1]^n$ to a topological space X and let $\{V_i\}_{i\in I}$ be an open cover of X. Then there exists an $N > 0$ such that for any $a_1,\ldots,a_n \in \{0,\ldots,N-1\}$ there exists an $i \in I$ such that*

$$f\big(\big[\tfrac{a_1}{N},\tfrac{a_1+1}{N}\big] \times \cdots \times \big[\tfrac{a_n}{N},\tfrac{a_n+1}{N}\big]\big) \subset V_i.$$

[41] Where do we use in the subsequent argument that $U_i \subsetneq K$?

More pictorially speaking the corollary says that if $f\colon [0,1]^n \to X$ is a map, and if we are given an open cover of X, then we can always find a small grid on the cube, such that each cube of the grid gets mapped into one of the open sets cover X.

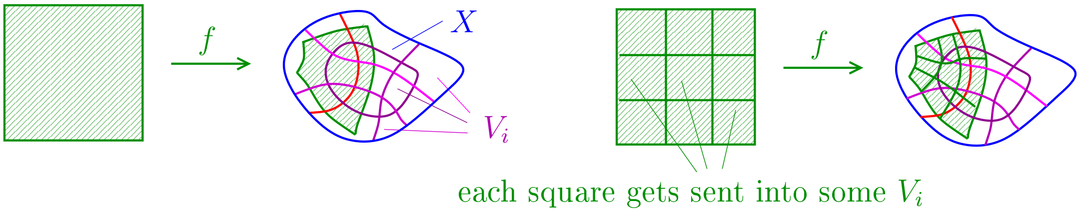


FIGURE 51. Illustration of Corollary 2.76

PROOF. Let $f\colon [0,1]^n \to X$ be a map from the cube $[0,1]^n$ to a topological space and let $\{V_i\}_{i\in I}$ be an open cover of X. By Proposition 2.21 we know that $[0,1]^n$ is compact. Thus we can apply Lemma 2.75 to the open cover $U_i := f^{-1}(V_i)$, $i \in I$ of $[0,1]^n$ and we obtain a $\delta > 0$ such that for each subset A with $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ with $A \subset U_i = f^{-1}(V_i)$, which means that $f(A) \subset V_i$. It follows from the example on page 141 that the diameter of the cube of side length $\frac{1}{m}$ equals $\frac{1}{\sqrt{m}}$. Thus for sufficiently large N any cube of side length $\frac{1}{N}$ has diameter less than δ. ■

Definition. Let $f\colon X \to Y$ be a map between two metric spaces X and Y. By Proposition 2.36 we know that

$$f \text{ is continuous} \quad \Longleftrightarrow \quad \underset{x\in X}{\forall} \ \underset{\epsilon>0}{\forall} \ \underset{\delta>0}{\exists} \ \underset{y\in B_\delta(x)}{\forall} \ d(f(x),f(y)) < \epsilon.$$

Now we define

$$f \text{ is \textit{uniformly continuous}} \quad :\Longleftrightarrow \quad \underset{\epsilon>0}{\forall} \ \underset{\delta>0}{\exists} \ \underset{x\in X}{\forall} \ \underset{y\in B_\delta(x)}{\forall} \ d(f(x),f(y)) < \epsilon.$$

The following proposition is the second main result of this section.

Proposition 2.77. *Let $f\colon X \to Y$ be a continuous map between two metric spaces. If X is compact, then f is uniformly continuous.*

PROOF (∗). Let $f\colon X \to Y$ be a continuous map between two metric spaces and let $\epsilon > 0$. The "⇐"-direction of Proposition 2.36 tells us in particular that given any $x \in X$ there exists a $\delta(x) > 0$ such that for all $y \in B_{\delta(x)}(x)$ we have $d(f(x),f(y)) < \frac{\epsilon}{2}$. Since the open balls $B_{\delta(x)/2}(x)$ form an open cover of X and since X is compact there exist $x_1,\dots,x_k$ such that $B_{\delta(x_1)/2}(x_1) \cup \dots \cup B_{\delta(x_1)/2}(x_1) = X$. We claim that

$$\delta \ := \ \min\left\{\frac{\delta(x_1)}{2},\dots,\frac{\delta(x_k)}{2}\right\}$$

has the desired properties. Thus let $x, y \in X$ with $d(x,y) < \delta$. There exists a $j \in \{1,\dots,k\}$ with $x \in B_{\delta(x_j)/2}(x_j)$. From the triangle inequality and from $\delta \le \frac{\delta(x_j)}{2}$ we obtain that $y \in B_{\delta(x_j)}(x_j)$. It follows that

$$d(f(x),f(y)) \ \underset{\substack{\uparrow \\ \text{triangle inequality}}}{\le} \ d(f(x_j),f(x)) + d(f(x_j),f(y)) \ \underset{\substack{\uparrow \\ \text{since } x\in B_{\delta(x_j)}(x_j) \text{ and } y \in B_{\delta(x_j)}(x_j)}}{<} \ \frac{\epsilon}{2} + \frac{\epsilon}{2} \ = \ \epsilon.$$

We have thus shown that δ does indeed have the desired properties. ■

2.13. **Paracompact topological spaces** ($*$)**.** In this short section we introduce the notion of a "partition of unity" and the notion of a paracompact topological space. Paracompactness is often useful for building interesting maps "out of" a given topological manifolds. Even though this material is important, it is best to skip this section and to return to it, once the actual need for the material arises.

Definition. Let X be a topological space.

(1) The *support* of a function $f\colon X \to \mathbb{R}$ is defined as
$$\operatorname{supp}(f) \ := \ \overline{\{x \in X \mid f(x) \neq 0\}}.$$

(2) A *partition of unity* on X is a family of maps[42] $\{f_i\colon X \to [0,1]\}_{i\in I}$ with the following properties:
   (a) For each $x \in X$ there exists an open neighborhood U of x such that f_i vanishes on U for all but finitely many i. In particular for each $x \in X$ there exist only finitely many $i \in I$ such that $x \in \operatorname{supp}(f_i)$.
   (b) For every $x \in X$ we have[43]
$$\sum_{i\in I} f_i(x) \ = \ 1.$$

(3) Let $\{U_i\}_{i\in I}$ be an open cover. A *partition of unity subordinate to the open cover* is a partition of unity $\{f_j\colon X \to [0,1]\}_{j\in J}$ such that for each $j \in J$ there exists an $i \in I$ with $\operatorname{supp}(f_j) \subset U_i$.

Example. In Figure 52 we consider the topological space $X = \mathbb{R}$ with the open cover given by the intervals $U_i = (i-1, i+1)$, $i \in \mathbb{Z}$. Also we illustrate a smooth partition of unity subordinate to this open cover.

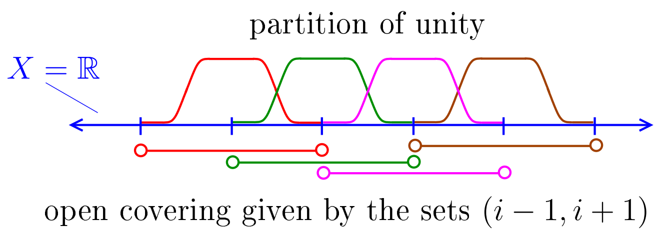


FIGURE 52

Definition. A topological space X is called *paracompact* if it is Hausdorff and if given any open cover $\{U_i\}_{i\in I}$ there exists a partition of unity subordinate to the open cover $\{U_i\}_{i\in I}$.

Example. As we will see, most topological spaces that are of interest to us, are in fact paracompact. For example in Theorems 2.82 and 6.57 and in Proposition 36.13 we will see that

[42]Note that by (a) there are only finitely many non-zero summands, hence the expression on the left-hand side makes sense even if I is infinite.

[43]As always, all maps are assumed to be continuous.

metric spaces, "topological manifolds" and "CW-complexes" are paracompact. Nonetheless, elementary topological psychology tells us that surely not every topological space is paracompact. Indeed in Proposition 7.11 we will see that the "long line", that we will define on page 334, is not paracompact.

The following lemma is sometimes useful for simplifying the notation.

Lemma 2.78. *Let X be a paracompact topological space and let $\{U_i\}_{i\in I}$ be an open cover of X. There exists a partition of unity $\{f_i\colon X \to [0,1]\}_{i\in I}$ such that for every $i \in I$ we have* $\operatorname{supp}(f_i) \subset U_i$.

Proof. Since X is paracompact we know, by definition, that there exists a partition of unity $\{g_j\colon X \to [0,1]\}_{j\in J}$ such that for each $j \in J$ there exists an $i(j) \in I$ with $\operatorname{supp}(g_j) \subset U_{i(j)}$. Now given $i \in I$ we define

$$\begin{array}{rcl} f_i\colon X & \to & [0,1] \\ x & \mapsto & \sum\limits_{j\in J \text{ with } i(j)=i} g_j(x). \end{array}$$

Note that f_i is locally a finite sum of maps, hence the map f_i is locally continuous, which implies that f_i is continuous. Furthermore note that

$$\operatorname{supp}(f_i) \underset{\substack{\uparrow \\ \text{follows easily from Condition (a) of a partition of unity}}}{=} \bigcup_{j\in J \text{ with } i(j)=i} \underbrace{\operatorname{supp}(g_j)}_{\subset U_i} \subset U_i.$$

It is now clear that the functions $\{f_i\colon X \to [0,1]\}_{i\in I}$ have all the desired properties. ■

In our next theorem we will given give a convenient characterization of paracompact topological spaces. To formulate the theorem we need the following straightforward definition.

Definition. Let X be a topological space and let $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ be two covers of X. We say that $\{V_j\}_{j\in J}$ is a *refinement of* $\{U_i\}_{i\in I}$ if each V_j is contained in some U_i.

Theorem 2.79. *Let X be a topological space that is Hausdorff. The following two statements are equivalent:*

1. *The topological space X is paracompact.*
2. *Every open cover $\{U_i\}_{i\in I}$ admits a locally finite open cover $\{V_j\}_{j\in J}$ that is a refinement of $\{U_i\}_{i\in I}$.*[44]

Remark.

1. In many textbooks, e.g. [**Mun75**, Chapter 41], [**Bre93**, Definition I.12.3], [**Kel75**, p. 156] or [**Dug66**, Chapter VIII.2], the condition stated in Theorem 2.79 is used as the definition of a paracompact topological space. Fortunately by Theorem 2.79 our definition agrees with the alternative definition.
2. In some books, see e.g. [**Jos83**, p. 373] the Hausdorff condition is dropped from the definition of paracompactness.

[44] In other words, there exists an open cover $\{V_j\}_{j\in J}$ such that each V_j is contained in some U_i and such that each $x \in X$ admits a neighborhood W that is contained in only finitely many V_j.

PROOF. The "(1)⇒(2)"-direction is basically trivial. Indeed let $\{U_i\}_{i\in I}$ be an open cover. Since X is paracompact there exists a partition of unity $\{f_j\colon X\to[0,1]\}_{j\in J}$ subordinate to the open cover $\{U_i\}_{i\in I}$. It follows immediately from the definitions that $\{f^{-1}((0,\infty])\}_{j\in J}$ is a locally finite open cover that is a refinement of $\{U_i\}_{i\in I}$.

The "(2)⇒(1)" is much more interesting. With some effort this direction can be proved using Urysohn's Lemma 2.50. Since we will not make use of this direction we will not provide the details. Instead we refer to [**Bre93**, Chaper I.12.8], or alternatively [**LW69**, Appendix I Proposition 6] and [**Dug66**, Chapter VIII.4] for details. ∎

The following, slightly technical proposition, sometimes makes it possible find a suitable countable open cover.

Proposition 2.80. (∗) *Let X be a topological space and let $\{U_i\}_{i\in I}$ be an open cover of X. If X is paracompact, then there exists a* countable *locally finite open cover $\{V_j\}_{j\in J}$ such that each component of any V_j is contained in some U_i.*

PROOF (∗). Since X is paracompact there exists a partition of unity $\{f_j\colon X\to[0,1]\}_{j\in J}$ subordinate to the given open cover $\{U_i\}_{i\in I}$. Given a finite subset $S\subset J$ we define

$$V_S \;:=\; \{x\in X\,|\,f_s(x)>f_j(x)\text{ for all }s\in S\text{ and }j\not\in s\}.$$

Since a partition of unity is locally finite one sees almost immediately that V_S is an open subset of X. Furthermore note that given any $s\in S$ we have $V_S\subset\operatorname{supp}(f_s)$. In particular there exists an $i\in I$ with $V_S\subset U_i$. Next note that for finite subsets S and T of J we have $V_S\subset V_T$ if and only if $S\subset T$. Finally note that each $x\in X$ is contained in V_S where S is the finite set $S=\{j\in J\,|\,f_j(x)>0\}$.

Now given $n\in\mathbb{N}_0$ we define V_n to be the union of all V_S with $\#S=n$. It follows from the above that the V_S with $\#S=n$ are disjoint and open. Thus each component of V_n is contained in some V_S which in turn, as we saw above, is contained in some U_i. It is now clear that the family $\{V_n\}_{n\in\mathbb{N}_0}$ has all the desired properties. ∎

The following proposition shows that a large and popular class of topological spaces is paracompact. The proposition perhaps also justifies somewhat the name "paracompact".

Proposition 2.81. *Let X be a topological space. If X is compact and Hausdorff, then it is also paracompact.*

PROOF. Let X be a topological space that is compact and Hausdorff. Furthermore let $\{U_i\}_{i\in I}$ be an open cover of X. We perform the following steps:

(1) Since X is Hausdorff we know by Lemma 2.13 that any subset of X consisting of a single point is closed.
(2) Since X is compact and Hausdorff we know by Lemma 2.46 that X is normal.
(3) Let $x\in X$. We pick an $i_x\in I$ with $x\in U_{i_x}$. It follows almost immediately from (1) and (2) that there exists an open neighborhood V_x of x with $\overline{V}_x\subset U_{i_x}$.
(4) Let $x\in X$. By Urysohn's Lemma 2.50, applied to $A=\{x\}$ and $B=X\setminus V_x$, there exists a continuous function $\varphi_x\colon X\to[0,1]$ with $\varphi_x(x)=1$ and $\varphi_x|_{X\setminus V_x}\equiv 0$. Note that by construction we have $\operatorname{supp}(\varphi_x)\subset U_{i_x}$.

(5) Evidently the open sets $\{\varphi_x^{-1}((0,1])\}_{x\in X}$ form an open cover of X. Since X is compact we know that there exist $y_1,\dots,y_s \in X$ such that the corresponding sets cover X.
(6) By construction for each $x \in X$ there exists a $j \in \{1,\dots,s\}$ with $\varphi_{y_j}(x) > 0$. In particular $\varphi_{y_1} + \dots + \varphi_{y_s}$ is non-zero everywhere.

It follows easily from the construction and (4), (5) and (6) that the maps $g_j := \frac{\varphi_{y_j}}{\varphi_{y_1}+\dots+\varphi_{y_s}}$, $j = 1,\dots,s$ form a partition of unity subordinate to the open cover $\{U_i\}_{i\in I}$. ■

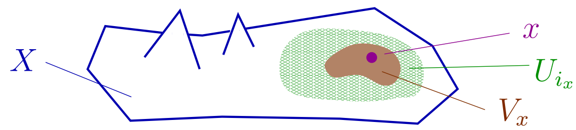


FIGURE 53. Illustration of the proof of Proposition 2.81.

For completeness' sake we also mention the following difficult theorem. We will not make use of it.

Theorem 2.82. (Stone's Theorem) *Every metric space is paracompact.*

PROOF. The theorem was first proved by Arthur Stone [**Stone48**] in 1948. A much shorter proof can be found in [**RudiM69**]. ■

This concludes our discussion of paracompactness. We refer to [**Mun75**, Chapter 6.4] for many other interesting results on paracompact topological spaces.

Exercises for Chapter 2.

Exercise 2.1. Show that for statements (1), (2) and (4) of Lemma 1.3 there are indeed examples such that the stated inclusion is indeed a strict inclusion.

Exercise 2.2. Let $f\colon X \to Y$ be a map between two sets. Let $A \subset X$ and let $B \subset Y$ be two subsets. Prove the following equivalence of statements:

$$f(A) \subset B \qquad \Longleftrightarrow \qquad A \subset f^{-1}(B).$$

Exercise 2.3. Let X be a topological space and let A be a subset of X. Show that the following three equalities hold:

$$\begin{array}{lrcl} (1) & \mathring{A} & = & \{x \in A \,|\, \text{ there exists a neighborhood } U \text{ of } x \text{ that is contained in } A\} \\ (2) & \overline{A} & = & \{x \in X \,|\, \text{ every neighborhood of } x \text{ contains at least one point in } A\} \\ (3) & \partial A & = & \{x \in X \,|\, \text{ every neighborhood of } x \text{ contains at least one point in } A \\ & & & \qquad\qquad\qquad \text{and one point that does not lie in } A\} \end{array}$$

Exercise 2.4. Let $n \in \mathbb{N}$. We fix a point $P \in \mathbb{R}^n$ and we consider the maps

$$\begin{array}{rcl} d_M\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ ((x_1,\dots,x_n),(y_1,\dots,y_n)) & \mapsto & \sum\limits_{i=1}^{n} |x_i - y_i| \end{array} \quad \text{and} \quad \begin{array}{rcl} d_S\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & \|x-P\| + \|y-P\|. \end{array}$$

(a) Show that both maps define metrics on $\mathbb{R}^n$.
Remark. The metric d_M is called the *Manhattan-metric* since it measures the distance between two points if one can go only vertically (avenues) and horizontally (streets). The Manhattan metric is sometimes also called the *taxicab metric*. The metric d_S is called the *SNCF-metric* since it models the distance in France where every train between two points necessarily has to go through Paris.

(b) Let $n = 2$. Sketch open balls in $\mathbb{R}^n$ with respect to the two metrics.

Manhattan metric

SNCF-metric

FIGURE 54. Illustration for Exercise 2.4.

Exercise 2.5. A norm on a real vector space V is a map $f\colon V \to \mathbb{R}_{\geq 0}$ that satisfies the following three properties:

(1) For every $v \in V$ we have $\quad f(v) = 0 \iff v = 0.$
(2) For every $v \in V$ and $\lambda \in \mathbb{R}$ we have $\quad f(\lambda \cdot v) = |\lambda| \cdot f(v).$
(3) For every $v, w \in V$ we have $\quad f(v + w) \leq f(v) + f(w).$

A norm f induces a metric on d_f by setting $d_f(v, w) := f(v - w)$, in particular it induces a topology on V. Show that any two norms on a *finite-dimensional* vector space give rise to the same topology.
Remark. In Exercise 2.5 we will see that the analogous statement for infinite-dimensional vector spaces does not necessarily hold.

Exercise 2.6. Let $C([0, 1])$ be the real vector space of continuous maps from the interval $[0, 1]$ to $\mathbb{R}$. We consider the

$$\text{supremum norm } s(f) := \sup\{f(x) \,|\, x \in [0, 1]\} \quad \text{and the integral norm} \quad i(f) := \int_0^1 f(x)\, dx.$$

We denote the topologies corresponding to these norms by $\mathcal{S}$ and $\mathcal{I}$. Show that one topology is contained in the other and show that the two topologies are not the same.
Remark. This shows in particular that the identity map $\mathrm{id}\colon (C([0, 1], \mathcal{S}) \to (C([0, 1], \mathcal{I})$ is not a homeomorphism.

Exercise 2.7. Let $C([0, 1])$ be the real vector space of continuous maps from the interval $[0, 1]$ to $\mathbb{R}$ and let $C^1([0, 1])$ be the real vector space of continuously differentiable maps from the interval $[0, 1]$ to $\mathbb{R}$. In this exercise we equip both vector spaces with the supremum norm $s(f) := \sup\{f(x) \,|\, x \in [0, 1]\}$. We consider the two maps

$$\begin{array}{rcl} D\colon C^1([0,1]) & \to & C([0,1]) \\ f & \mapsto & f' \end{array} \qquad \text{and} \qquad \begin{array}{rcl} I\colon C^1([0,1]) & \to & C([0,1]) \\ f & \mapsto & \left(x \mapsto \int_0^x f(t)\, dt\right). \end{array}$$

Note that D and I are inverses of one another, in particular both are isomorphisms of real vector spaces.

(a) Show that I is continuous.
(b) Show that D is not continuous.

Remark. This exercise shows that integration is usually a well-behaved operation whereas differentiation can be problematic.
Remark. This exercise is a variation on Exercise 2.6.

Exercise 2.8. Let U be an open subset of $\mathbb{R}^2$. Is it true that the interior of $\overline{U}$ equals U?

Exercise 2.9. Let U be an open subset of $\mathbb{R}^n$. We denote by $C = \overline{U}$ the closure of U.
(a) Suppose that U is convex. Show that $U = \mathring{C}$.
(b) Given an example of an open subset U of $\mathbb{R}^n$ where $U \neq \mathring{C}$.

Exercise 2.10. Let $n \in \mathbb{N}$. Show that there are precisely $n+2$ homeomorphism types of compact convex subsets of $\mathbb{R}^n$.
Remark. Use Proposition 2.52 (2).

Exercise 2.11. Let $f\colon X \to Y$ be a map between topological spaces and let $B \subset Y$ be a subset.
(a) Show that $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$.
(b) Given an example that shows that in general we do not have an equality in (a).
(c) Show that $f(f^{-1}(\overline{B})) = \overline{B}$.

Now, since was so much fun we also do it with the interiors instead of the closures:
(d) Show that $f^{-1}(\mathring{B})$ is contained in the interior of $f^{-1}(B)$.
(e) Given an example that shows that in general we do not have an equality in (e).
(f) Show that $f(f^{-1}(\mathring{B})) = \mathring{B}$.

Exercise 2.12. Does there exist a map $f\colon S^2 \to S^2$ with a single fixed point, i.e. with a unique point x such that $f(x) = x$?

Exercise 2.13. Let $\{U_i\}_{i\in I}$ be a family of subsets. If I is finite, then we know by Lemma 2.10 that the following two equalities hold:

$$\text{(a)} \quad \text{interior of } \bigcap_{i\in I} U_i \;=\; \bigcap_{i\in I} \mathring{U}_i \qquad \text{and} \qquad \text{(b)} \quad \text{closure of } \bigcup_{i\in I} U_i \;=\; \bigcup_{i\in I} \overline{U_i}.$$

Do these equalities also hold if I is infinite?

Exercise 2.14. Let (X,d) and (Y,e) be two metric spaces and let $\varphi\colon X \to Y$ be a map.
(a) Suppose that there exists some $C > 0$ such that for every choice of $x, x' \in X$ we have $e(\varphi(x), \varphi(x')) < C \cdot d(x, x')$. Show that d is continuous.
(b) Suppose that there exists some $c > 0$ such that for every choice of $x, x' \in X$ we have $e(\varphi(x), \varphi(x')) > c \cdot d(x, x')$. Furthermore suppose that φ is surjective. Does it follow that d is open?

Exercise 2.15. Show that every connected non-empty subset of $\mathbb{R}$ is an interval.

Exercise 2.16. Is every open subset of $\mathbb{R}$ the disjoint union of countably many open intervals?

Exercise 2.17. We consider $X = \mathbb{Q}$ with the subspace topology coming from $\mathbb{R}$. Show that the only connected subsets of $\mathbb{Q}$ are the subsets consisting of at most a point.

Exercise 2.18. Let $f\colon X \to Y$ be a map between topological spaces, let $U \subset X$ be a subset and let $A \subset Y$ be a subset.

(a) Suppose that A is closed and suppose that $f(U) \subset A$. Show that $f(\overline{U}) \subset A$.
(b) Suppose that A is open and that $A \subset f(U)$. Show that $A \subset f(\mathring{U})$.

Exercise 2.19. Let (X, d) be a compact metric space. A rather surprising theorem, see **[Gros64]** or alternatively **[Ris06]**, says that there exists a unique $r \in \mathbb{R}_{\geq 0}$ such that for any finite sequence of points $x_1, \ldots, x_n \in X$ there exists an $x' \in X$ with $\frac{1}{n}\sum\limits_{i=1}^{n} d(x_i, x') = r$. This number is called the *rendezvous value of* (X, d).

(a) Determine the rendezvous value of the interval $X = [0, 1]$ equipped with the metric $d(x, y) = |x - y|$.
(b) Determine the rendezvous value of the circle $X = S^1$ equipped with the Euclidean metric of $\mathbb{R}^2$.

Exercise 2.20.

(a) Show that "$\mathbb{R}^n$ with a point at infinity" is Hausdorff and compact.
(b) Show that "the line with two zeros" is neither Hausdorff nor compact.

Exercise 2.21. Let X be a topological space. We say a subset $A \subset X$ is a *retract of* X if there exists a retraction $r\colon X \to A$, i.e. a map with $r(a) = a$ for all $a \in A$. (Evidently we demand that r is continuous).

(a) Suppose that X is Hausdorff. Show that A is a closed subset of X.
(b) Show that the conclusion of (a) does not necessarily hold if we do not assume that X is Hausdorff.

Exercise 2.22. Let X be a topological space. Given a compact subset $K \subset X$ we define $m(K)$ to be the number of non-compact path-components of $X \setminus K$ that are *not* contained in a compact subset of X. We refer to

$$\operatorname{End}(X) \ := \ \max\{m(K) \,|\, K \text{ is a compact subset of } X\} \in \mathbb{N}_0 \cup \{\infty\}$$

as the *number of ends of* X.

(a) Given $n \in \mathbb{N}$ determine $m(\mathbb{R}^n)$.
Remark. This exercise shows that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.
(b) Can you find connected topological spaces with arbitrarily large number of ends?

Remark. We will introduce the "ends" of a topological space in Exercise 25.13. We will see that what we now call the "number of ends" equals the cardinality of the set of ends.

Exercise 2.23. Given $a, b \in \mathbb{N}$ we define

$$U_a(b) \ := \ \{b + n \cdot a \in \mathbb{N} \,|\, n \in \mathbb{Z}\} \ \subset \ \mathbb{N}.$$

Furthermore we set

$$\mathcal{C} \ := \ \{U_p(b) \,|\, p, b \in \mathbb{N} \text{ and } p \text{ prime}\} \ \subset \ \mathcal{P}(\mathbb{N}).$$

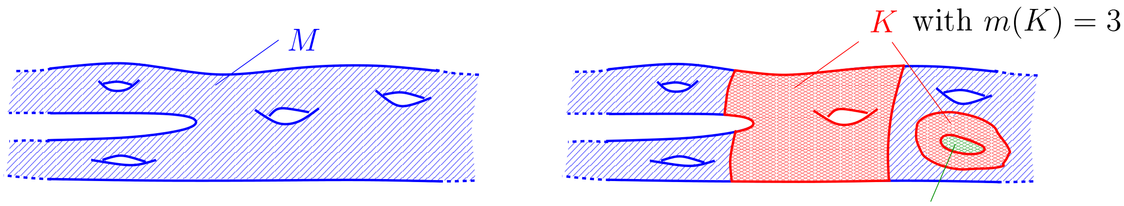


FIGURE 55. Illustration for Exercise 2.22.

The topology $\mathcal{T}$ on $\mathbb{N}$ generated by $\mathcal{C}$ is called the *prime integer topology*.

(a) Show that $(\mathbb{N}, \mathcal{T})$ is Hausdorff.
(b) A topological space X is called *regular* if given any $P \in X$ and given any closed subset $A \subset X$ there exist disjoint open subsets U and V with $P \in U$ and $A \subset V$. Show that $(\mathbb{N}, \mathcal{T})$ is not regular.

Exercise 2.24. Given $a \in \mathbb{Z} \setminus \{0\}$ and $b \in \mathbb{Z}$ we define

$$S(a, b) \; := \; \{a \cdot n + b \,|\, n \in \mathbb{Z}\} \; = \; a \cdot \mathbb{Z} + b.$$

Let $\mathcal{T}$ be all subsets of $\mathbb{Z}$ that are the union of sets of the form $S(a, b)$.

(a) Show that $\mathcal{T}$ is a topology on $\mathbb{Z}$. (This topology is called the *evenly spaced integer topology* on $\mathbb{Z}$).
(b) Show that each set of the form $S(a, b)$ is also closed.
(c) Let P be the set of prime numbers. Show that $\mathbb{Z} \setminus \{-1, 1\} = \bigcup\limits_{p \in P} S(0, p)$.
(d) Show that there exist infinitely many primes.

Remark. This proof that there exist infinitely many primes was found by Harry Furstenberg [**Furs55**] in 1955.

Exercise 2.25. Let $f\colon X \to Y$ be a map between topological spaces. Show that f is continuous if and only if for every $A \subset X$ we have $f(\overline{A}) \subset \overline{f(A)}$.

Exercise 2.26. Let $f\colon X \to Y$ be a continuous map between topological spaces. By Lemma 2.40 we know that the image of a compact subset of X under f is a compact subset of Y.

(a) Give an example of a continuous map $f\colon \mathbb{R} \to \mathbb{R}$ and an open subset $U \subset \mathbb{R}$ such that $f(U)$ is not an open subset of $\mathbb{R}$.
(b) Give an example of a continuous map $f\colon \mathbb{R} \to \mathbb{R}$ and a closed subset $U \subset \mathbb{R}$ such that $f(U)$ is not a closed subset of $\mathbb{R}$.

Exercise 2.27. Show that the maps

$$\begin{array}{rcl} \alpha\colon \mathbb{R}^2 & \to & \mathbb{R} \\ (x, y) & \mapsto & x + y \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \beta\colon \mathbb{R}^2 & \to & \mathbb{R} \\ (x, y) & \mapsto & x \cdot y \end{array}$$

are continuous.

Exercise 2.28. Let X be a topological space, let $A \subset X$ be a closed subset and let $K \subset X$ be a compact subset. Show that $A \cap K$ is also a compact subset of X.

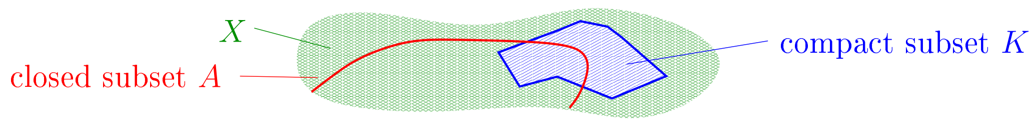


FIGURE 56

Exercise 2.29. Let $f\colon X \to Y$ be a continuous map from a topological space X to some metric space Y, e.g. $Y = \mathbb{R}^n$. Let $P \in Y$ be a point that is not contained in $f(X)$. We suppose that X is compact. Show that there exists an $r > 0$ such that $B_r(P) \cap f(X) = \varnothing$.

Exercise 2.30. Let $f, g\colon X \to Y$ be two maps between topological spaces.

(a) Suppose that Y is Hausdorff. Show that $\{x \in X \mid f(x) = g(x)\}$ is a closed subset of X.
(b) Show that the conclusion of (a) does not necessarily hold if we do not assume that Y is Hausdorff.

Exercise 2.31. Let $\mathbb{R}^n \cup \{\infty\}$ be "$\mathbb{R}^n$ with a point at infinity". Let $v \in \mathbb{R}^n$. Show that the map

$$\begin{array}{rcl} \mathbb{R}^n \cup \{\infty\} & \to & \mathbb{R}^n \cup \{\infty\} \\ Q & \mapsto & \begin{cases} Q + v, & \text{if } Q \in \mathbb{R}^n, \\ \infty, & \text{if } Q = \infty \end{cases} \end{array}$$

is continuous.

Exercise 2.32. Let X be a topological space and let $A \subset X$ be a subset.

(a) We suppose that X is Hausdorff. Show that $\partial A = \partial \overline{A}$.
(b) Does the conclusion in (a) also hold if X is not Hausdorff?

Exercise 2.33. Let f be a complex polynomial of degree ≥ 1. Show that the map

$$\begin{array}{rcl} \Theta(f)\colon \mathbb{C} \cup \{\infty\} & \to & \mathbb{C} \cup \{\infty\} \\ t & \mapsto & \begin{cases} f(t), & \text{if } t \in \mathbb{C}, \\ \infty, & \text{if } t = \infty \end{cases} \end{array}$$

is continuous.
Remark. By Lemma 2.44 we can make the identification $S^2 = \mathbb{C} \cup \{\infty\}$. Thus we see that complex polynomials of degree ≥ 1 give rise to interesting self-maps of S^2.
Remark. The analogous statement also holds for a polynomial f with real coefficients, it gives rise to a self-map $\Theta(f)$ of $\mathbb{R} \cup \{\infty\} = S^1$. We illustrate this situation in Figure 57.

Exercise 2.34. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & e^{2\pi i t} \end{array}$$

(a) Show that f is an open map.
(b) Is f also a closed map?

FIGURE 57. Illustration for Exercise 2.33.

Exercise 2.35. Let X be a set and let $\{A_i\}_{i \in I}$ be a family of subsets. Suppose that each A_i is equipped with a topology.

(a) We say $U \subset X$ is open if for each $i \in I$ the intersection $U \cap A_i$ is an open subset of A_i. Show that this defines a topology on X.
(b) We consider the set $X = [-2, 2]$.
  (i) We consider the subsets $A_1 = [-2, 0]$ and $A_2 = [0, 2]$.
  (ii) We consider the subsets $B_1 = [-2, 0]$ and $B_2 = (0, 2]$.
  (iii) We consider the subsets $C_1 = [-2, 0]$, $C_2 = (0, 2]$ and $C_3 = (-1, 1)$.
  (iv) We consider the subsets $D_1 = [-1, 1)$ and $D_2 = (0, 2]$.

  For each of these four settings we equip the subsets with the obvious topology and we equip $X = [-2, 2]$ with the corresponding topology. Describe the resulting four topological spaces. Are these topological spaces that we are familiar with?

We refer to Figure 58 for an illustration.

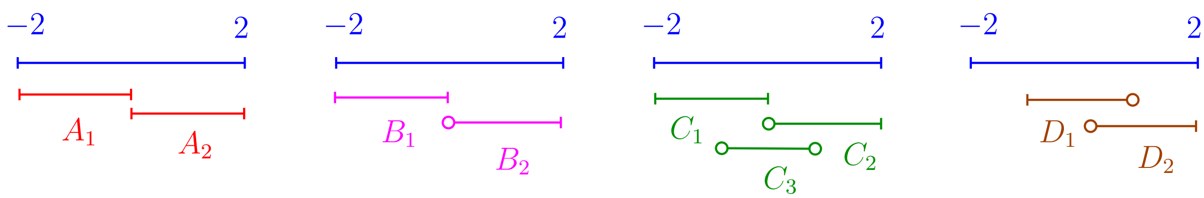


FIGURE 58

Exercise 2.36. Let $P_1, \dots, P_n \in \mathbb{R}^k$.

(a) Show that the convex hull of the set $\{P_1, \dots, P_n\}$ is given by the set

$$\Big\{ \sum_{i=1}^{n} t_i P_i \,\Big|\, t_1, \dots, t_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^{n} t_i = 1 \Big\}.$$

  Note: We make no hypothesis on the points $P_1, \dots, P_n$, in particular some of them could be identical and we do not assume say that the vectors $P_2 - P_1, \dots, P_n - 1$ are linearly independent.
(b) Show that the set in (a) is compact.
(c) Is the convex hull of a compact subset of $\mathbb{R}^k$ necessarily compact?
(d) Is the convex hull of an open subset of $\mathbb{R}^k$ necessarily an open subset?

Exercise 2.37. Let $f, g \colon X \to Y$ be two maps between topological spaces. Show that if f and g agree on a dense subset of X, then $f = g$.
Remark. Evidently by a "map" we mean as always a continuous map.

Exercise 2.38. Let X be a topological space. We say a subset $U \subset X$ is *large* if U is open and dense.

(a) Show that the intersection of finitely many large subsets of X is again large.
(b) Is it true that the intersection of countably many large subsets of X is necessarily large?

Exercise 2.39. Let $f, g\colon X \to Y$ be two maps between two topological spaces. Let $A \subset X$ be a subset such that f and g agree on A. Show that if A is dense, if Y is Hausdorff and if f and g are continuous, then f and g agree on all of X.

Exercise 2.40.

(a) Show that the half-open interval $[0,1)$ and the open interval $(0,1)$ are not homeomorphic.
(b) Is the half-open rational interval $[0,1)\cap\mathbb{Q}$ homeomorphic to the open rational interval $(0,1)\cap\mathbb{Q}$?

Exercise 2.41. Give an example of a bijective continuous map $f\colon X \to Y$ between two topological spaces where X is compact but such that f is nonetheless not a homeomorphism.

Exercise 2.42. Let X and Y be topological spaces and let $i\colon X \to Y$ be a map. We suppose that there exists a map $r\colon Y \to X$ with $r \circ i = \mathrm{id}_X$. Show that i is an embedding.

Exercise 2.43. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Let $0 \le a < b \le 1$. Show that f can only go finitely many times from a to b. More precisely, show that it is not possible to find numbers $x_1 < y_1 < x_2 < y_2 < x_3 < y_3 < \ldots$ in $[0,1]$ such that for each $i \in \mathbb{N}$ we have $f(x_i) = a$ and we have $f(y_i) = b$.

Exercise 2.44. Show that the topological space $[0,1]\cap\mathbb{Q}$, equipped with the subspace topology coming from $\mathbb{R}$, is not compact.

Exercise 2.45. We consider

$$\mathbb{R}^\infty \;:=\; \{(x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}} \,|\, \text{there exists an } i \in \mathbb{N} \text{ such that } x_j = 0 \text{ for } j \ge i\}.$$

We equip $\mathbb{R}^\infty$ with the Euclidean metric

$$\begin{aligned} d\colon \mathbb{R}^\infty \times \mathbb{R}^\infty &\to \mathbb{R}^\infty \\ ((x_1, x_2, \ldots), (y_1, y_2, \ldots)) &\mapsto \sqrt{\sum_{i\in\mathbb{N}} |x_i - y_i|^2}. \end{aligned}$$

Show that $(\mathbb{R}^\infty, d)$ is not compact.

Exercise 2.46.

(a) Let A be a real $m \times n$–matrix. Show that there exists a $C \in \mathbb{R}$ such that for all $v \in \mathbb{R}^n$ we have $\|Av\| \le C \cdot \|v\|$.
(b) Let $A \in \mathrm{GL}(n,\mathbb{R})$ be an invertible matrix. Show that there exists a $\lambda > 0$ such that $\|Av\| \ge \lambda\|v\|$ for all $v \in \mathbb{R}^n$.

Hint. In both cases use that S^{n-1} is compact.

Exercise 2.47. Let X be a topological space. A subset $A \subset X$ is called *precompact* if the closure $\overline{A}$ is compact. Let X be a subset of some topological space Y and let $A \subset X$ be a precompact subspace.

(a) Show that A is not necessarily precompact in Y.
(b) Show that if Y is Hausdorff, then A is in fact precompact in Y.

Remark. Dealing with closures can be quite subtle. For example, it seems to me that for (b) you want to use Lemma 2.4 (2) at some point.

Exercise 2.48. Let X be a topological space. We say that a family $\{A_i\}_{i\in I}$ of subsets has the *finite intersection property* if given any finite collection of indices $i_1, \dots, i_r \in I$ the corresponding intersection $A_{i_1} \cap \dots \cap A_{i_r}$ is non-empty.

(a) We assume that X is a compact topological space and that all the A_i are closed subsets. Show that $\bigcap\limits_{i\in I} A_i \neq \varnothing$.
(b) Show that in (a) we cannot drop the hypothesis that X is compact.

Exercise 2.49. Let A be an invertible real $(m \times m)$–matrix. Show that the map

$$\begin{array}{rcl} \mathbb{R}^m \cup \{\infty\} & \to & \mathbb{R}^m \cup \{\infty\} \\ x & \mapsto & \begin{cases} A \cdot x, & \text{if } x \in \mathbb{R}^m, \\ \infty, & \text{if } x = \infty \end{cases} \end{array}$$

is continuous.
Hint. Use Exercise 2.46.

Exercise 2.50. Is every open map necessarily continuous? More precisely, if $f\colon X \to Y$ is a map between topological spaces with the property that for any open $U \subset X$ the image $f(U)$ is an open subset of Y, does it follow that f is continuous?

Exercise 2.51. Let $n \in \mathbb{N}$. We consider the stereographic projection

$$\begin{array}{rcl} \Phi\colon S^n & \to & \mathbb{R}^n \cup \{\infty\} \\ (x_1, \dots, x_{n+1}) & \mapsto & \begin{cases} \left(\dfrac{x_1}{1 - x_{n+1}}, \dots, \dfrac{x_n}{1 - x_{n+1}}\right), & \text{if } x_{n+1} < 1, \\ \infty, & \text{if } x_{n+1} = 1 \end{cases} \end{array}$$

that we introduced in Lemma 2.44. Show that Φ is continuous.

Exercise 2.52. Show that the topologist's sine curve, that we defined on page 131 is connected and show that it is *not* path-connected.

Exercise 2.53. Let X be a topological space and let $\mathcal{U} \subset \mathcal{P}(X)$ be an open cover of X. Show that the following two statements are equivalent:

(1) The topological space X is connected.
(2) Given any $P, Q \in X$ there exist $U_0, \dots, U_k \in \mathcal{U}$ with $P \in U_0$, $Q \in U_k$ and such that $U_i \cap U_j = \varnothing$ if $|i - j| \geq 2$.

Remark. Some people refer to this statement as the *Sausage Lemma*.

Exercise 2.54. Let X be a topological space and let $A \subset X$ be a subset.

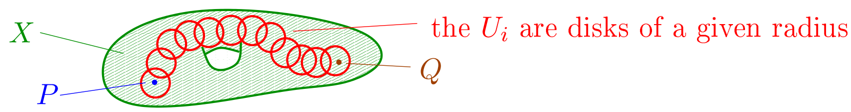


FIGURE 59. Illustration of Exercise 2.53.

(a) Suppose that X is path-connected and that ∂A is path-connected. Show that $X \setminus \mathring{A}$ is also path-connected.
(b) Suppose that X is connected and that ∂A is connected. Show that $X \setminus \mathring{A}$ is also connected.

Exercise 2.55. Let X be a topological space and let $\{B_i\}_{i\in I}$ be a family of *open* subsets such that the intersection $Y := \bigcap\limits_{i\in I} B_i$ is non-empty.

(a) We suppose that X and Y are path-connected. Show that each B_i is path-connected. *Hint.* Use the Lebesgue Lemma 2.75.
(b) We suppose that X and Y are connected. Show that each B_i is connected.
(c) Show that statements (a) and (b) are wrong if we drop the hypothesis that the B_i are open.
(d) Do statements (a) and (b) hold if each B_i is a closed subset?

Remark. This statement can be viewed as a partial converse to Lemma 2.62.

Exercise 2.56. Let X be a topological space and let $A \subset X$ be a subset. Show that if A is connected, then its closure $\overline{A}$ is also connected.

Exercise 2.57. Let $f\colon [0,1] \to [0,1]^2$ be an injective map such that $f([0,1])$ has non-zero 2-dimensional Lebesgue measure. Show that there exists an injective map $g\colon S^1 \to [0,1]^2$ such that $g(S^1)$ has non-zero 2-dimensional Lebesgue measure.
Remark. Note that this exercise provides the details for an easy step in the proof of Proposition 2.60 (2).

Exercise 2.58. A topological space X is called *arc-connected* if given any two points $P, Q \in X$ there exists a map $f\colon [0,1] \to X$ with $f(0) = P$, $f(1) = Q$ and which has the property that f is an embedding.[45]

(a) Show that the line with two zeros is not arc-connected.
(b) Let $n \geq 2$ and let X be the "$\mathbb{R}^n$ with two origins", which is defined in the hopefully obvious way. Is X arc-connected?

Remark. A rather amazing (non-trivial!) theorem says that any topological space that is path-connected *and which is Hausdorff* is in fact arc-connected. We refer to [**Wil70**, Corollary 31.6], [**Enge89**, p. 381] and [**HY88**, p. 118 and p. 129] for a proof of this statement.

[45] Sometimes, see e.g. [**Wil70, Enge89**], such topological spaces are also called *arcwise connected*. Rather confusingly some other books, e.g. [**Bre93**], use the term *arcwise connected* for what we call path-connected. The lesson is, that in any given book one always should look up the definitions.

Exercise 2.59.

(a) Let $n \in \mathbb{N}_{\geq 3}$. We consider
$$X_n \;:=\; \{r \cdot e^{2\pi i t k/n} \,|\, r \in [0,1) \text{ and } k \in \{0,\dots,n-1\}\}.$$
We write $P := 0$. Show that every self-homeomorphism $f\colon X_n \to X_n$ preserves P. We refer to Figure 60 for an illustration.

(b) Does the statement in (a) also hold for $n = 1$ and/or $n = 2$?

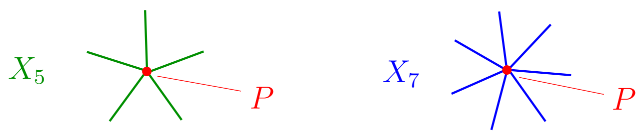


FIGURE 60

Exercise 2.60. Let $n \in \mathbb{N}$. We consider the general linear group
$$\mathrm{GL}(n,\mathbb{R}) \;=\; \{A \in \mathrm{M}(n \times n, \mathbb{R}) \,|\, \det(A) \neq 0\} \;\subset\; \mathrm{M}(n \times n, \mathbb{R}) \;=\; \mathbb{R}^{n^2}.$$
Show that two matrices lie in the same path-component of $\mathrm{GL}(n,\mathbb{R})$ if and only if the signs of their determinants agree. (Put differently, the goal is to prove the first part of Lemma 2.65.)

Exercise 2.61.

(a) Let $n \geq 1$. Show that the sphere S^n is path-connected.

(b) Is the line with two zeros connected?

Exercise 2.62. A subset of a topological space is sometimes called *clopen* if it is open and closed. Let $n \in \mathbb{N}$. Determine all clopen subsets of $\mathbb{R}^n$.

Exercise 2.63. Let X be a topological space that is locally path-connected. Show that if X is connected, then it is also path-connected.

Exercise 2.64. Let X be a path-connected subset of $\mathbb{R}^2$. Is it also necessarily regionally path-connected?

Exercise 2.65. We consider the set $X = (0,1)$ with the topology where the open sets are precisely the sets $\varnothing, X$ and the intervals $(0, 1 - \frac{1}{n})$. Show that X is compact and show that X is not regionally compact.

Exercise 2.66. Let X be a topological space. We consider the set $X^* := X \cup \{\infty\}$. We say $U \subset X^*$ is open if one of the following conditions is satisfied:

(1) U is an open subset of X, or

(2) $\infty \in U$ and there exists a compact subset K of X such that $U = X^* \setminus K$.

The inclusion $c\colon X \to X^*$ is called the *Alexandroff extension*. Show that the following statements hold:

(a) The open sets do indeed define a topology on X.

(b) The map c is continuous and open.

(c) Show that the topological space X^* is compact.

(d) Show that if X is non-compact, then $c(X) \subset X^*$ is dense.
(e) Show that X^* is Hausdorff if and only if X is Hausdorff and regionally compact.

Remark. The topological space X^* is called the *one-point compactification of* X or *Alexandroff compactification of* X.[46]

Remark. One can show easily that the one-point compactification of $\mathbb{R}^n$ equals the "$\mathbb{R}^n$ with a point at infinity" that we introduced on page 95.

Exercise 2.67. Let $X = \mathbb{N}$ with the discrete topology. Show that the one-point compactification of X, as defined in Exercise 2.66, is homeomorphic to a subset of $\mathbb{R}$.

Exercise 2.68.
(a) Show that every paracompact topological space is regular.
(b) Show that every paracompact topological space is normal.

Remark. These two statements can be viewed as a generalization of Lemma 2.46.

Exercise 2.69. Let X be a paracompact topological space and let $A \subset X$ be a closed subset. Show that A is also paracompact.

[46]Some authors use the expression "one-point compactification" only if X is non-compact, Hausdorff and regionally compact.

3. Constructions of topological spaces

In this chapter we will get to know many techniques for constructing new examples of topological spaces out of given ones.

3.1. **The disjoint union of topological spaces.** We start out this section with a short set-theoretic discussion. Sometimes, given two sets A and B one would like to consider a new set that is the "disjoint union of A and B". This leads us to the following, arguably slightly awkward definition.

Definition.

(1) Let A and B be two sets. We define the *disjoint union of A and B* as[47]

$$A \sqcup B \ := \ (A \times \{1\}) \cup (B \times \{2\}).$$

(2) If $\{A_i\}_{i\in I}$ is a family of sets, then we define the *disjoint union* as

$$\bigsqcup_{i\in I} A_i \ := \ \bigcup_{i\in I} (A_i \times \{i\}).$$

Convention. Let A and B be two sets. In most situations, when there is no danger of confusion, we use the obvious bijections $A = A \times \{1\}$ and $B = B \times \{2\}$ to identify A and B with the corresponding subsets of $A \sqcup B$. In particular, if A and B are already disjoint, then we make the identification $A \sqcup B = A \cup B$. The same convention applies to the disjoint union $\bigsqcup\limits_{i\in I} A_i$. At times the author of these notes will rely on the good will of the reader to correctly interpret a given statement.

Definition. Given a family $\{X_i\}_{i\in I}$ of topological spaces we equip the disjoint union $\bigsqcup\limits_{i\in I} X_i$ with the *disjoint union topology* that is defined by the condition that $U \subset \bigsqcup\limits_{i\in I} X_i$ is open if and only if for each $i \in I$ the intersection $U \cap X_i$ is an open subset of X_i.

For completeness we state the following lemma which follows immediately from the definition of the disjoint union topology together with Lemma 2.34.

Lemma 3.1. *Let $\{X_i\}_{i\in I}$ be a family of topological spaces. A subset $A \subset \bigsqcup\limits_{i\in I} X_i$ is closed if and only if for each $i \in I$ the intersection $A \cap X_i$ is a closed subset of X_i.*

The following lemma gives a criterion for showing that a union of subsets is actually the disjoint union of subsets.

Lemma 3.2. *Let X be a topological space and let A and B be two subsets of X. The following two statements are equivalent:*

(1) *There exist open neighborhoods U of A and V of B with $U \cap B = \varnothing$ and $V \cap A = \varnothing$.*

[47] In other words, we replace A and B by sets that are disjoint but that admit obvious bijections to A and B.

(2) *The map*

$$\begin{array}{rcl} A \cup B & \to & A \sqcup B \\ x & \mapsto & \begin{cases} (x,1), & \text{if } x \in A, \\ (x,2), & \text{if } x \in B \end{cases} \end{array}$$

is a homeomorphism.

The obvious generalization to an arbitrarily collection of subsets also holds.

Proof. We (i.e. you) will provide the proof in Exercise 3.1. ■

Examples.

(1) Using Lemma 3.2 one can easily verify that the topological space $S^1 \sqcup S^1$ is homeomorphic to the topological space $S^1(-2,0) \cup S^1(2,0) \subset \mathbb{R}^2$. We also refer to Figure 61 for an illustration.

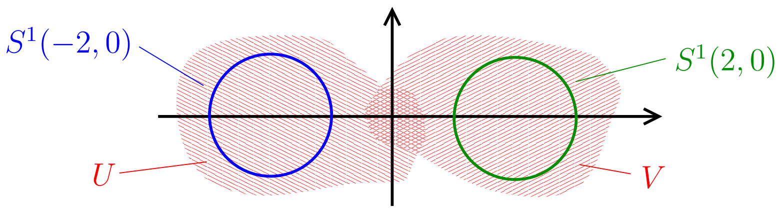


Figure 61

(2) Using Lemma 3.2 one can also easily show that the topological space $[-1,0] \sqcup (0,1]$ is homeomorphic to $[-2,-1] \cup (0,1] \subset \mathbb{R}$ and that it is not homeomorphic to $[-1,1]$.

(3) Let X be a topological space and let $\{U_i\}_{i\in I}$ be family of pairwise disjoint open subsets. It follows immediately from Lemma 3.2 that the map

$$\begin{array}{rcl} \bigcup\limits_{i\in I} U_i & \to & \bigsqcup\limits_{i\in I} U_i \\ x & \mapsto & (x,i) \quad \text{if } x \in U_i \end{array}$$

is a homeomorphism.

The following lemma summarizes a few very basic properties of the disjoint union of two topological spaces. We leave it to the reader to write down the dreadfully dull proof.

Lemma 3.3. $(*)$ *Let $\{X\}_{i\in I}$ be a family of topological spaces.*

(1) *For each $i \in I$ the inclusion map $X_i \to \bigsqcup\limits_{i\in I} X_i$ is an embedding which is open and closed.*

(2) *A map $f\colon \bigsqcup\limits_{i\in I} X_i \to Y$ to some topological space Y is continuous if and only if each restriction $f|_{X_i}$ is continuous. In particular the map*

$$\begin{array}{rcl} \{\textit{set of continuous maps } \bigsqcup\limits_{i\in I} X_i \to Y\} & \to & \prod\limits_{i\in I} \{\textit{set of continuous maps } X_i \to Y\} \\ f & \mapsto & (f|_{X_i})_{i\in I} \end{array}$$

is a bijection.

(3) *Let* $\{f_i\colon X_i \to Y_i\}_{i\in I}$ *be a family of maps between topological spaces. The map*

$$\begin{array}{rcl} \coprod_{i\in I} f_i\colon \coprod_{i\in I} X_i & \to & \coprod_{i\in I} Y_i \\ (x,i) & \mapsto & (f_i(x), i) \end{array}$$

is continuous if and only if each f_i *is continuous.*

(4) *The topological space* $\coprod_{i\in I} X_i$ *is compact if and only if each* X_i *is compact and if there are only finitely many* $i \in I$ *such that* $X_i \neq \emptyset$.

The above statements hold the same way for the disjoint union of two topological spaces. Furthermore, given topological spaces X, Y *and* Z *the following statements hold:*

(5) *The obvious bijection* $X \sqcup Y \to Y \sqcup X$ *is a homeomorphism.*

(6) *The obvious bijection* $(X \sqcup Y) \sqcup Z \to X \sqcup (Y \sqcup Z)$ *is a homeomorphism.*

Remark.

(1) The diligent reader could extend Lemma 3.3 ad infinitum with statements regarding the interaction of disjoint union with the subspace topology, product topologies (which we will introduce shortly) and quotient topologies.

(2) Let X be a topological space and let $\{X_i\}_{i\in I}$ be its set of components. Note that Lemma 2.68 says that as a set we have $X = \bigcup_{i\in I} X_i$. But, if we equip the right-hand side with the disjoint union topology that we introduced on page 159, then this is in general *not* an equality (or homeomorphism) of topological spaces. For example, an easy counterexample is given by $X = \mathbb{Q}$.

3.2. **The product of finitely many topological spaces I.** In the following two sections we introduce and discuss a suitable topology on the product of topological spaces. In this section and the following we deal with the product of *finitely* many topological spaces which is what we are mostly interested in. Afterwards we will quickly discuss a suitable topology on the product of arbitrarily many topological spaces.

Lemma 3.4. *Let* $X_1, \dots, X_k$ *be topological spaces. The set*

$$\mathcal{B} \;=\; \{U_1 \times \cdots \times U_k \,|\, \text{ each } U_i \text{ is open in } X_i\}$$

has the basis property from page 102.

Proof (∗). We need to verify that $\mathcal{B}$ satisfies the conditions (B1) and (B2) formulated on page 102.

(B1) We have $X_1 \times \cdots \times X_k \in \mathcal{B}$. Thus given any $(x_1, \dots, x_k) \in X_1 \times \cdots \times X_k$ there exists a set in $\mathcal{B}$, namely $X_1 \times \cdots \times X_k$, which contains $(x_1, \dots, x_k)$. Thus $\mathcal{B}$ satisfies (B1).

(B2) Let $U_1 \times \cdots \times U_k$ and $V_1 \times \cdots \times V_k$ be two sets in $\mathcal{B}$. Then

$$(U_1 \times \cdots \times U_k) \cap (V_1 \times \cdots \times V_k) \;=\; (U_1 \cap V_1) \times \cdots \times (U_k \cap V_k)$$

is again a set in $\mathcal{B}$. Thus $\mathcal{B}$ also satisfies (B2). ■

Definition. Let $X_1, \dots, X_k$ be topological spaces. The *product topology on* $X_1 \times \cdots \times X_k$ is the topology on $X_1 \times \cdots \times X_k$ that is generated by

$$\mathcal{B} \;=\; \{U_1 \times \cdots \times U_k \,|\, \text{ each } U_i \text{ is open in } X_i\}.$$

We refer to Figure 62 for an illustration.

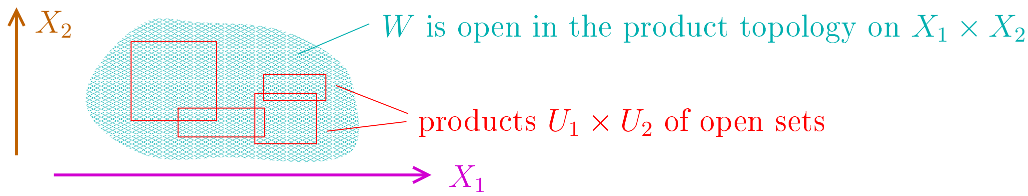


FIGURE 62

Convention. Given topological spaces $X_1, \dots, X_k$ we always equip $X_1 \times \cdots \times X_k$ with the product topology, unless we explicitly say something else.

The following lemma seems obvious, but as so many statements in general topology, the actual proof requires a little bit of thought.

Lemma 3.5. *Let $k_1, \dots, k_m \in \mathbb{N}_0$. The map*

$$\begin{array}{rcl} \mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_m} & \to & \mathbb{R}^{k_1+\cdots+k_m} \\ ((x_1, \dots, x_{k_1}), \dots, (x_1, \dots, x_{k_m})) & \mapsto & (x_1, \dots, x_{k_1}, \dots, x_1, \dots, x_{k_m}) \end{array}$$

is a homeomorphism. More generally, if we are given given $X_i \subset \mathbb{R}^{k_i}$, $i = 1, \dots, m$, then the restriction of the above map to

$$\underbrace{X^{k_1} \times \cdots \times X^{k_m}}_{\substack{\text{product of subspace}\\ \text{topologies}}} \to \underbrace{X^{k_1} \times \cdots \times X^{k_m}}_{\substack{\text{subspace topology}\\ \text{from } \mathbb{R}^{k_1+\cdots+k_m}}}$$

is a homeomorphism.

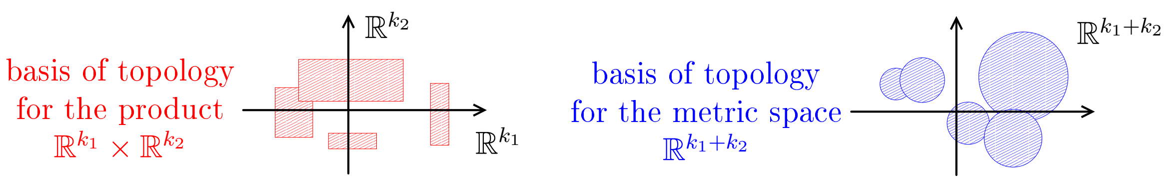


FIGURE 63. Illustration for Lemma 3.5.

PROOF (∗). It is clear that the map is a bijection. So it remains to show that the map and its inverse are continuous. The continuity of both maps can be shown using Proposition 2.37 and Lemma 2.5. We leave it to the reader to fill in the details. ∎

Convention. Henceforth, given $k_1, \dots, k_m \in \mathbb{N}_0$ we will always identify the product topological space $\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_m}$ with $\mathbb{R}^{k_1+\cdots+k_m}$. In particular, given $X_i \subset \mathbb{R}^{k_i}$, $i = 1, \dots, m$, we view $X_1 \times \cdots \times X_m$ as a subset of $\mathbb{R}^{k_1+\cdots+k_m}$.

Example. Let $s_1, \dots, s_n \in \mathbb{N}_0$. We set $s = s_1 + \cdots + s_n$. By the above convention we can view $B^{s_1} \times \cdots \times B^{s_n}$ as a subset of $\mathbb{R}^s$. By definition this is an open subset of $\mathbb{R}^s$ and one verifies easily that it is convex and bounded. Thus it follows from Proposition 2.52 that there exists a canonical homeomorphism $B^{s_1} \times \cdots \times B^{s_n}$ to the ball B^s that sends $(0, \ldots, 0)$ to the origin. The same statement holds for the corresponding closed balls.

Before we can discuss more examples we introduce the following convention:

Convention.

(1) Given $m \in \mathbb{N}_0$ we make the identification

$$\begin{array}{rcl} \mathbb{C}^m & \xrightarrow{=} & \mathbb{R}^{2m} \\ (x_1 + iy_1, \ldots, x_m + iy_m) & \mapsto & (x_1, y_1, x_2, y_2, \ldots, x_m, y_m). \end{array}$$

(2) Given $m, n \in \mathbb{N}_0$ we make the identification

$$\begin{array}{rcl} \mathrm{M}(m \times n, \mathbb{R}) & \xrightarrow{=} & \mathbb{R}^{m \cdot n} \\ (a_{ij})_{i=1,\ldots,m, j=1,\ldots,n} & \mapsto & (a_{11}, \ldots, a_{1n}, \ldots, a_{n1}, \ldots, a_{nm}) \end{array}$$

Similarly we identify $\mathrm{M}(m \times n, \mathbb{C})$ with $\mathbb{R}^{2mn}$.

(3) We use these maps to view (every subset of) $\mathbb{C}^m$, $\mathrm{M}(m \times n, \mathbb{R})$ and $\mathrm{M}(m \times n, \mathbb{C})$ as topological spaces.

Examples.

(1) It follows almost immediately from our conventions and the examples on page 107 that for any $k, l, m \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{M}(k \times l, \mathbb{R}) \times \mathrm{M}(l \times m, \mathbb{R}) & \to & \mathrm{M}(k \times m, \mathbb{R}) \\ (A, B) & \mapsto & A \cdot B \end{array}$$

is continuous. The same also applies to matrices over $\mathbb{C}$.

(2) It follows from (1) and Lemma 2.32 that

$$\mathrm{O}(n) = \{A \in \mathrm{M}(n \times n, \mathbb{R}) \mid A^T A = \mathrm{id}\} = f^{-1}(\mathrm{id}_n) \quad \text{with} \quad \begin{array}{rcl} \mathrm{M}(n \times n; \mathbb{R}) & \xrightarrow{f} & \mathrm{M}(n \times n; \mathbb{R}) \\ A & \mapsto & A \cdot A^T \end{array}$$

is a closed subset of $\mathbb{R}^{n^2}$. Since each column of an orthogonal matrix has norm one we see that $\mathrm{O}(n) \subset \mathbb{R}^{n^2}$ is a bounded subset. Thus it follows from the Heine-Borel Theorem 2.20 that $\mathrm{O}(n)$ is compact. Basically the same argument shows that the set $\mathrm{U}(n)$ of unitary $n \times n$-matrices is a compact topological space.

After these examples let us turn to the study of the more formal properties of the product topology. The following lemma states two properties of the product topology and it shows that the product topology is actually completely determined by these two properties.

Lemma 3.6. *Let $X_1, \ldots, X_k$ be topological spaces.*

(1) *For each $i \in \{1, \ldots, k\}$ the projection map*

$$\begin{array}{rcl} p_i \colon X_1 \times \cdots \times X_k & \to & X_i \\ (x_1, \ldots, x_k) & \mapsto & x_i \end{array}$$

is continuous.

(2) *Let W be a topological space. If $f_i\colon W \to X_i$, $i = 1, \dots, k$ are continuous maps, then the map*

$$\begin{array}{rcl} W & \to & X_1 \times \cdots \times X_k \\ w & \mapsto & (f_1(w), \dots, f_k(w)) \end{array}$$

is continuous.

(3) *The product topology on the set $X_1 \times \cdots \times X_k$ is the unique topology on $X_1 \times \cdots \times X_k$ that satisfies properties* (1) *and* (2).

Example. Let X be a topological space and let $\{P\}$ be the topological space consisting of a single point P (note that $\{P\}$ has a unique topological structure!). It follows from Lemma 3.6 that that the obvious bijection $X \to X \times \{P\}$ a homeomorphism. On many occasions we will use this homeomorphism to make the identification $X = X \times \{P\}$.

Proof.

(1),(2) These two statements follow easily from the definition.

(3) Let $\mathcal{P}$ be the product topology on $X_1 \times \cdots \times X_k$ and let $\mathcal{T}$ be some other topology on $X_1 \times \cdots \times X_k$ that satisfies (1) and (2). It follows easily from the fact that $\mathcal{T}$ satisfies (1) that $\mathcal{P} \subset \mathcal{T}$. Furthermore it follows from the fact that $\mathcal{T}$ satisfies (2) that the map

$$\begin{array}{rcl} (X_1 \times \cdots \times X_k, \mathcal{P}) & \to & (X_1 \times \cdots \times X_k, \mathcal{T}) \\ (x_1, \dots, x_k) & \mapsto & \underbrace{(p_1(x_1, \dots, x_k)), \dots, p_k(x_1, \dots, x_k))}_{=(x_1, \dots, x_k)} \end{array}$$

is continuous. Since this map is actually the identity we see that $\mathcal{T} \subset \mathcal{P}$. Thus we have shown that $\mathcal{T} = \mathcal{P}$. ■

The following remark can be viewed as a sibling of the future Remark 3.23.

Remark 3.7. *We consider maps in and out of a product space $X_1 \times X_2$:*

(1) *Lemma 3.6 gives us an easy to show that a given map $X \to X_1 \times X_2$ to a product space is continuous. More precisely, by Lemma 3.6 (1) we "just" need to show that the two coordinate maps $X \to X_1$ and $X \to X_2$ are continuous.*

(2) *There is usually no easy way to show that a map $X_1 \times X_2 \to X$ out of a product space is continuous.*

We move on to a longish list of further properties of the product topology.

Lemma 3.8.

(1) (a) *The product topology is "commutative" in the sense that given two topological spaces A and B the obvious map $A \times B \to B \times A$ is a homeomorphism.*

(b) *The product topology is "associative" in the sense that given three topological spaces A, B and C the identity map is a homeomorphism between $(A \times B) \times C$, $A \times B \times C$ and $A \times (B \times C)$.*

(c) *The product topology commutes with subspace topology. More precisely, let X and Y be two topological spaces and let $A \subset X$ and $B \subset Y$ be subsets. Then the subspace topology of $A \times B \subset X \times Y$ agrees with product topology of A and B.*

(d) *The product topology is "distributive" with respect to the disjoint union topology. More precisely, let X and Y be two topological spaces and let A be another topological space. The obvious map $(X \sqcup Y) \times A \to (X \times A) \sqcup (Y \times A)$ is a homeomorphism.*

(2) *Let $Y_1, \ldots, Y_k$ be topological spaces.*

(a) *Let $i \in \{1, \ldots, k\}$ and for $j \neq i$ let $y_j \in Y_j$. The inclusion map*

$$\begin{array}{rcl} Y_i & \to & Y_1 \times \cdots \times Y_i \times \cdots \times Y_k \\ y & \mapsto & (y_1, \ldots, y, \ldots, y_k) \end{array}$$

is continuous and it is an embedding.[48]

(b) *If $f_i \colon Y_i \to Z_i$, $i = 1, \ldots, k$ are continuous maps between topological spaces, then the map*

$$\begin{array}{rcl} Y_1 \times \cdots \times Y_k & \to & Z_1 \times \cdots \times Z_k \\ (y_1, \ldots, y_k) & \mapsto & (f_1(y_1), \ldots, f_k(y_k)) \end{array}$$

is continuous.

Example. It follows from Lemma 3.6 (1) that given any topological space X the *diagonal map*

$$\begin{array}{rcl} X & \to & X \times X \\ x & \mapsto & (x, x) \end{array}$$

is continuous. This fact will come in handy on many occasions.

Proof (∗).

(1) All these statement are easily verified by hand. Note that Statements (b) and (c) can also deduced from the defining property of the product topology that we stated in Lemma 3.6 (3).

(2) (a) It follows from Lemma 3.6 (2) and the observation that the identity map and the constant map are both continuous that the given map is continuous. Clearly the map is a bijection. Finally the inverse is given by the projection onto the i-th factor, which is continuous by Lemma 3.6 (1).

(b) It follows from Lemma 2.30 and Lemma 3.6 that the maps $Y_1 \times \cdots \times Y_k \to Z_i$ given by $(y_1, \ldots, y_k) \to f_i(y_i)$ are continuous. It now follows from Lemma 3.6 (2) that the given map $Y_1 \times \ldots Y_k \to Z_1 \times \cdots \times Z_k$ is continuous. ■

In the remainder of this section we will state several technical lemmas which will be used consciously and unconsciously on numerous occasions. Despite their usefulness they do not make for pleasurable reading, so the reader is encouraged to move on to the next section.

Here is the first technical lemma.

[48] Recall that "embedding" means that the map is a homeomorphism onto its image, in our setting this means that the inclusion map defines a homeomorphism $Y_i \to \{y_1\} \times \cdots \times \{y_{i-1}\} \times Y_i \times \{y_{i+1}\} \times \cdots \times \{y_k\}$.

Lemma 3.9. (∗) *Let $X_1, \dots, X_k$ be topological spaces.*

(1) *If $A_i \subset X_i$, $i = 1, \dots, k$ are closed subsets, then $A_1 \times \cdots \times A_k$ is a closed subset of $X_1 \times \cdots \times X_k$.*

(2) *Let $B_i \subset X_i$, $i = 1, \dots, k$ be subsets. The following equalities hold:*

$$\begin{aligned} \textit{closure of } B_1 \times \ldots \times B_k \textit{ in } X_1 \times \ldots \times X_k &= \overline{B}_1 \times \ldots \times \overline{B}_k \\ \textit{interior of } B_1 \times \ldots \times B_k \textit{ in } X_1 \times \ldots \times X_k &= \mathring{B}_1 \times \ldots \times \mathring{B}_k \\ \textit{boundary of } B_1 \times \ldots \times B_k \textit{ in } X_1 \times \ldots \times X_k &= \bigcup_{i=1}^{k} \overline{B}_1 \times \ldots \times \partial B_i \times \ldots \times \overline{B}_k. \end{aligned}$$

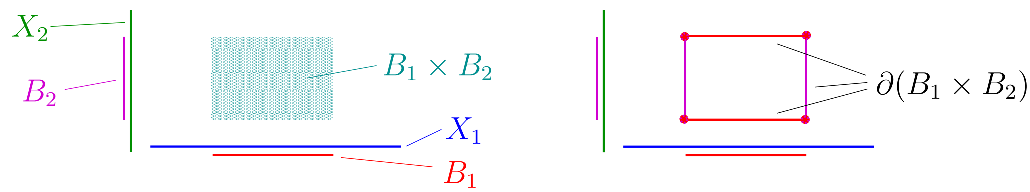


FIGURE 64. Illustration of Lemma 3.9 (2).

PROOF (∗).

(1) We have

$$(X_1 \times \cdots \times X_k) \setminus (A_1 \times \cdots \times A_k) \underset{\substack{\uparrow \\ \text{basic set theory}}}{=} \bigcap_{i=1}^{k} \underbrace{X_1 \times \cdots \times (X_i \setminus A_i) \times \cdots \times X_k}_{\text{open in } X_1 \times \cdots \times X_k \text{ since } A_i \text{ is closed}}.$$

Thus we see that $(X_1 \times \cdots \times X_k) \setminus (A_1 \times \cdots \times A_k)$ is the intersection of finitely many open sets, i.e. the set itself is open, which implies by definition that $A_1 \times \cdots \times A_k$ is closed.

(2) We will verify this statement in Exercise 3.5. ■

We move on to the following lemma which also gets used surprisingly often throughout these notes.

Lemma 3.10. (∗) *Let X, Y and Z be topological spaces and let $f\colon X \times Y \to Z$ be a map. Suppose we are given finitely many closed subsets $A_1, \dots, A_m$ of X with $\bigcup_{i=1}^{m} A_i = X$. If each map $A_i \times Y \to X \times Y \to Z$ is continuous, then f itself is continuous.*

PROOF (∗). By Lemma 3.8 the identity

$$\begin{array}{c} A_i \text{ equipped with the} \\ \text{subspace topology from } X \end{array} \times Y \longrightarrow \begin{array}{c} A_i \times Y \text{ equipped with the subspace} \\ \text{topology from } X \times Y \end{array}$$

is actually a homeomorphism. Together with our hypothesis this shows that for each $i \in \{1, \dots, m\}$ the map

$$f|_{A_i \times Y} \colon \underbrace{A_i \times Y}_{\substack{\uparrow \\ \text{equipped with subspace topology from } X \times Y}} \longrightarrow Z$$

is continuous. By Lemma 3.9 we know that $A_i \times Y$ is a closed subset of $X \times Y$. Now it follows from Lemma 2.35 (2) that f is indeed continuous. ■

We conclude this section with the following technical lemma.

Lemma 3.11. (∗) *Let $X_1, \dots, X_k$ be topological spaces and let $\mathcal{B}_1, \dots, \mathcal{B}_k$ be bases for these topological spaces. Then*

$$\mathcal{C} \;:=\; \{B_1 \times \cdots \times B_k \,|\, \text{ for each } i \in \{1, \dots, k\} \text{ we have } B_i \in \mathcal{B}_i\}$$

is a basis for $X_1 \times \cdots \times X_k$.

Example. On page 106 we saw that the *rational* intervals form a basis for the usual topology on $\mathbb{R}$. Together with Lemma 3.11 and Lemma 1.7 (5) it follows that the set $\mathcal{B}$ of rational hyperrectangles, i.e. that the set

$$\mathcal{B} \;=\; \{\text{all hyperrectangles } [a_1, b_1] \times \cdots \times [a_k, b_k] \text{ with } a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{Q}\} \;\subset\; \mathcal{P}(\mathbb{R}^k)$$

forms a basis for the standard topology on $\mathbb{R}^k$. In particular we see that $\mathbb{R}^k$ admits a basis for the topology that consists of only countably many sets.

Proof (∗). The main idea is to prove the lemma using Lemma 2.27 (1). Therefore let $W \subset X_1 \times \cdots \times X_k$ be an open set and let $(x_1, \dots, x_k) \in W$. By definition of the product topology there exist open sets $U_i \subset X_i$, $i = 1, \dots, k$ with $(x_1, \dots, x_k) \in U_1 \times \cdots \times U_k$ and $U_1 \times \cdots \times U_k \subset W$.

Let $i \in \{1, \dots, k\}$. Since $\mathcal{B}_i$ is a basis for X_i we see that there exists a $B_i \in \mathcal{B}_i$ with $x \in B_i \subset U_i$. It follows that

$$(x_1, \dots, x_k) \;\in\; B_1 \times \cdots \times B_k \;\subset\; U_1 \times \cdots \times U_k \;\subset\; W.$$

We deduce from Lemma 2.27 (1) that the given set $\mathcal{C}$ is indeed a basis for $X_1 \times \cdots \times X_k$. ■

Remark. After this discussion the reader might be convinced that the product topology on $X \times Y$ is obviously the correct topology to use. As we will see in Section 36.11, one could make a good case that a subtly different topology is more suitable. But throughout most of these lecture notes we will stick it out with the product topology we just introduced.

3.3. **The product of finitely many topological spaces II.** The only real goal of this section is to prove the following essential proposition.

Proposition 3.12. *Let $X_1, \dots, X_k$ be non-empty topological spaces.*

1. *The product $X_1 \times \cdots \times X_k$ is Hausdorff if and only if each X_i is Hausdorff.*
2. *The product $X_1 \times \cdots \times X_k$ is compact if and only if each X_i is compact.*
3. *The product $X_1 \times \cdots \times X_k$ is (path)-connected if and only if each X_i is (path)-connected.*

Remark. Proposition 3.12 (2) also shows how the case $n = 1$ in Proposition 2.21 implies the general case of Proposition 2.21.

Before we head to the proof of Proposition 3.12 let us first discuss some examples in detail.

Definition.

(1) We refer to $(S^1)^n$ as the *n-dimensional torus*. For $n = 2$ we just say *the torus*.

(2) We refer to $S^1 \times \overline{B}^2$ as the *solid torus* and we refer to $S^1 \times B^2$ as the *open solid torus*.

We deduce from Proposition 3.12 that the n-dimensional torus $(S^1)^n$ and the solid torus $S^1 \times \overline{B}^2$ are compact.

The following lemma allows us to view (solid) tori as subsets of $\mathbb{R}^3$.

Lemma 3.13. *We consider the map*[49]

$$\begin{aligned} \Theta\colon \mathbb{R}^2 \times S^1 &\to \mathbb{R}^3 \\ ((x,y), e^{i\varphi}) &\mapsto \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 1 + \frac{1}{2}x \\ 0 \\ -\frac{1}{2}y \end{pmatrix}}_{\substack{\text{describes circle} \\ \text{in } xz\text{-plane}}} \end{aligned}$$

The restrictions of Θ to $S^1 \times S^1$, $\overline{B}^2 \times S^1$ and $B^2 \times S^1$ are embeddings. (The restriction of Θ to $S^1 \times S^1$ is illustrated in Figure 65).

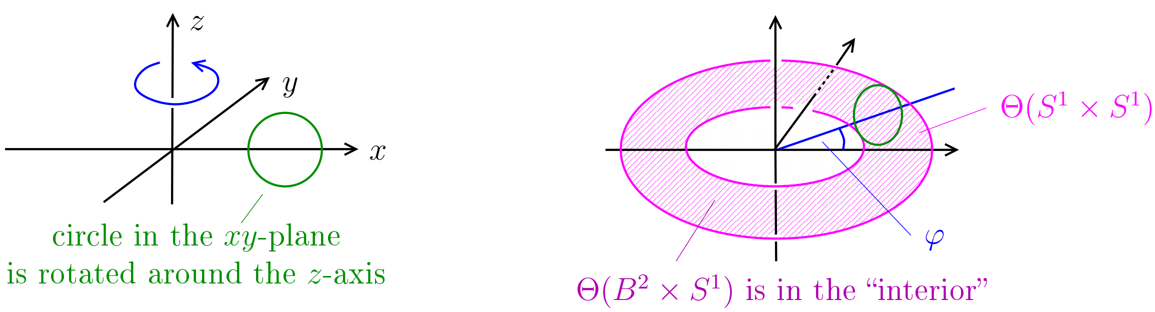


FIGURE 65

PROOF (∗). It follows easily from the example on page 163 that the map $\Theta\colon S^1 \times \mathbb{R}^2 \to \mathbb{R}^3$ is continuous. An elementary argument shows that the restriction of Θ to $\overline{B}^2 \times S^1$ is injective. Since the solid torus $\overline{B}^2 \times S^1$ is compact we obtain from Proposition 2.43 (3) that the restriction of Θ to $\Theta\colon \overline{B}^2 \times S^1 \to \Theta(S^1 \times \overline{B}^2)$ is a homeomorphism. But then it follows also that the restrictions of Θ to $S^1 \times S^1$ and $B^2 \times S^1$ are embeddings. ■

Lemma 3.13 allows us to view (solid) tori as subsets of $\mathbb{R}^3$. The typical picture is shown in Figure 66. We refer to the images under these maps as the *standard (solid) (open) torus in* $\mathbb{R}^3$. Unfortunately, in a picture, it is hard to distinguish between tori, solid tori and open solid tori.

Now the time of procrastination is over, we need to provide the proof of Proposition 3.12.

[49]The map is chosen in such a way that it is orientation-preserving (we will introduce the notions of orientation and orientation-preserving later on page 299) and such that $\Theta(\{(0,0)\} \times S^1) = S^1 \times \{0\}$.

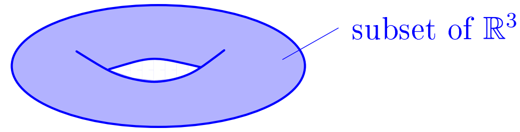
$S^1 \times S^1$ is homeomorphic to 

FIGURE 66

PROOF OF PROPOSITION 3.12 (1) ($*$). By induction we only have to deal with the case $k = 2$. Thus let X and Y be two topological spaces.

First we deal with the "$\Rightarrow$"-direction of statement (1). Thus we assume that X and Y are Hausdorff. Let (x, y) and (x', y') be two distinct points in $X \times Y$. Without loss of generality we can assume that $x \neq x'$. By our hypothesis there exist open neighborhoods U of x and U' of x' with $U \cap U' = \varnothing$. Note that

$$(U \times Y) \cap (U' \times Y) \;=\; (U \cap U') \times Y \;=\; \varnothing \times Y \;=\; \varnothing.$$

By definition of the product topology the sets $U \times Y$ and $U' \times Y$ are open in $X \times Y$. We have thus found disjoint open neighborhoods of (x, y) and (x', y'). Therefore $X \times Y$ is Hausdorff. (This argument is sketched in Figure 67 to the left.)

Now we deal with the '$\Leftarrow$"-direction of statement (1). Therefore we suppose that $X \times Y$ is Hausdorff. We want to show that X is Hausdorff. Thus let x and x' be two different points in X. Since Y is non-empty there exists a $y \in Y$. Furthermore, since $X \times Y$ is Hausdorff there exist disjoint open neighborhoods W and W' of (x, y) and (x', y). By definition of the product topology there exist open neighborhoods U of x and U' of x' and open neighborhoods V of y and V' of y such that $(U \times V) \cap (U' \times V') = \varnothing$. In particular we see that $(U \times \{y\}) \cap (U' \times \{y\}) = \varnothing$. But this implies that $U \cap U' = \varnothing$. In particular U and U' are disjoint open neighborhoods of x and x'. Thus we have shown that X is Hausdorff. (This argument is sketched in Figure 67 to the right.) The proof that Y is also Hausdorff is evidently basically the same. ■

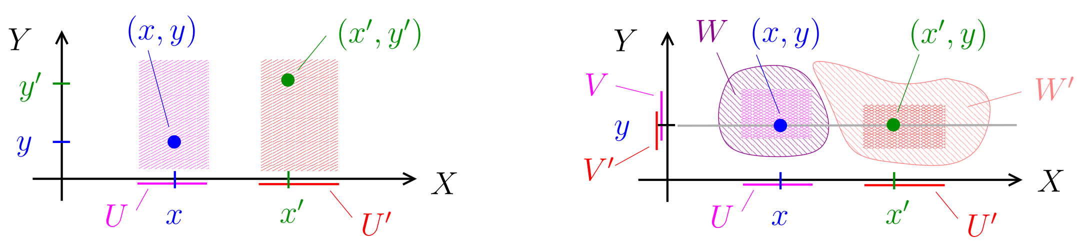


FIGURE 67. Illustration of the proof of Proposition 3.12 (1).

PROOF OF PROPOSITION 3.12 (2) ($*$). As in the proof of Statement (1) we only need to deal with the case $k = 2$. Thus let X and Y be two topological spaces. The '$\Rightarrow$"-direction of statement (2) is fortunately elementary, it follows immediately from Lemmas 3.8 and Lemma 2.40 (1).

Next we deal with '$\Leftarrow$"-direction of statement (2). This is by far the hardest part of the lemma. We start out with the following claim.

Claim. Let $x_0 \in X$. If N is an open set of $X \times Y$ that contains $\{x_0\} \times Y$, then there exists a neighborhood W of $x_0 \in X$ such that $W \times Y \subset N$.

By definition of the product topology there exist sets of the form $U_i \times V_i$, $i \in I$ with the following properties:

(1) each U_i is open in X and each V_i is open in Y,
(2) each U_i contains x_0, i.e. for each i we have $\{x_0\} \times V_i \subset U_i \times V_i$,
(3) each $U_i \times V_i$ is contained in N,
(4) $\{x_0\} \times Y$ is contained in the union of the $U_i \times V_i$.

Since $\{x_0\} \times Y$ is compact we can arrange that I is a finite set, i.e. we can assume that $I = \{1, \dots, n\}$. We set $W = U_1 \cap \dots \cap U_n$. This is an open subset of X that contains x_0. It follows immediately from (3) and the definition of W that $W \times Y \subset N$. ⊞

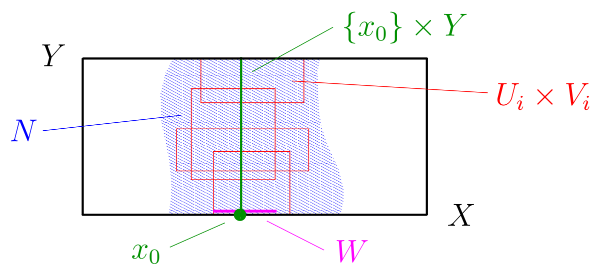


FIGURE 68. Illustration of the proof of Proposition 3.12 (2).

Now we turn to the actual proof that $X \times Y$ is compact. Thus suppose we are given an open cover $\{A_i\}_{i \in I}$ of $X \times Y$. Let $x \in X$. Since Y is compact there exists a finite set $I(x) \subset I$ such that $\{x\} \times Y \subset \bigcup\limits_{i \in I(x)} A_i$. By the above claim there exists an open neighborhood $W(x)$ of x such that $W(x) \times Y \subset \bigcup\limits_{i \in I(x)} A_i$.

All these open sets $\{W(x)\}_{x \in X}$ are an open cover of X. Since X is compact there exist $x_1, \dots, x_k \in X$ such that $X = W(x_1) \cup \dots \cup W(x_k)$. But then we see that

$$X \times Y \;=\; (W(x_1) \cup \dots \cup W(x_k)) \times Y \;=\; \bigcup_{j=1}^{k} W(x_j) \times Y \;\subset\; \bigcup_{j=1}^{k} \bigcup_{i \in I(x_j)} A_i \;\subset\; X \times Y.$$

Thus we have successfully covered $X \times Y$ by finitely many sets in the given open cover. ■

PROOF OF PROPOSITION 3.12 (3) (∗). As in the proof of Statement (1) we only need to deal with the case $k = 2$. Thus let X and Y be two topological spaces.

The '⇒"-direction of Statement (2) is once again basically elementary, Indeed, if $X \times Y$ is (path)-connected, then it follows immediately from Lemma 2.57 applied to the projection maps $X \times Y \to X$ and $X \times Y \to Y$, that X and Y are (path)-connected.[50]

Finally we deal with '⇐"-direction of statement (2). Now suppose that X and Y are path-connected. We need to show that $X \times Y$ is also path-connected. Thus let (x, y) and (x', y') be two points in $X \times Y$. Since X is path-connected we can find a path $p \colon [0, 1] \to X$

[50]Note that the projection maps are surjective since we assume that X and Y are non-empty.

from x to x' and similarly we can find a path $q\colon [0,1] \to Y$ from y to y'. Then

$$\begin{array}{rcl} [0,1] & \to & X \times Y \\ t & \mapsto & \begin{cases} (p(2t), y), & \text{if } t \in [0, \frac{1}{2}], \\ (x', q(2t-1)), & \text{if } t \in (\frac{1}{2}, 1] \end{cases} \end{array}$$

is a path from (x,y) to (x',y').[51]

Finally suppose that X and Y are connected. We want to show that $X \times Y$ is also connected. We pick $a \in X$ and $b \in Y$. Let $x \in X$. Note that the subsets $\{x\} \times Y$ and $X \times \{b\}$ are connected by our hypothesis. Since these have the point (x,b) in common we deduce from Lemma 2.62 (2) that $(\{x\} \times Y) \cup (X \times \{b\})$ is also connected. Finally note that

$$X \times Y \;=\; \bigcup_{x \in X} (\{x\} \times Y) \cup (X \times \{b\}).$$

All the sets on the right are connected and they have the point (a,b) in common. Thus it follows again from Lemma 2.62 (2) that the union on the right-hand side is connected. We refer to Figure 69 for an illustration. ■

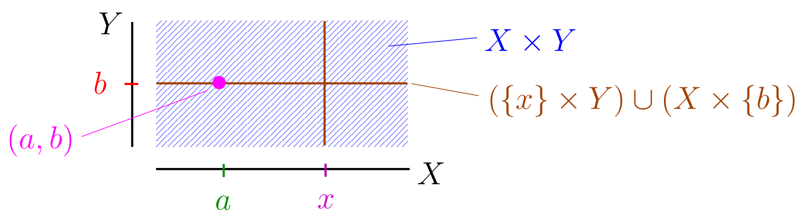


FIGURE 69. Illustration for the proof of Lemma 3.12 (3).

3.4. **The product of arbitrarily many topological spaces** (∗)**.** In this section we discuss the product of arbitrarily many topological spaces. In these notes we will only rarely deal with the case of the product of infinitely many topological spaces. Therefore we decided to deal with this case separately.

Let $\{X_i\}_{i \in I}$ a family of sets. The product of these sets is defined as the set

$$\prod_{i \in I} X_i \;:=\; \left\{ f\colon I \to \bigcup_{i \in I} X_i \;\middle|\; \text{for every } i \in I \text{ we have } f(i) \in X_i \right\}.$$

If the X_i are topological spaces, then we intend to equip the product with a suitable topology. Before we specify the topology we should decide what statements we want to hold. It seems reasonable to demand that the obvious analogue of Lemma 3.6 holds. This leads us to the following lemma.

Lemma 3.14. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. We set*

$$\mathcal{B} \;:=\; \left\{ \prod_{i \in I} U_i \;\subset\; \prod_{i \in I} X_i \;\middle|\; \begin{array}{ll} (1) & \textit{each } U_i \textit{ is open in } X_i \textit{ and} \\ (2) & \textit{there exist only finitely many } i\textit{'s with } U_i \neq X_i \end{array} \right\}.$$

[51] It follows from Lemma 3.6 (1) and Lemma 2.35 (2) that this map is indeed continuous.

The following statements hold:

(0) *The set $\mathcal{B}$ has the basis property that we introduced on page 102.*

(1) *Given any $i \in I$ the projection map*

$$\begin{array}{rcl} p_i \colon \prod\limits_{i\in I} X_i & \to & X_i \\ f & \mapsto & f(i) \end{array}$$

is continuous with respect to the topology from (0).

(2) *Given a topological space W and given a family $\{f_i \colon Y \to X_i\}_{i\in I}$ of continuous maps the resulting map*

$$\begin{array}{rcl} F \colon W & \to & \prod\limits_{i\in I} X_i \\ w & \mapsto & \left(\begin{array}{rcl} I & \to & \bigcup\limits_{i\in I} X_i \\ i & \mapsto & f_i(w) \end{array} \right). \end{array}$$

is continuous with respect to the topology from (0).

(3) *The topology introduced in* (0) *is the unique topology on the set $\prod\limits_{i\in I} X_i$ that satisfies properties* (1) *and* (2).

Proof. Let $\{X_i\}_{i\in I}$ be a family of topological spaces.

(0) The proof that $\mathcal{B}$ has the basis property is basically identical to the proof of Lemma 3.4.

(1) It follows immediately from the definitions that the projection maps p_i are continuous with respect to the given topology.

(2) Suppose we are given W and maps $f_i \colon W \to X_i$. We denote by F the map defined in (2). We claim that F is continuous. By Proposition 2.37 it suffices to verify that the preimage of any set in $\mathcal{B}$ is open. So let $\prod\limits_{i\in I} U_i$ be a set in $\mathcal{B}$. By definition there exists a finite set $J \subset I$ such that $U_i = X_i$ for any $i \notin J$. Then

$$F^{-1}\Big(\prod_{i\in I} U_i\Big) = \bigcap_{i\in I} f_i^{-1}(U_i) = \bigcap_{j\in j} \underbrace{f_j^{-1}(U_j)}_{\text{open in } Y} = \begin{array}{c}\text{intersection of } \textit{finitely} \\ \text{many open subsets of } Y.\end{array}$$

This concludes the proof that F is continuous.

(3) Finally the proof of the uniqueness statement is basically the same as the proof of Lemma 3.6 (3). ■

The lemma leads us to the following definition.

Definition. Let $\{X_i\}_{i\in I}$ a family of topological spaces. We refer to the topology on $\prod\limits_{i\in I} X_i$ that we defined in Lemma 3.14 as the *product topology* on $\prod\limits_{i\in I} X_i$.

On several occasions we will use, consciously or unconsciously, the following lemma. Some of the statements are the obvious generalizations of Lemma 3.8.

Lemma 3.15.

(1) (a) *If* $\{X_i\}_{i\in I}$ *is a finite family of topological spaces, i.e. if* I *is finite, then the above product topology agrees with the product topology introduced on page 162.*

(b) *If we are families of topological spaces* $\{\{X_i\}_{i\in I_j}\}_{j\in J}$*, then the obvious map*

$$\prod_{j\in J}\Big(\prod_{i\in I_j} X_i\Big) \;\to\; \prod_{i\in \bigsqcup\limits_{j\in J} I_j} X_i$$

is a homeomorphism.

(c) *The product topology commutes with subspace topology. More precisely, let* $\{X_i\}_{i\in I}$ *be a family of topological spaces and for each* $i\in I$ *let* A_i *be a subset. Then the subspace topology of* $\prod\limits_{i\in I} A_i \subset \prod\limits_{i\in I} X_i$ *agrees with product topology on* $\prod\limits_{i\in I} A_i$

(d) *Let* X *be a topological space and let* I *be a set. If we equip* I *with the discrete topology, then the map*

$$\begin{array}{ccc} \overbrace{X\,\times\, I}^{\substack{\text{product}\\ \text{topology}}} & \to & \overbrace{\bigsqcup_{i\in I} X}^{\substack{\text{disjoint union}\\ \text{topology}}} \\ (x,i) & \mapsto & (x,i) \end{array}$$

is a homeomorphism.

(2) *The obvious generalizations of Lemma 3.8 (2) hold.*

Proof. All statements follow easily from the definitions and/or from Lemma 3.14. We leave it to the reader to provide the details. ■

Finally we have the following generalization of Proposition 3.16.

Proposition 3.16. *Let* $\{X_i\}_{i\in I}$ *be a family of non-empty topological spaces.*

(1) *The product* $\prod\limits_{i\in I} X_i$ *is Hausdorff if and only if each* X_i *is Hausdorff.*

(2) *The product* $\prod\limits_{i\in I} X_i$ *is compact if and only if each* X_i *is compact.*

(3) *The product* $\prod\limits_{i\in I} X_i$ *is (path)-connected if and only if each* X_i *is (path)-connected.*

The proof of all statements of Proposition 3.16 is basically the same as the proof of Proposition 3.12, *except*, for the "if"-direction of Proposition 3.16. This proof is now considerably harder and thus the proposition gets upgraded to a theorem:

Theorem 3.17. (Tychonoff's Theorem[52]) *The product of arbitrarily many compact topological spaces is again a compact topological space.*

Sketch of proof. We will not really make use of Tychonoff's Theorem thus we limit ourselves to a sketch of the proof for $I=\mathbb{N}$. Thus let $\{X_i\}_{i\in\mathbb{N}}$ be a family of compact topological spaces. Furthermore let $\{U_j\}_{j\in J}$ be a family of open sets of $\prod\limits_{i\in I} X_i$ such that we

[52]Andrey Tikhonov (1906-1993) was a Russian mathematician. The spelling for the theorem is due to the fact that Tikhonov initially published in German and his name got transcribed from the Cyrillic alphabet to the Latin alphabet as Tychonoff.

cannot cover $\prod\limits_{i\in I} X_i$ by finitely many sets in the given open cover $\{U_j\}_{j\in J}$. We need to show that $\bigcup\limits_{j\in J} U_j$ is a proper subset of $\prod\limits_{i\in I} X_i$.

Claim. There exist $x_i \in X_i$, $i \in \mathbb{N}$ with the following property: Given any $k \in \mathbb{N}$ and given any open neighborhoods V_i of x_i, $i = 1, \dots, k$ the product

$$V_1 \times \cdots \times V_k \times X_{k+1} \times X_{k+2} \times \dots$$

cannot be covered by finitely many sets in the given open cover.

The sequence $\{x_i\}_{i\in\mathbb{N}}$ is defined iteratively. The key to the proof is that we already know by Proposition 3.12 (2) that any finite product $X_1 \times \cdots \times X_k$ is compact. We leave it to the reader to fill in the details. ⊞

The following claim concludes the proof of the theorem for $I = \mathbb{N}$.

Claim. The point $(x_1, x_2, \dots) \in \prod\limits_{i\in I} X_i$ does not lie in any of the U_j.

Suppose there exists a $j \in J$ with $(x_1, x_2, \dots) \in U_j$. By definition of the topology on $\prod\limits_{i\in I} X_i$ (see Lemma 2.22) there exists a $k \in \mathbb{N}$ and open neighborhoods V_i of x_i, $i = 1, \dots, k$ such that

$$V_1 \times \cdots \times V_k \times X_{k+1} \times X_{k+2} \times \dots \ \subset \ U_j.$$

But by construction of $(x_1, x_2, \dots)$ this is impossible. ⊞

The proof of the general case of the Tychonoff Theorem can be done in a very similar way by equipping the index set I with a total order. We refer to [**Wri94**] for details. Alternatively proofs of Tychonoff's Theorem are provided in most textbooks on general topology, see e.g. [**Mun75**, Chapter 5.1] or [**Jä05**, Chapter X]. A particularly concise proof is also given in [**Chf92**]. ■

We continue with a very instructive example.

Example. We equip $\{0, 2\}$ with the discrete topology. We consider the map

$$\begin{array}{rcl} \varphi\colon \prod\limits_{i\in\mathbb{N}} \{0,2\} & \to & C = \text{Cantor set from page 100} \\ (f\colon \mathbb{N} \to \{0,2\}) & \mapsto & \sum\limits_{i=1}^{\infty} \frac{f(i)}{3^i}. \end{array}$$

It is elementary to see that φ takes values in C. It is a refreshing exercise, left to the enthusiastic reader, to show that φ is a bijection. The left-hand side is compact by Tychonoff's Theorem 3.17 and the right-hand is evidently Hausdorff. In Exercise 3.19 we verify that φ is continuous. It thus follows from Proposition 2.43 (3) that φ is a homeomorphism.

Initially the definition of the Cantor set is nicely explicit, but the definition looks rather arbitrary. The previous example gives a much more natural definition of the Cantor set. Amazingly there exists a characterization of the Cantor set, up to homeomorphism, in terms of basic topological properties. To state the proposition we need two easy definitions.

Definition. Let X be a topological space. A point $x_0 \in X$ is called *isolated* if $\{x_0\}$ itself is an open subset.

Examples.

(1) We consider $X = \{\frac{1}{n} \,|\, n \in \mathbb{N}\} \cup \{0\}$. The only point in X that is not isolated is 0.
(2) No point in the Cantor set is isolated.
(3) No point in $\prod_{i\in\mathbb{N}} \{0,2\}$ is isolated.
(4) The topological spaces $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$ equipped with the subspace topology coming from $\mathbb{R}$ do not contain any isolated points.
(5) Let X be a topological space. If every point is isolated, then by definition X has the discrete topology. If X is furthermore finite, then it follows from Lemma 2.18 (1) that X is finite.

Definition. A topological space X is called *totally disconnected* if the only connected subsets are the empty set and the subsets consisting of a single point.

Example. The following topological spaces are totally disconnected:

(1) discrete topological spaces,
(2) the Cantor set,
(3) the infinite product $\prod_{i\in\mathbb{N}} \{0,2\}$,
(4) the topological spaces $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$ equipped with the subspace topology coming from $\mathbb{R}$.

Here (1) is obvious and (4) was is dealt with in Exercise 2.17. We will deal with the remaining two examples in Exercise 3.20.

We conclude this section with the following amusing proposition.

Proposition 3.18. *Let X be a topological space that is compact and totally disconnected. If X contains no isolated points, then X is homeomorphic to the Cantor set.*

PROOF. We will not make use of the proposition, hence we will not prove it. A proof is for example given in [**Val13**, Theorem 7.8] or [**Wil70**, Theorem 30.3]. ■

Example. By the above examples Proposition 3.18 implies not only that the Cantor set is homeomorphic to $\prod_{i\in\mathbb{N}} \{0,2\}$ but also to $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$. In particular we obtain the weird result that $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$ are homeomorphic.[53]

3.5. **Distance in metric spaces.** Before we can state our next lemma we need to introduce the following definition.

Definition. Let X be a metric space. Given two non-empty subsets A and B we refer to

$$d(A,B) \;=\; \inf\{d(a,b) \,|\, a \in A \text{ and } b \in B\}$$

as the *distance between A and B*. If $A = \{a\}$ consists of a single point, then we write $d(a,B) := d(\{a\},B)$.

[53]Can you prove directly that $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$ are homeomorphic?

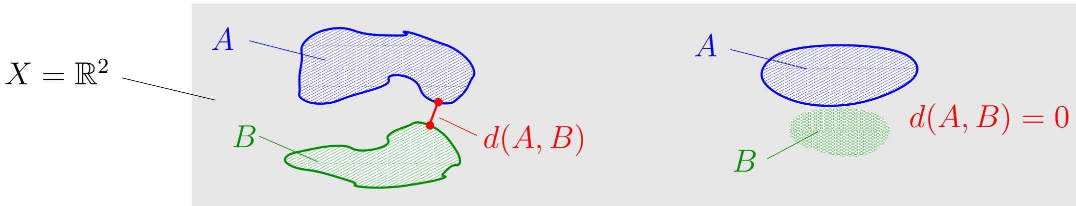


FIGURE 70

Example. If two subsets A and B intersect, then we evidently have $d(A, B) = 0$. But in general the converse does not hold. For example if we consider $X = \mathbb{R}$, then

$$d\big(\underbrace{\text{open interval } (0,1)}_{=A}, \underbrace{\text{closed interval } [1,2]}_{=B}\big) \;=\; 0,$$

but evidently the intervals do not intersect.

The reader might justly suspect that the phenomenon of the last example is due to the fact that one of the sets is not compact. In several arguments the following corollary will play a crucial role.

Corollary 3.19. *Let X be a metric space and let K and L be two non-empty subsets. We assume that one of the following holds:*

(i) *Both K and L are compact, or*
(ii) *$X = \mathbb{R}^n$, K is compact and L is a closed subset of $X = \mathbb{R}^n$.*

The following statements hold:

(1) *There exist $x \in K$ and $y \in L$ with $d(x, y) = d(K, L)$.*
(2) *If K and L are disjoint, then $d(K, L) > 0$.*

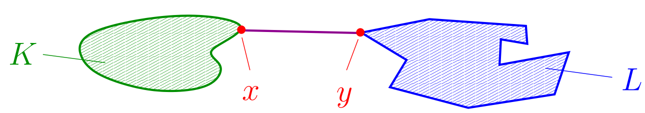


FIGURE 71

PROOF.

(1) First we suppose that we are in the setting (i). It follows immediately from our hypothesis and Proposition 3.12 that $K \times L$ is compact. As we pointed out above, the metric $d\colon K \times L \to \mathbb{R}_{\geq 0}$ is continuous. The desired statement is now an immediate consequence of Lemma 2.40 (2).

Now suppose that we are in the setting (ii). We will reduce this case to setting (i). To do so we set $s := \max\{\|x\| \,|\, x \in K\} + 1 + d(K, L)$. We set $L' := L \cap \overline{B}^n_s(0)$. Note that it follows from the Heine-Borel Theorem 2.20 together with Lemma 2.17 (2) that L' is compact. We leave it to the reader to verify that it follows easily from the choice of s and the triangle inequality that $L' \neq \varnothing$ and that $d(K, L') = d(K, L)$. We apply (i) to $X = \mathbb{R}^n$ together with the compact subsets K and L'. We obtain $x \in K$ and $y \in L'$ with $d(x, y) = d(K, L') = d(K, L)$.

(2) If $d(K,L)=0$, then it follows from (1) that there exists $x \in K$ and $y \in L$ with $d(x,y)=d(K,L)=0$. This implies that $x=y$, i.e. the intersection of K and L is non-empty. ∎

In Exercise 3.12 we will prove the following lemma.

Lemma 3.20. *Let X be a metric space. If $B \subset X$ is a non-empty subset, then the map*

$$\begin{array}{rcl} X & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & d(x,B) := \inf\{d(x,b) \mid b \in B\} \end{array}$$

is continuous.

3.6. **Quotients of topological spaces.** On page 74 we recalled the notion of an equivalence relation. Now we introduce a notation that we will use throughout these notes.

Notation. Given an equivalence relation $\sim$ on a set X and given $x \in X$ we denote by $[x]$ the unique equivalence class that contains x. We refer to the map

$$\begin{array}{rcl} X & \to & X/\sim \\ x & \mapsto & [x] \end{array}$$

as the *canonical projection*. Sometimes $X/\sim$ is called a *quotient* of X.

After these preliminaries we return to topological spaces.

Lemma 3.21. *Let $\sim$ be an equivalence relation on a topological space X. If we denote by $p\colon X \to X/\sim$ the canonical projection map from X onto the set of equivalence classes $X/\sim$, then the following statements hold:*

(1) *The set*

$$\mathcal{T} := \{U \subset X/\sim \mid p^{-1}(U) \text{ is open in } X\}$$

is a topology on $X/\sim$.[54]

(2) *Let $A \subset X/\sim$ be a subset. The set A is closed in $X/\sim$ if and only if the preimage $p^{-1}(A)$ is closed in X.*

(3) *The projection map $p\colon X \to X/\sim$ is continuous.*

(4) *If X is compact, then $X/\sim$ is also compact.*

(5) *If X is (path)-connected, then $X/\sim$ is also (path)-connected.*

Proof.

(1) This statement follows immediately from Lemma 1.3 (5) and (6).
(2) This statement follows easily from Lemma 1.3 (7).
(3) This statement is an immediate consequence of the definitions.
(4) This statement follows immediately from (3) and from Lemma 2.40.
(5) This statement follows immediately from (3) together with Lemma 2.57. ∎

The following definition will not come as a surprise.

[54] In other words, $U \subset X/\sim$ is open if and only if $p^{-1}(U) \subset X$ is open.

Definition. Let $\sim$ be an equivalence relation on a topological space X. We refer to the topology on $X/\sim$ from Lemma 3.21 (1) as the *quotient topology* and we refer to $X/\sim$ as a *quotient space* of X.

The following lemma is also so basic that later on we will no longer cite it explicitly.

Lemma 3.22. *Let $\sim$ be an equivalence relation on a topological space X and let $f\colon X \to Y$ be a map with the property that $f(x) = f(y)$ whenever $x \sim y$. Then the map*

$$\begin{array}{rcl} g\colon X/\sim & \to & Y \\ [x] & \mapsto & f(x) \end{array}$$

is well-defined and it is the unique map $g\colon X/\sim \to Y$, such that $f = g \circ p$, i.e. such that the following diagram of maps commutes:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "f"'] & X/\sim \arrow[d, "g"] \\ & Y. \end{tikzcd}$$

If f is continuous, then g is also continuous.

Definition. Let $f\colon X \to Y$ be a map as in Lemma 3.22.

(1) We refer to the map $g\colon X/\sim \to Y$ as the *induced map.*
(2) At times we denote the induced map by $\overline{f}$. Sometimes, when there is no danger of confusion we denote the induced map just by f itself.
(3) Sometimes we also say that the map $f\colon X \to Y$ *descends* to a map $X/\sim \to Y$.

Proof (∗). Let $\sim$ be an equivalence relation on a set X and let $f\colon X \to Y$ be a map with the property that $f(x) = f(y)$ whenever $x \sim y$. Let a be an element of $X/\sim$. By definition there exists an $x \in X$ with $a = [x]$. We set

$$g(a) \;:=\; f(x).$$

Evidently this definition does not depend on the choice of x. The map g satisfies $f = g \circ p$. Since the projection $X \to X/\sim$ is surjective we see that g is uniquely determined by this property.

Now suppose that f is continuous. We need to show that g is continuous as well. Let $U \subset Y$ be open. We need to show that $g^{-1}(U) \subset X/\sim$ is open. By definition of the quotient topology we need to show that $p^{-1}(g^{-1}(U))$ is open in X. We observe that

$$p^{-1}(g^{-1}(U)) \;=\; (g \circ p)^{-1}(U) \underset{\substack{\uparrow \\ \text{by the commutative diagram}}}{=} f^{-1}(U) \underset{\substack{\uparrow \\ \text{since } f \text{ continuous}}}{=} \text{open subset}\,.$$

We have thus shown that g is continuous. ■

The following remark can be viewed as a sibling of Remark 3.7.

Remark 3.23. *We consider maps in and out of a quotient space $X/\sim$:*

(1) *In practice it is usually easy to show that a given map $X/\sim \to Y$ out of the quotient space is continuous. More precisely, in practice it is often fairly clear that the map*

$X \to X/\sim \to Y$ is continuous, but then it follows from Lemma 3.22 that the original map $X/\sim \to Y$ is continuous.

(2) *There is usually no easy way to show that a map $Y \to X/\sim$ to a quotient space is continuous, unless the map factors through the projection $X \to X/\sim$.*

We continue with a particularly important source of quotient spaces.

Definition. Let X be a topological space and let $A \subset X$ be a subset.

(1) (a) First we suppose that A is a non-empty subset. For $P, Q \in X$ we define

$$P \sim Q \quad :\Longleftrightarrow \quad P = Q \ \text{ or } \ P, Q \text{ both lie in } A.$$

This is easily seen to be an equivalence relation on X. We write $X/A := X/\sim$ and we always equip X/A with the quotient topology. We refer to X/A as the *quotient of X by A*. Note that $A \subset X$ is an equivalence class, thus it defines a point X/A. In an attempt to avoid confusion we write $[A]$ for the point in X/A that is given by the equivalence class A.

(b) If $A = \varnothing$ is the empty set, then we define $X/\varnothing := X \sqcup \{\varnothing\}$, i.e. $X/\varnothing$ equals the disjoint union of the topological space X with the topological space consisting of a single point $\varnothing$. For consistency we denote this point by $[A] = [\varnothing]$ as well. Note that $\varnothing/\varnothing$ is a the topological space consisting of the single point $\{\varnothing\}$.[55]

(2) We refer to the obvious map $p\colon X \to X/A$ as the *projection map*.

Remark. Let us attempt to digest the definition of X/A. As a set X/A corresponds precisely to the points in $X \setminus A$ together with the point $[A]$. Furthermore, by definition of the quotient topology a set $U \subset X/A$ is open if and only if $p^{-1}(U)$ is an open subset of X. Note that given any subset $Z \subset X/A = (X \setminus A) \cup \{[A]\}$ we have[56]

$$p^{-1}(Z) \;=\; \begin{cases} Z, & \text{if } [A] \notin Z, \\ Z \cup A, & \text{if } [A] \in Z. \end{cases}$$

This shows that the open sets in X/A correspond precisely to the open sets of X that lie in $X \setminus A$ and the open sets of X that contain A.

Examples.

(1) We consider the quotient of $X = \mathbb{R}$ by the compact interval $A = [-1, 1]$. This example has two interesting features:

(a) The quotient $X/A = \mathbb{R}/[-1,1]$ is actually homeomorphic to an old favorite, namely one can easily verify that the maps

$$\begin{array}{rcl} f\colon \mathbb{R} & \to & \mathbb{R}/[-1,1] \\ x & \mapsto & \begin{cases} [x+1], & \text{if } x \geq 0 \\ [x-1], & \text{if } x < 0 \end{cases} \end{array} \quad \& \quad \begin{array}{rcl} g\colon \mathbb{R}/[-1,1] & \to & \mathbb{R} \\ P & \mapsto & \begin{cases} x-1, & \text{if } P = [x] \text{ with } x \geq 1 \\ x+1, & \text{if } P = [x] \text{ with } x < -1 \end{cases} \end{array}$$

are continuous and inverses of one another.

[55] The reader will have spotted that the case $A = \varnothing$ is a major nuisance. It is conceivable that later on in proofs we will subconsciously suppress this special case. Whenever this occurs we leave it to the reader to deal with the case $A = \varnothing$ separately.

[56] In an attempt to lighten the notation, given $x \in X \setminus A$ we do not distinguish between the point $x \in X \setminus A$ and the equivalence class $\{x\} \subset X$.

(b) Above we saw that projections $X \to X/\sim$ are basically by definition, continuous. One might think that projections should also be open. But in this example the projection $p\colon X = \mathbb{R} \to X/A = \mathbb{R}/[-1,1]$ is *not* an open map, for example it follows immediately from the previous remark that the image of the open set $(-1,1) \subset \mathbb{R}$ is not an open subset of $\mathbb{R}/[-1,1]$. We refer to Figure 72 for an illustration.

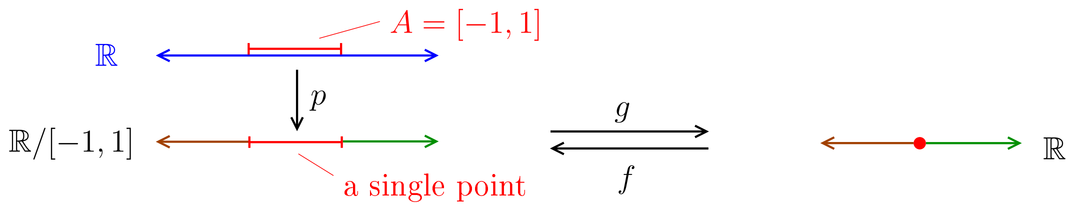


FIGURE 72

(2) We consider the quotient of $X = \mathbb{R}$ by the open interval $B = (-1,1)$. We denote by $p\colon \mathbb{R} \to \mathbb{R}/(-1,1)$ the projection map. Even though this example looks very similar to the previous one, it shows very different behavior. For example it turns out that the quotient $X/B = \mathbb{R}/(-1,1)$ is not Hausdorff. More precisely we consider the two points $[-1]$ and $[0]$ in $\mathbb{R}/(-1,1)$. The previous remark shows that any open neighborhood U of $[-1]$ in $\mathbb{R}/(-1,1)$ has the property that $p^{-1}(U)$ is an open subset of $\mathbb{R}$ that contains -1. In particular $p^{-1}(U)$ has non-empty intersection with B. But this implies that $U = p(p^{-1}(U))$ contains the point $[0]$. Thus $[-1]$ and $[0]$ cannot be separated by open neighborhoods. We refer to Figure 73 for an illustration.[57]

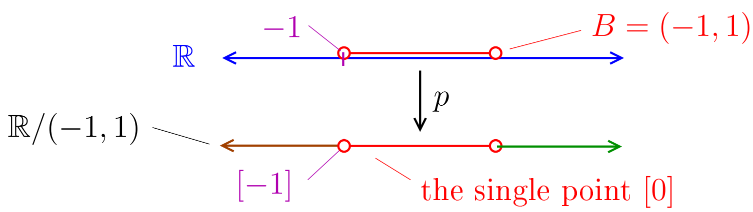


FIGURE 73

The following lemma is elementary but useful. On numerous occasions it will allow us to slightly shift our point of view.

Lemma 3.24. $(*)$ *Let X be a topological space and let $A \subset X$ be a subset. Furthermore let Y be another topological space and let $y_0 \in Y$. The map*

$$\begin{array}{rccc} \Phi\colon & \left\{ \begin{array}{c} \textit{set of continuous maps } f\colon X \to Y \\ \textit{such that } f(A) = \{y_0\} \end{array} \right\} & \to & \left\{ \begin{array}{c} \textit{set of continuous maps } f\colon X/A \to Y \\ \textit{such that } f([A]) = y_0 \end{array} \right\} \\ & (f\colon X \to Y) & \mapsto & (\overline{f}\colon X/A \to Y) \end{array}$$

is well-defined and it is a bijection.

PROOF $(*)$. It follows from Lemma 3.22 that given a continuous map $f\colon X \to Y$ with $f(A) = \{y_0\}$ the corresponding map $\overline{f}\colon X/A \to Y$ is continuous. This shows that the

[57] Note that in Lemma 3.26 we will give a criterion that ensures that the quotient of a topological space is Hausdorff.

map Φ is well-defined. It is clear that Φ is injective. Finally suppose we are given a map $g\colon X/A \to Y$ with $g([A]) = y_0$. We denote by $f\colon X \to X/A \to Y$ the map that is given by precomposing g with the obvious projection $X \to X/A$. It follows from Lemma 3.21 (3) that f is continuous. Clearly we have $\Phi(f) = g$. This shows that Φ is also a surjection. ■

Example.
Let $a < b$ be two real numbers. It follows easily from Lemma 3.21 (4), Lemma 3.22 and Proposition 2.43 (3) that the map

$$\begin{array}{rcl} \Omega\colon [a,b]/\{a,b\} & \to & S^1 \\ {[t]} & \mapsto & \exp\left(i \cdot \frac{2\pi(t-a)}{b-a}\right) \end{array}$$

is a homeomorphism.

We continue with the following innocuous lemma.

Lemma 3.25.

(1) *The map*

$$\begin{array}{rcl} ([0,1] \times S^n)/(\{0\} \times S^n) & \to & \overline{B}^{n+1} \\ {[(r,v)]} & \mapsto & r \cdot v \end{array}$$

is a homeomorphism.

(2) *Let X be a topological space and let $f\colon [0,1] \times S^n \to X$ be a continuous map. If $f(0,v) = f(0,w)$ for all $v, w \in S^n$, then the map*

$$\begin{array}{rcl} \overline{B}^{n+1} & \to & X \\ r \cdot v & \mapsto & f(r,v) \text{ with } r \in [0,1] \text{ and } v \in S^n \end{array}$$

is well-defined and continuous.

Proof.

(1) It follows from Lemma 3.22 that the given map is continuous. It is basically clear that the map is a bijection. The left-hand side is compact by Lemma 3.22 (4). It follows from Proposition 2.43 (3) that the map is a homeomorphism.
(2) It follows from the hypothesis that the map is well-defined. It follows from (1) and Lemma 3.22 that the map is continuous. ■

Example. It follows from Lemma 3.25, Lemma 3.22 and Proposition 2.43 (3) that the map

$$\begin{array}{rcl} \Omega\colon \overline{B}^n/S^{n-1} & \to & S^n \\ {[r \cdot v]} & \mapsto & \underbrace{\begin{pmatrix} 0 & (-1)^{n-1} & \dots \\ 0 & 0 & \mathrm{id}_{n-1} \\ -1 & \dots & 0 \end{pmatrix}}_{\in \mathrm{SO}(n+1)} \cdot \underbrace{\begin{pmatrix} \cos(\pi r) \\ \sin(\pi r) v \end{pmatrix}}_{\in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}} \text{ with } r \in [0,1] \text{ and } v \in S^{n-1} \end{array}$$

is a homeomorphism. Note that for $n=1$ and $n=2$ this homeomorphism is illustrated in Figure 74. This homeomorphism has the following properties which we will care about in the future:

(1) The map Ω sends the point $[S^{n-1}] \in \overline{B}^n/S^{n-1}$ to the North Pole $(0,\dots,0,1)$.
(2) The map sends the origin to the South Pole $(0,\dots,0,-1)$.
(3) The map $B^n \to \overline{B}^n/S^{n-1} \xrightarrow{\Omega} S^n$ is smooth and, if we equip B^n and S^n with the usual orientations, the map is orientation-preserving. We hope that the reader is not too fazed by the fact that we use some expressions that strictly speaking we will only introduce later on page 299.

Throughout this course we will use this homeomorphism to identify $\overline{B}^n/S^{n-1}$ with S^n.

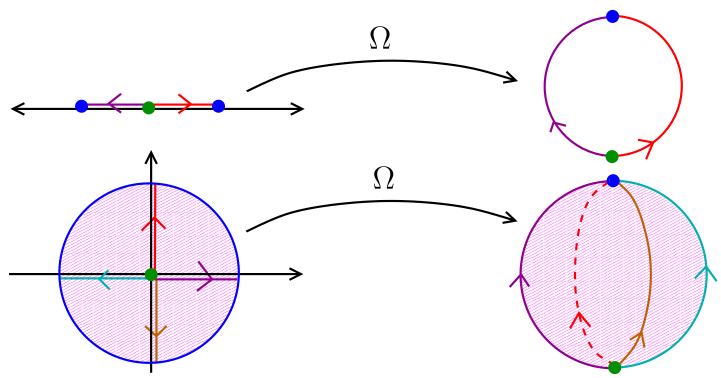


Figure 74

We conclude this section with a rather technical discussion. Namely, on page 180 we saw that the quotient of a topological space that is Hausdorff is not necessarily again Hausdorff. It is pretty clear that if one puts enough conditions on the equivalence relation, then one can avoid such misdemeanor. This leads us to the following definition.

Definition. Let X be a set and let $\sim$ be an equivalence relation on X.

(1) We refer to $\{(x,y) \in X \times X \,|\, x \sim y\}$ as the *graph of the equivalence relation.*
(2) Now suppose X is equipped with a topology. We say that the equivalence relation is *closed* if the graph is a closed subset of $X \times X$.

Now we can formulate the next lemma.

Lemma 3.26. $(*)$ *Let X be a topological space that is Hausdorff and compact. If $\sim$ is a closed relation, then $X/\sim$ is Hausdorff.*

Proof $(*)$. We denote by $p\colon X \to X/\sim$ the projection. We start out with the following claim.

Claim. Given any closed subset C of X the projection $p(C)$ is a closed subset of $X/\sim$.

Let C be a closed subset of X. We need to show that $p^{-1}(p(C))$ is closed in X. We denote by G the graph of the equivalence relation and we denote by $p_1, p_2\colon X \times X \to X$ the projection onto the first respectively the second factor. Then one easily sees that

$$p^{-1}(p(C)) \;=\; \{y \in X \,|\, \text{there exists an } x \in C \text{ with } y \sim x\} \;=\; p_2(p_1^{-1}(C) \cap G).$$

Note that $p_1^{-1}(C) = C \times X$ is a closed subset of $X \times X$.[58] Furthermore by our hypothesis G is a closed subset of $X \times X$. Thus we see that $p_1^{-1}(C) \cap G$ is closed. By Proposition 3.12 (2) and our hypothesis on X we know that $X \times X$ is compact. It follows from Lemma 2.17 (1) that $p_1^{-1}(C) \cap G$ is a compact subset of $X \times X$. But then it follows from Lemma 2.40 (1) that $p_2(p_1^{-1}(C) \cap G)$ is a compact subset of X. But since X is Hausdorff we obtain that from Lemma 2.17 (2) that is also a closed subset of X. ⊞

Now let $[x], [y] \in X/\sim$ be two disjoint points in $X/\sim$. Since X is Hausdorff we know from Lemma 2.17 (2) that $\{x\}$ and $\{y\}$ are closed subsets of X. By the claim we see that $\{[x]\}$ and $\{[y]\}$ are closed subsets of $X/\sim$. Since the projection map $p\colon X \to X/\sim$ is continuous we see that $p^{-1}(\{[x]\})$ and $p^{-1}(\{[y]\})$ are two disjoint closed subsets of X. Since X is compact and Hausdorff we know that X is normal. Thus there exist open disjoint neighborhoods U and V of $p^{-1}(\{[x]\})$ and $p^{-1}(\{[y]\})$. We write $U' = X \setminus U$ and $V' = X \setminus V$. These are now closed subsets of X. By the claim their images are closed subsets of $X/\sim$. Now we see that $(X/\sim) \setminus p(U')$ and $(X/\sim) \setminus p(V')$ are the desired open disjoint sets containing $[x]$ respectively $[y]$. ■

We conclude this section on quotient spaces with three technical lemmas.

Lemma 3.27. (∗) *Let X be a topological space and let $\sim$ be an equivalence relation on X. Furthermore let $\mathcal{B}$ be a basis of the topology of X. If the projection $p\colon X \to X/\sim$ is an open map, then*

$$p(\mathcal{B}) \;=\; \{p(B) \,|\, B \in \mathcal{B}\}$$

is a basis for the topology of $X/\sim$.

Proof (∗). First note that the sets in $p(\mathcal{B})$ are indeed open subsets of $X/\sim$ since we assume that the projection is open. We use the criterion provided by Lemma 2.27 (1) to show that $p(\mathcal{B})$ is a basis for the topology of $X/\sim$. Thus let $U \subset X/\sim$ be an open set and let $y \in U$. We need to show that there exists a $B \in \mathcal{B}$ with $y \in p(B) \subset U$.

We choose an $x \in X$ with $p(x) = y$. Since $\mathcal{B}$ is a basis for the topology of X and since $p^{-1}(U)$ is open in X there exists by Lemma 2.27 (1) a $B \in \mathcal{B}$ with $x \in B \subset p^{-1}(U)$. But then we also have that $y = p(x) \in p(B) \subset p(p^{-1}(U)) = U$. ■

A careful reading of these notes shows that the following lemma gets used on many occasions without giving its due credit.

Lemma 3.28. (∗)

(1) *Let X be a topological space and let $\sim_1$ be an equivalence relation on X. Furthermore let $\sim_2$ be an equivalence relation on $X/\sim_1$. Given $x, y \in X$ we define*

$$x \sim y \quad :\Longleftrightarrow \quad [x]_{\sim_1} \sim_2 [y]_{\sim_1}.$$

Then the following two statements hold:
(a) *“$\sim$” is an equivalence relation.*

[58]This follows from the observation that $(X \times X) \setminus (C \times X) = (X \setminus C) \times X$ is the product of two open sets.

(b) *The map*

$$\begin{array}{rcl} X/\sim & \to & (X/\sim_1)/\sim_2 \\ [x] & \mapsto & [[x]_{\sim_1}]_{\sim_2} \end{array}$$

is a homeomorphism.

(2) *Let $f\colon X \to Y$ be a map between two topological spaces and let $\sim_X$ and $\sim_Y$ be equivalence relations on X respectively Y. If $(f \times f)(\sim_X) \subset \sim_Y$, then*[59] *the map*

$$\begin{array}{rcl} X/\sim_X & \to & Y/\sim_Y \\ [x] & \mapsto & [f_*(x)] \end{array}$$

is continuous.

Proof.

(1) We leave the proof of this lemma as a gift to our dear reader.

(2) It follows from Lemma 3.21 (3) that the map $X \xrightarrow{f} Y \xrightarrow{y\mapsto[y]} Y/\sim_Y$ is continuous. It follows from $(f \times f)(\sim_X) \subset \sim_Y$ and Lemma 3.22 that this map descends to a continuous map $X/\sim_X \to Y/\sim_Y$. ■

Finally we have the following lemma, which might not sound particularly interesting at first sight, but which will used on many occasions.

Lemma 3.29. (∗) *Let X be a topological space and let A be a closed subset.*

(1) *For every open subset $U \subset X \setminus A$ the obvious map $U \to X/A$ is an open embedding.*

(2) *For every open neighborhood W of A the obvious map $W \setminus A \to W/A \setminus A/A$ is a homeomorphism.*

Figure 75. Illustration for Lemma 3.29.

Proof (∗). Let X be a topological space and let A be a closed subset. In the following we denote by $p\colon X \to X/A$ the projection.

(1) Let $U \subset X \setminus A$ be an open subset. We want to show that $q\colon U \to X/A$ is an open embedding. It follows from $U \subset X \setminus A$ that q is an injection. Furthermore note that by Lemma 3.21 (3) we know that q is continuous. By Lemma 2.42 (2) it remains to show that $q\colon U \to X/A$ is an open map. Now let $V \subset U$ be an open subset. We need to show that $q(V) \subset X/A$ is open. We break the argument into three easy steps:
   (a) It follows from Lemma 2.38 (1) and the fact that A is by hypothesis closed, that V is also an open subset of X.
   (b) Since $A \cap V = \varnothing$ we see that $p^{-1}(p(V)) = V$, which by (a) is in fact an open subset of X.
   (c) It follows from (b) and the definition of the quotient topology that $p(V)$ is an open subset of X/A.

[59] Recall that an equivalence relation on X is by definition a subset of $X \times X$, so it makes sense to ask whether $(f \times f)(\sim_X) \subset \sim_Y$.

(2) Let W be an open neighborhood of A. We set $U := W \setminus A$. Since A is closed we see that U is an open subset of $W \setminus A$. We want to show that the obvious map $q \colon W \setminus A \to W/A \setminus A/A$ is a homeomorphism. It is clear that q is a bijection and it follows from Lemma 3.28 (2) that q is continuous. By Lemma 2.42 (2) it remains to show that $q \colon W \setminus A \to W/A \setminus A/A$ is an open map. Thus let V be an open subset of $W \setminus A$. We have $q(V) = p(V) \cap (W/A \setminus A/A)$. It follows from (1c) and the definition of the subspace topology that $q(V)$ is an open subset of $W/A \setminus A/A$. ■

3.7. **Group actions on topological spaces.** We continue with the following definition that eventually will provide a rich source of examples of topological spaces.

Definition. Let X be a set and let G be a group with trivial element e.

(1) An *action* of G on X is a map
$$\begin{array}{rcl} G \times X & \to & X \\ (g,x) & \mapsto & g \cdot x \end{array}$$
with the following properties
$$\begin{array}{rcll} e \cdot x & = & x, & \text{for all } x \in X, \\ \underbrace{g \cdot (h \cdot x)}_{\text{group action}} & = & \underbrace{(g \cdot_G h)}_{\text{multiplication in the group } G} \cdot x, & \text{for all } x \in X \text{ and } g,h \in G. \end{array}$$
(2) The action is called *free*, if $g \cdot x = x$ for some $x \in X$ implies that $g = e$.
(3) We say G acts *transitively*, if for every x and y in X there exists a $g \in G$ with $g \cdot x = y$.
(4) If X is a topological space, then we say that the action is *continuous*, if for every $g \in G$ the map
$$\begin{array}{rcl} X & \to & X \\ x & \mapsto & g \cdot x \end{array}$$
is continuous.[60]

Remark. Sometimes what we call an "action" is referred to as a "left-action". A "right-action" is exactly the same as a left-action except that the second property gets replaced by $g \cdot (h \cdot x) = (h \cdot_G g) \cdot x$. Or, more intuitively, a right-action is a map $X \times G \to X$ such that we have $(x \cdot h) \cdot g = x(h \cdot_G g)$. Basically all the statements and definitions that we give for left-actions carry over, with minuscule modifications, to the setting of right-actions.

Examples.

(1) Given any group G the map
$$\begin{array}{rcl} G \times G & \to & G \\ (g,h) & \mapsto & ghg^{-1} \end{array}$$

[60] If the action is continuous, then the map $x \mapsto g \cdot x$ is in fact a homeomorphism with inverse map given by $x \mapsto g^{-1} \cdot x$.

defines an action of G on itself. Let G be a non-trivial group. Considering $h = e$ we see that this action is not free and it is also straightforward to see that the action is not transitive.

(2) Let $\mathbb{F}$ be a field and let V be a k-dimensional vector space over $\mathbb{F}$. We denote by B the set of all ordered bases of V. The map

$$\begin{array}{rcl} \mathrm{GL}(k,\mathbb{F}) \times B & \to & B \\ \big((a_{ij})_{i,j=1,\dots,k}, (v_1,\dots,v_k)\big) & \mapsto & \Big(\sum\limits_{i=1}^{k} a_{i1}v_i, \dots, \sum\limits_{i=1}^{k} a_{ik}v_i\Big) \end{array}$$

is a transitive action. It is an amusing linear algebra exercise to figure out under what circumstances this action is free.

(3) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is evidently free and continuous but it is not transitive.

(4) Basic linear algebra shows that the action of $\mathrm{O}(n)$ on S^{n-1} given by multiplication is transitive. Note thought that for $n \geq 3$ the action is not free.

We continue with the following definition.

Definition. Let X be a set and let G be a group that acts on X. It is straightforward to verify that

$$x \sim y \quad :\Longleftrightarrow \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y$$

is an equivalence relation on X. We write[61]

$$X/G \;:=\; X/\sim.$$

If X is a topological space, then we view X/G as a topological space equipped with the quotient topology.

Lemma 3.30. *Let X be a topological space and let G be a group that acts on X.*

(1) *The projection map $X \to X/G$ is continuous.*

(2) *If G acts continuously on X, then the projection map is also open.*

Proof. The first statement follows immediately from Lemma 3.21 (3). We turn to the proof of the second statement. Thus let $U \subset X$ be an open subset. We need to show that $p(U)$ is open in X/G. Thus we need to show that $p^{-1}(p(U))$ is an open subset of X. It follows easily from the definitions that

$$p^{-1}(p(U)) \;=\; \bigcup_{g\in G} gU.$$

Since G acts continuously the map $x \mapsto gx$ is a homeomorphism. Thus we see that $p^{-1}(p(U))$ is the union of open sets, thus it is open itself. ∎

The following lemma discusses one example in greater detail.

[61] It is arguably more logical to write $G\backslash X$ since we are dealing with a left-action and to reserve the notation X/G for right-actions. We stick to X/G since it is closer to our language "X mod G" and since we will not deal with right-actions at all.

Lemma 3.31. *The map*

$$\begin{array}{rcl} \mathbb{R}^n/\mathbb{Z}^n & \to & (S^1)^n \\ [(t_1,\dots,t_n)] & \mapsto & \left(e^{2\pi i t_1},\dots,e^{2\pi i t_n}\right) \end{array}$$

is a homeomorphism.

Convention. Often we use the homeomorphism of Lemma 3.31 to make the identification $\mathbb{R}^n/\mathbb{Z}^n = (S^1)^n$.

PROOF. We start out with the following observations:

(1) The map is continuous by Lemma 3.22 and it is easy to see that it is a bijection.
(2) The obvious map $[0,1]^n \to \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is continuous and it is easily seen to be surjective. By Proposition 2.21 we know that $[0,1]^n$ is compact. It follows from Lemma 2.40 that $\mathbb{R}^n/\mathbb{Z}^n$ is compact.
(3) From Proposition 3.12 we obtain that $(S^1)^n$ is Hausdorff.

Now it follows easily from Proposition 2.43 (3) that the given map is indeed a homeomorphism. ■

We continue with more examples of group actions and the resulting quotient topological spaces.

Examples.

(1) Let $X = S^n$ and $G = \{\pm 1\}$. The map
$$\begin{array}{rcl} \{\pm 1\}\times S^n & \to & S^n \\ (\epsilon, P) & \mapsto & \epsilon\cdot P \end{array}$$
defines an action that is free and continuous. We refer to the quotient space $S^n/\{\pm 1\}$ as the *n-dimensional real projective space* $\mathbb{R}\mathrm{P}^n$.[62] In Exercise 3.22 we will show that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to S^1.
(2) Let $X = \mathbb{R}$ and $G = \{\pm 1\}$. The map
$$\begin{array}{rcl} \{\pm 1\}\times \mathbb{R} & \to & \mathbb{R} \\ (\epsilon, x) & \mapsto & \epsilon\cdot x \end{array}$$
defines an action that is continuous. But this action is not free, since $(-1)\cdot 0 = 0$, but -1 is not the trivial element in the group $G = \{\pm 1\}$. In Exercise 3.24 we will show that $\mathbb{R}/\{\pm 1\}$ is homeomorphic to the half-open interval $[0,\infty)$.

We formulate the next example as a lemma.

Lemma 3.32. *Let $n\in\mathbb{N}$.*

(1) *The maps*
$$\begin{array}{rcl} \mathrm{O}(n+1)\times S^n & \to & S^n \\ (A,v) & \mapsto & A\cdot v \end{array} \quad \textit{and} \quad \begin{array}{rcl} \mathrm{O}(n+1)\times \overline{B}^{n+1} & \to & \overline{B}^{n+1} \\ (A,v) & \mapsto & A\cdot v \end{array}$$
define continuous actions of $\mathrm{O}(n+1)$ *on* S^n *and on* $\overline{B}^{n+1}$.

[62] For $n=2$ we sometimes refer to $\mathbb{R}\mathrm{P}^2$ as the real projective plane.

(2) *The action of* $\mathrm{SO}(n+1)$ *on* S^n *is free if and only if* $n=1$.
(3) *The action of* $\mathrm{O}(n+1)$ *on* S^n *is transitive.*
(4) *If* $n \geq 1$, *then the action of* $\mathrm{SO}(n+1)$ *on* S^n *is also transitive.*
(5) *The obvious action of* $\mathrm{SO}(n)$ *on the set of hyperplanes in* $\mathbb{R}^n$ *is transitive.*

We can use the above to draw the following two consequences:

(6) *For every* $P \in S^n$ *the complement* $S^n \setminus \{P\}$ *is homeomorphic to* $\mathbb{R}^n$.
(7) *Given any* $x, y \in S^n$ *there exists a homeomorphism* $\varphi\colon \overline{B}^{n+1} \to \overline{B}^{n+1}$ *with* $\varphi(x) = y$.

Proof. Statements (1) and (2) can be verified easily. Statements (3) and(4) fall in the realm of linear algebra. We leave it to the reader to undust their favorite linear algebra book and prove the statements. For $n \geq 2$ Statement (5) follows from Statement (4) and the observation that the map

$$\begin{array}{rcl} S^n & \to & \text{set of hyperplanes in } \mathbb{R}^{n+1} \\ v & \mapsto & v^\perp := \{w \in \mathbb{R}^{n+1} \,|\, \langle v, w\rangle = 0\} \end{array}$$

is a surjection. For $n=1$ Statement (5) is trivial. Next note that Statement (6) follows from Statement (3) together with Lemma 2.44. Finally Statement (7) follows from Statements (1) and (3). ∎

We consider our next example in greater detail.

Examples.

(1) Let $X = \mathbb{R} \times [-1,1]$ and $G = \mathbb{Z}$. The map

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n,(x,y)) & \mapsto & (x+n, (-1)^n \cdot y) \end{array}$$

defines an action that is free and continuous. This action is illustrated in Figure 76. We refer to the quotient $X/\mathbb{Z}$ as the *Möbius band*. The same argument as in the proof of Lemma 3.31 shows that the Möbius band is compact.

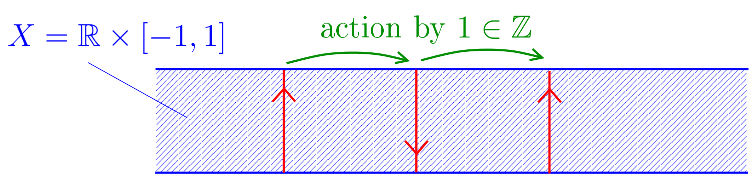


Figure 76

The name "Möbius band" conjures up a certain picture. We will now see that our name is justified. Given $\varphi, \psi \in \mathbb{R}$ and $r \in [-1,1]$ we set

$$A(\varphi) := \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v(r,\psi) := \begin{pmatrix} 2 + r\sin\psi \\ 0 \\ r\cos\psi \end{pmatrix}.$$

We consider the map

$$\begin{array}{rcl} \Psi\colon (\mathbb{R} \times [-1,1])/\mathbb{Z} & \to & \mathbb{R}^3 \\ [(\varphi, r)] & \mapsto & A(2\pi\cdot\varphi)\cdot v\big(r, \pi\varphi\big). \end{array}$$

The image of Ψ is shown in Figure 77. The argument of Lemma 3.31 shows that Ψ is an embedding. Thus the Möbius band, defined abstractly as above, is indeed homeomorphic to a subset of $\mathbb{R}^3$ that looks like what we had in mind.

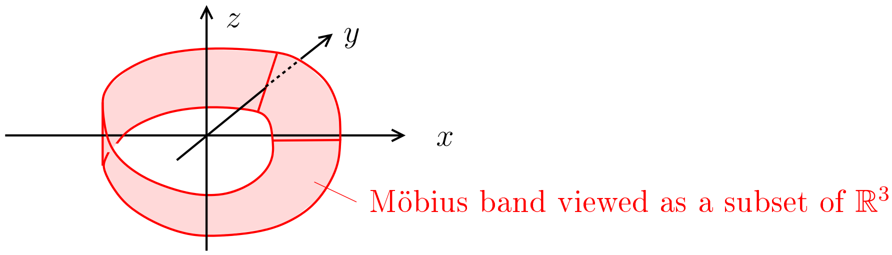


FIGURE 77

(2) On occasion we will also be interested in the *open Möbius band* which is basically defined as above, except that we define it as the quotient of $\mathbb{R} \times (-1,1)$ instead of the quotient of $\mathbb{R} \times [-1,1]$.

If X is a Hausdorff space and if G acts continuously on X, then it is unfortunately not necessarily true that X/G itself is Hausdorff. For example, consider the obvious action of $G = (\mathbb{Q},+)$ on $X = \mathbb{R}$. It is a moderately insightful exercise, see Exercise 3.25, to verify that the quotient space $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.

This example leads us to the following definition.

Definition. Let X be a topological space and let G be a group that acts on X. We say that G acts *properly* if for every two points x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \,|\, gU \cap V \neq \varnothing\}$ is finite.

Examples.

(1) The action of any finite group is proper by definition.
(2) One can show easily that the action of $\mathbb{Z}$ on $\mathbb{R} \times [-1,1]$ that we had considered on page 188 to define the Möbius band, is proper.
(3) It is straightforward to show that the action of $\mathrm{O}(n+1)$ on S^n from Lemma 3.32 is not proper.

We conclude this section with the following proposition.

Proposition 3.33. *Let G be a group that acts on a topological space X. If X is Hausdorff and if the action is proper and continuous, then the quotient space X/G is also Hausdorff.*

Unless one has an insatiable appetite for technical proofs it is best to move on to the next section.

The proof of Proposition 3.33 relies on the following two lemmas.

Lemma 3.34. $(*)$ *Let G be a group that acts continuously and properly on a topological space. If X is Hausdorff, then given any two points a and b in X there exist open neighborhoods A of a and B of b with the following property: for every $g \in G$ with $ga \neq b$ we also have $gA \cap B = \varnothing$.*

PROOF OF LEMMA 3.34 $(*)$. Let G be a group that acts continuously and properly on a topological space X that is Hausdorff. Let a and b be two points in X. Since G acts properly on X there exist open neighborhoods U of a and V of b such that $\{g \in G \,|\, gU \cap V \neq \varnothing\}$ is a finite set. We denote the elements of this finite set by $g_1, \ldots, g_r$. We set

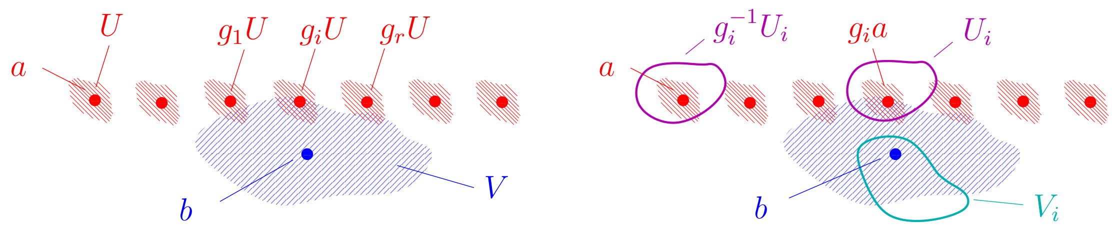


FIGURE 78. Illustration for the proof of Lemma 3.34 with $I = \{1, \ldots, r\}$.

$$I \;:=\; \{i = 1, \ldots, r \,|\, g_i a \neq b\}.$$

Since X is Hausdorff we can find for every $i \in I$ an open neighborhood U_i of $g_i a$ and an open neighborhood V_i of b such that $U_i \cap V_i = \varnothing$. We set

$$A \;:=\; U \cap \bigcap_{i \in I} g_i^{-1} U_i \quad \text{ and } \quad B \;:=\; V \cap \bigcap_{i \in I} V_i.$$

The sets A and B are the intersection of finitely many open subsets,[63] thus they are open. In particular A is an open neighborhood of a and B is an open neighborhood of b.

Now one can easily verify that A and B have the desired properties. Indeed, let $g \in G$ with $ga \neq b$. If $g \notin \{g_1, \ldots, g_r\}$, then $gU \cap V = \varnothing$, therefore $gA \cap B = \varnothing$. On the other hand, if $g \in \{g_1, \ldots, g_r\}$, then there exists an $i \in I$ with $g = g_i$. It follows that $g_i(g_i^{-1}U_i) \cap V_i = U_i \cap V_i = \varnothing$, hence $g_i A \cap B = \varnothing$. ■

Lemma 3.35. $(*)$ *Let G be a group that acts on a set X. We denote by $p\colon X \to X/G$ the projection. Let A and B be subsets of X. Then*

$$p(A) \cap p(B) \;=\; \varnothing \quad \Longleftrightarrow \quad \textit{for every } g \in G \textit{ we have } gA \cap B = \varnothing,$$

or equivalently,

$$p(A) \cap p(B) \;\neq\; \varnothing \quad \Longleftrightarrow \quad \textit{there exists a } g \in G \textit{ with } gA \cap B \neq \varnothing.$$

PROOF OF LEMMA 3.35$(*)$. We prove the second statement. The first statement is easily seen to be equivalent to the second statement. We have

$$\begin{aligned} p(A) \cap p(B) \;\neq\; \varnothing \;&\Longleftrightarrow\; \text{there exist } a \in A \text{ and } b \in B \text{ with } p(a) = p(b) \in X/G \\ &\Longleftrightarrow\; \text{there exist } a \in A,\ b \in B \text{ and } g \in G \text{ with } ga = b \\ &\Longleftrightarrow\; \text{there exists a } g \in G \text{ with } gA \cap B \neq \varnothing. \end{aligned}$$ ■

Now we turn to the actual proof of Proposition 3.33.

PROOF OF PROPOSITION 3.33 $(*)$. Let G be a group that acts on a topological space X. We assume that X is Hausdorff and that the action is proper and continuous. We want to show that the quotient space X/G is also Hausdorff. We denote by $p\colon X \to X/G$ the projection map.

[63]Here we use that G acts continuously, since this implies that the sets $h^{-1}U_i$ are also open.

Let x and y be two distinct points in X/G. We pick a and b in X with $p(a) = x$ and $p(b) = y$. Since $p(a) = x \neq y = p(b)$ we see that for every $g \in G$ we have $ga \neq b$. By Lemma 3.34 there exist open neighborhoods A of a and B of b such that $gA \cap B = \varnothing$ for every $g \in G$. It follows from Lemma 3.35 that $p(A)$ and $p(B)$ are disjoint.

Since A and B are open it follows from Lemma 3.30 that $p(A)$ and $p(B)$ are open subsets of X/G, in particular they are open neighborhoods of $x = p(a)$ and $y = p(b)$. We have thus found the desired disjoint open neighborhoods of x and y in X/G. ■

3.8. **Continuity of zeros** $(*)$**.** Group actions can also appear in unexpected situations. For example let us consider the following question.

Question 3.36. *Let $n \in \mathbb{N}$. Do the zeros of a polynomial of degree n vary continuously with the coefficients of the polynomial?*

The first question is, what does it mean that the "zeros of a polynomial" vary "continuously"? What are the topological spaces involved so that we can talk about continuity? The following proposition kills two birds at once: it makes the question precise and it gives a complete answer.

Proposition 3.37. *Let $n \in \mathbb{N}$. We consider the action by the permutation group S_n on $\mathbb{C}^n$ that is given by permuting the coefficients. The map*[64]

$$\begin{array}{rcl} f\colon \mathbb{C}^n & \to & \mathbb{C}^n/S_n \\ (v_0,\dots,v_{n-1}) & \mapsto & \left[\begin{array}{c} \textit{set of zeros of the polynomial} \\ v_0 + v_1 z + \cdots + v_{n-1}z^{n-1} + z^n, \\ \textit{counted with multiplicities} \end{array}\right] \end{array}$$

is continuous, in fact it is a homeomorphism.

Remark. For $n = 1, 2, 3, 4$ one can hope to prove Proposition 3.37 by the formulas for determining zeros of polynomials of degree n. But since no such formula exists for general polynomials of higher degree this approach is doomed for $n \geq 5$.

The proof of Proposition 3.37 makes use Rouché's Theorem from complex analysis.

Theorem 3.38. (Rouché's Theorem) *Let $U \subset \mathbb{C}$ be an open set, let $f, g\colon U \to \mathbb{C}$ be two holomorphic functions and let $z \in U$ and $r \in \mathbb{R}^+$ be such that $\overline{B}_r(z) = \{w \in \mathbb{C} \,|\, |w-z| \leq r\}$ is contained in U. If for all $w \in \partial\overline{B}_r(z)$ we have $|g(w)| < |f(w)|$, then f and $f+g$ have the same number of zeros (counted with multiplicities) within the open disk $B_r(z)$.*

Proof. This theorem is proved in every self-respecting book on complex analysis, see e.g. [**Lan99**, Theorem VI.1.6]. ■

Example. We consider $f(z) = z^5$ and the disk $\overline{B}_1(0)$. The polynomial $f(z)$ has one zero of multiplicity five in $\overline{B}_1(0)$. For any $a \in \mathbb{C}$ with $|a| < 1$ we have $|az| < |z^5|$ for all points $z \in \partial\overline{B}_1(0) = S^1$. Thus by Rouché's Theorem 3.38 we see that for such a the polynomial

[64] Slightly more rigorously the map f is defined as follows: given $(v_0,\dots,v_{n-1}) \in \mathbb{C}^n$ we define $f(v_0,\dots,v_{n-1}) := [(\alpha_1,\dots,\alpha_n)]$ where $\alpha_1,\dots,\alpha_n \in \mathbb{C}$ are determined uniquely (up to reordering) by the condition that $v_0 + v_1 z + \cdots + v_{n-1}z^{n-1} + z^n = (z-\alpha_1)\cdot\cdots\cdot(z-\alpha_n) \in \mathbb{C}[z]$.

$z^5 + az$ also has five zeros (counted with multiplicities) in $\overline{B}_1(0)$. In fact for most values of a the polynomial will have five distinct zeros, each with multiplicity one.

We will also need the following definition:

Definition. Given $w = [(w_1, \ldots, w_n)]$ and $z = [(z_1, \ldots, z_n)]$ in $\mathbb{C}^n/S_n$ we define

$$d(w, z) \;=\; \min_{\sigma \in S_n} \max_{1 \le j \le n} \left| w_j - z_{\sigma(j)} \right|.$$

The following lemma summarizes two key statements regarding the above definition.

Lemma 3.39.

(1) *The map*

$$\begin{array}{rcl} d\colon \mathbb{C}^n/S_n \times \mathbb{C}^n/S_n & \to & \mathbb{R}_{\ge 0} \\ (w, z) & \mapsto & d(w, z) \end{array}$$

is a metric on $\mathbb{C}^n/S_n$.

(2) *The topology defined by the metric* d *on* $\mathbb{C}^n/S_n$ *agrees with the quotient topology on* $\mathbb{C}^n/S_n$.

Proof. We hand over the task providing the proof of the lemma to the surely very meticulous reader. ■

Now we are in a position to prove Proposition 3.37.

Proof of Proposition 3.37. We consider the map[65]

$$\begin{array}{rcl} g\colon \mathbb{C}^n/S_n & \to & \mathbb{C}^n \\ [(\alpha_1, \ldots, \alpha_n)] & \mapsto & \text{coefficients of the polynomial } (z - \alpha_1) \cdot \cdots \cdot (z - \alpha_n). \end{array}$$

It is straightforward to check that g is continuous and that f and g are inverses of one another.

It remains to show that f is continuous. We introduce the following definitions: We denote by d the above metric on $\mathbb{C}^n/S_n$. It follows from Lemma 3.39 that we can view f as a map between the two metric spaces $(\mathbb{C}^n, \| - \|)$ and $(\mathbb{C}^n/S_n, d)$. Therefore it follows from Proposition 2.36 that it suffices to prove the following claim.

Claim.

$$\underset{v \in \mathbb{C}^n}{\forall} \; \underset{\epsilon > 0}{\forall} \; \underset{\delta > 0}{\exists} \; \underset{\substack{w \in \mathbb{C}^n \text{ with} \\ \|w\| < \delta}}{\forall} \quad d(f(v), f(v + w)) < \epsilon.$$

Before we start with the proof of the claim we introduce the following definition: Given $v = (v_0, \ldots, v_{n-1}) \in \mathbb{C}^n$ we denote by $p_v(z) := v_0 + v_1 z + \cdots + v_{n-1} z^{n-1} + z^n$ the corresponding polynomial. Now we turn to the proof of the claim. Thus let $v = (v_0, \ldots, v_{n-1}) \in \mathbb{C}^n$ and let $\epsilon > 0$. We denote by $\xi_1, \ldots, \xi_k$ the *distinct* zeros of the corresponding polynomial $p_v(z)$. (Note that if $p_v(z)$ has zeros of higher multiplicity, then $k < n$.) After possibly

[65] Slightly more precisely the map g is defined as follows: given $[(\alpha_1, \ldots, \alpha_n)] \in \mathbb{C}^n/S_n$ we have $f([(\alpha_1, \ldots, \alpha_n)]) = (v_0, \ldots, v_{n-1})$ where $v_0, \ldots, v_{n-1} \in \mathbb{C}$ are uniquely determined by the condition that $v_0 + v_1 z + \cdots + v_{n-1} z^{n-1} + z^n = (z - \alpha_1) \cdot \cdots \cdot (z - \alpha_n) \in \mathbb{C}[z]$.

replacing ϵ by a smaller non-negative number we can assume, without loss of generality, that $\epsilon < \frac{1}{2} \cdot |\xi_i - \xi_j|$ for all $i \neq j$.

We denote by Γ the union of the circles of the radius ϵ around $\xi_1, \dots, \xi_k$. Furthermore we set

$$\eta \;:=\; \inf_{z \in \Gamma} |p_v(z)| \;\in\; \mathbb{R}_{>0}.$$

It is elementary to show that there exists a $\delta > 0$ such that for all $w = (w_0, \dots, w_{n-1}) \in \mathbb{C}^n$ such that $\|w\| < \delta$ we have

$$|w_0 + w_1 + \cdots + w_{n-1} z^{n-1}| \;<\; \eta \text{ for all } z \in \Gamma.$$

We claim that this δ has the desired property. Thus let $w \in \mathbb{C}^n$ with $|w| < \delta$. It follows from the definition of δ and Rouché's Theorem 3.38 that the polynomials $p_v(z)$ and $p_{v+w}(z)$ have the same number of zeros (counted with multiplicities) in the disks $B_\epsilon(\xi_1), \dots, B_\epsilon(\xi_k)$. This means that the zeros of $p_v(z)$ and can be matched up with the zeros of $p_{v+w}(z)$ such that each zero is less then ϵ removed from its match. This statement implies immediately that $d(f(v), f(v+w)) < \epsilon$. ∎

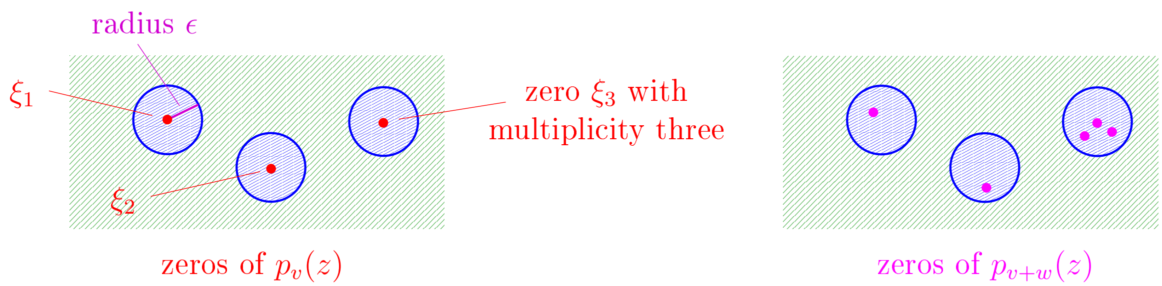


FIGURE 79. Illustration for the proof of Proposition 3.37.

Remark. With some effort one can also prove Proposition 3.37 without using Rouché's Theorem 3.38. We outline the key steps, using the notation that we introduced in the above proof of Proposition 3.37.

(1) By Lemma 2.42 (2) it suffices to show that the map $g\colon \mathbb{C}^n/S_n \to \mathbb{C}^n$ is open.
(2) By Lemma 2.39 it suffices to show that for each $\epsilon > 0$ and each $z \in \mathbb{C}^n/S_n$ the image of the open ball $B_\epsilon(z) \subset (\mathbb{C}^n/S_n, d)$ under g is open.
(3) Thus let $\epsilon > 0$ and $z \in \mathbb{C}^n/S_n$.
(4) We pick $C \in \mathbb{R}_{\geq 0}$ such that $g(\overline{B}_\epsilon(z)) \subset B_C(0) \subset \mathbb{C}^n$.
(5) There exists a $D \in \mathbb{R}_{\geq 0}$ such that for any $v \in \mathbb{C}^n$ which satisfies $\|v\| \leq C$ we have $f(v) \in B_D(0) \subset \mathbb{C}^n/S_n$. (This statement requires some unraveling and a little bit of thought. It can be proved using Rouché's Theorem 3.38, but with a worse bound it can also be proved by hand.)
(6) By Proposition 2.43 (3) we know that g restricted to $\overline{B}_D(0) \to g(\overline{B}_D(0))$ is a homeomorphism. Thus $g(B_\epsilon(z))$ is an open subset of $g(\overline{B}_D(0))$.
(7) By (5) and (6), and since f and g are inverses of one another, we know that $g(B_\epsilon(z))$ is also open in $B_D(0)$. But $B_D(0)$ is an open subset of $\mathbb{C}^n$, hence $g(B_\epsilon(z))$ is also open in $\mathbb{C}^n$.

3.9. **Projective spaces.** Let $n \in \mathbb{N}$. On page 187 we introduced the real projective space $\mathbb{RP}^n = S^n/\{\pm 1\}$. In this section we will give an alternative definition and we will introduce the complex projective spaces. Even though these projective spaces appear initially rather arbitrary they will on several occasions play an essential role in these notes.

Let $X = \mathbb{R}^{n+1} \setminus \{0\}$ and let $G = \mathbb{R} \setminus \{0\}$, where we view G as a group via multiplication. The map

$$\begin{array}{rcl} (\mathbb{R} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\}) & \to & \mathbb{R}^{n+1} \setminus \{0\} \\ (r, P) & \mapsto & r \cdot P \end{array}$$

defines an action that is free and continuous. It is straightforward to see that the map

$$\begin{array}{rcl} \mathbb{RP}^n = S^n/\{\pm 1\} & \to & (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\}) \\ [P] & \mapsto & [P] \end{array}$$

defines a homeomorphism.[66] In fact we will use this homeomorphism to make the identification $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\})$. Our new point of view on real projective spaces has the advantage that we can create a different class of examples of topological spaces by replacing $\mathbb{R}$ by $\mathbb{C}$. More precisely, this discussion leads us to the following definition:

Definition. Let $X = \mathbb{C}^{n+1} \setminus \{0\}$ and let $G = \mathbb{C} \setminus \{0\}$. The map

$$\begin{array}{rcl} (\mathbb{C} \setminus \{0\}) \times (\mathbb{C}^{n+1} \setminus \{0\}) & \to & \mathbb{C}^{n+1} \setminus \{0\} \\ (z, P) & \mapsto & z \cdot P \end{array}$$

defines an action that is free and continuous. The quotient space $(\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$ is called the *n-dimensional complex projective space* $\mathbb{CP}^n$.

Notation. Let $n \in \mathbb{N}_0$.

(1) For $(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ we denote by $[x_0 : \dots : x_n]$ the corresponding point in $\mathbb{RP}^n$. We also use the same notation for points in complex projective spaces.

(2) Let $m \le n$. We consider the maps

$$\begin{array}{rcl} i\colon \mathbb{RP}^m & \to & \mathbb{RP}^n \\ [x_0 : \dots : x_m] & \mapsto & [x_0 : \dots : x_m : 0 : \dots : 0] \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon \mathbb{CP}^m & \to & \mathbb{CP}^n \\ [z_0 : \dots : z_m] & \mapsto & [z_0 : \dots : z_m : 0 : \dots : 0]. \end{array}$$

It follows almost immediately from Proposition 2.43 (2) and Proposition 3.40 that both maps are embeddings. Thus we can and will view $\mathbb{RP}^m$ as a subset of $\mathbb{RP}^n$ and similarly we will view $\mathbb{CP}^m$ as a subset of $\mathbb{CP}^n$.

The following lemma shows that real and complex projective spaces have some of our favorite properties.

Proposition 3.40. *The real projective and the complex projective spaces are compact and Hausdorff.*

In the proof of Proposition 3.40 it is convenient to have the following description of complex projective spaces.

[66] For example an inverse is given by the map $(\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\}) \to S^n/\{\pm 1\}$ that is defined by $[x] \mapsto [\frac{x}{\|x\|}]$. More precisely, it follows from Lemma 3.22 and Lemma 3.21 (3) that both maps are continuous. Clearly they are inverses of one another.

Lemma 3.41. *Let $n \in \mathbb{N}_0$. We make the identification*

$$S^{2n+1} = \{(z_1,\dots,z_{n+1}) \in \mathbb{C}^{n+1} \,|\, |z_1|^2+\dots+|z_{n+1}|^2 = 1\}.$$

We consider the action

$$\begin{array}{rcl} S^1 \times S^{2n+1} & \to & S^{2n+1} \\ (w,(z_1,\dots,z_{n+1})) & \mapsto & (w\cdot z_1,\dots,w\cdot z_{n+1}). \end{array}$$

The maps

$$\begin{array}{rcl} f\colon S^{2n+1}/S^1 & \to & \mathbb{C}\mathrm{P}^n \\ {[z_1,\dots,z_{n+1}]} & \mapsto & [z_1:\ldots:z_{n+1}] \end{array} \quad \text{and} \quad \begin{array}{rcl} p\colon \mathbb{C}\mathrm{P}^n & \to & S^{2n+1}/S^1 \\ {[z_1:\ldots:z_{n+1}]} & \mapsto & \Big[\frac{1}{\sqrt{|z_1|^2+\dots+|z_{n+1}|^2}}(z_1,\dots,z_{n+1})\Big] \end{array}$$

are continuous and inverses of one another, in particular both maps are homeomorphisms. We will use these maps to make the identification $\mathbb{C}\mathrm{P}^n = S^{2n+1}/S^1$.

Lemma 3.41. It follows easily from the definition of the quotient topology on the complex projective space $\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1}\setminus\{0\})/(\mathbb{C}\setminus\{0\})$ and from Lemma 3.21 (3) together with Lemma 3.22 that both maps are continuous. It is easy to verify that the maps are inverses of one another. ■

Proof of Proposition 3.40 (∗). Let $n \in \mathbb{N}$. It follows from Lemma 2.40 and Proposition 3.33 that the real projective space $\mathbb{R}\mathrm{P}^n = S^n/\{\pm 1\}$ is compact and Hausdorff.

For the complex projective space $\mathbb{C}\mathrm{P}^n$ we need to work just a little harder. Throughout the remainder of this proof we will use the identification $\mathbb{C}\mathrm{P}^n = S^{2n+1}/S^1$ from Lemma 3.41. Note that since S^{2n+1} is compact it follows again from Lemma 2.40 that $\mathbb{C}\mathrm{P}^n = S^{2n+1}/S^1$ is compact.

It remains to show that $\mathbb{C}\mathrm{P}^n = S^{2n+1}/S^1$ is Hausdorff. A direct argument is sketched in Exercise 3.39. In the following we will provide a less direct but more general argument. Namely, we note that it follows from Lemma 3.26 that S^{2n+1}/S^1 is Hausdorff once we have shown that the graph of the equivalence relation, which is given by

$$G = \{(v_1,v_2) \in S^{2n+1}\times S^{2n+1} \,|\, \text{there exists } w \in S^1 \text{ with } w\cdot v_1 = v_2\},$$

is a closed subset of $S^{2n+1}\times S^{2n+1}$. The graph G is the image of the map

$$\begin{array}{rcl} \Phi\colon S^{2n+1}\times S^1 & \to & S^{2n+1}\times S^{2n+1} \\ (v,w) & \mapsto & (v, w\cdot v). \end{array}$$

This map is continuous and the left-hand side is compact by Proposition 3.12 (2). Thus the image of Φ is compact by Lemma 2.40. By Proposition 3.12 (1) we know that $S^{2n+1}\times S^{2n+1}$ is Hausdorff. Thus we obtain from Lemma 2.17 that the image of Φ is a closed subset of $S^{2n+1}\times S^{2n+1}$. ■

In the following lemma we give alternative descriptions of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^1$ that will be used on many occasions.

Lemma 3.42.

(1) *Let $n \in \mathbb{N}$. If we denote by $\sim$ the equivalence relation on $\overline{B}^n$ that is generated by $P \sim -P$ for $P \in S^{n-1} = \partial \overline{B}^n$, then the map*

$$\begin{array}{rcl} \mathbb{RP}^n = S^n/\{\pm 1\} & \xrightarrow{\cong} & \overline{B}^n/\sim \\ [(x_1,\dots,x_n,x_{n+1})] & \mapsto & \begin{cases} \ \ [(x_1,\dots,x_n)], & \text{if } x_{n+1} \geq 0, \\ [-(x_1,\dots,x_n)], & \text{if } x_{n+1} < 0 \end{cases} \end{array}$$

is a homeomorphism.

(2) *The map*

$$\begin{array}{rcl} \mathbb{CP}^1 & \to & \mathbb{C} \cup \{\infty\} = \mathbb{R}^2 \cup \{\infty\} \\ [z_0 : z_1] & \mapsto & \begin{cases} z_0 \cdot z_1^{-1}, & \text{if } z_1 \neq 0, \\ \infty, & \text{if } [z_0 : z_1] = [1 : 0] \end{cases} \end{array}$$

is a homeomorphism.

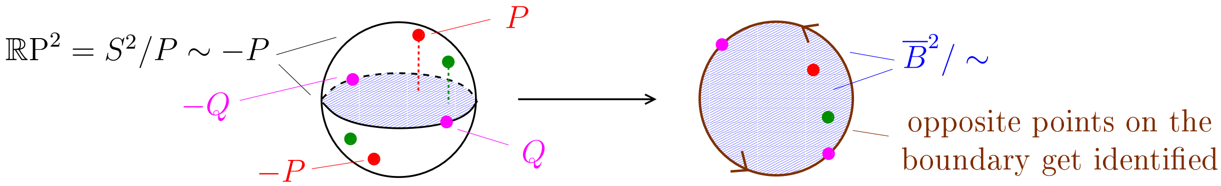


FIGURE 80. Illustration for Lemma 3.42 (1).

PROOF. First note that one can easily verify that both maps are actually well-defined and that they are bijections. Next we explain why the two maps are continuous:

(1) By Lemma 3.22 it suffices to show that the corresponding map $S^n \to \overline{B}^n/\sim$ is continuous. One can easily verify that the restriction to the closed subsets $S^n_{\geq 0}$ and $S^n_{\leq 0}$ are continuous. Thus it follows from Lemma 2.35 (2) that the map $S^n \to \overline{B}^n$ is continuous.

(2) It follows again from Lemma 3.22 that it suffices to show that the corresponding map $\Theta\colon \mathbb{C}^2 \setminus \{(0,0\} \to \mathbb{C} \cup \{\infty\}$ is continuous. Note that for any $C \in \mathbb{R}$ we have

$$\Theta^{-1}(\text{complement of } \overline{B}^n_C) \ = \ \{(z_0,z_1) \in \mathbb{C}^2 \setminus \{(0,0)\} \,|\, |z_0| > |z_1|\}$$

which is an open subset of $\mathbb{C}^2 \setminus \{(0,0)\}$. It follows quite easily from this discussion together with Proposition 2.37 and together with the discussion of the basis of $\mathbb{R}^2 \cup \{\infty\}$ on page 103 that the map Θ is continuous.

By Proposition 3.40 we know that each $\mathbb{RP}^n$ and $\mathbb{CP}^n$ is compact. Using Lemma 3.26 one can show that $\overline{B}^n/\sim$ is Hausdorff. Furthermore we know from Exercise 2.20 that $\mathbb{R}^2 \cup \{\infty\}$ is Hausdorff. Thus it follows from Proposition 2.43 (3) that both maps are homeomorphisms. ■

Convention.

(1) On many occasions we will use the homeomorphism from Lemma 3.42 to make the identification $\mathbb{RP}^n = \overline{B}^n/\sim$.

(2) In Lemma 2.44 we gave an explicit homeomorphism $S^2 \to \mathbb{R}^2 \cup \{\infty\}$. Combining this homeomorphism with the homeomorphism from Lemma 3.42 we obtain a homeomorphism $S^2 \to \mathbb{C}\mathrm{P}^1$ which we will often use to make the identification $S^2 = \mathbb{C}\mathrm{P}^1$. Note that under this identification the North Pole $(0,0,1) \in S^2$ corresponds to the point $[1:0] \in \mathbb{C}\mathrm{P}^1$.

We conclude this short section on projective spaces with two somewhat tangential remarks.

Remark.

(1) Here is a fun fact: if we consider $\mathbb{C}\mathrm{P}^2/\sim$, where $\sim$ denotes the equivalence relation defined by complex conjugation, i.e. where $[z:w] \sim [\overline{z}:\overline{w}]$, then the quotient space is homeomorphic (in fact diffeomorphic in a suitable sense) to S^4. This statement is known as the Arnold–Kuiper–Massey Theorem, see e.g. [**Hill17, Mass73, Sea06**].

(2) The definition of projective spaces makes sense for any field, even for fields of non-zero characteristic. For example, if $\mathbb{F}$ is a field of characteristic p we can still define the set $\mathbb{F}\mathrm{P}^n = (\mathbb{F}^{n+1} \setminus \{0\})/(\mathbb{F} \setminus \{0\})$. A priori this is just a set, but together with the Zariski topology, see [**Har92**, Lecture 2] or [**SKKT00**, Chapter 1], it actually becomes an unusual, but interesting topological space.
For any field the map

$$\begin{array}{rcl} \mathbb{F}\mathrm{P}^n = (\mathbb{F}^{n+1} \setminus \{0\})/(\mathbb{F} \setminus \{0\}) & \to & \{\text{all one-dimensional subspaces of } \mathbb{F}^{n+1}\} \\ {[v]} & \mapsto & \mathbb{F} \cdot v \end{array}$$

is a bijection. Thus we can view $\mathbb{F}\mathrm{P}^n$ as the set of all lines through the origin.

3.10. **The pushout construction.** In this section we consider a particularly important special case of a quotient topology which deserves its own section. Before we delve into topology we recall the following definition.

Definition. Suppose we are given a relation $\sim$ on a set X. We say $x, y \in X$ are *equivalent* if there exists a sequence $x = x_1, \dots, x_k = y$ of elements in X such that for all $i = 1, \dots, k-1$ the following holds: either $x_i \sim x_{i+1}$ or $x_{i+1} \sim x_i$. We write again $x \sim y$ if x and y are equivalent. It is straightforward to see that this is now indeed an equivalence relation. We say it is *generated* by the initial relation.

Example. The relations $x \sim (x+1)$ with $x \in \mathbb{R}$ on $\mathbb{R}$ generate the familiar equivalence relation $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$.

Definition. Let $f\colon X \to Y$ and $g\colon X \to Z$ be two maps between topological spaces. We define

$$Y \cup_X Z \ := \ (Y \sqcup Z)/\sim \qquad \text{where } f(x) \sim g(x) \text{ for all } x \in X.$$

We refer to $Y \cup_X Z$ as the *pushout of f and g*. Furthermore we refer to the maps

$$\begin{array}{rcl} i\colon Y & \to & Y \cup_X Z \\ y & \mapsto & [y] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} j\colon Z & \to & Y \cup_X Z \\ z & \mapsto & [z] \end{array}$$

as the two *natural maps*.

The notion of a pushout will play an important role throughout the lecture notes. We first summarize some key properties of pushouts, afterwards we will discuss an important special case

Lemma 3.43. $(*)$ *Let $f\colon X \to Y$ and $g\colon X \to Z$ be two maps between topological spaces. We obtain the following commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "i", "y\mapsto[y]"'] \\ Z \arrow[r, "j", "z\mapsto[z]"'] & Y\cup_X Z = (Y\sqcup Z)/\sim . \end{tikzcd}$$

The following statements hold:

(1) *The natural maps $i\colon Y \to Y\cup_X Z$ and $j\colon Z \to Y \cup_X Z$ are continuous.*

(2) (a) *If Y and Z are compact, then $Y\cup_X Z$ is also compact.*

(b) *If f is injective, then the "opposite map" j is injective. The same statement holds for "surjective" and "bijective".*

(c) *If f is an embedding, then the "opposite map" j is an embedding.*

(d) *If f is a homeomorphism, then the "opposite map" j is a homeomorphism.*

(e) *Suppose that at least one of the two maps f or g is injective.*

(α) *If the image of f is a closed subset of Y, then the image of the "opposite map" j is a closed subset of $Y\cup_X Z$.*

(β) *If the image of g is a closed subset of Z, then the image of the "opposite map" i is a closed subset of $Y\cup_X Z$.*

Both statements also hold with "closed" replaced by "open".

By symmetry the statements in (b), (c) *and* (d) *hold also for g and its "opposite map" i.*

(3) *Suppose we are given two maps $\alpha\colon Y \to W$ and $\beta\colon Z \to W$ to some topological space W such that $\alpha\circ f = \beta\circ g\colon X \to W$. The map*

$$\begin{aligned} \Theta\colon Y\cup_X Z &\to W \\ [P] &\mapsto \begin{cases} \alpha(x) & \text{if } P=[y] \text{ for some } y\in Y, \\ \beta(y) & \text{if } P=[z] \text{ for some } z\in Z \end{cases} \end{aligned}$$

is well-defined and it is the unique map $h\colon Y\cup_X Z \to W$ with $\Theta\circ i = \alpha$ and $\Theta\circ j = \beta$. Furthermore, if α and β are continuous, then Θ is also continuous. In particular we obtain the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "i", "y\mapsto[y]"'] \arrow[ddr, bend left, "\alpha"] & \\ Z \arrow[r, "j", "z\mapsto[z]"'] \arrow[drr, bend right, "\beta"'] & Y\cup_X Z = (Y\sqcup Z)/\sim \arrow[dr, "\Theta"] & \\ & & W. \end{tikzcd}$$

(4) *Suppose we are given a map $h\colon Z \to W$. Then the maps*

$$\begin{array}{rcl} Y \cup_X W & \to & (Y \cup_X Z) \cup_Z W \\ [P] & \mapsto & \begin{cases} [[P]], & \text{if } P \in Y, \\ [P], & \text{if } P \in W \end{cases} \end{array} \quad \text{and} \quad \begin{array}{rcl} (Y \cup_X Z) \cup_Z W & \to & Y \cup_X W \\ [P] & \mapsto & \begin{cases} [P], & \text{if } P \in Y, \\ [h(P)], & \text{if } P \in Z, \\ [P], & \text{if } P \in W \end{cases} \end{array}$$

are continuous and inverses of one another, in particular they are homeomorphisms.

Convention. We refer to a commutative diagram as in Lemma 3.43 as a *pushout diagram* or a *pushout square.*[67]

Example. Let $n \in \mathbb{N}_0$. We consider the maps[68]

$$\begin{array}{rcl} g_\pm\colon \overline{B}^n = \overline{B}^n_\pm & \to & S^n \\ (x_1, \ldots, x_n) & \mapsto & \big((-1)^n \cdot x_1, \ldots, x_n, \pm\sqrt{1 - x_1^2 - \cdots - x_n^2}\big). \end{array}$$

It follows from Lemma 3.43 (4) that the map

$$\begin{array}{rcl} \overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_- & \to & S^n \\ [P] & \mapsto & \begin{cases} g_+(P), & \text{if } P \in \overline{B}^n_+, \\ g_-(P), & \text{if } P \in \overline{B}^n_- \end{cases} \end{array}$$

is continuous. One can easily verify that it is a bijection. It follows from Lemma 3.43 (1) combined with Proposition 2.43 (3) that the map is a homeomorphism. We use this homeomorphism to make the identification $\overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_- = S^n$.

Proof (∗). Throughout the proof we denote by $p\colon Y \sqcup Z \to Y \cup_X Z$ the obvious projection.

(1) This statement follows from Lemma 3.3 (4) together with Lemma 3.21 (4).
(2) This statement follows immediately from Lemma 3.3 (1) and Lemma 3.21 (3).
(3) (a) If Y and Z are compact, then it follows from Lemma 3.3 (4) and Lemma 3.21 (4) that $Y \cup_X Z$ is compact.
 (b) Let $z \in Z$. It follows basically from the definitions that the set of elements in Z that are equivalent to z with respect to the equivalence relation "$\sim$" on $Y \sqcup Z$ equals precisely

$$\bigcup_{k \in \mathbb{N}_0} \underbrace{g(f^{-1}(f(g^{-1}(\ldots g(f^{-1}(f(g^{-1}(}_{\text{applied } k \text{ times}}\{z\})))) \ldots)))).$$

 If f is injective, then it follows immediately from Lemma 1.3 (1) and (2) that this set consists precisely of the element z. In other words, the map j is injective.
 (c) We suppose that $f\colon X \to Y$ is an embedding. By (1) we know that j is continuous and by (3b) we know that $j\colon Z \to Y \cup_X Z$ is an injection. By Lemma 2.42 (2) and

[67] There are various other names in the literature for this concept, e.g. "cocartesian square", "fibered coproduct" or "amalgamated sum".

[68] We put in the superfluous looking sign $(-1)^n$ for the following reason: once we have introduced orientations of smooth manifolds and once we have introduced the standard orientations of $\overline{B}^n$ and S^n this sign assures us that the map $g_+\colon \overline{B}^n \to S^n$ is orientation-preserving and it means that $g_-\colon \overline{B}^n \to S^n$ is orientation-reversing.

by definition of the subspace topology on $j(U)$ it remains to prove the following claim.

Claim. If U is an open subset of Z, then there exists an open subset W of $Y \cup_X Z$ such that $j(U) = W \cap j(Z)$.

Let U be an open subset of Z. Since g is continuous we know that $g^{-1}(U)$ is an open subset of X. Since f is an embedding we know that there exists an open subset $V \subset Y$ with $f(g^{-1}(U)) = f(X) \cap V$. We set $W := p(U \sqcup V)$. A short argument shows that $p^{-1}(W) = U \sqcup V$. Thus it follows from the definition of the quotient topology and the disjoint union topology that W is indeed an open subset of $Y \cup_X Z$. Basically by construction we have $j(U) = W \cap j(Z)$. ⊞

(d) Suppose $f\colon X \to Y$ is a homeomorphism. It follows immediately from (3b) and (3c) that $j\colon Z \to Y \cup_X Z$ is a homeomorphism.

(e) Without loss of generality we can assume that f is injective. The statements for "closed" follow almost immediately from the following observations:

(i) By Lemma 3.21 a subset W of $Y \cup_X Z$ is closed if and only if $p^{-1}(W)$ is closed.

(ii) Since f is injective it follows from the argument in (2a) that

$$p^{-1}(i(Y)) = Y \sqcup g(X) \quad \text{and} \quad p^{-1}(j(Z)) = f(g^{-1}(Z)) \sqcup Z = f(X) \sqcup Z.$$

(iii) By Lemma 3.1 a subset $U \sqcup V$ of $Y \sqcup Z$ is closed if and only if U and V are closed.

The proof for "open" is basically the same.

(4) This statement is an immediate consequence of the definitions and Lemma 3.22.

(5) It follows immediately from applying (3) altogether three times that the maps are continuous. It is basically clear that they are inverses of one another. Thus both are homeomorphisms. ■

Now, as promised we will discuss an important class of pushouts, namely where the map $f\colon X \to Y$ is just the inclusion of a subset. This leads us to the following definition.

Definition. Let Y and Z be topological spaces, let $A \subset Y$ be a subset and let $\varphi\colon A \to Z$ be a continuous map. (Note that φ does not need to be injective.) We write

$$Y \cup_\varphi Z \;:=\; Y \cup_{\varphi\colon A \to Z} Z \;:=\; (Y \sqcup Z)/\sim$$

where $\sim$ is the equivalence relation generated by $a \sim \varphi(a)$ for $a \in A$. We call $Y \cup_\varphi Z$ the topological space obtained by *gluing Y to Z via the map f*. The definition is illustrated in Figure 81.

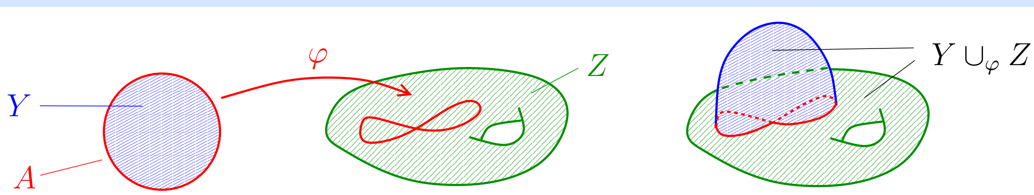


FIGURE 81

Example. Let Y be a topological space and let A be a subset of Y. If Z is the topological space that consists of a single point, then there exists evidently precisely one map $\varphi\colon A \to Z$ and it follows basically immediately from the definitions that $Y \cup_\varphi Z$ is homeomorphic to Y/A.

Out of a sense of duty we summarize the main properties of $Y \cup_{\varphi\colon A\to Z} Z$.

Lemma 3.44. (*) *Let Y and Z be two topological spaces, let $A \subset Y$ be a subset and let $\varphi\colon A \to Z$ be a map. We denote by $i\colon Y \to Y \cup_{\varphi\colon A\to Z} Z$ and $j\colon Z \to Y \cup_{\varphi\colon A\to Z} Z$ the obvious maps. For clarity we consider the following diagram:*

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "\varphi"] & Y \arrow[d, "i", "y\mapsto[y]"'] \\ Z \arrow[r, "j", "z\mapsto[z]"'] & Y \cup_{\varphi\colon A\to Z} Z = (Y \sqcup Z)/\sim . \end{tikzcd}$$

The following statements hold:

(1) *The maps $i\colon Y \to Y \cup_{\varphi\colon A\to Z} Z$ and $j\colon Z \to Y \cup_{\varphi\colon A\to Z} Z$ are continuous.*
(2) (a) *If A is a closed subset of Y, then $j(Z)$ is a closed subset of $Y \cup_{\varphi\colon A\to Z} Z$.*
(b) *If $\varphi(A)$ is a closed subset of Z, then $i(Y)$ is a closed subset of $Y \cup_{\varphi\colon A\to Z} Z$.*
The same statements hold with "closed" replaced by "open".
(3) (a) *If $B \subset Y \setminus A$ is a closed set, then $i(B)$ is a closed subset of $Y \cup_{\varphi\colon A\to Z} Z$.*
(b) *If $C \subset Z \setminus \varphi(A)$ is a closed set, then $j(C)$ is a closed subset of $Y \cup_{\varphi\colon A\to Z} Z$.*
The same statements hold with "closed" replaced by "open".
(4) *The map $j\colon Z \to Y \cup_{\varphi\colon A\to Z} Z$ is an embedding.*[69]
(5) *Let $f\colon Y \to W$ and $g\colon Z \to W$ be two maps to a topological space W. If we have $g(\varphi(a)) = f(a)$ for all $a \in A$, then*

$$\begin{array}{rcl} h\colon Y \cup_{\varphi\colon A\to Z} Z & \to & W \\ p & \mapsto & \begin{cases} f(x) & \text{if } p = [x] \text{ for some } x \in Y, \\ g(y) & \text{if } p = [y] \text{ for some } y \in Z \end{cases} \end{array}$$

is well-defined and it is the unique map $h\colon Y \cup_{\varphi\colon A\to Z} Z \to W$ with $h \circ i = f$ and $h \circ j = g$. Furthermore, if f and g are continuous, then h is also continuous.

Proof (*). Throughout the proof we will apply Lemma 3.43 to the embedding $i\colon A \to Y$ and the map $\varphi\colon A \to Z$.

(1) This statement is a special case of Lemma 3.43 (2).
(2) This statement is a special case of Lemma 3.43 (3d).
(3) As on earlier occasions we denote by $p\colon Y \sqcup Z \to Y \cup_{\varphi\colon A\to Z} Z$ the obvious projection. The proof of this statement is very similar to the proof of (2), we just need to notice that for $B \subset Y \setminus A$ we have $p^{-1}(i(B)) = B \sqcup \varnothing$ and that for $C \subset Z \setminus \varphi(A)$ we have $p^{-1}(j(C)) = \varnothing \sqcup C$.
(4) This statement is a special case of Lemma 3.43 (3b).
(5) This statement is a special case of Lemma 3.43 (4). ■

[69]Note that if φ is not an embedding, then the setting is not symmetric in Y and Z. For example, as we saw above, in general the map $i\colon Y \to Y \cup_{\varphi\colon A\to Z} Z$ is not even injective.

The following lemma is quite subtle and gets used, often inadvertently, on many occasions.

Lemma 3.45. *Let X be a topological space and let A and B be two subsets of X with $X = A \cup B$. We denote by $\varphi\colon A \cap B \to A \cap B$ the identity map. We suppose that one of the following two conditions hold:*

(1) *Both subsets A and B are closed.*
(2) *Both subsets A and B are open.*

Then the map

$$\begin{array}{rcl} A \cup_{\varphi\colon A\cap B \to A\cap B} B & \to & X \\ [x] & \mapsto & x \end{array}$$

is a homeomorphism.

Remark. In general the conclusion of the lemma does not hold without some assumptions on A and B. For example, consider $X = [-1,1]$ together with the subsets $A = [-1,0]$ and $B = (0,1]$. In this case $A \cap B = \varnothing$, thus $A \cup_\varphi B$ is the disjoint union of A and B. It follows from Lemma 3.3 (4) and the Heine-Borel 2.20 Theorem that $A \sqcup B$ is not compact. On the other hand $X = [-1,1]$ is compact by the Heine-Borel 2.20. Thus the two topological spaces $A \cup_\varphi B$ and X are not homeomorphic. We refer to Figure 82 for an illustration.

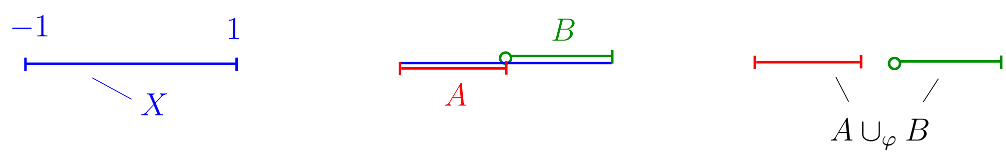


Figure 82

Proof. Let X be a topological space and let A and B be two subsets of X with $X = A \cup B$. We denote by $\varphi\colon A \cap B \to A \cap B$ the identity map. In the following we consider the case that A and B are both closed subsets of X. The case that both A and B are open subsets is dealt with in almost entirely the same way.

It follows from Lemma 3.44 (5) that the given map $h\colon A \cup_{\varphi\colon A\cap B \to A\cap B} B \to X$ is continuous. Furthermore, basically by definition the map h is a bijection. By Lemma 2.42 (2) it remains to show that the map h is closed. Thus let $W \subset A \cup_\varphi B$ be a closed subset. Note that $h(W) = (W \cap A) \cup (W \cap B)$. It follows from Lemma 3.21 (2), applied to the projection map

$$A \sqcup B \ \to \ A \cup_\varphi B = (A \sqcup B)/\sim,$$

that $A \cap W$ is closed in A and that $B \cap W$ is closed in B. Since A and B are by hypothesis closed in X we obtain from Lemma 2.38 (2) that $A \cap W$ and $B \cap W$ are closed in X. It follows that $h(W) = (W \cap A) \cup (W \cap B)$ is a closed subset of X. ■

We conclude this section with an amusing description of spheres. We introduce the following notation:

(1) We write $I = [-1,1]$.

(2) Given $k \in \mathbb{N}_0$ we write[70]

$$\partial I^k \;=\; \{(x_1,\dots,x_k) \in I^k \,|\, \text{ at least one } x_i \text{ lies in } \{\pm 1\}\}.$$

(3) We recall that given $k \in \mathbb{N}_0$ there exists by Proposition 2.52 a canonical homeomorphism

$$\varphi_k \colon (\overline{B}^k, S^{k-1}) \xrightarrow{\cong} (I^k, \partial I^k)$$

with $\varphi_k(0) = 0$.

Now let $m, n \in \mathbb{N}_0$. An elementary set-theoretic argument shows that

$$((\partial I^m) \times I^n) \,\cup\, (I^m \times \partial I^n) \;=\; \partial I^{m+n}$$

and the two subsets to the left coincide in $(\partial I^m) \times (\partial I^n)$. (We refer to Figure 83 for an illustration.) It follows from Lemma 3.45 that the inclusions induce a homeomorphism

$$((\partial I^m)\times I^n) \cup_{\mathrm{id}_{\partial I^m \times \partial I^n}} (I^m \times \partial I^n) \xrightarrow{\cong} \partial I^{m+n}.$$

Together with the above homeomorphisms we obtain a canonical homeomorphism

$$(S^{m-1} \times \overline{B}^n) \cup_{\mathrm{id}_{S^{m-1}\times S^{n-1}}} (\overline{B}^m \times S^{n-1}) \xrightarrow{\cong} S^{m+n-1}.$$

For example, if we take $m = n = 2$, then we see that we can write S^3 as the union of two solid tori. We will discuss this particular case in greater detail later in Section 27.1.

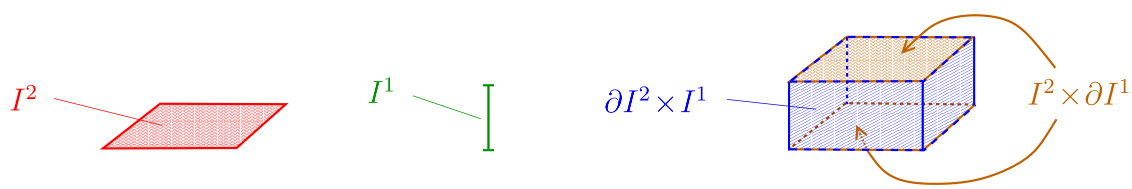


FIGURE 83

3.11. **Surfaces.** In this section we introduce some of my favorite topological spaces, namely surfaces.

(1) We consider $X = [0,1] \times [0,1] \subset \mathbb{R}^2$ and the equivalence relation which is generated by

$$(x,0) \;\sim\; (x,1) \qquad \text{for all } x \in [0,1].$$

The quotient topological space $X/\sim$ is obtained from the square $X = [0,1] \times [0,1]$ by identifying each point on the upper edge with the corresponding point on the lower edge. Put differently, "we glue the upper edge to the lower edge". Using Proposition 2.43 (3) one can easily show that the map

$$\begin{array}{rcl} X/\sim & \to & S^1 \times [0,1] \\ [(s,t)] & \mapsto & \big(e^{2\pi i t}, s\big) \end{array}$$

is a homeomorphism. We refer to $X/\sim$ and also to $S^1 \times [0,1]$ as the *cylinder* or the *annulus*. Sometimes we refer to $S^1 \times (0,1)$, which is homeomorphic to $(0,1) \times [0,1]/\sim$, as the *open cylinder* or *open annulus*.

[70] It is not difficult to see that ∂I^k is the boundary of I^k viewed as a subset of $\mathbb{R}^k$.

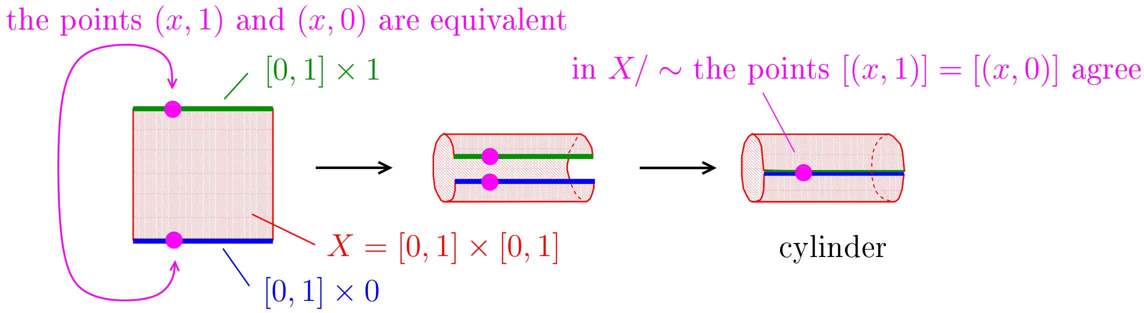


FIGURE 84. Construction of a cylinder by gluing two edges.

(2) Now we consider $X = [0,1] \times [0,1] \subset \mathbb{R}^2$, this time with the equivalence relation which is generated by

$$(0, y) \sim (1, 1-y) \quad \text{ for all } y \in [0,1].$$

The quotient space $X/\sim$ is obtained from the square X by gluing the edge on the left to the edge on the right, but this time "we glue the edges with a twist". We will now see that this topological space is homeomorphic to the Möbius band as defined on page 188. We consider the map

$$\begin{array}{rcl} ([0,1]\times[0,1])/\sim & \to & (\mathbb{R}\times[-1,1])/\mathbb{Z} \\ [(x,y)] & \mapsto & [(x, 2y-1)] \end{array}$$

The left-hand side is compact by Lemma 3.21 (4) and the right-hand side is Hausdorff by Proposition 3.33 and the discussion on page 189. The map is clearly continuous and a bijection. Thus it follows from the ever-popular Proposition 2.43 (3) that the map is a homeomorphism. We will use this homeomorphism to identify $([0,1]\times[0,1])/\sim$ with the Möbius band. It is straightforward to see that this homeomorphism restricts

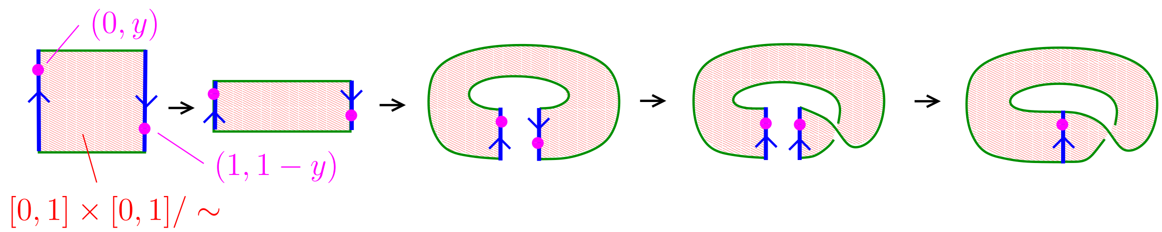


FIGURE 85. The Möbius band.

to a homeomorphism from $([0,1]\times(0,1))/(0,y)\sim(1,1-y)$ to the open Möbius band $(\mathbb{R}\times(-1,1))/\sim$.

(3) Now we consider again $X = [0,1]\times[0,1] \subset \mathbb{R}^2$, but this time with the equivalence relation which is generated by

$$(x,0) \sim (x,1) \quad \text{ for all } x \in [0,1]$$

and by

$$(0,y) \sim (1,y) \quad \text{ for all } y \in [0,1].$$

Put differently, we obtain the quotient topological space $X/\sim$ by gluing the top edge to the bottom edge and the left-hand edge to the right-hand edge, each time without adding a twist. Figure 86 illustrates that the quotient space $X/\sim$ is indeed a torus. As in the previous example one can show that the inclusion $X \to \mathbb{R}^2$ descends to

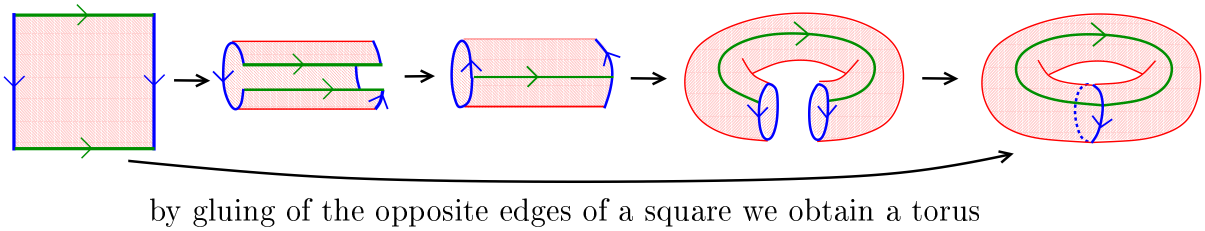


FIGURE 86

a homeomorphism $X/\sim \to \mathbb{R}^2/\mathbb{Z}^2$. In other words, we can make the identification $\mathbb{R}^2/\mathbb{Z}^2 = ([0,1]\times[0,1])/\sim$

(4) Now we consider $X = [0,1]\times[0,1] \subset \mathbb{R}^2$, this time with the equivalence relation which is generated by

$$(x,0) \sim (x,1) \qquad \text{for all } x \in [0,1]$$

and by

$$(0,y) \sim (1,1-y) \qquad \text{for all } y \in [0,1].$$

Thus we obtain the quotient space $X/\sim$ by gluing the upper edge to the lower edge and by gluing the edge on the left to the edge on the right with a twist. The resulting topological space is called the *Klein bottle*. In Figure 87 we show the Klein bottle to the left. Furthermore we indicate a map from the Klein bottle to $\mathbb{R}^3$. This particular map is not injective.

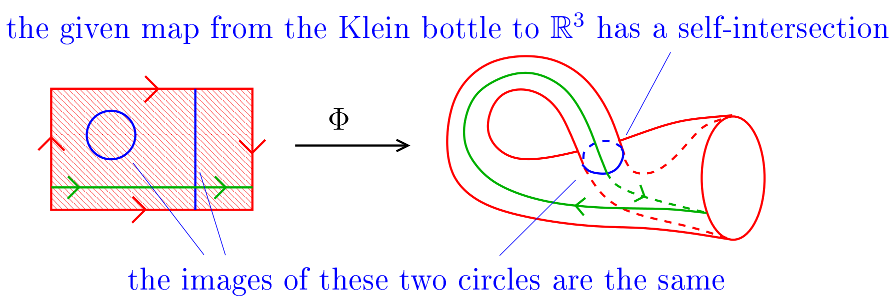


FIGURE 87. The definition of the Klein bottle.

(5) We denote by E_8 the regular octagon in $\mathbb{C}$ with the vertices $Q_k = e^{2\pi i k/16}$ where $k = 1,3,\dots,15$, i.e. E_8 is the convex hull of these eight points in $\mathbb{R}^2$. Given two points $A, B \in \mathbb{R}^2 = \mathbb{C}$ we denote by $\overline{AB}$ the Euclidean segment from A to B. Furthermore, given $\varphi \in \mathbb{R}$ we denote by $r_\varphi\colon \mathbb{C} \to \mathbb{C}$ the reflection in the Euclidean line $\{te^{i\varphi}\,|\,t \in \mathbb{R}\}$. We denote by $\sim$ the equivalence relation on E_8 which is generated by

$$P \in \overline{Q_{2k-1}Q_{2k+1}} \;\sim\; r_{2\pi(2k+2)/16}(P) \;\in\; \overline{Q_{2k+3}Q_{2k+5}}$$

for $k = 0,1,4,5$. These four relations are sketched in Figure 88. We refer to the topological space $E_8/\sim$ as the *surface of genus 2*. The equivalence relation is sketched in Figure 88. Furthermore, in Figure 89 we indicate how $E_8/\sim$ is related to the actual "physical" surface of genus 2 in $\mathbb{R}^3$. The same way we can define the *surface Σ_g of*

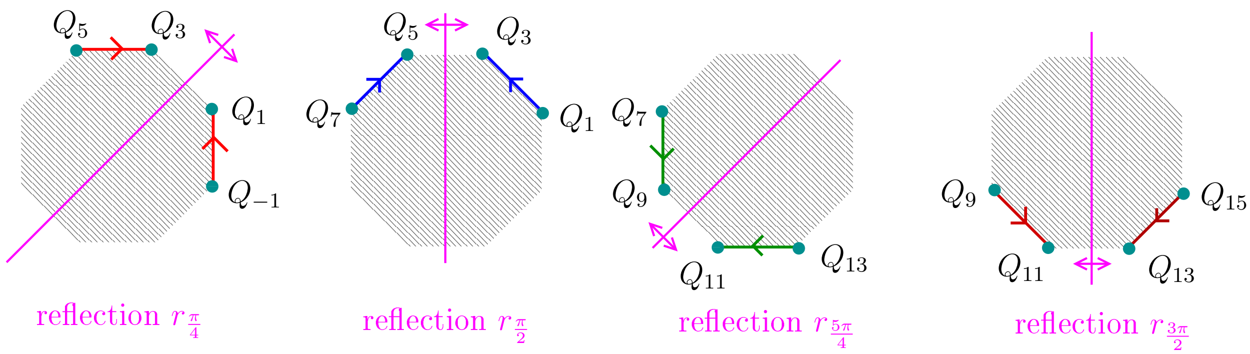


FIGURE 88

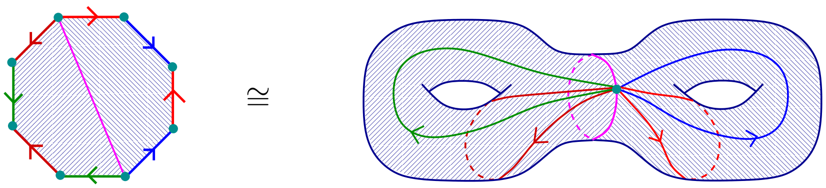


FIGURE 89

genus g for any $g \geq 3$. More precisely, we take a regular $4g$-gon in $\mathbb{C}$ and for $j = 1, \ldots, g$ we identify the $(4j+1)$-st edge with the $(4j+3)$-rd edge and the $(4j+2)$-nd edge with the $(4j+4)$-th edge. Here each identification is given by a reflection. For $g = 3$ this construction is sketched in Figure 90.

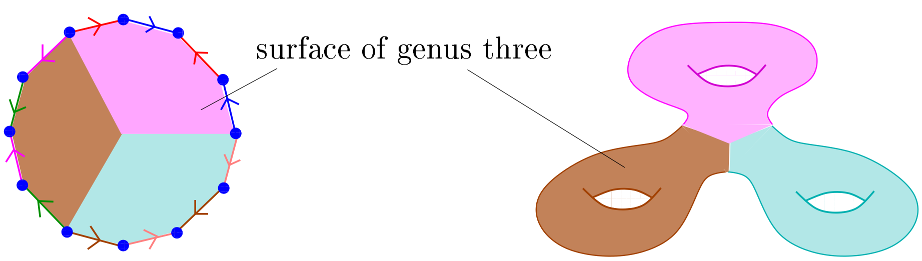


FIGURE 90. The surface of genus three.

Finally we refer to the sphere S^2 as the *surface of genus 0* and we refer to the torus $S^1 \times S^1$ as the *surface of genus 1*.

(6) Given $g \in \mathbb{N}_{\geq 2}$ we consider the regular $2g$-gon E_{2g} with the identification of the boundary segments shown in Figure 91. We refer to this topological space as the *non-orientable surface N_g of genus g*. (Later on in Lemma 6.48 we will see why N_g deserves its epithet.) Furthermore, we refer to the real projection place $\mathbb{RP}^2$ as the non-orientable surface of genus 1. In Exercise 3.43 we will see that the Klein bottle is homeomorphic to the non-orientable surface of genus 2. Furthermore, as a sneak preview, let us point out that later in Lemma 8.33 we will see that N_g is diffeomorphic to the connected sum of g copies of $\mathbb{RP}^2$.

We have now introduced many examples of topological spaces, some of them, for example the Möbius band, the torus and the surface of genus 2, can easily be pictured as non-orientable surface of genus 3

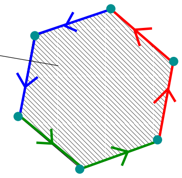


FIGURE 91

subsets of "our universe", i.e. they can be viewed as subsets of $\mathbb{R}^3$. This raises the following question.

Question 3.46. *Can the real projective space* $\mathbb{R}\mathrm{P}^2$ *and the Klein bottle, or more generally the non-orientable surfaces of some genus* ≥ 3*, be viewed*[71] *as subsets of* $\mathbb{R}^3$*?*

3.12. **The join of two topological spaces** $(*)$**.** In this section we introduce the join of two topological spaces. This construction will be useful on a few occasions, but it is difficult to argue that at this stage this is an essential topic in these notes.

Loosely speaking the join $X * Y$ is obtained by connecting each point in X to each point in Y via an interval. The following definition makes this idea precise.

Definition. Let X and Y be two non-empty topological spaces. We define the *join* $X*Y$ to be the topological space obtained from $X \times [0,1] \times Y$, equipped with the product topology, by performing the following two types of identifications:

(1) for every $x \in X$ we identify all points in $\{x\} \times \{0\} \times Y$ to a single point and
(2) for every $y \in Y$ we identify all points in $X \times \{1\} \times \{y\}$ to a single point.

Furthermore, if one of X or Y is the empty topological space, then we define the join $X*Y$ to be the other topological space.

For completeness we state the following lemma.

Lemma 3.47. $(*)$ *Let* X_1, Y_1, X_2, Y_2 *be four topological spaces and let* $f\colon X_1 \to X_2$ *and* $g\colon Y_1 \to Y_2$ *be maps. The corresponding map*

$$\begin{array}{rcl} f * g\colon X_1 * Y_1 & \to & X_2 * Y_2 \\ [(x,t,y)] & \mapsto & [(f(x),t,g(y))] \end{array}$$

is well-defined and continuous.

PROOF $(*)$. It is straightforward to verify that the map $f*g$ is well-defined. It follows easily from Lemma 3.22, Lemma 3.21 (3) and Lemma 3.8 (3c) that the map $f*g$ is continuous. ■

Before we deal with a few examples we give a criterion for showing that a map of the form $f\colon X * Y \to Z$ is a homeomorphism.

Lemma 3.48. *If* X *and* Y *are compact topological spaces, then the following two statements hold:*

(1) *The join* $X * Y$ *is compact.*

[71]The word "view" is intentionally vague.

(2) *Let $f\colon X \times [0,1] \times Y \to Z$ be a map to a topological space Z such that for any $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$ we have the equalities $f(x,0,y_1) = f(x,0,y_2)$ and $f(x_1,1,y) = f(x_2,1,y)$. If Z is Hausdorff and if the induced map*[72]

$$\begin{array}{rcl} \Psi\colon X * Y & \to & Z \\ [(x,t,y)] & \mapsto & f(x,y,t) \end{array}$$

is a bijection, then the map Ψ is a homeomorphism.

Proof.

(1) It follows immediately from our hypothesis and Proposition 3.12 together with Lemma 3.21 (4) that the join $X * Y = (X \times [0,1] \times Y)/\sim$ is compact.
(2) The map $\Psi\colon X * Y \to Z$ is continuous by Lemma 3.22. Thus we obtain from our hypotheses on Ψ and Z together with Proposition 2.43 (3) that Ψ is indeed a homeomorphism. ■

The following, otherwise broadly irrelevant lemma, justifies somewhat the idea that the join $X * Y$ is obtained by connecting each point in X to each point in Y via an interval.

Lemma 3.49. *Given $P, Q \in \mathbb{R}^n$ we denote by $\overline{PQ} = \{x \cdot (1-t) + y \cdot t \,|\, t \in [0,1]\}$ the Euclidean segment determined by P and Q. Let X and Y be two compact subsets of $\mathbb{R}^n$ which have the property that for any x, x' and y, y' the segments $\overline{xy}$ and $\overline{x'y'}$ intersect at most in the end points. Then the map*

$$\begin{array}{rcl} \Theta\colon X * Y = (X \times [0,1] \times Y)/\sim & \to & \bigcup\limits_{x \in X, y \in Y} \overline{xy} \\ [(x,t,y)] & \mapsto & x \cdot (1-t) + y \cdot t \end{array}$$

is a homeomorphism.

Proof. It follows almost immediately from Lemma 3.48 and our hypothesis that the map Θ is a homeomorphism. ■

We continue with another example of joins.

Example. We consider the join of the two intervals $X = [0,1]$ and $Y = [0,1]$. First let us consider Figure 92. The figure suggests that the join $X * Y = [0,1] * [0,1]$ should be homeomorphic to a 3-dimensional pyramid. In fact let us consider the map

$$\begin{array}{rcl} \overbrace{([0,1] \times [0,1] \times [0,1])/\sim}^{=[0,1]*[0,1]} & \to & \Delta = \{(t_1,t_2,t_3) \in [0,1]^3 \,|\, t_1 + t_2 + t_3 \leq 1\} \\ [(x,t,y)] & \mapsto & \frac{1}{2} \cdot \big(x \cdot t, y \cdot (1-t), t\big). \end{array}$$

It follows easily from Lemma 3.48 (2) that this map is a homeomorphism.

We record the next example of joins as a lemma.

[72] It follows from our hypothesis on f that the map f descends to a map on $X * Y = (X \times [0,1] \times Y)/\sim$.

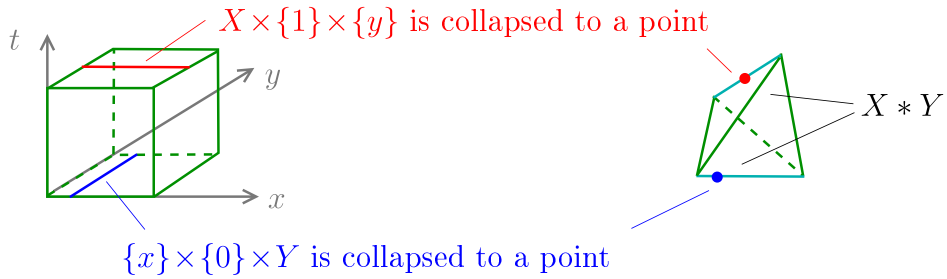


FIGURE 92

Lemma 3.50. *Given any $m,n\in\mathbb{N}_0$ the map*

$$\begin{array}{rcl} S^m * S^n & \to & S^{m+n+1} \\ [(x,t,y)] & \mapsto & \underbrace{\left(x\cdot\cos\left(\frac{\pi t}{2}\right), y\cdot\sin\left(\frac{\pi t}{2}\right)\right)}_{\in\mathbb{R}^{m+1}\times\mathbb{R}^{n+1}=\mathbb{R}^{m+n+2}} \end{array}$$

is a homeomorphism.

PROOF. Using Lemma 3.48 (2) one can fairly easily verify that the given map is a homeomorphism. ■

Convention. Let $m,n\in\mathbb{N}_0$. We use the homeomorphism from Lemma 3.50 to add the identification $S^m * S^n = S^{m+n+1}$ to our long list of avatars of the spheres.

For completeness we state the following fairly straightforward lemma which we will prove in Exercise 3.48.

Lemma 3.51. *Let X and Y be topological spaces. If X and Y are both Hausdorff, then the join $X*Y$ is also Hausdorff.*

Finally we consider the delicate question whether the join is associative.

Proposition 3.52. *Let X, Y and Z be non-empty topological spaces that are compact and Hausdorff. Then there exists a homeomorphism*

$$(X*Y)*Z \xrightarrow{\cong} X*(Y*Z).$$

Remark. As on page 82 we see that for finitely many topological spaces spaces $X_1,\dots,X_k$ that are non-empty compact and Hausdorff we can unambiguously define the join $X_1*\dots*X_k$.

It turns out that the proof of Proposition 3.52 is surprisingly difficult, remarkably we will only be able to give it later on on page 250.

Remark. The join $X*Y$, viewed as a set, can be equipped with another reasonable topology, namely the "initial topology", see [**BrownR06**, Chapter 5.7]. By [**BrownR06**, Corollary 5.7.2] these two topologies agree if X and Y are compact and Hausdorff. When dealing with joins it seems that it is safest to stick to the case that X and Y are compact and Hausdorff. For example Proposition 3.52 says that in this context the join is associative. But this statement is not true for all topological spaces. We refer to [**Cohe57**], [**BrownR06**, Chapter 5.7] and [**FF16**, Chapter 2.4] for details.[73]

3.13. **Topological groups.** In this section we consider the notion of a topological group which will crop up every now and then in these notes.

Definition. A *topological group* is a topological space X together with a group structure, such that the two maps[74]

$$\begin{array}{rcl} X\times X & \to & X \\ (x,y) & \mapsto & x\cdot y \end{array} \quad \text{and} \quad \begin{array}{rcl} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are continuous.

The most basic example of a topological group is given by taking any group and equipping it with the discrete topology. As we will see, somewhat surprisingly even this banal example can turn out to be interesting.

The following lemma gives us much more interesting examples of topological groups.

Lemma 3.53. *For any $n \in \mathbb{N}$ the matrix groups $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ are topological groups.*

Proof. As we pointed out on page 163, matrix multiplication is continuous. Furthermore, basic linear algebra, see e.g. [**HJ13**, Chapter 0.8.2], says that taking the inverse is given by the map

$$\begin{array}{rcl} \mathrm{GL}(n,\mathbb{R}) & \to & \mathrm{GL}(n,\mathbb{R}) \\ A & \mapsto & \frac{1}{\det(A)}\cdot \underbrace{\begin{array}{c}\text{matrix whose } (i,j)\text{-entry is given by taking the determinant} \\ \text{of the matrix that is obtained from } A \text{ by removing} \\ \text{the } i\text{-th column and the } j\text{-th row of } A.\end{array}}_{=\text{called the adjugate of } A} \end{array}$$

But this map is clearly continuous. This shows that $\mathrm{GL}(n,\mathbb{R})$ is a topological group. Basically the same argument also shows that $\mathrm{GL}(n,\mathbb{C})$ is a topological group. ■

We obtain more examples of topological groups by taking subgroups and products:

Lemma 3.54.

(1) *Let X be a topological group and let Y be a subgroup. If we equip Y with the subspace topology, then Y is also a topological group.*
(2) *Let $\{X_i\}_{i\in I}$ be a family of topological groups. If we equip $\prod_{i\in I} X_i$ with the product topology and the obvious group structure, then it is again a topological group.*

Proof.

(1) This statement follows immediately from Lemma 2.30 together with Lemma 3.8 (1b).
(2) This statement follows easily from Lemma 3.14 and Lemma 3.15 (2).

[73] As we will see in Lemma 5.14, products and quotients of topological spaces do not mix well. It is easy to write down “continuous looking” maps that are not continuous.

[74] Here we evidently equip $X \times X$ with the product topology.

■

Example. It follows from Lemmas 3.53 and 3.54 (1) that $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are topological groups.

Lemma 3.55. *Let G be a topological group.*

(1) *Given any $g \in G$ the two maps*

$$\begin{array}{rcl} l_g\colon G & \to & G \\ h & \mapsto & g \cdot h \end{array} \quad \textit{and} \quad \begin{array}{rcl} r_g\colon G & \to & G \\ h & \mapsto & h \cdot g \end{array}$$

are continuous.

(2) *If G is a topological group, then any two path-components of G are homeomorphic.*

Proof (∗).

(1) The map l_g is the composition of the two maps

$$\begin{array}{rcl} G & \to & G \times G \\ h & \mapsto & (g,h) \end{array} \quad \text{and} \quad \begin{array}{rcl} G \times G & \to & G \\ (x,y) & \mapsto & x \cdot y. \end{array}$$

The first map is continuous by Lemma 3.8 (2a) and the second map is continuous since G is a topological group. Thus l_g itself is continuous. The same argument shows that r_g is continuous.

(2) Let X, Y be two path-components of G. We pick $x \in X$ and $y \in Y$. We consider the map $\varphi\colon G \to G$ given by $g \mapsto yx^{-1}g$. By (1) we know that φ is continuous. Furthermore it follows from Lemma 2.70 that $\varphi(X) \subset Y$.

We also consider the map $\psi\colon G \to G$ given by $g \mapsto xy^{-1}g$. The same argument as above shows that ψ is continuous and that $\psi(Y) \subset X$. Clearly φ and ψ are inverses of one another. But this implies that φ and ψ are homeomorphisms between X and Y. ■

Exercises for Chapter 3.

Exercise 3.1. Let X be a topological space and let A and B be two subsets of X. Show that the following two statements are equivalent:

(1) There exist open neighborhoods U of A and V of B with $U \cap B = \varnothing$ and $V \cap A = \varnothing$.

(2) The map

$$\begin{array}{rcl} A \cup B & \to & A \sqcup B \\ x & \mapsto & \begin{cases} (x,1), & \text{if } x \in A, \\ (x,2), & \text{if } x \in B \end{cases} \end{array}$$

is a homeomorphism.

Remark. This is the content of Lemma 3.2.

Exercise 3.2. Let $C \subset \mathbb{R}$ be the Cantor set, as defined on page 100. Is the disjoint union $C \sqcup C$ homeomorphic to C itself?

Exercise 3.3. Show that there are non-homeomorphic topological spaces A and B and a non-empty topological space X such that $A \times X$ is homeomorphic to $B \times X$.
Hint. you could take $A = \{0\}$ and $B = \{0, 1\}$.

Exercise 3.4. Let X be a topological space. We consider the "diagonal"

$$\Delta \;:=\; \{(x, x) \in X \times X \,|\, x \in X\}.$$

We equip $X \times X$ with the product topology and we equip Δ with the subspace topology. Show that the map $X \to \Delta$ given by $x \mapsto (x, x)$ is a homeomorphism.

Exercise 3.5. Let $X_1, \dots, X_k$ be topological spaces and let $B_i \subset X_i$, $i = 1, \dots, k$ be subsets. Show that the following equalities hold:

$$\begin{array}{lrcl} \text{(a)} & \text{closure of } B_1 \times \ldots \times B_k \text{ in } X_1 \times \ldots \times X_k & = & \overline{B}_1 \times \ldots \times \overline{B}_k \\ \text{(b)} & \text{interior of } B_1 \times \ldots \times B_k \text{ in } X_1 \times \ldots \times X_k & = & \mathring{B}_1 \times \ldots \times \mathring{B}_k \\ \text{(c)} & \text{boundary of } B_1 \times \ldots \times B_k \text{ in } X_1 \times \ldots \times X_k & = & \bigcup\limits_{i=1}^{k} \overline{B}_1 \times \ldots \times \partial B_i \times \ldots \times \overline{B}_k. \end{array}$$

Exercise 3.6. Let X be a compact topological space. Let Y be a topological space, let $y_0 \in Y$ and let U be an open subset of $X \times Y$ that contains $X \times \{y_0\}$. Show that there exists an open neighborhood V of y_0 such that $X \times V \subset U$.
Remark. This statement is sometimes called the *Tube Lemma*.

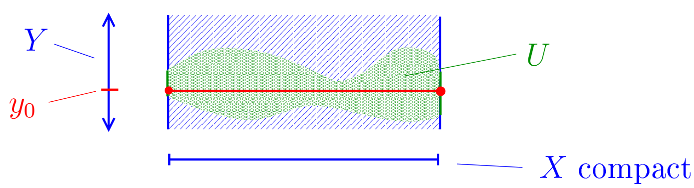


FIGURE 93. Illustration for Exercise 3.6.

Exercise 3.7. Let X and Y be topological spaces.

(a) Show that if X is compact, then the projection map $p\colon X \times Y \to Y$ is a closed map.
Remark. This exercise is related to Exercise 3.6.

(b) Give an example that shows that in general in (a) we cannot drop the hypothesis that X is compact.

Somewhat surprisingly, the converse to (a) holds. More precisely, if a topological space X has the property, that for any other topological space Y the projection map $p\colon X \times Y \to Y$ is a closed map, then X is actually compact. This non-trivial statement is sometimes known as the *Kuratowski-Mrówka Theorem*, it was first proved by Stanislaw Mrówka [**Mró59**, p. 21] in 1959. See also [**Herl96**, p. 3] and see also [**Enge89**, Theorem 3.1.16] under the extra hypothesis that X is Hausdorff.

Exercise 3.8. Let $X \subset \mathbb{R}^n$ be a compact convex non-empty subset. Show that there exists a $k \in \mathbb{N}_0$ such that X is homeomorphic to $\overline{B}^k$.

Exercise 3.9. Let $\{X_i\}_{i \in I}$ be a family of topological spaces.

(a) Suppose that for each $i \in I$ we are given a closed subset $A_i \subset X_i$. If I is finite, then it follows from Lemma 3.9 that $\prod_{i\in I} A_i$ is a closed subset of $\prod_{i\in I} X_i$. Does this conclusion also hold if I is infinite?

(b) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Suppose that for each $i \in I$ we are given a subset $A_i \subset X_i$. Show that

$$\prod_{i\in I} \overline{A_i} = \overline{\prod_{i\in I} A_i} \subset \prod_{i\in I} X_i.$$

Exercise 3.10. Let X and Y be topological spaces, let $W \subset X \times Y$ be an open subset and let K be a compact subset of Y. Show that $U := \{x \in X \,|\, \{x\} \times K \subset W\}$ is an open subset of X.

Exercise 3.11. Let (X, d) be a metric space.

(a) Show that the map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} \\ (x, y) & \mapsto & d(x, y) \end{array}$$

is continuous.

(b) Suppose that X is compact and that $f\colon X \to X$ is a map with $f(x) \neq x$ for all $x \in X$. Show that there exists an $\epsilon > 0$ such that $d(f(x), x) \geq \epsilon$ for all $x \in X$.

Exercise 3.12. Let (X, d) be a metric space and let $B \subset X$ be a non-empty subset. Show that the map

$$\begin{array}{rcl} X & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & d(x, B) := \inf\{d(x, b) \,|\, b \in B\} \end{array}$$

is continuous.

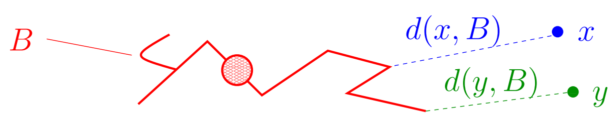


FIGURE 94. Illustration for Exercise 3.12.

Exercise 3.13. Let X be a metric space and let A, B be two non-empty subsets of X. Suppose that $\operatorname{diam}(A) + \operatorname{diam}(B) < d(A, B)$. Show that $A \cap B = \varnothing$.

Exercise 3.14. Let X and Y be topological spaces and let $M \subset X \times Y$ be a compact subset. Show that there exist compact subsets $K \subset X$ and $L \subset Y$ such that $M \subset K \times L$.

Exercise 3.15. Let X and Y be two topological spaces. Show that the product topology on $X \times Y$ is the smallest topology that has the property that the two natural projection maps $X \times Y \to X$ and $X \times Y \to Y$ are continuous.

Exercise 3.16. Let A be the annulus and let M be the Möbius band. Are the product topological spaces $A \times [0, 1]$ and $M \times [0, 1]$ homeomorphic?

Exercise 3.17. Let X be a topological space that is compact and Hausdorff. Let $f\colon X \to X$ be a map. We denote by $F := \{x \in X \,|\, f(x) = x\}$ the fixed point set of f. Show that F is a discrete subset of X if and only if F is finite.

Exercise 3.18. A topological space X is called *metrizable* if there exists a metric d on X such that the given topology agrees with the topology coming from d. Now suppose that X is a metrizable topological space. Show that given any closed set $A \subset X$ there exists a map $f\colon X \to \mathbb{R}$ such that $f^{-1}(\{0\}) = A$.
Hint. Use Exercise 3.12.

Exercise 3.19. Show that the map from $\prod\limits_{i\in\mathbb{N}} \{0, 2\}$ to the Cantor set, that we defined on page 174, is continuous.

Exercise 3.20.
(a) Show that the Cantor set is totally disconnected.
(b) Show that $\prod\limits_{i\in\mathbb{N}} \{0, 2\}$ is totally disconnected.

Exercise 3.21. Let X be a topological space. Show that

$$X \text{ is Hausdorff} \quad \iff \quad \text{the diagonal } \Delta = \{(x, x) \,|\, x \in X\} \text{ is a closed subset of } X \times X.$$

Exercise 3.22.
(a) Show that the real projective space $\mathbb{RP}^1$ is homeomorphic to S^1.
(b) Do you think that the real projective space $\mathbb{RP}^2 = (\mathbb{R}^3 \setminus \{0\})/(\mathbb{R} \setminus \{0\})$ is homeomorphic to S^2? As of now we lack the tools to answer this question. Nonetheless, what do you think the answer will be?

Exercise 3.23. Let $\sim$ be an equivalence relation on a topological space X and let $f\colon X \to Y$ be a map with the property that $f(x) = f(y)$ whenever $x \sim y$. Show that if $f\colon X \to Y$ is an open map, then the induced map $X/\sim\, \to Y$ is also an open map.

Exercise 3.24. Show that the topological space $\mathbb{R}/\{\pm 1\} = \mathbb{R}/x \sim -x$ is homeomorphic to the half-open interval $[0, \infty)$.

Exercise 3.25. Let the group $(\mathbb{Q}, +)$ act on the topological space $\mathbb{R}$ by addition. Show that $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.

Exercise 3.26. Let $\sim$ be the equivalence relation on $\mathbb{R}_{\geq 0} \times \mathbb{R}$ that is generated by
(1) $(0, \varphi) \sim (0, \varphi')$ for all $\varphi, \varphi' \in \mathbb{R}$ and by
(2) $(r, \varphi) \sim (r, \varphi + k \cdot 2\pi)$ for all $r \in \mathbb{R}_{\geq 0}$, $\varphi \in \mathbb{R}$ and $k \in \mathbb{Z}$.

We equip $(\mathbb{R}_{\geq 0} \times \mathbb{R})/\sim$ with the quotient topology.
(a) Show that the map

$$\begin{array}{rcl} \Theta\colon (\mathbb{R}_{\geq 0} \times \mathbb{R})/\sim & \to & \mathbb{C} \\ (r, \varphi) & \mapsto & r \cdot e^{\mathrm{i}\varphi} \end{array}$$

is a homeomorphism.
Hint. To show that the map is a homeomorphism you might want to use Proposition 2.45.

(b) Let $f\colon \mathbb{R}_{\geq 0} \times \mathbb{R} \to X$ be a continuous map to some topological space X such that $f(P) = f(Q)$ whenever $P \sim Q$. Show that the map

$$\begin{array}{rcl} \mathbb{C} & \to & X \\ re^{i\varphi} & \mapsto & f(r,\varphi) \end{array}$$

is continuous.

Remark. This exercise is a variation on Lemma 3.25.

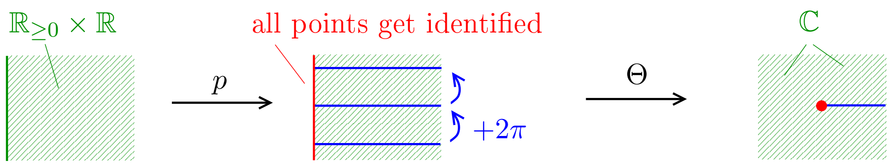


FIGURE 95

Exercise 3.27. Let $\gamma \in \mathbb{R}$ and let $s \in \mathbb{R}$. We consider the map

$$\begin{array}{rcl} f_s\colon \mathbb{C} \setminus \mathbb{R}_{>0} \cdot e^{i\gamma} & \to & \mathbb{C} \\ r \cdot e^{i\varphi} & \mapsto & r \cdot e^{i\varphi \cdot s}. \end{array}$$

(a) Show that f_s is well-defined and continuous.
(b) Suppose that $s \neq 0$. Show that f_s is a local homeomorphism.

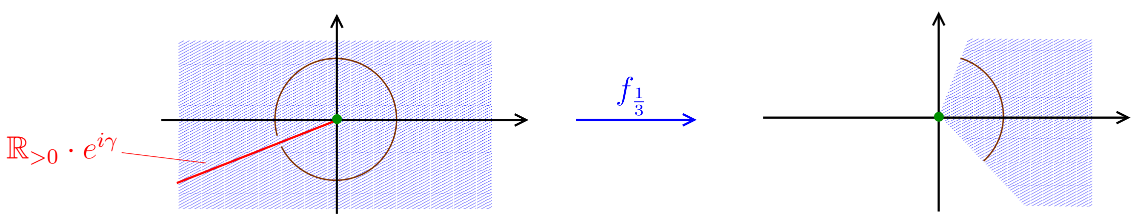


FIGURE 96. Illustration for Exercise 3.27.

Exercise 3.28.

(a) Show that any matrix $A \in \mathrm{O}(2n)$ with $\det(A) = -1$ has 1 as an eigenvalue.
Hint. Show that if $\lambda \in \mathbb{C}$ is an eigenvalue of A, then so is $\overline{\lambda}$.
(b) Let $G \subset \mathrm{O}(2n)$ be a subgroup that acts freely on S^{2n-1}. Show that G is contained in $\mathrm{SO}(2n)$.
(c) Show that the analogue of (b) does not hold for subgroups of $G \subset \mathrm{O}(2n-1)$.

Exercise 3.29. Let $n \in \mathbb{N}$ and let $\varphi\colon S^n \to S^n$ be a homeomorphism. We consider the map

$$\begin{array}{rcl} \Phi\colon \overline{B}^{n+1} & \to & \overline{B}^{n+1} \\ x & \mapsto & \begin{cases} \|x\| \cdot \varphi(\frac{x}{\|x\|}), & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

(a) Show that Φ is continuous.
(b) Show that Φ is a homeomorphism.

(c) Give any example of a *diffeomorphism* $\varphi\colon S^n \to S^n$ such that the map $\Phi\colon \overline{B}^{n+1} \to \overline{B}^{n+1}$ is *not* a diffeomorphism.

Hint. You might want to use Lemma 3.25.

Remark. The fact that we can extend φ to a homeomorphism of $\overline{B}^{n+1}$ is sometimes referred to as the Alexander trick. We will get to know a more fancy Alexander trick later on page 540.

Exercise 3.30. Given $v, w \in \mathbb{R}^2 \setminus \{(0,0)\}$ we define

$$\sphericalangle(v,w) \;:=\; \text{unique } \varphi \in (-\pi,\pi] \text{ such that } w = r\cdot\begin{pmatrix}\cos(\varphi) & -\sin(\varphi)\\ \sin(\varphi) & \cos(\varphi)\end{pmatrix}\cdot w \text{ for some } r \in \mathbb{R}_{>0}.$$

(a) Show that the map

$$\begin{array}{rcl}\Phi\colon (\mathbb{R}^2 \setminus \{(0,0)\}) \times (\mathbb{R}^2 \setminus \{(0,0)\}) & \to & (-\pi,\pi]\\ (v,w) & \mapsto & \sphericalangle(v,w)\end{array}$$

is not continuous.

(b) Show that the map

$$\begin{array}{rcl}\Phi\colon (\mathbb{R}^2 \setminus \{(0,0)\}) \times (\mathbb{R}^2 \setminus \{(0,0)\}) & \to & \mathbb{R}/2\pi\mathbb{Z}\\ (v,w) & \mapsto & [\sphericalangle(v,w)]\end{array}$$

is continuous. (Here $\mathbb{R}/2\pi\mathbb{Z}$ denotes the quotient of the topological space $\mathbb{R}$ by the group $(2\pi\mathbb{Z},+)$.)

Exercise 3.31. We consider the following subsets of $[0,1]$:

$$A \;=\; \{0,\tfrac{1}{3},\tfrac{2}{3},1\}, \qquad B \;=\; \{\tfrac{1}{n}\,|\, n \in \mathbb{N}\} \qquad \text{and} \qquad C \;=\; \{0\} \cup B.$$

(a) Show that the quotient $[0,1]/A$ is homeomorphic to a subset of $\mathbb{R}^2$. See e.g. Figure 97 for an idea.

(b) Is the quotient $[0,1]/B$ homeomorphic to a subset of $\mathbb{R}^2$?

(c) Is the quotient $[0,1]/C$ homeomorphic to a subset of $\mathbb{R}^2$?

FIGURE 97

Exercise 3.32. Let X be topological space and let A and B be two subsets with $A \cup B = X$. We write $C := A \cap B$. We consider the disjoint union $A \sqcup B = (A \times \{1\}) \cup (B \times \{2\})$ and we consider $Y = (A \sqcup B)/\sim$ where for each $c \in C$ we identify the point $(c,1) \in A \times \{1\}$ with the corresponding point $(c,2) \in B \times \{2\}$. We consider the map

$$p\colon Y = (A \sqcup B)/\sim \;\;\to\;\; X$$

that is given by

$$\begin{array}{rcll}[(a,1)] & \mapsto & a & \text{for } a \in A,\\ [(b,2)] & \mapsto & b & \text{for } b \in B.\end{array}$$

(a) Is the map p necessarily continuous?
(b) Is the map p necessarily open?
(c) Suppose that A and B are open subsets of X. Show that p is a homeomorphism.

Remark. The failure of p to be a homeomorphism can lead to nasty surprises.

Exercise 3.33. Let Σ be the surface of genus 4.

(a) Show that Σ admits a free and continuous action by $\mathbb{Z}_2$.
(b) Show that Σ admits a free and continuous action by $\mathbb{Z}_3$.
(c) Show that Σ admits a free and continuous action by $\mathbb{Z}_6$.

Remark. In Proposition 55.11 we will see that if a finite group G admits a free and continuous action on Σ, then the order of G divides 6.

Exercise 3.34. Give an example of maps $f\colon X \to Y$ and $X \to Z$ between infinite topological spaces such that the pushout $Y \cup_X Z$ consists of a single point.

Exercise 3.35. Let X be a topological space and let $A \subset X$ be a subset. We denote by $\{*\}$ the topological space with a single point. Let $i\colon A \to X$ be the inclusion map and let $p\colon A \to \{*\}$ be the unique map there is. Show that the corresponding pushout $X \cup_A \{*\}$ is homeomorphic to the quotient X/A.
Remark. Note that we also allow the case $A = \varnothing$.

Exercise 3.36. Let X be a topological space and let A and B be two disjoint subsets. We equip A with the subspace topology and we equip X/B with the quotient topology. We denote by $i\colon A \to X$ the inclusion map and we denote by $p\colon X \to X/B$ the projection.

(a) Show that the map $p \circ i\colon A \to X/B$ is continuous and injective.
(b) Show that the map $p \circ i\colon A \to X/B$ is not necessarily an embedding.
(c) Suppose that there exists an open subset $U \subset X$ with $A \subset U$ and $U \cap B = \varnothing$. Show that the map $p \circ i\colon A \to X/B$ is an embedding.

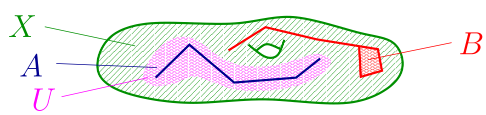


FIGURE 98. Illustration for Exercise 3.36.

Exercise 3.37. Similar to the definition of the Möbius band on page 188 we consider the action of the group $G = \mathbb{Z}$ on $X = \mathbb{R} \times \{-1, 1\}$ that is given by

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times \{-1,1\}) & \to & \mathbb{R} \times \{-1,1\} \\ (n,(x,y)) & \mapsto & (x+n, (-1)^n \cdot y). \end{array}$$

Show that $X/G = X/\mathbb{Z}$ is homeomorphic to S^1.

Exercise 3.38. Let

$$\begin{array}{rcl} \langle -,- \rangle\colon \mathbb{C}^n \times \mathbb{C}^n & \to & \mathbb{C} \\ ((v_1,\dots,v_n),(w_1,\dots,w_n)) & \mapsto & \sum\limits_{i=1}^n v_i \cdot \overline{w}_i \end{array}$$

be the standard complex hermitian inner product on $\mathbb{C}^n$. Show that for any $v, w \in \mathbb{C}^n$ the following two statements hold:

(a) $|\langle v, w\rangle| \le |v| \cdot |w|$.
(b) $|\langle v, w\rangle| = |v| \cdot |w|$ if and only if there exist $\lambda, \mu \in \mathbb{C}$ with $\lambda \cdot v = \nu \cdot w$ and such that $(\lambda, \mu) \neq (0, 0)$.

The first statement is known as the *Cauchy-Schwarz inequality*.
Remark. For any commutative ring R and any $(a_1, \ldots, a_n) \in R^n$ and $(b_1, \ldots, b_n) \in \mathbb{R}^n$ we have the equality

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 + \sum_{\{(i,j)\,|\,1 \le i < j \le n\}} (a_i b_j - a_j b_i)^2 = \Big(\sum_{i=1}^{n} a_i^2\Big) \cdot \Big(\sum_{i=1}^{n} b_i^2\Big).$$

Exercise 3.39. We consider the map

$$\begin{array}{rcl} (\mathbb{C} \setminus \{0\}) \times (\mathbb{C}^{n+1} \setminus \{0\}) & \to & \mathbb{C}^{n+1} \setminus \{0\} \\ (z, P) & \mapsto & z \cdot P. \end{array}$$

By definition we have $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$.

(1) Show that the obvious map $p \colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}\mathrm{P}^n$ is open.
(2) In Proposition 3.40 we showed that $\mathbb{C}\mathrm{P}^n$ is Hausdorff. In this exercise we want to give a more direct, down-to-earth proof that $\mathbb{C}\mathrm{P}^n$ is Hausdorff. Thus let $[x], [y] \in \mathbb{C}\mathrm{P}^n$ be two disjoint points. Let $\langle -, - \rangle$ be the usual hermitian inner product on $\mathbb{C}^{n+1}$ that we introduced in Exercise 3.38. We consider

$$U_x = \big\{ z \in \mathbb{C}^{n+1} \setminus \{0\} \,\big|\, |\langle z, y\rangle| < |\langle z, x\rangle| \big\} \;\;\&\;\; U_y = \big\{ z \in \mathbb{C}^{n+1} \setminus \{0\} \,\big|\, |\langle z, x\rangle| < |\langle z, y\rangle| \big\}.$$

(a) Show that U_x and U_y are open subsets of $\mathbb{C}^{n+1} \setminus \{0\}$.
(b) Show that $p(U_x)$ and $p(U_y)$ are disjoint. (Note that they are open by (1) and (2a).)
(c) Show that $[x] \in p(U_x)$ and $[y] \in p(U_y)$. (*Hint.* Use Exercise 3.38 (b).)

Exercise 3.40. Let $\sim$ be the equivalence relation on $\overline{B}^3$ that is given by $x \sim -x$ for $x \in S^2$. We consider the map

$$\begin{array}{rcl} F \colon \overline{B}^3/\sim & \to & \mathrm{SO}(3) \\ [\underbrace{r \cdot v}_{r \in [0,1] \text{ and } v \in S^2}] & \mapsto & \begin{array}{c}\text{matrix in SO(3) which represents rotation} \\ \text{by the angle } \pi \cdot r \text{ around the axis } \mathbb{R}_{\ge 0} \cdot v \\ \text{where the rotation direction is given by} \\ \text{the right-hand rule}\end{array} \end{array}$$

We refer to Figure 99 for an illustration.

(a) Give a rigorous description of F and show that F is continuous.
(b) Use your linear algebra skills to show that F is a bijection.
(c) Show that F is a homeomorphism.

Remark. On page 196 we showed that $\overline{B}^3/\sim$ is homeomorphic to $\mathbb{R}\mathrm{P}^3$. Thus in combination we have just shown the amusing and surprising fact that $\mathbb{R}\mathrm{P}^3$ is homeomorphic to SO(3). We will give a different proof of this fact later on in Theorem 51.2.

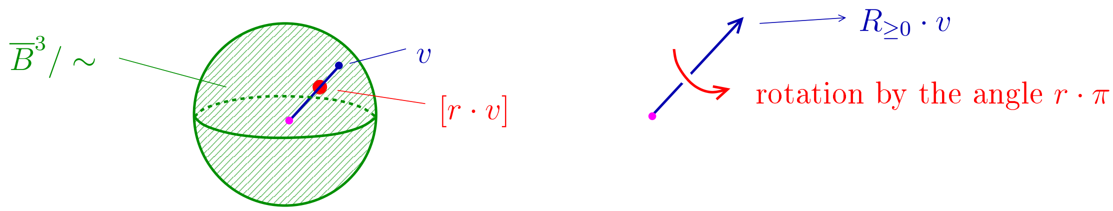


FIGURE 99. Illustration for Exercise 3.40.

Exercise 3.41. We say a group G *acts discretely* on a topological space X, if it acts continuously and if for each $x \in X$ there exists an open neighborhood U such that we have $U \cap gU$ for all $g \neq e$. Can the group $\mathbb{Z}$ act discretely on S^2?

Exercise 3.42. Let X be a Hausdorff space and let G be a group that acts discretely on X. (See Exercise 3.41 for the notion of a discrete action.) Furthermore let K be a compact subset of X.

(a) We assume that X is locally compact. Show that $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite.
(b) Does the conclusion of (a) also hold if we drop the hypothesis that X is locally compact? This statement is [**Scot78**, Lemma 1.2].

Exercise 3.43. Show that the Klein bottle, as defined on page 205, is homeomorphic to the non-orientable surface N_2 of genus 2, as defined on page 206. We refer to Figure 100 for a reminder of the definitions.

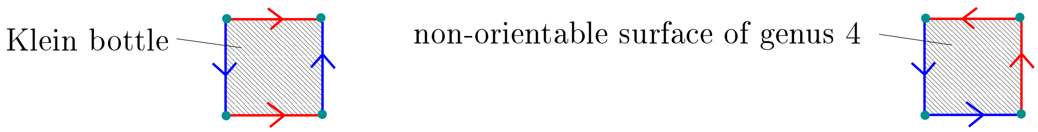


FIGURE 100. Illustration of Exercise 3.43.

Exercise 3.44. Show that for any choice of $m, n \in \mathbb{N}_0$ there exists a homeomorphism $\overline{B}^m * \overline{B}^n \xrightarrow{\cong} \overline{B}^{m+n+1}$.

Exercise 3.45. We let $X = S^1$ be the standard circle in the xy-plane of $\mathbb{R}^3$ and we let Y be the circle in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is given by the z-axis together with the point $\{\infty\}$. (See Figure 101 for a simple illustration.) Show that we can "continuously" connect every point on X to every point Y such that the paths fill out S^3. In other words, try to visualize the fact that $S^1 * S^1 \cong X * Y$ is homeomorphic to S^3.

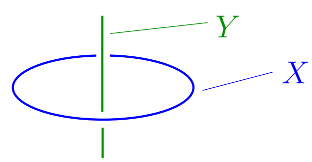


FIGURE 101. Illustration of Exercise 3.45.

Exercise 3.46. Build a Möbius band out of a strip of paper and cut it along the central curve. What topological space do you obtain that way?

Exercise 3.47. Let X and Y be topological spaces. Show that the join $X * Y$ is homeomorphic to the pushout of the maps

$$\begin{array}{ccc} X \times \{0,1\} \times Y & \longrightarrow & X \times [0,1] \times Y \\ \downarrow & & \\ X \sqcup Y & & \end{array}$$

where the horizontal map is the inclusion and the vertical map is defined by the two projections $X \times \{0\} \times Y \to X$ and $X \times \{1\} \times Y \to Y$.
Remark. This exercise explains in particular the initially slightly odd convention in the definition of the join for the special case that at least one of X or Y is the empty topological space.

Exercise 3.48. Let X and Y be topological spaces. Show that if X and Y are Hausdorff, then the join $X * Y$ is also Hausdorff.

4. Graphs

In this short chapter we will first introduce the purely combinatorial concept of an (undirected) abstract graph. Afterwards we introduce the corresponding topological realizations. These notions are of interest for various reasons. Abstract graphs are ubiquitous in mathematics and at times they will also play a useful role in our discussions. The corresponding topological realizations form fairly simple, but nonetheless interesting examples of topological spaces.

4.1. **Abstract graphs and topological graphs.** We start out with the following rather abstract definition.

Definition. An *abstract graph* G is a quadruple (V, E, i, t) where V is a set, E is a set and $i, t\colon E \to V$ are maps from E to V.

(1) The elements of V are called the *vertices of* G and the elements of E are called the *edges of* G.
(2) Given $e \in E$ the point $i(e)$ is called the *initial point of* e and $t(e)$ is called the *terminal point of* e. We refer to $i(e)$ and $t(e)$ as the *endpoints of* e.
(3) We say an edge $e \in E$ is a *loop* if $i(e) = t(e)$. Otherwise we call e a *non-loop*.
(4) We say G is *finite* (*countable*) if both V and E are finite (*countable*).

Remark. The origins of graph theory go back to Leonhard Euler[**Eul1736**] and his work in 1736 on the problem on the seven bridges of Königsberg, see Exercise 4.1.

Examples.

(A) We take $V = \{A, B, C\}$ and $E = \{e, f, g, h\}$. Furthermore let $i\colon E \to V$ be the map that is defined by $i(e) = A$, $i(f) = B$, $i(g) = B$ and $i(h) = A$ and let $t\colon E \to V$ be the map that is defined by $t(e) = B$, $t(f) = C$, $t(g) = C$ and $t(h) = A$. The quadruple (V, E, i, t) is an abstract graph.
(B) Let $a, b \in \mathbb{Z}$. We denote by $G_{a,b} = (V, E, i, t)$ the abstract graph with vertex set $V = \mathbb{Z}$, edge set $E = \mathbb{Z} \times \{0, 1\}$ and where $i(n, 0) = i(n, 1) = n$ and $t(n, 0) = n + a$ and $t(n, 1) = b$.

At this stage it is difficult to visualize the above definitions. Soon we will get to know a tool which rectifies this problem.

The following definition is somewhat topology-like.

Definition. We say that an abstract graph (V, E, i, t) is *connected* if for every two vertices $v \neq w \in V$ there exist vertices $v = v_0, v_1, \ldots, v_n = w$ and edges $e_0, \ldots, e_{n-1}$ such that for each $i \in \{0, \ldots, n-1\}$ the endpoints of e_i are v_i and v_{i+1}.

Examples. We want to determine whether or not the above examples (A) and (B) of abstract graphs are connected.

(A) This abstract graph is connected, for example if we take $v = A$ and $w = C$, then $v_0 = A$, $v_1 = B$ and $v_2 = C$ and $e_0 = e$ and $e_1 = f$ have the desired property.
(B) Let $a, b \in \mathbb{Z}$. Basically by definition the abstract graph $G_{a,b}$ is connected if and only if for every two integers $x, y \in \mathbb{Z}$ there exist $n, m \in \mathbb{Z}$ such that $x - y = ma + nb$. Elementary number theory says that this is the case if and only if a and b are coprime.

So far this definition has little to do with topology. We will change this sorry state of affairs by introducing the topological realization of a graph. The idea is, basically, that the topological realization of an abstract graph $G = (V, E, i, t)$ is given by one point for each vertex $v \in V$ and one interval for each edge $e \in E$ with endpoints given by $\varphi(e)$. The definition below is the formalization of that idea.

Definition. Let $G = (V, E, i, t)$ be an abstract graph.

(1) We define the *topological realization of* G as the topological space

$$|G| \;:=\; \big(V \;\sqcup\; (\underset{\substack{\uparrow\\ \text{here } E \text{ is equipped with the } \textit{discrete} \text{ topology}}}{E} \times [0,1])\big)/\sim \quad \begin{array}{l}\text{where for any } e \in E \text{ we have}\\ (e,0) \sim i(e) \text{ and } (e,1) \sim t(e).\end{array}$$

Note that on many occasions we will use implicitly that by Lemma 3.15 (3) we know that $[0,1] \times E = \bigsqcup_{e \in E} [0,1]$.

(2) Given $e \in E$ we refer to the map $\Phi_e \colon [0,1] \xrightarrow{x \mapsto (e,x)} \{e\} \times [0,1] \to |G|$ as the *characteristic map of the edge* e.

(3) Let $e \in E$. By a slight abuse of language we refer to $|e| = \Phi_e([0,1])$ also as an *edge of* $|G|$. Furthermore we refer to $\langle e \rangle := \Phi_e((0,1))$ as an *open edge of* $|G|$.

Before we attempt to give a better intuition for the definition of the topological realization we need to get the following lemma out of the way.

Lemma 4.1. *Let $G = (V, E, i, t)$ be an abstract graph.*

(1) *The obvious map $V \to |G|$ is an inclusion and the image is a discrete subset of $|G|$.*

(2) *$|G|$ is Hausdorff.*

(3) *Let $e \in E$.*

 (a) *The map $\Phi_e \colon [0,1] \to |G|$ is continuous and closed.*

 (b) *The map $\Phi_e \colon (0,1) \to |G|$ is an open embedding.*

 (c) *If $i(e) = t(e)$, then Φ_e induces a homeomorphism $[0,1]/0 \sim 1 \to |e|$.*

 (d) *If $i(e) \neq t(e)$, then Φ_e restricts to a homeomorphism $[0,1] \to |e|$.*

 The same statements (a), (b), (c) *and* (d) *also hold for the map $\widetilde{\Phi}_e \colon [0,1] \to |G|$ given by $t \mapsto \Phi_e(1-t)$. This map satisfies $\widetilde{\Phi}_e(0) = t(e)$ and $\widetilde{\Phi}_e(1) = i(e)$. For convenience we sometimes refer to $\widetilde{\Phi}_e$ also as a* characteristic map of the edge e.

(4) *G is finite if and only if $|G|$ is compact.*

(5) *The following statements are equivalent:*

 (a) *G is connected,*

 (b) *given any two distinct vertices v, v' there exists an embedding $\gamma \colon [0,1] \to |G|$ with $\gamma(0) = v$, $\gamma(1) = v'$ and such that $\gamma([0,1]) \subset |G|$ is the union of edges,*

 (c) *the topological realization $|G|$ is path-connected,*

 (d) *the topological realization $|G|$ is connected.*

PROOF $(*)$.

(1) This statement follows fairly easily from the definitions of the quotient topology and the disjoint union topology. We leave it to the reader to fill in the details.
(2) Ditto.
(3) Let $e \in E$.
  (a) It follows immediately from Lemma 3.3 (1) and Lemma 3.21 (3) that Φ_e is continuous. Furthermore it follows easily from the definitions together with Lemma 3.1 and Lemma 3.21 (3) that Φ_e is closed.
  (b) One can easily verify that $\Phi_e\colon (0,1) \to |G|$ is injective and that it is an open map. Thus it follows from Lemma 2.42 that it is an embedding.
  (c) We suppose that $i(e) = t(e)$. It follows easily from Lemma 3.28 that Φ_e induces a continuous map $[0,1]/0 \sim 1 \to |e|$ and that this map is a bijection. It follows from (2) and Proposition 2.43 (3) that the map $\Phi_e\colon [0,1]/0 \sim 1 \to |e|$ is in fact a homeomorphism.
  (d) Now we suppose that $i(e) \neq t(e)$. In this case the map is clearly injective. Thus it follows from (2) together with Proposition 2.43 (3) that $\Phi_e\colon [0,1] \to |e|$ is a homeomorphism.

  The statements regarding $\widetilde{\Phi}_e$ follow from the fact that the map $t \mapsto 1-t$ is a homeomorphism of $[0,1]$.
(4) First assume that G is finite. It follows almost immediately from Lemma 3.3 (4) together with Lemma 2.40 that $|G|$ is compact. In Exercise 4.6 we will show that any compact subset of the topological realization of a graph is contained in the topological realization of a "subgraph". This statement immediately implies that if $|G|$ is finite, then G is finite.
(5) We will prove this statement in Exercise 4.3. ■

Even though the above definition of the topological realization of an abstract graph is a perfectly fine, rigorous and precise definition, it is perhaps not the most intuitive definition. It would be much more fun to work with topological spaces that we can "see". This leads us to the following definition.

Definition. A *topological graph* is a triple $(X, G = (V,E,i,t), \Theta\colon |G| \to X)$ where the following holds:

(1) X is a topological space,
(2) $G = (V,E,i,t)$ is an abstract graph,
(3) $\Theta\colon |G| \to X$ is a homeomorphism between the topological realization $|G|$ of G and the topological space X.

Given a topological graph $(X, G = (V,E,i,t), \Theta\colon |G| \to X)$ we introduce the following extra definitions:

(1) Given $v \in V$ we refer to $\Theta(v)$ as a *vertex of the topological graph.*
(2) Given $e \in E$ we refer to $\Theta(|e|)$ as an *edge of the topological graph.* Given a loop e we refer to $|e|$ as a *loop of the topological graph.*
(3) We refer to the abstract graph $G = (V,E,i,t)$ as the *underlying abstract graph.*
(4) Sometimes we refer, somewhat incorrectly, to X as a *topological realization of G.*

Sometimes we will just say that X is a topological graph. Furthermore on occasions we will suppress $G = (V, E, i, t), \Theta\colon |G| \to X$ from the notation.

The following lemma gives us a way to construct topological graphs with a given underlying abstract graph.

Lemma 4.2. *Let $G = (V, E, i, t)$ be an abstract graph and let $n \in \mathbb{N}_0$. Suppose we are given the following data:*

(1) *An injective map $\alpha\colon V \to \mathbb{R}^n$.*
(2) *For each $e \in E$ we are given a map $\beta_e\colon [0,1] \to \mathbb{R}^n$ which is injective on $(0,1)$ and which satisfies $\beta_e(0) = \alpha(i(e))$ and $\beta_e(1) = \alpha(t(e))$.*
(3) *For $e \neq f \in E$ we have $\beta_e((0,1)) \cap \beta_f((0,1)) = \varnothing$.*

If G is infinite, then we also demand that the following condition is satisfied:

(4) *For every $r \in \mathbb{R}_{\geq 0}$ there exist only finitely many $v \in V$ with $\alpha(v) \cap B^n_r(0) \neq \varnothing$ and there exist only finitely many $e \in E$ with $\beta_e([0,1]) \cap B^n_r(0) \neq \varnothing$.*

Then the following map is an embedding:

$$\begin{array}{rcl} |G| = \big(V \;\sqcup\; (E \times [0,1])\big)\big/ \overset{\substack{\text{where } (e,0)\sim i(e) \text{ and } (e,1)\sim t(e)\\ \downarrow}}{\sim} & \to & \mathbb{R}^n \\ [x] & \mapsto & \begin{cases} \alpha(x), & \text{if } x \in V, \\ \beta_e(t), & \text{if } x = (e,t) \in E \times [0,1]. \end{cases} \end{array}$$

Proof. It follows easily from our hypotheses that the map is well-defined and that it is an injection. Furthermore we obtain from Lemma 3.22 that the map is continuous. If G is finite, then it follows from Lemma 4.1 together with Proposition 2.43 (3) that the given map $|G| \to \mathbb{R}^n$ is an embedding. In the general case the same conclusion follows from Proposition 2.45. ■

Examples. We return to the two examples of abstract graphs that we gave on page 221.

(A) As on page 221 we consider $V = \{A, B, C\}$ and $E = \{e, f, g, h\}$ together with the maps $i, t\colon E \to V$ given by $i(e) = A$, $i(f) = B$, $i(g) = B$ and $i(h) = A$ and given by $t(e) = B$, $t(f) = C$, $t(g) = C$ and $t(h) = A$. In Figure 102 we show two topological realizations[75] of this abstract graph.
(B) Let $a, b \in \mathbb{Z}$. As on page 221 we consider the abstract graph with vertex set $V = \mathbb{Z}$, edge set $\mathbb{Z} \times \{0,1\}$ and where $i(n,0) = i(n,1) = n$ and $t(n,0) = n+a$ and $t(n,1) = b$. In Figure 103 we show topological realizations of the graphs $G_{2,3}$ and $G_{0,2}$.

It is elementary[76] to show that given any finite abstract graph G there exists an embedding of $|G|$ into $\mathbb{R}^3$. We conclude this section with the following question.

[75] Here evidently we use the slightly sloppy language from above. More precisely, we do not show the actual topological realization $|G|$ but we show a topological graph X such that the underlying abstract graph equals G.

[76] When we say that it is "elementary to show something" this usually means that one can show the statement using basic methods without the need to use advanced results. This does not mean that this is necessarily easy. In this particular case a rigorous proof is given in [**CELR94**, p. 2].

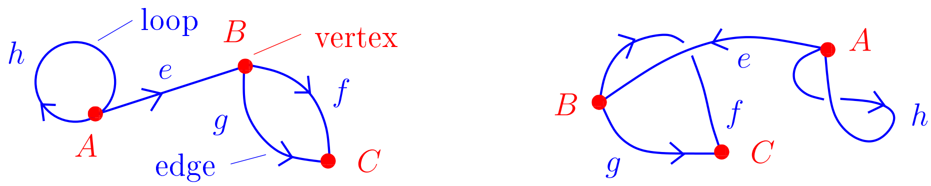
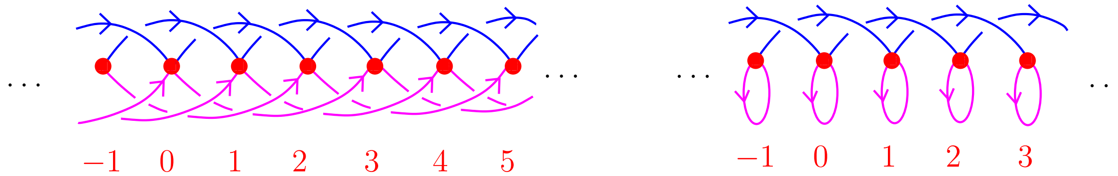


FIGURE 102



topological realization of $G_{2,3}$ topological realization of $G_{0,2}$

FIGURE 103

Question 4.3. *Let G be a finite abstract graph. Does there necessarily exist an embedding of $|G|$ into $\mathbb{R}^2$?*

Example. We consider the finite abstract graphs G, H and I that are (implicitly) illustrated in Figure 104. Do their topological realizations admit embeddings into $\mathbb{R}^2$?

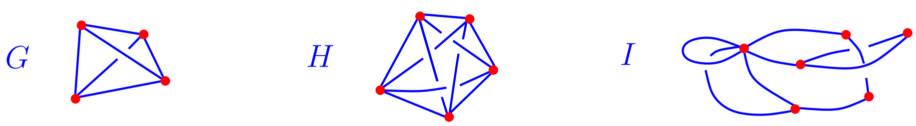


FIGURE 104

Surely the answer to Question 4.3 must be no, but it is not clear how one can prove this statement rigorously. We will return to this question in Section 56.3.

4.2. **Undirected abstract graphs and their topological realizations** $(*)$**.** In this section we will introduce the notion of an undirected abstract graph and the corresponding topological realizations. In many settings undirected abstract graphs are more suitable than the abstract graphs from the previous section. But as we will see, the topological realization of an undirected abstract graph is more unwieldy. Thus in these notes we will mostly work with abstract graphs and we will leave it to the reader to generalize some of the later results to the setting of undirected abstract graphs.

Without further ado, here is the definition.

Definition. An *undirected abstract graph* G is a triple (V, E, φ) where V is a set, E is a set and φ is a map

$$\varphi\colon E \;\to\; \{\text{subsets of } V \text{ with one or two elements}\}.$$

Similar to the discussion on page 221 we introduce the following extra definitions:

(1) The elements of V are called the *vertices of* G and the elements of E are called the *edges of* G.
(2) We say an edge $e \in E$ is a *loop* if $\#\varphi(e) = 1$. Otherwise we call e a *non-loop*.
(3) Given $e \in E$ the points in $\varphi(e)$ are called the *endpoints of* e.
(4) We say G has *multi-edges* if there exist distinct edges e and e' with $\varphi(e) = \varphi(e')$.
(5) We say G is *finite* (*countable*) if both V and E are finite (*countable*).

Examples. We take V to be the set of all subway stations in München and we take E to be the set of all segments of the subway system. The map φ then assigns to each $e \in E$ the two stations that are connected by the segment e. The triple (V, E, φ) is an undirected abstract graph.

Now comes the slightly intimidating definition of the topological realization of an undirected abstract graph.

Definition. Let $G = (V, E, \varphi)$ be an undirected abstract graph. We define the *topological realization of* G as the topological space[77]

$$|G| := \Big(V \sqcup \bigsqcup_{\substack{e \in E \\ \#\varphi(e)=1}} [0,1] \sqcup \bigsqcup_{\substack{e \in E \\ \#\varphi(e)=2}} [0,1] \times \mathrm{Bij}(\{0,1\}, \varphi(e))\Big) \Big/ \sim$$

where $\sim$ is generated by the following two relations:

(1) For each $e \in E$ with $\#\varphi(e) = 1$ we identify 0 and 1 in the corresponding copy of $[0,1]$ with the unique point $\varphi(e) \in V$.
(2) For each $e \in E$ with $\#\varphi(e) = 2$ we make the following identifications:
    (a) For each $\gamma \in \mathrm{Bij}(\{0,1\}, \varphi(e))$ we set $(0, \gamma) \sim \gamma(0)$ and $(1, \gamma) \sim \gamma(1)$.
    (b) For $\alpha \neq \beta \in \mathrm{Bij}(\{0,1\}, \varphi(e))$ and each $t \in [0,1]$ we set $(t, \alpha) \sim (1 - t, \beta)$.

Example. Using a variation on Lemma 4.2, or alternatively using Lemma 4.4 below one can see that the subset of $\mathbb{R}^2$ shown in Figure 105 is a topological realization of the subway system of München.

Lemma 4.4. *Let $G = (V, E, i, t)$ be an abstract graph.*

(1) *We define $\varphi\colon E \to \mathcal{P}(V)$ by $\varphi(e) = \{i(e), t(e)\}$. The triple $\widetilde{G} = (V, E, \varphi)$ is an undirected abstract graph.*
(2) *The map*

$$\begin{aligned} \Theta_G\colon |G| &\to |\widetilde{G}| \\ [x] &\mapsto \begin{cases} [x], & \text{if } x \in V, \\ [t \in [0,1]_e], & \text{if } x = (e,t) \in E \times [0,1] \text{ with } i(e) = t(e), \\ [(t,\gamma)], & \text{if } x = (e,t) \in E \times [0,1] \text{ with } i(e) \neq t(e), \\ & \text{here } \gamma(0) = i(e),\ \gamma(1) = t(e) \end{cases} \end{aligned}$$

is a homeomorphism.

[77] Let $e \in E$ be a vertex with $\#\varphi(e) = 2$. Note that in this case $\mathrm{Bij}(\{0,1\}, \varphi(e))$ has precisely two elements. The problem is that there is no "preferred" bijection, so we need to consider both in our definition.

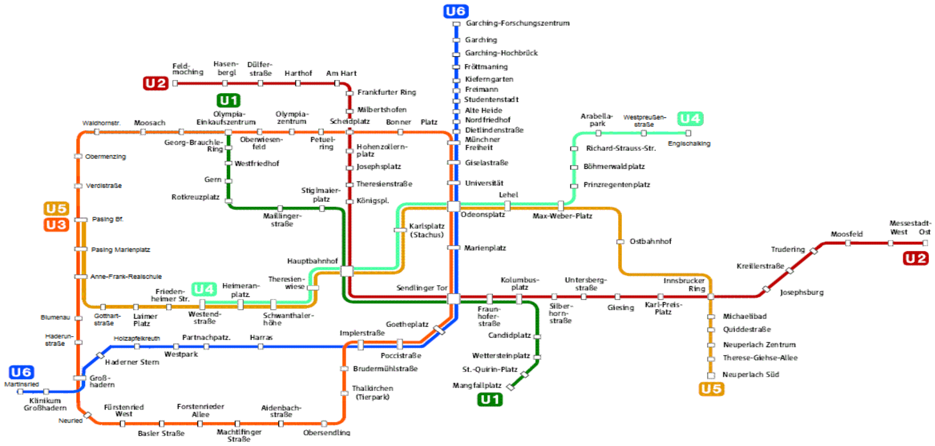


FIGURE 105. The subway system of München.

PROOF.

(1) This statement follows immediately from the definitions.
(2) It follows fairly easily from the definitions that the map is well-defined and a bijection. Furthermore one can deduce easily from Lemma 3.28 that the map is continuous. Working directly with the definitions one sees that the map is open. In summary it follows from Lemma 2.42 that the map is indeed a homeomorphism. ■

Remark. Perhaps by now it has become clear why the definition of the topological realization of an *undirected* abstract graph is more painful than the definition of the topological realization of an abstract graph. In both cases we want to attach one interval for each edge. If we are given a map $i, t\colon E \to V$, then we have a preferred way of doing so. If we are only given a map $\varphi\colon E \to \mathcal{P}(V)$ none of the two ways is preferred, thus we need to attach an interval in both ways and then we need to "anti-identify" the two intervals.

The following definition is almost verbatim the same as the definition of a connected abstract graph that we provided on page 221.

Definition. We say that an undirected abstract graph (V, E, φ) is *connected* if for every two vertices $v \neq w \in V$ there exist vertices $v = v_0, v_1, \ldots, v_n = w$ and edges $e_0, \ldots, e_{n-1}$ such that for each $i \in \{0, \ldots, n-1\}$ the endpoints of e_i are v_i and v_{i+1}.

The following totally elementary lemma can be viewed as an analogue of Lemma 2.61.

Lemma 4.5. *Let $G = (V, E, i, t)$ be an abstract graph or let $G = (V, E, \varphi)$ be an undirected abstract graph. Let $f\colon V \to X$ be a map to some set X such that for any $e \in E$ the values of f on the endpoints of e agree. If G is connected, then f is constant.*

PROOF. This lemma follows almost immediately from the definitions. ■

4.3. **Trees.** In this last section we study a basic but surprisingly useful class of abstract graphs.

Definition. Let $G = (V, E, i, t)$ be an abstract graph.

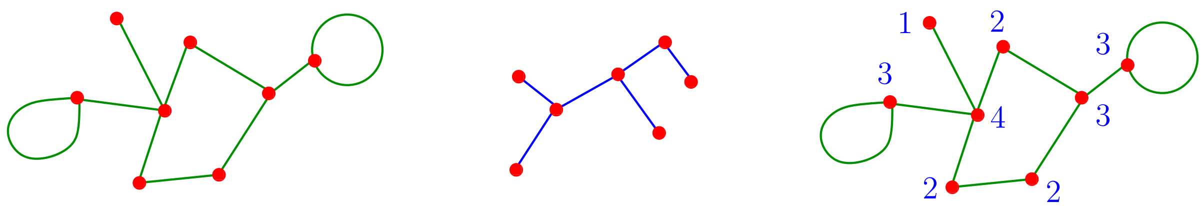
(1) If G is finite, then we refer to
$$\chi(G) \;=\; \#V \;-\; \#E \;=\; \text{number of vertices} \;-\; \text{number of edges}$$
as the *Euler characteristic of G*.

(2) A *tree* is a finite connected abstract graph with Euler characteristic 1.

(3) The *valence of a vertex v* is defined as
$$\text{valence}(v) \;:=\; \#\{e \in E \mid i(e) = v\} + \#\{e \in E \mid t(e) = v\}.$$

For undirected abstract graphs we introduce the obvious analogues of the above notions.

In a slightly less formal way, the valence of a vertex v is the number of edges at v, but hereby we have to count any loop at v twice, since it "goes twice into v".



Euler characteristic of G equals $\chi(G) = 8 - 10 = -2$

example of a tree

valences of the vertices

FIGURE 106

Lemma 4.6. *Let $G = (V, E, i, t)$ be an abstract graph or let $G = (V, E, \varphi)$ be an undirected abstract graph. If G is a tree and if G has at least one edge, then G admits at least two vertices of valence one.*

PROOF $(*)$. We prove Lemma 4.6 for abstract graphs, the proof for undirected abstract graphs is verbatim the same. Thus let $G = (V, E, i, t)$ be a tree that has at least one edge. Since G has at least one edge and since it is by definition connected, we see that the tree G does not have a vertex of valence 0. We calculate that

$$\begin{aligned}\sum_{\text{vertex } v} \underbrace{\text{valence}(v)}_{\geq 1} \;&=\; \sum_{\text{vertex } v} \big(\#\{e \in E \mid i(e) = v\} + \#\{e \in E \mid t(e) = v\}\big) \\ &\underset{\substack{\uparrow \\ \text{each edge contributes 2 to the sum}}}{=} 2 \cdot \#\text{ edges} \underset{\substack{\uparrow \\ \text{since } \chi(T) = 1}}{=} 2 \cdot \#\text{vertices} - 2.\end{aligned}$$

This equality can only hold if there are at least two vertices with valence less than 2. By the above the valence has to be at least one, so there exist at least two vertices with valence one. ■

On one occasion the following corollary will come in handy.

Corollary 4.7. *Let $G = (V, E, i, t)$ be an abstract graph and let $\widetilde{v} \in V$ be a vertex. If G is a tree, then there exist distinct edges $e_1, \ldots, e_k$ and distinct vertices $v_1, \ldots, v_k$ with the following properties:*

(1) *We have $V = \{\widetilde{v}, v_1, \ldots, v_k\}$ and $E = \{e_1, \ldots, e_k\}$.*
(2) *For each $i \in \{1, \ldots, k\}$ the vertex v_i is an endpoint of the edge e_i.*
(3) *If some vertex v_i is an endpoint of some e_j, then $j \leq i$.*

The same statement also holds for undirected abstract graphs. We refer to Figure 107 for an illustration.

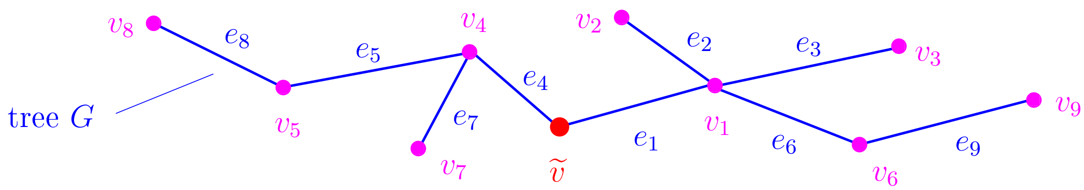


FIGURE 107. Illustration of Corollary 4.7.

PROOF. For abstract graphs we will provide the proof of the corollary in Exercise 4.5. For undirected abstract graphs the proof is verbatim the same. ■

Now we provide a few more definitions on graphs.

Definition. Let $G = (V, E, i, t)$ be an abstract graph.

(1) A *subgraph* of $G = (V, E, i, t)$ is an abstract graph $G' = (V', E', i', t')$ such that $V' \subset V$ and $E' \subset E$ are subsets such that $i' = i|_{E'}$ and $t' = t|_{E'}$ If G' is a tree, then we refer to G' as a *subtree*. Since i' and t' are determined by V' and E' we often suppress both of them from the notation.
(2) We say an edge e is *adjacent to a subgraph* G' if e has two endpoints where one of the endpoints lies on G' and the other does not.
(3) A *spanning tree* for G is a subgraph that is a tree and which does not admit an adjacent edge.

For undirected abstract graphs we introduce the obvious analogues of the above notions.

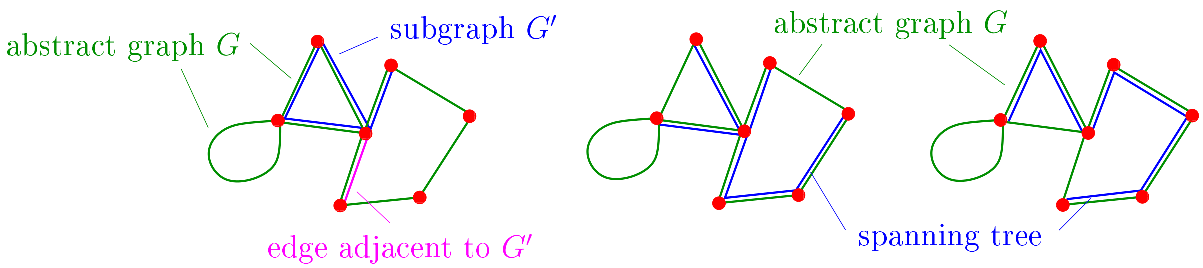


FIGURE 108

Remark. Let $G = (V, E, i, t)$ be an abstract graph.

(1) Every $v \in V$ gives rise to the rather trivial subgraph with a single vertex v and no edge.

(2) If an edge e of E is adjacent to some subgraph G', then it is not part of the subgraph. Indeed, by definition of a subgraph, for any edge that is part of the subgraph also its endpoints have to lie in the subgraph.
(3) The example of Figure 108 shows that an abstract graph usually has many different spanning trees.
(4) A spanning tree is often also called a *maximal tree*.

We conclude this chapter with the following proposition.

Proposition 4.8. *Let $G = (V, E, i, t)$ be a finite connected non-empty abstract graph.*
(1) *The graph G admits a spanning tree.*
(2) *Every spanning tree of G contains all vertices of G.*

The same statement also holds for undirected abstract graphs.

PROOF. In the following we provide the proof for abstract graphs. The proof for undirected abstract graphs is verbatim the same. Thus let $G = (V, E, i, t)$ be a finite connected abstract graph.

(1) We pick a vertex $v \in V$. We define $T_0 = \{v\}$ and view T_0 as a subtree of G. Now we construct iteratively a sequence of subtrees $T_0 \subset T_1 \subset \dots$ as follows: Suppose we have a subtree $T_i = (V_i, E_i)$. If there exists an edge $e \in E$ that is adjacent to T_i, then we "add e to T_i", more precisely, we define $T_{i+1} = (V_i \cup \varphi(e), E_i \cup e)$. Note that by definition of an adjacent edge we just added precisely one edge and precisely one vertex to T_i. Thus we see that $\chi(T_{i+1}) = \chi(T_i) = \dots = \chi(T_0) = 1$, i.e. T_{i+1} is indeed a subtree. Since G is finite this process stops after finitely many steps. Let T be the final subtree of our sequence. Note that by definition T is in fact a spanning tree. This argument is illustrated in Figure 109 on the left.
(2) Let T be a subgraph of G that is a tree. Suppose that there exists a vertex w that does not lie in T. We will show that T is not maximal. Let v be a vertex in T. Since G is connected there exists a sequence of edges $e_0, \dots, e_n$ with the following properties:
   (a) v is a vertex of e_0,
   (b) for each $i \in \{0, \dots, n-1\}$ a vertex of e_i is also a vertex of e_{i+1}, and
   (c) w is a vertex of e_n.

   Then there exists an $i \in \{0, \dots, n-1\}$ such that a vertex of e_i lies in T and the other one does not. But then e_i is an edge adjacent to T. This shows that T was not maximal. This argument is illustrated in Figure 109 on the right. ■

Exercises for Chapter 4.

Exercise 4.1. In Figure 110 we see the seven bridges of Königsberg. Is it possible to cross all seven bridges in a walk without crossing a single bridge twice?

Exercise 4.2. Let $G = (V, E, \varphi)$ and $G' = (V', E', \varphi')$ be two undirected abstract graphs. If the topological realizations $|G|$ and $|G'|$ are homeomorphic, does it follow that the undirected abstract graphs G and G' are isomorphic?

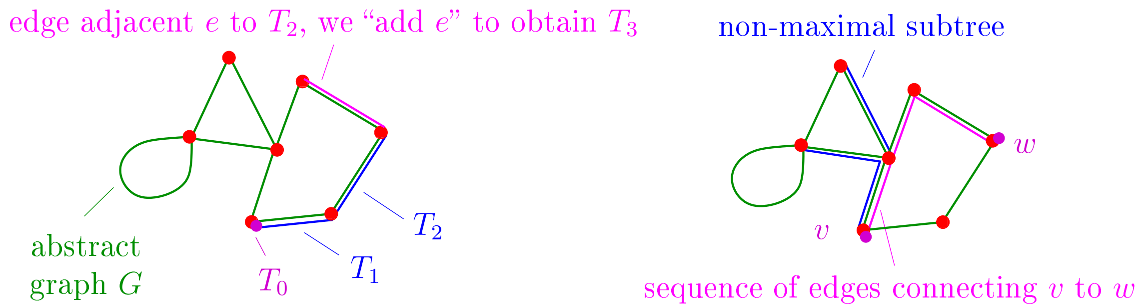


FIGURE 109. Illustration for the proof of Proposition 4.8.

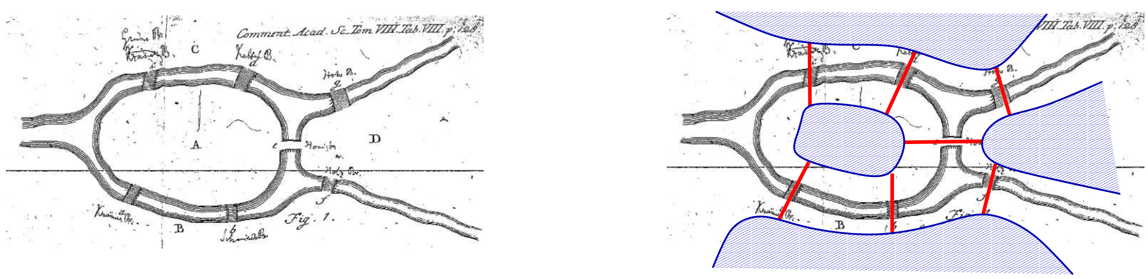


Illustration from Euler's paper **[Eul1736]**

FIGURE 110. Illustration of Exercise 4.1.

Exercise 4.3. Let G be an abstract graph. Show that the following are equivalent:

(a) G is connected,
(b) given any two distinct vertices v, v' there exists an embedding $\gamma\colon [0,1] \to |G|$ with $\gamma(0) = v$, $\gamma(1) = v'$ and such that $\gamma([0,1]) \subset |G|$ is the union of edges,
(c) the topological realization $|G|$ is path-connected,
(d) the topological realization $|G|$ is connected.

Exercise 4.4. Are the two topological graphs G and H shown in Figure 111 homeomorphic?

FIGURE 111

Exercise 4.5. Let $G = (V, E, i, t)$ be an abstract graph and let $\widetilde{v} \in V$ be a vertex. We suppose that G is a tree. Show that there exist distinct edges $e_1, \ldots, e_k$ and distinct vertices $v_1, \ldots, v_k$ with the following properties:

(1) We have $V = \{\widetilde{v}, v_1, \ldots, v_k\}$ and $E = \{e_1, \ldots, e_k\}$.
(2) For each $i \in \{1, \ldots, k\}$ the vertex v_i is an endpoint of the edge e_i.
(3) If some vertex v_i is an endpoint of some e_j, then $j \leq i$.

We refer to Figure 107 for an illustration.

Exercise 4.6. Let G be an abstract graph and let $K \subset |G|$ be compact subset. Show that there exists a finite subgraph H of G with $K \subset |H|$.

Exercise 4.7. Let $G = (V, E, i, t)$ be a countable connected non-empty abstract graph. We pick a vertex $v \in V$. Show that there exists a sequence $\{G_j = (V_j, E_j, i, t)\}_{j \in \mathbb{N}}$ of subgraphs with the following properties:

(1) Each G_j is finite and connected.
(2) Each G_j contains v.
(3) We have $G = \bigcup_{j \in \mathbb{N}} G_j$ in the sense that $V = \bigcup_{j \in \mathbb{N}} V_j$ and $E = \bigcup_{j \in \mathbb{N}} G_j$.

5. General topology: some delicate bits (∗)

In this chapter we will introduce the compact-open topology and we will discuss the very delicate interplay between product and quotient topologies. In all likelihood the reader is better off with consulting the chapter only when the results of this chapter are actually used.

5.1. **The compact-open topology** (∗).

So far we had mostly studied topological spaces that arose more or less directly from geometry. In this section we will see that topological spaces also arise naturally from more abstract set ups.

First, recall that we saw on page 96 that a topology on a given set X makes it possible to talk of the convergence of a sequence in X. From real analysis we know that it is not only interesting to think about convergence of sequences of points but also to talk of convergence of functions. In fact, in real analysis one often encounters two different types of convergence that we now recall.

Definition. Let $U \subset \mathbb{R}^n$ be a subset, let $\{f_n : U \to \mathbb{R}\}_{n\in\mathbb{N}}$ be a sequence of functions and let $f : U \to \mathbb{R}$ be a function. We define

$$(f_n)_{n\in\mathbb{N}} \textit{ converges pointwise to } f \quad :\Longleftrightarrow \quad \underset{x\in U}{\forall} \ \underset{\epsilon>0}{\forall} \ \underset{N\in\mathbb{N}}{\exists} \ \underset{n\geq N}{\forall} \ |f_n(x) - f(x)| < \epsilon$$

and

$$(f_n)_{n\in\mathbb{N}} \textit{ converges uniformly to } f \quad :\Longleftrightarrow \quad \underset{\epsilon>0}{\forall} \ \underset{N\in\mathbb{N}}{\exists} \ \underset{x\in U}{\forall} \ \underset{n\geq N}{\forall} \ |f_n(x) - f(x)| < \epsilon.$$

Example. We consider the sequence of functions

$$\begin{aligned} f_n : \mathbb{R} &\to \mathbb{R} \\ t &\mapsto \begin{cases} 0, & \text{if } t < 0 \text{ or } t > \frac{2}{n}, \\ n \cdot t, & \text{if } t \in [0, \frac{1}{n}], \\ 2 - n \cdot t, & \text{if } t \in (\frac{1}{n}, \frac{2}{n}]. \end{cases} \end{aligned}$$

We refer to Figure 112 for an illustration. One easily verifies that this sequence converges pointwise to the zero function but it does not converge uniformly to the zero function.

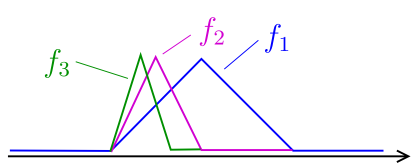


FIGURE 112

At times one also wants to study convergence of sequences of continuous[78] maps between general topological spaces. This leads us to the following definition.

Definition. Let X and Y be topological spaces.

(1) We denote by $X^Y = C(Y, X)$ the set of all continuous maps from Y to X.

[78]Here and in the following we sometimes talk explicitly about "continuous" maps, even though it is usually understood in these notes that all maps are continuous.

(2) The *compact-open topology* on X^Y is the topology generated, in the sense of the definition on page 105, by all the sets of the form
$$M(K,U) \ := \ \{f \in X^Y \mid f(K) \subset U\}$$
where $K \subset Y$ is compact and where $U \subset X$ is open. This definition is illustrated in Figure 113.

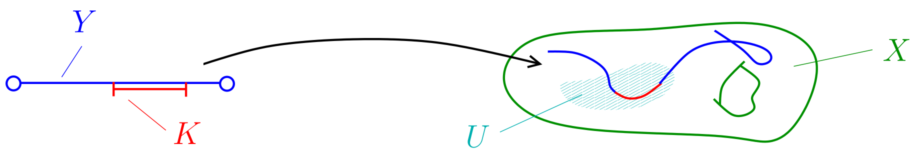


FIGURE 113

Examples.

(1) Let $Y \subset \mathbb{R}$ be an interval. We want to consider the compact-open topology on $C(Y,\mathbb{R}) = \mathbb{R}^Y$. As usual we picture elements of $C(Y,\mathbb{R})$ by their graphs. In Figure 114 we show on the left an open set $M(K,U)$ as above and on the right we show the graph of some function $f \in C(Y,\mathbb{R})$ and an open neighborhood thereof.

Now let $f_n \in C(Y,\mathbb{R})$, $n \in \mathbb{N}$ be a sequence of functions and let $f \in C(Y,\mathbb{R})$. We leave it as an amusing, albeit somewhat lengthy, exercise to show that the following holds:

$(f_n)_{n\in\mathbb{N}}$ converges[79] to f with respect to the compact-open topology	$\Longleftrightarrow$	for each compact subset $K \subset Y$ the sequence $(f_n\vert_K)_{n\in\mathbb{N}}$ converges uniformly to $f\vert_K$

Alternatively a proof for this statement is given in [**Wil70**, Theorem 43.7] or in [**Jos83**, Theorem 15.1.2].

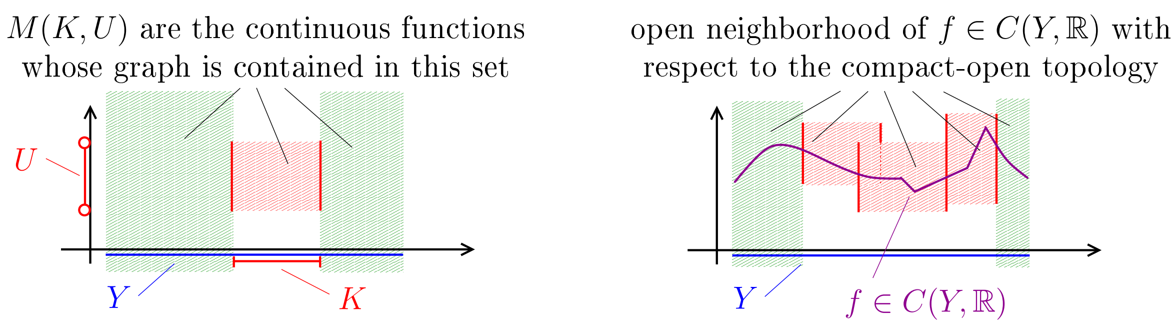


FIGURE 114

(2) We consider $Y = \{1,\ldots,n\}$ which we equip with the discrete topology. In Exercise 5.7 we will see that for any topological space X we have a homeomorphism $X^Y \cong X^n$.

The final example deserves its own lemma.

[79]We refer to page 96 for the definition of convergence of a sequence in a topological space.

Lemma 5.1. *Let $m, n \in \mathbb{N}$. We have the obvious inclusion*

$$\begin{array}{rcl} \mathrm{M}(m \times n, \mathbb{R}) & \to & C(\mathbb{R}^n, \mathbb{R}^m) = (\mathbb{R}^m)^{\mathbb{R}^n} \\ A & \mapsto & \left(\begin{array}{rcl} \mathbb{R}^n & \to & \mathbb{R}^m \\ v & \mapsto & Av \end{array} \right). \end{array}$$

The compact-open topology on $C(\mathbb{R}^n, \mathbb{R}^m)$ induces the usual topology on $\mathrm{M}(m \times n, \mathbb{R})$ that we introduced on page 163.

Sketch of proof. This statement can be proved, with some effort, using Lemma 2.23. Alternatively the statement can also be deduced from [**Hat02**, Proposition A.13] or [**Mun75**, Theorem 5.1]. ■

To get some practice with the compact-open topology we will prove the following two lemmas in Exercises 5.1 and 5.2.

Lemma 5.2. *Let X and Y be topological spaces. If Y is Hausdorff, then X^Y is also Hausdorff.*

Lemma 5.3. *If X and Y are non-empty topological spaces, then the map*

$$\begin{array}{rcl} X & \to & X^Y \\ x & \mapsto & \textit{constant map } y \mapsto x \end{array}$$

is continuous and it is an embedding.

We continue with a few technical statements on the compact-open topology. In most applications later on we will consider the case that $Y = [0,1]$. In particular the extra hypothesis in the subsequent propositions on the topological space Y will not be an issue in our future applications.

Proposition 5.4. *Let X and Y be topological spaces. If Y is regionally compact, then the following two statements hold:*

(1) *The evaluation map*[80]

$$\begin{array}{rcl} e\colon X^Y \times Y & \to & X \\ ((f\colon Y \to X), y) & \mapsto & f(y) \end{array}$$

is continuous.

(2) *For any $y \in Y$ the evaluation map*

$$\begin{array}{rcl} e_y\colon X^Y & \to & X \\ (f\colon Y \to X) & \mapsto & f(y) \end{array}$$

is continuous.

Proof (∗).

(1) Let U be an open subset of X. We need to show that $e^{-1}(U)$ is open. By Lemma 2.5 it suffices to show that given $(f,y) \in e^{-1}(U)$ there exists a neighborhood W of (f,y) with $W \subset e^{-1}(U)$. Thus let $f \in X^Y$ and $y \in Y$ with $e(f,y) = f(y) \in U$. Since f is continuous and since Y is regionally compact there exists a compact neighborhood

[80] Here we equip $X^Y \times Y$ with the product topology from page 162.

K of y which is contained in the open neighborhood $f^{-1}(U)$ of y. In other words, we have found a compact neighborhood K of y with $f(K) \subset U$. Therefore we have $f \in M(K,U)$ and the map e sends $W := M(K,U) \times K$ into U. By the definition of the product topology the set $W = M(K,U) \times K$ is in fact a neighborhood of (f,y) in $X^Y \times Y$.

(2) This statement follows from (1) and Lemma 3.8 (2a). ■

We continue with the following elementary but essential lemma.

Lemma 5.5. $(*)$ *Let X and Y be topological spaces.*

(1) *If $\varphi\colon X \to \widetilde{X}$ is a continuous map, then the induced map*

$$\begin{array}{rcl} \varphi^*\colon X^Y & \to & \widetilde{X}^Y \\ (f\colon Y \to X) & \mapsto & (\varphi \circ f\colon Y \to \widetilde{X}) \end{array}$$

is continuous.

(2) *If $\varphi\colon X \to \widetilde{X}$ is an embedding, then so is $\varphi^*\colon X^Y \to \widetilde{X}^Y$.*

(3) *Let $\varphi, \psi\colon X \to \widetilde{X}$ be continuous maps. If φ and ψ are homotopic*[81]*, then so are $\varphi^*, \psi^*\colon X^Y \to \widetilde{X}^Y$.*

(4) *If $\widetilde{X}$ is a topological space that is homotopy equivalent to X, then $\widetilde{X}^Y$ is also homotopy equivalent to X^Y.*

(5) *If $\psi\colon Y \to \widetilde{Y}$ is a continuous map, then the induced map*

$$\begin{array}{rcl} \psi_*\colon X^{\widetilde{Y}} & \to & X^Y \\ (f\colon \widetilde{Y} \to X) & \mapsto & (f \circ \psi\colon Y \to X) \end{array}$$

is continuous.

Remark. Lemma 5.5 (2) says in particular, that given topological spaces X and Y and a subset A of X, we can view A^Y as a subset of X^Y.

Proof $(*)$. To clarify the statements we replace the notation $M(K,U)$ by the notation $M_{X^Y}(K,U)$, with the obvious interpretation thereof.

(1) We check continuity using the continuity criterion of Proposition 2.37. Thus let $K \subset Y$ be a compact subset and let $U \subset \widetilde{X}$ be an open subset. We have

$$\begin{aligned} (\varphi^*)^{-1}(M_{\widetilde{X}^Y}(K,U)) &= \{f \in X^Y \mid \varphi^*(f) \in M_{\widetilde{X}^Y}(K,U)\} = \{f\colon Y \to X \mid \overbrace{(\varphi \circ f)(K)}^{=\varphi(f(K))} \subset U\} \\ &\underset{\substack{\uparrow \\ \text{by Exercise 2.2}}}{=} \{f\colon Y \to X \mid f(K) \subset \varphi^{-1}(U)\} \;=\; M_{X^Y}(K, \varphi^{-1}(U)). \end{aligned}$$

Since φ is continuous we see that $\varphi^{-1}(U)$ is an open subset of X. Since the sets $M(K,U)$ form by definition a basis for the topology of $\widetilde{X}^Y$ we deduce from Proposition 2.37 that φ^* is indeed continuous.

[81] We apologize to the reader for using the notion of being homotopic, even though it is only introduced on page 537. The reader who made it that far into this chapter will surely not be frightened by this inconvenience.

(2) Since φ is an embedding we can identify X with its image in $\widetilde{X}$. Put differently, it suffices to consider the case of an inclusion $i\colon X \to \widetilde{X}$. By Lemma 2.42 (1) together with Lemma 2.39 it suffices to show that for any compact subset $K \subset Y$ and any open subset $U \subset X$ the image $i_*(M_{X^Y}(K,U))$ is an open subset of $i_*(X^Y)$. Since $U \subset X$ is open there exists, by definition of the subspace topology, an open subset $V \subset \widetilde{X}$ such that $U = V \cap X$. We have

$$\begin{aligned} i_*(M_{X^Y}(K,U)) &= \{i \circ f\colon Y \to \widetilde{X} \mid f\colon Y \to X \text{ and } f(K) \subset U\} \\ &\underset{\substack{\uparrow \\ \text{since } f(Y)\subset X \text{ implies for any } x \in X \text{ that } f(x) \in U \text{ if and only if } f(x) \in V}}{=} \{f\colon Y \to X \mid f(K) \subset V\} \;=\; i_*(X^Y) \cap M_{\widetilde{X}^Y}(K,V). \end{aligned}$$

By the definition of the subspace topology this is an open subset of $i_*(X^Y) \subset \widetilde{X}^Y$.

(3) Let $F\colon X \times [0,1] \to Y$ be a homotopy between $F_0 = \varphi$ and $F_1 = \psi$. Using Proposition 2.37 and Lemma 2.5 it is not particularly challenging to show that the map

$$\begin{aligned} X^Y \times [0,1] &\to \widetilde{X}^Y \\ (g\colon Y \to X, t) &\mapsto \begin{pmatrix} Y & \to & \widetilde{X} \\ y & \mapsto & F(g(x),t) \end{pmatrix} \end{aligned}$$

is continuous. This is the desired homotopy between φ^* and ψ^*.

(4) This statement follows almost immediately from (3).

(5) Let $K \subset Y$ be a compact subset and let $U \subset X$ be an open subset. We have

$$\begin{aligned} (\psi_*)^{-1}(M_{X^Y}(K,U)) &= \{f \in X^{\widetilde{Y}} \mid \psi_*(f) \in M_{X^Y}(K,U)\} \\ &= \{f\colon \widetilde{Y} \to X \mid \underbrace{(f \circ \psi)(K)}_{=f(\psi(K))} \subset U\} \;=\; M_{X^{\widetilde{Y}}}(\psi(K),U). \end{aligned}$$

Since ψ is continuous we obtain from Lemma 2.40 that $\psi(K)$ is a compact subset of $\widetilde{Y}$. Thus we obtain again from Proposition 2.37 that ψ_* is indeed continuous. ■

On many occasions we will also need the following rather technical proposition.

Proposition 5.6. *Let Y be a topological space that is regionally compact. Furthermore let X and T be two topological spaces. Let $H\colon Y \times T \to X$ be a map. Given $t \in T$ we denote by $H_t\colon Y \to X$ the map defined by $H_t(y) := H(y,t)$. Then*

$$H\colon Y \times T \to X \text{ is continuous} \iff \begin{array}{l} (1) \text{ each } H_t\colon Y \to X \text{ is continuous, and} \\ (2) \text{ the map}^{82}\ T \to X^Y \text{ given by } t \mapsto H_t \text{ is continuous.} \end{array}$$

In other words, the map

$$\begin{aligned} X^{Y \times T} &\to (X^Y)^T \\ (H\colon Y \times T \to X) &\mapsto \begin{pmatrix} T & \to & X^Y \\ t & \mapsto & (y \mapsto H(y,t)) \end{pmatrix} \end{aligned}$$

is well-defined[83] *and it is a bijection.*

---

[82]Note that by (1) we know that $H_t \in X^Y$.

[83]Here "well-defined" means that the map $T \to X^Y$ actually defines an element in $(X^Y)^T$. More precisely, the map $T \to X^Y$ given by $z \mapsto (y \mapsto H(y,z))$ actually lies in $(X^Y)^T$.

Proof (∗). [84] Note that the "in other words" part of the proposition is indeed just a reformulation of the first part of the proposition. Thus it remains to prove the equivalence of the two statements in the beginning.

First we prove the "$\Leftarrow$"-direction. We point out that $H\colon Y\times T\to X$ is the composition of the two maps

$$\begin{array}{rcl} Y\times T & \to & X^Y\times Y \\ (y,t) & \mapsto & (H_t,y) \end{array} \quad \text{and the evaluation map} \quad \begin{array}{rcl} X^Y\times Y & \to & X \\ (f,y) & \mapsto & f(y). \end{array}$$

The first map is continuous by (2) and Lemma 3.8. Furthermore, since Y is regionally compact we obtain from Proposition 5.4 that the second map is continuous. Thus H itself is continuous.

Now we turn to the "$\Rightarrow$"-direction. Thus we suppose that $H\colon Y\times T\to X$ is continuous.

(1) Let $t\in T$. The map H_t is the composition of the inclusion $Y\to Y\times T$, $y\mapsto(y,t)$ with the map H. The first map is continuous by Lemma 3.8 (2a) and the second map is continuous by our hypothesis. Hence H_t is also continuous. In particular we now know that H_t defines an element in X^Y.

(2) We need to show that the map $T\to X^Y$ given by $t\mapsto H_t$ is continuous. By Proposition 2.37 and Lemma 2.5 it suffices to show that for any set of the form $M(K,U)\subset X^Y$ and any $t\in T$ with $H_t\in M(K,U)$ there exists a neighborhood W of $t\in T$ such that for all $t'\in W$ we have $H_{t'}\in M(K,U)$.

Thus let K be a compact subset of Y, let U be an open subset of X and let $t\in T$ with $H_t\in M(K,U)$. We start out with the following claim.

Claim. Given any $y\in K$ there exist open neighborhoods $V_y\subset Y$ of y and $W_y\subset T$ of t such that $H(V_y\times W_y)\subset U$.

By the continuity of $H\colon Y\times T\to X$ we know that $H^{-1}(U)$ is an open subset of $Y\times K$. Note that $H(y,t)=H_t(y)\in U$ since $H_t\in M(K,U)$ and $y\in K$. Thus $H^{-1}(U)$ is an open neighborhood of $(y,t)\in Y\times K$. The existence of V_y and W_y is now a consequence of the definition of the product topology of $Y\times T$. ⊞

It follows from the compactness of K and Lemma 2.15 that there exist points $y_1,\dots,y_n\in K$ with $K\subset V_{y_1}\cup\dots\cup V_{y_n}$. We set $W:=W_{y_1}\cap\dots\cap W_{y_n}\subset T$. Note that this is an open neighborhood of t. Then

$$H(K\times W)\underset{\substack{\uparrow\\ \text{since } K\subset\bigcup_{i=1}^n V_{y_i}}}{\subset} H\Big(\bigcup_{i=1}^n V_{y_i}\times W\Big)\underset{\substack{\uparrow\\ \text{since } W\subset W_{y_i}}}{\subset} H\Big(\bigcup_{i=1}^n V_{y_i}\times W_{y_i}\Big)=\bigcup_{i=1}^n\underbrace{H(V_{y_i}\times W_{y_i})}_{\subset U}\subset U.$$

It follows from this observation and the definition of $H_{t'}$ and $M(K,U)$ that for any $t'\in W$ we have $H_{t'}\in M(K,U)$. ■

On one later occasion we will also need the following equally technical proposition.

[84] I strongly advise against reading the proof.

Proposition 5.7. *Let X, Y and Z be topological spaces. If Y and Z are both regionally compact and Hausdorff, then the map*[85]

$$\begin{array}{rcl} X^{Y\times Z} & \to & (X^Y)^Z \\ (f\colon Y\times Z\to X) & \mapsto & \left(\begin{array}{rcl} Z & \to & X^Y \\ z & \mapsto & (y\mapsto f(y,z)) \end{array}\right) \end{array}$$

is a homeomorphism.

Remark. In fact a slightly stronger version of Proposition 5.7 holds. By [**Dug66**, Theorem XII.5.3] one only needs to assume that Y is regionally compact, with no further restrictions on X, Y or Z. We will not need this stronger statement.

Once again no sane person will ever voluntarily read the proof. For those readers who are still around we first provide the proof of the following lemma.

Lemma 5.8. *Let X and Y be topological spaces. Furthermore let $\mathcal{V}$ be a subbasis for X. We consider*

$$\mathcal{C} \;=\; \big\{ M(K,V)\in\mathcal{P}\big(X^Y\big) \,\big|\, K \text{ a compact subset of } Y \text{ and } V\in\mathcal{V}\big\}.$$

If Y is Hausdorff, then $\mathcal{C}$ is subbasis[86] *for the topology of X^Y.*

Proof. Let K be a compact subset of Y, let U be an open subset of X and let f be a point in $M(K,U)$, i.e. $f\colon Y\to X$ is a map with $K\subset f^{-1}(U)$. By Lemma 2.27 (2) we need to show that there exist finitely many $C_1,\dots,C_m\in\mathcal{C}$ with $f\in\bigcap\limits_{i=1}^{m} C_i\subset M(K,U)$.

Since $\mathcal{V}$ is a subbasis for the topology of X we can write $U=\bigcup\limits_{i\in I} U_i$ where each U_i is of the form $U_i=\bigcap\limits_{V\in P_i} V$ where P_i is a finite subset of $\mathcal{V}$. Since K is compact and since f is continuous it follows from Lemma 2.15 that there exists a finite subset $J\subset I$ such that $K\subset\bigcup\limits_{j\in J} f^{-1}(U_j)$.

Since K is compact and Hausdorff we obtain from Lemma 2.46 that K is normal. It follows from Lemma 2.49, together with Lemma 2.17 (1) and the hypothesis that Y is Hausdorff, that there exist compact subsets $K_j\subset f^{-1}(U_j)$, $j\in J$, with $K\subset\bigcup\limits_{j\in J} K_j$.

Thus we see that

$$f \;\underset{\substack{\uparrow \\ \text{since } K_j\subset f^{-1}(U_j)}}{\in}\; \bigcap_{j\in J} M(K_j,U_j) \;\underset{\substack{\uparrow \\ \text{definition of } P_j}}{=}\; \bigcap_{j\in J} M\Big(K_j,\bigcap_{V\in P_j} V\Big) \;\overset{\substack{\text{since } M(K,A\cap B)=M(K,A)\cap M(K,B) \\ \downarrow}}{=}\; \bigcap_{j\in J}\bigcap_{V\in P_j}\underbrace{M(K_j,V)}_{\in\mathcal{C}} \;\underset{\substack{\uparrow \\ \text{since } K\subset\bigcup_{j\in J} K_j \text{ and since each } V \text{ lies in } U}}{\subset}\; M(K,U).$$

The finitely many sets $\{M(K_j,V)\}_{j\in J,V\in P_j}$ in $\mathcal{C}$ thus do the trick. ■

Now we are finally ready to provide the proof of Proposition 5.7.

[85] By Proposition 5.6 we know that the map is well-defined and that it is a bijection.

[86] We refer to page 105 for the definition of a subbasis of a topological space.

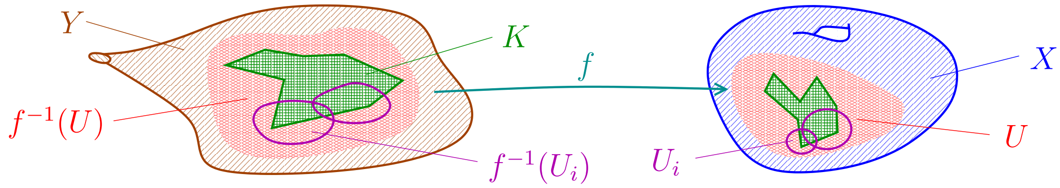


FIGURE 115. Illustration for the proof of Lemma 5.8.

PROOF OF PROPOSITION 5.7. Let X, Y and Z be topological spaces. We assume that Y and Z are both regionally compact and Hausdorff. We need to show that the map

$$\begin{array}{rcl} \Phi\colon X^{Y\times Z} & \to & (X^Y)^Z \\ (f\colon Y\times Z\to X) & \mapsto & \left(\begin{array}{ccl} Z & \mapsto & X^Y \\ z & \mapsto & (y\mapsto f(y,z)) \end{array}\right) \end{array}$$

is a homeomorphism. We start out with the following claim.

Claim. The set

$$\mathcal{C} = \left\{ M(A\times B, U) \in \mathcal{P}\big(X^{Y\times Z}\big) \;\middle|\; \begin{array}{l} A \text{ a compact subset of } Y,\ B \text{ a compact subset of } Z \\ \text{and } U \text{ an open subset of } X \end{array} \right\}$$

is a subbasis for the topological space $X^{Y\times Z}$.

Let $K\subset Y\times Z$ be a compact subset, let U be an open subset of X and let f be a point in $M(K,U)$, i.e. $f\colon Y\times Z\to X$ is a map with $K\subset f^{-1}(U)$. By Lemma 2.27 (2) we need to show that there exist finitely many $C_1,\dots,C_m\in\mathcal{C}$ with $f\in\bigcap_{i=1}^m C_i\subset M(K,U)$.

Let $k\in K\subset Y\times Z$. We write $k=(y,z)$. Since $f^{-1}(U)$ is an open subset of $Y\times Z$ it follows from the definition of the product topology on $Y\times Z$ that there exist open neighborhoods $V_k\subset Y$ of y and $W_k\subset Z$ of z such that $k=(y,z)\in V_k\times W_k\subset f^{-1}(U)$. Since Y and Z are regionally compact we can find a compact neighborhood $A_k\subset Y$ of y and a compact neighborhood $B_k\subset Z$ of z such that $k\in A_k\times B_k\subset V_k\times W_k\subset f^{-1}(U)$. Since K is compact it follows from Lemma 2.15 that there exist finitely many points $k_1,\dots,k_m\in K$ such that $K\subset\bigcup_{i=1}^m A_{k_i}\times B_{k_i}$. Now we see that

$$f \underset{\substack{\uparrow \\ \text{since } f(A_{k_i}\times B_{k_i})\subset U}}{\in} \bigcap_{i=1}^m \overbrace{M(A_{k_i}\times B_{k_I}, U)}^{\in\mathcal{C}} \overset{\substack{\text{since } M(V\cup W,U)=M(V,U)\cap M(W,U) \\ \downarrow}}{=} M\Big(\bigcup_{i=1}^m A_{k_i}\times B_{k_i}, U\Big) \underset{\substack{\uparrow \\ \text{since } K\subset \bigcup_{i=1}^m A_{k_i}\times B_{k_i}}}{\subset} M(K,U).$$

⊞

As we already remarked in Proposition 5.6, the given map $\Phi\colon X^{Y\times Z}\to(X^Y)^Z$ is clearly a bijection. Furthermore under this bijection the sets $M(A\times B,U)$ are precisely the sets $M(B,M(A,U))$. By the claim the former sets form a subbasis for $X^{Y\times Z}$. Thus it suffices to show that the latter sets form a subbasis for $(X^Y)^Z$. But that is an immediate consequence of Lemma 5.8, applied twice, and our hypothesis that Y and Z are Hausdorff. ■

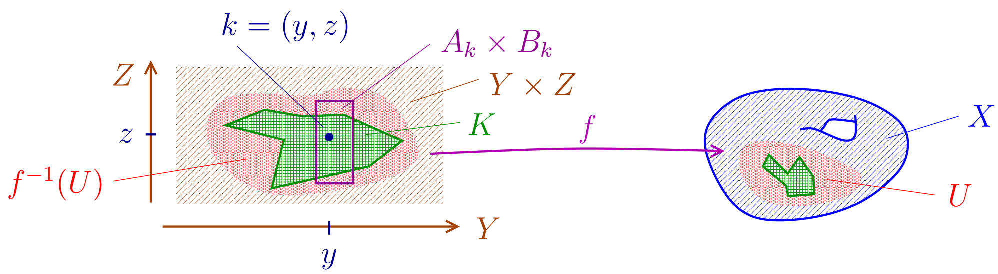


FIGURE 116. Illustration for the proof of Proposition 5.7.

The following proposition is also somewhat technical.

Proposition 5.9. *Let X, Y and Z be topological spaces. If Y is regionally compact, then the map*

$$\begin{array}{rcl}\Phi\colon X^Y \times Y^Z & \to & X^Z \\ (f, g) & \mapsto & f \circ g\end{array}$$

is continuous.

The proof of Proposition 5.9 rests on the following little lemma.

Lemma 5.10. (∗) *Let Y be a topological space that is regionally compact. If K is a compact subset that is contained in an open subset U of Y, then there exists a compact subset L with $K \subset \mathring{L} \subset L \subset U$.*

PROOF (∗). Let $x \in K$. Since Y is regionally compact there exists a compact neighborhood $L_x \subset U$ of x. Since K is compact there exist $x_1, \dots, x_m \in K$ with $K \subset \mathring{L}_{x_1} \cup \dots \cup \mathring{L}_{x_m}$. We set $L := L_{x_1} \cup \dots \cup L_{x_m}$. By Lemma 2.16 (1) we know that L is compact. We have

$$K \;\subset\; \bigcup_{i=1}^{m} \mathring{L}_{x_i} \;\underset{\substack{\uparrow \\ \text{Lemma 2.10 (1)}}}{\subset}\; \text{interior of } \bigcup_{i=1}^{m} L_x \;=\; \mathring{L}_{x_i} \;\subset\; L \;\subset\; U.$$

■

PROOF OF PROPOSITION 5.9 (∗). By Proposition 2.37 and Lemma 2.5 and the definition of the product topology on $X^Y \times Y^Z$ it suffices to prove the following claim.

Claim. Let K be a compact subset of Z and let V be an open subset of X. If $f \in X^Y$ and $g \in Y^Z$ satisfy $f \circ g \in M(K, V)$, then there exists an open subset $U \subset Y$ and a compact subset $L \subset Y$ with

$$\Phi\big(\underbrace{M(L, V)}_{\subset X^Y} \times \underbrace{M(K, U)}_{\subset Y^Z}\big) \;\subset\; M(K, V).$$

First note that $g(K) \subset Y$ is compact by Lemma 2.40 and that $f^{-1}(V) \subset Y$ is open since f is evidently continuous. Since Y is regionally compact we obtain from Lemma 5.10 that there exists a compact subset L with $g(K) \subset \mathring{L} \subset L \subset f^{-1}(U)$. It is clear that $U = \mathring{L}$ and L have the desired properties. ■

The following corollary gives us a very long list of new examples of topological groups.

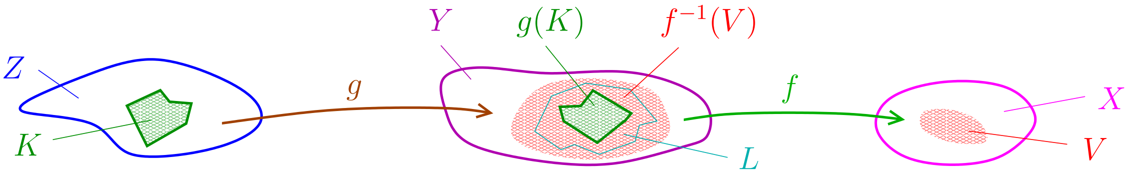


FIGURE 117. Illustration for the proof of Proposition 5.9.

Corollary 5.11. *Let X be a topological space. If X is compact and Hausdorff, then the group*

$$\mathrm{Homeo}(X) := \text{the set of all self-homeomorphisms of } X,$$

equipped with the subspace topology coming from X^X, is a topological group.

Remark. Let X be a topological space that is regionally compact and Hausdorff. In light of Proposition 5.9 one might expect that in this case $\mathrm{Homeo}(X)$ forms a group. In [**Dij05**] it is shown that this is in general *not* the case. On the other hand it is shown in [**Are46**, Theorem 4] that if X is furthermore regionally connected, then $\mathrm{Homeo}(X)$ is in fact a topological group.

PROOF (∗). Let X be a topological space that is compact and Hausdorff. First note that by Lemma 2.73 we know that X is regionally compact. Thus it follows from Proposition 5.9 that the multiplication map $\mathrm{Homeo}(X)\times\mathrm{Homeo}(X) \to \mathrm{Homeo}(X)$ is continuous. Therefore it suffices to show that the map

$$\begin{array}{rcl} \Phi\colon \mathrm{Homeo}(X) & \to & \mathrm{Homeo}(X) \\ f & \mapsto & f^{-1} \end{array}$$

is continuous. The proof of continuity rests on the following simple claim.

Claim. If $K \subset X$ and $U \subset X$ are subsets, then $\Phi^{-1}(M(K,U)) = M(X \setminus U, X \setminus K)$.

Let $h \in \mathrm{Homeo}(X)$. We have

$$\begin{array}{rlll} h \in \Phi^{-1}(M(K,U)) & \overset{\substack{\text{since } \Phi \text{ is a bijection and } \Phi(h)=h^{-1} \\ \downarrow}}{\Longleftrightarrow} h^{-1} \in M(K,U) & \Longleftrightarrow h^{-1}(K) \subset U & \overset{\substack{\text{since } h \text{ is a bijection} \\ \downarrow}}{\Longleftrightarrow} K \subset h(U) \\ & \Longleftrightarrow \underbrace{X \setminus h(U)}_{=h(X\setminus U)} \subset X \setminus K & \Longleftrightarrow h \in M(X \setminus U, X \setminus K). & \end{array}$$

⊞

By Proposition 2.37 it suffices to prove the following claim.

Claim. If $K \subset X$ is a compact subset and if $U \subset X$ is an open subset, then $\Phi^{-1}(M(K,U))$ is an open subset of X.

By the previous claim we know that $\Phi^{-1}(M(K,U)) = M(X \setminus U, X \setminus K)$. Since U is open we know that $X \setminus U$ is closed. Since X is compact we obtain from Lemma 2.17 (1) that $X \setminus U$ is compact. Furthermore, since K is compact and since X is Hausdorff we obtain from Lemma 2.17 (2) that $X \setminus K$ is open. Thus we see that $M(X \setminus U, X \setminus K)$ is an open subset of X^X. ■

We conclude this section with a different source of topological groups.

Lemma 5.12. *Let X be a topological space.*

(1) *Let $\widetilde{X}$ and Y be topological spaces. If Y is regionally compact, then the map*
$$\begin{array}{rcl}\Psi\colon X^Y\times\widetilde{X}^Y & \to & (X\times\widetilde{X})^Y\\ (f,\widetilde{f}) & \mapsto & \left(\begin{array}{rcl} f\times\widetilde{f}\colon Y & \to & X\times\widetilde{X}\\ y & \mapsto & (f(y),\widetilde{f}(y))\end{array}\right)\end{array}$$
is continuous.

(2) *Let G be a topological group. If G is regionally compact, then the topological space G^X together with the multiplication*
$$\begin{array}{rcl}G^X\times G^X & \to & G^X\\ (f,g) & \mapsto & \left(\begin{array}{rcl} X & \to & G\\ x & \mapsto & f(x)\cdot g(x)\end{array}\right)\end{array}$$
is a topological group.

Proof.

(1) By Proposition 2.37 it suffices to show that for any compact subset $K\subset Y$ and any open subset $W\subset X\times\widetilde{X}$ the preimage $\Psi^{-1}\big(M_{(X\times\widetilde{X})^Y}(K,W)\big)$ is an open subset of $X^Y\times\widetilde{X}^Y$. We will do so by appealing to Lemma 2.5.

Thus let $(f,\widetilde{f})\in\Psi^{-1}(M_{(X\times\widetilde{X})^Y}(K,W))$. Note that this means that we are given $f\colon Y\to X$ and $\widetilde{f}\colon Y\to\widetilde{X}$ with $(f\times\widetilde{f})(K)\subset W$. For each $y\in K$ we can and will pick open subsets U_y of X and $\widetilde{U}_y$ of $\widetilde{X}$ such that $(f\times\widetilde{f})(y)\in U_y\times\widetilde{U}_y\subset W$. By Lemma 3.6 (1) we know that $(f\times\widetilde{f})^{-1}(U_y\times\widetilde{U}_y)$ is an open subset of Y. Since Y is regionally compact there exists a compact neighborhood K_y of y such that $y\in K_y\subset(f\times\widetilde{f})^{-1}(U_y\times\widetilde{U}_y)$.

Since K is compact we obtain from Lemma 2.15 that there exist $y_1,\dots,y_m\in Y$ such that $K\subset K_{y_1}\cup\dots\cup K_{y_m}$. Now we see that
$$(f,\widetilde{f})\underset{\substack{\uparrow\\ \text{since }(f\times\widetilde{f})(K_{y_i})\subset U_{y_i}\times\widetilde{U}_{y_i}}}{\in}\bigcap_{i=1}^m M_{X^Y}(K_{y_i},U_{y_i})\times M_{\widetilde{X}^Y}(K_{y_i},\widetilde{U}_{y_i})\underset{\substack{\uparrow\\ \text{since }U_{y_i}\times\widetilde{U}_{y_i}\subset W\text{ and since }K\subset\bigcup K_{y_i}}}{\subset}\Psi^{-1}\big(M_{(X\times\widetilde{X})^Y}(K,W)\big)\subset X^Y\times\widetilde{X}^Y.$$

It follows from Lemma 2.5 that $\Psi^{-1}(M_{(X\times\widetilde{X})^Y}(K,U))$ is indeed an open subset of $X^Y\times\widetilde{X}^Y$. We refer to Figure 118 for an illustration.

(2) At this stage there is not much left that needs to be done, we outsource the argument to Exercise 5.4. ■

5.2. **Mixing product and quotient topologies** (∗)**.** In this, rather technical section, we will see that mixing product and quotient topologies can be quite delicate. We start out with the following elementary lemma.

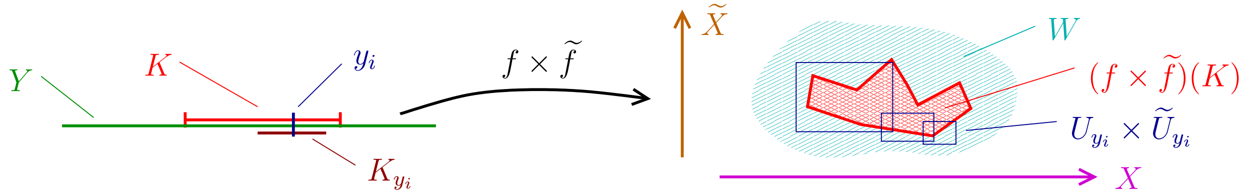


FIGURE 118. Illustration for the proof of Lemma 5.5 (6).

Lemma 5.13. *Let X and Y be two topological spaces. Suppose we are given an equivalence relation $\sim$ on X. We denote by $\sim_\times$ the equivalence relation on $X \times Y$ that is generated by $(x, y) \sim_\times (x', y)$ whenever $x \sim x'$. The following map is a bijection and it is continuous:*

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\sim_\times & \to & (X/\sim) \times Y \\ [(x,y)] & \mapsto & ([x], y). \end{array}$$

PROOF ($*$). It follows basically immediately from the definition of $\sim_\times$ that the map φ is a bijection. We will walk cautiously through the continuity argument.

We denote by $p\colon X \times Y \to (X \times Y)/\sim_\times$ and $q\colon X \to X/\sim$ the two obvious projection maps. We have the following commutative diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q \times \mathrm{id}_Y"] & \\ (X \times Y)/\sim_\times \arrow[rr, "\varphi"] & & (X/\sim) \times Y. \end{tikzcd}$$

Let $W \subset (X/\sim) \times Y$ be an open subset. We need to show that $\varphi^{-1}(W)$ is an open subset of $(X \times Y)/\sim_\times$. By definition of the quotient topology we need to show that $p^{-1}(\varphi^{-1}(W))$ is an open subset of $X \times Y$.

By Lemma 2.5 and the definition of the product topology it suffices to prove the following claim.

Claim. For each $(x, y) \in p^{-1}(\varphi^{-1}(W)) \subset X \times Y$ there exists an open neighborhood U_x of $x \in X$ and an open neighborhood V_y of $y \in Y$ such that $U_x \times V_y \subset p^{-1}(\varphi^{-1}(W))$.

Since $([x], y) \in W$ and since W is an open subset of $(X/\sim) \times Y$ we obtain, by definition of the product topology, an open neighborhood $\widetilde{U}_{[x]}$ of $[x] \in X/\sim$ and an open neighborhood V_y of $y \in Y$ with $\widetilde{U}_{[x]} \times V_y \subset W$. We set $U_x := q^{-1}(\widetilde{U}_{[x]})$. By definition of the quotient topology this is an open subset of X. It remains to show that $U_x \times V_y \subset p^{-1}(\varphi^{-1}(W))$. In other words, we need to show that $\varphi(p(U_x \times V_y)) \subset W$. This inclusion follows from the following short argument:

$$\varphi(p(U_x \times V_y)) = (\varphi \circ p)(U_x \times V_y) \underset{\substack{\uparrow \\ \text{by the above commutative triangle}}}{=} (q \times \mathrm{id}_Y)(U_x \times V_y) \underset{\substack{\uparrow \\ \text{since } U_x = q^{-1}(\widetilde{U}_{[x]})}}{=} q(q^{-1}(\widetilde{U}_{[x]})) \times V_y = \widetilde{U}_{[x]} \times V_y \subset W.$$

This concludes the proof of the claim and thus also of the lemma. ∎

One would expect that the map from Lemma 5.13 surely must be a homeomorphism. But perhaps rather shockingly, the next lemma shows that in general this is not the case:

Lemma 5.14. *There exist topological spaces X and Y and an equivalence relation $\sim$ on X with the following property: If we denote by $\sim_\times$ the equivalence relation on $X \times Y$ generated by $(x, y) \sim_\times (x', y)$ whenever $x \sim x'$, then the map*

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\sim_\times & \to & (X/\sim) \times Y \\ [(x, y)] & \mapsto & ([x], y) \end{array}$$

is a bijection and it is continuous, but it is not *a homeomorphism.*

Remark. We refer to [**Fab05**, Theorem 1], [**BrownR06**, p. 111] and [**Bou98**, p. 128] for other examples which also show that mixing product and quotient topologies is in general a very dangerous idea.

Proof. We consider $X = \mathbb{R} \times \mathbb{N} = \{(x, n) \,|\, x \in \mathbb{R}, n \in \mathbb{N}\}$, equipped with the product topology. Let $\sim$ be the equivalence relation on X that is generated by $(0, m) \sim (0, n)$ for all $m, n \in \mathbb{N}$. We set $Y := \prod_{i \in \mathbb{N}} \mathbb{R}$, equipped with the product topology introduced on page 172. By Lemma 5.13 we know that the map

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\sim_\times & \to & (X/\sim) \times Y \\ [(x, y)] & \mapsto & ([x], y) \end{array}$$

is a bijection and that it is continuous. It remains to show that φ is not a homeomorphism. We will do so by exhibiting an open subset V of $(X \times Y)/\sim_\times$ such that the image $\varphi(V)$ is not an open subset of $(X/\sim) \times Y$.

Before we can proceed we need to introduce the following notation:

(1) We denote by $q\colon X \to X/\sim$ the obvious projection.
(2) We write $f = q \times \mathrm{id}_Y\colon X \times Y \to (X/\sim) \times Y$.
(3) Given $n \in \mathbb{N}$ we set

$$U_n \;:=\; \{((t, n), (y_1, y_2, y_3, \dots)) \in X \times Y \,|\, |t \cdot y_n| < 1\}.$$

and we set $U := \bigcup_{n \in \mathbb{N}} U_n$.

The key to proving the lemma is the following claim.

Claim. The set U has the following properties:

(1) U is an open subset of $X \times Y$,
(2) $f^{-1}(f(U)) = U$,
(3) $f(U)$ is not open in $(X/\sim) \times Y$.

We set out to prove that U does indeed have the three promised properties.

(1) We want to show that U is an open subset of $X \times Y$. It suffices to show that each U_n is open. First note that the map

$$\begin{array}{rcl} \gamma\colon X \times Y = \mathbb{R} \times \mathbb{N} \times \prod_{i \in \mathbb{N}} \mathbb{R} & \to & \mathbb{R} \\ (t, n, y_1, y_2, \dots) & \mapsto & t \cdot y_n \end{array}$$

is continuous by Lemma 3.14. Furthermore note that $\{n\}$ is an open subset of $\mathbb{N}$. Now we see that

$$U_n \;=\; \gamma^{-1}((-1,1)) \cap \underbrace{\mathbb{R}\times\{n\}\times\prod_{i\in\mathbb{N}}\mathbb{R}}_{\text{open by definition of the product topology}}$$

is the intersection of two open sets, thus it is open itself.

(2) We need to show $f^{-1}(f(U)) = U$. By Lemma 1.3 (1) it remains to show that $f^{-1}(f(U)) \subset U$. In other words, we need to show that if v is an element of $X\times Y$ with $f(v)\in f(U)$, then $v\in U$. But this statement follows from the following claim.

Claim. Let $v = ((s,m),(y_1,y_2,\dots))$ and $w = ((t,n),(z_1,z_2,\dots))$ be two elements in $X\times Y$. If $f(v) = f(w)$ and if $w\in U$, then we also have $v\in U$.

If $v = w$, then there is nothing to show. On the other hand, if $v\neq w$, then it follows from the definition of the map f and the equivalence relation $\sim$ that $s=t=0$. But this implies that v and w lie in every U_n, in particular they both lie in U. ⊞

(3) We prove this statement by contradiction. Thus suppose that $f(U)$ is, contrary to our hopes, an open subset of $(X/\sim)\times Y$. Note that $((0,0),(0,0,\dots))\in U$. This implies that $f((0,0),(0,0,\dots)) = ([(0,0)],(0,0,\dots))$ is contained in $f(U)$. By definition of the product topology of $(X/\sim)\times Y$ and of the product topology of $Y = \prod_{i\in\mathbb{N}}\mathbb{R}$ there exists an open subset $V\subset X/\sim$ and open subsets $W_i\subset\mathbb{R}$ with $([(0,0)],(0,0,\dots))\in V\times\prod_{i\in\mathbb{N}}W_i\subset f(U)$ and such that all but finitely W_i are equal to $\mathbb{R}$. We pick an $N\in\mathbb{N}$ with $W_N=\mathbb{R}$. Since $[(0,0)]\in V$ we see that $q^{-1}(V)$ is an open subset of $X=\mathbb{R}\times\mathbb{N}$ that contains the subset $q^{-1}([(0,0)]) = \{0\}\times\mathbb{N}$. In particular $q^{-1}(V)$ contains some point (s,N) with $s\neq 0$. We pick a real number $y_N > \frac{1}{|s|}$ and we consider the point

$$P \;:=\; ((s,N),(0,\dots,0,\underset{\substack{\uparrow\\ \text{the }N\text{-th entry}}}{y_N},0,\dots)) \;\in X\times Y.$$

Note that

$$P \underset{\substack{\uparrow\\ \text{since }(s,N)\in q^{-1}(V)\\ \text{and since }W_N=\mathbb{R}}}{\in} q^{-1}(V)\times\prod_{i\in\mathbb{N}}W_i \underset{\substack{\uparrow\\ \text{since }f=q\times\mathrm{id}_Y}}{=} f^{-1}\Big(V\times\prod_{i\in\mathbb{N}}W_i\Big) \underset{\substack{\uparrow\\ \text{see above}}}{\subset} f^{-1}(f(U)) \underset{\substack{\uparrow\\ \text{by (2)}}}{=} U = \bigcup_{n\in\mathbb{N}}U_n.$$

By definition of the U_n's we see that the only possibility is that $P\in U_N$. But we have $|s\cdot y_N|>1$, i.e. P cannot possibly lie in U_N. We have thus obtained a contradiction. ⊞

It remains to prove the following claim.

Claim. We set $V := p(U)$.

(1) V is an open subset of $(X\times Y)/\sim_\times$,
(2) $\varphi(V)$ is not an open subset of $(X/\sim)\times Y$.

We denote by $p\colon X \times Y \to (X \times Y)/\sim_\times$ the obvious projection map. We have the following commutative diagram

$$\begin{tikzcd} X \times Y \arrow[rr, "f=q\times \mathrm{id}_Y"] \arrow[dr, "p"'] & & (X/\sim) \times Y \\ & (X \times Y)/\sim_\times . \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Note that

$$p^{-1}(V) \;=\; p^{-1}(p(U)) \underset{\substack{\uparrow \\ \text{since } p=\varphi^{-1}\circ f \text{ and since } \varphi \text{ is a bijection}}}{=} f^{-1}(f(U)) \underset{\uparrow}{=} U \underset{\uparrow}{=} \text{an open subset of } X \times Y.$$
$$\text{by the previous claim}$$

By the definition of the quotient topology on $(X \times Y)/\sim_\times$ this means that V is indeed an open subset of $(X \times Y)/\sim_\times$. Furthermore we have $\varphi(V) = \varphi(p(U)) = f(U)$. But by the previous claim we know that $f(U)$ is not an open subset of $(X/\sim) \times Y$. This concludes the proof of the claim and thus of the lemma. ■

The previous lemma leads us to the following cautionary remark:

Remark.

In general products and quotients of topological spaces do not mix well.

Definition. We say that a map $f\colon X \to Y$ between topological spaces is a *quotient map* if f is surjective and if a subset $U \subset Y$ is open in Y if and only if the preimage $f^{-1}(U)$ is open in X.

Examples.

(1) We consider the map $f\colon \mathbb{R} \to \mathbb{R}$ whose graph is shown in Figure 119. This map is continuous, it is *not* an open map, but the reader is invited to verify that the map is in fact a quotient map.

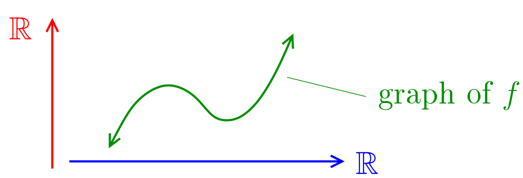


FIGURE 119

(2) Let $X = \{0, 1\}$ be a set of two elements $0, 1$. Let $\mathcal{S} = \{\varnothing, X\}$ be the trivial topology and let $\mathcal{T} = \{\varnothing, X, \{0\}, \{1\}\}$ be the discrete topology. Evidently the identity map $(X, \mathcal{T}) \to (X, \mathcal{S})$ is a continuous bijection. But clearly it is not a quotient map.

The following lemma summarizes a few elementary properties of quotient maps.

Lemma 5.15.

(1) *Let X be a topological space and let $\sim$ be an equivalence relation on X. The projection map $X \to X/\sim$ is a quotient map.*

(2) *If $f\colon X \to Y$ is a quotient map, then it is also continuous.*

(3) *If $f\colon X \to Y$ is a bijection between topological spaces and if it is a quotient map, then f is a homeomorphism.*

Proof.

(1) This statement follows immediately from the definition of the quotient topology on the quotient $X/\sim$.
(2) Let $f\colon X \to Y$ be a quotient map. If follows immediately from the "only if" bit of the definition of a quotient map that f is continuous.
(3) This statement follows immediately from (2) and Lemma 2.42 (2). ■

The following is the key technical theorem which in most reasonable situations will help us to navigate the choppy waters involving product and quotient maps.

Theorem 5.16. *Let $f\colon X \to Z$ be a map between topological spaces that is a quotient map. Furthermore let Y be a topological space. If Y is regionally compact (e.g. if $Y = [0,1]$), then the map*

$$f \times \mathrm{id}_Y\colon X \times Y \;\to\; Z \times Y$$

is also a quotient map.

Remark. This theorem is sometimes called the "Whitehead Theorem" since it was first proved by John Whitehead [**WhdJ48**, Lemma 4] in 1948. We reserve the name "Whitehead Theorem" for Theorem 119.9.

We will provide the proof of Theorem 5.16 in the next section. In the following we first state and prove some useful consequences of Theorem 5.16. In fairness to the reader one should perhaps say that it is arguably best to move on to the next chapter. It is only advisable to read the remainder of this section once one really faces acute technical problems involving the mix of product and quotient maps.

The following little lemma is the key to many applications of Theorem 5.16.

Lemma 5.17. *Suppose we are given a commutative diagram of maps between topological spaces:*

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "f"] & \\ Y \arrow[rr, "\varphi"'] & & Z. \end{tikzcd}$$

The following two statements hold:

(1) *If p and f are quotient maps, then φ is also a quotient map.*
(2) *If p is a quotient map and if f is continuous, then φ is continuous.*

Proof.

(1) Suppose that p and f are quotient maps. Let $V \subset Z$ be an open subset. We have the following implications:

$$V \text{ is open } \underset{\substack{\uparrow\\ \text{since } f \text{ is a quotient}\\ \text{map}}}{\Leftrightarrow} f^{-1}(V) \text{ is open } \underset{\substack{\uparrow\\ \text{since } f=\varphi\circ p \text{ we have}\\ f^{-1}(V)=p^{-1}(\varphi^{-1}(V))}}{\Leftrightarrow} p^{-1}(\varphi^{-1}(V)) \text{ is open } \underset{\substack{\uparrow\\ \text{since } p \text{ is a quotient map and}\\ \text{since } p(p^{-1}(\varphi^{-1}(V)))=\varphi^{-1}(V)}}{\Leftrightarrow} \varphi^{-1}(V) \text{ is open.}$$

Furthermore, it follows from the fact that the diagram commutes and the fact that f is surjective, that φ is surjective. We have thus verified that φ is a quotient map.

(2) Now we suppose that p is a quotient map and that f is continuous. We need to show that given an open subset $V \subset Z$ the preimage $\varphi^{-1}(V)$ is open. The argument is basically exactly the same as in (1), except that the first arrow is only a right arrow, since we only assume that f is continuous. ■

We have the following addendum to Lemma 5.13.

Lemma 5.18. *Let X and Y be two topological spaces. Suppose we are given an equivalence relation $\sim$ on X. We denote by $\sim_\times$ the equivalence relation on $X \times Y$ that is generated by $(x, y) \sim_\times (x', y)$ whenever $x \sim x'$.* If Y is regionally compact, *then the following map is a homeomorphism:*

$$\begin{array}{rcl} \varphi\colon (X \times Y)/\sim_\times & \to & (X/\sim) \times Y \\ [(x, y)] & \mapsto & ([x], y). \end{array}$$

Proof (∗). As we pointed out in the proof of Lemma 5.13, it is basically clear that φ is a bijection. By Lemma 5.15 it remains to show that φ is a quotient map. We denote by $p\colon X \times Y \to (X \times Y)/\sim_\times$ and $q\colon X \to X/\sim$ the two obvious projection maps. We have the following commutative diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "f=q\times \mathrm{id}_Y"] & \\ (X \times Y)/\sim_\times \arrow[rr, "\varphi"] & & (X/\sim) \times Y. \end{tikzcd}$$

By Lemma 5.15 (1) we know that p and q are quotient maps. Since Y is regionally compact we obtain from Theorem 5.16 that $q \times \mathrm{id}_Y$ is a quotient map. It follows from Lemma 5.17 (1) that φ is also a quotient map. ■

Later on we will use the following special case of Lemma 5.18.

Lemma 5.19. *Let X and Y be two topological spaces, let $A \subset X$ be a subset and let $f\colon A \to Y$ be a map. Using the notation from page 200, we see that the map*

$$\begin{array}{rcl} (X \times [0,1]) \cup_{f \times \mathrm{id}_{[0,1]}} (Y \times [0,1]) & \to & (X \cup_f Y) \times [0,1] \\ [(P, t)] & \mapsto & ([P], t) \end{array}$$

is a homeomorphism.

Proof. Once one unravels the definitions it is clear that this statement follows immediately from Lemma 5.18 together with the observation that $[0, 1]$ is regionally compact. ■

Now we can also finally prove the following proposition which says that the join of topological spaces is associative, at least for reasonable topological spaces.

Proposition 3.52. *Let X, Y and Z be non-empty topological spaces that are compact and Hausdorff. Then there exists a homeomorphism*

$$(X * Y) * Z \xrightarrow{\cong} X * (Y * Z).$$

Remark. Recall that we pointed out on page 210 that in general the join is not associative. In light of Lemma 5.14 this is perhaps no longer that shocking.

Proof. We define the join $X * Y * Z$ as follows: we consider

$$\{(r,x,s,y,t,z) \in [0,1] \times X \times [0,1] \times Y \times [0,1] \times Z \,|\, r+s+t=1\}/\sim$$

where "$\sim$" is generated by the following relations:

(1) $(0,x,s,y,t,z) \sim (0,x',s,y,t,z)$,
(2) $(r,x,0,y,t,z) \sim (r,x,0,y',t,z)$,
(3) $(r,x,s,y,0,z) \sim (r,x,s,y,0,z')$.

In the following we will use "$\sim$" with several means, we leave it to the reader to figure out any given moment what is meant by this symbol.

Claim. The map

$$\begin{array}{rcl} \Theta\colon \overbrace{(((X \times [0,1] \times Y)/\sim) \times [0,1] \times Z)/\sim}^{=(X*Y)*Z} & \to & \overbrace{(X \times [0,1] \times Y * [0,1] \times Z)/\sim}^{=X*Y*Z} \\ [([(x,s,y)],t,z)] & \mapsto & [((1-t)(1-s),x,(1-t)s,y,t,z)] \end{array}$$

is a homeomorphism.

We consider the following diagram

$$\begin{tikzcd}
(X \times [0,1] \times Y) \times ([0,1] \times Z) \arrow[r, "\mathrm{id}"] \arrow[d, two heads] & (X \times [0,1] \times Y) \times ([0,1] \times Z) \arrow[dd, two heads] \\
((X \times [0,1] \times Y) \times ([0,1] \times Z)/\sim \arrow[dr, "\Phi"] & \\
((X \times [0,1] \times Y)/\sim \times([0,1] \times Z) \arrow[u, "\varphi"', "{([u],v)\mapsto[(u,v)]}"] \arrow[r, "\Psi"] \arrow[d, two heads] & X * Y * Z. \\
(((X \times [0,1] \times Y)/\sim) \times [0,1] \times Z)/\sim \arrow[ur, "\Theta"'] &
\end{tikzcd}$$

We make the following clarifications and observations:

(1) All the maps are the obvious (projection) maps, except for φ which is the map given by Lemma 5.14. The diagram commutes basically by definition.
(2) It follows from Lemma 3.28 that Φ is continuous.
(3) Our hypothesis that Z is Hausdorff and compact implies by Proposition 3.12 together with Lemma 2.73 that $[0,1] \times Z$ is regionally compact. Thus we obtain from Lemma 5.18 that φ is actually a homeomorphism.
(4) It follows from (2) and (3) that Ψ is continuous.
(5) It follows from (4) together with Lemma 3.22 that Θ is continuous.
(6) It follows easily from the definitions that Θ is a bijection.
(7) By our hypotheses on X, Y and Z we obtain from Proposition 3.12 and Lemma 2.40 that the domain $(X*Y)*Z$ of the map Θ is compact. Furthermore, by an argument as in Exercise 3.48 one sees that the target $X*Y*Z$ of the map Θ is Hausdorff.

(8) It follows from (5), (6) and (7) together with Proposition 2.43 (3) that Θ is a homeomorphism. ⊞

Basically the same argument as in the above claim also shows that there exists a homeomorphism $X*(Y*Z) \to X*Y*Z$. In summary we see that there exists a homeomorphism $X*(Y*Z) \to (X*Y)*Z$. ■

We conclude this list of applications of Theorem 5.16 with the following proposition that gives a fairly explicit illustration of how Theorem 5.16 can be used.

Proposition 5.20. (∗) *Let X be a topological space, let A be a subset of X and let Y be a topological space that is regionally compact. Then the map*

$$\begin{array}{rcl} \varphi\colon (X/A)\times Y & \to & (X\times Y)/(A\times Y) \\ ([x],y) & \mapsto & [(x,y)] \end{array}$$

is a quotient map, in particular it is continuous.

Example. In Figure 120 we illustrate the map studied in Proposition 5.20 in the case that $X=\mathbb{R}$, $A=[a,b]$ and $Y=[0,1]$.

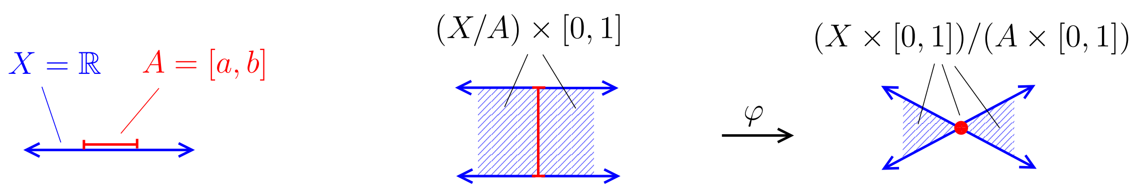


Figure 120

Proof (∗). We denote by $p\colon X\times Y \to (X\times Y)/(A\times Y)$ and $q\colon X\to X/A$ the two obvious projection maps. We have the following commutative diagram

$$\begin{tikzcd} & X\times Y \arrow[dl, "q\times \mathrm{id}_Y"'] \arrow[dr, "p"] & \\ (X/A)\times Y \arrow[rr, "\varphi"] & & (X\times Y)/(A\times Y). \end{tikzcd}$$

By Lemma 5.15 the maps p and q are quotient maps. Since Y is regionally compact we obtain from Theorem 5.16 that $q\times \mathrm{id}_Y$ is a quotient map. It follows from Lemma 5.17 that φ is also a quotient map. By Lemma 5.15 (2) this implies that φ is indeed continuous. ■

5.3. **Proof of Theorem 5.16** (∗)**.** Now that we procrastinated for as long as possible, we will finally run out of reasons not to turn to the proof of Theorem 5.16. In fact we provide two proofs.

The first proof of Theorem 5.16 cleverly reduces the proof to Proposition 5.6. This approach pushes all the set-theoretic nastiness into the proof of Proposition 5.6. The second proof of Theorem 5.16 is a bare-hands argument which can be read independently of the previous section on the compact-open topology.

First proof of Theorem 5.16, using Proposition 5.6. We write $W = Z \times Y$ and we equip W with the topology that turns

$$\begin{array}{rcl} g\colon X \times Y & \to & W \\ (x,y) & \mapsto & (f(x),y) \end{array}$$

into a quotient map. In other words, as a set we have $W = Z \times Y$, but this time equipped with the topology that is defined by the condition that $U \subset W$ is open if and only if $g^{-1}(U)$ is open. Note that with this topology on W the map g is in particular continuous. Next we consider the following diagram

$$\begin{tikzcd} X \times Y \arrow[rr, "f\times \mathrm{id}_Y"] \arrow[dr, "g(x,y)=(f(x),y)"'] & & Z \times Y \arrow[dl, "h(z,y)=(z,y)"] \\ & W & \end{tikzcd}$$

where $h\colon Z \times Y \to W$ is the identity. Since g is by definition a quotient map it remains to show that h is a homeomorphism. Evidently h is a bijection. Furthermore note that every open set in $Z \times Y$ is open in W since $f \times \mathrm{id}_Y$ is continuous. Thus it remains to show that h is continuous.

We consider the maps

$$\begin{array}{rcl} \widehat{g}\colon X & \to & W^Y \\ x & \mapsto & \left(\begin{array}{rcl} Y & \to & W \\ y & \mapsto & g(x,y) \end{array}\right) \end{array} \quad \text{and} \quad \begin{array}{rcl} \widehat{h}\colon Z & \to & W^Y \\ z & \mapsto & \left(\begin{array}{rcl} Y & \to & W \\ y & \mapsto & h(z,y) \end{array}\right) \end{array}$$

These maps fit into the following commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "\widehat{g}"] & \\ Z \arrow[rr, "\widehat{h}"'] & & W^Y. \end{tikzcd}$$

We make the following observations:

(1) Since $g\colon X \times Y \to W$ is continuous and since Y is by hypothesis regionally compact we obtain from Proposition 5.6 that $\widehat{g}\colon X \to W^Y$ is continuous.
(2) By our hypothesis the map f is a quotient map.
(3) It follows from (1) and (2), together with Lemma 5.17 (2), that the map $\widehat{h}\colon Z \to W^Y$ is continuous.
(4) It follows from Proposition 5.6 and (3) that h is indeed continuous. ■

Our second proof of Theorem 5.16 is independent of earlier results. But it does mean that we will have to get our hands somewhat dirty.

Definition. Let $f\colon X \to Y$ be a map between two sets. We say $U \subset X$ is *saturated with respect to* f if $U = f^{-1}(f(U))$. If f is understood, then we just say U *is saturated.*

Remark. Let $f\colon X \to Y$ be a map between two sets.

(1) Let $U \subset Y$ be a subset. By Lemma 1.3 (1) we know that $U \subset f^{-1}(f(U))$. Thus U is saturated if and only if $f^{-1}(f(U)) \subset U$.

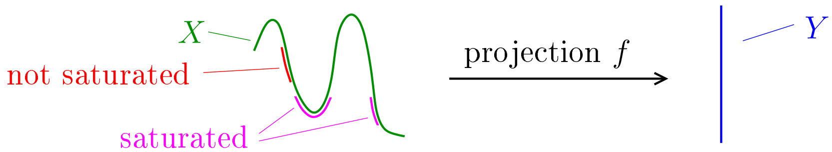


FIGURE 121

(2) If f is an injection, then it follows from Lemma 1.3 (1) that every subset of X is saturated.

The following is, besides Lemma 1.3, the only purely set-theoretic lemma of these notes.

Lemma 5.21. *Let $f\colon X \to Z$ be a map between two sets and let Y be another set. Let $\widetilde{W}$ be a subset of $X \times Y$ that is saturated with respect to the map $f \times \mathrm{id}_Y\colon X \times Y \to Z \times Y$. Let $K \subset Y$. We consider the set*

$$U \;:=\; \{x' \in X \,|\, \{x'\} \times K \subset \widetilde{W}\}.$$

The set U is saturated with respect to f.

PROOF. We write $h = f \times \mathrm{id}_Y$. By the above we need to show that $f^{-1}(f(U)) \subset U$. Thus let $x \in f^{-1}(f(U)) \subset X$. We have

$$h(\{x\}\times K) \underset{\substack{\uparrow\\ \text{definition of } h}}{=} (f\times\mathrm{id}_Y)(\{x\}\times K) = \{f(x)\}\times K \underset{\substack{\uparrow\\ \text{since } f(x)\in f(U) \text{ we know that}\\ f(x)=f(x') \text{ for some } x'\in U}}{=} \{f(x')\}\times K \underset{\substack{\uparrow\\ \text{definition of}\\ h=f\times\mathrm{id}_Y}}{=} h(\{x'\}\times K) \underset{\substack{\uparrow\\ \text{since } x'\in U}}{\subset} h(\widetilde{W}).$$

It follows that

$$\{x\} \times K \underset{\substack{\uparrow\\ \text{Lemma 1.3 (1)}}}{\subset} h^{-1}(h(\{x\} \times K) \underset{\substack{\uparrow\\ \text{by the above argument}}}{\subset} h^{-1}(h(\widetilde{W})) \underset{\substack{\uparrow\\ \text{since } \widetilde{W} \text{ is saturated}}}{=} \widetilde{W}.$$

We have thus shown that $x \in U$. ■

We will also need the following lemma which is again in the realm of topological spaces. It can be viewed as a minuscule generalization of the Tube Lemma from Exercise 3.6.

Lemma 5.22. (∗) *Let X and Y be topological spaces, let $x \in X$ and let $K \subset Y$ be a compact subset. Let $\widetilde{W} \subset X \times Y$ be an open subset that contains $\{x\} \times K$. There exist open neighborhoods $U \subset X$ of x and $V \subset Y$ of K such that $\{x\} \times K \subset U \times V \subset \widetilde{W}$.*

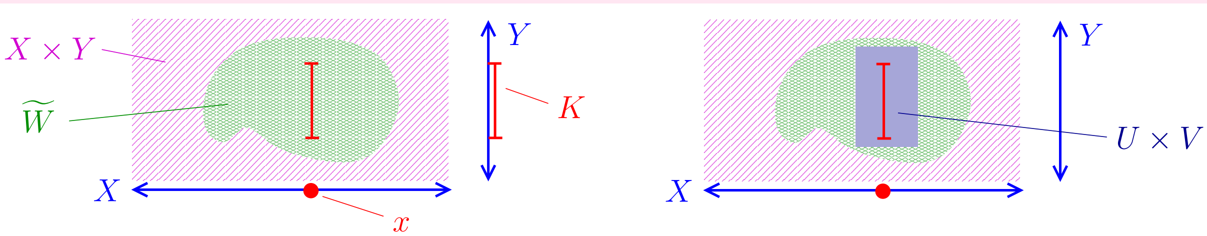


FIGURE 122

Proof of Lemma 5.22 (∗). Since $\widetilde{W} \subset X \times Y$ is open and since $\{x\} \times K \subset \widetilde{W}$ we can find for any $k \in K$ open neighborhoods A_k of $x \in X$ and B_k of $k \in Y$ such that $A_k \times B_k \subset \widetilde{W}$. Since K is compact there exist $k_1, \dots, k_m \in K$ such that $K \subset V := B_{k_1} \cup \dots \cup B_{k_m}$. We define $U := U_{k_1} \cap \dots \cap U_{k_m}$. Since U is the intersection of finitely many open sets each of which contain x we see that U is an open neighborhood of x. Clearly we have the inclusions $\{x\} \times K \subset U \times V \subset \widetilde{W}$. ■

For convenience we remind the reader of the following lemma.

Lemma 2.5. *Let X be a topological space and let $U \subset X$ be a subset. If given any $x \in U$ there exists a neighborhood V of x that is contained in U, then U is open.*

Now we can provide our alternative proof of Theorem 5.16.

Second proof of Theorem 5.16. Let $f\colon X \to Z$ be a map between topological spaces that is a quotient map. Furthermore let Y be a topological space that is regionally compact. We need to show that the map

$$f \times \mathrm{id}_Y\colon X \times Y \ \to\ Z \times Y$$

is also a quotient map.

Since f is a quotient map we know by Lemma 5.15 (2) that f is continuous. Therefore it follows from Lemma 3.8 (2b) that the map $f \times \mathrm{id}_Y$ is continuous. Thus it remains to show that if $W \subset Z \times Y$ is a subset such that the preimage $\widetilde{W} := (f \times \mathrm{id}_Y)^{-1}(W)$ is an open subset of $X \times Y$, then W itself is open.

We intend to use Lemma 2.5 to show that W is open. Thus let $(z, y) \in W$. We pick $x_0 \in f^{-1}(\{z\})$. We equip $\{x_0\} \times Y \subset X \times Y$ with the subspace topology. By Lemma 3.8 (2a) the map

$$\begin{array}{rcl} i\colon Y & \to & \{x_0\} \times Y \\ y & \mapsto & (x_0, y) \end{array}$$

is an embedding. In particular $Y_{x_0} := i^{-1}(\widetilde{W} \cap (\{x_0\} \times Y))$ is an open subset of Y. Note that $\{x_0\} \times Y_{x_0} \subset \widetilde{W}$. Since Y is regionally compact we can find a compact neighborhood K_{x_0} of x_0 that is contained in Y_{x_0}. By definition of a neighborhood there exists an open subset Y'_{x_0} of Y with $x_0 \in Y'_{x_0} \subset K_{x_0}$. We define

$$U_{x_0} \ := \ \{x \in X \,|\, \{x\} \times K_{x_0} \subset \widetilde{W}\}.$$

Claim. The set $f(U_{x_0})$ is an open subset of Z.

First we intend to use Lemma 2.5 to show that U_{x_0} is indeed an open subset of Y_{x_0}. Thus let $x \in U_{x_0}$. Since $\widetilde{W}$ is an open subset of $X \times Y$ and since K_{x_0} is compact it follows from Lemma 5.22 that there exist open subsets $A \subset X$ and $B \subset Y$ with $x \in A$ and $K_{x_0} \subset B$ such that $A \times B \subset \widetilde{W}$. Evidently this implies that $A \subset U_{x_0}$. Since $x \in A$ we have verified that the openness criterion of Lemma 2.5 is satisfied. We have thus shown that U_{x_0} is an open subset of Y_{x_0}.

Now we want to show that $V := f(U_{x_0})$ is an open subset of Z. By Lemma 5.21 we know that U_{x_0} is saturated with respect to f. In other words, we know that $U_{x_0} = f^{-1}(V)$.

Now we use that by hypothesis $f\colon X \to Z$ is a quotient map. Since $f^{-1}(V) = U_{x_0}$ is open by (1) we obtain that V itself is open. But that is what we had wanted to show. ⊞

By definition of the product topology we obtain that $f(U_{x_0}) \times Y'_{x_0}$ is an open subset of $X \times Y$. Evidently we have $(x, y) \in f(U_{x_0}) \times Y'_{x_0}$. It remains to prove the following claim.

Claim. We have $f(U_{x_0}) \times Y'_{x_0} \subset W$.

The claim follows from the following equalities and inclusions:

$$f(U_{x_0})\times Y'_{x_0} = (f\times \mathrm{id}_Y)(U_{x_0}\times Y'_{x_0}) \underset{\substack{\uparrow \\ \text{since } Y'_{x_0}\subset K_{x_0}}}{\subset} (f\times \mathrm{id}_Y)(U_{x_0}\times K_{x_0}) \underset{\substack{\uparrow \\ \text{definition of } U_{x_0}}}{\subset} (f\times \mathrm{id}_Y)(\widetilde{W}) \underset{\substack{\uparrow \\ \text{by Lemma 1.3 (2)} \\ \text{since by definition} \\ \widetilde{W} = (f\times \mathrm{id}_Y)^{-1}(W)}}{=} W.$$

■

Remark. A different proof of Theorem 5.16, based on Propositions 5.4 and 5.6, is given in [**Hat02**, Lemma A.17].

5.4. **Mixing product and quotient topologies II (∗).** In this section we prove a second statement regarding the interplay between product and quotient topologies. More precisely, we will prove the following proposition.

Proposition 5.23. (∗) *Let X and Y be topological spaces. Let B be a subset of Y. We denote by $q\colon Y \to Y/B$ the projection. If B is a compact subset of Y and if Y is Hausdorff, then the map*

$$\begin{aligned} \mathrm{id}_X \times q\colon X \times Y &\to X \times (Y/B) \\ (x, y) &\mapsto (x, [y]) \end{aligned}$$

is a quotient map.

Remark.

(1) At first glance the statement of Proposition 5.23 might sound quite similar to the statement of Theorem 5.16. But in a way it is quite different, this time there are no conditions at all on the "unaffected" factor X, but we do put a condition on the projection $Y \to Y/B$.
(2) According to [**tD08**, Chapter 2.2, Exercise 16] one does not need to assume that Y is Hausdorff.
(3) Finally we refer to [**BrownR06**, Chapter 4.3] for more information on the interplay between product and quotient topologies.

In the proof of Proposition 5.23 we need the following fairly elementary lemma.

Lemma 5.24. (∗) *Let X and Y be topological spaces and let B be a compact subset of Y. Let $x \in X$. If W is an open subset of $X \times Y$ that contains $\{x\} \times B$, then there exists an open subset U of X and an open subset $\widetilde{V} \subset Y$ with $\{x\} \times B \subset U \times \widetilde{V} \subset W$.*

Proof (∗). Let W be an open subset of $X \times Y$ that contains $\{x\} \times B$. By definition of the product topology we can find for each $b \in B$ an open neighborhood U_b of x in X and an open neighborhood V_b of b in Y such that $U_b \times V_b \subset W$. Since B is compact there exist

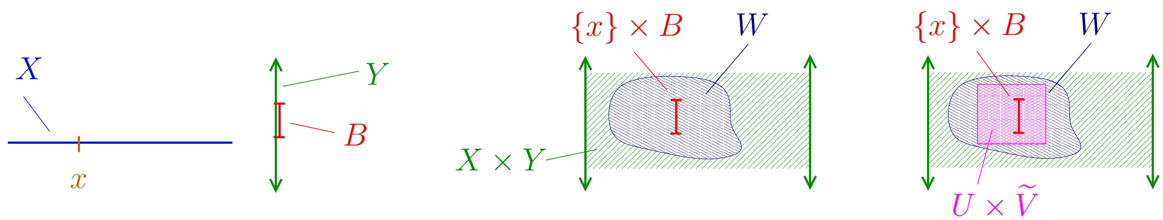


FIGURE 123. Illustration of Lemma 5.24.

$b_1, \ldots, b_m$ such that $B \subset V_{b_1} \cup \cdots \cup V_{b_m}$. One can easily verify that $U := U_{b_1} \cap \cdots \cap U_{b_m}$ and $\widetilde{V} := V_{b_1} \cup \cdots \cup V_{b_m}$ have the desired properties. ■

PROOF OF PROPOSITION 5.23 (∗). Let X and Y be topological spaces and let B be a compact subset of Y. We assume that Y is Hausdorff. We denote by $q\colon Y \to Y/B$ the obvious projection. Note that $p = \mathrm{id}_X \times q$. First note that it follows from Lemma 3.22 and Lemma 3.8 (2b) that p is continuous. This implies the "only if"-direction. It remains to show that "if"-direction. Thus let $W \subset X \times (Y/B)$ be a subset that has the property that $p^{-1}(W)$ is an open subset of $X \times Y$. We need to show that W is an open subset of $X \times (Y/B)$. By Lemma 2.5 and the definition of the product topology it suffices to prove the following claim.

Claim. For each $(x, [y]) \in W \subset X \times (Y/B)$ there exists an open neighborhood U_x of $x \in X$ and an open neighborhood $V_{[y]}$ of $[y] \in Y/B$ such that $U_x \times V_{[y]} \subset W$.

As in the proof of Proposition 5.20 we consider two cases separately. First we consider the case that $[y] = [B] \in Y/B$. It follows that $p^{-1}((x, [y])) = \{x\} \times B$. By hypothesis $p^{-1}(W)$ is an open subset of $X \times Y$ that contains $p^{-1}((x, [y])) = \{x\} \times B$. Therefore it follows from Lemma 5.24 there exists an open subset U_x of X and an open subset $\widetilde{V}_y \subset Y$ with $\{x\} \times B \subset U_x \times \widetilde{V}_y \subset p^{-1}(W)$. We denote by $V_{[y]}$ the image of $\widetilde{V}_y$ under the projection $q\colon Y \to Y/B$. Note that $V_{[y]}$ is an open neighborhood of $[y] \in Y/B$.[87] Finally note that

$$U_x \times V_{[y]} \;=\; (\mathrm{id}_X \times q)(U_x \times \widetilde{V}_y) \;=\; p(U_x \times \widetilde{V}_y) \subset p(p^{-1}(W)) \;=\; W.$$

It remains to consider the case that $[y] \neq [B] \in Y/B$. We see that $(x, y) \in W$. Since B is compact and since Y is Hausdorff we know by Lemma 2.17 that B is a closed subset of Y, i.e. $Y \setminus B$ is open. It follows immediately from the definition of the product topology on $X \times B$, the fact that $Y \setminus B$ is open, the fact that $p^{-1}(W)$ is an open neighborhood of (x, y) and the fact that $y \notin B$ that there exists an open neighborhood U_x of x and an open neighborhood $\widetilde{V}_y$ of y with $\widetilde{V}_y \subset X \setminus B$ and with $U_x \times \widetilde{V}_y \subset p^{-1}(W)$. One easily verifies that U_x and $V_y = q(\widetilde{V}_y)$ have the desired properties. ■

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[87] Indeed, since $B \subset \widetilde{V}_y$ we see that $q^{-1}(V_{[y]}) = q^{-1}(q(\widetilde{V}_y)) = \widetilde{V}_y \cup B = \widetilde{V}_y$, which is an open set. By the definition of the quotient topology on Y/B we see that $q^{-1}(V_{[y]})$ is indeed open.

Exercises for Chapter 5.

Exercise 5.1. Let X and Y be topological spaces. Suppose that Y is Hausdorff. Show that X^Y is also Hausdorff.

Exercise 5.2. Let X and Y be non-empty topological spaces. We consider the following map

$$\begin{array}{rcl} \varphi\colon X & \to & X^Y \\ x & \mapsto & \text{constant map } y \mapsto x \end{array}$$

(a) Show that φ is continuous.
(b) Show that φ is an embedding.

Exercise 5.3. Let X be a topological space that is compact and Hausdorff. Show that the group $\mathrm{Homeo}(X)$ of self-homeomorphisms of X, equipped with the compact-open topology, is a topological group.

Exercise 5.4. Let $X, \widetilde{X}$ and Y be topological spaces. We suppose that Y is regionally compact and Hausdorff. Show that the map

$$\begin{array}{rcl} X^Y \times \widetilde{X}^Y & \to & (X \times \widetilde{X})^Y \\ (f,g) & \mapsto & \left(\begin{array}{rcl} Y & \to & X \times \widetilde{X} \\ y & \mapsto & (f(y), \widetilde{f}(y)) \end{array} \right) \end{array}$$

is a homeomorphism.
Remark. We refer to [**Enge89**, Exercise 3.4B, p. 165] for a similar, and for the most part, much more general statement.

Exercise 5.5. Let X be a topological space and let G be a topological group. We suppose that G is regionally compact. Show that G^X with the multiplication

$$\begin{array}{rcl} G^X \times G^X & \to & G^X \\ (f,g) & \mapsto & \left(\begin{array}{rcl} X & \to & G \\ x & \mapsto & f(x) \cdot g(x) \end{array} \right) \end{array}$$

is a topological group.

Exercise 5.6. Let $n \in \mathbb{N}$. We consider the following map

$$\begin{array}{rcl} \Theta\colon \mathrm{SO}(n) & \to & \mathrm{Homeo}(S^{n-1}) := \text{self-homeomorphisms of } S^{n-1} \\ A & \mapsto & (P \mapsto A \cdot P). \end{array}$$

(a) Show that the map Θ is continuous.
(b) Show that the map Θ is an embedding.

Hint. Use Lemma 2.30 (4) and Lemma 5.1.

Exercise 5.7. Let $Y = \{1, \ldots, n\}$ be a finite discrete topological space with n elements and let X be a topological space. Show there exists a homeomorphism $X^Y \cong X^n$.

Part II

Differential Topology

6. Topological manifolds and smooth manifolds

In this chapter we introduce topological manifolds and study some basic properties. We also introduce smooth structures on topological manifolds. A topological manifold equipped with a smooth structure will just be called a smooth manifold. In the literature the term "smooth manifold" can be used with quite different conventions and meanings. So we try to state clearly the various definitions used throughout these notes. At the end, just for fun, we give the classification of 1-dimensional (topological) smooth manifolds.

6.1. Second-countable topological spaces.

Our goal is to give as soon as possible the definition of a topological manifold. Unfortunately we first have to get the following technical and slightly annoying definition out of the way.

Definition. A topological space is called *second-countable* if there exists a countable basis for the topology.

Remark.

(1) Some books say instead that such a topological space "satisfies the second axiom of countability".
(2) In case you were wondering, a topological space X is called *first-countable* if given any $P \in X$ there exists a countable family of open neighborhoods $\{U_i\}_{i\in I}$ of P such that each open neighborhood of P contains at least one U_i. We will not make use of this notion.

Of course not every topological space is second-countable. For example consider $X = \mathbb{R}$, but for once equipped with the discrete topology. Since $\mathbb{R}$ is uncountable one sees almost immediately that $X = \mathbb{R}$ is not second-countable. There are almost more subtle examples, for example we will see in Exercise 6.3 that there exist compact topological spaces that are not second-countable.

Nonetheless, we will use the next lemma to see that most topological spaces that we are interested in are actually second-countable.

Lemma 6.1.

(1) *Every subset of some $\mathbb{R}^n$ is second-countable. (Recall that, unless we say something else, we equip $\mathbb{R}^n$ with the standard topology introduced on page 85).*
(2) *If X is a topological space that is second-countable, then every subset is also second-countable.*
(3) *If $X_1, \dots, X_k$ are topological spaces that are second-countable, then $X_1 \times \dots \times X_k$ is second-countable.*
(4) *Let X be a topological space. If there exists a countable family of open subsets $\{U_i\}_{i\in I}$ such that $\bigcup\limits_{i\in I} U_i = X$ and if each U_i is second-countable, then X itself is second-countable.*
(5) *Let X be a topological space and let $\sim$ be an equivalence relation on X. If X is second-countable and if the projection $X \to X/\sim$ is an open map, then $X/\sim$ is also second-countable.*

Proof (∗).

(1) This follows from the discussion on page 167.
(2) This statement follows from Lemma 2.28 together with Lemma 1.7 (3).
(3) This statement follows from Lemma 3.11 together with Lemma 1.7 (5).
(4) For each U_i we pick a countable basis $\mathcal{B}_i$ of the topology of U_i. It is straightforward to verify that $\bigcup_{i\in I}\mathcal{B}_i$ is a basis for the topology of X. By Lemma 1.7 (2) this is a countable set.
(5) This statement follows from Lemma 3.27 together with Lemma 1.7 (3). ■

The following lemma gives us a convenient property of second-countable topological spaces.

Lemma 6.2. $(*)$ *Let X be a topological space and let $\{U_i\}_{i\in I}$ be an open cover of X. If X is second-countable, then there exists a countable subset $J \subset I$ with $X = \bigcup_{j\in J} U_j$.*

Proof $(*)$. Let $\mathcal{B}$ be a countable basis for the topology of X. We write

$$\mathcal{B}' := \{V \in \mathcal{B} \,|\, V \subset U_i \text{ for some } i \in I\}.$$

Given $V \in \mathcal{B}'$ we pick $j \in I$ with $V \subset U_j$. It follows from Lemma 1.7 that the set $J \subset I$ of all such j is countable. It remains to show that $X = \bigcup_{j\in J} U_j$.

Thus let $x \in X$. By hypothesis there exists an $i \in I$ with $x \in U_i$. Since $\mathcal{B}$ is a basis for the topology of X there exists a $V \in \mathcal{B}$ with $x \in V \subset U_i$. Note that $V \in \mathcal{B}'$. But this implies that $x \in U_j$ for some $j \in J$. ■

6.2. **Topological manifolds: definition and examples.** In this section we give the definition of a topological manifold. In these notes the study of topological manifolds will be one of our main objectives. First we need to introduce the following notation.

Notation. Given $n \in \mathbb{N}_0$ we write

$$\begin{aligned} H_n &= \{(x_1,\dots,x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\} \\ \partial H_n &= \{(x_1,\dots,x_n) \in \mathbb{R}^n \,|\, x_n = 0\}. \end{aligned}$$

We refer to H_n as the *upper half-space in* $\mathbb{R}^n$. We equip both sets with the subspace topology coming from $\mathbb{R}^n$. One can easily verify that ∂H_n is indeed the boundary of H_n viewed as a subset of $\mathbb{R}^n$. Evidently the map $\mathbb{R}^{n-1} \to \partial H_n$ given by $x \mapsto (x, 0)$ is a homeomorphism.

Now we can move on to the definition of a topological manifold.

Definition. Let X be a topological space.

(1) An *n-dimensional chart for X at a point $x \in X$* is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of x and
  (i) V is an open subset of $\mathbb{R}^n$ or
  (ii) V is an open subset of the upper half-space H_n and $\Phi(x)$ lies in ∂H_n.
  In the former case we say that Φ is a chart of type (i), in the latter case we say that Φ is a chart of type (ii). We refer to Figure 124 for an illustration.

(2) We say X is an *n-dimensional topological manifold*, if X is second-countable and Hausdorff, and if for every $x \in X$ there exists an n-dimensional chart $\Phi\colon U \to V$ at x.
(3) We say that a point x on a topological manifold is a *boundary point* if x does not admit a chart of type (i).[88] We denote by ∂X the set of all boundary points of X.
(4) We say a topological manifold X is *closed* if X is compact with $\partial X = \varnothing$.[89]
(5) An *n-dimensional atlas* for X is a family of n-dimensional charts $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ such that $\bigcup\limits_{i\in I} U_i = X$.

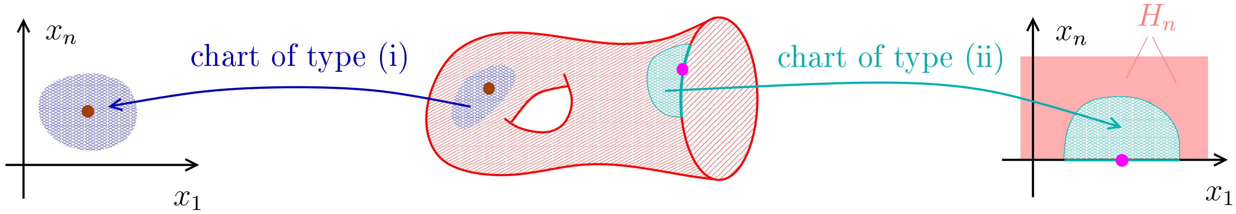

FIGURE 124

In the following we will give several examples of topological manifolds. When it comes to verifying that a topological space is a topological manifold the main focus is usually on establishing that any point admits a chart. It is easy to forget that one also needs to verify that the topological space is second-countable and Hausdorff. In many cases this task is taken care of by the following lemma.

Lemma 6.3. *Let $n \in \mathbb{N}$. Every subset of $\mathbb{R}^n$ is second-countable and Hausdorff.*

PROOF. Let A be a subset of $\mathbb{R}^n$. By Lemma 6.1 we know that A is second-countable. Furthermore, it follows from Proposition 2.11 and Lemma 2.12 that A is also Hausdorff. ■

For the record we also state the following elementary lemma.

Lemma 6.4. $(*)$ *Let X be a topological space.*

(1) *If X admits a finite atlas, then X is second-countable.*
(2) *If X is compact and if it admits an atlas, then X is second-countable.*

PROOF $(*)$. The first statement is an immediate consequence of Lemma 6.1 (4). The second statement in turn is an immediate consequence of the first statement since every compact topological space that admits an atlas also admits a finite atlas. ■

Now we turn to giving examples of topological manifolds.

Examples. Let $n \in \mathbb{N}$.

(1) We consider the sphere S^n. As usual we refer to $N := (0, \dots, 0, 1) \in S^n$ as the North Pole and similarly we refer to $S := (0, \dots, 0, -1) \in S^n$ as the South Pole.

[88]It might sound more natural to define a boundary point to be a point that admits a chart of type (ii). In fact we will see in Lemma 44.1 that a point on a topological manifold admits *either* a chart of type (i) or a chart of type (ii). But the proof of this statement is not trivial and as of now we can not prove this statement.

[89]The latter condition just means that all points in X admit a chart of type (i).

It follows quite easily from slight modifications of the arguments in the proof of Lemma 2.44 that the stereographic projection $S^n \setminus \{N\} \to \mathbb{R}^n$ from the North Pole together with the stereographic projection $S^n \setminus \{S\} \to \mathbb{R}^n$ from the South Pole are two n-dimensional charts of type (i) that cover all of S^n. By Lemma 6.3 this shows that S^n is an n-dimensional topological manifold.

Note that both charts are of type (i), thus we see that the boundary of the topological manifold S^n is the empty set.[90] Finally note that S^n is compact, thus S^n is a closed n-dimensional topological manifold.

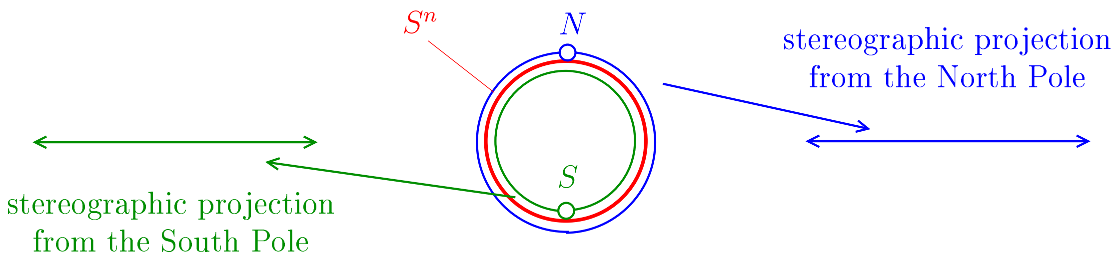

FIGURE 125

(2) Every open subset U of $\mathbb{R}^n$ is an n-dimensional topological manifold. This follows again from Lemma 6.3 and the observation that the identity map is a chart of type (i) for all points. In particular we see that the boundary of U is empty.

(3) Let M be an open subset of the upper half-space $H_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\}$. Again, using Lemma 6.3, it is straightforward to see that M is an n-dimensional topological manifold:
   (a) Given any point $X = (x_1, \dots, x_n) \in M$ with $x_n > 0$ the identity map on the open subset $U := \{(y_1, \dots, y_n) \in M \,|\, y_n > 0\}$ is a chart of type (i) around x.
   (b) Given any point $(x_1, \dots, x_n) \in M$ with $x_n = 0$ the identity map $M \to M$ is a chart of type (ii) around x.

   In this example we run into a major, and rather painful, subtlety. Namely one would think that the boundary of M as a topological manifold should be precisely the points in M with $x_n = 0$. But at the moment we cannot draw this conclusion. After all, in principle, such points also might admit charts of type (i). Only after having introduced and studied homology groups we will see that no such charts of type (i) exist.

(4) Next we want to show that the closed n-ball $\overline{B}^n$ is an n-dimensional topological manifold. We provide the following charts:
   (a) We consider the chart of type (i) that is given by the identity on B^n.

[90] The vigilant reader will have noticed that we have committed a major sin: we introduced an object called "boundary" twice, with quite different meanings:
(a) On page 92 we introduced the boundary ∂A of a subset A of a topological space X.
(b) On page 262 we introduced the boundary ∂M of a topological manifold M.
Usually it is clear from the context what we mean when we write ∂.

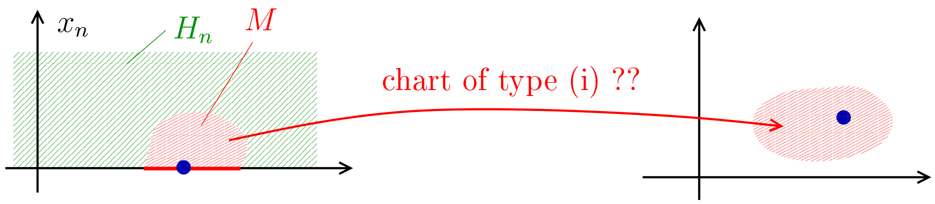

FIGURE 126

(b) We set $U := \{(x_1,\dots,x_n) \in \overline{B}^n \,|\, x_n > 0\}$ and we consider the map

$$\begin{array}{rcl} U & \to & H_n \\ (x_1,\dots,x_n) & \mapsto & (x_1,\dots,x_{n-1},\sqrt{x_1^2+\dots+x_{n-1}^2}-x_n). \end{array}$$

This is a chart of type (ii) for all points in $\{(x_1,\dots,x_n) \in S^{n-1} \,|\, x_n > 0\}$. We refer to Figure 127 for an illustration of this chart.

(c) We consider the obvious variations of the previous chart for $x_n < 0$ and for other coordinates.

These charts form an atlas for $\overline{B}^n$, together with Lemma 6.3 this shows that $\overline{B}^n$ is an n-dimensional topological manifold. We have found a chart of type (i) for each point in B^n and for the points on S^{n-1} we had found points of type (ii). This shows that $\partial\overline{B}^n \subset S^{n-1}$. Furthermore we have the strong suspicion that $\partial\overline{B}^n = S^{n-1}$ but as of right now we cannot confirm this. We will only be able to confirm our hunch by the time we have proved Proposition 44.2.

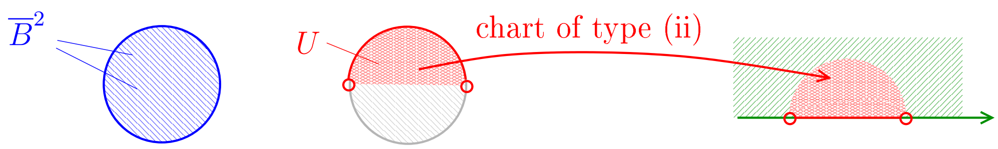

FIGURE 127

(5) Given $a < b$ in $\mathbb{R}$ the interval $[a,b]$ is easily seen to be a 1-dimensional topological manifold. Every point in (a,b) admits a chart of type (i). Using Exercise 2.40 one can show fairly easily that the two points $a,b \in [a,b]$ do *not* admit charts of type (i). Thus the boundary of $[a,b]$ consists, as expected, of the two points a and b.

(6) In Exercise 6.2 we will verify that the line with two zeros admits a chart of type (i) for every point. It is also easy to see that the line with two zeros is second-countable. But since the line with two zeros is not Hausdorff we see that it is not a topological manifold.

(7) We consider the topological space X that is given by the set $X = \mathbb{R}$ but for once we equip $X = \mathbb{R}$ with the discrete topology. Every point of X admits a 0-dimensional chart and X is obviously Hausdorff. But X is not second-countable, thus X is not a topological manifold.

(8) We conclude this list of examples with the simplest of all examples: the empty set is a topological manifold of any dimension.

The following proposition gives a useful method for constructing more topological manifolds.

Proposition 6.5. *Let M be an m-dimensional topological manifold with possibly non-empty boundary and let N be an n-dimensional topological manifold with empty boundary. Then the following hold:*

(1) *The product $M \times N$ is an $(m+n)$-dimensional topological manifold.*
(2) *We have*[91] $\partial(M \times N) \subset \partial M \times N$.
(3) *If M and N are closed, then $M \times N$ is also closed.*

The above statements also hold with the roles of M and N reversed.

Examples.

(1) It follows from Proposition 6.5 and the above examples that for any $n \in \mathbb{N}$ the n-dimensional torus $(S^1)^n$ is a closed n-dimensional topological manifold.
(2) It follows from Proposition 6.5 and the above examples that the annulus $S^1 \times [0,1]$ is a 2-dimensional topological manifold and that $S^1 \times \overline{B}^2$ is a 3-dimensional topological manifold.

Proof (∗).

(1) First note that it follows from Proposition 3.12 (1) that $M \times N$ is Hausdorff and it follows from Lemma 6.1 that $M \times N$ is second-countable. One can now easily use products of charts to show that $M \times N$ is an $(m+n)$-dimensional topological manifold.
(2) The product of charts of type (i) is again a chart of type (i). This immediately implies the second statement.
(3) This statement follows from (2) and Proposition 3.12 (2). ■

We conclude this chapter with the following two lemmas.

Lemma 6.6. (∗) *Let M and N be two topological manifolds.*

(1) *If $f\colon M \to N$ is a local homeomorphism, then $f(\partial M) = (\partial N) \cap f(M)$.*
(2) *If $f\colon M \to N$ is in fact a homeomorphism, then f restricts to a homeomorphism $\partial M \to \partial N$.*

Proof (∗). The lemma follows from the elementary observation that the restriction of a chart for a point on a topological manifold has the same type as the original chart. We leave it to the reader to fill in the minuscule details. ■

The following lemma is slightly more interesting than the previous lemma.

Lemma 6.7. (∗) *Let M be a topological manifold.*

(1) *Any union of components of ∂M is a closed subset of M. In particular ∂M itself is a closed subset of M.*
(2) *If M is compact, then ∂M is also compact.*

[91] Note that right now we can only prove the inclusion "⊂". Later on in Proposition 44.2 we will see that in fact we have an equality.

Proof (∗). Let M be a topological manifold.

(1) First note that, basically by definition, the set of points in M admitting a chart of type (i) is an open subset of M. It follows from the definition of ∂M that $M \setminus \partial M$ is an open subset. Thus the boundary ∂M is a closed subset of M. We denote by $\{Y_i\}_{i\in I}$ the components of ∂M. By Lemma 2.69 each Y_i is an open subset of ∂M. Thus for any $i \in I$ there exists, by definition of the subspace topology, an open set $Z_i \subset M$ with $Z_i \cap \partial M = Y_i$. For any $J \subset I$ we have

$$M \setminus \bigcup_{j\in J} Y_j \;=\; \underbrace{(M \setminus \partial M)}_{\text{open in } M} \;\cup\; \underbrace{\bigcup_{j\notin J} Z_j}_{\text{open in } M}.$$

We have thus shown that the union of components of ∂M corresponding to J is indeed a closed subset of M.

(2) Now suppose that M is compact. Thus ∂M is a closed subset of the compact topological space M. We obtain from Lemma 2.17 (1) that ∂M itself is compact. ■

6.3. **Surfaces are topological manifolds.** On pages 205 and 206 we introduced the (non-) orientable surfaces of some genus. The name already suggests that these should be 2-dimensional topological manifolds, and the following proposition confirms our impression.

Proposition 6.8.

(1) *For every $g \in \mathbb{N}_0$ the surface Σ_g of genus g is a closed 2-dimensional topological manifold.*

(2) *For every $g \in \mathbb{N}$ the non-orientable surface N_g of genus g is a closed 2-dimensional topological manifold.*

Proof (∗). To simplify the discussion we will only prove Statement (1) of the proposition. We leave it to the reader to modify the argument to deal with the non-orientable case.

We start out this proof with the discussion of the two cases $g = 0$ and $g = 1$ that we had basically dealt with already. First recall that on page 263 we saw that any S^n is a closed n-dimensional topological manifold. The surface of genus 0 is by definition the sphere S^2, thus we are done by the above. Furthermore, the surface of genus 1 is by definition the torus $S^1 \times S^1$. The desired statement thus follows from the above and Proposition 6.5.

It remains to deal with the case $g \geq 2$. To simplify the notation we prove the proposition for $g = 2$. The proof of the general case is totally analogous. We consider the octagon E_8 with the vertices $Q_k = e^{2\pi ik/16}$, $k = 1, 3, \dots, 15$ and the equivalence relation $\sim$ that we introduced on page 205. We illustrate the octagon together with the equivalence relation in Figure 128.

Note that E_8 is compact by the Heine-Borel Theorem 2.20. Thus we obtain from Lemma 3.21 (4) that $E_8/\sim$ is also compact. In the following we will see that $E_8/\sim$ admits a finite atlas, by Lemma 6.4 this implies in particular that $E_8/\sim$ is second-countable.[92] Note

[92]It would also not be difficult to prove second-countability "by hand".

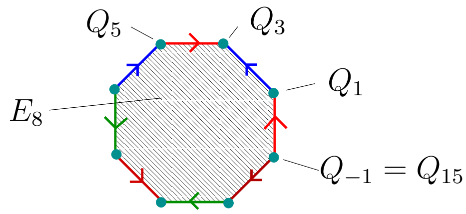

FIGURE 128

that E_8 is compact by the Heine-Borel Theorem 2.20. Thus we obtain from Lemma 2.40 that $E_8/\sim$ is also compact. We leave it to the reader to verify that $E_8/\sim$ is Hausdorff.[93]

Thus, to show that $E_8/\sim$ is a closed 2-dimensional topological manifold it remains to give a finite atlas for $E_8/\sim$ that consists only of charts of type (i).

First we consider the map

$$\begin{array}{rcl} \Phi\colon U := p\big(\mathring{E}_8\big) & \to & V = \mathring{E}_8 \ \subset \ \mathbb{R}^2 \\ p(X) & \mapsto & X. \end{array}$$

This is a chart of type (i) for each point in the interior of E_8.

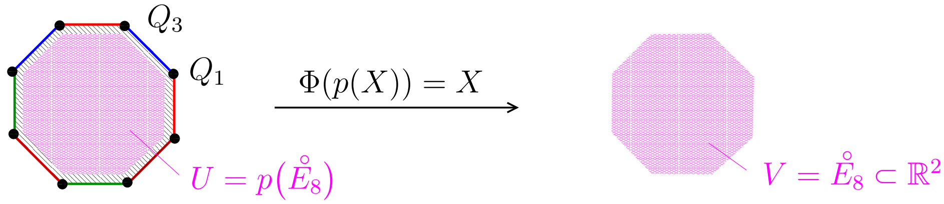

FIGURE 129

Next we introduce charts that cover the interiors of the edges. First we consider the edge given by Q_{-1} and Q_1. We denote by U and U' the half-disks that are illustrated in Figure 130. (Note that it is understood that the line segment is included but that the circle is not included in U and U'.) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the line that goes through Q_{-1} and Q_1. It is straightforward to verify that the map

$$\begin{array}{rcl} \Psi_1\colon p(U \ \cup \ U') & \to & \text{the open disk } U \cup f(r_{\frac{\pi}{4}}(U)) \ \subset \ \mathbb{R}^2 \\ p(X) & \mapsto & \begin{cases} X, & \text{if } X \in U \\ f(r_{\frac{\pi}{4}}(X)), & \text{if } X \in U' \end{cases} \end{array}$$

is well-defined. Furthermore, using Lemma 2.35 (2) one can show that Ψ_1 and its inverse are both continuous. Finally, it follows easily from the definitions that $p(U \cup U')$ is an open subset of $E_8/\sim$. Thus Ψ_1 is a chart of type (i) for all points in the interior of the edge from Q_{-1} to Q_1. Similarly we define charts Ψ_2, Ψ_3, Ψ_4 that take care of the other pairs of edges.

[93] This can either be done "by hand", or more elegantly but possibly with more work, by using Lemma 3.26.

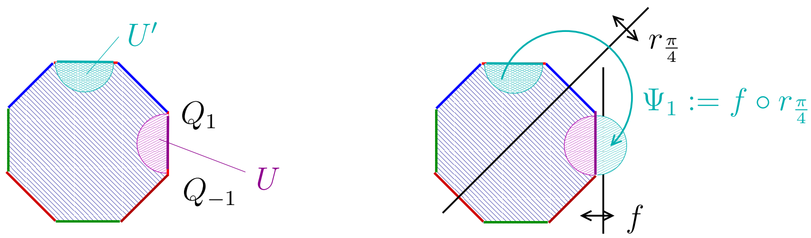

FIGURE 130

It remains to find a chart of type (i) that takes care of the point P in $E_8/\sim$ that is represented by the vertices. (It is a fun exercise to verify that all vertices get identified.) We pick an $\eta > 0$ such that the η-disks centered at $Q_1, \ldots, Q_{15}$ are disjoint. For $k = 1, 3, \ldots, 15$ we set $U_k := B_\eta(Q_k) \cap E_8$. The sets $U_1, U_3, \ldots, U_{15}$ are disjoint and one can easily verify that

$$p(U_1) \cup p(U_3) \cup \cdots \cup p(U_{15})$$

is an open neighborhood of $p(Q_1) = p(Q_3) = \cdots = p(Q_{15})$.

Before we write down the chart rigorously it might help to visualize the situation. To

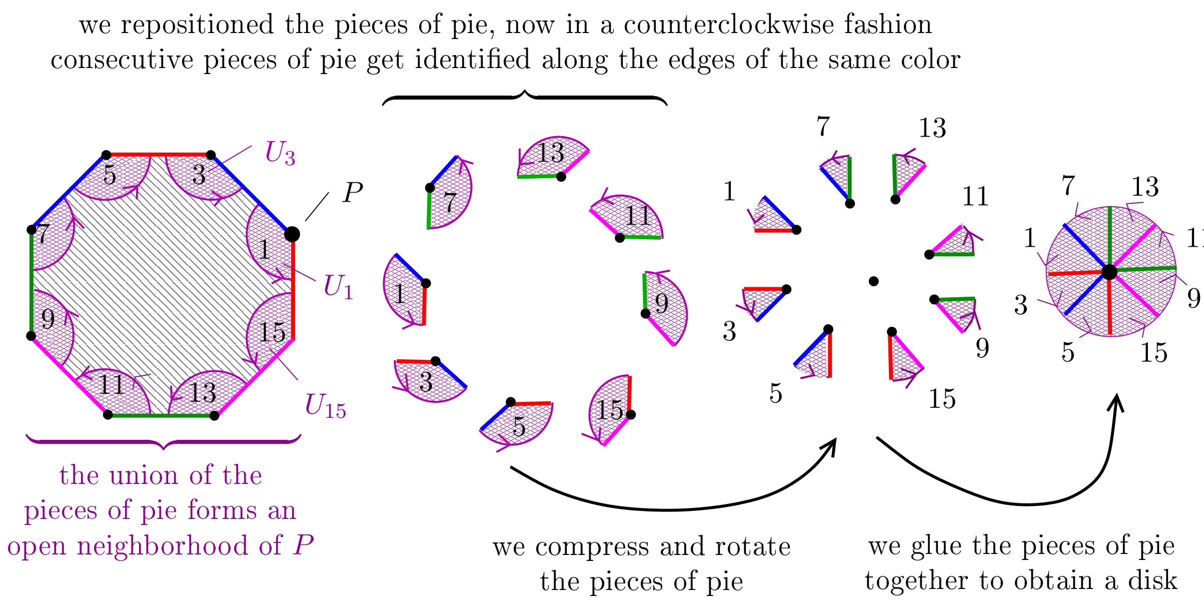

FIGURE 131

the left of Figure 131 we see the sets $U_1, \ldots, U_{15}$. Next we arrange these "pieces of pie" as in the second second part of Figure 131. Thus we see the "pieces of pie" with an angle of $\frac{3\pi}{4}$ where we show the various identifications of the edges. The map from the second to the third part is given by compressing the angle of the "pieces of pie" to a third and by rotating the "pieces of pie". These eight "pieces of pie", each with an angle of $\frac{\pi}{4}$, can be combined to give a disk in $\mathbb{R}^2$.

Now we want to turn the above idea into formal mathematics. We consider the map

$$\Omega\colon p(U_1)\cup\dots\cup p(U_{15}) \quad\to\quad B^2_\eta(0)$$

that is defined on each $p(U_k)$, $k\in\{1,3,\dots,15\}$ to be the map

$$p\big(\underbrace{Q_k+re^{\mathrm{i}\alpha}}_{\text{where } r\in[0,\eta) \text{ and } \alpha\in[\beta(k),\beta(k)+\frac{3}{4}\pi]}\big) \quad\mapsto\quad re^{\mathrm{i}\left(\frac{1}{3}(\alpha-\beta(k))+\gamma(k)\right)}.$$

Here $\beta(k)$ and $\gamma(k)$ are given by the following table:

k	1	3	5	7	9	11	13	15
$\beta(k)$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$
$\gamma(k)$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{1}{2}\pi$	$\frac{7}{4}\pi$	0	$\frac{1}{4}\pi$	$\frac{3}{2}\pi$

It is straightforward, albeit slightly cumbersome, to verify that Ω is a bijection. Furthermore, using Exercise 3.27 together with Lemma 2.35 (2) one can show fairly easily that Ω is continuous and that its inverse is also continuous. In other words, the map Ω is a homeomorphism. As pointed out above, it is straightforward to verify that $p(U_1)\cup\dots\cup p(U_{15})$ is an open subset of $E_8/\sim$. In summary, we have shown that Ω is a chart of type (i) for P.

The eight charts $\Phi,\Psi_1,\Psi_2,\Psi_3,\Psi_4$ and Ω are now charts of type (i) that cover all of $E_8/\sim$. We have thus found the desired atlas. ■

6.4. **Properties of topological manifolds.** Topological manifolds are particularly important and also particularly nice topological spaces. For example the next lemma shows in particular that when it comes to topological manifolds we do not have to worry about the two different definitions of connectedness.

Lemma 6.9.

(1) *Every n-dimensional topological manifold is locally homeomorphic to an open convex subset of $\mathbb{R}^n$ or H_n.*
(2) *Every topological manifold is locally path-connected.*
(3) *A topological manifold is connected if and only if it is path-connected.*
(4) *Every topological manifold is regionally compact.*

Throughout the lecture notes we will on many occasions make use of Lemma 6.9 (3) without referring to it explicitly.

Proof (∗). Let X be an n-dimensional topological manifold.

(1) Let M be an n-dimensional topological manifold and let $P\in M$. First assume that $P\notin\partial M$. This means that there exists a chart $\Phi\colon U\to V$ to some open subset V of $\mathbb{R}^n$. We set $Q=\Phi(P)$. Since V is open there exists an $r>0$ such that $B^n_r(Q)\subset V$. Evidently $B^n_r(Q)$ is convex and we see that $\Phi^{-1}(B^n_r(Q))$ is an open neighborhood of P that is homeomorphic to a convex subset of $\mathbb{R}^n$. Basically the same argument also works if $P\in\partial M$.
(2) Since convex sets are path-connected we obtain from (1) that topological manifolds are locally path-connected.

(3) This statement is an immediate consequence of (3) and of Lemma 2.72.
(4) On page 139 we saw that open subsets of $\mathbb{R}^n$ are regionally compact. The same argument shows that open subsets of H_n are regionally compact. Using Lemma 2.40 it is now easy to see that topological manifolds are regionally compact. ■

On several occasions we will need the following, slightly technical lemma.

Lemma 6.10. *Let M be a topological manifold and let $\Phi\colon U \to V$ be a chart. Let $A \subset V$ be a subset that is compact. Then the corresponding subset $\Phi^{-1}(A)$ is a closed subset of M.*[94]

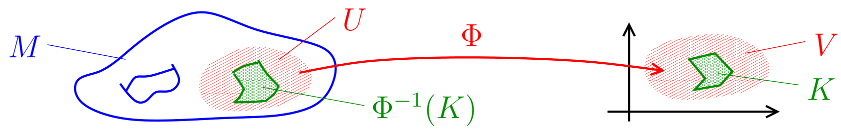

FIGURE 132. Illustration of Lemma 6.10.

PROOF. Let M be a topological manifold and let $\Phi\colon U \to V$ be a chart. We denote by $\Psi = \Phi^{-1}\colon V \to U$ be the inverse map. Let $A \subset V$ be a subset that is compact. It follows from Lemma 2.40 that $\Psi(A)$ is a compact subset of M. But M is Hausdorff by definition of a topological manifold. Hence it follows from Lemma 2.17 (2) that $\Psi(A) = \Phi^{-1}(A)$ is closed in M. ■

Next we specify what we mean by a submanifold.

Definition. Let M be an n-dimensional topological manifold.

(1) We say a subset $N \subset M$ is a *k-dimensional submanifold* if given any $P \in N$ one of the following holds:
(α) there exists a chart $\Phi\colon U \to V$ of type (i) for M and P such that[95]
$$\Phi(U \cap N) \;=\; V \cap \{(0,\dots,0,x_1,\dots,x_k) \,|\, x_i \in \mathbb{R}\},$$
(β) or there exists a chart $\Phi\colon U \to V$ of type (ii) for M and P such that $\Phi(P)$ lies in E_{n-1} and
$$\Phi(U \cap N) \;=\; V \cap \{(0,\dots,0,x_1,\dots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\},$$
(γ) or there exists a chart $\Phi\colon U \to V$ of type (i) for M and P such that $\Phi(P)$ lies in E_{n-1} and
$$\Phi(U \cap N) \;=\; V \cap \{(0,\dots,0,x_1,\dots,x_k) \in \mathbb{R}^n \,|\, x_k \geq 0\}.$$
We refer to such charts as *submanifold charts* of N.

(2) We say a submanifold N is *proper* if N is a closed subset of M and if for each $P \in N$ we can find submanifold charts of type (α) and (β).

We illustrate the three types of submanifold charts in Figure 133.

[94] It does not suffice that A is a closed subset of the topological space V. Indeed, $A = V$ is a closed subset of V, but $U = \Phi^{-1}(V)$ is in general not a closed subset of M.

[95] Why are the zeros to the left and not to the right?

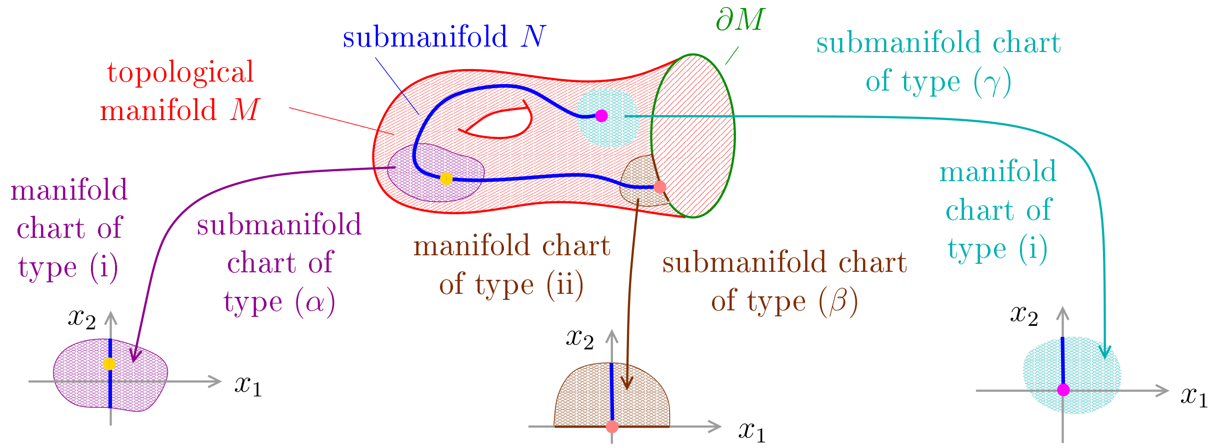

FIGURE 133

Examples.

(1) In Figure 134 we give typical examples of submanifolds, some of which are proper, some are not. We also show some subsets that are not submanifolds. In particular it is worth pointing out that a non-empty submanifold is never a submanifold.[96]

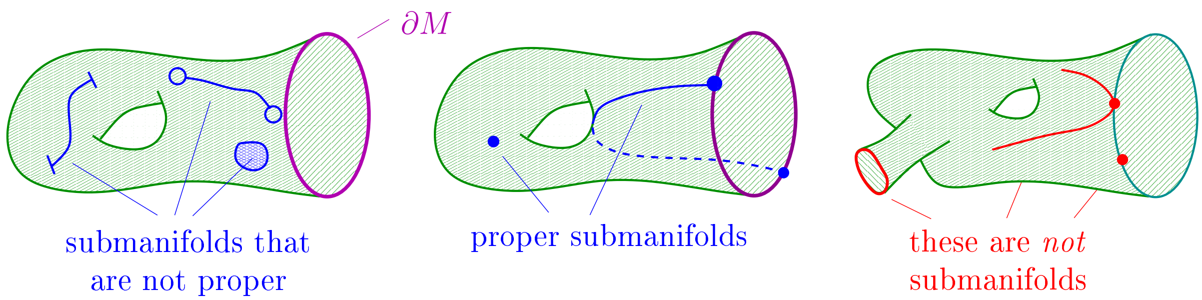

FIGURE 134

(2) Let $f\colon V \to W$ be a map between two topological manifolds with $f(V) \subset W \setminus \partial W$. In Exercise 6.10 we will see that the graph $\mathrm{Gr}(f) := \{(x, f(x)) \,|\, x \in V\}$ is a proper submanifold of the topological manifold $V \times W$.

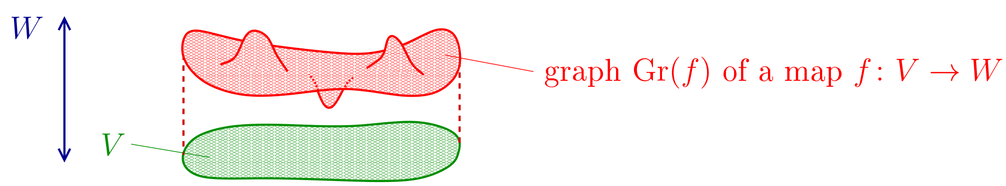

FIGURE 135

[96]Indeed, suppose that N is a non-empty submanifold of a topological manifold M. It follows immediately from the definitions of a submanifold that N contains a point P that is contained in a chart of type (i) for M. But by the definition of ∂M on page 262 this means that $P \notin \partial M$.

(3) Let M be a topological manifold. It follows easily from the definitions that a subset $X \subset M$ is a 0-dimensional submanifold of M if and only if X is a discrete subset of M.

Remark. The notion of a "submanifold" is used differently by different authors. Therefore, reading the literature it is a good idea to check what definition of a submanifold the authors use. In particular our definition of a "proper submanifold" is often used as the definition of a submanifold.

We conclude this section with the following lemma that is almost obvious.

Lemma 6.11. *Let M be a topological manifold. If $N \subset M$ is a k-dimensional submanifold, then it is also a k-dimensional topological manifold.*

Proof. It follows from Lemmas 2.12 and 6.1 that N is Hausdorff and second-countable. It is clear from the definitions that N admits a k-dimensional atlas. Thus N is indeed a k-dimensional topological manifold. ■

Remark. Let M be a topological manifold and let $U \subset M$ be an open subset. It follows almost immediately from the definitions that U is a codimension-zero submanifold of M. Furthermore, if we view U as a topological manifold in its own right, then it follows again basically immediately from the definitions that $\partial U = U \cap \partial M$.

6.5. **Smooth functions on subsets of $\mathbb{R}^n$.** In the next section we want to define the notion of a "smooth structure" on a topological manifold. In this section we want to recall same basic definitions and facts regarding "classical" smooth functions.

Before we discuss smooth functions we have to agree on what we mean by smooth maps on a subset of $\mathbb{R}^n$.

Definition.

(1) Let U be an open subset of $\mathbb{R}^n$. We say a map $f = (f_1, \ldots, f_m)\colon U \to \mathbb{R}^m$ is *smooth*, if all iterated partial derivatives $\frac{\partial}{\partial x_{k_1}} \ldots \frac{\partial}{\partial x_{k_d}} f_i$, of arbitrarily large degree, of each coordinate function f_i exist.

(2) Let A be any subset of $\mathbb{R}^n$. We say a map $f\colon A \to \mathbb{R}^m$ is *smooth*, if given any $a \in A$ there exists an open neighborhood U (in $\mathbb{R}^n$) of a and a smooth map $\widetilde{f}\colon U \to \mathbb{R}^m$ which coincides with f on $A \cap U$.

Example.

(1) The function $f\colon [0, \infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is not smooth, since it cannot be extended to a smooth function on any open interval (a, b) that contains 0.

(2) One can easily show that any function $f\colon \mathbb{Z} \to \mathbb{R}$ is smooth.

(3) In the following our main interest will lie in maps defined on open subsets of the upper half-space $H_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. Thus let U be an open subset of H_n and let $f\colon U \to \mathbb{R}^m$ be a function. By a more advanced version of Borel's Theorem, see [**Hör90a**, Theorem I.1.2.6], the map f is smooth in the above sense if and only if f is continuous, if $f|_{\mathring{H}_n \cap U}$ is smooth and if all the partial derivatives of $f|_{\mathring{H}_n \cap U}$ have continuous extensions to all of U. We will not make use of this fact.

For $n = 1$ the following proposition sometimes gives a useful criterion for showing that a given map on an interval is smooth in the above sense.

Proposition 6.12. *Let $a < b$ be two real numbers and let $f\colon [a, b] \to \mathbb{R}$ be a function. The following two statements are equivalent:*

(1) *The function $f\colon [a, b] \to \mathbb{R}$ is smooth in the above sense.*
(2) *All derivatives of $f\colon (a, b) \to \mathbb{R}$ are defined on the open interval (a, b) and they extend to continuous maps on the closed interval $[a, b]$.*

PROOF. The "(1)⇒(2)"-implication is basically clear. The "(2)⇒(1)"-implication is a consequence of Borel's Theorem, see [**Strg81**, Theorem 4.32]. ■

The following lemma is one of the key tools for constructing interesting smooth functions.

Lemma 6.13.

(1) *Suppose that we are given real numbers $a \leq b < c \leq d$ and $\epsilon > 0$. There exists a smooth function $f\colon [a, d] \to [0, 1]$ with the following properties:*
   (a) *$f|_{[a,b]} \equiv 0$ and $f|_{[c,d]} \equiv 1$.*
   (b) *f is strictly monotonously increasing on $[b, c]$.*
   (c) *Given any $t \in (a, d)$ we have $f'(t) < (1 + \epsilon) \cdot \frac{1}{c-b}$.*
(2) *Let $a < b$ be two positive real numbers and let $n \in \mathbb{N}$. There exists a smooth function $f\colon \mathbb{R}^n \to [0, 1]$ such that $f(x) = 1$ whenever $\|x\| \leq a$ and such that $f(x) = 0$ whenever $\|x\| > b$.*

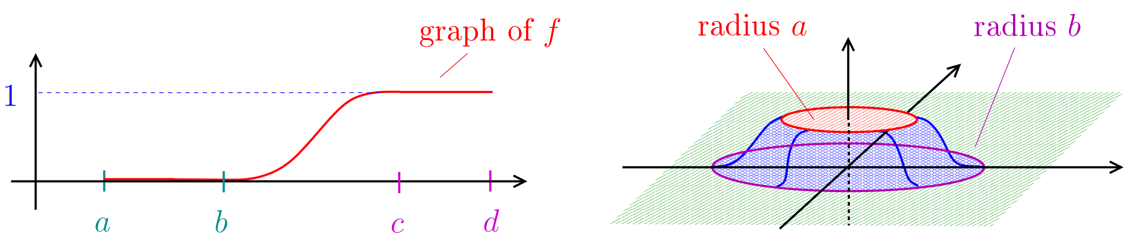

FIGURE 136. Illustration of Lemma 6.13.

PROOF.

(1) This statement is proved in any self-respecting analysis course, we refer to [**Kön04**, Chapter 9] for details.
(2) By (1) there exists a smooth function $\varphi\colon [0, \infty) \to [0, 1]$ such that $\varphi(x) = 0$ for $x \in [0, a]$ and such that $\varphi(x) = 1$ for $x \in [b, \infty)$. Now consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}^n & \to & [0,1] \\ x & \mapsto & 1 - f(\varphi(\|x\|)). \end{array}$$

Since $x \mapsto \|x\|$ is smooth on $\mathbb{R}^n \setminus \{0\}$ we see that f is smooth on $\mathbb{R}^n \setminus \{0\}$. Furthermore, since f is constant on a neighborhood of the origin we see that f is in fact smooth on all of $\mathbb{R}^n$. ■

Next we recall the definition of the differential.

Definition. Let $U \subset \mathbb{R}^n$ be an open subset, let $f = (f_1, \dots, f_m)\colon U \to \mathbb{R}^m$ be a smooth map and let $P \in U$. We refer to the $(m \times n)$-matrix $\mathrm{D}\, f_P := \left\{ \frac{\partial f_i}{\partial x_j}\Big|_P \right\}_{i=1,\dots,m, j=1,\dots,n}$ as the *differential of f at P.*

If $f\colon U \to \mathbb{R}^m$ is a smooth function on an open subset of $\mathbb{R}^n$, then given any $P \in U$ the map f is "close" to the affine linear map $w \mapsto \mathrm{D}\, f_P(P-w)+f(P)$. The following proposition, which is contained in every book on multivariable real analysis (see e.g. [**Fors08**, p. 70]), makes this notion precise.

Proposition 6.14. *Let $U \subset \mathbb{R}^n$ be an open subset and let $f\colon U \to \mathbb{R}^m$ be a smooth map. Given any $P \in U$ we have*

$$\lim_{v \to 0} \frac{\| f(P+v) - f(P) - \mathrm{D}\, f_P(v) \|}{\|v\|} \;=\; 0.$$

We also recall the chain rule for differentials.

Proposition 6.15. *Let $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^l$ be open subsets and let $f\colon U \to \mathbb{R}^l$ and $g\colon V \to \mathbb{R}^m$ be smooth maps with $f(U) \subset V$. Given any $P \in U$ we have*

$$\mathrm{D}(g \circ f)_P \;=\; \mathrm{D}\, f_{f(P)} \cdot \mathrm{D}\, f_P \qquad\qquad \text{“chain rule”}.$$

The next proposition says that locally, up to translation, one describe any smooth map as matrix multiplication where the coefficients of the matrix are allowed to vary.

Proposition 6.16. *Let $U \subset \mathbb{R}^n$ be an open convex subset that contains the origin. Furthermore let $f\colon U \to \mathbb{R}^m$ be a smooth map with $f(0) = 0$. There exists a smooth map $A\colon U \to \mathrm{M}(m \times n, \mathbb{R})$ such that $A(0) = \mathrm{D}\, f_0$ and such that*

$$f(x) \;=\; A(x) \cdot x \qquad \textit{for all } x \in U.$$

Proof. By considering the coordinate functions of a given map $f = (f_1, \dots, f_m)\colon U \to \mathbb{R}^m$ separately we see that it suffices to deal with the case $m = 1$. Thus let $f\colon U \to \mathbb{R}$ be a smooth map with $f(0) = 0$. Note that in this case we need to show that there exist smooth maps $a_1, \dots, a_n\colon U \to \mathbb{R}$ such that $a_i(0) = \frac{\partial f}{\partial x_i}(0)$ and with

$$f(x) \;=\; \sum_{i=1}^{n} a_i(x_i) \cdot x_i \qquad \text{for all } x = (x_1, \dots, x_n) \in U.$$

We observe that given any $x = (x_1, \dots, x_n) \in U$ we have

$$f(x) \;=\; f(x) - \underbrace{f(0)}_{=0} \;=\; \int_{t=0}^{t=1} \frac{d}{dt} f(tx)\, dt \;\underset{\substack{\uparrow \\ \text{chain rule}}}{=}\; \int_{t=0}^{t=1} \sum_{i=1}^{n} \frac{\partial f(tx)}{\partial x_i} \cdot x_i\, dt \;=\; \sum_{i=1}^{n} \underbrace{\int_{t=0}^{t=1} \frac{\partial f(tx)}{\partial x_i}\, dt}_{=: a_i(x)} \cdot x_i.$$

It follows from the above equality and the product rule that we have indeed the equality $a_i(0) = \frac{\partial f}{\partial x_i}(0)$. ■

We move on to the following definition.

Definition. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets. We say that a map $f\colon U \to V$ is a *diffeomorphism* if f is smooth, if f is a bijection and if $f^{-1}\colon V \to U$ is also smooth. If such a diffeomorphism exists, then we say that U and V are *diffeomorphic.*

Next we recall the following proposition from the analysis courses. Since it is one of the reasons why later on dealing with smooth manifolds is easier than dealing with topological manifolds we also give the proof.

Proposition 6.17. *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets.*

(1) *If $f: U \to V$ is a diffeomorphism, then for any $P \in U$ the differential $\mathrm{D}\, f_P$ is an invertible matrix.*
(2) *If U and V are diffeomorphic and non-empty, then $m = n$.*

Example. Suppose U is an open subset, let's say connected, of $\mathbb{R}^n$ and that $f: U \to \mathbb{R}^n$ is a smooth map such that $\mathrm{D}\, f_P$ is invertible for every $P \in U$. One might be tempted to think that $f: U \to f(U)$ is already a diffeomorphism itself. But this is not necessarily the case, a counterexample is for example given by

$$\begin{array}{rcl} f: \mathbb{C} \setminus \{0\} & \to & \mathbb{C} \setminus \{0\} \\ z & \mapsto & z^2. \end{array}$$

One can easily verify that $\mathrm{D}\, f_z$ is invertible for every $z \in \mathbb{C} \setminus \{0\}$. But f is evidently not injective.

Proof.

(1) Let $f: U \to V$ be a diffeomorphism between open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$. We denote by $g: V \to U$ the inverse of f. Let $P \in U$. It follows from the chain rule, i.e. from Proposition 6.15, that the matrices $\mathrm{D}\, f_P$ and $\mathrm{D}\, f_{f(P)}$ are inverses of one another, in particular $\mathrm{D}\, f_P$ is invertible.
(2) Since U is non-empty we can find a $P \in U$. By (1a) the $(m \times n)$–matrix $\mathrm{D}\, f_P$ is invertible, this means that the matrix is a square matrix, i.e. $m = n$. ■

Before we state the next theorem let us introduce the following harmless definition.

Definition. Let $f: U \to V$ be a smooth map between two open subsets of $\mathbb{R}^n$ and let $P \in U$. We say f is a *local diffeomorphism at P* if there exist open neighborhoods U' of P in U and V' of $f(P)$ in V such that $f|_{U'}: U' \to V'$ is a diffeomorphism.

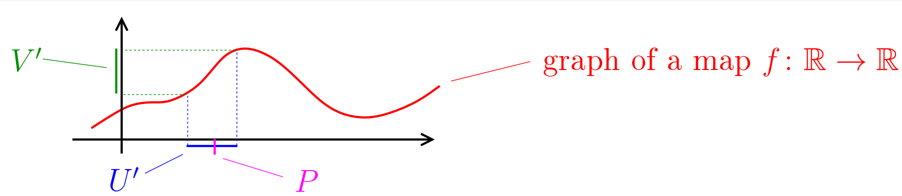

Figure 137

Now we can formulate the following theorem which can be viewed as a partial inverse to Proposition 6.26 (1a).

Theorem 6.18. (Inverse Function Theorem) *Let $f: U \to V$ be a smooth map between two open subsets of $\mathbb{R}^n$ and let $P \in U$ be a point. If $\mathrm{D}\, f_P$ is invertible, then f is a local diffeomorphism at P.*

Proof. This theorem is proved in most courses on multivariable real analysis. A proof is for example given in [**Lee02**, Corollary C.34]. ■

The Inverse Function Theorem 6.18 is one of the main results in real analysis. For example the proof of the following proposition rests heavily on the Inverse Function Theorem 6.18.

Theorem 6.19. (Smooth Invariance of Domain Theorem) *Let $U \subset \mathbb{R}^n$ be an open set and let $f\colon U \to \mathbb{R}^n$ be a smooth map such that the differential $\mathrm{D} f_P$ is invertible at every point P in the open set U. Then the following two statements hold:*

(1) *The map f is open, i.e. given any open subset $V \subset U$ the image $f(V)$ is an open subset of $\mathbb{R}^n$.*
(2) *If f is injective, then $f\colon U \to f(U)$ is a diffeomorphism.*

Remark. In the Topological Invariance of Domain Theorem 50.6 we will see that the statement of the Smooth Invariance of Domain Theorem 6.19 (1) also holds if we only assume that f is continuous and injective.

Proof. This proposition is a straightforward consequence of the Inverse Function Theorem 6.18. We leave it to the reader to fill in the few remaining details or alternatively we refer to [**Lee02**, Corollary C.36] for a proof. ■

On a few occasions we will need the following proposition that is proved in [**Pol01**, Theorem 2.1].

Proposition 6.20. *Let $U \subset \mathbb{R}^n$ be an open subset, let $f\colon U \to \mathbb{R}^n$ be a smooth map and let $P \in U$. If $\mathrm{D} f_P$ is invertible, then there exists an $\epsilon > 0$ such that $f(B^n_\epsilon(P))$ is convex.*

6.6. **Smooth manifolds.** In this section we will define the notion of a smooth structure on a topological manifold. We will refer to a topological manifold that is equipped with a "smooth structure" as a smooth manifold. One can easily fill many books just with the theory of smooth manifolds. Since it is not our goal to write a book on smooth manifolds we will only give a fairly quick introduction to smooth manifolds. We refer to the many books on smooth manifolds, e.g. [**Lee02, Jä93**] for details.

Now we can continue with the definition of a smooth manifold.

Definition.

(1) An atlas $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ for an n-dimensional topological manifold X is called *smooth* if for all $i, j \in I$ the *transition map*
$$\Phi_j \circ \Phi_i^{-1}\colon \underbrace{\Phi_i(U_i \cap U_j)}_{\text{open set in } \mathbb{R}^n \text{ or } H_n} \to \underbrace{\Phi_j(U_i \cap U_j)}_{\subset \mathbb{R}^n}$$
is smooth.
(2) An *n-dimensional smooth manifold* is a pair $(X, \mathcal{A})$ consisting of an n-dimensional topological manifold X together with a smooth atlas $\mathcal{A}$ for X. (As with many other definitions we usually suppress the choice of a smooth atlas from the notation.) We refer to X as the *underlying topological manifold.*

(3) We define the *boundary of a smooth manifold* $(X, \mathcal{A})$ as the set of all points that do not admit a chart of type (i) from the given atlas $\mathcal{A}$.[97]

Remark.

(1) In the literature the term "manifold" gets used with wildly different meanings, in particular often it gets to mean "topological manifold" or "smooth manifold". To prevent any misunderstanding we only rarely write "manifold" without any decorations, in we mostly do so in informal discussions where the precise meaning is irrelevant.
(2) Rather surprisingly in 1960 Michel Kervaire [**Kerv60**] showed that there exists a 10-dimensional topological manifold that does not admit a smooth atlas.

The following proposition gives us interesting examples of smooth manifolds.

Proposition 6.21. *The following topological spaces admit a smooth atlas:*

(1) *open subsets of* $\mathbb{R}^n$,
(2) *all closed balls* $\overline{B}^n$,
(3) *all spheres* $S^n \subset \mathbb{R}^{n+1}$,
(4) *the* n*-dimensional torus* $(S^1)^n$,
(5) *the surfaces of genus* $g \in \mathbb{N}_0$*, as defined on page 205,*
(6) *the non-orientable surfaces of genus* $g \in \mathbb{N}$*, as defined on page 206.*

More precisely, the atlases that we provided in Section 6.2 and in the proof of Proposition 6.8 are smooth.

Proof (∗). It is straightforward to verify that the atlases for examples (1) to (4) that we provided in Section 6.2 are smooth. It remains to consider the (non-orientable) surfaces of genus $g \geq 2$. The atlas that we provided in the proof of Proposition 6.8 looks more dubious. In fact all transition maps are compositions of (some of) the following types of maps:

(1) reflections in lines,
(2) translations,
(3) and for some $\eta > 0$ the map

$$\begin{array}{rcl} \{re^{\mathrm{i}\alpha}\,|\,r \in (0,\eta), \alpha \in (0, \frac{\pi}{4})\} & \to & \{re^{\mathrm{i}\alpha}\,|\,r \in (0,\eta), \alpha \in (0, \frac{3\pi}{4})\} \\ z = re^{\mathrm{i}\varphi} & \mapsto & re^{3\mathrm{i}\varphi} = \frac{1}{|z|^2} \cdot z^3, \end{array}$$

(4) the inverse of (3).

It is clear that the first three types of maps are smooth. It follows from Theorem 6.19 (2) that the fourth map is also smooth. Thus the transition maps in the atlas are smooth. ■

For completeness we state the following lemma which follows easily from the definitions.

[97] In fact in Proposition 44.2 (2) we will see that the boundary of a smooth manifold in the above sense agrees with the boundary of a topological manifold. But since so far we do not have the tools to prove this statement we prefer to use this definition.

Lemma 6.22. *Let M be a smooth manifold and let X be an open subset of M.*

(1) *The restrictions of the charts of the smooth atlas of M to X define a smooth atlas for X.*

(2) *The boundary ∂X of X viewed as a smooth manifold is given by $\partial X = X \cap \partial M$.*

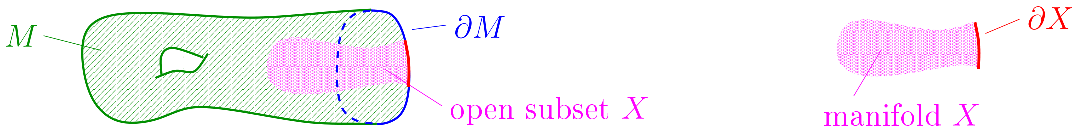

Figure 138. Illustration of Lemma 6.22.

We will constantly use Lemma 6.22 to view open subsets of a smooth manifold as smooth manifolds in their own right.

We give a few more standard definitions regarding smooth manifolds.

Definition.

(1) A map $f\colon M \to N$ between smooth manifolds is called *smooth* if for every chart Φ of M and any chart Ψ of N, in the respective atlases, the concatenation $\Psi \circ f \circ \Phi^{-1}$ is smooth.[98]

(2) Given two smooth manifolds M and N we denote by $C^\infty(M, N)$ the set of all smooth maps from M to N.

(3) We say a map $f\colon M \to N$ between smooth manifolds is a *diffeomorphism* if f is smooth, if f is a bijection and if $f^{-1}\colon N \to M$ is also smooth. If such a diffeomorphism exists, then we say that M and N are *diffeomorphic*.

(4) We say that two smooth atlases for a given topological manifold are *equivalent*, if the identity map is a diffeomorphism between the two smooth manifolds corresponding to the smooth atlases. A *smooth structure* on a topological manifold is a choice of an equivalence class. Usually we do not distinguish between a smooth atlas and the smooth structure it represents. In particular a smooth manifold, which by definition is a topological manifold together with a smooth atlas, is basically the same as a topological manifold together with a smooth structure.

(5) Let A be a subset of a smooth manifold M (in most applications $M = \mathbb{R}^n$), let N be a smooth manifold and let $f\colon A \to N$ be a map. We say f is *smooth* if given any $a \in A$ there exists an open neighborhood U in M of a and a smooth map $\widetilde{f}\colon U \to N$ which coincides with f on A.[99].

Remark.

(1) There exist smooth manifolds that are homeomorphic but not diffeomorphic. In fact John Milnor [**Miln56a**] showed in 1956 that there exist smooth manifolds that are homeomorphic to S^7 but that are not diffeomorphic to S^7. In other words, Milnor showed that S^7 admits at least two different smooth structures.

[98]If M is an m-dimensional smooth manifold and if N is an n-dimensional smooth manifold, then the map $\Psi\circ f\circ\Phi^{-1}$ is a map from an open subset of $\mathbb{R}^m$ or from an open subset of $H_m = \{(x_1, \dots, x_m) \,|\, x_m \geq 0\}$ to $\mathbb{R}^n$, hence the notion of smooth makes sense for such a map.

[99]This is of course just a generalization of the definition on page 272.

(2) It follows from [**KSi77**, p. 251] together with [**KeM63**, Theorem 1.2] that in dimension $n \neq 4$ every topological manifold admits at most a finite number of smooth structures. (We also refer to [**AMB07**, Chapter 7] and especially to [**Scor05**, p. 207-224] for more details.) Amazingly, this statement is wrong in dimension 4. More precisely there is a plethora of closed 4-dimensional smooth manifolds that admit an infinite number of smooth structures, see e.g. Theorem 102.13 for details.

The following harmless lemma contains three basic statements about smooth maps between smooth manifolds.

Lemma 6.23. *Let $f\colon M \to N$ be a map between two smooth manifolds.*

(1) *If there exists an open cover $\{U_i\}_{i\in I}$ of M such that each restriction $f|_{U_i}\colon U_i \to N$ is smooth, then f itself is smooth.*
(2) *If $f\colon M \to N$ is smooth it is also continuous.*
(3) *If $f\colon M \to N$ is a bijection (e.g. if it is a homeomorphism) and if it is a local diffeomorphism, then it is in fact a diffeomorphism.*

Proof.

(1) This statement follows from the observation that smoothness is an open condition.
(2) This statement is a straightforward consequence of Lemma 2.35 and the well-known fact that given an open set $U \subset \mathbb{R}^n$, any smooth map $U \to \mathbb{R}^m$ is also continuous. We refer to [**Lee02**, Proposition 2.4] for more details.
(3) Let $f\colon M \to N$ be a local diffeomorphism that is a bijection. We need to show that $f^{-1}\colon N \to M$ is also smooth. But that follows immediately from (1) and our hypothesis that f is a local diffeomorphism. ■

We continue with the following definition.

Definition. Given a smooth atlas $\mathcal{A}$ for a topological manifold M we define the *maximal smooth atlas* of $(M, \mathcal{A})$ to be the set of all charts of type (i) and (ii) from page 261 that are smooth maps with respect to the given smooth structure.[100]

The verification of the following lemma is left to the reader.

Lemma 6.24. *Let $(M, \mathcal{A})$ be a smooth manifold.*

(1) *The maximal atlas of $(M, \mathcal{A})$ is a smooth atlas for M that contains $\mathcal{A}$.*
(2) *The maximal atlas of $(M, \mathcal{A})$ equals the union of all smooth atlases of M that are equivalent to the atlas $\mathcal{A}$.*

In the previous section we introduced the notion of a submanifold of a topological manifold. We add the following convention.

Convention. If M is a smooth manifold, then we demand that in the definition of a "submanifold" on page 270 the charts come from the maximal smooth atlas.

[100] Let $\Phi\colon U \to V$ be a chart for M. By the convention from Lemma 6.22 we view U as a smooth manifold. Furthermore recall that we view any open subset of $\mathbb{R}^n$ and H_n as a smooth manifold with the obvious smooth atlas given by the identity. Thus it makes sense to ask whether or not $\Phi\colon U \to V$ is smooth.

Remark. Sometimes the above convention can be ambiguous. For example, if we consider $M = \mathbb{R}^2$ it is not clear whether we view M as a topological manifold or as a smooth manifold. If from the context it is not clear what we mean, then we distinguish in our language between a *topological submanifold* and a *smooth submanifold.* This distinction is illustrated in Figure 139.

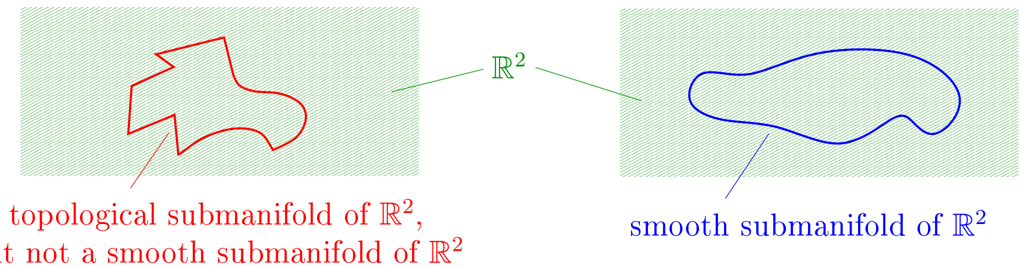

FIGURE 139

Examples.

(1) We consider the topological manifold $M = \mathbb{R}^2$ with the smooth atlas $\mathcal{A}$ given by the identity $\Phi = \mathrm{id}_M$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function. We consider the subset $N = \{(x, f(x) \,|\, x \in \mathbb{R}\}$. We cannot use the unique chart Φ in our atlas $\mathcal{A} = \{\Phi\}$ to show that N is indeed a submanifold of the smooth manifold $(\mathbb{R}^2, \mathcal{A})$. But the chart $\Psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\Psi(x, y) = (x, y - f(x))$ lies in the maximal atlas of $(\mathbb{R}^2, \mathcal{A})$ and we can use it to verify that N is indeed a submanifold of the smooth manifold $(\mathbb{R}^2, \mathcal{A})$.

(2) In Figure 140 we show a smooth manifold M with boundary together with two subsets K and L. In Exercise 6.11 we will see that both K and L are submanifolds of the topological manifold M but that they are not submanifolds if we view M as a smooth manifold.

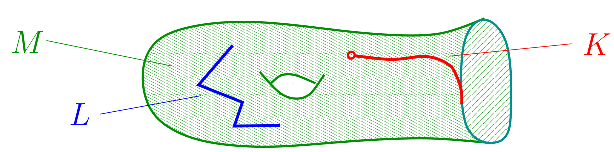

FIGURE 140

Remark. As we remarked on page 272, there are various subtly different definitions of a "submanifold" out in the literature. This is even more pronounced when it comes to submanifolds of smooth manifolds, especially if M and N are allowed to have boundary.

(1) Surely there is no textbook that would consider L in Figure 140 as a submanifold of the smooth manifold M. As we remarked above, if we follow our convention or the convention of [**Wall16**, p. 31], then K is not a submanifold of M. On the other

hand the definition used in [**Lee02**, p. 98] or [**Hirs76**, p. 30] views K as a smooth submanifold of the smooth manifold M.

(2) Also note that what we call a proper submanifold is called a neat submanifold in [**Hirs76**, p. 30].

The following lemma follows almost immediately from Lemma 6.11 and the definitions. We leave it to the reader to fill in the details.

Lemma 6.25. *Let M be a smooth manifold and let N be a submanifold of M. Then N admits a natural structure of a smooth manifold which has the property that the inclusion map $N \to M$ is a smooth map.*

Example. Let M be a smooth manifold and let $U \subset M$ be an open subset. As on page 272 one sees easily that U is a codimension-zero submanifold of M. Furthermore, if we use Lemma 6.25 to view U as a smooth manifold in its own right, then it follows again basically immediately from the definitions that $\partial U = U \cap \partial M$.

On many occasions we will develop the theory of topological and of smooth manifolds in parallel. In colloquial terms we sometimes use just the term "manifolds", without adjectives, to indicate that we are interested on both types or that the type is irrelevant. In actual definitions and statements we will never use the term "manifold" and we will specify explicitly use the cumbersome expressions "smooth manifold" and "topological manifold". Since the underlying topological space of a smooth manifold is a topological manifold, many results on topological manifolds hold by fiat also for smooth manifolds.

6.7. **Topological properties of smooth manifolds.** In this section we will employ several results from Section 6.5 to draw conclusions on smooth manifolds. The fact that real analysis provides us with such powerful tools is one of the reasons why dealing with smooth manifolds is much easier than dealing with topological manifolds.

Proposition 6.26. *Let M be a non-empty m-dimensional smooth manifold and let N be an n-dimensional smooth manifold. If M and N are diffeomorphic, then $m = n$.*

Proof. The proposition follows easily from Proposition 6.17 by considering charts. ■

Proposition 6.26 thus says, in particular, that the dimension of a non-empty smooth manifold is well-defined. Eventually, using homology groups, we will show that the analogous statements also holds for topological manifolds.

As mentioned earlier, the Inverse Function Theorem 6.18 and the resulting Theorem 6.19 are some of the main tools coming from real analysis. In particular one can use them to prove the following, slightly subtle proposition.

Proposition 6.27. *Let X be an n-dimensional smooth manifold, i.e. X is an n-dimensional topological manifold equipped with a smooth atlas. The following statements hold:*

(1) *Every point on X admits either a chart of type (i) from the given smooth atlas or it admits a chart of type (ii) from the given smooth atlas.*
(2) *The boundary ∂X has a canonical structure of an $(n-1)$-dimensional smooth manifold with $\partial(\partial X) = \varnothing$. Furthermore the inclusion map $\partial X \to X$ is a smooth map.*

(3) (a) *Any union of components of ∂X is a closed subset of X. In particular ∂X itself is a closed subset of X.*
(b) *If X is compact, then ∂X is also compact.*
(4) *If ∂X is non-empty, then $X \setminus \partial X$ is non-compact.*
(5) *If $N \subset X$ is a proper k-dimensional submanifold, then we have $N \cap \partial X = \partial N$[101], and $N \cap \partial X = \partial N$ is a proper $(k-1)$-dimensional submanifold of ∂X.*

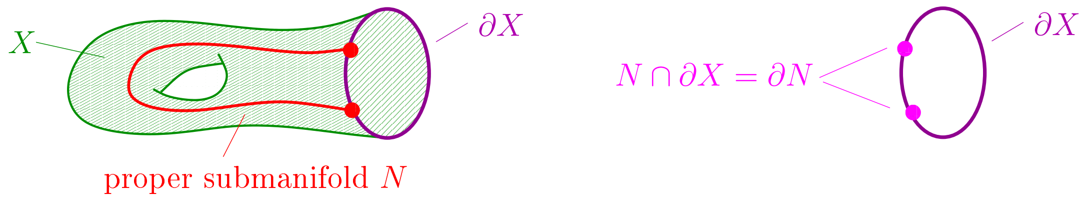

FIGURE 141. Illustration of Proposition 6.27 (5).

Remark. Here again we would expect that a similar statement should hold for topological manifolds. But since for topological manifolds we have fewer tools, e.g. right now we are missing an analogue of Theorem 6.19, we have to postpone a proof for topological manifolds to Proposition 44.2.

PROOF (∗). Let X be an n-dimensional topological manifold that is equipped with a smooth atlas $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$.

(1) With a little bit of effort this statement can be deduced from Theorem 6.19. We refer to [**Lee02**, Theorem 1.46] for the full details.
(2) First note that it follows from Lemma 2.12 and from Lemma 6.1 (2) that ∂X is Hausdorff and second-countable. Now it remains to show that ∂X has a canonical smooth structure. We start out with the following observation. Let $\Phi\colon U \to V$ be a chart of type (ii) for X. Note that it is a chart of type (i) precisely for the points in $\Phi^{-1}(\partial H_n)$. It follows from (1) and the definition of ∂X that $\Phi^{-1}(\partial H_n) = \partial X \cap U$.

Given $i \in I$ we now set $\widetilde{V}_i = V_i \cap \partial H_n$, we set $\widetilde{U}_i := \Phi_i^{-1}(\widetilde{V}_i)$ and we set $\widetilde{\Phi}_i := \Phi_i|_{\widetilde{U}_i}$. It follows fairly easily from the above discussion that that $\{\widetilde{\Phi}_i\colon \widetilde{U}_i \to \widetilde{V}_i\}_{i\in I}$ is a smooth atlas for ∂M and that with respect to this atlas every point in ∂X admits a chart of type (i). In other words, $\partial(\partial X) = \varnothing$.

Finally note that it follows easily from the fact that the inclusion $\mathbb{R}^{n-1} \xrightarrow{x\,\mapsto\,(x,0)} \mathbb{R}^n$ is smooth that the inclusion $\partial X \to X$ is a smooth map.
(3) The proof of the third statement is identical to the proof of Lemma 6.7.[102]
(4) So suppose that the boundary ∂X is non-empty. Let $P \in \partial X$. Given $r \in \mathbb{R}_{\geq 0}$ we write $V_r := \{(x_1, \ldots, x_n) \in B_r^n \,|\, x_n \geq 0\}$ and we denote by $\overline{V}_r$ the closure of V_r in $\mathbb{R}^n$. We write $V = V_1$. We can and will pick a chart $\Phi\colon U \to V$ of type (ii) with

[101]Here ∂N denotes the boundary of N, viewed as a smooth manifold in its own right.

[102]Note that this statement does not follow directly from Lemma 6.7 since our definition of the boundary of a smooth manifold, given on page 277, is a priori different from the definition of the boundary of a topological manifold. Only later in Proposition 44.2 (2) will we see that there is no difference.

$\Phi(P) = 0$. We write $\Psi := \Phi^{-1}\colon V \to U$. Given $n \in \mathbb{N}$ we consider

$$U_n \;=\; \underbrace{\big(X \setminus \Psi\big(\overline{V}_{\frac{1}{2n}}\big)\big)}_{\text{open by Lemmas 2.40 and 2.17 (2)}} \cap \overbrace{(X \setminus \partial X)}^{\text{open by (3a)}}$$

We make the following observations:

(a) As discussed, the sets U_n are open and it is clear that they cover $X \setminus \partial X$.

(b) It follows from (1) that $U \cap \partial X = \Psi(\partial V)$. This implies that no finite collection of these sets covers $X \setminus \partial X$.

(c) The combination of (a) and (b) shows $X \setminus \partial X$ is not compact.

(5) This statement follows now easily from the definition of a proper submanifold. We leave it to the reader to fill in the details. ■

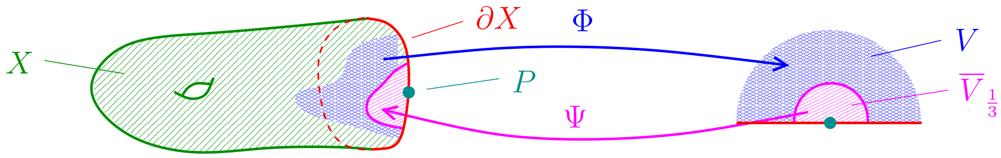

Figure 142. Illustration for the proof of Proposition 6.27 (4).

Next we introduce the following notation.

Notation. Let M be an n-dimensional smooth manifold and let X be an n-dimensional submanifold. We write[103]

$$\partial_0 X \;:=\; \partial X \setminus \partial M \qquad \text{and} \qquad \partial_1 X \;:=\; \partial X \cap \partial M.$$

We refer to Figure 143 for an illustration.

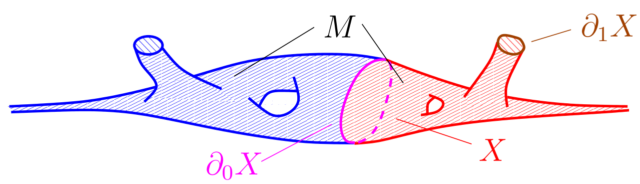

Figure 143

The following lemma is used frequently, often without noticing it.

Lemma 6.28. *Let M be an n-dimensional smooth manifold and let X be an n-dimensional submanifold. If X is a closed subset of M (e.g. if X is a compact subset), then the following statements hold:*

(1) *$\partial_0 X$ and $\partial_1 X$ are unions of components of ∂X.*

(2) *The boundary of X as a subset of M agrees with $\partial_0 X$.*

(3) *The interior $\mathring{X}$ of X as a subset of M agrees with $X \setminus \partial_0 X$.*

[103]Here ∂X denotes the boundary of X viewed as a smooth manifold in its own right.

Proof (∗). Let X be an n-dimensional submanifold of M that is a closed subset of M.

(1) It follows easily from Proposition 6.27 (1) that $\partial_1 X = \partial X \cap \partial M$ is an open subset of ∂X. Since X is a closed subset of M it also follows from Proposition 6.27 (3) that $\partial_1 X = \partial X \cap \partial M$ is a closed subset of ∂X. Thus it follows from Lemma 2.68 (4) that $\partial_1 X$ is a union of components of ∂X. Evidently $\partial_0 X$ is the union of the remaining components of ∂X.

(2) Let $P \in M$. The desired statement follows from the following observations:

$$\begin{array}{rcl} & \text{since } X \text{ is closed} & \\ & \downarrow & \\ \begin{array}{c} P \text{ lies in the boundary} \\ \text{of } X \text{ as a subset of } M \end{array} & \Longleftrightarrow & \begin{array}{c} P \text{ lies on } X \text{ and every open neighborhood} \\ \text{of } P \text{ contains a point in } M \setminus X \end{array} \\ & \Longleftrightarrow & P \text{ admits a submanifold chart of type } (\gamma) \\ & \uparrow & \\ & \text{since } X \text{ is a submanifold of codimension zero} & \\ & \Longleftrightarrow & \begin{array}{c} P \text{ lies on the boundary of } X \text{ viewed as a} \\ \text{smooth manifold but it does not lie on } \partial M. \end{array} \\ & \uparrow & \\ & \text{Proposition 6.27 (1)} & \end{array}$$

(3) This statement follows immediately from (2) together with Lemma 2.8 (6). ■

We continue with the following definition.

Definition. Let M be an n-dimensional smooth manifold. We say M is *decomposed into submanifolds* $\{A_i\}_{i \in I}$ if the following conditions hold:

(1) We have $M = \bigcup\limits_{i \in I} A_i$.
(2) Each A_i is an n-dimensional submanifold of M.
(3) Each A_i is a closed subset of M.
(4) For any $i, j \in I$ with $i \neq j$ the intersection $A_i \cap A_j$ is a union of boundary components of A_i and it is a union of boundary components of A_j.

This definition is illustrated in Figure 144.

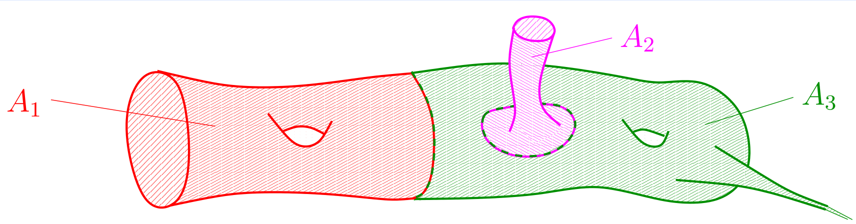

Figure 144

The following lemma shows that the Klein bottle and the real projective plane admit some interesting decompositions into submanifolds.

Lemma 6.29.

(1) *We can decompose the Klein bottle into two submanifolds M and N where M and N are both diffeomorphic to the Möbius band.*

(2) *We can decompose the real projective plane into two submanifolds D and M where D is diffeomorphic to the disk $\overline{B}^2$ and M is diffeomorphic to the Möbius band.*

PROOF.

(1) The proof of the first statement is sketched in Figure 145. In Figure 146 we attempt to sketch the equivalent statement that the Klein bottle is obtained from gluing two copies of the Möbius band along the common boundary.
(2) The proof of the second statement is sketched in Figure 147. Here, for $\mathbb{RP}^2$ we use the homeomorphism $\mathbb{RP}^2 \cong \overline{B}^2/\sim$ that we gave in Lemma 3.42. ■

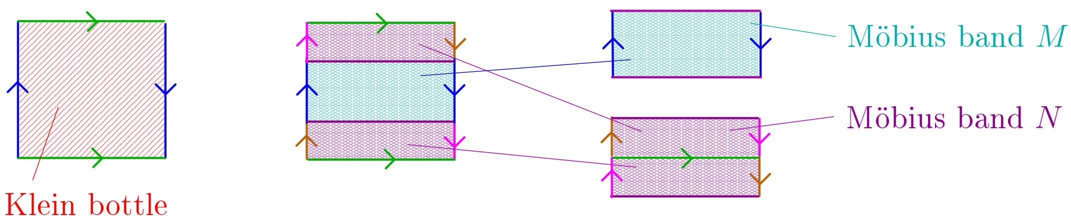

FIGURE 145

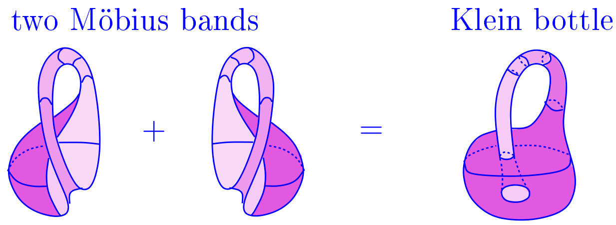

FIGURE 146

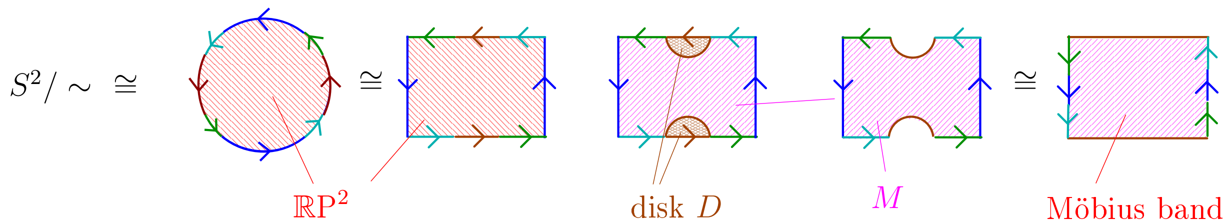

FIGURE 147

The following proposition will be helpful on several occasions.

Proposition 6.30. (∗) *Let M be an n-dimensional smooth manifold and let W be an n-dimensional submanifold of M with $W \cap \partial M = \varnothing$. If W is compact, then the following two statements hold:*

(1) *The smooth manifold M is decomposed into the submanifolds M and $X := M \setminus \mathring{W}$. (Recall that by Lemma 6.28 we know that the interior $\mathring{W}$ of W as a subset of M agrees with $W \setminus \partial W$ where ∂W denotes the boundary of W viewed as a smooth manifold in its own right.)*
(2) *If M is connected and if ∂W is connected, then $M \setminus \mathring{W}$ is also connected.*

(3) *If M is compact, then $M \setminus \mathring{W}$ is also compact.*

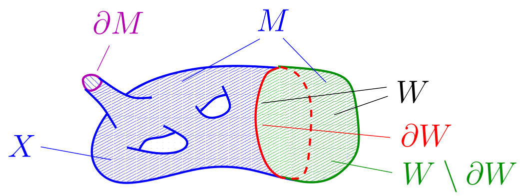

FIGURE 148. Illustration of Proposition 6.30.

PROOF (∗).

(1) First note that it is a consequence of Lemma 2.17 and the fact that smooth manifolds are Hausdorff that W is a closed subset of M. The statement is now a fairly immediate consequence of Lemma 6.28. We leave it to the reader to fill in the details.
(2) This statement follows immediately from Lemma 6.28 together with Lemma 2.63.
(3) Since $\mathring{W}$ is an open subset of M we see that $M \setminus \mathring{W}$ is a closed subset of M. If M is compact, then it follows from Lemma 2.17 that $M \setminus \mathring{W}$ is indeed compact. ■

We conclude this section with one more technical statement. To formulate this statement we need the following definition which we had already introduced in Exercise 2.47.

Definition. Let X be a topological space. A subset $A \subset X$ is called *precompact* if the closure $\overline{A}$ is compact.

Now we can formulate the last result of this section.

Proposition 6.31. (∗) *Given any n-dimensional topological manifold M there exists a countable basis $\{U_i\}_{i\in\mathbb{N}}$ of the topology with the following properties:*

(1) *Each U_i is open and each U_i is precompact.*
(2) *Given any $i \in \mathbb{N}$ there exists either a homeomorphism $\varphi_i \colon B^n \to M$ or a homeomorphism $\varphi_i \colon B^n \cap H_n \to M$.*

Furthermore, if M is in fact a smooth manifold, then the following condition is satisfied:

(2') *In (2) above, the maps φ_i can be chosen to be diffeomorphisms.*

SKETCH OF PROOF (∗).

(1) If M is an open subset of $\mathbb{R}^n$, then the statement follows immediately from Lemma 2.7. If M is an open subset of the upper half-space H_n, then the statement follows from a modest variation on the argument of Lemma 2.7.
(2) By Lemma 6.2 we know that M admits a countable atlas.
(3) The general statement is now a consequence of (1), (2) and Exercise 2.47.

We leave it to the reader to fill in the details. Alternatively, a somewhat more detailed account of the proof is given in [**Lee02**, Lemma 1.10]. ■

6.8. **Manifolds and group actions.** Now we return to group actions since these are one of the best ways to construct new topological spaces and manifolds.

Definition. Let M be a smooth manifold with an action by a group G. We say that the action is *smooth* if for every $g \in G$ the map

$$\begin{array}{rcl} M & \to & M \\ x & \mapsto & g \cdot x \end{array}$$

is smooth.

Example. Basically all the actions on smooth manifolds that we had written down so far are smooth. We will study some examples in greater detail after the next proposition.

The following proposition is the main result of this section.

Proposition 6.32. *If G is a group that acts freely, properly and continuously on an n-dimensional topological manifold M, then the following hold:*

(1) *The quotient space M/G is also an n-dimensional topological manifold and the projection map $p\colon M \to M/G$ is a local homeomorphism.*
(2) *If M is a smooth manifold and if G acts smoothly, then there exists a unique smooth structure on M/G such that the projection map $p\colon M \to M/G$ is a local diffeomorphism.*

Furthermore, in both cases the following three statements hold:

(a) *The G-action restricts to an action on ∂M and we have $\partial(M/G) = (\partial M)/G$.*
(b) *If M is compact, then M/G is compact.*
(c) *If M is closed, then M/G is closed.*

Examples.

(1) The action
$$\begin{array}{rcl} \mathbb{Z}^n \times \mathbb{R}^n & \to & \mathbb{R}^n \\ (z, v) & \mapsto & z + v \end{array}$$
from page 186 is free, proper and smooth. It follows from Proposition 6.32 that $\mathbb{R}^n/\mathbb{Z}^n$ is an n-dimensional smooth manifold.
(2) Using the action of $G = \mathbb{Z}$ on $M = \mathbb{R} \times [-1, 1]$ from page 188 we obtain that the Möbius band $(\mathbb{R} \times [-1, 1])/\mathbb{Z}$ is a 2-dimensional smooth manifold. Furthermore, we have
$$\partial(\text{Möbius band}) \underset{\substack{\uparrow \\ \text{Proposition 6.32 (a)}}}{=} (\mathbb{R} \times \{-1, 1\})/\mathbb{Z} \underset{\substack{\uparrow \\ \text{see Exercise 3.37}}}{\cong} S^1.$$
(3) Finally, using the action from page 187 of $G = \{\pm 1\}$ on $M = S^n$ we obtain that the real projective space $\mathbb{RP}^n = S^n/\{\pm 1\}$ is a closed n-dimensional smooth manifold. Now let $m \le n$. On page 194 we introduced the convention that we view $\mathbb{RP}^m$ as a subset of $\mathbb{RP}^n$.[104] We leave it as a voluntary exercise to verify that $\mathbb{RP}^m$ is in fact a submanifold of $\mathbb{RP}^n$.

The proof of Proposition 6.32 relies on the following two somewhat technical lemmas.

[104] We will give an explicit smooth atlas for $\mathbb{RP}^n$ in Exercise 6.22.

Lemma 6.33. (∗) *Let X be a topological space and let G be a group that acts freely, properly and continuously on X. If X is Hausdorff, then for every $x \in X$ there exists an open neighborhood U such that $gU \cap U = \varnothing$ for all $g \neq e$.*

Proof of Lemma 6.33 (∗) . Let X be a topological space that is Hausdorff. Furthermore let G be a group that acts freely, properly and continuously on X. Finally let $x \in X$. Since G acts freely we know that for every non-trivial $g \in G$ we have $gx \neq x$.

Since the action is proper and continuous we can apply Lemma 3.34 to $a = b = x$ and we obtain open neighborhoods A and B of x such that $gA \cap B = \varnothing$ for every non-trivial $g \in G$. Thus $U := A \cap B$ has the desired property ■

Lemma 6.34. (∗) *Let X be a topological space that is Hausdorff. Furthermore let G be a group that acts continuously on X. We denote by $p\colon X \to X/G$ the projection. If $U \subset X$ is an open subset such that the map $p\colon U \to X/G$ is injective, then the map $p\colon U \to p(U)$ is a homeomorphism.*

Proof of Lemma 6.34 (∗) . By Lemma 3.30 the projection $p\colon X \to X/G$ is continuous. Thus the restriction of p to U is also continuous. By our hypothesis the map $p\colon U \to p(U)$ is injective and evidently it is surjective. Thus the map is a bijection. By Lemma 3.30 (2) the map is furthermore open. It follows from Lemma 2.42 (2) that the map $f\colon U \to p(U)$ is in fact a homeomorphism. ■

Now we are in a position to prove Proposition 6.32. For clarity we deal with topological manifolds and smooth manifolds separately.

Proof of Proposition 6.32 for topological manifolds. Let M be an n-dimensional topological manifold and let G be a group that acts on M. We suppose that the action is continuous, free and proper. We want to show that M/G is an n-dimensional topological manifold. Thus we need to prove the following claim.

Claim.

(1) M/G is second-countable,
(2) M/G is Hausdorff, and
(3) every point $y \in M/G$ admits an n-dimensional chart.

The first statement of the claim follows immediately from Lemma 3.30 and 6.1 (5). The second statement follows from Proposition 3.33. Thus it remains to deal with the third statement. As usual we denote by $p\colon M \to M/G$ the projection. Let $y \in M/G$. We pick an $x \in M$ with $p(x) = y$. We make the following two observations:

(a) Since M is an n-dimensional topological manifold there exists an n-dimensional chart $\Phi\colon V \to W$ with $x \in V$.
(b) By Lemma 6.33 there exists an open neighborhood U of x such that $gU \cap U \neq \varnothing$ for every $g \neq e$. Note that this implies that the restriction of $p\colon M \to M/G$ to U is injective.

After possibly replacing U and V by $U \cap V$ we can assume that $U = V$. Note that by Lemma 3.30 (2) the projection $p(U)$ is an open neighborhood of x. Now we consider the maps

$$p(U) \xrightarrow{p^{-1}} U \xrightarrow{\Phi} W.$$

The map Φ is by definition a homeomorphism. Furthermore it follows from Lemma 6.34 that the map $p\colon U \to p(U) \subset M/G$ is a homeomorphism as well. Thus the above map $\Phi \circ p^{-1}\colon p(U) \to W$ is a homeomorphism, in particular it is an n-dimensional chart for M/G that contains y. This completes the proof of the claim and thus completes the proof that M/G is an n-dimensional topological manifold.

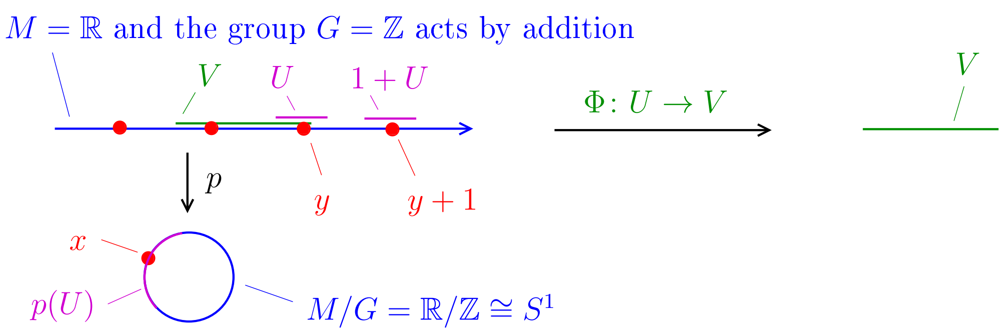

FIGURE 149. Illustration for the proof of Proposition 6.32.

It remains to prove the following claim.

Claim.

(1) The projection $p\colon M \to M/G$ is a local homeomorphism.
(2) The G-action on M restricts to an action on ∂M and we have $\partial(M/G) = (\partial M)/G$.
(3) If M is compact, then M/G is compact.
(4) If M is closed, then M/G is closed.

We turn to the proofs of these four statements.

(1) Implicitly we had already proved the first statement, indeed, given $x \in M$ we run through the above argument and we see, with the same notation, that $p(U)$ is an open subset of M/G and that $p\colon U \to p(U)$ is a homeomorphism.
(2) It follows from Lemma 6.6 (1) that the G-action on M restricts to an action on ∂M. Furthermore it follows easily from Lemma 6.6 (2) and the above statement (1) that $\partial(M/G) = (\partial M)/G$. We leave it to the reader to fill in the details.
(3) The third statement is of course just an immediate consequence of Lemma 2.40.
(4) This statement is an immediate consequence of statements (2) and (3).

This completes the proof of the second claim. ■

PROOF OF PROPOSITION 6.32 FOR SMOOTH MANIFOLDS $(*)$. Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold. Let G be a group that acts freely, properly and smoothly on M. We need to show that M/G admits a smooth atlas such that the projection map $p\colon M \to M/G$ is a local diffeomorphism.

(1) We run through the proof of Proposition 6.32 (1), but this time only using charts for M that lie in $\mathcal{A}$, and we obtain an atlas $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ for M/G. It is straightforward to see that any transition map $\Phi_j \circ \Phi_i^{-1}\colon \Phi_i(U_i \cap U_j) \to \Phi_j(U_i \cap U_j)$ can be written as the composition of (some) the following maps:
   (a) charts[105] and their inverses,
   (b) restrictions to open subsets,
   (c) the restriction of the action of $g \in G$ on M to some open subset.
   All these maps are smooth maps between smooth manifolds.[106] Thus the transition maps are smooth.

(2) Note that in the above proof of Proposition 6.32 (1) we already saw that the projection $p\colon M \to M/G$ is a local homeomorphism. It is straightforward to verify that, with our choice of a smooth atlas for M/G, the projection is in fact a local diffeomorphism.

The proof of the remaining statements (a), (b) and (c) is basically identical to the above proof for topological manifolds.

Finally, the statement that the smooth structure on M/G is uniquely determined by the smooth structure on M and the property that the projection $M \to M/G$ is a local diffeomorphism is a consequence of Lemma 6.23 (1) and the fact that the action is proper. We leave it to the reader to fill in the details. ■

We conclude this section with the following amusing lemma. It gives an example of a diffeomorphism that will pop up on several occasions.

Lemma 6.35. *Let $n \in \mathbb{N}$. If $A \in \mathrm{GL}(n,\mathbb{Z})$ is a matrix, then the map*

$$\begin{array}{rcl} f(A)\colon \mathbb{R}^n/\mathbb{Z}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ v & \mapsto & Av \end{array}$$

is a diffeomorphism. It is orientation-preserving if and only if $A \in \mathrm{SL}(n,\mathbb{Z})$.

Proof. We will provide the proof in Exercise 6.18. ■

Example. We consider the map $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. The corresponding self-diffeomorphism $f(A)$ of $\mathbb{R}^2/\mathbb{Z}^2 = ([0,1]\times[0,1])/\sim$ is illustrated in Figure 150.

6.9. **Tangent spaces of smooth manifolds.** In this and the following section we introduce tangent spaces and orientations on smooth manifolds. A more detailed account is given in most textbooks on smooth manifolds. The following discussion is based on [**Lee02**, Chapter 15], [**Jä93**, Chapter 4] and [**Frie16a**].

[105] Recall that we view open subsets of $\mathbb{R}^n$ as smooth manifolds. With this convention charts are in fact diffeomorphisms between smooth manifolds.

[106] Here the smooth manifolds are open subsets of $\mathbb{R}^n$ or of M.

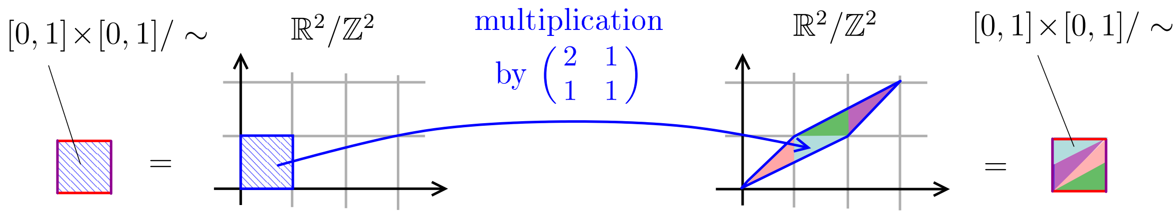

FIGURE 150

Definition. Let M be a smooth manifold and let $P \in M$.[107]

(1) A *derivation at P* is a linear map $v\colon C^\infty(M,\mathbb{R}) \to \mathbb{R}$ such that for every choice of $f, g \in C^\infty(M,\mathbb{R})$ we have
$$v(fg) \;=\; f(P)\cdot v(g) + v(f)\cdot g(P).$$
(2) The set of all derivations at P is called the *tangent space* $\mathrm{T}_P\, M$ *at* P. We refer to elements of $\mathrm{T}_P\, M$, i.e. we refer to derivations at P, also as *tangent vectors*.
(3) Let $P \in \partial M$. We say a tangent vector $v \in \mathrm{T}_P\, M$ *points outward*, if the following two conditions are satisfied:
   (a) For every smooth map $f\colon M \to \mathbb{R}_{\geq 0}$ with $f|_{\partial M} \equiv 0$ we have $v(f) \geq 0$.
   (b) There exists a smooth map $f\colon M \to \mathbb{R}_{\geq 0}$ with $f|_{\partial M} \equiv 0$ such that $v(f) > 0$.

   By replacing "$\geq$" and "$>$" by "$\leq$" and "$<$" we obtain the notion of an inward pointing tangent vector.

This definition of a tangent space might sound rather odd, but the following proposition should help in getting some intuition for the definition.

Proposition 6.36. *Let M be an open subset of $\mathbb{R}^n$ or of H_n. For any $P \in M$ the map*
$$\begin{array}{rcl} \mathbb{R}^n & \to & \mathrm{T}_P\, M \\ v & \mapsto & \left(\begin{array}{rcl} C^\infty(M,\mathbb{R}) & \to & \mathbb{R} \\ f & \mapsto & \frac{d}{dt}\big|_{t=0} f(P+tv)) \end{array}\right) \end{array}$$
is an isomorphism of vector spaces.

PROOF. The proof of this proposition is an amusing exercise in real analysis. A detailed proof for the case that M is an open subset of $\mathbb{R}^n$ is given in [**Lee02**, Proposition 3.2]. The case that M is an open subset of H_n follows fairly easily from the definition of a smooth map on H_n, see page 272. ■

Convention. Given an open subset M of $\mathbb{R}^n$ or of H_n and given $P \in M$ we will use the isomorphism $\mathbb{R}^n \xrightarrow{\cong} \mathrm{T}_P\, M$ from Proposition 6.36 to make the identification $\mathrm{T}_P\, M = \mathbb{R}^n$.

The following proposition summarizes some basic properties of the tangent space.

[107] We should stress that we allow P to be in the boundary of M.

Proposition 6.37. *Let M and N be two smooth manifolds.*

(1) *If $f\colon M \to N$ is a smooth map, then for any $P \in M$ the map*

$$\begin{array}{rcl} \mathrm{D}\, f_P = f_*\colon \mathrm{T}_P\, M & \to & \mathrm{T}_{f(P)}\, N \\ v & \mapsto & \left(\begin{array}{rcl} C^\infty(N,\mathbb{R}) & \to & \mathbb{R} \\ g & \mapsto & v(g\circ f) \end{array}\right) \end{array}$$

is linear.

(2) *The maps from (1) have the following two properties:*[108]

(a) *If $\mathrm{id}\colon M \to M$ is the identity map, then for any $P \in M$ the corresponding map $D\,\mathrm{id}_P\colon \mathrm{T}_P\, M \to \mathrm{T}_P\, M$ is also the identity.*

(b) *For any two smooth maps $f\colon L \to M$ and $g\colon M \to N$ between smooth manifolds and any $P \in L$ we have*

$$\mathrm{D}(g\circ f)_P \;=\; \mathrm{D}\, g_{f(P)} \circ \mathrm{D}\, f_P\colon\ \mathrm{T}_P\, L \;\to\; \mathrm{T}_{(g(f(P))}\, N.$$

(3) (a) *If $f\colon M \to N$ is a local diffeomorphism, then for any $P \in M$ the corresponding map $\mathrm{D}\, f_P\colon \mathrm{T}_P\, M \to \mathrm{T}_{f(P)}\, N$ is an isomorphism.*

(b) *If M is a codimension-zero submanifold of N, then for any $P \in M$ the corresponding map $\mathrm{D}\, f_P\colon \mathrm{T}_P\, M \to \mathrm{T}_{f(P)}\, N$ is an isomorphism.*

(4) *If M is an n-dimensional smooth manifold, then for any $P \in M$ the tangent space is an n-dimensional vector subspace of the vector space $\mathrm{Hom}(C^\infty(M,\mathbb{R}),\mathbb{R})$.*

Definition. Given a smooth map $f\colon M \to N$ between two smooth manifold and given $P \in M$ we refer to the map $\mathrm{D}\, f_P\colon \mathrm{T}_P\, M \to \mathrm{T}_{f(P)}\, N$ from Proposition 6.37 (1) as the *differential of f at P.*

Sketch of proof. The first three statements follow fairly easily from the definitions. Using charts one can reduce the proof of Statement (4) to the case that M is an open subset of $\mathbb{R}^n$ or H_n. But we had dealt with this case in Proposition 6.36. We refer to [**Lee02**, Proposition 3.10] and [**Lee02**, Proposition 3.6] for more details. ■

There are several alternative interpretations of the tangent space of a smooth manifold. In the following we will discuss a commonly used definition for submanifolds of $\mathbb{R}^k$.

Definition.

(1) Let M be an n-dimensional submanifold of $\mathbb{R}^k$. Given any $P \in M$ we define the *visual tangent space*

$$\mathrm{V}_P\, M \;:=\; \begin{array}{c} \text{all derivatives } \gamma'(0) \text{ of smooth maps } \gamma\colon (-\epsilon,0] \to M \text{ with } \gamma(0)=P \\ \text{and all derivatives } \gamma'(0) \text{ of smooth maps } \gamma\colon [0,\epsilon) \to M \text{ with } \gamma(0)=P. \end{array}$$

This definition is illustrated in Figure 151.

[108]The reader who is already familiar with categories will notice that this statement just says that the tangent space at a point is a functor from the category of pointed smooth manifolds to the category of real vector spaces.

(2) Let M be an n-dimensional submanifold of $\mathbb{R}^k$ and let $P \in \partial M$. We say a tangent vector $v \in \mathrm{V}_P\, M$ *points outward* if $v = \gamma'(0)$ for some smooth map $\gamma\colon (-\epsilon, 0] \to M$ and if $v \notin \mathrm{V}_P\, \partial M$. Similarly we also define the notion that v *points inward*.
(3) Let $f\colon M \to N$ be a smooth map between submanifolds of $\mathbb{R}^k$ respectively $\mathbb{R}^l$. Given any $P \in M$ we define[109]

$$\begin{array}{rcl} \mathrm{D}\, f_P = f_*\colon \mathrm{V}_P\, M & \to & \mathrm{V}_{f(P)}\, N \\ \gamma'(0) & \to & (f \circ \gamma)'(0) \end{array}$$

We refer to this map as the *differential of f at P*.[110]

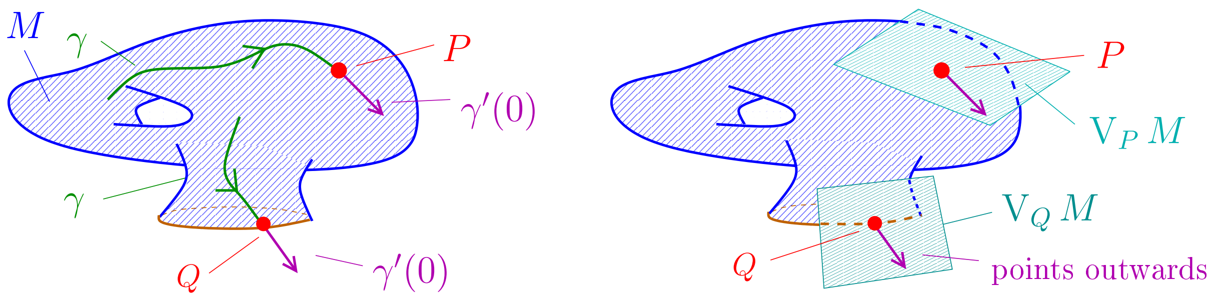

FIGURE 151

Remark. Let M be an n-dimensional submanifold of $\mathbb{R}^k$ and let $P \in M$. If $P \notin \partial M$, then using charts and using the definition of a smooth map on a half-open interval one can easily show that

$$\mathrm{V}_P\, M \ := \ \text{all derivatives } \gamma'(0) \text{ of smooth maps } \gamma\colon (-\epsilon, \epsilon) \to M \text{ with } \gamma(0) = P.$$

Examples.

(1) Let M be an open subset of $\mathbb{R}^n$ or of the half-plane H_n. For any $P \in M$ we have, more or less evidently, the equality $\mathrm{V}_P\, M = \mathbb{R}^n$.
(2) Let $f\colon M \to N$ be a smooth map between an open subset of $\mathbb{R}^m$ and an open subset of $\mathbb{R}^n$. We leave it as a minor exercise to show, using Proposition 6.15, that the map $\mathrm{D}\, f_P\colon \mathrm{V}_P\, M = \mathbb{R}^m \to \mathrm{V}_P\, N = \mathbb{R}^n$ is given precisely by multiplication by the differential $\mathrm{D}\, f_P \in \mathrm{M}(n \times m, \mathbb{R})$ that we introduced on page 274.
(3) For $M = S^n \subset \mathbb{R}^{n+1}$ one easily shows that for any $P \in S^n$ we have the equality $\mathrm{V}_P\, M = \{v \in \mathbb{R}^{n+1} \mid \langle v, P\rangle = 0\}$.

We formulate the next example as a lemma, so that later on it is easy to refer to the statement.

Lemma 6.38. $(*)$ *If $W \subset \mathbb{R}^n$ is a k-dimensional vector space, then W is a k-dimensional submanifold of $\mathbb{R}^n$ and for any $P \in W$ we have $\mathrm{V}_P\, W = W$.*

---

[109] Using charts and the chain rule one can show fairly easily that this map is well-defined.

[110] The slightly unfortunate fact we use the same symbol, namely $\mathrm{D}\, f_P$, for two a priori different objects will be dealt with shortly.

Proof (∗). It follows from elementary linear algebra that W is a k-dimensional submanifold of $\mathbb{R}^n$. We leave it as an elementary exercise to the reader to verify that for any $P \in W$ we have $\mathrm{V}_P\, W = W$. ■

The next proposition elucidates in particular the relationship between the tangent space $\mathrm{T}_P\, M$ and the visual tangent space $\mathrm{V}_P\, M$ of a submanifold M of $\mathbb{R}^k$.

Proposition 6.39.

(1) *The obvious analogues of the statements of Proposition 6.37 hold for the visual tangent spaces.*

(2) *Let M be an n-dimensional submanifold of $\mathbb{R}^k$ and let $P \in M$. The map*

$$\begin{array}{rcl} \Phi_M\colon\ \mathrm{V}_P\, M & \to & \mathrm{T}_P\, M \\ \gamma'(0) & \mapsto & \left(\begin{array}{rcl} C^\infty(M,\mathbb{R}) & \to & \mathbb{R} \\ f & \mapsto & \frac{d}{dt}\big|_{t=0} f(\gamma(t)) \end{array} \right) \end{array}$$

is an isomorphism of vector spaces.

(a) *The isomorphism restricts to a bijection between outward tangent vectors in $\mathrm{V}_P\, M$ and outward tangent vectors in $\mathrm{T}_P\, M$. The same statement also holds with "outward" replaced by "inward".*

(b) *If $f\colon M \to N$ is a smooth map between submanifolds of $\mathbb{R}^k$ respectively $\mathbb{R}^l$, then the following diagram commutes*

$$\begin{tikzcd} \mathrm{V}_P\, M \arrow[r, "\Phi_M"] \arrow[d, "\mathrm{D}\, f_P = f_*"'] & \mathrm{T}_P\, M \arrow[d, "f_* = \mathrm{D}\, f_P"] \\ \mathrm{V}_{f(P)}\, N \arrow[r, "\Phi_N"] & \mathrm{T}_{f(P)}\, N. \end{tikzcd}$$

Proof.

(1) The proof that the obvious analogues of the statements for Proposition 6.37 hold for the visual tangent spaces is fairly elementary and is left to the reader.

(2) First note that it follows almost immediately from the definitions that the given diagram does indeed commute. It remains to show that the horizontal maps are isomorphisms. Using charts it suffices to show this for open subsets of $\mathbb{R}^n$ and of H_n. But in this special case that is an immediate consequence of Proposition 6.36. ■

Convention. Given a submanifold M of some $\mathbb{R}^k$ and $P \in M$ we will use the isomorphism from Proposition 6.39 to make the identification $\mathrm{T}_P\, M = \mathrm{V}_P\, M$.

Remark. The above reinterpretation of the tangent space $\mathrm{T}_P\, M$ can be generalized to any smooth manifold. More precisely, if M is a smooth manifold and $P \in M \setminus \partial M$ one can interpret $\mathrm{T}_P\, M$ as equivalence classes of smooth paths, i.e. of equivalence classes of smooth maps $\gamma\colon (-\epsilon, \epsilon) \to M$ with $\gamma(0) = P$. We refer to [**Lee02**, p. 72] or [**Jä93**, Chapter 3.3] for details. The interpretation (or very often the definition) of the tangent space in terms of equivalence classes of smooth paths is very intuitive, but it has the disadvantage that it gets awkward to write down the definition of the tangent space at a point on the boundary.

The following definition is the obvious generalization of the concept introduced on page 275.

Definition. Let $f\colon M \to N$ be a smooth map between two smooth manifolds and let $P \in M \setminus \partial M$. We say f is a *local diffeomorphism at P* if there exist open neighborhoods U of P in M and V of $f(P)$ in N such that $f|_U\colon U \to V$ is a diffeomorphism.

By Proposition 6.37 we know that for a local diffeomorphism the differential at any point is an isomorphism. The following theorem gives the much more interesting converse. The theorem gives in particular a convenient criterion for showing that a map between smooth manifolds is a (local) diffeomorphism.

Theorem 6.40. (Inverse Mapping Theorem) $(*)$ *Let $f\colon M \to N$ be a smooth map between two smooth manifolds.*

(1) *Let $P \in M$ such that $\mathrm{D}\, f_P$ is invertible.*
  (a) *If $P \notin \partial M$, then f is a local diffeomorphism at P.*
  (b) *If $P \in \partial M$ and if there exists an open neighborhood U of P with $f(\partial M \cap U) \subset \partial N$, then f is a local diffeomorphism at P.*

(2) *If $\mathrm{D}\, f_P$ is invertible for every $P \in M$, if f is a bijection and if $f(\partial M) = \partial N$, then f is a diffeomorphism.*

Examples.

(1) We consider the inclusion $i\colon H = \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0\} \to \mathbb{R}^2$ together with the point $P = (0,0) \in \partial H$. We see that the differential $\mathrm{D}\, i_P$ is an isomorphism, but the inclusion is not a local diffeomorphism at P. This shows that in general in the formulation of the Inverse Mapping Theorem 6.40 we have to put an extra condition on f if $P \in \partial M$.

(2) We consider the map

$$\begin{aligned} f\colon \mathbb{R}^n &\to B^n \\ (x_1,\dots,x_n) &\mapsto \tfrac{x_1^2+\dots+x_n^2}{1+x_1^2+\dots+x_n^2} \cdot (x_1,\dots,x_n). \end{aligned}$$

One can easily verify that f satisfies the conditions of the Inverse Mapping Theorem 6.40 (2), thus we see that f is a diffeomorphism. This is actually quite interesting since at first glance it is not totally apparent why $f^{-1}\colon B^n \to \mathbb{R}^n$ is smooth at the origin.

(3) We consider the smooth map $f\colon M \to N$ that is illustrated in Figure 152. It is a bijection and for every $x \in M$ the differential $\mathrm{D}\, f_x\colon \mathrm{T}_x\, M \to \mathrm{T}_{f(x)}\, N$ is an isomorphism. But evidently f is not a diffeomorphism. This shows that we cannot drop the condition $f(\partial M) = \partial N$ in the statement of Inverse Mapping Theorem 6.40 (2).

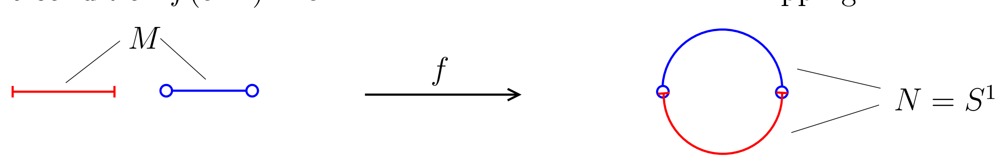

Figure 152

Sketch of proof.

(1) (a) This can be reduced fairly easily to the Inverse Function Theorem 6.18 that deals with maps between open subsets of some $\mathbb{R}^n$. We refer to [**Lee02**, Theorem 4.5] for more details.
   (b) This statement can also be deduced from the Inverse Function Theorem 6.18 and the definition of a smooth map on the half-plane H_n.

   In both cases we leave it to the reader to fill in the details.

(2) The second statement follows immediately from (1) and Lemma 6.23 (3). ■

6.10. **Orientations of real vector spaces.** In the following section we will introduce and study orientations of smooth manifolds. To prepare for that section we will introduce in this section the notion of an orientation of a finite-dimensional real vector space and we will study a few basic properties.

Definition. Let V be a finite-dimensional non-trivial real vector space.

(1) We say that two bases[111] of V are *equivalent* if the determinant of the base change matrix is positive.
(2) An equivalence class of bases is called an *orientation* for V. An *oriented vector space* is a vector space together with an orientation.
(3) Let $\mathcal{O}$ be an orientation for V. We say a basis $\{v_1, \dots, v_k\}$ of V is a *positive basis*, if $\{v_1, \dots, v_k\}$ lies in $\mathcal{O}$, otherwise we say that it is a negative basis.

Remark.

(1) It is straightforward to see that any finite-dimensional non-zero real vector space admits precisely two orientations. Given an orientation $\mathcal{O}$ for some vector space V we denote by $-\mathcal{O}$ the other orientation. Sometimes we refer to it as the *opposite orientation*.
(2) We always view $\mathbb{R}^n$ as equipped with the orientation such that the canonical basis $\{e_1, \dots, e_n\}$ is a positive basis.

Definition. Let V and W be oriented vector spaces. We say an isomorphism $\Phi\colon V \to W$ is *orientation-preserving* if the image of a positive basis of V is a positive basis of W. Otherwise we say that Φ is *orientation-reversing*.

The following lemma often gets used subconsciously:

Lemma 6.41. *Let $V_0, \dots, V_k$ be oriented vector spaces and let $f_i\colon V_i \to V_{i+1}$, $i = 0, \dots, k-1$ be isomorphisms. Then*

the map $f_{k-1} \circ \dots \circ f_0\colon V_0 \to V_k$ is orientation-preserving	$\Longleftrightarrow$	*the number of f_i which are orientation-reversing is even.*

Proof. The lemma is an elementary exercise in linear algebra that is left to the reader. ■

[111] Here and throughout the notes a basis of a finite-dimensional vector space is always understood to be an ordered basis.

Lemma 6.42. *Let U, V and W be a finite-dimensional real vector spaces and suppose we are given maps*

$$U \xhookrightarrow{\varphi} V \xrightarrow{\psi} W$$

such that φ is injective, ψ is surjective and $\mathrm{im}(\varphi) = \ker(\psi)$.[112] *We suppose that U and W are oriented. We pick a positive basis $\{u_1, \dots, u_k\}$ for U and we pick a positive basis $\{w_1, \dots, w_m\}$ for W. Furthermore we pick a right-inverse $s\colon W \to V$ of φ, i.e. we have $\psi \circ s = \mathrm{id}_W$. Then $\{u_1, \dots, u_k, s(w_1), \dots, s(w_l)\}$ is an oriented basis for V and the orientation it defines only depends on the orientations of U and W.*

Definition. In the setting of the above lemma we say that orientations of U, V and W are *compatible* if the orientations of U and W define, in the sense of Lemma 6.42, the orientation of V.

Sketch of proof. It is clear that the given vectors define a basis for V. Furthermore note that any other choice of a positive basis for U, choice of positive basis for W and right-inverse $W \to V$ leads to a base change matrix of the form

$$\begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

with $\det(A) > 0$ and $\det(B) > 0$. Thus the two sets of choices lead to equivalent bases for W. ∎

Remark. In Lemma 12.6 we will see that every finite-dimensional complex vector space comes with a canonical orientation. The algebraically inclined reader might want to figure out what this means and how this works.

6.11. **Orientations of smooth manifolds.** After the preparations from the last section we can now introduce the notion of an orientation on a smooth manifold.

Definition. Let M be a k-dimensional smooth manifold with $k \geq 1$.[113]

(1) An *orientation for M* is a map which assigns to every tangent space $\mathrm{T}_P M$, $P \in M$, an orientation such that the following condition is satisfied: [114]
   (∗) For every chart $\Phi\colon U \to V$ of the smooth structure of M and any component U' of U the map

$$\Phi_*\colon\ \mathrm{T}_P M \ \underset{\substack{\uparrow \\ \text{isomorphism by Proposition 6.37 (2)}}}{\xleftarrow{\cong}}\ \mathrm{T}_P U \ \underset{\uparrow}{\xrightarrow{\Phi_*}}\ \mathrm{T}_{\Phi(P)} V \ \underset{\substack{\uparrow \\ \text{convention on page 291}}}{=}\ \mathbb{R}^k$$

   is either orientation-preserving for all $P \in U'$ or it is orientation-reversing for all $P \in U'$.

(2) A smooth manifold together with an orientation is called an *oriented smooth manifold*.

[112] The experienced reader will notice that this just means that that $0 \to U \xrightarrow{\varphi} V \xrightarrow{\psi} W \to 0$ is a short exact sequence.

(3) A smooth manifold that admits an orientation is called *orientable*, otherwise we say that the smooth manifold is *non-orientable*.

We refer to figure 153 for an illustration.

Convention.

(1) Given a picture of a 1-dimensional submanifold M of $\mathbb{R}^2$ or $\mathbb{R}^3$ we indicate an orientation by drawing a positive vector in $\mathrm{T}_P M = \mathrm{V}_P M \subset \mathbb{R}^3$. (Note in this case a non-zero vector is already a basis for the tangent space).

(2) Given a picture of a 2-dimensional submanifold of $\mathbb{R}^3$ we often adopt one of the usual two conventions for indicating an orientation:

  (a) We draw part of a little circle together with a directional arrow. This indicates that a positive basis is given by v, w where w is obtained by rotating v into the indicated direction by an angle in $(0, \pi)$.

  (b) Alternatively we draw a normal vector n, in this case a basis v, w of the tangent space is positive if n, v, w is a positive basis for $\mathbb{R}^3$.

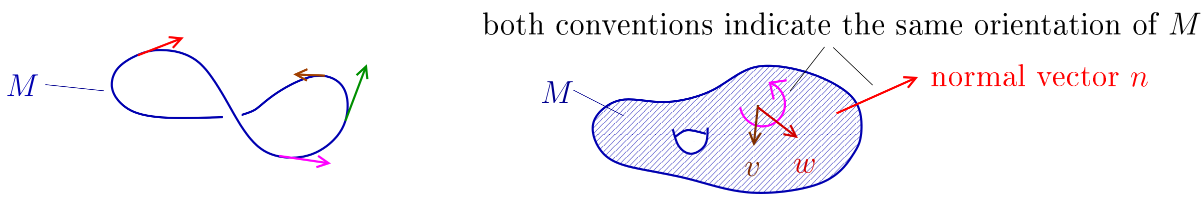

FIGURE 153

In practice the following lemma can be quite useful for verifying that orientations of tangent spaces actually define an orientation of a smooth manifold.

Lemma 6.43. *Let M be a k-dimensional smooth manifold with $k \geq 1$. Suppose that for each $P \in M$ we are given an orientation of $\mathrm{T}_P M$. The following statements are equivalent:*

(1) *The orientations of the tangent spaces defined an orientation M.*

(2) *Given any $P \in M$ there exists a chart $\Phi\colon U \to V$ in the given smooth structure of M such that for any component U' of U the map*

$$\Phi_*\colon\ \mathrm{T}_P M \ \underset{\substack{\uparrow \\ \text{isomorphism by Proposition 6.37 (2)}}}{\xleftarrow{\ \cong\ }}\ \mathrm{T}_P U \ \underset{\uparrow}{\xrightarrow{\ \Phi_*\ }}\ \mathrm{T}_{\Phi(P)} V \ \underset{\substack{\uparrow \\ \text{convention on page 291}}}{=}\ \mathbb{R}^k$$

is either orientation-preserving for all $P \in U'$ or it is orientation-reversing for all $P \in U'$.

PROOF (∗). The implication (1) ⇒ (2) is a tautology. The reverse implication (2) ⇒ (1) follows from the fact that a map that is locally constant, is actually constant. We leave it to the reader to fill in the details. ■

Examples.

[113]Later, on page 1718 we will introduce the notion of an orientation on a 0-dimensional smooth manifold.

[114]One can show fairly easily that it suffices to show that this condition holds for some family of charts that cover M. We do not need to verify it for all charts in the given atlas.

(1) Let U be an open subset of $\mathbb{R}^n$ with $n \in \mathbb{N}$. We equip U with the orientation where a positive basis for any tangent space $\mathrm{T}_P\, U = \mathbb{R}^n$ is given by the standard basis for $\mathbb{R}^n$. Since an atlas is given by the identity map we see that this defines an orientation on U.

(2) Let $n \in \mathbb{N}$. Given $P \in S^n$ we say that a basis $v_1, \dots, v_n \in \mathrm{T}_P\, S^n = \mathrm{V}_P\, S^n \subset \mathbb{R}^{n+1}$ is positive if $\det(P, v_1, \dots, v_n) > 0$.[115] Using Lemma 6.43 one can easily verify that these orientations define an orientation of the smooth manifold S^n. Unless we explicitly say something else we will always view S^n as an oriented smooth manifold with the above orientation.

(3) Let M be an oriented smooth manifold. If for each tangent space we pick the opposite orientation then evidently we obtain again an orientation, which we refer to as the *opposite orientation*. We denote the resulting oriented smooth manifold by $-M$.

orientation of S^1 orientation of S^2

FIGURE 154

The next definition is basically self-explanatory.

Definition. Let $f\colon M \to N$ be a local diffeomorphism between two oriented smooth manifolds.

(1) We say the map f is *orientation-preserving* if for every $P \in M$ the induced isomorphism $f_*\colon \mathrm{T}_P\, M \to \mathrm{T}_{f(P)}\, N$ is orientation-preserving.

(2) We say the map f is *orientation-reversing* if for every $P \in M$ the induced isomorphism $f_*\colon \mathrm{T}_P\, M \to \mathrm{T}_{f(P)}\, N$ is orientation-reversing.

Examples.

(1) Let $A \in \mathrm{O}(n)$. In Exercise 6.28 we will see that the map $S^n \to S^n$ given by $x \mapsto A \cdot x$ is orientation-preserving if and only if $\det(A) = 1$.

(2) Let $U \subset \mathbb{R}^n$ be an open subset. Let $A \in \mathrm{GL}(n)$ with $\det(A) > 0$ and let $v \in \mathbb{R}^n$. It follows easily from the definitions that the map f given by $f(x) = x \mapsto A \cdot x + v$ defines an orientation-preserving diffeomorphism $U \to f(U)$.

(3) Let $f\colon M \to N$ be a local diffeomorphism between two oriented smooth manifolds. If M is non-empty and *connected*, then it follows from Lemma 6.46 below that f is either orientation-preserving or orientation-reversing. On the other hand for disconnected M one can easily come up with maps that are neither orientation-preserving nor orientation-reversing.

The following lemma gives an easy criterion for showing that a diffeomorphism between two open subsets of $\mathbb{R}^n$ is orientation-preserving.

[115]How would you formulate this orientation convention purely in terms of derivations?

Lemma 6.44. *Let $f\colon X \to Y$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. Given $P \in X$ we have*[116]

$$f_* = \mathrm{D}\, f_P\colon\ \mathrm{T}_P\, X \to \mathrm{T}_{f(P)}\, Y \text{ is orientation-preserving} \quad\Longleftrightarrow\quad \det(\mathrm{D}\, f_P) > 0.$$

PROOF. By Proposition 6.39 we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{R}^n = \mathrm{V}_P\, X \arrow[r, "\Phi_X"] \arrow[d, "f_*=\mathrm{D}\, f_P"'] & \mathrm{T}_P\, X \arrow[d, "f_*=\mathrm{D}\, f_P"] \\
\mathbb{R}^n = \mathrm{V}_{f(P)}\, Y \arrow[r, "\Phi_Y"] & \mathrm{T}_{f(P)}\, Y.
\end{tikzcd}$$

where the horizontal maps are isomorphisms and where the horizontal maps are orientation-preserving by definition of the orientations of the oriented smooth manifolds U and V. Thus it follows from Lemma 6.41 that the left-hand vertical map is orientation-preserving if and only if the right-hand vertical map is orientation-preserving. But by definition the left-hand vertical map is orientation-preserving if and only if $\det(\mathrm{D}\, f_P) > 0$. ■

The following lemma gives another approach to defining an orientation on a given smooth manifold.

Lemma 6.45. $(*)$ *Let $(M, \mathcal{A})$ be a smooth manifold. Suppose that all transition maps between charts in $\mathcal{A}$ are orientation-preserving. Given $P \in M$ we define*

$$\textit{orientation of } \mathrm{T}_P\, M \;:=\; (\mathrm{D}\,\Phi_P)^{-1}(\textit{orientation of } \mathrm{T}_{\Phi(P)}\, V = \mathbb{R}^n)$$

where $\Phi\colon U \to V$ is a chart around P from the given atlas. The following statements hold:

(1) *The definition of the orientation of $\mathrm{T}_P\, M$ does not depend on the choice of the chart.*
(2) *The orientations of the tangent space $\mathrm{T}_P\, M$ define an orientation for M.*

SKETCH OF PROOF $(*)$. The first statement follows easily from the hypothesis that the transition maps are orientation-preserving. The second statement is a consequence of the observation in Footnote 114. ■

Example. We consider the surface Σ of genus 2. In Proposition 6.8 we gave an explicit atlas for Σ. In Proposition 6.21 we saw that the transition maps are smooth. We leave it to the reader to verify that all the transition maps are in fact orientation-preserving. Thus it follows from Lemma 6.45 that Σ is orientable. Basically the same argument shows that all surfaces of genus $g \geq 3$ are also orientable. Note that, as the name suggests, the same approach does not work for the non-orientable surfaces of some genus $g \in \mathbb{N}$.

The following lemma summarizes a few elementary properties of orientations.

Lemma 6.46. *Let $n \in \mathbb{N}$ and let M be an n-dimensional smooth manifold.*

(1) *If two orientations on M agree at a point and if M is path-connected, then the two orientations agree everywhere.*

[116] Unfortunately the symbol $\mathrm{D}\, f_P$ has several, subtly different meanings. To the left it stands for the induced map on tangent spaces. To the right it stands for the $n \times n$–matrix given by the partial derivatives of f.

(2) *If M is path-connected and non-empty, then M is either non-orientable or it admits precisely two orientations, which are opposites of one another.*
(3) *Let $f: M \to N$ be a diffeomorphism. If M is path-connected, then for any $P \in M$ we have*

$$f \text{ is orientation-preserving} \iff \mathrm{D}\, f_P\colon\ \mathrm{T}_P\, M \to \mathrm{T}_{f(P)}\, N \text{ is orientation-preserving.}$$

(4) *Suppose M is equipped with an orientation $\mathcal{O}$. Let $f: M \to M$ be a self-diffeomorphism. Then*

$$\begin{array}{c} f \text{ is orientation-preserving} \\ \text{with respect to } \mathcal{O} \end{array} \iff \begin{array}{c} f \text{ is orientation-preserving} \\ \text{with respect to } -\mathcal{O}. \end{array}$$

(5) *Let N be a submanifold of codimension zero. (In many applications N is just an open subset of M.) Then N admits a unique orientation such that the inclusion map $N \to M$ is orientation-preserving.*
(6) *If $M \setminus \partial M$ admits an orientation, then M admits a unique orientation such that the inclusion $M \setminus \partial M \to M$ is orientation-preserving.*

PROOF (∗).

(1) Suppose we are given two orientations on M. We denote by U the set of all points where the orientations agree and we denote by V the set of all points where the orientations disagree. It follows easily from the definitions that U and V are open. If the orientations agree at a point we see that $U \neq \varnothing$. It follows from Lemma 6.9 (3) that $U = M$.
(2) This statement follows easily from (1).
(3) The proof of this statement is very similar to the proof of the first statement. We leave it to the reader to fill in the details.
(4) This statement follows immediately from the definitions and Lemma 6.41.
(5) This statement follows easily from Proposition 6.37.
(6) Using charts it suffices to prove this statement for smooth manifolds that are open subsets of H_n, but in this case the statement is clear. ∎

We move on to constructing orientations via group actions. The first part of the following proposition can be viewed as a refinement of Proposition 6.32.

Proposition 6.47. *Let G be a group that acts freely, properly and smoothly on an n-dimensional smooth manifold M.*

(1) *If M is oriented and if for each $g \in G$ the map*

$$\begin{array}{ccc} M & \to & M \\ P & \mapsto & g \cdot P \end{array}$$

is orientation-preserving, then M/G admits an orientation such that the projection map $p: M \to M/G$ is orientation-preserving.

(2) *Conversely, if M/G admits an orientation, then there exists a unique orientation on M such that the projection map $p\colon M \to M/G$ is orientation-preserving. Furthermore this orientation has the property that the G-action on M is orientation-preserving.*

Remark. In Exercise 6.20 we will prove a partial converse to Proposition 6.47 (2).

Proof. We leave it to the reader to write down the fairly straightforward proof of the proposition. ■

Example. Note that the differential of any translation map on $\mathbb{R}^n$ is the identity. Thus it follows from Lemma 6.46 (3) that the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ that is given by addition is orientation-preserving. It follows from Proposition 6.47 that $\mathbb{R}^n/\mathbb{Z}^n$ is orientable.

We formulate the next examples as a lemma.

Lemma 6.48. *The following smooth manifolds are non-orientable:*

(1) *the Möbius band,*
(2) *the projective plane $\mathbb{R}\mathrm{P}^2$,*
(3) *the Klein bottle, and*
(4) *any non-orientable surface of genus ≥ 1.*

Proof. Let $X = \mathbb{R} \times (-1,1)$ and $G = \mathbb{Z}$. Very similar to the definition on page 188 we consider the action

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times (-1,1)) & \to & \mathbb{R} \times (-1,1) \\ (n,(x,y)) & \mapsto & (x+n, (-1)^n \cdot y). \end{array}$$

This action is again free, proper and smooth. As on page 189 we refer to the quotient as the open Möbius band.

Claim. The open Möbius band is non-orientable.

Suppose that the open Möbius band $X/\mathbb{Z}$ admits an orientation. It follows from Proposition 6.47 that $X = \mathbb{R} \times (-1,1)$ admits an orientation such that the $\mathbb{Z}$-action is orientation-preserving. It follows from Lemma 6.46 (3) and (4) that this $\mathbb{Z}$-action preserves the usual orientation on $X = \mathbb{R} \times (-1,1)$. Given $n \in \mathbb{Z}$ we consider the diffeomorphism of $X = \mathbb{R} \times (-1,1)$ that is given by the action by n, i.e. we consider the diffeomorphism

$$\begin{array}{rcl} \Phi_n\colon \mathbb{R} \times (-1,1) & \to & \mathbb{R} \times (-1,1) \\ (x,y) & \mapsto & (x+n, (-1)^n \cdot y). \end{array}$$

Given any $(x,y) \in X$ the differential of Φ_n is given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & (-1)^n \end{pmatrix}$. It follows from Lemma 6.44 that for n odd the diffeomorphism Φ_n is not orientation-preserving. We have thus obtained a contradiction. ⊞

We turn to the actual proof that the three given smooth manifolds are non-orientable. By Lemma 6.46 (5) it suffices to prove the following claim.

Claim. The open Möbius band is diffeomorphic to open subsets of the Möbius band, of the projective plane $\mathbb{R}\mathrm{P}^2$, of the Klein bottle and for any non-orientable surface of genus $g \geq 3$.

It is clear that the open Möbius band is an open subset of the Möbius band. The statements regarding the projective plane $\mathbb{RP}^2$ and the Klein bottle follow immediately from Lemma 6.29. Finally in Figure 155 we show that for any $g \geq 3$ the non-orientable surface of genus g admits an open subset that is diffeomorphic to the open Möbius band.
⊞ ■

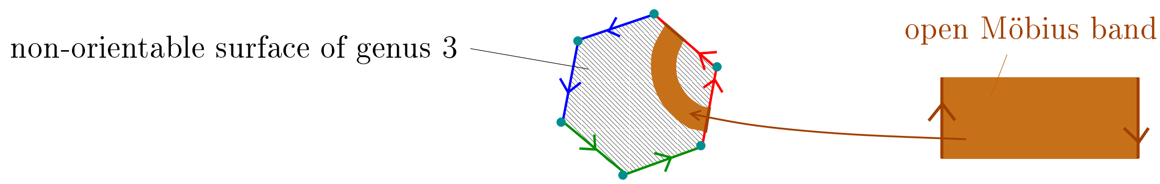

FIGURE 155

Remark. For most people starting out with mathematics the definition of a submanifold M of some $\mathbb{R}^n$ and the definition of a tangent space to M as a vector subspace of $\mathbb{R}^n$ are quite natural and reasonable. But initially it is perhaps not clear why one should introduce "abstract" smooth manifolds. Furthermore the definition on page 291 of the tangent space $\mathrm{T}_P\, M$ to such a smooth manifold traditionally leads to a lot of head scratching.

The above proof of Lemma 6.48 is a good example why it is useful to introduce these abstract notions. In principle it is possible to show that Möbius band is non-orientable purely by working in the explicit 3-dimensional setting, see e.g. [**Frie16a**, Lemma 9.16]. But this proof is rather messy and not very illuminating. In contrast, the fact that we can work with abstract smooth manifolds, in this case with $(\mathbb{R} \times [-1,1])/\mathbb{Z}$, and their tangent spaces liberates us from the constraints of the 3-dimensional setting and we obtain a much clearer and much more elegant proof that the Möbius band is non-orientable.

The following lemma, which we will prove in Exercise 6.12, generalizes Lemma 6.48 (3).

Lemma 6.49. *Let $n \in \mathbb{N}_0$. The real projective space $\mathbb{RP}^n$ is orientable if and only if n is odd.*

In the next lemma we will see that the boundary of an oriented smooth manifold inherits an orientation.

Lemma 6.50. *Let $n \geq 2$ and let M be an oriented n-dimensional smooth manifold. Let $P \in \partial M$. We denote by $w \in \mathrm{T}_P\, M$ a tangent vector which points outward in the sense of the definition on page 291. We say that a basis $v_1, \dots, v_{n-1}$ for $\mathrm{T}_P(\partial M)$ is a positive basis of $\mathrm{T}_P(\partial M)$ if $w, v_1, \dots, v_{n-1}$ is a positive basis of $\mathrm{T}_P\, M$. The following two statements hold:*

(1) *The definition of a positive basis for P does not depend on the choice of w.*
(2) *These orientations of the tangent spaces of ∂M define an orientation on ∂M.*

PROOF. We leave the fairly elementary verification of the lemma to the reader. Alternatively full details are given in [**Lee02**, Proposition 15.24]. ■

Convention. Let $n \geq 2$ and let M be an oriented n-dimensional smooth manifold. Unless we say something else we always equip ∂M with the orientation given by Lemma 6.50. We refer to this orientation of ∂M as the *boundary orientation*.

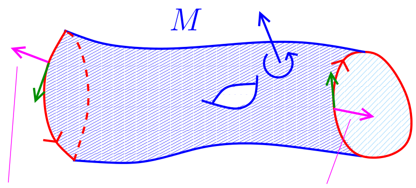

FIGURE 156. Illustration of Lemma 6.50.

Examples.

(1) If we equip the smooth manifold $\overline{B}^{n+1}$ with the usual orientation coming from $\mathbb{R}^{n+1}$, then the orientation of $\partial\overline{B}^{n+1} = S^n$ agrees with the above orientation of S^n.

(2) Given an oriented smooth manifold N we denote by $-N$ the smooth manifold equipped with the opposite orientation. It follows almost immediately from the definitions that $\partial(-N) = -\partial N$.

We conclude this section with the following proposition.

Proposition 6.51. (∗) *Let M be a smooth manifold with possibly non-empty boundary and let N be a smooth manifold with empty boundary. Given $Q \in N$ and $P \in M$ we consider the inclusion maps*

$$\begin{array}{rcl} i_Q\colon M & \to & M \times N \\ x & \mapsto & (x, Q) \end{array} \quad \text{and} \quad \begin{array}{rcl} j_P\colon N & \to & M \times N \\ y & \mapsto & (P, y). \end{array}$$

The following statements hold:

(1) *The product $M \times N$ admits a unique smooth manifold structure such that products of smooth charts for M and N lie in the smooth atlas for $M \times N$.*

(2) *With the smooth manifold structure as in (1) the following statements hold:*
   (a) *The projections from $M \times N$ onto M and onto N are smooth.*
   (b) *All the inclusion maps i_P and j_Q are smooth.*

(3) (a) *Let $P \in M$ and $Q \in N$. The map*

$$(i_Q)_* \oplus (i_P)_*\colon\ \mathrm{T}_P M \oplus \mathrm{T}_Q N \ \to\ \mathrm{T}_{(P,Q)}(M \times N)$$

is an isomorphism. Sometimes we use this isomorphism to identify the vector spaces to the left and right.

   (b) *If M and N are oriented and of dimension ≥ 1, then we obtain an orientation on $M \times N$ by defining*

$$\text{positive basis of } \mathrm{T}_{(P,Q)}(M \times N) \ :=\ \begin{array}{c} (i_Q)_*(\text{positive basis of } \mathrm{T}_P M), \text{ followed by} \\ (j_P)_*(\text{positive basis of } \mathrm{T}_Q N). \end{array}$$

We refer to this orientation as the product orientation on $M \times N$.

(4) *We have*[117] $\partial(M \times N) = (\partial M) \times N$.

(5) *We suppose that M and N are oriented and that both are of dimension ≥ 1.*
   (a) *If the dimension of M is at least two, then the equality $\partial(M \times N) = (\partial M) \times N$ from (4) is in fact an equality of oriented smooth manifolds.*

(b) *In one-dimensional setting we have the following equality of oriented smooth manifolds:*[118]

$$\partial([a,b]\times N) \;=\; -(\{a\}\times N) \;\sqcup\; (\{b\}\times N)$$

with the obvious variations if we work with half-open intervals instead.

All of the above statements, except for (5), *hold also with the roles of M and N reversed.*[119]

Remark.

(1) In Proposition 80.9 we will deal with the product of two smooth manifolds where both manifolds are allowed to have non-empty boundary.
(2) On page 1718 we will introduce orientations of 0-dimensional manifolds and we will extend the definition of the product orientation to this setting.

Proof (∗).

(1),(2) By Proposition 6.5 we already know that $M\times N$ is a topological manifold. Furthermore it is very easy to see that the products of charts from the smooth atlases for M and for N define a smooth atlas for $M\times N$ which has the properties specified in (2).
(3) The elementary proof of this statement is left to the reader.
(4) First we want to argue that we have the equality $\partial(M\times N)=\partial M\times N$. In Proposition 6.5 we could only show the inclusion "$\subset$". The situation is much better in the present setting of smooth manifolds since now we have Proposition 6.27 (1) at our disposal. Using this proposition one can show[120] fairly easily that we have $\partial(M\times N)=M\times\partial N$. We refer to [**Lee02**, Proposition 1.45 and 2.12] for more details.
(5) It is elementary (albeit potentially slightly confusing) to deduce this statement regarding the orientations from the definitions. ■

This concludes our introduction to the notion of an orientation of a smooth manifold. Perhaps somewhat inintuitively there does also exist a perfectly fine notion of the orientation of a topological manifold. We will introduce this notion in Chapter 86.

6.12. **Intersections of submanifolds** (∗)**.** In this section we want to study the following question: given a smooth manifold M and two submanifolds X and Y, is the intersection $X\cap Y$ again a submanifold of M? It is not difficult to show that in general the answer is negative. For example consider $M=\mathbb{R}^2$ with the two submanifolds X and Y that are shown in Figure 157. The intersection $X\cap Y$ has three components, one of which is a submanifold of dimension one, and the other two components consist of a single point. We

[117]Note that here, when we are dealing with smooth manifolds, we can determine the boundary. This is in contrast to the slightly unfortunate situation in Proposition 6.5.

[118]This statement has a rather ad hoc flair. In fact in Section 68.3 we will give a more elegant treatment. But for the time being this formulation will satisfy all our needs.

[119]In Exercise 6.26 we will determine the relationship between the orientations of $\partial(M\times N)$ and $M\times\partial N$ if N is allowed to have non-empty boundary.

[120]Here we use our super subtle definition of the boundary of a smooth manifold, see page 277.

see that $X \cap Y$ is not a smooth manifold (after all, what would its dimension be?), in particular it is not a submanifold of M.

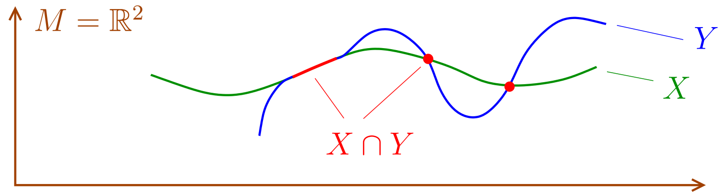

FIGURE 157

If one spends some time pondering the question why in the previous example the intersection $X \cap Y$ is not a submanifolds, then one is inexorably led towards the following definition.

Definition. Let M be an m-dimensional smooth manifold.[121] Let X and Y be proper submanifolds of M.

(1) Let $P \in X \cap Y$. We say P is a *transverse intersection point* if $P \notin \partial M$ and if X and Y *intersect transversally in* P, i.e. if the equality[122]
$$\mathrm{T}_P\, X + \mathrm{T}_P\, Y \;=\; \mathrm{T}_P\, M$$
holds.

(2) We say that X and Y *intersect transversally* if each intersection point is a transverse intersection point.[123]

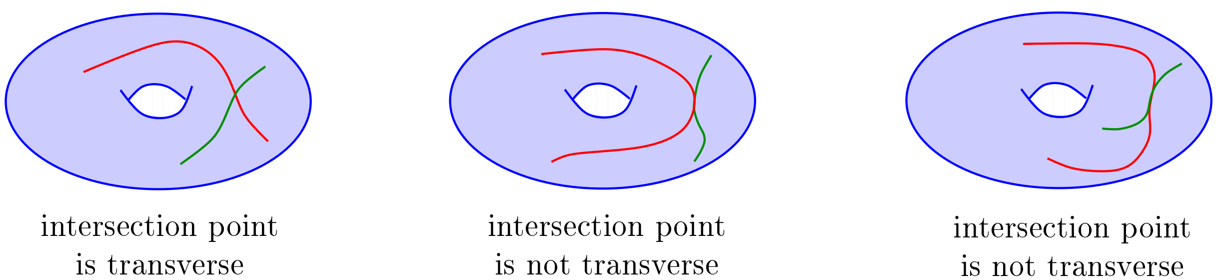

FIGURE 158

The following lemma is basically an amusing real analysis exercise.

Lemma 6.52. *Let M be an m-dimensional smooth manifold, let $X \subset M$ a proper submanifold of codimension k and let Y be a proper submanifold of M of codimension l.*

(1) *If $P \in M$ is a transverse intersection point of X and Y, then there exists a chart $\Phi\colon U \to V$ around P with*
$$\Phi(U \cap X) = V \cap (\mathbb{R}^{m-k} \times \{0\}) \quad \text{and} \quad \Phi(U \cap Y) = V \cap (\{0\} \times \mathbb{R}^{m-l}).$$

[121] Note that in the following we work with smooth manifolds, we will need the smooth atlas in the definition of a transverse intersection point.

[122] Since X is a submanifold of M we can view the tangent space $\mathrm{T}_P\, X$ as a subspace of $\mathrm{T}_P\, M$.

[123] Note that if X and Y intersect transversally, then $X \cap Y \cap \partial M = \varnothing$.

(2) *If the submanifolds X and Y intersect transversally, then $X \cap Y$ is a submanifold of codimension $k+l$.*
(3) *If W is compact codimension-zero submanifold of M and if Y intersects ∂W transversally, then $Y \cap W$ is a proper submanifold of W.*

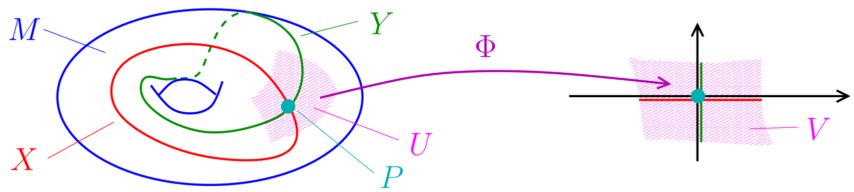

FIGURE 159. Illustration of Lemma 6.52.

PROOF.

(1) The first statement is proved in [**Kos93**, Theorem IV.1.6] or implicitly also in [**Lee02**, Theorem 6.30 (b)].
(2) The second statement is an immediate consequence of the first statement.
(3) The last statement can also be deduced without too much effort from the first statement. ■

Example. In Figure 160 we see two infinite cylinders X and Y in the smooth manifold $\mathbb{R}^3$. The cylinders are submanifolds of codimension one. They intersect transversally and the intersection consists of two circles, i.e. $X \cap Y$ is a one-dimensional submanifold of $\mathbb{R}^3$. Put differently, $X \cap Y$ is a submanifold of codimension two.

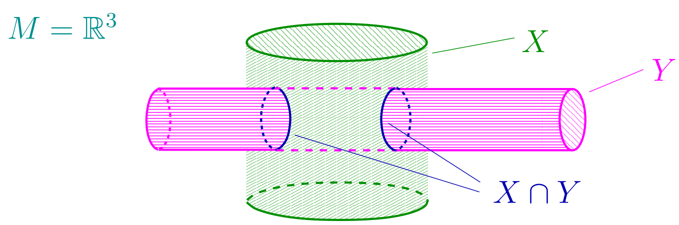

FIGURE 160

6.13. **The Regular Value Theorem I.** In general it can be quite painful to show directly that a given object is a smooth manifold. For example, we will shortly see that the set of orthogonal matrices $\mathrm{O}(n)$ form a smooth manifold. But only the most enterprising reader will feel like writing down an explicit atlas.

In this section we will state the Regular Value Theorem which is a convenient way for showing that certain objects are indeed smooth manifolds. Not surprisingly the formulation of the Regular Value Theorem is preceded by the definition of a regular value. This notion might be familiar to the reader from an earlier course on real analysis.

Definition. Let M and N be smooth manifolds (possibly with boundary). Let $f\colon M \to N$ be a smooth map.

(1) Let $P \in M \setminus \partial M$. We say P is a *regular point* if the corresponding differential $\mathrm{D} f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is an epimorphism. Otherwise we say that P is a *critical point.*

(2) Let $P \in \partial M$. We say P is a *regular point* if the differential $\mathrm{D} f_P\colon \mathrm{T}_P(\partial M) \to \mathrm{T}_{f(P)} N$ is an epimorphism.[124] Otherwise we say that P is a *critical point.*

(3) We say $Q \in N$ is a *regular value* if all points in the preimage $f^{-1}(Q)$ are regular. Otherwise we call Q a *critical value.*

The following elementary observation will be used several times without explicitly mentioning it.

Observation. Let $f\colon M \to N$ be a smooth map between smooth manifolds.

(1) If $Q \in N$ is a point with $f^{-1}(Q) = \varnothing$, then Q is a regular value.

(2) If $\dim(M) < \dim(N)$, then the converse to (1) holds, thus a point $P \in N$ is a regular value if and only if $f^{-1}(P) = \varnothing$.

The following theorem is often a convenient way for showing that a given subset of a smooth manifold, e.g. a given subset of some $\mathbb{R}^n$, is in fact a smooth manifold.

Theorem 6.53. (Regular Value Theorem) *Let M be an m-dimensional smooth manifold, let N be an n-dimensional smooth manifold without boundary, let $f\colon M \to N$ be a smooth map and let $s \in N$ be a regular value of f.*

(1) *The preimage $X := f^{-1}(\{s\})$ is a proper $(m-n)$-dimensional submanifold of M.*

(2) *If M is orientable, then X is also orientable.*

(3) *The intersection of X with any compact set of M is compact. In particular, if M is compact, then X is also compact.*

Remark. In Section 11.1 we will formulate many generalizations, extensions and variations on the Regular Value Theorem 6.53.

Sketch of proof. The first statement is proved in basically every book on smooth manifolds, see e.g. [**Miln65a**, p. 11] or [**Lee02**, Corollary 5.14]. The second statement is unfortunately not written down explicitly in the literature, but it can be proved without too much effort by modifying the proof of (1). The third statement follows from the observation that $f^{-1}(\{s\})$ is a closed subset of the compact topological space M, hence it itself is compact. ■

Examples.

(1) In Figure 161 we show the torus M as a submanifold of $\mathbb{R}^3$ and we consider the "height function" $f\colon M \to \mathbb{R}$ given by $f(x, y, z) = z$. This function has precisely four critical points which are indicated in Figure 161 on the left. On the right we show a regular value s and its preimage and we show a critical value t and its preimage $f^{-1}(\{t\})$ which in this case is *not* a submanifold.

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[124] Note that the restriction of a map on a smooth manifold to its boundary is also a smooth map.

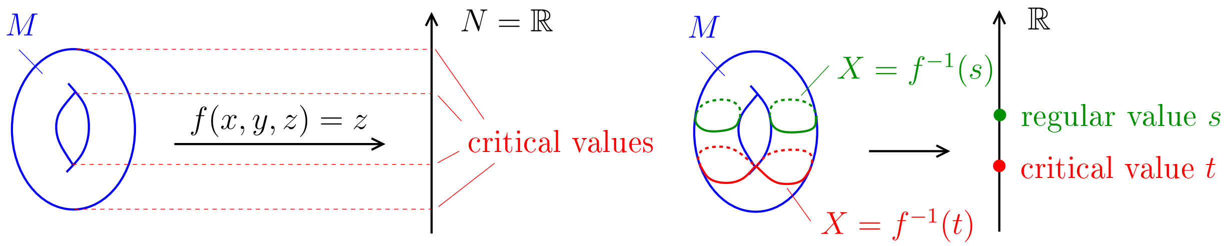


FIGURE 161. Illustration of the Regular Value Theorem 6.53.

(2) Similarly to (1) we again consider a smooth manifold $M \subset \mathbb{R}^3$ together with the "height function". This time M has boundary and $f^{-1}(\{s\})$ is a proper submanifold of M with non-empty boundary.

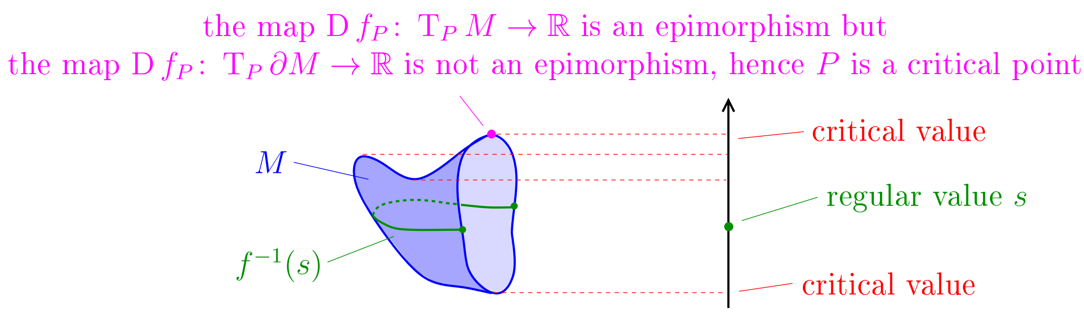


FIGURE 162. Illustration of the Regular Value Theorem 6.53.

(3) Let $g \in \mathbb{N}_0$. According to [**Hirs76**, p. 28] there exists a polynomial function $f\colon \mathbb{R}^3 \to \mathbb{R}$ such that 0 is a regular value and such that $f^{-1}(\{0\})$ is a surface of genus g. For example, if we take

$$f(x,y,z) \;=\; (4x^2(1-x^2)-y^2)^2 + z^2 - \tfrac{1}{4},$$

then [**Hirs76**, p. 28] states that $f^{-1}(\{0\})$ is a surface of genus 2.

Lemma 6.54. *Let $f\colon M \to N$ be a smooth map between two smooth manifolds of the same dimension.*

(1) *For any regular point $x \in M \setminus \partial M$ the map f is a local diffeomorphism around x.*
(2) *If $y \in N$ is a regular value and if M is compact, then $f^{-1}(\{y\})$ is finite subset of M.*

PROOF (∗).

(1) Let $x \in M \setminus \partial M$ be a regular value of f. This means that $\mathrm{D}\, f_x\colon\ \mathrm{T}_{f(x)}\, M \to \mathrm{T}_y\, N$ is an epimorphism. Since M and N have the same dimension this implies that $\mathrm{D}\, f_x$ is an isomorphism. It follows from the hypothesis that $x \in M \setminus \partial M$ together with the Inverse Mapping Theorem 6.40 that f is a local diffeomorphism around x.
(2) Let y be a regular value of f. It follows from the Regular Value Theorem 6.53 (1) and (3) that $f^{-1}(\{y\})$ is a compact 0-dimensional submanifold of M. It follows from the discussion on page 272 together with Lemma 2.18 and our hypothesis that M is compact that $f^{-1}(\{x\})$ is a finite subset of M. ■

The following lemma can be viewed as a particularly interesting application of the Regular Value Theorem 6.53.

Lemma 6.55. *Let $n \in \mathbb{N}$. The set*

$$\mathrm{O}(n) \;=\; \{A \in \mathrm{M}(n \times n, \mathbb{R}) \mid A^T A = \mathrm{id}\}$$

of orthogonal matrices (and thus also its component $\mathrm{SO}(n)$*) is a* $\frac{1}{2}n(n-1)$*-dimensional submanifold of* $\mathrm{M}(n \times n, \mathbb{R}) = \mathbb{R}^{n^2}$*. Furthermore the set of unitary matrices*

$$\mathrm{U}(n) \;=\; \{A \in \mathrm{M}(n \times n, \mathbb{C}) \mid \overline{A}^T A = \mathrm{id}\}$$

is an n^2*-dimensional submanifold of* $\mathrm{M}(n \times n, \mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$ *and the set of special unitary matrices*

$$\mathrm{SU}(n) \;=\; \{A \in \mathrm{M}(n \times n, \mathbb{C}) \mid \overline{A}^T A = \mathrm{id} \;\text{ and }\; \det(A) = 1\}$$

is an (n^2-1)*-dimensional submanifold of* $\mathrm{M}(n \times n, \mathbb{C}) = \mathbb{C}^{n^2} = (\mathbb{R}^2)^{n^2} = \mathbb{R}^{2n^2}$*. All of the above examples, viewed as smooth manifolds in their own right, are closed.*

PROOF. We start with a preamble: Throughout the proof we use Proposition 6.39 to identify the abstract tangent spaces $\mathrm{T}_P\, M$ with the visual tangent spaces $\mathrm{V}_P\, M$.
Now we proceed to the actual proof of the lemma. First we prove the desired statements for $\mathrm{O}(n)$.

Claim.

(1) The set

$$\mathrm{Sym}(n, \mathbb{R}) \;=\; \{B \in \mathrm{M}(n{\times}n, \mathbb{R}) \mid B^T = B\} \;=\; \{(b_{ij}) \in \mathrm{M}(n{\times}n, \mathbb{R}) \mid b_{ij} = b_{ji}\}$$

is a submanifold of $\mathrm{M}(n \times n, \mathbb{R}) = \mathbb{R}^{n^2}$ of dimension $\frac{1}{2}n(n+1)$.

(2) The map

$$\begin{aligned} f\colon\ \mathrm{M}(n \times n, \mathbb{R}) &\to \mathrm{Sym}(n, \mathbb{R}) \\ A &\mapsto A^T A \end{aligned}$$

is smooth and id is a regular value.

The first statement follows immediately from Lemma 6.38. We turn to the proof of the second statement. It should be clear that f is smooth and that f does indeed take values in $\mathrm{Sym}(n, \mathbb{R})$. So it remains to show that id is a regular value.

In the following we again use Lemma 6.38 which implies that the tangent spaces of the submanifolds $\mathrm{Sym}(n, \mathbb{R})$ and $\mathrm{M}(n \times n, \mathbb{R})$ are just given by these subvector spaces. So let $A \in \mathrm{M}(n \times n, \mathbb{R})$ with $A^T A = \mathrm{id}$ and let $C \in \mathrm{T}_{\mathrm{id}}\, \mathrm{Sym}(n, \mathbb{R}) = \mathrm{Sym}(n, \mathbb{R})$. We need to show that there exists a B in $\mathrm{T}_A\, \mathrm{M}(n \times n, \mathbb{R}) = \mathrm{M}(n \times n, \mathbb{R})$ with $\mathrm{D}\, f_A B = C$.

We start out with the following little calculation: Given $B \in \mathrm{M}(n, \times n, \mathbb{R})$ we have

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} f(A + tB) &= \frac{d}{dt}\Big|_{t=0} (A^T + tB^T) \cdot (A + tB) \\ &= \frac{d}{dt}\Big|_{t=0} \big(A^T A + tA^T B + tB^T A + t^2 B^T B\big) \;=\; A^T B + B^T A. \end{aligned}$$

Using that $A^T \cdot A = \mathrm{id}$ and $C = C^T$ it is now easy to see that $B := \frac{1}{2} A \cdot C$ satisfies $\mathrm{D}\, f_A B = C$. $\boxplus$

It follows from the claim, together with the Regular Value Theorem 6.53, that $\mathrm{O}(n)$ is a proper submanifold of $\mathrm{M}(n\times n,\mathbb{R})=\mathbb{R}^{n^2}$ of dimension $n^2-\frac{1}{2}n(n+1)=\frac{1}{2}n(n-1)$. Since $\mathbb{R}^{n^2}$ has no boundary and since $\mathrm{O}(n)$ is a proper submanifold we see that the boundary of $\mathrm{O}(n)$ is also empty. Finally note that we already saw on page 163 that $\mathrm{O}(n)$ is compact. These last two observations imply that $\mathrm{O}(n)$ is in fact a closed smooth manifold.

The proof for $\mathrm{U}(n)$ is almost verbatim the same as the proof for $\mathrm{O}(n)$. The proof for $\mathrm{SU}(n)$ requires just a little bit of extra thought. We will deal with it in Exercise 6.33. ■

For later usage we introduce the notion of a submersion:

Definition. A map $f\colon M\to N$ between smooth manifolds is called a *submersion* if f is smooth and if given any $P\in M$ the map $\mathrm{D}\,f_P\colon \mathrm{T}_P\,M\to \mathrm{T}_{f(P)}\,N$ is an epimorphism.

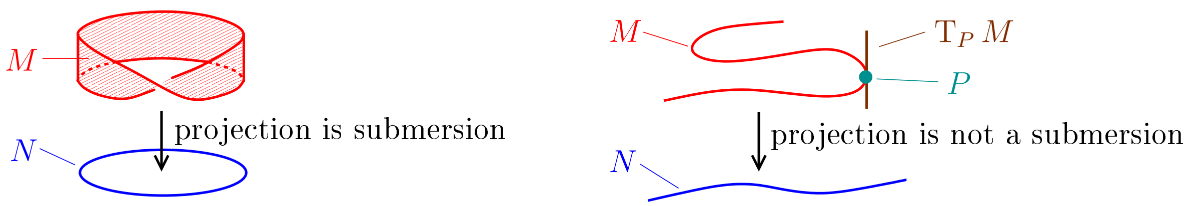


FIGURE 163

Any projection of the form $f\colon M\times N\to N$ is evidently a submersion. The following theorem says that if C is a convex subset of $\mathbb{R}^n$, then given any compact smooth manifold M any submersion $f\colon M\to N$ is basically precisely of that form.

Theorem 6.56. (Submersion Theorem) *Let M be a compact m-dimensional smooth manifold, let C be an n-dimensional submanifold of $\mathbb{R}^n$ and let $f\colon M\to C$ be a surjective map. If C is a convex subset of $\mathbb{R}^n$ and if f is a submersion, then there exists a closed smooth manifold Y and a diffeomorphism $\varphi\colon M\to C\times Y$ such that the following diagram commutes:*

$$\begin{tikzcd} M \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & C\times Y \arrow[dl, "(c,y)\mapsto c"] \\ & C. & \end{tikzcd}$$

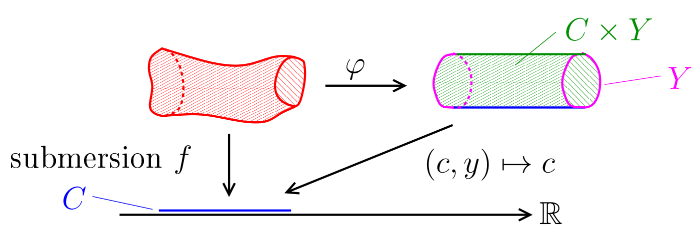


FIGURE 164. Illustration of the Submersion Theorem 6.56.

PROOF. This theorem follows from the proof of [**Dun18**, Theorem 8.5.10] or alternatively the proof of [**Ebe07**, Theorem 4.1]. ■

6.14. **Partitions of unity on manifolds.** In this short section we show in particular that manifolds are paracompact. This fact will be a key technical tool for constructing interesting functions on smooth manifolds.

First we recall the following definitions from page 144.

Definition. Let X be a topological space.

(1) The *support* of a function $f\colon X \to \mathbb{R}$ is defined as $\operatorname{supp}(f) := \overline{\{x \in X \mid f(x) \neq 0\}}$.
(2) A *partition of unity* on X is a family of maps $\{f_i\colon X \to [0,1]\}_{i\in I}$ with the following properties:
    (a) For each $x \in X$ there exists an open neighborhood U of x such that f_i vanishes on U for all but finitely many i.
    (b) For every $x \in X$ we have $\sum\limits_{i\in I} f_i(x) = 1$.
(3) Let $\{U_i\}_{i\in I}$ be an open cover. A *partition of unity subordinate to the open cover* is a partition of unity $\{f_j\}_{j\in J}$ such that for each $j \in J$ there exists an $i \in I$ with $\operatorname{supp}(f_j) \subset U_i$.
(4) We say X is *paracompact* if X is Hausdorff and if given any open cover $\{U_i\}_{i\in I}$ there exists a partition of unity subordinate to the open cover.[125]

The following theorem guarantees the existence of many interesting (smooth) partitions of unity.

Theorem 6.57.

(1) *Every topological manifold is paracompact.*
(2) *Let M be a smooth manifold. For every open cover of M there exists a* smooth *partition of unity subordinate to the given open cover.*

Before we provide the proof of Theorem 6.57 we first show how one can use the existence of partitions of unity to construct two types of useful functions on manifolds.

Definition. Let X be a topological space. An *exhaustion function* is a map $f\colon X \to \mathbb{R}_{\geq 0}$ such that for every $c \in \mathbb{R}_{\geq 0}$ the preimage $f^{-1}([0,c])$ is a compact subset of X.

Examples.

(1) Let $M = [0,1)$ the half-open interval viewed as a 1-dimensional smooth manifold. In this case the function defined by $f(x) = \frac{1}{1-x}$ has the desired properties
(2) We consider the surface of infinite genus illustrated in Figure 165. In this case the function $f(x,y,z) = x$ has the desired properties of Lemma 6.58.

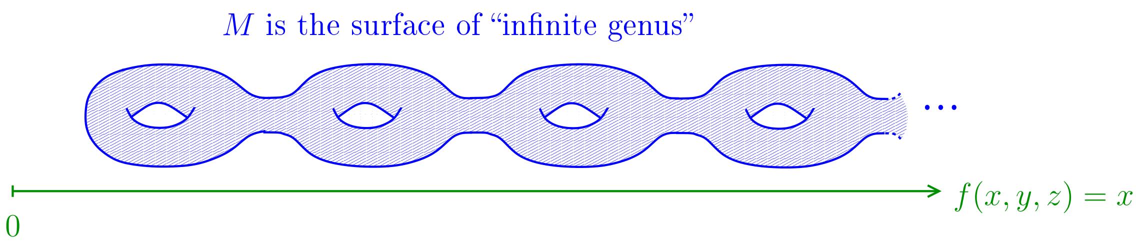


FIGURE 165

[125] See page 144 for the definition of a partition of unity subordinate to a given open cover.

Lemma 6.58.

(1) *Every topological manifold admits an exhaustion function.*
(2) *Every smooth manifold admits a smooth exhaustion function.*

Proof. We prove the statement for smooth manifolds. The proof for topological manifolds is almost identical. Thus let M be a smooth manifold. By Proposition 6.31 there exists a countable cover $\{U_i\}_{i\in\mathbb{N}}$ of M by precompact sets. Furthermore, by Theorem 6.57 there exists a smooth partition of unity $\{\psi_i\}_{i\in\mathbb{N}}$ subordinate to this open cover. We define

$$\begin{aligned} f\colon M &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \sum_{i\in\mathbb{N}} i\cdot\psi_i(x). \end{aligned}$$

Since locally this is a finite sum of smooth functions we see that f is smooth. It remains to prove the following claim.

Claim. For any $c\in\mathbb{R}$ the preimage $f^{-1}([0,c])$ is compact.

Let $c\in\mathbb{R}$. We pick $N\in\mathbb{N}$ with $N\geq c$. We start out with the following observation: Given any point $x\notin \overline{U}_1\cup\dots\cup\overline{U}_N$ we have

$$f(x) = \sum_{i=1}^{\infty} i\cdot\psi_i(x) \underset{\uparrow}{=} \sum_{i=N+1}^{\infty} i\cdot\psi_i(x) \geq \sum_{i=N+1}^{\infty} N\cdot\psi_i(x) \underset{\uparrow}{=} N\cdot\underbrace{\sum_{i=1}^{\infty}\psi_i(x)}_{=1} = N > c.$$

since $x\notin \overline{U}_1\cup\dots\cup\overline{U}_N$ and since $\operatorname{supp}\psi_i\subset U_i\subset\overline{U}_i$

It follows from this observation that

$$f^{-1}([0,c]) = \underbrace{f^{-1}([0,c])}_{\text{closed}} \cap \big(\underbrace{\overline{U}_1\cup\dots\cup\overline{U}_N}_{\text{compact}}\big).$$

It follows from Exercise 2.28 that $f^{-1}([0,c])$ is indeed compact. ■

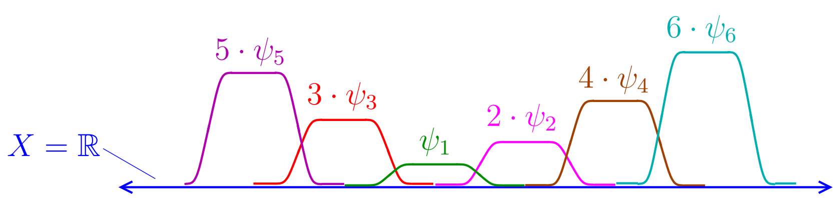


Figure 166. Illustration for the proof of Lemma 6.58.

We conclude this section with the following lemma that will be proved in Exercise 6.34.

Lemma 6.59. *Let M be a smooth manifold. For any continuous function $f\colon M\to\mathbb{R}_{>0}$ there exists a smooth function $g\colon M\to\mathbb{R}_{>0}$ such that $g(x)\in(0,f(x)]$ for all $x\in M$.*

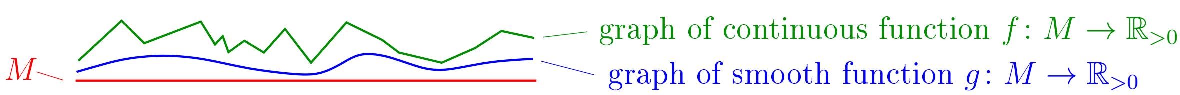


Figure 167. Illustration of Lemma 6.59.

Now we turn to the proof of Theorem 6.57. As we will see shortly, the proof is a consequence of the following proposition which sounds technical, but which is interesting in its own right.

Proposition 6.60. *Given any n-dimensional topological manifold M and given any open cover $\{U_i\}_{i\in I}$ there exists an atlas $\{\Phi_j\colon V_j \to W_j\}_{j\in J}$ with the following properties:*

(1) *The index set J is countable. If M is compact, then J can be chosen to be finite.*
(2) *Given any $j \in J$ there exists an i with $V_j \subset U_i$.*
(3) *Each W_j is either the open n-ball B^n or the open half-ball $B^n \cap H_n$.*
(4) *Given any point $x \in M$ there exists an open neighborhood W of x which has the property that $W \cap W_j = \varnothing$ for all but finitely many $j \in J$.*
(5) *We have $M = \bigcup\limits_{j\in J} \Phi_j^{-1}(W_j \cap \overline{B}^n_{\frac{1}{2}})$.*

If M is in fact a smooth manifold, then we can also arrange that the following extra condition is satisfied:

(6) *The charts are smooth, i.e. they lie in the maximal atlas of M.*

Remark.

(1) An atlas with Property (4) is sometimes called *locally finite*.
(2) With more effort, see e.g. [**Mun75**, Exercise 7.9 on p. 315] one can arrange for any topological manifold (not necessarily compact) that $\#J \leq n+1$.

Let us first show how we can deduce Theorem 6.57 from Proposition 6.60.

Proof of Theorem 6.57 assuming Proposition 6.60. Let M be a topological or a smooth manifold with a given open cover $\{U_i\}_{i\in I}$ of M. We apply Proposition 6.60 and we use the notation of Proposition 6.60. Note that according to Lemma 6.13 there exists a smooth function $\gamma\colon B^n \to [0,1]$ such that $\gamma(x)$ whenever $\|x\| \leq \frac{1}{2}$ and such that $\gamma(x) = 0$ whenever $\|x\| > \frac{3}{4}$. For each $j \in J$ we consider the function

$$\begin{array}{rcl} g_j\colon M & \to & [0,1] \\ x & \mapsto & \begin{cases} \gamma(\Phi_j(x)), & \text{if } x \in V_j, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

We make the following observations:

(a) Let $j \in J$. Note that the restriction of g_j to the open sets V_j and $M \setminus \Phi_j^{-1}(\overline{B}^n_{\frac{3}{4}})$ is continuous, hence g_j is continuous by Lemma 2.35 (2). Using (6) and Lemma 6.23 (1) we also see that the g_j are smooth, if we start out with a smooth manifold.
(b) It follows from (4) that given any $x \in M$ there exists an open neighborhood W of x such that $g_j(x) = 0$ for all but finitely many $j \in J$.
(c) It follows from (2) that for each $j \in J$ there exists an $i \in I$ with $\operatorname{supp}(g_j) \subset U_i$.
(d) It follows from (5) that for each $x \in M$ there exists a $j \in J$ with $g_j(x) \neq 0$.

It follows from (b) that it makes sense to consider the function $G\colon M \to \mathbb{R}_{\geq 0}$ that is given by $G(x) := \sum\limits_{j\in J} g_j(x)$ and that this function is smooth. It is now clear that the functions $x \mapsto \frac{g_j(x)}{G(x)}$ are a (smooth) partition of unity subordinate to the given open cover. ■

Lemma 6.61. *Let M be a topological manifold. There exists a family of compact subsets $\{C_n\}_{n\in\mathbb{N}}$ of M and a family of open subsets $\{D_{n+\frac{1}{2}}\}_{n\in\mathbb{N}}$ of M such that the following two conditions are satisfied:*

(1) *for every $n\in\mathbb{N}$ we have $C_n \subset D_{n+\frac{1}{2}} \subset C_{n+1}$,*

(2) *we have $M = \bigcup\limits_{n\in\mathbb{N}} C_n$.*

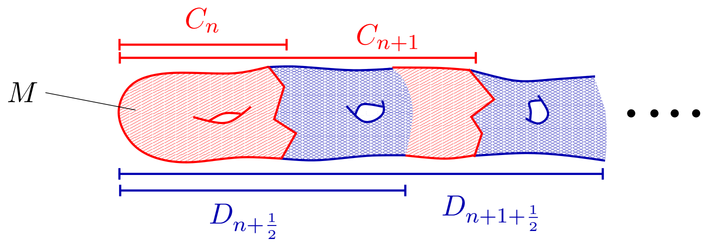


FIGURE 168. Illustration of Lemma 6.61.

PROOF. Let M be a topological manifold. It follows easily from Lemma 6.9 (4) and Lemma 6.2 that there exists a family $\{A_n\}_{n\in\mathbb{N}}$ of compact subsets such that $M = \bigcup\limits_{n\in\mathbb{N}} A_n$. It suffices to prove the following claim.

Claim. There exists a sequence of compact subsets $C_1, C_2, \dots$ of M and a sequence of open subsets $D_{1+\frac{1}{2}} \subset D_{2+\frac{1}{2}} \subset \dots$ such that for each $n\in\mathbb{N}$ we have $A_n \subset C_n \subset D_{n+\frac{1}{2}} \subset C_{n+1}$.

We set $C_1 := A_1$. Suppose that we already defined $C_1,\dots,C_n$ and $D_{1+\frac{1}{2}},\dots,D_{n-1+\frac{1}{2}}$. Since M is regionally compact we know that for each $x\in C_n\cup A_{n+1}$ there exists a compact neighborhood K_x. By definition of a neighborhood there exists an open neighborhood U_x of x with $U_x\subset K_x$. These open sets U_x cover the subset $C_n\cup A_{n+1}$ which is compact by Lemma 2.16 (1). Thus there exist $x_1,\dots,x_m$ such that $C_n\cup A_{n+1}\subset U_{x_1}\cup\dots\cup U_{x_m}$. We set $D_{n+\frac{1}{2}} := U_{x_1}\cup\dots\cup U_{x_m}$ and we set $C_{n+1} = K_{x_1}\cup\dots\cup K_{x_m}$. Note that C_{n+1} is compact by Lemma 2.16 (1). We have thus extended our sequences by one. Iterating this procedure gives us the desired sequences. ■

Now we are finally ready to provide the proof of Proposition 6.60.

PROOF OF PROPOSITION 6.60. To simplify the discussion we only treat the case that M is a topological manifold with empty boundary. The proof of all other cases can is basically the same and is left to the reader.

Thus let M be an n-dimensional topological manifold with empty boundary and let $\{U_i\}_{i\in I}$ be an open cover of M. We pick a family of compact subsets $\{C_n\}_{n\in\mathbb{N}}$ and a family of open subsets $\{D_{n+\frac{1}{2}}\}_{n\in\mathbb{N}}$ with the properties stated in Lemma 6.61.

Note that it follows from Lemma 2.17 (2) that each $D_{n+\frac{1}{2}}\setminus C_{n-1}$ is an open of M. Furthermore, by design these open subsets cover all of M. Using this observation it is elementary to see that given any $x\in M$ there exists a chart $\Phi_x\colon V_x\to W_x$ with the following property:

(1) V_x is an open neighborhood of x which is contained in some U_i and for which there exists an $n\in\mathbb{N}$ with $V_x\subset D_{n+\frac{1}{2}}\setminus C_{n-1}$,

(2) W_x is the open n-ball B^n of radius 1.

For each $x \in M$ we set $\widetilde{U}_x := \Phi_x^{-1}(B^n_{\frac{1}{2}})$. Evidently these are open sets which cover all of M. It follows from Lemma 2.15 that for each $n \in \mathbb{N}$ there exists a finite set $J_n \subset M$ such that $C_n \subset \bigcup_{x \in J_n} \widetilde{U}_x$. We set $J := \bigcup_{n \in \mathbb{N}} J_n$. One can now easily verify that the atlas $\{\Phi_j : V_j \to W_j\}_{j \in J}$ has all the desired properties. We leave this straightforward task to the reader. ■

6.15. **Sets of measure zero and Sard's Theorem.** One can ask, whether a smooth map between smooth manifolds always admits regular values. In this section we will formulate Sard's Theorem which gives an affirmative answer to this question.

To formulate the theorem we need the notion of a "subset of measure zero" that we will now introduce. The precise definition is not all that important, what matters in practice are the properties stated in the subsequent proposition.

Definition.

(1) A *cuboid* is any subset of $\mathbb{R}^m$ of the form $[c_1, d_1] \times \cdots \times [c_m, d_m]$. We define the volume of such a cuboid as

$$\mathrm{vol}([c_1, d_1] \times \cdots \times [c_m, d_m]) = \prod_{i=1}^{m} (d_i - c_i).$$

(2) Let $X \subset \mathbb{R}^m$ be a subset. We say that X *has measure zero* if given any $\epsilon > 0$ there exists a countable family $\{A_i\}_{i \in I}$ of cuboids such that $X \subset \bigcup_{i \in I} A_i$ and such that $\sum_{i \in I} (\nu)(A_i) < \epsilon$. This definition is illustrated in Figure 169.

(3) Let M be an m-dimensional smooth manifold.
   (a) We say that a subset $X \subset M$ is a *subset of measure zero* if for every chart $\Phi : U \to V$ from the smooth atlas the set $\Phi(X \cap U)$ is a subset of measure zero of $\mathbb{R}^m$.
   (b) We say that a subset $X \subset M$ is a *subset of full measure* if $M \setminus X$ is a subset of measure zero.

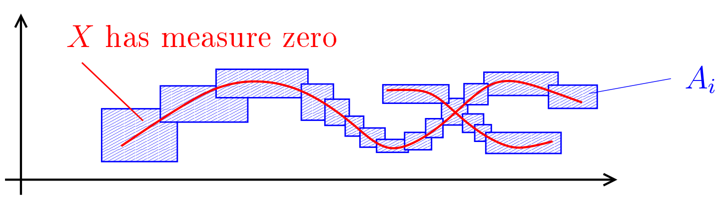


FIGURE 169

The following proposition summarizes key properties of subsets of measure zero.

Proposition 6.62. *Let M be an m-dimensional smooth manifold.*

(1) (a) *The union of countably many subsets of measure zero is again a subset of measure zero.*
   (b) *The intersection of countably many subsets of full measure is again a subset of full measure.*

(2) (a) *Given any subset X of measure zero of M the complement $M \setminus X$ is dense in M.*
(b) *Every subset of M of full measure is dense in M.*

(3) *Let X be a subset of M. If there exist charts $\{\Phi_i \colon U_i \to V_i\}_{i \in I}$ from the smooth atlas such that $\bigcup_{i \in I} U_i = M$ and such that each $\Phi_i(U_i \cap X)$ is a subset of measure zero of $\mathbb{R}^m$, then X is a subset of measure zero.*

(4) *If $f \colon M \to N$ is a smooth map between smooth manifolds and if $\dim(M) < \dim(N)$, then $f(M)$ is a subset of measure zero in N. In other words, $N \setminus f(M)$ is a subset of full measure.*

(5) (a) *Let $f \colon M \to N$ be a smooth map between smooth manifolds of the same dimension. If X is a subset of measure zero in M, then $f(X)$ has measure zero in N.*
(b) *If $f \colon M \to N$ is a diffeomorphism, then $X \subset M$ has full measure if and only if $f(X)$ does.*

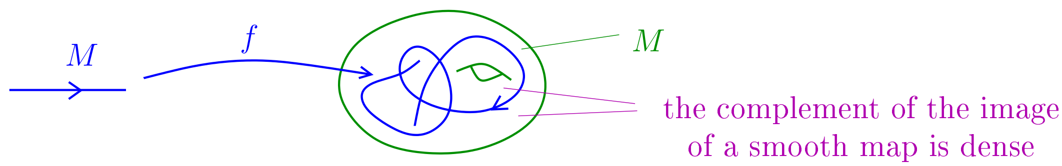


FIGURE 170. Illustration of Proposition 6.62 (2) and (4).

SKETCH OF PROOF.

(1) Let $\{X_i\}_{i \in I}$ be a countable family of subsets of measure zero of $\mathbb{R}^n$. Using the fact that $\sum_{i \in \mathbb{N}} \frac{1}{2^i} = 1$ it is straightforward to show that $\bigcup_{i \in I} X_i$ is also a subset of measure zero. Statement (1a) is an easy consequence of this observation. Statement (1b) is just a convenient reformulation of Statement (1a).

(2) Statement (2a) follows easily from the observation that a subset of $\mathbb{R}^m$ that contains an open ball $B^m_r(z)$ of radius $r > 0$ does not have measure zero. We refer to [**Lee02**, Proposition 6.8] for more details. Once again Statement (2b) is just a reformulation of Statement (2a).

(3) This statement is proved in [**Lee02**, Lemma 6.6].

(4) We write $m = \dim(M)$ and we write $n = \dim(N)$. We pick a countable family $\{\Phi_i \colon U_i \to V_i\}_{i \in I}$ of charts for M from the maximal smooth atlas for M such that $M = \bigcup_{i \in I} U_i$.[126] It follows easily from [**Lee02**, Proposition 6.5] and the hypothesis that $m < n$ that every $f(\Phi_i(U_i))$ is a subset of measure zero of N. It follows from (1) that $f(M)$ is subset of measure zero of N.

(5) Statement (5a) is [**Lee02**, Theorem 6.9]. Finally note that Statement (5b) follows easily from Statement (5a). ■

[126] In Exercise 6.19 (2) we will show that such a countable atlas exists.

Now we can formulate Sard's Theorem which shows that a smooth map between smooth manifolds always admits a plethora of regular values. The theorem is named after Arthur Sard [**Sar42**] who first proved a similar statement in 1942.

Theorem 6.63. (Sard's Theorem) *If $f\colon M \to N$ is a smooth map between smooth manifolds (possibly with boundary)*[127]*, then the set of critical values of f is of measure zero in N.*

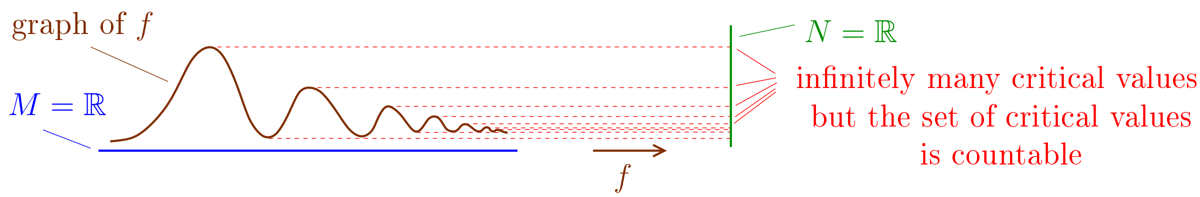


FIGURE 171. Illustration of Sard's Theorem 6.63.

Remark.

(1) As we pointed out on page 308, if $f\colon M \to N$ is a smooth map between smooth manifolds with $\dim(M) < \dim(N)$, then a point $P \in N$ is a regular value if and only if $f^{-1}(P) = \varnothing$. It follows from this observation that Sard's Theorem 6.63 together with Proposition 6.62 (2) implies Proposition 6.62 (4).

(2) Sard's Theorem 6.63 can also be used to give one more proof of the fundamental theorem of algebra, we refer to [**Bre93**, Corollary II.6.4] or [**Miln65a**, p. 8] for details.

(3) Sometimes Sard's Theorem is also called *Morse-Sard Theorem*, since the case that $N = \mathbb{R}$ was first proved by Anthony Morse [**MorsA39**] in 1939.

PROOF. The theorem is a straightforward consequence of [**Lee02**, Theorem 6.10]. ■

We conclude this section with the following proposition which often allows us to reduce a statement about non-compact smooth manifolds to a statement about compact smooth manifolds.

Proposition 6.64. *Let W be an n-dimensional smooth manifold with compact boundary. There exists a sequence $X_1, X_2, \ldots$ of n-dimensional smooth submanifolds of W with the following four properties:*

(1) *The sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$.*
(2) *Each X_i is compact and connected.*
(3) *The X_i are submanifolds in the sense of the definition on page 270.*
(4) *We have $\bigcup\limits_{i\in\mathbb{N}} \mathring{X}_i = W$.*

Remark. There is an analogue of Proposition 6.64 if ∂W is non-compact. Since that statement is more technical to formulate we postpone it to Proposition 11.4.

PROOF. Let W be an n-dimensional smooth manifold. If W is empty, then there is nothing to show, so we might as well assume that W is non-empty. We pick a point $w_0 \in W$. By

[127] We stress that M and N are allowed to have boundary, since (a) we need these cases, and (b) since many proofs in the literature assume that M and N do not have boundary.

Lemma 6.58 there exists a smooth exhaustion function on ∂W, i.e. there exists a smooth function $f\colon W \to \mathbb{R}_{\geq 0}$ with the property that for any $c \in \mathbb{R}_{\geq 0}$ the preimage $f^{-1}([0,c])$ is a compact subset of W. Since ∂W is compact it follows from Lemma 2.40 that there exists a $C \in \mathbb{R}_{\geq 0}$ such that $f(\partial W) \subset (-\infty, C]$ and such that $C \geq f(w_0)$.

It follows from Sard's Theorem 6.63 and Proposition 6.62 (2) that there exists an increasing sequence $C < C_1 \leq C_2 \leq C_3 \leq \dots$ of regular values of f such that $\lim\limits_{i\to\infty} C_i = \infty$. For each $i \in \mathbb{N}$ we denote by X_i the path-component of $f^{-1}((-\infty, C_i])$ that contains w_0. By Exercise 6.31, which is a reasonably straightforward consequence of the Regular Value Theorem 6.53, and by our choice of f we know that each X_i is a compact n-dimensional submanifold. We have now shown that the X_i satisfy Properties (1), (2) and (3).

Now let $P \in W$. Since W is path-connected there exists a path $\gamma\colon [0,1] \to W$ from w_0 to P. Since $[0,1]$ is compact it follows again from Lemma 2.40 and from $\lim\limits_{i\to\infty} C_i = \infty$ that there exists an i with $\gamma([0,1]) \subset f^{-1}((-\infty, C_i))$. This implies almost immediately that $P \in \mathring{X}_i$. We have thus shown that (4) is also satisfied. ■

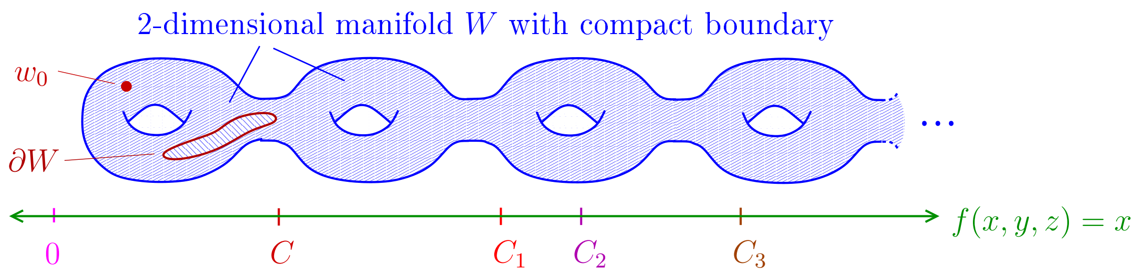


FIGURE 172. Illustration of the proof of Proposition 6.64

Exercises for Chapter 6.

Exercise 6.1. Given $I \subset \mathbb{R}$ we denote by $C^b(I,\mathbb{R})$ the set of all bounded maps $I \to \mathbb{R}$ and given $f \in C^b(I,\mathbb{R})$ we set

$$\|f\| \;:=\; \sup\{f(x) \,|\, x \in I\}.$$

Note that $(C^b(I,\mathbb{R}), \|-\|)$ is a metric space. We will equip $C^b(I,\mathbb{R})$ with the corresponding topology.

(a) Show that if I is a compact interval, then $C^b(I,\mathbb{R})$ is second-countable.
(b) Show that if I is a non-compact interval, then $C^b(I,\mathbb{R})$ is not second-countable.

Exercise 6.2. Show that the line with two zeros admits a chart of type (i) for every point.

Exercise 6.3. Let I be an uncountable set. By the Tychonoff Theorem 3.17 we know that $[0,1]^I$ is compact. Show that $[0,1]^I$ is not second-countable.

Exercise 6.4. A topological space is called *separable* if it admits a subset that is countable and dense.

(a) Show that every second-countable topological space is separable.

(b) Let X be a set. We equip X with the cofinite topology that we defined on page 84.
 (i) Show that X is separable.
 (ii) Suppose that X is uncountable. Show that X is not second-countable.
 Remark. This example is taken from [**SS78**, Example 19].

(c) Let X be a topological space that is separable *and* Hausdorff. Show that X is second-countable.

Exercise 6.5. Does there exist a compact topological space that satisfies all axioms of a closed 1-dimensional topological manifold except for the Hausdorff axiom?

Exercise 6.6. Let $f\colon M \to N$ be a map between two topological manifolds. Show that given any $x \in M$ there exist neighborhoods U of x and V of $f(x)$ with $f(U) \subset V$ such that both U and V are homeomorphic to closed balls.

Exercise 6.7. We consider the topological space $X = \{(x,y) \in \mathbb{R}^2 \,|\, xy = 0\}$ shown in Figure 173. Show that X is not a topological manifold.
Remark. It is clear that things go awry at the origin P. But it is somewhat tricky to nail down why there is no chart around that point.

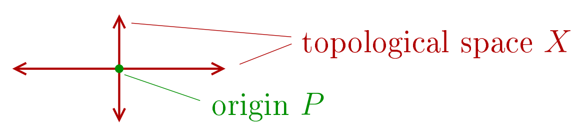


FIGURE 173

Exercise 6.8. Let M be a smooth manifold and let W be a submanifold of M. Show that for every component C of ∂W we have either $C \subset \partial M$ or $C \subset M \setminus \partial M$.

Exercise 6.9. Show that there exists a smooth map $f = (f_1, f_2)\colon [0,1] \to [0,1] \times [0,1]$ with the following properties:

(a) For $t \in [0, \frac{1}{4}]$ we have $f(t) = (0, 1-t)$.
(b) For $t \in [\frac{3}{4}, 1]$ we have $f(t) = (1, t)$.
(c) The map $f_1\colon [0,1] \to [0,1]$ is monotonously increasing.
(d) For each $t \in [0,1]$ we have $f'(t) \neq 0 \in \mathbb{R}^2$.
(e) For $t \neq \{0,1\}$ we have $f_2(t) \in (0,1)$.

Hint. use Lemma 6.13.

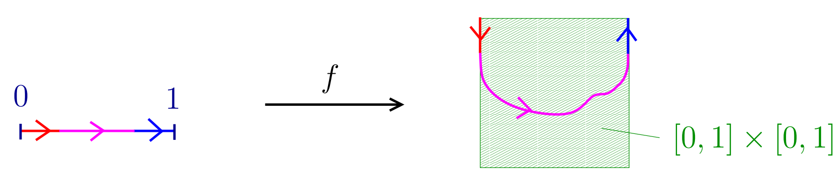


FIGURE 174. Illustration for Exercise 6.9.

Exercise 6.10. Let $f\colon V \to W$ be a map between two topological manifolds such that $f(V) \subset W \setminus \partial W$. Show that the corresponding graph $\mathrm{Gr}(f) := \{(x, f(x)) \,|\, x \in V\}$ is a proper submanifold of the topological manifold $V \times W$.

Exercise 6.11. In Figure 175 we show $M = \{(x, y) \mid y \geq 0\}$ together with two subsets $K := \{(x, x^2) \mid x \geq 0\}$ and $L := \{(x, |x| + 1) \mid x \in \mathbb{R}\}$. Show that both K and L are submanifolds of the topological manifold M but that they are not submanifolds if we view M as a smooth manifold.

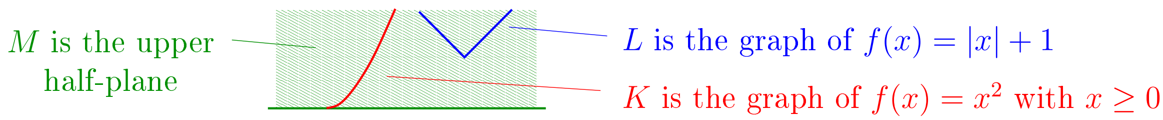


FIGURE 175

Exercise 6.12. Let $n \in \mathbb{N}_0$.

(a) Show that the action of $G = \{\pm 1\}$ on S^n is orientation-preserving if and only if n is odd.
Hint. Use Proposition 6.39 (2) and Lemma 6.46 (3).

(b) Show that the real projective space $\mathbb{RP}^n$ is orientable if and only if n is odd.
Hint. Use (a) together with Lemma 6.46 and Proposition 6.47.

Exercise 6.13. Let $n \in \mathbb{N}$. We consider $M = \mathbb{R}^n \cup \{\infty\}$. We denote by 0 the origin of $\mathbb{R}^n$.

(a) We consider the maps $\Phi = \mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$ and

$$\begin{array}{rcl} \Psi\colon M \setminus \{0\} & \to & \mathbb{R}^n \\ x & \mapsto & \begin{cases} \frac{1}{\|x\|^2} \cdot x, & \text{if } x \neq \infty, \\ 0, & \text{if } x = \infty. \end{cases} \end{array}$$

Show that Φ and Ψ form a smooth atlas for M.

(b) We consider the map

$$\begin{array}{rcl} f\colon M & \to & M \\ x & \mapsto & \begin{cases} 3x, & \text{if } x \neq \infty, \\ \infty, & \text{if } x = \infty \end{cases} \end{array}$$

Show that f smooth and show that $\mathrm{D}\, f_0 = 3 \cdot \mathrm{id}$ and $\mathrm{D}\, f_\infty = \frac{1}{3} \cdot \mathrm{id}$.

Exercise 6.14. Let U be an open convex neighborhood of $0 \in \mathbb{R}^n$ and let $f\colon U \to \mathbb{R}$ be a smooth map with $f(0 = 0$. We assume that 0 is a critical point, i.e. we assume that the differentialD f_0 is zero. Show that there exist smooth functions $h_{ij}\colon U \to \mathbb{R}$, $i, j = 1, \ldots, n$ such that $h_{ij} = h_{ji}$ and such that

$$f(x) \;=\; \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij}(x) \cdot x_i \cdot x_j \qquad \text{for all } x = (x_1, \ldots, x_n) \in U.$$

Hint. Make use of Proposition 6.16.

Exercise 6.15. Let $M \subset \mathbb{R}^n$ be a k-dimensional smooth submanifold and let $P \in M \setminus \partial M$.

(a) Show that there exists an $A \in \mathrm{O}(n)$ with $A \cdot \mathrm{V}_P\, M = \{(x, 0) \in \mathbb{R}^n \mid x \in \mathbb{R}^k\}$.

(b) Show that there exists an neighborhood U of $P \subset M$ such that the map

$$\begin{array}{rcl} U & \to & \mathrm{V}_P\, M \xrightarrow{x\mapsto A\cdot x} \{(x,0)\,|\,x\in\mathbb{R}^k\} \xrightarrow{(x,0)\mapsto x} \mathbb{R}^k \\ x & \mapsto & \text{orthogonal projection} \\ & & \text{of } x-P \text{ onto } \mathrm{V}_P\, M \end{array}$$

is a chart.

Exercise 6.16. Let M and N be two smooth manifolds of dimension ≥ 1. Show that $M \times N$ is orientable if and only if M and N are both orientable.

Exercise 6.17.

(a) Let $p\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ be the stereographic projection from Lemma 2.44. It restricts to a diffeomorphism $\varphi\colon S^n \setminus \{(0,\dots,0,1)\} \to \mathbb{R}^n$. We equip both sides with the usual orientation. Show that φ is orientation-preserving if and only if n is odd.

(b) We consider the projection

$$\begin{array}{rcl} \psi\colon S^n_{>0} = \{(x_1,\dots,x_{n+1}) \in S^n \,|\, x_{n+1} > 0\} & \to & B^n \\ (x_1,\dots,x_{n+1}) & \to & (x_1,\dots,x_n). \end{array}$$

We equip both sides with the usual orientation. Show that ψ is orientation-preserving if and only if n is even.

Exercise 6.18. Let $n \in \mathbb{N}$ and let $A \in \mathrm{GL}(n,\mathbb{Z})$ be a matrix. Show that the map

$$\begin{array}{rcl} f(A)\colon \mathbb{R}^n/\mathbb{Z}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ v & \mapsto & Av \end{array}$$

is a diffeomorphism.

Exercise 6.19.

(a) Let M be a topological manifold. Show that M admits a countable atlas.
Hint. Use that topological manifolds are second-countable and use Lemma 2.25.

(b) Let M be a smooth manifold. Show that M admits a countable atlas such that each chart lies in the maximal smooth atlas of M.

Exercise 6.20. Let G be a group that acts freely, properly and smoothly on an n-dimensional oriented smooth manifold M. Suppose that M/G is orientable.

(a) Suppose that M is path-connected. Show that every $g \in G$ acts on M in an orientation-preserving way.
Hint. Use Proposition 6.47 (2).

(b) Show that the conclusion of (a) does not hold if M is not path-connected.

Remark. This exercise can be viewed as a converse to Proposition 6.47 (1).

Exercise 6.21. Let M be a smooth manifold and let $N \subset M$ be a smooth submanifold with $\dim(N) < \dim(M)$.

(a) Show that the interior of N in M is the empty set.

(b) Show that the closure of $M \setminus N$ in M equals M.

Exercise 6.22. Let $n \in \mathbb{N}_0$. As on page 194 we identify the real projective space $\mathbb{R}\mathrm{P}^n$ with $\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\})$.

(a) Let $i \in \{0, \dots, n\}$. Show that $V_i := \{[x_0 : \dots : x_n] \in \mathbb{R}\mathrm{P}^n \mid x_i \neq 0\}$ is an open subset of $\mathbb{R}\mathrm{P}^n$.

(b) Show that the maps

$$\begin{array}{rcl} \Phi_i \colon \{[x_0 : \dots : x_n] \in \mathbb{R}\mathrm{P}^n \mid x_i \neq 0\} & \to & \mathbb{R}^n \\ [x_0 : \dots : x_n] & \mapsto & \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right). \end{array}$$

form a smooth atlas for $\mathbb{R}\mathrm{P}^n$.

Exercise 6.23. Let $M := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \leq 0\}$, $N := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$ and $H := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$. Let $X \subset M$ be a k-dimensional topological submanifold with $\partial X = X \cap H$ and let $Y \subset N$ be a k-dimensional topological submanifold with $\partial Y = Y \cap H$. We suppose that $X \cap H = Y \cap H$. Show that $X \cup Y$ is a k-dimensional topological submanifold of $\mathbb{R}^n$.

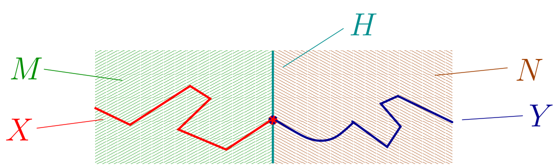


FIGURE 176. Illustration for Exercise 6.23.

Exercise 6.24. Let M be a compact smooth manifold and let N be a proper codimension-zero submanifold of M. Show that if M is connected and if $N \neq \varnothing$, then $M = N$.

Exercise 6.25. Let M be a topological manifold, let $P \in M$ and let $f \colon K \to M$ be a map from a compact topological space K to M. Show that if $P \notin f(K)$, then there exists an open neighborhood U of P with $f(K) \cap U = \varnothing$.

Exercise 6.26. Let M be an m-dimensional smooth manifold with possibly non-empty boundary and let N be an n-dimensional smooth manifold with empty boundary. We equip the product smooth manifolds $M \times N$ and $N \times M$ with the product orientations as defined in Proposition 6.5.

(a) Show that the diffeomorphism

$$\begin{array}{rcl} M \times N & \to & N \times M \\ (x, y) & \mapsto & (y, x) \end{array}$$

is orientation-preserving if and only if $m \cdot n$ is even.

(b) We equip ∂M and $N \times M$ with the boundary orientation and we equip $N \times \partial M$ with the product orientation. Show that we have the following equality of *oriented* smooth manifolds:

$$\partial(N \times M) \;=\; \begin{cases} N \times \partial M, & \text{if } n \text{ is even}, \\ -(N \times \partial M), & \text{if } n \text{ is odd}. \end{cases}$$

Exercise 6.27. We consider the smooth manifold $M = \mathbb{R} \times [-1,1]$ together with the submanifolds

$$W_t \;:=\; \{(t + \tfrac{1}{y^2-1}, y) \,|\, y \in (-1,1)\} \quad \text{where } t \in \mathbb{R}.$$

Show that there is no smooth path $\gamma\colon [-1,1] \to M$ that connects the bottom $\mathbb{R} \times \{-1\}$ with the top $\mathbb{R} \times \{1\}$ such that γ is transverse to every W_t, i.e. such that for any $s \in (-1,1)$ with $\gamma(s) \in W_t$ we have $\gamma'(s) \notin V_{\gamma(s)}W_t$.
Remark. This decomposition of the interior of $M = \mathbb{R} \times [-1,1]$ into submanifolds is sometimes called a *Reeb foliation*.

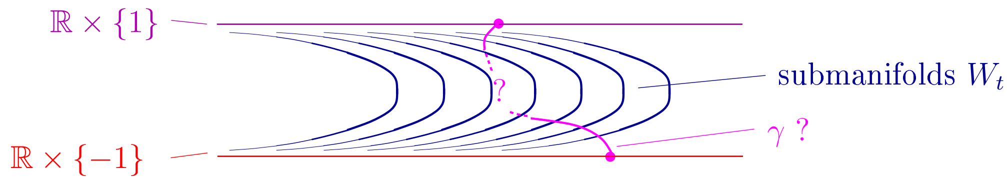


Figure 177. Illustration of Exercise 6.27.

Exercise 6.28. Let $A \in \mathrm{O}(n)$. Show that the map $S^n \to S^n$ given by $x \mapsto A \cdot x$ is orientation-preserving if and only if $\det(A) = 1$.

Exercise 6.29. Let $f\colon M \to N$ be a smooth map between two smooth manifolds. We assume that $\partial M = \varnothing$.

(a) Show that the set of regular points of f is an open subset of M.
Hint. You can use the fact that a real $(m \times n)$-matrix A has rang m if and only if $\det(A \cdot A^T) \neq 0$.
(b) Let $v \in N$ be a regular value such that $f^{-1}(v)$ is finite. We assume that M is compact. Show that there exists an open neighborhood U of v such that each point in U is a regular value.
(c) Show that in general the set of regular values of f is not an open subset of N.

Exercise 6.30. Let $f\colon M \to N$ be a map between two closed n-dimensional smooth manifolds which is injective and which has the property that for $P \in M$ the differential $\mathrm{D}\, f_P$ is an isomorphism. Show that f is a diffeomorphism.

Exercise 6.31. Let M be an m-dimensional smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function.

(a) Let $s \in \mathbb{R}$ be a regular value of f such that for each component C of ∂M we have either $f^{-1}(\{s\}) \cap C = C$ or $f^{-1}(\{s\}) \cap C = \varnothing$. Show that $X := f^{-1}((-\infty, s])$ is a smooth codimension-zero submanifold of M with $\partial_0 X = f^{-1}(\{s\}) \setminus \partial M$ and $\partial_1 X = X \cap \partial M$.
(b) Let $s < t \in \mathbb{R}$ be regular values such that for each component C of ∂M we have either $f^{-1}(\{s\}) \cap C = C$ or $f^{-1}(\{s\}) \cap C = \varnothing$ and such that for each component C of ∂M we have either $f^{-1}(\{t\}) \cap C = C$ or $f^{-1}(\{t\}) \cap C = \varnothing$. Show that $X := f^{-1}([s,t])$ is a smooth codimension-zero submanifold of M with $\partial_0 X = (f^{-1}(\{s\}) \cup f^{-1}(\{t\})) \setminus \partial M$ and $\partial_1 X = X \cap \partial M$.

Hint. Use the Regular Value Theorem 6.53.
Remark. Later in the Regular Value Theorem 11.2 for $\mathbb{R}$ we will prove a refinement of the above statement.

Exercise 6.32. Let $n \in \mathbb{N}$.

(a) Show that
$$\mathrm{GL}(n,\mathbb{R}) \;=\; \{A \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, \det(A) \neq 0\}$$
is a smooth manifold of dimension n^2.

(b) Show that
$$\mathrm{SL}(n,\mathbb{R}) \;=\; \{A \in \mathrm{M}(n\times n,\mathbb{R}) \,|\, \det(A) = 1\}$$
is a smooth manifold of dimension n^2-1.

Exercise 6.33. Show that
$$\mathrm{SU}(n,\mathbb{R}) \;=\; \left\{A \in \mathrm{M}(n\times n,\mathbb{C}) \,\middle|\, \overline{A}^T A = \mathrm{id} \text{ and } \det(A) = 1\right\}$$
is a smooth manifold of dimension n^2-1.

Exercise 6.34. Let M be a smooth manifold and let $f\colon M \to \mathbb{R}_{>0}$ be a continuous function. Show that there exists a smooth function $g\colon M \to \mathbb{R}_{>0}$ such that $g(x) \in (0, f(x)]$ for all $x \in M$.
Hint. Use a suitable partition of unity.

Exercise 6.35. Let M be a compact n-dimensional smooth manifold and let K and L be two compact subsets of M. Show that there exists a smooth map $f\colon M \to \mathbb{R}$ such that $\sup\{f\colon A \to \mathbb{R}\} < \inf\{f\colon B \to \mathbb{R}\}$.
Hint. Use Theorem 6.57 (2).

Exercise 6.36. Let M be a closed n-dimensional smooth manifold.

(a) Let $\{U, V\}$ be an open cover of M. Show that there exist compact n-dimensional submanifolds A and B of M such that $A \subset U$, $V \subset B$, $A \cup B = M$ and such that $A \cap B = \partial A = \partial B$.
Hint. Use Exercises 6.31 and 6.35.

(b) Let K and L be compact subsets of M. Show that there exist compact n-dimensional submanifolds A and B of M such that $K \subset \mathring{A}$, $L \subset \mathring{B}$, $A \cup B = M$ and such that $A \cap B = \partial A = \partial B$.

FIGURE 178. Illustration of Exercise 6.36.

7. 1-dimensional manifolds and the long line

In this section we will consider “1-dimensional objects”. More precisely, we will do the following:

(1) We will give the classification of 1-dimensional (topological) smooth manifolds.
(2) We will construct the *long line*, this is a path-connected topological space that satisfies all properties of a 1-dimensional topological manifold, except that it is not second-countable.

7.1. The classification of 1-dimensional (topological) smooth manifolds $(*)$.

One of the goals of topology is to completely classify suitable classes of topological spaces. For example one could try to classify all connected topological and smooth manifolds of a given dimension. In this short section we give a complete classification in the 1-dimensional setting.

In the following we first deal with 1-dimensional *topological* manifolds.

Theorem 7.1. *Let M be a non-empty connected 1-dimensional topological manifold. Then the homeomorphism type of M is given by the following table*[128]

	without boundary	*with non-empty boundary*
compact	*the circle* S^1	*the closed interval* $[0,1]$
non-compact	*the open interval* $(0,1)$	*the half-open interval* $[0,1)$.

Note that the adjectives “Hausdorff” and “second-countable” in the definition of a topological manifold are both necessary:

(1) The line with two zeros satisfies all axioms of a smooth manifold except for the Hausdorff axiom.
(2) If one drops the condition that a topological manifold has to be second-countable, then there a two more examples of connected 1-dimensional topological manifolds, namely the “open long ray $\mathbb{L}^+$ and the “long line $\mathbb{L}$” which we will introduce on page 334.

In the following proposition we first deal with the case that M is closed. Afterwards we will outline how one can obtain the general statement.

Proposition 7.2. *Let M be a closed connected non-empty 1-dimensional topological manifold. Then M is homeomorphic to S^1.*

In the proof of Proposition 7.2 we will need the following lemma.

Lemma 7.3. *Let M be a 1-dimensional topological manifold and let $\Phi\colon U \to (-2,2)$ be a chart. Let $[a,b] \subset (-2,-2)$ be a compact interval. Then $\Phi^{-1}([a,b])$ is a closed subset of M.*

Proof. Since Φ is a homeomorphism the map $\Phi^{-1}\colon (-2,2) \to U$ is also continuous. Since $[a,b]$ is compact it follows from Lemma 2.40 that $\Phi^{-1}([a,b])$ is also compact. But by

[128] By Exercise 2.40 we know that the half-open interval $[0,1)$ is not homeomorphic to the open interval $(0,1)$.

Lemma 2.17 (2) any compact subspace of a Hausdorff space is closed. But M is a topological manifold, in particular M is Hausdorff. Therefore $\Phi^{-1}([a,b])$ is a closed subset of M. ■

Definition. Let M be a 1-dimensional topological manifold.

(1) A *c-parametrization* for M is a homeomorphism $\Psi\colon [-1,1] \to C$ from the compact interval $[-1,1]$ to a closed subset C of M. We say the c-parametrization is *open* if $\Psi((-1,1))$ is an open subset of M.
(2) We say that two c-parametrizations Ψ and Ψ' are *almost disjoint* if

$$\Psi((-1,1)) \cap \Psi'((-1,1)) \; = \varnothing.$$

(3) A *c-atlas* is a family of c-parametrizations $\{\Psi_i\colon [-1,1] \to C_i\}_{i\in I}$ of M such that the union of the C_i's equals M. We call the c-atlas *open* if all c-parametrizations in the c-atlas are open. We say the c-atlas *almost disjoint* if any two different c-parametrizations in the c-atlas are almost-disjoint.

Using Lemma 7.3 one can prove the following lemma that is illustrated in Figure 179. We leave it to the reader to provide the proof.

Lemma 7.4. *Let M be a 1-dimensional topological manifold and let $\Psi_1\colon [-1,1] \to C_1$ and $\Psi_2\colon [-1,1] \to C_2$ be two open c-parametrizations such that $C_1 \cap C_2$ is non-empty but such that C_2 is not contained in C_1. Then*

$$[-1,1] \setminus \Psi_1^{-1}\big(\Psi_2((-1,1) \cap C_1)\big) \; = \; [a,b]$$

for some $-1 \le a < b \le 1$.

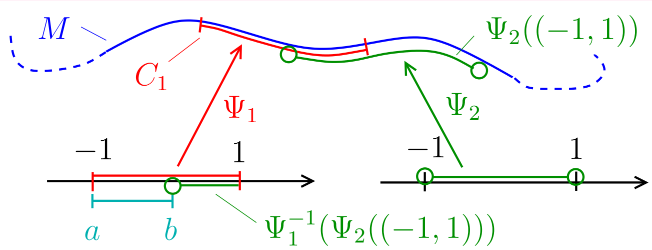


FIGURE 179. Illustration of Lemma 7.4.

Now we can provide the proof of Proposition 7.2.

PROOF OF PROPOSITION 7.2. Let M be a closed connected 1-dimensional topological manifold. We start out with the following claim.

Claim. There exists a finite open almost-disjoint c-atlas $\Psi_i\colon [-1,1] \to C_i$ with $i = 1,\dots,k$.

Since M is assumed to be closed there exists for each $P \in M$ a chart $\Phi_P\colon U_P \to (-2,2)$ of type (i) such that $\Phi_P(P) = 0$. The open sets $\Phi_P^{-1}((-1,1))$ are an open cover of M.[129] Since M is compact there exist $P_1,\dots,P_k$ such that the open sets $\Phi_{P_i}^{-1}((-1,1))$, $i = 1,\dots,k$ are an open cover of M. It follows that the union of the sets $C_i := \Phi_{P_i}^{-1}([-1,1])$, $i = 1,\dots,k$ is also all of M. It follows from Lemma 7.3 that each C_i is a closed subset of M.

[129]Here we use implicitly the following fact, which is for example an immediate consequence of Lemma 2.38: if U is open in M, and if V is open in U, then V is also open in M. Why is that?

For $i = 1, \dots, k$ we now consider the homeomorphism Ψ_i that is given by

$$\Psi_i := \big(\Phi_{P_i}|_{C_i}\big)^{-1} \colon [-1,1] \xrightarrow{\cong} C_i.$$

By the above discussion these are open c-parametrizations. After possibly removing some c-parametrizations we can assume that no C_i is contained in any other C_j.

Now we make the following modifications to our c-atlas. If for some $i < j$ we have $\Psi_i((-1,1)) \cap \Psi_j((-1,1)) \neq \varnothing$, then it follows from Lemma 7.4 that

$$[-1,1] \setminus \Psi_i^{-1}\big(\Psi_j((-1,1)) \cap C_i\big) \;=\; [a,b]$$

for some $-1 \le a < b \le 1$. Now we replace Ψ_i by the map

$$[-1,1] \xrightarrow{\text{linear homeomorphism}} [a,b] \xrightarrow{\Psi_i} M.$$

Repeating this process we obtain finitely many open c-parametrizations that are furthermore almost-disjoint. These c-parametrizations form the desired c-atlas. ⊞

In the following we say that a c-atlas is *simple* if given any $P \in M$ there exist at most two c-parametrizations that contain P.

Claim. Every almost-disjoint open c-atlas is simple.

This follows from the fact that any point P on a 1-dimensional topological manifold has the property, that given any open neighborhood U there exists an open neighborhood V of x, contained in U, such that $V \setminus \{P\}$ contains precisely two components. If a point $P \in M$ is contained in k c-parametrizations of an open almost-disjoint c-atlas, then one can easily show that P admits an open neighborhood U such that for any open neighborhood V of P contained in U the set $V \setminus \{P\}$ has at least k components. ⊞

Now let $\Psi_i \colon [-1,1] \to C_i$ and $\Psi_j \colon [-1,1] \to C_j$ be two c-parametrizations of our c-atlas such that $C_i \cap C_j$ consists of a single point. Since the c-parametrizations are almost disjoint we have $C_i \cap C_j \subset \Psi_i(\{\pm 1\})$ and $C_i \cap C_j \subset \Psi_j(\{\pm 1\})$. After possibly replacing $\Psi_i(t)$ by $\Psi_i(-t)$ and/or $\Psi_j(t)$ by $\Psi_j(-t)$ we can assume that $\Psi_i(1) = \Psi_j(-1)$. We consider the map

$$\begin{array}{rcl} \Psi \colon [-1,1] & \to & C_1 \cup C_2 \\ x & \mapsto & \begin{cases} \Psi_i(2x+1), & \text{if } x \in [-1,0], \\ \Psi_j(2x-1), & \text{if } x \in [0,1]. \end{cases} \end{array}$$

It is straightforward to verify that Ψ is a bijection and continuous. It follows from Proposition 2.43 (3) that Ψ is in fact a homeomorphism.[130]

By iteratively applying the procedure from the previous claim to our c-atlas we obtain a finite almost-disjoint simple c-atlas[131] $\Psi_i \colon C_i \to [-1,1]$, $i = 1, \dots, k$ such that for any $i \neq j$ either $C_i \cap C_j$ is empty or it consists of two points.

Now pick a c-parametrization $\Psi_i \colon [-1,1] \to C_i$ from the c-atlas. Since M is closed and since $[-1,1]$ is not closed we see that Ψ_i is not a homeomorphism,[132] in particular $C_i \neq M$.

[130]Here we use that M is Hausdorff, hence the subset $C_i \cup C_j$ is also Hausdorff.

[131]Note that we no longer claim that the resulting c-atlas is open.

[132]Careful reading shows that here we make use of the fact that a half-open interval is not homeomorphic to an open interval.

Since M is connected and since each C_j is a closed of M we see that there exists another c-parametrization $\Psi_j\colon [-1,1] \to C_j$ such that $C_i \cap C_j \neq \varnothing$. By our choice of c-atlas the set $C_i \cap C_j$ consists of two points.

After possibly replacing $\Psi_i(t)$ by $\Psi_i(-t)$ we can assume that $\Psi_i(1) = \Psi_j(-1)$ and $\Psi_i(-1) = \Psi_j(1)$. We consider the map

$$\begin{aligned} \Psi\colon [-1,1]/-1\sim 1 &\to C_i \cup C_j \\ x &\mapsto \begin{cases} \Psi_i(2x+1), & \text{if } x \in [-1,0], \\ \Psi_j(2x-1), & \text{if } x \in [0,1]. \end{cases} \end{aligned}$$

It is straightforward to verify that ψ is a bijection.[133] Furthermore note that it follows from $\Psi_i(-1) = \Psi_j(1)$ and Lemma 3.22 that Ψ is continuous. Thus it follows from Proposition 2.43 (3) that Ψ is in fact a homeomorphism.

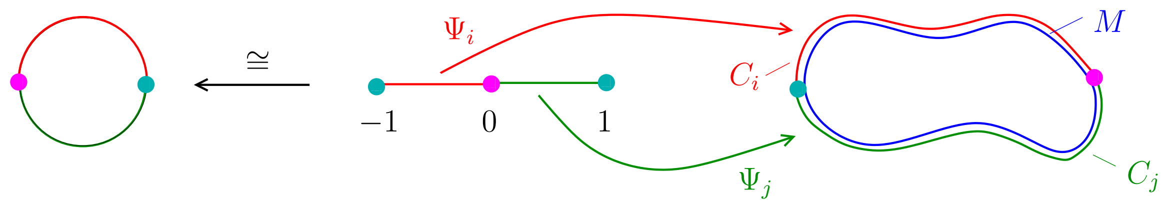


FIGURE 180

Thus we have found a homeomorphism from $[-1,1]/-1 \sim 1 = S^1$ to $C_i \cup C_j$. It remains to prove the following claim.

Claim. We have $C_i \cup C_j = M$.

The c-atlas we work with is simple and almost-disjoint. Since $\Psi_i(\pm 1) = \Psi_j(\mp 1)$ it follows that C_i and C_j did not intersect any other C_k. This shows that $C_i \cup C_j$ and $\bigcup\limits_{k \neq i,j} C_k$ are disjoint. Since both are closed and since M is connected and since $C_i \cup C_j$ is non-empty it follows that $M = C_i \cup C_j$. ■

Now we outline the proof of Theorem 7.1.

PROOF. Let M be a non-empty connected 1-dimensional topological manifold. Recall that we need to show that the homeomorphism type of M is given by the following table:

	without boundary	with non-empty boundary
compact	the circle S^1	the closed interval $[0,1]$
non-compact	the open interval $(0,1)$	the half-open interval $[0,1)$.

So let us deal with these cases separately:

(1) If M is closed, then we know by Proposition 7.2 that M is homeomorphic to S^1.

(2) If M is compact and with non-empty boundary, then we can consider the "double" $\operatorname{D} M := M \cup_{\partial M = \partial M'} M'$ where M' is a second copy of M. One can easily verify that $\operatorname{D} M$ is a compact 1-dimensional topological manifold without boundary. Furthermore one sees that it follows from $\partial M \neq \varnothing$ that $\operatorname{D} M$ is connected. Thus by (1) we

[133] Here we use that we identify -1 and 1 on the left-hand side.

know that $\mathrm{D}\,M$ is homeomorphic to S^1. Using Exercise 2.15 one can easily show that any compact proper submanifold of S^1 is homeomorphic to $[0,1]$.

(3) Suppose that M is non-compact with empty boundary. It follows from Lemma 6.58 (1) that M admits an exhaustion function $f\colon M \to \mathbb{R}_{\geq 0}$. (Note that this is the point where we use that according to our definition topological manifolds are second-countable.) In Exercise 7.1 we will use this exhaustion function, together with (2), to show that M is homeomorphic to $(0,1)$.

(4) Finally assume that M is non-compact with non-empty boundary. As in (2) we consider the double $\mathrm{D}\,M$. This time it is straightforward to see that $\mathrm{D}\,M$ is a non-compact connected 1-dimensional topological manifold with empty boundary. By (3) we know that $\mathrm{D}\,M$ is homeomorphic to $(0,1)$. It follows easily from Exercise 2.15 that M is homeomorphic to a half-open interval. ■

If we consider smooth manifolds instead of topological manifolds and diffeomorphisms instead of homeomorphisms we get the same answer. More precisely the following proposition holds.

Theorem 7.5. *Let M be a non-empty connected 1-dimensional smooth manifold. Then the diffeomorphism type of M is given by the following table*

	without boundary	*with non-empty boundary*
compact	*the circle* S^1	*the closed interval* $[0,1]$
non-compact	*the open interval* $(0,1)$	*the half-open interval* $[0,1)$.

Proof. It is not that difficult to modify the proof of Theorem 7.5 to prove the desired result. One just needs need to be a little more cautious with the various constructions of maps.

For compact smooth manifolds we will also provide a different proof, using "Morse theory", in Proposition 108.1. As in the setting of topological manifolds the non-compact case can again be reduced to the compact case by using the smooth analogue of Exercise 7.1. We leave it to the reader to fill in the details.

Finally we also refer to the appendix of [**Miln65a**] for a different proof of Theorem 7.5. ■

7.2. **Interlude: The order topology.** In the next section we will introduce the "long line", which is a topological space, that has all the properties of a connected 1-dimensional topological manifold, except that it is not second-countable. The construction of the "long line" makes use of the order topology which we will introduce in this section.

First we recall some definitions. At least this particular author invariably forgets the precise meanings of the different type of orders that exist.

Definition. A *partial order* on a set X is a relation $\leq$ that has the following properties:

(1) For every $x \in X$ we have $x \leq x$.
(2) Whenever $x \leq y$ and $y \leq x$, then we have $x = y$.
(3) Whenever $x \leq y$ and $y \leq z$, then we also have $x \leq z$.

A *total order* is a partial order that also satisfies the following condition:

(4) For every $x, y \in X$ we have $x \leq y$ or $y \leq x$.

Finally a *well-order* is a total order that satisfies the following condition:

(5) Every non-empty subset has a minimal element.

Such a pair $(X, \leq)$ is called a *partially ordered set*, *totally ordered set* respectively *well-ordered set*.

Next we introduce some self-explanatory notation.

Notation. Let $(X, \leq)$ be a totally ordered set. As usual, given $x, y \in X$ we write $x < y$ if $x \leq y$ and $x \neq y$. Given $x, y \in X$ we write

$$\begin{array}{rclcrcl} (x,y) & := & \{a \in X \,|\, x < a < y\}, & & (x,y] & := & \{a \in X \,|\, x < a \leq y\} \\ [x,y) & := & \{a \in X \,|\, x \leq a < y\} & \text{and} & [x,y] & := & \{a \in X \,|\, x \leq a \leq y\}. \end{array}$$

We refer to these subsets in the obvious way as *open intervals*, *half-open intervals* and *closed intervals*.

Definition. Let $(X, \leq)$ be a totally ordered set on X. Let $\mathcal{B}$ be the subset of the power set $\mathcal{P}(X)$ given by the subsets of X of the following four types:

(1) All intervals (a, b) in X.
(2) If x_0 is a smallest element of $(X, \leq)$, then we also consider the intervals $[x_0, b)$.
(3) If x_1 is a largest element of $(X, \leq)$, then we also consider the intervals $(a, x_1]$.
(4) The total set X.

We leave it to the reader to verify that $\mathcal{B}$ has the basis property, as defined on page 102.[134] By Lemma 2.22 the basis $\mathcal{B}$ defines a topology on X. We refer to this topology as the *order topology on X*.

Examples.

(1) If we equip $X = \mathbb{R}$ with the usual total order $\leq$, then the resulting order topology equals, basically by definition, the usual topology on X.
(2) We equip $X = \mathbb{R} \times \mathbb{R}$ with the lexicographic order induced from the usual well-order on the two $\mathbb{R}$-factors. Now we equip X with the corresponding order topology. In Figure 181 we show some open intervals in $\mathbb{R} \times \mathbb{R}$. We will discuss this fun example in greater detail in Exercise 7.3.

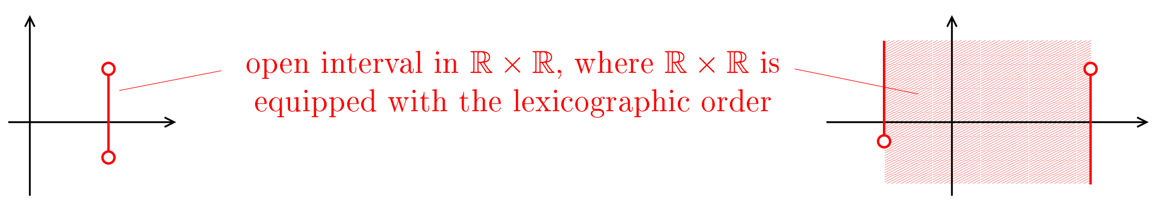


FIGURE 181

[134] At first glance it might look unnecessary to explicitly throw in X into the set $\mathcal{B}$. But if X should happen to consist of a single element, then the sets of type (1), (2) and (3) actually do not satisfy property B1.

Convention. In this chapter, given a totally ordered set $(X, \leq)$ we equip X with the corresponding order topology. On numerous occasions we will use the obvious statement that if two totally ordered sets $(X_1, \leq_1)$ and $(X_2, \leq_2)$ are of the same order type, then the corresponding topological spaces are homeomorphic.

On several occasions we will use the following lemma, which we will prove in Exercise 7.2, without explicitly referring to it.

Lemma 7.6. *Let $(X, \leq)$ be a totally ordered set. Given any interval $I = (a,b)$, $[a,b)$, $(a,b]$ and $[a,b]$ in X the order topology on $(I, \leq)$ agrees with the subspace topology coming from X.*

Example. Later in this section the half-open interval $[0,1) \subset \mathbb{R}$, equipped with the usual total order $\leq$, will play an important role. It follows from Lemma 7.6 that the resulting order topology equals the usual topology on $X \subset \mathbb{R}$.

We conclude this section with the following fairly straightforward lemma.

Lemma 7.7. *Let $(X, \leq)$ be a totally ordered set. The topological space X that is given by the order topology is Hausdorff.*

Proof. Let $(X, \leq)$ be a totally ordered set. We start out with the following claim.

Claim. Given any $x \in X$ the sets

$$(-\infty, x) := \{a \in X \,|\, a < x\} \qquad \text{and} \qquad (x, \infty) := \{b \in X \,|\, x < b\}$$

are open subsets in the order topology.

If X has a minimal element x_0, then evidently we have $(-\infty, x) = [x_0, x)$. By definition this set is open. If X does not have a minimal element, then we have $(-\infty, x) = \bigcup\limits_{a<x} (a, x)$ which is of course an open subset of X. Basically the same argument also shows that (x, ∞) is open. ⊞

Now let $x \neq y$ be two distinct points in X. Since $(X, \leq)$ is totally ordered we know that $x < y$ or $y < x$. Without loss of generality we can assume that $x < y$. If there exists an $s \in (x, y)$, then we set $U := (-\infty, s)$ and $V := (s, \infty)$. Otherwise we set $U := (-\infty, y)$ and $V := (x, \infty)$. By the claim U and V are open. It is now clear that these are disjoint open neighborhoods of x and y. ■

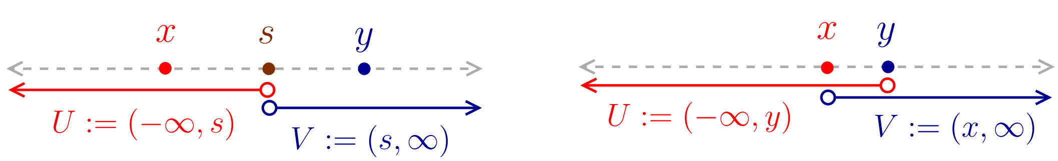


Figure 182. Illustration for the proof of Lemma 7.7.

7.3. **The long line.** Let us start with the following dull definition.

Definition. Let $(X, \leq)$ be an ordered set. Given $x \in X$ we refer to $S_x := \{y \in X \,|\, y < x\}$ as the *x-section of* $(X, \leq)$.

We consider the well-ordered set $(\mathbb{N}, \leq)$. Evidently the set is *countable*, but every section is actually *finite*. The following, quite remarkably, theorem says that the same type of mismatch between the cardinalities of all sections and the cardinality of the actual well-ordered set can also happen for other cardinalities.

Theorem 7.8.

(1) *There exists an uncountable well-ordered set such that every section is countable.*
(2) *If X and Y are two well-ordered sets as in* (1)*, then X and Y have the same order type.*

We denote by ω_1 any well-ordered set as in (1).

Remark.

(1) The alert reader will have noticed that statement (1) of Theorem 7.8 is precisely the content of Exercise 1.3.
(2) The order type of such a well-ordered set as in Theorem 7.8 is often called the *first uncountable ordinal*. We refer to [**Je03**] and [**Dev93**, Chapter 3] for an introduction into the concept of "ordinals".
(3) The construction of the well-ordered set ω_1 is a product of the ever-fertile mind of our friends the set-theorists. To the best of my knowledge there is no "down to earth" construction of ω_1.
(4) The set well-ordered set ω_1 is studied, in "everyday language", in [**Mun75**, p. 67] and [**Gaul14**, Chapter B.2].

Proof.

(1) By the Well-ordering Theorem 1.6 there exists a well-order on any set. In particular there exists a well-order $\leq$ on the uncountable set $X = \mathbb{R}$. If every section of $(X, \leq)$ is countable, then we are done since $\mathbb{R}$ is uncountable. If not, then we consider the following non-empty subset
$$S \;:=\; \{x \in X \,|\, \text{ the } x\text{-section is uncountable}\}.$$
Recall that by definition of a well-order we know that the non-empty subset S of X has a minimal element z. Finally we consider the corresponding z-section $Y := S_z := \{x \in X \,|\, x < z\}$. Note that it follows from the construction of z that $Y = S_z$ is uncountable. Furthermore note that it follows immediately from the definition of z that every section of $Y = S_z$ is actually countable.
(2) We will not make use of this statement, thus we will not provide a proof. Instead we refer to [**Mun75**, p. 73] or alternatively to [**Je03**, Theorem I.2.12] for a proof. ■

Later on we will need the following simple lemma.

Lemma 7.9. *The well-ordered set ω_1 does not have a maximal element.*

Proof. Let $x \in \omega_1$. We need to show that x is not a maximal element. By one of the two defining properties of ω_1 we know that the section $S_x = \{y \in X \,|\, y < x\}$ is countable. It follows from Lemma 1.7 (4) that $\{y \in X \,|\, y \leq x\} = S_x \cup \{x\}$ is also countable. Since ω_1 is uncountable we see that x cannot be a maximal element. ■

Now we are finally ready to give the definition of the mythical long line.

Definition.

(1) We equip $[0,1)$ with the usual well-order. Next we equip $\mathbb{L}_{\geq 0} := \omega_1 \times [0,1)$ with the lexicographic order that we introduced on page 76 and we equip $\mathbb{L}_{\geq 0}$ with the corresponding order topology. We refer to this topological space $\mathbb{L}_{\geq 0}$ as the *closed long ray*.
(2) We denote by $* \in \omega_1$ the minimal element. We refer to $\mathbb{L}^+ := \mathbb{L}_{\geq 0} \setminus \{(*,0)\}$, equipped with the subspace topology coming from $\mathbb{L}_{\geq 0}$, as the *open long ray*.
(3) We write $\mathbb{L}_{\leq 0} := L_{\geq 0}$. We refer to the topological space $\mathbb{L} := \mathbb{L}_{\leq 0} \sqcup_{(*,0)=(*,0)} \mathbb{L}_{\geq 0}$ as the *long line*. (In the literature the long line is sometimes also called the *Alexandroff line*.)

Remark. In many references, see e.g. [**Mun66a**, p. 158], [**SS78**, p. 71] and [**Kra09**, p. 66], the open long ray is called the long line.

The following proposition shows in particular that the open long ray $\mathbb{L}^+$ and the long line $\mathbb{L}$ both satisfy all properties of a 1-dimensional topological manifold, except that neither is second-countable.

Theorem 7.10. *The open long ray $\mathbb{L}^+$ and the long line $\mathbb{L}$ both have the following properties:*

(1) *they are Hausdorff,*
(2) *they are locally homeomorphic to $\mathbb{R}$,*
(3) *they are path-connected and*
(4) *they are not second-countable.*

Remark.

(1) The proof we give below is based on the sketch given in [**Mun66a**, p. 158]. An alternative account of the proof is given in [**Gaul14**, Chapter 1.2].
(2) The discovery of the long line $\mathbb{L}$ goes back to work of Georg Cantor [**Cant1883**] from 1885.
(3) A slightly less otherworldly example of a topological space that satisfies all properties of a path-connected topological manifold, except that it is not second-countable, is given in [**Bre93**, Corollary 17.5] and [**Spr57**, p. 56].

We will prove Theorem 7.10 in the next section. In the remainder of this section we will discuss the open long ray and long line in greater detail. First of all the following proposition lists several weird and surprising properties of the open long ray and the long line.

Proposition 7.11.

(1) *Every map $f\colon \mathbb{L}^+ \to \mathbb{R}$ is eventually constant, i.e. there exists an $(x,s) \in \mathbb{L}^+$ such that $f((y,t)) = f(x,s)$ for every $(y,t) \geq (x,s)$.*
(2) *The topological space $\mathbb{L}^+$ is not paracompact.*

For the remaining statements we need the notion of maps being homotopic, which we only introduce on page 537. Furthermore we need the notion of a topological space to be contractible, this notion will be introduced on page 547.

(3) *Every bounded map $f\colon \mathbb{L}^+ \to \mathbb{L}^+$ is homotopic to a constant map.*
(4) *Every unbounded map $f\colon \mathbb{L}^+ \to \mathbb{L}^+$ is homotopic to the identity map.*
(5) *An unbounded map $f\colon \mathbb{L}^+ \to \mathbb{L}^+$ is never homotopic to bounded map.*
(6) *The open long line $\mathbb{L}^+$ is not contractible.*

PROOF.

(1) It is a great exercise to try to prove this statement. If, inexplicably, the reader is not inclined to provide the arguments, then we refer to [**Gaul14**, Proposition 1.17] for a proof.
(2) By Theorem 2.79 it suffices to show that $\mathbb{L}^+$ admits an open cover that does not admit a locally finite open refinement. In fact one can show that the open subsets $U_{(x,s)} := ((*,0),(x,s))$ of $\mathbb{L}^+$ is such an open cover. Since we will not make use of this statement we will not show that this cover does not admit a locally finite open refinement. The statement can be proved using a slight variation on the argument given on [**Jos83**, Theorem 15.3.2(vi)].

(3)-(5) These statements are proved in [**Gaul14**, Proposition 1.12].
(6) This statement follows immediately from (5) and Lemma 18.12. ■

At this stage one might fear that now, when we no longer restrict ourselves to second-countable topological spaces, the genie is out of the bottle, and that thus any type of classification result is completely out of reach. Rather stunningly this is not the case.

The following theorem can now be viewed as a generalization of Theorem 7.1.

Theorem 7.12. *Let X be a topological space that is Hausdorff and path-connected.*

(1) *If X is locally homeomorphic to an open subset of $\mathbb{R}$, then X is homeomorphic to one of the following: S^1, $\mathbb{R}$, $\mathbb{L}^+$ and $\mathbb{L}$.*
(2) *If X is locally homeomorphic to an open subset of $\mathbb{R}_{\geq 0}$, then either X is homeomorphic to one of the above, or one of the following: $[0,1)$, $[0,1]$ and $\mathbb{L}_{\geq 0}$.*

PROOF.

(1) This statement is proved in proved in [**Knes58**].
(2) This statement can be deduced easily from (1) by first removing the "boundary points". We leave it to the reader to provide the details. ■

By Theorem 7.10 we now know in particular that the long line $\mathbb{L}$ admits an atlas. In fact, as is shown in [**Gaul14**, Proposition 7.9], there exists an atlas on $\mathbb{L}$ that is smooth, i.e. such that all transition maps are smooth. Much more impressively, it turns out that there are uncountable many mutually non-diffeomorphic smooth structures on $\mathbb{L}$. We refer to [**Gaul14**, Theorem 7.14] and [**Nyi92**] for a precise formulation and a proof of this assertion.

7.4. **Proof of Theorem 7.10.** In this section we finally provide the proof of Theorem 7.10. We start out with some general definitions and statements regarding well-ordered sets.

Definition. Let $(X, \leq)$ be a well-ordered set. A subset A of X is called *inductive* if for every $a \in X$ with $S_a \subset A$ we also have $a \in A$.

Examples.

(1) Let $X = \mathbb{R}_{\geq 0}$. The interval $A = [0, 2)$ is not inductive whereas $A = [0, 2]$ is inductive.
(2) One can easily show that the only inductive subset of $\mathbb{N}_0$ is $\mathbb{N}_0$.

The grandiose name of the following theorem is designed to impress friends and family.

Theorem 7.13. (Principle of Transfinite Induction) *Let $(X, \leq)$ be a well-ordered set. If $A \subset X$ is an inductive subset of X, then $A = X$.*

PROOF. Let $A \subset X$ be an inductive subset. We suppose that $A \neq X$. Since X is well-ordered the set $X \setminus A$ has a minimal element y. Since y is the minimal element of $X \setminus A$ we see that for every $x < y$ we have $x \in A$. But this means that the y-section S_y is contained in A. But since A is inductive this implies that $y \in A$. We have thus obtained a contradiction. ■

In the proof of Theorem 7.10 we will need the following fairly straightforward lemma.

Definition. Let $(X, \leq)$ be a partially ordered set.

(1) Let $x \in X$. We say that $a \in X$ is an *immediate predecessor of x* if $a < x$ and if there is no $b \in X$ with $a < b < x$.
(2) Let $x \in X$. We say that $a \in X$ is an *immediate successor of x* if $x < a$ and if there is no $b \in X$ with $x < b < a$.
(3) Let $A \subset X$ be a subset. We say that $x \in X$ is an *upper bound of A* if for every $a \in A$ we have $a \leq x$.
(4) Let $A \subset X$ be a subset. We say that $x \in X$ is a *supremum of A* if x is an upper bound of A and if there is no upper bound of A that is less than x.
(5) Let $x_1 < x_2 < \dots$ be an increasing sequence of elements in X. If $b = \sup\{x_1, x_2, \dots\}$ then we say that b is a *limit of the sequence*.

Examples.

(1) Every element in $\mathbb{Z}$ has an immediate predecessor and an immediate successor.
(2) No element in $\mathbb{R}$ has an immediate predecessor and no element in $\mathbb{R}$ has an immediate successor.
(3) If $(X, \leq)$ is a well-ordered set, then it follows easily from the definitions that every proper subset $A \subsetneq X$ admits a unique supremum.

Lemma 7.14. *Let $(X, \leq)$ be a well-ordered set and let $x \in X$ such that the corresponding section S_x is countable. Precisely one of the following three statements holds:*

(1) *x is the minimal element of X.*
(2) *There exists an immediate predecessor of x.*
(3) *x is the limit of an increasing sequence $x_0 < x_1 < x_2 <$ of elements of X.*

PROOF. Let $(X, \leq)$ be a well-ordered set and let $x \in X$ such that the corresponding section $S_x = \{w \in X \,|\, w < x\}$ is countable. We only need to consider the case that $x \in X$ is not the minimal element. This means that the section S_x is non-empty. If the section S_x is finite, then it admits a maximal element, which is clearly an immediate predecessor of x. Next we suppose that the section S_x is infinite. By hypothesis on x we know that S_x is also countable. Thus there exists a bijection $\varphi\colon \mathbb{N}_0 \to S_x$. Now given any $n \in \mathbb{N}_0$ we set $y_n := \max\{\varphi(0), \dots, \varphi(n)\}$. By construction we have $y_1 \leq y_2 \leq y_3 \leq \dots$. If the sequence $y_1, y_2, \dots$ is eventually constant, i.e. if there exists an N such that $y_n = Y_N$ for every $n \geq N$, then Y_N is easily seen to be an immediate predecessor of x. Otherwise there exists a subsequence $y_{s_1} < y_{s_2} < y_{s_3} < \dots$ which is strictly increasing. By construction x is the limit of this sequence.

Finally we leave it to the reader to verify that three possibilities are mutually exclusive. ■

Example. We consider the subset $X := \mathbb{N}_0 \cup \{n - \frac{1}{k} \mid n \in \mathbb{N}_0 \text{ and } k \in \mathbb{N}_{\geq 2}\}$ of $\mathbb{R}$, equipped with the usual partial order. One can easily verify that $(X, \leq)$ is actually well-ordered. The number 0 is evidently the minimal element. Every x of the form $n - \frac{1}{k}$ with $n \in \mathbb{N}$ and $k \in \mathbb{N}_{\geq 2}$ has an immediate successor, namely $n - \frac{1}{k-1}$. Finally every $n \in \mathbb{N}$ is the limit of the increasing sequence $x_i = n - \frac{1}{1+i}$.

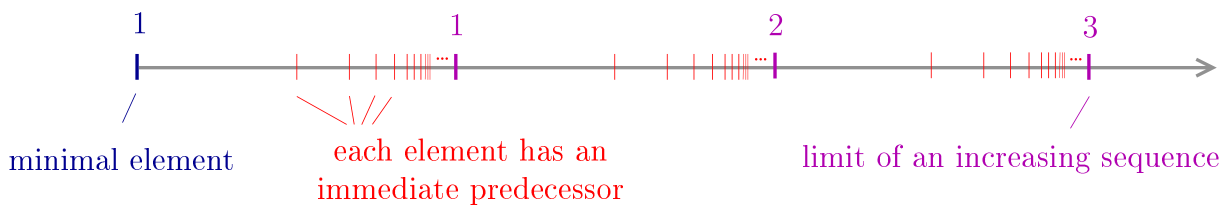


FIGURE 183

Lemma 7.15. *Let $(X, \leq)$ be a totally ordered set.*

(1) *Let $a < b < c$ be three elements of X. The interval $\overbrace{[a,c)}^{\subset X}$ has the order type of $\overbrace{[0,1)}^{\subset \mathbb{R}}$ if and only if each of the two intervals $[a,b)$ and $[b,c)$ has the order type of $[0,1)$.*
(2) *Let $b \in X$ be the limit of an increasing sequence $x_0 < x_1 < \dots$ of elements of X. If each $[x_n, x_{n+1})$ has the order type of $[0,1)$, then so does $[x_0, b)$.*

PROOF. Throughout the proof we will make use of the following simple observation: if $(A, \leq)$ and $(B, \leq)$ are two totally ordered sets, then they have the same order type if and only if there exists an order-preserving bijection $f\colon A \to B$. In other words, the fact that A and B are totally ordered implies that the inverse $f^{-1}\colon B \to A$ is automatically order-preserving. Now we turn to the proofs of the two statements.

(1) Let $a < b < c$ be three elements of X. First suppose that there exist order-preserving bijections $f\colon [a,b) \to [0,1)$ and $g\colon [b,c) \to [0,1)$. It is clear that the map
$$\begin{aligned} [a,c) &\to [0,1) \\ t &\mapsto \begin{cases} \frac{1}{2} \cdot f(t), & \text{if } t \in [a,b), \\ \frac{1}{2} + \frac{1}{2} \cdot g(t), & \text{if } t \in [b,c) \end{cases} \end{aligned}$$
is an order-preserving bijection. Conversely suppose that there exists an order-preserving bijection $h\colon [a,c) \to [0,1)$. We point out that $h([a,b)) = [0, h(b))$ and

$h([b,c)) = [h(b),1)$. Evidently the intervals $[0,h(b))$ and $[h(b),1)$ have the order type of $[0,1)$. Thus we are done.

(2) Now let $b \in X$ be the limit of an increasing sequence $x_0 < x_1 < \dots$ of elements of X. We suppose that for each $n \in \mathbb{N}_0$ there exists an order-preserving bijection $\varphi_n \colon [x_n, x_{n+1}) \to [0,1)$. One can now easily verify that the map

$$\begin{array}{rcccl} [x_0,b) & \underset{\substack{\uparrow \\ \text{since } b \text{ is the limit of the} \\ \text{increasing sequence } x_0<x_1<\dots}}{=} & \bigsqcup\limits_{n\in\mathbb{N}_0} [x_n,x_{n+1}) & \to & [0,1) \\ & & t & \mapsto & \frac{1}{2^{n+1}} \cdot \varphi_n(t) + \sum\limits_{i=0}^{n-1} \frac{1}{2^{i+1}} \quad \text{for } t \in [x_n,x_{n+1}) \end{array}$$

is an order-preserving bijection. ■

The next lemma contains the heart of the proof of Theorem 7.10.

Lemma 7.16. *As above we denote by $*$ the minimal element of ω_1 and let $x \in \omega_1 \setminus \{*\}$.*

(1) *The half-open interval $[(*,0),(x,0)) \subset \mathbb{L}_{\geq 0}$ has the order type of $[0,1)$.*

(2) *The open interval $((*,0),(x,0)) \subset \mathbb{L}_{\geq 0}$ has the order type of $(0,1)$.*

Proof. We consider the set

$$B \;:=\; \big\{ b \in \omega_1 \setminus \{*\} \,\big|\, \text{the set } [(*,0),(b,0)) \text{ has order type of } [0,1) \big\}.$$

We want to show that $B = \omega_1 \setminus \{*\}$. Note that it follows from the fact that ω_1 is well-ordered that the complement of the minimal element $*$, i.e. the set $\omega_1 \setminus \{*\}$, is again well-ordered. Thus it follows from Theorem 7.13, i.e. it follows from the Principle of Transfinite Induction, that it suffices to show that B is inductive.

Now let $b \in B$ such that $S_b \subset B$. By Lemma 7.14 we know that one of the following holds:

(1) b is the minimal element of B.
(2) b has an immediate predecessor a.
(3) b is the limit of an increasing sequence $x_0 < x_1 < \dots$ of elements of X.

We will now treat these three cases separately.

(1) Suppose that b is the minimal element of $\omega_1 \setminus \{*\}$. In this case we see easily that $[(*,0),(b,0)) = \{(*,t) \,|\, t \in [0,1)\}$, which evidently has the order type of $[0,1)$.

(2) Suppose that b has an immediate predecessor a. Since $S_b \subset B$ and $a < b$ we see that $a \in B$, i.e. we see that $[(*,0),(a,0))$ has order type $[0,1)$. Next note that the fact that a is an immediate predecessor of b implies that $[(a,0),(b,0)) = \{(a,x) \,|\, x \in [0,1)\}$. Thus we see that this interval also has order type $[0,1)$. It follows from Lemma 7.15 (1) that $[(*,0),(b,0))$ has order type $[0,1)$.

(3) Now suppose that b is the limit of an increasing sequence $x_0 < x_1 < \dots$ of elements of X. Since $S_b \subset B$ and since for each x_n we have $x_n < b$ we see as in (2) that each $[(*,0),(x_n,0))$ has order type $[0,1)$. It follows from Lemma 7.15 (1) that each interval $[(x_n,0),(x_{n+1},0))$ has order type $[0,1)$. Finally it follows from Lemma 7.15 (2) that the interval $[(*,0),(b,0))$ has order type $[0,1)$. ■

Now we can finally provide the proof of Theorem 7.10.

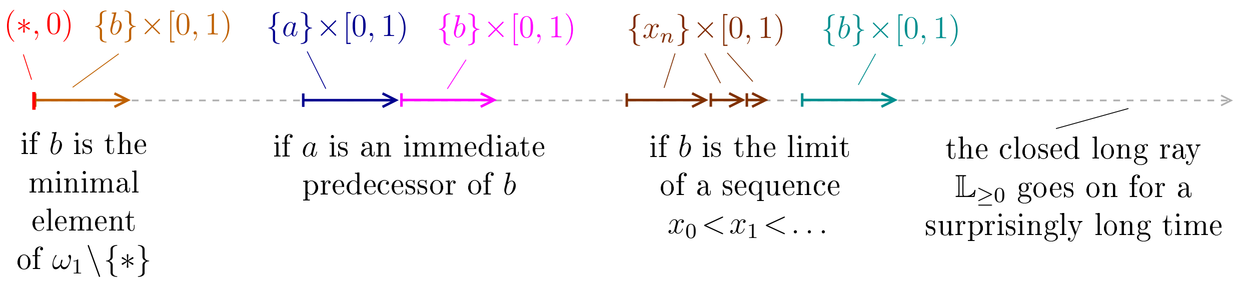


FIGURE 184. Illustration for the proof of Lemma 7.16.

PROOF OF THEOREM 7.10. We will prove the desired statements only for the open long ray $\mathbb{L}^+$. We leave it to the reader to make the minor modifications need to show that the analogous statements also hold for the long line $\mathbb{L}$. Now recall that we need to show the following four statements regarding the open long ray $\mathbb{L}^+$.

(1) $\mathbb{L}^+$ is Hausdorff,
(2) $\mathbb{L}^+$ is locally homeomorphic to an open subset of $\mathbb{R}$,
(3) $\mathbb{L}^+$ is path-connected,
(4) $\mathbb{L}^+$ is not second-countable.

We turn to the proofs of these four statements.

(1) This statement conveniently enough follows immediately from Lemma 7.7.
(2) Let $(x,t) \in \mathbb{L}^+$. By Lemma 7.9 we can pick some $y > x$. Since $y > x$ we see that $U := ((*,0),(y,0))$ is an open neighborhood of (x,t). Note that by Lemma 7.16 we know that the interval $[(*,0),(y,0))$ has the order type of $[0,1)$. It follows that the open interval $U := ((*,0),(y,0))$ is homeomorphic to $(0,1)$.
(3) Let (x,s) and (y,t) be two points in $\mathbb{L}^+$. By Lemma 7.9 we can pick $z \in X$ with $z > x$ and $z > y$. As in (2) we obtain from Lemma 7.16 that the open interval $U := ((*,0),(z,0))$ is homeomorphic to $(0,1)$. Since the open interval contains the points (x,s) and (y,t) and since $(0,1)$ is path-connected we see that there exists a path $\gamma\colon [0,1] \to \mathbb{L}^+$ that connects (x,s) and (y,t).
(4) Let $\mathcal{B}$ be a basis for the topological space $\mathbb{L}^+$. We need to show that $\mathcal{B}$ is uncountable. For each $x \in \omega_1$ we consider the open interval $U_x := ((x,\frac{1}{4}),(x,\frac{3}{4}))$. It follows from the definition of a basis that given any $x \in X$ there exists a set $B_x \in \mathcal{B}$ with $(x,\frac{1}{2}) \in B_x \subset U_x$. Since the U_x are disjoint we see that the B_x are distinct. Since X is uncountable we obtain from Lemma 1.7 (1) that $\mathcal{B}$ is uncountable. ■

Exercises for Chapter 7.

Exercise 7.1. Let M be a non-compact connected non-empty 1-dimensional topological manifold with empty boundary.

(a) Show that there exists a sequence $X_0 \subset X_1 \subset \dots$ of compact connected 1-dimensional submanifolds of M such that for each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$ and such that

$M = \bigcup_{i \in \mathbb{N}_0} X_i$.
Hint. Use Lemma 6.58.

(b) Show that M is homeomorphic to $\mathbb{R}$.
Remark. You can use that on page 330 we proved that every compact connected non-empty 1-dimensional topological manifold is homeomorphic to $[0, 1]$.

Exercise 7.2.

(a) Let $(X, \leq)$ be a totally ordered set. Show that given any interval $I = (a, b)$, $[a, b)$, $(a, b]$ and $[a, b]$ in X the order topology on $(I, \leq)$ agrees with the subspace topology coming from X.
(b) Let $(X, \leq)$ be a totally ordered set and let $A \subset X$ be a subset. Show that the order topology on $(A, \leq)$ does not necessarily agree with the subspace topology coming from X.

Exercise 7.3. We consider $X := \mathbb{R} \times \mathbb{R}$, for once equipped with the order topology $\mathcal{O}$.

(a) Let $\mathcal{T}$ be the usual topology on $X = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. Do we have an inclusion $\mathcal{O} \subset \mathcal{T}$? Do we have an inclusion $\mathcal{T} \subset \mathcal{O}$?
(b) Show that the topological space $(X, \mathcal{O})$ is Hausdorff.
(c) Is the topological space $(X, \mathcal{O})$ second-countable?
(d) Is $(X, \mathcal{O})$ locally homeomorphic to open subsets of $\mathbb{R}$?
(e) Is $(X, \mathcal{O})$ homeomorphic to the long line?

Exercise 7.4. Let ω_1 be the well-ordered set introduced in Theorem 7.8.

(a) Show that for every $x \in \omega_1$ the subset $\{y \in \omega_1 \mid x < y\}$ is uncountable.
(b) Let $X = \{x \in \omega_1 \mid x \text{ has no immediate successor}\}$. Show that X is uncountable.

Exercise 7.5.

(a) We consider the well-ordered set $\mathbb{L}^+ := \omega_1 \times [0, 1)$. Show that every increasing sequence in $\mathbb{L}_{\geq 0}$ converges.
(b) A topological space that has the property that every sequence admits a convergent subsequence is called sequentially compact. Show that the closed long ray $\mathbb{L}^+$ is sequentially compact.

Exercise 7.6. Show that the open long ray $\mathbb{L}^+$ and the long line $\mathbb{L}$ are not homeomorphic.
Hint. What happens if you remove a point?

Exercise 7.7.

(a) Let $(X, \leq)$ be a well-ordered set. Let $x \in X$. Show that either x is the largest element of X or that X has an immediate successor.
(b) Give an example of a partially ordered set $(X, \leq)$ that is not well-ordered but that nonetheless has the property that every $x \in X$ has an immediate successor.

Exercise 7.8. Let $(X, \leq)$ be a partially ordered set and let $x \in X$.

(1) We say x has *depth* -1 if x is a minimal element of X.
(2) We say x has *depth* 0 if x has an immediate predecessor.

Now we iteratively define what it means for $x \in X$ to have depth $k \in \mathbb{N}_{\geq 2}$.

(3) We say x has *depth* k if x is not of depth $k-1$ but if there exists an increasing sequence $x_1 < x_2 < \dots$ in X, such that each x_i has depth $< k$, and such that x is a limit of this sequence.

Show that there exists a countable well-ordered subset of $\mathbb{R}$ that admits elements of depth two.

Exercise 7.9. Does the long line $\mathbb{L}$ admit an embedding into $\mathbb{R}^2$?

8. Differential topology

Differential topology is the study of smooth manifolds and their submanifolds viewed as topological spaces. In this section we will first give several basic definitions in differential topology. Afterwards we will state, without proof but with precise references, some technical results which later on will make our life much easier.

8.1. **Immersions and smooth embeddings.** We start out with some definitions that will crop up frequently throughout these lecture notes.

Definition. Let N be a compact topological manifold and let M be any topological manifold. A map $\varphi\colon N \to M$ is called *proper* if $\varphi^{-1}(\partial M) = \partial N$.

We illustrate the definition of a proper map in Figure 185.

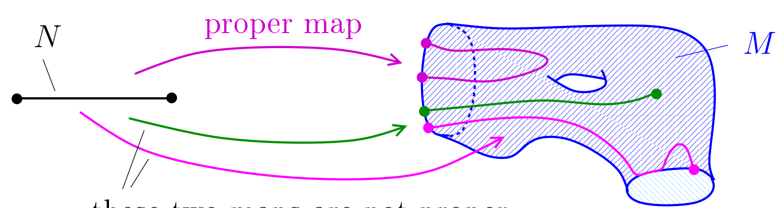


Figure 185

The following definition is related in spirit to the notion of a submersion that we had introduced on page 311.

Definition. A map $\varphi\colon N \to M$ between two smooth manifolds is called an *immersion* if the following conditions are satisfied:

(1) The map φ is smooth.
(2) For each $P \in N$ the induced map $\varphi_*\colon\ \mathrm{T}_P\, N \to \mathrm{T}_{\varphi(P)}\, M$ is a monomorphism.
(3) For each $P \in N \setminus \partial N$ we have $\varphi(P) \in M \setminus \partial M$.
(4) For each $P \in \partial N$ with $\varphi(P) \in \partial M$ the image $\varphi_*(\mathrm{T}_P\, N)$ is *not* contained in $\mathrm{T}_{\varphi(P)}(\partial M)$.[135]

An immersion that also satisfies the following extra condition

(5) $\varphi\colon N \to M$ is an embedding,[136][137]

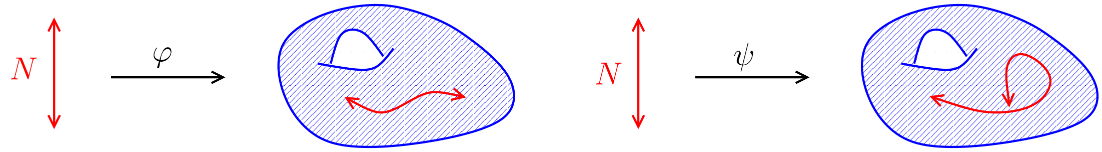
is called a *smooth embedding*.

The following proposition relates the notion of smooth embeddings to the notion of submanifolds.

[135]Another way of expressing this condition is that the map $\mathrm{T}_P\, N \to \mathrm{T}_{\varphi(P)}\, M/\,\mathrm{T}_{\varphi(P)}\,\partial M$ is an epimorphism.

[136]Recall that "embedding" means that the map $\varphi\colon N \to \varphi(N)$ is a homeomorphism where $\varphi(N) \subset M$ is endowed with the subspace topology.

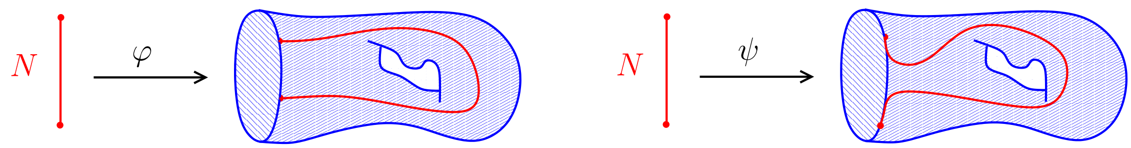
[137]If N is compact, then it follows from Proposition 2.43 (2) that it suffices to verify that φ is injective and continuous.



φ is a smooth embedding

ψ is not a smooth embedding since
$\varphi\colon N \to \varphi(N)$ is not a homeomorphism

FIGURE 186



φ is a smooth embedding

ψ is not a smooth embedding since
condition (4) is not satisfied

FIGURE 187

Proposition 8.1. *Let N be an n-dimensional smooth manifold and let M be a smooth manifold.*

(1) *Let $\varphi\colon N \to M$ be a smooth embedding such that $\varphi(N \setminus \partial N) \subset M \setminus \partial M$ and such that the following condition is satisfied:*
   $(*)$ *For every component C of ∂N we have either $\varphi(C) \subset \partial M$ or $\varphi(C) \subset M \setminus \partial M$.*
   Then $\varphi(N)$ is an n-dimensional submanifold of M and the map $\varphi\colon N \to \varphi(N)$ is a diffeomorphism.
(2) *Suppose N is compact. If $\varphi\colon N \to M$ is a proper smooth embedding, then $\varphi(N)$ is a proper n-dimensional submanifold of M and the map $\varphi\colon N \to \varphi(N)$ is a diffeomorphism.*

Remark.

(1) Let M be a smooth manifold and let W be a submanifold. It follows fairly easily from the definitions and Exercise 6.8 that the inclusion map $\iota\colon W \to M$ is a smooth embedding and that it satisfies Condition $(*)$ from Proposition 8.1 (1).
(2) In Figure 188 we show a smooth embedding $\varphi\colon N \to M$ that does not satisfy Condition $(*)$ from Proposition 8.1 (1). This author has to admit that he had initially overlooked that $(*)$ was a necessary condition. This shows once again that statements, for which detailed proofs are *not* given, tend to be incorrectly phrased.

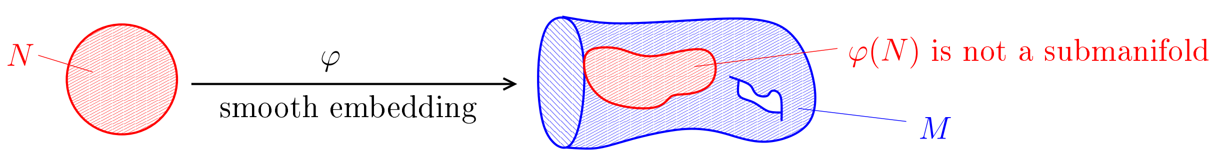


FIGURE 188

Proof. For the proof of (2) we need to make the following observation: If $\varphi\colon N \to M$ is a map and N is compact, then it follows from Lemma 2.40 and Lemma 2.17 that $\varphi(N)$ is a closed subset of M.

For the actual proof of (1) and (2) we have to refer to the literature. For smooth manifolds without boundary the proposition is proved in [**Lee02**, Proposition 5.2] and [**BG88**, Corollary 2.6.11]. The proof for smooth manifolds with boundary is quite similar. We leave it to the reader to fill in the details. Note that [**Lee02**, Proposition 5.2] actually deals with smooth manifolds with boundary. But as we pointed out on page 281, the definition of a submanifold with boundary in [**Lee02**] differs from our definition and also the notion of a smooth embedding differs from our definition. ■

Examples.

(1) It is straightforward to verify that the map $\Psi\colon \mathbb{R}^2 \times S^1 \to \mathbb{R}^3$ given in Lemma 3.13 restricts to a smooth embedding $\Psi\colon S^1 \times S^1 \to \mathbb{R}^3$ and that it also restricts to a smooth embedding $\Psi\colon \overline{B}^2 \times S^1 \to \mathbb{R}^3$. By Proposition 8.1 this allows us to view $S^1 \times S^1$ and $\overline{B}^2 \times S^1$ as submanifolds of $\mathbb{R}^3$.

(2) Let $n \in \mathbb{N}_0$. In Lemma 2.44 we used the stereographic projection to give an explicit homeomorphism $\Phi\colon S^n \to \mathbb{R}^n \cup \{\infty\}$. We leave it to the reader to verify that the map $\mathbb{R}^n \to \mathbb{R}^n \cup \{\infty\} \xrightarrow{\Phi^{-1}} S^n$ is a smooth embedding. We will use this embedding to view $\mathbb{R}^n$ as a submanifold of the smooth manifold S^n.

In the following proposition we consider immersions and smooth embeddings between smooth manifolds of the same dimension.

Proposition 8.2. *Let M and N be smooth manifolds of the same dimension and furthermore let $\varphi\colon N \to M \setminus \partial M$ be a map.*

(1) *If $\varphi\colon N \to M \setminus \partial M$ is an immersion, then $\varphi(N \setminus \partial N)$ is an open subset of M.*

(2) *If $\varphi\colon N \to M \setminus \partial M$ is a smooth embedding and if N is compact, then the following three statements hold:*

  (a) *M is decomposed, in the sense of the definition on page 284, into the submanifolds $\varphi(N)$ and $X := M \setminus \varphi(N \setminus \partial N)$.*

  (b) *The boundary of $M \setminus \varphi(N \setminus \partial N)$ as a smooth manifold in its own right is given by $\partial M \cup \varphi(\partial N)$.*

  (c) *If ∂N is connected and if M is connected, then X is also connected.*

  (d) *If M is compact, then $M \setminus \varphi(N \setminus \partial N)$ is also compact.*

  (*We refer to Figure 189 for an illustration.*)

(3) *If M and N are closed and connected and if $\varphi\colon N \to M$ is a smooth embedding, then φ is a diffeomorphism.*

Sketch of proof.

(1) Let $\varphi\colon N \to M \setminus \partial M$ be an immersion. It is not difficult to deduce from Theorem 6.19 that $\varphi(N \setminus \partial N)$ is an open subset of M. We leave it to the reader to fill in the details.

(2) This statement follows immediately from Propositions 8.1 and 6.30.

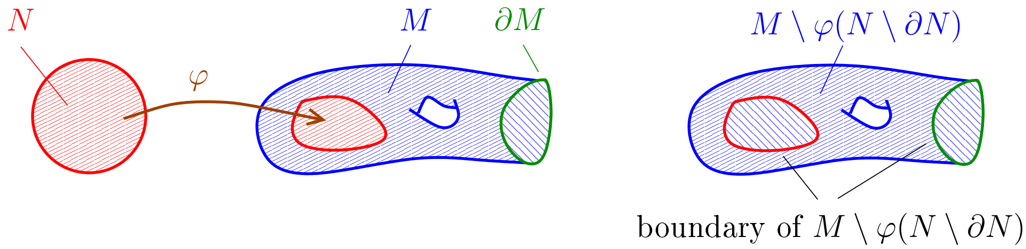


FIGURE 189. Illustration of Proposition 8.2 (2).

(3) We already know by Proposition 8.1 that $\varphi\colon N \to \varphi(N)$ is a diffeomorphism. Thus it suffices to show that $\varphi(N) = M$. First note that it follows from Lemma 2.40 and Lemma 2.17 (2) that $\varphi(N)$ is a closed subset of M. Furthermore, it follows easily from Theorem 6.19 that $\varphi(N)$ is an open subset of M. Since M is connected we see that $M = \varphi(N)$. ■

We conclude this discussion of immersions and smooth embeddings with the following instructive example which is dressed up as a lemma.

Lemma 8.3. *Given $\lambda \in \mathbb{R}$ we consider the following map*

$$\begin{array}{rcl} \varphi_\lambda\colon \mathbb{R} & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ t & \mapsto & [(t, \lambda \cdot t)]. \end{array}$$

The following statements hold:

(1) *The map φ_λ is an immersion if and only if $\lambda \neq 0$.*
(2) *The map φ_λ is injective if and only if λ is irrational.*
(3) *If λ is irrational, then $\varphi_\lambda\colon \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ is* not *an embedding, i.e. φ_λ is not a smooth embedding.*
(4) *Suppose λ is rational. We write $\lambda = \frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that p and q are coprime. Then the map*

$$\begin{array}{rcl} \varphi_\lambda\colon S^1 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ e^{2\pi i t} & \mapsto & [(qt, \lambda \cdot qt)] \end{array}$$

is a smooth embedding.

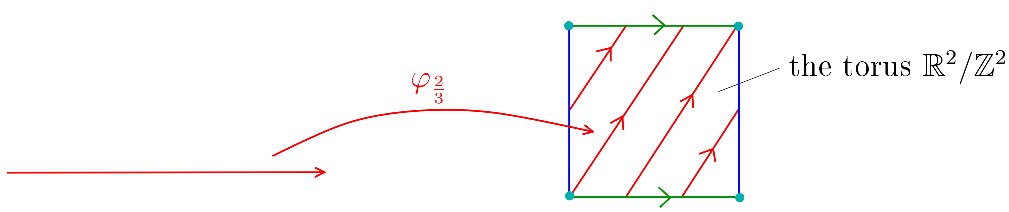


FIGURE 190

Proof. Statements (1), (2) and (4) are amusing exercises. Statement (3) requires the simple number theoretic fact that given any $\alpha \in \mathbb{R}$ and any $N \in \mathbb{N}$ there exist $n, m \in \mathbb{Z}$ with $1 \leq n \leq N$ and with $|n\alpha - m| < \frac{1}{N}$.[138] We leave the details to the reader. ■

Examples.

(1) It is straightforward to verify that the map

$$\begin{array}{rcl} S^1 \times S^1 & \to & \mathbb{R}^3 \\ (e^{\mathrm{i}\theta}, e^{\mathrm{i}\varphi}) & \mapsto & ((2+\sin(\theta))\cdot\cos(\varphi), (2+\sin(\theta))\cdot\sin(\varphi), \cos(\theta)) \end{array}$$

is a smooth embedding of the 2-dimensional torus into $\mathbb{R}^3$.

(2) Let M be a closed non-empty n-dimensional topological manifold. In Corollary 50.8 we showed that there is no embedding of M into $\mathbb{R}^n$.

8.2. **Smooth homotopies.** Now suppose we are given two smooth maps $f_0, f_1 \colon M \to N$ between smooth manifolds. On page 540 we defined a homotopy between f_0 and f_1 to be a map $F \colon M \times [0,1] \to N$ with $F_0 = f_0$ and $F_1 = f_1$. As we will see shortly, on many occasions this definition is not very useful. It is much more reasonable to consider "smooth" homotopies, namely we want F itself to be a smooth map.

But this leads to a slightly annoying question: What does it mean for $F \colon M \times [0,1] \to N$ to be smooth? If M is closed, then by Proposition 6.51 (1) we can view $M \times [0,1]$ as a smooth manifold and we know what it means for a map $F \colon M \times [0,1] \to N$ to be smooth. But what should we do if M has non-empty boundary?

This leads us to the following definition.

Definition. Let M and N be smooth manifolds and let $a < b$ be two real numbers. We say a map $\varphi \colon M \times [a,b] \to N$ is *smooth* if the following conditions are satisfied:

(1) The map $(M \setminus \partial M) \times [a,b] \to N$ is smooth.

(2) Given any $x \in \partial M$ there exists an open neighborhood U of x in M, an $\epsilon \in (0, b-a)$ and a smooth map $\psi \colon U \times (a-\epsilon, a+\epsilon) \to N$ that agrees with φ on $U \times [a, a+\epsilon)$.

(3) Given any $x \in \partial M$ there exists an open neighborhood U of x in M, an $\epsilon \in (0, b-a)$ and a smooth map $\psi \colon U \times (b-\epsilon, b+\epsilon) \to N$ that agrees with φ on $U \times (b-\epsilon, b]$.

Note that in all three cases it makes sense to talk of a map being smooth since by Proposition 6.51 (4) we know that $(M \setminus \partial M) \times [a,b]$ is a smooth manifold with boundary $(M \setminus \partial M) \times \{a,b\}$, that $U \times (a-\epsilon, a+\epsilon)$ is a smooth manifold with boundary $\partial U \times (a-\epsilon, a+\epsilon)$ and finally $U \times (b-\epsilon, b+\epsilon)$ is a smooth manifold with boundary $\partial U \times (b-\epsilon, b+\epsilon)$.

Remark.

(1) If M is a smooth manifold without boundary, then the cases (2) and (3) do not occur and the above definition of a smooth map $\varphi \colon M \times [a,b] \to N$ is precisely the one we had always used.

(2) Let M be a smooth manifold. We could view $M \times [0,1]$ as a smooth manifold with "corners", see e.g. [**Wall16**, p. 30] or [**Lee02**, Chapter 16] for a definition. The definition of a smooth map on a smooth manifold with corners is very similar to the

[138] This statement is known as Dirichlet's Approximation Theorem, but it is actually elementary to prove, see e.g. [**Lee02**, Lemma 4.21].

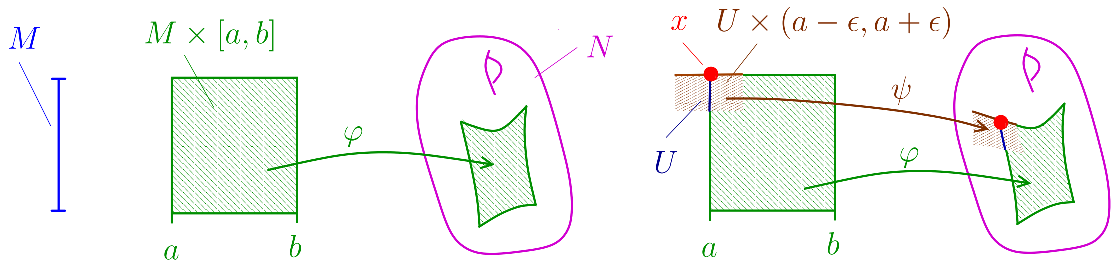


FIGURE 191

definition for smooth manifolds with boundary, which in turn relies on the definition of a smooth map on an arbitrary subset of $\mathbb{R}^n$ that we gave on page 272. It follows easily from the definitions that these two definitions of a smooth map on $M \times [0,1]$ agree.

Now we can define the notion of a smooth homotopy.

Definition. Let M and N be smooth manifolds.

(1) A *smooth homotopy* is a smooth map $F: M \times [0,1] \to N$.
(2) We say that two smooth maps $f_0, f_1: M \to N$ are *smoothly homotopic* if there exists a smooth homotopy $F: M \times [0,1] \to N$ with $F_0 = f_0$ and $F_1 = f_1$.

Example. Let M and N be smooth manifold and let $F: M \times [0,1] \to N$ be a homotopy. If there exists an $\epsilon > 0$ and a smooth map $M \times (-\epsilon, 1+\epsilon) \to N$ that restricts to F on $M \times [0,1]$, then F is by definition smooth. For example, it follows easily from this observation that the map

$$\begin{array}{rcl} S^1 \times [0,1] & \to & \mathbb{R}^2 \\ (z,t) & \mapsto & z \cdot (1-t) \end{array}$$

is a smooth homotopy from the inclusion map $S^1 \to \mathbb{R}^2$ to a constant map.

The following elementary lemma is sometimes useful when one tries to combine smooth homotopies.

Lemma 8.4. $(*)$ *Let M and N be smooth manifolds and let $F: M \times [0,1] \to N$ be a smooth homotopy. There exists an $\epsilon \in (0, \frac{1}{2})$ and a smooth homotopy $G: M \times [0,1] \to N$ with $G_0 = F_0$ and $G_1 = F_1$ and such that $G_t = G_0$ for all $t \in [0,\epsilon]$ and $G_t = G_1$ for all $t \in [1-\epsilon, 1]$.*

PROOF. Using Lemma 6.13 one can easily find a smooth map $\varphi: [0,1] \to [0,1]$ such that $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi(t) = 0$ for all $t \in [0,\epsilon]$ and $\varphi(t) = 1$ for all $t \in [1-\epsilon, 1]$. The map $(x,t) \mapsto F(x, \varphi(t))$ has the desired properties. ■

We continue with our next definition.

Definition. Let M and N be smooth manifolds.

(1) A (*proper*) *smooth isotopy of M in N* is a smooth[139] map
$$\begin{array}{rcl} F\colon M\times[0,1] & \to & N \\ (z,t) & \mapsto & F(z,t) \end{array}$$
such that for each $t\in[0,1]$ the map $F_t\colon M\to N$ is a (proper) smooth embedding.

(2) We say that two (proper) smooth embeddings $f_0,f_1\colon M\to N$ are (*properly*) *smoothly isotopic* if there exists a (proper) smooth isotopy $F\colon M\times[0,1]\to N$ with $F_0=f_0$ and $F_1=f_1$.

If $M=N$ and if each F_t is a diffeomorphism, then we refer to a smooth isotopy as a *diffeotopy* and we say *diffeotopic* instead of *smoothly isotopic*.

The following lemma gives in particular an example of a diffeotopy.

Lemma 8.5. *Let $n\in\mathbb{N}$.*

(1) *If $\gamma\colon[0,1]\to\mathrm{O}(n)$ is a smooth path, then the map*
$$\begin{array}{rcl} F\colon \overline{B}^n\times[0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \gamma(t)\cdot x \end{array}$$
is a diffeotopy rel the origin.

(2) *Let $A\in\mathrm{SO}(n)$. If $n\geq 2$, then there exists a diffeotopy of $\overline{B}^n$ from $\mathrm{id}_{\overline{B}^n}$ to the diffeomorphism of $\overline{B}^n$ that is given by multiplication by A.*

The same statement holds with $\overline{B}^n$ replaced by $\mathbb{R}^n$ and S^{n-1}.

Proof.

(1) The first statement is basically obvious.
(2) Suppose that $n\geq 2$ and let $A\in\mathrm{SO}(n)$. By Lemma 2.65 the smooth manifold $\mathrm{SO}(n)$ is path-connected. Therefore, by Exercise 18.7 there exists a smooth path in $\mathrm{SO}(n)$ from id to A. The statement now follows from (1). ■

Remark. The term “diffeotopic” is (alas!) used differently by different authors. Our definition agrees with the definition given in most textbooks, see e.g. [**Wall16**, p. 49] and [**BJ82**, Definition 9.3]. But it differs from the definition provided in [**Hirs76**, Chapter 8], there the map on $M\times[0,1]$ only needs to be continuous, but each F_t needs to be a diffeomorphism. At least a priori this is a weaker condition.

We leave the proof of the following lemma as a voluntary exercise to the eager reader.

Lemma 8.6. *Let M be an oriented connected smooth manifold and let $F\colon M\times[0,1]\to M$ be a diffeotopy. Then either all maps $F_t\colon M\to M$ are orientation-preserving or all F_t are orientation-reversing.*

For later purposes we state the following lemma.

[139] Note that we defined on page 346 what it means for the map $F\colon M\times[0,1]\to N$ to be smooth. Also note that the notion of a smooth isotopy is stronger than of an isotopy $M\times[0,1]\to N$ which happens to be a smooth map. We also require each F_t to be a (proper) smooth embedding. We will discuss this in Exercise 8.4.

Lemma 8.7. (∗) *Let N be a smooth manifold, let M be another smooth manifold and let $f, g, h\colon M \to N$ be proper smooth embeddings.*

(1) *If f is smoothly isotopic to g and g is smoothly isotopic to h, then f is also smoothly isotopic to h. The same statement also holds for "diffeotopic" instead of "smoothly isotopic".*
(2) *Being "smoothly isotopic" and being "diffeotopic" is an equivalence relation.*

Proof (∗). The only non-trivial aspect of Statement (2) is contained in Statement (1). So let us prove Statement (1). Let $\Phi\colon M \times [0,1] \to N$ be a smooth isotopy from f to g. Furthermore let $\Psi\colon M \times [0,1] \to N$ be a smooth isotopy from g to h. The first idea is of course to "stack" the two smooth isotopies together, i.e. to consider the map

$$\begin{array}{rcl} M \times [0,1] & \to & N \\ (P,t) & \mapsto & \begin{cases} \Phi(P, 2\cdot t), & \text{if } t \in [0,\frac{1}{2}], \\ \Psi(P, 2\cdot t - 1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

By Lemma 14.3 we know that this map is continuous. The problem with this approach is that this map is not necessarily smooth at $M \times \{\frac{1}{2}\}$. Thus we have to modify the above approach slightly. It follows from Lemma 6.13 that there exist a smooth function $f\colon [0,1] \to [0,1]$ with $f(0) = 0$ such that $f(t) = 1$ for all $t \in [\frac{1}{2}, 1]$ and a smooth function $g\colon [0,1] \to [0,1]$ with $g(1) = 1$ such that $g(t) = 0$ for all $t \in [0, \frac{1}{2}]$. It is now straightforward to verify that the map

$$\begin{array}{rcl} M \times [0,1] & \to & N \\ (P,t) & \mapsto & \begin{cases} \Phi(P, f(2\cdot t)), & \text{if } t \in [0,\frac{1}{2}], \\ \Psi(P, g(2\cdot t - 1)), & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \end{array}$$

is a smooth isotopy from f to h. The proof for "diffeotopic" instead of "smoothly isotopic" is verbatim the same. ■

The following lemma gives an interesting example of a smooth deformation retraction.

Lemma 8.8. (∗) *Let $n \in \mathbb{N}_0$ and let $k \in \{1, \dots, n\}$. We write*

$$\begin{array}{rcl} \mathrm{GL}(n,k) & = & \textit{set of } k\textit{-tuples of linearly independent vectors of } \mathbb{R}^n \\ \mathrm{O}(n,k) & = & \textit{set of } k\textit{-tuples of orthonormal vectors of } \mathbb{R}^n. \end{array}$$

(Note that for $k = n$ we have $\mathrm{GL}(n,k) = \mathrm{GL}(n)$ and $\mathrm{O}(n,k) = \mathrm{O}(n)$.) We denote by $r\colon \mathrm{GL}(n,k) \to \mathrm{O}(n,k)$ the map given by the Gram-Schmidt orthonormalization process.[140] *The following statements hold:*

(1) *The above map $r\colon \mathrm{GL}(n,k) \to \mathrm{O}(n,k)$ is realized by a smooth deformation retraction*[141] *$R\colon \mathrm{GL}(n,k) \times [0,1] \to \mathrm{O}(n,k)$ which has the following extra property:*
    (∗) *For any k-dimensional vector subspace V of $\mathbb{R}^n$ and any $t \in [0,1]$ we have*
    $$R(\textit{basis of } V, t) \subset \textit{basis of } V.$$
    In particular r is smooth and it is a retraction.
(2) *The inclusion $\mathrm{O}(n,k) \to \mathrm{GL}(n,k)$ is a homotopy equivalence.*

The analogous statement hold if we replace the pair $\mathrm{O}(n,k)$ *and* $\mathrm{GL}(n,k)$ *by one of the following four pairs:*

(a) $\mathrm{SO}(n)$ *and* $\mathrm{GL}_+(n,\mathbb{R})$*,*
(b) $\mathrm{U}(n)$ *and* $\mathrm{GL}(n,\mathbb{C})$*,*
(c) $\mathrm{SO}(n)$ *and* $\mathrm{SL}(n,\mathbb{R})$*.*
(d) $\mathrm{SU}(n)$ *and* $\mathrm{SGL}(n,\mathbb{C})$*.*

Sketch of proof Proof (∗).

(1) One easily verifies that each of the $2k$ steps of the Gram-Schmidt process is realized by a smooth deformation retraction that satisfies (∗). The proof of Lemma 8.7 shows that we can combine these smooth deformation retractions to obtain the desired smooth deformation retraction that satisfies (∗).
(2) This statement follows immediately from (1) and Lemma 18.14.

We consider the four extra pairs :

(a) The given smooth deformation retraction restricts to a deformation retraction from $\mathrm{GL}_+(n,\mathbb{R})$ to $\mathrm{SO}(n)$.
(b) The proof in this case is basically identical to the proof concerning the pair $\mathrm{O}(n)$ and $\mathrm{GL}(n,\mathbb{R})$.
(c) The given smooth deformation retraction does *not* restrict to a deformation retraction from $\mathrm{SL}(n,\mathbb{R})$ to $\mathrm{SO}(n)$. The problem is that in the Gram-Schmidt process we have to successively normalize the column vectors to length one. This process does not preserve the determinant. The solution is that when we normalize the i-th column vector by a factor r, then we compensate by simultaneously rescaling the $(i+1)$-st column vector by r^{-1}. This issue does not arise when we have arrived at the last column vector, since starting out with a matrix of determinant one there is no need to rescale the last column vector.
(d) The proof in this case is basically identical to the proof of (c). ■

For general culture we cite the following difficult theorem due to Jean Cerf [**Cer70**]. We could not find results in the literature that deal with the lower dimensional cases.

Theorem 8.9. (Cerf Theorem) (∗) *Let* $n \geq 6$*. Every orientation-preserving diffeomorphism of* $\overline{B}^n$ *is diffeotopic to the identity.*

We conclude this dense section with a short discussion of submanifolds.

Definition. Let M_0, M_1 be two (proper) submanifolds of a smooth manifold N.

(1) A *(proper) smooth isotopy* is a (proper) smooth isotopy $F\colon M_0 \times [0,1] \to N$ with $F_0 = \mathrm{id}_{M_0}$ and with $F_1(M_0) = M_1$.
(2) We say that the two submanifolds are *(properly) smoothly isotopic* if there exists a (proper) smooth isotopy between M_0 and M_1.

[140] This map is defined in any course on linear algebra, see e.g. [**Cu74**, Theorem 15.9].

[141] In Exercise 8.18 we will see that $\mathrm{GL}(n,k)$ is an open subset of $\mathbb{R}^{nk}$. Therefore the word "smooth" makes sense in the most naive sense.

If M_0 and M_1 are oriented smooth manifolds, then we demand that the map $F_1\colon M_0 \to M_1$ is orientation-preserving.

Example. In Figure 192 we consider some oriented 1-dimensional submanifolds of a surface Σ. In some cases it is clear that submanifolds are smoothly isotopic, in other cases this looks dubious.

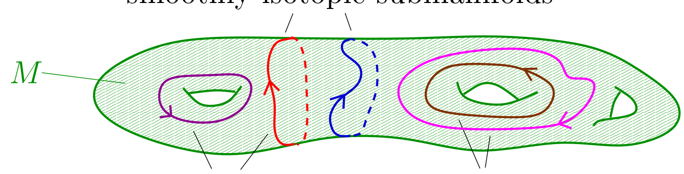


FIGURE 192

For completeness' sake we state the following lemma.

Lemma 8.10. *Let N be a smooth manifold. The property of being (properly) smoothly isotopic is an equivalence relation on the set of (proper) submanifolds of (oriented) submanifolds of M.*

PROOF. It follows from Proposition 8.1 (1) that being (properly) smoothly isotopic is a symmetric relation. It now follows as before from Lemma 8.7 that being (properly) smoothly isotopic is an equivalence relation. ■

8.3. **The Collar Neighborhood Theorem.** After the preparations from the last two sections we now have the language to state many useful results from differential topology.

Definition. Let M be a smooth manifold. A *collar* of ∂M is defined as a smooth embedding $f\colon [0,1] \times \partial M \to M$ that has the following two properties:

(a) the restriction of f to $\{0\} \times \partial M = \partial M$ is the identity,
(b) the image $f([0,1] \times \partial M)$ is a closed subset of M.

We refer to the image of f as a *collar neighborhood*.

Example. Let $m \in \mathbb{N}$ and let W be a closed smooth manifold. The map

$$\begin{array}{rcl} f\colon [0,1] \times (S^{m-1} \times W) & \to & \overline{B}^m \times W \\ (t,P,w) & \mapsto & ((1-\frac{1}{2}t)\cdot P, w) \end{array}$$

is evidently a collar for the smooth manifold $\overline{B}^m \times W$. We refer to this map as the *standard collar of* $\overline{B}^m \times W$ and we refer to the image of f as the *standard collar neighborhood of* $\overline{B}^m \times W$.

Convention.

(1) We continue with the notation from the definition. By abuse of notation we usually denote the image of f by $[0,1] \times \partial M$ as well. In particular we make the identification $\{0\} \times \partial M = \partial M$. Sometimes we refer to $[0,1) \times \partial M$ as an *open collar neighborhood* of ∂M.

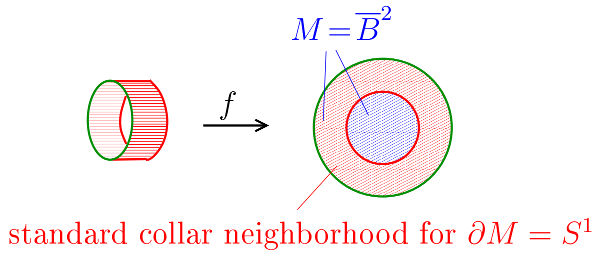


FIGURE 193

(2) Let $m, n \in \mathbb{N}$ and let W be a closed smooth manifold. Unless we say something else we will equip $\overline{B}^m \times W$ with the above standard collar.

Remark.

(1) It follows from the consideration of orientations on product smooth manifolds, see Proposition 6.51 (4), that it is better to consider products $[0,1] \times \partial M$ instead of products $\partial M \times [0,1]$. For example, if ∂M is oriented, then it follows from our conventions that $\partial([0,1) \times \partial M) = \{0\} \times \partial M$ as oriented smooth manifolds.
(2) Let $[0,1] \times \partial M$ be a collar neighborhood. It follows from Lemmas 2.6 and 3.9 that any product $[a,b] \times \partial M$ is also a closed subset of M.
(3) Our condition (b), namely that a collar neighborhood needs to be closed, is non-standard. If ∂M is compact, which is anyway the case in most situations, the image of any smooth embedding $[0,1] \times \partial M \to M$ is a closed subset by Lemma 2.17. Thus the condition only has content if ∂M is non-compact. It seems to me that in the case of non-compact boundary it is often implicitly assumed that any subset that satisfies (a) is actually closed. But in general this is not the case. For example, in Figure 194 we consider $M = \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0\}$ and the image of a smooth embedding $[0,1] \times \partial M$ that does not contain a certain point P, but such that P lies in the closure of the smooth embedding. In other words, the image of $[0,1] \times \partial M$ is not a closed subset of M.

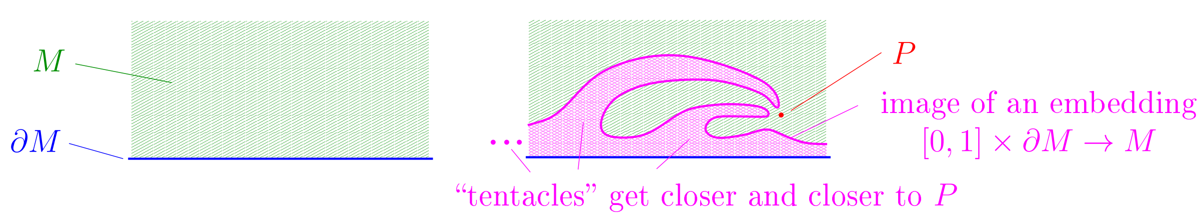


FIGURE 194

The following lemma justifies the names "collar neighborhood" and "open collar neighborhood".

Lemma 8.11. *Let M be a smooth manifold and let $f\colon [0,1] \times \partial M \to M$ be a collar. For any $t \in (0,1]$ the set $[0,t) \times \partial M$ is an open subset of M, in particular it is in fact a neighborhood of ∂M in the sense of the definition on page 89.*

Proof. The proof of this lemma is quite similar to the proof of Proposition 8.2, i.e. the proof is a reasonably straightforward consequence of Theorem 6.19. Once again we leave it to the reader to fill in the details. ■

Now we can state the Collar Neighborhood Theorem.

Theorem 8.12. (Collar Neighborhood Theorem) *Every smooth manifold admits a collar neighborhood.*

Remark.

(1) In Theorem 8.20 we will state a refinement of the Collar Neighborhood Theorem 8.12 which can also take into account submanifolds.
(2) In the Topological Collar Neighborhood Theorem 44.5 we will generalize the above theorem to the setting of topological manifolds.

Proof (∗). Let M be a smooth manifold. Basically any book on differential topology shows that there exists a smooth embedding $f\colon [0,1]\times\partial M \to M$ such that f is the identity map on the subset $\{0\}\times\partial M = \partial M$. For instance this statement is proved in [**Wall16**, Theorem 1.5.5] or alternatively [**Lee02**, Theorem 9.25], [**BJ82**, Theorem 13.6], [**Muk15**, Theorem 7.2.13]. It is also not that difficult to rewrite our proof of the Topological Collar Neighborhood Theorem 44.5 for the setting of smooth manifolds. Furthermore, if ∂M is compact one can also prove the existence of such a smooth embedding using the Smooth Manifold Product Theorem 104.8.

In the following we will outline, following [**Lee02b**], how one can modify the smooth embedding f to obtain a smooth embedding $g\colon [0,1]\times\partial M \to M$ such that the image is in fact a closed subset of M. By Lemma 6.58 there exists a map $\mu\colon M \to [0,\infty)$ with the property that given any $C\in\mathbb{R}$ the preimage $f^{-1}([0,C])$ is a compact subset of M. We consider the map

$$\begin{array}{rcl}\varphi\colon \partial M & \to & [0,1] \\ x & \mapsto & \sup\big\{t\in[0,1]\,|\,\mu(f(x,s)) > \mu(\underbrace{f(x,0)}_{=x}) - 1 \text{ for all } s\in[0,t]\big\}.\end{array}$$

Using charts one can show fairly easily that the map φ is continuous and that $\varphi(x) > 0$ for every $x\in\partial M$. By Lemma 6.59 there exists a smooth map $\psi\colon \partial M \to [0,1]$ such that $\psi(x)\in(0,\varphi(x)]$ for all $x\in\partial M$. It suffices to prove the following claim:

Claim. The map

$$\begin{array}{rcl} g\colon [0,1]\times\partial M & \to & M \\ (t,x) & \mapsto & f(\psi(x)\cdot t, x)\end{array}$$

is a smooth embedding with $g|_{\{0\}\times M} = \mathrm{id}$ and such that $g([0,1]\times\partial M)$ is a closed subset of M.

The only statement which requires some thought is to show that $g([0,1]\times\partial M)$ is a closed subset of M. We have

$$M\setminus g([0,1]\times\partial M) = \underbrace{\Big(\bigcup_{C\in\mathbb{R}}\mu^{-1}([0,C))\Big)}_{=M}\setminus g([0,1]\times\partial M) = \bigcup_{C\in\mathbb{R}}\mu^{-1}([0,C))\setminus g([0,1]\times\partial M)$$

$$\underset{\substack{\uparrow\\ \text{since } \mu(g(t,x))>\mu(x,0)-1 \text{ for all } (t,x)\in[0,1]\times\partial M}}{=} \bigcup_{C\in\mathbb{R}}\underbrace{\mu^{-1}([0,C))}_{\text{open}}\setminus\underbrace{g([0,1]\times(\overbrace{\partial M\cap\overbrace{\mu^{-1}([0,C+1])}^{\text{compact}}}^{\text{compact by Prop. 6.27 (3a) and Exercise 2.28}}))}_{\text{closed by Lemma 2.40 and Lemma 2.17}}.$$

Thus we have shown that $M\setminus g([0,1]\times\partial M)$ is the union of open subsets of M, thus $M\setminus g([0,1]\times\partial M)$ itself is open, i.e. $g([0,1]\times\partial M)$ is closed. ■

The following proposition says, that under a fairly mild hypothesis, collar neighborhoods are essentially unique.

Proposition 8.13. $(*)$ *Let M be a smooth manifold and let $f_0,f_1\colon[0,1]\times\partial M\to M$ be two smooth embeddings that are the identity map on $\partial M\times\{0\}=\partial M$. If ∂M is compact[142], then there exists a diffeotopy*

$$H\colon M\times[0,1]\ \to\ M$$

rel ∂M[143] from $H_0=\mathrm{id}_M$ to a diffeomorphism $\Phi=H_1\colon M\to M$ with $\Phi\circ f_0=f_1$. In particular Φ restricts to a diffeomorphism

$$\Phi\colon f_0([0,1]\times\partial M))\ \to\ f_1([0,1]\times\partial M).$$

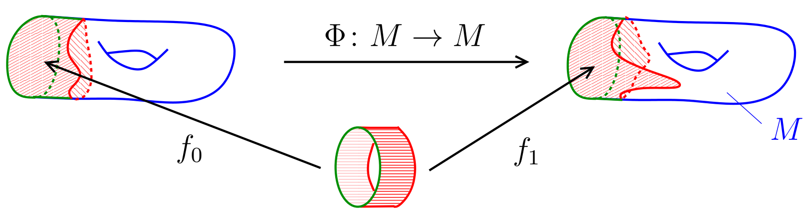


FIGURE 195

PROOF. The proposition is proved in [**Wall16**, Proposition 2.5.7]. A very similar statement is also proved in [**BJ82**, Theorem 13.7]. ■

For later we record the following corollary to the Collar Neighborhood Theorem 8.12.

[142]To get a uniqueness statement for non-compact boundary one needs to consider smooth embeddings such that the image is closed. Unfortunately it seems like nobody every proved a uniqueness statement in this more general context. Both references prefer to assume that the boundary is compact. For most purposes it is irrelevant whether a more general version of Proposition 8.13 holds, but it s nonetheless somewhat unsatisfactory that the general case has not been treated yet.

[143]Recall that this means that $H(P,t)=P$ for all $P\in\partial M$ and $t\in[0,1]$.

Corollary 8.14. $(*)$ *Let M be an n-dimensional smooth manifold. Let $[0,1]\times\partial M$ be a collar neighborhood.*

1. *The subset $W := M\setminus([0,1)\times\partial M)$ is an n-dimensional smooth manifold with boundary given by $\{1\}\times\partial M$.*
2. *The subset $W = M\setminus([0,1)\times\partial M)$ is a deformation retract of M and it is a deformation retract of $M\setminus\partial M$.*
3. *If M is non-empty, then M is homotopy equivalent to a non-compact n-dimensional smooth manifold without boundary.*

PROOF.

1. This statement follows fairly easily from Proposition 6.27. We leave it to the reader to fill in the details.
2. We write $W := M\setminus([0,1)\times\partial M)$ and we set $M' := M\setminus\partial M$. We consider the map
$$\begin{array}{rcl} F\colon M\times[0,1] & \to & M \\ (Q,t) & \mapsto & \left\{\begin{array}{ll} (t\cdot s+(1-t),R), & \text{if } Q=(s,R)\in[0,1]\times\partial M, \\ Q, & \text{if } Q\notin M\setminus[0,2)\times\partial M. \end{array}\right. \end{array}$$
Note that by definition the collar neighborhood $[0,1]\times\partial M$ is a closed subset of M and that we know by Lemma 8.11 that $M\setminus([0,1)\times\partial M)$ is also a closed subset of M. Since the restriction of F to each of these two closed subsets is continuous we obtain from Lemma 2.35 (2) that the map $F\colon M\times[0,1]\to M$ is indeed continuous. This map defines a deformation retraction from M to W and also from $M' = M\setminus\partial M$ to W.
3. It follows from (2) and Lemmas 18.14 and 18.11 that M and M' are homotopy equivalent. Finally note that it follows from Proposition 6.27 that M' is non-compact, that it is an open subset of M (in particular it is an n-dimensional smooth manifold) and that the smooth manifold M' does not have any boundary. ■

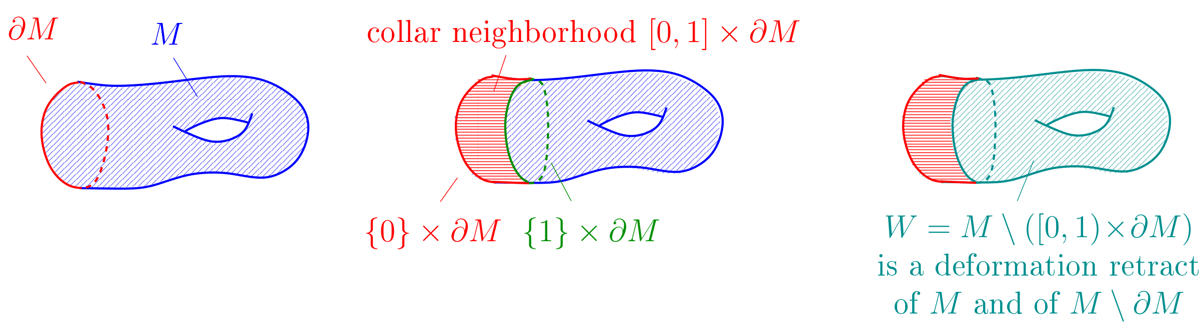


FIGURE 196. Illustration for the proof of Corollary 8.14.

Remark. We proved Corollary 8.14 (3) by considering an "internal" collar $[0,1]\times\partial M$. An alternative approach would be to consider M together with an "external" collar, namely $W := M\cup_{\partial M=\partial M\times\{0\}}([0,1)\times\partial M)$. In this approach one needs to show that W is indeed an n-dimensional smooth manifold, that it is non-compact and that M is a deformation retract of W. Even though this sounds very reasonable, it also comes with its share of technical difficulties, for example, what is the quickest argument for showing that W is second-countable?

8.4. **Gluing smooth manifolds.** The following proposition says that if, given a (possibly) disconnected smooth manifold, we glue boundary components via a diffeomorphism, then the quotient is canonically a smooth manifold. For future reference we write down the full gory details, even though this makes the proposition somewhat unreadable.

Proposition 8.15. *Let M be an n-dimensional smooth manifold (we do not assume that M is connected). Let A and B be disjoint unions of boundary components of M. Furthermore let $f\colon A \to B$ be a diffeomorphism.*[144]

(1) (a) *The topological space*
$$M/a \sim f(a) \quad \text{for } a \in A$$
is an n-dimensional topological manifold.
 (b) *Any choice of a collar*[145] *$\varphi\colon [0,1] \times \partial M \to M$ for ∂M equips the topological manifold $M/a \sim f(a)$ canonically*[146] *with the structure of an n-dimensional smooth manifold such that $M \setminus (A \cup B)$ is a submanifold and such that the image of A under the map $A \to M \to M/a \sim f(a)$ is a codimension one submanifold.*
 (c) *Let φ and ψ be two collars. By (1b) these two collars give rise to two smooth structures $\mathcal{A}_\varphi$ and $\mathcal{A}_\psi$ on the topological manifold $M/a \sim f(a)$. If ∂M is compact*[147]*, then given any neighborhood U of the image of A in $M/a \sim f(a)$ there exists a homeotopy $H\colon (M/a \sim f(a)) \times [0,1] \to M/a \sim f(a)$ with the following three properties:*
  (i) $H_0 = \mathrm{id}$.
  (ii) $H_1\colon (M/a \sim f(a), \mathcal{A}_\varphi) \to (M/a \sim f(a), \mathcal{A}_\psi)$ *is a diffeomorphism.*
  (iii) $H(x,t) = x$ *for all* $x \notin U$.
 (d) *If ∂M is compact, then the diffeomorphism type of $M/a \sim f(a)$ does not depend on the choice of the collar.*

(2) *If M consists of two components X and Y with $A \subset X$ and $B \subset Y$, then the following statements hold:*
 (a) *The obvious maps $i\colon X \to M/a \sim f(a)$ and $j\colon Y \to M/a \sim f(a)$ are smooth embeddings.*
 (b) *The images $i(X)$ and $j(Y)$ are closed subsets of $M/a \sim f(a)$ and they are codimension-zero submanifolds of $M/a \sim f(a)$.*

(3) *If M is compact, then $M/a \sim f(a)$ is also compact.*

(4) *The boundary of $M/a \sim f(a)$ is given by $\partial M \setminus (A \cup B)$.*

(5) *If each component of M contains at least one component of A or B, then $M/a \sim f(a)$ is connected.*

(6) *Assume that M is oriented. We equip A and B with the boundary orientations.*
 (a) *If f is orientation-reversing, then $M/a \sim f(a)$ admits an orientation such that the inclusion $M \setminus (A \cup B) \to M/a \sim f(a)$ is orientation-preserving.*
 (b) *If M is connected and if f is not orientation-reversing, then $M/a \sim f(a)$ is non-orientable.*

(7) *Let $\widetilde{M}$ be a compact n-dimensional smooth manifold, let $\widetilde{A}$ and $\widetilde{B}$ be disjoint unions of boundary components of $\widetilde{M}$ and let $f\colon \widetilde{A} \to \widetilde{B}$ be a diffeomorphism. Furthermore let $\varphi\colon M \to \widetilde{M}$ be a diffeomorphism such that $\varphi(A) = \widetilde{A}$, $\varphi(B) = \widetilde{B}$ and such that for any $a \in A$ we have $\varphi(f(a)) = \widetilde{f}(\varphi(a))$. The induced map*

$$\begin{array}{rcl}\overline{\varphi}\colon M/a \sim f(a) & \to & \widetilde{M}/a \sim \widetilde{f}(a) \\ {[x]} & \mapsto & [\varphi(x)]\end{array}$$

is a homeomorphism[148] *and this homeomorphism is homeotopic to a diffeomorphism.*

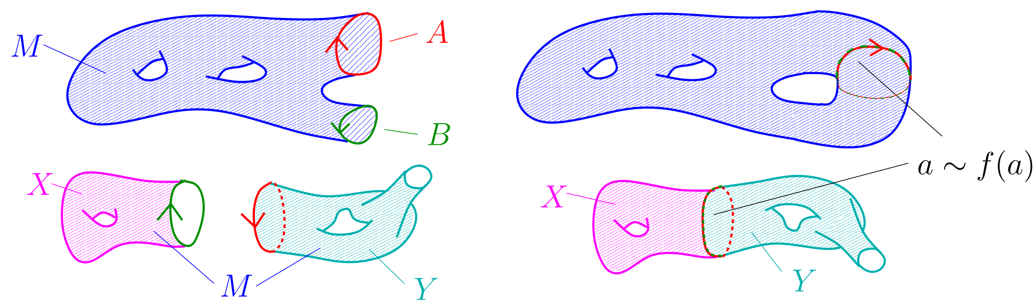


FIGURE 197. Illustration of Proposition 8.15.

Remark.

(1) We will prove the analogue of Proposition 8.15 for topological manifolds later on in Proposition 44.8.
(2) The fact that the differential structure on $M/a \sim f(a)$ depends on the choice of a collar, and that only the diffeomorphism type is well-defined, is an awkward nuisance. Like many other sources we usually gloss over this issue.
(3) In Exercise 8.25 we will prove more statements about the tricky question under which circumstances $M/a \sim f(a)$ is orientable.

For better readability we first provide the proof of Statement (1) of Proposition 8.15, before we prove the remaining statements.

PROOF OF PROPOSITION 8.15 (1). In the following we write $W := M/a \sim f(a)$ and $M' := M \setminus (A \cup B)$. It follows from Proposition 6.27 (3a) that A and B are closed subsets of M. Thus we see that M' is an open subset of M and of W. We denote by $p\colon M \to W$

[144]Recall that by Proposition 6.27 the boundary components A and B are canonically $(n-1)$-dimensional smooth manifolds.

[145]By the Collar Neighborhood Theorem 8.12 we know that a collar always exists.

[146]This statement is slightly vague, the totally precise statement can be found in the proof.

[147]The slightly annoying restriction to the case that the boundary ∂M is compact is due to the fact that we need to apply Proposition 8.13. It might well be that the statement holds without any restrictions on A and B. Also note that in practice one deals almost always with the case of compact boundary components.

[148]It is basically clear that $\overline{\varphi}$ is a homeomorphism. But note that $\overline{\varphi}$ is in general not a diffeomorphism since in general is not smooth along the image of A in $M/a \sim \widetilde{f}(a)$. So the real statement is that this flaw can be rectified by a homeotopy.

the projection map. Let $\alpha\colon [0,1]\times\partial M\to M$ be a collar. First we intend to provide a smooth atlas for W. We do so as follows:

(α) We restrict all the charts in the given atlas of M to the open subset M'.

(β) We equip A with the smooth $(n-1)$-dimensional atlas $\{\Phi_i\colon U_i\to V_i\}_{i\in I}$ that is provided by Proposition 6.27. Given $i\in I$ we define

$$\begin{array}{rcl}\overbrace{p([0,1)\times U_i)\cup p([0,1)\times f(U_i))}^{\text{open subset of } M/a\sim f(a)} & \xrightarrow{\beta_i} & (-1,1)\times V_i \\ x & \mapsto & \begin{cases}(t,\Phi_i(y)), & \text{if } x=p(\alpha(t,y)) \text{ with } \\ & \quad t\in[0,1) \text{ and } y\in U_i, \\ (-t,\Phi_i(f^{-1}(\{y\}))), & \text{if } x=p(\alpha(t,y)) \text{ with } \\ & \quad t\in[0,1) \text{ and } y\in f(U_i).\end{cases}\end{array}$$

Using (a) and (b) one easily verifies that the charts from (α) and (β) define an n-dimensional smooth atlas, let's call it $\mathcal{A}_\beta$, for W. (We refer to Figure 198 for an illustration.) As usual we still have to deal with the slightly annoying task of showing that W is Hausdorff and second-countable:

(i) We leave it to the reader to show, using (a) and (b) above, that W is Hausdorff.

(ii) Let $\mathcal{B}=\{B_i\}_{i\in I}$ be a countable basis for the topology of M. Once again we leave it to the reader to verify, using (a) and (b) above, that

$$\big\{B_i\cap M'\big\}_{i\in I}\ \cup\ \big\{p\big([0,\tfrac{1}{2})\times((B_i\cap A)\cup(f(B_i\cap A)))\big)\big\}_{i\in I}$$

is a basis for the topology of W. Since this is a countable family of sets we see that W is second-countable.

We have now shown that $M/a\sim f(a)$ is a topological manifold and that the choice of the collar α provides $M/a\sim f(a)$ canonically with a smooth structure $\mathcal{A}_\alpha$. This concludes the proof of (1a) and (1b).

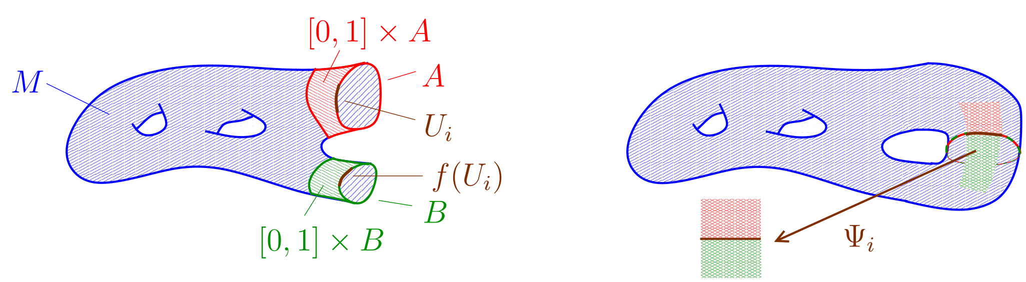


Figure 198

We move on to (1c). Now we assume that ∂M is compact. Suppose we are given two collars $\alpha\colon [0,1]\times\partial M\to M$ and $\beta\colon [0,1]\times\partial M\to M$. By Proposition 8.13 there exists a diffeotopy $H\colon M\times[0,1]\to M$ rel ∂M from $H_0=\mathrm{id}_M$ to a diffeomorphism $\Phi=H_1\colon M\to M$ with $\Phi\circ\alpha=\beta\colon [0,1]\times\partial M\to M$. We consider the map

$$\begin{array}{rcl}F\colon (M/a\sim f(a))\times[0,1] & \to & M/a\sim f(a) \\ (p(Q),t) & \mapsto & [p(Q,t)].\end{array}$$

Note that it follows from Proposition 18.20 that this map is actually continuous. It is basically clear that each F_t is a homeomorphism. Finally note that F_1 defines a diffeomorphism from $(M/a \sim f(a), \mathcal{A}_\alpha)$ to $(M/a \sim f(a), \mathcal{A}_\beta)$. We leave it to the reader to refine the argument to ensure that the map F is a constant homotopy outside of a given open neighborhood U of the image of A in $M/a \sim f(a)$.

Finally not that (1d) is an immediate consequence of (1c). ■

Proof of Proposition 8.15 (2)–(7) (∗).

(2) It follows from Lemma 3.44 (4) that the obvious maps $i\colon X \to M/a \sim f(a)$ and $j\colon Y \to M/a \sim f(a)$ are embeddings. Furthermore it follows almost immediately from inspecting the atlas we had just constructed that the images $i(X)$ and $j(Y)$ are codimension-zero submanifolds of $M/a \sim f(a)$. Finally it follows easily from the aforementioned fact that A and B are closed subsets, that $i(X)$ and $j(Y)$ are closed subsets of $M/a \sim f(a)$. We leave the details to the reader.
(3) This statement follows immediately from Lemma 3.21 (4).
(4) The atlas that we gave above for $W = M/a \sim f(a)$ has a chart of type (i) for every point in $M \setminus (\partial M \setminus (A \cup B))$ and a chart of type (ii) for every point in $\partial M \setminus (A \cup B)$. Therefore it follows from Proposition 6.27 (1) that the boundary of $W = M/a \sim f(a)$ is given by $\partial M \setminus (A \cup B)$.
(5) This statement follows, without too much effort, from Lemma 2.62.
(6) This statement can be proved once again using the Collar Neighborhood Theorem 8.12. We leave the verification of this statement as a slightly annoying exercise to the dedicated reader.
(7) It is clear that the induced map $\overline{\varphi}\colon M/a \sim f(a) \to \widetilde{M}/a \sim \widetilde{f}(a)$ is a homeomorphism. It remains to prove that $\overline{\varphi}$ is homeotopic to a diffeomorphism. We pick disjoint collar neighborhoods $[0,1] \times A$ and $[0,1] \times B$. We equip $\widetilde{A}$ and $\widetilde{B}$ with the collar neighborhoods $\varphi([0,1] \times A)$ and $\varphi([0,1] \times B)$. Using Lemma 6.13 (1) one can easily find a diffeotopy from φ to a diffeomorphism $\psi\colon M \to M$ that is "constant" along $[0, \frac{1}{4}] \times A$ and $[0, \frac{1}{4}] \times B$, i.e. which satisfies $\psi(x,t) = (\psi(x,0),t)$ for all $x \in A \cup B$ and all $t \in [0, \frac{1}{4}]$. This map again descends to a map $\overline{\psi}\colon M/a \sim f(a) \to \widetilde{M}/a \sim \widetilde{f}(a)$. It is straightforward to verify that $\overline{\psi}$ is actually a diffeomorphism and that the diffeotopy from φ to ψ descends to a homeotopy from $\overline{\varphi}$ to $\overline{\psi}$. ■

For many purposes and constructions it does not make a difference if we modify a map by a diffeotopy. For example we have the following lemma which often gets used implicitly.

Lemma 8.16. *Let M be an n-dimensional smooth manifold (we do not assume that M is connected) with compact boundary. Let A and B be two distinct compact boundary components of M. Furthermore let f_0 and f_1 be two diffeomorphisms from A to B. If f_0 and f_1 are diffeotopic, then the results of gluing via f_0 respectively via f_1 are diffeomorphic, i.e. we have a diffeomorphism*

$$M/x \sim f_0(x) \xrightarrow{\cong} M/x \sim f_1(x).$$

Furthermore, given a neighborhood of A and given a neighborhood of B one can find a diffeomorphism that is the identity outside of the chosen neighborhoods. Finally, if M is oriented and if f_0 and f_1 are orientation-reversing, then the above diffeomorphism is in fact orientation-preserving.

SKETCH OF PROOF. Since f_0 and f_1 are diffeotopic and since they are diffeomorphisms there exists a diffeotopy $F\colon A \times [0,1] \to A$ from $f_1 \circ f_0^{-1}$ to id_A. With the same approach as in the proof of Lemma 8.7 we can arrange that $F_t = \mathrm{id}_A$ for $t \in [\frac{1}{2}, 1]$.

Next note that by the Collar Neighborhood Theorem 8.12 we can pick a collar neighborhood $[0,1] \times \partial M$. (Since A is compact we can, after possibly rescaling, arrange that $[0,1] \times \partial A$ is contained in a given neighborhood of A.) One can now easily verify that

$$\begin{array}{rcl} M/x \sim f_0(x) & \to & M/x \sim f_1(x) \\ y & \mapsto & \begin{cases} y, & \text{if } y \in M \setminus ([0,1] \times A), \\ F(a,t) & \text{if } y = (t,a) \in [0,1] \times A \end{cases} \end{array}$$

is a diffeomorphism. Furthermore it is clear that this diffeomorphism has all the required properties. ■

8.5. **Cutting along submanifolds.** In this section we will introduce the operation of cutting along a codimension-one submanifold. This construction can be seen as the inverse of the gluing operation from the last section.

Definition. Let M be an oriented smooth manifold and let $F \subset M$ be a compact oriented proper submanifold of codimension one. By the General Tubular Neighborhood Theorem 10.5 together with Proposition 10.6 we know that there exists an orientation-preserving tubular map $\Phi\colon [-1,1] \times F \to M$. We define[149]

$$M \backslash\backslash F \;:=\; (M \setminus F) \sqcup (F \times \{\pm 0\})$$

with the topology that is generated by the following sets:

(1) the usual open subsets of $M \setminus F$,
(2) the sets of the form $\Phi(U \times (0,\epsilon)) \cup (U \times \{+0\})$ where U is an open subset of F and where $\epsilon \in (0,1)$,
(3) the sets of the form $\Phi(U \times (-\epsilon,0)) \cup (U \times \{-0\})$ where U is an open subset of F and where $\epsilon \in (0,1)$.

We refer to $M \backslash\backslash F$ as M *cut along* F. This definition is illustrated in Figure 199. We also introduce the following extra notation:

(4) We denote by $F_\pm$ the images of $F \times \{\pm 0\}$ in $M \backslash\backslash F$. Furthermore we denote by $i_\pm\colon F_\pm \to M \backslash\backslash F$ the obvious inclusion maps.

The following proposition summarizes some of the key properties of this construction.

[149] Here ± 0 are just understood to be the two symbols $+0$ and -0 with no further meaning.

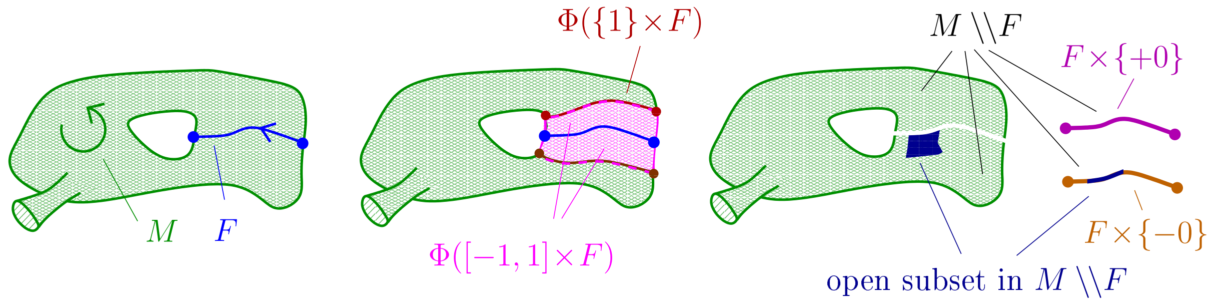


FIGURE 199

Proposition 8.17. *Let M be an orientable n-dimensional smooth manifold and let $F \subset M$ be a compact orientable proper submanifold of codimension one.*

(1) *The definition of the topological space $M \setminus\!\setminus F$ does not depend on the choice of the tubular map.*[150]
(2) *The topological space $M \setminus\!\setminus F$ admits a canonical structure of an n-dimensional smooth manifold such that F_- and F_+ are boundary components.*
(3) *If $\Phi\colon [-1,1] \times F \to M$ is an orientation-preserving tubular map, then the complement*[151] *$M \setminus \Phi((-1,1) \times F)$ is diffeomorphic to $M \setminus\!\setminus F$ and $M \setminus \Phi((-1,1) \times F)$ is a deformation retract of $M \setminus\!\setminus F$.*
(4) *The map*

$$\begin{aligned} \chi_F\colon M \setminus\!\setminus F &\to M \\ P &\mapsto \begin{cases} P, & \text{if } P \in M \setminus F, \\ Q, & \text{if } P = [(Q, \pm 0)] \text{ for some } Q \in F. \end{cases} \end{aligned}$$

is smooth and it descends to a diffeomorphism[152]

$$\overline{\chi_F}\colon (M \setminus\!\setminus F)/i_-(\underset{\substack{\uparrow \\ \text{with } x \in F}}{x}) = i_+(x) \to M.$$

Remark.

(1) In the figures, see e.g. Figure 200, we will use Proposition 8.17 (3) as an excuse to draw $M \setminus\!\setminus F$ the same way that we would draw $M \setminus \Phi((-1,1) \times F)$.
(2) In the literature the smooth manifold $M \setminus \Phi((-1,1)\times F)$ sometimes is also called "M cut along F". This name is partly justified by Proposition 8.17 (3). But one should note that our notion of "M cut along F" is much more elegant since the definition does not depend on the choice of a tubular neighborhood.
(3) A slightly different approach to defining the operation of "cutting along a codimension one manifold" is given in [**Wall16**, p. 64].

SKETCH OF PROOF.

[150]Note that here "does not depend" is meant in the strong sense, that different choices of the trivial tubular neighborhood give precisely the same topological space.

[151]Note that we know by Proposition 8.2 that the left-hand side is a smooth manifold.

[152]Note that the left-hand side is a smooth manifold by Proposition 8.15.

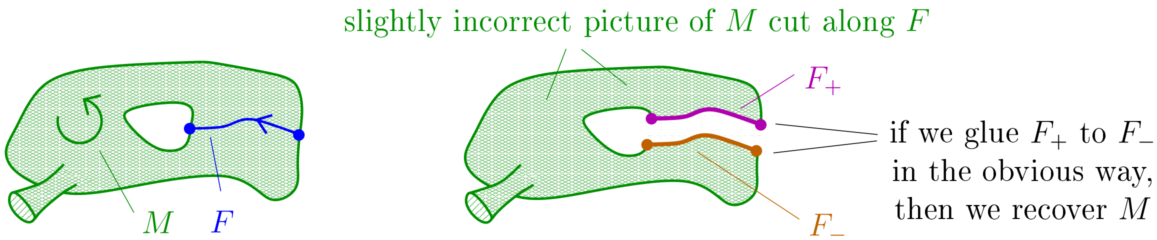


FIGURE 200

(1) This statement follows easily from the uniqueness result on tubular maps, see Proposition 10.7.

(2) Using an explicit tubular map it should be pretty clear to the reader how to show that $M \backslash\backslash F$ is actually an n-dimensional topological manifold and how to write down a smooth atlas. It follows again from the uniqueness of tubular neighborhoods, see Proposition 10.7, that the smooth structure on $M \backslash\backslash F$ does not depend on the choice of the tubular map.

(3) We pick a tubular map $\Psi\colon [-1,1] \times F \to M$. Evidently $\Phi(t,P) := \Psi(\frac{1}{2}t, P)$ also defines a tubular neighborhood. By the above it suffices to prove the statement for Φ. We consider the map

$$\begin{array}{rcl} \Theta\colon M \backslash\backslash F & \to & M \setminus \Phi((-1,1)\times F) \\ P & \mapsto & \begin{cases} P, & \text{if } P \in M \setminus \Psi((-1,1) \times F), \\ \Psi(\frac{1}{2}t + \frac{1}{2}, Q), & \text{if } P = \Psi(t,Q) \text{ with } t \in (0,1), \\ \Psi(\frac{1}{2}t - \frac{1}{2}, Q), & \text{if } P = \Psi(t,Q) \text{ with } t \in (-1,0), \\ \Psi(\pm\frac{1}{2}, Q), & \text{if } P = (\pm 0, Q) \text{ for some } Q \in F. \end{cases} \end{array}$$

Using Lemma 2.35 (2) one can easily show that Θ is continuous. We leave it to the reader to write down an explicit inverse to Θ which then shows that Θ is a homeomorphism. As always it is not difficult to slightly adjust the map Θ to turn it into a diffeomorphism.

It is an elementary, but admittedly rather messy, task to write down an explicit deformation retraction from $M \backslash\backslash F$ to $M \setminus \Phi((-1,1) \times F)$. Since we will not make use of this statement we cannot be bothered with providing all the details.

(4) We pick a tubular map $\Phi\colon [-1,1] \times F \to M$. It is straightforward to verify that the map χ_F is continuous on the open subsets $M \setminus F$ and $\Phi((-1,1) \times F) \cup F_- \cup F_+$. It follows from Lemma 2.35 (1) that χ_F is continuous. Furthermore it follows from Lemma 3.22 that the induced map $\overline{\chi}_F\colon (M \backslash\backslash F)/\sim \to M$ is also continuous. It is clear that the map $\overline{\chi}_F$ is a bijection. Furthermore using Lemma 2.39 one can easily show that the map $\overline{\chi}_F$ is open. Thus it follows from Lemma 2.42 that the map $\overline{\chi}_F$ is in fact a homeomorphism. Finally it is not hard to see that the map is actually a diffeomorphism.

We leave it to the insatiable appetite of the reader for technical proofs to fill in the details. ■

Proposition 8.17 (4) together with the following proposition shows that cutting and gluing are indeed inverse operations.

Proposition 8.18. *Let M be an orientable n-dimensional smooth manifold and let A and B be disjoint unions of boundary components of M. Furthermore let $f\colon A \to B$ be an orientation-preserving diffeomorphism. Let $W = M/a \sim f(a)$ be the smooth manifold that is obtained from gluing A to B. We denote by F the image of A under the map $A \to M \to W = M/a \sim f(a)$. By Proposition 8.15 we know that F is a codimension one submanifold of W. The result of cutting W along F is diffeomorphic to M.*

PROOF. We leave it to the reader to verify that the obvious maps are indeed diffeomorphisms. ∎

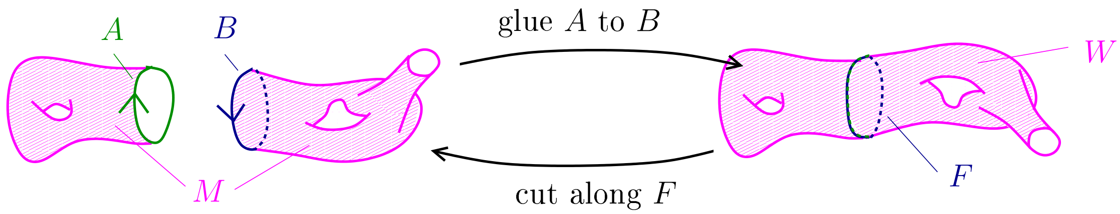


FIGURE 201. Illustration of Proposition 8.18.

8.6. **Gluing submanifolds** (∗)**.** In the previous section we introduced the notion of a collar neighborhood of the boundary of a given smooth manifold M. In many situations one is also given a proper submanifold N and one would like the collar neighborhood and the submanifold to live in harmony. In this section we will deal with this situation and show that harmony can indeed be achieved. The results in this section are even more technical than in the previous section. Thus it is best to skip this section at first reading and only to look at it when we actually use it later on.

In Proposition 8.15 we saw that if we are given a smooth manifold M (which in many examples will be disconnected) and a diffeomorphism $f\colon \partial M \to \partial M$, then any choice of a collar neighborhood of ∂M lets us view $M/x \sim f(x)$ again as a smooth manifold. Now suppose that we are given a proper submanifold N of M with $f(\partial N) = \partial N$. We would like to view $N/x \sim f(x)$ as a submanifold of $M/x \sim f(x)$. But it is easy to find examples where this is not the case, since in all likelihood $N/x \sim f(x)$ will not be smooth at the gluing points. We refer to Figure 202 for an illustration where we consider $M = ((-\infty, 0] \times \mathbb{R}) \sqcup ([1, \infty) \times \mathbb{R})$ with the obvious gluing and the obvious collar neighborhoods.

If one thinks about this issue for a few minutes then one is almost inexorably lead to the following definition.

Definition. Let M be a smooth manifold with a collar neighborhood $[0, 1] \times \partial M$, let N be a proper submanifold of M. Furthermore let $\epsilon \in [0, 1]$. We say that N *is a product with respect to* $[0, \epsilon] \times \partial M$ if the map $[0, \epsilon] \times \partial N \to [0, \epsilon] \times M$ defines a diffeomorphism $[0, \epsilon] \times \partial N \to ([0, \epsilon] \times M) \cap N$.

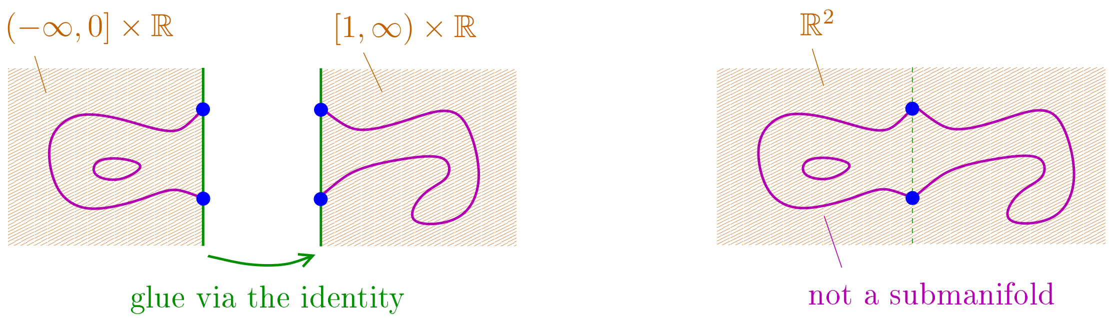


FIGURE 202

Example. We consider $M = [0, \infty) \times \mathbb{R}$ with the obvious collar neighborhood $[0, 1] \times \mathbb{R}$. Furthermore we consider the proper submanifold N that is illustrated in Figure 203. Note that N is a product with respect to $[0, \frac{1}{2}] \times \mathbb{R}$ but that it is not a product with respect to $[0, 1] \times \mathbb{R}$.

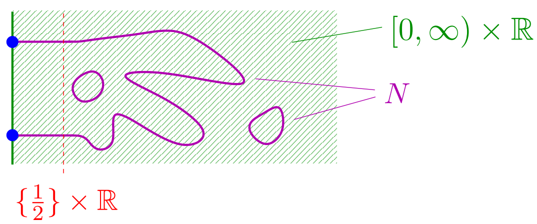


FIGURE 203

The following proposition gives an approach to circumventing this issue. For better readability it is formatted as if it was statement (7) of Proposition 8.15.

Proposition 8.19. *Let M be an n-dimensional smooth manifold (we do not assume that M is connected). Let A and B be two distinct boundary components of M. Furthermore let $f\colon A \to B$ be a diffeomorphism.*

(7) *Let N be a proper submanifold of M such that the map f restricts to a diffeomorphism $A \cap N \to B \cap N$. If there exists a collar neighborhood of M such that N is a product with respect to $[0, \epsilon] \times \partial N$ for some $\epsilon > 0$ and if we use this collar neighborhood to define the smooth manifold structure on $M/a \sim f(a)$, then $N/a \sim f(a)$ is a submanifold of $M/a \sim f(a)$.*

PROOF. The proof follows easily from the definition of the smooth structure on $M/a \sim f(a)$ as provided in the proof of Proposition 8.15 (1). We will not bore the reader with unnecessary details. ■

The question arises whether we can always find such a nice collar neighborhood. The following theorem, which is a refined version of the above Collar Neighborhood Theorem 8.12, gives an affirmative answer.

Theorem 8.20. (Collar Neighborhood Theorem) *Let M be a smooth manifold. If N is a proper submanifold of M, then M admits a collar neighborhood $[0,1] \times \partial M$ such that N is a product with respect to $[0,1] \times \partial M$.*

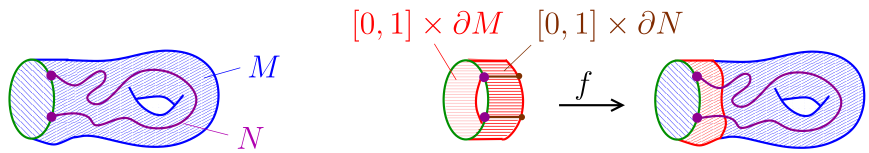


FIGURE 204. Illustration of the Collar Neighborhood Theorem 8.20.

PROOF. This statement is precisely [**Wall16**, Proposition 1.5.6 (ii)]. ■

At times one is given a collar neighborhood for the smooth manifold M that one is quite happy with. The following proposition says that we can modify the given submanifold to still obtain the desired statement.

Proposition 8.21. $(*)$ *Let M be a smooth manifold, let N be a compact proper submanifold of M and let $[0,1] \times \partial M$ be a collar neighborhood for M. There exists a diffeotopy F of M rel ∂M and an $\epsilon > 0$ such that $F_0 = \mathrm{id}$ and such that $F_1(N)$ is a product with respect to $[0,\epsilon] \times \partial M$.*

SKETCH OF PROOF $(*)$. If ∂M is compact, then we can take $\epsilon = 1$ and the statement follows immediately from the above Collar Neighborhood Theorem 8.20 and the uniqueness statement given by Proposition 8.13.

It ∂M is non-compact (and later in Proposition 8.22 we will be interested in such cases), then we can not use the above "of the shelf argument" and we need to work a little harder. Note that it follows from the hypothesis that N is a proper submanifold that the map

$$p\colon N \cap ([0,1] \times \partial M) \xrightarrow{\text{inclusion}} [0,1] \times \partial M \xrightarrow{(t,x)\mapsto t} [0,1]$$

is a submersion in a neighborhood of ∂N. Since N is compact we see that there exists an $\eta > 0$ such that p is in fact a submersion on $p^{-1}([0,\eta])$. It follows from the Submersion Theorem 6.56 there exists a smooth embedding $\varphi\colon [0,\eta] \times \partial N \to [0,\eta] \times \partial M$ with the following properties:

(1) $\varphi([0,\eta] \times \partial N) = V \cap ([0,\eta] \times \partial M)$,
(2) for each $t \in [0,\eta]$ we have $\varphi(\{t\} \times \partial N) \subset \{t\} \times \partial M$.

Similar to Lemma 6.13 there exists a smooth function $f\colon [0,\eta] \to [0,\eta]$ such that $f(t) = 0$ for $t \in [0, \frac{\eta}{4}]$ and such that $f(t) = t$ for $t \in [\frac{3\eta}{4}, \eta]$. One can now easily verify that

$$N' \ := \ \{(\varphi(P, f(t)), t) \,|\, P \in \partial N \text{ and } t \in [0,\eta]\} \cup (N \setminus (N \cap ([0,\eta) \times \partial M)))$$

is a submanifold that is product with respect to $[0, \frac{\eta}{4}] \times \partial M$. We refer to Figure 205 for an illustration of the argument.

It is not difficult to write down an explicit proper isotopy $E\colon [0,1] \times N \to M$ rel ∂N with $E_0 = \mathrm{id}$ and $E_1(N) = N'$. The existence of the desired isotopy F of M is now a consequence of the Isotopy Extension Theorem 8.27 (2) below. ■

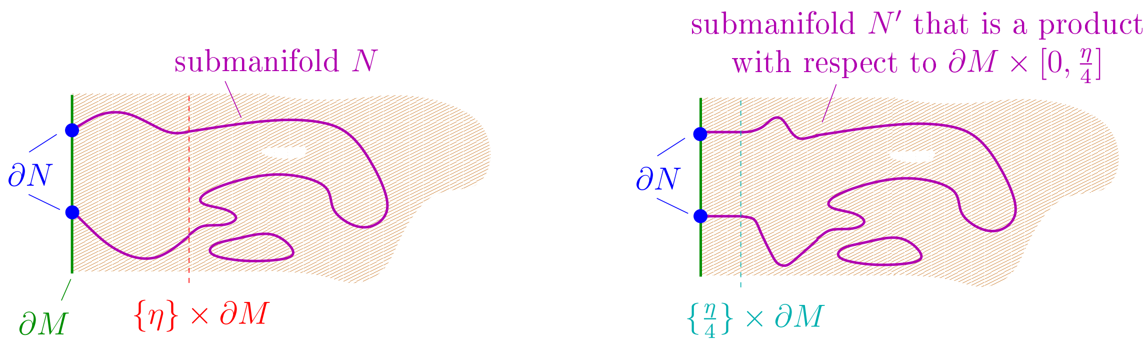


FIGURE 205

The above results now allow us to prove the following proposition which will be used on several occasions.

Proposition 8.22. *Let M and N be two n-dimensional smooth manifolds with boundary components $A \subset M$ and $B \subset N$ and let $f\colon A \to B$ be a diffeomorphism. Suppose we have chosen collar neighborhoods for ∂M and ∂N such that $M \cup_f N$ comes with a smooth manifold structure. Furthermore let $X \subset M$ and $Y \subset N$ be two proper m-dimensional submanifolds such that $f(X \cap A) = Y \cap B$. Then there exists a proper submanifold $Z \subset M \cup_f N$ with the following three properties:*

1. *$Z \cap A = X \cap A$ (and thus also $Z \cap B = Y \cap B$),*
2. *outside of any previously chosen neighborhood of $A = B \subset M \cup_f N$ we have the equality $Z = X \cup_f Y$, and*
3. *the submanifold Z is homeomorphic to $X \cup_f Y$.*

We refer to Figure 206 for an illustration.

PROOF. The proposition can be deduced quite easily from Propositions 8.19 and 8.21. We leave it to the reader to fill in the details. ■

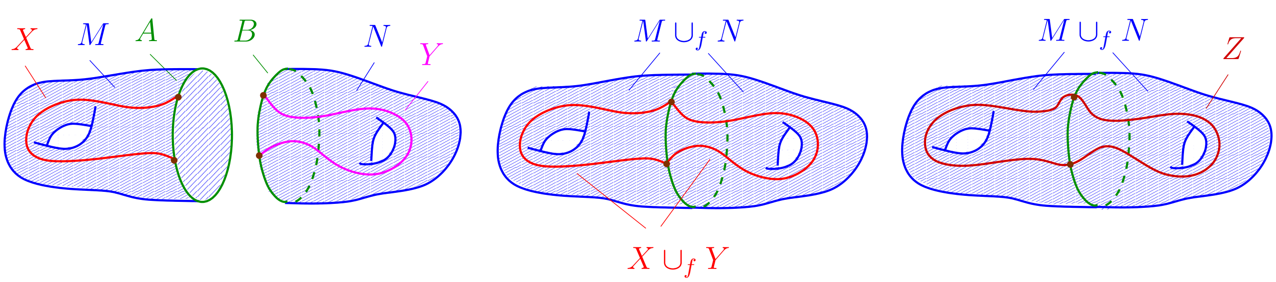


FIGURE 206. Illustration of Proposition 8.22.

8.7. **The push-in of a submanifold** (∗).

In this short section we introduce the following quite natural definition.

Definition. Let W be a smooth manifold with compact boundary and let X be a compact submanifold of the boundary ∂W. Note that by the Collar Neighborhood Theorem 8.12 we can pick a collar neighborhood $\mu\colon [0,1]\times\partial W\to W$ and we can pick a collar neighborhood $\nu\colon [0,1]\times\partial X\to X$. Furthermore by Lemma 6.13 we can pick a smooth monotonously increasing function $\varphi\colon [0,1]\to[0,1]$ with $\varphi(0)=0$, $\varphi(t)>0$ for $t>0$ and $\varphi(t)=1$ for $t\in[\frac{1}{2},1]$. We consider the map

$$\begin{array}{rcl} f\colon X & \to & W \\ P & \mapsto & \begin{cases} \mu(\varphi(t),P), & \text{if } P=\nu(t,Q) \text{ for some } t\in[0,1] \text{ and some } Q\in\partial X, \\ \mu(1,P), & \text{otherwise.} \end{cases} \end{array}$$

We call $f(X)$ the *push-in of X*.[153] We refer to Figure 207 for an illustration.

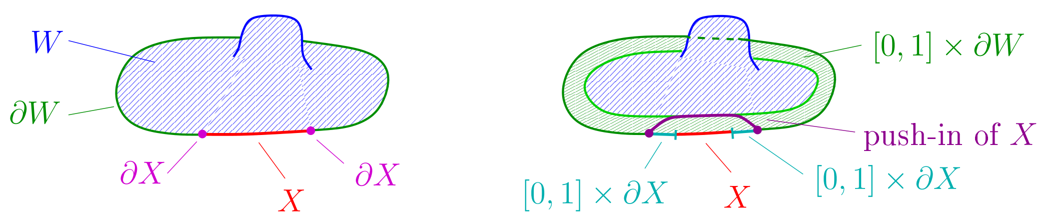


FIGURE 207

The following lemma summarizes the key properties of push-ins.

Lemma 8.23. *Let W be a smooth manifold with compact boundary and let X be a compact submanifold of ∂W. The push-in Y of X is a compact proper submanifold of W whose boundary equals $\partial Y=\partial X$.*

PROOF. We continue with the notation from the definition of the push-in. First we claim that f is smooth. This can be seen as follows: It follows easily from the choice of φ that the map f is evidently smooth on $[0,1)\times\partial X$ and $X\setminus([0,\frac{3}{4}]\times\partial X)$. Since these sets are open and since they cover all of X we see that f is indeed smooth.

Note that one can easily verify that f is a proper smooth embedding. It follows from Proposition 8.1 (2) that $f(X)$ is a proper submanifold. It is now straightforward to verify that $Y:=f(X)$ has the desired properties. ■

8.8. **The Tubular Neighborhood Theorem I.** The following definition resembles the definition of the collar of the boundary of a smooth manifold.

Definition. Let M be an m-dimensional smooth manifold and let N be a closed[154] n-dimensional submanifold.

(1) A *tubular map for N* is a smooth embedding $F\colon \overline{B}^{m-n}\times N\to M\setminus\partial M$ such that for any $P\in N$ we have $F(0,P)=P$. The image of a tubular map is called a *tubular neighborhood for N*. (Note that it follows immediately from Proposition 8.2 that a tubular neighborhood of N is in fact a neighborhood of N in the sense of the definition on page 89.)
(2) If M and N are oriented, then we say that a tubular map for N is *orientation-preserving* if the smooth embedding $F\colon \overline{B}^{m-n}\times N\to M$ is orientation-preserving where we equip $\overline{B}^{m-n}\times N$ with the product orientation introduced on page 304.

[153]It follows from Proposition 8.13 that the push-in of X is unique in a suitable sense.

Examples.

(1) We consider $M = S^1 \times S^1$ together with the submanifold $N = S^1 \times \{1\}$. The map

$$\begin{array}{rcl} \overbrace{[-1,1]}^{=\overline{B}^1} \times \overbrace{S^1}^{=N} & \to & S^1 \times S^1 \\ (t,z) & \mapsto & (e^{\mathrm{i}t}, z) \end{array}$$

is a tubular map for N. We refer to Figure 208 for an illustration.

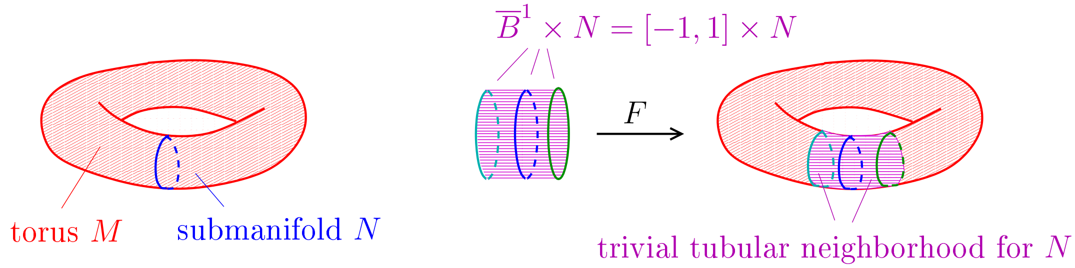


Figure 208

(2) We consider the m-dimensional smooth manifold $\mathbb{R}^m$ and the n-dimensional submanifold $K := \{(x,0) \in \mathbb{R}^m \,|\, x \in \mathbb{R}^{n+1}\}$. In this case the following is an example of a tubular map:

$$\begin{array}{rrcl} \Theta\colon & \overline{B}^{m-n} \times K & \to & \mathbb{R}^m \\ & (\underbrace{(x,y)}_{\substack{\uparrow \\ \text{with } x \in \mathbb{R} \text{ and } y \in \mathbb{R}^{m-n-1}}}, P) & \mapsto & \big((1 + \frac{x}{2}) \cdot P, y\big). \end{array}$$

The following Tubular Neighborhood Theorem guarantees the existence of (orientation-preserving) tubular maps in some situations.

Theorem 8.24. (Tubular Neighborhood Theorem) *Let M be an m-dimensional smooth manifold and let N be a closed n-dimensional submanifold. Suppose we are in one of the following four situations:*

(1) *N is zero-dimensional,*
(2) *N is one-dimensional and M is orientable,*
(3) *N is of codimension one and M and N are both orientable,*
(4) *we have $M = \mathbb{R}^m$ or $M = S^m$ and N is a closed orientable submanifold of codimension two,*
(5) *we have $M = \mathbb{R}^m$ or $M = S^m$ and N is diffeomorphic to a sphere of dimension less than $\frac{m}{2}$,*

then N admits a tubular map $F\colon \overline{B}^{m-n} \times N \to M$. Furthermore, if M and N are oriented, then we can find an orientation-preserving tubular map $F\colon \overline{B}^{m-n} \times N \to M$.[155]

[154]Here by "closed" we mean that N is closed as a smooth manifold, i.e. N is compact and $\partial N = \varnothing$.

[155]If N is 0-dimensional, then $\overline{B}^{m-n} \times N$ is just a finite union of copies of $\overline{B}^m$ and we give it the obvious orientation.

Example. In Figure 209 we illustrate the first three of the above cases. Note that in the third example we consider the case that M is a 3-dimensional ball and N is a 2-dimensional disk. In particular in this case N actually has a boundary. This artistic freedom is due to the fact that it is rather challenging to draw a 3-dimensional smooth manifold together with a tubular neighborhood of a closed 2-dimensional submanifold.

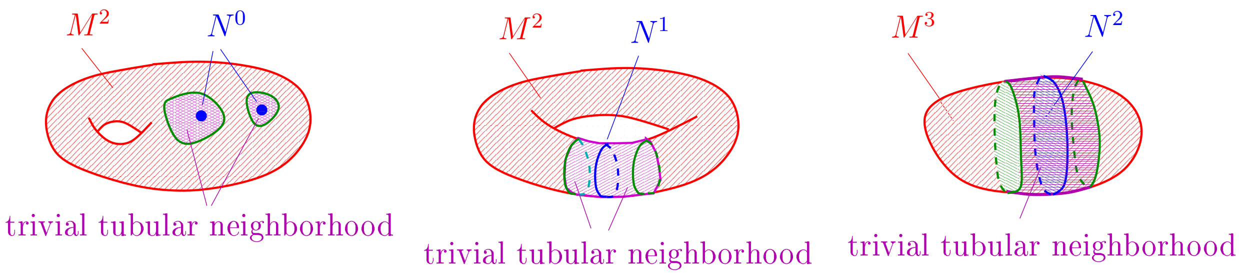


FIGURE 209

Remark. It is not entirely clear in what range of dimensions the conclusion of Theorem 8.24 holds. For example it follows from [**Kerv59**, Theorem 8.2] that if N is diffeomorphic to S^k with $k \leq 6$, then N admits a tubular map, regardless of the dimension m. On the other hand it is shown by André Haefliger [**HSL65**, p. 174] and Jerry Levine [**Lev65b**, Table 7.2] that there exist smooth embeddings $\varphi\colon S^{11} \to \mathbb{R}^{17}$ such that $\varphi(S^{11})$ has a "non-trivial normal bundle", which implies that $\varphi(S^{11})$ does not admit a tubular map.

Our sketch of a proof of the Tubular Neighborhood Theorem 8.24 relies very much on the following proposition, which will come in handy at a few other occasions.

Proposition 8.25. *Let N be a closed k-dimensional smooth submanifold of $\mathbb{R}^n$. Given $P \in N$ we write*

$$(\mathrm{V}_P\, N)^\perp \;=\; \{w \in \mathbb{R}^n \,|\, \langle v, w\rangle = 0 \textit{ for all } v \in \mathrm{V}_P\, N\}.$$

There exists an $\epsilon > 0$ such that the following conditions are satisfied:

(1) *The subset*
$$Z(N, \epsilon) \;:=\; \{P + w \,|\, P \in N \textit{ and } w \in (\mathrm{V}_P\, N)^\perp \textit{ with } \|w\| \leq \epsilon\}$$
is a compact n-dimensional submanifold of $\mathbb{R}^n$.

(2) *$Z(N, \epsilon)$ is a neighborhood of N.*

(3) *The map*
$$\begin{array}{rcl} Z(N,\epsilon) & \to & N \\ P + w & \mapsto & P \textit{ where } P \in N \textit{ and } w \in (\mathrm{V}_P\, N)^\perp \end{array}$$
is well-defined and smooth.

SKETCH OF A PROOF OF PROPOSITION 8.25. Let N be a closed k-dimensional submanifold of $\mathbb{R}^n$. We consider

$$WN \;:=\; \{(w, P) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, P \in N \text{ and } w \in (\mathrm{V}_P\, N)^\perp\}.$$

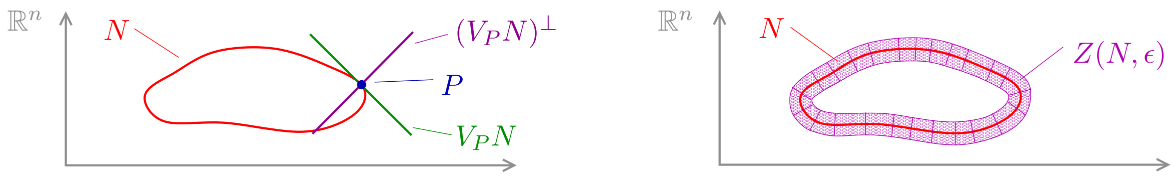


FIGURE 210. Illustration of Proposition 8.25.

Without too much effort one can equip WN with an "obvious" structure of an n-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. We consider the map

$$\begin{array}{rcl} f\colon WN & \to & \mathbb{R}^n \\ (w,P) & \mapsto & P+w. \end{array}$$

One can easily verify that this map is smooth and that for each $P \in N$ the differential of f at $(0,P)$ is an isomorphism. It follows from the Inverse Mapping Theorem 6.40 that for each $P \in N$ there exists an open neighborhood U_P of P and an $\epsilon_P > 0$ such that f restricted to $\{(Q,w)\,|\,Q \in U_P,$ and $w \in (\mathrm{V}_Q\, N)^\perp$ with $\|w\| < \epsilon_P\}$ is a smooth embedding. But since N is compact this implies that there exists an $\epsilon > 0$ such that the restriction of f to

$$Z \;:=\; \{(w,P)\,|\,P \in N \text{ and } w \in (\mathrm{V}_P\, N)^\perp \text{ with } \|w\| \le \epsilon\}$$

is a smooth embedding. Note that $f(Z) = Z(N,\epsilon)$. It is now straightforward to verify that $Z(N,\epsilon)$ has all the desired properties. We leave it to the reader to fill in the details. We refer to Figure 211 for an illustration. ■

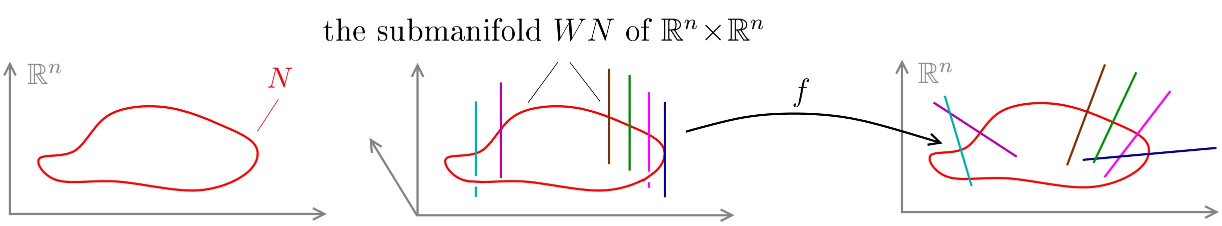


FIGURE 211. Illustration for the proof of Proposition 8.25.

We move on to provide an even more sketchy proof of the Tubular Neighborhood Theorem 8.24.

SKETCH OF A PROOF OF TUBULAR NEIGHBORHOOD THEOREM 8.24 (1)-(4). In the following we provide a sketch of the argument if $M = \mathbb{R}^n$ and if N is a closed k-dimensional submanifold of $\mathbb{R}^n$. In this special situation we apply Proposition 8.25 and we obtain an $\epsilon > 0$ together with the submanifold $Z(N,\epsilon)$. We claim that there exists a tubular map $F\colon \overline{B}^{n-k} \times N \to M$ such that $F(\overline{B}^{n-k} \times N)$ equals precisely $Z(N,\epsilon)$.

As in the proof of Proposition 8.25 we consider

$$WN \;=\; \{(w,P) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, P \in N \text{ and } w \in (\mathrm{V}_P\, N)^\perp\}$$

together with the map

$$\begin{array}{rcl} f\colon WN & \to & \mathbb{R}^n \\ (w,P) & \mapsto & P+w. \end{array}$$

Claim. In our four situations there exists a diffeomorphism

$$\Phi\colon \mathbb{R}^{n-k} \times N \;\to\; WN$$

such that for each P the map restricts to an *isometry* $\mathbb{R}^{n-k} \times \{P\} \to (\mathrm{V}_P\, N)^{\perp} \times \{P\}$ of vector spaces.

We consider the cases (1)–(4) separately:

(1) This case is trivial.
(2) This case follows from [**Hau14**, Proposition 9.2.3].
(3) This case follows from [**MiS74**, Axiom 3] together with [**Hau14**, Proposition 9.2.3].
(4) This case is dealt with in [**Bre93**, Corollary VII.14.6], alternatively see [**Er69**, p. 100] and [**Rosm00**, Corollary 6.2]. ■

Sketch of a proof of Tubular Neighborhood Theorem 8.24 (5). We deal with the last case of the Tubular Neighborhood Theorem 8.24 “by hand”. In the following we consider the case $M = \mathbb{R}^m$ and the case that N is diffeomorphic to the sphere S^r, with $r < \frac{m}{2}$. The case $M = S^m$ is treated the same way.

First note that the case $r = 1$ is dealt with in (2). Thus we now assume that $r \geq 2$. Let $\varphi\colon S^r \to N \subset \mathbb{R}^m$ be a diffeomorphism. We denote by $\theta\colon S^r \to \mathbb{R}^m$ the standard smooth embedding that is given by $x \mapsto (x, 0)$. We set $K := \theta(S^r)$. On page 368 we saw that K admits a tubular map $\Theta\colon \overline{B}^{m-r} \times K \to \mathbb{R}^m$.

Next note, since $r > 2m$ and $r \geq 2$ we obtain from Haefliger’s Unknotting Theorem 27.35[156] that there exists a smooth isotopy $F\colon S^r \times [0,1] \to \mathbb{R}^m$ with $F_0 = \theta$ and $F_1 = \varphi$. By the Isotopy Extension Theorem 8.27 there exists a diffeotopy $G\colon \mathbb{R}^m \times [0,1] \to \mathbb{R}^m$ with $G_1 \circ \theta = \varphi$. The map $\Phi := G_1 \circ \Theta \circ (\mathrm{id} \times \varphi^{-1})\colon \overline{B}^{N-r} \times N \to \mathbb{R}^m$ is a tubular map for N.

Finally note that the Statement (5) is also proved in [**Kerv59**, Theorem 8.2] or alternatively in [**MW17**, Proposition 2.10]. ■

Remark. As many readers might already know, there exists a generalization of the Tubular Neighborhood Theorem 8.24 that deals with any compact proper submanifold N. The precise definition of a tubular neighborhood is in general somewhat lengthy and technical (and frankly often done imprecisely in the literature). Thus we postpone the formulation to Section 10.2 when we really need the more general case.

The following proposition tells us to what degree tubular maps are unique.

Proposition 8.26. (Uniqueness of tubular maps and tubular neighborhoods) *Let M be an m-dimensional smooth manifold and let N be a closed n-dimensional submanifold. Furthermore let $F, G\colon \overline{B}^{m-n} \times N \to M$ be two tubular maps for N. Then there exists a diffeotopy*

$$H\colon M \times [0,1] \;\to\; M$$

[156] In the case that $r > 2m+1$ we could also use the more elementary Proposition 11.11 instead of the fancy Haefliger’s Unknotting Theorem 27.35.

rel N[157] *from $H_0 = \mathrm{id}_M$ to a diffeomorphism $\Phi = H_1$ such that the following two statements hold:*

(1) *The diffeomorphism $\Phi = H_1$ restricts to diffeomorphisms*

$$\begin{array}{rcl}\Phi\colon F(\overline{B}^{m-n} \times N) & \to & G(\overline{B}^{m-n} \times N) \\ \Phi\colon M \setminus F(B^{m-n} \times N) & \to & M \setminus G(B^{m-n} \times N).\end{array}$$

(2) *For all $P \in N$ the diffeomorphism $\Phi = H_1$ restricts to a boundary preserving diffeomorphism*

$$\Phi\colon F(\overline{B}^{m-n} \times \{P\}) \;\to\; G(\overline{B}^{m-n} \times \{P\})$$

and the self-diffeomorphism $G^{-1} \circ \Phi \circ F$ of $\overline{B}^{m-n} \times \{P\} = \overline{B}^{m-n}$ is given by multiplication by a matrix $A \in \mathrm{O}(m-n)$.

If M and N are oriented and if F and G are orientation-preserving, then all the above diffeomorphisms are orientation-preserving, in particular the matrix A mentioned in (2) *lies in* $\mathrm{SO}(m-n)$.

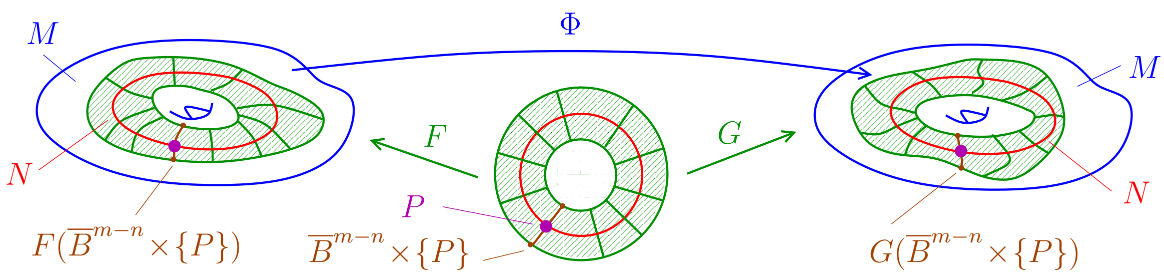


FIGURE 212. Illustration of Proposition 8.26.

PROOF. A proof for the proposition is given in [**Wall16**, Theorem 2.5.5 and Theorem 2.5.8]. We refer to [**BJ82**, Theorem 12.13], [**Kos93**, Corollary III.3.2] and [**Lan01**, Theorem IV.6.2] for alternative approaches, which, with a little bit of mental gymnastics, can also be used to prove the proposition.

The last statement that, under the given hypotheses, the diffeomorphisms are orientation-preserving is a consequence of Lemma 8.6 and the fact that id_M is orientation-preserving. ■

Before we can state the next theorem we need to introduce one more innocuous definition.

Definition. Let M be a smooth manifold. We say that a diffeotopy $F\colon M \times [0,1] \to M$ has *compact support* if there exists a compact subset $K \subset M$ such that $F(x,t) = x$ for all $x \notin K$ and all $t \in [0,1]$.

Examples.

(1) Evidently every diffeotopy of a compact smooth manifold has compact support.
(2) The diffeotopy $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ given by $F(x,t) = x + t$ does *not* have compact support.
(3) In Figure 213 we sketch a non-trivial diffeotopy of $\mathbb{R}$ with compact support.

[157] Recall that this means that $H(P,t) = P$ for all $P \in N$ and $t \in [0,1]$.

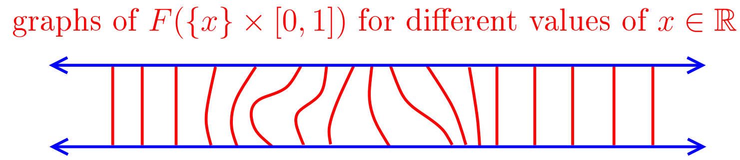


FIGURE 213

The following theorem is one of the key technical tools for dealing with smooth manifolds.

Theorem 8.27. (Isotopy Extension Theorem) *Let M be a smooth manifold and let N be a compact smooth manifold.*

(1) *Let $f\colon N \to M \setminus \partial M$ be a smooth embedding. If $F\colon N \times [0,1] \to M \setminus \partial M$ is a smooth isotopy from f to some smooth embedding f', then there exists a diffeotopy $G\colon M \times [0,1] \to M$ rel some neighborhood of ∂M with $G_0 = \mathrm{id}$ such that*
$$G(f(x),t) \;=\; F(x,t) \quad \text{for all } x \in N \text{ and } t \in [0,1].$$
(2) $(*)$ *Let $f\colon N \to M$ be a proper smooth embedding. If $F\colon N \times [0,1] \to M$ is a proper smooth isotopy from f to some proper smooth embedding f', then there exists a diffeotopy $G\colon M \times [0,1] \to M$ with $G_0 = \mathrm{id}$ such that*
$$G(f(x),t) \;=\; F(x,t) \quad \text{for all } x \in N \text{ and } t \in [0,1].$$
Furthermore, if F is an isotopy rel ∂N, then we can choose G to be a diffeotopy rel ∂M.

In both cases we can furthermore arrange that the diffeotopy G has compact support.

Examples.

(1) Let M be a smooth manifold. The Isotopy Extension Theorem 8.27 often gets applied as follows: N is a submanifold in $M \setminus \partial M$ and we start out with the inclusion map $i\colon N \to M$. The Isotopy Extension Theorem 8.27 thus loosely says, that if we deform N in M, then we can do so by "dragging M" along.
(2) In fact often we apply (1) to the special case that $N = \{P\}$ consists of a single point $P \in M \setminus \partial M$. In this case a smooth isotopy of P is just a smooth path $F\colon [0,1] \to M \setminus \partial M$ with $F(0) = P$. The Isotopy Extension Theorem 8.27 says that there exists a diffeotopy $G\colon M \times [0,1] \to M$ rel a neighborhood of ∂M from the identity to a diffeomorphism G_1 with $G(P,t) = F(t)$ for all $t \in [0,1]$. We refer to Figure 214 for an illustration.

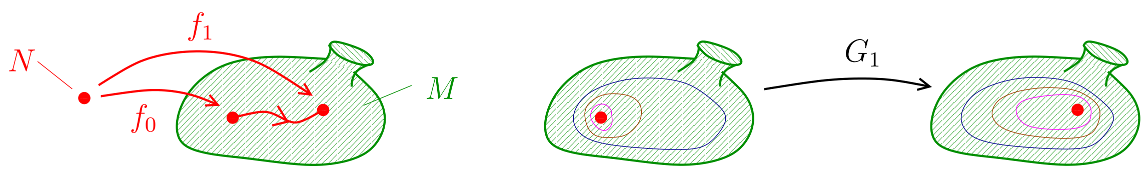


FIGURE 214. Illustration of the Isotopy Extension Theorem 8.27.

Remark. Just for reference we point out that there is a version of the Isotopy Extension Theorem 8.27 for topological manifolds. We refer to [**EKi71**, Corollary 1.4] and [**FNOP19**, Theorem 2.10] for details. We have no intention to make use of it.

Proof of the Isotopy Extension Theorem 8.27. The theorem follows almost immediately from the statements and the proofs of Theorem 2.4.2 and Theorem 2.4.6 of [**Wall16**]. Note that the results of [**Wall16**] do not explicitly say that we can assume that $G_0 = \mathrm{id}$, but a quick look at the proof of [**Wall16**, Theorem 2.4.2] shows that this is the case. Similarly in (2) the fact that in the given situation we can choose G to be a diffeotopy rel ∂M is not mentioned in [**Wall16**], but it does follow from the proof. We also refer to [**Kos93**, Theorem II.5.2], [**Hirs61**, Theorem 8.1.3] and [**Pal60b**, Chapter 6] for related results. ■

The following corollary to the Isotopy Extension Theorem 8.27 is quite useful.

Corollary 8.28. *Let M be a smooth manifold without boundary and let N be a compact smooth manifold. Let $f\colon N \to M$ and $g\colon N \to M$ be two smooth embeddings. If there exists a smooth isotopy $F\colon N \times [0,1] \to M$ from f to g, then there exists a diffeomorphism*[158] *from $M \setminus f(N)$ to $M \setminus g(N)$.*

Proof. First note that by the Isotopy Extension Theorem 8.27 there exists a diffeotopy $G\colon M \times [0,1] \to M$ with $G_0 = \mathrm{id}$ such that $G_1(f(N)) = g(N)$. But this implies that $G_1\colon M \to M$ is a diffeomorphism with $G_1(f(N)) = g(N)$. It follows that G_1 restricts to a diffeomorphism $M \setminus f(N) \to M \setminus g(N)$. ■

Example. The statement of Corollary 8.28 and the statement of the Isotopy Extension Theorem 8.27 are quite subtle. In Figure 215 we illustrate two smooth embeddings $f, g\colon S^1 \to \mathbb{R}^3$ and we illustrate a map $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ with $F_0 = f$, $F_1 = g$ and with the property that every single $F_t\colon S^1 \to \mathbb{R}^3$ is a smooth embedding. But as should be pretty clear (and we will give a rigorous proof in Section 27.2) the complements $\mathbb{R}^3 \setminus f(S^1)$ and $\mathbb{R}^3 \setminus g(S^1)$ are *not* diffeomorphic. This shows that in the formulations of Corollary 8.28 and the Isotopy Extension Theorem 8.27 it is essential that the map F is smooth on all of $N \times [0,1]$.

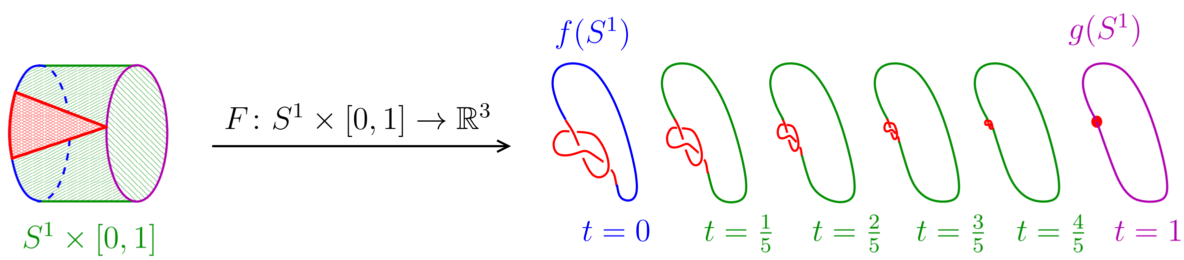


Figure 215

We also have the following proposition which is at times quite useful.

[158] Since N is compact we know by Lemmas 2.40 and 2.17 that $M \setminus f(N)$ and $M \setminus g(N)$ are open subsets of M, in particular they are smooth manifolds in their own right.

Proposition 8.29. *Let M be a connected n-dimensional smooth manifold.*

(1) *Given any two points P and Q in $M \setminus \partial M$ there exists a diffeotopy rel a neighborhood of ∂M from the identity to a diffeomorphism $\Phi\colon M \to M$ with $\Phi(P) = Q$.*
(2) *If $n \geq 2$, then given distinct points $P_1, \dots, P_m$ and distinct points $Q_1, \dots, Q_m$ in $M \setminus \partial M$ there exists a diffeotopy rel a neighborhood of ∂M from the identity to a diffeomorphism $\Phi\colon M \to M$ with $\Phi(P_i) = Q_i$ for $i = 1, \dots, m$.*

In both cases we can furthermore arrange that the diffeotopy has compact support. (Note that if M is oriented, then it follows from Lemma 8.6 that in both cases Φ is in fact orientation-preserving.)

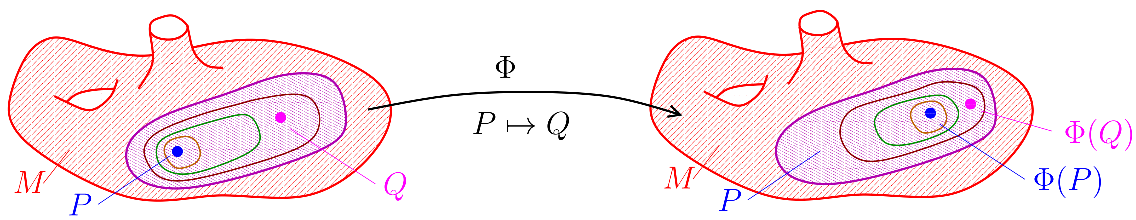


FIGURE 216. Illustration of Proposition 8.29.

In the proof of Proposition 8.29 we will make use of the following lemma which will be useful in its own right.

Lemma 8.30. *Let M be a smooth manifold.*

(1) *Any path $\gamma\colon [0,1] \to M \setminus \partial M$ is path-homotopic in $M \setminus \partial M$ to a smooth path. (Here recall "path-homotopic" means that we keep the endpoints fixed).*
(2) *If M is connected, then any two points P, Q in $M \setminus \partial M$ can be connected by a smooth path in $M \setminus \partial M$.*

PROOF OF LEMMA 8.30.

(1) The first statement follows from Exercise 18.7 applied to $W := M \setminus \partial M$.
(2) Now suppose that M is connected. From Lemma 6.9 (3) and Exercise 8.12 we obtain that $M \setminus \partial M$ is path-connected. Thus the second statement follows from the first statement. ■

Now we can move on to the actual proof of Proposition 8.29.

PROOF OF PROPOSITION 8.29. Let M be a connected n-dimensional smooth manifold.

(1) Let P, Q be two points in $M \setminus \partial M$. By Lemma 8.30 there exists a smooth path $F\colon [0,1] \to M \setminus \partial M$ from P to Q. The desired statement follows now from the Isotopy Extension Theorem 8.27, using the above example.
(2) $(*)$ We prove this statement by induction on m. We dealt with the case $m = 1$ in (1). Now suppose that we know the statement for some $m \in \mathbb{N}_{\geq 2}$. Thus let $P_1, \dots, P_{m+1}$ and $Q_1, \dots, Q_{m+1}$ be two sets of distinct points in $M \setminus \partial M$. By induction we can without loss of generality already assume that $P_i = Q_i$ for $i = 1, \dots, m$.[159] Now pick disjoint closed n-balls around $P_1, \dots, P_m$ that are disjoint from $P := P_{m+1}$ and

[159]The super vigilant reader will have noticed that implicitly we are using Lemma 8.7.

$Q := Q_{m+1}$. We denote by W the smooth manifold obtained from M by removing the corresponding open balls.[160] Since $n \geq 2$ we obtain from Proposition 6.30 that W is still connected. Thus we can apply (1) to P and Q in W to obtain a diffeotopy of W with all the desired properties. Since the diffeotopy is the identity near the boundary of W we can extend it via the identity to a diffeotopy of all of M. ■

On one occasion later on we will need the following proposition.

Proposition 8.31. $(*)$ *Let M be a connected* non-orientable *n-dimensional smooth manifold, let $P_1, \dots, P_m$ be points in $M \setminus \partial M$ and let $I \subset \{1, \dots, m\}$ be a subset. There exists a diffeotopy rel a neighborhood of ∂M from the identity to a diffeomorphism $\Phi \colon M \to M$ with the following properties:*

(1) *for all $i \in \{1, \dots, m\}$ we have $\Phi(P_i) = P_i$,*
(2) *the diffeotopy fixes all P_i with $i \notin I$,*
(3) *the map $\mathrm{D}\,\Phi_{P_i} \colon \mathrm{T}_{P_i} M \to \mathrm{T}_{P_i} M$ is orientation-preserving if and only if $i \in I$.*

Furthermore can assume that the diffeotopy has compact support.

Proof. We leave it to the reader to use the paths from Lemma 17.5 to modify the proof of Proposition 8.29 to obtain a proof of the proposition. ■

We conclude with the following theorem which follows easily from [**Wall16**, Proposition 2.4.7].

Theorem 8.32. (Extension Theorem) $(*)$ *Let $f \colon N \to M$ be a smooth embedding of a smooth manifold with boundary into a smooth manifold M such that $f(N) \subset M \setminus \partial M$. Then there exists a smooth embedding*[161]

$$f \colon \underbrace{\big(N \sqcup (\partial N \times [0,1])\big)/_{\partial N = \partial N \times 0}}_{\text{smooth manifold by Proposition 8.15}} \to M$$

which restricts to the original f on N and such that the image lies in $M \setminus \partial M$.

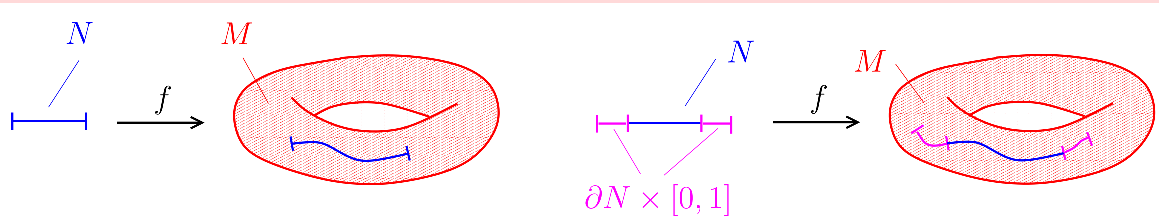


Figure 217. Illustration of the Extension Theorem 8.32.

8.9. **The connected sum operation for smooth manifolds.** In this section we introduce the connected sum of two smooth manifolds. As we will see this construction combines features from the last two sections.

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[160] Here we implicitly use Lemma 6.28.

[161] Here the topological space $\big(N \sqcup (\partial N \times [0,1])\big)/_{\partial N = \partial N \times 0}$ is understood to be the topological space $(N \sqcup (\partial N \times [0,1]))/\sim$ where $P \in \partial N$ is equivalent to $P \times \{0\}$ on $\partial N \times [0,1]$.

Definition. Let $n \in \mathbb{N}$. Let M and M' be two connected non-empty n-dimensional smooth manifolds.

(1) Suppose that at least one of M or M' is non-orientable. We pick two smooth embeddings $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$ and we define the *connected sum of M and M'* as

$$M \# M' := \big(M \setminus \varphi(B^n)\big) \sqcup \big(M' \setminus \varphi'(B^n)\big) / \sim \text{ where } \varphi(P) \sim \varphi'(P) \text{ for all } P \in S^{n-1}.$$

(2) If both M and M' are orientable, then we pick orientations for M and M'. Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$ be smooth embeddings such that φ is orientation-preserving and such that φ' is orientation-reversing. As before we define the *connected sum of the oriented smooth manifolds M and M'* as

$$M \# M' := \big(M \setminus \varphi(B^n)\big) \sqcup \big(M' \setminus \varphi'(B^n)\big) / \sim \text{ where } \varphi(P) \sim \varphi'(P) \text{ for all } P \in S^{n-1}.$$

For completeness we define the connected sum of an empty smooth manifold with another connected smooth manifold M to be M, i.e. $\varnothing \# M := M \# \varnothing := M$.[162]

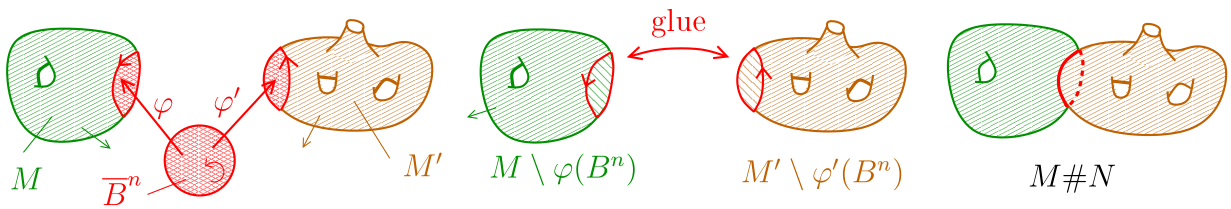

FIGURE 218

Remark. In Section 100.5 we will introduce the connected sum of topological manifolds.

In Proposition 8.35 we will discuss the properties of the connected sum. In particular we will see that in a suitable sense the connected sum $M \# N$ is well-defined, i.e. independent of any of the choices. So we will talk of "*the* connected sum" instead of "*a* connected sum". Before we go to the technicalities let us consider the connected sum in the 2-dimensional setting.

Lemma 8.33.

(1) *Let $g \in \mathbb{N}$. The iterated connected sum of g tori is diffeomorphic to the surface of genus g, as defined on page 206.*
(2) *Let $k \in \mathbb{N}$. The iterated connected sum of k copies of $\mathbb{RP}^2$ is diffeomorphic to the non-orientable surface of genus k, as defined on page 206.*

Remark. On page 2569 we will provide an alternative proof of Lemma 8.33 (2)

Examples.

(1) In Figure 220 we give a pictorial argument why the connected sum of two tori is diffeomorphic to the surface of genus two. We will provide a somewhat more formal argument in the proof of Lemma 8.33 (1).

[162]The reader will surely benignly overlook the fact that in some of the proofs down the line we do not deal with this special case separately.

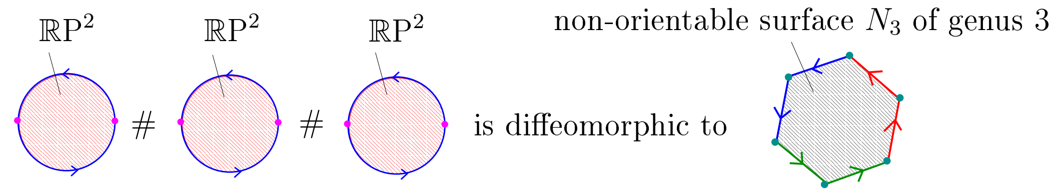

FIGURE 219. Illustration of Lemma 8.33 (2).

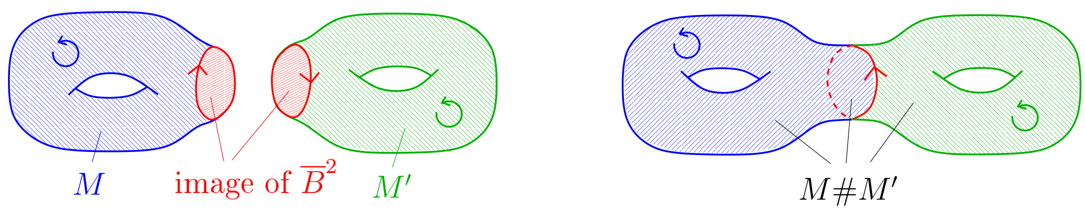

FIGURE 220

(2) By Exercise 3.43 we know that the Klein bottle is diffeomorphic to the non-orientable surface of genus 2. Thus Lemma 8.33 (2) implies that the connected sum $\mathbb{RP}^2\#\mathbb{RP}^2$ is diffeomorphic to the Klein bottle. It is instructive to do this example separately. First in Figure 221 on the top we recall the argument why $\mathbb{RP}^2$ minus an open disk is diffeomorphic to the Möbius band. This shows that $\mathbb{RP}^2\#\mathbb{RP}^2$ is given by gluing two Möbius bands along their common boundary. In Figure 221 on the bottom we see a closed curve in the Klein bottle that decomposes the Klein bottle into two Möbius bands. Put differently, we see that both $\mathbb{RP}^2\#\mathbb{RP}^2$ and the Klein bottle are the result of gluing two Möbius bands along their boundary. It is now straightforward to see that the Klein bottle is diffeomorphic to $\mathbb{RP}^2\#\mathbb{RP}^2$.

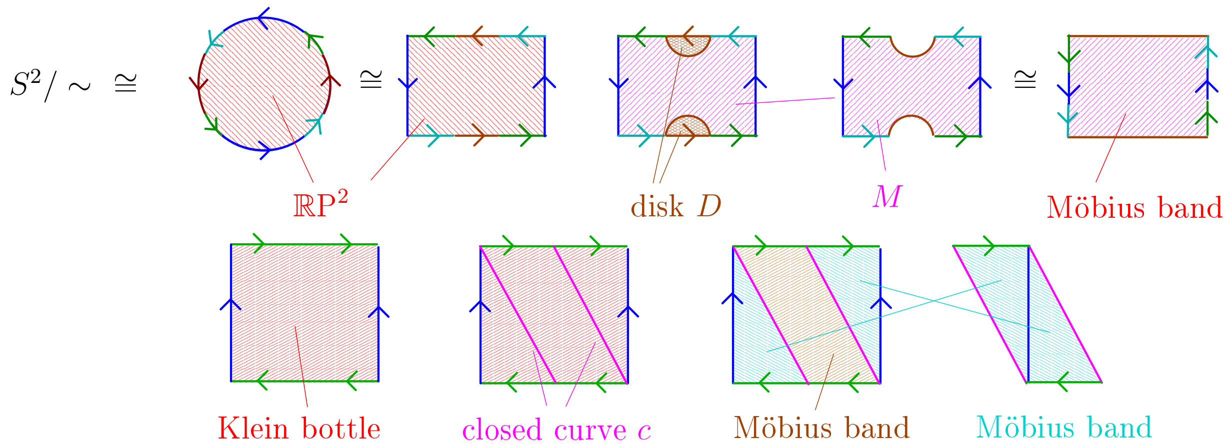

FIGURE 221

SKETCH OF PROOF.

(1) We prove the statement for $g = 2$. More precisely, in Figure 222 to the left we show two copies of the torus and two embeddings of $\overline{B}^2$, one is orientation-preserving

and the other one is orientation-reversing. Furthermore we indicate a map from the corresponding connected sum to the surface of genus 2. With some good will one can see that this gives rise to the desired diffeomorphism. We leave it to the reader to deal with the case $g \geq 3$.

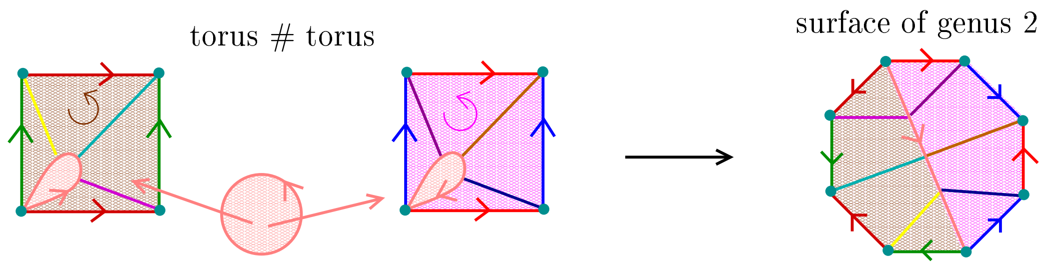

FIGURE 222. Illustration for the proof of Lemma 8.33.

(2) We outsource the proof of this statement to Exercise 8.21. ■

Our final example of a connected sum works in all dimensions.

Lemma 8.34. *The connected sum of an n-dimensional smooth manifold M with S^n is diffeomorphic to M.*

PROOF. Indeed, this can be seen as follows: in the construction of the connected sum $M\#S^n$ we consider the ball $S^n_{\geq 0}$ in S^n. Since $S^n \setminus S^n_{>0} = S^n_{\leq 0}$ is again a closed ball it is very believable that $M\#S^n$ is again diffeomorphic to M. Later on, in Lemma 8.37, we will turn this cartoon of an argument into a proper proof. ■

Now we turn to the technical proposition that contains all the statements anybody ever wanted to know about the topology of the connected sum of two smooth manifolds.

Proposition 8.35. *Let $n \in \mathbb{N}$. Let M and M' be two connected n-dimensional smooth manifolds. (If both M and M' are orientable, then we demand that M and M' are oriented.) Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$ be smooth embeddings. If M and M' are orientable, then we demand that φ is orientation-preserving and that φ' is orientation-reversing. The following seven statements hold:*

(1) *The resulting connected sum $M\#M'$ is a n-dimensional topological manifold that admits a canonical smooth structure which has the property that $M \setminus \varphi(B^n)$ and $M' \setminus \varphi'(B^n)$ are submanifolds.*
(2) (a) *The boundary of $M\#M'$ is given by $\partial M \sqcup \partial M'$.*
    (b) *If M and M' are compact, then $M\#M'$ is also compact.*
    (c) *If M and M' are closed, then $M\#M'$ is also closed.*
(3) *If $n \geq 2$ or if at least one of M or M' is closed, then $M\#M'$ is also connected.*[163]
(4) *The diffeomorphism type of $M\#M'$ does not depend on the choice of φ and φ'.*
(5) *If M and M' are oriented, then the smooth manifold $M\#M'$ admits a unique orientation that coincides with the orientations of $M \setminus \varphi(B^n)$ and $M' \setminus \varphi'(B^n)$.*

(6) *The connected sum operation is symmetric in the sense that the smooth manifolds $M\#M'$ and $M'\#M$ are diffeomorphic. If M and M' are oriented, then there exists in fact an orientation-preserving diffeomorphism $M\#M' \to M'\#M$.*

(7) *If $n \geq 2$,*[164] *then the connected sum operation is associative, in the sense that if we are furthermore given a connected (oriented) n-dimensional smooth manifold M'', then there exists an (orientation-preserving) diffeomorphism from $(M\#M')\#M''$ to $M\#(M'\#M'')$.*

Remark.

(1) Note that the connected sum $\mathbb{R}\#\mathbb{R}$ of $\mathbb{R}$ with itself is the disjoint union of two copies of $\mathbb{R}$, thus it is disconnected. This shows that in Proposition 8.35 (3) we cannot drop the hypothesis that $n \geq 2$.

(2) In Exercise 8.17 we will show that if the connected sum of two smooth manifolds is orientable, then each summand is orientable. This can be viewed as a converse to Proposition 8.35 (5).

(3) Proposition 8.35 (7), together with the discussion on page 82, allows us to unambiguously define the connected sum of any finite non-empty set of smooth manifolds of dimension $n \geq 2$. For safety we define the connected sum of the empty set of n-dimensional smooth manifolds as S^n.

(4) As discussed in Proposition 8.15 (1), the precise smooth atlas on $M\#M'$ will depend on the choice of certain collar neighborhoods, but the diffeomorphism type of $M\#M'$ is well-defined.

(5) A slightly different approach to defining the connected sum of two smooth manifolds is presented in [**KeM63**, p. 505] and [**Kos93**, p. 90]. In that reference the gluing is performed along open subsets which reduces some of the issues that arise while gluing along boundary components, i.e. while gluing along subsets that are not open.

In the proof of Proposition 8.35 (4) we will need the following theorem which was originally proved by Richard Palais [**Pal60a**] in 1960.

Theorem 8.36. *Let M be an n-dimensional smooth manifold. (If M is orientable, then we pick an orientation for M.) In the following let $\varphi_1, \dots, \varphi_m \colon \overline{B}^n \to M \setminus \partial M$ and $\psi_1, \dots, \psi_m \colon \overline{B}^n \to M \setminus \partial M$ be two sets of m smooth embeddings of $\overline{B}^n$ with disjoint images. (If M is orientable, then we demand that all φ_i and ψ_i are orientation-preserving or that all are orientation-reversing.) If $n \geq 2$ or if $m = 1$, then there exists a diffeotopy*

$$F \colon M \times [0,1] \ \to \ M$$

[164] It is a fun exercise to figure out why the hypothesis $n \geq 2$ is actually necessary.

rel some neighborhood of ∂M from the identity $F_0 = \mathrm{id}_M$ to a diffeomorphism $F_1 \colon M \to M$ such that for each $i \in \{1, \dots, m\}$ the following diagram commutes:

$$\begin{tikzcd} & \overline{B}^n \arrow[dl, "\varphi_i"'] \arrow[dr, "\psi_i"] & \\ M \arrow[rr, "F_1", "\cong"'] & & M. \end{tikzcd}$$

Furthermore we can assume that F has compact support. (*Note that if M is orientable, then the diffeomorphism F_1 is in fact orientation-preserving by Lemma 8.6*).

Remark.

(1) In Exercise 8.8 we will give an alternative proof of Theorem 8.36.
(2) In Theorem 100.10 we will state an analogue of Theorem 8.36 in the setting of topological manifolds.

Proof of Theorem 8.36. Before we start with the actual proof we want to point out that on several occasions we implicitly use Lemma 8.6 which allows us to "compose" smooth isotopies and diffeotopies. We also introduce the notation that given $A \in \mathrm{O}(n)$ we denote by $\rho(A)$ the diffeomorphism of $\overline{B}^n$ that is given by multiplication by A.

For $i = 1, \dots, m$ we write $P_i := \varphi_i(0)$ and $Q_i := \psi_i(0)$. If $n \geq 2$ or if $m = 1$, then we can apply Proposition 8.29 so that we can arrange that $P_i = Q_i$ for $i = 1, \dots, m$.

For a short period we consider the cases that M is orientable and non-orientable separately.

(1) Suppose that M is orientable. Recall that in this case we assume that M is in fact oriented. Furthermore we assume that all the φ_i and ψ_i are either orientation-preserving or orientation-reversing. In particular for each $i \in \{1, \dots, m\}$ the map $\mathrm{D}(\psi_i^{-1} \circ \varphi_i) \colon \mathrm{T}_0 \overline{B}^n \to \mathrm{T}_0 \overline{B}^n$ is orientation-preserving.
(2) Suppose that M is non-orientable. We denote by $I \subset \{1, \dots, m\}$ the set of i's for which the map $\mathrm{D}(\psi_i^{-1} \circ \varphi_i) \colon \mathrm{T}_0 \overline{B}^n \to \mathrm{T}_0 \overline{B}^n$ is orientation-reversing. We apply Proposition 8.31 to I. Using the resulting diffeotopy we can henceforth assume that each map $\mathrm{D}(\psi_i^{-1} \circ \varphi_i) \colon \mathrm{T}_0 \overline{B}^n \to \mathrm{T}_0 \overline{B}^n$ is in fact orientation-preserving.

We continue with the common discussion.

We observe that $\varphi_1, \dots, \varphi_m$ and $\psi_1, \dots, \psi_m$ are tubular maps for the closed 0-dimensional submanifold $N = \{P_1, \dots, P_m\}$. Thus it follows from Proposition 8.26 that there exist matrices $A_1, \dots, A_m \in \mathrm{O}(n)$ and a smooth isotopy rel $N = \{P_1, \dots, P_m\}$ from $\varphi_1 \cup \dots \cup \varphi_m$ to $\psi_1 \circ \rho(A_1) \sqcup \dots \sqcup \psi_m \circ \rho(A_m)$. Since each $\mathrm{D}(\psi_i^{-1} \circ \varphi_i) \colon \mathrm{T}_0 \overline{B}^n \to \mathrm{T}_0 \overline{B}^n$ is orientation-preserving we see that each A_i lies in fact in $\mathrm{SO}(n)$.

Therefore it follows from Lemma 8.5 that there exists a smooth isotopy from the map $\psi_1 \circ \rho(A_1) \sqcup \dots \sqcup \psi_m \circ \rho(A_m)$ to the map $\psi_1 \sqcup \dots \sqcup \psi_m$. Thus we have found a smooth isotopy from $\varphi_1 \sqcup \dots \sqcup \varphi_m$ to $\psi_1 \sqcup \dots \sqcup \psi_m$. The existence of the desired diffeotopy follows from the Isotopy Extension Theorem 8.27. ■

Now we can provide the proof of Proposition 8.35.

Proof of Proposition 8.35.

(1) We write $W := M \setminus \varphi(B^n)$ and $W' := M \setminus \varphi'(B^n)$. By Proposition 8.2 (2) these are both n-dimensional smooth manifolds with boundary components $\varphi(S^{n-1})$ and $\varphi'(S^{n-1})$. Note that by definition we have

$$M\#M' \;=\; (W \sqcup W')/a \sim \varphi'(\varphi^{-1}(a)) \quad \text{ where } a \in \varphi(S^{n-1}).$$

It follows from Proposition 8.15 that $M\#M'$ admits a canonical structure of an n-dimensional smooth manifold such that W and W' are submanifolds.

(2) (a) It follows from Proposition 8.15 (4) that $\partial(M\#M') = \partial M \cup \partial M'$.

(b) Suppose that M and M' are compact. It follows from Proposition 8.15 (5) together with Proposition 8.2 that $M\#M'$ is compact.

(c) This statement is an immediate consequence of (a) and (b).

(3) (a) Let $n \in \mathbb{N}_{\geq 2}$. In this case S^{n-1} is connected. It follows from Proposition 8.1 that W and W' are connected. We obtain from Proposition 8.15 (5) that $M\#M'$ is indeed connected.

(b) Now suppose that $n = 1$ and that at least one of M or M' is closed. Let us say that M is closed. It follows from the classification of compact 1-dimensional smooth manifolds, see Theorem 7.5, that M is diffeomorphic to S^1 and that M' is diffeomorphic either to S^1 or to $\overline{B}^1$. One can now show "by hand" that $M\#M'$ is connected.

(5) Now we suppose that M and M' are oriented. Recall that in this case we suppose that the map $\varphi\colon \overline{B}^n \to M \setminus \partial M$ is orientation-preserving and furthermore that the map $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$ is orientation-reversing. We continue with the notation from (1). It follows from Lemma 6.46 that W and W' inherit an orientation from M and M'. The gluing map $\varphi' \circ \varphi^{-1}\colon \varphi(S^{n-1}) \to \varphi'(S^{n-1})$ is orientation-reversing. Thus the desired statement follows from Proposition 8.15.

(4) We need to show that the definition of $M\#M'$ is independent of the choices of the smooth embeddings φ and φ'. Given smooth embeddings $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$ we now introduce the following cumbersome notation:

$$(M\#M')(\varphi, \varphi') \;:=\; \big(M \setminus \varphi\big(B^n\big)\big) \sqcup \big(M' \setminus \varphi'\big(B^n\big)\big) \,/\, \varphi(P) \sim \varphi'(P) \text{ for all } P \in S^{n-1}.$$

Below we show independence of φ and φ'.

(a) First we consider the case that both M and M' are orientable. Therefore let $\varphi, \psi\colon \overline{B}^n \to M \setminus \partial M$ be two orientation-preserving smooth embeddings and furthermore let $\varphi', \psi'\colon \overline{B}^n \to M' \setminus \partial M'$ be two orientation-reversing smooth embeddings. By Theorem 8.36 there exist in particular diffeomorphisms $F\colon M \to M$ with $F \circ \varphi = \psi$ and $F'\colon M' \to M'$ with $F' \circ \varphi' = \psi'$. One can now easily verify that the map

$$\begin{array}{rcl} (M\#M')(\varphi, \varphi') & \to & (M\#M')(\psi, \psi') \\ x & \mapsto & \begin{cases} F(x) & \text{if } x \in M \setminus \varphi(B^n), \\ F'(x) & \text{if } x \in M' \setminus \varphi'(B^n) \end{cases} \end{array}$$

is a diffeomorphism.

(b) Next we consider the case that M and M' are both non-orientable. Then basically the same argument as in (a) applies. The only difference is that we now apply

Theorem 8.36 to the case of non-orientable smooth manifolds, so we no longer have to make any assumptions on whether or not φ, ψ and φ', ψ' are orientation-preserving.

(c) Next we consider the case that M is orientable and that M' is non-orientable. Thus let $\varphi, \psi\colon \overline{B}^n \to M \setminus \partial M$ be two smooth embeddings. Furthermore let $\varphi', \psi'\colon \overline{B}^n \to M' \setminus \partial M'$ be two smooth embeddings. If φ and ψ are both orientation-preserving or if both are orientation-reversing, then the same argument as in (a) and (b) applies. Now suppose that this is not the case. We denote by $\tau\colon \overline{B}^n \to \overline{B}^n$ any orientation-reversing diffeomorphism (e.g. τ could be the reflection in one of the coordinate axes.) Then as before we see that $(M\#M')(\varphi, \varphi')$ is diffeomorphic to $(M\#M')(\psi\circ\tau, \psi'\circ\tau)$. But $(M\#M')(\psi\circ\tau, \psi'\circ\tau)$ is, basically by definition, the same as $(M\#M')(\psi, \psi')$.

(d) The case that M is non-orientable and that M' is orientable is of course dealt with the same way as in (c).

(6) We need to show that the connected sum operation is symmetric. The only slight asymmetry that occurs in the definition is if both M and M' are oriented. Thus we only need to deal with this case. Now let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ be an orientation-preserving embedding and let $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$ be an orientation-reversing embedding. Let $\rho\colon \overline{B}^n \to \overline{B}^n$ be a reflection in some hyperplane. Note that the map $\varphi' \circ \rho\colon \overline{B}^n \to M' \setminus \partial M'$ is now an orientation-preserving embedding and that the map $\varphi \circ \rho\colon \overline{B}^n \to M \setminus \partial M$ is now an orientation-reversing embedding. If we use these two maps to define $M'\#M$ we see almost immediately that the "obvious" map $M\#M' \to M'\#M$ is an orientation-preserving diffeomorphism.

(7) Finally we still need to show that the connected sum operation is actually associative. To do so we pick smooth embeddings (with appropriate orientation conditions) $\varphi\colon \overline{B}^n \to M\setminus\partial M$, $\varphi'\colon \overline{B}^n \to M'\setminus\partial M'$, $\psi'\colon \overline{B}^n \to M'\setminus\partial M'$ and $\psi''\colon \overline{B}^n \to M''\setminus\partial M''$ such that φ' and ψ' have disjoint images. Note that φ' also defines a smooth embedding $\overline{B}^n \to M\#M'$ and that ψ' also defines a smooth embedding $\overline{B}^n \to M'\#M''$. If we use all these smooth embeddings it is clear that there exists an (orientation-preserving) diffeomorphism from $(M\#M')\#M''$ to $M\#(M'\#M'')$. ∎

Remark. The fact that the diffeomorphism type of the connected sum is well-defined is rather subtle.

(1) If M and M' are two *orientable* smooth manifolds, then the diffeomorphism type of the connected sum is defined only if both smooth manifolds are in fact *oriented*, i.e. they are equipped with an orientation. In general the definition does depend on the choice of orientation. More precisely, there exist oriented smooth manifolds M and M' such that $M\#M'$ is not diffeomorphic (in fact not even homeomorphic) to $M\#(-M')$, where $-M'$ denotes as usual the smooth manifold M' with the opposite orientation. We will see such examples later, namely in Corollary 101.12, once we have introduced the cup product on cohomology groups.

(2) Often the connected sum operation gets abbreviated as follows: *Given two smooth manifolds we remove an open ball from each smooth manifold and then glue M and*

M' together, along the boundary spheres that we had just created, in such a way that the orientations of the boundaries are opposite. This description is rather dangerous since the diffeomorphism type of the connected sum is only well-defined if we glue the boundary components via the diffeomorphism given by the smooth embeddings of the closed balls. In Proposition **??** we will see that the above shortened description of the connected sum operation is incorrect.

The following lemma now gives the promised proof of the statement that performing the connected sum with S^n does not alter the diffeomorphism type.

Lemma 8.37. $(*)$ *Let M be a non-empty n-dimensional smooth manifold M. The connected sum $M\#S^n$ is diffeomorphic to M.*

Proof $(*)$. First we pick an orientation-preserving embedding $\varphi\colon \overline{B}^n \to M \setminus \partial M$. Next let $g_-\colon \overline{B}^n \xrightarrow{\cong} S^n_{\leq 0} \subset S^n$ and $g_+\colon \overline{B}^n \xrightarrow{\cong} S^n_{\geq 0} \subset S^n$ be the two embeddings from page 199. The map

$$\begin{array}{rcl} M\#M' = (M\setminus\varphi(B^n)) \sqcup \underbrace{(S^n\setminus g_-(B^n))}_{=S^n_{\geq 0}}/\sim & \to & M \\ P & \mapsto & \begin{cases} P, & \text{if } P \in M \setminus \varphi(B^n), \\ \varphi(g_+^{-1}(Q)), & \text{if } P \in S^n \setminus g_-(B^n) \end{cases} \end{array}$$

is now a diffeomorphism.[165] It follows from Proposition 8.35 (4) that we also have $M\#S^n$ for any choice of orientation-reversing embedding $\overline{B}^n \to S^n$. ■

In Exercise 8.27 we will use Theorem 8.36 to prove the following corollary which can be viewed as a strengthening of the result obtain in Exercise 18.7.

Corollary 8.38. *Let M be a connected smooth manifold and let P, Q be two distinct points in M. Then there exists a smooth embedding $\gamma\colon [0,1] \to M$ with $\gamma(0) = P$ and $\gamma(1) = Q$.*

We conclude this section with the following lemma.

Lemma 8.39. *Let $n \geq 1$.*

(1) *Given two connected non-empty n-dimensional smooth manifolds X and Y the connected sums $X\#Y$ and $Y\#X$ are diffeomorphic.*
(2) *Given three connected non-empty n-dimensional smooth manifolds X, Y and Z the connected sums $(X\#Y)\#Z$ and $X\#(Y\#Z)$ are diffeomorphic.*

Remark. Let $n \geq 1$. It follows from Lemma 8.39 and the remark on page 82 that given connected n-dimensional smooth manifolds $M_1, \dots, M_k$ we can unambiguously define the diffeomorphism type of the connected sum $M_1\# \dots \#M_k$ without having to worry in what order we perform the connected sum of two smooth manifolds.

Proof. The first statement is just an immediate consequence of the definitions, whereas the second statement is a fairly straightforward consequence of Theorem 8.36. We leave it to the reader to fill in the details. ■

[165] The reader who is equipped with a logical magnifying class will spot that it requires a little bit of thought why this map is really a diffeomorphism, in particular one needs to spend a little bit of time thinking about the question why the map is smooth in the first place.

8.10. **Knots and their complements.** Before we introduce knots we recall several different ways how we can describe the 3-sphere S^3. More precisely, by definition and Lemma 2.44 we have

$$S^3 = \big\{(z,w)\in\mathbb{C}^2\,\big|\,|w|^2+|z|^2=1\big\} = \big\{(w,x,y,z)\in\mathbb{R}^4\,\big|\,w^2+x^2+y^2+z^2=1\big\} \underset{\substack{\uparrow\\ \text{via the stereographic projection from Lemma 2.44}}}{\cong} \mathbb{R}^3\cup\{\infty\}.$$

We will go back and forth between these models without mentioning these maps. In particular, as discussed on page 344, we will always view $\mathbb{R}^3$ as a submanifold of the smooth manifold $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

Definition. A *knot* is a submanifold of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is diffeomorphic to S^1.[166] We say that the knot is *oriented* if it is oriented as a 1-dimensional smooth manifold.

Remark.

(1) It follows from Proposition 8.1 that a subset of S^3 is a knot if and only if it is the image of a smooth embedding $S^1 \to S^3$.[167]

(2) The definition of a knot is supposed to model the "physical objects" that we have in mind and that are sketched in Figure 223. It is therefore perhaps at first not clear why we consider knots in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ instead of knots in $\mathbb{R}^3$. The reason is that topologists prefer, if possible, to work with compact spaces. In particular the compact space $S^3 = \mathbb{R}^3 \cup \{\infty\}$ is often strongly preferable to the non-compact space $\mathbb{R}^3$.

We start out with the most boring but also the most important knot.

Definition. The *trivial knot* is defined as $\{(x, y, 0) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\} \subset \mathbb{R}^3 \cup \{\infty\} = S^3$.

In Figure 223 we show the trivial knot together with two other examples of knots. More precisely, we show 1-dimensional submanifolds of $\mathbb{R}^3$ that we view as 1-dimensional submanifolds of $\mathbb{R}^3 \cup \{\infty\} = S^3$.[168]

Now we want to say that two knots are "the same" if one can be "deformed" into the other. Fortunately we already have the language to make this picture precise: The notion of a smooth isotopy between (oriented) submanifolds that we introduced on page 350 makes this notion precise. For the reader's convenience we repeat the definition of "smoothly isotopic" in our context.

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[166] Sometimes, to distinguish this knot from the higher dimensional siblings that we will introduce on pages 803 and **??** we also refer to such a knot as a *classical knot*.

[167] To be more precise, the "if"-direction is a consequence of Proposition 8.1. On the other hand, if $K \subset S^3$ is a knot, then by definition there exists a diffeomorphism $\varphi\colon S^1 \to K \subset S^3$ and it follows easily from the definitions that the map φ viewed as a map from $S^1 \to S^3$ is a smooth embedding whose image is precisely K.

[168] In principle it is possible to give a precise description of all these three knots. For example we just gave a definition of the trivial knot as $\{(x, y, 0) \,|\, x^2+y^2 = 1\} \subset S^3 = \mathbb{R}^3\cup\{\infty\}$. Later, on page 777, we will give a precise definition of the trefoil. It is also clear that one can give a precise description of the figure-8 knot, but this description would be painful to write down and it would not add to our understanding. We therefore stick with the picture, with the understanding, that if somebody was challenging us, we could write down a precise description in coordinates. But it is considered very impolite to challenge a topologist to give a rigorous description.

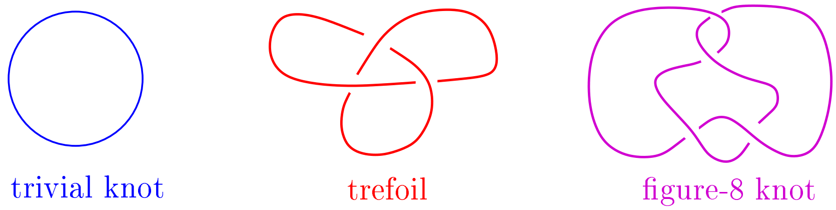


FIGURE 223. The trivial knot, the trefoil and the figure-8 knot.

Definition. We say that *two knots K and J are smoothly isotopic* if there exists a smooth isotopy from K to J, i.e. a smooth map

$$\begin{array}{rcl} F\colon S^1 \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & F(z,t) \end{array}$$

such that the following hold:

(1) we have $F(S^1 \times \{0\}) = K$ and $F(S^1 \times \{1\}) = J$, and
(2) for each $t \in [0,1]$ the map $S^1 \to S^3$ given by $z \mapsto F(z,t)$ is a smooth embedding.

If K and J are oriented, then we demand that the map $K \to J$ given by $x \mapsto F_1(F_0^{-1}(x))$ is orientation-preserving.

Remark. It follows from Lemma 8.7 that the property of being smoothly isotopic defines an equivalence relation on the set of knots.

For example the three knots shown in Figure 224 are smoothly isotopic. Usually we do not distinguish two knots if they are smoothly isotopic. For example, any knot that is smoothly isotopic to a trivial knot is called *trivial knot*. Sometimes we also say that a knot that is smoothly isotopic to a trivial knot is *trivial*.

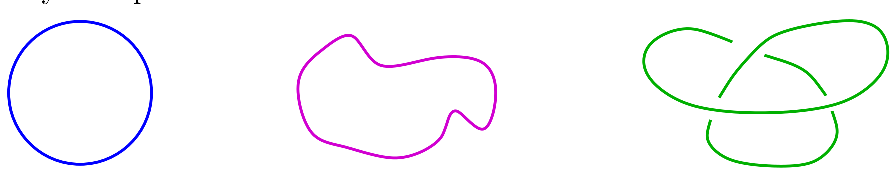


FIGURE 224. The three knots are smoothly isotopic.

Playing around with pictures for some time shows that it might be quite difficult to show that the trefoil is smoothly isotopic to the trivial knot. This arouses the suspicion, that the trefoil is in fact not smoothly isotopic to the trivial knot. This raises the following question.

Question 8.40. *How can we show that the trefoil is not smoothly isotopic to the trivial knot?*

It is also fun to consider orientations. This leads us to the following definition.

Definition. A knot K is called *reversible* if the two orientations on K give smoothly isotopic oriented knots.[169]

[169]In the literature what we call a reversible knot is often called an invertible knot. We reserve the adjective invertible for a related, but different notion, see page 2388.

Example. As we see in Figure 225, it is basically clear that the trivial knot is reversible. In Exercise 8.29 we will see that the trefoil and the figure-8 knot are also reversible.

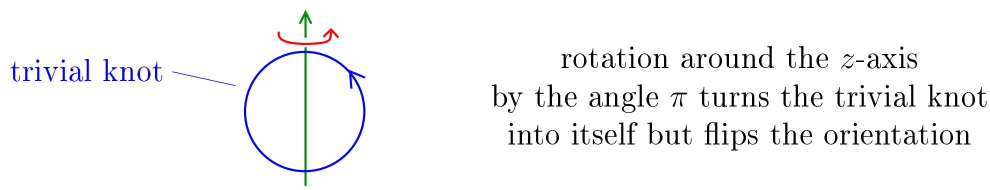


FIGURE 225

This leads us to the following question which we will answer on page 2390.

Question 8.41. *Is every knot reversible?*

In the remainder of this section we will ignore orientations and instead we will discuss the following generalization of Question 8.40.

Question 8.42. *Given a knot K[170], how can we determine whether or not K is smoothly isotopic to the trivial knot?*

The following proposition allows us to turn these two questions into the question, whether two topological spaces are diffeomorphic.

Proposition 8.43. *Let K and J be two knots. If they are smoothly isotopic, then the following two statements hold:*

(1) *There exists an orientation-preserving diffeomorphism $\Phi\colon S^3 \to S^3$ with $\Phi(K) = J$. If K and J are oriented, then Φ restricts to an orientation-preserving diffeomorphism $K \to J$.*
(2) *There exists an orientation-preserving diffeomorphism between the knot complements $S^3 \setminus K$ and $S^3 \setminus J$.*

PROOF. Let

$$\begin{array}{rcl} F\colon S^1 \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & F(z,t) \end{array}$$

be a smooth isotopy from a knot K to a knot J. By the Isotopy Extension Theorem 8.27 we can extend the smooth isotopy F to a diffeotopy of S^3. This means in particular that there exists a diffeotopy

$$\begin{array}{rcl} G\colon S^3 \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & G(z,t) \end{array}$$

from the identity to a diffeomorphism $\Phi = G_1\colon S^3 \to S^3$ with $\Phi|_K = F_1$, in particular with $\Phi(K) = J$. By Lemma 8.6 the diffeomorphism Φ is orientation-preserving. The map Φ restricts to an orientation-preserving diffeomorphism $\Phi\colon S^3 \setminus K \to S^3 \setminus J$. ■

It is natural to ask to what degree the converse of Proposition 8.43 holds. It turns out that in both cases the converse holds, but in both cases these are difficult theorems. First we have the following theorem that was proved in 1969 by Jean Cerf [**Cer68**].

[170]The phrase "given a knot K" is consciously slightly vague.

Theorem 8.44. (Cerf Theorem) *Every orientation-preserving diffeomorphism of S^3 is diffeotopic to the identity.*

Now we obtain the promised converse to Proposition 8.43 (1).

Corollary 8.45. *Let K and J be two knots. If there exists an orientation-preserving diffeomorphism $\Phi\colon S^3 \to S^3$ with $\Phi(K) = J$, then K and J are smoothly isotopic.*

Proof. Let $\Phi\colon S^3 \to S^3$ be an orientation-preserving diffeomorphism with $\Phi(K) = J$. By the Cerf Theorem 8.44 there exists a diffeotopy $F\colon S^3 \times [0,1] \to S^3$ with $F_0 = \mathrm{id}$ and $F_1 = \Phi$. The restriction of F to $[0,1] \times K$ is the desired smooth isotopy from K to J. ■

The following theorem also gives us the converse to Proposition 8.43 (2).

Theorem 8.46. (Gordon-Luecke) *Let K and J be two knots in S^3. If there exists an orientation-preserving diffeomorphism between the knot complements $S^3 \setminus K$ and $S^3 \setminus J$, then K and J are smoothly isotopic.*

Proof. This theorem is significantly more difficult to prove than Proposition 8.43. In fact a proof was given only in the 1980s by Cameron Gordon and John Luecke [**GL89**, Theorem 1]. The proof builds on the work of William Thurston [**Thu82**] for which he got the fields medal in 1982. ■

The following proposition gives a neat characterization of trivial knots.

Proposition 8.47. *Let K be a knot. The following two statements are equivalent:*

(1) *The knot K bounds a disk, i.e. there exists a smooth embedding $\varphi\colon \overline{B}^2 \to S^3$ with $\varphi(S^1) = K$.*
(2) *The knot K is smoothly isotopic to the trivial knot.*

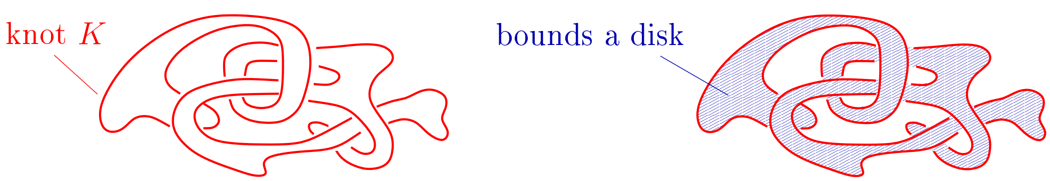


Figure 226. Illustration of Proposition 8.47.

In the proof of Proposition 8.47 we will need the following lemma which is interesting in its own right.

Lemma 8.48. *Let $n \in \mathbb{N}$ and let $\varphi\colon \overline{B}^n \to \mathbb{R}^n$ be an embedding with $\varphi(0) = 0$.*

(1) *If φ is orientation-preserving, then φ is smoothly isotopic, rel 0, to the identity.*
(2) *If φ is orientation-reversing, then φ is smoothly isotopic, rel 0, to the reflection $\rho\colon \overline{B}^n \to \mathbb{R}^n$ that is given by $(x_1, x_2, \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$.*

Proof of Lemma 8.48. This lemma is a special case of Proposition 8.26. But it is more fun to prove the lemma “by hand”. We will do so in Exercise 8.6. ■

Sketch of a proof of Proposition 8.47. Let K be a knot. We need to prove the equivalence of (1) and (2).

First we prove the almost trivial "(2)⇒(1)"-direction. It is clear that the trivial knot $\{(x,y,0) \in \mathbb{R}^3 \,|\, x^2+y^2=1\} \subset \mathbb{R}^3 \cup \{\infty\} = S^3$ bounds a disk. It follows immediately from the Isotopy Extension Theorem 8.27 that any knot that is smoothly isotopic to the trivial knot also bounds a disk.

Now we turn to the proof of the much more interesting "(2)⇒(1)"-direction. Thus let $\varphi\colon \overline{B}^2 \to S^3 = \mathbb{R}^3 \cup \{\infty\}$ be a smooth embedding with $\varphi(S^1) = K$. We write $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ and we make the identification $\mathbb{C} = \mathbb{C} \times \{0\} \subset \mathbb{R}^3$. It suffices to show that there exists a smooth isotopy from φ to a smooth embedding $\psi\colon \overline{B}^2 \to \mathbb{R}^3$ such that $\psi(S^1) = S^1 \subset \mathbb{C}$. We perform the following steps:

(0) By Lemma 2.65 (1) we know that SO(4) is path-connected. Together with Exercise 18.7 we see that we can find a smooth isotopy such that afterwards we have $\infty \notin \varphi(\overline{B}^2)$.

(1) Using that $\mathrm{SL}(3,\mathbb{R})$ is path-connected, see again Lemma 2.65 (2), and using again Exercise 18.7 we can find a smooth isotopy such that afterwards we have $\varphi(0) = 0$ and $(\mathrm{D}\,\varphi_0)(\mathrm{T}_0\,\overline{B}^2) = \mathbb{C} \times \{0\}$.

(2) We can shrink the disk, i.e. we can precompose the map $\varphi\colon \overline{B}^2 \to \mathbb{C} \times \mathbb{R}$ with the smooth isotopy $\overline{B}^2 \times [\epsilon, 1] \to \overline{B}^2$ that is given by $(p,t) \mapsto p \cdot t$ for some $\epsilon \in (0,1]$, such that composition $p \circ \varphi$ of φ with the projection $p\colon \mathbb{R}^3 = \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ is an embedding.

(3) We write $\varphi = \varphi_{\mathbb{C}} \times \varphi_{\mathbb{R}}\colon \overline{B}^2 \to \mathbb{C} \times \mathbb{R}$. It follows from (2) that the map

$$\begin{array}{rcl} \overline{B}^2 \times [0,1] & \to & \mathbb{C} \times \mathbb{R} \\ (x,t) & \mapsto & (\varphi_{\mathbb{C}}(x), \varphi_{\mathbb{R}}(x) \cdot t) \end{array}$$

is a smooth isotopy. This shows that we only need to deal with a smooth embedding $\varphi\colon \overline{B}^2 \to \mathbb{C}$ with $\varphi(0) = 0$.

(4) Now we can apply Lemma 8.48.

We illustrate the steps (2) to (4) in Figure 227. We leave it to the reader to fill in all the technical details and to make all steps rigorous. ■

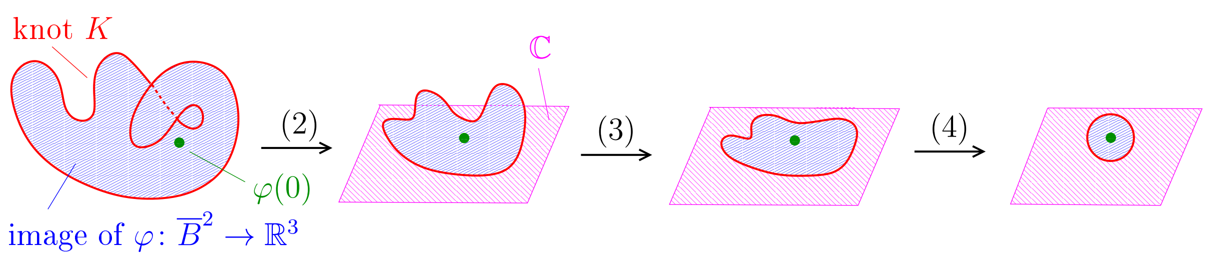


FIGURE 227

We continue our discussion of knots with a definition.

Definition. Let K be an oriented knot in S^3. By the Tubular Neighborhood Theorem 8.24 there exists an orientation-preserving smooth embedding

$$F\colon \overline{B}^2 \times K \;\to\; S^3$$

such that $F(0,P) = P$ for all $P \in K$. For any $P \in K$ we refer to the oriented submanifold[171] $F(S^1 \times \{P\})$ as a *meridian* of K.

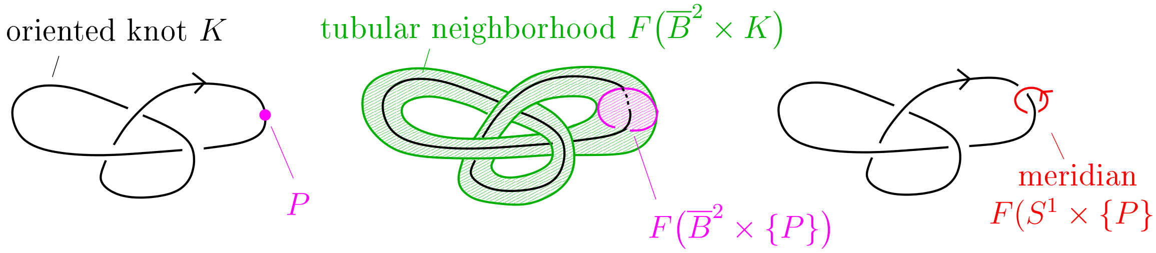


FIGURE 228

A priori the definition of the meridian depends on the choice of the tubular neighborhood. The following lemma, which is an immediate consequence of Proposition 8.26, says that the meridian is well-defined up to smooth isotopy.

Lemma 8.49. *The meridian of an oriented knot in S^3 is well-defined up to smooth isotopy in $S^3 \setminus K$.*

The following proposition gives a convenient criterion for showing that a given closed oriented curve is indeed a meridian.

Proposition 8.50. *Let K be an oriented knot in S^3. Let $\mu \subset S^3 \setminus K$ be a closed oriented curve. Suppose there exists a smooth embedding $\varphi\colon \overline{B}^2 \subset S^3$ with the following two properties:*

(1) *the $\varphi|_{S^1}$ defines an orientation-preserving diffeomorphism $S^1 \to \mu$,*
(2) *the image $\varphi(\overline{B}^2)$ intersects K transversally in a single point P,*
(3) *a positive basis for $\mathrm{T}_P(\varphi(\overline{B}^2))$ together with a positive basis for $\mathrm{T}_P\,K$ gives a positive basis for $\mathrm{T}_P\,S^3$,*[172]

then μ is a meridian of K.

Remark. More casually speaking, Proposition 8.50 says that a meridian is any closed curve which circles once around a knot where the orientation is given by the right-hand-rule, i.e. if the thumb points into the direction of the knot, then the fingers point into the direction of the meridian. This rule is illustrated in Figure 229.

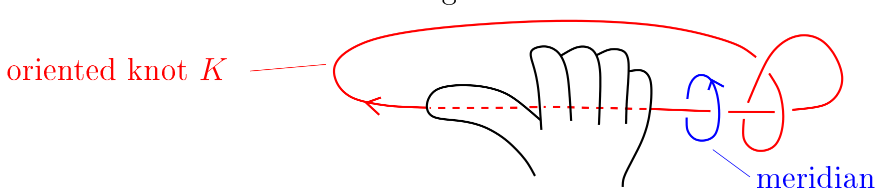


FIGURE 229

SKETCH OF PROOF. By the Tubular Neighborhood Theorem 8.24 there exists an orientation-preserving smooth embedding $F\colon \overline{B}^2 \times K \to S^3$ such that $F(0,P) = P$ for all $P \in K$. For

[171]It follows immediately from Proposition 8.1 that $F(S^1 \times \{P\})$ is indeed a submanifold of $S^3 \setminus K$.

[172]Later, on page 2271, we will say that P is a positive intersection point of $\varphi(\overline{B}^2)$ and K.

convenience we denote the image of F by $\overline{B}^2 \times K$. We perform the following steps which are quite similar to the steps in the proof of Proposition 8.47.

(a) We pick a point $Q \neq P$ on K. We can shrink the disk such that $\varphi(\overline{B}^2)$ is contained in $\overline{B}^2 \times (K \setminus \{Q\})$.
(b) It follows from our hypothesis (1) and the definition of differentiability that we can shrink the disk even further, such that the map
$$\overline{B}^2 \xrightarrow{\varphi} \overline{B}^2 \times (K \setminus \{Q\}) \xrightarrow{(v,z)\mapsto(v,P)} \overline{B}^2 \times \{P\}$$
is an embedding. It follows from the hypothesis (2) that this map is actually orientation-preserving.
(c) Similar to Step (3) of the proof of Proposition 8.47 we see that φ is smoothly isotopic, rel $(0, P)$, to an orientation-preserving embedding into $\overline{B}^2 \times \{P\}$.
(d) The statement now follows from Lemma 8.48.

As in the proof of Proposition 8.47 we leave it to the reader to fill in the details. ■

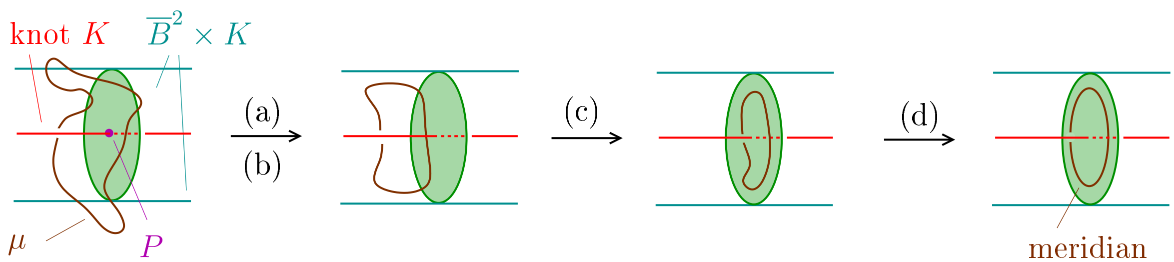


FIGURE 230. Illustration of the proof of Proposition 8.50.

Later on, in Chapter 27, we will study the fundamental group of a knot complement. Implicitly we will make use of the following lemma.

Lemma 8.51. *Given any knot $K \subset S^3$ the complement $S^3 \setminus K$ is path-connected.*

PROOF. We will provide the proof in Exercise 8.28. ■

Exercises for Chapter 8.

Exercise 8.1. Let $f \colon M \to N$ be a smooth map between two smooth manifolds. Let $P \in M \setminus \partial M$. We suppose that $\mathrm{D} f_P \colon \mathrm{T}_P M \to \mathrm{T}_{f(P)} N$ is a monomorphism. Show that there exists an open neighborhood U of p such that the restriction of f to a map $U \to N$ is a smooth embedding.
Hint. Use the Inverse Function Theorem 6.18

Exercise 8.2. Let X and Y be closed topological manifolds and let $f \colon X \to Y$ an embedding in the sense of the definition on page 116, in other words, we assume that $f \colon X \to Y$ is a homeomorphism onto its image. Is $f(X)$ necessarily a submanifold of Y?
Remark. In other words, we are wondering whether the obvious analogue of Proposition 8.1 also holds in the context of topological manifolds.

Exercise 8.3. Let M and N be smooth manifolds and let $f\colon M \to N$ be an immersion. We denote by $\Delta = \{(x,x) \in M \times M \,|\, x \in M\}$ the "diagonal" of $M \times M$.

(a) Show that there exists an open neighborhood U of the diagonal Δ such that for any $(x,y) \in U \setminus \Delta$ we have $f(x) \neq f(y)$.
(b) We suppose that M is compact. Show that f admits only finitely many double points, i.e. that there exist only finitely many distinct pairs $(x_1,y_1),\dots,(x_n,y_n)$ in $M \times M \setminus \Delta$ such that $f(x_i) = f(y_i)$ for $i = 1,\dots,n$.

Exercise 8.4. Give an example of smooth manifolds M and N together with an isotopy $F\colon M \times [0,1] \to N$ which is smooth, but which is not a smooth isotopy in the sense of the definition on page 348.

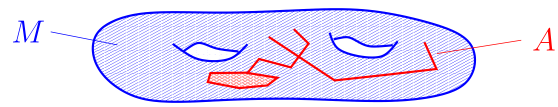


FIGURE 231. Illustration for Exercise 9.3.

Exercise 8.5. Let $f\colon M \to N$ be an immersion between two smooth manifolds without boundary.

(a) Show that given any $P \in M$ there exists an open neighborhood U of P such that the restriction $f|_U\colon U \to M$ is a smooth embedding.
Hint. Use the Inverse Mapping Theorem 6.40.
(b) We suppose that M and N are compact, non-empty and connected. Show that f is a covering map.
(c) Does the conclusion of (b) still hold if instead we only assume that M and N are connected and that f is surjective?

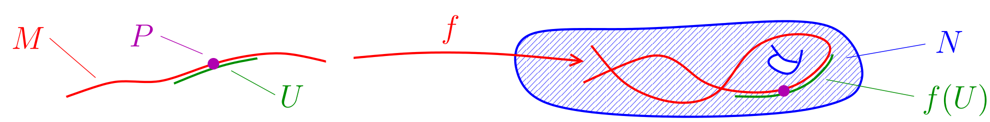


FIGURE 232. Illustration of Exercise 8.5.

Exercise 8.6. Let $n \in \mathbb{N}$ and let $\varphi\colon \overline{B}^n \to \mathbb{R}^n$ be an embedding with $\varphi(0) = 0$. The goal of this exercise is to show the following two statements:

(1) If φ is orientation-preserving, then φ is smoothly isotopic, rel 0, to the identity.
(2) If φ is orientation-reversing, then φ is smoothly isotopic, rel 0, to the reflection $\rho\colon \overline{B}^n \to \mathbb{R}^n$ that is given by $(x_1,x_2,\dots,x_n) \mapsto (-x_1,x_2,\dots,x_n)$.

We will prove the two statements in a few easy steps:

(a) (i) Show that if φ is orientation-preserving, then φ is smoothly isotopic rel 0 to a map ψ such that $\mathrm{D}\,\psi_0 = \mathrm{id}$.
(ii) Show that if φ is orientation-reversing, then φ is smoothly isotopic rel 0 to a map ψ such that $\mathrm{D}\,\psi_0 = D\rho_0$.
Hint. Use Lemma 2.65 together with Exercise 18.7.

(b) Let $\psi\colon \overline{B}^n \to \mathbb{R}^n$ be a map with $\mathrm{D}\,\psi_0 = \mathrm{id}$. Show that there exists an $\epsilon > 0$ such that the map

$$\begin{array}{rrcl} F\colon & \overline{B}^n_\epsilon \times [0,1] & \to & \mathbb{R}^n \\ & (z,t) & \mapsto & \psi(z)\cdot(1-t) + z\cdot t \end{array}$$

is a smooth isotopy.
Hint. Consider the function $\overline{B}^n_\epsilon \times \mathbb{R} \to \mathbb{R}^{n+1} \times \mathbb{R}$ given by $(z,t) \mapsto (F(z,t),t)$ and apply the Inverse Function Theorem 6.18 to the points $(0,t)$ and use that $[0,1]$ is compact. Alternatively use Proposition 6.16.

(c) Prove the original statements (1) and (2).
Remark. At some point you surely want to use Lemma 8.7.

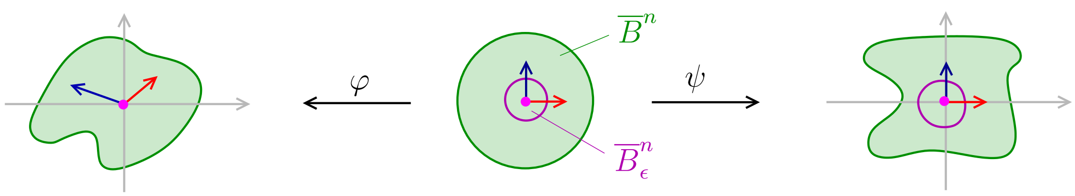


FIGURE 233. Illustration for Exercise 8.6.

Exercise 8.7. Let M be a connected n-dimensional smooth manifold.

(a) Suppose that M is oriented. Let $\varphi, \psi\colon \overline{B}^n \to M$ be two orientation-preserving smooth embeddings. Show that there exists a smooth isotopy $F\colon \overline{B}^n \times [0,1] \to M$ with $F_0 = \varphi$ and $F_1 = \psi$.
Hint. First use Exercise 8.6 to deal with the case that there exists a chart $\Theta\colon U \to B^n$ for M with $\varphi(\overline{B}^n) \subset U$ and $\psi(\overline{B}^n) \subset U$. Use the Sausage Lemma, stated and proved in Exercise 2.53, to deal with the general case.

(b) Prove (a) for the case that M is non-orientable.
Hint. Let $p\colon \widetilde{M} \to M$ be the orientation covering that we introduced in Proposition 17.3. By (a) we know that the conclusion holds for $\widetilde{M}$. Now Exercise 17.6 comes in handy.

Exercise 8.8. Let M be a connected n-dimensional smooth manifold.

(a) Suppose that M is oriented. Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\psi\colon \overline{B}^n \to M \setminus \partial M$ be two orientation-preserving smooth embeddings of $\overline{B}^n$. Show that there exists a diffeotopy $F\colon M\times[0,1] \to M$ rel some neighborhood of ∂M with $F_0 = \mathrm{id}_M$ and with $F_1\circ\varphi = \psi$.

(b) Suppose that M is non–orientable. Let $\varphi\colon \overline{B}^n \to M\setminus\partial M$ and $\psi\colon \overline{B}^n \to M\setminus\partial M$ be two smooth embeddings of $\overline{B}^n$. Show that there exists a diffeotopy $F\colon M\times[0,1] \to M$ rel some neighborhood of ∂M with $F_0 = \mathrm{id}_M$ and with $F_1 \circ \varphi = \psi$.

Hint. Use Exercises 8.6 and 8.7 together with the Isotopy Extension Theorem 8.27.
Remark. Applying this theorem iteratively one can now easily provide an alternative proof of Theorem 8.36.

Exercise 8.9. Let M and N be two smooth manifolds with empty boundary. We suppose that $\dim(M) < \dim(N)$. Furthermore let W be a submanifold of N without boundary.

Finally we suppose that for every point $P \in M$ with $f(P) \in W$ we have the equality $\mathrm{D}\, f_P(\mathrm{T}_P\, M) + \mathrm{T}_{f(P)}\, W = \mathrm{T}_{f(P)}\, N$. Show that $W \setminus f(M)$ has full measure in W.
Hint. Use Exercise 8.5, Lemma 6.52 (2) and Proposition 6.62.

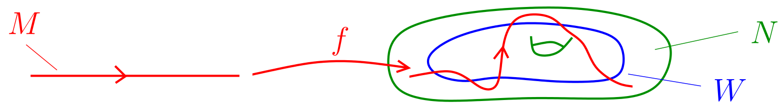


FIGURE 234

Exercise 8.10. Give an example of a map $f\colon M \to N$ between smooth manifolds with $\dim(N) \geq \dim(M)$ which is not homotopic to an immersion.

Exercise 8.11.

(a) Show that the two oriented 1-dimensional submanifolds A and B of $S^1 \times S^1$ that are shown in Figure 235 on the left are not smoothly isotopic.
(b) We consider $\mathbb{R}^2$ with the two oriented 1-dimensional submanifolds X and Y that are shown in Figure 235 on the right. Are these X and Y smoothly isotopic?

Remark. Beware of base points.

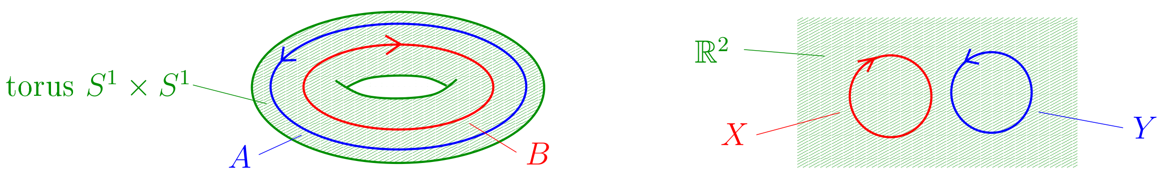


FIGURE 235

Exercise 8.12. Let M be a smooth manifold. Show that M is path-connected if and only if $M \setminus \partial M$ is path-connected.
Hint. Use the Collar Neighborhood Theorem 8.12.

Exercise 8.13. Let M be a compact smooth manifold and let C be a union of components of ∂M. Let $\varphi\colon C \to C$ be a diffeomorphism.

(a) Show that the smooth manifold

$$(M \sqcup ([0,1] \times C))/\sim \qquad \text{where } x \sim \varphi(x,0) \text{ for } x \in C$$

is diffeomorphic to M.
Remark. Recall that the definition of the smooth structure depend on the choice of a collar $[0,1] \times C$.
(b) In fact show that given any neighborhood U of $C \subset M$ there exists a diffeomorphism that is the identity on the common subset $M \setminus U$.

Remark. $[0,1] \times C$ is sometimes called an *external collar.*

Exercise 8.14. For $i = 1, 2$ let M_i be an n-dimensional smooth manifold, let A_i and B_i be two distinct boundary components of M_i and let $f_i\colon A_i \to B_i$ be a diffeomorphism.

Furthermore suppose we are given a diffeomorphism $\Phi\colon M_1 \to M_2$ such that the following diagram commutes:

$$\begin{tikzcd} A_1 \arrow[r, "\Phi|_{A_1}"] \arrow[d, "g_1"'] & A_2 \arrow[d, "g_2"] \\ B_1 \arrow[r, "\Phi|_{B_1}"] & B_2. \end{tikzcd}$$

We define the manifolds $N_i := M_i/\sim$ as in Proposition 8.15.

(a) Show that the "obvious" map $N_1 \to N_2$ given by $[P] \mapsto [\Phi(P)]$ is a homeomorphism.
(b) Give an example where the map from (a) is not a diffeomorphism.
(c) Show that N_1 and N_2 are diffeomorphic.
Hint. Use the Collar Neighborhood Theorem 8.12.

Exercise 8.15. Let M be a compact smooth manifold and let $f\colon M \to M$ be a diffeomorphism such that $f|_{\partial M}\colon \partial M \to \partial M$ is diffeotopic to $\mathrm{id}_{\partial M}$. Show that f is diffeotopic to a map $g\colon M \to M$ such that $g|_{\partial M} = \mathrm{id}$.

Exercise 8.16. Let $n \in \mathbb{N}_0$. On page 199 we showed that $\overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_-$ is homeomorphic to S^n. By Proposition 8.15 we should now view $\overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_-$ as a smooth manifold. Show that this smooth manifold is diffeomorphic to S^n.

Exercise 8.17. Let M and N be connected non-empty n-dimensional smooth manifolds. Suppose that a connected sum $M\#N$ is orientable. Show that M and N are also orientable.

Exercise 8.18. Let $n \in \mathbb{N}_0$ and let $k \in \{1,\dots,n\}$. We write

$$\mathrm{GL}(n,k) \;=\; \underbrace{\text{set of } k\text{-tuples of linearly independent vectors of } \mathbb{R}^n}_{\mathbb{R}^n\times\cdots\times\mathbb{R}^n\,=\,\mathbb{R}^{nk}}.$$

Show that $\mathrm{GL}(n,k)$ is an open subset of $\mathbb{R}^{nk}$.

Exercise 8.19. Let M be a connected n-dimensional smooth manifold. Let $P \in M$. Show that if $n \geq 2$, then $M \setminus \{P\}$ is also connected.

Exercise 8.20. We consider the Möbius band M in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ as shown in Figure 8.20. It should be pretty clear that there exists an isotopy F which turns ∂M into the unknot K. By the Isotopy Extension Theorem 8.27 this isotopy F can be extended to a diffeotopy G of S^3. In particular we obtain a map $G_1\colon S^3 \to S^3$ with $G_1(\partial M) = K$. What does $G_1(M)$ look like? Put differently, we just saw that there exists an embedded Möbius band in S^3 with boundary given by K. What does it look like?

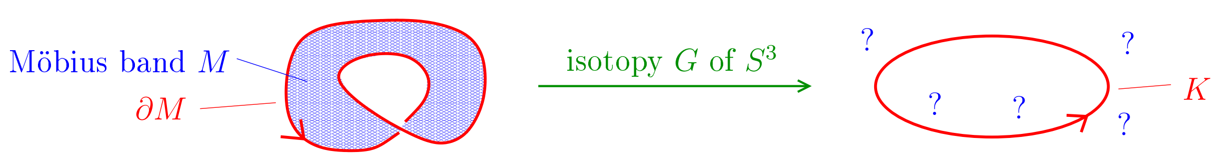


FIGURE 236. Illustration of Exercise 8.20.

Exercise 8.21. Let $g \in \mathbb{N}_{\geq 2}$. Show that the connected sum of g copies of $\mathbb{RP}^2$ is diffeomorphic to the non-orientable surface of genus g that we introduced on page 206.
Remark. We refer to Figure 219 for an illustration.

Exercise 8.22. Does the statement of the Tubular Neighborhood Theorem 8.24 (2) also hold if the smooth manifold M is non-orientable? More precisely, let M be a non-orientable m-dimensional smooth manifold and let N be a 1-dimensional submanifold. Does there exist a smooth embedding

$$F\colon \overline{B}^{m-1} \times N \;\to\; M$$

such that $F(0, P) = P$ for all $P \in N$?

Exercise 8.23. Let $K = \{(x, y, 0) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\}$ be the trivial knot in $\mathbb{R}^3 \cup \{\infty\} = S^3$. Let $X = S^3 \setminus K$ and let $Y = \mathbb{R}^3 \setminus K$. One of the two topological spaces X or Y is homotopy equivalent to S^1. Which one?

Exercise 8.24. We consider the disk $\overline{B}^2$ with two distinct points P and Q. By Proposition 8.29 there exists an orientation-preserving diffeomorphism f of $\overline{B}^2$ that swaps P and Q and that is the identity on S^1. Try to visualize such a map. (See also Figure 237 for a mental aid.)

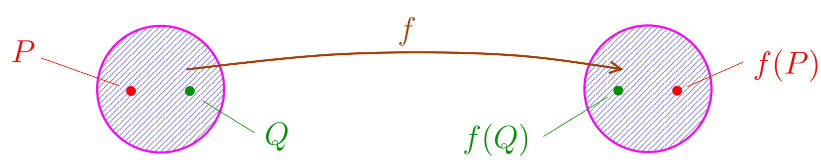


FIGURE 237

Exercise 8.25. Let M and N be two connected oriented n-dimensional smooth manifolds, let A be a union of boundary components of M and let B be a union of boundary components of N and let $\varphi\colon A \to B$ be a diffeomorphism.

(a) Show that if φ is orientation-preserving or orientation-reversing, then $M \sqcup_\varphi N$ admits an orientation such that the map $M \to M \sqcup_\varphi N$ is an orientation-preserving smooth embedding.
(b) Show that if $M \cup_\varphi N$ admits an orientation, then the map $A \to B$ is orientation-preserving or orientation-reversing.

Exercise 8.26. Show that there exists a smooth deformation retraction from $\mathrm{GL}_+(n, \mathbb{R})$ to $\mathrm{SL}(n, \mathbb{R})$.

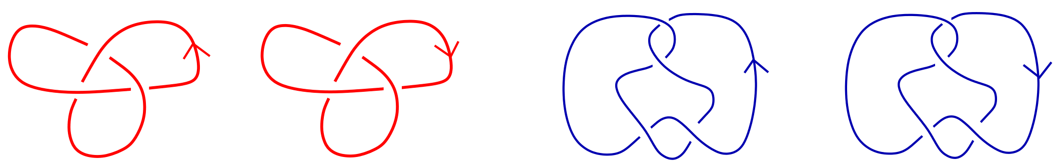
Exercise 8.27. Let M be a connected n-dimensional smooth manifold.

(a) Let P, Q be two distinct points in $M \setminus \partial M$. Show that there exists a smooth embedding $\gamma\colon [0, 1] \to M \setminus \partial M$ with $\gamma(0) = P$ and $\gamma(1) = Q$.
Hint. If P and Q lie in an open n-ball, then it is clear how to prove the statement. Use Theorem 8.36 to show that P and Q do indeed lie in an open n-ball.

(b) Let P, Q be two distinct points in M Show that there exists a smooth embedding $\gamma\colon [0,1] \to M$ with $\gamma(0) = P$ and $\gamma(1) = Q$.
Hint. Use (a) and the Collar Neighborhood Theorem 8.12.

Exercise 8.28. Let $K \subset S^3$ be a knot. Show that the knot complement $S^3 \setminus K$ is path-connected.
Remark. It is not that easy to give a rigorous proof for this statement. You should use a tubular neighborhood of K to get some control over the situation. Furthermore you might want to have a look at Exercise 2.55.

Exercise 8.29. Show that the trefoil and the figure-8 knot are reversible. In other words, show that the two oriented knots shown in Figure 238 to the left, respectively to the right, are smoothly isotopic.



trefoil with the two orientations   figure-8 knot with the two orientations

FIGURE 238. Illustration for Exercise 8.29.

Exercise 8.30. Let M be a closed k-dimensional smooth manifold.

(a) Let $n \in \mathbb{N}_{\geq k+1}$ and let $f\colon M \to S^n = \mathbb{R}^n \cup \{\infty\}$ be an embedding. Show that there exists a smooth isotopy $F\colon M \times [0,1] \to S^n = \mathbb{R}^n \cup \{\infty\}$ with $F_0 = f$ and such that $F_1(M) \subset \mathbb{R}^n$.
Hint. Use Proposition 6.62 (4), Lemma 3.32 (4) and Lemma 8.5 (2).
(b) Let $n \in \mathbb{N}_{\geq k+2}$ and let $f_0, f_1\colon M \to \mathbb{R}^n$ be two embeddings. Suppose there exists a smooth isotopy $F\colon M \times [0,1] \to S^n = \mathbb{R}^n \cup \{\infty\}$ with $F_0 = f_1$ and $F_1 = f_1$. Show that there exists also a smooth isotopy $G\colon M \times [0,1] \to \mathbb{R}^n$ with $G_0 = f_1$ and $G_1 = f_1$.
Remark. The basic idea of the proof is the same as in (a). But this time the argument requires a little more ingenuity.
(c) Show that the dimension restrictions on n in (a) and (b) are necessary.

Remark. The exercise, applied to $M = S^1$ and $n = 3$, shows in particular that the "theory of knots in $\mathbb{R}^3$" is the same as the "theory of knots in S^3".

9. Maps between smooth manifolds

9.1. **Embeddings of smooth (topological) manifolds into $\mathbb{R}^n$ I.** In this section we will show that every topological respectively smooth manifold can be embedded into some $\mathbb{R}^n$. Later, in Chapter 11, we will discuss smooth embeddings of manifolds into $\mathbb{R}^n$ in much greater detail, for example we will study how small n can be chosen. But in the intermediate future the following proposition will be good enough for us.

Proposition 9.1.

(1) *For any smooth manifold M there exists some $n \in \mathbb{N}$ and a smooth embedding $\varphi\colon M \to \mathbb{R}^n$ such that $\varphi(M)$ is a closed subset of $\mathbb{R}^n$.*
(2) *Given any topological manifold M there exists an $n \in \mathbb{N}$ and an embedding $M \to \mathbb{R}^n$ such that $\varphi(M)$ is a closed subset of $\mathbb{R}^n$.*

Proof. We will only provide a proof for the proposition if M is compact. The case of non-compact smooth manifolds is dealt with in [**Wall16**, Corollary 4.7.8]. Furthermore, non-compact topological manifolds are dealt with in [**Mun75**, p. 315]. Note that in both settings the non-compact is significantly harder than the argument we provide below.

(1) First let M be a compact m-dimensional smooth manifold. Since M is compact we can pick a finite smooth atlas $\{\Phi_i\colon U_i \to V_i\}_{i=1,\dots,s}$. By Theorem 6.57 we can find a smooth partition of unity subordinate to the open cover $\{U_i\}_{i=1,\dots,s}$ of M. Put differently, we can find smooth functions $\nu_i\colon M \to [0,1]$, $i = 1,\dots,s$ such that for every $i \in \{1,\dots,s\}$ we have $\operatorname{supp}(\nu_i) \subset U_i$ and such that for every $x \in M$ we have $\nu_1(x) + \dots + \nu_s(x) = 1$. For $i = 1,\dots,s$ we define

$$\begin{array}{rcl} \widetilde{\Phi}_i\colon M & \to & \mathbb{R}^m \\ x & \mapsto & \begin{cases} \Phi_i(x)\cdot\nu_i(x), & \text{if } x \in U_i, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

It follows easily[173] from Lemma 6.23 (1) that each of these maps $\widetilde{\Phi}_i$ is smooth. Finally it remains to prove the following claim.

Claim. The map

$$\begin{array}{rcl} \varphi\colon M & \to & \mathbb{R}^{s\cdot m+s} \\ x & \mapsto & (\widetilde{\Phi}_1(x),\dots,\widetilde{\Phi}_s(x),\nu_1(x),\dots,\nu_s(x)) \end{array}$$

is a smooth embedding.

For completeness' sake we provide the proof of the claim, even though the interest in the full details might be quite muted.

(i) First we show that φ is injective. Let $x, x' \in M$. We pick i with $\nu_i(x) \neq 0$. If $\nu_i(x') = 0$, then evidently $\varphi(x) \neq \varphi(x')$. If $\nu_i(x) \neq 0$, then $x, x' \in U_i$. Note that $\Phi_i(x) \neq \Phi_i(x')$, which is equivalent to $\nu_i(x)^{-1}\cdot\widetilde{\Phi}_i(x) \neq \nu_i(x')^{-1}\cdot\widetilde{\Phi}_i(x')$. But this means that $(\widetilde{\Phi}_i(x),\nu_i(x)) \neq (\widetilde{\Phi}_i(x'),\nu_i(x'))$, which implies that $\varphi(x) \neq \varphi(x')$.

[173] Hmm, how does the smoothness of $\widetilde{\Phi}_i$ follow from Lemma 6.23 (1)?

(ii) It follows from (i), the hypothesis that M is compact and Proposition 2.43 (2) that the map φ is an embedding.

(iii) We need to show that for each $x \in M$ the differential $\mathrm{D}\,\varphi_x\colon \mathrm{T}_x\,M \to \mathbb{R}^{s\cdot m+s}$ is a monomorphism. We pick i with $\nu_i(x) \neq 0$. We consider the map

$$\begin{array}{ccccc} U_i & \to & \{(v,t) \in \mathbb{R}^m \times \mathbb{R} \,|\, t \neq 0\} & \to & \mathbb{R}^m \\ y & \mapsto & (\widetilde{\Phi}_i(y), \nu_i(y)) & & \\ & & (v,t) & \mapsto & \frac{1}{t}\cdot v. \end{array}$$

Note that the composition is just the diffeomorphism Φ_i, thus the differential at x is a monomorphism. But this implies that the differential of the first map at x is a monomorphism. Finally this note that this implies that the differential of φ at x is a monomorphism. ⊞

(2) The proof for compact topological manifolds is basically the same as in (a). The only difference is that from Theorem 6.57 we only obtain a continuous partition of unity subordinate to the open cover $\{U_i\}_{i=1,\ldots,s}$ of M. In particular now the map $\varphi\colon M \to \mathbb{R}^{s\cdot m+s}$ that we constructed in (a) is only an embedding. ■

We conclude this short section with an amusing corollary.

Definition. A topological space X is called *metrizable* if there exists a metric d on X such that the given topology agrees with the topology coming from d.

Now we can state the promised corollary.

Corollary 9.2. *Every topological manifold M is* metrizable.

Proof. Let M be a topological manifold. By Proposition 9.1 (2) there exists an $n \in \mathbb{N}$ and a map $\varphi\colon M \to \mathbb{R}^n$ which is an embedding. In other words, M is homeomorphic to a subset of $\mathbb{R}^n$. This means that the usual Euclidean metric d defines a metric on $\varphi(M)$ such that the topology coming from d agrees with the topology given by $\varphi(M)$. ■

9.2. **The Whitney Approximation Theorem.** As we will see later on, there are many ways to construct interesting continuous maps between two smooth manifolds. The following theorem allows us to “replace” such a continuous map by a smooth map.

Theorem 9.3. (Whitney Approximation Theorem) *Let M be smooth manifold (with or without boundary) and let N be a smooth manifold (with or without boundary).*

(1) *Let $f\colon M \to N$ be a map that is smooth on a (possibly empty) closed subset A.*[174] *There exists a homotopy $F\colon M \times [0,1] \to N$ with the following properties:*
   (a) *The homotopy is a homotopy rel A.*
   (b) *We have $F_0 = f$.*
   (c) *The map $F_1\colon M \to N$ is smooth.*

   Suppose that we are given a continuous function $\rho\colon M \to \mathbb{R}_{>0}$ and suppose that we are given a metric d on N. Then we can find a homotopy F as above that also satisfies the following extra condition:
   (d) *For every $(x,t) \in M \times [0,1]$ we have $d(F(x,t), f(x)) \leq \rho(x)$.*

(2) *Let $f, g\colon M \to N$ be two smooth maps that agree on a (possibly empty) closed subset A. If f and g are homotopic rel A, then there exists also a smooth homotopy between f and g that is constant on A.*

Remark. Suppose we are in the setting of the Whitney Approximation Theorem 9.3. Let $f\colon M \to N$ be a map that is injective (respectively surjective, bijective). Now we know that we can find a homotopy from f to some smooth map $F_1 = g\colon M \to N$, but the theorem does not give us any control over whether the smooth map $F_1 = g$ is also injective (respectively surjective, bijective).

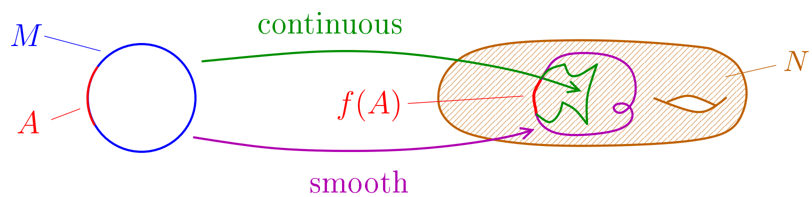


FIGURE 239. Illustration of Theorem 9.3 (1).

PROOF.

(1) The first statement, without Condition (d), is the content of [**Lee02**, Theorem 6.26] and [**Lee02**, Theorem 9.27]. The full statement including Condition (d) follows by almost the same argument as in the proof of [**Lee02**, Theorem 6.26] and [**Lee02**, Theorem 9.27], one just needs to make full use of [**Lee02**, Theorem 6.21]. We refer to [**MTo97**, Lemma A.9] for a closely related argument. A weaker version of (1) is also proved in [**Hirs76**, Lemma 5.1.5] and in [**DFN85**, Theorem 10.1.2].
(2) The proof of Statement (2) can reduced to Statement (1) without too many problems. We refer to [**Lee02**, Theorem 6.29] or alternatively to [**DFN85**, Theorem 12.1.4] for details. ■

We continue with the following pleasant corollary to the Whitney Approximation Theorem 9.3.

Corollary 9.4. *Let $n \in \mathbb{N}$. We pick a base point $*$ on S^n. Furthermore let M be a smooth manifold without boundary. We pick a base point x_0. The obvious map*

$$\{\textit{all smooth maps } f\colon (S^n, *) \to (M, x_0) \textit{ up to smooth homotopies}\} \;\to\; \pi_n(M, x_0)$$

is an isomorphism.

Remark. In Exercise 9.1 we will use this isomorphism, together with Proposition 6.62 to give a quick proof of Proposition 14.14, i.e. we will reprove that for $n \geq 2$ we have $\pi_1(S^n) = 0$.

PROOF. Note that it follows immediately from the definition on page 278 that the restriction of any map $S^n \to M$ to any one-point subset $\{*\}$ is smooth. Thus the corollary follows immediately from the Whitney Approximation Theorem 9.3, applied to $A = \{*\}$. ■

[174] Let M and N be smooth manifolds and let $A \subset M$ be an arbitrary subset. On page 278 we defined what it means for a map $f\colon A \to N$ to be smooth.

We conclude this section with two and a half more technical consequences of the Whitney Approximation Theorem 9.3. These results are interesting in their own right, but the reader might want to skip them initially.

Proposition 9.5. ($*$) *Let M be smooth manifold (with or without boundary) and let N be a smooth manifold without boundary. Furthermore let $f\colon M \to N$ be a map. Finally let A and X be disjoint closed subsets such that the restriction of f to A is smooth and let $U \subset N$ be an open subset that contains $f(M \setminus X)$. If $X \neq \varnothing$, then there exists a homotopy $F\colon M \times [0,1] \to N$ with the following properties:*

1. *The homotopy is a homotopy rel $X \cup A$.*
2. *We have $F_0 = f$.*
3. *The restriction of F_1 to the open subset $M \setminus X$ is smooth.*[175]
4. *For each $t \in [0,1]$ we have $F_t(M \setminus X) \subset U$.*

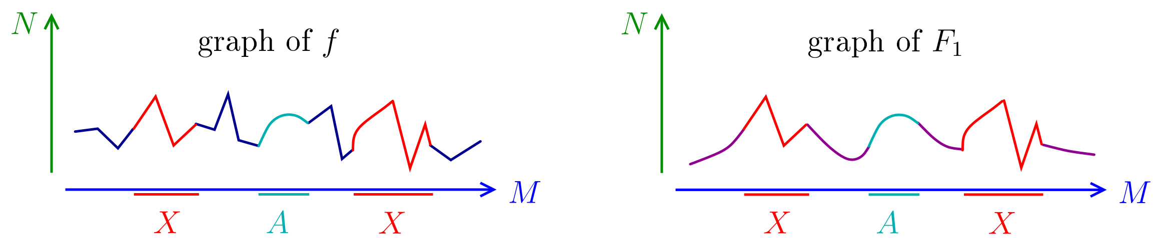


FIGURE 240. Illustration of Proposition 9.5.

PROOF. First we consider the case that $U = N$. By Corollary 9.2 we can pick a metric c on M such that the topology from the metric c agrees with the given topology on M. The same way we can pick a metric d on N. We consider the open subset $W := M \setminus X$ and we view it as a smooth manifold. Since $X \neq \varnothing$ we can consider the function

$$\begin{array}{rcl} \rho\colon W & \to & \mathbb{R}_{\geq 0} \\ w & \mapsto & \rho(w) := c(w, X) = \inf\{c(w,x) \,|\, x \in X\}. \end{array}$$

Since X is closed and since the topology defined by c agrees with the topology on M we see that $\rho(w) > 0$ for every $w \in W$. By Lemma 3.20 we know that $\rho\colon M \to \mathbb{R}_{\geq 0}$ is continuous. Since N is a smooth manifold without boundary we can apply the Whitney Approximation Theorem 9.3 to the map $f\colon W \to N$ and we obtain that there exists a homotopy $G\colon W \times [0,1] \to N$ with the following properties:

(a) The homotopy is a homotopy rel A.
(b) We have $G_0 = f|_W$.
(c) The map $G_1\colon W \to N$ is smooth.
(d) For every $(w,t) \in W \times [0,1]$ we have $d(G(w,t), f(w)) \leq \rho(w)$.

[175]Since X is closed we see that $M \setminus X$ is an open subset of M, thus we can view $M \setminus X$ as a smooth manifold in its own right.

It follows easily from (d) and the definition of ρ that the map

$$\begin{array}{rcl} F\colon M\times[0,1] & \to & N \\ (x,t) & \mapsto & \begin{cases} G(x,t), & \text{if } x\in W, \\ f(x), & \text{if } x\in X \end{cases} \end{array}$$

is continuous. Now it is straightforward to verify that this map has all the desired properties.

Finally we consider the case that $U\neq N$. In this case we consider the map

$$\begin{array}{rcl} \rho\colon W & \to & \mathbb{R}_{\geq 0} \\ w & \mapsto & \rho(w):=\min\{c(w,X),d(f(w),N\setminus U)\}. \end{array}$$

Since U is open and since $f(W)\subset U$ we see that ρ is again non-zero every. The rest of the argument is basically the same and it follows from (d) that for each $t\in[0,1]$ we have $F_t(M\setminus X)\subset U$. ■

Loosely speaking the following proposition says that a map $M\to X$ from a smooth manifold to a topological space X can be made smooth with respect to open balls B^n contained in X.

Proposition 9.6. *Let M be a smooth manifold and let $f\colon M\to X$ be a map to a Hausdorff space X. Let $\varphi\colon \overline{B}^n\to X$ be an injective map such that $\varphi(B^n)$ is an open subset of X. Given any $r\in(0,1)$ there exists a map $g\colon M\to X$ with the following properties:*

(1) *g is homotopic to f rel the complement of $f^{-1}(\varphi(B^n_r))$ in M,*

(2) *the map*

$$\begin{array}{rcl} f^{-1}(\varphi(B^n_r)) & \to & B^n_r \\ x & \mapsto & \varphi^{-1}(g(x)) \end{array}$$

is smooth.

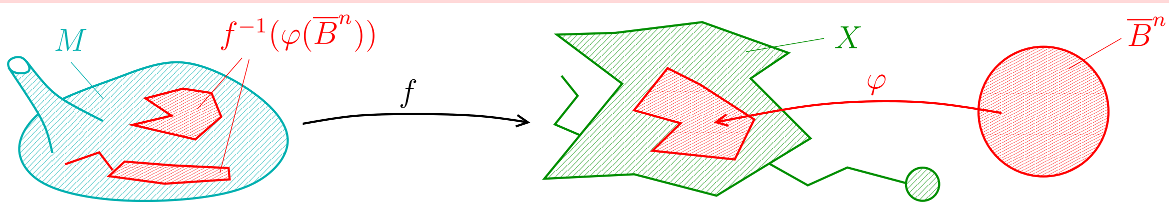


FIGURE 241

PROOF. Since $\overline{B}^n$ is compact, since X is Hausdorff and since $\varphi\colon\overline{B}^n\to X$ is injective we know by Proposition 2.43 (2) that φ is a closed embedding. Thus we might as well identify $\overline{B}^n$ with its image under the map $\varphi\colon\overline{B}^n\to X$. Now let $r\in\mathbb{R}_{>0}$. By Lemma 2.38 we know that B^n_r is also an open subset of X. Furthermore, note that it follows from the fact that X is Hausdorff, together with Lemma 2.17 (1), that $\overline{B}^n_r$ and $\overline{B}^n$ are a closed subsets of X. We consider the map

$$\begin{array}{rcl} \alpha\colon M & \to & \overline{B}^n/S^{n-1} \\ x & \mapsto & \begin{cases} f(x), & \text{if } x\in f^{-1}(B^n), \\ [S^{n-1}], & \text{if } x\in M\setminus f^{-1}(B^n). \end{cases} \end{array}$$

Note that α is continuous.[176] As discussed on page 182 we can view $\overline{B}^n/S^{n-1}$ as a smooth manifold such that B^n is a submanifold. Therefore we can apply Proposition 9.5 to the map $\alpha\colon M \to \overline{B}^n/S^{n-1}$ with $A = \emptyset$, with the closed subset $X = M \setminus f^{-1}(B^n_r)$[177] and with the open subset $U = B^n_r \subset N$. We obtain a homotopy $F\colon M \times [0,1] \to \overline{B}^n/S^{n-1}$ with the following properties:

(1) The homotopy is a homotopy rel $X = M \setminus f^{-1}(B^n_r)$.
(2) We have $F_0 = \alpha$.
(3) The restriction of F_1 to $M \setminus X = f^{-1}(B^n_r)$ is smooth.
(4) For each $t \in [0,1]$ we have $F_t(f^{-1}(B^n_r)) \subset B^n_r$.

Finally we consider the map

$$\begin{array}{rcl} G\colon M \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} F(x,t), & \text{if } x \in f^{-1}(B^n_r) \\ f(x), & \text{if } x \in M \setminus f^{-1}(B^n). \end{cases} \end{array}$$

Note that the restrictions of G to the closed subsets $(M \setminus f^{-1}(B^n_r)) \times [0,1]$ and $f^{-1}(\overline{B}^n_r) \times [0,1]$ are continuous.[178] Thus we obtain from Lemma 3.10 that G is continuous. It is now straightforward to verify that $g := G_1$ has all the desired properties. ■

The following proposition is a slight variation on Proposition 9.6.

Proposition 9.7. *Let M be a smooth manifold and let $f\colon M \to X$ be a map to a Hausdorff space X. Let $n \in \mathbb{N}$ and let $* \in S^n$ be a point. Finally let $\varphi\colon S^n \to X$ be an injective map such that $\varphi(S^n \setminus \{*\})$ is an open subset of X. There exists a map $g\colon M \to X$ with the following properties:*

(1) g is homotopic to f rel the complement of $f^{-1}(\varphi(S^n \setminus \{*\}))$ in M,
(2) the map

$$\begin{array}{rcl} f^{-1}(\varphi(S^n \setminus \{*\})) & \to & S^n \setminus \{*\} \\ x & \mapsto & \varphi^{-1}(g(x)) \end{array}$$

is smooth.

Proof. We leave it to the reader to modify the proof of Proposition 9.6 to obtain the desired result. As the reader will notice, in this case there is no need to work with some $r \in (0,1)$. In particular the conclusion we get is slightly more satisfactory. ■

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[176] Indeed, the restrictions of α to the closed subset $M \setminus f^{-1}(B^n)$ is a constant map and the restriction of α to the closed subset $f^{-1}(\overline{B}^n)$ is given by the composition of the map f with the projection map $\overline{B}^n \to \overline{B}^n/S^{n-1}$. Therefore we obtain from Lemma 2.35 (2) that α is indeed continuous.

[177] Note the discrete subscript "r".

[178] For nervous people, like the author, let us verify in slightly greater detail that the restriction of G to $f^{-1}(\overline{B}^n_r) \times [0,1]$ is indeed continuous. First note that by (4) the restriction of the map F to $f^{-1}(B^n_r) \times [0,1]$ takes values in B^n_r. It follows from Exercise 2.11 (3) and Exercise 3.9 that the restriction of the map F to $f^{-1}(\overline{B}^n_r) \times [0,1]$ takes values in the closure of B^n_r in $\overline{B}^n/S^{n-1}$. Since $r \in (0,1)$ we see that the closure equals $\overline{B}^n_r \subset B^n$. Now we see that the restriction of the map G to $f^{-1}(\overline{B}^n_r) \times [0,1]$ equals the composition of the map $F\colon f^{-1}(\overline{B}^n_r) \times [0,1] \to \overline{B}^n_r$ with the inclusion $\overline{B}^n_r \to X$.

Example. Let $\bigvee_{i=1}^{k} S^{n_i}$ be a wedge of spheres. We denote by $*$ the wedge point. Let M be a smooth manifold and let $f\colon M \to \bigvee_{i=1}^{k} S^{n_i}$ be a map. By iteratively applying Proposition 9.7 altogether k times we see that f is homotopic rel $f^{-1}(\{*\})$ to a map that is smooth on the open sets $f^{-1}(S^{n_i} \setminus \{*\})$, $i = 1, \dots, k$.

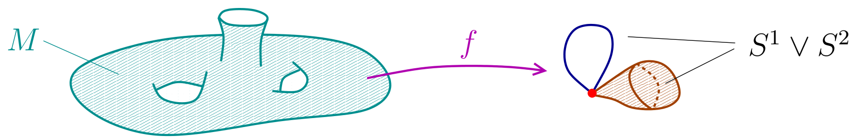


FIGURE 242

Lemma 9.8. *Let M be an orientable n-dimensional smooth manifold, let $f\colon M \to X$ be a map to some Hausdorff space X, let $\varphi\colon B^k \to X$ be a map and let $x \in \varphi(B^k)$ be a point such that the following conditions are satisfied:*

1. *the image $\varphi(B^k)$ is an open subset of X,*
2. *the restriction of f to $f^{-1}(\varphi(B^k))$ is a smooth map,*
3. *the point x defines a closed subset of X.*[179]

Then $f^{-1}(\{x\})$ is a proper $(n-k)$-dimensional submanifold of M.

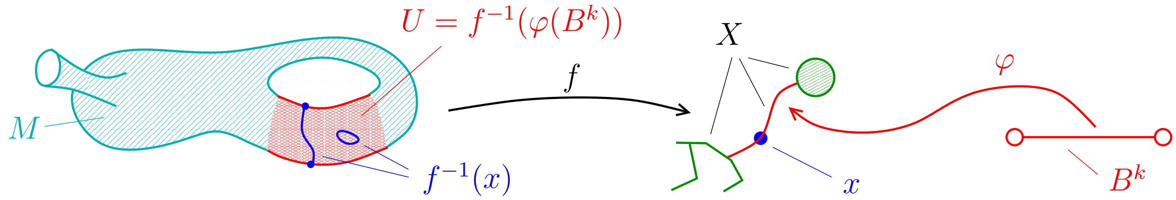


FIGURE 243. Illustration of Lemma 9.8.

PROOF ($*$). Note that $U := f^{-1}(\varphi(B^k))$ is an open subset of M, in particular, as discussed in Lemma 6.22, it is a smooth manifold in its own right. It follows from this observation and (3) that we can apply the Regular Value Theorem 6.53. We obtain that $f^{-1}(\{x\})$ is a proper orientable $(n-k)$-dimensional submanifold of U. In particular $f^{-1}(\{x\})$ admits submanifold charts of type (α) and (β), as defined on page 270, with respect to the smooth manifold U. Since U is an open subset of M we see that these submanifold charts are also submanifold charts of the same type for the set $f^{-1}(\{x\})$, viewed as a subset of M.

Since X is by hypothesis Hausdorff we know, by Lemma 2.13 that $\{x\}$ is a closed subset of X. Finally, since f is of course understood to be continuous on all of M we obtain that $f^{-1}(\{x\})$ is a closed subset of M. In other words, we have shown that $f^{-1}(\{x\})$ satisfies all the conditions, as set out on page 270, of being a proper submanifold of M. ■

9.3. **The Transversality Theorem.** The following definition is a harmless generalization of the definition from page 306.

[179] Wait, does this perhaps follow from our other hypotheses?

Definition. Let M be a smooth manifold and let Y be a proper submanifold of M. Let X be some other smooth manifold and let $f: X \to M$ be a smooth map. We say f *intersects* Y *transversally* if for every $P \in X$ with $f(P) \in Y$ the following two conditions are satisfied:

(1) We have $P \notin \partial X$ and $f(P) \notin \partial Y$.
(2) We have the following equality[180]

$$f_*(\mathrm{T}_P X) + \mathrm{T}_{f(P)} Y \;=\; \mathrm{T}_{f(P)} M.$$

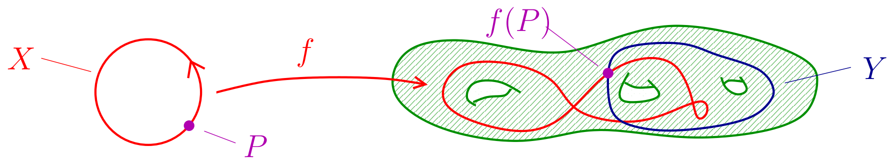


FIGURE 244

Examples.

(1) Let M be a smooth manifold and let X and Y be two proper submanifolds of M. We denote by $i: X \to M$ the inclusion map. It follows immediately from the definitions that i intersects Y transversally if and only if X intersects Y transversally in the sense of the definition on page 306.
(2) Let X be a smooth manifold and let $f: X \to \mathbb{R}^n$ be a smooth map. Furthermore let $y \in \mathbb{R}^n$. By definition the map f is transverse to the 0-dimensional submanifold $\{y\}$ if and only if y is a regular value of f.

The following lemma might ring a bell. Considering the above examples we see that it is quite close to the statement of Lemma 9.9 and to the statement of the Regular Value Theorem 6.53.

Lemma 9.9. *Let M be a smooth manifold and let Y be a proper submanifold of M of codimension l. Let X be some other smooth manifold and let $f: X \to M$ be a smooth map. If f intersects Y transversely, then the following statements hold:*

(1) *The preimage $f^{-1}(Y)$ is a proper submanifold of X of codimension l.*
(2) *If M, X and Y are orientable, then $f^{-1}(Y)$ is orientable.*
(3) *If X is compact, then so is $f^{-1}(Y)$.*

PROOF. The proof of this lemma is not overly hard. For space reasons we skip the proof and instead we refer to [**Lee02**, Theorem 6.30] for a proof of (1). We leave the task of verifying the remaining statements to the reader. ∎

The following definition is a variation on the definition that we gave on page 348.

Definition. Let M be a smooth manifold.

(1) Let W, W' be two proper submanifolds and let Z be (possibly empty) subset of W. We say that W and W' are (*smoothly*) *isotopic rel* Z, if there exists a (*smooth*)

[180]Since Y is a submanifold of M we can view the tangent space $\mathrm{T}_P Y$ as a subspace of $\mathrm{T}_P M$.

isotopy from W to W' rel Z, i.e. a map

$$\begin{array}{rcl} H\colon W\times[0,1] & \to & M \\ (z,t) & \mapsto & H(z,t) \end{array}$$

such that the following hold:

(a) for each $P \in W$ we have $H(P,0) = P$,

(b) for each $P \in Z$ and any $t \in [0,1]$ we have $H(P,t) = P$,

(c) the map

$$\begin{array}{rcl} W & \to & M \\ z & \mapsto & H(z,1) \end{array}$$

defines a homeomorphism (diffeomorphism) from W to W',

(d) for each $t \in [0,1]$ the map $W \to M$ given by $P \mapsto H(P,t)$ is a proper embedding (is a proper smooth embedding).

If W and W' are two oriented proper submanifolds, then we also want the following condition to be satisfied:

(e) the diffeomorphism $W \to W'$ in (c) is orientation-preserving.

(2) Let X and Y be two submanifolds of M. We say *X and Y can be made transverse* (*respectively disjoint*) if there exist transverse (respectively disjoint) submanifolds X' and Y' of M such that X' is smoothly isotopic to X and Y' is smoothly isotopic to Y.

The next theorem says in particular that we can make any two proper submanifolds of a given smooth manifold transverse.

Theorem 9.10. (Transversality Theorem) *Let M be a smooth manifold and let Y be a proper submanifold of M.*[181]

(1) *If X is a smooth manifold and if $f\colon X \to M$ is a smooth map, then there exists a smooth homotopy from f to a smooth map that is transverse to Y.*

(2) *Let X be a proper submanifold of M. Given any neighborhood U of X there exists a smooth isotopy $F\colon X \times [0,1] \to U$, such that $F(P,t) \in \partial M$ for all $P \in \partial X$ and $t \in [0,1]$, from X to a proper submanifold X' that is transverse to Y.*

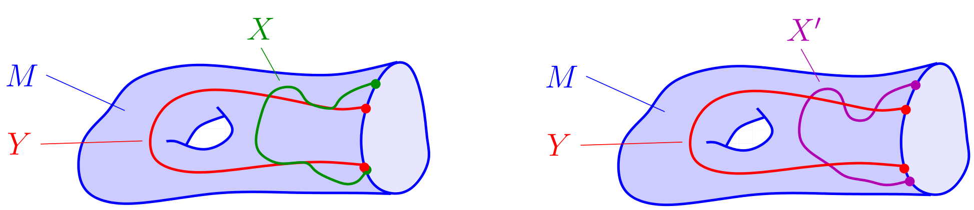


FIGURE 245. Illustration of Theorem 9.10 (2).

PROOF.

(1) The first part of this statement is proved in [**Lee02**, Theorem 6.36]. A slightly less general statement is proved in [**GP74**, p. 70].

[181]Note that the two statements are similar, but at least a priori they are independent in the sense that neither implies the other statement.

(2) The theorem follows from Proposition 4.4.4, Theorem 4.5.6 and Proposition 4.5.7 of [**Wall16**].[182] ∎

Example. Let M be a smooth manifold. Furthermore let X and Y be two proper submanifolds. If $\dim(X)+\dim(Y)<\dim(M)$, then one deduces easily from the definitions that X and Y intersect transversally if and only if X and Y are disjoint. With this observation the Transversality Theorem 9.10 (2) says in particular that any two proper 1-dimensional submanifolds X and Y of the 3-dimensional ball $\overline{B}^3$ can be made disjoint after an isotopy rel boundary. This statement is illustrated in Figure 246.

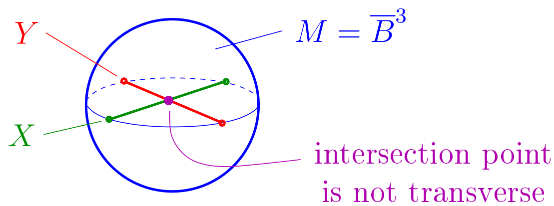


FIGURE 246

9.4. **The Self-Transversality Theorem.** We conclude this section with a short discussion of self-transverse maps.

Definition. Let M and X be smooth manifolds and let $\varphi\colon (X,\partial X)\to(M,\partial M)$ be a proper smooth map. We say φ *is self-transverse* if for every two points $P,Q\in X$ with $\varphi(P)=\varphi(Q)$ we have

$$\varphi_*(\mathrm{T}_P\,X)+\varphi_*(\mathrm{T}_Q\,Y) \;=\; \mathrm{T}_{\varphi(P)}\,M$$

and if for every two points $P,Q\in\partial X$ with $\varphi(P)=\varphi(Q)$ we have

$$\varphi_*(\mathrm{T}_P\,X)+\varphi_*(\mathrm{T}_Q\,Y) \;=\; \mathrm{T}_{\varphi(P)}(\partial M).$$

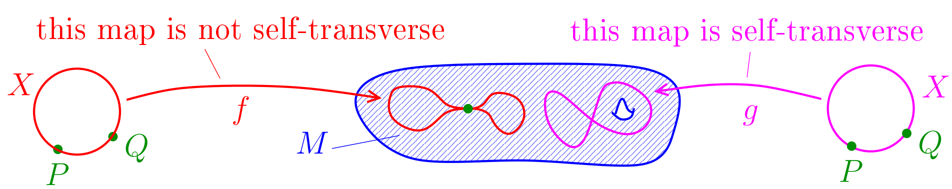


FIGURE 247

The proof of the following lemma is similar to the proof of Lemma 6.52. We leave it to the reader to make the necessary adjustments of the argument.

Lemma 9.11. *Let M be an n-dimensional smooth manifold and let a X be an m-dimensional smooth manifold. Furthermore let $\varphi\colon (X,\partial X)\to(M,\partial M)$ be a proper smooth map. If φ is a self-transverse immersion, then the set*

$$\{P\in X\,|\,\text{there exists a } Q\neq P \text{ with } \varphi(P)=\varphi(Q)\}$$

is a proper submanifold of M of dimension $2m-n$.[183]

[182] A first look at [**Wall16**, Theorem 4.5.6] might be slightly confusing since it demands that Y is a "submanifold of $J^r(M,N)$" for some $r\in\mathbb{N}_0$. But as is explained on page 101 of [**Wall16**], we have $J^0(M,X)=M\times X$, and $Y\times\{*\}$ is clearly a submanifold.

In the following we will discuss to what degree we can “upgrade” a map between smooth manifolds. First we see that if the codimension is large enough, then we can turn a given map into an immersion via a homotopy.

Theorem 9.12. *Let M be a compact smooth manifold and let N be a smooth manifold. If $\dim(M) \geq 2 \cdot \dim(N)$, then any map $f\colon M \to N$ is homotopic to an immersion.*

Proof. This theorem follows from [**GG73**, Theorem II.5.6] and the argument in [**Wall16**, Proof of Proposition 4.4.4]. ■

Remark. Note that the conclusion of Theorem 9.12 does not hold without a condition on the dimensions. For example the map $S^1 \to \mathbb{R}$ that is given by $(x, y) \mapsto x$ is not homotopic to an immersion. A more subtle example of such a phenomenon is given in Figure 254.

To formulate the last theorem we need to introduce a self-explanatory definition.

Definition. Let $f\colon X \to Y$ be a map between two sets. We say f is a map *without triple points*, if there are no pairwise distinct points $P, Q, R \in X$ with $f(P) = f(Q) = f(R)$.

The following is a variation on the Transversality Theorem 9.10 (2). It says, that if we are already given an immersion, then we can actually make is self-transverse.

Theorem 9.13. (Self-Transversality Theorem) *Let M be a smooth manifold, let X be a smooth manifold, let $\varphi\colon (X, \partial X) \to (M, \partial M)$ be an* immersion *and let U be a neighborhood of $\varphi(X)$.*

1. *There exists a smooth isotopy $F\colon X \times [0, 1] \to U$ from φ to a self-transverse immersion.*[184]
2. *If M is compact and if $\dim(M) = 2 \cdot \dim(X)$, then there exists a smooth isotopy $F\colon X \times [0, 1] \to U$ from φ to a self-transverse immersion without triple points.*

Proof.

1. This statement is a straightforward consequence of [**Wall16**, Proposition 4.6.6].
2. With a little bit of an effort one can show that this statement follows from (1) together with [**Adac93**, Theorem II.6.2.8]. ■

Remark. Let M and X be closed smooth manifolds and let $\varphi\colon X \to M$ be a continuous map. The combination of Theorem 9.12 together with the Self-Transversality Theorem 9.13 gives us the following two statements:

1. If $\dim(M) > 2 \cdot \dim(X)$, then φ is homotopic to a smooth embedding.
2. If $\dim(M) = 2 \cdot \dim(X)$, then φ is homotopic to a self-transverse immersion without triple points. We refer to Figure 248 for an illustration.

We discuss these statements in greater detail in Theorem 9.15.

The following is an often used consequence of the above transversality theorems.

[183] In other words, the codimensions add up, i.e. the codimension of X is $n - m$, the codimension of the set of double points if $2(n - m)$, i.e. the dimension is $n - 2(n - m) = 2m - n$.

[184] To avoid confusion, note that we start out with the hypothesis that we are given an immersion. If we have do not know that φ is an immersion, then we first need to apply Theorem 9.12, which requires that $\dim(M) \geq 2 \cdot \dim(X)$.

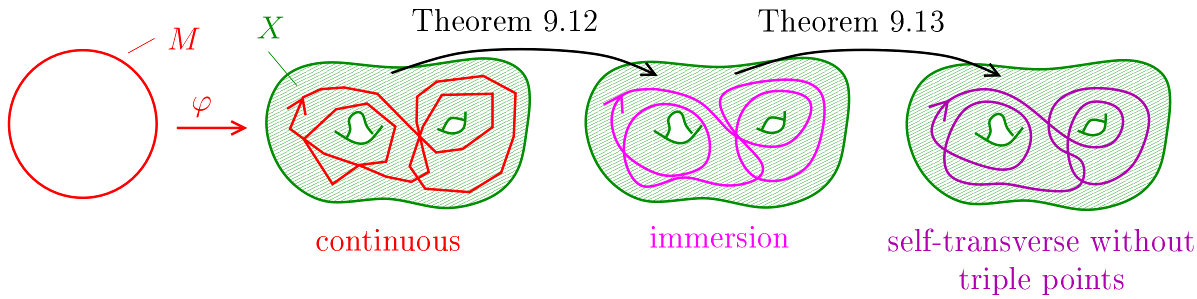


FIGURE 248

Corollary 9.14. *Let M be a connected n-dimensional smooth manifold and let Y be a proper k-dimensional smooth submanifold of M. If $k \leq n-2$, then $M \setminus Y$ is path-connected.*

PROOF. We leave the basically trivial case $n = 2$ to the reader. Furthermore, to simplify the discussion we assume that $\partial M = \varnothing$. We leave it to the reader to modify the argument below to deal with the case that $\partial M \neq \varnothing$.

Thus in the following we assume that $n \geq 3$ and that $\partial M = \varnothing$. Let $P, Q \in M \setminus Y$. Since Y is proper we know that Y is in particular a closed subset of M. This implies that we can find smooth embeddings $\Phi\colon \overline{B}^n \to M \setminus Y$ and $\Psi\colon \overline{B}^n \to M \setminus Y$ with $\Phi(0) = P$, $\Psi(0) = Q$ and such that the images are disjoint. Now we consider $N := M \setminus (\Phi(B^n) \cup \Psi(B^n))$. Note that it follows from Proposition 8.2 that N is a connected n-dimensional smooth manifold.

We pick points $P' \in \Phi(S^{n-1}) \subset \partial N$ and $Q' \in \Psi(S^{n-1}) \subset \partial N$. Since N is connected we know by Lemma 6.9 that N is also path-connected. This means that there exists a path $\gamma\colon [0,1] \to N$ with $\gamma(0) = P'$ and $\gamma(1) = Q'$. By Exercise 18.7 we can arrange that γ is actually an immersion. [185] Furthermore, since $n \geq 3$ we obtain from the Self-Transversality Theorem 9.13 that we can arrange that γ is actually an embedding. By the Transversality Theorem 9.10 (2) we know that γ is homotopic to a map $\widetilde{\gamma}\colon [0,1] \to N$ which is transverse to Y. Since $k \leq n-2$ this means that $\widetilde{\gamma}([0,1])$ is actually disjoint from Y. Now pick a path in $\Phi(\overline{B}^n)$ from P to $\widetilde{\gamma}(0)$, concatenate it with $\widetilde{\gamma}$ and then concatenate it with a path in $\Psi(\overline{B}^n)$ from $\widetilde{\gamma}(1)$ to Q. This construction gives us the desired path in $M \setminus Y$ from P to Q. ■

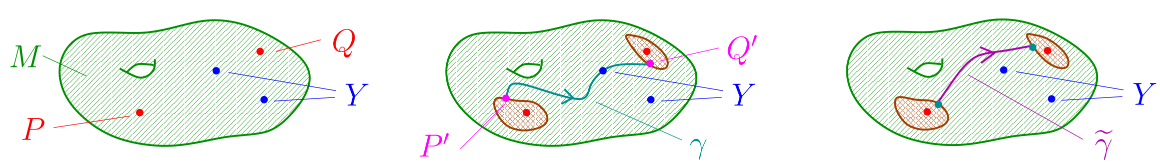


FIGURE 249. Illustration for the proof of Corollary 9.14.

[185] It would be tempting to use the Whitney Approximation Theorem 9.3 together with Theorem 9.12 to obtain this statement, but unfortunately for the latter statement we do not have a relative statement which allows us to keep endpoints fixed.

9.5. **Homotopies to proper immersions and embeddings.** We start out with the following definition that is a slight generalization of the concept introduced on page 410.

Definition. Let M be a compact topological manifold, let A be a submanifold of M and let X be any topological manifold. A map $\varphi\colon X \to M$ is called *proper rel A* if $\varphi^{-1}(\partial M \cup A) = \partial X$.

On many occasions the following theorem allows us to "upgrade" a map between smooth manifolds to a map with nicer properties.

Theorem 9.15. *Let M be a smooth manifold, let $A \subset M \setminus \partial M$ be a (possibly empty) submanifold of M and let X be a compact smooth manifold. Furthermore let $f\colon X \to M$ be a map with $f(\partial X) \subset \partial M \cup A$.*

(1) *The map $f\colon (X, \partial X) \to (M, \partial M \cup A)$ is homotopic*[186]*, to a smooth map $\widetilde{f}\colon X \to M$ that is proper rel A.*[187]
(2) *If $\dim(M) \geq 2 \cdot \dim(X)$, then the map $f\colon (X, \partial X) \to (M, \partial M \cup A)$ is homotopic to an immersion $\widetilde{f}\colon X \to M$ that is proper rel A and which has furthermore the property that there exist points $P_1, \dots, P_k$ in $M \setminus \partial M$ with the following properties:*
   (a) *for each i the preimage $\widetilde{f}^{-1}(P_i)$ consists of precisely two points $Q_i^{\pm}$ which furthermore satisfy that $f_*(\mathrm{T}_{Q_i^+} X) + f_*(\mathrm{T}_{Q_i^-} X) = \mathrm{T}_{P_i}(M)$,*
   (b) *the restriction of $\widetilde{f}$ to $X \setminus \{Q_1^{\pm}, \dots, Q_k^{\pm}\}$ is injective.*
(3) *If $\dim(M) \geq 2 \cdot \dim(X) + 1$, then the map $f\colon (X, \partial X) \to (M, \partial M \cup A)$ is homotopic to a smooth embedding $\widetilde{f}\colon X \to M$ that is proper rel A.*

Furthermore, if f is already smooth on ∂X or if f is already a smooth embedding on ∂X, then in (1) and (3) we can find a homotopy rel ∂X.

Examples.

(1) In Figure 250 we consider a 2-dimensional smooth manifold M together with a map $f\colon [-1, 1] \to M$. We first see that f is homotopic to a smooth map and then we see that it is homotopic to a smooth map without triple points and such that all double points are "transverse".

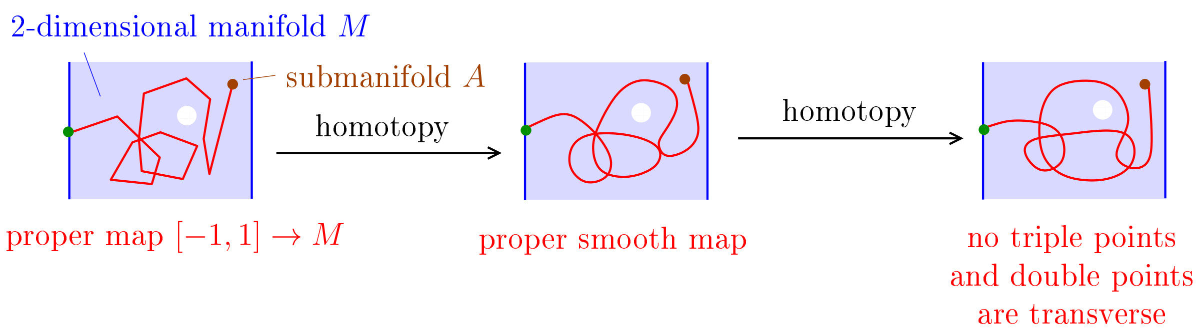


Figure 250. Illustration of Theorem 9.15 (2).

[186]Here and in the following it is always understood that these are homotopies of maps between pairs of topological spaces. We refer to page 571 for the definition of this notion.
[187]This statement is very close to the statement of the Whitney Approximation Theorem 9.3.

(2) In Figure 251 we consider $M = [-1,1] \times \mathbb{R}^2$ and a map $f\colon [-1,1] \to M$. We first see that f is homotopic to a smooth map and then we see that it is homotopic to a smooth embedding.

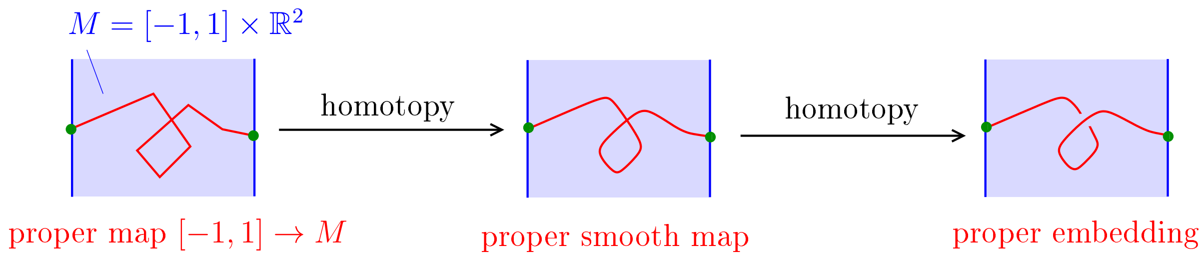


FIGURE 251. Illustration of Theorem 9.15 (3).

(3) Note that every smooth map $S^1 \to \mathbb{R}$ necessarily has a maximum and minimum, which implies that such a map cannot be an immersion. This shows that in general we cannot drop the hypothesis that $\dim(M) \geq 2 \cdot \dim(X)$ in Theorem 9.15.
(4) In Figure 252 we show a map f from S^1 to some 2-dimensional smooth manifold M which is "clearly" not homotopic to an embedding. This shows that in general we cannot drop the hypothesis that $\dim(M) \geq 2 \cdot \dim(X) + 1$ in Theorem 9.15.

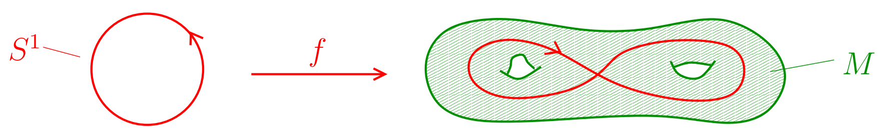


FIGURE 252

PROOF. The statements are modest generalizations of [**Hirs76**, Chapter 2, Theorem 2.6] and [**Hirs76**, Chapter 2, Theorem 2.13], except that the reference does not talk explicitly about homotopies, instead it says that given a map f one can find a smooth map $\widetilde{f}$ with the desired properties which is "close to f" in a suitable sense.

In the following we will sketch an argument why two maps that are "close" are actually homotopic. To simplify the discussion we assume that M is closed. By Proposition 9.1 we can view M as a submanifold of some $\mathbb{R}^n$. Careful reading of the above references shows that it remains to prove the following claim.

Claim. There exists some $\epsilon > 0$ with the following property: any two maps $f, \widetilde{f}\colon X \to M$ with $\sup\{\|f(x) - \widetilde{f}(x)\|_{\mathbb{R}^n} \mid x \in X\} < \epsilon$ are homotopic.

On page 369, in our sketch of the proof of Proposition 8.25, we showed that there exists a closed neighborhood N of $M \subset \mathbb{R}^n$ together with a deformation retraction $p\colon N \to M$. Since N is a neighborhood and since M is compact we obtain easily that there exists an $\epsilon > 0$ such that for every $y \in M$ we have $q(\overline{B}_\epsilon(y)) \subset N$.

Now let $f, g\colon X \to M$ be two maps with $\sup\big\{\|f(x) - \widetilde{f}(x)\| \,|\, x \in X\big\} < \epsilon$. We consider the following map:

$$\begin{array}{rcl} F\colon X \times [0,1] & \to & M \\ (x,t) & \mapsto & (p \circ q)\big(\underbrace{(1-t)\cdot f(x) + t \cdot \widetilde{f}(x)}_{\in \overline{B_\epsilon(f(x))}}\big). \end{array}$$

By choice of ϵ we see that F is actually defined. It is now straightforward to verify that F is a proper homotopy between f and $\widetilde{f}$. ■

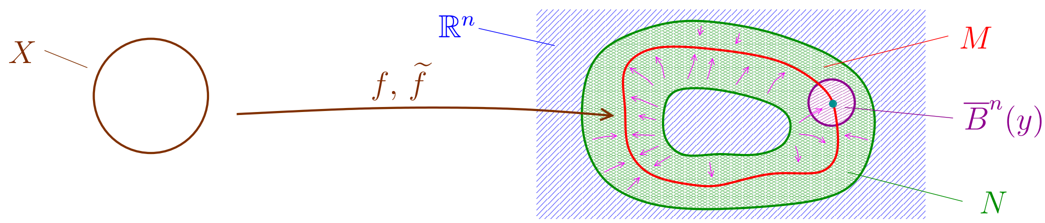


FIGURE 253. Illustration for the proof of Theorem 9.15.

Remark. We conclude this chapter with a word of caution. Let M and N be two smooth manifolds with $\dim(M) < 2 \cdot \dim(N)$. It is quite difficult to see what type of self-intersections can appear in a proper map $f\colon M \to N$. A good example of an overlooked type of self-intersection is provided by the proper map $f\colon \overline{B}^2 \to M$ from the disk $\overline{B}^2$ to a 3-dimensional ellipsoid M that is shown in Figure 254. The topic of singularities that can arise by mapping a surface into a 3-dimensional smooth manifold are studied in greater detail in [**Carte95**, Chapter 1.3].

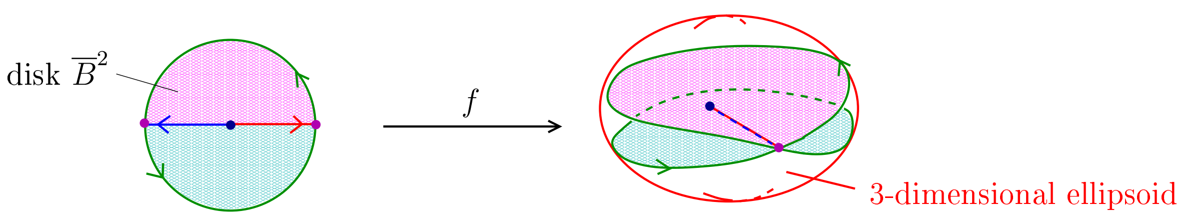


FIGURE 254

Exercises for Chapter 9.

Exercise 9.1. Let $n \in \mathbb{N}_{\geq 2}$. Use the Whitney Approximation Theorem 9.3 and Proposition 6.62 to give a quick proof that $\pi_1(S^n) = 0$.

Exercise 9.2. Provide a proof for Proposition 9.7.

Exercise 9.3. Let M be a compact topological manifold and let $A \subset M$ be a non-empty closed subset with $A \neq M$. Show that $M \setminus A$ is not compact.
Hint. Use Corollary 9.2 and Exercise 3.18.

Exercise 9.4. Let M be a closed n-dimensional smooth manifold and let $x_1, \dots, x_k \in M$ be disjoint points. Show that there exists a smooth embedding $\varphi\colon M \to \mathbb{R}^m$ such that $\varphi(M)$ is "flat" near $\varphi(x_1), \dots, \varphi(x_k)$, in the sense that there exist disjoint neighborhoods $U_1, \dots, U_k$ of $x_1, \dots, x_k$ and points $y_1, \dots, y_k \in \mathbb{R}^{m-n}$ such that for each $i \in \{1, \dots, k\}$ we have $\varphi(U_i) \subset \mathbb{R}^n \times \{y_i\}$.

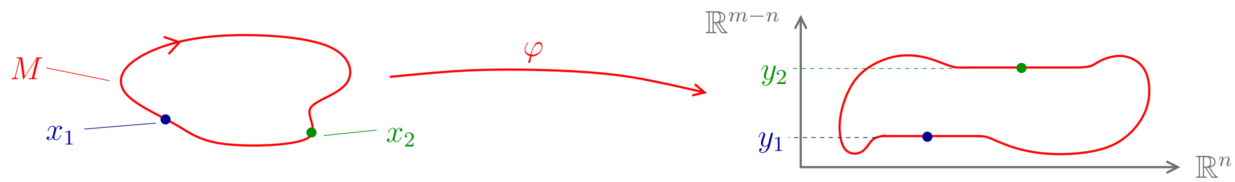


FIGURE 255. Illustration of Exercise 9.4.

Exercise 9.5. Let M be a compact smooth manifold and let $\alpha\colon S^k \to M$ be a map. Show that there exists an $r \in \mathbb{N}_0$ and a smooth embedding $\varphi\colon S^k \to \partial(M \times \overline{B}^r)$ such that the maps $S^k \xrightarrow{\alpha} M \xrightarrow{x \mapsto (x,0)} M \times \overline{B}^r$ and $\varphi\colon S^k \to M \times \overline{B}^r$ are homotopic.

10. Regular and Tubular Neighborhoods

In this chapter we will introduce the notion of a regular neighborhood of a subset of a smooth manifold and we will show that submanifolds admit tubular neighborhoods and we will show that tubular neighborhoods are regular. The results of this chapter can be viewed as a generalization of the Tubular Neighborhood Theorem 8.24. For the most part it might be OK to just skip this chapter and work with a naive understanding of regular and tubular neighborhoods. Nonetheless, the reader should be aware that these notions can be defined properly, that the definitions are not entirely self-evident, and that existence of such neighborhoods is non-trivial.

10.1. Regular neighborhoods.

In this section we will introduce the notion of a regular neighborhood of a compact subset of a topological manifold. This notion is a variation on the eponymous concept that we defined on page 1603 and it is a variation on the notion of a tubular neighborhood that we encountered on page 367.

Definition. Let M be an m-dimensional topological manifold and let K be a compact subset. A *regular neighborhood* of K is an m-dimensional compact submanifold N with corner of M with the following properties:

(1) K is contained in the interior $\mathring{N}$ of N,
(2) K is a deformation retract of N,
(3) $\partial_0 N$ is a deformation retract of $N \setminus K$.

We refer to $\mathring{N}$ as an *open regular neighborhood of K*.

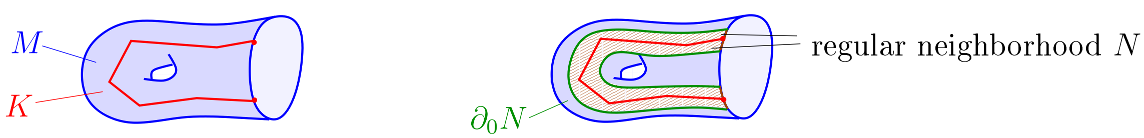


Figure 256

Example. Let M be a PL-manifold and let $K \subset M \setminus \partial M$ be a subcomplex. In Section 64.3 we defined the regular neighborhood $N_K(M)$ of K in M. By Proposition 64.12 we know that $N_K(M)$ does indeed have the properties demanded in the above definition of a regular neighborhood. In other words, by Proposition 64.12 we know that K has a regular neighborhood in the above sense.

We continue this section with the following lemma which can be viewed as a generalization of Lemma 68.16 (3).

Lemma 10.1. *Let K be a compact subset of a topological manifold M. If N is a regular neighborhood of K, then the following statements hold:*

(1) *$M \setminus \mathring{N}$ is a deformation retract of $M \setminus K$.*
(2) *The inclusion $M \setminus \mathring{N} \to M \setminus K$ is a homotopy equivalence, in particular it induces an isomorphism of homotopy groups (for suitable base points) and homology groups.*
(3) *$M \setminus \mathring{N}$ is path-connected if and only if $M \setminus K$ is path-connected.*

(4) *The inclusion map* $M \setminus \mathring{N} \to M \setminus K$ *induces isomorphisms of fundamental groups and (co-) homology groups.*

Proof.

(1) By definition of a regular neighborhood there exists a deformation retraction r from $N \setminus K$ to $\partial_0 N$. We extend this deformation retraction to all of $M \setminus K$ in the obvious way. More precisely, we consider the map

$$\begin{array}{rcl} f\colon (M \setminus K) \times [0,1] & \to & M \setminus K \\ (P,t) & \mapsto & \begin{cases} P, & \text{if } P \notin N, \\ r(P,t), & \text{if } P \in N \setminus K. \end{cases} \end{array}$$

Using Lemma 3.10 and Proposition 8.2 one can easily show that this map is indeed continuous. Once continuity is verified it is straightforward to show that f is indeed a deformation retraction from $M \setminus K$ to $M \setminus \mathring{N}$.

(2) It follows from (1) and Lemma 18.14 that the inclusion is a homotopy equivalence. The statement regarding homotopy groups and homology groups follow from Proposition 40.7 (2) and Corollary 42.8.

(3) This statement follows from (2) and Lemma 18.13.

(4) This last statement follows from (2) together with Proposition 18.16 (2), Corollary 42.8 and Lemma 73.13 (3). ∎

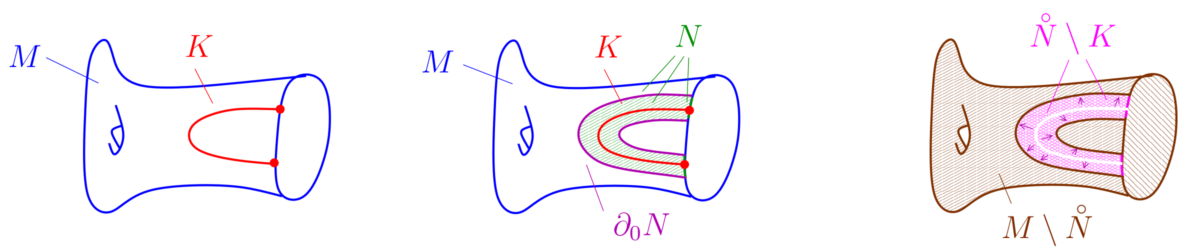


Figure 257. Illustration of Lemma 10.1.

The following corollary, which in the literature is often used implicitly or explicitly, gives one application of the theory of regular neighborhoods.

Lemma 10.2. *Let M be a compact topological manifold and let $K \subset M$ be a compact subset. If K admits a regular neighborhood, then the following two statements hold:*

(1) *the fundamental group $\pi_1(M \setminus K)$ is finitely presented, and*

(2) *for each $k \in \mathbb{N}_0$ the homology group $\mathrm{H}_k(M \setminus K;\mathbb{Z})$ is finitely generated.*

Proof. Note that by Lemma 10.1 the topological space $M \setminus K$ is homotopy equivalent to $M \setminus \mathring{N}$. Since M is compact it follows from Lemma 89.1 (3) that $M \setminus \mathring{N}$ is also compact. Furthermore, by Lemma 89.1 (4) we know that $M \setminus \mathring{N}$ is an n-dimensional smooth manifold. Summarizing we have seen that $M \setminus \mathring{N}$ is a compact n-dimensional smooth manifold. The lemma now follows from Proposition 85.13 and Lemma 10.1 (4). ∎

In the Tubular Neighborhood Theorem 8.24 we showed that if M is an orientable smooth manifold, then every closed 1-dimensional submanifold admits a trivial tubular neighborhood, which is easily seen to be a regular neighborhood. It is thus natural to guess that something similar should also hold for higher-dimensional submanifolds. In fact we have the following theorem.

Theorem 10.3. (Regular Neighborhood Theorem) *Let M be a smooth manifold. Every compact proper submanifold of M admits a regular neighborhood.*

An outline of the proof will be sketched in the next section.

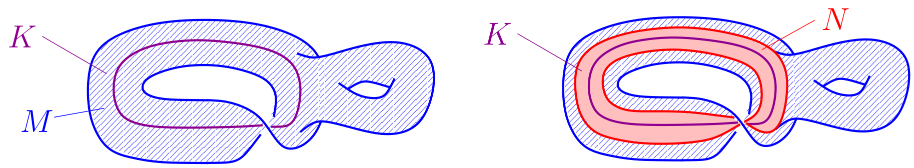


FIGURE 258. Illustration of the Regular Neighborhood Theorem 10.3.

10.2. **The Tubular Neighborhood Theorem II.** In Section 8.8 we introduced the notion of a "trivial tubular neighborhood" of a submanifold and we saw in Theorem 8.24 that some types of submanifolds admit trivial tubular neighborhoods.

In this section we will first introduce the general notion of a tubular neighborhood of a submanifold. We will then discuss existence and uniqueness, before we then show that tubular neighborhoods give rise to regular neighborhoods.

Definition. Let $m \in \mathbb{N}_0$.

(1) Let K be a smooth manifold. A *smooth $\overline{B}^m$-bundle over K* is a smooth manifold N together with a smooth map $p\colon N \to K$ with the following property: given any $x \in K$ there exists a *trivialization of p around x*, i.e. there exists an open neighborhood U of x and a diffeomorphism $\Phi\colon p^{-1}(U) \to \overline{B}^m \times U$ such that the following diagram commutes:
$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\Phi", "\cong"'] \arrow[dr, "p"'] & & \overline{B}^m \times U \arrow[dl, "(x,v)\mapsto x"] \\ & U. & \end{tikzcd}$$
A smooth $\overline{B}^1$-bundle is often called a *smooth interval bundle* and a smooth $\overline{B}^2$-bundle is often called a *smooth disk bundle*.

(2) An *isomorphism* between two smooth $\overline{B}^m$-bundles $p\colon M \to K$ and $q\colon N \to K$ is defined to be a diffeomorphism $\Phi\colon M \to N$ such that the following diagram commutes
$$\begin{tikzcd} M \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & N \arrow[dl, "q"] \\ & K. & \end{tikzcd}$$
If such an isomorphism exists we say that the two smooth $\overline{B}^m$-bundles are *isomorphic*.

The following gives two trivial but nonetheless extremely important examples of constructions of smooth $\overline{B}^m$-bundles.

Example.

(1) Let K be a smooth manifold and let $m \in \mathbb{N}_0$. We denote by $p\colon \overline{B}^m \times K \to K$ the obvious projection map. This map is evidently a smooth $\overline{B}^m$-bundle over K and we refer to it as the *trivial smooth $\overline{B}^m$-bundle over K*.

(2) Let $p\colon N \to K$ be a smooth $\overline{B}^m$-bundle. If $Y \subset K$ is a submanifold, then the restriction $p\colon p^{-1}(Y) \to Y$ is also a smooth $\overline{B}^m$-bundle.

Definition.

(1) Let W be a smooth manifold and let $p\colon \overline{B}^m \times W \to W$ and $q\colon \overline{B}^m \times W \to W$ be two smooth $\overline{B}^m$-bundles. We say an isomorphism $\Phi\colon \overline{B}^m \times W \to \overline{B}^m \times W$ of smooth $\overline{B}$-bundles is *linear* if there exists a map $g\colon W \to \mathrm{O}(m)$ such that for any $(v,w) \in \overline{B}^m \times W$ we have

$$\Phi(v,w) \;=\; (g(w)\cdot v, w).$$

(2) A *linear $\overline{B}^m$-bundle* is a smooth $\overline{B}^m$-bundle $p\colon N \to K$ together with a family of trivializations $\{(U_i, \Phi_i\colon p^{-1}(U_i) \to \overline{B}^m \times U_i)\}_{i\in I}$ such that for any $i,j \in I$ the isomorphism $\Phi_j \circ \Phi_i^{-1}\colon \overline{B}^m \times (U_i \cap U_j) \to \overline{B}^m \times (U_i \cap U_j)$ of smooth $\overline{B}^m$-bundles over $U_i \cap U_j$ is linear. Henceforth, by a trivialization of a linear $\overline{B}^m$-bundle we always mean a trivialization from the above family.

(3) An *isomorphism of two linear $\overline{B}^m$-bundles* $p\colon M \to K$ and $q\colon N \to K$ is defined as an isomorphism $\Theta\colon M \to N$ of smooth $\overline{B}^m$-bundles such that for any trivializations $(U, \Phi\colon p^{-1}(U) \to \overline{B}^m \times U)$ of p and $(V, \Psi\colon q^{-1}(V) \to \overline{B}^m \times V)$ of q the isomorphism $\Phi \circ \Theta \circ \Psi^{-1}\colon \overline{B}^m \times (U \cap V) \to \overline{B}^m \times (U \cap V)$ of smooth $\overline{B}^m$-bundles over $U \cap V$ is linear.

The trivializations of a linear $\overline{B}^m$-bundle play a role similar to the smooth atlas of a smooth manifold. In the case of a smooth manifold we usually suppress the specific atlas from the notation. This leads us to the following convention.

Convention. In the following we will usually suppress the family of trivializations in the notation of a linear $\overline{B}^m$-bundle.

Examples.

(1) Any trivial $\overline{B}^m$-bundle is a linear $\overline{B}^m$-bundle in the obvious way.

(2) We consider the Möbius band $M = ([0,1] \times [-1,1])/(0,x) \sim (1,-x)$ together with the projection map

$$\begin{array}{rcl} p\colon M = ([0,1]\times[-1,1])/\sim & \to & [0,1]/0\sim 1 \\ {[(x,y)]} & \mapsto & [x]. \end{array}$$

In Exercise 11.7 we will verify that this map defines a linear interval bundle. We refer to Figure 259 for an illustration.

For future reference we formulate the following lemma that classifies isomorphisms of some linear $\overline{B}^m$-bundles.

Lemma 10.4.

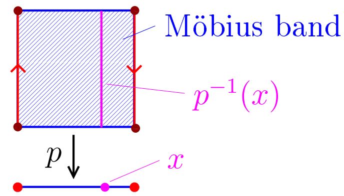


FIGURE 259

(1) *Let N be an oriented n-dimensional smooth manifold and let $k \in \mathbb{N}$. The map*

$$\begin{aligned} \{maps\ N \to \mathrm{SO}(k)\}/homotopy &\to \left\{ \begin{array}{c} orientation\text{-}preserving \\ isomorphisms\ of\ \overline{B}^k \times N \end{array} \right\} \Big/ isotopy \\ [f\colon N \to \mathrm{SO}(k)] &\to \left(\begin{array}{rcl} \overline{B}^k \times N & \to & \overline{B}^k \times N \\ (v, P) & \mapsto & (\underbrace{f(P)}_{\in \mathrm{SO}(k)} \cdot v, P) \end{array} \right) \end{aligned}$$

is a bijection.

(2) *Let N be an oriented n-dimensional smooth manifold. The identity is the only orientation-preserving linear $\overline{B}^1$-isomorphism of the trivial bundle $\overline{B}^1 \times N$.*

(3) *Up to isotopy every isomorphism of the linear bundle $\overline{B}^2 \times S^1$ is of the form*

$$\begin{array}{rcl} \overline{B}^2 \times S^1 & \to & \overline{B}^2 \times S^1 \\ (w, z) & \mapsto & (w \cdot z^k, w) \end{array}$$

for some $k \in \mathbb{Z}$.

SKETCH OF PROOF.

(1) This statement follows fairly easily from the definitions. We leave the details to the reader.
(2) This follows from (1) and the trivial observation that $\mathrm{SO}(1) = \{1\}$, thus there exists precisely one map from N to $\mathrm{SO}(1)$.
(3) This statement follows from (1) and the observation that any map $S^1 \to \mathrm{SO}(2) = S^1$ is homotopic to a map of the form $z \mapsto z^k$ for some $k \in \mathbb{Z}$. This in turn is a fairly straightforward consequence of the fact that $\pi_1(S^1, 1) \cong \mathbb{Z}$, see Proposition 16.17.[188]

■

Now we can formulate the definition that we had been aiming for.

Definition. Let M be an n-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold. A *tubular neighborhood* for K is a pair $(N, p\colon N \to K)$ with the following properties:

(1) N is a compact codimension-zero submanifold with corner.
(2) N is a neighborhood of K.
(3) The map $p\colon N \to K$ is a linear $\overline{B}^{n-k}$-bundle such that $p(x) = x$ for all $x \in K$.

[188]So why don't we have to worry about base points?

(4) We have $\partial_1 N = p^{-1}(\partial K)$.[189]

Remark. There are many books that introduce the notion of a tubular neighborhood, see e.g. [**Kos93, BJ82, Bre93, Lan02**] and [**Wall16**]. The definitions and results tend to be subtly different and it requires some effort to go back and forth between these books. Our treatment is, by accident, very similar to the one given in [**Wall16**, Chapter 2.5].

Example.

(1) Let M be an n-dimensional smooth manifold and let K be a *closed* k-dimensional submanifold. On page 367 we defined a tubular map to be a smooth embedding $F\colon \overline{B}^{m-k} \times K \to M$ such that for any $P \in K$ we have $F(0, P) = P$. Furthermore we referred to the image $\Phi(\overline{B}^{m-k} \times K)$ as tubular neighborhood for K. One can easily verify that $N := \Phi(\overline{B}^{n-k} \times K)$ with the obvious map to K defines a tubular neighborhood in the above sense.

(2) As above we consider the Möbius band $M = ([0,1] \times [-1,1])/\sim$. Furthermore let $K = [0,1] \times \{0\} \subset M$ be the submanifold that is given by the central circle.[190] In Figure 260 we illustrate a tubular neighborhood for K.

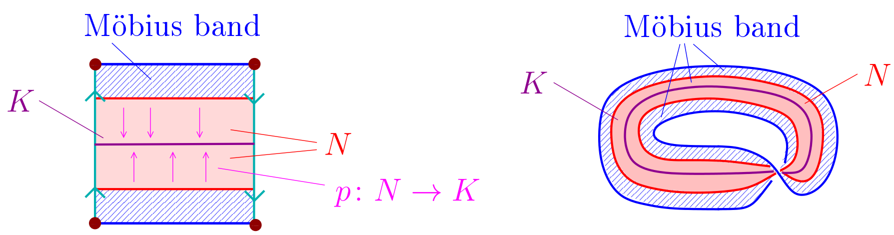


FIGURE 260

The following theorem is a generalization of the earlier Tubular Neighborhood Theorem 8.24.

Theorem 10.5. (General Tubular Neighborhood Theorem) *Let M be a smooth manifold and let K be a proper submanifold. If K is compact, then K admits a tubular neighborhood.*

PROOF. The sketch for the existence of a tubular neighborhood that we provided for the previous Tubular Neighborhood Theorem 8.24 is also valid in this more general context. As before a full proof is provided in [**Wall16**, Theorem 2.3.3]. Closely related and similar results are also proved in [**Kos93**, Section III.2], [**Bre93**, Theorem II.11.14], [**Lee02**, Theorem 6.24], [**BJ82**, Theorem 12.11] and [**Lan02**, Chapter IV]. ■

In Theorem 8.24 we already saw that certain types of submanifolds admit a trivial tubular neighborhood. To this list we add a few more examples.

[189] Note that in Proposition 44.2 we showed that for the proper submanifold K we have $\partial K = \partial M \cap K$.

[190] Could we not just take all of M to be a tubular neighborhood?

Proposition 10.6. (∗) *Let M be an orientable smooth manifold and let N be an orientable proper submanifold. Suppose one of the following holds:*

(1) *N is one-dimensional,*
(2) *N is of codimension one,*
(3) *N is 2-dimensional and every component of N has non-empty boundary.*

Then K admits a trivial tubular neighborhood.

Proof. By the General Tubular Neighborhood Theorem 10.5 it remains to show that a tubular neighborhood is already trivial. For (1) and (2) this gets taken care of by the same references as in the Tubular Neighborhood Theorem 8.24. It remains to deal with (3). Note that with our hypothesis the topological space N admits a retraction onto a 1-dimensional CW-complex.[191] Now it follows fairly easily from Propositions 9.1.1 and Proposition 9.2.3 of [**Hau14**] that the tubular neighborhood is indeed trivial. ■

Example. Let $M = \overline{B}^2 \times S^1$ be the solid torus. We can view the Möbius band as a proper submanifold N of M. Note that $M \setminus N$ is actually connected. From this observation it follows fairly easily that N does *not* admit a trivial tubular neighborhood. This shows that in Proposition 10.6 (2) we cannot drop the hypothesis that N is orientable.

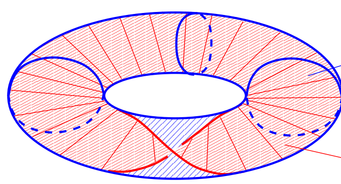


Figure 261

The following proposition gives an appropriate uniqueness of tubular neighborhoods. The proposition can be viewed as a generalization of Proposition 8.26.

Proposition 10.7. (Uniqueness of tubular neighborhoods) *Let M be an n-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold. Furthermore let $p\colon U \to K$ and $q\colon V \to K$ be two tubular neighborhoods of K. Then there exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:*

(1) *$F(P,0) = P$ for all $P \in M$,*
(2) *$F(P,t) = P$ for all $P \in K$ and $t \in [0,1]$,*
(3) *the diffeomorphism $\Phi = F_1\colon M \to M$ restricts to diffeomorphisms $\Phi\colon M \setminus \mathring{U} \to M \setminus \mathring{V}$ and $\Phi\colon U \to V$, and the latter map is an isomorphism of linear $\overline{B}^{n-k}$-bundles,*
(4) *the diffeotopy F has compact support.*

Proof. As in the case of Proposition 8.26 a proof is provided in [**Wall16**, Chapter 2.5]. ■

Convention. By the uniqueness of tubular neighborhoods that we had just proved in Proposition 10.7 we usually do not have to worry about the precise choice of a tubular

[191] To be super-precise, this follows from the classification of 2-dimensional smooth manifolds with boundary, see the Surface Classification Theorem 23.4 and Lemma 23.12.

neighborhood. Given a compact proper submanifold of a smooth manifold M we usually denote by $N(K)$ some choice of a tubular neighborhood.

Remark. The General Tubular Neighborhood Theorem 10.5 is stated for smooth manifolds instead of the more general context of topological manifolds. With slight modifications the definition of a tubular neighborhood also makes sense for a k-dimensional proper submanifold K of an n-dimensional topological manifold M.

(1) By [**FNOP19**, Theorem 4.6] tubular neighborhoods exist for any compact proper submanifold of a 4-dimensional topological manifold. If the submanifold is one-dimensional and the topological manifold is orientable, then there exists a trivial tubular neighborhood. Furthermore, by [**FNOP19**, Theorem 4.7] such a tubular neighborhood is unique in an appropriate sense.
(2) If $k = n - 2$, i.e. if K is a codimension 2 submanifold, then some type of tubular neighborhood seems to exist, see e.g. [**DV09**, Theorem 6.8.1] and [**FQ90**, p. 137] for details. We also refer to [**Ped77**] and [**Joha72**] for some positive results. Note though that when it comes to topological manifolds one should be very cautious with references. It is advisable to very carefully check the definitions used in papers on topological manifolds.
(3) If $k = n - 3$, i.e. if K is a codimension 3 submanifold, then, according to [**DV09**, p. 347] and [**Edw09**, p. 9], a tubular neighborhood does not exist in general.

Fortunately for the most part we will not deal with topological manifolds and the General Tubular Neighborhood Theorem 10.5 will be good enough for our purposes.

Definition.

(1) A linear $\overline{B}^m$-bundle $p\colon N \to K$ is called *trivial* if it is isomorphic, as a linear $\overline{B}^m$-bundle, to the trivial $\overline{B}^m$-bundle $\overline{B}^m \times K \to K$.
(2) Let M be an m-dimensional smooth manifold and let K be a k-dimensional submanifold.
    (a) A tubular neighborhood $(N, p\colon N \to K)$ of K is called *trivial* if the linear $\overline{B}^m$-bundle $p\colon N \to K$ is trivial.
    (b) A *tubular map* is a map $F\colon \overline{B}^{m-k} \times K \to N$ with the following properties:
        (i) for every $P \in K$ we have $F(P, 0) = P$,
        (ii) we have $F(\overline{B}^{m-k} \times K) \cap \partial N = F(\overline{B}^{m-k} \times \partial K)$,
        (iii) the image $F(\overline{B}^{m-k} \times K)$ is a submanifold with corner of N.

    It is clear that if F is tubular map, then $N := F(\overline{B}^{m-k} \times K)$ with the obvious map to K is a trivial tubular neighborhood.

Example. As on page 419 we consider again the Möbius band $M = ([0, 1] \times [-1, 1])/\sim$ and the submanifold $K = [0, 1] \times \{0\} \subset M$. At least from the picture it is pretty clear that K does not admit a trivial tubular neighborhood.[192]

[192] What would be a rigorous proof for the statement that K does not admit a trivial tubular neighborhood?

The following lemma gives some other instances where the tubular neighborhood is non-trivial.[193]

Lemma 10.8. *Let M be an orientable smooth manifold and let K be a proper submanifold. If K is non-orientable, then the tubular neighborhood is non-trivial.*

Sketch of proof. We prove the converse, if the K admits a trivial tubular neighborhood, then K is orientable. So suppose that K is a proper k-dimensional submanifold of an orientable m-dimensional smooth manifold M and suppose that K admits a trivial neighborhood $\overline{B}^{m-k} \times K$. Let $P \in K \subset M$. We identify $\mathrm{T}_P K$ and $\mathrm{T}_0 \overline{B}^{m-k}$ with its images in $\mathrm{T}_{(0,P)} M$ under the obvious inclusion maps $K \to \{0\} \times K \to M$ and $\overline{B}^{m-k} \to \overline{B}^{m-k} \times \{P\} \to P$. We define

$$\mathcal{B} \subset \mathrm{T}_P K \text{ is a positive basis} \quad :\Longleftrightarrow \quad \begin{array}{c}\text{a positive basis of } \mathrm{T}_0 \overbrace{\overline{B}^{m-k}}^{=\mathbb{R}^{m-k}} \text{ followed by } \mathcal{B} \\ \text{is a positive basis of } \mathrm{T}_{(0,P)} M.\end{array}$$

One easily verifies that this defines an orientation for K. ■

If we get away from the pesky non-orientable (sub-) smooth manifolds we get some more positive results.

Proposition 10.9. *Let M be an oriented[194] m-dimensional smooth manifold and let K be a compact oriented proper submanifold.*

(1) *If K is of dimension one and $m \geq 2$, then there exists a trivial tubular neighborhood $N(K) = \overline{B}^{m-1} \times K$ such that the product orientation of $\overline{B}^{m-1} \times K$ matches the orientation of $N(K) \subset M$.*

(2) *If K is of codimension one, then there exists a trivial tubular neighborhood of the form $N(K) = [-1, 1] \times K$ such that the product orientation of $[-1, 1] \times K$ matches the orientation of M.*

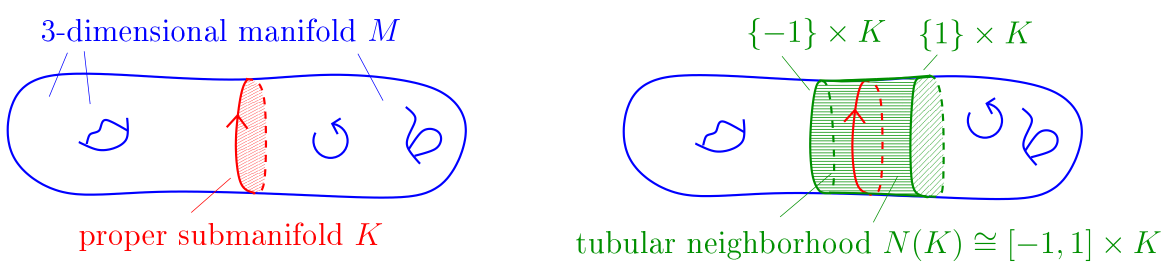


Figure 262. Illustration of Proposition 10.9.

[193] To be more precise, we still need examples of say closed non-orientable smooth manifolds that are embedded in orientable smooth manifolds. We will shortly see that we can embed any real projective space $\mathbb{RP}^n$ into $\mathbb{R}^{2n}$.

[194] Is the hypothesis that K is orientable, necessary? Put differently, is there an example of an orientable smooth manifold and a compact proper codimension-one submanifold such that the tubular neighborhood is non-trivial?

PROOF. In both cases it follows from the General Tubular Neighborhood Theorem 10.5 together with [**Hau14**, Proposition 9.2.3] and [**MiS74**, Axiom 3] that K admits a trivial neighborhood.[195] The statement regarding the orientations can be achieved by possibly using an orientation-reversing diffeomorphism of $\overline{B}^{m-1}$ or $[-1,1]$. ■

Proposition 10.9 raises the following question.

Question 10.10. *Let M be an orientable smooth manifold and let K be a compact orientable proper submanifold. Is the tubular neighborhood $N(K)$ necessarily trivial?*

After this long discussion of tubular neighborhoods one can be excused for having lost track of what they are actually supposed to be good for. The following proposition finally implies Theorem 10.3.

Proposition 10.11. *Let M be a smooth manifold and let K be a compact proper submanifold of M. Every tubular neighborhood of K is also a regular neighborhood of K.*

PROOF. Let M be an n-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold of M. Let $(N, p\colon N \to K)$ be a tubular neighborhood of K. Recall that this means that the following three statements are satisfied:

(1) N is a compact codimension-zero submanifold with corner of M that is a neighborhood of K.
(2) The map $p\colon N \to K$ is a linear $\overline{B}^{n-k}$-bundle such that for all $x \in K$ we have $p(x) = x$. We denote by $\{(U_i, \Phi_i\colon p^{-1}(U_i) \to \overline{B}^{n-k} \times U_i)\}_{i\in I}$ the corresponding family of trivializations.
(3) We have $\partial_1 N = p^{-1}(\partial K)$, put differently, we have $\partial M \cap N = p^{-1}(\partial M \cap K)$.

We start out the actual proof of the proposition with the following definition and construction. Let $v \in N$. We write $x := p(v)$. We pick an $i \in I$ with $p(v) \in U_i$ and we write $\Phi_i(v) = (w, x)$ where $w \in \overline{B}^m$.

(1) Given $s \in [0,1]$ we write $\varphi_s(v) := \Phi_i^{-1}(s \cdot w, x)$.
(2) Suppose $v \notin K$. This is equivalent to saying that $w \neq 0$. Then given $t \in [0,1]$ we write $\psi_t(v) := \Phi_i^{-1}\big((1-t)\cdot w + t \cdot \frac{w}{\|w\|}, x\big)$.

We leave it to the reader to verify that these definitions are independent of the choice of the trivialization Φ_i.[196] From this independence it also follows easily that the maps

$$\begin{array}{rcl} \Phi\colon N \times [0,1] & \to & N \\ (v,s) & \mapsto & \varphi_s(v) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \Psi\colon (N \setminus K) \times [0,1] & \to & N \setminus K \\ (v,t) & \mapsto & \psi_t(v) \end{array}$$

are continuous.[197] We refer to Figure 263 for an illustration of the definition of Φ and Ψ. Finally recall that by definition of a regular neighborhood we have to prove the following claim.

[195] We had mentioned the case $\dim(K) = 1$ explicitly in the original Tubular Neighborhood Theorem 8.24.

[196] At this point it is essential that we work with a linear $\overline{B}^{n-k}$-bundle.

[197] Evidently we only have to show continuity locally. So let $v \in N$. We can pick a trivialization that contains x and then it follows immediately from the definitions that the maps Φ and Ψ are continuous in the neighborhood of x corresponding to the trivialization.

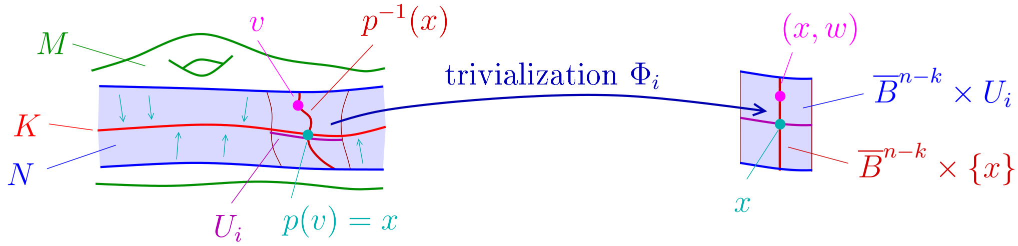


FIGURE 263. Illustration for the proof of Proposition 10.11.

Claim.

(1) K is contained in the interior of N,
(2) K is a deformation retract of N,
(3) $\partial_0 N$ is a deformation retract of $N \setminus K$.

We turn to the proof of the claim.

(1) By definition of a neighborhood we know that K is contained in the interior of N.
(2) The above map $\Phi\colon N \times [0,1] \to N$ is easily seen to be a deformation retraction from N to K.
(3) The above map $\Psi\colon (N \setminus K) \times [0,1] \to N \setminus K$ is easily seen to be a deformation retraction from $N \setminus K$ to $\partial_0 N$.

This concludes the proof of the claim and thus also of the proposition. ■

Remark. One of the frustrating aspects about tubular neighborhoods is that every book seems to work with a different definition. IMHO a tubular neighborhood should have the following two features:

(1) it should be a regular neighborhood, and
(2) it should satisfy a uniqueness statement.

By Propositions 10.7 and 10.11 our definition has these two properties. It is not always apparent that the definitions in the literature have these two features.

We conclude this chapter with a lemma which says that for transverse submanifolds one can find "transverse" tubular neighborhoods. We will not really make use of the lemma in these notes, but it seems to me that this lemma often gets used implicitly in the literature.

Lemma 10.12. (∗) *Let M be a smooth manifold and let X and Y be proper submanifolds of M. If the submanifolds X and Y are compact and if they intersect transversally, then there exist tubular neighborhoods $p\colon U \to X$ and $q\colon V \to Y$ such that*[198]

$$U \cap V \;=\; p^{-1}(V \cap X) = q^{-1}(U \cap Y).$$

PROOF. I could not find a reference for the third statement in any of the standard books on differential topology [**Lee02, Wall16, Kos93, GP74**]. But the standard proof for the

[198]From Figure 264 it should be pretty clear what we want to say. But does the statement of the lemma capture this idea? It is at times difficult to turn a picture into a precise mathematical statement.

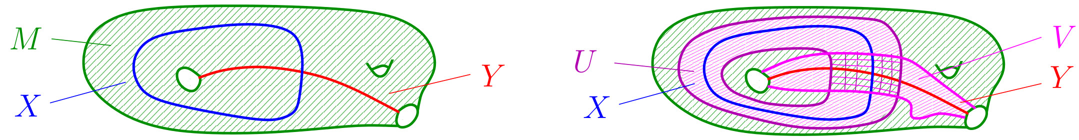


FIGURE 264. Illustration of Lemma 10.12.

existence of tubular neighborhoods can easily be adapted to prove the statement. The restriction to compact X and Y is almost certainly not necessary. ■

10.3. **Spatial graphs** $(*)$**.** Note that in the definition of a regular neighborhood we did not demand that K is a submanifold, in particular K could a priori be any subset. As we will see in Exercise 10.4 it is not difficult to come up with compact subsets of even just $\mathbb{R}$ which do not regular neighborhoods. On the other hand one might hope that reasonable subsets of some $\mathbb{R}^n$ admit regular neighborhoods. For example consider the "spatial graphs" shown in Figure 265. It seems reasonable to expect that these actually admit regular neighborhoods.

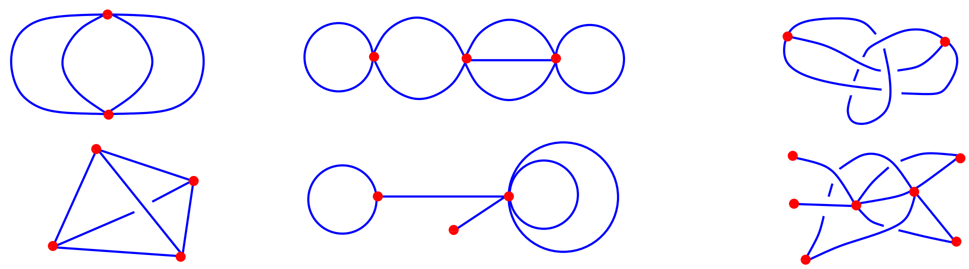


FIGURE 265

To make sense of this hunch we first need a definition of a "spatial graph" which captures the above examples. A first attempt might be to attempt to define a "a spatial" as the image of an embedding of a (say finite) topological into $\mathbb{R}^3$. But ever since on page 2065 and in Exercise 85.4 we learned of wild knots and its strange behaviors we know that we need some control over such embeddings to end up with reasonable objects.

Some leisurely thinking leads us to the following definition is modelled on the notion of a topological graph that we introduced on page 223.

Definition. A *spatial graph* is a triple $(G = (V, E, i, t), \Theta\colon |G| \to \mathbb{R}^3)$ where the following holds:

(1) The quadruple $G = (V, E, i, t)$ is a finite abstract graph.
(2) The map $\Theta\colon |G| \to \mathbb{R}^3$ is an embedding of the topological realization $|G|$ into $\mathbb{R}^3$.
(3) For every edge e with characteristic map $\Phi_e\colon [0,1] \to |G|$ the following statements hold:
   (a) The map $f := \Theta \circ \Phi_e\colon [0,1] \to \mathbb{R}^3$ is smooth. Note that by Proposition 6.12 this means that the derivative f' is defined on all of $(0,1)$ and that it extends continuously to a map $f'\colon [0,1] \to \mathbb{R}^3$.
   (b) The first derivative $f'(t)$ is non-zero for all $t \in [0,1]$,

Sometimes we just refer to $\Theta(|G|)$ as a "spatial graph" and we just ignore the extra data involved in the proper definition of a spatial graph.

Example. It should hopefully be clear that the examples sketched in Figure 265 are spatial graphs.

Now we can formulate the following proposition.

Proposition 10.13. *Every finite spatial graph G in $\mathbb{R}^3$ admits a regular neighborhood.*

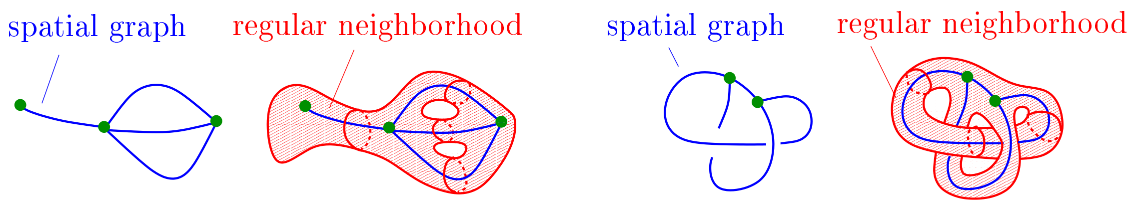


FIGURE 266. Illustration of Proposition 10.13.

PROOF. This proposition is proved in **[FH18, FH19]**. Note that in **[FH18, FH19]** it is also shown that there is a type of regular neighborhood which is unique in an appropriate sense. ■

Exercises for Chapter 10.

Exercise 10.1. Let K be the Klein bottle. Show that there exists a closed *orientable* 3-dimensional smooth manifold M which contains K as a submanifold.
Hint. You could first try to find a compact 3-dimensional smooth manifold with toroidal boundary that contains M as a submanifold.

Exercise 10.2. Let M be a 2-dimensional smooth manifold. Furthermore let $f\colon [0,1] \to M$ be a smooth embedding such that $f(0) \in \partial M$ and such that $f((0,1]) \subset M \setminus \partial M$. Show that the inclusion $M \setminus f([0,1]) \to M$ is a homotopy equivalence.
Hint. Use the Extension Theorem 8.32 and the General Tubular Neighborhood Theorem 10.5.

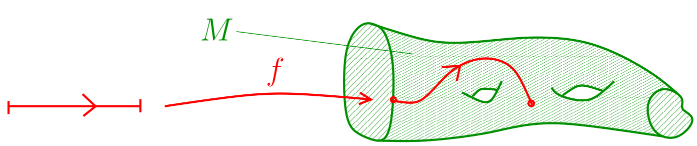


FIGURE 267

Exercise 10.3. Let M be an orientable n-dimensional smooth manifold and let K be a compact proper k-dimensional submanifold.

(1) Suppose K admits a trivial tubular neighborhood $\overline{B}^{n-k} \times K$.
  (a) Show that the tubular neighborhood can be extended to a trivial tubular neighborhood $2\overline{B}^{n-k} \times K$.
  Hint. Apply the General Tubular Neighborhood Theorem 10.5 and Proposition 10.6 to the proper $(n-1)$-dimensional submanifold $S^{n-k-1} \times K$.

(b) Show that there exists a smooth isotopy $F\colon (M \setminus K) \times [0,1] \to M \setminus K$ with $F_0 = \mathrm{id}$ and $F_1(M \setminus K) = M \setminus (\overline{B}^{n-k} \times K)$.

Note that by Proposition 8.1 this implies in particular that $M \setminus \overline{B}^{n-k} \times K$ is diffeomorphic to $M \setminus K$.

(2) Now suppose that K does not admit a trivial tubular neighborhood. Formulate the appropriate generalization of (1) and prove it.

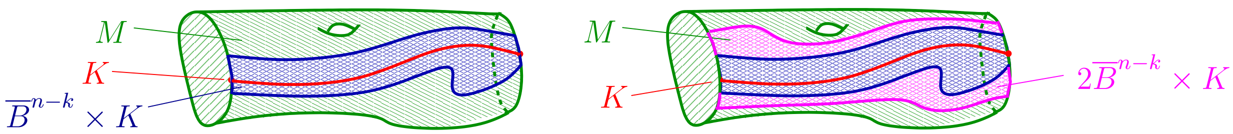


FIGURE 268. Illustration for Exercise 10.3.

Exercise 10.4. Show that the subset $K := \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ of $\mathbb{R}$ does not admit a regular neighborhood.

Exercise 10.5. Let G and G' be two spatial graphs in $\mathbb{R}^3$ with regular neighborhoods $N(G)$ and $N(G')$. Which of the following two implications is correct?

$$G \text{ and } G' \text{ are isotopic} \quad \Longleftrightarrow \quad N(G) \text{ and } N(G') \text{ are isotopic?}$$

Exercise 10.6. We consider the spatial graph G shown in Figure 269.

(a) Determine a presentation for $\pi_1(\mathbb{R}^3 \setminus G)$.
(b) What is the abelianization of $\pi_1(\mathbb{R}^3 \setminus G)$?

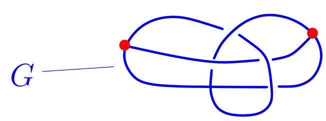


FIGURE 269. Illustration of Exercise 10.6.

We continue with our attempt to expand our toolkit for dealing with smooth manifolds.

11.1. **The Regular Value Theorem II** $(*)$. In this section we formulate several refinements of the Regular Value Theorem 6.53. The refinements will play an important role later on, but perhaps it is best to skip the section at first reading.

Theorem 11.1. (Regular Value Theorem) *Let M be an m-dimensional smooth manifold, let N be an n-dimensional smooth manifold without boundary, let $f\colon M \to N$ be a smooth map and let $s \in N$ be a regular value of f.*

(1) *The preimage $X := f^{-1}(\{s\})$ is a proper $(m-n)$-dimensional submanifold of M.*
(2) *If M is orientable, then X is also orientable.*
(3) *If M is compact, then X is also compact.*

For the last statement we assume that M is compact and orientable.

(4) *There exists an open neighborhood U of $s \in N$ such that for any smooth embedding $\varphi\colon \overline{B}^n \to U$ with $\varphi(0) = s$ there exists a tubular map $\Phi\colon \overline{B}^n \times X \to M$ such that the following two statements hold:*
   (a) *we have $f^{-1}(\varphi(\overline{B}^n)) = \Phi(\overline{B}^n \times X)$,*
   (b) *the following diagram commutes:*

$$\begin{array}{ccc} \overline{B}^n \times X & \xrightarrow{\quad\Phi\quad} & M \\ \big\downarrow{\scriptstyle (p,x)\mapsto p} & & \big\downarrow{\scriptstyle f} \\ \overline{B}^n & \xrightarrow{\quad\varphi\quad} & N. \end{array}$$

   If M is oriented, then the orientation on X can be chosen such that Φ is orientation-preserving.

The first three statements of the above theorem are precisely the content of the original Regular Value Theorem 6.53.

Example. In Figure 161 we show the torus M as a submanifold of $\mathbb{R}^3$ and we consider the "height function" $f\colon M \to \mathbb{R}$ given by $f(x, y, z) = z$. This function has precisely four critical points which are indicated in Figure 161 on the left. On the right we show a regular value s and its preimage and we show a critical value t and its preimage $f^{-1}(\{t\})$ which in this case is *not* a submanifold.

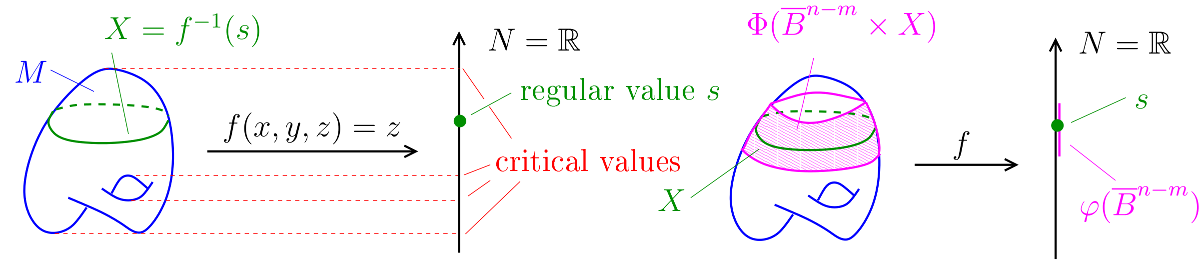


Figure 270. Illustration of the Regular Value Theorem 11.2.

SKETCH OF PROOF. As mentioned above, the first three statements of the theorem form precisely the content of the original Regular Value Theorem 6.53. So it remains to deal with the final statement. This can be deduced quite easily from the Submersion Theorem 6.56. We leave it to the reader to fill in the details. ■

For the reader's convenience we discuss the cases that $N = \mathbb{R}$ and $N = S^1$ separately. These two special cases also allow us to elucidate the somewhat confusing fourth statement of the Regular Value Theorem 11.1.

Theorem 11.2. (Regular Value Theorem for $N = \mathbb{R}$) *Let M be an n-dimensional smooth manifold, let $f\colon M \to \mathbb{R}$ be a smooth map and let $s \in \mathbb{R}$ be regular value of f. Then beyond statements (1), (2), (3) of the Regular Value Theorem 11.1 the following two statements hold:*

(4) *If M is compact and orientable, then there exists an $\epsilon > 0$ and a tubular neighborhood $[-\epsilon, \epsilon] \times X$ of X such that $f^{-1}([s-\epsilon, s+\epsilon]) = [-\epsilon, \epsilon] \times X$ and such that the following diagram commutes*

$$\begin{tikzcd} {[-\epsilon,\epsilon] \times X} \arrow[r, hook] \arrow[d, "(t,x)\mapsto t"] & M \arrow[d, "f"] \\ {[-\epsilon,\epsilon]} \arrow[r, "t\mapsto t+s"] & \mathbb{R}. \end{tikzcd}$$

If M is oriented, then the orientation on X can be chosen such that the smooth embedding $[-\epsilon, \epsilon] \times X \to M$ is orientation-preserving.

(5) *The preimage $W := f^{-1}((-\infty, s])$ is an n-dimensional submanifold of M with corner where*

$$\partial_0 W = f^{-1}(\{s\}), \quad \partial_1 W = W \cap \partial M \quad \text{and} \quad \partial_c W = f^{-1}(\{s\}) \cap \partial M.$$

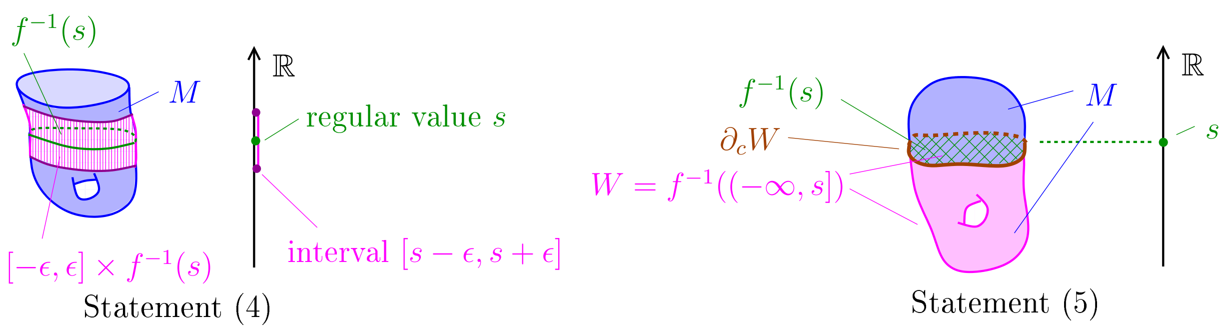


FIGURE 271. Illustration of the Regular Value Theorem 11.2.

Remark. The Regular Value Theorem 11.2 can be viewed as a refinement of Exercise 6.31.

SKETCH OF PROOF.

(4) Note that given any neighborhood U of $s \in \mathbb{R}$ there exists an $\epsilon > 0$ such that $[s-\epsilon, s+\epsilon] \subset U$. The statement now follows immediately from the Regular Value Theorem 11.1 (4) applied to the map $\varphi\colon [-\epsilon, \epsilon] \to \mathbb{R}$ given by $t \mapsto t + s$.

(5) If one takes a look at the proof of the Regular Value Theorem 11.1 (1) one sees that it actually provides submanifold charts of type (β) and (δ)[199] for $W := f^{-1}((-\infty, s])$ and it provides submanifold charts of type (β) for any point on $f^{-1}(\{s\})$. Also note that $f^{-1}((-\infty, s))$ is an open subset of M. We leave it to the reader to assemble from this observations the full proof of the statement. ■

We also mention the following variation on the Regular Value Theorem 11.1.

Theorem 11.3. (Regular Value Theorem for $N = S^1$) *Let M be an n-dimensional smooth manifold, let $f\colon M \to S^1$ be a smooth map and let $z = e^{2\pi \mathrm{i} t} \in S^1$ be a regular value. Then beyond statements (1), (2), (3) of the Regular Value Theorem 11.1 the following statement holds:*

(4) *If M is compact and orientable, then there exists an $\epsilon > 0$ and a tubular neighborhood $[-\epsilon, \epsilon] \times X$ of X such that $f^{-1}(\{e^{2\pi \mathrm{i}(t+s)} \,|\, t \in [-\epsilon, \epsilon]\}) = [-\epsilon, \epsilon] \times X$ and such that the following diagram commutes*

$$\begin{tikzcd} {[-\epsilon,\epsilon] \times X} \arrow[r, hook] \arrow[d, "(t,x)\mapsto t"'] & M \arrow[d, "f"] \\ {[-\epsilon,\epsilon]} \arrow[r, "s\mapsto e^{2\pi \mathrm{i}(t+s)}"] & S^1. \end{tikzcd}$$

If M is oriented, then the orientation on X can be chosen such that the smooth embedding $[-\epsilon, \epsilon] \times \to M$ is orientation-preserving.

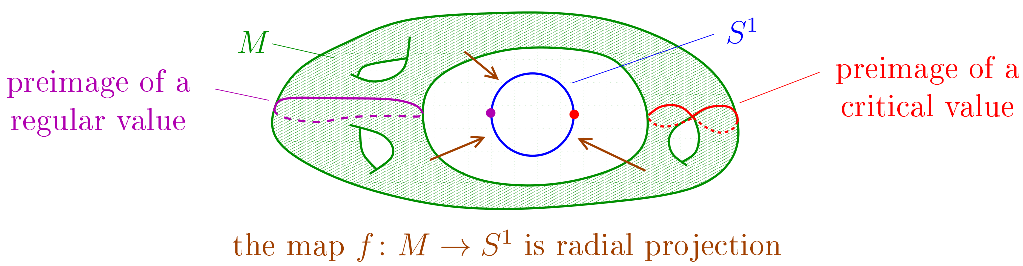


FIGURE 272. Illustration of the Regular Value Theorem 11.3.

Example. Let N be a smooth manifold and let $\varphi\colon N \to N$ be a diffeomorphism. We consider the mapping torus $M := \mathrm{Tor}(N, \varphi) = (N \times [0,1])/(x, 0) \sim (\varphi(x), 1)$ together with the canonical projection $f\colon M \to S^1$ given by $f([p, x)]) = e^{2\pi \mathrm{i} x}$. Then every $z = e^{2\pi \mathrm{i} x} \in S^1$ is a regular value and the preimage $f^{-1}(z) = N \times \{x\}$ is just a "parallel" copy of N.

PROOF. As in the case of Theorem 11.2 (4) this statement can be deduced easily from the Regular Value Theorem 11.1 (4). ■

The following proposition is a generalization of Proposition 6.64 to the case of smooth manifolds with non-compact boundary.

[199] Recall that these notions are defined on pages 270 and 2183.

Proposition 11.4. *Let W be an n-dimensional smooth manifold. There exists a sequence $X_1, X_2, \ldots$ of n-dimensional submanifolds* with corner *of W with the following three properties:*

(1) *The sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$.*
(2) *Each X_i is compact and connected.*
(3) *We have $\bigcup\limits_{i\in\mathbb{N}} \mathring{X}_i = W$.*

If ∂W is compact, then we can furthermore arrange that the following holds:

(4) *The corner set of the submanifolds with corner X_i is the empty set, in other words, the X_i are submanifolds in the sense of the definition on page 270.*

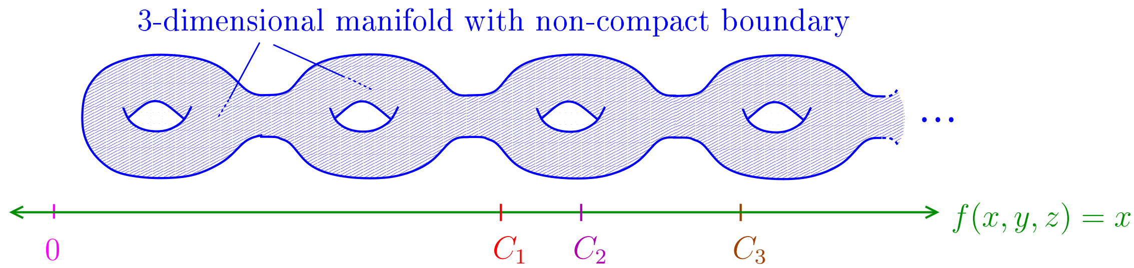


FIGURE 273. Illustration of the proof of Proposition 11.4

PROOF. The proof of Proposition 11.4 is almost identical to the proof of Proposition 6.64, for the most part we just need to replace the Exercise 6.31 by Regular Value Theorem 11.2. In particular we again pick a suitable smooth function $f: W \to \mathbb{R}$, an increasing sequence of regular values C_i with $\lim\limits_{i\to\infty} \mathrm{C}_i = \infty$ and given $i \in \mathbb{N}$ we define X_i to be a suitable path-component of $f^{-1}((-\infty, \mathrm{C}_i])$. We leave it to the reader to fill in the details. ■

We conclude this section with the following corollary which basically says that one can "wrap" any compact subset of a compact subset of a smooth manifold in a suitable codimension-zero compact submanifold.

Lemma 11.5. $(*)$ *Let M be a smooth manifold.*

(1) *Let K be a compact subset of $M \setminus \partial M$ and let U be a neighborhood of K in M. There exists a submanifold X such that $K \subset \mathring{X} \subset X \subset U$.*
(2) *Let K be any compact subset of M, let U be a neighborhood of K in M and let V be some neighborhood of $K \cap \partial M$ (evidently one could take $V = \partial M$). Then there exists a submanifold X with corner such that $K \subset \mathring{X} \subset X \subset U$ and such that $K \cap \partial M \subset \mathring{X} \cap \partial M \subset X \cap \partial M \subset V$.*

PROOF $(*)$. Let M be a smooth manifold. Recall that by Proposition 6.27 we know that ∂M is a closed subset of M.

(1) Let K be a compact subset of $M \setminus \partial M$. Basically by definition of a neighborhood we might as well assume that U is an open subset of M. Since ∂M is closed we can furthermore assume that U is an open subset of $M \setminus \partial M$. Since U is an open subset of $M \setminus \partial M$ it follows from Lemma 6.22 that we can view it as a smooth manifold in

its own right and that $\partial U = \varnothing$. Thus we can apply Proposition 11.4 to the smooth manifold U. It follows from the fact that K is compact, from Lemma 2.15 and from the fact that the submanifolds provided by Proposition 11.4 are nested that there exists a submanifold X of U with $K \subset \mathring{X} \subset X \subset U$. Since U is an open subset of M one obtains basically immediately that X is also a submanifold of M.

(2) We consider $W := U \setminus (\partial M \setminus V)$. It follows from the fact that ∂M is a closed subset of M, the fact that V is an open subset of ∂M and Lemma 2.38 that W is an open subset of M. By Lemma 6.22 we can view X as a smooth manifold in its own right with $\partial W = W \cap \partial M = \partial M \setminus (U \cup V)$. The proof of (2) is now almost identical to the proof of (1). We leave it to the reader to make the minute modifications. ■

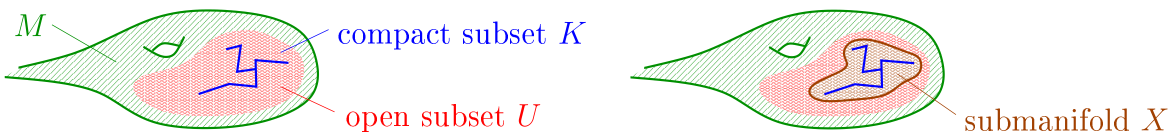


FIGURE 274. Illustration for Lemma 11.5 (1).

11.2. **Embeddings of (topological) smooth manifolds into $\mathbb{R}^n$ II.** In this and the coming section we study the existence of smooth embeddings of (topological) smooth manifolds into some sufficiently high-dimensional $\mathbb{R}^n$. For the reader's convenience we recall the following definitions from page 342.

Definition.

(1) A map $\varphi\colon N \to M$ between two topological manifolds is called a *embedding* if $\varphi\colon N \to \varphi(N)$ is a homeomorphism and if $\varphi(N)$ is a submanifold of M. (Here we view $\varphi(N)$ as a topological space equipped with the subspace topology coming from M.)

(2) A map $\varphi\colon N \to M$ between two smooth manifolds is called an *immersion* if the following conditions are satisfied:
   (a) φ is smooth,
   (b) for each $P \in N$ the induced map $\varphi_*\colon \mathrm{T}_P N \to \mathrm{T}_{\varphi(P)} M$ is a monomorphism,
   (c) for each $P \in N \setminus \partial N$ we have $\varphi(P) \in M \setminus \partial M$,
   (d) for each $P \in \partial N$ with $\varphi(P) \subset \partial M$ the image $\varphi_*(\mathrm{T}_P N)$ is *not* contained in $\mathrm{T}_{\varphi(P)}(\partial M)$.[200]

   An immersion that also satisfies the following extra condition
   (e) $\varphi\colon N \to \varphi(N)$ is an embedding,
   is called an *smooth embedding*.

Remark.

(1) Let $\varphi\colon N \to M$ be an injective map between topological manifolds. As is shown in Figure 276 such a map is not necessarily a smooth embedding. But it follows from Lemma 2.40 that such a map is an embedding under the added hypothesis that

[200]Another way of expressing this condition is that the map $\mathrm{T}_P N \to \mathrm{T}_{\varphi(P)} M / T_{\varphi(P)} \partial M$ is an epimorphism.

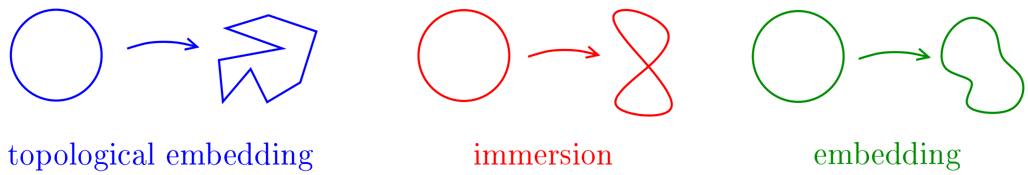


FIGURE 275

N is compact. On several occasions we will use this observation without saying it explicitly.

(2) The super-meticulous reader will have noticed that the above definition of a smooth embedding between smooth manifolds differs slightly from the definition we gave on page 342. In the original definition we had not demanded that the image of a smooth embedding is (a topological) submanifold, but instead we saw in Proposition 8.1 that this property follows already from the a priori slightly weaker notion of a smooth embedding that we had introduced on page 342. So any fears of incoherence in these notes just mercifully dissipated.

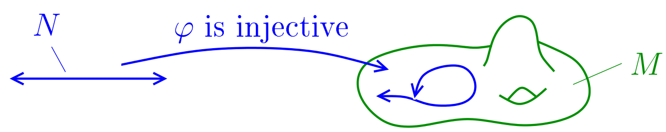


FIGURE 276

Examples.

(1) It is straightforward to verify that the map

$$\begin{array}{rcl} S^1 \times S^1 & \to & \mathbb{R}^3 \\ (e^{i\theta}, e^{i\varphi}) & \mapsto & ((2+\sin(\theta))\cdot\cos(\varphi), (2+\sin(\theta))\cdot\sin(\varphi), \cos(\theta)) \end{array}$$

is a smooth embedding of the 2-dimensional torus into $\mathbb{R}^3$.

(2) Let M be a closed non-empty n-dimensional topological manifold. In Corollary 50.8 we showed that there is no embedding of M into $\mathbb{R}^n$.

For the reader's convenience we recall the following proposition.

Proposition 9.1.

(1) *Given any smooth manifold M there exists some $n \in \mathbb{N}$ and a smooth embedding $M \to \mathbb{R}^n$.*

(2) *Given any topological manifold M there exists an $n \in \mathbb{N}$ and a map $\varphi\colon M \to \mathbb{R}^n$ which is an embedding.*

The question arises, how small n can be chosen with respect to the dimension of the manifold? For smooth manifolds we have the following result.

Proposition 11.6. *If M is an m-dimensional smooth manifold, then there exists a smooth embedding $M \to \mathbb{R}^{2n+1}$.*

The proof of Proposition 11.6 requires the following lemma, which is of interest in its own right.

Lemma 11.7. *Let M be an m-dimensional submanifold of $\mathbb{R}^n$. If $n > 2m+1$, then there exists a non-zero vector $v \in \mathbb{R}^n$ which does not lie in any tangent space*[201] $\mathrm{T}_P M$ *and that is not secant to M, in other words, for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to v.*

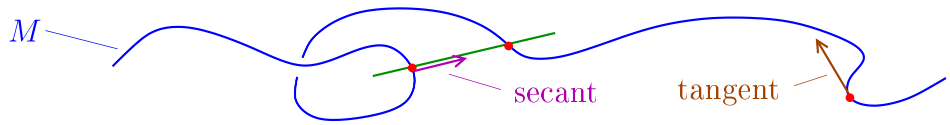


FIGURE 277. Illustration for Lemma 11.7.

PROOF OF LEMMA 11.7. Let M be an m-dimensional submanifold of $\mathbb{R}^n$. We suppose that $n > 2m+1$. We consider the maps

$$\begin{array}{rcl} f\colon \{(P,w) \,|\, P \in M, w \in \mathrm{T}_P M \setminus \{0\}\} & \to & \mathbb{RP}^{n-1} \\ (P,w) & \mapsto & [w] \end{array}$$

and

$$\begin{array}{rcl} g\colon \{(x,y) \in M \times M \,|\, x \neq y\} & \to & \mathbb{RP}^{n-1} \\ (x,y) & \mapsto & [x-y]. \end{array}$$

By the discussion on page 287 we can view $\mathbb{RP}^{n-1}$ as an $(n-1)$-dimensional smooth manifold. One can easily verify that the left-hand sides of both f and g are smooth manifolds of dimension $2m$ and that the two maps are smooth. Since $2m < n-1$ we obtain from Proposition 6.62 that the complement of the images of f and g is dense, in particular non-empty. This means that there exists a $v \in \mathbb{RP}^{n-1}$ such that the preimages of v under f and g are both the empty set. But that means that v has the desired properties. ■

Now we turn to the proof of Proposition 11.6.

PROOF OF PROPOSITION 11.6. Let M be a compact m-dimensional smooth manifold. We claim that we can can embed M into $\mathbb{R}^{2m+1}$. By Proposition 9.1 we can assume that M is already a submanifold of some $\mathbb{R}^n$. If $n \leq 2m+1$ there is nothing to prove, so we may assume that $n > 2m+1$. By Lemma 11.7 there exists a non-zero vector $v \in \mathbb{R}^n$ which does not lie in any tangent space $\mathrm{T}_P M$ and that is not secant to M, in other words, for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to v.

We consider the projection of $\mathbb{R}^n$ onto the orthogonal complement of v, i.e. we consider the map

$$\begin{array}{rcl} \mathbb{R}^n & \to & \{w \in \mathbb{R}^n \,|\, \langle w, v\rangle = 0\} \\ u & \mapsto & u - \langle u, v\rangle \cdot \frac{v}{\|v\|^2}. \end{array}$$

It follows easily from the claim that the restriction of the projection to M is a smooth embedding. Thus we have embedded M into a real vector space of dimension $n-1$. Iterating this process we get a smooth embedding of M into $\mathbb{R}^{2m+1}$. ■

In the remainder of this section and in the subsequent section we will discuss refinements of Proposition 11.6 for smooth manifolds. Many of the results also have more or less obvious

[201] Here we use Proposition 6.39 to interpret the tangent space $\mathrm{T}_P M$ as a vector subspace of $\mathbb{R}^n$.

analogues for topological manifolds, but to simplify the discussion we will not spell out the statements for topological manifolds.

Our first goal is to consider smooth manifolds with boundary in greater detail. We recall the following definition and notation.

Definition.

(1) Let N be a compact smooth manifold and let M be any smooth manifold. A map $\varphi\colon N \to M$ is called *proper* if $\varphi^{-1}(\partial M) = \partial N$.
(2) Given $n \in \mathbb{N}$ we write
$$H_n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\}.$$

For smooth manifolds with boundary the following proposition refines the statement of Proposition 9.1.

Proposition 11.8. *Let M be a compact m-dimensional smooth manifold.*

(1) *Given any $n \geq 2m+1$ there exists a proper smooth embedding $M \to H_n$.*
(2) *Suppose we are given a smooth embedding $f\colon \partial M \to \mathbb{R}^k$ of the boundary of M. Given any $n \geq 2m+k+1$ there exists a proper smooth embedding $F\colon M \to H_n$ such that the following diagram commutes:*

$$\begin{array}{ccc} \partial M & \hookrightarrow & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbb{R}^k & \xrightarrow{x \mapsto (x,0)} & H_n. \end{array}$$

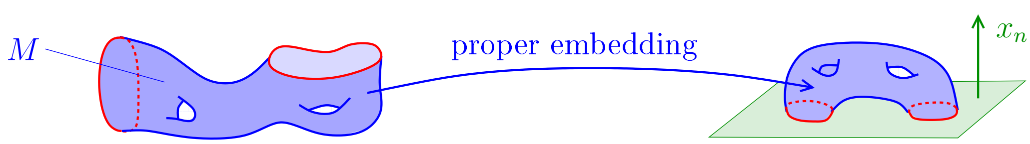


FIGURE 278. Illustration of Proposition 11.8 (1).

PROOF (∗). Both statements of the proposition follow from the following claim.[202]

Claim. Let M be a compact m-dimensional smooth manifold and let Y be a (possibly empty) union of components of ∂M. Suppose we are given a smooth embedding $f\colon Y \to \mathbb{R}^k$ of the boundary of M. Let $n \geq 2m+k+1$. Then there exists a proper smooth embedding $F\colon M \to H_n$ such that the following diagram commutes:

$$\begin{array}{ccc} Y & \hookrightarrow & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbb{R}^k & \xrightarrow{x \mapsto (x,0)} & H_n. \end{array}$$

[202] Indeed, the first statement of the proposition follows from $Y = \varnothing$ and $k = 0$. The second statement follows from setting $Y = \partial M$.

Evidently the proof of the claim is quite similar to the proofs of Proposition 9.1. and Proposition 11.6 Therefore we will only indicate in what ways the proof needs to be modified.

(a) By the Collar Neighborhood Theorem 8.12 we can pick a collar $[0,1] \times \partial M$. Note that using Lemma 6.13 one can easily construct a smooth non-decreasing function $\eta\colon [0,1] \to [0,1]$ with $\eta(t) = t$ for all $t \in [0, \frac{1}{4}]$ and with $\eta(t) = 1$ for all $t \in [\frac{1}{2}, 1]$. We define[203]

$$\begin{array}{rcl} \mu\colon M & \to & [0,1] \\ P & \mapsto & \begin{cases} \eta(t), & \text{if } P = (t, Q) \text{ with } t \in [0,1] \text{ and } Q \in \partial M, \\ 1, & \text{otherwise} \end{cases} \end{array}$$

and we define

$$\begin{array}{rcl} \widetilde{f}\colon M & \to & \mathbb{R}^k \\ P & \mapsto & \begin{cases} f(Q) \cdot (1 - \eta(t)), & \text{if } P = (t, Q) \text{ with } t \in [0,1] \text{ and } Q \in \partial M, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

In the following we use the same notation and definitions as in the proof of Proposition 9.1. In particular we pick a finite atlas $\{\Phi_i\colon U_i \to V_i\}_{i=1,\dots,s}$, we pick a corresponding smooth partition of unity $\nu_i\colon M \to [0,1]$, $i = 1, \dots, s$ and we define $\widetilde{\Phi}_i\colon M \to \mathbb{R}^m$, $i = 1, \dots, s$ the same way as before. Similar to the proof of Proposition 9.1 one can verify that the map

$$\begin{array}{rcl} F\colon M & \to & \mathbb{R}^k \times \mathbb{R}^{s\cdot m+s} \times \mathbb{R}_{\geq 0} = H_{k+s\cdot m+s+1} \\ x & \mapsto & \big(\underbrace{\widetilde{f}(x)}_{\in \mathbb{R}^k}, \underbrace{\mu(x) \cdot \big(\widetilde{\Phi}_1(x), \dots, \widetilde{\Phi}_s(x), \nu_1(x), \dots, \nu_s(x)\big)}_{\in \mathbb{R}^{s\cdot m+s}}, \underbrace{\mu(x)}_{\in \mathbb{R}_{\geq 0}}\big) \end{array}$$

is a proper smooth embedding. It is clear that the map F makes the diagram in the claim commute.

(b) To prove the claim it now remains to show that given such a proper smooth embedding $F\colon M \to H_n$ with $n > 2m+k+1$ we can find a lower-dimensional proper smooth embedding with the same properties. Without loss of generality we can assume that M is already a submanifold of H_n.

Similar to the proof of Proposition 11.6 we can find a non-zero vector $v \in \mathbb{R}^n$ with the following three properties:
(a) the vector does not lie in any tangent space $T_x M$,
(b) for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to v,
(c) the vector is orthogonal to $\mathbb{R}^k \times \{0\}$.[204]
Note that projection onto the hyperplane that is orthogonal to v is the identity on $\mathbb{R}^k \times \{0\}$. Thus as in the proof of Proposition 11.6 we can lower the dimension by one without losing any of the desired properties. ■

[203] It follows from Lemma 6.23 (2) that these two maps are smooth.
[204] Here we need that we have "extra k dimensions", i.e. we need that $n > 2m + 1 + k$.

As happens so often in mathematics, the proof of a result immediately invites new questions. For example, in light of Propositions 9.1, 11.6 and 11.8 the following question arises.

Question 11.9. *For a given (topological) smooth manifold M, e.g. $M = \mathbb{R}\mathrm{P}^n$, what is the minimal $k \in \mathbb{N}$ such that M admits a (topological) smooth embedding into $\mathbb{R}^k$?*

We will address this question in the following chapter.

We conclude this section with a discussion of the uniqueness of smooth embeddings of a given smooth manifold M into some $\mathbb{R}^n$. Let us consider the three smooth embeddings $f_0, f_1, f_2 \colon S^1 \to \mathbb{R}^3$ that are shown in Figure 279. The smooth embeddings f_0, f_1 are "essentially the same", in the sense that they are smoothly isotopic. On the other hand it follows from Proposition 27.6, Exercise 20.9 and the Isotopy Extension Theorem 8.27 that the two smooth embeddings f_0 and f_2 are not smoothly isotopic.

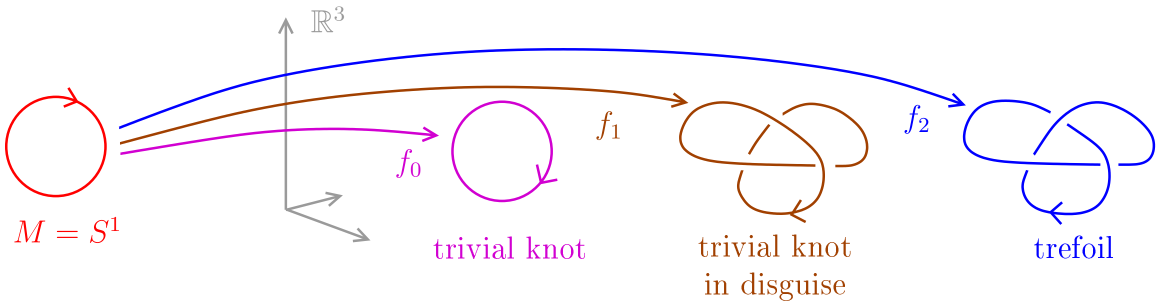


FIGURE 279

Now we identify $\mathbb{R}^3$ with $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$. Therefore we can view the above smooth embeddings f_0, f_1, f_2 as smooth embeddings $f_0, f_2 \colon S^1 \to \mathbb{R}^4$. In Figure 280 we do our best to illustrate an isotopy $F \colon S^1 \times [0,1] \to \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \subset \mathbb{R}^4$ from f_2 to f_1. The key to the understanding of the figure is that we use "pinkness" as the fourth coordinate, i.e. two points in $\mathbb{R}^3$ with different levels of "pinkness" differ in the fourth coordinate. With this convention the map $F \times S^1 \times [0,1] \to \mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ shown in Figure 280 has the property that each F_t is a smooth embedding. We have thus shown that f_2, f_1, viewed as smooth embeddings into $\mathbb{R}^4$, are smoothly isotopic.[205]

In the above longish example we have seen that smooth embeddings which are not smoothly isotopic can become smoothly isotopic once we add extra dimensions. The following proposition shows that this phenomenon holds in general. More precisely, the proposition says that any two smooth embeddings of a given closed smooth manifold into some $\mathbb{R}^k$ become smoothly isotopic if we are allowed to increase the dimensions.

Proposition 11.10. *Let M be a closed smooth manifold. Suppose we are given two smooth embeddings $f_0, f_1 \colon M \to \mathbb{R}^k$. There exists a smooth isotopy $F \colon M \times [0,1] \to \mathbb{R}^{2k}$ such that*

[205]In fact in Proposition 11.11 we will give a much more formal proof that in fact any two knots in $\mathbb{R}^3$ become smoothly isotopic once we view them as knots in $\mathbb{R}^4$.

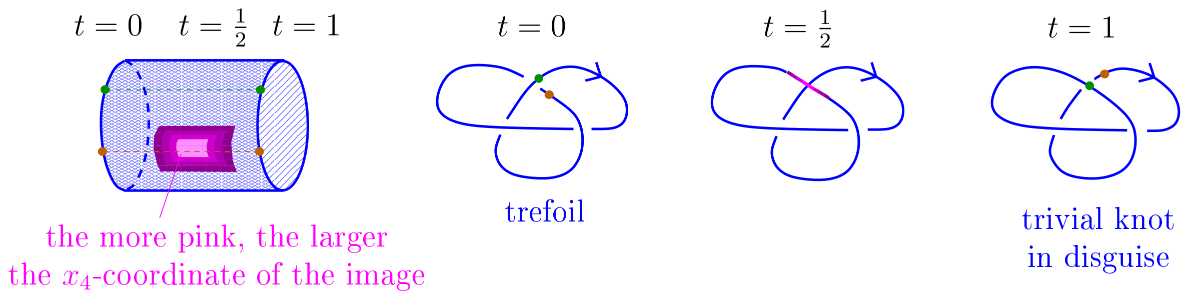


FIGURE 280

for $i=0,1$ the following diagram commutes

$$\begin{tikzcd} & M \arrow[dl, "f_i"'] \arrow[dr, "F_i"] & \\ \mathbb{R}^k \arrow[rr, "x\mapsto(x,0)"'] & & \mathbb{R}^{2k}. \end{tikzcd}$$

PROOF. By Lemma 6.13 there exists a smooth function $\nu\colon[0,1]\to[0,1]$ with $\nu(t)=1$ for $t\in[0,\frac{1}{4}]$ and $\nu(t)=0$ for $t\in[\frac{3}{4},1]$. We consider the map

$$\begin{array}{rcl} F\colon M\times[0,1] & \to & \mathbb{R}^{2k} \\ (x,t) & \mapsto & \big(\underbrace{f_0(x)\cdot\nu(t)+f_1(x)\cdot(1-\nu(t))}_{\in\mathbb{R}^k},\underbrace{f_0(x)\cdot(1-\nu(t))\cdot\nu(t)}_{\in\mathbb{R}^k}\big). \end{array}$$

It is straightforward to verify that F is a smooth isotopy and that F makes the given diagram commute for $i=0,1$. ■

Proposition 11.11. *Let M be a closed n-dimensional smooth manifold.*

(1) *Given any two smooth embeddings $f_0,f_1\colon M\to\mathbb{R}^{2n+2}$ there exists a smooth isotopy $F\colon M\times[0,1]\to\mathbb{R}^{2n+2}$ from f_0 to f_1.*
(2) *Given any two smooth embeddings $f_0,f_1\colon M\to S^{2n+2}$ there exists a smooth isotopy $F\colon M\times[0,1]\to S^{2n+2}$ from f_0 to f_1.*

Remark.

(1) In Proposition 27.6 we will see that there exist "non-trivial knots", more precisely, we will see that there exist two smooth embeddings $S^1\to S^3$ which are not smoothly isotopic. This shows that the conclusion of the dimension range in Proposition 11.11 is optimal, at least for $n=1$.
(2) Proposition 11.11 implies in particular that any two smooth embeddings $S^1\to\mathbb{R}^4$ are smoothly isotopic. Thus we have now obtained a formal proof that any two knots in $\mathbb{R}^3=\mathbb{R}^3\times\{0\}$ become smoothly isotopic once we view them as knots in $\mathbb{R}^4$.
(3) As usual, under extra hypotheses one can obtain better results. One such instance is given by Haefliger's Unknotting Theorem 27.35. More examples of such uniqueness statements are given in [**Wall16**, Chapter 6.4].

Proof of Proposition 11.11 (1). Let M be a closed n-dimensional smooth manifold. Suppose we are given two smooth embeddings $f_0, f_1 \colon M \to \mathbb{R}^{2n+2}$. By Proposition 11.10 there exists an $N \geq 2n+2$ and a smooth isotopy $F \colon M \times [0,1] \to \mathbb{R}^N$ such that for $i = 0, 1$ the following diagram commutes

$$\begin{tikzcd} & M \arrow[dl, "f_i"'] \arrow[dr, "F_i"] & \\ \mathbb{R}^{2n+2} \arrow[rr, "x \mapsto (x,0)"'] & & \mathbb{R}^N. \end{tikzcd}$$

If $N = 2n+2$, then we are done. Now suppose that $N > 2n+2$. To deal with this case we need to introduce some notation. We consider the map $G \colon M \times [0,1] \to \mathbb{R}^N \times [0,1]$ that is given by $(P, t) \mapsto (F(P,t), t)$. Since F is a smooth isotopy one can show easily that G is in fact a proper smooth embedding. Therefore $X := G(M \times [0,1])$ is a proper submanifold of $\mathbb{R}^N \times [0,1]$.

Claim. There exists a non-zero vector $v \in \mathbb{R}^N$ with

(1) v does not lie in $\mathbb{R}^{2n+2} \times \{0\}$,

and furthermore, such that for every $t \in [0,1]$ the following conditions are also satisfied by v:

(2) Given any $P \in M$ we have $v \notin \mathrm{T}_P(F_t(M))$.
(3) Given any two distinct points $x, y \in F_t(M)$ the difference $x - y$ is not parallel to v.

Evidently the idea for the proof of the claim is to tweak the argument in the proof of Lemma 11.7. First of all, we denote by $p \colon X \to [0,1]$ the projection map $(P,t) \mapsto t$. Next we consider the maps

$$\begin{array}{rcl} f \colon \{(P,w) \mid P \in X, w \in \mathrm{T}_P X \setminus \{0\} \text{ and } p_*(w) = 0\} & \to & \mathbb{RP}^{N-1} \\ (P,w) & \mapsto & [w] \end{array}$$

and

$$\begin{array}{rcl} g \colon \{(x,y) \in X \times X \mid x \neq y \text{ and } p(x) = p(y)\} & \to & \mathbb{RP}^{N-1} \\ (\underbrace{(x,t)}_{\in X}, \underbrace{(y,t)}_{\in X}) & \mapsto & [x-y]. \end{array}$$

By the discussion on page 287 we can view $\mathbb{RP}^{N-1}$ as an $(N-1)$-dimensional smooth manifold. It is fairly elementary to verify that the left-hand sides of both f and g are smooth manifolds of dimension $2 \cdot \dim(X) - 1 = 2n+1$ and that the two maps f and g are smooth. Our hypothesis that $N > 2n+2$ implies that $2n+1 < N-1$. Thus we obtain from Proposition 6.62 that the complement of the images of f and g is dense. This implies that there exists a $[v] \in \mathbb{RP}^{N-1}$ such that $v \notin \mathbb{R}^{2n+2} \times \{0\}$ and such that the preimages of $[v]$ under f and g are both the empty set. One can easily verify that this means that v has the desired properties. ⊞

Now we proceed with the following steps:

(i) Since v does not lie in $\mathbb{R}^{2n+2} \times \{0\}$ we can pick a codimension one subvector space $W \subset \mathbb{R}^N$ which contains $\mathbb{R}^{2n+2} \times \{0\}$ and which has the property that $W \oplus \mathbb{R} \cdot v = \mathbb{R}^N$.
(ii) We denote by $q \colon W \oplus \mathbb{R} \cdot v = \mathbb{R}^N \to W$ the projection.

(iii) Since W contains $\mathbb{R}^{2n+2} \times \{0\}$ we can pick an isomorphism $\Phi \colon W \to \mathbb{R}^{N-1}$ such that $\Phi(P, 0) = (P, 0)$ for all $P \in \mathbb{R}^{2n+2}$.

Next we consider the map

$$\begin{array}{rcl} \widetilde{F} \colon M \times [0,1] & \to & \mathbb{R}^{N-1} \\ (P,t) & \mapsto & (\Phi \circ q)(F(P,t)). \end{array}$$

We make the following observations:

(a) It follows immediately from (iii) that for any $P \in M$ we have $\widetilde{F}_0(P) = (f_0(P), 0)$ and $\widetilde{F}_1(P) = (f_1(P), 0)$.

(b) It follows from Properties (2) and (3) of v that $\widetilde{F}$ is a smooth isotopy.

Thus we have shown that f_0 and f_1 are isotopic in $\mathbb{R}^{N-1}$. By iterating this process we see that there exists a smooth isotopy between f_0 and f_1 in $\mathbb{R}^{2n+2}$. ■

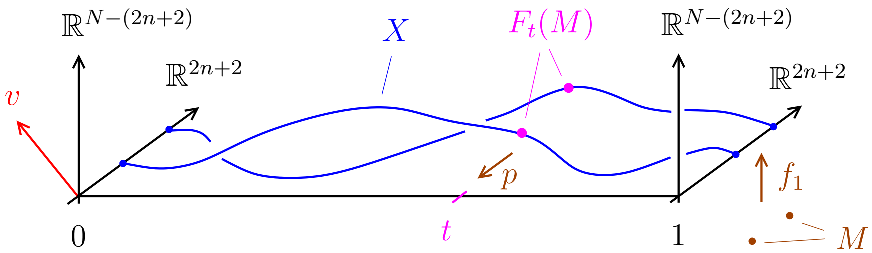


Figure 281. Illustration for the proof of Proposition 11.11 (1).

Proof of Proposition 11.11 (2). The case of two smooth embeddings $M \to S^{2n+2}$ can be reduced easily to the case of smooth embeddings $M \to \mathbb{R}^{2n+2}$. We will work out the details in Exercise 11.9. ■

The final proposition of this section is the "boundary version" of Proposition 11.10. This proposition will play an important role later in Section 117.1. But till then it is best to ignore it.

Proposition 11.12. $(*)$ *Let M be a compact smooth manifold and let Y be a (possibly empty) union of boundary components of M. Suppose we are given two proper smooth embeddings $f_0, f_1 \colon M \to H_k$.*

(1) *If $f_0|_Y = f_1|_Y$, then there exists a proper smooth isotopy $F \colon M \times [0,1] \to H_{2k}$ rel Y such that for $i = 0, 1$ the following diagram commutes*

$$\begin{tikzcd} & M \arrow[dl, "f_i"'] \arrow[dr, "F_i"] & \\ H_k \arrow[rr, "x \mapsto (0,x)"'] & & H_{2k}. \end{tikzcd}$$

(2) *If $f_0(Y) = f_1(Y)$, then there exists a proper smooth isotopy $F \colon M \times [0,1] \to H_{2k}$ rel Y such that for $i = 0, 1$ the image of $f_i(M)$ under the obvious inclusion $H_k \to H_{2k}$ agrees with $F_i(M)$.*

The analogous statements also hold if we replace H_k by $[0,1]\times\mathbb{R}^{k-1}$ and if we replace H_{2k} by $[0,1]\times\mathbb{R}^{2k-1}$.

Proof (∗).

(1) We suppose that $f_0|_Y = f_1|_Y$. As in the proof of Proposition 11.10 we pick a smooth function $\nu\colon [0,1]\to[0,1]$ with $\nu(t)=1$ for $t\in[0,\frac14]$ and $\nu(t)=0$ for $t\in[\frac34,1]$. Furthermore, as in the proof of Proposition 11.8 we pick a collar $[0,1]\times\partial M$, we pick a non-decreasing smooth function $\eta\colon[0,1]\to[0,1]$ with $\eta(t)=t$ for all $t\in[0,\frac14]$ and with $\eta(t)=1$ for all $t\in[\frac12,1]$ and we consider the smooth functions

$$\begin{array}{rcl} \mu\colon M & \to & [0,1] \\ P & \mapsto & \begin{cases} \eta(t), & \text{if } P=(t,Q) \text{ with } t\in[0,1] \text{ and } Q\in Y, \\ 1, & \text{otherwise} \end{cases} \end{array}$$

and

$$\begin{array}{rcl} \epsilon\colon M & \to & [0,1] \\ P & \mapsto & \begin{cases} \eta(t), & \text{if } P=(t,Q) \text{ with } t\in[0,1] \text{ and } Q\in \partial M, \\ 1, & \text{otherwise.} \end{cases} \end{array}$$

Finally we consider the map

$$\begin{array}{rcl} M\times[0,1] & \xrightarrow{F} & \mathbb{R}^k\times H_k = H_{2k} \\ (x,t) & \mapsto & (\underbrace{\mu(x)\cdot(f_0(x)\cdot(1-\eta(t))\cdot\eta(t))}_{\in\mathbb{R}^k}, \underbrace{f_0(x)\cdot\eta(t)+f_1(x)\cdot(1-\eta(t))+\epsilon(x)\cdot e_k}_{\in H_k}). \end{array}$$

It is straightforward to verify that F is a proper smooth isotopy rel Y and that F makes the given diagram commute for $i=0,1$.

(2) Now we only suppose that $f_0(Y)=f_1(Y)$. We consider the map $\varphi := f_1^{-1}\circ f_0\colon M\to M$. By Proposition 8.1 (2) we know that φ is a diffeomorphism of M. Note that the smooth embeddings $f_0\colon M\to H_k$ and $f_1\circ\varphi\colon M\to H_k$ agree on Y. Thus we can apply (1). The proper smooth isotopy that we obtain this way evidently has the desired property.

The above proofs also apply, with minuscule modifications, if we replace H_k by $[0,1]\times\mathbb{R}^{k-1}$ and if we replace H_{2k} by $[0,1]\times\mathbb{R}^{2k-1}$. ■

Example. We consider the two smooth embeddings $f_0, f_1\colon M=[0,1]\to H_3$ shown in Figure 282. They agree on $Y=\{0\}$. By Proposition 11.12 these two smooth embeddings become smoothly isotopic rel Y if we view f_0 and f_1 as smooth embeddings into H_6.

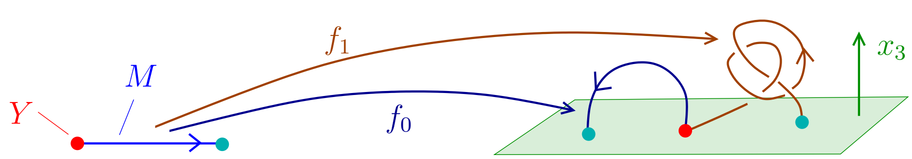


Figure 282

11.3. **The Whitney Embedding Theorem** ($*$)**.** In this section we state several propositions and theorems that generalize the results from the previous section to larger classes of smooth manifolds and that give slightly improved bounds on the dimensions required to obtain (unique) smooth embeddings. We will not make use of the results, therefore we will not provide any proofs.

The following proposition can be viewed as a moderately difficult generalization of Proposition 11.6.

Proposition 11.13.

(1) *Every n-dimensional smooth manifold M without boundary admits a proper smooth embedding $f\colon M \to \mathbb{R}^{2n+1}$ such that $f(M)$ is a closed subset of $\mathbb{R}^{2n+1}$.*
(2) *Every n-dimensional smooth manifold M with non-empty boundary admits a proper smooth embedding $f\colon M \to H_{2n+1}$ such that $f(M)$ is a closed subset of H_{2n+1}.*

Proof. As indicated above, it is an entertaining and slightly challenging exercise to modify the proofs of Proposition 11.6 and 11.8 to obtain the results. The argument is carried out in [**Why36**, Theorem 1], [**Bre93**, Theorem II.10.8], [**Hirs76**, Theorems 1.3.5 and 1.4.3] and [**Lee02**, Theorem 6.15]. ■

Remark. Note that in Proposition 11.13 it is essential that smooth manifolds are, by definition, second-countable. If we did not make this assumption, then any set with the discrete topology would be a 0-dimensional smooth manifold, but there are sets whose cardinality is bigger than the cardinality of any $\mathbb{R}^n$. More precisely, by Cantor's Theorem, see page 80, the power set of $\mathbb{R}$ does not admit an injection into any $\mathbb{R}^n$.

In Propositions 11.6, 11.8 and 11.13 we had in particular seen that every n-dimensional smooth manifold admits a smooth embedding into $\mathbb{R}^{2n+1}$. The following theorem decreases the dimension by one. Even though this looks like a minor achievement, the proof of the Whitney Embedding Theorem is quite intricate.

Theorem 11.14. (Whitney Embedding Theorem) *Every n-dimensional smooth manifold admits a smooth embedding into $\mathbb{R}^{2n}$.*

Proof. This result was first proved by Hassler Whitney [**Why44a**, p. 236] in 1944. For closed smooth manifolds more recent accounts of the proof can be found in [**Adac93**, Theorem II.2.11] or [**Pra07**, Chapter 6.3]. ■

Remark.

(1) Under some extra conditions one can obtain better results. For example we saw on page 309 that every closed orientable 2-dimensional smooth manifold embeds into $\mathbb{R}^3$. Furthermore Hirsch [**Hirs61**] and Wall [**Wall65b**] showed that every 3-dimensional smooth manifold embeds into $\mathbb{R}^5$. Finally for $n > 4$ Haefliger-Hirsch [**HH63**, Theorem 1.2] showed that every closed orientable n-dimensional smooth manifold embeds into $\mathbb{R}^{2n-1}$.
(2) In [**Mun75**, p. 315] it is shown that every n-dimensional topological manifold admits an embedding into $\mathbb{R}^{2n+1}$. At least for compact topological manifolds one can actually

find an embedding into $\mathbb{R}^{2n}$. We refer to [**MeS06**, Corollary 4.14] and [**BrM99**] for details.

(3) If M and N are two smooth manifolds that can be embedded (respectively immersed) into $\mathbb{R}^n$, then the construction in [**Hae61a**, Chapter 1] [**Hae67**, Chapter 3] shows that the connected sum $M \# N$ can also be embedded in $\mathbb{R}^n$.

(4) Let M be a compact connected smooth manifold and let N be a simply connected smooth manifold. If $\dim(N) = 2 \cdot \dim(M)$ and $\dim(M) \geq 3$, then [**Wall16**, Theorem 6.3.4] says that any map $M \to N$ is homotopic to a smooth embedding. As is explained on [**Wall16**, p. 182] the various hypothesis on M, N and the dimensions are necessary.

The following theorem is a refinement of Propositions 11.10, 11.11 and 11.12.

Theorem 11.15.

(1) *Let M be a closed n-dimensional smooth manifold. If $m \geq 2n+1$ and $m \geq n+3$, then any two smooth embeddings $f_0, f_1 \colon M \to \mathbb{R}^m$ are smoothly isotopic.*

(2) *Let M be a compact n-dimensional smooth manifold. If $m \geq 2n+1$ and $m \geq n+3$, then any two proper smooth embeddings $f_0, f_1 \colon M \to H_m$ are related by a proper smooth isotopy.*

Proof. Both statements follow from [**Hud72**, Theorem 4]. For connected smooth manifolds the first statement also follows from [**Hae61a**, p. 47] or alternatively from [**Wall16**, Theorem 6.4.11]. ■

Remark.

(1) We consider again the two smooth embeddings $f_0, f_2 \colon S^1 \to \mathbb{R}^3$ from Figure 279. As we pointed out on page 437, these two smooth embeddings are not smoothly isotopic. This shows that in Theorem 11.15 we not only need the condition that $m \geq 2n+1$ (which is satisfied in this example) but also the condition that $m \geq n+3$.

(2) The statements of Theorem 11.15 can be generalized considerably, in particular the target smooth manifolds do not necessarily need to be $\mathbb{R}^m$ and H_m. We refer to [**Hae61a**, p. 47], [**Hud72**], [**Sko**] and [**Adac93**, Theorem II.2.3] for many generalizations.

Sometimes it is also interesting to look at immersions instead of embeddings. In most cases one gets analogues of the above statements, except that one can lower the dimension of the target by one. For example the following variation on Proposition 11.6 holds.

Proposition 11.16. *Every n-dimensional smooth manifold admits an immersion into $\mathbb{R}^{2n}$.*

Proof. We will prove Proposition 11.16 in Exercise 11.8. ■

Furthermore, with significantly more work one can prove the following theorem which is the analogue of the Whitney Embedding Theorem 11.14.

Theorem 11.17. (Whitney Immersion Theorem) *Every smooth manifold of dimension $n \geq 2$ admits an immersion into $\mathbb{R}^{2n-1}$.*

PROOF. This theorem is proved in [**Why44b**, p. 247] and [**Wall16**, Theorem 6.3.6]. ■

For the record we also state the following theorem which is the analogue of Theorem 11.15 (1).

Theorem 11.18. *Let M be a closed connected n-dimensional smooth manifold and let $N = \mathbb{R}^k$ or $N = S^k$. Furthermore let $f_0, f_1: M \to N$ be two smooth embeddings. If $k \geq 2n$, then there exists a smooth homotopy $F: M \times [0,1] \to N$ from f_0 to f_1 and a finite subset $T \subset [0,1]$ with the following properties:*

(1) *For each $t \in [0,1] \setminus T$ the map $F_t: M \to N$ is an embedding.*
(2) *For each $t \in T$ the map $F_t: M \to N$ is a self-transverse immersion without triple points. (We refer to pages 407 and 408 for the definition of "self-transverse" and "without triple points".)*

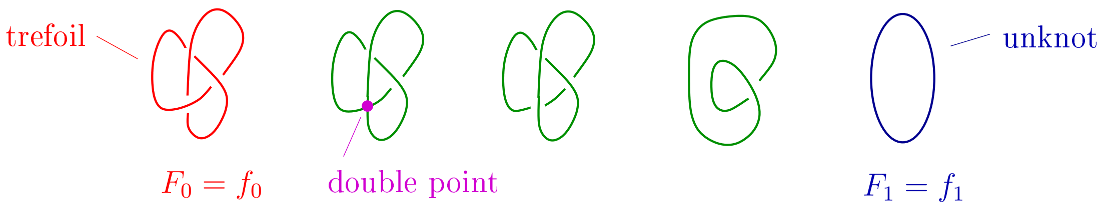


FIGURE 283

PROOF. This theorem can be proved following the techniques of the proofs of [**Wall16**, Theorem 6.4.11] and [**Adac93**, Theorem II.6.2.8]. We leave the verification of this somewhat bold assertion to the reader. ■

Remark. It does not take much effort to use Theorem 11.18 (1) to give an alternative proof that the Gordian distance between knots, as defined on page 809 is actually finite.

Exercises for Chapter 11.

Exercise 11.1. Let M be a compact smooth manifold. We want to show that there is no retraction from M to ∂M. Thus suppose that there exists a retraction from M to ∂M.

(a) Use the Whitney Approximation Theorem 9.3 to show that there exists a smooth retraction g from M to ∂M.
(b) By Sard's Theorem 6.63 we know that g admits a regular value y. What does Theorem 11.1 say about $g^{-1}(\{y\})$? Show that this contradicts the classification of 1-dimensional smooth manifolds that we gave in Theorem 7.5.

Remark. In Corollary 87.28 we will give an alternative proof of the fact that there is no retraction from M to ∂M.

Exercise 11.2. Let M be a compact n-dimensional smooth manifold and let K and L be compact subsets of M. Show that there exist compact n-dimensional submanifolds with corner A and B of M such that $K \subset \mathring{A}$, $L \subset \mathring{B}$, $A \cup B = M$ and such that $A \cap B = \partial A = \partial B$.
Remark. This exercise is a generalization of Exercise 6.36 (2). In particular you should make use of Exercise 6.35.

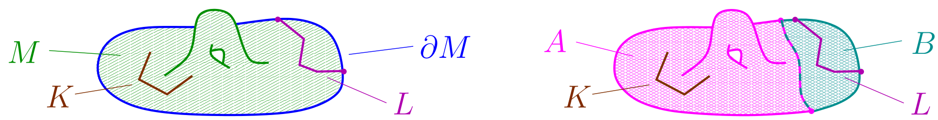


FIGURE 284. Illustration of Exercise 11.2.

Exercise 11.3. Let M be a smooth manifold and let V and W be two disjoint closed subsets. Show that for any $\epsilon > 0$ there exists a smooth function $f\colon M \to \mathbb{R}$ such that $f(V) \subset (-\epsilon, \epsilon)$ and $f(W) \subset (1-\epsilon, 1+\epsilon)$.
Hint. Use Urysohn's Lemma 2.50.

Exercise 11.4. Let M be a smooth manifold, let U be an open subset of M and let K be a compact subset of M that is contained in U. Show that there exists a compact codimension-zero submanifold N of M with $K \subset N \subset U$.

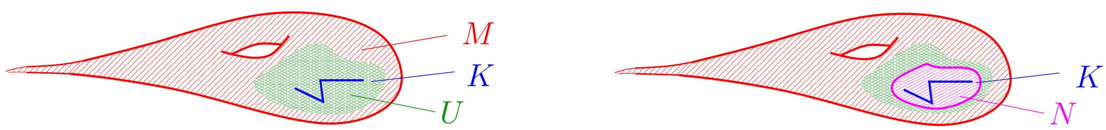


FIGURE 285

Exercise 11.5. Let $f_0, f_1\colon S^1 \to \mathbb{R}^3$ be two smooth embeddings. As usual we identify $\mathbb{R}^3$ with $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$. By Proposition 11.11 we know that $f_0, f_1\colon S^1 \to \mathbb{R}^4$ are isotopic. Show that f_0 and f_1 are also isotopic as maps $S^1 \to \mathbb{R}^3 \times \mathbb{R}_{\geq 0}$.

Exercise 11.6. Let M be a compact m-dimensional smooth manifold. Show by hand that there exists an immersion $f\colon M \to \mathbb{R}^{2m}$.
Hint. Modify the proof of Proposition 11.6.

Exercise 11.7. We consider the Möbius band $M = ([0,1] \times [-1,1])/(0,x) \sim (1,-x)$ together with the projection map

$$\begin{array}{rcl} p\colon M = ([0,1]\times[-1,1])/\sim & \to & [0,1]/0\sim 1 \\ {[(x,y)]} & \mapsto & [x]. \end{array}$$

Show that this map defines a linear interval bundle.

Exercise 11.8. Let M be an m-dimensional smooth manifold. Show that there exists an immersion $M \to \mathbb{R}^{2m}$.
Hint. Modify the proof of Proposition 11.6.

Exercise 11.9.

(a) Let M be a closed m-dimensional smooth manifold and let $f\colon M \to S^l = \mathbb{R}^l \cup \{\infty\}$ be a smooth embedding. Show that if $m < l$, then f is smoothly isotopic to a smooth embedding $gf\colon M \to \mathbb{R}^l$.

(b) Let M be a closed n-dimensional smooth manifold. Show that given any two smooth embeddings $f_0, f_1\colon M \to S^{2n+2}$ there exists a smooth isotopy $F\colon M \times [0,1] \to S^{2n+2}$ from f_0 to f_1.
Hint. Use Proposition 11.11 (1).

12. Complex manifolds

In this chapter we start out with a very short introduction to complex manifolds.

12.1. **Holomorphic functions.** First we recall the definition of a holomorphic function.

Definition. Let $U \subset \mathbb{C}$ be an open subset. A function $f\colon U \to \mathbb{C}$ is *holomorphic* if for every $z_0 \in U$ the limit

$$\frac{d}{dz} f(z_0) \;:=\; f'(z_0) \;:=\; \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \;\in \mathbb{C}$$

exists.

Example. In any textbook on complex analysis it is shown that polynomials, the exponential function, and more generally functions defined by power series are holomorphic. Moreover products, fractions, sums and compositions of holomorphic functions are again holomorphic.

The definition of a complex function being holomorphic is almost verbatim the same as the definition of a real function being differentiable. But rather surprisingly being holomorphic is significantly more restrictive, as is shown in the following two standard results in complex analysis, see e.g. [**Jä11**] or [**Lan99**].

Theorem 12.1. (Liouville's Theorem) *Every bounded holomorphic map $f\colon \mathbb{C} \to \mathbb{C}$ is constant.*

Theorem 12.2. (Maximum Principle) *Let U be an open connected subset of $\mathbb{C}$ and let $f\colon U \to \mathbb{C}$ be a holomorphic function. If the function $|f|\colon U \to \mathbb{R}_{\geq 0}$ has a local maximum, then f is constant.*

The following proposition also belongs to the usual repertoire of a course on complex analysis, see e.g. [**Lan99**, Chapter IX.1].

Proposition 12.3. (Schwarz Reflection Principle) *Let U be an open subset of the upper half-plane $\{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\}$ and let $f\colon U \to \mathbb{C}$ be a continuous function with the following properties:*

(1) *f is holomorphic on $\mathring{U} = \{z \in U \mid \operatorname{Im}(z) > 0\}$,*
(2) *f only assumes real values on $U \cap \mathbb{R}$.*

We set $U' := \{\overline{z} \mid z \in U\}$, i.e. U' is the reflection of U in the x-axis. Then the function

$$\begin{aligned} \widetilde{f}\colon U \cup U' &\to \mathbb{C} \\ z &\mapsto \begin{cases} f(z), & \text{if } z \in U, \\ \overline{f(\overline{z})}, & \text{if } z \in U', \end{cases} \end{aligned}$$

is holomorphic.

12.2. **Complex manifolds.** Given a topological manifold we now introduce the notion of a holomorphic atlas and a complex manifold. The definition is almost the same as for a smooth atlas and a smooth manifold, except that we have to replace "smooth" by "holomorphic".

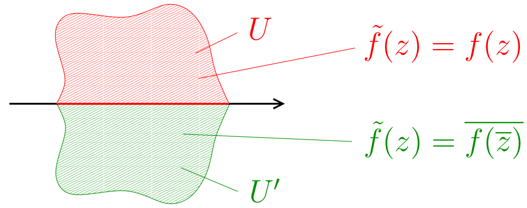


FIGURE 286. Illustration of the Schwarz reflection principle.

Definition.

(1) Let $U \subset \mathbb{C}^m$ be open. We say that a map $f = (f_1, \dots, f_n)\colon U \to \mathbb{C}^n$ is *holomorphic* if for every $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$ and any $(w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_m) \in \mathbb{C}^{m-1}$ the map

$$\begin{array}{rcl} \overbrace{\{z \in \mathbb{C} \mid (w_1, \dots, w_{i-1}, z, w_{i+1}, \dots, w_m) \in U\}}^{\text{open subset of } \mathbb{C}} & \to & \mathbb{C} \\ z & \mapsto & f_j(w_1, \dots, w_{i-1}, z, w_{i+1}, \dots, w_m) \end{array}$$

is holomorphic

(2) Let M be a $2n$-dimensional topological manifold M without boundary. An atlas $\{\Phi_i\colon U_i \to V_i\}_{i \in I}$ for M is called *holomorphic* if for all $i, j \in I$ the transition map

$$\Phi_j \circ \Phi_i^{-1}\colon \underbrace{\Phi_i(U_i \cap U_j)}_{\text{open in } \mathbb{R}^{2n} = \mathbb{C}^n} \to \underbrace{\Phi_j(U_i \cap U_j)}_{\subset \mathbb{R}^{2n} = \mathbb{C}^n}$$

is holomorphic.

(3) An *n-dimensional complex manifold* is a pair $(M, \mathcal{A})$ consisting of a $2n$-dimensional topological manifold M without boundary, together with a holomorphic atlas for M.

(4) We define the *maximal holomorphic atlas* of a complex manifold the same way as we defined the maximal smooth atlas of a smooth manifold, see page 279.

Remark. Since holomorphic maps are in particular smooth we see that every n-dimensional complex manifold is in particular a $2n$-dimensional smooth manifold.

Examples.

(1) Let U be an open subset of $\mathbb{C}^n$. The identity map is an atlas for U and it is evidently a holomorphic atlas. Thus U is an n-dimensional complex manifold.

(2) We consider the action

$$\begin{array}{rcl} (\mathbb{Z} \oplus \mathbb{Z}\mathrm{i}) \times \mathbb{C} & \to & \mathbb{C} \\ ((a + b\mathrm{i}), z) & \mapsto & z + a + b\mathrm{i}. \end{array}$$

The same argument as in the proof of Proposition 6.32 shows that $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\mathrm{i})$ is a 1-dimensional complex manifold. The topological space $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\mathrm{i})$ is evidently diffeomorphic to the torus $\mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$. Thus we can view the torus $S^1 \times S^1$ as a complex manifold.

Definition. We say that a map $f\colon M \to N$ between two complex manifolds is *holomorphic* if f is continuous and if for every chart $\Phi\colon U \to V$ in the holomorphic atlas of M and every chart $\Psi\colon W \to Y$ in the holomorphic atlas of N the map

$$\Phi(f^{-1}(W) \cap U) \xrightarrow{\Psi\circ f\circ\Phi^{-1}} Y$$

is holomorphic.

Example. It follows easily from the definitions that the projection map $\mathbb{C} \to \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\mathrm{i})$ is holomorphic.

The following proposition will be proved in Exercise 12.1.

Proposition 12.4. *Let M be a compact 1-dimensional complex manifold. Then any holomorphic function $f\colon M \to \mathbb{C}$ is constant.*

Remark. If $M = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\mathrm{i})$, then we can also give another, arguably easier proof. More precisely, let $f\colon \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\mathrm{i}) \to \mathbb{C}$ be a holomorphic function. Since $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\mathrm{i})$ is compact the function f is bounded. We denote by $p\colon \mathbb{C} \to \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\mathrm{i})$ the projection map. Then the composition $f \circ p\colon \mathbb{C} \to \mathbb{C}$ is a bounded holomorphic function, thus constant by the Liouville Theorem 12.1. Since p is surjective it follows that f is also constant.

In the next definition we generalize the notion of (local) diffeomorphism in the obvious way to the complex setting.

Definition.

(1) We say that a map $f\colon M \to N$ between two complex manifolds is a *biholomorphism* if f is a bijection and if both f and f^{-1} are holomorphic.
(2) We say that a map $f\colon M \to N$ between two complex manifolds is a *local biholomorphism* if given any $P \in M$ there exists an open neighborhood U of P and an open neighborhood V of $f(P)$ such that $f\colon U \to V$ is a biholomorphism.
(3) We say that two holomorphic atlases for a given topological manifold are *equivalent*, if the identity map is a biholomorphism between the two complex manifolds corresponding to the complex atlases. A *complex structure* on a topological manifold is a choice of an equivalence class.

Examples.

(1) In some cases one can explicitly write down a biholomorphism. For example, for the upper half-plane $\mathbb{H} = \{x + \mathrm{i}y \mid y > 0\}$ and the open disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ it is straightforward to verify that the maps

$$\begin{array}{rccc} \Phi\colon \mathbb{H} & \to & \mathbb{D} \\ z & \mapsto & \frac{z-\mathrm{i}}{z+\mathrm{i}} \end{array} \quad \text{and} \quad \begin{array}{rccc} \Psi\colon \mathbb{D} & \to & \mathbb{H} \\ w & \mapsto & \frac{i+\mathrm{i}w}{1-w} \end{array}$$

are biholomorphisms that are inverse to one another.
(2) The proof of Proposition 6.32 shows that $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\mathrm{i})$ has the structure of a 1-dimensional complex manifold in such a way that the projection map $\mathbb{C} \to \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ is a local biholomorphism.

(3) Let $a, b \in \mathbb{C}$ be two complex numbers that are linearly independent over $\mathbb{R}$. We write $T(a,b) = \mathbb{C}/(\mathbb{Z}a + \mathbb{Z}b)$. The same argument as in (2) shows that this is a 1-dimensional complex manifold such that the projection map $p\colon \mathbb{C} \to T(a,b)$ is a local biholomorphism. Now suppose that $v, w \in \mathbb{C}$ are also two complex numbers that are linearly independent over $\mathbb{R}$. We can form the torus $T(v,w) = C/(\mathbb{Z}v + \mathbb{Z}w)$. The tori $T(a,b)$ and $T(v,w)$ are diffeomorphic. But the argument of [**Bal06**, p. 15] shows that the 1-dimensional complex manifolds $T(a,b)$ and $T(v,w)$ are biholomorphic only if $ab^{-1} = vw^{-1} \in \mathbb{C}$ or if $ab^{-1} = wv^{-1} \in \mathbb{C}$.

The next lemma gives us an example of a complex manifold that will play a major role later on.

Lemma 12.5. *Let $n \in \mathbb{N}$. On page 194 we introduced the complex projective space*

$$\mathbb{CP}^n \;=\; (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\}).$$

For $i = 0, \dots, n$ we consider the map

$$\begin{aligned} \Phi_i\colon \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n \mid z_i \neq 0\} &\to \mathbb{C}^n \\ [z_0 : \cdots : z_n] &\mapsto \left(\tfrac{z_0}{z_i}, \dots, \tfrac{z_{i-1}}{z_i}, \tfrac{z_{i+1}}{z_i}, \dots, \tfrac{z_n}{z_i}\right). \end{aligned}$$

These maps form a holomorphic atlas for $\mathbb{CP}^n$ that turn the complex projective space $\mathbb{CP}^n$ into a closed n-dimensional complex manifold, in particular we can view $\mathbb{CP}^n$ as a closed $2n$-dimensional oriented real smooth manifold.

PROOF. In Proposition 3.40 (see also Exercise 3.39) we saw that each $\mathbb{CP}^n$ is compact and Hausdorff. As we will show below, $\mathbb{CP}^n$ admits a finite atlas, thus it follows from Lemma 6.4 that $\mathbb{CP}^n$ is second-countable. For the remaining statements it is easy to overlook that first one needs to prove the following claim.

Claim. If $i \in \{0, \dots, n\}$, then

$$V_i \;:=\; \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n \mid z_i \neq 0\}$$

is an open subset of $\mathbb{CP}^n$.

The claim is equivalent to showing that the complement $\mathbb{CP}^n \setminus V_i$ is a closed subset of $\mathbb{CP}^n$. We note that

$$\mathbb{CP}^n \setminus V_i \;=\; \text{the image of the map } \begin{aligned} f\colon \mathbb{CP}^{n-1} &\to \mathbb{CP}^n \\ [w_0 : \ldots : w_{n-1}] &\mapsto [w_0 : \ldots : w_{i-1} : 0 : w_i : \ldots : w_{n-1}]. \end{aligned}$$

The map f is easily seen to be continuous. Since $\mathbb{CP}^{n-1}$ is compact we obtain from Lemma 2.40 that $f(\mathbb{CP}^{n-1})$ is compact. But since $\mathbb{CP}^n$ is Hausdorff we obtain from Lemma 2.17 (2) that $f(\mathbb{CP}^{n-1}) = \mathbb{CP}^n \setminus V_i$ is in fact a closed subset of $\mathbb{CP}^n$. ⊞

Using this claim it is now straightforward to verify that the given maps $\Phi_0, \dots, \Phi_n$ form a holomorphic atlas for $\mathbb{CP}^n$. This shows that $\mathbb{CP}^n$ is a complex n-dimensional smooth manifold. Note that the charts are of type (i). This shows that $\mathbb{CP}^n$ has no boundary points, in particular since it is compact, it is closed. Finally the last statement is an immediate consequence of Proposition 12.7. ■

Remark. The notion of a submanifold also extends in the obvious way to the setting of complex manifolds. Let $m \leq n$. On page 194 we introduced the convention that we view $\mathbb{C}\mathrm{P}^m$ as a subset of $\mathbb{C}\mathrm{P}^n$. Using the above atlas one can easily that $\mathbb{C}\mathrm{P}^m$ is in fact a complex submanifold of $\mathbb{C}\mathrm{P}^n$.

12.3. **The canonical orientation of a complex manifold.**

Definition. Let V be an n-dimensional complex vector space. We say that a set of vectors $\{v_1, \dots, v_n\}$ in V is *totally real* if the vectors $v_1, \mathrm{i}v_1, \dots, v_n, \mathrm{i}v_n$ form a basis of V, viewed as a $2n$-dimensional real vector space.

Lemma 12.6.

(1) *Let V be an n-dimensional complex vector space.*
   (a) *V admits a set of totally real vectors.*
   (b) *If $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ are two sets of totally real vectors for a complex vector space V, then the bases $\{v_1, \mathrm{i}v_1, \dots, v_n, \mathrm{i}v_n\}$ and $\{w_1, \mathrm{i}w_1, \dots, w_n, \mathrm{i}w_n\}$ are equivalent in the sense of page 296.*

(2) *Let $f\colon V \to W$ be an isomorphism of n-dimensional complex vector spaces. If we equip V and W, viewed as real vector spaces, with the orientation from* (1)*, then $f\colon V \to W$ is an orientation-preserving isomorphism.*

Definition. Given a finite-dimensional complex vector space we refer to the orientation introduced in Lemma 12.6 (1) as the *canonical orientation.*

Example. Let $n \in \mathbb{N}_0$. If we make the usual identification $\mathbb{C}^n = (\mathbb{R}^2)^n = \mathbb{R}^{2n}$, then the corresponding canonical orientation on $\mathbb{C}^n = \mathbb{R}^{2n}$ agrees with the standard orientation.

Proof.

(1) (a) This statement follows immediately from the observation that V is isomorphic, as a complex vector space, to $\mathbb{C}^n$, and for $\mathbb{C}^n$ a totally real basis is given by vectors $e_i = (0, \dots, 1, \dots, 0)$, $i = 1, \dots, n$.
   (b) First we consider the case $n = 1$. In this case we are given two non-zero vectors v and w. Then $v = (a + b\mathrm{i})w$ for some $a + b\mathrm{i} \neq 0$. The base change matrix from the real basis $\{v, \mathrm{i}v\}$ to the real basis $\{w, \mathrm{i}w\}$ is given by the real 2×2–matrix
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$
   Evidently the determinant of this matrix is positive. This concludes the proof of the claim for $n = 1$. The general statement is basically just an exercise in linear algebra. This exercise though turns out to be slightly harder than one might expect.[206] A proof is for example given in [**CCL99**, Theorem 7.2.1].

[206]Basically the problem is as follows: Let A and B be two real $n\times n$-matrices such that $\det(A+iB) \in \mathbb{C}$ is non-zero. Why does it follow that the determinant of the real matrix
$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$
is positive?

(2) This statement follows immediately from (1). ∎

The following proposition says that complex manifolds admit a canonical orientation, in particular they are orientable.

Proposition 12.7.

(1) *Let M be an n-dimensional complex manifold.*
   (a) *Given any $P \in M$ we can use charts from the holomorphic atlas to equip $\mathrm{T}_P M$ with the structure of an n-dimensional vector space.*
   (b) *If we equip each complex vector space $\mathrm{T}_P M$ with the corresponding canonical orientation, then these orientations define an orientation on M, viewed as a $2n$-dimensional real smooth manifold. We refer to this orientation as the* canonical orientation of the complex manifold M..

(2) *If $f\colon M \to N$ is a local biholomorphism between two complex manifolds, then f is an orientation-preserving diffeomorphism between M and N, viewed as oriented smooth manifolds.*

Sketch of a proof.

(1) (a) Let M be an n-dimensional complex manifold and let $P \in M$. We pick a chart $\Phi\colon U \to V$ from the holomorphic atlas around P. We get an induced isomorphism $\mathrm{D}\,\Phi_P\colon\, \mathrm{T}_P M \to \mathrm{T}_P V = \mathbb{C}^n$. We use this isomorphism to equip $\mathrm{T}_P M$ with the structure of a complex vector space. Since Φ comes from a holomorphic atlas one obtains from Exercise 12.2 that this complex vector space structure does not depend on the choice of the chart Φ.
   (b) This statement follows easily from the definitions. We leave it to the reader to fill in the details.

(2) The proof is identical to the proof of (1b). ∎

Corollary 12.8. *The Klein bottle and the real projective plane $\mathbb{RP}^2$ do not admit the structure of a complex manifold.*

Proof. In Lemma 6.48 we showed that neither the Klein bottle nor the real projective plane $\mathbb{RP}^2$ are orientable. This is in contrast to the fact, shown in Proposition 12.7, that complex manifolds are orientable. ∎

12.4. **Properties of complex manifold.** The following theorem is the complex analogue of the Regular Value Theorem 6.53. As in the real case is again a machine for providing examples of complex manifolds.

Theorem 12.9. (Regular Value Theorem for Complex Manifolds) *Let M be an m-dimensional complex manifold and let N be an n-dimensional complex manifold. Furthermore $f\colon M \to N$ be a holomorphic map and let $s \in N$ be a regular value of f.*

(1) *The preimage $X := f^{-1}(\{s\})$ is an $(m-n)$-dimensional complex submanifold of M.*
(2) *The intersection of X with any compact set of M is compact. In particular, if M is compact, then X is also compact.*

Example. Let $n \in \mathbb{Z}$. In Exercise 12.3 we will use the Regular Value Theorem 12.9 for Complex Manifolds) to show that $\{[x : y : z] \in \mathbb{CP}^2 \,|\, x^n + y^n + z^n = 0\}$ is a closed 1-dimensional complex manifold.

PROOF. The proof is almost identical to the proof of the "real" Regular Value Theorem 6.53. We refer to [**Huy05**, p. 11 and p. 58] for some more details. ■

As we had already pointed out on page 197, the one-dimensional projective space $\mathbb{CP}^1$ is homeomorphic to S^2. This shows that S^2 can be viewed as a 1-dimensional complex manifold. We had just seen that the 2-sphere and the torus are complex manifolds and we saw that non-orientable smooth manifolds cannot be complex manifolds. The following question now arises:

Question 12.10. *Let $g \geq 2$. Does the surface of genus g have the structure of a complex manifold?*

We will discuss this question in Proposition 33.2.

Remark. It is often very hard to determine whether or not a given orientable even-dimensional smooth manifold can also be a complex manifold. For example, Armand Borel and Jean-Pierre Serre [**BoS53**][207] showed in 1953 that for $n \neq 2, 6$ the n-dimensional sphere S^n cannot be a complex manifold. On the other hand it is still an open question whether S^6 is a complex manifold.

In general it is hard to write down an explicit biholomorphism between two complex manifolds. One of the main sources of biholomorphisms is the Riemann Mapping Theorem.

Theorem 12.11. (Riemann Mapping Theorem) *Let $U \subset \mathbb{C}$ be a simply connected open subset U of $\mathbb{C}$. If $U \neq \mathbb{C}$, then U is biholomorphic to $\mathbb{D}$.*

The Riemann Mapping Theorem was first stated by Bernhard Riemann in 1851 his thesis. The first rigorous proof was given by William Osgood in 1900, see [**Wals73**]. The Riemann mapping theorem is proved in most textbooks on complex analysis, see e.g. [**Jä11**] or alternatively [**Lan99**, Chapter X].

In many applications it is also useful to know that the biholomorphism from the Riemann Mapping Theorem extends to the boundary of U and $\mathbb{D}$. To state the corresponding theorem we need one more definition.

Definition.

(1) We write $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which we equip with the topology that we defined on page 95.
(2) Given $U \subset \overline{\mathbb{C}}$ we denote by $\overline{U}$ the closure of U in $\overline{\mathbb{C}}$.
(3) A *Jordan curve* is the image of an injective map $S^1 \to \overline{\mathbb{C}}$.

The following theorem was proved in 1913 by Constantin Carathéodory.[208] A proof is for example provided in [**GM05**, Chapter I.3].

[207]Armand Borel (1923–2003) was a Swiss mathematician, Jean-Pierre Serre (*1926) is a French mathematician who was awarded a Fields medal in 1954.

[208]Constantin Carathéodory (1873-1950) was a Greek mathematician who was in particular a professor at the LMU in Munich.

Theorem 12.12. (Carathéodory Theorem) *Let $U \subsetneq \mathbb{C}$ be a simply connected open subset U of $\mathbb{C}$ and let $\phi\colon U \to \mathbb{D}$ be a biholomorphism. If $\partial\overline{U}$ is a Jordan curve, then ϕ extends to a homeomorphism $\overline{U} \to \mathbb{C}$.*

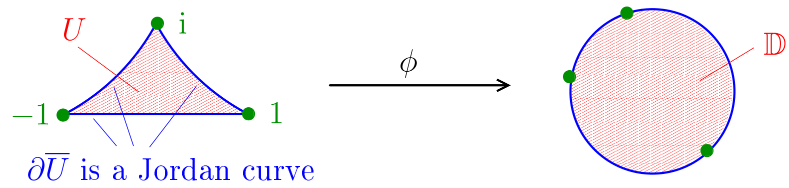


FIGURE 287

Example. We consider the open subset U of $\mathbb{C}$ that is shown in Figure 287. It is a simply connected open proper subset of $\mathbb{C}$ and the boundary is clearly a Jordan curve. Thus it follows from the Riemann Mapping Theorem and from Carathéodory's Theorem that there exists a homeomorphism $\phi\colon \overline{U} \to \overline{\mathbb{D}}$ which restricts to a biholomorphism $\phi\colon U \to \mathbb{D}$.

Exercise for Chapter 12.

Exercise 12.1. Let M be a compact connected 1-dimensional complex manifold. Show that any holomorphic function $f\colon M \to \mathbb{C}$ is constant.
Hint. Use the Maximum Principle, i.e. use Theorem 12.2.

Exercise 12.2. Let $U \subset \mathbb{C}^n$ be an open set and let $f\colon U \to \mathbb{C}^m$ be a holomorphic map. Show that given any $P \in U$ the differential $\mathrm{D}\, f_P$ lies in $\mathrm{M}(m \times n, \mathbb{C}) \subset \mathrm{M}(2m \times 2n, \mathbb{R})$.

Exercise 12.3. Let $n \in \mathbb{Z}$. Show that $\{[x : y : z] \in \mathbb{C}\mathrm{P}^2 \,|\, x^n + y^n + z^n = 0\}$ is a closed 1-dimensional complex submanifold of $\mathbb{C}\mathrm{P}^2$.

Part III

Fundamental Groups

13. How can we show that two topological spaces are (not) homeomorphic?

In the previous two sections we collected many examples of topological spaces. In many cases it was relatively straightforward to see that two topological spaces are homeomorphic. For example we wrote down an explicit homeomorphism from $\mathbb{R}^n/\mathbb{Z}^n$ to $(S^1)^n$. Sometimes it is rather painful to explicitly write down a homeomorphism, for example it was not particularly easy to find a homeomorphism from the open cube $(0,1)^n$ to the open ball $B^n = \{x \in \mathbb{R}^n \,|\, \|x\| < 1\}$. Nonetheless, if two topological spaces are homeomorphic, then usually one can write down an explicit homeomorphism.

On the other hand, how can we tell that two topological spaces are not homeomorphic? There are a couple of trivial criteria: being compact, connected, Hausdorff are properties of topological spaces that are shared by homeomorphic topological spaces. This implies for example that the closed cube $[0,1]^n$ is not homeomorphic to the open cube $(0,1)^n$, since the former topological space is compact and the latter is not.

But these criteria are of no use when it comes to studying more subtle examples. For example they do not suffice to address the following questions:

(1) Can surfaces with different genera be homeomorphic?
(2) Given $k \in \mathbb{Z}_{\geq 0}$ we denote by U the complement of k points in $\mathbb{R}^2$. If $k \neq l$, is it possible that U_k and U_l are homeomorphic?
(3) Is the complement of the trefoil homeomorphic to the complement of the trivial knot?

In each case we suspect that the answer is no. Certainly we do not succeed in finding a homeomorphism, but how can we show that this is not due to lack of imagination, but that there is indeed no homeomorphism?

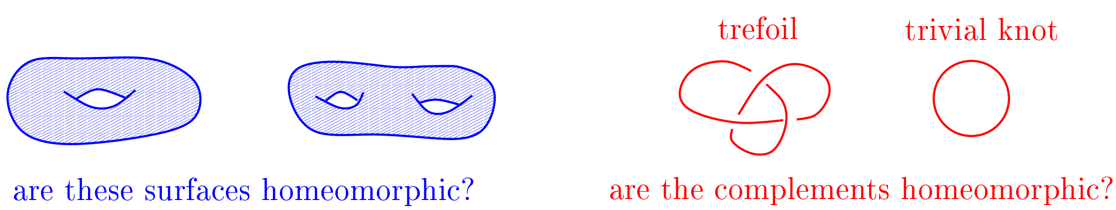


Figure 288

The reader might have encountered some ideas for tackling these questions in one or both of the following topics:

(a) Differential forms on smooth manifolds, in particular de Rham cohomology, as developed say in [**BoT82, Lee02**].
(b) Complex analysis, see e.g. [**Lan99, Jä11**].

For the reader who has seen differential forms and de Rham cohomology we recall that each smooth manifold M and each $k \in \mathbb{Z}_{\geq 0}$ we can assign the de Rham cohomology group $H^k_{\mathrm{dR}}(M)$. It follows more or less from the definition that diffeomorphic smooth manifolds have isomorphic de Rham cohomology groups. This approach has several disadvantages:

(1) the de Rham cohomology groups are only defined for smooth manifolds,

(2) the de Rham cohomology groups can only be used to show that two smooth manifolds are not *diffeomorphic*, a priori they cannot be used to show that two smooth manifolds are not *homeomorphic*,
(3) at least just starting from the definition it is not clear how one can calculate the de Rham cohomology groups of a given smooth manifold.

Later on, in Chapter 73, we will introduce "singular cohomology" which will take care of all of these three issues.

Next we give some definitions that are familiar to the reader who has taken a course on complex analysis.

(1) Given a subset U of $\mathbb{C}$, we defined a *path* in U to be a map $\gamma\colon [a,b] \to U$. If $\gamma(a) = \gamma(b)$, then we say that γ is a *loop*.
(2) Two paths $\gamma_0, \gamma_1\colon [a,b] \to U$ with the same starting point $P := \gamma_0(a) = \gamma_1(a)$ and the same endpoint $Q := \gamma_0(b) = \gamma_1(b)$ are called *path-homotopic in* U, if there exists a map

$$\begin{array}{rcl} \Gamma\colon [a,b]\times[0,1] & \to & U \\ (t,s) & \mapsto & \Gamma(t,s), \end{array}$$

with the following properties
(a) for every $t \in [a,b]$ we have

$$\Gamma(t,0) \;=\; \gamma_0(t) \text{ and } \Gamma(t,1) \;=\; \gamma_1(t),$$

(b) for every $s \in [0,1]$ we have

$$\Gamma(a,s) \;=\; P \qquad \text{and } \Gamma(b,s) \;=\; Q.$$

Put differently, a path-homotopy between two paths consists of a "continuous" family of paths $\{\Gamma(-,s)\}_{s\in[0,1]}$ from P to Q which interpolates between the paths γ_0 and γ_1. In Figure 289 we illustrate a path-homotopy between two paths.
(3) A loop $\gamma\colon [a,b] \to U$ is called *null-homotopic* if it is path-homotopic to the constant path given by $\delta(t) := \gamma(a)$ for all $t \in [a,b]$.
(4) We say U is *simply connected* if U is non-empty and if each loop is null-homotopic.

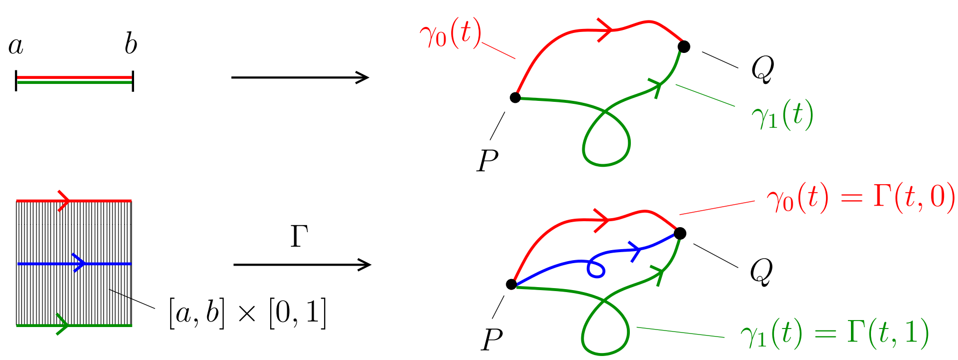


FIGURE 289. Schematic image of a path-homotopy Γ between two paths γ_0 and γ_1 from P to Q.

Example. Every loop in $\mathbb{R}^2 = \mathbb{C}$ is null-homotopic. Indeed, we let $\gamma\colon [0,1] \to \mathbb{C}$ be a loop with starting and endpoint P. Then

$$\begin{array}{rcl} \Gamma\colon [a,b]\times[0,1] & \to & \mathbb{R}^n \\ (t,s) & \mapsto & \gamma(t)\cdot(1-s) + P\cdot s \end{array}$$

is a path-homotopy between the loop γ and the constant path at P. The fact that every loop in $\mathbb{C}$ is null-homotopic is illustrated in Figure 290.

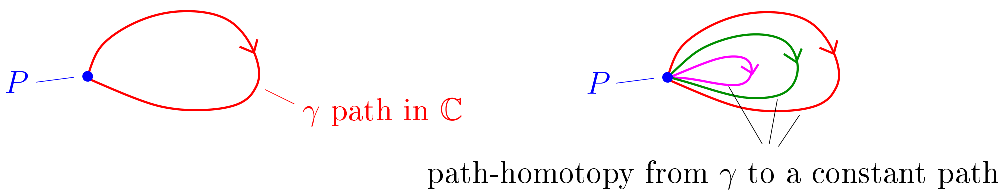


FIGURE 290

Now we recall the following definition from complex analysis.

Definition. Let $U \subset \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a holomorphic function and let $\gamma\colon [a,b] \to U$ be a smooth path. We define the *path integral* of f along γ as follows:

$$\int_\gamma f(z)\,dz \;:=\; \int_{t=a}^{t=b} f(\gamma(t))\cdot\gamma'(t)\,dt \;\in \mathbb{C}.$$

Examples.

(1) Let $U = \mathbb{C}\setminus\{0\}$, let $f(z) = \frac{1}{z}$ and let $\gamma\colon [0,1] \to \mathbb{C}\setminus\{0\}$ be the loop given by $\gamma(t) = e^{2\pi\mathrm{i}t}$, i.e. γ is a loop that "goes once around the origin". Then

$$\int_\gamma f(z)\,dz \;=\; \int_{t=0}^{t=1} f(\gamma(t))\cdot\gamma'(t)\,dt \;=\; \int_{t=0}^{t=1} \tfrac{1}{e^{2\pi\mathrm{i}t}}\cdot e^{2\pi\mathrm{i}t}\cdot 2\pi\mathrm{i}\,dt \;=\; 2\pi\mathrm{i}.$$

(2) If γ is a constant path, then $\gamma'(t) \equiv 0$. This shows that the path-integral along a constant path is zero.

The following proposition is one of the main results in complex analysis, see e.g. [**Lan99**, Theorem III.5.1].

Proposition 13.1. *Let $U \subset \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a holomorphic function and let $\gamma, \delta\colon [a,b] \to U$ be two smooth paths. If γ and δ are path-homotopic, then*

$$\int_\gamma f(z)\,dz \;=\; \int_\delta f(z)\,dz.$$

The following corollary is a consequence of Proposition 13.1 and of the above calculations.

Corollary 13.2. *The loop $\gamma\colon [0,1] \to \mathbb{C}\setminus\{0\}$ given by $\gamma(t) = e^{2\pi\mathrm{i}t}$ is* not *path-homotopic to a constant path, i.e. γ is not null-homotopic.*

Remark. In the following Chapter 13.2 we will use Corollary 13.2 so that we have at least one non-trivial topological statement in our toolbox. Somewhat later, after somewhat

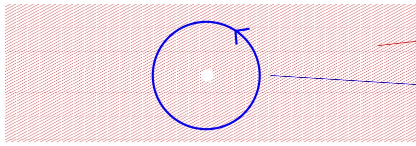


FIGURE 291

lengthy preparations, we will proof a general result, namely Corollary 16.18, which in particular will provide us with a new proof of Corollary 13.2.

Example. If two subsets of $\mathbb{C}$ are homeomorphic, then evidently either both are simply connected or none is. From Corollary 13.2 it now follows immediately that $\mathbb{R}^2 = \mathbb{C}$ and $\mathbb{R}^2 \setminus \{(0,0)\} = \mathbb{C} \setminus \{0\}$ are not homeomorphic.

This second approach to showing that two topological spaces are different also has several disadvantages over the first approach:

(1) A priori these notions are defined only for subsets of $\mathbb{C}$,
(2) this approach does not allow us to show that the two subsets $U = \mathbb{C} \setminus \{0\}$ and $V = \mathbb{C} \setminus \{0, 1\}$ are not homeomorphic, since neither is simply connected.

The first objection is of course easy to rectify, since the definitions make sense for any topological space. But if we work with more general situations a new problem arises: how can we show, without using complex analysis, that a topological space is not simply connected? In the following chapters we will address all these issues.

In particular, when it comes to distinguishing $U = \mathbb{C} \setminus \{0\}$ and $V = \mathbb{C} \setminus \{0, 1\}$, then naively we would say that V has "more" loops that are not null-homotopic than U. In the Chapter 14 we will in particular make precise what "more" means.

14. The fundamental group

14.1. **Path-homotopy classes of paths.** In this section we extend some of the ideas and notions about paths that we introduced in the previous section to the more general setting of paths in topological spaces.

Definition. Let X be a topological space.

(1) A *path* in X is a[209] map $\gamma\colon [a,b] \to X$. We call $\gamma(a)$ the *starting point of* γ and we call $\gamma(b)$ the *endpoint of* γ. Often we say that γ is a *path from* $\gamma(a)$ *to* $\gamma(b)$.
(2) A *loop* is a path for which the starting point and the endpoint coincide.

Convention. In the following, if we do not specify the domain of a path or a loop, then it is understood to be $[0,1]$.

Definition. Given a path $\gamma\colon [0,1] \to X$ we refer to the path $\overline{\gamma}\colon [0,1] \to X$ that is given by $\overline{\gamma}(t) = \gamma(1-t)$ as the *inverse path of* γ.

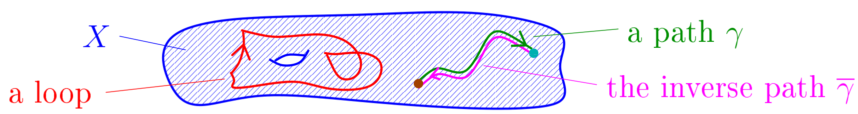


Figure 292

Definition. Let X be a topological space.

(1) Let $\gamma_0, \gamma_1\colon [a,b] \to X$ be two paths with the same starting point P and the same endpoint Q. A *path-homotopy between the paths* γ_0 *and* γ_1 is a map
$$\begin{array}{rcl}\Gamma\colon [a,b]\times[0,1] & \to & X \\ (t,s) & \mapsto & \Gamma(t,s),\end{array}$$
with the following properties
(a) for every $t \in [a,b]$ we have
$$\Gamma(t,0) \;=\; \gamma_0(t) \text{ and } \Gamma(t,1) \;=\; \gamma_1(t),$$
(b) for every $s \in [0,1]$ we have
$$\Gamma(a,s) \;=\; P \qquad \text{and } \Gamma(b,s) \;=\; Q.$$
(2) If there exists a path-homotopy between two paths γ_0 and γ_1, then we say that γ_0 and γ_1 are *path-homotopic*, and we write $\gamma_0 \simeq \gamma_1$.
(3) We say a loop $\gamma\colon [a,b] \to X$ is *null-homotopic*, if it is path-homotopic to the constant path δ given by $\delta(t) = \gamma(a)$ for all $t \in [a,b]$.

Examples.

(A) Let $\gamma_0, \gamma_1\colon [a,b] \to \mathbb{R}^n$ be two paths in $\mathbb{R}^n$ with same starting point and the same endpoint. Then
$$\begin{array}{rcl}\Gamma\colon [a,b]\times[0,1] & \to & \mathbb{R}^n \\ (t,s) & \mapsto & \gamma_0(t)\cdot(1-s) + \gamma_1(t)\cdot s\end{array}$$

[209]Recall that all maps are understood to be continuous, unless we say something else.

is a path-homotopy between γ_0 and γ_1. We had already illustrated this fact in Figure 290.

(B) Let $X = \mathbb{C} \setminus \{0\} = \mathbb{R}^2 \setminus \{(0,0)\}$. As we had just seen in Corollary 13.2, the path

$$\begin{array}{rcl}\gamma\colon [0,1] & \to & X \\ t & \mapsto & e^{2\pi i t} = (\cos(2\pi t), \sin(2\pi t))\end{array}$$

is not null-homotopic in X.

(C) Let $Y = S^1 \times (0,2)$ and consider the loop

$$\begin{array}{rcl}\widetilde{\gamma}\colon [0,1] & \to & Y = S^1 \times (0,2) \\ t & \mapsto & (e^{2\pi i t}, 1).\end{array}$$

This loop is not null-homotopic. This can be seen as follows: we consider the map

$$\begin{array}{rcl}\Phi\colon Y = S^1 \times (0,2) & \to & X = \mathbb{C} \setminus \{0\} \\ (z,t) & \mapsto & z \cdot t.\end{array}$$

If $\widetilde{\gamma}$ was null-homotopic in Y, then there would exist a path-homotopy

$$\widetilde{F}\colon [0,1] \times [0,1] \to Y$$

from $\widetilde{\gamma}$ to the constant path $\widetilde{\delta}(t) := (1,1)$, $t \in [0,1]$. But then

$$F := \Phi \circ \widetilde{F}\colon [0,1] \times [0,1] \to X$$

would be a path-homotopy in $X = \mathbb{C} \setminus \{0\}$ from the path $\gamma := \Phi \circ \widetilde{\gamma}$ to the constant path $\delta := \Phi \circ \widetilde{\delta}$. But $\gamma = \Phi \circ \widetilde{\gamma}$ is precisely the path γ from example (B) which we showed not to be null-homotopic in $X = \mathbb{C} \setminus \{0\}$.

(D) We consider the two loops in Figure 293. In both cases it is hard to see how these loops could possibly be path-homotopic to a constant path, but it is also difficult to find an argument why that should not be possible.[210]

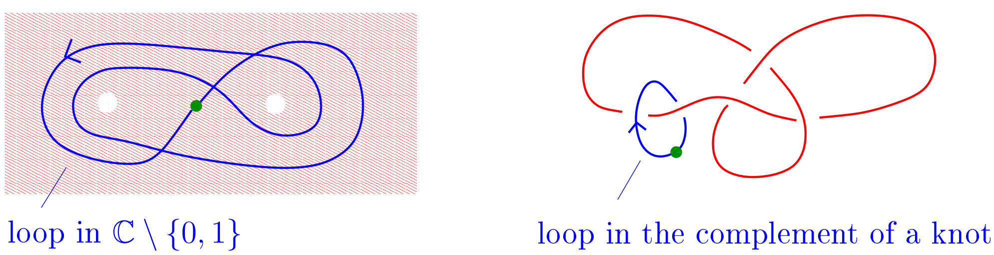


FIGURE 293

(E) We consider the real projective space $\mathbb{RP}^2$. Recall that in Lemma 3.42 we gave an identification $\mathbb{RP}^2 = \overline{B}^2/\sim$ where $\sim$ is the equivalence relation on $\overline{B}^2$ generated by $P \sim -P$ for $P \in S^1 = \partial \overline{B}^2$. Then

$$\begin{array}{rcl}\gamma\colon [0,1] & \to & \overline{B}^2/\sim \\ t & \mapsto & [(\cos(\pi t), \sin(\pi t))]\end{array}$$

is a loop in $\mathbb{RP}^2 = \overline{B}^2/\sim$, see Figure 294.[211] But is it null-homotopic?

[210]For example the given loop in $\mathbb{C}\setminus\{0,1\}$ is actually null-homotopic in $\mathbb{C}\setminus\{0\}$ and it is null-homotopic in $\mathbb{C} \setminus \{1\}$. In particular our trick of using Corollary 13.2 does not work in this case.

[211]Why is it a loop?

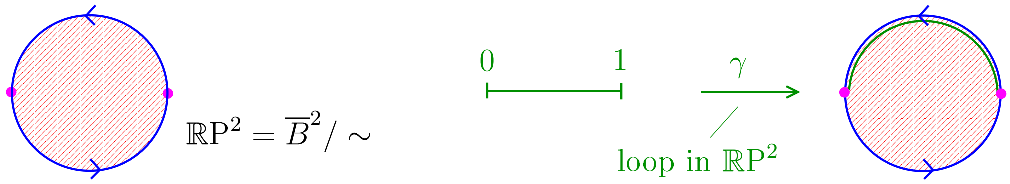


FIGURE 294

(F) On page 107 we considered the pseudocircle, namely the set $X = \{A, B, C, D\}$ with a slightly unusual topology. Furthermore we gave an example of a continuous non-constant loop $f\colon S^1 \to X$. The question arises, is this loop null-homotopic?

Remark. Let X be a topological space and let $x_0 \in X$. It follows from Lemma 3.24 that the maps

$$\begin{array}{rcl}\{\text{loops } \gamma\colon [0,1] \to X \text{ with } \gamma(0) = \gamma(1) = x_0\} & \Leftrightarrow & \{\text{maps } f\colon S^1 \to X \text{ with } f(1) = x_0\} \\ (\gamma\colon [0,1] \to X) & \mapsto & (z = e^{2\pi i t} \mapsto \gamma(t)) \\ (t \mapsto f(e^{2\pi i t})) & \leftarrow\!\shortmid & (f\colon S^1 \to X)\end{array}$$

are evidently bijections. Thus we can think of loops in X where the starting and endpoint is x_0 as maps from S^1 to X that send 1 to x_0. We will use this bijection to go back and forth between these two points of view.

The following lemma gives a useful criterion for a loop to be null-homotopic.

Lemma 14.1. *Let X be a topological space and let $x_0 \in X$. Let $\gamma\colon [0,1] \to X$ be a loop with $\gamma(0) = \gamma(1) = x_0$. We denote by*

$$\begin{array}{rcl}\varphi\colon S^1 & \to & X \\ z = e^{2\pi i t} & \mapsto & \gamma(t)\end{array}$$

the corresponding map from S^1 to X. Then γ is null-homotopic if and only if there exists a map $\Phi\colon \overline{B}^2 \to X$ so that $\Phi|_{S^1} = \varphi$.

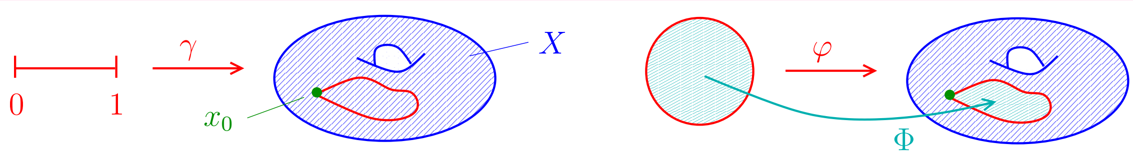


FIGURE 295. Illustration of Lemma 14.1.

PROOF ($*$). We start out this proof with a few elementary preparations. We write $A := ([0,1] \times \{0\}) \cup (\{0,1\} \times [0,1])$ and we consider the map

$$\begin{array}{rcl}g\colon [0,1] \times [0,1] & \to & \overline{B}^2 \\ (t,s) & \mapsto & se^{2\pi i t} + 1 - s.\end{array}$$

Note that this map has the property that all points in A get sent to $1 \in S^1$. Using this observation, Lemma 3.22 and Proposition 2.43 (3) we see that the map

$$\begin{array}{rcl}h\colon ([0,1] \times [0,1])/A & \to & \overline{B}^2 \\ [P] & \mapsto & g(P)\end{array}$$

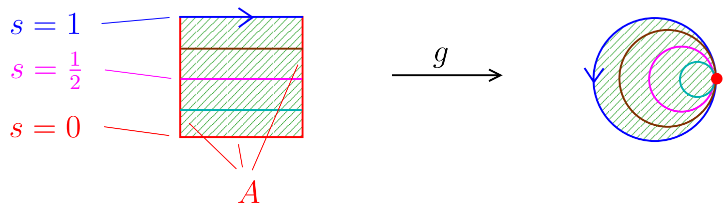


FIGURE 296

is a homeomorphism.

Now we turn to the actual proof of the lemma. Thus let X be a topological space, let $x_0 \in X$ and let $\gamma\colon [0,1] \to X$ be a loop with $\gamma(0) = \gamma(1) = x_0$. We define $\varphi\colon S^1 \to X$ via $\varphi(e^{2\pi i t}) = \gamma(t)$.

First we suppose that γ is null-homotopic. This means that there exists a path-homotopy $F\colon [0,1] \times [0,1] \to X$ such that $F(t,0) = x_0$ and $F(t,1) = \gamma(t)$ for all $t \in [0,1]$ and such that $F(0,s) = F(1,s) = x_0$ for all $s \in [0,1]$. In other words, F is constant on A. We consider the map[212]

$$\begin{array}{rcl} \Phi\colon \overline{B}^2 \underset{\cong}{\xleftarrow{h}} ([0,1] \times [0,1])/A & \to & X \\ [P] & \mapsto & F(P). \end{array}$$

One can easily verify that we have $\Phi|_{S^1} = \varphi$.

Conversely, if there exists a map $\Phi\colon \overline{B}^2 \to X$ so that $\Phi|_{S^1} = \varphi$, then one can easily verify that $\Phi \circ g\colon [0,1] \times [0,1] \to X$ is a path-homotopy from the loop γ to the constant loop c_{x_0}.[213] ■

Before we continue with the examples we need to gather a few more elementary facts about path-homotopies of paths.

Proposition 14.2. *Let X be a topological space and let $x, y \in X$. Then "path-homotopy" is an equivalence relation on the set of paths $\gamma\colon [a,b] \to X$ from x to y.*

The proof of Proposition 14.2 makes use of the following lemma which we will use, implicitly or explicitly, surprisingly often.

Lemma 14.3. $(*)$ *Let X be a topological space and let $a < b < c$ be real numbers. Let $f\colon X \times [a,c] \to Y$ be a map to some topological space Y. If the restrictions of f to $X \times [a,b]$ and to $X \times [b,c]$ are continuous, then f itself is continuous.*

PROOF OF LEMMA 14.3. This statement is an immediate consequence of Lemma 2.35 (2) and the observation, see Lemma 3.9, that $X \times [a,b]$ and $X \times [b,c]$ are in fact closed subsets of $X \times [a,c]$. ■

[212]Note that we implicitly use Lemma 3.22 to argue that the map $([0,1] \times [0,1])/A \to X$ is well-defined and continuous.

[213]Why did we not just define g to be the map that sends the horizontal lines to concentric circles where each of them is centered around the origin?

Proof of Proposition 14.2. In order to simplify the notation we only work with paths defined on the interval $[0,1]$. We have to show that "path-homotopy" satisfies the three properties of an equivalence relation. Let $\alpha, \beta, \gamma\colon [0,1] \to X$ be three paths from x to y.

(1) The "constant path-homotopy" given by $\Gamma(t,s) := \gamma(t)$ defines a path-homotopy between γ and γ.

(2) Let Γ be a path-homotopy from α to β. Then the map Γ' defined by

$$\Gamma'(t,s) \; := \; \Gamma(t, 1-s)$$

is a path-homotopy from β to α.

(3) Let Γ be a path-homotopy from α to β and let Δ be a path-homotopy from β to γ. We consider the map

$$\begin{array}{rcl} [0,1] \times [0,1] & \to & X \\ (t,s) & \mapsto & \begin{cases} \Gamma(t,2s), & \text{if } s \in \left[0, \tfrac{1}{2}\right], \\ \Delta(t, 2s-1), & \text{if } s \in \left(\tfrac{1}{2}, 1\right] \end{cases} \end{array}$$

It follows from Lemma 14.3 that this map is continuous. It is now clear that this map is a path-homotopy from α to γ. This part of the proof is illustrated in Figure 297. ■

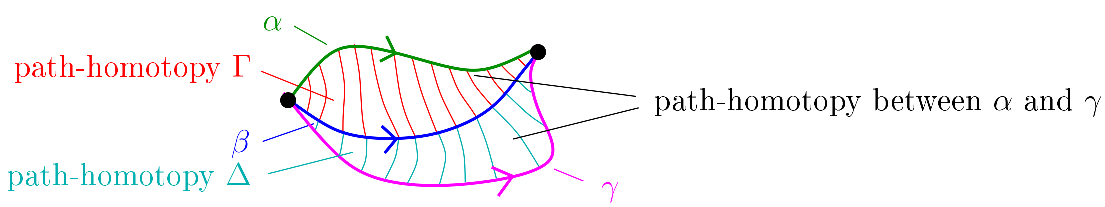


Figure 297

Definition. Given a path $f\colon [0,1] \to X$ in a topological space X we denote by $[f]$ the equivalence class of f with respect to the equivalence relation which is given by path-homotopies. We call $[f]$ the *path-homotopy class of* f.

Definition. Let $\gamma\colon [a,b] \to X$ be a path in a topological space and let $\varphi\colon [a,b] \to [a,b]$ be a map[214] with $\varphi(a) = a$ and $\varphi(b) = b$. We refer to $\gamma \circ \varphi\colon [a,b] \to X$ as a *reparametrization of* γ.

The following lemma says that reparametrizing does not change the path-homotopy class of a path.

Lemma 14.4. *Let $\gamma\colon [a,b] \to X$ be a path in a topological space. If a path $\delta\colon [a,b] \to X$ is obtained from γ by a reparametrization, then γ and δ are path-homotopic.*

Proof. Let $\gamma\colon [a,b] \to X$ be a path in a topological space and let $\varphi\colon [a,b] \to [a,b]$ be a map with $\varphi(a) = a$, $\varphi(b) = b$. Then a path-homotopy between γ and $\gamma \circ \varphi$ is given by[215]

$$\begin{array}{rcl} P\colon [a,b] \times [0,1] & \to & X \\ (t,s) & \mapsto & \gamma(t \cdot s + \varphi(t) \cdot (1-s)). \end{array}$$ ■

[214] Note that φ is not required to be a homeomorphism.

[215] Where do we actually use the hypothesis that $\varphi(a) = a$ and $\varphi(b) = b$?

It is pictorially clear that we can concatenate two paths if the endpoint of the first path agrees with the starting point of the second path. We use this observation to define the product of two paths.

Definition. Let X be a topological space and let $\alpha\colon [a,b] \to X$ and $\beta\colon [c,d] \to X$ be two paths with $\alpha(b) = \beta(c)$. We define the *product of* α *and* β as the path $\alpha * \beta$ which is given by[216]

$$\begin{array}{rcl} \alpha * \beta\colon [0,1] & \to & X \\ t & \mapsto & \begin{cases} \alpha(a + 2t(b-a)), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(c + (2t-1)(d-c)), & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases} \end{array}$$

The product of two paths is thus given by "first running along α" and then "running along β". Here we reparametrized the path so that the domain of $\alpha * \beta$ is $[0,1]$. This convention might look slightly odd, but later on we will mostly work with paths that are defined on the interval $[0,1]$.

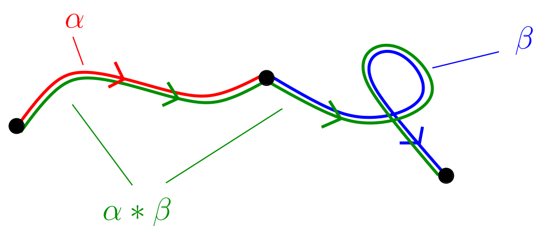


FIGURE 298. Definition of the product of two paths.

We continue with the following lemma, the statement of which is illustrated in Figure 299.

Lemma 14.5. *Let $\alpha, \alpha'\colon [a,b] \to X$ and $\beta, \beta'\colon [c,d] \to X$ be two pairs of paths in a topological space X such that $\alpha(b) = \alpha'(b) = \beta(c) = \beta'(c)$. Then*

$$\alpha \simeq \alpha' \text{ and } \beta \simeq \beta' \quad \Longrightarrow \quad \alpha * \beta \simeq \alpha' * \beta'.$$

Put differently, if $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$, then $[\alpha * \beta] = [\alpha' * \beta']$.

PROOF. We leave it to the reader to provide the fairly elementary proof. ∎

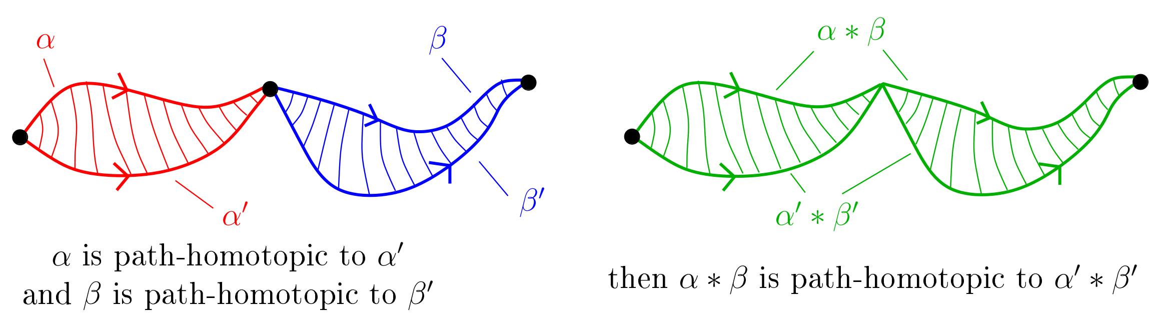


FIGURE 299. Illustration of Lemma 14.5.

[216]It follows from Lemma 2.35 (2) and our hypothesis $\alpha(b) = \beta(c)$ that $\alpha * \beta\colon [0,1] \to X$ is a continuous map.

Definition. Let $[\alpha]$ and $[\beta]$ be two equivalence classes of paths in a topological space X with $\alpha(1)=\beta(0)$. We define the[217] *product of the equivalence classes* $[\alpha]$ *and* $[\beta]$ as

$$[\alpha]\cdot[\beta] \;=\; [\alpha*\beta].$$

The following proposition says that the product of equivalence classes of paths shares many properties of the product structure of a group.

Proposition 14.6. *Let $\alpha,\beta,\gamma\colon[0,1]\to X$ be three paths in a topological space X. Then the following hold:*

(1) *If $\alpha(1)=\beta(0)$ and $\beta(1)=\gamma(0)$, then*[218]

$$[\alpha]\;\cdot\;([\beta]\cdot[\gamma]) \;=\; ([\alpha]\cdot[\beta])\;\cdot\;[\gamma].$$

(2) *For $x\in X$ we denote by e_x the constant path that is given by $e_x(t):=x$, $t\in[0,1]$. Then the following equality holds*

$$[e_{\alpha(0)}]\cdot[\alpha] \;=\; [\alpha] \;=\; [\alpha]\cdot[e_{\alpha(1)}].$$

(3) *We denote by $\overline{\alpha}$ the inverse path. Then the following equalities hold*

$$[\alpha]\cdot[\overline{\alpha}] \;=\; [e_{\alpha(0)}] \quad \textit{and} \quad [\overline{\alpha}]\cdot[\alpha] \;=\; [e_{\alpha(1)}].$$

Proof $(*)$. Let $\alpha,\beta,\gamma\colon[0,1]\to X$ be three paths in a topological space X.

(1) Suppose that $\alpha(1)=\beta(0)$ and $\beta(1)=\gamma(0)$. We need to show that

$$\alpha*(\beta*\gamma) \;\simeq\; (\alpha*\beta)*\gamma.$$

We consider the map

$$\begin{aligned}\Phi\colon[0,1] &\to X\\ t &\mapsto \begin{cases}\alpha(3t), & \text{if } t\in[0,\tfrac13],\\ \beta(3t-1), & \text{if } t\in[\tfrac13,\tfrac23],\\ \gamma(3t-2), & \text{if } t\in[\tfrac23,1].\end{cases}\end{aligned}$$

It follows from our hypothesis and Lemma 2.35 (2) that this map is indeed continuous. We consider the maps

$$\begin{aligned}p\colon[0,1] &\to [0,1]\\ t &\mapsto \begin{cases}\tfrac23 t, & \text{if } t\in[0,\tfrac12],\\ \tfrac43 t-\tfrac13, & \text{if } t\in(\tfrac12,1]\end{cases}\end{aligned} \quad\text{and}\quad \begin{aligned}q\colon[0,1] &\to [0,1]\\ t &\mapsto \begin{cases}\tfrac43 t, & \text{if } t\in[0,\tfrac12],\\ \tfrac23 t+\tfrac13, & \text{if } t\in(\tfrac12,1]\end{cases}\end{aligned}$$

Then

$$\alpha*(\beta*\gamma) \underset{\substack{\uparrow\\ \text{by definition}}}{=} \Phi\circ p \underset{\substack{\uparrow\\ \text{by Lemma 14.4}}}{\simeq} \Phi \underset{\uparrow}{\simeq} \Phi\circ q \underset{\substack{\uparrow\\ \text{by definition}}}{=} (\alpha*\beta)*\gamma$$

---

[217] According to Lemma 14.5 this definition does not depend on the choice of the representatives α and β of the equivalence classes.

[218] Put differently, the paths $\alpha*(\beta*\gamma)$ and $(\alpha*\beta)*\gamma$ are path-homotopic. Note that these paths are similar in the sense that both paths "run through" the points on α, β and γ, but at different speeds. In general the paths are not equal, for example we have $(\alpha*(\beta*\gamma))(\tfrac12)=\alpha(1)$ and $((\alpha*\beta)*\gamma)(\tfrac12)=\beta(1)$.

(2) We consider the map

$$\begin{aligned} q\colon [0,1] &\to [0,1] \\ t &\mapsto \begin{cases} 0, & \text{if } t\in[0,\frac{1}{2}], \\ 2t-1, & \text{if } t\in[\frac{1}{2},1]. \end{cases} \end{aligned}$$

Then

$$e_{\alpha(0)} * \alpha \underset{\substack{\uparrow \\ \text{by definition}}}{=} \alpha\circ q \underset{\substack{\uparrow \\ \text{by Lemma 14.4}}}{\simeq} \alpha.$$

This shows that $[e_{\alpha(0)}]\cdot[\alpha] = [e_{\alpha(0)} * \alpha] = [\alpha]$. Almost the same argument also implies that $[\alpha] = [\alpha]\cdot[e_{\alpha(1)}]$.

(3) We write $P = \alpha(0)$. Note that $\alpha * \overline{\alpha}$ is a loop in P. We consider the map

$$\begin{aligned} F\colon [0,1]\times[0,1] &\to X \\ (t,s) &\mapsto \begin{cases} \alpha(2t\cdot s), & \text{if } t\in[0,\frac{1}{2}], \\ \alpha(2s-2t\cdot s), & \text{if } t\in[\frac{1}{2},1]. \end{cases} \end{aligned}$$

It follows from Lemma 14.3 that this map is indeed continuous. For $s=0$ we obtain the constant path at P and for $s=1$ we obtain the path $\alpha*\overline{\alpha}$. For any $s\in[0,1]$ we furthermore have $F(0,s) = F(1,s) = P$. Thus F is a path-homotopy from the constant path to $\alpha*\overline{\alpha}$ which implies that $[e_{\alpha(0)}] = [\alpha*\overline{\alpha}] = [\alpha]\cdot[\overline{\alpha}]$. Almost the same argument shows that $[\overline{\alpha}*\alpha] = [e_{\alpha(1)}]$. ■

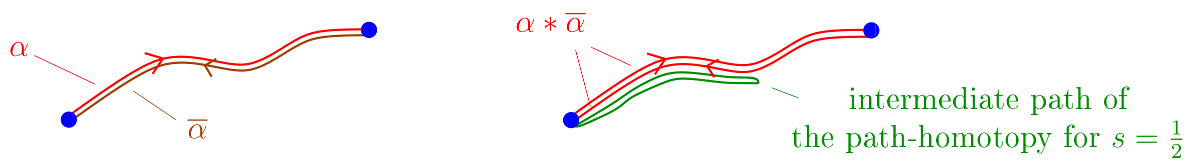


FIGURE 300. Illustration of the proof of Proposition 14.6 (3).

14.2. **The fundamental group of a pointed topological space.** In Proposition 14.6 we saw that the product of path-homotopy classes of paths shares many properties of the multiplication in a group. Nonetheless the path-homotopy classes of paths do not form a group, since the product of two paths is not defined, unless the endpoint of the first path agrees with the starting point of the second path.

We will resolve this problem by restricting our attention to loops.

Definition. Let X be a topological space and let $x_0\in X$ be a point. A *loop in* (X,x_0) is a path $f\colon[0,1]\to X$[219] with $f(0)=f(1)=x_0$, i.e. x_0 is the starting as well as the endpoint of f.

Proposition 14.7. *Let X be a topological space and $x_0\in X$. The set*

$$\pi_1(X,x_0) := \{\textit{path-homotopy classes of loops in } (X,x_0)\}$$

together with the product map $[\alpha]\cdot[\beta] := [\alpha*\beta]$ forms a group where the neutral element is represented by the constant path at x_0 and where the inverse of $[\gamma]$ is given by $[\overline{\gamma}]$.

[219]Note that part of the definition of "loop in (X,x_0)" is that the domain is the interval $[0,1]$.

PROOF. It is clear that the product of any two loops in (X, x_0) is defined. According to Lemma 14.5 the product of paths descends to a well-defined map

$$\begin{array}{rcl} \pi_1(X,x_0) \times \pi_1(X,x_0) & \to & \pi_1(X,x_0) \\ ([\alpha],[\beta]) & \mapsto & [\alpha]\cdot[\beta] := [\alpha * \beta]. \end{array}$$

According to Proposition 14.6 (1) this product map satisfies the associativity axiom. The trivial element is according to Proposition 14.6 (2) given by the path-homotopy class of the constant loop $[e_{x_0}]$. Furthermore according to Proposition 14.6 (3) the inverse of the path-homotopy class of a loop $\gamma\colon [0,1] \to X$ in x_0 is given by the path-homotopy class of the loop defined by $\overline{\gamma}(t) := \gamma(1-t)$, $t \in [0,1]$. ■

Definition. We call $\pi_1(X, x_0)$ the *fundamental group of X with respect to the base point x_0*.

One of our main goals in the intermediate future will be to determine the fundamental group of the topological spaces that we introduced in the earlier sections.

Examples.

(1) We consider $X = \mathbb{R}^n$ and let $x_0 \in \mathbb{R}^n$ be an arbitrary point. As we pointed out on page 460, all loops in (X, x_0) are null-homotopic. More precisely, if $\gamma\colon [0,1] \to \mathbb{R}^n$ is a loop in $(\mathbb{R}^n, x_0)$, then

$$\begin{array}{rcl} F\colon [0,1]\times[0,1] & \to & \mathbb{R}^n \\ (t,s) & \mapsto & \gamma(t)\cdot(1-s) + x_0\cdot s \end{array}$$

is a path-homotopy between the loop γ and the constant loop e_{x_0}. Thus it follows that $\pi_1(X, x_0) = 0$.[220]

(2) We say that a subset $X \subset \mathbb{R}^n$ is *star-shaped* if there exists an $x \in X$ such that for all $y \in X$ the segment $\{xt + y(1-t) \,|\, t \in [0,1]\}$ lies in X. For example open and closed balls and open and closed rectangles are star-shaped. Furthermore all convex subsets are star-shaped. Precisely the same proof as in (1) shows that the $\pi_1(X, x) = 0$.

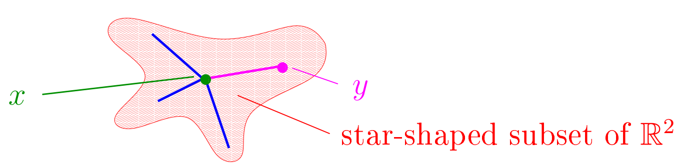


FIGURE 301

Next we state the following theorem from complex analysis.

[220] If $\pi_1(X, x_0)$ is the trivial group, then we usually write $\pi_1(X, x_0) = 0$. This notation is commonly used, even though it is not entirely logical: we use the multiplicative notation for the product structure on the group $\pi_1(X, x_0)$, and it would therefore make more sense to write $\pi_1(X, x_0) = 1$ if the group $\pi_1(X, x_0)$ is trivial.

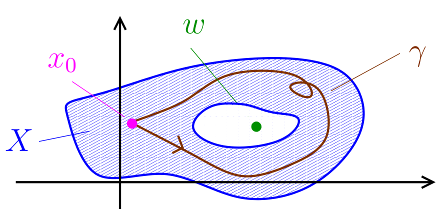
Theorem 14.8. *Let X be a proper subset*[221] *of $\mathbb{C}$, let $x_0 \in X$ be a point and let $w \in \mathbb{C} \setminus X$ be a point in the complement of X. The map*

$$\begin{array}{rcl} \pi_1(X, x_0) & \to & \mathbb{Z} \\ [\gamma] & \mapsto & \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{1}{z-w}\, dz \end{array}$$

is a well-defined homomorphism.

Remark. As in the case of Corollary 13.2, we do not really need Theorem 14.8 and its subsequent Corollary 14.9. If one has seen complex analysis, or if one is willing to take Theorem 14.8 on face value, it allows us to show that some fundamental groups are non-trivial, before we have developed the full machinery for calculating fundamental groups. But eventually we will supplant Corollary 14.9 by a much stronger statement proved using purely topological methods.

Proof. A vigilant reader might first have noticed that the statement of Theorem 14.8 contains a slight cheat. We use the path integral over a continuous path γ, whereas so far on page 458 we gave only the definition for the integral over a smooth path. The more general definition of a path integral is given in [**Lan99**, Chapter III.4]. The fact that the given integral takes values in $\mathbb{Z}$ is proved in [**Lan99**, Lemma IV.4.1]. Furthermore the fact that this integral defines a well-defined map on the fundamental group is a consequence of [**Lan99**, Theorem III.5.1]. Finally it follows almost immediately from the definitions that the given map is a homomorphism. ■



the map

$$\begin{array}{rcl} \pi_1(X, x_0) & \to & \mathbb{Z} \\ [\gamma] & \mapsto & \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{1}{z-w}\, dz \end{array}$$

is a homomorphism

Figure 302

The previous example together with Theorem 14.8 give us the following neat corollary.

Corollary 14.9. *There exists an epimorphism $\pi_1(S^1, 1) \to \mathbb{Z}$.*

Proof. The corollary follows from Theorem 14.8 and the calculation on page 458. ■

This raises the following question.

Question 14.10. *Is the above epimorphism $\pi_1(S^1, 1) \to \mathbb{Z}$ in fact an isomorphism?*

The definition of the fundamental group of a topological space X relies on the choice of a base point x_0. The following proposition shows that, at least for path-connected topological spaces, the choice of the base point does not affect the isomorphism type of the fundamental group.

[221] We say A is a proper subset of B if $A \subset B$ and if $A \neq B$.

Proposition 14.11. *Let X be a topological space, let x_0 and x_1 be two points in X and let $p\colon [0,1] \to X$ be a path from x_0 to x_1. Then the following five statements hold:*

(1) *The map*

$$\begin{array}{rcl} p_*\colon \pi_1(X,x_1) & \to & \pi_1(X,x_0) \\ [\gamma] & \mapsto & [p * \gamma * \overline{p}] \end{array}$$

is well-defined and it is a group isomorphism.

(2) *If $x_0 = x_1$, then the map p_* is just conjugation by $[p] \in \pi_1(X,x_0)$, i.e. for any $g \in \pi_1(X,x_0) = \pi_1(X,x_1)$ we have*

$$p_*(g) \;=\; [p]\cdot g \cdot [p]^{-1} \;\in\; \pi_1(X,x_0).$$

(3) *If $q\colon [0,1] \to X$ is a path from x_1 to another point x_2, then*

$$(p*q)_* \;=\; p_* \circ q_*\colon \pi_1(X,x_2) \;\to\; \pi_1(X,x_0).$$

(4) *If $\pi_1(X,x_0)$ is abelian, then for any two paths $p,q\colon [0,1] \to X$ from x_0 to x_1 we have $p_* = q_*\colon \pi_1(X,x_1) \to \pi_1(X,x_0)$.*

(5) *If $f\colon X \to Y$ is a map to another topological space, then the following diagram commutes:*

$$\begin{tikzcd} \pi_1(X,x_0) \arrow[r, "p_*"] \arrow[d, "f_*"] & \pi_1(X,x_1) \arrow[d, "f_*"] \\ \pi_1(Y,f(x_0)) \arrow[r, "(f\circ p)_*"] & \pi_1(Y,f(x_1)). \end{tikzcd}$$

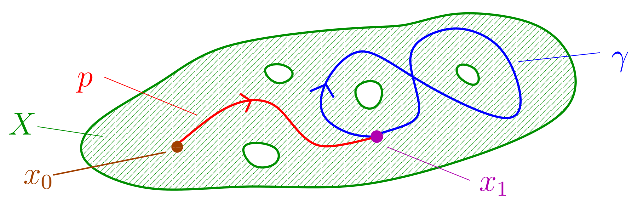


FIGURE 303. Schematic sketch of Proposition 14.11 (1).

PROOF (∗). Let X be a topological space and let x_0 and x_1 be two points in X that are connected via a path $p\colon [0,1] \to X$ from x_0 to x_1.

(1) Let γ be a loop in (X,x_1). Then $p*\gamma*\overline{p}$ is evidently a loop in (X,x_0). From Lemma 14.5 and Proposition 14.6 it follows that the path-homotopy class of $p*\gamma*\overline{p}$ depends only on the choice of the path-homotopy class of γ, i.e. the map

$$\begin{array}{rcl} \Phi\colon \pi_1(X,x_1) & \to & \pi_1(X,x_0) \\ [\gamma] & \mapsto & [p * \gamma * \overline{p}] \end{array}$$

is well-defined. Furthermore for two loops γ,δ in (X,x_1) we have

$$\begin{array}{rclcl} \Phi([\gamma])*\Phi([\delta]) & = & [p*\gamma*\overline{p}]*[p*\delta*\overline{p}] & = & [p*\gamma*\overline{p}*p*\delta*\overline{p}] \\ & \underset{\substack{\uparrow \\ \text{Proposition 14.6 (3) and Lemma 14.5}}}{=} & [p*\gamma*e_{x_1}*\delta*\overline{p}] & \underset{\substack{\uparrow \\ \text{Proposition 14.6 (2) and Lemma 14.5}}}{=} & [p*(\gamma*\delta)*\overline{p}] \;=\; \Phi([\gamma*\delta]). \end{array}$$

Thus we have shown that Φ is a group homomorphism. Furthermore Φ is even a group isomorphism, since an inverse map is given by[222]

$$\begin{array}{rcl}\Psi\colon \pi_1(X,x_0) & \to & \pi_1(X,x_1)\\ [\delta] & \mapsto & [\overline{p}*\delta*p].\end{array}$$

(2),(3) These two statements follow basically immediately from the definitions.

(4) This statement is a straightforward consequence of (1), (2) and (3). We will provide the proof in Exercise 14.3.

(5) This statement again follows immediately from the definitions. ■

We obtain the following corollary:

Corollary 14.12. *Let X be a path-connected topological space and let $x_0, x_1 \in X$. Then the groups $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$ are isomorphic.*

Example. Let X be a star-shaped subset of $\mathbb{R}^n$. Above we saw that there exists an $x \in X$ such that $\pi_1(X,x) = 0$. It now follows from Corollary 14.12 that $\pi_1(X,y) = 0$ for any point $y \in X$. Put differently, the fundamental group of every star-shaped subset of $\mathbb{R}^n$ with respect to any base point is trivial.

If X is a path-connected topological space, then Corollary 14.12 states that the isomorphism type of the fundamental group does not depend on the choice of the base point. Sometimes we suppress the base point from the notation and we denote by $\pi_1(X)$ the isomorphism type of the fundamental group.

Definition. Let X be a non-empty path-connected topological space. If $\pi_1(X) = 0$[223], then we say that X is *simply connected.*[224]

The following lemma will be proved in Exercise 14.2.

Lemma 14.13. *Let X be a topological space and let P, Q be two points in X. If X is simply connected, then any two paths from P to Q are path-homotopic.*

We already saw on page 468 that $\mathbb{R}^n$ and more generally, all star-shaped subsets of $\mathbb{R}^n$ are simply connected. The following statement is much more interesting:

Proposition 14.14. *For any $n \geq 2$ the sphere S^n is simply connected.*

In the proof of Proposition 14.14 we will need the following lemma.

Lemma 14.15. *For any $P \in S^n$ the topological space $S^n \setminus \{P\}$ is simply connected.*

Proof. In Lemma 2.44 we showed that there exists a homeomorphism $f\colon S^n \to \mathbb{R}^n \cup \{\infty\}$. We set $Q := f^{-1}(\infty)$. Thus f restricts to homeomorphism $f\colon S^n \setminus \{Q\} \to \mathbb{R}^n$. Now let P be any other point on S^n. There exists a homeomorphism[225] $g\colon S^n \to S^n$ with $g(P) = Q$.

---

[222] Why is $\Phi \circ \Psi$ the identity on $\pi_1(X,x_0)$ and why is $\Psi \circ \Phi$ the identity on $\pi_1(X,x_1)$?

[223] Since X is a path-connected topological space it is irrelevant which base point we consider.

[224] It is straightforward to see that for subsets of $\mathbb{C}$ we obtain the same notion of "simply connectedness" as on page 457.

[225] The existence of the homeomorphism follows from Proposition 8.29. But one can also easily give an elementary proof of that fact. More precisely, using basic facts from linear algebra one can show that there

Then $f \circ g\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ restricts to a homeomorphism $S^n \setminus \{P\} \to \mathbb{R}^n$. But $\mathbb{R}^n$ is simply connected, hence $S^n \setminus \{P\}$ is also simply connected. ■

Now we can turn to the proof of Proposition 14.14.

Proof of Proposition 14.14.

> Here is a very short "proof": Let $x_0 \in S^n$ and let $\gamma\colon [0,1] \to S^n$ be a loop in x_0. We have to show that γ is null-homotopic. We pick a point P that does not lie in the image of γ. By Lemma 14.15 the topological space $S^n \setminus \{P\}$ is simply connected, hence γ is null-homotopic in $S^n \setminus \{P\}$, in particular it is null-homotopic in S^n.
> This sounds convincing, except for one crucial gap: we did not justify why there exists a point P that does not lie in the image of γ. In fact, as we saw in Proposition 2.60, such a point does not need to exist since there are surjective maps from an interval to any S^n.

We write

$$N \;:=\; (0,\dots,0,1) \quad \text{“North Pole”} \quad \text{and} \quad U \;:=\; S^n \setminus \{N\},$$

and also

$$S \;:=\; (0,\dots,0,-1) \quad \text{South Pole} \quad \text{and} \quad V \;:=\; S^n \setminus \{S\}.$$

We fix a base point x_0 on S^n that is neither N nor S. By Lemma 14.15 it suffices to show that any loop in (S^n, x_0) is path-homotopic to a loop in $U = S^n \setminus \{N\}$.

So let γ be a loop in (S^n, x_0).

Claim. There exists a subdivision

$$0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = 1,$$

such that the following hold:

(1) for each $i \in \{0,\dots,k-1\}$ we have $\gamma([t_i,t_{i+1}]) \subset U$ or $\gamma([t_i,t_{i+1}]) \subset V$, and
(2) for each $i \in \{1,\dots,k-1\}$ we have $\gamma(t_i) \neq N$ and $\gamma(t_i) \neq S$.

By Corollary 2.76 applied to $\gamma\colon [0,1] \to S^n$ and $S^n = U \cup V$ there exists a k such that for any $i \in \{0,\dots,k-1\}$ the image $\gamma([\frac{i}{k}, \frac{i+1}{k}])$ lies in U or in V. In particular the subdivision $t_i = \frac{i}{k}$, $i = 0,\dots,k$ satisfies condition (1).

Now suppose that there exists an $i \in \{1,\dots,k-1\}$ with $\gamma(t_i) = N$. Then neither $\gamma([t_{i-1},t_i])$ nor $\gamma([t_i,t_{i+1}])$ can be contained in $U = S^n \setminus \{N\}$. So by (1) both have to be contained in V, in particular we have $\gamma([t_{i-1},t_{i+1}]) \subset V$. But then we can remove t_i from the subdivision. Iterating this process we can assume that for each every i we have $\gamma(t_i) \neq N$. Similarly we can arrange for every i we have $\gamma(t_i) \neq S$. ⊞

As mentioned before, by Lemma 14.15 it now suffices to prove the following claim.

Claim. The loop $\gamma\colon [0,1] \to S^n$ in (S^n, x_0) is path-homotopic to a path that does not hit the North Pole N.

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exists an orthogonal matrix A with $A \cdot P = Q$. Then the map $g\colon S^n \to S^n$ that is given by multiplication by A has the desired property.

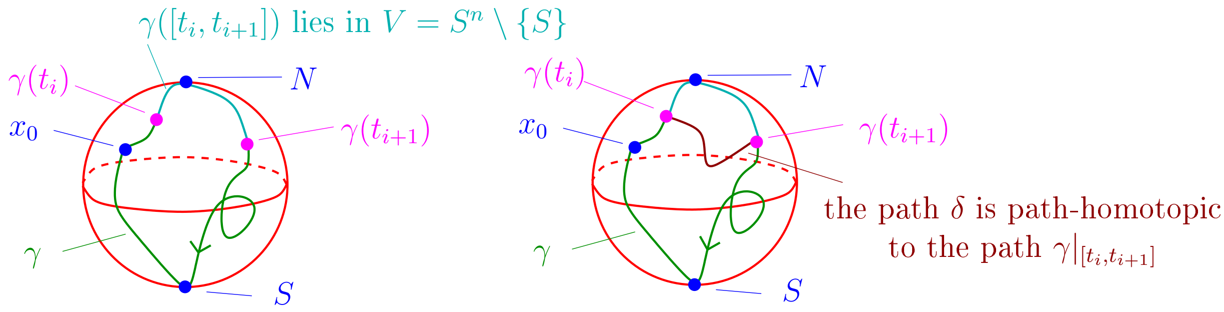

FIGURE 304. Illustration of Proposition 14.14.

We pick a subdivision as in the previous claim. Suppose there exists an $i \in \{0, \ldots, k-1\}$ such that $\gamma([t_i, t_{i+1}])$ hits N. By the choice of our subdivision we know that $\gamma([t_i, t_{i+1}])$ lies entirely in $V = S^n \setminus \{S\}$.[226] Since $S^n \setminus \{N, S\}$ is path-connected and since $\gamma(t_i) \neq N, S$, $\gamma(t_{i+1}) \neq N, S$ we can pick a path $\delta\colon [t_i, t_{i+1}] \to U \cap V = S^n \setminus \{N, S\}$ that connects $\gamma(t_i)$ and $\gamma(t_{i+1})$. Since the paths $\gamma|_{[t_i,t_{i+1}]}$ and δ both lie in the simply connected set $V = S^n \setminus \{S\}$ they are path-homotopic by Lemma 14.13. In the loop γ we now replace $\gamma([t_i, t_{i+1}])$ by δ. We obtain a new loop that is path-homotopic to the original loop and that does not hit N on the interval $[t_i, t_{i+1}]$. Iterating this procedure gives us the desired path-homotopy to a loop that does not hit N.[227] ■

Exercises for Chapter 14.

Exercise 14.1. Consider the torus $T = S^1 \times S^1$ and consider the loop

$$\begin{array}{rcl} [0,1]/0 \sim 1 & \to & T = S^1 \times S^1 \\ t & \mapsto & (e^{2\pi \mathrm{i} t}, 1). \end{array}$$

Is the loop null-homotopic? If yes, provide a path-homotopy to a constant path. If not, show that the path is not null-homotopic.

Exercise 14.2. Let X be a topological space and let P, Q be two points in X. We suppose that X is simply connected. Show that any two paths from P to Q are path-homotopic.

Exercise 14.3. Let X be a topological space and let x_0 and x_1 be two points in X. Furthermore let $p, q\colon [0,1] \to X$ be two paths from x_0 to x_1. We suppose that $\pi_1(X, x_0)$ is abelian. Show that $p_* = q_*\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$.

Exercise 14.4.
(a) Let $f\colon X \to Y$ be a surjective map of topological spaces and let $x \in X$. Is the induced map $f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ necessarily an epimorphism?
(b) Let $f\colon X \to Y$ be an injective map of topological spaces and let $x \in X$. Is the induced map $f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ necessarily a monomorphism?

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[226] We know it lies in U or V, but since it hits N it cannot lie in U.
[227] Where did we use in the proof that $n \geq 2$?

Exercise 14.5. Given a group π we define its *automorphism group* $\operatorname{Aut}(\pi)$ as the group of all self-isomorphisms of π. We define the *inner automorphism group* as

$$\operatorname{Inn}(\pi) \ := \ \{\varphi \in \operatorname{Aut}(\pi) \,|\, \text{ there exists an } h \in \pi \text{ such that } \varphi(g) = hgh^{-1} \text{ or all } g \in G\}.$$

One can easily verify that $\operatorname{Inn}(\pi)$ is a normal subgroup of $\operatorname{Aut}(\pi)$. Thus we can define the *outer automorphism group* to be the group

$$\operatorname{Out}(\pi) \ := \ \operatorname{Aut}(\pi)/\operatorname{Inn}(\pi).$$

Now let X be a path-connected topological space, let $x_0 \in X$ and let $n \in \mathbb{N}$.

(a) Show that the map

$$\begin{array}{rcl} \Theta\colon \operatorname{Homeo}(X) & \to & \operatorname{Out}(\pi_1(X,x_0)) \\ (f\colon X \to X) & \mapsto & (\pi_1(X,x_0) \xrightarrow{f_*} \pi_1(X,f(x_0)) \underset{\uparrow}{\xrightarrow{\gamma_*}} \pi_1(X,x_0)) \end{array}$$

pick any path γ from x_0 to $f(x_0)$ and consider the corresponding isomorphism given by Proposition 14.11

is well-defined, i.e. independent of the choice of the paths.

(b) Show that $\Theta\colon \operatorname{Homeo}(X) \to \operatorname{Out}(\pi_1(X,x_0))$ is a group homomorphism.

15. Categories, functors and natural transformations

In this section we introduce the notion of a category, of a functor and of a natural transformation that many readers will already have encountered in different courses. These notions might initially appear to be rather abstract, but they play an essential role in algebraic topology and they are regularly used in all branches of pure mathematics.

15.1. **Definition and examples of categories.**

Definition. A *category* $\mathcal{C}$ consists of the following data:

(1) A class[228] $\mathrm{Ob}(\mathcal{C})$ of mathematical objects which are called the *objects of the category*,
(2) for each pair (X,Y) of objects there exists a set[229] of *morphisms* $\mathrm{Mor}_{\mathcal{C}}(X,Y)$,
(3) for any three objects X,Y and Z there exists a map

$$\begin{array}{rcl}\mathrm{Mor}_{\mathcal{C}}(X,Y)\times\mathrm{Mor}_{\mathcal{C}}(Y,Z) & \to & \mathrm{Mor}_{\mathcal{C}}(X,Z)\\ (f,g) & \mapsto & g\circ f,\end{array}$$

that satisfies the following axioms:

(A) (associativity): For every $f\in\mathrm{Mor}_{\mathcal{C}}(W,X)$, $g\in\mathrm{Mor}_{\mathcal{C}}(X,Y)$ and $h\in\mathrm{Mor}_{\mathcal{C}}(Y,Z)$ we have

$$(h\circ g)\circ f \;=\; h\circ(g\circ f)\;\in\;\mathrm{Mor}_{\mathcal{C}}(W,Z).$$

(B) (identity): For every object X there exists a morphism $\mathrm{id}_X\in\mathrm{Mor}_{\mathcal{C}}(X,X)$ with the property that

$$\begin{array}{rcll}\mathrm{id}_X\circ f & = & f & \text{for all } f\in\mathrm{Mor}_{\mathcal{C}}(Z,X),\quad\text{and}\\ f\circ\mathrm{id}_X & = & f & \text{for all } f\in\mathrm{Mor}_{\mathcal{C}}(X,Y).\end{array}$$

We adopt the following (naming) conventions:

(a) If the category is clear from the context, then we just write $\mathrm{Mor}(X,Y)$ instead of $\mathrm{Mor}_{\mathcal{C}}(X,Y)$.
(b) We say a morphism $\varphi\in\mathrm{Mor}_{\mathcal{C}}(X,Y)$ is a *morphism from X to Y*.

In the following we introduce all kinds of categories. Many of those will be faithful companions throughout these notes.

Definition.

[228] As many "working mathematicians" the author has a slightly guilty conscience for not knowing what a "set" and what a "class" is. Therefore the reader is referred to [**Cie97, FPr85**] for details. Truth be told, most mathematicians learn from logicians at some point when to say "class" instead of "set", but many mathematicians do not know exactly what they are doing at that point.

[229] One could ask why $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ has to be a set, why not just a class? To the best of my knowledge the axioms on morphisms that we write down also make sense if $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ is only required to be a class. Here's a long discussion which gives a satisfactory answer to people in category theory:

`http://mathoverflow.net/questions/48810/why-need-the-morphisms-to-form-a-set`

I myself do not understand the answer.

(a) We refer to the category $\mathcal{Set}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Set}) &:= \text{all sets}, \\ \mathrm{Mor}(X,Y) &:= \text{all maps from } X \text{ to } Y, \end{aligned}$$

together with the usual composition of maps the *category of sets*.

(b) Let $\mathbb{K}$ be a field. We refer to the category $\mathcal{Vec}_{\mathbb{K}}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Vec}_{\mathbb{K}}) &:= \text{all } \mathbb{K}\text{-vector spaces}, \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}_{\mathbb{K}}(X,Y) = \text{all } \mathbb{K}\text{-homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of homomorphisms as the *category of* $\mathbb{K}$*-vector spaces*.

(c) We refer to the category $\mathcal{Gr}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Gr}) &:= \text{all groups}, \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}(X,Y) = \text{all group homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of group homomorphisms as the *category of groups*.

(d) We refer to the category $\mathcal{AbGr}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{AbGr}) &:= \text{all abelian groups}, \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}(X,Y) = \text{all group homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of group homomorphisms as the *category of abelian groups*.

(e) We refer to the category $\mathcal{Ring}$ with[230]

$$\begin{aligned} \mathrm{Ob}(\mathcal{Ring}) &:= \text{all rings}, \\ \mathrm{Mor}(R,S) &:= \mathrm{Hom}(R,S) = \text{all ring homomorphisms from } R \text{ to } S, \end{aligned}$$

with the usual composition of ring homomorphisms as the *category of rings*. Similarly we define the category $\mathcal{CRing}$ of commutative rings.

(f) We refer to the category $\mathcal{Top}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Top}) &:= \text{all topological spaces}, \\ \mathrm{Mor}(X,Y) &:= C(X,Y) := \text{all continuous maps from } X \text{ to } Y, \end{aligned}$$

with the usual composition of maps as the *category of topological spaces*. A *pointed topological space* is a pair (X, x_0), where X is a topological space and x_0 is a point in X. We refer to the category $\mathcal{PTop}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{PTop}) &:= \text{all pointed topological spaces}, \\ \mathrm{Mor}((X,x_0),(Y,y_0)) &:= \text{all continuous maps } f \text{ from } X \text{ to } Y \text{ with } f(x_0) = y_0 \end{aligned}$$

with the usual composition of maps as the *category of pointed topological spaces*.

(g) We refer to the category $\mathcal{TopGr}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{TopGr}) &:= \text{all topological groups}, \\ \mathrm{Mor}(X,Y) &:= \text{all homomorphisms from } X \text{ to } Y \text{ that are continuous}, \end{aligned}$$

with the usual composition of maps as the *category of topological groups*.

(h) We refer to the category $\mathcal{SmMfd}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{SmMfd}) &:= \text{ all smooth manifolds,} \\ \mathrm{Mor}(X,Y) &:= \text{ all smooth maps from } X \text{ to } Y, \end{aligned}$$

with the usual composition of maps as the *category of smooth manifolds.* We define the category $\mathcal{PSmMfd}$ of *pointed smooth manifolds* the same way as we defined the category of *pointed topological spaces.* Finally we refer to the category $\mathcal{OrSmMfd}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{OrSmMfd}) &:= \text{ all oriented smooth manifolds,} \\ \mathrm{Mor}(X,Y) &:= \text{ all orientation-preserving diffeomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of maps as the *category of oriented smooth manifolds.*

(i) A *pair of topological spaces* is a pair (X, A), where X is a topological space and A is a subset of X. We refer to the category $\mathcal{PairTop}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{PairTop}) &:= \text{ all pairs of topological spaces,} \\ \mathrm{Mor}((X,A),(Y,B)) &:= \text{ all continuous maps } f \text{ from } X \text{ to } Y \text{ with } f(A) \subset B \end{aligned}$$

with the usual composition of maps as the *category of pairs of topological spaces.*

(j) Let $G = (V, E, i\colon E \to V, t\colon E \to V)$ and $G' = (V', E', i'\colon E' \to V', t'\colon E' \to V')$ be two abstract graphs in the sense of the definition on page 221. We define a *map $G \to G'$ of abstract graphs* to be a pair $(\alpha\colon V \to V', \beta\colon E \to E')$ of maps such that for each $e \in E$ we have $i'(\beta(e)) = \alpha(i(e))$ and $t'(\beta(e)) = \alpha(t(e))$. We refer to the category $\mathcal{AbsGraph}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{AbsGraph}) &:= \text{ all abstract graphs,} \\ \mathrm{Mor}(G,G') &:= \text{ all maps from } G \text{ to } G' \text{ in the above sense} \end{aligned}$$

with the usual composition of maps as the *category of abstract graphs.* Similarly, we consider the closely related setup of undirected abstract graphs which we introduced on page 226. We define *a map $G \to G'$ between two undirected abstract graphs $G = (V, E, \varphi\colon E \to \mathcal{P}(V))$ and $G' = (V', E', \varphi'\colon E' \to \mathcal{P}(V'))$* to be a pair $(\alpha\colon V \to V', \beta\colon E \to E')$ of maps such that for every $e \in E$ we have the equality $\varphi'(\beta(e)) = \alpha(\varphi(e))$. We refer to the category $\mathcal{UndirAbsGraph}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{UndirAbsGraph}) &:= \text{ all undirected abstract graphs,} \\ \mathrm{Mor}(G,G') &:= \text{ all maps from } G \text{ to } G' \text{ in the above sense} \end{aligned}$$

with the usual composition of maps as the *category of undirected abstract graphs.*

The name "morphisms" suggests that morphisms are necessarily maps. But as we will see in the following two examples, this is not necessarily always the case.

Example.

(k) We consider the category $\mathcal{C}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}) &:= \text{ all sets,} \\ \mathrm{Mor}(X,Y) &:= \begin{cases} \{\mathrm{id}_X\}, & \text{if } X = Y \\ \varnothing, & \text{if } X \neq Y. \end{cases} \end{aligned}$$

[230]We refer to page 81 for our definition of a ring.

The composition map only needs to be defined if $X = Y = Z$[231], and in this case we define $\mathrm{id}_X \circ \mathrm{id}_X := \mathrm{id}_X$. One can easily convince oneself that this defines a category.

(l) Let G be an arbitrary group. We consider

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{C}) & := & \{\text{a set with a unique element } *\}, \\ \mathrm{Mor}(*,*) & := & G. \end{array}$$

We define

$$\begin{array}{rcl} \mathrm{Mor}(*,*) \times \mathrm{Mor}(*,*) & \to & \mathrm{Mor}(*,*) \\ (f,g) & \mapsto & g \circ f := gf \end{array}$$

via the group structure on $\mathrm{Mor}(*,*) = G$. The axioms of a category can be easily deduced from the group axioms of G.

Convention. If the category we are working with at a given moment is understood, then we just write "map" instead of "morphisms". For example we might write "let $f\colon X \to Y$ be a map", if X and Y are groups, then it is understood that the map is in fact a homomorphism, whereas if X and Y are topological spaces, then it is understood that f is continuous. This convention thus generalizes the convention that we introduced on page 115.

The following definition is almost self-explanatory.

Definition. Let $\mathcal{C}$ be a category. A *subcategory* is a category $\mathcal{D}$ such that $\mathrm{Ob}(\mathcal{D}) \subset \mathrm{Ob}(\mathcal{C})$ and such that for each $X, Y \in \mathrm{Ob}(\mathcal{D})$ we have $\mathrm{Mor}_{\mathcal{D}}(X,Y) \subset \mathrm{Mor}_{\mathcal{C}}(X,Y)$. We say $\mathcal{D}$ is a *full subcategory* if for each $X, Y \in \mathrm{Ob}(\mathcal{D})$ we have $\mathrm{Mor}_{\mathcal{D}}(X,Y) = \mathrm{Mor}_{\mathcal{C}}(X,Y)$.

Examples.

(1) The category $\mathcal{AbGr}$ of abelian groups is a full subcategory of the category of groups $\mathcal{Gr}$.
(2) The category $\mathcal{Vec}_{\mathbb{R}}$ of $\mathbb{R}$-vector spaces is a subcategory of the category $\mathcal{AbGr}$ of abelian groups, but it is not a full subcategory.

We conclude this introduction with the following definition and a short discussion thereof.

Definition. Let $\mathcal{C}$ be a category.

(1) In the following we call a morphism $F \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ *invertible* if there exists a morphism $G \in \mathrm{Mor}_{\mathcal{C}}(Y,X)$ such that $F \circ G = \mathrm{id}_Y$ and $G \circ F = \mathrm{id}_X$. Sometimes we refer to invertible morphisms also as *isomorphisms*.
(2) We denote by $\mathrm{Mor}_{\mathcal{C}}(X,Y)_{\mathrm{inv}}$ the set of all invertible morphisms.

Examples.

(1) In the category of topological spaces the invertible morphisms are precisely the homeomorphisms.
(2) In the category of smooth manifolds the invertible morphisms are precisely the diffeomorphisms.
(3) In the category considered in (k) every morphism is an isomorphism.

We leave it to the reader to prove the following elementary lemma.

[231] If $X \neq Y$ then $\mathrm{Mor}(X,Y) = \varnothing$, but then $\mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z)$ is also the empty set for any Z.

Lemma 15.1. *Let $\mathcal{C}$ be a category. If $f \in \operatorname{Mor}_{\mathcal{C}}(X,Y)$ is an invertible morphism, then for any $Z \in \operatorname{Ob}(\mathcal{C})$ the maps*

$$\begin{array}{rcl} \operatorname{Mor}_{\mathcal{C}}(Y,Z) & \to & \operatorname{Mor}_{\mathcal{C}}(X,Z) \\ g & \mapsto & g \circ f \end{array} \quad \textit{and} \quad \begin{array}{rcl} \operatorname{Mor}_{\mathcal{C}}(Z,X) & \to & \operatorname{Mor}_{\mathcal{C}}(Z,Y) \\ g & \mapsto & f \circ g \end{array}$$

are bijections.

15.2. **Functors.** In this section we introduce the notions of covariant and contravariant functors.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *covariant functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of a map

$$F\colon \operatorname{Ob}(\mathcal{C}) \ \to \ \operatorname{Ob}(\mathcal{D})$$

and for all $X, Y \in \mathcal{C}$ there exists furthermore a map

$$F\colon \operatorname{Mor}_{\mathcal{C}}(X,Y) \ \to \ \operatorname{Mor}_{\mathcal{D}}(F(X),F(Y)),$$

such that following axioms are satisfied

(F1) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \operatorname{Ob}(\mathcal{C})$,
(F2) for $\phi \in \operatorname{Mor}_{\mathcal{C}}(X,Y)$ and $\psi \in \operatorname{Mor}_{\mathcal{C}}(Y,Z)$ the following equality holds

$$F(\psi \circ \phi) \ = \ F(\psi) \circ F(\phi) \ \in \operatorname{Mor}_{\mathcal{D}}(F(X),F(Z)).$$

Convention. For a covariant functor F and $\phi \in \operatorname{Mor}_{\mathcal{C}}(X,Y)$ we often write ϕ_* instead of $F(\phi)$. The second axiom then becomes $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

Now we give some examples of functors.

Examples.

(1) As above let $\mathcal{Vec}_{\mathbb{R}}$ be the category of real vector spaces and let W be a real vector space. Then the map

$$\begin{array}{rcl} F\colon \operatorname{Ob}(\mathcal{Vec}_{\mathbb{R}}) & \to & \operatorname{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ V & \mapsto & \operatorname{Hom}_{\mathbb{R}}(W,V), \end{array}$$

together with the maps

$$\begin{array}{rcl} \operatorname{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(U,V) & \to & \operatorname{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(F(U),F(V)) \\ (\phi\colon U \to V) & \mapsto & \begin{pmatrix} \operatorname{Hom}_{\mathbb{R}}(W,U) & \to & \operatorname{Hom}_{\mathbb{R}}(W,V) \\ f & \mapsto & \phi \circ f \end{pmatrix} \end{array}$$

defines a covariant functor.

(2) Let $\mathcal{PSmMfd}$ be the category of pointed smooth manifolds and let $\mathcal{Vec}_{\mathbb{R}}$ be the category of real vector spaces. By Proposition 6.37 (2) we know that the map

$$\begin{array}{rcl} F\colon \operatorname{Ob}(\mathcal{PSmMfd}) & \to & \operatorname{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ (M,P) & \mapsto & \mathrm{T}_P\, M, \end{array}$$

together with the maps

$$\begin{array}{rcl} \operatorname{Mor}_{\mathcal{P}}((M,P),(N,Q)) & \to & \operatorname{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(\mathrm{T}_P\, M, \mathrm{T}_Q\, N) \\ f & \mapsto & f_* = \mathrm{D}\, f_P \end{array}$$

is a covariant functor from the category of pointed smooth manifolds to the category of real vector spaces.

(3) Let $\mathcal{Top}$ be the category of topological spaces and let $X \in \mathrm{Ob}(\mathcal{Top})$. It follows easily from Lemma 3.8 (2b) that the map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{Top}) & \to & \mathrm{Ob}(\mathcal{Top}) \\ Y & \mapsto & X \times Y, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Top}}(Y,Z) & \to & \mathrm{Mor}_{\mathcal{Top}}(X \times Y, X \times Z) \\ f & \mapsto & \mathrm{id}_X \times f \end{array}$$

is a covariant functor from the category $\mathcal{Top}$ of topological spaces to itself.

(4) Let $\mathcal{Top}$ be the category of topological spaces and let $\mathcal{Set}$ be the category of sets. It follows easily from Lemma 2.70 that the map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{Top}) & \to & \mathrm{Ob}(\mathcal{Set}) \\ X & \mapsto & \pi_0(X), \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Top}}(X,Y) & \to & \mathrm{Mor}_{\mathcal{Set}}(\pi_0(X), \pi_0(Y)) \\ f & \mapsto & \left(\begin{array}{rcl} \pi_0(X) & \to & \pi_0(Y) \\ C & \mapsto & f(C) \end{array} \right) \end{array}$$

is a covariant functor from the category of topological spaces to the category of sets.

(5) Let $\mathcal{AbsGraph}$ be the category of abstract graphs and let $\mathcal{UndirAbsGraph}$ be the category of undirected abstract graphs. It follows easily from the definitions that the map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{AbsGraph}) & \to & \mathrm{Ob}(\mathcal{UndirAbsGraph}) \\ (V, E, i\colon E \to V, t\colon E \to V) & \mapsto & \left(V, E, \begin{array}{rcl} E & \to & \mathcal{P}(V) \\ e & \mapsto & \{i(e), t(e)\} \end{array} \right) \end{array}$$

together with the hopefully maps from the morphisms in $\mathcal{AbsGraph}$ to the morphisms in $\mathcal{UndirAbsGraph}$ is a covariant functor from the category $\mathcal{AbsGraph}$ of abstract graphs to the category $\mathcal{UndirAbsGraph}$ of undirected abstract graphs. We make the following two comments:

(a) The reader will surely remember that implicitly we studied this functor in Lemma 4.4. There, given an abstract graph G we gave an explicit homeomorphism $|G| \to |F(G)|$ between the topological realization of the abstract graph G and the topological realization of the undirected abstract graph $F(G)$.

(b) It follows from the Axiom of Choice that given any undirected abstract graph G there exists an G' with $F(G') = G$.

(6) Let $\mathcal{OrSmMfd}$ be the category of oriented smooth manifolds. The map

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{OrSmMfd}) & \to & \mathrm{Ob}(\mathcal{OrSmMfd}) \\ M & \mapsto & \partial M \text{ equipped with the orientation from Lemma 6.50} \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{OrSmMfd}}(M,N) & \mapsto & \mathrm{Mor}_{\mathcal{OrSmMfd}}(\partial M, \partial N) \\ (f\colon M \to N) & \mapsto & (f|_{\partial M}\colon \partial M \to \partial N) \end{array}$$

is a covariant functor.

We state our next example as a lemma.

Lemma 15.2. *The following two statements hold:*

(1) *Let $G = (V, E, i, t)$ and $G' = (V', E', i', t')$ be two abstract graphs. Furthermore let $f := (\alpha\colon V \to V', \beta\colon E \to E')$ be a map between the two abstract graphs. Recall that, by the definition on page 477, this means that for each $e \in E$ we have the equalities $i'(\beta(e)) = \alpha(i(e))$ and $t'(\beta(e)) = \alpha(t(e))$. Now we consider the following map between the topological realizations of G and G':*

$$\begin{array}{rcl} |f|\colon |G| = \big(V \sqcup (E \times [0,1])\big)/\underset{\substack{\text{where } (e,0)\sim i(e) \text{ and } (e,1)\sim t(e)}}{\sim} & \to & |G'| = \big(V' \sqcup (E' \times [0,1])\big)/\underset{\substack{\text{similarly } (e,0)\sim i'(e) \text{ and } (e,1)\sim t'(e)}}{\sim} \\ {[x]} & \mapsto & \begin{cases} \alpha(x), & \text{if } x \in V, \\ (\beta(e), t), & \text{if } x = (e,t) \in E \times [0,1]. \end{cases} \end{array}$$

This map $|f|\colon |G| \to |G'|$ is well-defined and continuous.

(2) *The map*

$$\begin{array}{rcl} F\colon \operatorname{Ob}(\mathcal{AbsGraph}) & \to & \operatorname{Ob}(\mathcal{Top}) \\ G & \mapsto & |G|, \end{array}$$

together with the maps

$$\begin{array}{rcl} \operatorname{Mor}_{\mathcal{AbsGraph}}(G, G') & \to & \operatorname{Mor}_{\mathcal{Top}}(|G|, |G'|) \\ f & \mapsto & |f| \end{array}$$

is a covariant functor from the category $\mathcal{AbsGraph}$ of abstract graphs to the category $\mathcal{Top}$ of topological spaces.

PROOF.

(1) It follows almost immediately from the definitions that the map is well-defined. We leave it to the reader to verify, e.g. using Lemma 3.28, that the map is continuous.
(2) This statement follows easily from the definitions. ■

We formulate our last example also as a lemma.

Lemma 15.3. *Let $\mathcal{D}$ be a category and let $\mathcal{Set}$ be the category of sets. Furthermore let $W \in \operatorname{Ob}(\mathcal{D})$. The map*

$$\begin{array}{rcl} F\colon \operatorname{Ob}(\mathcal{D}) & \to & \operatorname{Ob}(\mathcal{Set}) \\ X & \mapsto & \operatorname{Mor}_{\mathcal{D}}(W, X) \end{array}$$

together with the maps

$$\begin{array}{rcl} \operatorname{Mor}_{\mathcal{D}}(Y, Z) & \to & \operatorname{Mor}_{\mathcal{Set}}(\operatorname{Mor}_{\mathcal{D}}(W, Y), \operatorname{Mor}_{\mathcal{D}}(W, Z)) \\ f & \mapsto & \left(\begin{array}{rcl} \operatorname{Mor}_{\mathcal{D}}(W, Y) & \to & \operatorname{Mor}_{\mathcal{D}}(W, Z) \\ g & \mapsto & f \circ g \end{array} \right) \end{array}$$

is a covariant functor.

PROOF. The statement follows easily from the axioms of a category. ■

We also have the following definition which looks almost the same as the previous definition.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *contravariant functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of a map

$$F\colon \operatorname{Ob}(\mathcal{C}) \;\to\; \operatorname{Ob}(\mathcal{D})$$

and for all $X, Y \in \mathcal{C}$ there exists furthermore a map

$$\operatorname{Mor}_{\mathcal{C}}(X,Y) \;\to\; \operatorname{Mor}_{\mathcal{D}}(F(Y), F(X))$$

such that following axioms are satisfied:

(F1) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \operatorname{Ob}(\mathcal{C})$,

(F2) for any $\phi \in \operatorname{Mor}_{\mathcal{C}}(X,Y)$ and $\psi \in \operatorname{Mor}_{\mathcal{C}}(Y,Z)$ we have

$$F(\psi \circ \phi) \;=\; F(\phi) \circ F(\psi) \;\in \operatorname{Mor}_{\mathcal{D}}(F(Z), F(X)).$$

The definition of a contravariant functor looks very much the same as the definition of a covariant functor. The difference is that a contravariant functor "reverses the direction of maps" and the orders of $F(\psi)$ and $F(\phi)$ in the second axiom are swapped.

Convention. For a contravariant functor F and $\phi \in \operatorname{Mor}_{\mathcal{C}}(X,Y)$ we often write ϕ^* instead of $F(\phi)$. The second axiom then becomes $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

Examples.

(1) Let $\mathcal{Vec}_{\mathbb{R}}$ be the category of real vector spaces. Then the map

$$\begin{array}{rcl} F\colon \operatorname{Ob}(\mathcal{Vec}_{\mathbb{R}}) & \to & \operatorname{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ V & \mapsto & V^* := \operatorname{Hom}_{\mathbb{R}}(V,\mathbb{R}) =: \text{dual vector space of } V, \end{array}$$

together with the maps

$$\begin{array}{rcl} \operatorname{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(U,V) & \to & \operatorname{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(V^*,U^*) \\ (\phi\colon U \to V) & \mapsto & \left(\begin{array}{rcl} \operatorname{Hom}_{\mathbb{R}}(V,\mathbb{R}) & \to & \operatorname{Hom}_{\mathbb{R}}(U,\mathbb{R}) \\ f & \mapsto & f \circ \phi \end{array}\right) \end{array}$$

is a contravariant functor.

(2) Let $\mathcal{Top}$ be the category of topological spaces and let $\mathcal{Vec}_{\mathbb{R}}$ be the category of real vector spaces. Then the map

$$\begin{array}{rcl} F\colon \operatorname{Ob}(\mathcal{Top}) & \to & \operatorname{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ X & \mapsto & C(X,\mathbb{R}) := \text{set of continuous functions } X \to \mathbb{R}, \end{array}$$

together with the maps

$$\begin{array}{rcl} \operatorname{Mor}_{\mathcal{Top}}(X,Y) & \to & \operatorname{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(C(Y,\mathbb{R}), C(X,\mathbb{R})) \\ (\phi\colon X \to Y) & \mapsto & \left(\begin{array}{rcl} C(Y,\mathbb{R}) & \to & C(X,\mathbb{R}) \\ f & \mapsto & f \circ \phi \end{array}\right) \end{array}$$

is a contravariant functor.

(3) Let $\mathcal{SmMfd}$ be the category of smooth manifolds, let $\mathcal{Vec}_{\mathbb{R}}$ be the category of real vector spaces and let $k \in \mathbb{Z}_{\geq 0}$. In [**BoT82**, Chapter 2] or [**Lee02**, Corollary 17.3] it

is shown that the map

$$\begin{array}{rcl} F\colon \operatorname{Ob}(\mathcal{SmMfd}) & \to & \operatorname{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ M & \mapsto & H^k_{\mathrm{dR}}(M) := k\text{-th de Rham cohomology group of } M, \end{array}$$

together with the maps

$$\begin{array}{rcl} \operatorname{Mor}_{\mathcal{SmMfd}}(M,N) & \to & \operatorname{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(H^k_{\mathrm{dR}}(N), H^k_{\mathrm{dR}}(M)) \\ (f\colon M \to N) & \mapsto & \left(\begin{array}{rcl} f^*\colon H^k_{\mathrm{dR}}(N) & \to & H^k_{\mathrm{dR}}(M) \\ {[\omega]} & \mapsto & [f^*\omega] \end{array} \right) \end{array}$$

is a contravariant functor.

Again we formulate our last example as a lemma.

Lemma 15.4. *Let $\mathcal{D}$ be a category and let $\mathcal{Set}$ be the category of sets. Furthermore let $W \in \operatorname{Ob}(\mathcal{D})$. The map*

$$\begin{array}{rcl} F\colon \operatorname{Ob}(\mathcal{D}) & \to & \operatorname{Ob}(\mathcal{Set}) \\ X & \mapsto & \operatorname{Mor}_{\mathcal{D}}(X, W) \end{array}$$

together with the maps

$$\begin{array}{rcl} \operatorname{Mor}_{\mathcal{D}}(Y,Z) & \to & \operatorname{Mor}_{\mathcal{Set}}(\operatorname{Mor}_{\mathcal{D}}(Z,W), \operatorname{Mor}_{\mathcal{D}}(Y,W)) \\ f & \mapsto & \left(\begin{array}{rcl} \operatorname{Mor}_{\mathcal{D}}(Z,W) & \to & \operatorname{Mor}_{\mathcal{D}}(Y,W) \\ g & \mapsto & g \circ f \end{array} \right) \end{array}$$

is a contravariant functor.

Proof. Again the statement follows easily from the axioms of a category. ■

We will not see any other examples of contravariant functors in the near future. We will come back to them when we will study cohomology groups.

We conclude this section with the following basically obvious lemma.

Lemma 15.5. *Every functor (regardless of whether it is covariant or contravariant) sends invertible morphisms to invertible morphisms.*

15.3. **The fundamental group as functor.** Now, after recalling the definition of categories and functors, we turn back to the study of fundamental groups.

Let $f\colon X \to Y$ be a map between topological spaces and let $x_0 \in X$ be a base point. If $\gamma\colon [0,1] \to X$ is a loop in the point x_0, then $f \circ \gamma\colon [0,1] \to Y$ is a loop in the point $f(x_0)$. It is straightforward to see that

$$\begin{array}{rcl} f_*\colon \pi_1(X, x_0) & \to & \pi_1(Y, f(x_0)) \\ {[\gamma]} & \mapsto & [f \circ \gamma] \end{array}$$

is a well-defined map, i.e. $f_*([\gamma])$ is independent of the choice of the representative of the path-homotopy class.[232] Furthermore it follows easily from the definitions[233] that f_* is a

[232]Indeed, suppose that γ and δ are two loops in (X, x_0) that are path-homotopic. Let F be a path-homotopy between γ and δ. Then $f \circ F$ is a path-homotopy between $f \circ \gamma$ and $f \circ \delta$. We had used almost the same argument on page 461.

[233]If γ and δ are loops in x_0, then it follows immediately from the definitions that $(f \circ \gamma) * (f \circ \delta) = f \circ (\gamma * \delta)$.

group homomorphism. We call f_* the *induced map*. It also follows easily from the definitions that

$$\begin{array}{rcll} (\mathrm{id}_X)_* & = & \mathrm{id}_{\pi_1(X,x_0)}, & \text{for all pointed pairs } (X,x_0), \\ (g\circ f)_* & = & g_*\circ f_*, & \text{for all maps } f\colon (X,x_0)\to(Y,y_0) \text{ and } g\colon (Y,y_0)\to(Z,z_0). \end{array}$$

Summarizing we just convinced ourselves of the validity of the following proposition.

Proposition 15.6. *Let $\mathcal{PTop}$ be the category of pointed topological spaces and let $\mathcal{Gr}$ be the category of groups. Then the map*

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PTop}) & \to & \mathrm{Ob}(\mathcal{Gr}) \\ (X,x_0) & \mapsto & \pi_1(X,x_0) \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}((X,x_0),(Y,y_0)) & \to & \mathrm{Mor}(\pi_1(X,x_0),\pi_1(Y,y_0)) \\ (f\colon (X,x_0)\to(Y,y_0)) & \mapsto & \left(\begin{array}{rcl} f_*\colon \pi_1(X,x_0) & \to & \pi_1(Y,y_0) \\ {[\gamma]} & \mapsto & [f\circ\gamma] \end{array}\right) \end{array}$$

is a covariant functor.

Next we will see that the functoriality of the fundamental groups can be helpful for studying certain questions.

Definition. Let X be a topological space and let $A\subset X$ be a subspace. We say A is a *retract of* X if there exists a retraction $r\colon X\to A$, i.e. a map with $r(a)=a$ for all $a\in A$.

Examples.

(1) Let X be a topological space. Then any point $x_0\in X$ is a retract of X, indeed, the map $r(x)=x_0$ for $x\in X$ is a retraction.
(2) The circle $S^1\times\{1\}$ is a retract of the torus $S^1\times S^1$, in fact a retraction is given by $r(z,w)=(z,1)$.
(3) The circle S^1 is a retract of $\overline{B}^2\setminus\{(0,0)\}$, indeed, the map

$$\begin{array}{rcl} \overline{B}^2\setminus\{(0,0)\} & \to & S^1 \\ z & \mapsto & \frac{z}{|z|} \end{array}$$

is a retraction. (Here we consider $\overline{B}^2$ as subset of $\mathbb{C}$.)
(4) The set $S^0=\{-1,1\}$ is not a retract of $\overline{B}^1=[-1,1]$. Indeed, since $[-1,1]$ is connected and since S^0 is discrete it follows from Lemma 2.61 that any map from $[-1,1]$ to S^0 has to be constant. Thus there does not exist a map $f\colon[-1,1]\to S^0$ with $f(-1)=-1$ and $f(1)=1$.
(5) Finally we consider the knotted curve C in the solid torus $\overline{B}^2\times S^1$ that is shown in Figure 305. We invite the reader to think about the question whether or not C is a retract of $\overline{B}^2\times S^1$. We will answer the question in Section 119.3.

Now we can push the second-to-last example up by one dimension.

Lemma 15.7. *The circle S^1 is not a retract of $\overline{B}^2$.*

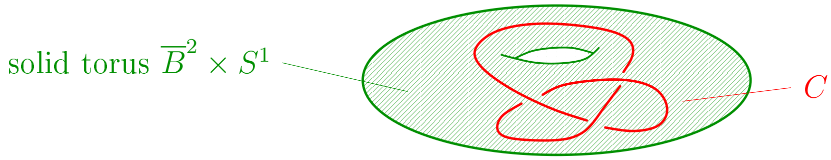


FIGURE 305

PROOF. We denote by $i\colon S^1 \to \overline{B}^2$ the inclusion map. Suppose there exists a retraction $r\colon \overline{B}^2 \to S^1$. By definition this means that $r \circ i = \mathrm{id}_{S^1}$. We obtain the following commutative diagram of maps between topological spaces

$$\begin{tikzcd} & \overline{B}^2 \arrow[dr, "r"] & \\ S^1 \arrow[ur, "i"] \arrow[rr, "r\circ i=\mathrm{id}_{S^1}"'] & & S^1 \end{tikzcd} \quad \text{which induces} \quad \begin{tikzcd} & \pi_1(\overline{B}^2,1) \arrow[dr, "r_*"] & \\ \pi_1(S^1,1) \arrow[ur, "i_*"] \arrow[rr, "(r\circ i)_*=(\mathrm{id}_{S^1})_*"'] & & \pi_1(S^1,1). \end{tikzcd}$$

It follows from the functoriality of fundamental groups that $r_* \circ i_* = (r \circ i)_*$, i.e. the diagram on the right-hand side commutes. By the functoriality we know that $(\mathrm{id}_{S^1})_*$ is the identity of $\pi_1(S^1, 1)$.

We saw on page 471 that $\pi_1(\overline{B}^2, 1)$ is the trivial group whereas we saw in Corollary 14.9 that $\pi_1(S^1, 1)$ is not the trivial group. The lower horizontal map in the second diagram is thus an isomorphism of a non-trivial group, but the upper map factors through the trivial group, i.e. the composition of i_* and r_* cannot be an isomorphism of a non-trivial group.[234] Thus we have obtained a contradiction. ■

The last example and the last lemma now raise the following question.

Question 15.8. *Let $n \geq 2$. Is S^n a retract of $\overline{B}^{n+1}$?*

At the moment we cannot answer this question. We will first need to develop new invariants before we can return to this question. We will not be able to answer this question for many months.

Almost exactly the same argument as in the proof of Lemma 15.7 gives a proof of the following lemma which we will prove in Exercise 15.3.

Lemma 15.9. *Let X be a topological space and let $A \subset X$ be a retract of X. Let $a \in A$. We denote by $i\colon A \to X$ the inclusion map.*

(1) *There exists an epimorphism $\varphi\colon \pi_1(X, a) \to \pi_1(A, a)$ such that $\varphi \circ i_*$ is the identity on $\pi_1(A, a)$.*
(2) *The inclusion induced map $\pi_1(A, a) \to \pi_1(X, a)$ is a monomorphism.*

Example. We had just seen on page 484 that the circle $S^1 \times \{1\}$ is a retract of the torus $S^1 \times S^1$. Furthermore we saw in Corollary 14.9 that $\pi_1(S^1, 1)$ is not the trivial group. It follows from Lemma 15.9 that the fundamental group of the torus is also non-trivial. In

[234] Why not?

Proposition 14.14 we saw that $\pi_1(S^2) = 0$. Thus we see that the torus $S^1 \times S^1$ and the sphere S^2 are not homeomorphic.

Lemma 15.10. $(*)$ *Let E_8 be the regular octagon from the definition of the surface of genus 2, see page 205 for details, and let $\Sigma = E_8/\sim$ be the surface of genus 2. Let P and Q be two adjacent vertices of E_8. Then the following hold:*

(1) *The map*

$$\begin{array}{rcl} \gamma\colon S^1 & \to & E_8/\sim \\ e^{2\pi i t} & \mapsto & [t\cdot P + (1-t)\cdot Q] \end{array}$$

is an embedding.

(2) *There exists a map $\Phi\colon E_8/\sim \to S^1 \times S^1$ such that the following diagram commutes*

$$\begin{tikzcd} & S^1 \arrow[dl, "\gamma"'] \arrow[dr, "z\mapsto(z,1)"] & \\ E_8/\sim \arrow[rr, "\Phi"] & & S^1 \times S^1. \end{tikzcd}$$

(3) *The subset $\gamma(S^1)$ is a retract of $\Sigma = E_8/\sim$.*

(4) *There exists an epimorphism $\pi_1(\Sigma, P) \to \mathbb{Z}$ that sends $[\gamma]$ onto 1.*[235]

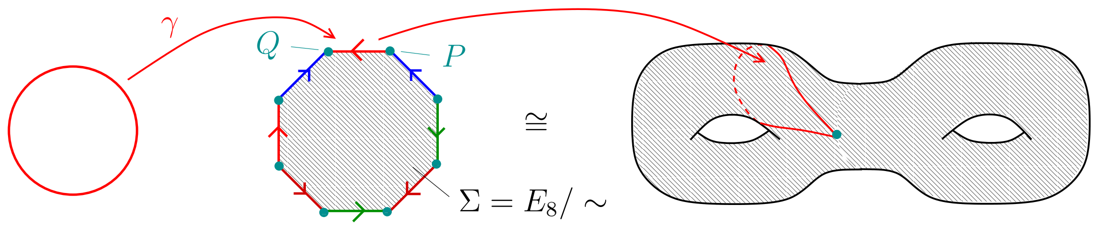


FIGURE 306. Illustration of Lemma 15.10.

Remark. The lemma shows in particular that the four loops that we drew in Figure 89 on the surface of genus 2 represent non-trivial elements in the fundamental group. Almost the same way one can show that in the surface of genus g, $g \geq 2$, defined as a quotient of a regular $4g$-gon, each edge gives rise to a non-trivial element of the fundamental group.

PROOF $(*)$.

(1) It is clear that γ is continuous and injective. By Proposition 6.8 we know that $\Sigma = E_8/\sim$ is Hausdorff. It follows from Proposition 2.43 (2) that γ is a closed embedding.
(2) The map Φ is sketched in Figure 307 on the top. It is given by sending all points in the lower pentagon to a single point, such a map can be extended continuously to the upper pentagon in such a way that the remaining four sides of the upper pentagon

[235]Recall that by the bijection on page 462 we view maps $f\colon S^1 \to \Sigma$ with $f(1) = P$ as elements in $\pi_1(\Sigma, P)$.

get sent homeomorphically to the four sides of the square.[236] [237] This map Φ has the desired property.

(3) The retraction is now given by the concatenation of the maps

$$\Sigma \xrightarrow{\Phi} S^1 \times S^1 \xrightarrow{(z,w)\mapsto z} S^1.$$

(4) This statement is a consequence of (3), Lemma 15.9 and Corollary 14.9. ■

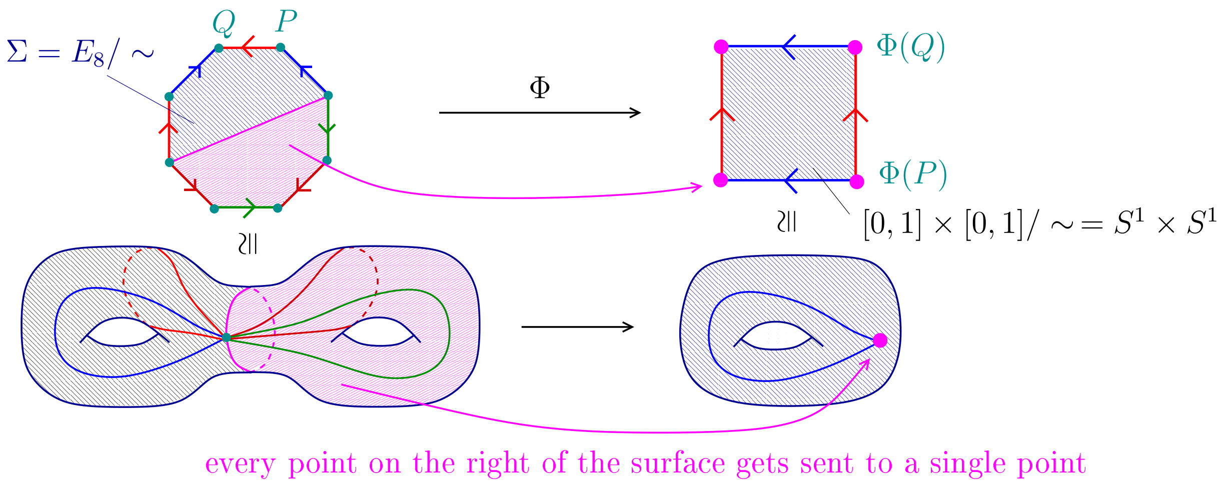


FIGURE 307

In many applications we will want to know whether a given map between topological spaces induces an isomorphism of fundamental groups. The following lemma says that in many settings the property of being an isomorphism is independent of the choice of base point.

Proposition 15.11. *Let $\varphi\colon X \to Y$ be a map between topological spaces and let $x_0 \in X$.*

(1) *If $\gamma\colon [0,1] \to X$ is a path from x_0 to some point x_1, then the following diagram commutes*[238]

$$\begin{tikzcd}
\pi_1(X,x_1) \arrow[r, "\gamma_*", "\cong"'] \arrow[d, "\varphi_*"'] & \pi_1(X,x_0) \arrow[d, "\varphi_*"] \\
\pi_1(Y,\varphi(x_1)) \arrow[r, "(\varphi\circ\gamma)_*", "\cong"'] & \pi_1(Y,\varphi(x_0)).
\end{tikzcd}$$

---

[236] It is clearly slightly painful to write down this map explicitly. It is easier to write down a similar map, which maps a square X onto a triangle Y in such a way that one side of the square gets sent to a vertex of the triangle and such that the remaining three sides of the square get sent to the three sides of the triangle. For example the map

$$\begin{array}{rcl} X := [0,1]\times[0,1] & \to & Y := \{(x,y)\in\mathbb{R}^2 \,|\, x\in[0,1], y\in[0,x]\} \\ (x,y) & \mapsto & (x,xy) \end{array}$$

is of that form.

[237] In the lower part of Figure 307 we illustrate the map for the surfaces viewed as subsets of $\mathbb{R}^3$. There the map Φ is given by crushing the right-hand half of the surface of genus 2 to a single point.

(2) *If* $\varphi_*\colon \pi_1(X,x_0) \to \pi_1(Y,\varphi(x_0))$ *is an isomorphism (epimorphism, monomorphism) for some base point* $x_0 \in X$*, then for any other base point* x_1 *in the same path component of* X *the map* $\varphi_*\colon \pi_1(X,x_1) \to \pi_1(Y,\varphi(x_1))$ *is also an isomorphism (epimorphism, monomorphism).*

Proof.

(1) This statement follows immediately from the definitions.
(2) This statement follows from (1) and the fact that the horizontal maps are isomorphisms. ■

15.4. **Natural transformations.** In this last section we introduce the notion of a natural transformation between two functors. This notion will initially not play a major role, but once we introduce homology groups this notion will become increasingly important to state results precisely. As of right now we do not have enough functors at our hand to perhaps truly appreciate the value of the notion of a natural transformation.

After this long preamble, let us give the definition of natural transformations.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F,G\colon \mathcal{C} \to \mathcal{D}$ be two covariant functors. A *natural transformation between the functors* F *and* G assigns to each $X \in \mathrm{Ob}(\mathcal{C})$ a morphism $\Phi_X\colon F(X) \to G(X)$ in $\mathcal{D}$ such that for each morphism $f\colon X \to Y$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "F_*(f)"] \arrow[d, "\Phi_X"'] & F(Y) \arrow[d, "\Phi_Y"] \\
G(X) \arrow[r, "G_*(f)"] & G(Y).
\end{tikzcd}$$

If for every $X \in \mathrm{Ob}(\mathcal{C})$ the morphism $\Phi_X\colon F(X) \to G(X)$ is an isomorphism, then we refer to the natural transformation as a *natural isomorphism.* Given $X \in \mathrm{Ob}(\mathcal{C})$ we often say that $F(X)$ and $G(X)$ are *naturally* isomorphic.[240] The same way we can of course also define the notion of a natural transformation between two contravariant functors.

Remark. The notion of a natural transformation can also be summarized in the following diagram:

$$\begin{tikzcd}
& F(X) \arrow[rr, "F_*(f)"] \arrow[dd, "\Phi_X"] & & F(Y) \arrow[dd, "\Phi_Y"] \\
X \arrow[ur, squiggly] \arrow[dr, squiggly] \arrow[rr, "f"] & & Y \arrow[ur, squiggly] \arrow[dr, squiggly] & \\
& G(X) \arrow[rr, "G_*(f)"] & & G(Y).
\end{tikzcd}$$

[239] Here the horizontal maps are the isomorphisms that we introduced in Proposition 14.11 (1).

[240] In the literature the words "natural isomorphism" and "canonical isomorphism" are often used indiscriminately, but whereas the expression "natural isomorphism" has a precise meaning, the expression "canonical isomorphism" is not a well-defined mathematical concept.

Here the wiggly arrows are not maps but they are just meant to remind us which objects a functor associates to the objects X and Y.

Now we will see that we have encountered natural transformations and natural isomorphisms much more often than one might have initially thought.

Examples.

(1) Let $\mathcal{CRing}$ be the category of commutative rings and let $\mathcal{Gr}$ be the category of groups. Let $n \in \mathbb{N}$. We consider the functors

$$\begin{array}{rccc} F\colon & \mathcal{CRing} & \to & \mathcal{Gr} \\ & R & \mapsto & \mathrm{GL}(n,R) \end{array} \quad \text{and} \quad \begin{array}{rccc} G\colon & \mathcal{CRing} & \to & \mathcal{Gr} \\ & R & \mapsto & R^* := \text{units in } R. \end{array}$$

For each commutative ring R we consider the group homomorphism

$$\begin{array}{rccc} \Phi_R\colon & \mathrm{GL}(n,R) & \to & R^* \\ & A & \mapsto & \det(A). \end{array}$$

We claim that these group homomorphisms define a natural transformation from F to G. Thus let $f\colon R \to S$ be a homomorphism between commutative rings and let $A \in F(R)$, i.e. $A = (a_{ij})$ is a matrix in $\mathrm{GL}(n,R)$. We have

$$\begin{aligned} \Phi_S(F_*(f)(A)) &= \det((f(a_{ij}))_{i,j=1,\dots,n}) \overset{\substack{\text{Leibniz formula} \\ \downarrow}}{=} \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \cdot \prod_{i=1}^{n} f(a_{i,\sigma(i)}) \\ &\underset{\substack{\uparrow \\ f \text{ is a ring homomorphism}}}{=} f\Big(\sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \cdot \prod_{i=1}^{n} a_{i,\sigma(i)}\Big) \underset{\substack{\uparrow \\ \text{Leibniz formula}}}{=} f(\det(A)) = G_*(f)(\Phi_R(A)). \end{aligned}$$

But that means, by definition, that the Φ_R define a natural transformation from F to G.

(2) Let $\mathcal{Vec}_{\mathbb{R}}$ be the category of real vector spaces and consider the functor F given by[241]

$$\begin{array}{rcl} V & \mapsto & (V^*)^* \\ (f\colon V \to W) & \mapsto & ((f^*)^*\colon (V^*)^* \to (W^*)^*). \end{array}$$

and

For each real vector V we denote by Φ_V the homomorphism

$$\begin{array}{rcl} V & \to & (V^*)^* \\ v & \mapsto & \left(\begin{array}{rcl} V^* & \to & \mathbb{R} \\ (\varphi\colon V \to \mathbb{R}) & \mapsto & \varphi(v) \end{array} \right). \end{array}$$

It is straightforward to verify that these maps define a natural transformation from the identity functor to the functor F.[242]

[241]As usual, given a real vector space V we denote by $V^* = \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$ the dual space. Therefore $(V^*)^*$ is the dual space of the dual space of V. The functor given by $V \mapsto V^*$ is contravariant, but the "double functor" $V \mapsto (V^*)^*$ is again covariant.

[242]Is this in fact a natural isomorphism, i.e. is V always isomorphic to $(V^*)^*$?

(3) Let $\mathcal{Ring}$ be the category of rings. We consider the functors

$$\begin{array}{rcl} F\colon \mathcal{Ring} & \to & \mathcal{Ring} \\ R & \mapsto & \mathbb{Z} \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{Ring} & \to & \mathcal{Ring} \\ R & \mapsto & R. \end{array}$$

For each commutative ring R we consider the ring homomorphism

$$\begin{array}{rcl} \Phi_R\colon \mathbb{Z} & \to & R \\ n & \mapsto & n \cdot 1_R. \end{array}$$

This is a natural transformation from F to G. Thus, given a commutative ring, we refer to this ring homomorphism $\mathbb{Z} \to R$ as the *natural ring homomorphism.*

Perhaps in this instance it is also important to give a "non-example".

Example. Let $\mathcal{C}$ be the category of finitely generated abelian groups. Given an abelian group A we denote by $\operatorname{Tor}(A)$ its torsion subgroup and we denote by $\mathrm{F}A := A/\operatorname{Tor}(A)$ the maximal torsion-free quotient. We consider the following two covariant functors:[243]

$$\begin{array}{rcl} F\colon \mathcal{C} & \mapsto & \mathcal{C} \\ A & \mapsto & A \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{C} & \to & \mathcal{C} \\ A & \mapsto & \operatorname{Tor}(A) \oplus \mathrm{F}A. \end{array}$$

It follows from the classification of finitely generated abelian groups, see Theorem 19.4, that for any finitely generated abelian group A the groups $F(A) = A$ and $G(A) = \operatorname{Tor}(A) \oplus \mathrm{F}A$ are isomorphic. Now we claim that nonetheless there is *no* natural isomorphism between F and G. Indeed, suppose there was such a natural isomorphism Φ. Let us consider the group $A = \mathbb{Z}_2 \oplus \mathbb{Z}$ and the homomorphism

$$\begin{array}{rcl} f\colon A = \mathbb{Z}_2 \oplus \mathbb{Z} & \to & \mathbb{Z}_2 \oplus \mathbb{Z} = A \\ ([a], b) & \mapsto & ([b], 0). \end{array}$$

Since we assume that Φ is a natural isomorphism we obtain the following commutative diagram

$$\begin{tikzcd} F(A) \arrow[r, "\Phi_A", "\cong"'] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(A) \arrow[r, "\Phi_A", "\cong"'] & G(A) \end{tikzcd}$$

which specializes in our setting to the following commutative diagram

$$\begin{tikzcd} \mathbb{Z}_2 \oplus \mathbb{Z} \arrow[r, "\Phi_A", "\cong"'] \arrow[d, "F(f)=f"'] & \mathbb{Z}_2 \oplus \mathbb{Z} \arrow[d, "G(f)=0"] \\ \mathbb{Z}_2 \oplus \mathbb{Z} \arrow[r, "\Phi_A", "\cong"'] & \mathbb{Z}_2 \oplus \mathbb{Z}. \end{tikzcd}$$

Note that the vertical map to the left is non-trivial whereas a close inspection shows that the vertical map to the right is trivial. But this collides with the statement that the horizontal maps are isomorphisms.

On many occasions we, and many authors, say that a certain map is "natural". This means that the map is in fact a natural transformation between two suitable functors between two suitable categories. Sometimes it takes some thought to figure out what the right context is.

---

[243]Let us explain why G is in fact a covariant functor. Let $f\colon A \to B$ be a homomorphism between two abelian groups. Since a homomorphism sends torsion elements to torsion elements we see that f restricts to a homomorphism $\operatorname{Tor}(A) \to \operatorname{Tor}(B)$. This also implies that f descends to a homomorphism $\mathrm{F}\,A = A/\operatorname{Tor}(A) \to \mathrm{F}\,B = B/\operatorname{Tor}(B)$. Summarizing we see that $f\colon A \to B$ induces a homomorphism $G(f)\colon G(A) = \operatorname{Tor}(A) \oplus \mathrm{F}\,A \to G(B) = \operatorname{Tor}(B) \oplus \mathrm{F}\,B$.

Example. On page 292, given a submanifold M of some $\mathbb{R}^n$ and a point $P \in M \setminus \partial M$, we introduced the visual tangent space $\mathrm{V}_P\, M$ which is a vector subspace $\mathrm{V}_P\, M$ of $\mathbb{R}^n$. In Proposition 6.39 (2) we showed that the map

$$\begin{array}{rcl} \Phi_M\colon \mathrm{V}_P\, M & \to & \mathrm{T}_P\, M \\ \gamma'(0) & \mapsto & \left(\begin{array}{rcl} C^\infty(M,\mathbb{R}) & \to & \mathbb{R} \\ f & \mapsto & \frac{d}{dt}\big|_{t=0} f(\gamma(t)) \end{array} \right) \end{array}$$

is an isomorphism of vector spaces. Now we want to say that this is a “natural isomorphism”. What does this mean? Let $\mathcal{P}$ be the category of pointed submanifolds of some $\mathbb{R}^n$. More precisely, let $\mathcal{P}$ be the category with

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{P}) & = & \{(M,P) \mid M \text{ is a submanifold of some } \mathbb{R}^n \text{ and } P \in M \setminus \partial M\} \\ \mathrm{Mor}((M,P),(N,Q)) & = & \{\text{all smooth maps } f\colon M \to N \text{ with } f(P) = Q\}. \end{array}$$

It follows easily from the definitions that the maps Φ_M define a natural transformation from the functor

$$\begin{array}{rcl} \mathcal{P} & \to & \mathcal{V}ec_{\mathbb{R}} \\ (M,P) & \mapsto & \mathrm{V}_P\, M \end{array} \quad \text{to the functor} \quad \begin{array}{rcl} \mathcal{P} & \to & \mathcal{V}ec_{\mathbb{R}} \\ (M,P) & \mapsto & \mathrm{T}_P\, M. \end{array}$$

All of the above examples of natural transformations are perhaps not particularly impressive. But as we will see, the concept of a natural transformation will eventually become one of the key notions in this course.

Exercises for Chapter 15.

Exercise 15.1. Show that $\pi_1(S^1 \times S^1)$ admits an epimorphism onto $\mathbb{Z}^2$.

Exercise 15.2. Show that there exists a homeomorphism of the torus $S^1 \times S^1$ that does not extend to a diffeomorphism of the solid torus $S^1 \times \overline{B}^2$.

Exercise 15.3. Let $a \in A$. We denote by $i\colon A \to X$ the inclusion map.

(a) Show that there exists an epimorphism $\varphi\colon \pi_1(X,a) \to \pi_1(A,a)$ such that $\varphi \circ i_*$ is the identity on $\pi_1(A,a)$.
(b) Show that the inclusion induced map $\pi_1(A,a) \to \pi_1(X,a)$ is a monomorphism.

Exercise 15.4. As above let $\mathcal{T}op$ be the category of topological spaces, let $\mathcal{A}bs\mathcal{G}raph$ be the category of abstract graphs and let $\mathcal{U}ndir\mathcal{A}bs\mathcal{G}raph$ be the category of undirected abstract graphs. In Lemma 15.2 we defined a functor $|-|$ from $\mathcal{A}bs\mathcal{G}raph$ to $\mathcal{T}op$. Furthermore on page 480 we defined a functor F from $\mathcal{A}bs\mathcal{G}raph$ to $\mathcal{U}ndir\mathcal{A}bs\mathcal{G}raph$ such that for any abstract graph the homeomorphism $\Theta_G\colon |G| \to |F(G)|$ between the topological realizations is a homeomorphism.

Is it possible to define a functor Ψ from $\mathcal{U}ndir\mathcal{A}bs\mathcal{G}raph$ to $\mathcal{T}op$ such that the homeomorphisms Θ_G are natural?
Remark. Beware of loops.

Exercise 15.5. Let $\mathcal{C}$ be a category.

(1) A morphism $g \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ is called a *monomorphism* if for every two morphisms $f_1, f_2 \in \mathrm{Mor}_{\mathcal{C}}(W,X)$ with $g \circ f_1 = g \circ f_2$ we also have $f_1 = f_2$.

(2) A morphism $g \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ is called an *epimorphism* if for every two morphisms $h_1, h_2 \in \mathrm{Mor}_{\mathcal{C}}(Y,Z)$ with $h_1 \circ g = h_2 \circ g$ we also have $h_1 = h_2$.

We consider these notions in two different categories.

(a) Show that in the category $\mathcal{AbGr}$ of abelian groups the monomorphisms are precisely the injective homomorphisms and that the epimorphisms are precisely the surjective homomorphisms.
Remark. The same statements also hold in the category $\mathcal{Gr}$ of all groups. The proof for monomorphisms is straightforward, but the proof for epimorphisms is non-trivial, we refer to [**Lin70**] and [**Pare70**, p. 16] for details.

(b) Show that in the category $\mathcal{Top}$ of topological spaces a monomorphism is not necessarily injective and show that an epimorphism is not necessarily surjective.

(c) Show that in the category $\mathcal{Ring}$ of rings the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism.

(d) An abelian group G is called *divisible* if for every $g \in G$ and $n \in \mathbb{N}$ there exists an $h \in G$ with $n \cdot h = g$. We define the category of divisible abelian groups in the obvious way. Show that in this category the obvious homomorphism $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism.

Exercise 15.6. Let $\mathcal{C}$ be a category. An object $X \in \mathrm{Ob}(\mathcal{C})$ is called *initial* if for each object $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism from X to Y. Similarly an object $X \in \mathrm{Ob}(\mathcal{C})$ is called *terminal* if for each object $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism from Y to X.

(a) Show that if an initial object exists, then it is unique up to isomorphism. The same statement of course holds for terminal objects.

(b) Show that the category $\mathcal{Top}$ of topological spaces admits an initial and a terminal object.

(c) Give an example of a category that contains an initial object but not a terminal object.

(d) Give an example of a category that contains a terminal object but that does not contain an initial object.

16. Fundamental groups and coverings

We have now introduced the fundamental group of a (pointed) topological space and we have proved several basic facts. But the only proof that fundamental groups are non-trivial relied on a fact from complex analysis. In particular, so far we have not yet managed to compute a single fundamental group that is not the trivial group. For example, we know that $\pi_1(S^1)$ is non-trivial, but we do not know what group it is.

In this chapter we will compute many non-trivial fundamental groups using the theory of covering spaces.

16.1. Coverings of topological spaces.

Now we return to the study of topological spaces.

Definition. Let $p\colon X \to B$ be a map between topological spaces.

1. We say an open subset $U \subset B$ is *uniformly covered*, if $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ with the property, that the restriction of p to each subset V_i is a homeomorphism.
2. We say the map $p\colon X \to B$ is a[244] *covering*, if it is surjective and if for every $b \in B$ there exists an open neighborhood U of b which is uniformly covered.

Remark. If U is a uniformly covered subset of B, then clearly any open subset of U is also uniformly covered. For example it follows that if $p\colon X \to B$ is a covering and if B is a smooth manifold, then any point $b \in B$ admits also a *connected* uniformly covered neighborhood.[245]

Examples.

(a) The map

$$\begin{array}{rcl} p\colon \mathbb{R} & \to & S^1 \\ \varphi & \mapsto & e^{\mathrm{i}\varphi} \end{array}$$

is a covering. Indeed, let $P = e^{\mathrm{i}\alpha}$ be a point in S^1. We pick the open neighborhood

$$U \quad := \quad \big\{ e^{\mathrm{i}\varphi} \,\big|\, \varphi \in \big(\alpha - \tfrac{\pi}{4}, \alpha + \tfrac{\pi}{4}\big)\big\}.$$

Then

$$p^{-1}(U) \quad := \quad \bigsqcup_{j\in\mathbb{Z}} \underbrace{\big(\alpha - \tfrac{\pi}{4} + 2\pi j, \alpha + \tfrac{\pi}{4} + 2\pi j\big)}_{=:V_j},$$

and for each $j \in \mathbb{Z}$ the restriction of p to $V_j \to U$ is a homeomorphism. This example is illustrated in Figure 308.

[244]On page 89 we introduced the notion of a “cover”. As we mentioned on page 89, such a “cover” is sometimes also called a “covering”. Now we introduce the notion of a “covering”. Rather maddeningly, in the literature this notion often also gets called a “cover”. Fortunately, from the context, it is usually clear what is meant by “cover” and “covering”. These issues do not arise in other languages, e.g. in German and French what we call “cover” is referred to as “Überdeckung” and “recouvrement” and what we call “covering” is referred to as “Überlagerung” and “revêtement”.

[245]Why does this follow for smooth manifolds? Why does this not hold for all topological spaces B?

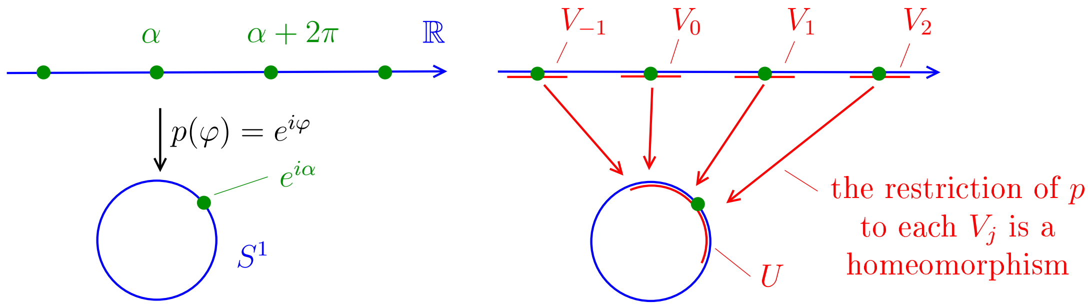


FIGURE 308

(b) We consider the "infinite helix"

$$H \quad := \quad \left\{ (re^{\mathrm{i}t}, t) \,\middle|\, r \in (1,2) \text{ and } t \in \mathbb{R} \right\}$$

together with the "annulus"

$$A \quad := \quad \left\{ re^{\mathrm{i}t} \,\middle|\, r \in (1,2) \text{ and } t \in \mathbb{R} \right\}.$$

The map

$$\begin{array}{rcl} p\colon H & \to & A \\ (re^{\mathrm{i}t}, t) & \mapsto & re^{\mathrm{i}t} \end{array}$$

is a covering. This statement is illustrated in Figure 309. The proof of the statement is similar to the proof in the previous example.

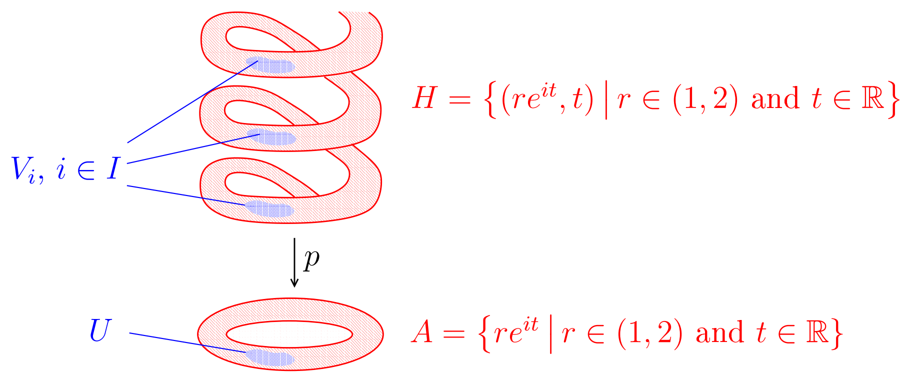


FIGURE 309

(c) The map

$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ x & \mapsto & [x] = x + \mathbb{Z}^n \end{array}$$

is a covering. Indeed, let $[x] = [(x_1, \ldots, x_n)]$ be a point in $\mathbb{R}^n/\mathbb{Z}^n$. We consider

$$W \;:=\; \left\{ (y_1, \ldots, y_n) \,\middle|\, |y_i - x_i| < \tfrac{1}{2} \right\} \subset \mathbb{R}^n$$

and we put $U := p(W)$. Then U is an open[246] neighborhood of $[x]$. Furthermore

$$p^{-1}(U) \;=\; \bigcup_{i \in \mathbb{Z}^n} (W + i),$$

[246] The set U is open since the projection map $p\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is open by Lemma 3.30 (2).

and for each i the restriction of p to $W+i\subset\mathbb{R}^n$ is a homeomorphism.[247]

(d) For every $n\in\mathbb{N}$ the map
$$\begin{array}{rcl} p\colon S^n & \to & \mathbb{RP}^n = S^n/x\sim -x \\ x & \mapsto & [x]=\{x,-x\} \end{array}$$
is a covering. Indeed, for $[x]\in\mathbb{RP}^n$ we set
$$W \;=\; \{y\in S^n \mid \underbrace{x\cdot y}_{\text{scalar product}} >0\}.$$
Then $p(W)$ is an open neighborhood of $[x]$ and it is uniformly covered by the two open sets W and $-W$.

(e) The projection map
$$\begin{array}{rcl} p\colon S^1 & \to & [-1,1] \\ (x,y) & \mapsto & x \end{array}$$
is not a covering. Indeed, the point $b=1\in[-1,1]$ has no open neighborhood that is uniformly covered. This example is illustrated in Figure 310.

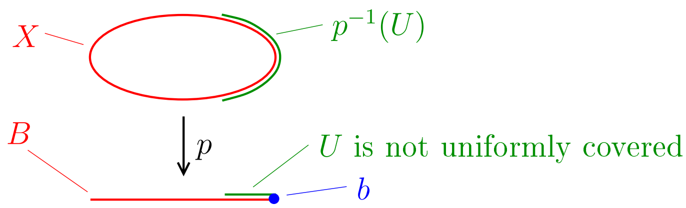


FIGURE 310

Now we can formulate the following lemma that will be proved in Exercise 16.1.

Lemma 16.1. *Let $p\colon X\to B$ be a covering of topological spaces. If B is a path-connected topological space, then for any two points P and Q in B we have*
$$\#p^{-1}(P) \;=\; \#p^{-1}(Q).$$

Example. For a general map $p\colon X\to B$ between topological spaces the number of preimages of a point depends of course on the point chosen in B. For instance in the above example (e) we have $\#p^{-1}(0)=2$ but $\#p^{-1}(1)=1$.

Definition. Let $p\colon X\to B$ be a covering of a path-connected topological space.

(1) We define the *degree* $[X:B]$ *of* p as the cardinality of $\#p^{-1}(b)$ for some $b\in B$.[248]
(2) If the degree is a finite number n, then we say that p is an *n-fold covering* of B. Otherwise we say that p is an *infinite covering*. If the cardinality is countable, then sometimes we call it a *countable covering*.

Examples.

[247]It is straightforward to verify that the map $p\colon V_i:=W+i\to U$ is a bijection. It follows from Lemma 3.30 (1) that the projection map p is continuous and as we had just mentioned in the previous footnote, the projection map is also open. It follows that the map $p\colon V_i=W+i\to U$ is a homeomorphism.

[248]It follows from Lemma 16.1 that this definition does not depend on the choice of b.

(1) We had just seen that the map $p\colon S^n \to S^n/\sim$ is a 2-fold covering of the real projective space $S^n/\sim = \mathbb{RP}^n$.

(2) Let $n \in \mathbb{N}$. As usual we consider $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ as a subset of $\mathbb{C}$. Then the map $p\colon S^1 \to S^1$ given by $p(z) := z^n$ is a covering. Indeed, let $z = e^{2\pi i\varphi}$ be a point in S^1. Then

$$U \quad := \quad \left\{e^{2\pi i\psi} \,\middle|\, \psi \in \left(\varphi - \tfrac{1}{4}, \varphi + \tfrac{1}{4}\right)\right\}$$

has the desired property. Indeed, we have

$$p^{-1}(U) \quad = \quad \bigsqcup_{j=1}^{n} \underbrace{\left\{e^{2\pi i\psi} \,\middle|\, \psi \in \left(\tfrac{1}{n}(\varphi - \tfrac{1}{4}) + \tfrac{j}{n}, \tfrac{1}{n}(\varphi + \tfrac{1}{4}) + \tfrac{j}{n}\right)\right\}}_{=: V_j},$$

and the restriction of p to each set V_j is easily seen to be a homeomorphism. This shows that p is a covering and it is clear that p is an n-fold covering.

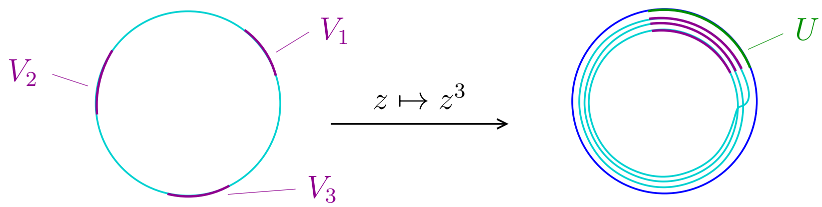


FIGURE 311

(3) Let $n \in \mathbb{N}$. Noe that basically the same argument as before shows that the map $p\colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ given by $p(z) = z^n$ is an n-fold covering.

(4) We consider the map

$$\begin{array}{rcl} p\colon \overbrace{\mathbb{R} \times [-1,1]/(x,y) \sim (x+1,y)}^{\text{annulus}} & \to & \overbrace{\mathbb{R} \times [-1,1]/(x,y) \sim (x+\tfrac{1}{2},-y)}^{\text{Möbius band}} \\ [(x,y)] & \mapsto & [(x,y)]. \end{array}$$

It is straightforward to show that p is a 2-fold covering. Thus we can view the annulus as a 2-fold covering of the Möbius band. This covering is illustrated in Figure 312.

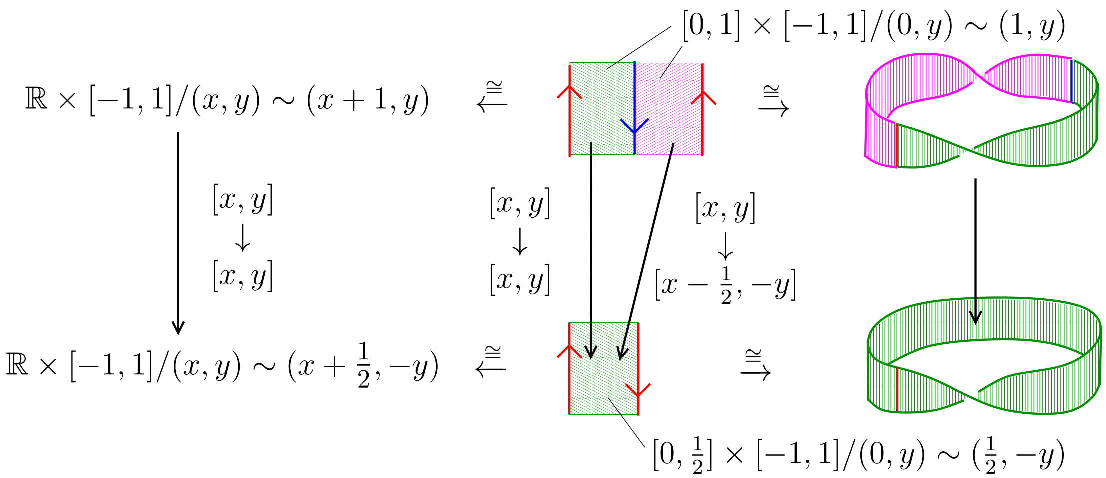


FIGURE 312

(5) If in the previous example we replace the interval $[-1,1]$ by the circle $[-1,1]/-1 \sim 1$, then on the left-hand side we obtain the torus and on the right-hand side we obtain the Klein bottle. This shows that there exists a degree 2 covering map from the torus to the Klein bottle.

(6) In Figure 313 we sketch three coverings of topological graphs. The coverings are hereby given by sending edges of a certain color, with the indicated direction, to the edge of the same color with the indicated direction.[249]

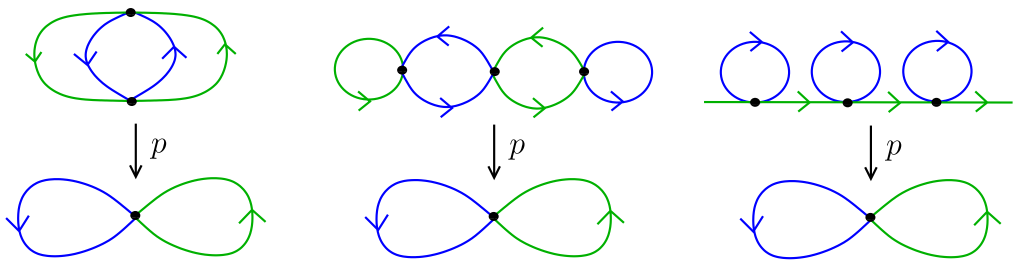


FIGURE 313

Question 16.2. *We had just seen that there exists a covering map $S^2 \to \mathbb{R}P^2$ of degree two and that there exists a covering map from the torus to the Klein bottle of degree two. This raises the question: between which surfaces do there exist covering maps? More precisely we can ask the following:*

(1) *Does there exist a covering map from the torus $S^1 \times S^1$ to the sphere S^2?*
(2) *Does there exist a covering map from the sphere S^2 to the torus $S^1 \times S^1$?*

The following lemma summarizes some basic facts about coverings.

Lemma 16.3. *Let $p\colon X \to B$ be a covering of topological spaces. Then the following hold:*

(1) *The map $p\colon X \to B$ is open.*[250]
(2) *For any $b \in B$ the preimage $p^{-1}(b)$ is a discrete subset of X.*
(3) *If B is Hausdorff, then X is also Hausdorff.*
(4) *If p is a finite covering and if B is compact, then X is also compact.*
(5) *If p is an infinite covering, then X is non-compact.*

PROOF (∗). Let $p\colon X \to B$ be a covering of topological spaces.

(1) Let U be an open subset of X. We need to show that $p(U)$ is open. It suffices to show that for each $P \in p(U)$ there exists an open neighborhood V such that $P \in V \subset U$. Since p is a covering there exists an open neighborhood W of P such that $p^{-1}(W)$ is the union of disjoint open subsets $\{W_i\}_{i\in I}$ with the property that the restriction of p to each subset W_i is a homeomorphism.

Since $P \in p(U)$ there exists a $Q \in U$ with $p(Q) = U$. There exists an $i \in I$ with $Q \in W_i$. Since W_i is open and since U is open the set $U \cap W_i$ is also open in X. Since

[249]It is a good exercise to convince oneself, that the indicated maps are in fact coverings.

[250]Recall that this means that the image of each open set in X is open in B.

$p\colon W_i \to W$ is a homeomorphism and since $U \cap W_i$ is open in W_i the set $p(U \cap W_i)$ is also open in $p(W_i) = W$. But since W is open the set $V := p(U \cap W_i)$ is also an open subset of B. It is the desired open neighborhood of P with $P \in V \subset U$.

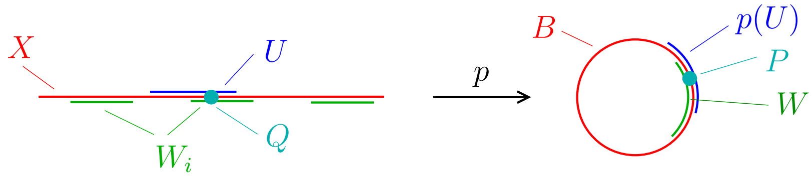


FIGURE 314. Illustration for the proof of Lemma 16.3 (1).

(2) Let $b \in B$. Since $p\colon X \to B$ is a covering there exists an open neighborhood U of b such that $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ with the property that the restriction of p to each subset V_i is a homeomorphism. For each $i \in I$ there exists a unique point $x_i \in V_i$ with $p(x_i) = b$ and the preimage $p^{-1}(b)$ consists of precisely the x_i, $i \in I$. Since each V_i is open in X it follows from the definition of the subspace topology that each $x_i \in p^{-1}(b)$ is an open subset of $p^{-1}(b)$. Put differently, the subspace topology on $p^{-1}(b)$ is the discrete topology. But this is equivalent to saying that $p^{-1}(b)$ is a discrete subset of X.

(3) This statement is relatively easy to prove and it in Exercise 16.5.

(4) Let $p\colon X \to B$ be an n-fold covering of a compact space B. Let $\{U_i\}_{i\in I}$ be an open cover of X. Let $b \in B$. We pick an open connected neighborhood M of b that is uniformly covered. For each $x \in p^{-1}(b)$ we denote by N_x the component of $p^{-1}(M)$ that contains x. For each point in $x \in p^{-1}(b)$ we pick an $i_x \in I$ with $x \in U_{i_x}$. We write

$$W_b \;:=\; \bigcap_{x\in p^{-1}(b)} p(U_{i_x} \cap N_x).$$

Claim. The preimage $p^{-1}(W_b)$ is contained in the union of $n = \#p^{-1}(b)$ subsets of our open cover $\{U_i\}_{i\in I}$.

Since M is uniformly covered the map $p\colon N_x \to M$ is a homeomorphism for any $x \in p^{-1}(y)$. We denote by $p_x\colon N_x \to M$ this homeomorphism. We write $F := p^{-1}(b)$. We then have

$$p^{-1}(W_b) = \bigcup_{y\in F} p_y^{-1}\Big(\bigcap_{x\in F} p_x(U_{i_x}\cap N_x)\Big) = \bigcup_{y\in F}\bigcap_{x\in F} \overbrace{\underbrace{p_y^{-1}(p_x(U_{i_x}))}_{=\,U_{i_y}\text{ if } y=x} \cap \underbrace{p_y^{-1}(p_x(N_x))}_{=N_y}}^{\subset U_{i_y}\cap N_y} \subset \bigcup_{y\in F} U_{i_y}.$$

⊞

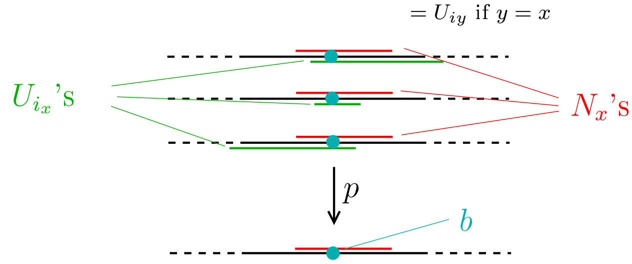


FIGURE 315. Illustration for the proof of Lemma 16.3 (4).

By (1) each $p(U_{i_x} \cap N_x)$ is an open subset of B and since $p^{-1}(b)$ is a finite set we see that W_b is an open subset of B. Furthermore b is contained in each $p(U_{i_x} \cap N_x)$, so W_b is in fact an open neighborhood of $b \in B$. Since B is compact we can cover B by finitely many of these open sets, i.e. there exist $b_1, \dots, b_k \in B$ with $B = W_{b_1} \cup \dots \cup W_{b_k}$. The preimages $p^{-1}(W_{b_1}), \dots, p^{-1}(W_{b_k})$ cover X. By the above claim each preimage $p^{-1}(W_{b_i})$ is contained in the union of $n = \#p^{-1}(b_i)$ subsets of our open cover $\{U_i\}_{i\in I}$. Summarizing we have shown that we can cover X by $n \cdot k$ sets out of the collection $\{U_i\}_{i\in I}$.

(5) Let $p\colon X \to B$ be an infinite covering. If B is non-compact, then it follows from Lemma 2.40 that X is also non-compact. So now suppose that B is compact. By definition we can cover B by open sets that are uniformly covered. Since B is compact we can cover B by finitely many open sets $U_1, \dots, U_k$ that are uniformly covered. Without loss of generality we can assume that U_k is not contained in $U_1 \cup \dots \cup U_{k-1}$.[251] Now let P be a point in $U_k \setminus (U_1 \cup \dots \cup U_{k-1})$. Furthermore let V_i, $i \in I$ be the disjoint open subsets of X such that $p^{-1}(U_k)$ is the union of the V_i's and such that the restriction of p to each V_i is a homeomorphism. For $i \in I$ we denote by $Q_i \in V_i$ the unique point with $p(Q_i) = P$. Now we consider the open cover

$$X \;=\; p^{-1}(U_1) \,\cup\, \dots \,\cup\, p^{-1}(U_{k-1}) \,\cup\, \bigcup_{i\in I} V_i.$$

Note that for $i \in I$ the point Q_i is contained only in V_i. This means that we cannot remove any of the open sets V_i, $i \in I$ from the open cover. Since I is infinite it follows that X does not satisfy the definition of a compact set, i.e. X is not compact. ■

We conclude this section with the following lemma that we will prove in Exercise 16.4.

Lemma 16.4. *Let $p\colon \widetilde{X} \to X$ be a 2-fold covering. We define*

$$\begin{array}{rcl} f\colon \widetilde{X} & \to & \widetilde{X} \\ Q & \mapsto & \text{the unique other point in } p^{-1}(p(Q)). \end{array}$$

(The definition of f is illustrated in Figure 316). This map is continuous and it has the property that the following diagram commutes:

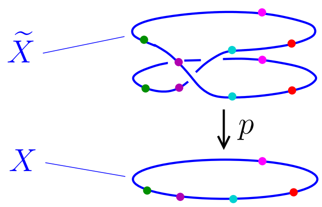
$$\begin{tikzcd} \widetilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \widetilde{X} \arrow[dl, "p"] \\ & X. & \end{tikzcd}$$

16.2. **Group actions and covering spaces.** We recall two definitions about actions from pages 185 and 189 and we introduce one new definition.

Definition. Let G be a group which acts continuously on a topological space X.

(1) We say G acts *freely* if $g \cdot x = x$ for $g \in G$ and $x \in X$ implies that $g = e$.

[251]Indeed, if U_k was contained in $U_1 \cup \dots \cup U_{k-1}$, then we could remove U_k from the collection of open sets and we continue with the collection $U_1, \dots, U_{k-1}$.



$f\colon \widetilde{X} \to \widetilde{X}$ swaps the points of the same color

FIGURE 316. Illustration of Lemma 16.4.

(2) We say G acts *properly* if for every two points x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \,|\, gU \cap V \neq \varnothing\}$ is finite.
(3) We say G *acts discretely*, if for each $x \in X$ there exists an open neighborhood U so that
$$U \cap gU \;=\; \varnothing \text{ for all } g \neq e.$$

The following lemma summarizes the relationship between these different definitions.

Lemma 16.5. *Let G be a group which acts continuously on a topological space X.*

(1) *If G is finite, then the action is proper.*
(2) *If G acts discretely, then it also acts freely.*
(3) *If X is a Hausdorff space and if the action by G is free and proper, then the action is also discrete.*

PROOF. The first two statements follow immediately from the definitions. The third statement is the content of Lemma 6.33. ■

Examples.

(A) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is discrete. Indeed, for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the open neighborhood
$$U \;=\; \big\{y = (y_1, \ldots, y_n) \in \mathbb{R}^n \,\big|\, \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\} < \tfrac{1}{2}\big\}$$
has the desired property.

(B) Let $X = \mathbb{R} \times [-1, 1]$ and $G = \mathbb{Z}$. The action
$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n, (x, y)) & \mapsto & (x + n, (-1)^n y) \end{array}$$
is discrete. Indeed, for any $(x, y) \in X$ the open neighborhood
$$U \;=\; \big\{(a, b) \,\big|\, |a - x| < \tfrac{1}{2} \text{ and } b \in [-1, 1]\big\}$$
has the desired property.

(C) Let $X = S^n$ and $G = \{\pm 1\}$. The map
$$\begin{array}{rcl} \{\pm 1\} \times S^n & \to & S^n \\ (\epsilon, P) & \mapsto & \epsilon \cdot P \end{array}$$
defines an action that is discrete. Indeed, for any $P \in S^n$ the open hemisphere
$$U \;=\; \{Q \in S^n \,|\, \underbrace{P \cdot Q}_{\text{scalar product}} > 0\}$$

has the desired property.[252]

(D) Every discrete action is free, but as we will see in this example, the converse does not hold. Let $X = \mathbb{R}^{n+1} \setminus \{0\}$ and $G = \mathbb{R} \setminus \{0\}$. The map

$$\begin{array}{rcl} (\mathbb{R} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\}) & \to & \mathbb{R}^{n+1} \setminus \{0\} \\ (r, P) & \mapsto & r \cdot P \end{array}$$

defines an action. It is straightforward to see that the action is free. But the action is not discrete. Indeed, it is easy to see that for each open non-empty set U there exist uncountably many real numbers $r \neq 1$ with $U \cap rU \neq \emptyset$.

(E) We consider the following two self-homeomorphisms of $\mathbb{R}^2$:

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x, y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x, y) & \mapsto & (x, y+1). \end{array}$$

We denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. This means[253]

$$G = \begin{array}{c} \text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written} \\ \text{as a finite concatenation of the maps } A, B, A^{-1} \text{ and } B^{-1}. \end{array}$$

The group G acts by definition on $\mathbb{R}^2$. In Exercise 16.7 we will see that this action of G on $\mathbb{R}^2$ is discrete.

We consider again the square $X = [0,1] \times [0,1]$ with the equivalence relation that is generated by $(0, y) \sim (1, 1-y)$ and $(x, 0) \sim (x, 1)$. It is straightforward to see that the map

$$\begin{array}{rcl} X/\sim & \to & \mathbb{R}^2/G \\ [(x, y)] & \mapsto & [(x, y)] \end{array}$$

is a homeomorphism. Put differently, $\mathbb{R}^2/G$ is homeomorphic to the Klein bottle.

(F) Let

$$X = S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

and let $G = \mathbb{Z}_p$ for some $p \in \mathbb{Z} \setminus \{0\}$. Furthermore let $q \in \mathbb{Z}$ with $\gcd(p, q) = 1$. Then the map

$$\begin{array}{rcl} \mathbb{Z}_p \times S^3 & \to & S^3 \\ (k + p\mathbb{Z}, (z_1, z_2)) & \mapsto & (e^{2\pi \mathrm{i} k/p} z_1, e^{2\pi \mathrm{i} kq/p} z_2) \end{array}$$

defines a smooth, orientation-preserving and discrete action of $\mathbb{Z}_p$ on S^3.[254] We denote by $L(p, q) = S^3/\mathbb{Z}_p$ the quotient space. It follows from Propositions 6.32 and that $L(p, q)$ is again a closed 3-dimensional smooth manifold. Furthermore it follows from Proposition 6.47 that $L(p, q)$ is orientable, in fact it has an orientation such that the projection map $p\colon S^3 \to L(p, q)$ is orientation-preserving. We will always

[252] Alternatively one can argue, that the action is obviously free and S^n is Hausdorff, hence the action is discrete by Lemma 16.5 (1) and (3).

[253] For example the map $B \circ A \circ A \circ B^{-1} \circ A$ lies in G.

[254] It is straightforward to verify that the map does indeed define an action. It is clear that the action is smooth and a direct calculation shows that the action is orientation-preserving. One can also easily verify that the action is free. It follows from Lemma 16.5 (1) and (3) that the action is discrete.

view $L(p,q)$ as a smooth manifold equipped with this orientation. We refer to the smooth manifolds $L(p,q)$ as *lens spaces*.[255] We consider two special cases:

(a) For $p=1$ we only have the action of the trivial group on S^3 and the quotient space $L(1,q)$ is therefore just the original sphere S^3.
(b) For $p=2$ we have q is odd, so for $k=0$ we have $e^{2\pi i k/2}=e^{2\pi i kq/2}=1$ and for $k=1$ we have $e^{2\pi i k/2}=e^{2\pi i kq/2}=-1$. Put differently, the action of $\mathbb{Z}_2$ on S^3 is the same as the action of $\mathbb{Z}_2=\{\pm 1\}$ that we had used in the definition of the real projective space $\mathbb{RP}^n$. This shows that $L(2,q)=\mathbb{RP}^n$.

We discuss the last type of examples in slightly more detail. For $p\in\mathbb{N}$ and $q\in\mathbb{N}$ with $\gcd(p,q)=1$ we just introduced a new topological space, namely the lens space $L(p,q)$. It is natural to ask, for which parameters (p,q) are these topological spaces homeomorphic?

Lemma 16.6. *Let $p\in\mathbb{N}$ and furthermore let $q,r\in\mathbb{Z}$ with $\gcd(p,q)=\gcd(p,r)=1$. If $q\equiv \pm r^{\pm 1} \bmod p$,[256] then $L(p,q)$ and $L(p,r)$ are diffeomorphic.*[257]

Proof. Let $p\in\mathbb{N}$ and let $q,r\in\mathbb{Z}$ with $\gcd(p,q)=\gcd(p,r)=1$. We suppose that $q\equiv \pm r^{\pm 1} \bmod p$. We define a map

$$L(p,q) \to L(p,r)$$

as follows:

$$\begin{array}{llrcl}
\text{Case 1: } & q\equiv r \bmod p. & [(z_1,z_2)] & \mapsto & [(z_1,z_2)] \\
\text{Case 2: } & q\equiv -r \bmod p. & [(z_1,z_2)] & \mapsto & [(z_1,\overline{z_2})] \\
\text{Case 3: } & q\equiv r^{-1} \bmod p. & [(z_1,z_2)] & \mapsto & [(z_2,z_1)] \\
\text{Case 4: } & q\equiv -r^{-1} \bmod p. & [(z_1,z_2)] & \mapsto & [(z_2,\overline{z_1})]
\end{array}$$

In all four cases it is straightforward to verify that the given map is well-defined and that it is in fact a diffeomorphism. ■

In the proof of Lemma 16.6 it was quite straightforward to write down the homeomorphism. This raises the question whether there are homeomorphisms between more lens spaces that are harder to find, or whether all other pairs of lens spaces are non-homeomorphic. More precisely, we have the following question.

Question 16.7. *Let $p,\widetilde{p}\in\mathbb{N}$ and let $q,\widetilde{q}\in\mathbb{Z}$ with $\gcd(p,q)=\gcd(\widetilde{p},\widetilde{q})=1$. If $L(p,q)$ and $L(\widetilde{p},\widetilde{q})$ are homeomorphic, does it follow that $p=\widetilde{p}$ and $q\equiv\pm\widetilde{q}^{\pm 1} \bmod p$?*

This question will keep us busy for a very long time.

We had just seen on page 500 that the group $\{\pm 1\}$ acts freely and continuously on any sphere. We had also seen that any finite cyclic group $\mathbb{Z}_p$ acts freely and continuously on the 3-sphere. This raises the question, whether such an action by $\mathbb{Z}_p$ also exists on the 2-sphere. More generally we can ask the following question.

Question 16.8. *Given $n\in\mathbb{N}$, which finite cyclic groups can act freely and continuously on the n-dimensional sphere S^n?*[258]

[255] The name "lens space" comes from an alternative description of these smooth manifolds.

[256] By our hypothesis we have $\gcd(p,r)=1$. This implies that there exists a multiplicative inverse of r modulo p, i.e. there exists an s with $r\cdot s \bmod p$. We write $r^{-1}:=s$.

[257] In particular any lens space is of the form $L(p,q)$ with $q\in\{0,\dots,p-1\}$.

We will give a complete answer to this question in e introduced homology groups, see Exercise 45.21 and Proposition 55.10.

The following proposition connects the notion of discrete actions to the theory of covering spaces.

Proposition 16.9. *Let X be a topological space together with a discrete and continuous action by a group G. Then the canonical projection $p\colon X \to X/G$ is a covering.*

Example. This proposition, combined with examples (A) and (C), gives in particular new proofs that the maps $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ and $S^n \to S^n/\sim = \mathbb{R}\mathrm{P}^n$ are covering maps.

Proof. Let G be a group which acts discretely and continuously on a topological space X. We want to show that the projection map $p\colon X \to X/G$ is a covering. Let $[x] \in X/G$. Since the group acts discretely there exists an open neighborhood V of $x \in X$ such that

$$V \cap gV \;=\; \varnothing \text{ for all } g \neq e.$$

We put $U := p(V)$. It follows from Lemma 3.30 that the set U is an open neighborhood of $[x] = p(x)$. The preimage $p^{-1}(U)$ is the disjoint union of the open subsets[259] $gV, g \in G$. The restriction of the map p to any gV is a continuous, bijective[260] map $gV \to U$ and according to Lemma 3.30 this map is furthermore open. It follows that the restriction of p to any gV is in fact a homeomorphism. Thus we see that U is uniformly covered. Since this holds for any point in X/G we have shown that the projection map $p\colon X \to X/G$ is a covering. ■

The following proposition is a slight generalization of the previous proposition.

Proposition 16.10. *Let X be a topological space together with a discrete and continuous action by a group G. Furthermore let H be a subgroup of G. Then the canonical projection*

$$\begin{aligned} p\colon X/H &\to X/G \\ [x] &\mapsto [x] \end{aligned}$$

is a covering.

For H the trivial group we recover the statement of Proposition 16.9. In fact the proof of Proposition 16.10 is almost identical to the proof of Proposition 16.9. We leave the details to the reader.

Examples.

(1) Let $X = \mathbb{R} \times [-1,1]$ and $G = \mathbb{Z}$. Once again we consider the discrete action

$$\begin{aligned} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) &\to \mathbb{R} \times [-1,1] \\ (n,(x,y)) &\mapsto \big(x + \tfrac{1}{2}n, (-1)^n y\big). \end{aligned}$$

[258] Any cyclic group $\mathbb{Z}_k$, $k \neq 0$ admits a non-trivial action on S^n where $j \in \mathbb{Z}_k$ acts by rotation around the x-axis by the angle $2\pi\frac{j}{k}$. But this action is not free, since the points $(-1,0,\dots,0)$ and $(1,0,\dots,0)$ are fixed.

[259] Why is $p^{-1}(U) = \bigcup\limits_{g\in G} gV$ and why are the sets gV disjoint?

[260] Why is the map bijective?

Let $H = 2\mathbb{Z}$. According to Proposition 16.10 the projection map $p\colon X/H \to X/G$ is a covering. Put differently, all the horizontal maps in the following commutative diagram are coverings:

$$\begin{array}{ccc}
(\mathbb{R}\times[-1,1])/2\mathbb{Z} & \longrightarrow & (\mathbb{R}\times[-1,1])/\mathbb{Z} \\
\downarrow{=} & & \downarrow{=} \\
(\mathbb{R}\times[-1,1])/(x,y)\sim(x+\tfrac{1}{2}2k,(-1)^{2k}y) & \longrightarrow & (\mathbb{R}\times[-1,1])/(x,y)\sim(x+\tfrac{1}{2}k,(-1)^{k}y) \\
\downarrow{=} & & \downarrow{=} \\
\underbrace{(\mathbb{R}\times[-1,1])/(x,y)\sim(x+k,y)}_{=\text{the annulus}} & \longrightarrow & \underbrace{(\mathbb{R}\times[-1,1])/(x,y)\sim(x+\tfrac{1}{2}k,(-1)^{k}y).}_{=\text{Möbius band}}
\end{array}$$

Each point on X/G has precisely two preimages, so the covering is in fact a 2-fold covering. Thus we recover the result from page 496 that the annulus is a 2-fold covering of the Möbius band.

(2) We consider again the group G of self-homeomorphisms of $\mathbb{R}^2$ that is generated by

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1,1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x,y+1). \end{array}$$

We let H be the subgroup of G generated by A^2 and B. Note that A^2 is the map given by

$$\begin{array}{rcccl} A^2\colon \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1,1-y) & \mapsto & \underbrace{((x+1)+1,(1-(1-y)).}_{=(x+2,y)} \end{array}$$

Note that A^2 and B are two homeomorphisms that commute, so any element in H is in fact of the form

$$\begin{array}{rcl} A^{2k}\circ B^l\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+2k,y+l) \end{array}$$

for some $k,l\in\mathbb{Z}$. By Proposition 16.10 the projection map $p\colon \mathbb{R}^2/H \to \mathbb{R}^2/G$ is a covering map. We already know that $\mathbb{R}^2/G$ is the Klein bottle. It follows immediately from the above description of H, $\mathbb{R}^2/H = \mathbb{R}^2/(2\mathbb{Z}\oplus\mathbb{Z})$, so $\mathbb{R}^2/H$ is just the 2-dimensional torus. As on page 497 we see that we can view the 2-dimensional torus as a 2-fold covering of the Klein bottle.

16.3. **Lifting of paths.**

Definition. Let $p\colon X\to B$ be a map between topological spaces and let $f\colon [a,b]\to B$ be a path. Let $x\in X$ be a point with $p(x)=f(a)$. A path $g\colon [a,b]\to X$ is called a *lift of f to the starting point x*, if $g(a)=x$ and if $p\circ g=f$, i.e. if the following diagram of maps commutes:

$$\begin{tikzcd} & X \arrow[d, "p"] \\ {[a,b]} \arrow[ur, "g"] \arrow[r, "f"'] & B. \end{tikzcd}$$

Examples.

(1) We consider the map

$$\begin{array}{rcl} p\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & e^{\mathrm{i}t} \end{array}$$

and the path

$$\begin{array}{rcl} f\colon [0,b] & \to & S^1 \\ t & \mapsto & e^{2\mathrm{i}t} \end{array}$$

with starting point $f(0) = 1$. Let $x = 4\pi$. Then $p(x) = p(4\pi) = 1$ and

$$\begin{array}{rcl} g\colon [0,b] & \to & \mathbb{R} \\ t & \mapsto & 4\pi + 2t \end{array}$$

is a lift of f to the starting point 4π. Note that if $b = \pi$, then f is a loop in $(S^1, 1)$, but the lift is not a loop in $(\mathbb{R}, 4\pi)$. Put differently, the lift of a loop is not necessarily again a loop. This example is illustrated in Figure 317.

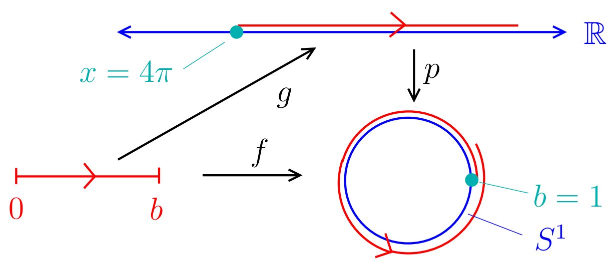


FIGURE 317

(2) In Figure 318 we illustrate the lifting of a path in the covering $p\colon H \to A$ of an infinite helix over an annulus that we had considered on page 494.

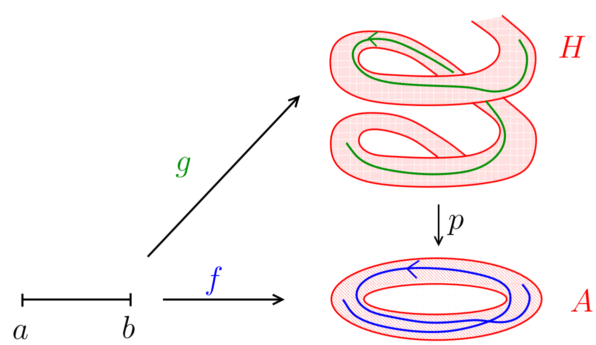


FIGURE 318

Proposition 16.11. *Let $p\colon X \to B$ be a covering and let $f\colon [0,1] \to B$ be a path. Then given any $x \in X$ with $p(x) = f(0)$ there exists a unique lift $g\colon [0,1] \to X$ of f to the starting point x.*

Proof. Let $p\colon X \to B$ be a covering, let $f\colon [0,1] \to B$ be a path and let x be a point in X with $p(x) = f(0)$. We first show the existence of a lift and then we prove the uniqueness of the lift.

(a) We consider

$$S \ := \ \big\{ s \in [0,1] \,\big|\, \text{there exists a lift of } f|_{[0,s]} \text{ to the starting point } x \big\}.$$

Note that S is non-empty since 0 has the desired property. We put $t := \sup(S)$. We want to show that $t = 1$ and that $t \in S$. It suffices to show that there exists an $\epsilon > 0$ such that $[t, t+\epsilon) \cap [0,1]$ still lies in S. We set $b := f(t)$. Since p is a covering there exists an open neighborhood U of $b \in B$, which is uniformly covered. As a reminder, this means that $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i \in I}$ with the property that the restriction of p to each subset V_i is a homeomorphism. Since U is open and since f is continuous there exists an $\epsilon > 0$ such that $f((t-\epsilon, t+\epsilon) \cap [0,1]) \subset U$.

By the definition of t there exists an $s \in (t-\epsilon, t] \cap S$. We denote by g the lift of $f|_{[0,s]}$ the starting point x. There exists a unique $i \in I$ such that $g(s) \in V_i$. We denote by $p_i\colon V_i \to U$ the homeomorphism which is given by the restriction of p to V_i.

We consider the map

$$\begin{aligned} h\colon [0, t+\epsilon) \cap [0,1] \ &\to \ X \\ z \ &\mapsto \ \begin{cases} g(z), & \text{if } z \in [0,s], \\ p_i^{-1}(f(z)), & \text{if } z \in (s, t+\epsilon) \cap [0,1]. \end{cases} \end{aligned}$$

Since $g(s) = p_i^{-1}(f(s))$ we see that this map is indeed continuous. It is clear that this defines a lift of f to the starting point x.

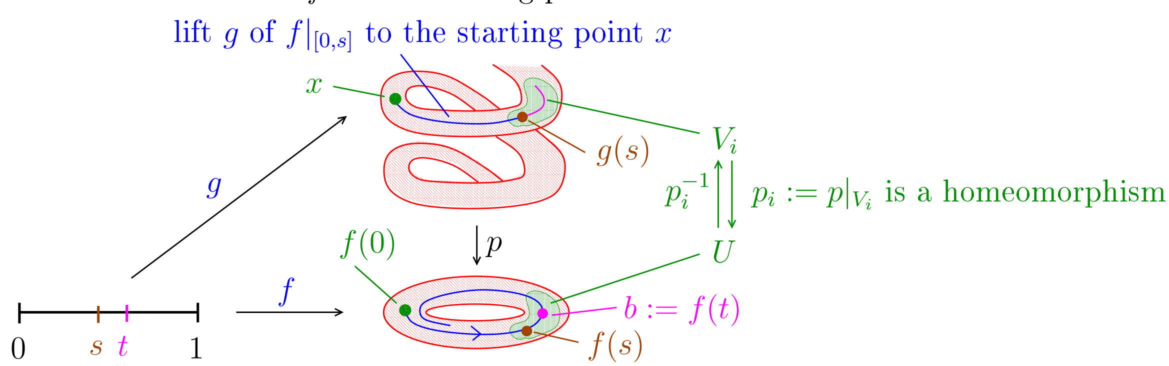


Figure 319

(b) Now we show the uniqueness of the lift. The proof of that statement is similar to the proof of the existence of a lift. So suppose that $g, h\colon [0,1] \to X$ are two lifts of f to the starting point x. We want to show that $g = h$. We consider

$$S \ := \ \{ s \in [0,1] \,|\, g(t) = h(t) \text{ for all } t \in [0,s] \}.$$

Note that $0 \in S$, i.e. S is non-empty. We put $t := \sup(S)$. We want to show that $t = 1$ and that $t \in S$. Again it suffices to show that there exists an $\epsilon > 0$ such that $[t, t+\epsilon) \cap [0,1]$ still lies in S.

We put $b := f(t)$. Since p is a covering there exists an open neighborhood U of $b \in B$ which is uniformly covered. Since U is open and since f is continuous there exists an $\epsilon > 0$ such that $f((t-\epsilon, t+\epsilon) \cap [0,1]) \subset U$. We pick an $s \in (t-\epsilon, t] \cap [0,1]$. There exists a unique $i \in I$ such that $g(s) \in V_i$. We denote by $p_i \colon V_i \to U$ the homeomorphism which is given by the restriction of p to V_i.

Since $g(s) = h(s)$ it follows that $h(s) \in V_i$. Then it follows that $g(z) \in V_i$ and $h(z) \in V_i$ for all $z \in (t-\epsilon, t+\epsilon) \cap [0,1]$. It follows from the definitions that

$$g(z) \;=\; p_i^{-1}(f(z)) \quad \text{and} \quad h(z) \;=\; p_i^{-1}(f(z))$$

for all $z \in (t-\epsilon, t+\epsilon) \cap [0,1]$. In particular $[t, t+\epsilon) \cap [0,1]$ lies in S. ■

16.4. **Lifting of path-homotopies.** In the last chapter we saw that given a covering map $p \colon X \to B$ any path $f \colon [0,1] \to B$ can be lifted to a path $g \colon [0,1] \to X$. In this chapter we want to study the question whether the lifts of path-homotopic paths are still path-homotopic.

Before we start this discussion we first generalize the notion of a lift.

Definition. Let $p \colon X \to B$ be a covering and let $f \colon Y \to B$ be a map between topological spaces. A *lift of f to X* is a map[261] $\widetilde{f} \colon Y \to X$ such that $p \circ \widetilde{f} = f$, i.e. such that the following diagram of maps commutes:

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & B. \end{tikzcd}$$

Proposition 16.12. *Let $p \colon X \to B$ be a covering, let B be a topological space, let Y be a topological manifold*[262] *and let $f \colon Y \times [0,1] \to B$ be a map. Furthermore let $\widetilde{f} \colon Y \times 0 \to X$ be a lift of $f|_{Y \times 0}$. Let*

$$\widetilde{f} \colon Y \times [0,1] \;\to\; X$$

be the map that is uniquely determined[263] *by the property that for each $y \in Y$ the path $t \mapsto \widetilde{f}(y,t)$ is the lift of the path $t \mapsto f(y,t)$ to the starting point $\widetilde{f}(y,0)$. Then the map $\widetilde{f} \colon Y \times [0,1] \to X$ is continuous.*

The statement of Proposition 16.12 is sketched in Figure 320. One way of formulating Proposition 16.12 is that a "continuous" family of paths in B lifts again to a "continuous" family of paths in X.

[261] Once again, all maps are understood to be continuous.

[262] Later on we will apply the proposition to the case that $Y = [0,1]$ and that $Y = \overline{B}^n$ for some $n \in \mathbb{N}_0$.

[263] The existence and the uniqueness of this map is an immediate consequence of Proposition 16.11.

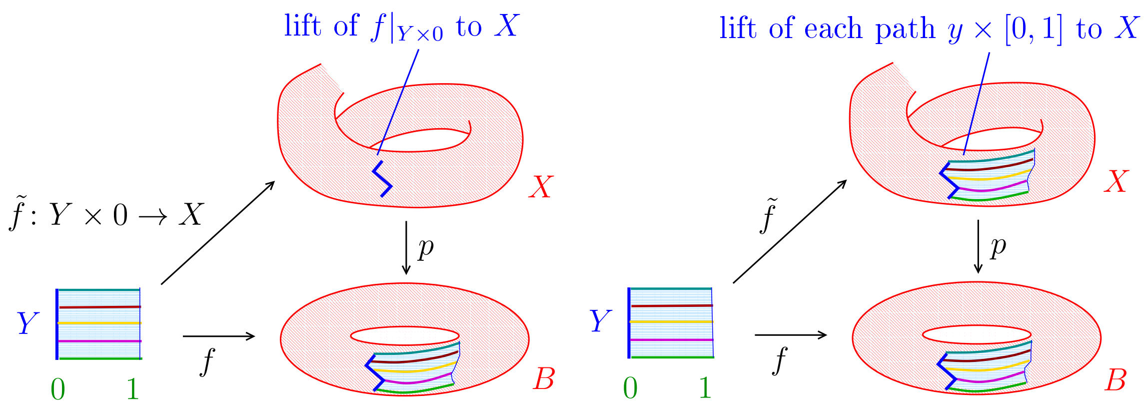


FIGURE 320. Schematic image for the lift of f.

PROOF (∗). Let $p\colon X\to B$ be a covering, let $f\colon Y\times[0,1]\to B$ be a continuous map and let $\widetilde{f}\colon Y\times 0\to X$ be a continuous lift of $f|_{Y\times 0}$. (Here and throughout this proof we do not suppose that all maps are continuous.) We begin with the following definitions:

(1) A *good set* $U\times I$ consists of a connected subset $U\subset Y$ and an open interval[264] $I\subset[0,1]$ such that $f(U\times I)$ lies in a uniformly covered subset of B.

(2) We say a good set $U\times I$ is *very good*, if there exists a $t\in I$ such that the restriction of $\widetilde{f}$ to $U\times\{t\}$ is continuous.

(3) Let $y\in Y$ and $t\in[0,1]$. A (*very*) *good neighborhood of* (y,t) is a (very) good set $U\times I$ with $y\in U$ and $t\in I$.

It follows easily[265] from the continuity of f, the hypothesis that Y is a topological manifold and the fact that $p\colon X\to B$ is a covering that each point (y,t) admits a good neighborhood.[266]

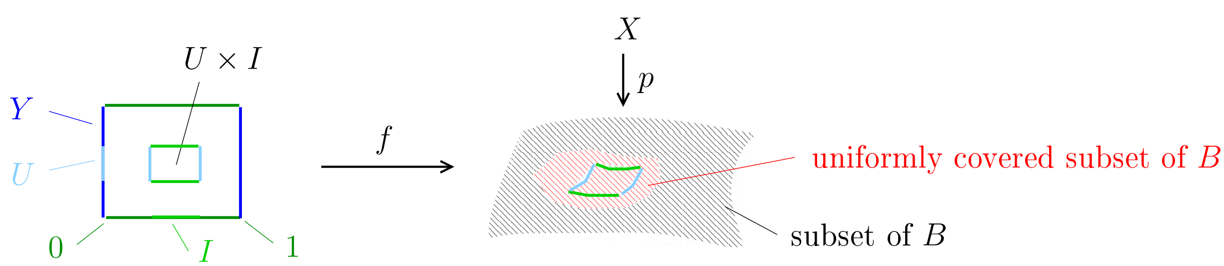


FIGURE 321. Schematic image of a good neighborhood.

Claim. Let $U\subset Y$ open and let $I\subset[0,1]$ be an open interval such that $U\times I$ is very good. Then $\widetilde{f}$ is continuous on $U\times I$.

[264]We say $A\subset[0,1]$ is an open interval, if A is an interval and if A is open with respect to the subspace topology of $[0,1]$. Put differently, $A=(a,b)$, or $A=[0,a)$ or $A=(b,1]$ or $A=[0,1]$ for $0\le a\le b\le 1$.

[265]Why does this follow?

[266]Here we use the fact, established in Corollary 2.56 and Lemma 6.9 (2), that the topological manifold Y is locally connected.

Let $U \subset Y$ be an open connected subset and let $I \subset [0,1]$ be an interval such that $U \times I$ is very good. This means in particular that there exists a $t \in I$ such that

$$\begin{array}{rcl} U & \to & X \\ y & \mapsto & \widetilde{f}(y,t) \end{array}$$

is continuous.

According to our hypothesis there exists a uniformly covered subset $W \subset B$ such that $f(U \times I) \subset W$. Since $U \times I$ is connected we can without loss of generality suppose that W is connected. Since W is uniformly covered the preimage $p^{-1}(W)$ is the union of disjoint open subsets $\{V_j\}_{j \in J}$, with the property, that the restriction of p to each subset V_j is a homeomorphism.

Now we want to show that there exists a $j \in J$ with $\widetilde{f}(U \times I) \subset V_j$. We choose a $y \in U$. Then there exists a unique $j \in J$ such that $\widetilde{f}(y,t) \in V_j$. Since $\widetilde{f}$ is continuous $U \times \{t\}$ and since U is connected it follows from Lemma 2.70[267] that $\widetilde{f}(U \times \{t\}) \subset V_j$. Now let $z \in U$. We just showed that $\widetilde{f}(z,t) \in V_j$. By the construction of $\widetilde{f}$ the map $\widetilde{f}$ is continuous on $\{z\} \times I$. Since I is connected it follows again from Lemma 2.70 that $\widetilde{f}(\{z\} \times I) \subset V_j$. We have thus shown that $\widetilde{f}(U \times I) \subset V_i$. This part of the argument is sketched in Figure 322.

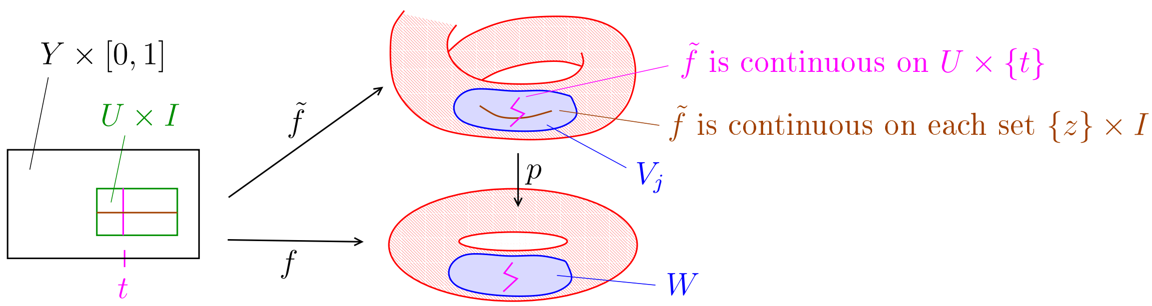


FIGURE 322

We denote by $p_j \colon V_j \to W$ the homeomorphism which is given by the restriction of p to V_j. Since $\widetilde{f}(U \times I) \subset V_j$ it follows that on $U \times I$ the map $\widetilde{f}$ is given by $\widetilde{f} = p_j^{-1} \circ f$. In particular the map $\widetilde{f}$ is a composition of continuous maps, so it is continuous on $U \times I$. ⊞

Claim. Let $y \in Y$. We consider

$$T \;:=\; \{t \in [0,1] \,|\, \text{there exists a very good neighborhood of } (y,t)\}.$$

We claim that $T = [0,1]$.

Thus let $y \in Y$. Since $[0,1]$ is connected it suffices to show the following three statements:

(1) $T \neq \varnothing$,
(2) T is open, and
(3) T is closed.

[267]We apply Lemma 2.70 to the continuous map $U \times \{t\} \to p^{-1}(W)$ from the connected set U to the set $p^{-1}(W)$ with components V_j, $j \in J$.

Now we will prove these three statements:

(1) As we pointed out above, every point in $Y \times I$, in particular the point $(y, 0)$, admits a good neighborhood. By our hypothesis $\widetilde{f}$ is continuous on $Y \times \{0\}$. It follows that the good neighborhood is in fact very good.
(2) It is easy to see that T is open. Indeed, suppose that we are given a very good neighborhood $U \times I$ of (y, t). Then $U \times I$ is also a very good neighborhood for any $(y, s) \in U \times I$. Therefore the open neighborhood I of t also lies in T.
(3) It remains to show that T is closed. Put differently, we need to show that T equals its closure $\overline{T}$. Thus let $t \in \overline{T}$. As we already pointed out, there exists a good neighborhood $U \times I$ of (y, t). Since $t \in \overline{T}$ there exists an $s \in T \cap I$. Since $s \in T$ it follows from the above claim that there exists a very good open neighborhood $V \times J$ of (y, s). By the previous claim the restriction of $\widetilde{f}$ to $V \times J$ is continuous, in particular it is continuous on $(U \cap V) \times \{s\}$. We denote by W the component of $U \cap V$ that contains y. Then $W \times I$ is a very good neighborhood of (y, t). This step is illustrated in Figure 323. ⊞

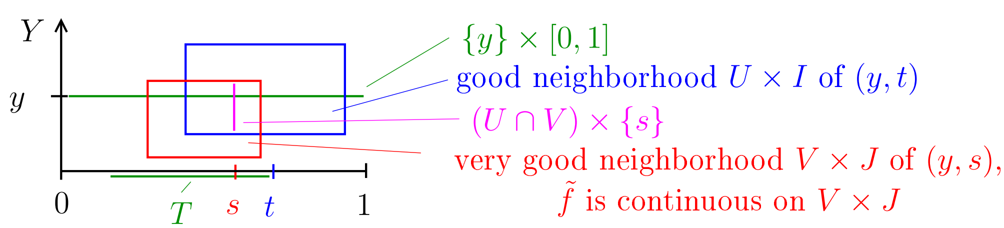


FIGURE 323

We have just shown that given any $y \in Y$ and any $t \in [0, 1]$ there exists a very good neighborhood $U_{y,t}$ of (y, t). By the first claim the map $\widetilde{f}$ is continuous on each $U_{y,t}$. But the open subsets $U_{y,t}$ cover all of $Y \times [0, 1]$. It follows from Lemma 2.74 that $\widetilde{f}$ is continuous on $Y \times [0, 1]$. ■

Corollary 16.13. *Let $p: X \to B$ be a covering and let $f, g: [0, 1] \to B$ be two paths. Let $\widetilde{f}, \widetilde{g}: [0, 1] \to X$ be two lifts with the same starting point. If f and g are path-homotopic, then the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree and the paths $\widetilde{f}$ and $\widetilde{g}$ are path-homotopic.*

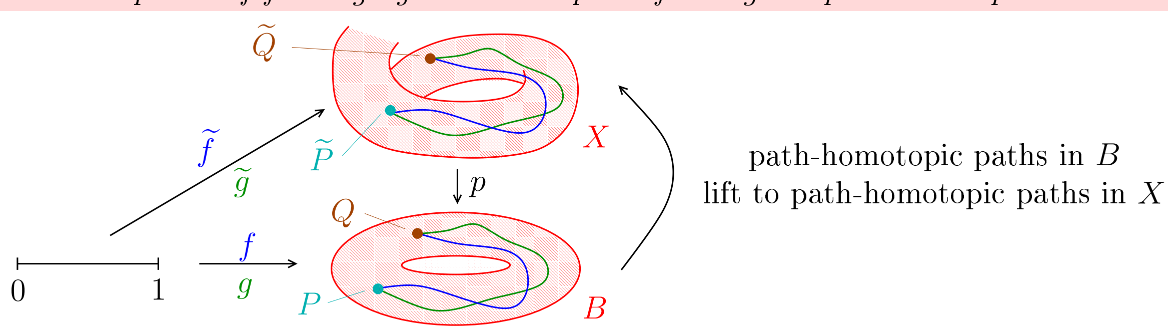


FIGURE 324. Illustration of Corollary 16.13.

PROOF (*). We write $P := f(0) = g(0)$ and $Q := f(1) = g(1)$. We denote by $\widetilde{P} := \widetilde{f}(0) = \widetilde{g}(0)$ the common starting point of the lifts and we write $\widetilde{Q} := \widetilde{f}(1)$. Let

$$\begin{array}{rcl} H\colon [0,1]\times[0,1] & \to & B \\ (t,s) & \mapsto & H(t,s) \end{array}$$

be a path-homotopy between the paths f and g. As a reminder, H is a map such that

$$H(t,0) \;=\; f(t) \quad\text{and}\quad H(t,1) \;=\; g(t) \quad \text{for all } t\in[0,1],$$

and such that

$$H(0,s) \;=\; P \quad\text{and}\quad H(1,s) \;=\; Q \quad \text{for all } s\in[0,1].$$

On $\{0\}\times[0,1]$ we consider the lift $\widetilde{H}$ of H which is given by $\widetilde{H}(0,s) = \widetilde{P}$. Furthermore we denote by $\widetilde{H}\colon [0,1]\times[0,1]\to X$ the map which is defined as follows: for each $s\in[0,1]$ the path $t\mapsto \widetilde{H}(t,s)$ is the unique lift of the path $t\mapsto H(t,s)$ to the starting point $\widetilde{P}$ lift. According to Proposition 16.12 the map $\widetilde{H}$ is in fact continuous. By definition of $\widetilde{H}$ we have

$$\widetilde{H}(t,0) \;=\; \widetilde{f}(t) \quad\text{and}\quad \widetilde{H}(t,1) \;=\; \widetilde{g}(t) \text{ for all } t\in[0,1],$$

For each $s\in[0,1]$ we have $p(\widetilde{H}(1,s)) = Q$. Since $\widetilde{H}$ is continuous we obtain a map

$$\begin{array}{rcl} [0,1] & \to & p^{-1}(Q) \;\subset\; X \\ s & \mapsto & \widetilde{H}(1,s). \end{array}$$

Since $[0,1]$ is connected it follows from Lemma 2.61 and Lemma 16.3 that $\widetilde{H}(1,s) = \widetilde{Q}$ for all $s\in[0,1]$. In particular we obtain that

$$\widetilde{f}(1) \;=\; \widetilde{H}(1,0) \;=\; \widetilde{Q} \;=\; \widetilde{H}(1,1) \;=\; \widetilde{g}(1).$$

Thus it follows that $\widetilde{H}$ is indeed a path-homotopy between the paths $\widetilde{f}$ and $\widetilde{g}$. ■

Definition. Let $p\colon X\to B$ be a map between topological spaces and let $x_0\in X$ and $b_0\in B$. We say a map $p\colon X\to B$ is *a covering of pointed topological spaces*, if $p\colon X\to B$ is a covering and if $p(x_0) = b_0$.

Corollary 16.14. *For any covering $p\colon (X,x_0)\to(B,b_0)$ of pointed topological spaces the induced map $p_*\colon \pi_1(X,x_0)\to\pi_1(B,b_0)$ is a monomorphism.*

PROOF. We have[268]

$$\begin{array}{rlcl} [f]\in\ker\big(\pi_1(X,x_0)\xrightarrow{p_*}\pi_1(B,b_0)\big) & \Longrightarrow\; [p\circ f] = e\in\pi_1(B,b_0) & \Longrightarrow & p\circ f\simeq e_{b_0} \\ & \underset{\uparrow}{\Longrightarrow}\; f\;\simeq\; e_{x_0} & \Longrightarrow & [f] = e\in\pi_1(X,x_0). \end{array}$$

by Corollary 16.13, since f is the lift of $p\circ f$ and e_{x_0} is the lift of e_{b_0} to the starting point x_0 ■

Example. In Question 16.2 (1) we had asked whether there exists a covering map from the torus $S^1\times S^1$ to the sphere S^2. By Proposition 14.14 we know that $\pi_1(S^2) = 0$ whereas we saw on page 486 that $\pi_1(S^1\times S^1)$ is non-trivial. Using Corollary 16.14 we can now answer Question 16.2 (1) in the negative: there is no covering map from $S^1\times S^1$ to S^2.

[268]We use the notation from page 466: for $y\in Y$ we denote by e_y the constant path $e_y(t) := x$, $t\in[0,1]$.

We conclude the section with the following lemma.

Lemma 16.15. *Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces.*

(1) *Let f be a loop in (B, b_0). We denote by $\widetilde{f}$ the lift of f to the starting point x_0. Then the following holds:*
$$\widetilde{f} \text{ is a loop in } (X, x_0) \quad \Longleftrightarrow \quad [f] \text{ lies in } p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0).$$
In particular, if $\widetilde{f}$ is not a loop in (X, x_0), then $[f]$ is a non-trivial element of $\pi_1(B, b_0)$.

(2) *Now we suppose that X is simply connected. Let f and g loops in (B, b_0) and let $\widetilde{f}$ and $\widetilde{g}$ be the lifts of f and g to the starting point x_0. Then the following holds:*
$$f \text{ and } g \text{ are path-homotopic loops} \quad \Longleftrightarrow \quad \text{the endpoints of } \widetilde{f} \text{ and } \widetilde{g} \text{ agree.}$$

(3) *If X is path-connected, then $[X : B] = [\pi_1(B, b_0) : p_*(\pi_1(X, x_0))]$.*

Remark. In Exercise 16.9 we will use Lemma 16.15 (3) to answer Question 16.2 (2) in the negative.

PROOF (∗).

(1) We prove the "⇒"-direction. So let f be a loop in (B, b_0). We denote by $\widetilde{f}$ the lift of f to the starting point x_0. If $\widetilde{f}$ is a loop in (X, x_0), then it defines an element in $\pi_1(X, x_0)$ and we have $[f] = [p \circ \widetilde{f}] = p_*([\widetilde{f}]) \in p_*(\pi_1(X, x_0))$. The proof of the "⇐"-direction is Exercise 16.8.[269]

(2) The "⇒"-direction is an immediate consequence of Corollary 16.13. The proof of the "⇐"-direction is done once again in Exercise 16.8. For that direction we need that X is simply connected.

(3) This statement is proved in Exercise 16.8. ■

Example. The last sentence of Lemma 16.15 (1) can at times be surprisingly useful for showing that a given element in a fundamental group. As proof of concept, we consider the pointed topological space (B, b_0) and we show a loop $f\colon [0, 1] \to B$ shown in Figure 325. We also show a covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces together with the lift $\widetilde{f}$ of f to the starting point x_0. Since $\widetilde{f}$ is not a loop we see that $[f] \in \pi_1(B, b_0)$ is non-trivial. In Exercise 16.10 we will use the same approach to show that $\pi_1(B, b_0)$ is actually non-abelian.

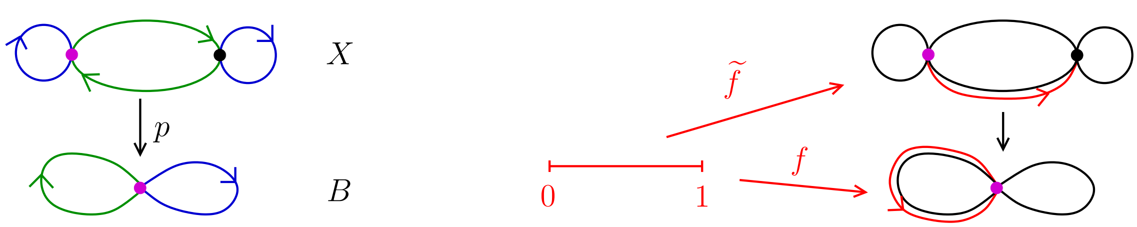


FIGURE 325

[269]Note that $[f]$ lies in $p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0)$ means that there exists a loop g in (X, x_0) such that $p \circ g$ is *path-homotopic* to f. A priori it does not mean that there exists a loop g with $p \circ g = f$.

16.5. **Group actions and fundamental groups.** The following proposition finally allows us to determine many fundamental groups. This proposition is the reward for the many rather theoretical results of the previous sections.

Theorem 16.16. *Let X be a topological space and let G be a group which acts continuously and discretely on X. We choose an $x \in X$. We denote by $p\colon X \to X/G$ the canonical projection map.*

(1) *If X is simply connected, then the map*
$$\begin{array}{rcl} G & \to & \pi_1(X/G,[x]) \\ g & \mapsto & [p \circ (\text{path in } X \text{ from } x \text{ to } g \cdot x)] \end{array}$$
is a well-defined isomorphism of groups. In particular we have $\pi_1(X/G) \cong G$.

(2) $(*)$ *Regardless of whether or not X is simply connected, the subgroup $p_*(\pi_1(X,x))$ is normal in $\pi_1(X/G,[x])$ and the map*
$$\begin{array}{rcl} G & \to & \pi_1(X/G,[x])/p_*(\pi_1(X,x)) \\ g & \mapsto & [p \circ (\text{path in } X \text{ from } x \text{ to } g \cdot x)] \end{array}$$
is a well-defined isomorphism of groups.

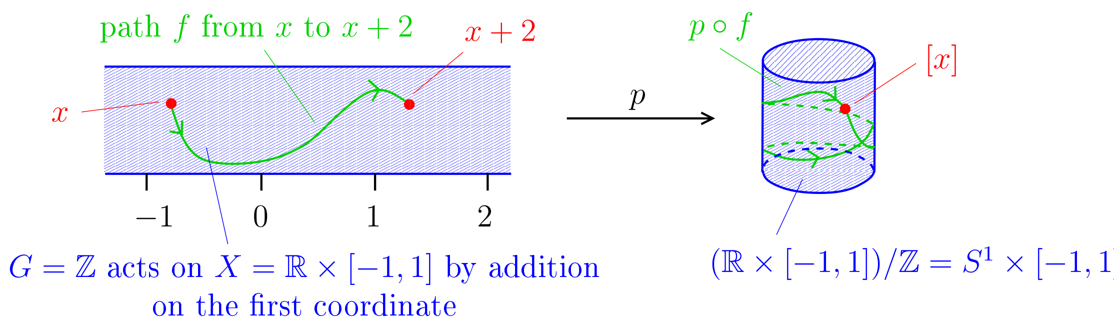


FIGURE 326. Illustration of Theorem 16.16 (1).

Before we provide the proof of Theorem 16.16, let us impatiently use it to finally confirm our suspicion that the fundamental group of S^1 is isomorphic to $\mathbb{Z}$.

Proposition 16.17.

(1) *The maps*
$$\begin{array}{rcl} \Phi\colon \mathbb{Z} & \to & \pi_1(\mathbb{R}/\mathbb{Z},[0]) \\ n & \mapsto & \underbrace{\left[\begin{array}{rcl} [0,1] & \to & \mathbb{R}/\mathbb{Z} \\ t & \mapsto & p(nt)=[nt] \end{array}\right]}_{\substack{\text{the map } t\mapsto nt \text{ is a} \\ \text{path from } 0 \text{ to } 0+n}} \end{array} \quad \textit{and} \quad \begin{array}{rcl} \Psi\colon \mathbb{Z} & \to & \pi_1(S^1,1) \\ n & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & S^1 \\ t & \mapsto & e^{2\pi i n t} \end{array}\right] \end{array}$$
are isomorphisms of groups.

(2) *Let $n \in \mathbb{N}$. We consider the map $f\colon S^1 \to S^1$ that is given by $z \mapsto z^n$. The induced map on $\pi_1(S^1,1)$ is given by multiplication by n.*

(3) *The map*

$$\begin{array}{rcl}\Theta\colon \pi_1(S^1,1) & \to & \mathbb{Z} \\ [\gamma] & \mapsto & \frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma} \frac{1}{z}\,dz\end{array}$$

is an isomorphism.

Proof.

(1) We consider the action of $G=\mathbb{Z}$ on the simply connected topological space $X=\mathbb{R}$. We denote by $p\colon \mathbb{R}\to\mathbb{R}/\mathbb{Z}$ the canonical projection map. It follows immediately from Theorem 16.16 that the given map $\Phi\colon \mathbb{Z}\to\pi_1(\mathbb{R}/\mathbb{Z},[0])$ is an isomorphism. We denote by $\phi\colon \mathbb{R}/\mathbb{Z}\to S^1$ the homeomorphism given by $\phi([t])=e^{2\pi\mathrm{i}t}$. If we compose the above isomorphism with the isomorphism $\phi_*\colon \pi_1(\mathbb{R}/\mathbb{Z},[0])\to\pi_1(S^1,1)$, then we see that the second map $\Psi\colon \mathbb{Z}\to\pi_1(S^1,1)$ is also an isomorphism.

(2) This statement follows easily from the definitions. To practice the definitions we will prove this statement in Exercise 16.14.

(3) It is straightforward to see that $\Theta\circ\Psi\colon \mathbb{Z}\to\pi_1(S^1,1)\to\mathbb{Z}$ is the identity. By (1) the first map Ψ is an isomorphism. Thus $\Theta=\Phi^{-1}$ is also an isomorphism. ■

We will provide many more examples for Theorem 16.16 immediately after the proof of Theorem 16.16.

Proof of Theorem 16.16 (∗). To simplify the discussion we only provide the proof of Theorem 16.16 (1). The proof of Theorem 16.16 (2) is almost the same. We leave it to the reader to make the necessary modifications.

Let X be a simply connected topological space and let G be a group which acts continuously and discretely on X.[270] We choose an $x\in X$. We denote by $p\colon X\to X/G$ the canonical projection map. We consider the map

$$\begin{array}{rcl}\Phi\colon G & \to & \pi_1(X/G,[x]) \\ g & \mapsto & [p\circ(\text{path in } X \text{ from } x \text{ to } gx)].\end{array}$$

We need to prove the following four statements:

(1) Φ is well-defined,
(2) Φ is a group homomorphism,
(3) Φ is surjective,
(4) Φ is injective.

Now we provide the proofs for these four claims.

(1) We first show that the map Φ is well-defined. Let $g\in G$. Since X is in particular path-connected there exists a path from x to gx. Now let f be an arbitrary path in X from x to gx. From $p(x)=[x]=[gx]=p(gx)$ it follows that $p\circ f$ is a loop in $(X/G,[x])$. In particular $p\circ f$ does indeed define an element in $\pi_1(X/G,[x])$.

Now we need to show that this element does not depend on the choice of f. So let f' be another path in X from x to gx. Since X is simply connected it follows from Lemma 14.13 that the paths f and f' are path-homotopic in X, i.e. there exists

[270] It is perhaps helpful to read the proof with the example of the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ in mind.

a path-homotopy $H\colon [0,1]\times[0,1]\to X$ from f to f'. But then $p\circ H\colon [0,1]\times[0,1]\to X/G$ is a path-homotopy from the loop $p\circ f$ to the loop $p\circ f'$. This shows that the map Φ is well-defined.

(2) Now we show that Φ is a group homomorphism. Let $g,h\in G$ and pick be a path k from x to gx and pick a path l from x to hx. Then gl[271] is a path from gx to ghx and $k*gl$ is a path from x to ghx. It follows that

$$\begin{aligned}\Phi(gh) \underset{\substack{\uparrow\\ \text{since } k*gl \text{ is a path from } x \text{ to } ghx}}{=} [p\circ(k*gl)] &= [(p\circ k)*(p\circ gl)] \overset{\substack{\text{since } p(gx)=p(x)\in X/G \text{ for all } x\in X\\ \downarrow}}{=} [(p\circ k)*(p\circ l)]\\ &\underset{\substack{\uparrow\\ \text{definition of } \cdot \text{ in } \pi_1(X/G,[x])}}{=} [p\circ k]\cdot[p\circ l] = \Phi(g)\cdot\Phi(h).\end{aligned}$$

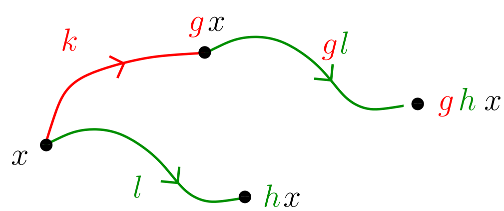


FIGURE 327. Schematic image for the proof that Φ is a group homomorphism.

Before we continue let us recall that according to Proposition 16.9 the projection map $p\colon X\to X/G$ is a covering.

(3) Now we prove that Φ is surjective. We choose an arbitrary element $c\in\pi_1(X/G,[x])$ and represent it by a loop f in $(X/G,[x])$. According to Proposition 16.11 there exists a lift $\widetilde{f}\colon[0,1]\to X$ of the path f to the starting point x. By definition we have $p\circ\widetilde{f}=f$. In particular we obtain for the endpoint of $\widetilde{f}$ that $p(\widetilde{f}(1))=[x]$, therefore it follows that $\widetilde{f}(1)=gx$ for some $g\in G$. Thus we have proved that

$$\Phi(g) = \big[p\circ(\text{path from } x \text{ to } gx)\big] = [p\circ\widetilde{f}] = [f] = c.$$

(4) We conclude the proof of the proposition by showing that Φ is injective. Let $g\in G$. Then the following holds:

$$\begin{aligned}\Phi(g)=e \;&\Longrightarrow\; [p\circ(\text{path } f \text{ from } x \text{ to } gx)] = e\in\pi_1(X/G,[x])\\ &\Longrightarrow\; p\circ f\simeq e_{[x]} \text{ in } X/G\\ &\underset{\substack{\uparrow\\ \text{by Corollary 16.13, since } f \text{ is a lift of } p\circ f \text{ and } e_x\\ \text{is a lift of } e_{[x]} \text{ to the starting point } x}}{\Longrightarrow} f(1) = e_x(1) \;\Longrightarrow\; gx = x \underset{\substack{\uparrow\\ \text{since the action is discrete,}\\ \text{hence free}}}{\Longrightarrow} g=e.\end{aligned}$$

This shows that Φ is indeed injective.

We have thus shown that Φ is a well-defined isomorphism. ■

Now we can determine the fundamental groups of many of the topological spaces that we have encountered so far.

[271] Here gl denotes the path $[0,1]\to X$ that is given by $t\mapsto g\cdot l(t)$.

Corollary 16.18. *We have the following isomorphisms of fundamental groups*

$$\begin{array}{llcll}
\text{(A)} & \pi_1(n\text{-}dimensional\ torus) & = & \pi_1(\mathbb{R}^n/\mathbb{Z}^n) & \cong \mathbb{Z}^n \\
& \textit{in particular} & & & \\
& \pi_1(S^1) & = & \pi_1(\mathbb{R}/\mathbb{Z}) & \cong \mathbb{Z}, \\
\text{(B)} & \textit{for any } n \neq 0,1 & & & \\
& \pi_1(\textit{real projective space } \mathbb{R}\mathrm{P}^n) & = & \pi_1(S^n/\{\pm 1\}) & \cong \mathbb{Z}_2, \\
\text{(C)} & \pi_1(\textit{Möbius band }) & = & \pi_1((\mathbb{R}\times[-1,1])/\mathbb{Z}) & \cong \mathbb{Z}, \\
\text{(D)} & \pi_1(\textit{lens space } L(p,q)) & = & \pi_1(S^3/\mathbb{Z}_p) & \cong \mathbb{Z}_p.
\end{array}$$

PROOF (∗). By Proposition 14.14 we know that for any $n \geq 2$ the sphere S^n is simply connected. Furthermore we know by the discussion on page 468 that $\mathbb{R}^n$, $n \geq 1$, and that the strip $\mathbb{R} \times [-1,1]$ are simply connected. We saw on page 500 that all the actions are discrete. The actions are of course also continuous. The corollary is now an immediate consequence of Theorem 16.16. ■

Thus we have now finally determined the fundamental groups of many spaces. In Proposition 14.14 we saw that $\pi_1(S^2) = 0$ whereas we have now seen that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$. The fundamental groups are not isomorphic, hence the sphere S^2 and the torus $S^1 \times S^1$ are not homeomorphic.

We also obtained a partial answer to Question 16.7: if two lens spaces $L(p,q)$ and $L(\widetilde{p},\widetilde{q})$ are homeomorphic, then $p = \widetilde{p}$. But as of right now we cannot say anything about the parameters q and $\widetilde{q}$.

Examples. Theorem 16.16 not only gives us the fact that $\pi_1(X/G)$ is isomorphic to the group G, but it also gives us an explicit isomorphism. Now we will study this isomorphism in two examples to explicitly determine non-trivial elements in the fundamental group of the torus and the real projective space.

(1) As a reminder, the torus is defined as[272]

$$T \;=\; \mathbb{R}^2/\mathbb{Z}^2 \;=\; [0,1]\times[0,1]/(x,0)\sim(x,1) \text{ and } (0,y)\sim(1,y).$$

The explicit isomorphism of Theorem 16.16

$$\Phi\colon \mathbb{Z}^2 \;\to\; \pi_1(T,(0,0))$$

assumes in particular the following values

$$\Phi((1,0)) \;=\; x \;:=\; \Bigg[\underbrace{\begin{array}{rcl} [0,1] & \to & ([0,1]\times[0,1])/\sim \\ t & \mapsto & [(t,0)] \end{array}}_{t\mapsto(t,0) \text{ is a path from } (0,0) \text{ to } (1,0)}\Bigg]$$

and

$$\Phi((0,1)) \;=\; y \;:=\; \Bigg[\underbrace{\begin{array}{rcl} [0,1] & \to & ([0,1]\times[0,1])/\sim \\ t & \mapsto & [(0,t)] \end{array}}_{t\mapsto(0,t) \text{ is a path from } (0,0) \text{ to } (0,1)}\Bigg].$$

[272] Here we are slightly generous in our usage of the equality sign. More precisely, strictly speaking $\mathbb{R}^2/\mathbb{Z}^2$ is not the same as $([0,1]\times[0,1])/\sim$. But there exists a canonical homeomorphism between these two spaces that we use to identify these spaces.

In the description of the torus as a subset of $\mathbb{R}^3$, see Figure 328, these two loops correspond to the "longitude" and the "meridian".

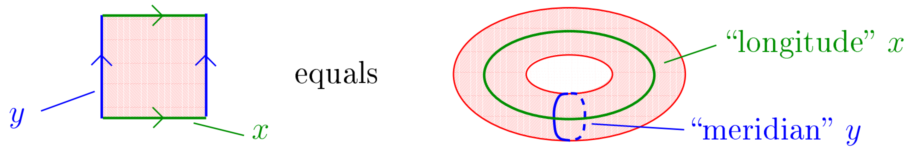


FIGURE 328

(2) Now we consider the real projective space $\mathbb{RP}^2 = S^2/\{\pm 1\} = S^2/x \sim -x$ and as usual we denote by $p\colon S^2 \to \mathbb{RP}^2$ the projection. We put $Q := (1,0,0)$. We consider the following path in S^2:

$$\begin{array}{rcl} f\colon [0,1] & \to & S^2 \\ t & \mapsto & (\cos(\pi t), \sin(\pi t), 0). \end{array}$$

This is a path from Q to $-Q = (-1)\cdot Q$. Since -1 is the one non-trivial element of the group $G = \{\pm 1\}$ it follows from the explicit isomorphism of Theorem 16.16 that the equivalence class of $p \circ f$ is precisely the one non-trivial element of the fundamental group $\pi_1(\mathbb{RP}^2, p(Q)) \cong \{\pm 1\}$.[273] Since $\pi_1(\mathbb{RP}^2, p(Q))$ is isomorphic to $\mathbb{Z}_2$ we see that $p \circ f$ represents an element of order 2, i.e. $(p\circ f) * (p \circ f)$ represents the trivial element in the fundamental group. Put differently, $(p \circ f) * (p \circ f)$ is null-homotopic. This one can see quite explicitly. More precisely, we have

$$\begin{array}{rcl} [p \circ f] \cdot [p \circ f] & = & [(p \circ f) * (p \circ f)] \;=\; [(p \circ (-f)) * (p \circ f)] \;=\; [p \circ (f * (-f))] \\ & = & \left[p \circ \underbrace{\left(\begin{array}{rcl} f\colon [0,1] & \to & S^2 \\ t & \mapsto & (\cos(2\pi t), \sin(2\pi t), 0) \end{array} \right)}_{\text{loop } g \text{ in } S^2} \right]. \end{array}$$

The loop g in (S^2, Q) is clearly null-homotopic.[274] But then the loop $p \circ g$ is also null-homotopic. This example is illustrated in Figure 329.

Example. We consider the Klein bottle. Recall that on page 501 we introduced the group G that is given by all self-homeomorphisms of $\mathbb{R}^2$ that can be written as concatenations of the two self-homeomorphisms

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1) \end{array}$$

and their inverses. In Exercise 16.7 we will see G acts discretely on $\mathbb{R}^2$ and we saw on page 501 that $\mathbb{R}^2/G$ is homeomorphic to the Klein bottle. It follows from Theorem 16.16 that

$$\pi_1(\text{Klein bottle}) \;\cong\; G = \;\text{group that is generated by the homeomorphisms } A \text{ and } B.$$

[273] In particular we have now shown that the loop in $\mathbb{RP}^2$ given on page 461 is not null-homotopic.

[274] More precisely, it is straightforward to write down an explicit path-homotopy from g to the constant path e_Q. Alternatively one can also deduce that g is null-homotopic from our calculation $\pi_1(S^2, Q) = 0$.

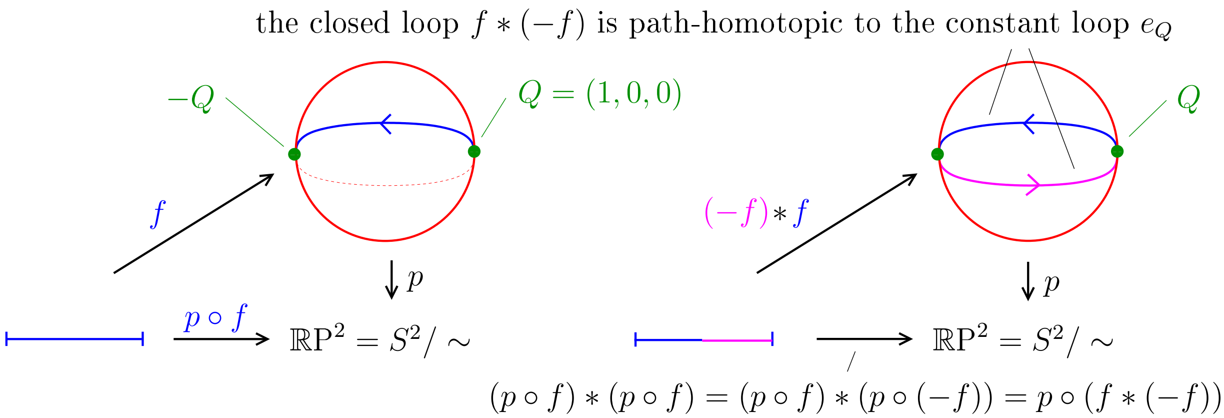


FIGURE 329

The maps A and B do not commute. Indeed, the map $A \circ B \colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$(x, y) \mapsto (x+1, (1-(y+1))) = (x+1, -y)$$

whereas the map $B \circ A \colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$(x, y) \mapsto (x+1, (1-y)+1) = (x+1, 2-y).$$

Thus we see that the fundamental group of the Klein bottle is a non-abelian group.

In Figure 330 we show the two elements b and c of the fundamental group of the Klein bottle $K = ([0,1] \times [0,1])/\sim$ which correspond to B and $C = B^{-1} \circ A$ under the isomorphism of Theorem 16.16 and the homeomorphism from page 501. Since the two homeomorphisms $B(x, y) = (x, y+1)$ and $C(x, y) = (x+1, -y)$ do not commute, the corresponding elements $[b]$ and $[c]$ of $\pi_1(K)$ do not commute either.

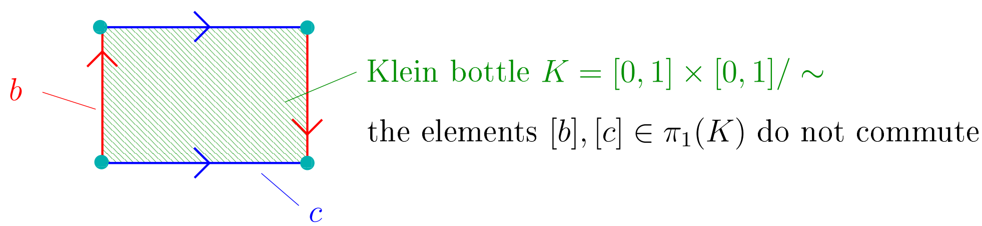


FIGURE 330

The previous example showed that fundamental groups can be non-abelian groups. In fact, as we will see later in Chapter 20, many, actually arguably "most" fundamental groups are non-abelian. This will pose several major problems. For example, what is a good way to describe a non-abelian group? How can we decide whether or not two non-abelian groups are isomorphic?

We conclude this section with a more unusual topological space. Namely we remark in Exercise 16.13 we will use Theorem 16.16 to show that the fundamental group of the pseudocircle, i.e. of the topological space $X = \{A, B, C, D\}$ from page 107, is isomorphic to $\mathbb{Z}$. In particular we show that the loop given on page 107 is not null-homotopic. This answers the question we had posed on page 462.

16.6. **The fundamental group of the product of two topological spaces.** The next proposition says that the fundamental groups of the product $A \times B$ of two topological spaces A and B is the direct product of the fundamental groups of A and B. Before we state the proposition explicitly we recall the definition of the direct product of two groups.

Definition. Let G and H two groups. We can form the cartesian product

$$G \times H \;:=\; \{(g,h)\,|\,g \in G \text{ and } h \in H\}.$$

This is again a group with the group structure given by

$$(g_1,h_1)\cdot(g_2,h_2) \;:=\; (g_1g_2,h_1h_2)$$

for $g_1, g_2 \in G$ and $h_1, h_2 \in H$. We call the group $G \times H$ the *direct product of the groups G and H.*

We start out with the following lemma, which strictly speaking we will not need, but which is worth knowing.

Lemma 16.19. $(*)$ *Let $\varphi_1\colon G_1 \to \pi$ and $\varphi_2\colon G_2 \to \pi$ be two group homomorphisms. If for every $g_1 \in G_1$ and $g_2 \in G_2$ the images $\varphi_1(g_1) \in \pi$ and $\varphi_2(g_2) \in \pi$ commute, i.e. if we have $\varphi_1(g_1)\cdot\varphi_2(g_2) = \varphi_2(g_2)\cdot\varphi_1(g_1)$, then the map*

$$\begin{aligned} \varphi_1 \times \varphi_2\colon G_1 \times G_2 &\to \pi \\ (g_1,g_2) &\mapsto \varphi_1(g_1)\cdot\varphi_2(g_2) \end{aligned}$$

is a homomorphism.[275]

Proof $(*)$. We write $\Phi = \varphi_1 \times \varphi_2$. Let $g_1, g_1' \in G_1$ and let $g_2, g_2' \in G_2$. We have

$$\begin{aligned} \Phi(g_1,g_2)\cdot\Phi(g_1',g_2') &= \varphi_1(g_1)\cdot\varphi_2(g_2)\cdot\varphi_1(g_1')\cdot\varphi_2(g_2') \\ &\underset{\substack{\uparrow \\ \text{by our hypothesis}}}{=} \varphi_1(g_1)\cdot\varphi_1(g_1')\cdot\varphi_2(g_2)\cdot\varphi_2(g_2') = \varphi_1(g_1g_1')\cdot\varphi_2(g_2g_2') \\ &= \Phi(g_1g_1', g_2g_2'). \end{aligned}$$

■

Now we can formulate the promised proposition.

Proposition 16.20. *Let A and B be two topological spaces and let $a_0 \in A$ and $b_0 \in B$. We consider the inclusion maps*

$$\begin{aligned} i\colon A &\to A \times B \\ a &\mapsto (a,b_0) \end{aligned} \quad \textit{and} \quad \begin{aligned} j\colon B &\to A \times B \\ b &\mapsto (a_0,b) \end{aligned}$$

and we consider the projection maps

$$\begin{aligned} p\colon A \times B &\to A \\ (a,b) &\mapsto a \end{aligned} \quad \textit{and} \quad \begin{aligned} q\colon A \times B &\to B \\ (a,b) &\mapsto b. \end{aligned}$$

Then the maps

$$\begin{aligned} \Phi\colon \pi_1(A,a_0) \times \pi_1(B,b_0) &\to \pi_1(A \times B,(a_0,b_0)) \\ (x,y) &\to i_*(x)\cdot j_*(y) \end{aligned}$$

[275]In fact the converse also holds, if $\varphi_1 \times \varphi_2$ is a homomorphism, then the for every $g_1 \in G_1$ and $g_2 \in G_2$ the images $\varphi_1(g_1) \in \pi$ and $\varphi_2(g_2) \in \pi$ commute. Why is that?

and

$$\begin{array}{rcl}\Psi\colon \pi_1(A\times B,(a_0,b_0)) & \to & \pi_1(A,a_0)\times\pi_1(B,b_0)\\ z & \to & (p_*(z),q_*(z))\end{array}$$

are both homomorphisms[276] *and they are inverses of one another, in particular both are isomorphisms.*

Example. It follows from Proposition 16.20 and Corollary 16.18 that

$$\pi_1(S^1\times L(p,q)) \;\cong\; \pi_1(S^1)\times\pi_1(L(p,q)) \;\cong\; \mathbb{Z}\times\mathbb{Z}_p.$$

Similarly, we see, using Proposition 14.14, that for any $k\geq 2$ we have

$$\pi_1(S^1\times S^k) \;\cong\; \pi_1(S^1)\times\pi_1(S^k) \;\cong\; \mathbb{Z}\times 0 \;\cong\; \mathbb{Z}.$$

Of course we can also apply Proposition 16.20 several times and we see once again that

$$\pi_1\big(\underbrace{S^1\times\cdots\times S^1}_{n\text{ times}}\big) \;\cong\; \underbrace{\pi_1(S^1)\times\ldots\times\pi_1(S^1)}_{n\text{ times}} \;\cong\; \underbrace{\mathbb{Z}\times\cdots\times\mathbb{Z}}_{n\text{ times}} \;\cong\; \mathbb{Z}^n.$$

Finally it follows from Propositions 14.14 and 16.20 that for any $k,l\geq 2$ we have

$$\pi_1(S^k\times S^l) \;\cong\; \pi_1(S^k)\times\pi_1(S^l) \;\cong\; 0\times 0 \;=\; 0.$$

In particular we see that the fundamental group cannot distinguish the products of spheres of dimension ≥ 2. For example at the moment we cannot show that the two 6-dimensional smooth manifolds $S^3\times S^3$ and $S^2\times S^2\times S^2$ are not homeomorphic.

In the proof of Proposition 16.20 we will need the following purely group-theoretic lemma.

Proof of Proposition 16.20 (∗). It is clear that

$$\Psi=(p_*,q_*)\colon \pi_1(A\times B,(a_0,b_0)) \;\to\; \pi_1(A,a_0)\times\pi_1(B,b_0)$$

is a homomorphism. Thus it suffices to show that Φ and Ψ are inverses of one another. Once we have shown that, it follows for free that Φ is also a homomorphism.

To show that Φ and Ψ are inverses of one another we introduce another map, namely we consider[277]

$$\begin{array}{rcl}\Theta\colon \pi_1(A,a_0)\times\pi_1(B,b_0) & \to & \pi_1(A\times B,(a_0,b_0))\\ ([f\colon (S^1,1)\to(A,a_0)],[g\colon (S^1,1)\to(B,b_0)]) & \mapsto & \left[\begin{array}{rcl}(S^1,1) & \to & (A\times B,(a_0,b_0))\\ z & \mapsto & (f(z),g(z))\end{array}\right].\end{array}$$

It is almost a tautology that Θ and Ψ are inverses of one another. Thus it remains to prove the following claim.

Claim. The two maps $\Phi,\Theta\colon \pi_1(A,a_0)\times\pi_1(B,b_0)\to\pi_1(A\times B,(a_0,b_0))$ agree.

We use the convention from page 462 to change our view of fundamental groups. Thus let $f\colon [0,1]\to A$ be a loop in (A,a_0) and $g\colon [0,1]\to B$ be a loop in (B,b_0). We need to show that

$$i_*([f])\cdot j_*([g]) \;=\; [(t\mapsto(f(t),g(t))]\;\in\;\pi_1(A\times B,(a_0,b_0).$$

[276]Note that a priori it is not clear that Φ is a homomorphism since it is not clear whether the hypothesis of Lemma 16.19 is satisfied.

[277]One can easily verify that Θ is well-defined.

In other words, we need to show that the two loops

$$\begin{array}{rcl} [0,1] & \to & A\times B \\ t & \mapsto & \begin{cases} (f(2t),b_0), & \text{if } t\in[0,\frac12] \\ (a_0,g(2t-1)), & \text{if } t\in[\frac12,1] \end{cases} \end{array} \quad \text{and} \quad \begin{array}{rcl} [0,1] & \to & A\times B \\ t & \mapsto & (f(t),g(t)) \end{array}$$

are path-homotopic. In fact an explicit path-homotopy is given by the map[278]

$$\begin{array}{rcl} F\colon [0,1]\times[0,1] & \to & A\times B \\ (t,s) & \mapsto & \begin{cases} \big(f\big(\frac{2t}{1+s}\big),g(st)\big), & \text{if } t\in[0,\frac12] \\ \big(f(st),g\big(\frac{2t-1+s}{1+s}\big)\big), & \text{if } t\in[\frac12,1]. \end{cases} \end{array}$$

■

Summarizing we have now finally managed to calculate many non-trivial fundamental groups. Nonetheless, the list of topological spaces for which we cannot yet calculate the fundamental group is still pretty long, it includes

(1) topological graphs,
(2) knot complements,
(3) the complement of more than one point in $\mathbb{C}$, e.g. $\mathbb{C}\setminus\{0,1\}$,
(4) surfaces of genus ≥ 2.

In the following chapters we will introduce more methods for determining the fundamental group of a given topological space. But first we give in the next section two applications of our results.

16.7. **The Fundamental Theorem of Algebra and the Borsuk-Ulam Theorem ❀.** As a first application of our methods we give a proof of the Fundamental Theorem of Algebra. Somewhat surprisingly, contrary to the standard proof [**Lan99**, Corollary III.7.6], this proof does not make any use of complex analysis.

Theorem 16.21. (Fundamental Theorem of Algebra) *Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$.*

Proof. Let $q(z)$ be a polynomial with complex coefficients with $q(z)\neq 0$ for all $z\in\mathbb{C}$ with $|z|=r$. We consider the path

$$\begin{array}{rcl} f_r^{q(z)}\colon [0,1] & \to & S^1 \\ s & \mapsto & \dfrac{q(re^{2\pi i s})/q(r)}{\big|q(re^{2\pi i s})/q(r)\big|} \end{array}$$

Note that $f_r^{q(z)}(s=0)=f_r^{q(z)}(s=1)=1$, i.e. $f_r^{q(z)}$ is a loop in $(S^1,1)$. Now let

$$p(z) \quad = \quad a_nz^n+a_{n-1}z^{n-1}+\dots+a_1z+a_0$$

be a polynomial of degree $n\geq 1$. We assume that $p(z)$ has no zeros in $\mathbb{C}$. For $t\in[0,1]$ we also consider the polynomial

$$p_t(z) \quad := \quad a_nz^n+t(a_{n-1}z^{n-1}+\dots+a_1z+a_0).$$

[278]Note that this is indeed a path-homotopy of loops.

Note that $p_1(z) = p(z)$ and $p_0(z) = a_n z^n$. Since $a_n \neq 0$ there exists an $s > 0$ such that[279]

$$|a_n z^n| \;>\; |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0|$$

for any z with $|z| = s$. This inequality implies that for any $t \in [0,1]$ and any z with $|z| = s$ we have $p_t(z) \neq 0$, i.e. $p_t(z)$ has no zeros on the circle $|z| = s$.

We obtain the following sequence of path-homotopic loops:

follows from the definition | follows from $p_0(z) = a_n z^n$

$$\text{constant loop at 1} \overset{\downarrow}{=} f_0^{p(z)} \underset{\uparrow}{\simeq} f_s^{p(z)} = f_s^{p_1(z)} \underset{\uparrow}{\simeq} f_s^{p_0(z)} \overset{\downarrow}{=} \text{the loop } \begin{array}{rcl} [0,1] & \to & S^1 \\ t & \mapsto & e^{2\pi i n t} \end{array}.$$

the maps $f_r^{p(z)}$ with $r \in [0,s]$ are defined, since the polynomials $p(z)$ has no zeros they form a path-homotopy from $f_0^{p(z)}$ to $f_s^{p(z)}$ | the maps $f_s^{p_t(z)}$ with $t \in [0,1]$ are defined, since the polynomials $p_t(z)$ have no zeros on $|z| = s$, and and they form a path-homotopy from $f_r^{p_1(z)}$ to $f_r^{p_0(z)}$

By Proposition 16.17 we had know that the loop on the right defines a non-trivial element in the fundamental group $\pi_1(S^1, 1)$, but we had just shown that it is path-homotopic to the constant path. Thus we have obtained a contradiction. ■

The following theorem goes back to the Polish mathematicians Karol Borsuk (1905-1982) and Stanislaw Ulam (1909-1984).[280] It was first proved in 1932.

Theorem 16.22. (Borsuk-Ulam Theorem) *For every map $f\colon S^2 \to \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.*

Examples.

(1) It follows from the Borsuk-Ulam Theorem that there exist two antipodal points on earth with the same temperature and the same barometric pressure.
(2) The Borsuk-Ulam Theorem implies in particular that there is no injective map from $S^2 \to \mathbb{R}^2$.
(3) The statement of the Borsuk-Ulam Theorem also makes sense for 2 replaced by any $n \in \mathbb{N}$. For $n = 1$ one can prove the statement using elementary methods.[281] We will address the problem for $n \geq 3$ once we have introduced homology groups.

Proof. We prove the theorem by contradiction. So suppose there exists a map $f\colon S^2 \to \mathbb{R}^2$ such that $f(x) \neq f(-x)$ for all $x \in S^2$. This allows to define the map

$$\begin{array}{rcl} g\colon S^2 & \to & S^1 \\ x & \mapsto & \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}. \end{array}$$

Note that $g(-x) = -g(x)$ for all $x \in S^2$. Now let η be the loop that is defined by

$$\begin{array}{rcl} \eta\colon [0,1] & \to & S^2 \\ s & \mapsto & \eta(s) = (\cos(2\pi s), \sin(2\pi s), 0) \end{array}$$

[279] In fact we can write it down explicitly: it follows easily from the triangle inequality that any $s \in \mathbb{R}$ such that $s \cdot |a_n| > |a_{n-1}| + \cdots + |a_0|$ will work.

[280] Ulam became much more famous through his work on the hydrogen bomb.

[281] How would you prove the statement?

and let $h = g \circ \eta \colon [0,1] \to S^1$ be the composition of η with g. We pick a $c \in \mathbb{R}$ such that $h(0) = e^{2\pi \mathrm{i} c}$. We denote by $p \colon \mathbb{R} \to S^1$, $t \mapsto e^{2\pi \mathrm{i} t}$, the usual covering map. We note that for $z, w \in \mathbb{R}$ we have

$$p(z) \;=\; p(w) \quad \Longleftrightarrow \quad z - w \in \mathbb{Z} \quad \text{and} \quad p(z) \;=\; -p(w) \quad \Longleftrightarrow \quad z - w \in \tfrac{1}{2} + \mathbb{Z}.$$

By Proposition 16.11 we can lift the path $h \colon [0,1] \to S^1$ to a path $\widetilde{h} \colon [0,1] \to \mathbb{R}$ with $\widetilde{h}(0) = c$. We refer to Figure 331 for an illustration of all the definitions.

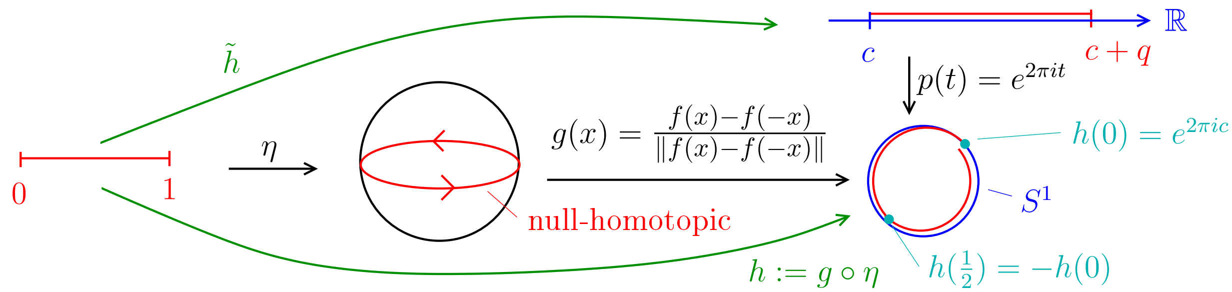


FIGURE 331. Illustration of the proof of the Borsuk-Ulam Theorem 16.22

Given any $s \in [0, \frac{1}{2}]$ we have

$$p\big(\widetilde{h}\big(s + \tfrac{1}{2}\big)\big) \;=\; h\big(s + \tfrac{1}{2}\big) \;=\; g\big(\eta\big(s + \tfrac{1}{2}\big)\big) \underset{\substack{\uparrow \\ \text{by the definition of } \eta}}{=} g(-\eta(s)) \underset{\substack{\uparrow \\ \text{since } g(-x) = -g(x) \text{ for all } x \in S^2}}{=} -g(\eta(s)) \;=\; -h(s) \;=\; -p\big(\widetilde{h}(s)\big).$$

By the above discussion of p this implies that

$$\widetilde{h}\big(s + \tfrac{1}{2}\big) - \widetilde{h}(s) \;=:\; \tfrac{q(s)}{2}$$

with $q(s)$ an odd integer. The map $q \colon [0, \frac{1}{2}] \to \mathbb{Z}$ is the difference of two continuous maps, hence continuous. It follows from Lemma 2.61 that it is constant. We denote this constant value by q. Since q is odd it is in particular non-zero.

It follows from the above that

$$\widetilde{h}(1) \;=\; \widetilde{h}\big(\tfrac{1}{2}\big) + \tfrac{q}{2} \;=\; \widetilde{h}(0) + \tfrac{q}{2} + \tfrac{q}{2} \;=\; \widetilde{h}(0) + q.$$

Since q is non-zero we obtain from Lemma 16.15 that $h = g \circ \eta$ is not null-homotopic. But h was the composition $h = g \circ \eta \colon [0,1] \to S^2 \to S^1$. By Proposition 14.14 the loop η is null-homotopic in S^2 which implies by Proposition 15.6 that $h = g \circ \eta$ is null-homotopic in S^1. Thus we have obtained a contradiction. ■

Exercises for Chapter 16.

Exercise 16.1. Let $p \colon X \to B$ be a covering of topological spaces. We suppose that B is a path-connected topological space. Show that for any two points P and Q in B we have

$$\#p^{-1}(P) \;=\; \#p^{-1}(Q).$$

Exercise 16.2.

(a) Let $p\colon X \to B$ be a map between topological spaces that is two-to-one, i.e. such that for any $b \in B$ the preimage $p^{-1}(b)$ consists of precisely two points. Is p necessarily a covering map?

(b) Is the map
$$\begin{array}{rcl} \mathbb{C} & \to & \mathbb{C} \setminus \{0\} \\ z & \mapsto & \exp(z) \end{array}$$
a covering map? *Hint.* $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$.

(c) Let $p\colon X \to B$ be a surjective map between topological spaces. Let $f\colon [0,1] \to B$ be a map. Does there always exist a lift from f to X, i.e. does there exist a map $g\colon [0,1] \to X$ such that $p \circ g = f$? Give a short justification for your answer.

Exercise 16.3.

(a) Show that there exists a map $p\colon X \to B$ be a map between topological spaces that is a local homeomorphism and that is surjective but which is not a covering map.

(b) Let $p\colon X \to B$ be a map between topological spaces that is a local homeomorphism and that is surjective. Furthermore we suppose that p is proper, i.e. we assume that the preimage of each compact set is compact. Is p necessarily a covering map?

Exercise 16.4. Let $p\colon \widetilde{X} \to X$ be a 2-fold covering. We define
$$\begin{array}{rcl} f\colon \widetilde{X} & \to & \widetilde{X} \\ Q & \mapsto & \text{the unique other point in } p^{-1}(p(Q)). \end{array}$$
Show that this map is continuous.

Exercise 16.5. Let $p\colon X \to B$ be a covering of topological spaces. Show that if B is Hausdorff, then X is also Hausdorff.

Exercise 16.6. Consider the 5-fold covering map
$$\begin{array}{rcl} p\colon X = S^1 & \to & B = S^1 \\ z & \mapsto & z^5. \end{array}$$
Let $f\colon [0,1] \to B = S^1$ be the loop given by $t \mapsto e^{2\pi\mathrm{i}kt}$. By Proposition 6.11 this loop lifts to a path $g\colon [0,1] \to X = S^1$. For which k is the lift g again a loop?

Exercise 16.7. We consider the following two self-homeomorphisms of $\mathbb{R}^2$:
$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1). \end{array}$$
We denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. This means
$$G = \begin{array}{c} \text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written} \\ \text{as a finite concatenation of the maps } A, B, A^{-1} \text{ and } B^{-1}. \end{array}$$

(a) For $n \in \mathbb{Z}$, what is $A^{-n}BA^n$? (Perhaps start out with $n = 1$.)

(b) We write $C = A^2$. We denote by H the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by C and B. This means

$$H = \begin{array}{c}\text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written}\\ \text{as a finite concatenation of the maps } C, B, C^{-1} \text{ and } B^{-1}.\end{array}$$

Does A lie in H?
Hint. Consider which self-diffeomorphisms of $\mathbb{R}^2$ are orientation-preserving.
(c) Show that any element X of G lies in H or it is of the form $G = AY$ with $Y \in H$.
(d) Show that the group G acts discretely on $\mathbb{R}^2$.

Exercise 16.8. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces.

(a) Let f be a loop in (B, b_0). We denote by $\widetilde{f}$ the lift of f to the starting point x_0. Show that if $[f]$ lies in $p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0)$, then $\widetilde{f}$ is a loop in (X, x_0).
(b) Now we suppose that X is simply connected. Let f and g loops in (B, b_0) and let $\widetilde{f}$ and $\widetilde{g}$ be the lifts of f and g to the starting point x_0. Show that if the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree, then f and g are path-homotopic loops.
(c) Show that if X is path-connected, then $[X : B] = [\pi_1(B, b_0) : p_*(\pi_1(X, x_0))]$.

Exercise 16.9. Show that there is no covering map from the sphere S^2 to the torus $S^1 \times S^1$.
Hint. Use Lemma 16.15.

Exercise 16.10. We consider the pointed topological space (B, b_0) and we consider the loops $\alpha, \beta\colon [0,1] \to B$ shown in Figure 325.

(a) Show that there exists a covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces such that the loop $\alpha * \beta * \overline{\alpha} * \overline{\beta}$ does not lift to a loop in (X, x_0).
(b) Use Lemma 16.15 (1) to show that $\pi_1(B, b_0)$ is non-abelian.

FIGURE 332

Exercise 16.11.

(a) Does there exist a covering map $p\colon K \to T$ from the Klein bottle to the 2-dimensional torus?
(b) Let X be a topological space and let $b \in \pi_1(X, x_0)$. Does there necessarily exist a map $f\colon (S^1, 1) \to (X, x_0)$ with $b \in f_*(\pi_1(S^1, 1))$?
(c) Let X be a topological space and let $b, c \in \pi_1(X, x_0)$. Does there necessarily exist a map $f\colon (S^1 \times S^1, (1,1)) \to (X, x_0)$ with $b, c \in f_*(\pi_1(S^1 \times S^1, (1,1)))$?

Exercise 16.12. Which of the maps between topological graphs that are shown in Figure 333 are covering maps?

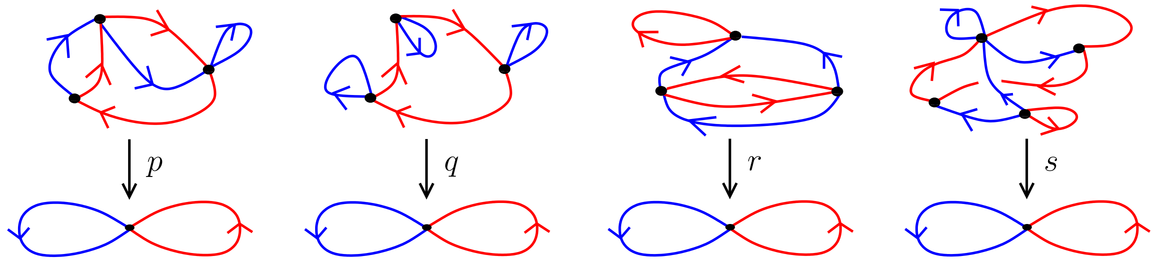


FIGURE 333

Exercise 16.13.

(a) Use Theorem 16.16 to show that the fundamental group of the pseudocircle, i.e. of the topological space $X = \{A, B, C, D\}$ from page 107, is isomorphic to $\mathbb{Z}$.

(b) Let $f\colon S^1 \to X$ be the map that we introduced on page 107. Show that the induced map $f_*\colon \pi_1(S^1, 1) \to \pi_1(X, A)$ is an isomorphism.

Exercise 16.14. Let $n \in \mathbb{N}$. We consider the map $f\colon S^1 \to S^1$ that is given by $z \mapsto z^n$. Show that the induced map on $\pi_1(S^1, 1)$ is given by multiplication by n.

Exercise 16.15. Let A and B be topological spaces and let $a_0 \in A$ and $b_0 \in B$. We consider the maps

$$\begin{array}{rcl} f\colon A & \to & A \times B \\ a & \mapsto & (a, b_0) \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon B & \to & A \times B \\ b & \mapsto & (a_0, b) \end{array}$$

and the map

$$\begin{array}{rcl} \Phi\colon \pi_1(A, a_0) \times \pi_1(B, b_0) & \to & \pi_1(A \times B, (a_0, b_0)) \\ (x, y) & \to & f_*(x) \cdot g_*(y) \end{array}$$

(a) Show that Φ is a monomorphism, i.e. that it is injective.
Hint. Consider the maps

$$\begin{array}{rcl} p\colon A \times B & \to & A \\ (a, b) & \mapsto & a \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon A \times B & \to & B \\ (a, b) & \mapsto & b. \end{array}$$

(b) Show that Φ is an epimorphism, i.e. that it is surjective.
Hint. Let $h(t) = (\alpha(t), \beta(t))$ be a loop in $(A \times B, (a_0, b_0))$. Consider the map

$$\begin{array}{rcl} \Psi\colon [0,1] \times [0,1] & \to & A \times B \\ (s, t) & \mapsto & (\alpha(s), \beta(t)). \end{array}$$

Hint. You might want to use ideas similar to the proof that the map Φ is a homomorphism.

Exercise 16.16. Let $A \in \mathrm{GL}(n, \mathbb{Z})$ be a matrix. In Lemma 6.35 we saw that the map

$$\begin{array}{rcl} f(A)\colon \mathbb{R}^n/\mathbb{Z}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ v & \mapsto & Av \end{array}$$

is a homeomorphism. We identify $\pi_1(\mathbb{R}^n/\mathbb{Z}^n,0)$ with $\mathbb{Z}^n$ as in Theorem 16.16. Show that the following diagram

$$\begin{tikzcd}
\pi_1(\mathbb{R}^n/\mathbb{Z}^n,0) \arrow[r, "f(A)_*"] \arrow[d, "="] & \pi_1(\mathbb{R}^n/\mathbb{Z}^n,0) \arrow[d, "="] \\
\mathbb{Z}^n \arrow[r, "v\mapsto Av"] & \mathbb{Z}^n
\end{tikzcd}$$

commutes.

Exercise 16.17. Let (X,d) be a metric space and let $p\colon \widetilde{X}\to X$ be a covering. Given $x,y\in\widetilde{X}$ we define

$$\widetilde{d}(x,y) \ := \ \inf\left\{ \sum_{i=0}^{n-1} d(w_i,w_{i+1}) \ \middle|\ \begin{array}{c}\text{with } w_0,\dots,w_n\in X,\ p(w_0)=x,\ p(w_n)=y \text{ and such}\\ \text{that for each } i\in\{0,\dots,n-1\} \text{ there exists a uniformly}\\ \text{covered open subset that contains } w_i \text{ and } w_{i+1}\end{array}\right\}.$$

(a) Show that $\widetilde{d}$ defines on $\widetilde{X}$ and that the corresponding topology equals the topology of $\widetilde{X}$.
(b) Show that $p\colon \widetilde{X}\to X$ is a local isometry.

Remark Recall that a topological space X is called *metrizable* if there exists a metric d on X such that the given topology agrees with the topology coming from d. The exercise shows in particular that every covering of a metrizable topological space is again metrizable.

17. Coverings of smooth and topological manifolds

17.1. **Coverings of manifolds are manifolds.** In this section we show that covering spaces of (topological) smooth manifolds are themselves (topological) smooth manifolds.

Proposition 17.1. *Let $p\colon \widetilde{X} \to X$ be a countable covering*[282] *of a connected n-dimensional topological manifold.*

(1) *The covering space $\widetilde{X}$ is also an n-dimensional topological manifold. Furthermore, $\partial\widetilde{X} = p^{-1}(\partial X)$.*
(2) (a) *If X admits a smooth atlas, then $\widetilde{X}$ also admits a smooth atlas such that p is a local diffeomorphism,*[283] *in particular such that p is a smooth map.*
    (b) *If X is equipped with a smooth structure*[284] *then $\widetilde{X}$ admits a* unique *smooth structure such that $p\colon \widetilde{X} \to X$ is a local diffeomorphism.*
(3) *If X is an oriented smooth manifold, then $\widetilde{X}$ also admits an orientation such that $p\colon \widetilde{X} \to X$ is orientation-preserving.*
(4) *If X is equipped with a complex structure, then $\widetilde{X}$ admits a unique complex structure such that $p\colon \widetilde{X} \to X$ is a local biholomorphism.*

Convention. If X is a smooth manifold, i.e. if X is a topological manifold equipped with a smooth structure, and if we have constructed a finite covering $p\colon \widetilde{M} \to M$, then we always equip $\widetilde{M}$ with the unique smooth structure given by Proposition 17.1 (2b). The same convention also applies to coverings of complex manifolds.

In the proof of Proposition 17.1 we will need the following technical lemma.

Lemma 17.2. $(*)$ *Let $p\colon \widetilde{X} \to X$ be a countable covering of a topological space X. If X is second-countable, then $\widetilde{X}$ is also second-countable.*

Proof $(*)$. Let $p\colon \widetilde{X} \to X$ be a countable covering of a topological space X. Recall that we say $U \subset X$ is uniformly covered, if $p^{-1}(U)$ is the union of disjoint open subsets $\{\widetilde{U}_i\}_{i\in I}$ with the property that the restriction of p to each subset $\widetilde{U}_i$ is a homeomorphism. If U is uniformly covered, then every open subset of U is also uniformly covered.

Let $\mathcal{B}$ be a countable basis for the topology of X. We denote by $\mathcal{U} = \{U_j\}_{j\in J}$ the family of all open subsets of X that are uniformly covered. Since p is a covering we have $\bigcup\limits_{j\in J} U_j = X$. It follows from Lemma 2.25 that

$$\mathcal{B}' \ := \ \{B \in \mathcal{B} \mid \ \text{there exists a } j \in J \text{ with } B \subset U_j\}$$

is also a basis for the topology of X.

---

[282] Recall that a covering $p\colon \widetilde{X} \to X$ of a connected topological space is called countable if the degree, i.e. the cardinality of the preimage of a point in X, is countable.

[283] Recall that a map $f\colon M \to N$ between two smooth manifolds is called a *local diffeomorphism* if given any $P \in M$ there exists an open neighborhood U of P and open neighborhood V of $f(P)$ such that $f\colon U \to V$ is a diffeomorphism.

[284] We refer to page 278 for the definition of a smooth structure on a smooth manifold.

Now let $B \in \mathcal{B}'$. Since open subsets of uniformly covered subsets are uniformly covered we know that $p^{-1}(B)$ is the union of disjoint open sets B_i, $i \in I_B$ such that the restriction of p to each B_i is a homeomorphism. Since p is a countable covering the index set I_B is countable.

Claim.

$$\widetilde{\mathcal{B}} \ := \ \{B_i \mid B \in \mathcal{B}' \text{ and } i \in I_B\}$$

is a basis for the topology of $\widetilde{X}$.

It is straightforward to see that $\widetilde{\mathcal{B}}$ has the basis property. Every set in $\widetilde{\mathcal{B}}$ is of course open in $\widetilde{X}$. By Lemma 2.24 it suffices to show that given any open set W of $\widetilde{X}$ and any point $x \in W$ there exists a set B_i in $\widetilde{\mathcal{B}}$ with $x \in B \subset W$. So let $x \in W$. Since $\mathcal{B}'$ is a basis for the topology of X there exists a $B \in \mathcal{B}'$ with $p(x) \in B \subset p(W)$. But then there exists an $i \in I_B$ with $x \in B_i \subset W$. This completes the proof of the claim.

Since $\mathcal{B}'$ is countable and since each I_B is countable we see that $\widetilde{\mathcal{B}}$ consists of countably many open sets. This shows that $\widetilde{X}$ is second-countable. ■

Now we can provide the proof of Proposition 17.1.

Proof of Proposition 17.1 (∗). Let $p\colon \widetilde{X} \to X$ be a countable covering of a connected n-dimensional topological manifold X.

(1) First we note that it follows from the hypothesis that X is a topological manifold that X is Hausdorff. It follows from Lemma 16.3 (3) that $\widetilde{X}$ is also Hausdorff. Similarly it follows from our hypothesis that X is a topological manifold and that p is a countable covering together with Lemma 17.2 that $\widetilde{X}$ is second-countable.

Since $p\colon \widetilde{X} \to X$ is a covering there exists a family of open subsets $\{U_i\}_{i\in I}$ of X such that each U_i is uniformly covered. Furthermore, since X is an n-dimensional smooth manifold there exists an atlas $\{\Phi_j\colon V_j \to W_j\}_{j\in J}$ for X. After possibly replacing each V_j by the family $\{V_j \cap U_i\}_{i\in I}$ we can without loss of generality assume that each V_j is uniformly covered. Thus for each j the preimage $p^{-1}(V_j)$ is the union of disjoint open subsets $\{\widetilde{V}_{ji}\}_{i\in I_j}$ with the property, that the restriction of p to each subset $\widetilde{V}_{ji}$ is a homeomorphism. It is clear that $\{\Phi_j \circ p\colon \widetilde{V}_{ji} \to W_j \mid j \in J \text{ and } i \in I_j\}$ is an atlas for $\widetilde{X}$.

Finally we intend to determine the boundary of $\widetilde{X}$. Let $y \in \widetilde{X}$. Using the fact that $p\colon \widetilde{X} \to X$ is a local homeomorphism one can easily show that the point $y \in \widetilde{X}$ does not admit a chart of type (i) in the sense of page 261 if and only if $p(y) \in X$ does not admit such a chart. This implies that $\partial\widetilde{X} = p^{-1}(\partial X)$.

(2) (a) The transition maps of the atlas from (1) are precisely given by restrictions to open subsets of the transition maps of the atlas $\{\Phi_j\colon V_j \to W_j\}_{j\in J}$ for X. Thus if we start out with a smooth atlas for X, then the atlas we just constructed for $\widetilde{X}$ is also smooth.

We also know that a covering map is a local homeomorphism. It follows immediately from the definitions that with this smooth atlas for $\widetilde{X}$ these homeomorphisms are in fact diffeomorphisms.

(b) This statement follows easily from (a), together with Lemma 6.23. We leave it to the reader to fill in the details.

(3) This statement follows easily from the fact that p is a local diffeomorphism.

(4) The proof of this statement is almost identical to the proof of (2). ■

17.2. **The orientation covering of a non-orientable smooth manifold.** We saw that many of our examples of non-orientable smooth manifolds admit a 2-fold covering such that the covering space is an orientable smooth manifold.

(1) For example the projection map $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/\{\pm 1\}$ shows that the real projective plane is covered by the sphere.
(2) On page 496 we had considered the map

$$\begin{array}{rcl} p\colon \overbrace{\mathbb{R}\times[-1,1]/(x,y)\sim(x+1,y)}^{\text{annulus}} & \to & \overbrace{\mathbb{R}\times[-1,1]/(x,y)\sim(x+\tfrac{1}{2},1-y)}^{\text{Möbius band}} \\ [(x,y)] & \to & [(x,y)] \end{array}$$

and we remarked this is a 2-fold covering. Thus the non-orientable Möbius band admits a 2-fold covering such that the covering space is orientable.
(3) Almost the same map as in (2) also shows that the torus is a 2-fold covering of the Klein bottle.

Now we will see that any non-orientable smooth manifold admits a unique 2-fold covering such that the covering space is orientable.

Proposition 17.3.

(1) *To each smooth manifold M we can canonically*[285] *associate an oriented smooth manifold $\widetilde{M}$ together with a 2-fold covering $p\colon \widetilde{M} \to M$ and to each local diffeomorphism $f\colon M \to N$ we can associate an orientation-preserving local diffeomorphism $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$ such that the following diagram commutes:*[286]

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{N} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{f} & N. \end{array}$$

(2) *Let M be a connected smooth manifold and let $p\colon \widetilde{M} \to M$ be the 2-fold covering constructed in* (1).
   (a) *If M is orientable, then there exists an orientation-preserving diffeomorphism to $\Theta\colon \widetilde{M} \to M \sqcup -M$ such that the following diagram commutes:*

$$\begin{tikzcd} \widetilde{M} \arrow[rr, "\Theta", "\cong"'] \arrow[dr, "p"'] & & M \sqcup -M \arrow[dl, "q"] \\ & M & \end{tikzcd}$$

   where q is the obvious map given by the identity on each component of $M \sqcup -M$.
   (b) *If M is non-orientable, then $\widetilde{M}$ is connected.*
   (c) *In either case, the oriented smooth manifold $\widetilde{M}$ admits an orientation-reversing self-diffeomorphism $\Xi_M\colon \widetilde{M} \to \widetilde{M}$ such that the following diagram commutes:*

$$\begin{tikzcd} \widetilde{M} \arrow[rr, "\Xi_M", "\cong"'] \arrow[dr, "p"'] & & \widetilde{M} \arrow[dl, "p"] \\ & M. & \end{tikzcd}$$

   Furthermore, given an orientation-preserving local diffeomorphism $f\colon M \to N$ we have $\widetilde{f} \circ \Xi_M = \Xi_N \circ \widetilde{f}$.

(3) *Let M be a connected non-orientable smooth manifold. Suppose that $p\colon \widetilde{M} \to M$ and $q\colon \widehat{M} \to M$ are two connected 2-fold coverings such that $\widetilde{M}$ and $\widehat{M}$ are orientable. Then there exists a diffeomorphism $\Phi\colon \widetilde{M} \to \widehat{M}$ such that the following diagram commutes:*

$$\begin{tikzcd} \widetilde{M} \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & \widehat{M} \arrow[dl, "q"] \\ & M. & \end{tikzcd}$$

If $\widetilde{M}$ and $\widehat{M}$ are oriented, then we can arrange that Φ is orientation-preserving.

Remark. In Proposition 86.15 we will prove an analogue of Proposition 17.3 for topological manifolds.

Definition. Let M be a connected non-orientable smooth manifold. We refer to the covering $p\colon \widetilde{M} \to M$ from Proposition 17.3 (1) as the *orientation covering of M*. By Proposition 17.3 (2) this covering is unique in an appropriate sense, which justifies speaking of *the* orientation covering.

PROOF. We start out with a general construction. Let W be an n-dimensional smooth manifold. We consider the set

$$\widetilde{W} \;=\; \{(Q, O) \mid Q \in W \text{ and } O \text{ is an orientation of } \mathrm{T}_Q W\}$$

[285]Here canonically means that we give an explicit construction in the proof, but we do not to burden the reader with the explicit construction at this very moment.

[286]We leave it to the reader to express this statement in categorical language.

together with the map

$$\begin{array}{rcl} p = p_W\colon \widetilde{W} & \to & W \\ (Q,O) & \mapsto & Q. \end{array}$$

Now we have to put a topology on $\widetilde{W}$. We use the following notation:

(i) We denote by $O_{\mathbb{R}^n}$ the standard orientation of $\mathbb{R}^n$.
(ii) Given an orientation O for an n-dimensional vector space we denote by $-O$ the unique other orientation.
(iii) Given an isomorphism $\varphi\colon V \to W$ between vector spaces and given an orientation O on W we denote by φ^*O the corresponding orientation on V.

Given a chart $\Phi\colon U \to V$ of W and $\epsilon \in \{-,+\}$ we define[287]

$$B_\epsilon(\Phi) \; := \; \{(Q, \epsilon \cdot \Phi^* O_{\mathbb{R}^n}) \mid Q \in U\} \; \subset \; \widetilde{W}.$$

It is straightforward to show that the collection of these sets $B_\pm(\Phi)$ satisfies the basis properties (B1) and (B2) from page 102.[288] Now we endow $\widetilde{W}$ with the topology generated by the sets $B_\pm(\Phi)$. We leave it to the reader to show that $p\colon \widetilde{W} \to W$ is continuous. Next we show that p is a 2-fold covering. So let $Q \in W$. We pick a connected chart[289] $\Phi\colon U \to V$ around Q. Then $p^{-1}(U) = B_-(\Phi) \cup B_+(\Phi)$. It follows from Lemma 6.46 (1) applied to the connected smooth manifold U that $B_-(\Phi)$ and $B_+(\Phi)$ are disjoint. Furthermore it is straightforward to see that the two projections maps $p\colon B_\pm(\Phi) \to U$ are homeomorphisms. This shows that U is uniformly covered by two sets, i.e. p is a 2-fold covering.

In the proof of Proposition 17.1 we saw that a smooth atlas for $\widetilde{W}$ is given by all maps $B_\pm(\Phi) \xrightarrow{p} U \xrightarrow{\Phi} V$ for all connected charts $\Phi\colon U \to V$ of W. In particular $\widetilde{W}$ is again an n-dimensional smooth manifold and with this atlas the projection map $p\colon \widetilde{W} \to W$ is a local diffeomorphism.

Now we show that $\widetilde{W}$ is orientable, in fact we will equip $\widetilde{W}$ with a canonical orientation. So let $(Q,O) \in \widetilde{W}$. The map $p\colon \widetilde{W} \to W$ is a local diffeomorphism. Now we equip $\mathrm{T}_{(Q,O)}\widetilde{W}$ with the orientation p^*O.[290] By Footnote 114 it suffices to consider the continuity of the orientation with respect to the above atlas. So let $\Phi\colon U \to V$ be a connected chart of W. Let $\epsilon \in \{-,+\}$ and denote by $\Phi_\epsilon\colon B_\pm(\Phi) \to V$ the corresponding chart of $\widetilde{W}$. But then by construction we know that Φ_ϵ is orientation-preserving for all points if $\epsilon = +$ and it is orientation-reversing for all points if $\epsilon = -$.

Finally we turn to the actual proof of the proposition.

(1) Let M be a smooth manifold. We define $\widetilde{M}$ as above. It follows from the above discussion that the smooth manifold $\widetilde{M}$ is an oriented smooth manifold and that $p_M\colon \widetilde{M} \to M$ is a 2-fold covering. Furthermore, to a local diffeomorphism $f\colon M \to N$

[287]Strictly speaking we should write $(\mathrm{D}\,\Phi_P)^* O_{\mathbb{R}^n}$ instead of just $\Phi^* O_{\mathbb{R}^n}$.
[288]OK, so why does it satisfy (B2)?
[289]Here we say a chart $\Phi\colon U \to V$ is connected if U is connected.
[290]Put differently, using p we can identify $\mathrm{T}_{(Q,O)}\widetilde{W}$ with $\mathrm{T}_Q W$ and we equip it with the orientation at hand, namely O.

between smooth manifolds we assign the following commutative diagram

$$\begin{tikzcd} \widetilde{M} \arrow[r, "(Q,O)\mapsto(f(Q),f_*(O))", "=:\widetilde{f}"'] \arrow[d, "p"'] & \widetilde{N} \arrow[d, "p"] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

It follows easily from the definitions that the top map $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$ is an orientation-preserving local diffeomorphism.

(2) Let M be a connected smooth manifold.

(a) If M is orientable, then there exists an obvious orientation-preserving diffeomorphism $\Theta\colon \widetilde{M} \to M \sqcup -M$ that has the desired properties.

(b) Now suppose that M is non-orientable. We need to show that $\widetilde{M}$ is connected. If it was not connected, then it would consist of at least two components $\{\widetilde{M}_i\}_{i\in I}$. Since M is connected the restriction of each $p\colon \widetilde{M} \to M$ to $p_i\colon \widetilde{M}_i \to M$ would also be a covering. Clearly $2 = [\widetilde{M} : M] = \sum\limits_{i\in I}[\widetilde{M}_i : M]$. It follows that $\#I = 2$ and that each $p_i\colon \widetilde{M}_i \to M$ is a covering of degree one, i.e. each p_i is a homeomorphism. Since p_i is a local diffeomorphism it follows that $p_i\colon \widetilde{M}_i \to M$ is in fact a diffeomorphism. But that is not possible since M was assumed to be non-orientable whereas $\widetilde{M}_i$ is orientable.

(c) It is pretty straightforward to verify that the map $\Xi_M\colon \widetilde{M} \to \widetilde{M}$ that is given by $(Q,O) \mapsto (Q,-O)$ has all the desired properties.

(3) Let M be a connected non-orientable smooth manifold. Suppose we are given two connected 2-fold coverings of M by oriented smooth manifolds. We can assume that one of the two coverings is the covering constructed in (1). Let $q\colon N \to M$ be some other connected 2-fold covering such that N is orientable. It follows easily from the fact that q is a local diffeomorphism and the discussion in (1) that there exists a 2-fold covering $\widetilde{q}\colon \widetilde{N} \to \widetilde{M}$ which makes the following diagram commute

$$\begin{tikzcd} \widetilde{N} \arrow[dr, "\widetilde{q}"] \arrow[d, "p_N"'] & \\ N \arrow[dr, "q"'] & \widetilde{M} \arrow[d, "p_M"] \\ & M. \end{tikzcd}$$

From (2) and the fact that N is connected and orientable we obtain that $\widetilde{N}$ consists of two components $\widetilde{N}_1$ and $\widetilde{N}_2$. The restriction of p_N to $\widetilde{N}_1$ and the restriction of $\widetilde{q}$ to $\widetilde{N}_1$ are 1-fold coverings, hence diffeomorphisms. It is clear that $\widetilde{q}\circ(p_N|_{\widetilde{N}_1})^{-1}\colon N \to \widetilde{M}$ has the desired properties. If N is oriented, then one can easily arrange that the map $N \to \widetilde{M}$ is orientation-preserving. ■

Remark. A similar proof of Proposition 17.3 (1) with slightly more detail is also given in [**Lee09**, Chapter 8.7].

Corollary 17.4.

(1) *If M is a connected non-orientable smooth manifold, then there exists an epimorphism $\pi_1(M) \to \mathbb{Z}_2$.*
(2) *Every simply connected smooth manifold is orientable.*

PROOF. The second statement of the corollary is an immediate consequence of the first statement. Thus it suffices to prove the first statement.

Let M be a connected non-orientable smooth manifold. By Proposition 17.3 there exists a connected 2-fold covering $p\colon \widetilde{M} \to M$ such that $\widetilde{M}$ is orientable. It follows from Lemma 16.15 that $p_*(\pi_1(\widetilde{M}))$ is an index 2 subgroup of $\pi_1(M)$. In Exercise 17.1 we will show that every index 2 subgroup of a group π is normal. In particular $p_*(\pi_1(\widetilde{M}))$ is a normal index 2 subgroup of $\pi_1(M)$. Note that this implies that we obtain an epimorphism $\pi_1(M) \to \Gamma := \pi_1(M)/p_*(\pi_1(\widetilde{M}))$. The group Γ has two elements, but any group with two elements is isomorphic to $\mathbb{Z}_2$. Thus we found the desired epimorphism $\pi_1(M) \to \mathbb{Z}_2$. ■

We conclude this section with a discussion of orientation-preserving and reversing loops.

Definition. Let M be a connected non-orientable smooth manifold, let $x_0 \in M$ be a point and let $\gamma\colon [0,1] \to M$ be a loop in x_0. We denote by $p\colon \widetilde{M} \to M$ the orientation covering provided by Proposition 17.3. We pick $\widetilde{x}_0 \in p^{-1}(x_0)$ and we denote by $\widetilde{g}\colon [0,1] \to \widetilde{M}$ the unique lift of γ, provided by Proposition 16.11, to the starting point x_0. We say γ is *orientation-preserving* if $\widetilde{\gamma}(1) = \widetilde{\gamma}(0)$, otherwise we say that γ is *orientation-reversing*.[291]

Example. Let M be the Möbius band and let $\widetilde{M}$ be the annulus. As on page 496 we consider the 2-fold covering $p\colon \widetilde{M} \to M$ which is just an incarnation of the orientation covering. We pick a point in M and we denote by γ a loop that goes "once around M" The lift of γ is not a loop, hence γ is orientation-reversing. We refer to Figure 334 for an illustration.

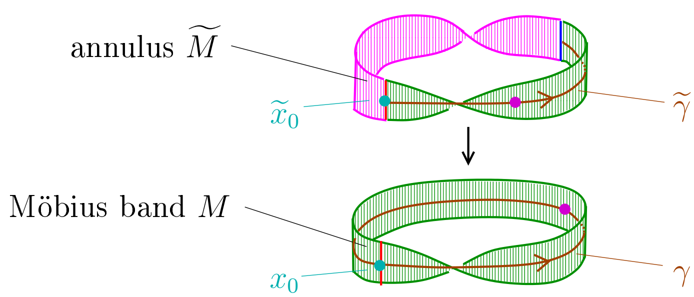


FIGURE 334

We conclude this chapter with the following lemma which gives an alternative proof to Corollary 17.4 (1).

[291] It follows easily from Lemma 16.4 that this definition does not depend on the choice of $\widetilde{x}_0$.

Lemma 17.5. *Let M be a connected non-orientable smooth manifold and let $x_0 \in M$ be a point. The map*

$$\begin{array}{rcl} \Phi\colon \pi_1(M,x_0) & \to & \{\pm 1\} \\ [\gamma\colon [0,1]\to M] & \mapsto & \begin{cases} 1, & \text{if } \gamma \text{ is orientation-preserving,} \\ -1, & \text{if } \gamma \text{ is orientation-reversing} \end{cases} \end{array}$$

is well-defined and it is an epimorphism.

PROOF. We denote by $p\colon \widetilde{M} \to M$ the orientation covering. The following three easy observations imply the lemma.

(1) It follows immediately from Corollary 16.13 that Φ is well-defined.
(2) Using Lemma 16.4 one easily verifies that Φ is a homomorphism.
(3) We denote by $\widetilde{x}_0'$ the other point in $p^{-1}(x_0)$. By Proposition 17.3 there exists a path $\delta\colon [0,1] \to \widetilde{M}$ from $\widetilde{x}_0$ to $\widetilde{x}_0'$. It follows basically immediately from the definitions that $\Phi([p\circ\delta]) = -1$. This shows that Φ is an epimorphism. ■

Exercises for Chapter 17.

Exercise 17.1. Let π be a group. Show that every index 2 subgroup of π is normal.

Exercise 17.2. Let M be a connected non-orientable smooth manifold and let $p\colon \widetilde{M} \to M$ be the orientable 2-fold covering from Proposition 17.3. Show that

$$\begin{array}{rcl} \#\,\text{boundary components of } \widetilde{M} & = & \#\,\text{non-orientable boundary components of } M \\ & & +2\cdot\#\,\text{orientable boundary components of } M. \end{array}$$

Exercise 17.3. Let $f\colon S^1 \to S^1$ be a map with $f(1) = 1$. Let $p\colon \mathbb{R} \to S^1$ be the covering that is given by $t \mapsto e^{it}$.

(a) Show that there exists a unique map $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \mathbb{R} \arrow[d, "p"] \\ S^1 \arrow[r, "f"] & S^1. \end{tikzcd}$$

(b) Suppose that f is a diffeomorphism. Show that the map $\widetilde{f}$ from (a) is also a diffeomorphism.

Exercise 17.4.

(a) We consider the following commutative diagram:

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \widetilde{N} \arrow[d, "q"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

where $\widetilde{M}$, M, $\widetilde{N}$ and N are smooth manifolds and where p and q are covering maps of the same finite degree. Show that f is a diffeomorphism if and only if $\widetilde{f}$ is a diffeomorphism.

(b) Let M and N be smooth manifolds and let $f\colon M \to N$ be a diffeomorphism. We denote by $p\colon \widetilde{M} \to M$ and $q\colon \widetilde{M} \to M$ the orientation coverings of M and N. Show that there exists an orientation-preserving diffeomorphism $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$ such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \widetilde{N} \arrow[d, "q"] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

Exercise 17.5. Let $p\colon T \to K$ be the 2-fold covering of the Klein bottle by the torus that we discussed on page 497. Give an example of a smooth map $f\colon K \to K$ such that there is no map $\widetilde{f}\colon T \to T$ which makes the following diagram commute:

$$\begin{tikzcd} T \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & T \arrow[d, "p"] \\ K \arrow[r, "f"] & K. \end{tikzcd}$$

Exercise 17.6. Let M be an oriented n-dimensional smooth manifold, let N be a non-orientable connected n-dimensional smooth manifold and let $f\colon M \to N$ be a smooth embedding. We denote by $p\colon \widetilde{N} \to N$ the orientation-covering from Proposition 17.3. Show that there exists an orientation-preserving smooth embedding $g\colon M \to \widetilde{N}$ with $p \circ g = f$.

18. Homotopies and homotopy equivalent topological spaces

In this chapter we will see that "homotopy equivalent" topological spaces have isomorphic fundamental groups. In some instances this fact will allow us to reduce the calculation of fundamental groups to some of our earlier calculations.

The definition of being "homotopy equivalent" is slightly subtle and rests on the notion of two maps being homotopic.

18.1. **Homotopic maps.**

In its most basic form a homotopy is a continuous deformation of maps between two topological spaces. To make this notion precise it is convenient to introduce the following notation.

Notation. Let X and Y be two topological spaces and let $a < b$ be two real numbers. Given a map $F\colon X \times [a, b] \to Y$[292] and given $t \in [a, b]$ we sometimes denote by F_t the map[293]

$$\begin{array}{rcl} F_t\colon X & \to & Y \\ x & \mapsto & F(x, t). \end{array}$$

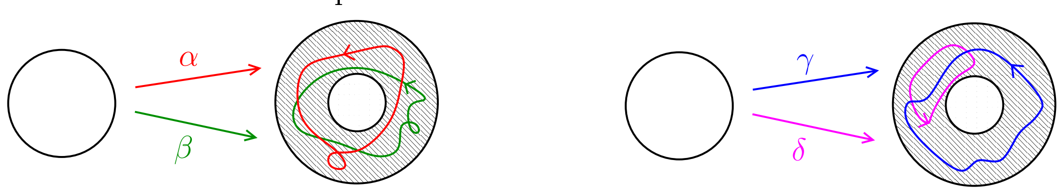
Now we move on to the actual definition we are interested in.

Definition. Let X and Y be two topological spaces.

(1) A *homotopy from X to Y* is a map $F\colon X \times [0, 1] \to Y$.
(2) Let $f_0, f_1\colon X \to Y$ be two maps. A *homotopy between the maps f_0 and f_1* is a homotopy $F\colon X \times [0, 1] \to Y$ with $F_0 = f_0$ and $F_1 = f_1$. If there exists a homotopy between the maps f_0 and f_1, then we say that f_0 and f_1 are *homotopic* and sometimes we write $f_0 \simeq f_1$.

Examples.

(1) In Figure 335 we show four maps α, β, γ and δ from the circle S^1 to the annulus $Y = B_2(0) \setminus B_1(0)$. The maps α and β are homotopic whereas the maps γ and δ look as if they are not homotopic. We will develop methods for showing that these maps are indeed not homotopic.



the maps α and β are homotopic   the maps γ and δ appear not to be homotopic

Figure 335

[292] Recall that all maps between topological spaces are understood to be continuous.
[293] Note that the map F_t is indeed continuous.

(2) Given an orthogonal matrix $A \in \mathrm{O}(n)$ we denote by $\rho(A)$ the map $f\colon S^{n-1} \to S^{n-1}$ that is given $x \mapsto A \cdot x$.

(a) If $\gamma\colon [0,1] \to \mathrm{SO}(n)$ is a path from some matrix A to some matrix B, then the map

$$\begin{array}{rcl} S^{n-1} \times [0,1] & \to & S^{n-1} \\ (x,t) & \mapsto & \gamma(t) \cdot x \end{array}$$

is a homotopy from $\rho(A)$ to $\rho(B)$.

(b) Let $A \in \mathrm{O}(n)$. If $\det(A) = 1$, i.e. if $A \in \mathrm{SO}(n)$, then it follows from Lemma 2.65 (2) that there exists a path $\gamma\colon [0,1] \to \mathrm{SO}(n)$ with $\gamma(0) = A$ and $\gamma(1) = \mathrm{id}$. It follows from (a) that $\rho(A)$ is homotopic to the identity. Much later, on page 1182 we will see that if $\det(A) = -1$, then f is actually not homotopic to the identity.

The next example is written down as a lemma.

Lemma 18.1. *Let X be a topological space and let $Y \subset \mathbb{R}^n$ be a subset. Furthermore let $f_0, f_1\colon X \to Y$ be two maps. If Y is convex, then*

$$\begin{array}{rcl} F\colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & \underbrace{f_0(x) \cdot (1-t) + f_1(x) \cdot t}_{\in Y \text{ since } Y \text{ is convex}} \end{array}$$

is a homotopy between f_0 and f_1.

Proof ($*$). Basically by definition we have $F_0 = f_0$ and $F_1 = f_1$. But we are not done yet, the conscientious reader will not have failed to notice that we need to show that the map F is actually continuous. The map F can be written as the composition of the following two maps:

$$X \times [0,1] \xrightarrow{(x,t)\mapsto(f_0(x),f_1(x),t)} \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \xrightarrow{(x,y,t)\mapsto(x\cdot(1-t)+y\cdot t)} \mathbb{R}^n.$$

The first map is continuous by Lemma 3.8 (3) and the second map is continuous by basic results of real analysis. ■

Remark. Let $\gamma_0, \gamma_1\colon [a,b] \to Y$ be two maps from an interval to a topological space Y. We just said that the two maps are homotopic if there exists a map $\Gamma\colon [a,b] \times [0,1] \to Y$ such that

(i) $\Gamma(t,0) = \gamma_0(t)$ and $\Gamma(t,1) = \gamma_1(t)$ for all $t \in [a,b]$.

We can also view these two maps as paths. If they have the same same starting point $\gamma_0(a) = \gamma_1(a) =: P$ and the same endpoint $\gamma_0(b) = \gamma_1(b) =: Q$, then we said on page 457 that the paths are path-homotopic if there exists a map $\Gamma\colon [a,b] \times [0,1] \to Y$ that satisfies (i) and that satisfies also

(ii) $\Gamma(a,s) = P$ and $\Gamma(b,s) = Q$ for all $s \in [0,1]$.

In the literature sometimes “path-homotopic” is also called “homotopic”. To distinguish the two notions of homotopies what we call “homotopic” is sometimes called “freely homotopic”. Clearly two path-homotopic paths are also homotopic. But in general the converse does not hold. For example, consider the two paths

$$\begin{array}{rcl} \alpha\colon [0,1] & \to & Y = \mathbb{C}\setminus\{0\} \\ x & \mapsto & e^{2\pi i x} \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon [0,1] & \to & Y = \mathbb{C}\setminus\{0\} \\ x & \mapsto & 1. \end{array}$$

Here α goes counterclockwise once around the origin and β is the constant path. We saw in Corollary 13.2 that α and β are not path-homotopic. On the other hand they are homotopic. Indeed a homotopy is given by

$$\begin{array}{rcl} F\colon [0,1]\times[0,1] & \to & Y = \mathbb{C}\setminus\{0\} \\ (x,t) & \mapsto & e^{2\pi i x(1-t)}. \end{array}$$

This free is illustrated in Figure 336.

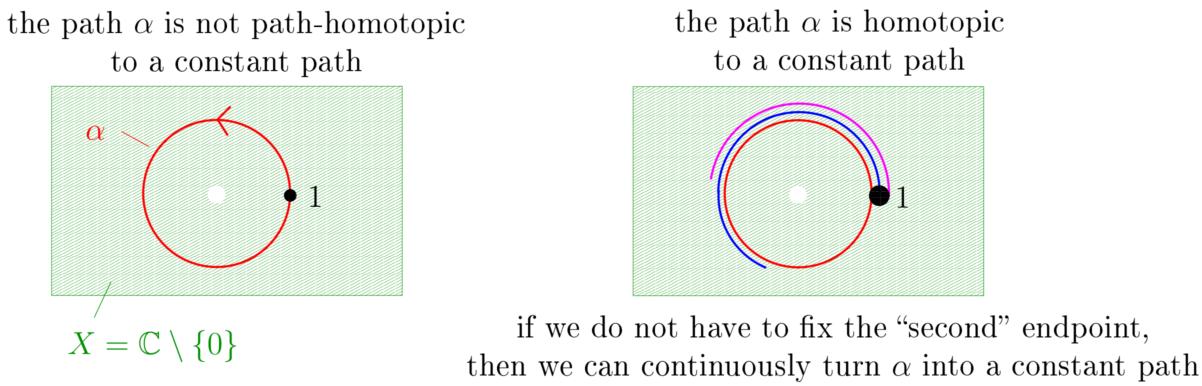


FIGURE 336

The difference between a path-homotopy and a homotopy of paths is that in the former context we want to "keep the endpoints fixed". Sometimes when we study other maps between topological spaces we also might want to keep a suitable subset fixed under a homotopy. This leads us to the following definition.

Definition. Let X and Y be topological spaces and let $A \subset X$ be a subset.

(1) We say a map $F\colon X\times[0,1]\to Y$ is a *homotopy rel A* if for all $x\in A$ and $s,t\in[0,1]$ we have $F(x,s)=F(x,t)$. Sometimes we also say that such a homotopy is *stationary on A*.
(2) We say that two maps $f,g\colon X\to Y$ with $f|_A = g|_A$ are *homotopic rel A* if there exists a homotopy F rel A with $F_0 = f$ and $F_1 = g$.

Examples.

(1) By definition two paths $\alpha,\beta\colon[0,1]\to X$ in a topological space X are path-homotopic if and only if the two maps are homotopic rel $\{0,1\}$.
(2) We consider the two maps $f,g\colon \overline{B}^2 \to S^2$ that are shown in Figure 337. These maps agree on S^1. It is fairly easy to see that the two maps are homotopic but it seems rather unlikely that they are homotopic rel S^1. Unfortunately right now we lack the tools for showing the latter statement.

We continue with the next variation on the notion of homotopy.

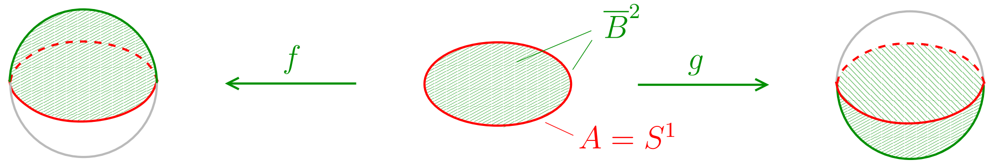


FIGURE 337

Definition. Let X and Y be two topological spaces.

(1) An *isotopy* is a homotopy $F: X \times [0,1] \to Y$ such that for each $t \in [0,1]$ the map $F_t: X \to Y$ is an embedding. A *homeotopy* is a homotopy $F: X \times [0,1] \to Y$ such that for each $t \in [0,1]$ the map $F_t: X \to Y$ is a homeomorphism.

(2) Let $f_0, f_1: X \to Y$ be two maps that are embeddings. We say f_0 and f_1 are *isotopic* if there exists an isotopy $F: X \times [0,1] \to Y$ with $F_0 = f_0$ and $F_1 = f_1$.

Examples.

(1) In Figure 338 we show two maps $f_0, f_1: [0,1] \to \mathbb{R}^2$ that are embeddings. We also indicate two homotopies F and G between f_0 and f_1. The map F is an isotopy, the map G is not.

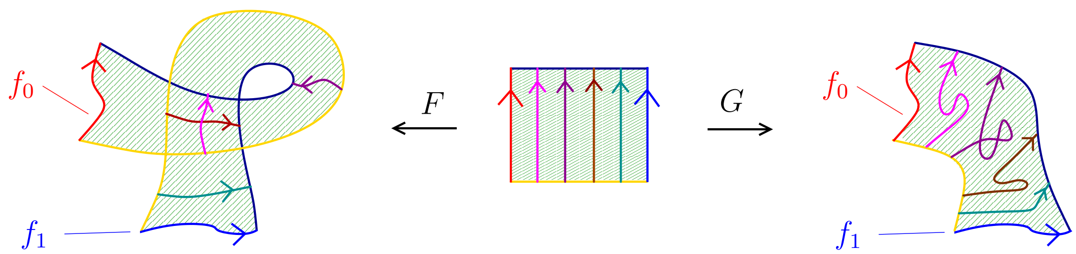


FIGURE 338

(2) Let $n \in \mathbb{N}$. Let $f: \overline{B}^n \to \overline{B}^n$ be a homeomorphism. If $f|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$, then f is isotopic to $\mathrm{id}_{\overline{B}^n}$. Indeed, in Exercise 18.3 we will see that the map

$$\begin{array}{rcl} H: \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \begin{cases} t \cdot f(\frac{x}{t}), & \text{if } 0 \le \|x\| < t, \\ x, & \text{otherwise} \end{cases} \end{array}$$

is continuous and that each H_t is a homeomorphism. It is clear that $H_0 = \mathrm{id}$ and $H_1 = f$. Thus we see that H is the desired isotopy. The fact that f is isotopic to $\mathrm{id}_{\overline{B}^n}$ is often referred to as the *Alexander trick*.[294] The definition of the homotopy H is illustrated in Figure 339.

Now we can formulate the following lemma that is a refined version of Proposition 2.52 (2).

[294] The existence of such a homotopy was first observed by a James Alexander in 1923, see [**Al24**].

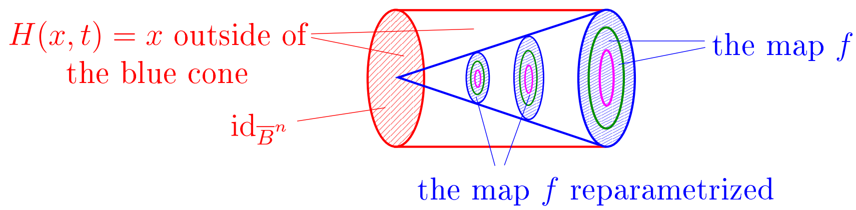


FIGURE 339

Lemma 18.2. $(*)$ *Let A be a bounded closed convex subset of $\mathbb{R}^n$ with non-empty interior. Let Q be a point in the interior of A. There exists a canonical isotopy $F\colon A \times [0,1] \to \mathbb{R}^n$ with the following three properties:*

1. *we have $F_0 = \mathrm{id}_A$,*
2. *the map F_1 is a homeomorphism from A to $\overline{B}^n$ and this map restricts to a homeomorphism $\partial A \to S^{n-1}$,*
3. *for any $t \in [0,1]$ we have $F(Q,t) = (1-t)\cdot Q$.*[295]

PROOF $(*)$. First consider the case that Q is the origin. In the following we use the notation that we introduced in the proof of Proposition 2.53 (2). Similar to the discussion on page 128 we consider the map

$$\begin{array}{rcl} F\colon A \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \begin{cases} x \cdot \frac{1}{(1-t)+t\cdot\rho(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{array}$$

It is straightforward to verify that this map has all the desired properties.

We leave it to the reader to modify the above proof to deal with the case that Q is not the origin. ■

Sometimes it is useful to combine two homotopies or isotopies. This leads us to the following definition.

Lemma 18.3. $(*)$ *Let X and Y be topological spaces and let $A \subset X$ be a (possibly empty) subset of X. Let $G, H\colon X \times [0,1] \to Y$ be two homotopies rel A. If $G_1 = H_0$, then the map*

$$\begin{array}{rcl} G * H\colon X \times [0,1] & \to & Y \\ (P,t) & \mapsto & \begin{cases} G(P,2t), & \text{if } t \in [0,\frac{1}{2}], \\ H(P,2t-1), & \text{if } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

is a homotopy rel A from G_0 to H_1. Furthermore, if G and H are isotopies, then $G * H$ is also an isotopy. We refer to $G * H$ as the combination of G and H.

PROOF $(*)$. We only need to verify that $G * H$ is indeed continuous, but that is a consequence of Lemma 14.3 together with our hypothesis that $G_1 = H_0$. ■

[295] In particular, if Q is already the origin, then F is a homotopy rel the origin.

In Proposition 14.2 we showed that being path-homotopic is an equivalence relation on the set of paths between two points. The first part of the following lemma is the obvious generalization to "homotopy of maps".

Lemma 18.4.

(1) *Let X be and Y topological spaces and let A be a (possibly empty) subset of X. Then "homotopy rel A of maps" forms an equivalence relation on the set of maps from X to Y.*

(2) *Let $f, g\colon X \to Y$, let $\alpha\colon W \to X$ and $\beta\colon Y \to Z$ be maps between topological spaces. The following conclusion holds:*

$$f \simeq g \quad \Longrightarrow \quad f \circ \alpha \sim g \circ \alpha \quad \text{and} \quad \beta \circ f \sim \beta \circ g.$$

Proof.

(1) Not surprisingly the proof of Lemma 18.4 is basically identical to the earlier proof of Proposition 14.2. The only change is that we need to use Lemma 18.3 to show that the combination of two homotopies is indeed a homotopy, i.e. continuous.
(2) This statement is basically obvious, we just need to precompose and postcompose a homotopy H between f and g by α respectively β. ■

We continue with a rather classical type of an example.

Definition. Let $n \in \mathbb{N}$ and let $V \subset \mathbb{R}^n$ be a hyperplane, i.e. V is a vector subspace of codimension one. We pick a vector v of unit length orthogonal to V. We refer to the map

$$\begin{aligned} \rho_V\colon \mathbb{R}^n &\to \mathbb{R}^n \\ w &\mapsto w - 2 \cdot \langle v, w\rangle \cdot v \end{aligned}$$

as the *reflection in the hyperplane V*. We refer to Figure 340 for an illustration.

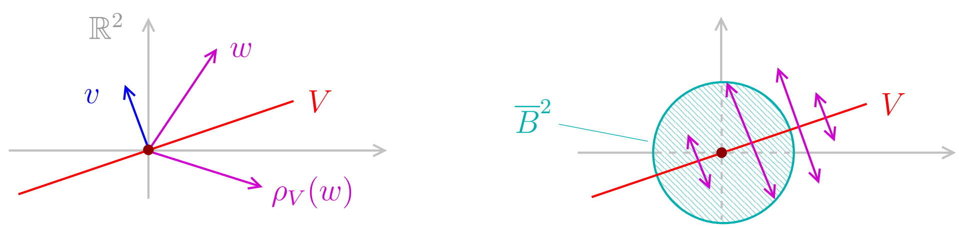


Figure 340

Lemma 18.5. *Let $n \in \mathbb{N}$.*

(1) *Any reflection in a hyperplane of $\mathbb{R}^n$ has the following properties:*
   (a) *it is a diffeomorphism,*
   (b) *it is an isometry (i.e. it preserves the Euclidean distance) and preserves the origin,*
   (c) *it restricts to diffeomorphisms of $\overline{B}^n$ and S^{n-1}.*
(2) *Any two reflections in hyperplanes of $\mathbb{R}^n$ are in fact related by a smooth isotopy*[296] *$F\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ such that each F_t is also a reflection. In particular the isotopies are isotopies rel the origin.*

PROOF. Let $n \in \mathbb{N}$.

(1) Let ρ be a reflection in a hyperplane of $\mathbb{R}^n$. Clearly ρ is smooth and it satisfies $\rho \circ \rho = \mathrm{id}_{\mathbb{R}^n}$. This implies immediately that ρ is a diffeomorphism. A short amusing calculation shows that ρ preserves the scalar product, hence length. Since $\rho(0) = 0$ we see that ρ restricts to diffeomorphisms of $\overline{B}^n$ and S^{n-1}.

(2) Suppose we are given two hyperplanes V and W of $\mathbb{R}^n$. By Lemma 3.32 there exists (5) an orthogonal matrix $A \in \mathrm{SO}(n)$ with $A \cdot V = W$. By Lemma 2.65 (2) together with Exercise 18.7 we know that there exists a smooth path $\gamma\colon [0,1] \to \mathrm{O}(n)$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = A$. One can now easily verify that the map

$$\begin{array}{rcl} \mathbb{R}^n \times [0,1] & \to & \mathbb{R}^n \\ (P,t) & \mapsto & \rho_{\gamma(t)\cdot V}(P) \end{array}$$

is the desired smooth isotopy. ■

We return to more theoretical material.

Lemma 18.6. *Let X, Y and Z be topological spaces. Let $f, f'\colon X \to Y$ and $g, g'\colon Y \to Z$ be maps.*

(1) *If f and f' are homotopic and if g and g' are homotopic, then the map $g \circ f$ is homotopic to $g' \circ f'$.*

(2) *If there exists a subset A of X such that f and f' are homotopic rel A and if there exists a subset B of Y such that $f(A) = f'(A) \subset B$ and such that g and g' are homotopic rel B, then $g \circ f$ is homotopic to $g' \circ f'$ rel A.*

PROOF.

(1) Let $F\colon X \times [0,1] \to Y$ be a homotopy between f and f' and let $G\colon Y \times [0,1] \to Z$ be a homotopy between g and g'. We consider the map

$$\begin{array}{rcl} H\colon X \times [0,1] & \to & Z \\ (x,t) & \mapsto & G(\underbrace{F(x,t)}_{\in Y}, t). \end{array}$$

This map is a homotopy between $g \circ f$ and $g' \circ f'$.

(2) One can easily verify that the homotopy in (1) also does the trick for (2). ■

Definition. Given two topological spaces X and Y we define

$$[X, Y] := \{\text{maps } X \to Y\}/\sim \quad \text{where } f \sim g \text{ if there exists a homotopy between } f \text{ and } g.$$

Sometimes, given a map $f\colon X \to Y$ we denote by $[f]$ the equivalence class it represents in $[X, Y]$.

Example. We write $I = [0,1]$. In Exercise 18.9 we will see that for any path-connected topological space X the set $[I, X]$ consists of a single element, i.e. all maps $I = [0,1] \to X$ are homotopic.

[296]Technically speaking we did not introduce the notion of a smooth isotopy yet, we will do so on page 348, but presumably that does not pose a problem to the reader.

The following lemma is an immediate consequence of Lemmas 18.11 and 18.6.

Lemma 18.7. *Let X, Y and Z be topological spaces. The map*

$$\begin{array}{rcl} [X,Y]\times[Y,Z] & \to & [X,Z] \\ ([f],[g]) & \mapsto & [g\circ f] \end{array}$$

is well-defined.

Lemma 18.7 leads us to the following definition.

Definition. We define the *homotopy category $\mathcal{HomTop}$ of topological spaces* by

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{HomTop}) & := & \text{all topological spaces,} \\ \mathrm{Mor}(X,Y) & := & [X,Y] \end{array}$$

with the composition

$$\begin{array}{rcl} \mathrm{Mor}(X,Y)\times\mathrm{Mor}(Y,Z) & \to & \mathrm{Mor}(X,Z) \\ ([f],[g]) & \mapsto & [g\circ f]. \end{array}$$

Note that in this category the morphisms are not maps but they are equivalence classes of maps.

We conclude this section with the following elementary but useful lemma.

Lemma 18.8.

(1) *Let $f\colon X\to Y$ be a map between topological spaces. The element $[f]\in\mathrm{Mor}(X,Y)$ is an isomorphism*[297] *in the category $\mathcal{HomTop}$ if and only if f is a homotopy equivalence.*
(2) *Let $f\colon X_1\to X_2$ be a map between topological spaces and let A be a topological space. If f is a homotopy equivalence, then the induced maps*

$$\begin{array}{rcl} [X_2,A] & \to & [X_1,A] \\ [g] & \mapsto & [g\circ f] \end{array} \quad \text{and} \quad \begin{array}{rcl} [A,X_1] & \to & [A,X_2] \\ [g] & \mapsto & [f\circ g] \end{array}$$

are bijections.

Proof.

(1) This statement is a tautology, it follows immediately from the definitions.
(2) This statement follows immediately from (1) together with Lemma 15.1. ■

18.2. **Closed oriented curves and fundamental groups.** (∗) The following definition gets used frequently throughout the notes.

Definition. Let M be a smooth manifold. A *closed (oriented) curve in M* is a closed (oriented) connected 1-dimensional submanifold either of ∂M or of M.

Convention. Often we just write "curve" instead of "closed curve".

Remark.

(1) Note that the boundary ∂M a smooth manifold M is *not* a submanifold of M. In particular, if M is a 2-dimensional smooth manifold and C is a closed boundary

[297] We refer to page 478 for the definition of an isomorphism in a category.

component, then C is not a submanifold of M, but it is a closed curve in M in the above sense.

(2) On occasions, when we deal with explicit examples, we will use gentle generalizations of the above notion of a closed oriented curve, which in practice should not cause any confusion.

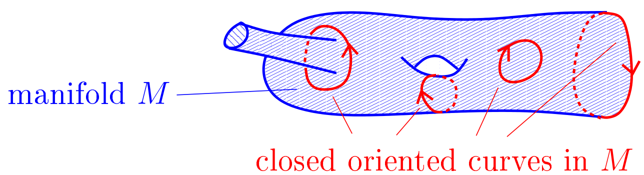


FIGURE 341

Closed (oriented) curves in a given smooth manifold M have the nice feature that they are easy to draw. But strictly speaking they do not represent elements in $\pi_1(M)$ since elements in $\pi_1(M)$ are represented by *maps* $S^1 \to M$ and not just the images of such maps. Thus we need to clarify the connection between curves and homotopy classes of maps.

The key to this issue is the following lemma, which is also of independent interest.

Lemma 18.9.

(1) *Let $f\colon [0,1] \to [0,1]$ be a self-homeomorphism.*[298]
  (a) *If f is monotonously increasing, then f is isotopic to the map $f(x) = x$.*
  (b) *If f is monotonously decreasing, then f is isotopic to the map $f(x) = -x$.*

(2) *Every self-homeomorphism of S^1 is isotopic to the map $f(z) = z$ or to the map $g(z) = \overline{z}$.*

(3) *Let $f\colon S^1 \to S^1$ be a diffeomorphism. If f is orientation-preserving, then f is diffeotopic to the identity, otherwise f is diffeotopic to the reflection $r\colon S^1 \to S^1$ given by $(x, y) \mapsto (x, -y)$.*

PROOF. We will provide the proof of Statements (1) and (2) in Exercise 18.4. The proof of Statement (3) is quite similar to the proof of Statement (2). One just needs to work a little harder since one needs to make sure that a map is actually smooth. Alternatively we refer to the proof that we will give in Proposition 30.1 which cleverly sidesteps these technical issues by working with covering spaces. ■

Corollary 18.10. *Let M be a smooth manifold and let $C \subset M$ be a closed oriented curve in M. Furthermore let $x_0 \in C$. For any two orientation-preserving diffeomorphisms $\alpha, \beta\colon S^1 \to C$ with $\alpha(1) = \beta(1) = x_0$ we have $[\alpha] = [\beta] \in \pi_1(M, x_0)$.*

PROOF. It is fairly straightforward to prove the corollary using Lemmas 18.9 (1) and 14.4. We leave the task of filling in the details to the reader. ■

[298]It is an elementary real analysis exercise to show that every self-homeomorphism $f\colon [0,1] \to [0,1]$ is either monotonously increasing or monotonously decreasing.

Convention. Let M be a smooth manifold and let $C \subset M$ be a closed oriented curve in M. Throughout these notes we use that Corollary 18.10 allows us to view C unambiguously as an element in $\pi_1(M, x_0)$ for any base point $x_0 \in C$.

18.3. **Homotopy equivalent topological spaces.** The following definition is one of the key definitions in topology. Unfortunately it usually takes some time to fully digest it.

Definition. Let X be and Y topological spaces.

(1) A map $f: X \to Y$ is a *homotopy equivalence between X and Y*, if f admits a *homotopy inverse*, i.e. a map $g: Y \to X$ with $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.
(2) If there exists a homotopy equivalence between X and Y, then we say that X and Y are *homotopy equivalent.*
(3) Given a topological space X we refer to the class of all topological spaces that are homotopy equivalent to X as the *homotopy type of X*.

The name "homotopy equivalence" already suggests that this is an equivalence relation on the class of topological spaces. This is indeed the case.

Lemma 18.11.

(1) *Let $f: X \to Y$ and $g: Y \to Z$ be maps between topological spaces. If two out of the three maps f, g and $g \circ f$ are homotopy equivalences, then so is the third.*
(2) *The relation "homotopy equivalence" is an equivalence relation on the class of all topological spaces.*[299]
(3) *Let $f, g: X \to Y$ be two maps between topological spaces. If f is a homotopy equivalence and if g is homotopic to f, then g is also a homotopy equivalence.*
(4) *Let X, Y and Z be topological spaces.*
   (a) *Let $f: X \to Y$ and let $g_0, g_1: Y \to Z$ be maps such that $g_0 \circ f \simeq g_1 \circ f$. If f is a homotopy equivalence, then $g_0 \simeq g_1$.*
   (b) *Let $f_0, f_1: X \to Y$ and let $g: Y \to Z$ be maps such that $g \circ f_0 \simeq g \circ f_1$. If g is a homotopy equivalence, then $f_0 \simeq f_1$.*

Proof (∗).

(1) Let $f: X \to Y$ and $g: Y \to Z$ be maps between topological spaces. There are three slightly different statements that need to be proved.
   (a) First suppose that the maps $f: X \to Y$ and $g: Y \to Z$ admit homotopy inverses $f': Y \to X$ and $g': Z \to Y$. Then we see that

$$(f' \circ g') \circ (g \circ f) \;=\; f' \circ (g' \circ g) \circ f \underset{\substack{\uparrow \\ \text{by Lemma 18.4 (2) since } g' \circ g \simeq \mathrm{id}_Y}}{\simeq} f' \circ \mathrm{id}_Y \circ f \;=\; f' \circ f \;\simeq\; \mathrm{id}_X\,.$$

[299]The perspicacious reader might notice that here things get logically a little dicey. More precisely, on page 74 we had recalled the notion of an equivalence relation. But we had only defined the notion of an "equivalence relation" on a set and "all topological spaces" do not form a set they "only" form a class. One can also define the notion of an "equivalence relation" on a class, see e.g. [**FPr85**, Kapitel 2]. But we do not want to get into the discussion "what is a class?". As a cheap way out we interpret the second statement of the lemma as saying that reflexivity, symmetry and transitivity hold for "homotopy equivalence" with the obvious interpretation.

Similarly one shows that $(g \circ f) \circ (f' \circ g') \simeq \mathrm{id}_Z$. Thus we have shown that $g \circ f$ is a homotopy equivalence.

(b) Next suppose that $f\colon X \to Y$ and $g \circ f\colon X \to Z$ admit homotopy inverses $f'\colon Y \to X$ and $h'\colon Z \to X$. Then we see that

$$(f \circ h') \circ g \underset{\substack{\uparrow \\ \text{by Lemma 18.4 (2) since } f \circ f' \simeq \mathrm{id}_Y}}{\simeq} f \circ h' \circ (g \circ f) \circ f' \underset{\substack{\uparrow \\ \text{since } h' \text{ is a homotopy inverse of } g \circ f}}{\simeq} f \circ \mathrm{id}_X \circ f' \simeq \mathrm{id}_Y .$$

Similarly one shows that $g \circ (f \circ h') \simeq \mathrm{id}_Z$. Thus we have shown that g is a homotopy equivalence.

(c) A slight variation on the argument of (1b) shows that if g and $g \circ f$ are homotopy equivalence, then so is f.

(2) It is clear that the only property of an equivalence relation that requires any thought is transitivity. But this follows from (1).

(3) We leave the elementary verification of the last statement to the reader.

(4) The proof is very similar to the proof of (1). Again we leave the details to the reader. ■

Before we give examples of homotopy equivalent topological spaces we introduce the following definition.

Definition. Let X be a topological space. We say X is *contractible*, if X is homotopy equivalent to a topological space which consists of precisely one point.

The following elementary lemma gives a handy criterion for showing that a topological space is contractible.

Lemma 18.12. *Let X be a topological space.*

(1) *X is contractible* $\iff$ (2) *there exists an $x_0 \in X$ such that the inclusion $\{x_0\} \to X$ is a homotopy equivalence*

$\iff$ (3) *there exists an $x_0 \in X$ such that the identity map id_X is homotopic to the constant map c_{x_0}.*

Remark. In Exercise 18.20 we will see that if statement (2) or (3) of Lemma 18.12 holds for a single point, then it holds for all points in X.

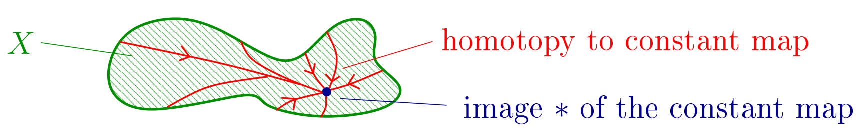


FIGURE 342

PROOF (∗).

(1)⇒(2) First suppose that X is contractible. Let $Y = \{*\}$ be the topological space consisting of a single element. Since X is contractible we know that there exists a map $g\colon \{*\} \to X$ that is a homotopy equivalence. We set $x_0 := g(*)$. Basically by definition the inclusion map $\{x_0\} \to X$ is also a homotopy equivalence.

(2)⇒(3) Let $x_0 \in X$ be a point such that the inclusion map $f\colon \{x_0\} \to X$ is a homotopy equivalence. Let $g\colon X \to \{x_0\}$ be the unique map. Since f is a homotopy equivalence we know that $f \circ g$ is homotopic to id_X. But $f \circ g = c_{x_0}$. Thus we have shown that id_X is homotopic to c_{x_0}.

(3)⇒(1) Finally we suppose that the identity map of X is homotopic to a constant map $f\colon X \to X$. Let $*$ be the unique point in the image of $*$. We set $Y := \{*\}$. Let $g\colon \{*\} \to X$ be the inclusion map. By hypothesis we have $g \circ f \simeq \mathrm{id}_X$. Furthermore we evidently have $f \circ g = \mathrm{id}_Y$. This shows that X is homotopic equivalent to $Y = \{*\}$. ■

We give two basic examples of homotopy equivalent topological spaces.

Examples.

(1) Let X be a star-shaped subset of $\mathbb{R}^n$ and let $x_0 \in X$ be a point such that for any $x \in X$ the segment from x_0 to x lies in X. Basically by the same argument as in Lemma 18.1 we see that the map

$$\begin{array}{rcl} f\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & x_0 \cdot (1-t) + x \cdot t \end{array}$$

is a homotopy from id_X to a constant map. By Lemma 18.12 this shows that X is contractible.

(2) It follows from the previous example together with Lemma 18.11 that for any $n \in \mathbb{N}$ and any $k \in \mathbb{N}$ the topological spaces $\mathbb{R}^n$ and $\mathbb{R}^k$ are homotopy equivalent.

We leave the proof of the following lemma as an edifying exercise to the reader, see Exercise 18.23.

Lemma 18.13. *Let $f\colon X \to Y$ be a map between topological spaces. If f is a homotopy equivalence, then f induces a bijection $\pi_0(X) \to \pi_0(Y)$. In particular, X is path-connected if and only if Y is path-connected.*

Definition. Let X be a topological space and let A be a subset.

(1) A homotopy $F\colon X \times [0,1] \to X$ is called *stationary on A*, if $F(a,t) = a$ for all $a \in A$ and all $t \in [0,1]$.
(2) A *deformation retraction* from X to A is a homotopy $F\colon X \times [0,1] \to X$ which has the following three properties:[300]
   (a) $F_0 = \mathrm{id}_X$,
   (b) the homotopy is stationary on A,
   (c) $F_1(X) \subset A$.
(3) We say A is a deformation retract of X if there exists a deformation retraction from X to A.[301]
(4) We say a retraction $r\colon X \to A$ is *realized by a deformation retraction* if there exists a deformation retraction F from X to A with $F_1 = r$.

We say a point $a \in X$ is a deformation retract of X if the subset $\{a\}$ if a deformation retract of X.

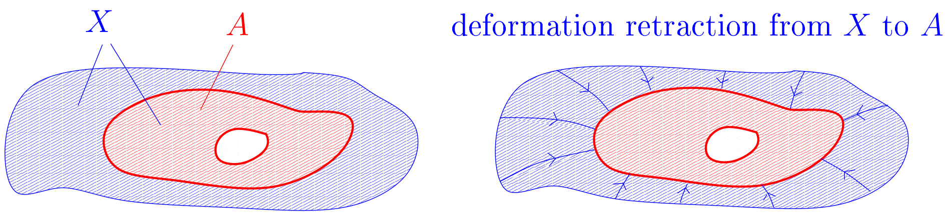


FIGURE 343. Schematic picture of a deformation retraction.

Examples.

(1) For any topological space X and any interval $I \subset \mathbb{R}$ containing 0 the subspace $X \times \{0\}$ is a deformation retract of $X \times I$. In fact the deformation retraction is given[302] by

$$\begin{array}{rcl} (X \times I) \times [0,1] & \to & X \times I \\ ((x,s),t) & \mapsto & (x, s \cdot (1-t)). \end{array}$$

(2) Let $n \in \mathbb{N}$. In Exercise 18.24 we will prove the following two statements:
   (a) The sphere S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.
   (b) The hemispheres $S^{n-1}_{\leq 0}$ is a deformation retract of $\overline{B}^n$.
   (c) Given any three non-collinear points $A, B, C \in \mathbb{R}^n$ the triangle Δ_{ABC} admits a deformation retraction to the union of any two sides of the triangle.

   These three examples are illustrated in Figure 344.

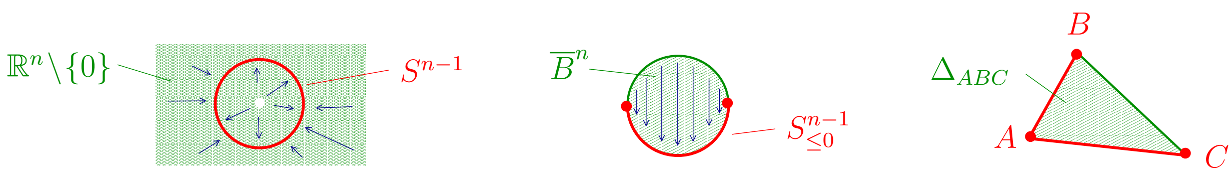


FIGURE 344

Remark. Let X be a topological space and let $A \subset B \subset X$ be a subspaces. It follows almost immediately from Lemma 18.3 that if A is a deformation retract of B and B is a deformation retract of X, then A is a deformation retract of X.

In Exercise 18.25 we will prove the following elementary lemma.

---

[300]Note that $F_1 \colon X \to A$ is retraction from X to A.

[301]Unfortunately in the literature the term "deformation retract" is used differently by different authors. Our notion of a "deformation retract" agrees with the definition in [**Hat02**, p. 2] and [**Mun84**, p. 208]. What we call a "deformation retract" is called a "strong deformation retract" in [**Bre93**, p. 45] and [**Rot88**, p. 209]. In these books this notion is used to distinguish it from the notation of a "weak deformation retract", which means that there exists a map $F \colon X \times [0,1] \to X$ such that

(1) for every $x \in X$ we have $F(x,0) = x$ and $F(x,1) \in A$,
(2) for every $a \in A$ we have $F(a,1) = a$.

It turns out that the two concepts are indeed different. For example, let Z be the infinite zigzag comb from Exercise 18.11. Every point is a weak deformation retract but no point is a strong deformation retract of X.

[302]It follows from the discussion on page 109 and Lemma 3.8 (2b) that this map is actually continuous.

Lemma 18.14. *Let A be a subset of a topological space X which admits a deformation retraction F from X to A. Then the inclusion map*

$$\begin{array}{rcl} A & \to & X \\ a & \mapsto & a \end{array} \quad \textit{and the map} \quad \begin{array}{rcl} X & \to & A \\ x & \mapsto & F(x,1) \end{array}$$

are both homotopy equivalences. In particular the deformation retract A is homotopy equivalent to the total space X.

The lemma often gives a convenient way for showing that two spaces are homotopy equivalent. For example the lemma, together with the previous example implies that the sphere S^{n-1} and $\mathbb{R}^n \setminus \{0\}$ are homotopy equivalent.

The notions of a topological space being contractible and admitting a deformation retraction to a point sound quite similar. The following lemma gives the precise relationship between these two notions.

Lemma 18.15.

(1) *If a topological space X admits a deformation retraction to a point $x_0 \in X$, then X is homotopy equivalent to $\{x_0\}$, in particular X is contractible.*
(2) *If a topological space X is contractible, then given any point $x_0 \in X$ there exists a homotopy $H\colon X \times [0,1] \to X$ such that $H_0 = \mathrm{id}_X$ and $H_1(x) = x_0$ for all $x_0 \in X$.*
(3) *The "infinite zigzag comb" shown in Figure 345 is contractible, but it does not admit a deformation retraction to a point.*

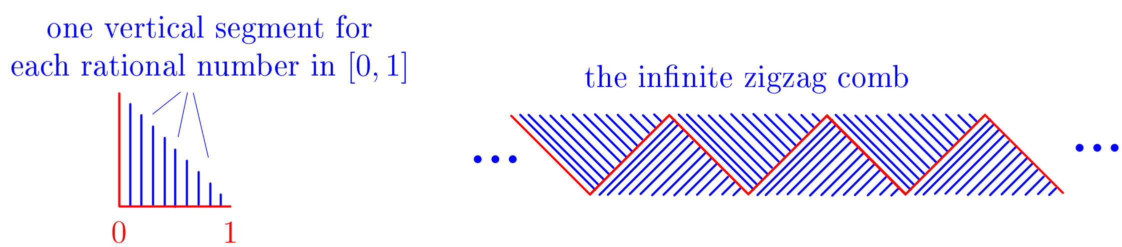


Figure 345

Proof.

(1) This statement follows immediately from Lemma 18.14.
(2) Suppose X is a topological space that is contractible. By Lemma 18.12 we know that there exists a homotopy $G\colon X \times [0,1] \to X$ from id_X to a constant map. Let x_1 be the image of this constant map. Recall that we know from Lemma 18.13 that X is path-connected. Thus there exists a path $\gamma\colon [0,1] \to X$ from x_1 to x_0. The desired homotopy is now given by the map[303]

$$\begin{array}{rcl} H\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} G(x, 2\cdot t), & \text{if } t \in [0,\frac{1}{2}], \\ \gamma(2\cdot t - 1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

(3) This statement will be proved in Exercise 18.11. ∎

[303]This map is continuous by Lemma 2.35 (2).

18.4. **The fundamental groups of homotopy equivalent topological spaces.** Now we will see that homotopy equivalent topological spaces have isomorphic fundamental groups. On numerous occasions this fact will often greatly simplify the calculation of fundamental groups.

Proposition 18.16.

(1) *Let $f, g\colon X \to Y$ be two maps between topological spaces and let $x_0 \in X$. If f and g are homotopic, then there exists a path $\alpha\colon [0,1] \to Y$ from $f(x_0)$ to $g(x_0)$ such that the following diagram commutes:*

$$\begin{tikzcd} & \pi_1(Y, g(x_0)) \arrow[d, "\alpha_*"', "{[s] \mapsto [\alpha * s * \overline{\alpha}]}"] \\ \pi_1(X, x_0) \arrow[ur, "g_*"] \arrow[r, "f_*"'] & \pi_1(Y, f(x_0)). \end{tikzcd}$$

Here the vertical map is the isomorphism given by Proposition 14.11. In particular

$$\begin{array}{ccc} f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) & & g_*\colon \pi_1(X, x_0) \to \pi_1(Y, g(x_0)) \\ \textit{is an isomorphism} & \Longleftrightarrow & \textit{is an isomorphism.} \end{array}$$

The analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism". Furthermore, the following two refinements hold:

(a) *If f and g are homotopic rel x_0, then $f_* = g_*$.*

(b) *If $f(x_0) = g(x_0)$ and if the maps f and g are homotopic, then there exists an $\alpha \in \pi_1(Y, f(x_0))$ such that $f_*(z) = \alpha \cdot g_*(z) \cdot \alpha^{-1}$ for every $z \in \pi_1(X, x_0)$.*

(2) *Let $f\colon X \to Y$ be a map between topological spaces and let $x_0 \in X$. If f is a homotopy equivalence, then the induced map*

$$f_*\colon \pi_1(X, x_0) \;\to\; \pi_1(Y, f(x_0))$$

is an isomorphism.

(3) *If X is contractible topological space, then for any $x_0 \in X$ we have $\pi_1(X, x_0) = 0$.*

Example. For any $n \geq 2$ and any $Q \in \mathbb{R}^n$ we have

$$\pi_1(\mathbb{R}^n \setminus \{Q\}) \underset{\substack{\uparrow \\ \text{since } \mathbb{R}^n \setminus \{Q\} \text{ is} \\ \text{homeomorphic to } \mathbb{R}^n \setminus \{0\}}}{\cong} \pi_1(\mathbb{R}^n \setminus \{0\}) \underset{\substack{\uparrow \\ \text{by Lemma 18.14, Exercise 18.24} \\ \text{and Proposition 18.16 (2)}}}{\cong} \pi_1(S^{n-1}) \underset{\substack{\uparrow \\ \text{by Corollary 16.18} \\ \text{and Proposition 14.14}}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{if } n \geq 3. \end{cases}$$

Proof (∗).

(1) Let $f, g\colon X \to Y$ be two maps between topological spaces and let $x_0 \in X$. Let $H\colon X \times [0,1] \to Y$ be a homotopy between f and g. We denote by

$$\begin{aligned} \alpha\colon [0,1] &\to Y \\ t &\mapsto H(x_0, t) \end{aligned}$$

the path with which the homotopy H connects the points $H(x_0, 0) = f(x_0)$ and $H(x_0, 1) = g(x_0)$.

Claim. The maps

$$\begin{array}{rcl} f_*\colon \pi_1(X,x_0) & \to & \pi_1(Y,f(x_0)) \\ [s] & \mapsto & [f\circ s] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \pi_1(X,x_0) & \to & \pi_1(Y,f(x_0)) \\ [s] & \mapsto & [\alpha * (g\circ s) * \overline{\alpha}] \end{array}$$

are identical.

Let $s\colon [0,1]\to X$ be a loop in (X,x_0). We need to show that the loops $f\circ s$ and $\overline{\alpha} * (g\circ s) * \alpha$ are path-homotopic. We consider the map

$$\begin{array}{rcl} G\colon [0,1]\times[0,1] & \to & X \\ (r,t) & \mapsto & \begin{cases} \alpha(4r), & \text{if } r\in \big[0,\tfrac{1}{4}t\big], \\ H\Big(s\Big(\dfrac{r-\frac{1}{4}t}{1-\frac{1}{2}t}\Big),1-t\Big), & \text{if } r\in (\tfrac{1}{4}t, 1-\tfrac{1}{4}t), \\ \overline{\alpha}(4r-3), & \text{if } r\in \big[1-\tfrac{1}{4}t,1\big]. \end{cases} \end{array}$$

Using Lemma 14.3 it is straightforward to verify that G is indeed a continuous map. Furthermore it follows easily from the definition that the map G satisfies

$$G(0,t) \;=\; G(1,t) \;=\; f(x_0) \quad \text{ for all } t\in[0,1]$$

and that it satisfies

$$G(r,0) \;=\; (f\circ s)(r) \quad \text{ and } \quad G(r,1) \;=\; (\overline{\alpha} * (g\circ s) * \alpha)(r) \quad \text{ for all } r\in[0,1].$$

This observation shows that G is in fact a path-homotopy between the loops $f\circ s$

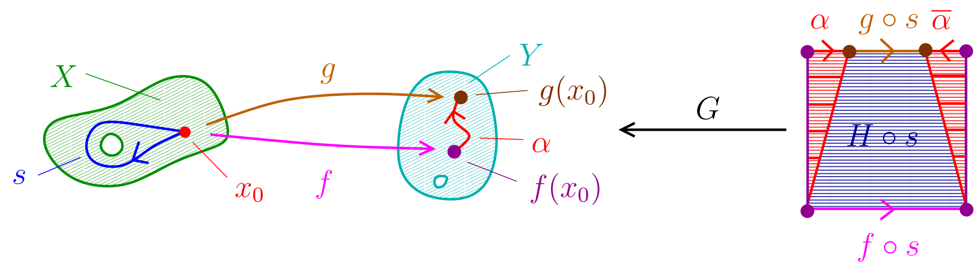


FIGURE 346. Schematic image of H and G.

and $\alpha * (g\circ s) * \overline{\alpha}$[304] in $(Y,f(x_0))$. ⊞

As in the statement of the proposition we consider the following diagram:

$$\begin{tikzcd} & \pi_1(Y,g(x_0)) \arrow[d, "\alpha_*"', "{[s]\mapsto[\alpha*s*\overline{\alpha}]}"] \\ \pi_1(X,x_0) \arrow[ur, "g_*"] \arrow[dr, "f_*"'] & \\ & \pi_1(Y,f(x_0)). \end{tikzcd}$$

We make the following observations:

(a) It follows from the claim that the diagram commutes.
(b) By Proposition 14.11 we know that the vertical map is an isomorphism.
(c) If follows from (a) and (b) that f_* is an isomorphism if and only if g_* is an isomorphism.

[304]To be precise, G is a homotopy between $f\circ s$ and a suitable parametrization of $\alpha * (g\circ s) * \overline{\alpha}$.

(d) Now suppose that f and g are homotopic rel x_0. By definition this means that there exists a homotopy H with $H(x_0,t) = f(x_0)$ for all $t \in [0,1]$. In other words, $\alpha(t) := H(x_0,t)$ is a constant path. It follows from the above that $f_* = g_*$.

(e) If $f(x_0) = g(x_0)$, then α is a loop in $(Y, f(x_0))$, i.e. it defines an element in $\pi_1(Y, y_0)$. Thus we have $[\alpha * (g \circ s) * \overline{\alpha}] = [\alpha] \cdot g_*(s) \circ [\alpha]^{-1}$. By the above commutative diagram we obtain the desired equality.

The analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism".

(2) Let $f\colon X \to Y$ be a map between topological spaces which is a homotopy equivalence. Let $x_0 \in X$ be a point. We pick a homotopy inverse $g\colon Y \to X$ of f. According to our hypothesis we have $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. It follows from (1) and the functoriality of fundamental groups that $g_* \circ f_*$ is an isomorphism of $\pi_1(X, x_0)$ and that $f_* \circ g_*$ is an isomorphism of $\pi_1(Y, f(x_0))$. Thus it remains to prove the following purely group theoretic claim.

Claim. Let $\alpha\colon A \to B$ and $\beta\colon B \to A$ be group homomorphisms. If $\alpha \circ \beta$ and $\beta \circ \alpha$ are isomorphisms, then α is an isomorphism.

First note that the fact that $\beta \circ \alpha\colon A \to A$ is an isomorphism implies that α is a monomorphism. Furthermore, the fact that $\alpha \circ \beta\colon B \to B$ is an isomorphism implies that α is an epimorphism. Thus we have shown that α is a monomorphism and an epimorphism, therefore it is an isomorphism. ⊞

(3) This statement is just a frequently used special case of the second statement. ■

In Lemma 2.59 we already saw that the topological spaces $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. Now we can prove a similar statement in one dimension higher.

Lemma 18.17. *The topological spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic.*

Unfortunately we are still not able to settle the question whether for $k \neq l \geq 3$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ can be homeomorphic. Before we can settle this question we will have to develop a completely new set of tools.

Proof. Let us suppose that there exists a homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^3$. Let $P \in \mathbb{R}^2$ be a point. We put $Q := f(P)$. Then f restricts to a homeomorphism from $\mathbb{R}^2 \setminus \{P\}$ to $\mathbb{R}^3 \setminus \{Q\}$. But then we obtain that

$$\mathbb{Z} \underset{\substack{\uparrow \\ \text{page 551}}}{\cong} \pi_1(\mathbb{R}^2 \setminus \{P\}) \underset{\substack{\uparrow \\ \text{since } f \text{ is a homeomorphism}}}{\cong} \pi_1(\mathbb{R}^3 \setminus \{Q\}) \underset{\substack{\uparrow \\ \text{page 551}}}{=} 0.$$

Thus we obtained a contradiction. ■

18.5. **The boundary of 2-dimensional topological manifolds** $(*)$**.** The goal of this section is to prove the a subtle statement that the boundary of a 2-dimensional topological manifold is what you think it should be. To make sense of this it is perhaps worth reminding us of the definitions from page 261.

Definition. Let F be a n-dimensional topological manifold.

(a) An *n-dimensional chart for F at a point $x \in X$* is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of x and V is one of the following:
   (i) V is an open subset of $\mathbb{R}^n$ or
   (ii) V is an open subset of the upper half-space $H := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and $\Phi(x)$ lies on $\partial H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$.

   In the former case we say that Φ is a chart of type (i) in the latter case we say that Φ is a chart of type (ii).

(b) We say that a point x on a topological manifold is a *boundary point* if x does not admit a chart of type (i). We denote by ∂X the set of all boundary points of X.

Proposition 18.18. *Let F be an 2-dimensional topological manifold. The following statements hold:*

(1) *Every point on F admits either a chart of type (i) or it admits a chart of type (ii).*
(2) *If F is equipped with a smooth atlas, i.e. if we view F as a smooth manifold, then the boundary of F as a topological manifold agrees with the boundary of F viewed as a topological manifold.*
(3) *The boundary ∂F is an 1-dimensional topological manifold with empty boundary, i.e. we have $\partial(\partial F) = \varnothing$.*

Remark. It might come as a surprise that we do not yet know the statement of the proposition. In fact in Proposition 6.27 we saw that the boundary of any n-dimensional smooth manifold is an $(n-1)$-dimensional smooth manifold. But we did not show the corresponding statement for topological manifolds. In fact at the moment we can prove the statement only for topological manifolds of dimension *two*. We will deal with the case of higher-dimensional topological manifolds in Proposition 44.2 once we introduced and studied homology groups.

Example. In Figure 347 we show once again the Möbius band M and the annulus A. Proposition 18.18, together with Proposition 6.27, says that the boundary of the Möbius band M and the annulus A, *as a topological manifold*, is exactly what we think it should be. In particular we see that ∂M consists of one component whereas ∂A consists of two components. Since homeomorphisms of topological manifolds induced homeomorphisms of their boundaries and since ∂M is not homeomorphic to ∂A we see that the Möbius band M is not homeomorphic to the annulus A.

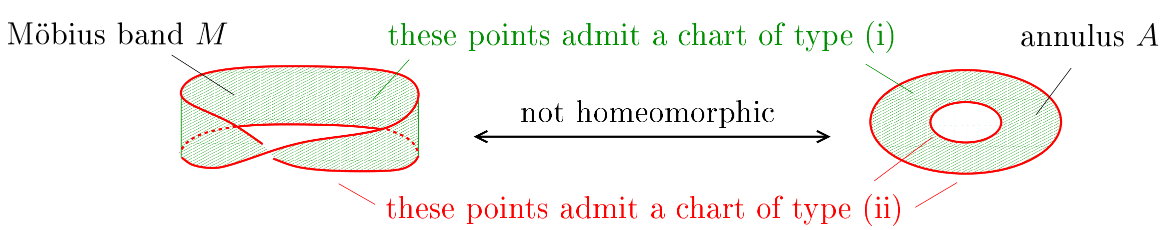


Figure 347

The key ingredient in the proof of Proposition 18.18 is the following lemma.

Lemma 18.19. *Let $H = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ be the upper half-space in $\mathbb{R}^2$. If V is an open subset of H with $V \cap \partial H \neq \varnothing$, then V is not homeomorphic to an open subset of $\mathbb{R}^2$.*

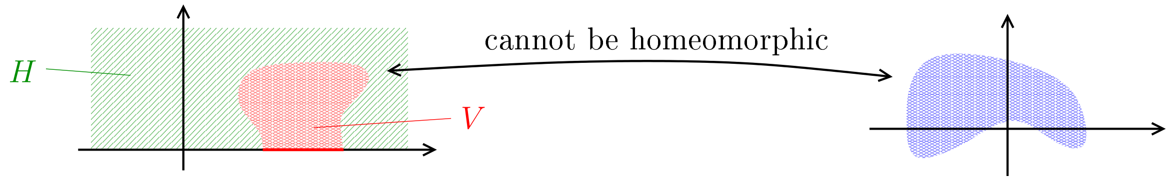


FIGURE 348. Illustration of Lemma 18.19.

PROOF. Let V be an open subset of the upper half-space H which contains at least one point $Q \in \partial H = \{(x, 0) \mid x \in \mathbb{R}\}$. Furthermore let U be an open subset of $\mathbb{R}^2$. Suppose there exists a homeomorphism $\Phi\colon U \to V$. We write $\Psi := \Phi^{-1}$ and we write $P = \Psi(Q)$.

Since U is open in $\mathbb{R}^2$ we can pick a $\sigma > 0$ with $B_\sigma(P) \subset U$. Furthermore, since V is open in H and since Ψ is continuous there exists an $\epsilon > 0$ with $B_\epsilon(Q) \cap H \subset V$ and with $\Psi(B_\epsilon(Q)) \subset B_\sigma(P)$. Finally, by the continuity of Φ there exists an $\eta > 0$ with $\Phi(B_\eta(P)) \subset B_\epsilon(Q))$. We refer to Figure 349 for an illustration. We consider the following commutative diagram of maps

$$\begin{tikzcd} & (B_\epsilon(Q) \setminus \{Q\}) \cap H \arrow[dr, "\Psi = \Phi^{-1}"] & \\ B_\eta(P) \setminus \{P\} \arrow[ur, "\Phi"] \arrow[rr, "i"'] & & B_\sigma(P) \setminus \{P\}, \end{tikzcd}$$

where $i\colon B_\eta(P) \setminus \{P\} \to B_\sigma(P) \setminus \{P\}$ denotes the inclusion map. Now we pick a base point $x \in B_\eta(P) \setminus \{P\}$. By the functoriality of fundamental groups, the above commutative diagram of maps gives rise to the following commutative diagram

$$\begin{tikzcd} & \pi_1((B_\epsilon(Q) \setminus \{Q\}) \cap H, \Phi(x)) \arrow[dr, "\Psi_* = \Phi_*^{-1}"] & \\ \pi_1(B_\eta(P) \setminus \{P\}, x) \arrow[ur, "\Phi_*"] \arrow[rr, "i_*"'] & & \pi_1(B_\sigma(P) \setminus \{P\}, x). \end{tikzcd}$$

We make the following observations:

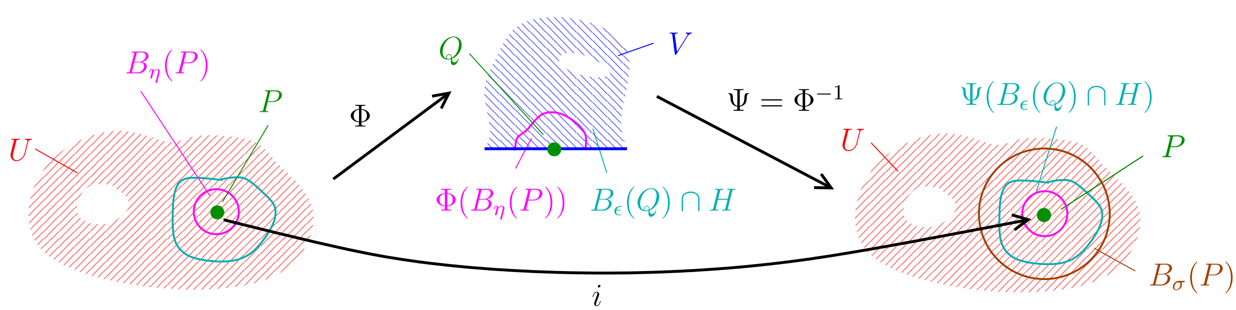


FIGURE 349. Illustration for the proof of Lemma 18.19.

(1) the topological spaces on the left and on the right are homotopy equivalent to S^1, in particular the fundamental groups are non-trivial,
(2) by Exercise 18.22 the inclusion map $B_\eta(P) \setminus \{P\} \to B_\sigma(P) \setminus \{P\}$ is a homotopy equivalence, therefore it follows from Proposition 18.16 (2) that the inclusion induced map $i_*\colon \pi_1(B_\eta(P) \setminus \{P\}, x) \to \pi_1(B_\sigma(P) \setminus \{P\}, x)$ is an isomorphism,
(3) the topological space $(B_\epsilon(Q) \setminus \{Q\}) \cap H$ is star-shaped[305], which implies by the discussion on page 471 that $\pi_1((B_\epsilon(Q) \setminus \{Q\}) \cap H, \Phi(x)) = 0$.

Summarizing we just showed that an isomorphism of non-trivial groups factors through the trivial group. This is not possible and hence we have obtained the desired contradiction. ■

Now we can turn to the proof of Proposition 18.18.

Proof of Proposition 18.18. Let F be an 2-dimensional topological manifold.

(1) Let x be a point on F. We need to show that x admits either a chart of type (i) or it admits a chart of type (ii). Suppose that x admits a chart $\Phi\colon U \to V$ of type (i) and also a chart $\Psi\colon X \to Y$ of type (ii). Then $\Phi \circ \Psi^{-1}\colon \Psi(U \cap X) \to \Phi(U \cap X)$ would give rise to a homeomorphism from an open set of H containing a point on $\partial H = \{(x, 0) \mid x \in \mathbb{R}\}$ to an open set in $\mathbb{R}^2$. But by Lemma 18.19 such a homeomorphism cannot exist.
(2) Now suppose that F is equipped with a smooth atlas. For the purpose of this proof we write $\partial^{\text{top}} F$ and $\partial^{\text{sm}} F$. By the definitions on pages 262 and 3 we have

$$\begin{aligned} \partial^{\text{top}} F &= \text{all points that do not admit a chart of type (i)} \\ \partial^{\text{sm}} F &= \text{all points that do not admit a chart of type (i) from the smooth atlas.} \end{aligned}$$

By definition we have $\partial^{\text{top}} F \subset \partial^{\text{sm}} F$. Now let $P \in \partial^{\text{sm}} F$. This means that P does not admit a chart of type (i) from the smooth atlas. By definition of a smooth manifold it admits a chart of type (ii). But by (1) this means that it cannot admit any chart of type (i). Thus we see that $P \in \partial^{\text{sm}} F$.
(3) We need to show that the boundary ∂F is a 1-dimensional topological manifold which has no boundary, i.e. such that $\partial Y = \varnothing$. First let $P \in F$ be a point on the boundary. Let $\Phi\colon U \to V$ be a chart of type (ii) for P. It follows from the (1) and the definitions that $\partial F \cap U = \Phi^{-1}(\partial H \cap V)$. Thus we see that Φ restricts to a chart $\Phi\colon \partial F \cap U \to \partial H \cap V$ of type (i), here we view $\partial H \cap V$ as an open subset of $\partial H = \mathbb{R}$. This shows that ∂F admits a 1-dimensional atlas where all charts are charts of type (i). Since F is second-countable and Hausdorff we obtain from Lemma 6.1 (2) and Lemma 2.12 that ∂F is also second-countable and Hausdorff. ■

18.6. **Homotopies on quotients and pushouts** $(*)$**.** In this section we collect several subtle lemmas which show that, under the right circumstances, homotopies descend to quotient spaces and that given a “gluing $X \cup_f Y$” we can also glue homotopies. This section is much less pleasant to read than the previous section and it is arguably best if the reader only takes a peek once the results are actually used.

[305]Why is that?

Before we state the two technical lemmas, we point out that the proof of the lemmas eventually boils down to the rather subtle and delicate Theorem 5.16. One might view both lemmas “obvious”, but they are not.

After this preamble, let us move to the key result of this section which shows that homotopies descend to quotients.

Proposition 18.20. *Let X and Y be topological spaces. Furthermore let $\sim_X$ be an equivalence relation on X and let $\sim_Y$ be an equivalence relation on Y. If $F\colon X \times [0,1] \to Y$ is a homotopy such that $F(x,t) \sim_Y F(x',t)$ whenever $x \sim_X x'$, then the map*[306]

$$\begin{array}{rcl} \varphi\colon (X/\sim_X) \times [0,1] & \to & Y/\sim_Y \\ ([x],t) & \mapsto & [F(x,t)] \end{array}$$

is continuous.

Proof. We denote by $p\colon X \to X/\sim_X$ and $q\colon Y \to Y/\sim_Y$ the obvious projection maps. We have the following commutative diagram

$$\begin{tikzcd} & X \times [0,1] \arrow[dl, "p\times \mathrm{id}_{[0,1]}"'] \arrow[dr, "q\circ F"] & \\ (X/\sim_X) \times [0,1] \arrow[rr, "\varphi", "{([x],t)\mapsto[F(x,t)]}"'] & & Y/\sim . \end{tikzcd}$$

By Lemma 5.15 (1) the map p is a quotient map. Since $[0,1]$ is regionally compact we obtain from Theorem 5.16 that $p \times \mathrm{id}_{[0,1]}$ is also a quotient map. Note that by Lemma 3.21 (3) and by hypothesis the map $q \circ F$ is continuous. Thus we obtain from Lemma 5.17 (1) that φ is continuous. ■

Examples.

(1) We consider $X = Y = [0,1] \times [-1,1]$ and the equivalence relation $\sim=\sim_X=\sim_Y$ that is given by
$$(0,y) \sim (1,-y) \quad \text{for all } y \in [-1,1].$$
Basically the same argument as on page 204 shows that we can view $X/\sim$ as the Möbius band. We consider the map
$$\begin{array}{rcl} H\colon (X/\sim) \times [0,1] & \to & X/\sim \\ ([(x,y)],t) & \mapsto & [(x, y\cdot(1-t))]. \end{array}$$
It follows from Proposition 18.20 that this map H is in fact continuous. One sees easily that H is deformation retraction from the Möbius band $([0,1] \times [-1,1])/\sim$ to the central circle $[0,1] \times \{0\}$ of the Möbius band. It follows from this discussion together with Lemma 18.14 that the Möbius band is homotopy equivalent to S^1. We refer to Figure 350 for an illustration.

(2) Let $n \in \mathbb{N}$ and let $\varphi\colon [0,1] \to \mathrm{GL}(n,\mathbb{R})$ be a map. It follows from Proposition 18.20 that the map
$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^{n-1} \times [0,1] & \to & \mathbb{R}\mathrm{P}^{n-1} \\ ([P],t) & \mapsto & [\varphi(t)\cdot P] \end{array}$$

[306] It follows from our hypothesis on F that the map φ is well-defined.

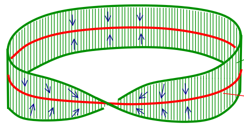


Figure 350

is continuous. The same conclusion also holds if we replace "$\mathbb{R}$" by "$\mathbb{C}$".

The following lemma is a hands-on special case of Proposition 18.20.

Lemma 18.21. *Let X be a topological space. Furthermore let $A \subset B$ be two subsets of X. If $F\colon B \times [0,1] \to X$ is a homotopy with $F(A \times [0,1]) \subset A$, then the induced map*[307]

$$\begin{array}{rcl} \varphi\colon (B/A) \times [0,1] & \to & X/A \\ ([x], t) & \mapsto & [F(x,t)] \end{array}$$

is continuous.

Proof. This statement follows immediately from Proposition 18.20 and the fact that quotients of topological spaces are defined via equivalence relations. ■

It is perhaps easier to appreciate the following useful corollary.

Corollary 18.22. *Let X be a topological space and let A be a subset of X. If A is a deformation retract of X, then the point A/A is a deformation retract of X/A. In particular X/A is contractible.*

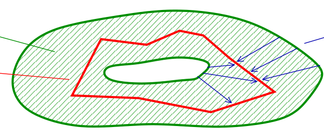


Figure 351. Illustration for Corollary 18.22.

Proof. Let $r\colon X \times [0,1] \to X$ be a deformation retraction from X to A. Since r is deformation retraction we obtain from Lemma 18.21 that the map

$$\begin{array}{rcl} (X/A) \times [0,1] & \to & X/A \\ ([x], t) & \mapsto & [r(x,t)]. \end{array}$$

is well-defined and continuous. Now it is clear that this map is a deformation retraction from X/A to A/A. ■

We conclude this section with the following lemma which shows how to define homotopies on pushouts.

Lemma 18.23. *Let $f\colon X \to Y$ and $g\colon X \to Z$ be maps between topological space. As on page 197 we define the pushout*

$$Y \cup_X Z \;:=\; (Y \sqcup Z)/\sim \qquad \text{where } f(x) \sim g(x) \text{ for all } x \in X.$$

[307]Since F is a homotopy rel A we see that the map is well-defined.

Furthermore let W be another topological space. Suppose that we are given two homotopies $G\colon Y \times [0,1] \to W$ and $H\colon Z \times [0,1] \to W$ such that for any $x \in X$ and any $t \in [0,1]$ we have $G(f(x),t) = H(g(x),t)$. Then the map

$$\begin{array}{rcl} F\colon (Y \cup_X Z) \times [0,1] & \to & W \\ ([P],t) & \mapsto & \begin{cases} G(P,t), & \textit{if } P \in Y, \\ H(P,t), & \textit{if } P \in Z \end{cases} \end{array}$$

is well-defined and it is continuous.

Proof. It follows from Lemma 3.8 that the map

$$\begin{array}{rcl} (Y \sqcup Z) \times [0,1]) & \to & W \\ (P,t) & \mapsto & \begin{cases} G(P,t), & \text{if } P \in Y, \\ H(P,t), & \text{if } P \in Z \end{cases} \end{array}$$

is continuous. The statement now follows from Proposition 18.20 and the definition of the pushout as a quotient of $Y \sqcup Z$. ■

18.7. The wedge of topological spaces.

Definition. Let $\{(X_i,x_i)\}_{i\in I}$ be a family of pointed topological spaces.

(1) We refer to
$$\bigvee_{i\in I} X_i \;:=\; \bigvee_{i\in I}(X_i,x_i) \;:=\; \Big(\bigsqcup_{i\in I} X_i\Big)/\sim \quad \text{where } x_i \sim x_j \text{ for all } i,j \in I$$
as the *wedge of the pointed topological spaces* (X_i,x_i). We refer to the point given by the identifications as the *wedge point*. If we do not say anything else it will be denoted by $*$.

(2) Let Y be a topological space and suppose that we are given a family of maps $\{f_i\colon X_i \to Z\}_{i\in I}$ such that for every $i,j \in I$ we have $f_i(x_i) = f_j(x_j)$. We define the map
$$\begin{array}{rcl} \bigvee_{i\in I} f_i\colon \bigvee_{i\in I}(X_i,x_i) & \to & Z \\ [y] & \mapsto & f_i([y]) \text{ if } y \in X_i. \end{array}$$
It follows from Lemma 3.22 that this map is indeed continuous.

(3) If $i = \{i_1,\dots,i_k\}$ is a finite family, then often we just write $(X_{i_1},x_{i_1})\vee\cdots\vee(X_{i_k},x_{i_k})$ and similarly we just write $f_{i_1}\vee\cdots\vee f_{i_k}$.

Examples.

(1) The wedge $(A,a)\vee(B,b)$ of two pointed topological spaces (A,a) and (B,b) should be viewed as the result of gluing A and B along the points a and b. The wedge of two pointed circles is shown in Figure 352.

(2) Let $n \in \mathbb{N}_0$ and let $G = (V = \{v\}, E, i, t)$ be an abstract graph with a *single* vertex. It follows quite easily from Lemma 3.28 (1) together with the fact that $[0,1]/0 \sim 1$

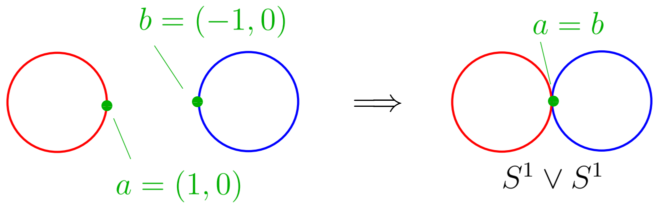


FIGURE 352. The wedge of two circles.

is homeomorphic to S^1 that the map

$$\begin{array}{rcl} |G| = (V \sqcup (E \times [0,1]))/\sim & \to & \bigvee\limits_{e \in E} S^1_e \\ [x] & \mapsto & \begin{cases} *, & \text{if } x = v, \\ e^{2\pi i t} \in S^1_e, & \text{if } x = (e,t) \in E \times [0,1] \end{cases} \end{array}$$

is a homeomorphism. Evidently this also shows that every topological graph with one vertex and n edges is homeomorphic to the wedge of n circles.

(3) In general the homeomorphism type of the wedge depends on the choice of base points. This becomes particularly clear if we consider disconnected topological spaces. For example it is pretty clear that the wedges $(A, a) \vee (B, b)$ and $(A, a) \vee (B, \widetilde{b})$ considered in Figure 353 are not homeomorphic.

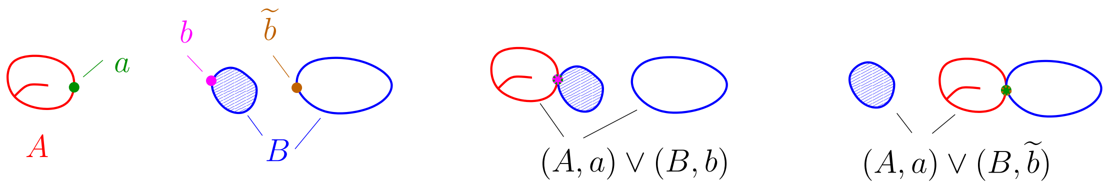


FIGURE 353

Even though we just saw that in general the homeomorphism type of the wedge of two topological spaces depends on the choice of the base points, there are also many other situations where the choice is basically irrelevant. For example we have the following lemma.

Lemma 18.24. *Let $\{X_i\}_{i \in I}$ be a family of smooth manifolds. We suppose that for each $i \in I$ we are given points $x_i, y_i \in X_i \setminus \partial X_i$. If each X_i is connected, then there exists a homeomorphism*

$$\bigvee_{i \in I} (X_i, x_i) \xrightarrow{\cong} \bigvee_{i \in I} (X_i, y_i).$$

PROOF. The lemma is an almost immediate consequence of Proposition 8.29. We leave it to the reader to fill in the details. ■

This leads us to the following convention which is often safe, but which can also be ambiguous.

Convention. Let $\{X_i\}_{i\in I}$ be a family of non-empty topological spaces. For each $i \in I$ we pick a point $x_i \in X_i$ and we write

$$\bigvee_{i\in I} X_i \;:=\; \bigvee_{i\in I}(X_i, x_i).$$

Examples.

(1) On numerous occasions we will talk about the wedge $S^1 \vee S^1$ of two circles. By Lemma 18.24 there is usually no need to specify which points in the two copies of S^1 get identified.

(2) In Exercise 18.26 we will see that the wedge of two circles is homotopy equivalent to $\mathbb{C} \setminus \{\pm 1\}$. More precisely, we will show that

$$S^1 \vee S^1 \;=\; \underbrace{\{z \in \mathbb{C} \,|\, |z+1| = 1\} \cup \{z \in \mathbb{C} \,|\, |z-1| = 1\}}_{\text{two circles that meet at the origin}}$$

is a deformation retract of $\mathbb{C} \setminus \{\pm 1\}$.

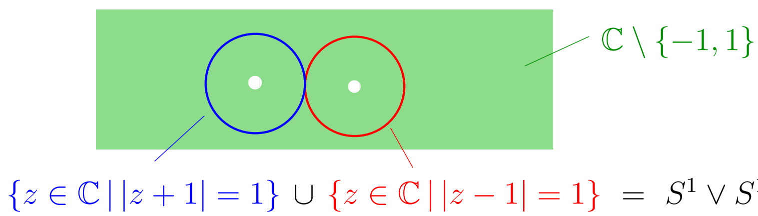


FIGURE 354

Definition. Let $\{(X_j, x_j)\}_{j\in J}$ be a family of pointed topological spaces. Given $k \in J$ we refer to the maps

$$\begin{array}{rcl} i_k\colon X_k & \to & \bigvee_{j\in J} X_j \\ x & \mapsto & [x] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} p_k\colon \bigvee_{j\in J} X_j & \to & X_k \\ [x] & \mapsto & \begin{cases} x, & \text{if } x \in X_k, \\ x_k, & \text{otherwise} \end{cases} \end{array}$$

as the *natural inclusion* respectively the *natural projection*.

Lemma 18.25. $(*)$ *Let $\{X_j\}_{j\in J}$ be a family of topological spaces and suppose that for each $j \in J$ we are given a point $x_j \in X_j$. Let $k \in J$.*

(1) *The natural inclusion i_k is an embedding.*

(2) *A map $f\colon \bigvee_{j\in J} X_j \to Y$ to a topological space is continuous if and only if for each $k \in J$ the map $f \circ i_k\colon X_k \to Y$ is continuous.*

(3) *The natural projection p_k is continuous.*

Remark. In most cases Lemma 18.25 allows us to safely view each X_k as a subset of the wedge $\bigvee_{j\in J} X_j$.

PROOF $(*)$.

(2) This statement follows easily from Lemma 3.3 and Lemma 3.22.

(3) This is an immediate consequence of (2).

(1) It follows easily from Lemma 3.3 and Lemma 3.21 (3) that the natural inclusion i_k is continuous. It is clear that the natural inclusion i_k is an injection. Evidently we have $p_k \circ i_k = \mathrm{id}_{X_k}$. Since the natural projection p_k is continuous we see that the natural inclusion i_k is indeed an embedding. ■

The following lemma collects a few basic properties of the wedge of topological spaces. It is entirely possible that later on we use the lemma without giving it credit.

Lemma 18.26. *Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces.*

(1) *If I is finite and if each X_i is compact, then $\bigvee\limits_{i\in I} X_i$ is compact.*

(2) *If each X_i is Hausdorff, then $\bigvee\limits_{i\in I} X_i$ is Hausdorff.*

(3) *If each X_i is (path)-connected, then $\bigvee\limits_{i\in I} X_i$ is (path)-connected*

Proof (∗).

(1) It follows immediately from Lemma 18.25 (3) and Lemma 3.3 (2) that the map $\bigsqcup\limits_{i\in I} p_i \colon \bigsqcup\limits_{i\in I} X_i \to \bigvee\limits_{i\in I} X_i$ is continuous. Thus the desired statement follows from Lemma 3.3 (4) together with Lemma 2.40.
(2) We will prove this statement in Exercise 18.27.
(3) We will prove this statement in Exercise 18.28. ■

We conclude this section with two technical lemmas which can safely be skipped at a first reading.

Lemma 18.27. (∗) *Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces and let (Z, z_0) be a pointed topological space. Let $\{f_i \colon (X_i, x_i) \to (Z, z_0)\}_{i\in I}$ and $\{g_i \colon (X_i, x_i) \to (Z, z_0)\}_{i\in I}$ be two families of maps. The following statement holds:*

for every $i \in I$ there exists a homotopy rel x_i from f_i to g_i $\iff$ *the maps $\bigvee\limits_{i\in I} f_i$ and $\bigvee\limits_{i\in I} g_i$ from $\bigvee\limits_{i\in I} X_i$ to Z are homotopic rel the wedge point.*

Proof. The "⇐"-direction of the lemma follows immediately from Lemma 18.25 (1). If the index set I has two elements, then the "⇒"-direction lemma is a straightforward consequence of Lemma 18.23 which eventually reduces the problem to Lemma 5.18. For a general family the "⇒"-direction lemma can also be reduced easily to Lemma 5.18.

It is worth pointing out that the proof of the "⇒"-direction is not obvious. As we just pointed out, it all boils down to Lemma 5.18. This lemma in turn relies on the delicate, frequently overlooked, underappreciated Theorem 5.16. ■

The following lemma gives in particular a nice little application of Lemma 18.27.

Lemma 18.28. (∗) *Let (A, a_0) and (B, b_0) be two pointed topological spaces. We consider the wedge $A\vee B = (A, a_0)\vee(B, b_0)$. If $\{a_0\}$ is a deformation retract of A, then $B = \{a_0\}\vee B$ is a deformation retract of $A \vee B$.*

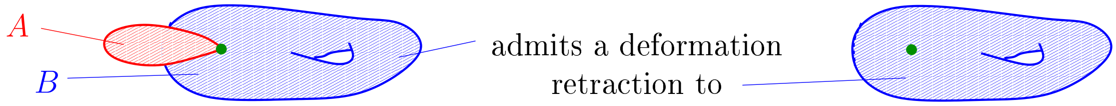


FIGURE 355. Illustration of Lemma 18.28.

PROOF (∗). Let $F\colon A\times[0,1]\to A$ be a deformation retraction from A to $\{a_0\}$. Let G be the trivial homotopy on B. It follows again from Lemma 18.23 that we can combine the two homotopies to get the desired deformation retraction. ■

18.8. **Topological graphs.** In this section we have another look at abstract graphs and their topological incarnations. First we recall the following definitions from page 228.

Definition. Let $G=(V,E,i,t)$ be an abstract graph.

(1) If G is finite, i.e. if V and E are finite, then we refer to
$$\chi(G) \;=\; \#V-\#E \;=\; \text{number of vertices} - \text{number of edges}$$
as the *Euler characteristic of* G.

(2) A *tree* is a finite connected abstract graph with Euler characteristic 1.

(3) The *valence of a vertex* v is defined as
$$\operatorname{valence}(v) \;:=\; \#\{e\in E\,|\,i(e)=v\}+\#\{e\in E\,|\,t(e)=v\}.$$

Given a topological graph $(X,G=(V,E,i,t),\Theta\colon|G|\to X)$ we define the properties of X to be the properties of G, e.g. we define $\chi(X):=\chi(G)$ and we say that X is a tree if G is a tree.[308]

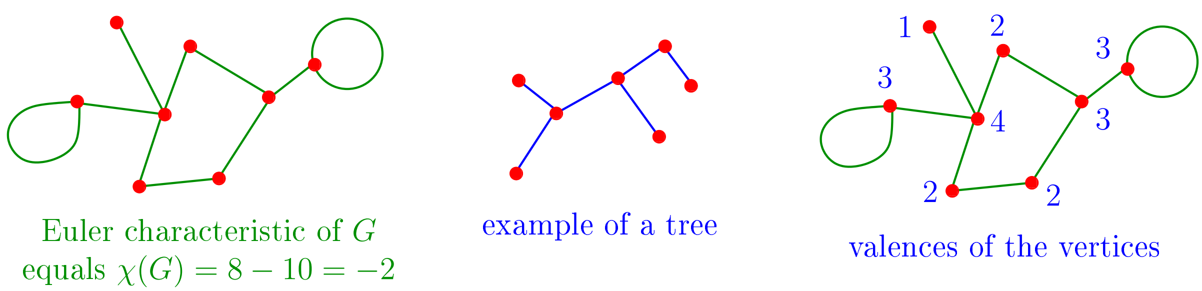


FIGURE 356

Proposition 18.29. *Let T be a topological graph that is a tree. Then the following hold:*

(1) *If T has at least one edge, then it admits at least two vertices of valence one.*

(2) *The tree T admits a deformation retraction to a vertex, in particular it is contractible.*

(3) *The fundamental group $\pi_1(T)$ is trivial.*

PROOF (∗). Basically by definition of a topological graph it suffices to deal throughout with the topological realizations of a tree. Thus let $T=(V,E,i,t)$ be a tree.

(1) This statement is precisely the content of Lemma 4.6.

[308]On many occasions we will use the fact, established in Lemma 4.1 (5), that a topological graph is connected if and only if the underlying abstract graph is connected.

(2) We will prove that any tree admits a deformation retraction to a vertex. We will prove this statement by induction on the number of edges. If a tree has zero edges, then it follows from the connectedness of T that T consists of precisely one vertex. So we are done.

Now suppose that we know that all trees with n edges admit a deformation retraction to a vertex. Let $T = (V, E, i, t)$ be a tree with $n+1$ edges. By (1) we know that T admits a vertex v of valence one. We denote by e the unique edge with v as an endpoint. We set $T' = (V \setminus \{v\}, E \setminus \{e\}, i|_{E\setminus\{e\}}, j|_{E\setminus\{e\}})$. Note that T' is again an abstract graph. Furthermore note that we reduced the number of vertices and of edges by one, so the Euler characteristic is unchanged, i.e. $\chi(T') = \chi(T) = 1$, so T' is again a tree.

We will now show that $|T'|$ is a deformation retract of $|T|$. We denote by u the other endpoint of e. We consider the maps

$$f\colon |T| \to |T'| \qquad x \mapsto \begin{cases} u, & \text{if } x \in |e|, \\ x, & \text{otherwise} \end{cases} \qquad \text{and} \qquad g\colon |T'| \to |T| \qquad x \mapsto x.$$

It follows quite easily from Lemma 3.28 (2) together with Lemma 3.3 that both maps are continuous.

We claim that g is a homotopy inverse to f. Evidently $f \circ g = \mathrm{id}_{|T'|}$. We need to show that $g \circ f \simeq \mathrm{id}_{|T|}$. By definition and by Lemma 4.1 (3) we can pick a characteristic map $\Phi_e\colon [0,1] \to |e|$ with $\Phi_e(0) = u$ and $\Phi_e(1) = v$. Next we consider the map

$$H\colon |T| \times [0,1] \to |T| \qquad (x,t) \mapsto \begin{cases} \Phi_e(\Phi_e^{-1}(x)\cdot t), & \text{if } x \in |e|, \\ x, & \text{otherwise.} \end{cases}$$

We leave it to the reader to verify, say using Proposition 18.20 and Lemmas 3.28 (2) and 3.3, that H is continuous. Now it is clear that H is a homotopy from $g \circ f$ to id_T.

We have thus shown that $|T'|$ is a deformation retract of $|T'|$. By our induction hypothesis $|T'|$ admits a deformation retraction to a vertex $v \in T'$. So it follows from Lemma 18.3 that $|T|$ also admits a deformation retraction to the vertex v. Finally it follows from this fact together with Lemma 18.15 (1) that T is contractible.

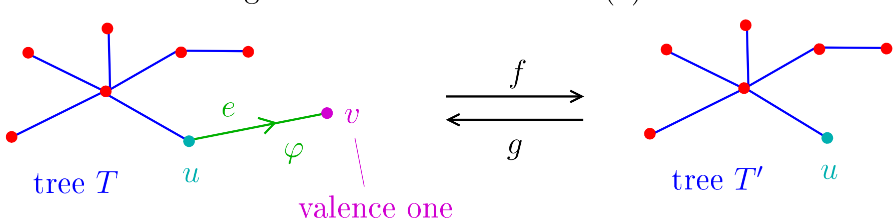


FIGURE 357. Illustration for the proof that every tree is contractible.

(3) It follows from (2) and Proposition 18.16 (3) that the fundamental group of T is trivial. ∎

Next we recall a few more definitions from page 229.

Definition. Let $G = (V, E, i, t)$ be an abstract graph.

(1) A *subgraph* of $G = (V, E, i, t)$ is an abstract graph $G' = (V', E', i', t')$ such that $V' \subset V$ and $E' \subset E$ are subsets such that $i' = i|_{E'}$ and $t' = t|_{E'}$.
(2) A *spanning tree* for G is a subgraph that is a tree and which does not admit an adjacent edge.

We extend the above definitions in an obvious way to topological graphs.

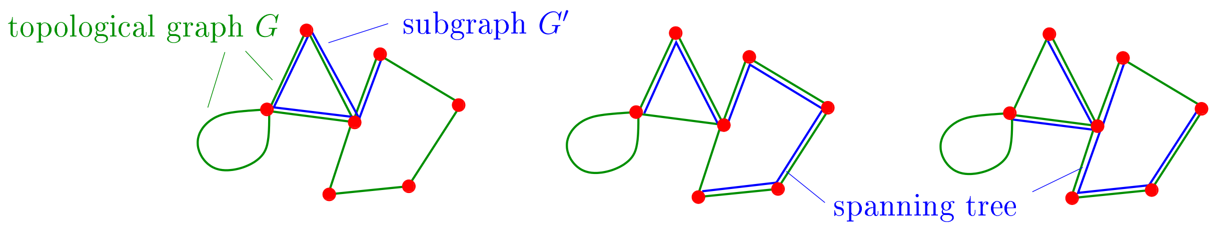


FIGURE 358

The following lemma summarizes a few basic properties of subgraphs.

Lemma 18.30. *Let $G = (V, E, i, t)$ be an abstract graph and let $G' = (V', E', i', t')$ be a subgraph.*

(1) *The inclusions $V' \to V$ and $E' \to E$ define a map $\iota\colon G' \to G$ of abstract graphs in the sense of the definition on page 477.*
(2) *The induced map $|\iota|\colon |G'| \to |G|$ between the topological realization is a closed embedding.*

In the following we will use (2) to view $|G'|$ as a closed subset of $|G|$.

PROOF.

(1) This statement follows immediately from the definitions.
(2) By the discussion on page 481 we know that the map $|\iota|$ is continuous. By Lemma 2.42 it remains to show that $|\iota|$ is a closed map. We leave it to the reader to show, using Lemma 3.1 and Lemma 3.21 (3), that $|\iota|$ is closed. ∎

We conclude this discussion of abstract and topological graphs with the following proposition.

Proposition 18.31. *Let $(X, G = (V, E, i, t), \Theta\colon |G| \to X)$ be a finite topological graph.*

(1) *For every spanning tree T of X the projection $X \to X/T$ is a homotopy equivalence.*
(2) *If X is connected and non-empty, then the topological graph X is homotopy equivalent to the wedge of $1 - \chi(X)$ circles.*
(3) *Two finite connected topological graphs with the same Euler characteristic are homotopy equivalent.*

Remark.

(1) In Proposition 37.8 and Corollary 39.10 we will provide generalizations of Proposition 18.31.

(2) Later, in Proposition 20.5 we will study the question, whether the converse to statement (3) holds: if two finite connected topological graphs are homotopy equivalent, does it follow that the Euler characteristics are the same?

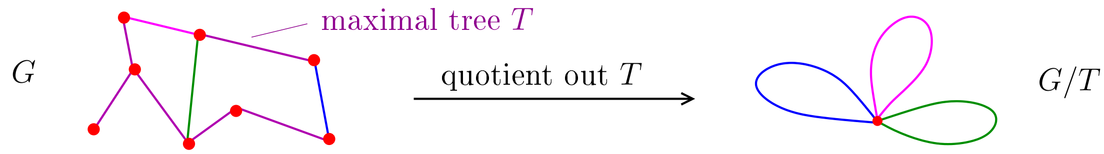


FIGURE 359. Illustration of Proposition 18.31 (1).

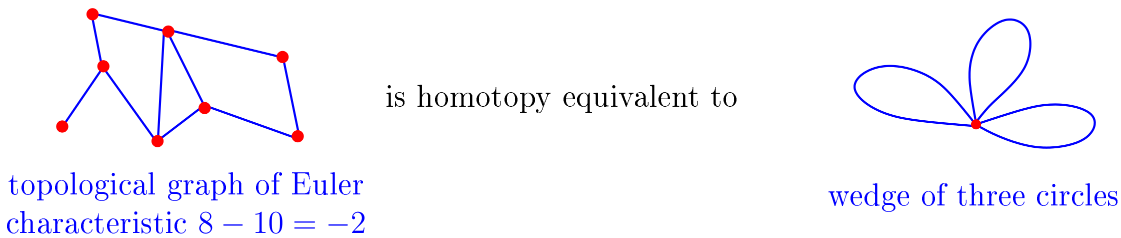


FIGURE 360. Illustration of Proposition 18.31 (2).

For the proof of Proposition 18.31 it is convenient to introduce the following definition.

Definition. Let $G=(V,E,i,t)$ be an abstract graph and let $H=(W,F)$ be a non-empty subgraph.

(1) We define $\sim$ to be the equivalence relation generated by $v\sim v'$ if v and v' are both vertices in H. We denote by $*$ the equivalence class defined by the vertices in H.
(2) We denote by $\rho\colon V\to V/\sim$ the map given by $v\mapsto [v]$.
(3) Finally we define the abstract graph

$$G/H \;:=\; (V/\sim, E\setminus F, \rho\circ i\colon E\setminus F\to V/\sim, \rho\circ t\colon E\setminus F\to V/\sim).$$

The following lemma summarizes a few properties of the abstract graph G/H.

Lemma 18.32. *Let $G=(V,E,i,t)$ be an abstract graph and let $H=(W,F)$ be a non-empty subgraph. We continue with the above notation.*

(1) *We have $\chi(G/H)=\chi(G)-\chi(H)+1$.*
(2) *We consider the map*

$$p\colon |G| = (V\sqcup(E\times[0,1]))/\overset{\substack{\text{where } (e,0)\sim i(e) \text{ and } (e,1)\sim t(e)\\ \downarrow}}{\sim} \;\to\; |G/H| = (V/\sim \sqcup((E\setminus F)\times[0,1]))/\overset{\substack{\text{similarly } (e,0)\sim \rho(i(e)) \text{ and } (e,1)\sim\rho(t(e))\\ \downarrow}}{\sim}$$
$$[x]\;\mapsto\;\begin{cases} [*], & \text{if } x\in|H|,\\ [\rho(x)], & \text{if } x\in V,\\ [(e,t)], & \text{if } x=(e,t) \text{ with } e\in E\setminus F \text{ and } t\in[0,1].\end{cases}$$

This map has the following two properties:
(a) *The map $p\colon |G|\to|G/H|$ is well-defined and continuous.*
(b) *If the abstract graph G is finite, then the map $p\colon|G|\to|G/H|$ descends to a homeomorphism*[309] *$|G|/|H|\to|G/H|$.*

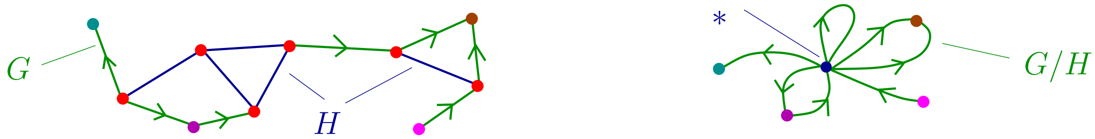


FIGURE 361. Illustration of Lemma 18.32

PROOF.

(1) We calculate that

$$\chi(G/H) = \#(V/\sim) - \#(E\setminus F) = (\#V - \#W + 1) - (\#E - \#F) = \chi(G) - \chi(H) + 1.$$

(2) (a) Once one has entangled all the definitions one sees that the map is indeed well-defined and one sees, using Lemma 3.28, that the map is indeed continuous.

(b) Since G is finite we know by Lemma 4.1 (4) and Lemma 2.40 that $|G|/|H|$ is compact. Furthermore, by Lemma 4.1 (2) and we know that $|G/H|$ is Hausdorff. It is elementary to see that the induced map $p\colon |G|/|H| \to |G/H|$ is a bijection. Thus we obtain from Proposition 2.43 (3) that $p\colon |G|/|H| \to |G/H|$ is a homeomorphism. ■

PROOF OF PROPOSITION 18.31. Basically by definition we only need to deal with the topological realization of an abstract graph. Thus let $G = (V, E, i, t)$ be a finite connected abstract graph.

(1) Let $T = (W, F)$ be a spanning tree of G. We consider the following little commutative diagram

$$\begin{tikzcd} |G| \arrow[rr, "p"] \arrow[dr, "x\mapsto[x]"'] & & |G/T| \\ & |G|/|T| \arrow[ur] & \end{tikzcd}$$

where p is the map introduced in Lemma 18.32 (2). We need to show that the projection map $|G| \to |G|/|T|$ to the left is a homotopy equivalence. By Lemma 18.32 (2b) we know that the diagonal map to the right is a homeomorphism. Thus it suffices to show that the horizontal map $p\colon |G| \to |G/T|$ is a homotopy equivalence.

Note that by Proposition 18.29 (2) we know that $|T|$ admits a deformation retraction $R\colon |T| \times [0,1] \to |T|$ to some vertex $t_0 \in T$. Put differently, there exists a homotopy $R\colon |T| \times [0,1] \to |T|$ such that $R_1 = \mathrm{id}_{|T|}$ and such that $R_0 = c_{t_0}$ is the constant map.

Let f be an edge of G that is not contained in T. If $i(f) = t(f)$, then it follows from Lemma 4.1 that there exists a map $\Phi_f\colon [-2,2] \to |G|$ which induces a homeomorphism $[-2,2]/-2 \sim 2 \to |f|$. Furthermore, if $i(f) \neq t(f)$, then it follows from Lemma 4.1 that there exists a map $\Phi_f\colon [-2,2] \to |G|$ which restricts to a homeomorphism $[-2,2] \to |f|$.[310] We write $a_f = \Phi_f(-2)$ and $b_f = \Phi_f(2)$. Note that it

[309]Note that it follows from Lemma 18.30 (2) that it is legitimate to view $|H|$ as a subset of $|G|$.

[310]In other words, the map Φ_f is just a rescaling of characteristic maps of f.

follows from our hypothesis that T is a spanning tree together with Proposition 4.8 that $a_f, b_f \in |T|$.

Next we consider the quotient graph G/T. Let $p\colon |G| \to |G/T|$ be the projection. For every edge f of G/T, i.e. for every $f \in E \setminus F$, we denote by $\Psi_f\colon [-2,2] \to |G/T|$ the map $p \circ \Phi_f$. We consider the map

$$\begin{aligned} q\colon |G/T| &\to |G| \\ x &\mapsto \begin{cases} R(a_f, -s-1), & \text{if } x = \Psi_f(s) \text{ for some } s \in [-2,-1], \\ \Psi_f(2 \cdot s), & \text{if } x = \Psi_f(s) \text{ for some } s \in [-1,1], \\ R(b_f, s-1), & \text{if } x = \Psi_f(s) \text{ for some } s \in [1,2]. \end{cases} \end{aligned}$$

It follows fairly easily from Lemma 3.28 that q is continuous. (The maps p and q are illustrated, to the best of the author's abilities, in Figure 362.) It remains to prove the following claim.

Claim. The maps p and q are homotopy inverses of one another.

The proof of the claim is fairly elementary, just notationally a little messy. First we consider the following map

$$\begin{aligned} H\colon |G/T| \times [0,1] &\to |G/T| \\ (x,t) &\mapsto \begin{cases} *, & \text{if } x = \Psi_f(s) \text{ for some } s \in [-2,-1-t], \\ \Psi_f(\frac{2s}{1+t}), & \text{if } x = \Psi_f(s) \text{ for some } s \in [-1-t,1+t], \\ *, & \text{if } x = \Psi_f(s) \text{ for some } s \in [1+t,2]. \end{cases} \end{aligned}$$

It follows, reasonably easily, from the combination of Lemma 3.28 and Proposition 18.20 that H is continuous. It follows immediately from the definitions that $H_0 = p \circ q$ and $H_1 = \mathrm{id}_{|G/T|}$. Next we consider the following map

$$\begin{aligned} \widetilde{H}\colon |G| \times [0,1] &\to |G| \\ (x,t) &\mapsto \begin{cases} R(x,t), & \text{if } x \in T, \\ R(a_f, s+2+t), & \text{if } x = \Phi_f(s) \text{ for some } s \in [-2,-1-t], \\ \Phi_f(\frac{2s}{1+t}), & \text{if } x = \Phi_f(s) \text{ for some } s \in [-1-t,1+t], \\ R(b_f, -s+2+t), & \text{if } x = \Phi_f(s) \text{ for some } s \in [1+t,2]. \end{cases} \end{aligned}$$

Again, using Lemma 3.28 and Proposition 18.20 one can show that $\widetilde{H}$ is continuous. Furthermore it follows again immediately from the definitions that $\widetilde{H}_0 = q \circ p$ and $\widetilde{H}_1 = \mathrm{id}_{|G|}$. ⊞

(2) Note that by Proposition 4.8 (1) we know that G admits a spanning tree T. Furthermore note that in (1) we showed that the map $p\colon |G| \to |G/T|$ is a homotopy equivalence. Thus it remains to show that $|G/T|$ is homeomorphic to the wedge of $1 - \chi(G)$ circles. We make the following observation:
  (a) By Proposition 4.8 the spanning tree T contains all vertices of G. In other words, the abstract graph G/T contains only a single vertex.
  (b) It follows from Lemma 18.32 (2) and the definition of a tree in terms of the Euler characteristic that $\chi(G/T) = \chi(G) - \chi(T) + 1 = \chi(G)$.
  (c) By (1) and (2) we know that G/T has $1 - \chi(G)$ edges.

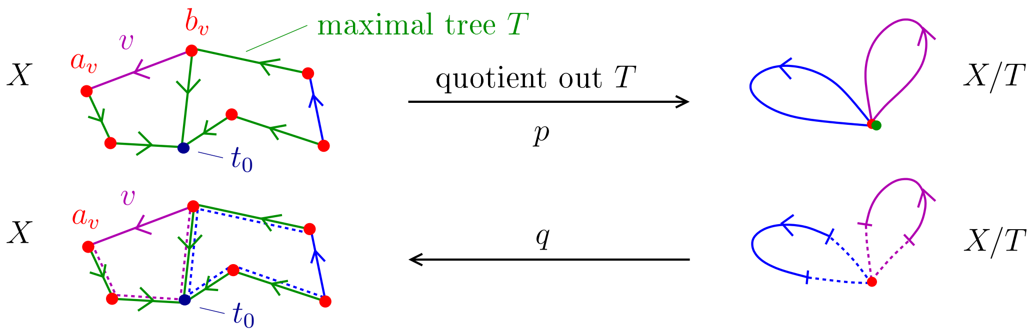


FIGURE 362. Illustration for the proof of Proposition 18.31 (1).

(d) It follows from (1) and (3), together with the discussion on page 560 that $|G/T|$ is indeed homeomorphic to the wedge of $1-\chi(G)$ circles.

(3) The last statement is an immediate consequence of (2) and Lemma 18.11. ■

18.9. **Homotopy classes of loops** $(*)$**.** In the last section of this chapter we want to study the question, when are two maps $S^1 \to X$ to a topological space homotopic? In fact the following proposition gives in some sense a complete answer to the above question.

Proposition 18.33. *Let X be a topological space and let $f_0, f_1 \colon S^1 \to X$ be two maps. We write $x_0 = f_0(1)$ and $x_1 = f_1(1)$.*

(1) *The following statement holds:*[311]

$$f_0 \text{ and } f_1 \text{ are homotopic} \iff \begin{array}{c}\text{there exists a path } \gamma \text{ from } x_0 \text{ to } x_1 \\ \text{such that } [\gamma * f_1 * \overline{\gamma}] = [f_0] \in \pi_1(X, x_0).\end{array}$$

(2) *If $x_0 = x_1$, then the following statement holds:*

$$f_0 \text{ and } f_1 \text{ are homotopic} \iff [f_0] \text{ and } [f_1] \text{ are conjugate in } \pi_1(X, x_0).$$

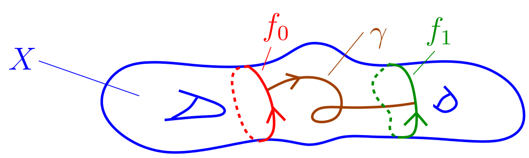


FIGURE 363. Illustration of Proposition 18.33.

PROOF. Note that the second statement is an immediate consequence of the first statement and the observation that if $x_0 = x_1$ any path from x_0 to $x_1 = x_0$ is in fact a loop and thus defines an element in $\pi_1(X, x_0)$. Therefore it suffices to prove the first statement.

Now let X be a topological space. We take the view that maps $S^1 \to X$ are the same as loops, i.e. maps $[0,1] \to X$ such that 0 and 1 get sent to the same point. Thus let $f_0, f_1 \colon [0,1] \to X$ be two loops. We write $x_0 = f_0(1)$ and $x_1 = f_1(1)$.

"$\Rightarrow$" Suppose there exists a homotopy $F \colon [0,1] \times [0,1] \to X$ with $F(0,t) = F(1,t)$ for all $t \in [0,1]$ and such that $F_0 = f_0$ and $F_1 = f_1$. We denote by $\gamma \colon [0,1] \to X$ the

[311]Sometimes two maps $S^1 \to X$ that are homotopic are called *freely homotopic* to stress the fact that no base points are fixed.

path that is given by $\gamma(t) = F(0,t)$ for $t \in [0,1]$. We consider the surjective map $\Phi\colon [0,1]\times[0,1] \to [0,1]\times[0,1]$ that is shown in Figure 364. The map $\Phi \circ F$ is easily seen to be a homotopy between the loops f_0 and $\gamma * f_1 * \overline{\gamma}$. But that is exactly what we needed to show.

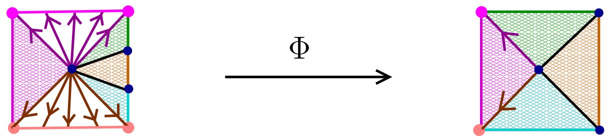


FIGURE 364. First illustration for the proof of Proposition 18.33.

"$\Leftarrow$" In the following we now suppose that there exists a path γ from x_0 to x_1 such that $[\gamma * f_1 * \overline{\gamma}] = [f_0] \in \pi_1(X, x_0)$. By definition this means that there exists a homotopy $F\colon [0,1]\times[0,1] \to X$ with $F(0,t) = F(1,t) = x_0$ for all $t \in [0,1]$ and such that $F_0 = f_0$ and $F_1 = \gamma * f_1 * \overline{\gamma}$. We consider the homeomorphism $\Psi\colon [0,1]\times[0,1] \to [0,1]\times[0,1]$ that is shown in Figure 365. The map $\Psi \circ F$ is easily seen to be a homotopy between $c_{x_0} * f_0 * c_{x_0}$ and f_1, viewed as maps $S^1 \to X$. But the map $c_{x_0} * f_0 * c_{x_0}$ is easily seen to be homotopic to the map f_0. Since being homotopic is transitive we have indeed shown that f_0 and f_1 are homotopic. ■

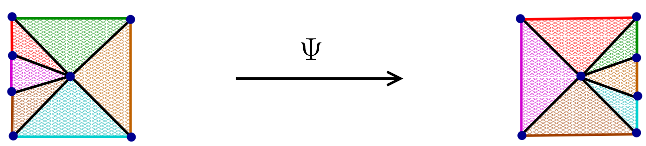


FIGURE 365. Second illustration for the proof of Proposition 18.33.

18.10. **Homotopies for pairs of topological spaces.** On page 537 we defined what it means for two maps $f, g\colon X \to Y$ between topological spaces to be homotopic. On many occasions one would like to restrict the freedom of a homotopy by fixing some subset pointwise or setwise. This idea leads us to two relative versions of "homotopy". The first version we already saw on page 539. For the reader's convenience we recall the definition.

Definition. Let $f, g\colon X \to Y$ be two maps between topological spaces and let $A \subset X$ be a subset such that $f|_A = g|_A$.

(1) A *homotopy between f and g rel A* is a homotopy $H\colon X \times [0,1] \to Y$ between f and g such that $H(a,t) = f(a) = g(a)$ for all $t \in [0,1]$ and all $a \in A$.
(2) We say *f and g are homotopic rel A*, if such a homotopy rel A exists.

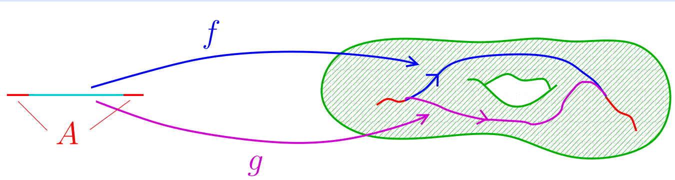


FIGURE 366. The maps f and g are homotopic, but they are not homotopic rel A.

We continue with another notion of a "relative homotopy".

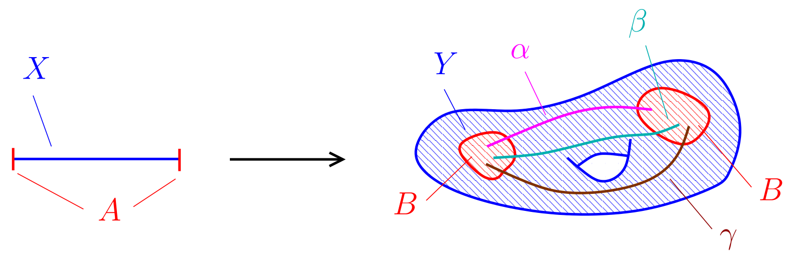
Definition. Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets.

(1) A map $f\colon (X,A) \to (Y,B)$ is defined as a map[312] $f\colon X \to Y$ such that $f(A) \subset B$.

Now let $f, g\colon (X,A) \to (Y,B)$ be two maps.

(2) A *homotopy between f and g* is a map[313]
$$H\colon X \times [0,1] \to Y,$$
such that the following two conditions are satisfied:
(a) $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$, and
(b) $H(a,t) \in B$ for all $a \in A$ and all $t \in [0,1]$.

(3) If there exists a homotopy between f and g, then we say that f and g are *homotopic* and we write $f \simeq g$.



$\alpha, \beta\colon (X,A) \to (Y,B)$ are homotopic but $\alpha, \gamma\colon (X,A) \to (Y,B)$ are not homotopic

FIGURE 367

For completeness we also mention the following definition which is the obvious relative analogue of the concepts that we introduced on page 546.

Definition. Let (X,A) and (Y,B) be two pairs of topological spaces.

(1) A map $f\colon (X,A) \to (Y,B)$ is a *homotopy equivalence between (X,A) and (Y,B)*, if there exists a map $g\colon (Y,B) \to (X,A)$ such that the following hold:
(a) $g \circ f$ and id_X are homotopic as maps of (X,A) to itself,
(b) $f \circ g$ and id_Y are homotopic as maps of (Y,B) to itself.

(2) If there exists a homotopy equivalence between (X,A) and (Y,B), then we say that (X,A) and (Y,B) are *homotopy equivalent.*

Examples.

(1) The two pairs of topological spaces (X,A) and (Y,B) shown in Figure 368 are homotopy equivalent.
(2) Let X be a topological space and let $A \subset B$ be subsets of X. If A is a deformation retract of B, then the inclusion map $(X,A) \to (X,B)$ is easily seen to be a homotopy equivalence of pairs of topological spaces.
(3) Let $A \subset Y$ be subsets of X. If Y is a deformation retract of X, then the inclusion map $(Y,A) \to (X,A)$ is again easily seen to be a homotopy equivalence of pairs of topological spaces.

[312] As usual all maps between topological spaces are understood to be continuous.
[313] For $A = B = \varnothing$ we get the usual notion of a homotopy between maps $f, g\colon X \to Y$.

FIGURE 368

The above notions can be generalized even further. For example, one could define triples of topological spaces and there is an obvious definition of what it means for maps $f, g\colon (X, A, B) \to (Y, C, D)$ between triples of topological spaces to be homotopic. Similarly, given pairs of topological spaces (X, A) and (Y, B) and a subset Z of A there is an obvious definition for what it means for two maps $f, g\colon (X, A) \to (Y, B)$ to be homotopic rel Z. We will not spell out all the definitions. From the context it should always be clear what we mean.

Exercises for Chapter 18.

Exercise 18.1. Let X be a topological space and let $f, g\colon X \to S^1$ be maps. We assume that for any $x \in X$ we have $f(x) \neq -g(x) \in S^1$. Show that f and g are homotopic.
Hint. Consider the map $S^1 \to S^1$ that is given by $z \mapsto f(z) \cdot g(z)^{-1}$.

Exercise 18.2. Let X be a topological space and let $f\colon X \to \mathbb{R}^n \setminus \{0\}$ be a map. Show that f is homotopic to the map $X \to \mathbb{R}^n \setminus \{0\}$ given by $x \mapsto \frac{f(x)}{\|f(x)\|}$.

Exercise 18.3. Let $n \in \mathbb{N}$ and let $f\colon \overline{B}^n \to \overline{B}^n$ be a homeomorphism with $f|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$. We consider the map

$$\begin{array}{rcl} H\colon \overline{B}^n \times [0,1] & \to & \overline{B}^n \\ (x,t) & \mapsto & \begin{cases} t \cdot f(\frac{x}{t}), & \text{if } 0 \leq \|x\| < t, \\ x, & \text{otherwise.} \end{cases} \end{array}$$

(a) Show that H is continuous.
(b) Show that for each $t \in [0,1]$ the map $H_t\colon \overline{B}^n \to \overline{B}^n$ is a homeomorphism.

Remark. We have thus shown that every homeomorphism of S^n extends to a homeomorphism of $\overline{B}^{n+1}$. In Proposition **??** we will see that exists a diffeomorphism $\varphi\colon S^6 \to S^6$ that does not extend to a diffeomorphism of $\overline{B}^7$.

Exercise 18.4.

(a) Let $f\colon [0,1] \to [0,1]$ be a self-homeomorphism.
  (i) Show that f is monotonously increasing or monotonously decreasing.
  (ii) Suppose f is monotonously increasing. Show that f is isotopic to the map $f(x) = x$.
  (iii) Suppose f is monotonously decreasing. Show that f is isotopic to the map $f(x) = -x$.

(b) Show that any self-homeomorphism of S^1 is isotopic to the map $f(z) = z$ or to the map $g(z) = \overline{z}$.
Hint. Let $f\colon S^1 \to S^1$ be a self-homeomorphism. First consider the case that $f(1) = 1$.

Exercise 18.5. Let h be a complex polynomial of degree ≥ 1. In Exercise 2.33 we showed that the map

$$\begin{array}{rcl}\Theta(h)\colon \mathbb{C}\cup\{\infty\} & \to & \mathbb{C}\cup\{\infty\} \\ z & \mapsto & \begin{cases} h(z), & \text{if } z\in\mathbb{C}, \\ \infty, & \text{if } z=\infty \end{cases}\end{array}$$

is continuous. Now let f and g be two polynomials with complex coefficients of degree ≥ 1. Suppose that $\deg(f) = \deg(g)$. Show that the maps $\Theta(f)$ and $\Theta(g)$ are homotopic.
Remark. It is quite easy to write down a candidate for a homotopy. But why is the map you wrote down actually continuous?

Exercise 18.6. Let $n \in \mathbb{N}$.

(a) Show that $\mathrm{SL}(n,\mathbb{R})$ is a retract of $\mathrm{GL}(n,\mathbb{R})$ and show that it is not a deformation retract of $\mathrm{GL}(n,\mathbb{R})$.
(b) Solve (a) with $\mathbb{C}$ instead of $\mathbb{R}$.

Exercise 18.7. Let M be a smooth manifold without boundary. We say that a path $\gamma\colon [a,b] \to M$ is *piecewise smooth* if there exists a subdivision $a = s_0 < s_1 < \cdots < s_m = b$ such that for $i = 0,\ldots,m-1$ the map $\gamma|_{[s_i,s_{i+1}]}$ is smooth.

(a) Show that any path $\alpha\colon [0,1] \to M$ is path-homotopic to a piecewise smooth path β.
(b) Show that any piecewise smooth path $\beta\colon [0,1] \to M$ is path-homotopic to a smooth path $\gamma\colon [0,1] \to M$.
Hint. Use Lemmas 6.13 and 14.4
(c) Suppose that $\dim(M) \geq 2$. Show that any path $\alpha\colon [0,1] \to M$ is path-homotopic to a smooth path δ such that $\delta'(t) \neq 0$ for every $t \in [0,1]$. (Later on we will say that δ is an immersion.)

Exercise 18.8. Using Lemma 2.44 we can make the identification $S^1 = \mathbb{R}\cup\{\infty\}$. Now let f be a polynomial with real coefficients of degree $n \geq 1$. In Exercise 2.33 we showed that the map

$$\begin{array}{rcl}F\colon S^1 = \mathbb{R}\cup\{\infty\} & \to & \mathbb{R}\cup\{\infty\} = S^1 \\ t & \mapsto & \begin{cases} f(t), & \text{if } t\in\mathbb{R}, \\ \infty, & \text{if } t=\infty \end{cases}\end{array}$$

is continuous. (See Figure 57 for an illustration.) Show that F is homotopic to one of the following three types of self-maps of S^1:

$$(1) \quad a(z) = z \qquad (2) \quad b(z) = \overline{z} \qquad (3) \quad c(z) = 1.$$

Exercise 18.9. Let X be a topological space and let Y be a contractible topological space.

(a) Show that all maps $X \to Y$ are homotopic.
(b) We suppose that X is path-connected. Show that all maps $Y \to X$ are homotopic.

Exercise 18.10. Let $U \subset \mathbb{R}^n$ be a neighborhood of 0 and let $f\colon U \to \mathbb{R}^n$ be map with $f^{-1}(\{0\}) = \{0\}$. Furthermore let $g\colon U \to \mathbb{R}^n$ be a map with $f(x) - g(x) = O(\|x\|^2)$, i.e. we assume that there exists a $C \geq 0$ and a $\delta > 0$ such that $\|f(x) - g(x)\| < M \cdot \|x\|^2$ for all $x \in \overline{B}^n_\delta$. Show that there exists an $s > 0$ such that the maps f and g are homotopic as maps of pairs $(\overline{B}^n_s, \overline{B}^n_s \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$.

Exercise 18.11.

(a) We consider the topological space
$$X \;=\; \{(x,y) \,|\, x \in [0,1] \cap \mathbb{Q}, y \in [0, 1-x]\} \cup ([0,1] \times \{0\})$$
that is shown in Figure 369 to the left.
  (i) Show that given any point P in $[0,1] \times \{0\}$ there exists a deformation retraction from X to P.
  (ii) Show that for any point Q in X that does not lie on $[0,1] \times \{0\}$ there is no deformation retraction from X to Q.

(b) We consider the infinite zigzag comb Z that is shown in Figure 369 to the right.
  (i) Show that Z is contractible.
  (ii) Show that Z does not admit a deformation retraction to any point in Z.

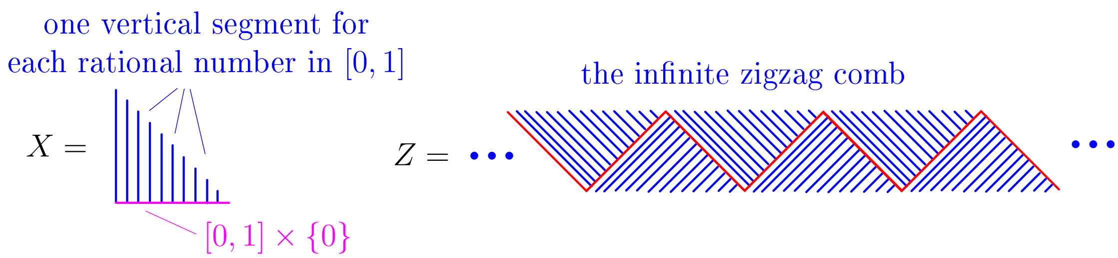


FIGURE 369

Exercise 18.12. Let $X = \{a, b\}$ be the topological space with two elements a and b and where the open sets are precisely $\varnothing, \{a\}, X$.

(a) Is $\{a\}$ a deformation retract of X?
(b) Is $\{b\}$ a deformation retract of X?

Exercise 18.13.

(a) Let $r > 0$. Show that $\mathbb{R}^n \setminus B^n_r(0)$ is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.
(b) Let A be a convex subset of $\mathbb{R}^n$ which contains the origin 0. Show that $\mathbb{R}^n \setminus B^n_r(0)$ is a deformation retract of $\mathbb{R}^n \setminus A$.
Remark. This sounds like one should use Proposition 2.53, but it turns out that this proposition is a not necessary.

Exercise 18.14. Let X be a topological space and let $H\colon X \times [0,1] \to X$ be a deformation to a subset A. We consider
$$\begin{array}{rcl} \phi\colon X & \to & [0,1] \\ x & \mapsto & \inf\{t \in [0,1] \,|\, H(x,t) \in A\}. \end{array}$$
Is the map ϕ necessarily continuous?

Exercise 18.15. Let X be a topological space and let Y be a topological space that is regionally compact. Let $f_0, f_1\colon Y \to X$ be two maps. Then the following holds:

$$f_0 \text{ and } f_1 \text{ are homotopic} \quad\Longleftrightarrow\quad f_0 \text{ and } f_1 \text{ lie in the same path component of } Y^X.$$

Remark. Use Proposition 5.6.

Exercise 18.16. Show that the topological spaces

$$X = \{\tfrac{1}{n} \,|\, n \in \mathbb{N}\} \cup \{0\} \quad \text{and} \quad Y = \{\tfrac{1}{n} \,|\, n \in \mathbb{N}\} \cup \{-1\}$$

are not homotopy equivalent.

FIGURE 370. Illustration of Exercise 18.16.

Exercise 18.17. We consider the two maps $f, g\colon [0,1] \to \mathbb{R}^3$ shown in Figure 371. Note that $f(0) = g(0)$ and $f(1) = g(1)$.

(a) Show that the maps f and g are homotopic rel $\{0\}$.
(b) Are the two maps f and g homotopic rel $\{0,1\}$?

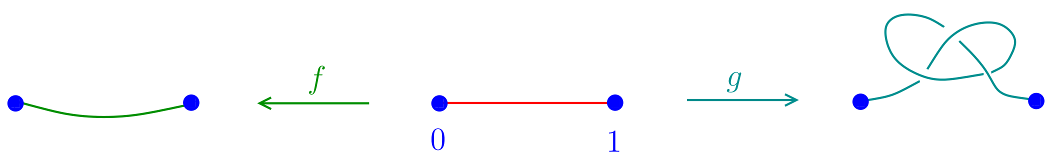


FIGURE 371

Exercise 18.18. Let $f\colon X \to Y$ be a map between topological spaces. A map $g\colon Y \to X$ is called a *homotopy left inverse of* f if $g \circ f$ is homotopic to id_X. Furthermore, a map $h\colon Y \to X$ is called a *homotopy right inverse of* f if $f \circ h$ is homotopic to id_Y. Show that if f admits a homotopy left inverse g and a homotopy right inverse h, then f is in fact a homotopy equivalence and both g and h are homotopy inverses.

Exercise 18.19. Let $f\colon X \to Y$ and $g\colon A \to B$ be maps between topological spaces. We say f *is a retract of* g if there exists a commutative diagram of maps

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "f"] \arrow[rr, bend left, "\mathrm{id}"] & A \arrow[r, "q"] \arrow[d, "g", "\simeq"'] & X \arrow[d, "f"] \\
Y \arrow[r, "r"] \arrow[rr, bend right, "\mathrm{id}"'] & B \arrow[r, "s"] & Y.
\end{tikzcd}$$

such that $q \circ p = \mathrm{id}_X$ and $s \circ r = \mathrm{id}_Y$. Show that if g is a homotopy equivalence, then f is also a homotopy equivalence.
Hint. Use Exercise 18.18.

Exercise 18.20. Let X be a path-connected topological space. Suppose that there exists an $x_0 \in X$ such that the inclusion $\{x_0\} \to X$ is a homotopy equivalence. Show that for any $x_1 \in X$ the inclusion $\{x_1\} \to X$ is a homotopy equivalence.

Exercise 18.21. Does there exist a topological space with three elements that is homotopy equivalent to the topology space $X = \{*\}$ given by a single point?

Exercise 18.22. Let $0 < r < s$ be two real numbers. Show that the obvious inclusion map $B_r^n(0) \setminus \{0\} \to B_s^n(0) \setminus \{0\}$ is a homotopy equivalence,

Exercise 18.23. Let $f\colon X \to Y$ be a map between topological spaces. Show that if f is a homotopy equivalence, then f induces a bijection $\pi_0(X) \to \pi_0(Y)$.

Exercise 18.24. Let $n \in \mathbb{N}$. Prove the following three statements:

(a) The sphere S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.
(b) The hemispheres $S^{n-1}_{\leq 0}$ is a deformation retract of $\overline{B}^n$.
(c) Given any three non-collinear points $A, B, C \in \mathbb{R}^n$ the triangle Δ_{ABC} admits a deformation retraction to the union of any two sides of the triangle.
Hint. You could first consider the case that $A = (0,0)$, $B = (0,1)$ and $C = (1,0)$.

These three examples are illustrated in Figure 344.

Exercise 18.25. Let A be a subset of a topological space X which admits a deformation retraction F from X to A. Show that the inclusion map

$$\begin{array}{rcl} A & \to & X \\ a & \mapsto & a \end{array} \qquad \text{and the map} \qquad \begin{array}{rcl} X & \to & A \\ x & \mapsto & F(x,1) \end{array}$$

are both homotopy equivalences.

Exercise 18.26. Show that

$$S^1 \vee S^1 \;=\; \underbrace{\{z \in \mathbb{C} \,|\, |z+1| = 1\} \cup \{z \in \mathbb{C} \,|\, |z-1| = 1\}}_{\text{two circles that meet at the origin}}$$

is a deformation retract of $\mathbb{C} \setminus \{\pm 1\}$.

Exercise 18.27. Let $\{(X_i, x_i)\}_{i \in I}$ be a family of pointed topological spaces. We suppose that each X_i is Hausdorff. Show that $\bigvee_{i \in I} X_i$ is also Hausdorff.

Exercise 18.28. Let $\{(X_i, x_i)\}_{i \in I}$ be a family of pointed topological spaces.

(a) We suppose that each X_i is path-connected. Show that $\bigvee_{i \in I} X_i$ is path-connected.
(b) We suppose that each X_i is connected. Show that $\bigvee_{i \in I} X_i$ is connected.

Exercise 18.29. Let $X := \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$. We denote by $\| - \|$ the usual Euclidean norm on X. We denote by $P = (0,0)$ the center of X. The *Paris metric* on X is given by

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} \\ (A,B) & \mapsto & \|A - P\| + \|B - P\|. \end{array}$$

This metric takes its name from the fact that all (OK, many) train lines in France go towards Paris. So if you want to go from A to B you invariably (OK, often) have to go through Paris.

(a) Show that the map

$$\begin{array}{rccl} \varphi\colon \bigvee_{z\in S^1}[0,1]_z & \to & X & \\ [r] & \mapsto & r\cdot z & \text{if } r\in[0,1]_z. \end{array}$$

is not a homeomorphism.

(b) Does there exist a homeomorphism between $\bigvee_{z\in S^1}[0,1]_z$ and X?

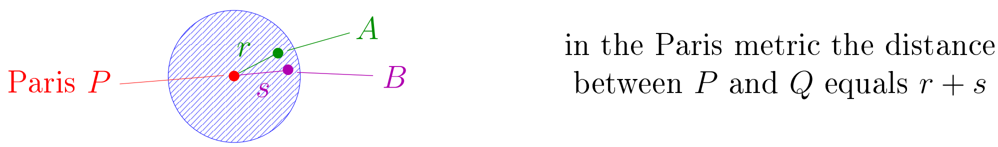


FIGURE 372. Illustration of Exercise 18.29.

Exercise 18.30. Let (X,x_0) and (Y,y_0) be two pointed topological spaces. We use the base points to define the wedge $X\vee Y$.

(a) Show that the map

$$\begin{array}{rcl} f\colon X\vee Y & \to & X\times Y \\ [P] & \mapsto & \begin{cases} (P,y_0), & \text{if } P\in X, \\ (x_0,P), & \text{if } P\in Y \end{cases} \end{array}$$

is an embedding.

Remark. We do not assume that X and Y have any nice properties.

(b) Using (a) we can view $X\vee Y$ as a subspace of $X\times Y$. We refer to the quotient

$$X\wedge Y \ := \ (X,x_0)\wedge(Y,y_0) \ := \ (X\times Y)/(X\vee Y)$$

as the *smash product of* (X,x_0) *and* (Y,y_0). Show that for any $m,n\in\mathbb{N}_0$ the smash product $S^m\wedge S^n$ is homeomorphic to S^{m+n}.

We refer to Figure 373 for an illustration.

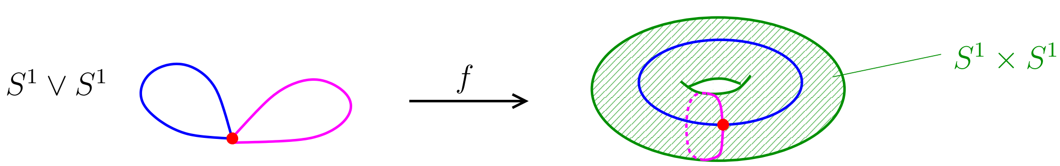


FIGURE 373. Illustration for Exercise 18.30.

Exercise 18.31. We consider the following two subsets of $\mathbb{C}$:

$$\begin{array}{rcl} X & := & \underbrace{\{z\in\mathbb{C}\,|\,|z-2|=1\}}_{=:A} \cup [-1,1] \cup \underbrace{\{z\in\mathbb{C}\,|\,|z+2|=1\}}_{=:B} \quad \text{and} \\ Y & := & \underbrace{\{z\in\mathbb{C}\,|\,|z-1|=1\}}_{=:C} \cup \underbrace{\{z\in\mathbb{C}\,|\,|z+1|=1\}}_{=:D}. \end{array}$$

Show that X and Y are homotopy equivalent.

Remark. It follows basically from Proposition 18.31 that X and Y are homotopy equivalent. The exercise thus consists of showing “by hand” that X and Y are homotopy equivalent.

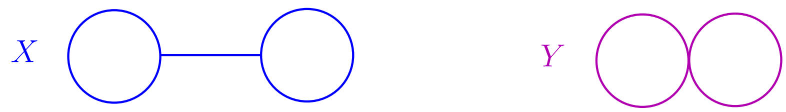


FIGURE 374. Illustration for Exercise 18.31.

Exercise 18.32. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Suppose that for each $i \in I$ we are given a point $x_i \in X_i$ with the property that $\{x_i\}$ is a closed subset of X_i. We use these points to form the wedge $\bigvee\limits_{i\in I} X_i$. We denote by $*$ the wedge point. Suppose that for each $i \in I$ we are given an open neighborhood U_i of x_i in X_i such that x_i is a deformation retract of U_i. We set $U := \bigcup\limits_{i\in I} U_i$.

(a) Show that U is an open neighborhood of $*$.
(b) Let $J \subset I$ be a subset. Show that $\bigvee\limits_{i\in J} X_i$ is a deformation retract of $U \cup \bigvee\limits_{i\in J} X_i$. (In particular, with $J = \varnothing$ this implies that $*$ is a deformation retract of U.)
Remark. You need to show that the wannabe deformation retraction which you wrote down is continuous. You might want to use a result along the lines of Lemma 5.18.

Exercise 18.33. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Suppose that for each $i \in I$ we are given a point $x_i \in X_i$. We use these points to form the wedge $\bigvee\limits_{i\in I} X_i$. Let $K \subset \bigvee\limits_{i\in I} X_i$ be a compact subset.

(a) We suppose that each X_i is Hausdorff. Show that there exists a finite subset $J \subset I$ such that $K \subset \bigvee\limits_{i\in J} X_i$.
Hint. For any $i \in I$ with $K \cap (X_i \setminus \{x_i\}) \neq \varnothing$ pick some point $a_i \in K \cap (X_i \setminus \{x_i\})$. We denote by A the union of these points. Show that A is a discrete subset of K.
(b) Show that in general the conclusion of (a) does not hold if we do not assume that the X_i are Hausdorff.

Exercise 18.34. We consider the subset X of $\mathbb{R}^2$ that is shown in Figure 375. It is the union of infinitely many circles $A_i \subset \mathbb{R}^2$, $i \in \mathbb{N}$ that meet in the origin. Furthermore let $Y = \bigvee\limits_{i\in\mathbb{N}} S^1$. There exists a fairly obvious bijection $f\colon X \to Y$ which sends each A_i to the i-th factor of Y.

(a) Is the map f continuous?
(b) Is the map f^{-1} continuous?

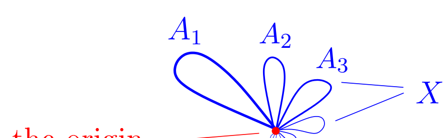


FIGURE 375. Illustration of Exercise 18.34.

Exercise 18.35. In the following we consider the wedge $X := \bigvee_{i\in\mathbb{N}_0} [0,1]_i$ given by "wedging" the points $0 \in [0,1]_i$, $i \in \mathbb{N}_0$.

(a) Show that the sequences of points $x_i := \frac{1}{2^i} \in [0,1]_i \subset X$, $i \in \mathbb{N}_0$ does not converge to the wedge point.
(b) Show that X is not metrizable, i.e. show that it does not admit a metric such that the topology given by the metric agrees with the topology of X.

Exercise 18.36. Let $f, g\colon (X, x_0) \to (Y, y_0)$ be two maps between pointed topological spaces.

(a) Show that if f and g are homotopic rel $\{x_0\}$, then $f_* = g_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$.
(b) Does the conclusion of (a) hold if we only demand that f and g are homotopic, but we no longer demand that they are homotopic rel the base point?

Exercise 18.37. Let $f\colon X \to Y$ be a map between two topological spaces. Let $x_0 \in X$ be a base point. We write $y_0 = f(x_0)$ and we denote by $i\colon f(X) \to Y$ the inclusion map. Suppose that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism. Does it follow that $i_*\colon \pi_1(f(X), y_0) \to \pi_1(Y, y_0)$ is also an isomorphism?

Exercise 18.38.

(a) Show that every topological graph is locally contractible.
(b) Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces. We suppose that each X_i is locally contractible. Show that the wedge $\bigvee_{i\in I} X_i$ is also locally contractible.

Remark. If you get away without using Proposition 18.20, then you should go carefully over your continuity arguments.

Exercise 18.39. Let T be a tree and let $v \in T$ be any vertex. Show that T admits a deformation retraction to v.
Remark. This is a slight refinement of the statement of Proposition 18.29 (2).

19. Basics of group theory

Before we continue with developing the theory of fundamental groups of topological spaces we want to introduce several basic constructions in group theory and we want state several facts which we will use throughout the remainder of this course.

19.1. Free abelian groups and finitely generated abelian groups.

The following definition will stay with us throughout these notes.

Definition.

(1) Let S be a non-empty set. We write
$$\mathbb{Z}^S \ := \ \text{all maps from } S \text{ to } \mathbb{Z}.$$
Furthermore, for S the empty set we write $\mathbb{Z}^S := \{0\}$. We consider $\mathbb{Z}^S$ with the obvious group structure, i.e. the group structure that is given by
$$\begin{array}{rcl} \mathbb{Z}^S \times \mathbb{Z}^S & \to & \mathbb{Z}^S \\ (f,g) & \mapsto & \left(\begin{array}{rcl} S & \to & \mathbb{Z} \\ s & \mapsto & f(s)+g(s) \end{array} \right). \end{array}$$

(2) Given any set S, we are mostly interested in the subgroup
$$\mathbb{Z}^{(S)} \ := \ \text{all maps from } S \text{ to } \mathbb{Z} \text{ which are non-zero for only finitely many } s \in S.$$
We refer to $\mathbb{Z}^{(S)}$ as the *free abelian group generated by* S or as the *free abelian group on the generating set* S.

(3) We refer to the cardinality of S as the *rank of* $\mathbb{Z}^{(S)}$.

When working with free abelian groups it is usually helpful to use the following notation.

Notation. Let S be a set.

(1) Given $s \in S$ we denote by s also the map
$$\begin{array}{rcl} S & \to & \mathbb{Z} \\ t & \mapsto & \begin{cases} 1, & \text{if } t = s, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$
This map lies of course in $\mathbb{Z}^{(S)}$. Thus we can view S as a subset of $\mathbb{Z}^{(S)}$.

(2) Given elements $s_1, \dots, s_k \in S$ and $n_1, \dots, n_k \in \mathbb{Z}$ we obtain the corresponding element $n_1 s_1 + \dots + n_k s_k$ of $\mathbb{Z}^{(S)}$. We refer to $n_1 s_1 + \dots + n_k s_k$ as a *formal linear combination* of elements in S. Every element of $\mathbb{Z}^{(S)}$ can be written as such a formal linear combination of finitely many elements of S. Indeed, for $f \in \mathbb{Z}^{(S)}$ we can write[314]
$$f \ = \sum_{s \in S \text{ with } f(s) \neq 0} f(s) \cdot s.$$

The following lemma says that the free abelian group generated by a set S has a "universal property". The lemma also says that possessing such a universal property classifies free abelian groups up to isomorphism.

[314] It is straightforward to see that both sides define the same map $S \to \mathbb{Z}$, so they are the same element of $\mathbb{Z}^{(S)}$.

Lemma 19.1.

(1) *Let S be a set and let $\phi\colon S \to G$ be a map to an abelian group G. Then there exists a unique homomorphism $\psi\colon \mathbb{Z}^{(S)} \to G$ that makes the following diagram commute*

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, "\phi"'] & \mathbb{Z}^{(S)} \arrow[d, "\psi"] \\ & G. \end{tikzcd}$$

(2) *Conversely let π be a group that admits a subset S with the following universal property: given any map $\phi\colon S \to G$ to an abelian group G there exists a unique homomorphism $\psi\colon \pi \to G$ that makes the following diagram commute*

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "\phi"'] & \pi \arrow[d, "\psi"] \\ & G. \end{tikzcd}$$

Then π is isomorphic to $\mathbb{Z}^{(S)}$.

Lemma 19.1 suggests the following definition.

Definition.

(1) We say that a group π is a *free abelian group on a generating set S*, if π has the universal property with respect to S. In other words, by Lemma 19.1, G needs to be isomorphic to $\mathbb{Z}^{(S)}$.
(2) We say that a group π is *free abelian* if it is a free abelian group on some generating set S.

Proof.

(1) We consider the map[315]

$$\begin{array}{rcll} \psi\colon \mathbb{Z}^{(S)} & \to & G & \\ f = \sum\limits_{i=1}^{n} n_i s_i & \mapsto & \psi(f) & := \sum\limits_{i=1}^{n} n_i \cdot \phi(s_i). \end{array}$$

It is straightforward to verify that ψ has the desired property and that it is the unique such homomorphism.

(2) Let π be a group that has the universal property with respect to S. We need to show that π is isomorphic to $\mathbb{Z}^{(S)}$. It is more elegant to show that it is isomorphic to any other group π' that has the universal property with respect to S.

This argument is similar to many other "universal property" proofs in algebra. We consider the inclusion maps $\varphi\colon S \to \pi$ and $\varphi'\colon S \to \pi'$. By the universal property of φ and φ' there exist homomorphisms $\psi\colon \pi \to \pi'$ and $\psi'\colon \pi' \to \pi$ that make the

[315]Here we assume that G is written as an additive group. If G is a multiplicative group we would need to replace $n_i \cdot \phi(s_i)$ by $\phi(s_i)^{n_i}$ and we would need to replace the sum by a product.

following diagrams commute:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & \pi \arrow[d, "\psi"] \\ & \pi' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S \arrow[r, "\varphi'"] \arrow[dr, "\varphi"'] & \pi' \arrow[d, "\psi'"] \\ & \pi. \end{tikzcd}$$

In particular we get the commutative diagrams

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[dr, "\varphi'"] \arrow[ddr, "\varphi"'] & \pi \arrow[d, "\psi"] \\ & \pi' \arrow[d, "\psi'"] \\ & \pi \end{tikzcd}$$

which simplifies to

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[dr, "\varphi"'] & \pi \arrow[d, "\psi'\circ\psi"] \\ & \pi \end{tikzcd}$$

and of course we also have the commutative diagram

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[dr, "\varphi"'] & \pi \arrow[d, "\mathrm{id}_\pi"] \\ & \pi. \end{tikzcd}$$

It follows from the uniqueness part of the universal property applied to the two diagrams on the right that $\psi' \circ \psi = \mathrm{id}_\pi$. The same way one shows that $\psi \circ \psi' = \mathrm{id}_{\pi'}$. In particular ψ and ψ' are isomorphisms. ■

Lemma 19.2.

(1) *Any subgroup G of a free abelian group F is again a free abelian group.*
(2) *If $\varphi\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ is a monomorphism, then*[316] $\#S \leq \#T$.
(3) *If $\varphi\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ is an epimorphism, then $\#S \geq \#T$.*

Proof. The first two statements are proved in [**Rot88**, Theorem 9.3]. It remains to prove the third statement. Now let $\varphi\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ be an epimorphism. For each $t \in T$ there exists therefore some $\phi(t) \in S$ with $\varphi(\phi(t)) = t$. By Lemma 19.1 there exists a unique homomorphism $\psi\colon \mathbb{Z}^{(T)} \to \mathbb{Z}^{(S)}$ with $\psi(t) = \phi(t)$ for all $t \in T$. Therefore we have $\varphi(\psi(t)) = \varphi(\phi(t)) = t$ for all t, so $\varphi \circ \psi = \mathrm{id}$, in particular ψ is a monomorphism. It follows from (2) that $\#T \leq \#S$. ■

Definition.

(1) If F is a free abelian group, then we define the *rank of F* as the cardinality of any set S with $F \cong \mathbb{Z}^{(S)}$.[317]
(2) Given an abelian group G we say that $S \subset G$ is a *generating set* if the canonical homomorphism $\mathbb{Z}^{(S)} \to G$ is an epimorphism.[318]
(3) Given an abelian group G we say that $S \subset G$ is a *basis* if the canonical homomorphism $\mathbb{Z}^{(S)} \to G$ is an isomorphism.
(4) We say that an abelian group G is *finitely generated* if it admits a finite generating set.

[316]Here we denote by $\#S$ and $\#T$ the cardinalities of the sets S and T, as defined on page 78.

Example. Let S be a finite set with m elements $s_1, \ldots, s_m$. Let $\phi\colon S \to \mathbb{Z}^m$ be the map given by $\phi(s_i) = e_i$. Then the resulting homomorphism $\Psi\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^m$ is an isomorphism. Put differently, $\mathbb{Z}^m$ is a free abelian group of rank m.

We leave it to the reader to provide the proof of the following useful characterizations of generating sets.

Lemma 19.3. *Let G be an abelian group and let $S \subset G$ be a subset. Then the following two statements are equivalent:*

(1) *S is a generating set of G.*
(2) *Any subgroup H of G that contains S is already equal to G.*

We use this opportunity to recall the classification of finitely generated abelian groups which had been proved in the algebra course. A proof is for example also provided in [**Hun80**, Chapter II.2].

Theorem 19.4. (Classification of Finitely Generated Abelian Groups) *Let A be a finitely generated abelian group. There exist non-zero natural numbers $a_1, \ldots, a_k$ with $a_i | a_{i+1}$ for $i = 1, \ldots, k-1$ and an $r \in \mathbb{N}_0$ such that*

$$A \;\cong\; \mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{a_i}.$$

Furthermore the numbers $a_1, \ldots, a_k$ and r are uniquely determined by A.

Definition. Given a finitely generated abelian group A we denote the direct sum of the cyclic groups on the right-hand side of Theorem 19.4 as the *canonical form of A*.

Example. The Chinese Remainder Theorem says that for any coprime m and n we have an isomorphism

$$\begin{array}{rcl} \mathbb{Z}_{mn} & \xrightarrow{\cong} & \mathbb{Z}_m \times \mathbb{Z}_n = \mathbb{Z}_m \oplus \mathbb{Z}_n \\ k + mn\mathbb{Z} & \mapsto & (k + m\mathbb{Z}, k + n\mathbb{Z}). \end{array}$$

For example consider $A = \mathbb{Z}_3 \oplus \mathbb{Z}_5$. It is not in the form of Theorem 19.4, but we have the isomorphism $\mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{15}$, which is of the desired form. Thus $\mathbb{Z}_{15}$ is the canonical form of $\mathbb{Z}_3 \oplus \mathbb{Z}_5$.

We will not provide a proof for Theorem 19.4, but we will sketch a proof for the following lemma. The third statement of the lemma is a somewhat weaker version of Theorem 19.4.

Lemma 19.5.

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[317] It follows from Lemma 19.2 and the Bernstein-Schröder Theorem 1.8 that the rank of a free abelian group is well-defined.

[318] Here we mean by "canonical map $\mathbb{Z}^{(S)} \to G$" the unique homomorphism $\psi\colon \mathbb{Z}^{(S)} \to G$ from Lemma 19.1 that makes the following diagram commute

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, hook, "\phi"'] & \mathbb{Z}^{(S)} \arrow[d, "\psi"] \\ & G \end{tikzcd}$$

where $S \to G$ is the inclusion map.

(1) *Let V be an $m \times n$–matrix over $\mathbb{Z}$. For matrices $A \in \mathrm{GL}(m,\mathbb{Z})$ and $B \in \mathrm{GL}(n,\mathbb{Z})$ we have an isomorphism*

$$\mathbb{Z}^m/AVB\mathbb{Z}^n \;\cong\; \mathbb{Z}^m/V\mathbb{Z}^n.$$

(2) *Let V be an $m \times n$–matrix over $\mathbb{Z}$. There exist two matrices $A \in \mathrm{GL}(m,\mathbb{Z})$ and $B \in \mathrm{GL}(n,\mathbb{Z})$*[319] *such that*[320]

$$A \cdot V \cdot B \;=\; \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & 0_{(m-t)\times(n-t)} \end{pmatrix} \;\in\; \mathrm{M}(m \times n, \mathbb{Z})$$

for some $p_1, \dots, p_t \in \mathbb{N}$.

(3) *Let A be a finitely generated abelian group. Then there exists an $r \geq 0$ and there exist $p_1, \dots, p_t \in \mathbb{N}$ such that*

$$A \;\cong\; \mathbb{Z}^r \oplus \bigoplus_{i=1}^{t} \mathbb{Z}_{p_i}.$$

As mentioned before, we will only provide sketch for the proof of the lemma.

Sketch of proof.

(1) The proof of the first statement is elementary. We leave it to the reader to fill in the details.

(2) First note that if an $m\times n$–matrix P is obtained from an $m\times n$–matrix Q by swapping two rows, then P is obtained by left-multiplying Q by an $m\times m$–matrix of the form[321]

$$\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Similarly if P is obtained from an $m \times n$–matrix Q by adding the k-th row to the l-th row, then P is obtained by left-multiplying Q by an $m \times m$–matrix of the form

$$\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}$$

[319] The matrices A and B are invertible over $\mathbb{Z}$, i.e. their determinants are ± 1.

[320] Here the last row and column of zeros can be arbitrarily large.

[321] Here is the reality check whether that is really correct: we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

where the extra 1 sits in the $(1,k)$-entry. Both these $m\times m$-matrices lie in $\mathrm{GL}(m,\mathbb{Z})$. If we do the same operations for columns, then this corresponds to right multiplication by the same type of matrices in $\mathrm{GL}(n,\mathbb{Z})$ as above.

Now let V be an $m\times n$–matrix over $\mathbb{Z}$. If V is the zero matrix, then there is nothing to show. So suppose that V is not the zero matrix. We perform the following steps:

(a) After swapping rows and columns we can arrange that the absolute value of the $(1,1)$-entry is less or equal than the absolute value of all other entries. After possibly multiplying the first column by -1 we can assume that the $(1,1)$-entry v_{11} is positive.
(b) By adding and subtracting the first column from the other columns we can arrange that all entries in the first row, except for the $(1,1)$-entry, lie between 0 and $v_{11}-1$.
(c) The same way as in (2) we can arrange that all entries in the first column, except for the $(1,1)$-entry, lie between 0 and $v_{11}-1$.

Now we have to distinguish two cases:

(a) If there exists outside of the $(1,1)$-entry another entry in the first row or column that is non-zero, then we start again with (1). Note that the new entry $(1,1)$-entry will be less than before. So this process will come to a halt after finitely many steps.
(b) If all entries in the first row and column, except for the $(1,1)$-entry are zero, then

$$V \;=\; \begin{pmatrix} v_{11} & 0 \\ 0 & V' \end{pmatrix}$$

for some $(m-1)\times(n-1)$–matrix V' and we perform the same operations on the smaller matrix V'.

(3) Let A be a finitely generated abelian group. By the definition of a finitely generated abelian group there exists an epimorphism $\psi\colon \mathbb{Z}^m \to A$. Since $\mathbb{Z}$ is a Noetherian ring we know that $\ker(\psi)$ is again a finitely generated free abelian group of rank $n\le m$. In particular there exists an isomorphism $v\colon \mathbb{Z}^n \to \ker(\psi)$. We denote by V the $m\times n$–matrix that represents the resulting map $\mathbb{Z}^n \to \ker(\psi)\subset \mathbb{Z}^m$ with respect to the standard bases. Then $A\cong \mathbb{Z}^m/V\mathbb{Z}^n$. We pick matrices A and B as in (2). From (1) it follows that $\mathbb{Z}^m/V\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^m/AVB\mathbb{Z}^n$. But the latter is isomorphic to $\mathbb{Z}_{p_1}\oplus\cdots\oplus\mathbb{Z}_{p_t}\oplus\mathbb{Z}^{m-n}$. ■

Definition. Let A be an abelian group.

(1) If A is finitely generated, then we refer to the r in the statement of Theorem 19.4 as the *rank* of the finitely generated abelian group A. We denote it by $\mathrm{rank}(A)$.[322]
(2) We refer to

$$\mathrm{Tor}(A) \;=\; \{a\in A \mid \text{ there exists an } n\in\mathbb{N} \text{ with } na=0\}$$

as the *torsion subgroup of A*.[323] If $\mathrm{Tor}(A)$ is the trivial subgroup, then we say that A is *torsion-free*.

In the following three lemmas we collect several basic facts about finitely generated abelian groups. Most of the statements follow more or less directly from Theorem 19.4. We will not prove the lemmas. We leave it to the reader to deduce the statements from Theorem 19.4 and the algebra course the reader took.

Lemma 19.6. *Let*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of abelian groups[324] *such that at least two of these groups are finitely generated. Then the following holds:*

(1) *all three groups are finitely generated abelian groups,*
(2) $\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C)$,
(3) *if A and C are torsion-free, then B is also torsion-free.*

Remark. Let B be a finitely generated group and let $\varphi\colon B \to C$ be an epimorphism. Note that C is evidently also finitely generated. If B is abelian it thus follows from Lemma 19.6 (1) that the kernel $\ker(\varphi)$ is also finitely generated. Note that if B is not abelian, then this conclusion does not necessarily hold. For example in Exercise 19.16 we will give an example of a group epimorphism $\varphi\colon B \to C$ from a non-commutative group B on two generators onto an abelian group C, such that the kernel is infinitely generated.

Lemma 19.7. *Let $A \in \mathrm{M}(n \times n, \mathbb{Z})$ be a matrix. Then*

$$\mathbb{Z}^n/A\mathbb{Z}^n \text{ is finite} \iff \det(A) \neq 0.$$

Furthermore, if $\det(A) \neq 0$, *then*

$$0 \to \mathbb{Z}^n \xrightarrow{A\cdot} \mathbb{Z}^n \to \mathbb{Z}^n/A\mathbb{Z}^n \to 0$$

is a short exact sequence and $\mathbb{Z}^n/A\mathbb{Z}^n$ is a finite group with $|\det(A)|$ elements.

We also have the following lemma.

Lemma 19.8.

(1) *If A, B and C are finitely generated abelian groups, then*

$$A \oplus C \cong B \oplus C \implies A \cong B.$$

(2) *Let A be a subgroup of an abelian group B.*
   (a) $\operatorname{Tor}(A)$ *is a subgroup of* $\operatorname{Tor}(B)$.
   (b) *If B is finitely generated, then A is also finitely generated, and we have the inequality* $\operatorname{rank}(A) \le \operatorname{rank}(B)$.
(3) *If A is a subgroup of a free abelian group of rank r, then A is a free abelian group with rank less or equal than r.*[325]

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[322] If A is a finitely generated free abelian group, then this definition of rank agrees with the definition given for free abelian groups on page 580.

[323] In Exercise 19.15 we will see that A is indeed a subgroup.

[324] Recall that this means that the map $A \to B$ is injective, that $B \to C$ is surjective and that the image of $A \to B$ agrees with the kernel of $B \to C$.

(4) *If A is a subgroup of a free abelian group B and if* $\operatorname{rank}(A) < \operatorname{rank}(B)$*, then A is a proper subgroup, i.e. $A \neq B$.*
(5) *Let $\varphi\colon A \to B$ be a homomorphism between two finitely generated abelian groups that are abstractly isomorphic.*[326] *If φ is an epimorphism, then φ is already an isomorphism.*
(6) *If A is a finitely generated abelian group and $\varphi\colon A \to B$ is an epimorphism, then B is also finitely generated and* $\operatorname{rank}(A) \geq \operatorname{rank}(B)$.

Abelian groups that are not finitely generated are not classified and they tend to be significantly harder to deal with. For example the following lemma shows that not every torsion-free abelian group is a free abelian group.

Lemma 19.9. *The abelian group $(\mathbb{Q}, +)$ is torsion-free but not free abelian.*

Proof. Clearly $(\mathbb{Q}, +)$ is torsion-free. If it was free abelian, then there would exist an epimorphism to $\mathbb{Z}$. In Exercise 19.1 we will show that such an epimorphism cannot exist. ■

In Exercise 19.3 (1) we will see that the conclusion of Lemma 19.8 (1) does not hold for general abelian groups. But the following statement actually does hold.

Proposition 19.10. *Let A and B be any two abelian groups. If C is a finitely generated abelian group, then*
$$A \oplus C \cong B \oplus C \quad \Longrightarrow \quad A \cong B.$$

Proof. We will consider a special case in Exercise 19.3 (b). The general case is proved in [**Cohn56, Walka56**]. ■

19.2. **The free product of groups.** We first recall the definition of the direct product of two groups G and H that we already gave on page 519. The direct product of two groups G and H is defined as
$$G \times H \ := \ \{(g, h) \mid g \in G \text{ and } h \in H\}$$
together with the group structure that is given by
$$(g_1, h_1) \cdot (g_2, h_2) \ := \ (g_1 g_2, h_1 h_2)$$
for $g_1, g_2 \in G$ and $h_1, h_2 \in H$. We call the group $G \times H$ the *direct product of the groups G and H*. Via the monomorphisms
$$\begin{array}{rcl} G & \to & G \times H \\ g & \mapsto & (g, e) \end{array} \quad \text{and} \quad \begin{array}{rcl} H & \to & G \times H \\ h & \mapsto & (e, h) \end{array}$$
we can view G and H as subgroups of $G \times H$. Note that these subgroups commute, i.e. for every $g \in G$ and $h \in H$ we have the equality $(g, e) \cdot (e, h) = (g, h) = (e, h) \cdot (g, e)$.

Now we will introduce the “free product” $G * H$ of two groups G and H. The definition is perhaps initially harder than the definition of the “direct product” $G \times H$. But for many, perhaps even most applications, the free product is more useful.

[325] In fact this statement is already contained in Lemma 19.2.

[326] For example A and B could be two free abelian groups of the same finite rank.

Definition. Let G be and H be two groups. We consider the set $G*H$ of all finite sequences $(x_1,\dots,x_m)$ such that the following conditions are satisfied:

(a) each x_i lies in one of the groups G or H,
(b) no x_j is the neutral element of G or of H,
(c) any two consecutive x_j's lie in two different groups.

Here we also allow the "empty sequence" (). Such sequences are sometimes called *reduced words in G and H.*

Let $(x_1,\dots,x_m)$ and $(y_1,\dots,y_n)$ be two elements in $G*H$. We define the product of these two elements as follows:

(1) First suppose that x_m and y_1 lie in different groups. Then we define
$$(x_1,\dots,x_m)\cdot(y_1,\dots,y_n) \;:=\; (x_1,\dots,x_m,y_1,\dots,y_n).$$
(2) Now suppose that x_m and y_1 lie in the same group. Let $j\in\{1,\dots,\min\{m,n\}\}$ be the maximal element such that $x_{m+1-i}=y_i^{-1}$ for $i=1,\dots,j$. If $j<\min\{m,n\}$, then we define[327]
$$(x_1,\dots,x_m)\cdot(y_1,\dots,y_n) \;:=\; (x_1,\dots,\underbrace{x_{m-j}\cdot y_{j+1}}_{\neq e},\dots,y_n).$$
If $j=m$, then we define
$$(x_1,\dots,x_m)\cdot(y_1,\dots,y_n) \;:=\; (y_{m+1},\dots,y_n).$$
We proceed similarly if $j=n$.

Another way of stating the product formula is as follows: given two sequences $(x_1,\dots,x_m)$ and $(y_1,\dots,y_n)$ we stack them together $(x_1,\dots,x_m,y_1,\dots,y_n)$ and then we delete any occurrence of a subsequence of the form a,a^{-1} for $a\in G$ or $a\in H$ and if a subsequence is of the form a,b with $a,b\in G$ or $a,b\in H$, then we replace it by ab.

We henceforth refer to $G*H$ together with this product structure as the *free product* of G and H.

We have the following important lemma.

Lemma 19.11. *The free product of two groups G and H is again a group. The neutral element of $G*H$ is hereby given by the empty sequence () and the inverse of an element $(x_1,\dots,x_m)\in G*H$ is given by*
$$(x_1,\dots,x_m)^{-1} \;=\; (x_m^{-1},\dots,x_1^{-1}) \;\in\; G*H.$$

Proof. It is clear that the empty sequence is a neutral element. It follows immediately from the definition of the product that the inverse of an element $(x_1,\dots,x_m)\in G*H$ is given by $(x_m^{-1},\dots,x_1^{-1})$. It remains to show that the multiplication satisfies associativity.

In principle one can show associativity naively "by hand", but there are surprisingly many cases one needs to distinguish, so that the proof becomes rather painful.

[327]Since x_m and y_1 lie in the same group, the elements x_{m-j} and y_{j+1} also lie in the same group, so it makes sense to consider $x_{m-j}\cdot y_{j+1}$.

We will therefore follow a different approach. We write $W = G * H$ and we denote by $(P(W), \circ)$ the group of all permutations of W, i.e. $P(W)$ is the set of all bijections from W to W and the group structure is given by composition of maps. For $g \in G$ we consider the map[328]

$$\begin{array}{rcl} \lambda_g \colon W & \to & W \\ (x_1, \dots, x_m) & \mapsto & \begin{cases} (g, x_1, x_2, \dots, x_m), & \text{if } x_1 \in H, \\ (gx_1, x_2, \dots, x_m), & \text{if } x_1 \in G \text{ and } g \neq x_1^{-1}, \\ (x_2, \dots, x_m), & \text{if } g = x_1^{-1}. \end{cases} \end{array}$$

It is now straightforward to see that for any $g, g' \in G$ and any $(x_1, \dots, x_m) \in W$ we have $g(g' \cdot (x_1, \dots, x_m)) = (gg') \cdot (x_1, \dots, x_m)$. This means that $\lambda_g \circ \lambda_{g'} = \lambda_{gg'}$. It is now straightforward to see that the map

$$\begin{array}{rcl} \lambda \colon G & \to & P(W) \\ g & \mapsto & \lambda_g \end{array}$$

is a group homomorphism. Similarly we define $\lambda \colon H \to P(W)$. Now we consider the map

$$\begin{array}{rcl} \lambda \colon W & \to & P(W) \\ (x_1, \dots, x_n) & \mapsto & \lambda_{x_1} \circ \dots \circ \lambda_{x_n}. \end{array}$$

This map is injective since $\lambda(x_1, \dots, x_n)$ applied to the empty word $()$ gives the word $(x_1, \dots, x_n)$. Hence $\lambda(x_1, \dots, x_n)$ is not the identity map from $W \to W$, i.e. $\lambda(x_1, \dots, x_n)$ is non-trivial in $P(W)$.

Furthermore the map $\lambda \colon W \to P(W)$ satisfies $\lambda(x \cdot y) = \lambda_x \circ \lambda_y$. Put differently, λ defines an injective map $\lambda \colon (W, \cdot\,) \to (P(W), \circ)$ that preserves the product structure. But since $(P(W), \circ)$ satisfies the associativity law we now see that $(W, \cdot\,)$ also satisfies the associativity law. ■

The following lemma summarizes two main properties of the free product of two groups.

Lemma 19.12. *Let G and H be two groups. The following two statements hold:*

(1) *The maps*

$$\begin{array}{rcl} i \colon G & \to & G * H \\ g & \mapsto & \begin{cases} (g), & \text{if } g \neq e, \\ (), & \text{if } g = e \end{cases} \end{array} \quad \text{and} \quad \begin{array}{rcl} j \colon H & \to & G * H \\ h & \mapsto & \begin{cases} (h), & \text{if } h \neq e, \\ (), & \text{if } h = e \end{cases} \end{array}$$

are monomorphisms.

(2) *For any two group homomorphisms $\alpha \colon G \to A$ and $\beta \colon H \to A$ there exists a unique group homomorphism $\gamma \colon G * H \to A$ such that the following diagram commutes*

$$\begin{tikzcd} G * H \arrow[dr, dashed, "\gamma"] & G \arrow[l, "i"'] \arrow[d, "\alpha"] \\ H \arrow[u, "j"] \arrow[r, "\beta"'] & A. \end{tikzcd}$$

[328] It is straightforward to see that the sequences on the right do indeed lie in $G * H$.

Example. Let G and H be groups. We denote by $\alpha\colon G \to G \times H$ and $\beta\colon H \to G \times H$ the obvious inclusion maps. It follows from Lemma 19.12 there exists a unique homomorphism $\gamma\colon G * H \to G \times H$ which makes the following diagram commute:

$$\begin{tikzcd} G * H \arrow[dr, dashed, "\gamma"] & G \arrow[l, "i"'] \arrow[d, "\alpha"] \\ H \arrow[u, "j"] \arrow[r, "\beta"'] & G \times H. \end{tikzcd}$$

Remark. In the literature the direct product of two groups often gets introduced in a slightly different way, namely by a "universal property". More precisely, let G be and H be two groups. The *direct product of* G *and* H is then defined as a group K together with two homomorphisms $i\colon G \to K$ and $j\colon G \to K$ which satisfy the following universal property: for every two group homomorphisms $\alpha\colon G \to A$ and $\beta\colon H \to A$ there exists a *unique* group homomorphism $\gamma\colon K \to A$ such that the following diagram commutes

$$\begin{tikzcd} K \arrow[dr, dashed, "\gamma"] & G \arrow[l, "i"'] \arrow[d, "\alpha"] \\ H \arrow[u, "j"] \arrow[r, "\beta"'] & A. \end{tikzcd}$$

Lemma 19.12 can now be reinterpreted as saying that the direct product of two groups always exists. An argument that is very similar to the one we gave on page 582 shows that the direct product is unique in the following sense: If $(K, i\colon G \to K, j\colon H \to K)$ and $(K', i'\colon G \to K', j'\colon H \to K')$ both satisfy the above universal property, then there exists a unique isomorphism $\Phi\colon K \to K'$ such that $i' = \Phi \circ i$ and $j' = \Phi \circ j$. [329]

Proof. It is clear that the maps i and j are injective, furthermore it follows immediately from the definition of the group structure on $G*H$ that the maps are group homomorphisms.

Now let $\alpha\colon G \to A$ and $\beta\colon H \to A$ be two group homomorphisms. For an element $(g_1, h_1, \ldots, g_k, h_k) \in G * H$ with $g_i \in G, h_i \in H$, $i = 1, \ldots, n$ we define

$$\gamma(g_1, h_1, \ldots, g_k, h_k) \ := \ \alpha(g_1) \cdot \beta(h_1) \cdot \cdots \cdot \alpha(g_k) \cdot \beta(h_k).$$

The same way we define γ for all other elements in $G * H$. Now one can show easily that γ is a group homomorphism.

It remains to show the uniqueness of γ. Any element in $G * H$ is a product of elements of the form $i(g)$, $g \in G$ and $j(h)$, $h \in H$. It follows that any homomorphism $\delta\colon G * H \to A$ is already uniquely determined by the values of δ on all elements of the form $i(g)$, $g \in G$ and $j(h)$, $h \in H$. Put differently, there is at most one homomorphism $\delta\colon G * H \to A$ with $\delta(i(g)) = \alpha(g)$ for all $g \in G$ and $\delta(i(h)) = \beta(h)$ for all $h \in H$. ■

Remark.

(1) We will use the monomorphisms from Lemma 19.12 to view G and H as subgroups of $G * H$.

[329] Why does $K = G \times H$ with the obvious maps $i\colon G \to K$ and $j\colon H \to K$ not satisfy this universal property? Or does it?

(2) If there is no danger of confusion, then we often shorten $(x_1, \ldots, x_m) \in G * H$ to the notation $x_1 \ldots x_m$. This notation is coherent with the convention (1). Note that if $g \in G$ and $h \in H$ are non-trivial elements, then the sequences (g, h) and (h, g) both lie in $G * H$ and they are different. With the convention that we just introduced that means that $g \cdot h \neq h \cdot g$. Put differently, a non-trivial element of G never commutes with a non-trivial element of H.

(3) The construction of the free product of two groups generalizes in an obvious way to the free product of finitely many, in fact even to infinitely many groups.

Let t be a symbol. We define

$$\langle t \rangle \;:=\; \{\ldots, t^{-2}, t^{-1}, e, t, t^2, \ldots\}.$$

The set $\langle t \rangle$ admits an obvious group structure that is given by $t^i \cdot t^j := t^{i+j}$. This group is called *the infinite cyclic group generated by* t. The map $\mathbb{Z} \to \langle t \rangle$, $n \mapsto t^n$ is evidently an isomorphism. Now we come to one of the main definitions in group theory.

Definition. For a set S we refer to

$$\langle S \rangle \;:=\; \text{free product of the groups } \langle s \rangle,\ s \in S$$

as the *free group on the (generating) set* S. Via the map $s \mapsto (s)$ we can and will view S as a subset of $\langle S \rangle$.

Example. If $S = \{t_1, \ldots, t_k\}$, then we obtain the group

$$\langle t_1, \ldots, t_k \rangle \;:=\; \langle t_1 \rangle * \cdots * \langle t_k \rangle$$

which we also refer to as the *free group on the generators* $t_1, \ldots, t_k$. Each element of the free group $\langle t_1, \ldots, t_k \rangle$ is thus of the form

$$t_{r_1}^{s_1}\, t_{r_2}^{s_2}\, \ldots\, t_{r_l}^{s_l}$$

with $r_i \neq r_{i+1}$ for $i = 1, \ldots, l-1$ and $s_i \neq 0$ for $i = 1, \ldots, l$.

Convention. Sometimes we denote the free group on k generators just by F_k.

Example. We consider the free group $\langle x, y \rangle$ on the two generators x and y. Elements in this group are of the form $xy^3x^{-2}y$ or $y^{-2}xy$. The multiplication is the "obvious one", for example

$$(xy^3x^{-1}y^2) \cdot (y^{-2}x^3yxy^{-1}) \;=\; xy^3x^2yxy^{-1}.$$

The group is not abelian since $xy \neq yx$.[330]

We continue with the following lemma that we will prove in Exercise 19.4.

Lemma 19.13. *Let S be a set. The free group $\langle S \rangle$ is torsion-free. In other words, every non-trivial element in the free group $\langle S \rangle$ has infinite order.*

The following innocent looking lemma summarizes the key property of free groups.

Lemma 19.14. *Let S be a set, let G be an arbitrary group and let $g\colon S \to G$ be a map. Then there exists a unique homomorphism $\varphi\colon \langle S \rangle \to G$ with $\varphi(s) = g(s)$ for every $s \in S$.*

[330]Why is $xy \neq yx$?

Proof. For $s \in S$ the map

$$\begin{array}{rcl} \langle s \rangle & \to & G \\ s^n & \mapsto & g(s)^n \end{array}$$

is evidently the unique homomorphism $\varphi_s \colon \langle s \rangle \to G$ with $\varphi_s(s) = g(s)$. It now follows from the obvious generalization of Lemma 19.12 to the case of the free product of arbitrarily many groups that there exists a unique homomorphism $\varphi \colon \langle S \rangle \to G$ with $\varphi(s) = \varphi_s(s) = g(s)$ for $s \in S$. ■

Remark. Sometimes in the literature free groups are defined via a universal property. More precisely, let S be a set. A *free group on* S is sometimes defined as a group K together with a map $f \colon S \to K$ which satisfies the following universal property: for any map $g \colon S \to G$ from the set S to a group G[331] there exists a unique group homomorphism $\varphi \colon K \to G$ such that the following diagram commutes

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "g"'] & K \arrow[d, "\varphi"] \\ & G. \end{tikzcd}$$

In this language our above Lemma 19.14 says that the obvious map $S \to \langle S \rangle$ is a free group on S in the above sense. As before, a slight variation on the argument on page 582 shows that the free group on S is unique in an appropriate sense.

Definition.

(1) A group G is called *free*, if it is isomorphic to the free group $\langle S \rangle$ on some set S. We say a subset $S \subset G$ is a *basis*, if the natural map[332] $\langle S \rangle \to G$ is an isomorphism.
(2) Given group G we say that $S \subset G$ is a *generating set*if the natural homomorphism $\langle S \rangle \to G$ is an epimorphism.Sometimes we also say that the elements in S are *generators of* G.
(3) We say that a group G is *finitely generated* if it admits a finite generating set.

Examples.

(1) As on page 501 we consider the following two self-homeomorphisms of $\mathbb{R}^2$:

$$\begin{array}{rcl} A \colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B \colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1) \end{array}$$

[331] Note that in contrast to Lemma 19.1, the group G is now not necessarily abelian.

[332] Here we mean by "canonical map $\langle S \rangle \to G$" the unique homomorphism $\psi \colon \langle S \rangle \to G$ from Lemma 19.14 that makes the following diagram commute

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, hook, "\phi"'] & \langle S \rangle \arrow[d, "\psi"] \\ & G \end{tikzcd}$$

where $S \to G$ is the inclusion map.

and we denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. This means that

$$G = \begin{array}{c}\text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written} \\ \text{as a finite concatenation of the maps } A, B, A^{-1} \text{ and } B^{-1}.\end{array}$$

We saw on page 501 that $\mathbb{R}^2/G$ equals the Klein bottle K and using Theorem 16.16 we saw on page 518 that $\pi_1(\text{Klein bottle}) \cong G$. The group G is finitely generated in the above sense. More precisely, it follows immediately from the definition of G that $S = \{A, B\} \subset G$ is a generating set for G.

(2) For any set S there exists by Lemma 19.14 a unique homomorphism $\langle S\rangle \to \mathbb{Z}^{(S)}$ that sends $s \in S$ to s viewed as an element in $\mathbb{Z}^{(S)}$. Since the set $S \subset \mathbb{Z}^{(S)}$ generates $\mathbb{Z}^{(S)}$ we see that this homomorphism is an epimorphism. It follows that any finitely generated abelian group in the sense of page 582 is also a finitely generated group in the above sense.

The subgroup of a free abelian group is again a free abelian group. Does the same hold for free groups? We record this as a question for future reference.

Question 19.15. *Is every subgroup of a free group again a free group?*

The next lemma says in particular that free groups on two sets are isomorphic if and only if the cardinalities of the sets are the same.

Lemma 19.16. *Let S and T be two sets.*

(1) *If there exists an epimorphism $\langle S\rangle \to \langle T\rangle$, then $\#S \geq \#T$.*
(2) *If $\langle S\rangle$ is isomorphic to $\langle T\rangle$, then $\#S = \#T$.*
(3) *If given $n \in \mathbb{N}$ we denote by F_n the free group on n generators, then for $k \neq l$ the groups F_k and F_l are not isomorphic.*

Proof.

(1) We first recall that we had just seen in the previous example that for any set U there exists a unique epimorphism $\psi_U \colon \langle U\rangle \to \mathbb{Z}^{(U)}$ that sends $u \in U$ to u viewed as an element in $\mathbb{Z}^{(U)}$.

Now let S and T be two sets and let $\varphi \colon \langle S\rangle \to \langle T\rangle$ be an epimorphism. By Lemma 19.1 there exists a unique homomorphism $\widetilde{\varphi} \colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ that sends $s \in S$ to $\psi_T(\varphi(s)) \in \mathbb{Z}^{(T)}$. We obtain the following diagram of homomorphisms

$$\begin{tikzcd} \langle S\rangle \arrow[r, two heads, "\varphi"] \arrow[d, two heads, "\psi_S"'] & \langle T\rangle \arrow[d, two heads, "\psi_T"] \\ \mathbb{Z}^{(S)} \arrow[r, "\widetilde{\varphi}"] & \mathbb{Z}^{(T)}. \end{tikzcd}$$

It follows immediately from the definition of $\widetilde{\varphi}$ that this diagram commutes.[333] Since the top horizontal is an epimorphism and since the vertical maps are epimorphisms

[333] Here we use that a homomorphism $\gamma \colon \langle S\rangle \to G$ is uniquely determined by the values of γ on S. Put differently, if $\alpha \colon \langle S\rangle \to G$ and $\beta \colon \langle S\rangle \to G$ are two homomorphisms with $\alpha(s) = \beta(s)$ for all $s \in S$, then it follows immediately from the uniqueness statement of Lemma 19.14 that $\alpha = \beta$.

we see that the bottom horizontal map is also an epimorphism. It follows from Lemma 19.2 that $\#S \geq \#T$.

(2) Let $f\colon \langle S\rangle \to \langle T\rangle$ be an isomorphism. We apply (1) to f and to f^{-1} and we obtain that $\#S \geq \#T$ and $\#T \geq \#S$. It follows from the Bernstein-Schröder Theorem 1.8 that $\#S = \#T$.

(3) This statement is an immediate consequence of (2).[334] ■

19.3. **An alternative definition of the free product of groups.** For some purposes it is sometimes useful to work with a different model for the free product of two groups.

Let A and B be two groups. We set[335]

$$S(A,B) \;:=\; \{\text{all finite sequences } (g_1,\dots,g_k) \text{ with each } g_i \text{ in } A \text{ or } B\}$$

and we denote by $\sim$ the equivalence relation on $S(A,B)$ that is generated by the relations

$$(g_1,\dots,g_l,e,g_{l+2},\dots,g_k) \;\sim\; (g_1,\dots,g_l,g_{l+2},\dots,g_k)$$

and by the relations

$$(g_1,\dots,g_l,g_{l+1},\dots,g_k) \;\sim\; (g_1,\dots,g_l g_{l+1},\dots,g_k)$$

if both g_l and g_{l+1} lie in A or both lie in B. It is now straightforward to verify that the map

$$\begin{array}{rcl} S(A,B)/\sim \;\times\; S(A,B)/\sim & \to & S(A,B)/\sim \\ ([(g_1,\dots,g_k)],[h_1,\dots,h_l]) & \mapsto & [(g_1,\dots,g_k,h_1,\dots,h_l)] \end{array}$$

is well-defined and that it defines a group structure on $S(A,B)/\sim$.[336]

Proposition 19.17. *Let A and B be two groups. Then the map*[337]

$$\begin{array}{rcl} \Phi\colon S(A,B)/\sim & \to & A*B \\ [(g_1,\dots,g_k)] & \mapsto & g_1\cdot\dots\cdot g_k \end{array}$$

is well-defined and it is an isomorphism. The inverse map is given by

$$\begin{array}{rcl} \Psi\colon A*B & \to & S(A,B)/\sim \\ (g_1,\dots,g_k) & \mapsto & [(g_1,\dots,g_k)]. \end{array}$$

In the proof of Proposition 19.17 we will need the following little lemma.

Lemma 19.18. *Let $\Phi\colon G\to H$ and $\Psi\colon H\to G$ be two group homomorphisms such that $\Phi\circ\Psi = \mathrm{id}_H$ and such that Ψ is an epimorphism. Then Φ is an isomorphism.*

[334] Note that if we just want to prove statement (3), then in the proof of (2) we could replace the slightly esoteric Lemma 19.2 by the more conventional Lemma 19.6 that was proved in the Algebra course.

[335] This definition is similar to the definition of $A*B$, but now we allow that elements of the sequence are trivial and we allow that consecutive elements lie in the same group.

[336] The neutral element is again represented by the empty sequence ( ) and the inverse of $[(g_1,\dots,g_k)]$ is given by $[(g_k^{-1},\dots,g_1^{-1})]$. This time associativity is not a problem. Why not?

[337] Recall that we use the canonical monomorphisms from Lemma 19.12 to view A and B as subgroups of $A*B$. If we want to be more precise, then we could say that Φ is given by $[(g_1,\dots,g_k)]\mapsto (g_1)\cdot\dots\cdot(g_k)$.

The lemma can be summarized as follows, if we are given a commutative diagram of the form

$$\begin{tikzcd} & G \arrow[dr, "\Phi"] & \\ H \arrow[ur, "\Psi"] \arrow[rr, "\mathrm{id}"'] & & H, \end{tikzcd}$$

then Φ is an isomorphism.[338]

Proof. It follows from $\Phi \circ \Psi = \mathrm{id}_H$ that Φ is an epimorphism. It remains to show that Φ is a monomorphism. So let $g \in \ker(\Phi)$. Since Ψ is an epimorphism there exists an $h \in H$ with $\Psi(h) = g$. Since $\Phi \circ \Psi = \mathrm{id}_H$ we see that $h = \Phi(\Psi(h)) = \Phi(g) = e$. Thus $h = e$ and thus $g = \Psi(h) = \Psi(e) = e$. ■

Proof. It is straightforward to verify that the map

$$\begin{array}{rcl} \Phi\colon S(A,B)/\sim & \to & A * B \\ [(g_1,\dots,g_k)] & \mapsto & g_1 \cdot \dots \cdot g_k \end{array}$$

is well-defined and that it is a homomorphism. This homomorphism is surjective since $A * B$ is a subset of $S(A, B)$. Next we consider the map[339]

$$\begin{array}{rcl} \Psi\colon A * B & \to & S(A,B)/\sim \\ (g_1,\dots,g_k) & \mapsto & [(g_1,\dots,g_k)]. \end{array}$$

It is straightforward to see that Ψ is a homomorphism. It is clear that $\Phi \circ \Psi = \mathrm{id}_{A*B}$. It is not so obvious that $\Psi \circ \Phi = \mathrm{id}_{S(A,B)/\sim}$. In fact we will proceed in a slightly different fashion: to show that Φ is an isomorphism it suffices, according to Lemma 19.18, to show that Ψ is surjective. This in turn follows from the following claim.

Claim. Any element in $S(A, B)$ is equivalent to an element in $A * B$.

We prove the claim by induction on the length of a sequence in $S(A, B)$. Clearly the claim holds for all sequences of length 0. Suppose we already know the claim for all sequences of length $\leq k-1$. Let $(g_1, \dots, g_k)$ be a sequence of length k. If $(g_1, \dots, g_k) \in A*B$, then we are done. Now suppose that this is not the case. Then one of the following occurs:

(1) either there exists an $i \in \{1, \dots, k\}$ with $g_i = e$, or
(2) there exists an $i \in \{1, \dots, k-1\}$ with $g_i, g_{i+1} \in A$, or
(3) there exists an $i \in \{1, \dots, k-1\}$ with $g_i, g_{i+1} \in B$.

In the first case the sequence is equivalent to $(g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_k)$. In the second and third case the sequence is equivalent to $(g_1, \dots, g_{i-1} \cdot g_{i+1}, \dots, g_k)$. In all three cases we have thus shown that our original sequence is equivalent to a sequence of length $k-1$. So we are done by our induction hypothesis. ■

[338] Is Ψ also necessarily an isomorphism?

[339] Put differently, $A * B \to S(A, B)/\sim$ is the homomorphism that is given by the universal property of $A * B$.

Let A and B be two groups. In the following we will use the isomorphism of Proposition 19.17 to identify the group $S(A,B)/\sim$ with the free product $A*B$. Both groups have their advantages:

(1) In the group $S(A,B)/\sim$ it is easier to write down the product of two elements, we just need to juxtapose sequences of elements in A and B.
(2) On the other hand in the group $A*B$ it is trivial to check whether two elements are the same. For example for two non-trivial elements $a \in A$ and $b \in B$ the elements (a,b) and (b,a) of $A*B$ are by definition different. But from the definition of $S(A,B)/\sim$ it is not immediately clear that $[(a,b)]$ and $[(b,a)]$ are indeed different.

19.4. **The Grushko-Neumann Theorem.** We start out with the following natural definition.

Definition. Let G be a finitely generated group. We denote by $d(G) \in \mathbb{N}_0$ the minimal number of elements in a generating set for G.

Remark. It is an unfortunate fact of life that in the literature the number $d(G)$ is usually referred to as the rank of the group G. But for abelian groups the invariants $\operatorname{rank}(G)$ and $d(G)$ can differ quite dramatically. For example for $G = (\mathbb{Z}_2)^n$ we have $\operatorname{rank}(G) = 0$ but $d(G) = n$.

The following theorem was proved by Igor Grushko [**Gru40**] in 1940 and independently by Bernhard Neumann [**Neum43**] in 1940.

Theorem 19.19. (Grushko-Neumann Theorem) *Given any two finitely generated groups A and B we have*

$$d(A*B) \;=\; d(A) + d(B).$$

Proof. The proof is evidently given in the above two references, but it can also be found in any self-respecting book on combinatorial group theory, see e.g. [**LS77**, Corollay IV.1.9] and [**MKS76**, p. 192]. ■

We already saw that some groups can be written as a direct product of two groups in different ways. For example it follows from the Chinese Remainder Theorem that the groups $\mathbb{Z}_3 \times \mathbb{Z}_{35}$ and $\mathbb{Z}_5 \times \mathbb{Z}_{21}$ are isomorphic. The following theorem says that the situation is much more rigid for free products. Before we can state the theorem we have to introduce the following definition.

Definition. A group G is *indecomposable* if it is not isomorphic to the free product of two non-trivial groups.

The following lemma will be proved in Exercise 19.6.

Lemma 19.20. *Abelian groups and finite groups are indecomposable.*

Now we can state the Grushko[340] Decomposition Theorem.

[340]Igor Grushko was a Russian mathematician. The theorem was proved in 1940.

Theorem 19.21. *Let G be a finitely generated group.*

(1) *The group G is isomorphic to the free product $G = A_1 * \cdots * A_k$ of non-trivial indecomposable groups.*[341]
(2) *The isomorphism types of $A_1, \ldots, A_k$ are unique up to permutation. More precisely, if $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$ are indecomposable finitely generated groups such that*

$$A_1 * \cdots * A_k \;\cong\; B_1 * \cdots * B_l,$$

then $k = l$ and furthermore there exists a permutation $\sigma \in S_k$ such that $A_i \cong B_{\sigma(i)}$ for $i = 1, \ldots, k$.

PROOF.

(1) This statement follows easily from induction on $d(G)$ and the Grushko-Neumann Theorem 19.19. Indeed, if $d(G) = 0$ then there is nothing to prove. Next suppose that the statement is known for all groups G with $d(G) < k$. Let G be a group with $d(G) = k$. If G is indecomposable, we are done. If G is not indecomposable, then we can write G as the free product of two non-trivial groups A and B. By the Grushko-Neumann Theorem 19.19 we have $d(G) = d(A) + d(B)$. Since A and B are non-trivial we have $d(A) \geq 1$ and $d(B) \geq 1$. Thus we have $d(A) < k$ and $d(B) < k$ and by induction we can find decompositions for A and B. These two decompositions put together give the desired decomposition for $G = A * B$.
(2) The uniqueness statement was proved by Aleksandr Kurosh in 1934, see [**Kuro34**, Isomorphismsatz] and [**Kuro60**, p. 27]. Alternative proofs are given in [**Stal75**, p. 168f] and [**Nel**, Theorem 4.5]. Also see [**CM82**, Chapter II.4] for more context. ■

Example. It follows from Lemma 19.20 and from the Grushko Decomposition Theorem 19.21 that the free product $\mathbb{Z}_3 * \mathbb{Z}_{35}$ is not isomorphic to the free product $\mathbb{Z}_5 * \mathbb{Z}_{21}$.

Exercises for Chapter 19.

Exercise 19.1. Show that $\mathbb{Q}$ does not admit an epimorphism onto $\mathbb{Z}$.

Exercise 19.2. Given $n \in \mathbb{Z}$ we consider the matrix $B(n) = \begin{pmatrix} 0 & 3 \\ 3 & n \end{pmatrix}$ and we consider the abelian group $A_n := \mathbb{Z}^2/B(n)\mathbb{Z}^2$. Complete the following sentence: given $m, n \in \mathbb{Z}$ the abelian groups A_m and A_n are isomorphic if and only if ....

Exercise 19.3.

(a) Show that there exists an abelian group G with $G \oplus G \cong G$.
(b) Let A be a finitely generated abelian group. Show that if B is an abelian group with $B \oplus A \cong A$, then $B \cong 0$.
Hint. Arguably it is easiest to start out with the special case that $A = \mathbb{Z}$.

Exercise 19.4. . Let S be a set. Show that every non-trivial element in the free group $\langle S \rangle$ has infinite order.

[341] If G is trivial then $k = 0$ and the free product of zero groups is by convention the trivial group.

Exercise 19.5. Let M be a connected smooth manifold, let $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ be points on M. Does there exist an action by a free group F on M such that for any $i \in \{1, \dots, n\}$ there exists a $g \in F$ with $g \cdot P_i = Q_i$?

Exercise 19.6. Show that abelian groups and finite groups are indecomposable.

Exercise 19.7. Let G be an additive abelian group. We say that an element $g \in G$ is *primitive* if there does not exist an $h \in G$ and an $n \geq 2$ with $n \cdot h = g$.

(1) Suppose G is finitely generated. Let $g \in G$. Show that the following statements are equivalent:
   (a) g is a primitive element.
   (b) We can write $G = \mathbb{Z} \cdot g \oplus H$ for some abelian group G.
   (c) There exists a homomorphism $\varphi\colon G \to \mathbb{Z}$ with $\varphi(g) = 1$.

(2) Which of the implications in (1) also hold for infinitely generated abelian groups?

Exercise 19.8. Let $p, q, r \in \mathbb{N}_{\geq 2}$ be coprime. By the Chinese Remainder Theorem we know that $\mathbb{Z}_{pq} \times \mathbb{Z}_r \cong \mathbb{Z}_p \times \mathbb{Z}_{qr}$. Show, without making use of the Grushko Decomposition Theorem, that $\mathbb{Z}_{pq} * \mathbb{Z}_r$ is not isomorphic to $\mathbb{Z}_p * \mathbb{Z}_{qr}$.
Perhaps it is notationally easier to write $A = \{e, a, \dots, a^{pq-1}\}$, $B = \{e, b, \dots, b^{r-1}\}$, $C = \{e, c, \dots, c^{p-1}\}$ and $D = \{e, d, \dots, d^{qr-1}\}$, where A, B, C and D have the obvious multiplicative group structure. Then the problem is to show that $A * B$ is not isomorphic to $C * D$.

Exercise 19.9. Let S and T be sets. If there exists a monomorphism $\varphi\colon \langle S \rangle \to \langle T \rangle$, does it follow that $\#S \leq \#T$?

Exercise 19.10. A group G is called *locally free* if every finitely generated subgroup is free. Is every locally free group also a free group?

Exercise 19.11. Let G be a group and let $g \in G$. The *centralizer* of $g \in G$ is defined as

$$C_G(g) \ := \ \{h \in G \,|\, gh = hg\}.$$

(a) Let A and B be groups and let $g \in A * B$. We view A and B as subgroups of $A * B$ in the obvious way. Let $g \in A * B$. Show that

$$C_G(g) \ = \ \begin{cases} C_A(g), & \text{if } g \in A, \\ C_B(g), & \text{if } g \in B, \\ \langle h \rangle, \quad h \in A * B \text{ of infinite order} & \text{otherwise.} \end{cases}$$

(b) Let S be a set. Show that every abelian subgroup of the free group $\langle S \rangle$ is isomorphic to $\mathbb{Z}$.
   Remark. Note that S is not necessarily finite.

Exercise 19.12. Let G be a group and let $x, y \in G$. Suppose there exists an action of G on some set X and two disjoint subsets A and B of X such that the following two conditions are satisfied:

(1) for every $n \in \mathbb{Z} \setminus \{0\}$ we have $x^n \cdot A \subset B$, and
(2) for every $n \in \mathbb{Z} \setminus \{0\}$ we have $y^n \cdot B \subset A$.

Show that x and y generate a free subgroup of G, i.e. show that the map $\langle x\rangle * \langle y\rangle \to G$ is a monomorphism.
Hint. Show that any non-trivial element $(g_1, \dots, g_k) \in \langle x\rangle * \langle y\rangle$ acts non-trivially on X.
Remark. This statement is known as the *Ping-Pong Lemma*.

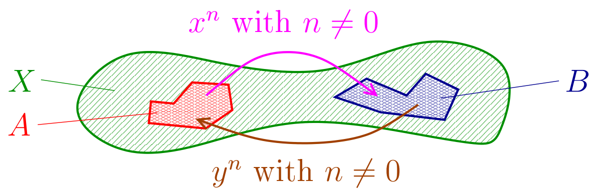

FIGURE 376. Illustration of Exercise 19.12.

Exercise 19.13. We consider the group $G = \mathrm{SL}(2,\mathbb{Z})$ and we consider the two matrices $x = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Use the Ping-Pong Lemma from Exercise 19.12 to show that x and y generate a free subgroup of G.
Hint. Consider $X = \mathbb{R}^2$ together with the subsets $A = \{(x,y) \in \mathbb{R}^2 \,|\, |y| > |x|\}$ and $B = \{(x,y) \in \mathbb{R}^2 \,|\, |x| > |y|\}$.

Exercise 19.14. A group π is called *residually finite* if given any non-trivial element $g \in \pi$ there exists a homomorphism $\alpha\colon \pi \to G$ to a finite group G such that $\alpha(g)$ is non-trivial.

(a) Show that for any $n \in \mathbb{N}$ the group $\mathrm{GL}(n,\mathbb{Z})$ is residually finite.
(b) Show that the free group on two generators is residually finite.
Remark. Use Exercise 19.13.

Exercise 19.15. Let G be a group. We consider the subset

$$\mathrm{Tor}(G) \;:=\; \{g \in G \,|\, \text{ there exists an } n \in \mathbb{N} \text{ with } g^n = e\}.$$

(a) Suppose that G is abelian. Show that $\mathrm{Tor}(G)$ is a subgroup of G. We refer to it as the *torsion subgroup of* G.
(b) Give an example of a group G such that $\mathrm{Tor}(G)$ is not a subgroup of G.

Exercise 19.16.

(a) Let $\phi\colon \langle x, y\rangle \to \mathbb{Z}$ be the homomorphism given by $\phi(x) = 1$ and $\phi(y) = 0$. Show that $\ker(\phi)$ is infinitely generated.
Remark. In Proposition 21.14 we will see that any non-trivial *normal* subgroup of infinite index of a free group is infinitely generated.
(b) Let $\psi\colon \langle s, t\rangle \times \langle x, y\rangle \to \mathbb{Z}$ be the homomorphism given by $\psi(s) = 1$, $\psi(t) = 0$, $\psi(x) = 1$ and $\psi(y) = 0$. Show that $\ker(\psi)$ is finitely generated.

Exercise 19.17. We consider the free group $F = \langle x, y\rangle$. Let $a = (a_1, a_2, \dots, a_m)$ be a reduced word in $\langle x\rangle$ and $\langle y\rangle$. We denote by $\overline{a} = (a_m, \dots, a_1)$ the reverse word. Given any reduced word $z = (z_1, \dots, z_n)$ we write

$$|z|_a \;=\; \text{number of non-overlapping subsequences of } z_1, \dots, z_n \text{ that agree with } a_1, \dots, a_m.$$

Now we consider the following map

$$\begin{array}{rcl} \varphi_a \colon \langle x, y\rangle & \to & \mathbb{Z}_{\geq 0} \\ z & \mapsto & |z|_a - |z|_{\overline{a}}. \end{array}$$

(a) Show that in general φ_a is not a homomorphism.
(b) Show that there exists a $D > 0$ such that for every $g, h \in \langle x, y\rangle$ we have

$$|\varphi_a(g \cdot h) - \varphi_a(g) - \varphi_a(h)| \ \leq \ D.$$

Remark. Any map $\varphi\colon G \to \mathbb{R}$ from a group G to $\mathbb{R}$ that admits a D as above is called a *quasi-morphism*. The above quasi-morphism is called the *Brooks quasi-morphism* [**Brooks81**].

Exercise 19.18. Let $A \in \mathrm{M}(m \times n; \mathbb{Z})$.

(a) Show that there exist matrices $P \in \mathrm{GL}(m, \mathbb{Z})$ and $Q \in \mathrm{GL}(n, \mathbb{Z})$ such that PAQ is of the form

$$PAQ \ = \ \begin{pmatrix} D_{k\times k} & 0_{k\times(n-k)} \\ 0_{k\times(m-k)} & 0_{(n-k)\times(m-k)} \end{pmatrix}$$

where $D = \mathrm{diag}(d_1, \dots, d_k)$ is a diagonal $(k\times k)$-matrix with diagonal entries $d_1, \dots, d_k \in \mathbb{N}$.
(b) Show that in (a) we can find P and Q such that for each $i \in \{1, \dots, k-1\}$ we have $d_i | d_{i+1}$.
Remark. The corresponding matrix PAQ is called the *Smith normal form* of A. The above result is sometimes called the *Smith Normal Form Theorem*. We refer to [**NewM72**, Chapter 15] and [**Nor12**, Chapter 1] for more information.

Exercise 19.19. Let G and H be two groups.

(a) Suppose there exist monomorphisms $\varphi\colon G \to H$ and $\psi\colon H \to G$. Does it follow that G and H are isomorphic?
(b) Suppose there exist epimorphisms $\varphi\colon G \to H$ and $\psi\colon H \to G$. Does it follow that G and H are isomorphic?

20. The basic Seifert–van Kampen Theorem

We are still lacking the computation of fundamental groups of even fairly simple topological spaces. Now we will take a new approach to calculating fundamental groups. Namely in the following two chapters we will formulate and prove results which allow us to calculate the fundamental group of a given topological space from the fundamental groups of suitable subspaces.

20.1. **The formulation of the basic Seifert–van Kampen Theorem.** The idea of the Seifert–van Kampen theorem is to reduce the calculation of the fundamental group of a topological space X to the determination of fundamental groups of "simpler subsets" of X.

Let X be a topological space and let $U, V \subset X$ be two subsets with $X = U \cup V$ and with $U \cap V \neq \varnothing$. We choose a base point $x_0 \in U \cap V$ and we consider the following inclusion maps and the corresponding induced homomorphisms of the fundamental groups:

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d, "i"] \\ V \arrow[r, "j"'] & X = U \cup V \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \pi_1(U \cap V, x_0) \arrow[r] \arrow[d] & \pi_1(U, x_0) \arrow[d, "i_*"] \\ \pi_1(V, x_0) \arrow[r, "j_*"'] & \pi_1(X, x_0) = \pi_1(U \cup V, x_0). \end{tikzcd}$$

Our goal is to determine the fundamental group of the total space $X = U \cup V$ in terms of the fundamental groups of U, V, $U \cap V$ and the induced maps $\pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$.

We start out with the following lemma:

Lemma 20.1. *Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and such that $U \cap V \neq \varnothing$. We choose a base point $x_0 \in U \cap V$. If $U \cap V$ is path-connected, then the homomorphism*

$$\pi_1(U, x_0) * \pi_1(V, x_0) \;\to\; \pi_1(X, x_0),$$

which is induced by the homomorphisms $\pi_1(U, x_0) \to \pi_1(X, x_0)$ and $\pi_1(V, x_0) \to \pi_1(X, x_0)$ and by Lemma 19.12 (2), is surjective.

Examples.

(1) We consider the sphere

$$X \;=\; S^2 \;=\; \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\},$$

with the covering $X = U \cup V$ given by the two open subsets

$$\begin{aligned} U &= \{(x, y, z) \mid x^2 + y^2 + z^2 = 1 \text{ and } z > -\tfrac{1}{2}\}, \\ V &= \{(x, y, z) \mid x^2 + y^2 + z^2 = 1 \text{ and } z < \tfrac{1}{2}\}. \end{aligned}$$

We pick the base point $x_0 = (1, 0, 0) \subset U \cap V$. Here the intersection $U \cap V$ is path-connected, therefore we can apply Lemma 20.1. It says that we have an epimorphism

$$\pi_1(U, x_0) * \pi_1(V, x_0) \;\to\; \pi_1(S^2, x_0).$$

The subspaces U and V are homeomorphic to an open disk,[342] thus the fundamental groups are trivial. It follows that the trivial group[343] surjects onto $\pi_1(S^2,x_0)$, thus $\pi_1(S^2,x_0)$ itself is trivial. Precisely the same argument shows that $\pi_1(S^n)=0$ for $n\geq 3$.

We had already obtained this result in Proposition 14.14. In fact, as we will see shortly, the proof of Lemma 20.1 has clear parallels to the proof of Proposition 14.14.

(2) We consider the sphere

$$X \;=\; S^1 \;=\; \{e^{\mathrm{i}t}\mid t\in\mathbb{R}\},$$

with the covering $X=U\cup V$ given by the two open subsets

$$\begin{aligned} U &:= \{e^{\mathrm{i}t}\mid t\in(-\tfrac{\pi}{4},\tfrac{5\pi}{4})\}\\ V &:= \{e^{\mathrm{i}t}\mid t\in(\tfrac{3\pi}{4},\tfrac{9\pi}{4})\}. \end{aligned}$$

We pick the base point $x_0=1$. The subsets U and V are both homeomorphic to an open interval, hence $\pi_1(U,x_0)=\pi_1(V,x_0)=0$, but of course we know from Corollary 16.18 that $\pi_1(X,x_0)\cong\mathbb{Z}$. Thus we see that $\pi_1(U,x_0)*\pi_1(V,x_0)\to\pi_1(X,x_0)$ is not an epimorphism. This is not a contradiction to Lemma 20.1, since in this example the intersection $U\cap V$ is not path-connected.

Now we turn to the proof of Lemma 20.1.

Proof of Lemma 20.1. Let X be a topological space and let $U,V\subset X$ be two open subsets with $X=U\cup V$ and such that $U\cap V\neq\varnothing$. We choose a base point $x_0\in U\cap V$. It suffices to prove the following claim.[344]

Claim. Every loop $s\colon[0,1]\to X=U\cup V$ in x_0 is path-homotopic to a loop which is the product of finitely many loops in (U,x_0) or in (V,x_0).

So let $s\colon[0,1]\to X=U\cup V$ be a loop in x_0. Since U and V are open we can use Corollary 2.76 to find a subdivision

$$0 \;=\; t_0 \;<\; t_1 \;<\; t_2 \;<\; \ldots \;<\; t_k \;<\; t_{k+1} \;=\; 1$$

of the interval $[0,1]$ such that the image of each $s_i:=s|_{[t_i,t_{i+1}]}$ lies in U or in V. The same argument as in the proof of Proposition 14.14 shows that we can assume that each $s(t_i)$ lies in $U\cap V$.[345] The problem is that even though the paths s_i lie in U respectively in V, they are not necessarily loops in x_0. But since $U\cap V$ is path-connected we can find for each $i\in\{1,\ldots,k\}$ a path p_i in $U\cap V$ which connects $s(t_i)\in U\cap V$ to the base point $x_0\in U\cap V$.

[342] A homeomorphism is for example given by stereographic projection.

[343] Here we use the obvious fact that the free product of two copies of the trivial group is again the trivial group.

[344] Why does it suffice to prove this claim?

[345] Indeed, if $s(t_i)$ lies in $U\setminus V$, then $s([t_{i-1},t_i])$ and $s([t_i,t_{i+1}])$ cannot lie in V, hence they both lie in U. This implies that $s_{([t_{i-1},t_{i+1}])}$ lies in U, and therefore we can remove t_i from the subdivision.

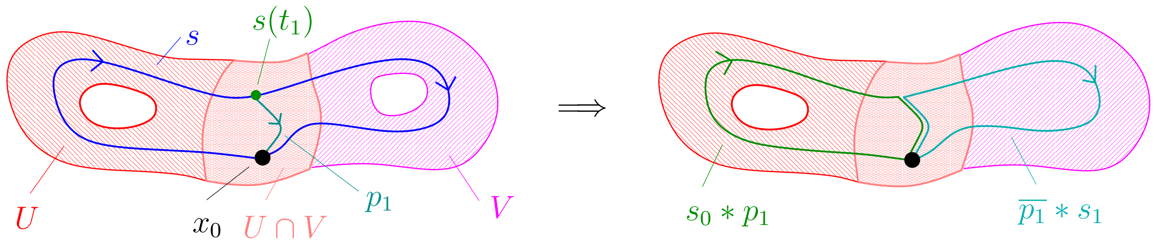

FIGURE 377. Schematic image for the proof of Lemma 20.1.

Now we have the following path-homotopies:

$$
\begin{aligned}
s \;&\overset{\substack{\text{by Lemma 14.4, since } s\\ \text{is a reparametrization}\\ \text{of } s_0*\cdots*s_k\\ \downarrow}}{\simeq}\; s_0 * \cdots * s_k \;\overset{\substack{\text{since each } p_i*\overline{p_i}\\ \text{is null-homotopic}\\ \downarrow}}{\simeq}\; s_0 * p_1 * \overline{p_1} * s_1 * p_2 * \overline{p_2} * s_2 * p_3 * \overline{p_3} * \ldots * \overline{p_k} * s_k \\
&\simeq \Big(\underbrace{s_0 * p_1}_{\substack{\text{loop in } (U,x_0)\\ \text{or in } (V,x_0)}}\Big) * \Big(\underbrace{\overline{p_1} * s_1 * p_2}_{\substack{\text{loop in } (U,x_0)\\ \text{or in } (V,x_0)}}\Big) * \Big(\underbrace{\overline{p_2} * s_2 * p_3}_{\substack{\text{loop in } (U,x_0)\\ \text{or in } (V,x_0)}}\Big) * (\overline{p_3} * \ldots * (\overline{p_k} * s_k).
\end{aligned}
$$

Thus we have shown that s is path-homotopic to a product of loops in (U, x_0) and (V, x_0).[346] ■

Theorem 20.2. (Seifert–van Kampen)[347] *Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and with $U \cap V \neq \emptyset$. We choose a base point $x_0 \in U \cap V$. If $U \cap V$ is simply connected, then the map*

$$\pi_1(U, x_0) * \pi_1(V, x_0) \;\to\; \pi_1(X, x_0)$$

that is induced by $\pi_1(U, x_0) \to \pi_1(X, x_0)$ and $\pi_1(V, x_0) \to \pi_1(X, x_0)$ and Lemma 19.12 (2), is an isomorphism.

Remark. One might wonder whether the hypothesis that $U \cap V$ is simply connected is really necessary. But this is indeed the case. For example we could take $X = S^1 \times [-1, 1]$ with $U = S^1 \times [-1, \frac{1}{2})$ and $V = S^1 \times (-\frac{1}{2}, 1]$. In this case we have $\pi_1(X) \cong \mathbb{Z}$ which is not isomorphic to the non-abelian group $\pi_1(U) * \pi_1(V) \cong \mathbb{Z} * \mathbb{Z}$.

We postpone the proof of the Seifert–van Kampen Theorem 20.2 to the next section and we first discuss several examples and applications of the theorem.

[346]The proof of Lemma 20.1 is arguably easier to follow than the proof of Proposition 14.14. The reason is that in the special case Proposition 14.14 we introduced more notation, e.g. North Pole and South Pole, than was perhaps strictly necessary. All we really needed was that we could cover S^2 by two open simply connected subsets U and V such that $U \cap V$ is path-connected. The precise choice of U and V is irrelevant.

[347]The theorem was proved independently by Herbert Seifert (1907-1996) and Egbert van Kampen (1908-1942).

Let A and B be two topological spaces and let $a \in A$ and $b \in B$ be two points. Recall that on page 559 we defined the wedge $A \vee B$ of A and B as

$$A \vee B \;:=\; A \sqcup B \;/\; a = b.$$

We illustrate this definition in Figure 378. In the following we will often identify A and B with their respective images in $A \vee B$. Another way of thinking about the wedge of A and B is to say that the wedge $X = A \vee B$ can be decomposed into two subsets A and B such that $A \cap B$ is a point. Since $A \cap B$ is a point it is in particular simply connected. Thus it sounds like one should be able to apply the Seifert–van Kampen Theorem 20.2. This is almost correct, except that in the Seifert–van Kampen Theorem we have to deal with decompositions of a topological space X into *open* subsets. We therefore need to introduce a somewhat technical definition.

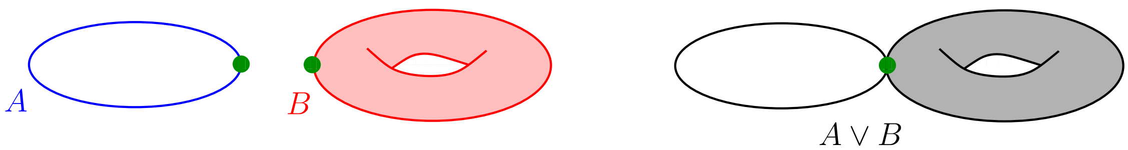

FIGURE 378. The wedge of a circle with a torus.

Definition. We say a point x in a topological space X is *good*, if the following two conditions are satisfied:

(1) $\{x\}$ is a closed subset of X and
(2) there exists an open neighborhood U of x such that x is a deformation retract of U.

Remark.

(1) If X is Hausdorff, then it we know by Lemma 2.13 that any subset consisting of a single point is closed. Thus the first condition in the definition of a good point is satisfied “for free”.
(2) What we call a “good point” is sometimes called a “nondegenerate point”, see e.g. [**Lee00**, p. 212]. On the other hand, the term “nondegenerate point” is defined in a subtly different way, see e.g. [**Spa95**, p. 380] or [**Bre93**, p. 435]. Also note that a point $x \in X$ is good if the pair $(X, \{x\})$ is good in the sense of [**Hat02**, p. 114].

Examples.

(1) Let X be an n-dimensional topological manifold (e.g. a sphere). We claim that every point is good. Since X is by definition Hausdorff we only need to show that Condition (2) is always satisfied. Thus let $x \in X$. Using a chart around x one easily sees that x admits an open neighborhood that is homeomorphic to the open ball B^n or to the “open half-ball” $\{(x_1, \ldots, x_n) \in B^n \,|\, x_n \geq 0\}$. In both cases x is evidently a deformation retract of that open neighborhood.
(2) We consider $X = \mathbb{C} \setminus \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Morally speaking the point $x = 0$ is not good, “since any open neighborhood of 0 has non-trivial fundamental group”. In Exercise 20.1 we will provide a proper proof that $x = 0$ is not a good point of X.

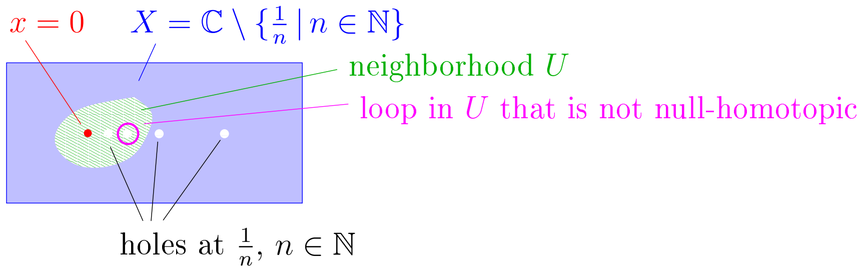

FIGURE 379. Example of a point that is not good.

Proposition 20.3. *Let A and B be two path-connected topological spaces and let $a \in A$ and $b \in B$ be good points. Then the inclusion maps induce an isomorphism*[348]

$$\pi_1(A, a) * \pi_1(B, b) \xrightarrow{\cong} \pi_1(A \vee B, a = b).$$

Remark. In the definition of a "good point" we demanded that the one-point subset given by the point is closed. Strictly speaking we do not need this fact in the proof of Proposition 20.3.

PROOF. Let A and B be two path-connected topological spaces and let $a \in A$ and $b \in B$ be good points. We write $X = A \vee B$ and $x_0 = \{a, b\} \in A \vee B$. By Lemma 18.25 we can view A and B as subsets of $X = A \vee B$ in the obvious way.

We pick an open neighborhood C in A of a that deformation retracts to a. Furthermore we pick an open neighborhood D of b in B that deformation retracts to b. We consider $U := A \vee D$ and $V := B \vee C$, each viewed as subsets of $X = A \vee B$.

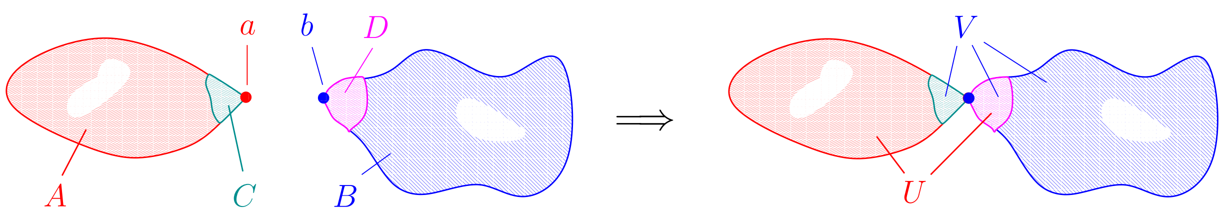

FIGURE 380. Schematic image for the proof of Proposition 20.3.

Now we consider the following diagram

$$\begin{tikzcd}
\pi_1(A, a) * \pi_1(B, b) \arrow[r] \arrow[d] & \pi_1(X, x_0) \arrow[d, "="] \\
\pi_1(U, x_0) * \pi_1(V, x_0) \arrow[r] & \pi_1(X, x_0).
\end{tikzcd}$$

[348]More precisely, the two natural inclusions $A \to A \vee B$ and $B \to A \vee B$ induce homomorphisms $\pi_1(A, a) \to \pi_1(A \vee B, a = b)$ and $\pi_1(B, b) \to \pi_1(A \vee B, a = b)$. Combined with Lemma 19.12 (2) we obtain the homomorphism $\pi_1(A, a) * \pi_1(B, b) \to \pi_1(A \vee B, a = b)$.

We make the following observations:

(1) The vertical and horizontal maps are all induced by the various inclusions of topological spaces. In particular the diagram commutes.
(2) It follows from the definition of the quotient topology on $A \vee B = (A \sqcup B)/\sim$ that U and V are open subsets of X.
(3) It follows from Lemma 18.28 that A is a deformation retract of $U = A \vee C$.
(4) As in (3) we see that B is a deformation retract of $V = B \vee D$.
(5) Furthermore, very similar to the argument in (3), we see that $A \cap B$ is a deformation retract of $U \cap V = C \vee D$. It follows from Lemma 18.14 and Proposition 18.16 (2) that $U \cap V$ is simply connected.
(6) By (2) and (5) we can apply the Seifert–van Kampen Theorem 20.2 to the decomposition $A \vee B = U \cup V$ and we obtain that the bottom horizontal map of the above diagram is an isomorphism.
(7) It follows from (3), (4) and Lemma 18.14 that the inclusions $A \to U = A \cup D$ and $B \to V = B \cup C$ are homotopy equivalences.
(8) It follows from (6) together with Proposition 18.16 (2) that the two inclusion induced maps $\pi_1(A, a) \to \pi_1(U, a)$ and $\pi_1(B, b) \to \pi_1(V, b)$ are isomorphisms, which implies that the left vertical map in the above commutative diagram is an isomorphism.
(9) We deduce that the top horizontal map is, as desired, also an isomorphism. ■

Examples.

(1) We consider again the wedge $S^1 \vee S^1$ with $a = (1, 0)$ and $b = (-1, 0)$. These points are simple.[349] By Proposition 16.17 we know that $\pi_1(S^1, a) = \langle x \rangle$[350] and $\pi_1(S^1, b) = \langle y \rangle$

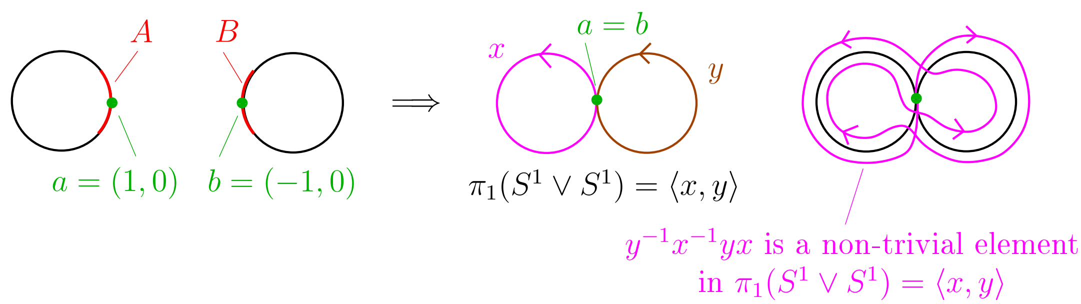

FIGURE 381. The wedge of two circles.

where x and y are represented by loops that go "once around the circle", as illustrated in Figure 381. Proposition 20.3 says that the inclusion maps induce an isomorphism

$$\langle x, y \rangle \;=\; \langle x \rangle * \langle y \rangle \;=\; \pi_1(S^1, a) * \pi_1(S^1, b) \;\xrightarrow{\cong}\; \pi_1(S^1 \vee S^1, a = b).$$

(2) We consider the loop γ from page 461 in $\mathbb{C} \setminus \{\pm 1\}$ that is also illustrated in Figure 382 on the left. We had already pointed out on page 561 that there exists a deformation

[349]This follows either from the general statement that any point on a topological manifold is simple, or more simply, by taking A and B to be small "intervals" around a and b.

[350]Recall that $\langle x \rangle$ denotes the free group generated by x, i.e. $\langle x \rangle = \{\ldots, x^{-1}, e = x^0, x, x^2, \ldots\}$.

retraction f from $\mathbb{C} \setminus \{\pm 1\}$ to the wedge of two circles that is given by

$$S^1 \vee S^1 = \underbrace{\{z \in \mathbb{C} \mid |z+1| = 1\} \cup \{z \in \mathbb{C} \mid |z-1| = 1\}}_{\text{two circles that meet at the origin}}.$$

We then have $f_*([\gamma]) = [f \circ \gamma] = \langle y^{-1}x^{-1}yx \rangle \in \langle x, y \rangle = \pi_1(S^1 \vee S^1)$, which is non-trivial. Thus we see that $f \circ \gamma$ is not null-homotopic in $S^1 \vee S^1$ which implies that γ is not null-homotopic in $\mathbb{C} \setminus \{\pm 1\}$.[351]

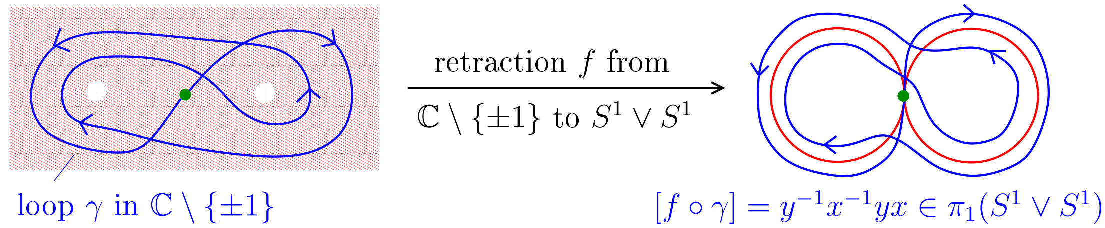

FIGURE 382

Before we continue we condense the above example into a lemma.

Lemma 20.4. *Let $k \in \mathbb{N}_0$. The map*

$$\langle x_1, \dots, x_k \rangle \to \pi_1\Big(\bigvee_{i=1}^k S^1, *\Big)$$

that is given by

$$x_i \mapsto \left[\begin{array}{rcl} [0,1] & \to & S^1_i \\ t & \mapsto & e^{2\pi i t} \end{array} \right]$$

is an isomorphism.

Convention. We refer to the images of $x_1, \dots, x_k$ under the isomorphism of Lemma 20.4 as the *standard generating set* of $\pi_1\Big(\bigvee_{i=1}^k S^1, *\Big)$.

PROOF. Similar to the above example the lemma follows from applying Proposition 20.3 altogether $k-1$ times. ■

Proposition 20.5.

(1) *Let $G = (V, E, i, t)$ be a finite connected non-empty abstract graph and let T be a maximal tree of G. Let $e_1, \dots, e_n$ be the edges not contained in T. We pick some $v \in V$. For $j = 1, \dots, n$ we pick a path $\alpha_j \colon [0,1] \to |T|$ from v to $i(e_j)$, we denote by $\beta_j \colon [0,1] \to [(e_j, t)] \in |G|$ the obvious path from $i(e_j)$ to $t(e_j)$, and we pick a path*

[351]We could also have applied Seifert–van Kampen Theorem 20.2 to the covering $\mathbb{C} \setminus \{\pm 1\} = U \cup V$ where $U = \{z \in \mathbb{C} \setminus \{-1\} \mid \mathrm{Re}(z) < \frac{1}{2}\}$ and $V = \{z \in \mathbb{C} \setminus \{1\} \mid \mathrm{Re}(z) > -\frac{1}{2}\}$. Both sets U and V are open and they are homotopy equivalent to a circle.

$\gamma_j\colon [0,1] \to |T|$ from $t(e_j)$ to v. Then the map

$$\Theta_G\colon \langle x_1, \dots, x_n\rangle \;\to\; \pi_1(|G|, v)$$

that is given by

$$x_j \;\mapsto\; [\alpha_j * \beta_j * \gamma_j]$$

is an isomorphism.

(2) *Let X be a finite connected non-empty topological graph with v vertices and e edges. Then*

$$\pi_1(X) \;\cong\; \text{free group on } 1 - \chi(X) = 1 + e - v \text{ generators.}$$

(3) *Two finite connected topological graphs are homotopy equivalent if and only if they have the same Euler characteristic.* ■

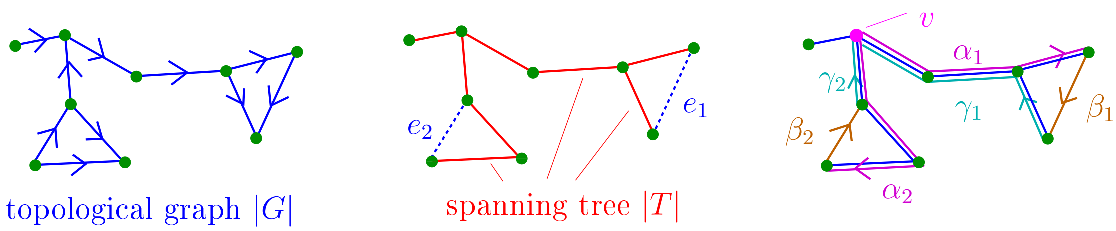

FIGURE 383. Illustration of Proposition 20.5 (1).

PROOF.

(1) We consider the maps

isomorphism induced by the homeomorphism on page 560

$$\langle x_1, \dots, x_n\rangle \xrightarrow{\Theta_G} \pi_1(|G|, v) \;\to\; \pi_1(|G|/|T|, v) \xleftarrow{\cong} \pi_1\Big(\bigvee_{j=1}^{n} S^1_j\Big) \xleftarrow{\cong} \langle y_1, \dots, y_n\rangle.$$

isomorphism by Propositions 18.31 and 18.16 (2) isomorphism given by Lemma 20.4

It follows fairly easily from the explicit descriptions of the various maps that for each $j \in \{1, \dots, n\}$ the images of x_j and y_j in $\pi_1(|G|/|T|, v)$ are represented by homotopic loops, in particular the images of x_j and y_j in $\pi_1(|G|/|T|, v)$ are the same. But now it follows from the above and the fact that all other maps are isomorphisms that Θ_G itself is indeed an isomorphism.

(2) Basically by definition we only need to prove the statement if $X = |G|$ is the topological realization of a finite connected non-empty abstract graph $G = (V, E, i, t)$. By Proposition 4.8 (1) we know that G admits a spanning tree $T = (W, F)$. By Proposition 4.8 we know that T contains all vertices of V. Furthermore, since T is a tree we know by definition of a tree that $\#W - \#F = 1$. Thus it follows that there are precisely $\#E - \#F = e - (\#W - 1) = e - v + 1$ edges of G that are not contained in T. The desired statement now follows from (1).

(3) Let X and X' be two finite connected topological graphs. If $\chi(X) = \chi(X')$, then we already saw in Proposition 18.31 (3) that X and X' are homotopy equivalent. On the other hand, if $\chi(X) \neq \chi(X')$, then it follows from (1) together with Lemma 19.16 that X and X' are not homotopy equivalent. ■

As a last example in this section we would like to determine the fundamental group of
$$X = \big(\mathbb{R} \sqcup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n,(0,0,-1)).$$
The space X consists of an infinite line where to each $n \in \mathbb{Z}$ we attach a 2-dimensional sphere. The space X is illustrated in Figure 384. Given $k \in \mathbb{N}$ we denote by X_k the topological space that is obtained by X by restricting oneself to the interval $(-k-\frac{1}{2}, k+\frac{1}{2})$, i.e.

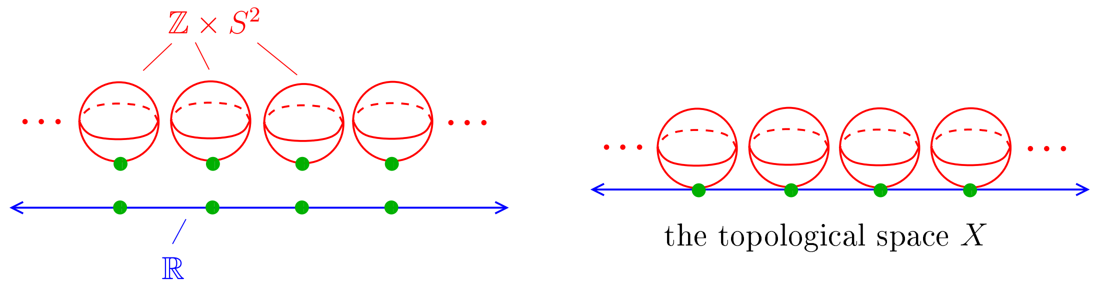

Figure 384

$$X_k = \Big(\big(-k-\tfrac{1}{2}, k+\tfrac{1}{2}\big) \sqcup (\{-k,\dots,k-1,k\} \times S^2)\Big) \,/\, n \sim (n,(0,0,-1)).$$
Note that X_k can be viewed as the wedge of an open interval with $2k+1$ spheres. It follows immediately from Proposition 14.14, Proposition 20.3 and a straightforward induction argument that $\pi_1(X_k) = 0$ for all k. Furthermore one can easily convince oneself that the sets X_k are open. It now follows from the following lemma that $\pi_1(X)$ is also trivial.

Lemma 20.6. *Let X be a path-connected topological space and let $x_0 \in X$. Furthermore let $X_k, k \in \mathbb{N}$ be a sequence of subsets of X such that the following hold:*

1. *each X_k is open,*
2. *each X_k is simply connected,*
3. *the sequence X_k is nested, i.e. for each k we have $X_k \subset X_{k+1}$,*
4. *we have $\bigcup_{k\in\mathbb{N}} X_k = X$.*

Then $\pi_1(X, x_0) = 0$.

The proof of the lemma, which will make essential use of Lemma 2.41, is the content of Exercise 20.2.

We conclude this section with the following generalization of Proposition 20.3.

Proposition 20.7. $(*)$ *Let $\{A_k\}_{k\in K}$ be a family of topological spaces. For each $k \in K$ suppose that we are given a good point $a_k \in A_k$. Given $j \in K$ we denote by*
$$i_j\colon A_j \to \bigvee_{k\in K} A_k \quad \textit{respectively} \quad p_j\colon \bigvee_{k\in K} A_k \to A_j$$

the natural inclusion map respectively the natural projection map. We denote by $*$ the wedge point of $\bigvee_{i\in I} X_i$. For every $n\in\mathbb{N}_0$ the projections p_k induces an isomorphism

$$\mathop{\ast}_{k\in K} p_{k*}\colon \pi_1\Big(\bigvee_{k\in K} A_k, *\Big) \;\to\; \textit{free product of the groups } \{\pi_1(A_k,a_k)\}_{k\in K}.$$

The inverse is induced by the inclusions i_k.

Proof ($*$). Let $\{A_k\}_{k\in K}$ be a family of topological spaces. For each $k\in K$ suppose that we are given a good point $a_k\in A_k$. We use these points to form the wedge $\bigvee_{k\in K} A_k$. Recall that the fact that a_k is good means that

(1) $\{a_k\}$ is a closed subset of A_k and
(2) that there exists an open neighborhood U_k of a_k in A_k such that a_k is a deformation retract of U_k.

The trick in the proof of Proposition 20.7 is to reduce the statement to the case of Proposition 20.7.

We will show that the map

$$\mathop{\ast}_{k\in K} i_k\colon \mathop{\ast}_{k\in K}\pi_1(A_k) \;\to\; \pi_1\Big(\bigvee_{k\in K} A_k\Big)$$

is an isomorphism. (Here and throughout out we work with the obvious base points, which we suppress from the notation.) Once we will have established this fact it is straightforward to show that the inverse is given by $\mathop{\ast}_{k\in K} p_{k*}$.

The idea of the proof is to show that the map is a monomorphism and an epimorphism.

Claim. The map

$$\mathop{\ast}_{k\in K} i_k\colon \mathop{\ast}_{k\in K}\pi_1(A_k) \;\to\; \pi_1\Big(\bigvee_{k\in K} A_k\Big)$$

is an epimorphism.

We introduce the following notation:

(1) We write $B:=\bigvee_{k\in K} A_k$ and we set $U:=\bigcup_{k\in K} U_k\subset B$.
(2) Given a subset $J\subset K$ we write $B_J:=\bigvee_{j\in J} A_j$ and we write $\widehat{B}_J := U\cup B_J$.

Given a finite subset $J\subset K$ we have the following commutative diagram

$$\begin{tikzcd}
\mathop{\ast}_{j\in J}\pi_1(A_j) \arrow[rr, "\cong"] \arrow[d, hook] & & \pi_1(B_J) \arrow[d] \arrow[dr, "\cong"] & \\
& & & \pi_1(\widehat{B}_J) \arrow[dl] \\
\mathop{\ast}_{k\in K}\pi_1(A_k) \arrow[rr] & & \pi_1(B). &
\end{tikzcd}$$

We make the following observations and comments:

(1) The maps are either the obvious algebraic inclusion maps or they are the obvious maps induced by inclusions of topological spaces. It follows almost immediately from the definitions that the diagram commutes.
(2) Since J is finite we obtain, from applying Proposition 20.3 finitely many times, that the top horizontal map is an isomorphism.
(3) By Exercise 18.32 we know that B_J is a deformation retract of $\widehat{B}_J$. It follows from Proposition 18.16 (2) that the top diagonal map is an isomorphism.
(4) By Exercise 18.33 we know that given any compact subset $L \subset B$ there exists a finite subset $J \subset K$ with $L \subset \widehat{B}_J$.

Now we show that the above map $\underset{k\in K}{\ast} i_{k*}$ is an epimorphism. So suppose we are given some $[\sigma\colon S^1 \to B] \in \pi_1(B)$. It follows from (4) there exists a finite subset $J \subset K$ such that $\sigma \in C_n(\widehat{B}_J)$. But considering the above diagram, and using the fact that the top horizontal map is an isomorphism we see that $[\sigma]$ actually lies in the image of $\underset{k\in K}{\ast} i_k$. ⊞

Claim. The map

$$\underset{k\in K}{\ast} i_k\colon \underset{k\in K}{\ast} \pi_1(A_k) \;\to\; \pi_1\Big(\bigvee_{k\in K} A_k\Big)$$

is a monomorphism.

We continue with the notation that we introduced in the proof of the previous claim. Given any two finite subsets $J \subset \widetilde{J} \subset K$ we have the following commutative diagram

$$\begin{tikzcd}
\underset{j\in J}{\ast}\pi_1(A_j) \arrow[rrr, "\cong"] \arrow[drr, hook] \arrow[ddr, hook] & & & \pi_1(B_J) \arrow[dr] \arrow[dd] \arrow[drr, bend left] & & \\
& & \underset{j\in \widetilde{J}}{\ast}\pi_1(A_j) \arrow[rr, "\cong"] \arrow[dl, hook'] & & \pi_1(B_{\widetilde{J}}) \arrow[r, "\cong"] \arrow[dl] & \pi_1(\widehat{B}_{\widetilde{J}}). \arrow[dll, bend left] \\
& \underset{k\in K}{\ast}\pi_1(A_k) \arrow[rr] & & \pi_1(B) & &
\end{tikzcd}$$

In Exercise 20.7 we will use this diagram to show that the map $\underset{k\in K}{\ast} i_{k*}$ is also a monomorphism. ■

20.2. **Proof of the Seifert–van Kampen Theorem 20.2.** After this long list of applications we finally provide the proof of the Seifert–van Kampen Theorem 20.2.

PROOF. Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and with $U \cap V \neq \emptyset$ such that $U \cap V$ is simply connected. We choose a base point $x_0 \in U \cap V$. We write $A := \pi_1(U, x_0)$ and $B := \pi_1(V, x_0)$. We need to show that the map

$$\Phi\colon A * B \;=\; \pi_1(U, x_0) * \pi_1(V, x_0) \;\to\; \pi_1(X, x_0)$$

that is induced by the inclusions $U \to X$ and $V \to X$ is an isomorphism. Taking into account Lemma 20.1 it suffices to show that the map is injective. To prove injectivity it is enough to show that there exists a map $\Psi\colon \pi_1(X,x_0) \to A * B$ such that $\Psi \circ \Phi = \mathrm{id}_{A*B}$. We will prove the existence of such a map Ψ in three steps:

(1) we first construct a map $\overline{\Psi}\colon \{\text{loops in } (X,x_0)\} \to A * B$,
(2) then we show that $\overline{\Psi}$ descends to a well-defined map

$$\begin{array}{rcl} \Psi\colon \pi_1(X,x_0) & \to & A * B \\ {[g]} & \mapsto & \overline{\Psi}(g) \end{array}$$

(3) and finally we show that $\Psi \circ \Phi = \mathrm{id}_{A*B}$.

For the remainder of the proof we will identify $A * B$ with $S(A,B)/\sim$ using the isomorphism given by Proposition 19.17.

We start out with step (1). Let $g\colon [0,1] \to X$ be a loop in (X,x_0). We apply Corollary 2.76 to the map $g\colon [0,1] \to X$ and the open cover $X = U \cup V$. We obtain an N such that for every $i \in \{0,\dots,N-1\}$ the image of the interval $[\frac{i}{N}, \frac{i+1}{N}]$ under the map g lies in U or in V. We introduce a few definitions.

(a) We say a subinterval of $[0,1]$ can be *colored green* if the image of the subinterval under the map g lies in U and we say an interval can be *colored blue* if the image of the subinterval lies in V. Note that the image of a subinterval lies in $U \cap V$ if and only if the subinterval can be colored green as well as blue.
(b) A *coloring* is a map $\alpha\colon \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\} \to \{\text{blue}, \text{green}\}$ such that each subinterval $[i, i + \frac{1}{N}]$ can be colored by $\alpha(i)$. This definition is illustrated in Figure 385.

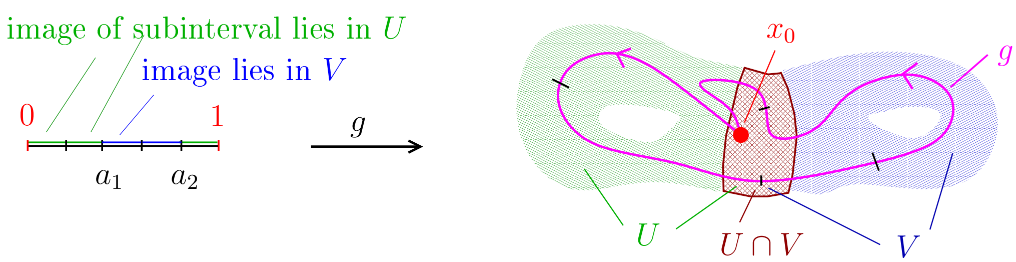

Figure 385

(c) Given a coloring α we denote by $0 < a_1 < \dots < a_{k-1} < 1$ the numbers where the coloring of the subintervals changes from one color to another. This definition is also illustrated in Figure 385. Note that for each a_i the intervals $[a_i - \frac{1}{N}, a_i]$ and $[a_i, a_i + \frac{1}{N}]$ can be colored in two different colors, which implies that $g(a_i) \in U \cap V$.
(d) A *basing* for a coloring α is a collection of paths $p_1, \dots, p_{k-1}$ in $U \cap V$ such that each path p_i goes from x_0 to $g(a_i)$.[352]

[352]Such basing exists since each $g(a_i)$ lies in $U \cap V$ and since $U \cap V$ is path-connected.

Now given a coloring α and a basing $p_1, \dots, p_{k-1}$ we define[353] [354]

$$\overline{\Psi}(g) \;:=\; \Big[\big[\underbrace{g|_{[0,a_1]} * \overline{p}_1}_{\substack{\text{loop in }(U,x_0)\text{ or}\\ \text{loop in }(V,x_0)}}\big], \big[\underbrace{p_1 * g|_{[a_1,a_2]} * \overline{p}_2}_{\substack{\text{loop in }(U,x_0)\text{ or}\\ \text{loop in }(V,x_0)}}\big], \dots, \big[\underbrace{p_{k-1} * g|_{[a_{k-1},1]}}_{\substack{\text{loop in }(U,x_0)\text{ or}\\ \text{loop in }(V,x_0)}}\big]\Big] \in S(A,B)/\sim\, = A * B.$$

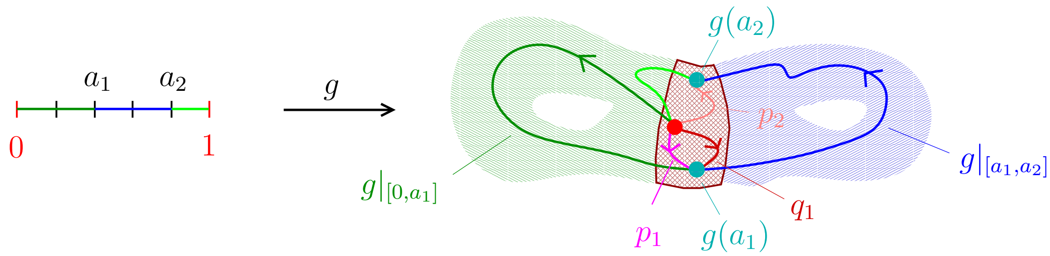

FIGURE 386

Our next goal is to prove the following claim.

Claim. The definition of $\overline{\Psi}(g)$ does not depend on any of the choices made.

We need to prove the following three statements:

(a) given N and a coloring the definition does not depend on the choice of the basings,
(b) given N the definition does not depend on the choice of the coloring,
(c) the definition does not depend on the choice of N.

We first fix a coloring α and we show that the definition of $\overline{\Psi}(g)$ does not depend on the choice of the basing. First we pick a different path q_1 from x_0 to $g(a_1)$ instead of the original path p_1. (See Figure 386 for an illustration.) In this case we have

$$\begin{aligned}
&\text{definition of } \overline{\Psi}(g) \text{ using the basing } p_1, p_2, \dots, p_{k-1} \;= \\
&\quad = \Big[[g|_{[0,a_1]} * \overline{p}_1],\ [p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\Big] \\
&\quad \underset{\substack{\uparrow \\ \text{since } q_1 * \overline{q}_1 \text{ is null-homotopic}}}{=} \Big[[g|_{[0,a_1]} * \overline{p}_1 * q_1 * \overline{q}_1],\ [q_1 * \overline{q}_1 * p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\Big] \\
&\quad \underset{\substack{\uparrow \\ \overline{p}_1 * q_1 \text{ and } \overline{q}_1 * p_1 \text{ are null-homotopic since } U \cap V \text{ is simply connected}^{355}}}{=} \Big[[g|_{[0,a_1]} * \overline{q}_1],\ [q_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\Big] \\
&\quad = \text{definition of } \overline{\Psi}(g) \text{ using the basing } q_1, p_2, \dots, p_{k-1}.
\end{aligned}$$

Precisely the same argument shows that the choices of the other paths p_i are also irrelevant. This concludes the proof of (a).

[353] Each path $p_m * g|_{[a_m,a_{m+1}]} * \overline{p}_{m+1}$ is a loop in (U, x_0) or (V, x_0), so each path represents an element in $\pi_1(U, x_0)$ or $\pi_1(V, x_0)$. Thus we obtain a sequence of elements in $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$. The equivalence class of this sequence defines an element in $S(A,B)/\sim\, = A * B$.

[354] This definition is almost the same as in the proof of Lemma 20.1.

[355] This is the only time that in the proof we use the hypothesis that $U \cap V$ is simply connected. In fact if one studies the argument carefully one sees that we only used that the maps $\pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$ are trivial.

Now we show that the definition does not depend on the choice of the coloring. So suppose we have two colorings α and β. We can turn one coloring into the other by a finite sequence of color swaps of a single subinterval. Therefore it suffices to consider the case that there exists a single i where the values for $\alpha(i)$ and $\beta(i)$ differ.

We consider the case, sketched in Figure 387, with $\alpha(i) =$ blue and $\beta(i) =$ green such that the common coloring to the left is green and the common coloring to the right is blue. All other cases are dealt with in a similar way.[356]

Let $0 = a_0 < a_1 < \cdots < a_k = 1$ be the decomposition for α. We pick a basing $p_1, \ldots, p_{k-1}$ for α. There exists a unique m with $a_m = i$. The corresponding decomposition for the coloring β is given by $0 = a_0 < a_1 < \cdots < a_m + \frac{1}{N} < a_{m+1} < a_k = 1$. We write $q = g_{[a_m, a_m + \frac{1}{N}]}$. Since $[a_m, a_m + \frac{1}{N}]$ can be colored green and blue we see that q lies actually in $U \cap V$. We have

$$\begin{aligned}
&\text{definition of } \overline{\Psi}(g) \text{ using the coloring } \alpha \text{ and the basing } p_1, \ldots, p_{k-1} \;= \\
&\quad = \big[[g|_{[0,a_1]} * \overline{p}_1], \ldots, [p_{m-1} * g|_{[a_{m-1},a_m]} * \overline{p}_m], [p_m * g|_{[a_m,a_{m+1}]} * \overline{p}_{m+1}], \ldots\big] \\
&\quad \underset{\substack{\uparrow \\ \text{since } g_{[a_{m-1},a_m]} \simeq g_{[a_{m-1},a_m+\frac{1}{N}]} * \overline{q} \text{ and } g_{[a_m,a_m+\frac{1}{N}]} \simeq q * g_{[a_m+\frac{1}{N},a_{m+1}]}}}{=} \big[[g|_{[0,a_1]} * \overline{p}_1], \ldots, [p_{m-1} * g|_{[a_{m-1},a_m+\frac{1}{N}]} * \overline{q} * \overline{p}_m], [p_m * q * g|_{[a_m+\frac{1}{N},a_{m+1}]} * \overline{p}_{m+1}], \ldots\big] \\
&\quad = \text{definition of } \overline{\Psi}(g) \text{ using the coloring } \beta \text{ and basing } p_1, \ldots, p_{m-1}, p_m * q, p_{m+1}, \ldots, p_{k-1}.
\end{aligned}$$

(We refer to Figure 387 for an illustration of the proof.) This concludes the proof of (b).

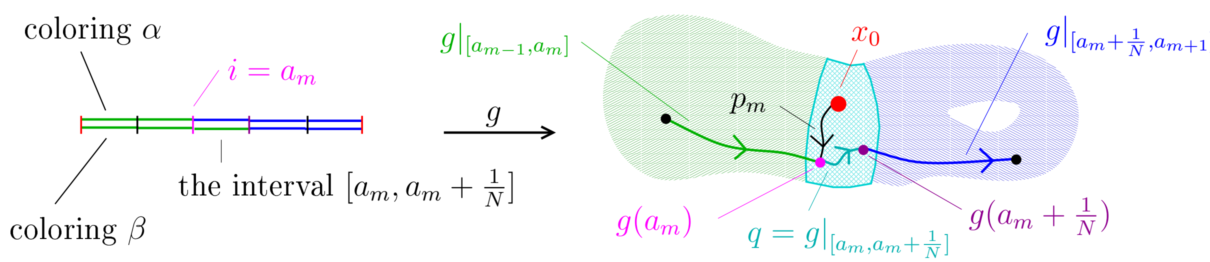

FIGURE 387

It remains to prove (c). First suppose that instead of dividing $[0, 1]$ into N intervals we divide it into $m \cdot N$ intervals for some $m \in \mathbb{N}$. A coloring for the subdivision into N subintervals then also gives rise to a coloring into $m \cdot N$ subintervals, and it follows immediately from the definitions that we end up with the same element of $A * B$. Now if had chosen a different N' instead, then we can replace N and N' by $N \cdot N'$ and by the argument we just made we obtain the same element of $A * B$. This concludes the proof of (c). Summarizing, we are now done with the proof of the claim.

We turn to step (2) of the above program. We need to prove the following claim.

Claim. If $g, g' \colon [0, 1] \to X$ are two path-homotopic loops in (X, x_0), then $\overline{\Psi}(g) = \overline{\Psi}(g')$.

[356] In fact the only case which requires a slightly modified argument is that the coloring to the left and right are the same, i.e. either both are blue or both are green. We leave it to the reader to perform that argument.

Let $g, g' \colon [0,1] \to X$ be two path-homotopic loops in (X, x_0). We pick a path-homotopy

$$\begin{array}{rcl} H\colon [0,1]\times[0,1] & \to & X \\ (t,s) & \mapsto & H^s(t). \end{array}$$

from the loop g to the loop g'. We apply Corollary 2.76 to the map $H\colon [0,1]\times[0,1] \to X$ and the open cover $X = U \cup V$. We obtain an $N \in \mathbb{N}$ such that for every choice of $i, j \in \{0, \frac{1}{N}, \dots, 1-\frac{1}{N}\}$ the image of the square $[i, i+\frac{1}{N}] \times [j, j+\frac{1}{N}]$ under the map H lies in U or in V. Given $j \in \{0, \frac{1}{N}, \dots, 1\}$ we now define

$$\begin{array}{rcl} g(j)\colon [0,1] & \to & X \\ t & \mapsto & H^j(t) \end{array}$$

Note that each $g(j)$ is a loop in x_0 and that $g(0) = g$ and $g(1) = g'$. Therefore it suffices to show that for any $j \in \{0, \frac{1}{N}, \dots, 1-\frac{1}{N}\}$ we have $\overline{\Psi}(g(j)) = \overline{\Psi}(g(j+\frac{1}{N}))$. So let $j \in \{0, \frac{1}{N}, \dots, 1-\frac{1}{N}\}$.

As above we say that a subset of $[0,1]\times[0,1]$ can be *colored green* if the image of the subset under the map H lies in U and we say the subset can be *colored blue* if the image lies in V. We pick a map $\alpha\colon \{0, \frac{1}{N}, \dots, 1-\frac{1}{N}\} \to \{\text{blue}, \text{green}\}$ such that for each i the value $\alpha(i)$ is a color with which the square $[i, i+\frac{1}{N}] \times [\frac{j}{N}, \frac{j+1}{N}]$ can be colored. Note that α also defines a coloring for the loops $g(j)$ and $g(j+\frac{1}{N})$.

We define $0 = a_0 < a_1 < \dots < \dots a_k = 1$ as above for the coloring α. Furthermore we pick a basing p_i, $i = 1, \dots, k-1$ for the loop $g(j)$. For $i = 0, \dots, k$ we write $q_i = H^{[j,j+\frac{1}{N}]}|_{a_i}$. Put differently, q_j connects a point on the loop $g(j)$ to the corresponding point on the loop $g(j+\frac{1}{N})$. Now we have

$$\begin{array}{rcl}
\overline{\Psi}(g(j)) & = & \overline{\Psi}(g(j)) \text{ defined using the coloring } \alpha \text{ and the basing } p_1, \dots, p_{k-1} \\
& = & \big[\cdots * [p_m * \underbrace{H^j|_{[a_m,a_{m+1}]}}_{=g(j)|_{[a_m,a_{m+1}]}} * \overline{p}_{m+1}] * \dots\big] \\
& \underset{\uparrow}{=} & \big[\cdots * [p_m * q_m * \overline{q}_m * H^j|_{[a_m,a_{m+1}]} * q_{m+1} * \overline{q}_{m+1} * \overline{p}_{m+1}] * \dots\big] \\
\multicolumn{3}{l}{\text{since each } q_m * \overline{q}_m \text{ is null-homotopic}} \\
& \underset{\uparrow}{=} & \big[\cdots * [p_m * q_m * \underbrace{H^{j+\frac{1}{N}}|_{[a_m,a_{m+1}]}}_{=g(j+\frac{1}{N})|_{[a_m,a_{m+1}]}} * \overline{q}_{m+1} * \overline{p}_{m+1}] * \dots\big] \\
\multicolumn{3}{l}{\text{since } H \text{ restricts to path-homotopies from } \overline{q}_m * H^j|_{[a_m,a_{m+1}]} * q_{m+1}} \\
\multicolumn{3}{l}{\quad\text{to } H^{j+\frac{1}{N}}|_{[a_m,a_{m+1}]} \text{ for } m = 0, \dots, k-1} \\
& = & \overline{\Psi}(g(j+\frac{1}{N})) \text{ defined using } \alpha \text{ and the basing } p_1 * q_1, \dots, p_{k-1} * q_{k-1}.
\end{array}$$

This last part of the proof is illustrated in Figure 388. This concludes the proof of the claim and thus of part (2) of our three step program.

Summarizing we now showed that the map $\overline{\Psi}$ descends to a well-defined map

$$\begin{array}{rcl} \Psi\colon \pi_1(X, x_0) & \to & A * B \\ [g] & \mapsto & \big[\overline{\Psi}(g)\big]. \end{array}$$

It remains to prove the following claim.

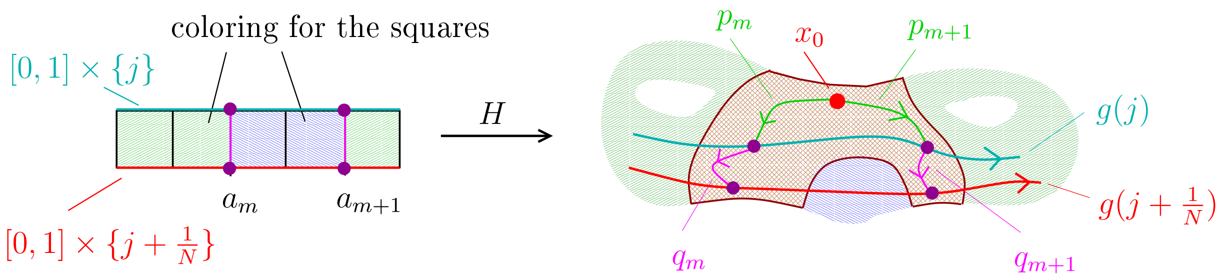

FIGURE 388

Claim. We have $\Psi\circ\Phi=\mathrm{id}_{A*B}$.

So let $[y_1]\cdot\dots\cdot[y_k]\in A*B$ where $y_1,\dots,y_k$ are loops in (U,x_0) and (V,x_0). Then

$$\begin{aligned}(\Psi\circ\Phi)([y_1]\cdot\dots\cdot[y_k]) &\overset{\substack{\text{definition of }\Phi\colon A*B=\pi_1(U,x_0)*\pi_1(V,x_0)\to\pi_1(X,x_0)\\ \downarrow}}{=} \Psi([y_1*\dots*y_k]) = \big[\overline{\Psi}(y_1*\dots*y_k)\big] \underset{\substack{\uparrow\\ \text{by definition of }\overline{\Psi}\text{, where we use }N=k\text{ and where}\\ \text{we can use the constant paths as basings since}\\ y_1,\dots,y_k\text{ are already loops in }(U,x_0)\text{ and }(V,x_0)}}{=} [y_1*\dots*y_k]\\ &= [y_1]\cdot\dots\cdot[y_k].\end{aligned}$$

This concludes the proof of the claim and thus also the proof of the Seifert–van Kampen Theorem 20.2 ■

20.3. **More examples: surfaces and the connected sum of smooth manifolds.** Given $g\in\mathbb{N}_0$ we denote by Σ_g the surface of genus g. In this section we first want to prove the following proposition.

Proposition 20.8. *For $g\neq h$ the surface Σ_g of genus g is not homeomorphic to the surface Σ_h of genus h.*

As on several earlier occasions we want to prove this result by considering what happens if we remove a point from the topological spaces.

Lemma 20.9. *Let $g\in\mathbb{N}_0$. We denote by Σ_g the surface of genus g.*

(1) *For any point P the complement $\Sigma_g\setminus\{P\}$ is homotopy equivalent to the wedge of $2g$ circles. In particular*
$$\pi_1(\Sigma_g\setminus\{P\})\;\cong\;\langle x_1,y_1,\dots,x_g,y_g\rangle.$$

(2) *Let $f\colon\overline{B}^2\to\Sigma_g$ be a smooth embedding. The complement $\Sigma_g\setminus f(B^2)$ is homotopy equivalent to the wedge of $2g$ circles. Furthermore, if $Q\in f(S^1)$ is a point, then there exists an isomorphism*
$$\begin{aligned}\pi_1(\Sigma_g\setminus f(B^2),Q)&\;\xrightarrow{\cong}\;\langle x_1,y_1,\dots,x_g,y_g\rangle\\ [f\colon S^1\to\Sigma_g\setminus f(B^2)]&\;\mapsto\;[x_1,y_1]\cdot\dots\cdot[x_g,y_g]\end{aligned}$$
with

where $[x_i, y_i] := x_i y_i x_i^{-1} y_i^{-1}$ is the commutator of x_i and y_i. In particular the map $f\colon S^1 \to \Sigma_g \setminus f(B^2)$ represents a non-trivial element in $\pi_1(\Sigma_g \setminus f(B^2), Q)$.

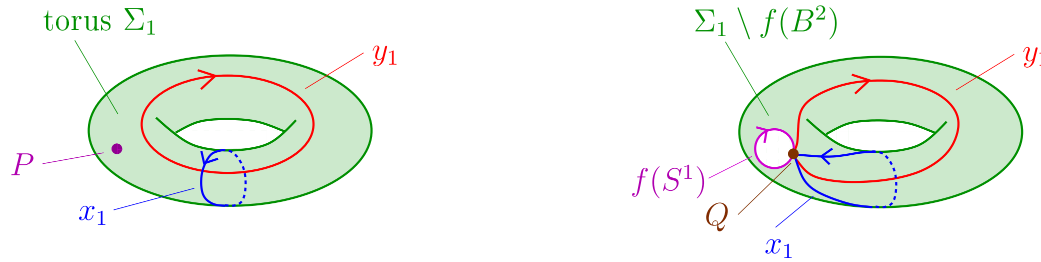

FIGURE 389. Illustration of Lemma 20.9.

PROOF OF LEMMA 20.9. We prove the lemma for the surface Σ of genus 2. The proof for $g \geq 3$ is almost identical. So let $\Sigma = E_8/\sim$ as sketched in Figure 390.

(1) We denote by O the origin in $E_8 \subset \mathbb{C}$. We first show that $\Sigma \setminus \{O\}$ is homotopy equivalent to the wedge of 4 circles.

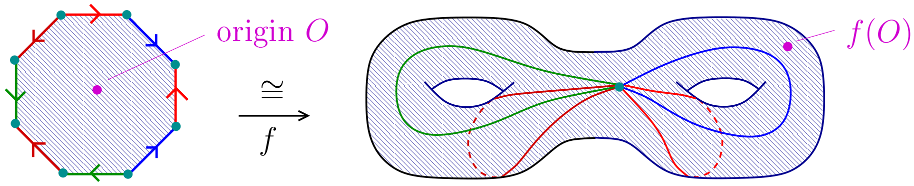

FIGURE 390

Claim. There exists a deformation retraction from $\Sigma \setminus \{O\} = (E_8 \setminus \{O\})/\sim$ to the subset $\partial E_8/\sim$.

First we consider the map[357][358]

$$\begin{array}{rcl} r\colon E_8 \setminus \{O\} & \to & \partial E_8 \\ re^{i\varphi} & \mapsto & \text{unique point } se^{i\varphi} \text{ on } \partial E_8. \end{array}$$

This map "projects points radially outward towards the boundary". The deformation retraction is now given by "pushing outward radially", i.e. it is given by the map

$$\begin{array}{rcl} ((E_8 \setminus \{O\})/\sim) \times [0,1] & \mapsto & (E_8 \setminus \{O\})/\sim \\ ([P], t) & \mapsto & [P \cdot (1-t) + r(P) \cdot t]. \end{array}$$

⊞

Claim. The space $\partial E_8/\sim$ is homeomorphic to the wedge of 4 circles.

A homeomorphism is given by sending the point on $\partial E_8/\sim$ that corresponds to the vertices to the common point of the four circles, and by sending the edges that get identified to the same circle. We refer to Figure 391 for an illustration. We leave it to the reader to write down the proof in detail.

[357]Note that this map cannot be extended continuously to O.

[358]Here we use polar coordinates, i.e. r, s are positive real numbers.

The two claims put together show that $\Sigma \setminus \{O\}$ is homotopy equivalent to the wedge of 4 circles. Now we obtain that

$$\pi_1(\Sigma_2 \setminus \{P\}) \underset{\substack{\uparrow \\ \text{Proposition 18.16 (2)}}}{\cong} \pi_1(\text{wedge of 4 circles}) \underset{\substack{\uparrow \\ \text{by Lemma 20.4}}}{\cong} \langle x_1, y_1, x_2, y_2 \rangle.$$

Now let P be some other point on Σ. By Proposition 8.29 there exists a homeomorphism $\Phi\colon \Sigma \to \Sigma$ with $\Phi(P) = O$.[359] This shows that $\Sigma \setminus \{P\}$ is homeomorphic to $\Sigma \setminus \{O\}$. We can thus appeal to the above calculation to get the desired result.

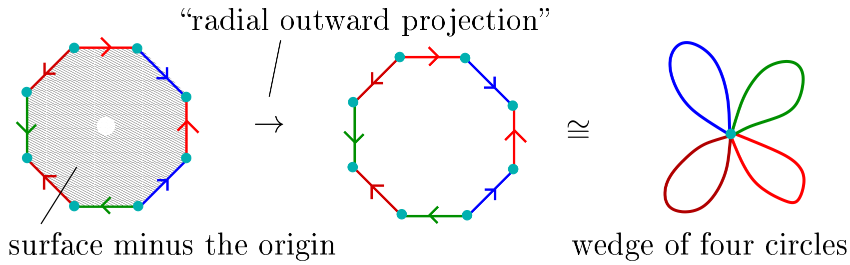

FIGURE 391

(2) The proof of this statement is very similar to the proof of the first statement. First remove an open disk centered at O and we perform almost the same calculation as above. One easily sees that the boundary curve corresponds to the product of commutators. It follows from Theorem 8.36 that the same calculation also works for any other disk. We leave the details to the reader. ■

Now we can turn to the proof of Proposition 20.8.

PROOF OF PROPOSITION 20.8. Let S and T be two surfaces of genus g and h. Suppose that there exists a homeomorphism $f\colon S \to T$. Pick a point $P \in S$. We write $Q = f(P)$. The homeomorphism f restricts to a homeomorphism $f\colon S \setminus \{P\} \to T \setminus \{Q\}$. It now follows that

$$\langle x_1, \dots, x_{2g} \rangle \underset{\substack{\uparrow \\ \text{Lemma 20.9 (1)}}}{\cong} \pi_1(S \setminus \{P\}) \xrightarrow[f_*]{\cong} \pi_1(T \setminus \{Q\}) \underset{\substack{\uparrow \\ \text{Lemma 20.9 (1)}}}{\cong} \langle y_1, \dots, y_{2h} \rangle.$$

We obtain from Lemma 19.16 that $g = h$. ■

For many applications the following theorem, which is a variation on the Seifert–van Kampen Theorem 20.2, is very useful.

Theorem 20.10. (Seifert–van Kampen Theorem for topological manifolds) *Let M be an m-dimensional topological manifold and let $R, S \subset M$ be two m-dimensional submanifolds such that the following hold:*

(1) $M = R \cup S$,
(2) $R \cap S$ *is a component of* ∂R *and it is a component of* ∂S,
(3) $R \cap S$ *is non-empty and simply connected,*

[359] Note that here we use Proposition 6.21 which says that Σ is a smooth manifold with empty boundary.

(4) *R and S are closed subsets of M. (Note that by Lemma 2.17 (2) this condition is satisfied if R and S are compact.)*

Then for any base point $x_0 \in R \cap S$ the inclusion induced map

$$\pi_1(R, x_0) * \pi_1(S, x_0) \;\to\; \pi_1(M, x_0)$$

is an isomorphism.

We will of course prove Theorem 20.10 by reducing it to Theorem 20.2. As we will see, the proof is very similar to the proof of Proposition 20.3 where we determined the fundamental group of a wedge of two spaces.

The proof of Theorem 20.10 relies on the following lemma which we will use on several occasions.

Lemma 20.11. $(*)$ *Let X be a topological space. Furthermore let A and B be two subsets with $X = A \cup B$. If $A \cap B$ is a deformation retract of B and if A and B are closed subsets of X, then A is a deformation retract of X.*

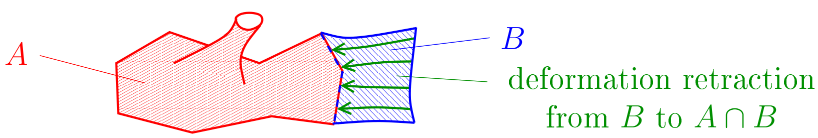

FIGURE 392. Illustration of Lemma 20.11.

PROOF $(*)$. Lemma 20.11 can also be viewed as a consequence of Lemma 3.45 and the slightly scary Lemma 18.23. Since our situation is fairly straightforward we also give a direct, fairly elementary proof.

We pick a deformation retraction $F\colon B \times [0,1] \to B$ from B to $A \cap B$. We consider the map

$$\begin{array}{rcl} G\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x \in A, \\ F(x,t), & \text{if } x \in B. \end{cases} \end{array}$$

Since A and B are closed subsets of X we obtain from Lemma 3.10 that G is continuous. It is now clear that G is the desired deformation retraction from X to A. ■

Remark. On two occasions we are going to struggle a lit bit to verify the hypothesis of Lemma 20.11 that A and B need to be closed. Thus it is good to remind ourselves of the fact that hypothesis are usually added not to vex the reader but since they are necessary. For example, let us consider the situation shown in Figure 393. Here $A \cap B$ is a deformation retract of B, but it should be pretty clear that A is not a deformation retract of $A \cup B = S^1$.

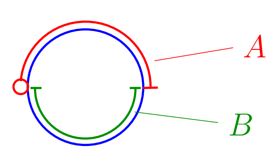

FIGURE 393

Proof of Theorem 20.10 (∗). Let M be an m-dimensional topological manifold and let $R, S \subset M$ be two m-dimensional submanifolds such that the following hold:

(1) $M = R \cup S$,
(2) $R \cap S$ is a union of components of ∂R and it is a union of components of ∂S,
(3) $R \cap S$ is simply connected,
(4) R and S are closed subsets of M.

We pick a base point[360] $x_0 \in R \cap S$. Furthermore we pick maps $f\colon [0,1] \times \partial R \to R$ and $g\colon [0,1] \times \partial S \to S$ that are provided by the Collar Neighborhood Theorem 44.5[361] It follows from (2) that we can set

$$U \;:=\; R \cup \underbrace{[0,1) \times g((R \cap S))}_{\subset [0,1) \times S} \quad \text{and} \quad V \;:=\; S \cup \underbrace{[0,1) \times f((R \cap S))}_{\subset [0,1) \times R}.$$

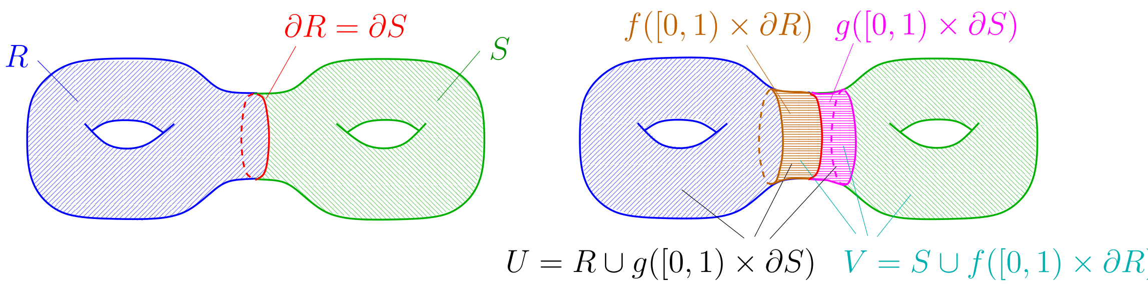

Figure 394. Illustration of the proof of Theorem 20.10.

We consider the following diagram

$$\begin{array}{ccc} \pi_1(R, x_0) * \pi_1(S, x_0) & \longrightarrow & \pi_1(M, x_0) \\ \downarrow & & \downarrow = \\ \pi_1(U, x_0) * \pi_1(V, x_0) & \longrightarrow & \pi_1(M, x_0). \end{array}$$

The following steps are similar to the ones we performed in the proof of Proposition 20.3.

(1) Since the vertical maps are induced by inclusions we see that the diagram commutes.
(2) We claim that U is an open subset of $M = R \cup S$. Since R and S are by our hypothesis (4) closed subsets of M we obtain from Lemma 2.6 (2a) it suffices to show that $U \cap R$ is open in R and that $U \cap S$ is open in S. Evidently $U \cap R = R$ is open in R. Furthermore, $U \cap S = g([0,1] \times (R \cap S))$ is an open subset of S by (2) and Lemma 8.11.
(3) We want to use Lemma 20.11 to show that R is a deformation retract of U. We need to verify that the hypotheses of Lemma 20.11 are satisfied, i.e. we need to show that R and $g([0,1) \times (R \cap S))$ are closed subsets of U. By our hypothesis R and S are closed subsets of M. It follows from Lemma 2.4 (2) that $R = R \cap U$ and $g([0,1] \times (R \cap S)) = S \cap U$ are closed subsets of U.
(4) The obvious analogues of (2) and (3) hold for V and S.

[360]Recall that being simply connected entails being non-empty, hence such a point exists.
[361]If we are dealing with a smooth manifold we can also use the Collar Neighborhood Theorem 8.12.

(5) We obtain from (2), (3), (4) and Proposition 18.16 (2) that the vertical map on the left is an isomorphism.
(6) Similar to (3) one can show that $R \cap S$ is a deformation retract of $U \cap V$. It follows from (3), Lemma 18.14 and Proposition 18.16 (2) that $U \cap V$ is simply connected.
(7) It follows from the Seifert–van Kampen Theorem 20.2 that the bottom horizontal map of the above commutative diagram is an isomorphism.
(8) Since the vertical maps are isomorphisms we see that the top horizontal map is also an isomorphism. ■

We conclude this section by considering the connected sum of two smooth manifolds of dimension ≥ 3. More precisely, we have the following proposition:

Proposition 20.12. *Let $n \geq 3$ and let M and M' be two oriented connected non-empty n-dimensional smooth manifolds. Then*[362]

$$\pi_1(M\#M') \;\cong\; \pi_1(M) * \pi_1(M').$$

Examples.

(1) Let p, q be two coprime integers and let r, s be another two coprime integers. We can then consider the lens spaces $L(p,q)$ and $L(r,s)$. It follows from Proposition 20.12 and Corollary 16.18 that
$$\pi_1(L(p,q)\#L(r,s)) \;\cong\; \mathbb{Z}_p * \mathbb{Z}_r.$$
It follows for example from this calculation of fundamental groups, from Lemma 19.20 and from the Grushko Decomposition Theorem 19.21 that the two topological spaces $L(3,1)\#L(35,1)$ and $L(5,1)\#L(21,1)$ are not homeomorphic.
(2) Let $n \geq 3$ and let M and N be two oriented connected n-dimensional smooth manifolds. We denote by $-N$ the same smooth manifold as N but with the opposite orientation. It follows from Proposition 20.12 that $\pi_1(M\#N) \cong \pi_1(M\#(-N))$.

Proof of Proposition 20.12. Let M and M' be two oriented connected non-empty n-dimensional smooth manifolds and let $\varphi\colon \overline{B}^n \to M$ and $\varphi'\colon \overline{B}^n \to M'$ be two smooth embeddings where φ is orientation-preserving and φ' is orientation-reversing. Recall that the connected sum of M and M' is defined as

$$M\#M' \;:=\; \big(\underbrace{(M \setminus \varphi(B^n))}_{=:X} \sqcup \underbrace{(M' \setminus \varphi'(B^n))}_{=:X'}\big)\,/\,\varphi(P) = \varphi'(P) \text{ for } P \in S^{n-1}.$$

We pick a base point x_0 that lies in the intersection of the images of X and X' in $M\#M'$.

Claim. The inclusion induced maps $\pi_1(X, x_0) \to \pi_1(M, x_0)$ and $\pi_1(X', x_0) \to \pi_1(M', x_0)$ are isomorphisms.

We show that the inclusion induced map $\pi_1(X, x_0) \to \pi_1(M, x_0)$ is an isomorphism. We write $Y = \varphi(\overline{B}^n)$. It follows from Proposition 8.1 that X and Y are n-dimensional

[362]Note that by Proposition 8.35 (3) we know that $M\#M'$ is also connected, thus we do not need to worry about base points.

submanifolds of M. Note that $X \cup Y = M$ and that $X \cap Y = \partial Y \cong S^{n-1}$. Since $n \geq 3$ we obtain from Proposition 14.14 that $X \cap Y$ is simply connected. We obtain that

$$\pi_1(X, x_0) \xrightarrow{\cong} \pi_1(X, x_0) * \pi_1(\overline{B}^n, x_0) \xrightarrow{\cong} \pi_1(M, x_0).$$

$$\underset{\text{since } \pi_1(\overline{B}^n, x_0) = \{e\}}{\uparrow} \qquad \underset{\text{Seifert–van Kampen Theorem 20.10}}{\uparrow}$$

Since the horizontal maps are induced by the inclusion maps we see that the inclusion induced map $\pi_1(X, x_0) \to \pi_1(M, x_0)$ is an isomorphism. The same way we can show that the inclusion induced map $\pi_1(X', x_0) \to \pi_1(M', x_0)$ is an isomorphism. ⊞

Now we see that[363]

$$\pi_1(M\#M, x_0) \xleftarrow{\cong} \pi_1(X, x_0) * \pi_1(X', x_0) \xrightarrow{\cong} \pi_1(M, x_0) * \pi_1(M', x_0).$$

$$\underset{\substack{\text{by Theorem 20.10, since} \\ X \cap X' \cong S^{n-1} \text{ is simply connected}}}{\uparrow} \qquad \underset{\text{above claim}}{\uparrow}$$

■

Remark. We cannot apply the Seifert–van Kampen Theorem 20.2 to the connected sum of two surfaces, since in this case the fundamental group of the overlap $R \cap R'$ in the proof of Proposition 20.12 is not simply connected. In particular we cannot use this approach to determine the fundamental group of the surface of genus 2, which by Figure 220 is the connected sum of two tori. Our next goal is to formulate and prove a generalization of the Seifert–van Kampen Theorem 20.2 which will allow us to determine the fundamental group of the surface of genus g for $g \geq 2$.

Exercises for Chapter 20.

Exercise 20.1. We consider $X = \mathbb{C} \setminus \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Show that the point $x = 0$ is not good, i.e. show that there does not exist an open neighborhood U of $x = 0$ such that $x = 0$ is a deformation retract of U.

Exercise 20.2.

(1) Let X be a path-connected topological space and let $x_0 \in X$. Furthermore let $X_k, k \in \mathbb{N}$ be a sequence of subsets of X such that the following hold:
  (a) each X_k is open,
  (b) each X_k is simply connected,
  (c) the sequence X_k is nested, i.e. for each k we have $X_k \subset X_{k+1}$,
  (d) we have $\bigcup_{k \in \mathbb{N}} X_k = X$.

  Show that $\pi_1(X, x_0) = 0$.
  Hint. Use Lemma 2.41.

(2) Does the conclusion of (1) also hold if the sets X_k are not open?

Exercise 20.3. Let M be a *topological* manifold of dimension $n \geq 3$. Let P be a point and let $x_0 \in M \setminus \{P\}$ be a base point. Show that the inclusion induced map

$$\pi_1(M \setminus \{P\}, x_0) \;\to\; \pi_1(M, x_0)$$

---

[363]Here the directions of the arrow indicate in which direction the obvious inclusion induced maps point.

is an isomorphism.

Exercise 20.4.

(a) How can one hang a painting with two nails provided, such that if one removes either of the two nails, the painting falls down?
(b) What is a solution to (a) if we use three nails instead of two nails?

Hint. What does this have to do with free groups?

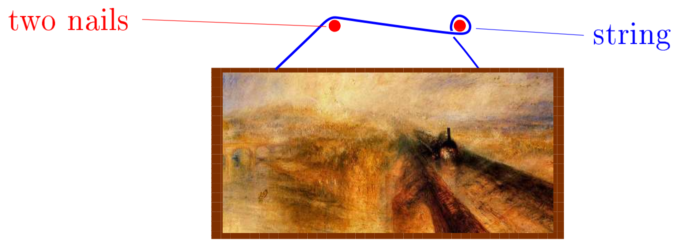

FIGURE 395. Illustration of Exercise 20.4.

Exercise 20.5. Show that condition (4) in the statement of the Seifert–van Kampen Theorem 20.10 for smooth manifolds cannot be dropped. More precisely, give an example of an m-dimensional smooth manifold and m-dimensional submanifolds R and S such that conditions (1), (2) and (3) are satisfied, but such that $\pi_1(R,x_0)*\pi_1(S,x_0)$ is not isomorphic to $\pi_1(M,x_0)$.
Hint. The dimension m does not have to be taken particularly large.

Exercise 20.6. We consider the two topological manifolds X and Y that are shown in Figure 396. Show that X and Y are not homeomorphic.
Remark. As a smooth manifold X is orientable and Y is non-orientable. But as of right now we do not know whether orientability is preserved under homeomorphisms.
Hint. You could use Proposition 18.18 and consider the interaction between the topological manifolds and their boundaries.

FIGURE 396. Illustration of Exercise 20.6.

Exercise 20.7. In the proof of Proposition 20.7, given on page 610, show that the map

$$\mathop{\ast}_{k\in K} i_{k*}\colon \mathop{\ast}_{k\in K} \pi_1(A_k) \;\to\; \pi_1\Big(\bigvee_{k\in K} A_k\Big)$$

is a monomorphism.
Hint. Consider the commutative diagram shown on page 611.

Exercise 20.8. Let Σ be the 2-dimensional smooth manifold that is shown in Figure 397. We denote by C the boundary of Σ.

(a) What is the homeomorphism type of the topological space that we obtain from gluing a disk $\overline{B}^2$ to Σ along the boundary C?
(b) Is C a retract of Σ?

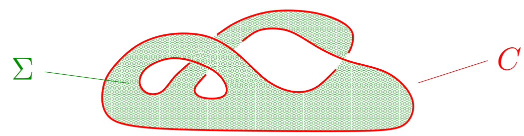

FIGURE 397. Illustration for Exercise 20.8.

Exercise 20.9.
(a) Let $n \geq 3$ and furthermore let $K \subset \mathbb{R}^n$ be a compact subset. We view K also as a subset of $S^n = \mathbb{R}^n \cup \{\infty\}$. Let $x_0 \in \mathbb{R}^n \setminus K$. Show that the inclusion induced map $\pi_1(\mathbb{R}^n \setminus K, x_0) \to \pi_1(S^n \setminus K, x_0)$ is an isomorphism.
(b) Does the conclusion of (a) also hold for $n = 2$?
(c) Does the conclusion of (c) also hold for non-compact subsets of $\mathbb{R}^n$?

Exercise 20.10. We consider the two topological spaces

$$X = \mathbb{R}^2 \setminus \{(\tfrac{1}{n}, 0) \,|\, n \in \mathbb{N}\} \text{ and } \quad \text{and } Y = \mathbb{R}^2 \setminus \{(\tfrac{1}{n}, y) \,|\, n \in \mathbb{N}, y \in [-1, 1]\}.$$

We write $x_0 = (0, 0)$. We refer to Figure 398 for an illustration.
(a) Show that $\pi_1(X, x_0)$ is uncountable.
Hint. For $n \in \mathbb{N}$ we write $Z_n := \mathbb{R}^2 \setminus \{\frac{1}{n}, 0\}$ and we denote by $f_n \colon X \to Z_n$ the inclusion map. Use the induced maps $f_{n*} \colon \pi_1(X, x_0) \to \pi_1(Z_n, x_0) \cong \mathbb{Z}$ to find a homomorphism from $\pi_1(X, x_0)$ to a group such that the image is uncountable.
(b) Show that $\pi_1(Y, x_0)$ is countable.
Hint. Use Exercise 2.43 to find a monomorphism from $\pi_1(Y, x_0)$ into a countable group.

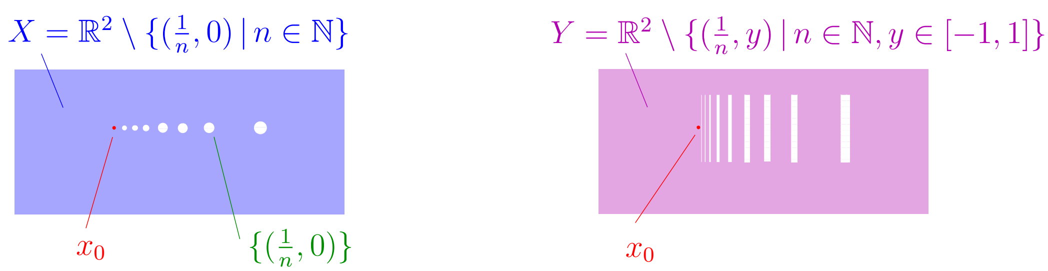

FIGURE 398

Exercise 20.11. Show that given any $k \in \mathbb{N}$ and given any $n \geq 3$ there exists a closed orientable connected n-dimensional smooth manifold M such that $\pi_1(M) \cong \langle x_1, \ldots, x_k \rangle$.

Exercise 20.12. We consider the wedge $S^1 \vee S^1$ of two circles and we denote by $*$ the wedge point. We consider the map $f \colon S^1 \vee S^1 \to S^1 \vee S^1$ that is shown in Figure 399.
(a) Show that f is a homotopy equivalence.
(b) Show that every homeomorphism of $S^1 \vee S^1$ preserves the wedge point $*$.

(c) Determine

$$\{\varphi \in \operatorname{Aut}(\pi_1(S^1 \vee S^1, *)) \mid \text{ there exists a homeomorphism } g \text{ with } \varphi = g_*\}.$$

(d) Show that f is not homotopic to a self-homeomorphism of $S^1 \vee S^1$.

Remark. In Proposition **??** we will show that there exist self-homotopy equivalences of smooth manifolds that are not homotopic to homeomorphisms.

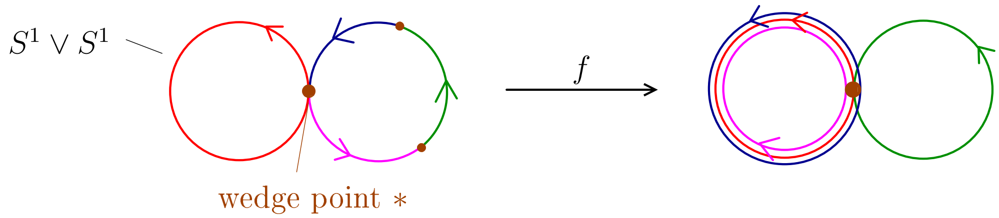

FIGURE 399. Illustration for Exercise 20.12.

We have now made some progress, for example we have shown that surfaces of different genus are not homeomorphic. But this proof is perhaps slightly unsatisfactory in so far as we used the trick of removing a point, instead of dealing directly with the fundamental groups of surfaces.

In the next section we will cover the generalized Seifert–van Kampen Theorem. It will allow us to compute many more fundamental groups. In particular we will determine the fundamental groups of surfaces and knot complements.

Before we can state the generalized Seifert–van Kampen Theorem we need to introduce more concepts from group theory. We will do so in this chapter.

21.1. **Basic definitions in group theory.** We recall a few basic definitions from group theory.

Definition. Let G be a group.

(1) A subgroup H is called *normal* if for every $g \in G$ we have $g^{-1}Hg = H$. If $H \subset G$ is normal, then we often write $H \lhd G$. If $\{H_i\}_{i\in I}$ is a family of normal subgroups of G, then it is straightforward to see that the intersection $\bigcap\limits_{i\in I} H_i$ is again a normal subgroup of G.

(2) Let H be a subgroup of G. The subgroup H defines an equivalence relation on G by defining $g \sim \widetilde{g}$ if there exists an $h \in H$ with $gh = \widetilde{g}$. Put differently, we have $g \sim \widetilde{g}$ if $\widetilde{g}^{-1}g \in H$. (Note that the equivalence classes are precisely the sets of the form $gH = \{gh \mid h \in H\}$ for some $g \in G$.) We denote the set of equivalence classes by G/H.

Remark. Let H be a normal subgroup of G.

(1) For any $h \in H$ and $a, b \in G$ we have

$$ahbH \;=\; ab\cdot \underbrace{b^{-1}hb}_{\text{in } H \text{ since } H \text{ is normal}} H \;=\; abH.$$

(2) If $aH = bH$, then for any $c, d \in G$ we have

$$cadH \;=\; cb\underbrace{(b^{-1}a)}_{\in H}dH \;\underset{\substack{\uparrow \\ \text{above remark (1)}}}{=}\; cbdH.$$

Now we will see that the group structure on G descends to a group structure on G/H. More precisely, given aH and bH we set

$$aH\cdot bH \;:=\; abH.$$

It follows immediately from the above remark (1) that this definition does not depend on the representatives a and b. It is now straightforward to see that this defines indeed a group structure on G/H. The following lemma plays the same role as Lemma 3.22 does for topological spaces.

Lemma 21.1. *Let G be a group and let H be a normal subgroup. Let $\varphi\colon G \to \Gamma$ be a group homomorphism with the property that $H \subset \ker(\varphi)$. Then there exists a unique homomorphism $\psi\colon G/H \to \Gamma$ such that the following diagram commutes*

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "\varphi"'] & G/H \arrow[d, "\psi"] \\ & \Gamma. \end{tikzcd}$$

Proof. For $gH \in G/H$ we put $\psi(gH) := \varphi(g)$. Since $H \subset \ker(\varphi)$ this definition is independent of the choice of the representatives g.[364] It is straightforward to verify that ψ is a homomorphism and that it is uniquely determined by the desired property. ■

21.2. **Presentations of groups.**

Definition. Let G be a group and let $A \subset G$ be a subset. We define

$$\text{subgroup of } G \text{ \textit{generated by} } A \;:=\; \langle A\rangle \;:=\; \bigcap_{\substack{A\subset H\subset G\\ H \text{ subgroup}}} H,$$

i.e. $\langle A\rangle$ is the intersection of all subgroups H of G which contain A. This is again a subgroup of G, namely the smallest[365]subgroup of G which contains A. Similarly we define

$$\text{subgroup of } G \text{ \textit{normally generated by} } A \;:=\; \langle\langle A\rangle\rangle \;:=\; \bigcap_{A\subset H\triangleleft G} H,$$

i.e. $\langle\langle A\rangle\rangle$ is the intersection of all *normal* subgroups H of G which contain A. This is again a normal subgroup of G, namely the smallest normal subgroup of G which contains A.

The following lemma follows immediately from the definitions and from the fact that in an abelian group every subgroup is normal.

Lemma 21.2. *Let G be a group and let $A \subset G$ be a subset.*

(1) *If A is a subgroup, then $\langle A\rangle = A$.*
(2) *If A is a normal subgroup, then $\langle\langle A\rangle\rangle = A$.*
(3) *If G is abelian, then $\langle\langle A\rangle\rangle = \langle A\rangle$.*
(4) *For any $B \subset G$ with $A \subset B \subset \langle\langle A\rangle\rangle$ we have $\langle\langle B\rangle\rangle = \langle\langle A\rangle\rangle$.*
(5) *Suppose that for each $a \in A$ we are given $g_a \in A$. Then $\langle\langle \{g_a a g_a^{-1}\}_{a\in A}\rangle\rangle = \langle\langle A\rangle\rangle$.*

The following lemma gives a more explicit description of $\langle A\rangle$ and $\langle\langle A\rangle\rangle$.

Lemma 21.3. *Let G be a group and let $A \subset G$ be a subset.*

(1) *Then $\langle A\rangle$ equals the set of all elements of G which can be written as products of elements in A and their inverses, i.e.*

$$\langle A\rangle \;=\; \big\{a_1^{\epsilon_1}\cdot\dots\cdot a_k^{\epsilon_k} \mid a_1,\dots,a_k \in A \text{ and } \epsilon_1,\dots,\epsilon_k \in \{-1,1\}\big\}.$$

[364]Indeed, if $gH = \widetilde{g}H$, then $g = \widetilde{g}h$ for some $h \in H$, therefore $\varphi(g) = \varphi(\widetilde{g}h) = \varphi(\widetilde{g})\cdot\varphi(h) = \varphi(\widetilde{g})$.

[365]Here "smallest" means that if B is another subgroup that contains A, then $\langle A\rangle \subset B$. This statement is obvious, since B appears in the definition of $\langle A\rangle$ as one of the H's.

(2) *The subset $\langle\langle A\rangle\rangle$ equals the set of all elements of G which can be written as products of conjugates of elements in A and their inverses, i.e.*

$$\langle\langle A\rangle\rangle = \big\{g_1a_1^{\epsilon_1}g_1^{-1}\dots g_ka_k^{\epsilon_k}g_k^{-1}\,|\,a_1,\dots,a_k\in A, g_1\dots,g_k\in G \text{ and } \epsilon_1,\dots,\epsilon_k\in\{-1,1\}\big\}.$$

Example. On page 501 we introduced a subgroup of the group H of all self-homeomorphisms of $\mathbb{R}^2$ that was generated by two given self-homeomorphisms A and B. The lemma shows that the definition of "being generated" by a subset used on page 501 agrees with the above more formal definition.

Proof. We only prove the second statement. The proof of the first statement is very similar. So let G be a group and let $A \subset G$ be a subset. The subgroup $\langle\langle A\rangle\rangle$ contains all elements of A and their inverses. The normal subgroup $\langle\langle A\rangle\rangle$ then contains also all elements of the form $ga^{\pm 1}g^{-1}, g\in G, a\in A$. Since $\langle\langle A\rangle\rangle$ is a subgroup it is also closed under multiplication, i.e. it contains products of such elements. It follows that $\langle\langle A\rangle\rangle$ contains the right-hand side of (2). Conversely it is straightforward to see that the right-hand side is indeed a normal[366] subgroup of G that contains A, hence it is contained in the left-hand side of (2). ■

Lemma 21.4. *Let G be a group and let $A \subset G$ be a subset. Let $\alpha\colon G\to\Gamma$ be a group homomorphism such that $\alpha(a) = e\in\Gamma$ for all $a\in A$. Then there exists a unique group homomorphism $\beta\colon G/\langle\langle A\rangle\rangle\to\Gamma$ such that the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "\alpha"'] & G/\langle\langle A\rangle\rangle \arrow[d, "\beta"] \\ & \Gamma. \end{tikzcd}$$

Proof. Let G be a group and let $A\subset G$ be a subset. Furthermore let $\alpha\colon G\to\Gamma$ be a group homomorphism such that $\alpha(a) = e\in\Gamma$ for all $a\in A$. This means that $A\subset\ker(\alpha)$. Since $\ker(\alpha)$ is a normal subgroup it follows from the definition of $\langle\langle A\rangle\rangle$ that $\langle\langle A\rangle\rangle\subset\ker(\alpha)$. The proposition now follows immediately from Lemma 21.1. ■

The following definition will subsequently play an important role:

Definition. Let X be a set and let R be a subset of the free group $\langle X\rangle$ generated by the set X. We define

$$\langle X \mid R\rangle \;:=\; \langle X\rangle\,/\,\langle\langle R\rangle\rangle.$$

If $X=\{x_1,\dots,x_k\}$ and $R=\{r_1,\dots,r_l\}$, then we use the convenient shorthand

$$\langle x_1,\dots,x_k\,|\,r_1,\dots,r_l\rangle \;:=\; \langle\{x_1,\dots,x_k\}\,|\,\{r_1,\dots,r_l\}\rangle.$$

Examples.

[366] Why is the right-hand side normal?

(1) For any $k \in \mathbb{N}_0$ we have

$$\langle g \mid g^k \rangle \;=\; \langle g \rangle / \langle\langle g^k \rangle\rangle \underset{\substack{\uparrow \\ \text{since } \langle g \rangle \text{ is abelian it follows from Lemma 21.2 that } \langle\langle g^k \rangle\rangle = \langle g^k \rangle}}{=} \langle g \rangle / \langle g^k \rangle \;\cong \mathbb{Z}_k.$$

(2) The group $\langle x_1, \dots, x_k \mid \rangle$ with $R = \varnothing$ is just the free group on $x_1, \dots, x_k$.

Before we continue we formulate an elementary but useful lemma.

Lemma 21.5. *Let* $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ *be a presentation.*

(1) *Given any* $a, b \in \langle x_1, \dots, x_k \rangle$, *given any* $i \in \{1, \dots, l\}$ *and given any* $\epsilon \in \{-1, 1\}$ *we have*

$$a \cdot r_i^\epsilon \cdot b \;=\; a \cdot b \;\in\; \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle.$$

(2) *Let* $a, b \in \langle x_1, \dots, x_k \rangle$ *and* $c, d \in \langle x_1, \dots, x_k \rangle$. *If* $a = b \in \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$, *then*

$$c \cdot a \cdot d \;=\; c \cdot b \cdot d \;\in\; \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle.$$

Proof. The first statement is just a special case of Remark (1) on page 626 and the second statement is just a special case of Remark (2) on page 626. ■

This lemma allows us to prove the following lemma.

Lemma 21.6.

(1) *There exists a unique homomorphism*

$$\beta \colon \pi = \langle x, y \rangle / \langle\langle [x, y] \rangle\rangle = \langle x, y \mid [x, y] \rangle \;\to\; \mathbb{Z}^2$$

with $\beta(x) = (1, 0)$ *and* $\beta(y) = (0, 1)$. *Furthermore this homomorphism is an isomorphism.*

(2) *Let* $m \in \mathbb{N}$. *There exists a unique isomorphism*

$$\beta \colon \langle x_1, \dots, x_m \mid [x_i, x_j] = e \textit{ for all } i, j \in \{1, \dots, m\} \rangle \;\xrightarrow{\cong}\; \mathbb{Z}^m$$

which sends each x_i *to the* i*-th standard basis vector* e_i *of* $\mathbb{Z}^m$.

Proof. By Lemma 19.14 there exists a unique homomorphism $\alpha \colon \langle x, y \rangle \to \mathbb{Z}^2$ which satisfies $\alpha(x) = (1, 0)$ and $\alpha(y) = (0, 1)$. Note that

$$\alpha([x, y]) \;=\; \alpha(xyx^{-1}y^{-1}) \;=\; \alpha(x) + \alpha(y) - \alpha(x) - \alpha(y) \;=\; 0.$$

Thus by Lemma 21.4 there exists a unique group homomorphism

$$\beta \colon \pi = \langle x, y \rangle / \langle\langle [x, y] \rangle\rangle = \langle x, y \mid [x, y] \rangle \;\to\; \mathbb{Z}^2$$

with $\beta(x) = (1, 0)$ and $\beta(y) = (0, 1)$. It is clear that β is an epimorphism. Thus it remains to show that α is a monomorphism. This follows immediately from the following claim.

Claim. Given any $w \in \langle x, y \rangle$ there exist $m, n \in \mathbb{Z}$ with $w = x^m y^n \in \langle x, y \mid [x, y] \rangle$.

First we note that we have the following equalities in $\langle x, y \mid [x, y] \rangle$

$$yx \underset{\substack{\uparrow \\ \text{Lemma 21.5 (1)}}}{=} xyx^{-1}y^{-1} \cdot yx \;=\; xy \qquad \text{and} \qquad xy^{-1} \;=\; y^{-1} \cdot yx \cdot y^{-1} \underset{\substack{\uparrow \\ \text{calculation on the left and Lemma 21.5 (2)}}}{=} y^{-1} \cdot xy \cdot y^{-1} \;=\; y^{-1}x.$$

In Exercise 21.4 we will show that the equalities $yx^{-1} = x^{-1}y$ and $x^{-1}y^{-1} = y^{-1}x^{-1}$ hold in $\pi = \langle x, y \mid [x, y]\rangle$. This means that in π an arbitrary product of powers of x and y can be reordered to equal a product of the $x^m y^n$. For example in π we have the equality

$$x^3y^2x^{-1}y = x^3y \cdot yx^{-1} \cdot y \underset{\substack{\uparrow \\ \text{since } yx^{-1} = x^{-1}y}}{=} x^3y \cdot x^{-1}y \cdot y = x^3 \cdot yx^{-1} \cdot y^2 \underset{\substack{\uparrow \\ \text{since } yx^{-1} = x^{-1}y}}{=} x^3 \cdot x^{-1}y \cdot y^2 = x^2y^3. \qquad \blacksquare$$

We consider one more example.

Example. We consider the following three self-homeomorphisms of $\mathbb{R}^2$:

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s+1, 1-t) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s, t+1) \end{array} \quad \text{and} \quad \begin{array}{rcl} \widetilde{A}\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s+1, -t). \end{array}$$

We already know the first two maps from page 501. We denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. We are interested in the group G since $\mathbb{R}^2/G$ is the Klein bottle K and by Theorem 16.16 we have $\pi_1(K) \cong G$.

Note that $\widetilde{A} = B^{-1}A$. Let $X = \{x, y\}$ be the set that consists of two elements. By Lemma 19.14 there exists a unique homomorphism $\alpha\colon \langle x, y\rangle \to G$ with $\alpha(x) = \widetilde{A}$ and $\alpha(y) = B$. Now let R be the subset of $\langle X\rangle = \langle x, y\rangle$ that consists of the single element $yxyx^{-1}$. We want to consider $\alpha(yxyx^{-1}) \in G$, which is by definition a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. For any point $(s, t) \in \mathbb{R}^2$ this map is given by

$$\begin{aligned} \alpha(yxyx^{-1})(s,t) &= B\widetilde{A}B\widetilde{A}^{-1}(s,t) = B\widetilde{A}B(s-1, -t) \\ &= B\widetilde{A}(s-1, -t+1) = B(s, t-1) = (s,t). \end{aligned}$$

This shows that $\alpha(yxyx^{-1})$ is the identity map of $\mathbb{R}^2$, i.e. it is the trivial element in G. It follows from Lemma 21.4 that α descends to a homomorphism

$$\beta\colon \langle x, y\rangle / \langle\langle yxyx^{-1}\rangle\rangle = \langle x, y \mid yxyx^{-1}\rangle \to G.$$

This homomorphism is an epimorphism, since the image of β is the group generated by $\widetilde{A} = B^{-1}A$ and B, but this group is the same as the group generated by A and B, i.e. the image of β is G. Is the homomorphism β also a monomorphism? We will come back to this question later on page 657.

We continue with the following definitions.

Definition.

(1) A *presentation* is a set X together with a subset $R \subset \langle X\rangle$. We call the elements of X the *generators* and we call the elements of R the *relators*[367] of the presentation.

(2) Let π be a group. A *presentation for* π is a presentation (X, R) together with an isomorphism

$$\langle X \mid R\rangle \xrightarrow{\cong} \pi.$$

Very often one just says $\langle X \mid R\rangle$ is a presentation for π and the explicit isomorphism is suppressed from the notation.

(3) We say that a group π is *finitely generated*, if π admits a presentation with finitely many generators.[368]

(4) We say a group π is *finitely presented*, if π admits a presentation with finitely many generators and finitely many relators.

Now we give some examples of finitely generated and finitely presented groups.

Examples.

(1) Every group admits a presentation. Indeed, let π be a group. Then we set $X := \pi$, i.e. X is the set of elements of π. By Lemma 19.14 there exists a unique homomorphism $\alpha\colon \langle X\rangle \to \pi$ with $\alpha(g) = g$ for all $g \in X = \pi$. We set $R := \ker(\alpha)$. Then

$$\langle X \mid R\rangle \;=\; \langle X\rangle/\langle\langle R\rangle\rangle \;=\; \langle X\rangle/\langle\langle \ker(\alpha)\rangle\rangle \underset{\substack{\uparrow \\ \text{by Lemma 21.2 since } R=\ker(\alpha) \text{ is a normal subgroup}}}{=} \langle X\rangle/\ker(\alpha) \xrightarrow[\alpha]{\cong} G.$$

(2) We had just see in Lemma 21.6 (2) that

$$\mathbb{Z}^n \;\cong\; \langle x_1,\dots,x_n \mid [x_i,x_j] = e \text{ with } i,j \in \{1,\dots,n\}\rangle$$

which shows that $\mathbb{Z}^n$ is a finitely presented group.

(3) In fact every finitely generated abelian group is also finitely presented. Indeed, let A be a finitely generated abelian group. Then

$$A \underset{\substack{\uparrow \\ \text{Theorem 19.4} \\ \text{for some } a_i \in \mathbb{N}_0}}{\cong} \bigoplus_{i=1}^n \mathbb{Z}_{a_i} \underset{\substack{\uparrow \\ \text{as in (2)}}}{=} \langle x_1,\dots,x_n \mid [x_i,x_j] = e \text{ for } i,j=1,\dots,n, x_i^{a_i}=e \text{ for } i=1,\dots,n\rangle.$$

(4) There exist groups which are finitely generated but not finitely presented. In fact there are countably many isomorphism types of finitely presented groups,[369] whereas there are uncountably many isomorphism types of finitely generated groups.[370] A proof for both statements is given in Chapter 14 of [**Rob95**].

(5) We saw on page 593 that the fundamental group of the Klein bottle is finitely generated. But is it also finitely presented?

21.3. **Tietze transformations.**

Definition. Let π be a group and let

$$\phi\colon \langle x_1,\dots,x_k \,|\, r_1,\dots,r_l\rangle \xrightarrow{\cong} \pi$$

be a presentation. We obtain new presentations as follows:

(1) For any element s of $\langle\langle r_1,\dots,r_l\rangle\rangle \subset \langle X\rangle$ we obtain the presentation

$$\langle x_1,\dots,x_k \mid r_1,\dots,r_l,s\rangle \;=\; \langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle \xrightarrow{\phi} \pi.$$

[367] The relators are often also called relations.

[368] This definition of a finitely generated group is equivalent to the one we gave on page 592. Why is that?

[369] That statement is pretty obvious.

[370] This statement is much less obvious. The idea is that in a free group $\langle x_1,\dots,x_k\rangle$ there are uncountably many infinite sequences of elements $\{r_1,r_2,\dots\}$. The difficulty is to show that there exist uncountably many sequences that give raise to non-isomorphic groups.

(2) For any $x \neq x_1, \ldots, x_k$ and any two words s, t in $x_1, \ldots, x_k$ we obtain the presentation[371]

$$\begin{array}{rcl} \langle x_1, \ldots, x_k, x \mid r_1, \ldots, r_l, s \cdot x \cdot t \rangle & \to & \langle x_1, \ldots, x_k \mid r_1, \ldots, r_l \rangle \xrightarrow{\phi} \pi. \\ x_i & \mapsto & x_i \\ x & \mapsto & s^{-1} \cdot t \end{array}$$

The above two methods, and their inverses, of obtaining new presentations out of a given one are called *Tietze transformations.*[372]

The above Tietze transformations show that from a given finite presentation we can obtain infinitely many new finite presentations. This fact is not that surprising. What is much more interesting is the following theorem which was first proved by Heinrich Tietze [**Tie1908**] in 1908.

Theorem 21.7. (Tietze Theorem) *Any two finite presentations for a given group are related by a finite sequence of Tietze transformations.*

In the proof of the Tietze Theorem 21.7 we will need the following four pretty elementary preparatory lemmas.

Lemma 21.8. *Let $\theta \colon G \to H$ be a group homomorphism and let X be a generating set for G. The subgroup of $G * H$ that is generated by $\{g \cdot \theta(g^{-1}) \,|\, g \in G\}$ equals that the subgroup of $G * H$ that is normally generated by $\{x \cdot \theta(x^{-1}) \,|\, x \in X\}$.*

Proof of Lemma 21.8. We set $N := \{x \cdot \theta(x^{-1}) \,|\, x \in X\}$. We only need to show that given any $g \in G$ we have $g \cdot \theta(g^{-1}) \in \langle\langle N \rangle\rangle$. This statement follows easily by induction on the length of g as a word in the generating set X and the observation that for $x, y \in X$ we have the following equality:

$$xy \cdot \theta((xy)^{-1}) \;=\; xy \cdot \theta(y^{-1}) \cdot \theta(x^{-1}) \;=\; x \cdot \underbrace{y \cdot \theta(y)^{-1}}_{\in N} \cdot x^{-1} \cdot \underbrace{x \cdot \theta(x^{-1})}_{\in N}.$$

■

Lemma 21.9. *Let G and H be two groups, let $\theta \colon G \to H$ be an isomorphism and let $\alpha \colon G \to \pi$ be an isomorphism. We consider the homomorphism*

$$\Psi \colon G * H \;\to\; \pi$$

defined by α on G and by $\alpha \circ \theta^{-1}$ on H. The kernel of the homomorphism Ψ is generated by $\{g \cdot \theta(g^{-1}) \,|\, g \in G\}$.

Proof of Lemma 21.9. We set $M := \{g \cdot \theta(g^{-1}) \,|\, g \in G\}$. We need to show that the kernel of the homomorphism Ψ is generated by M. Let $g_1 \cdot h_1 \cdot \cdots \cdot g_k \cdot h_k \in \ker(\Psi) \subset G * H$, where we use self-explanatory notation.

If $k = 1$, then it follows from $e = \Psi(g_1 \cdot h_1) = \alpha(g_1) \cdot \alpha(\theta^{-1}(g_1)) = \alpha(g_1 \cdot \theta^{-1}(g_1))$ and the fact that α and θ are isomorphisms that $g_1 \cdot h_1 = g_1 \cdot \theta(g_1{}^{-1})$.

Now suppose that $k > 1$. If $g_1 \neq e$, then we can write

$$g_1 \cdot h_1 \cdot g_2 \cdot h_2 \cdot \cdots \cdot g_k \cdot h_k \;=\; \underbrace{g_1 \cdot \theta(g_1{}^{-1})}_{\in M} \cdot \underbrace{\theta(g_1) \cdot h_1}_{\in H} \cdot g_2 \cdot h_2 \cdot \ldots g_k \cdot h_k.$$

[372] The map to the left is an isomorphism, in fact an inverse is given by $x_i \mapsto x_i$.

If $g_1 = e$, then we write

$$h_1 \cdot g_2 \cdot h_2 \cdot \cdots \cdot g_k \cdot h_k \;=\; \underbrace{(\theta^{-1}(h_1) \cdot \theta(\theta^{-1}(h_1)^{-1}))^{-1}}_{\in M} \cdot \underbrace{\theta^{-1}(h_1) \cdot g_2}_{\in G} \cdot h_2 \cdot \ldots g_k \cdot h_k.$$

Iterating this procedure we obtain the desired statement. ■

Lemma 21.10. *Let*

$$\phi\colon \langle x_1, \ldots, x_k \,|\, r_1(x_1, \ldots, x_k), \ldots, r_l(x_1, \ldots, x_k)\rangle \;\to\; \pi$$

be a presentation. For pairwise different $y_1, \ldots, y_k$ the presentation

$$\begin{array}{rcl} \langle y_1, \ldots, y_k \,|\, r_1(y_1, \ldots, y_k), \ldots, r_l(y_1, \ldots, y_k)\rangle & \to & \pi \\ y_i & \mapsto & \phi(x_i) \end{array}$$

can be obtained from the original one by Tietze transformations.

Proof of Lemma 21.10. We consider the Tietze transformations

$$\begin{array}{rl} \langle x_1, \ldots, x_k \,|\, r_1(x_1, \ldots, x_k), \ldots\rangle \overset{(2)}{\rightsquigarrow} & \langle x_1, \ldots, x_k, y_1 \,|\, r_1(x_1, \ldots, x_k), \ldots, y_1 \cdot x_1^{-1}\rangle \\ \overset{(1)}{\rightsquigarrow} & \langle x_1, \ldots, x_k, y_1 \,|\, r_1(x_1, \ldots, x_k), \ldots, y_1 \cdot x_1^{-1}, r_1(y_1, \ldots, x_k), \ldots\rangle \\ \overset{(1)}{\rightsquigarrow} & \langle x_1, \ldots, x_k, y_1 \,|\, y_1 \cdot x_1^{-1}, r_1(y_1, \ldots, x_k), \ldots\rangle \\ \overset{(1)}{\rightsquigarrow} & \langle x_1, \ldots, x_k, y_1 \,|\, x_1 \cdot y_1{}^{-1}, y_1 \cdot x_1^{-1}, r_1(y_1, \ldots, x_k), \ldots\rangle \\ \overset{(1)}{\rightsquigarrow} & \langle x_1, \ldots, x_k, y_1 \,|\, x_1 \cdot y_1{}^{-1}, r_1(y_1, \ldots, x_k), \ldots\rangle \\ \overset{(2)}{\rightsquigarrow} & \langle x_2, \ldots, x_k, y_1 \,|\, r_1(y_1, \ldots, x_k), \ldots\rangle. \end{array}$$

Iterating this process leads to the desired sequence of Tietze transformations. ■

Lemma 21.11. *Let $\alpha\colon A \to B$ and $\beta\colon B \to C$ be group epimorphisms. Let $X \subset A$ be a subset such that $\ker(\alpha) = \langle\langle X\rangle\rangle$ and let $Y \subset B$ be a subset such that $\ker(\beta) = \langle\langle \alpha(Y)\rangle\rangle$. Then $\ker(\beta \circ \alpha\colon A \to C) = \langle\langle X \cup Y\rangle\rangle$.*

Proof of Lemma 21.11. It is clear that $X \cup Y \subset \ker(\beta \circ \alpha\colon A \to C)$. To prove the reverse inclusion let $g \in \ker(\beta \circ \alpha\colon A \to C)$. We have $\alpha(g) \in \ker(\beta) = \langle\langle \alpha(Y)\rangle\rangle$. Since α is an epimorphism the map $\alpha\colon \langle\langle Y\rangle\rangle \to \langle\langle \alpha(Y)\rangle\rangle$ is an epimorphism. Thus there exists an $h \in \langle\langle Y\rangle\rangle$ with $\alpha(g) = \alpha(h)$. But this implies that $g \cdot h^{-1} \in \ker(\alpha) = \langle\langle X\rangle\rangle$. It follows that $g \in \langle\langle X \cup Y\rangle\rangle$. ■

Proof of the Tietze Theorem 21.7. Let π be a group and let $\phi_1\colon \langle X_1 \,|\, R_1\rangle \xrightarrow{\cong} \pi$ and $\phi_2\colon \langle X_2 \,|\, R_2\rangle \xrightarrow{\cong} \pi$ be two finite presentations for π. We need to show that they are related by Tietze transformations. First note that it follows from Lemma 21.10 that by applying a sequence of Tietze transformations we can arrange that $X_1 \cap X_2 = \emptyset$.

We denote by $p_1\colon \langle X_1\rangle \to \langle X_1 \,|\, R_1\rangle$ and $p_2\colon \langle X_2\rangle \to \langle X_2 \,|\, R_2\rangle$ the obvious epimorphisms. We set $X := X_1 \sqcup X_2$. We consider the homomorphism

$$\Psi\colon \langle X_1\rangle * \langle X_2\rangle = \langle X\rangle \;\to\; \pi$$

that is induced by $\phi_1 \circ p_1$ on $\langle X_1\rangle$ and $\phi_2 \circ p_2$ on $\langle X_2\rangle$.

Note that the restriction of Ψ to $\langle X_2\rangle$ is an epimorphism. Therefore we can pick for each $x \in X_1$ a $w_x \in \langle X_2\rangle$ with $\Psi(x) = \Psi(w_x)$. We write $S_1 := \{x \cdot w_x^{-1} \,|\, x \in X_1\} \subset \langle X\rangle$. The same way we define $S_2 \subset \langle X\rangle$.

Claim. We have $\ker(\Psi) = \langle\langle R_2 \cup S_1\rangle\rangle$.

First note that it follows almost immediately from Lemmas 21.8, 21.9 and 21.11 that $\ker(\Psi) = \langle\langle R_1 \cup R_2 \cup S_1\rangle\rangle$. Thus it remains to show that $\langle\langle R_1 \cup R_2 \cup S_1\rangle\rangle = \langle\langle R_2 \cup S_1\rangle\rangle$. Put differently, given $r_1 \in R_1$ we need to show that $r_1 \in \langle\langle R_2 \cup S_1\rangle\rangle$. Let $\theta\colon \langle\langle X_1\rangle\rangle \to \langle\langle X_2\rangle\rangle$ be the homomorphism that is given by $x \mapsto w_x$ for $x \in X_1$. Note that we have the following commutative diagram:

$$\begin{tikzcd}
\langle X_1\rangle \arrow[rr, "\theta"] \arrow[dr] \arrow[dd, two heads, "p_1"'] & & \langle X_2\rangle \arrow[dl] \arrow[dd, two heads, "p_2"] \\
& G & \\
\langle X_1 \,|\, R_1\rangle \arrow[ur, "\phi_1", "\cong"'] & & \langle X_2 \,|\, R_2\rangle. \arrow[ul, "\phi_2"', "\cong"]
\end{tikzcd}$$

In particular we have $(\phi_2 \circ p_2)(\theta(r_1)) = (\phi_1 \circ p_1)(r_1) = 1$. This implies that $\theta(r_1) \in \langle\langle R_2\rangle\rangle$. On the other hand it follows from Lemma 21.8 that $r_1 \cdot \theta(r_1)^{-1} \in \langle\langle S_1\rangle\rangle$. Thus we see that $r_1 = r_1 \cdot \theta(r_1)^{-1} \cdot \theta(r_1) \in \langle\langle S_1 \cup R_2\rangle\rangle$. ⊞

After these preparations we turn to the actual proof of the theorem.

(a) First note that we can turn the presentation $\langle X_2 \,|\, R_2\rangle$ into $\langle X \,|\, R_2 \cup S_1\rangle$ by a sequence of Tietze transformations (2), where we add one by one the elements of X_1 and for each element of X_1 we add the corresponding relation in S_1.

(b) Note that $R_1 \cup S_2 \subset \ker(\Psi)$. By the claim this implies that $R_1 \cup S_2 \subset \langle\langle R_2 \cup S_1\rangle\rangle$. In particular, using Tietze transformations (1) we can turn the presentation $\langle X \,|\, R_2 \cup S_1\rangle$ into the presentation $\langle X \,|\, R_1 \cup S_2 \cup R_2 \cup S_1\rangle$.

(c) By exactly the same reasoning as in (a) and (b) we can also turn the presentation $\langle X_1 \,|\, R_1\rangle$ into the presentation $\langle X \,|\, R_1 \cup S_2 \cup R_2 \cup S_1\rangle$ via a sequence of Tietze transformations.

Since $\langle X_1 \,|\, R_1\rangle$ and $\langle X_2 \,|\, R_2\rangle$ can be turned into precisely the same presentation using Tietze transformations, the two presentations are in fact related by Tietze transformations. ■

Remark. Alternative proofs of the Tietze Theorem 21.7 can be found in [**Tie1908**] and also [**LS77**, Proposition II.2.2] and [**MKS76**, Theorem 1.5].

The following lemma often works well in practice to show that two presentations correspond to isomorphic groups.

Lemma 21.12. *Let $X = \{x_1, \dots, x_k\}$ be a finite set and furthermore let $r = \{r_1, \dots, r_l\}$ be a finite subset of $\langle X\rangle = \langle x_1, \dots, x_k\rangle$. Suppose that we can write each $r_i(x_1, \dots, x_k)$ as a word $s_i(x_1, \dots, x_{k-1}, x_k \cdot w)$ where w is a word in $x_1, \dots, x_{k-1}$, then the homomorphism*

$$\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l\rangle \;\to\; \langle x_1, \dots, x_{k-1}, t \,|\, s_1, \dots, s_l\rangle$$

that is induced by $x_i \mapsto x_i$, $i = 1, \dots, k-1$, and $x_k \mapsto tw^{-1}$ is an isomorphism.

Definition. We refer to the isomorphism of Lemma 21.12 as the *isomorphism given by the substitution* $t = x_k \cdot w$.

Example. A picky reader might decide that the formulation of Lemma 21.12 is not completely precise. But the following example should make the meaning clear:

$$\langle a, b \,|\, a^{-1}ba^{-1}bab^{-1} \rangle \;=\; \langle a, b \,|\, \underbrace{a^{-1}b \cdot a^{-1}b \cdot a \cdot (a^{-1}b)^{-1} \cdot a^{-1}}_{=a^{-1}ba^{-1}bab^{-1}} \rangle \underset{\substack{\uparrow \\ \text{substitution } c = a^{-1}b}}{=} \langle a, c \,|\, c \cdot c \cdot a \cdot c^{-1} \cdot a^{-1} \rangle.$$

Proof. It follows from Lemma 21.13 that there exists in fact a unique homomorphism

$$\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle \;\to\; \langle x_1, \ldots, x_{k-1}, t \,|\, s_1, \ldots, s_l \rangle$$

that has the property that $x_i \mapsto x_i$, $i = 1, \ldots, k-1$ and that $x_k \mapsto tw^{-1}$. Furthermore, we have a map from the right to left by the reverse substitution $t \mapsto x_k \cdot w$. It is straightforward to verify that these homomorphisms are inverses of one another, hence both are isomorphisms. ■

As we will now see, finite presentations of groups are useful for finding group homomorphisms. To formulate the corresponding lemma we need the following definition.

Definition. Let $X = \{x_1, \ldots, x_k\}$ be a set and let $r \in \langle X \rangle$. Furthermore let G be a group and let $g_1, \ldots, g_k$ be elements of G. We say that $g_1, \ldots, g_k$ *satisfy the relation* r if the homomorphism[373] $\langle x_1, \ldots, x_k \rangle \to G$ that is given by $x_i \mapsto g_i$, $i = 1, \ldots, k$ sends r to the trivial element.

Examples.

(1) We consider $\{x_1, x_2\}$ and the word $r = x_1 x_2 x_1^{-1} x_2^{-1}$. If a group is abelian, then any two elements g_1, g_2 satisfy the relation $x_1 x_2 x_1^{-1} x_2^{-1}$.
(2) Let G be a finite group with d elements. We consider the set $\{x\}$. It follows from Lagrange's Theorem, see e.g. [**KuS04**, p. 8], that every element in G satisfies the relation x^d.

Lemma 21.13. *Let π be a group with a finite presentation $\pi = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$. For any group G the map*

$$\begin{array}{rcl} \operatorname{Hom}(\pi, G) & \to & \{\textit{all } (g_1, \ldots, g_k) \in G^k \textit{ that satisfy } r_1, \ldots, r_l\} \\ f & \mapsto & (f(x_1), \ldots, f(x_k)) \end{array}$$

is a bijection.

Remark. Note that if G is a finite group, then the right-hand side of Lemma 21.13 is a finite set. In practice it is usually straightforward, albeit tedious, to determine the right-hand side.

Proof. This lemma follows immediately from Lemma 21.4 and Lemma 19.14. ■

Next, recall that in Exercise 19.16 we gave an explicit example of a group epimorphism $\varphi\colon \langle x, y \rangle \to \mathbb{Z}$ such that the kernel $\ker(\varphi)$ is infinitely generated. The next proposition shows that this was not a fluke.

[373] The fact that there exists a unique such homomorphism is a consequence of Lemma 19.14.

Proposition 21.14. *Every non-trivial normal subgroup of a free group is infinitely generated.*

Proof. A purely group theoretic proof is given in [**LS77**, Proposition I.3.12]. Alternatively a proof using topological methods is provided in [**dlH00**, p. 45]. ■

Fortunately the following proposition, which we obtain from [**DrK18**, Lemma 7.29], gives us at least some control over the "size" of a kernel.

Proposition 21.15. *Let $\varphi\colon \pi \to G$ be a group epimorphism. If π is finitely generated and if G is finitely presented, then $\ker(\varphi)$ is finitely normally generated. In other words, there exists a finite subset $S \subset \ker(\varphi)$ such that $\langle\langle S\rangle\rangle = \ker(\varphi)$.*

Example. Let $\varphi\colon \langle x, y\rangle \to \mathbb{Z}^2$ be the epimorphism that is given by $\varphi(x) = (1,0)$ and $\varphi(y) = (0,1)$. In Exercise 21.8 we will see that $\ker(\varphi)$ is normally generated by a single element, namely the commutator $xyx^{-1}y^{-1}$.

We conclude this section with an interesting little lemma. For the formulation we need the following harmless definition.

Definition. Let G and H be groups. We say *H is a retract of G* if there exist a group homomorphisms $\varphi\colon H \to G$ and $\rho\colon G \to H$ such that $\rho \circ \varphi = \mathrm{id}_H$.

Lemma 21.16. $(*)$ *Every retract of a finitely presented group is also finitely presented.*

Proof $(*)$. Let H be a group, let G be a finitely presented group and let $\varphi\colon H \to G$ and $\rho\colon G \to H$ be group homomorphisms such that $\rho \circ \varphi = \mathrm{id}_H$. We need to show that H is also finitely presented.

To do so we pick a finite presentation $G = \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_l\rangle$ for the group G. We write $F = \langle g_1, \ldots, g_k\rangle$. In the following we denote by $p\colon F \to G$ the obvious epimorphism. For each $i \in \{1, \ldots, k\}$ we pick $w_i \in F$ with $p(w_i) = (\varphi \circ \rho \circ p)(g_i) \in G$. It suffices to prove the following claim.

Claim. A finite presentation for H is given by $\langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_l, g_1^{-1}w_1, \ldots, g_k^{-1}w_k\rangle$.

Let L be the group that is defined by the finite presentation of the claim. We denote by $u\colon G \to L$ and $q\colon F \to L$ the obvious projections. Note that for each $j \in \{1, \ldots, l\}$ we have $\rho(p(r_i)) = \rho(e) = e$ and for each $i \in \{1, \ldots, k\}$ we have

$$\rho(p(g_i^{-1}w_i)) \;=\; \rho(p(g_i^{-1})) \cdot \rho(p(w_i)) \underset{\substack{\uparrow \\ \text{since } p(w_i) = (\varphi\circ\rho\circ p)(g_i) \text{ and since } \rho\circ\varphi = \mathrm{id}_H}}{=} \rho(p(g_i^{-1})) \cdot \rho(p(g_i)) \;=\; e.$$

It follows from Lemma 21.4 that the homomorphism $p \circ \varphi\colon F \to H$ factors through a homomorphism $v\colon L \to H$.

We want to show that $u \circ \varphi$ is an isomorphism. First note that

$$v \circ (u \circ \varphi) \;=\; (v \circ u) \circ \varphi \underset{\substack{\uparrow \\ \text{since } \rho = v \circ u}}{=} \rho \circ \varphi \;=\; \mathrm{id}_H\,.$$

By Lemma 19.18 it remains to show that $u\circ\varphi$ is an epimorphism. Note that for $i=1,\dots,k$ we have

$$(u\circ\varphi)(\underbrace{(\rho\circ p)(w_i)}_{\in H}) \underset{\substack{\uparrow\\ \text{since } \varphi\circ\rho=\mathrm{id}_H}}{=} u(p(w_i)) = q(w_i) \underset{\substack{\uparrow\\ \text{since } q(w_i)=g_i\in L}}{=} g_i \in L.$$

Since $g_1,\dots,g_k$ is a generating set for L we see that $u\circ\varphi$ is indeed an epimorphism.

For convenience we summarize all objects in the following diagram

$$\begin{tikzcd}
H \arrow[r, "\varphi"] & G=\langle g_1,\dots,g_k\,|\,r_1,\dots,r_l\rangle \arrow[l, bend right, "\rho"'] \arrow[d, "u"] & F=\langle g_1,\dots,g_k\rangle \arrow[l, "p"'] \arrow[dl, "q"] \\
& L:=\langle g_1,\dots,g_k\,|\,r_1,\dots,r_l,g_1^{-1}w_1,\dots,g_k^{-1}w_k\rangle. \arrow[ul, "v"] &
\end{tikzcd}$$

■

21.4. **The Andrews-Curtis Conjecture** (∗)**.** In this section we want to introduce Nielsen transformations and we want to present the Andrews-Curtis Conjecture. Both are mostly peripheral to the content of these notes. But for general culture it is perhaps interesting to know that in this supposedly fairly elementary context of finite presentations there are difficult conjectures lurking in the background.

First we introduce the following notation.

Notation. Given a set S and $n\in\mathbb{N}_0$ we refer to an element $(g_1,\dots,g_n)\in S^n$ as an *ordered set of elements in* S.[374]

Now we can introduce the definition we are really interested in.

Definition. Let π be a group and let $(g_1,\dots,g_n)$ be an ordered set of elements in π. A *Nielsen operation* is one of the following three ways of obtaining a new ordered set of elements in π:

(1)	$(g_1,g_2,g_3,\dots,g_n)$	$\rightsquigarrow$	$(g_1^{-1},g_2,g_3,\dots,g_n)$
(2)	$(g_1,g_2,g_3,\dots,g_n)$	$\rightsquigarrow$	$(g_2,g_1,g_3,\dots,g_n)$
(3)	$(g_1,g_2,g_3,\dots,g_n)$	$\rightsquigarrow$	$(g_1\cdot g_2,g_2,g_3,\dots,g_n)$.

The following theorem that was proved by Jakob Nielsen [**Nie24**] (see also [**LS77**, Proposition I.4.1] and [**MKS76**, Theorem 3.2]) in 1924.

Theorem 21.17. (Nielsen 1924) *Let $F=\langle x_1,\dots,x_n\rangle$ be a free group. If $y_1,\dots,y_n\in F$ form a basis*[375] *for the free group F, then $(y_1,\dots,y_n)$ can be obtained from $(x_1,\dots,x_n)$ through a finite sequence of Nielsen operations* (1)*,* (2) *and* (3)*.*

Remark.

(1) Let $F=\langle g_1,\dots,g_n\rangle$ be a finitely generated free group. It follows from Lemma 19.14 that any Nielsen operation, applied to $(g_1,\dots,g_n)$, gives rise to an endomorphism of

[374]Note that we do not assume that $g_1,\dots,g_n$ are pairwise different.

[375]Recall that by the definition on page 592 the elements $y_1,\dots,y_n$ form a basis if the natural map $\langle y_1,\dots,y_n\rangle\to F$ is an isomorphism.

the free group F. One can easily show that these endomorphisms are in fact automorphisms of the free group. We refer to any such automorphism as a *Nielsen automorphism*. Nielsen's Theorem 21.17 implies almost immediately that any automorphism of the free group is in fact a composition of finitely many Nielsen automorphisms. In other words, the group $\operatorname{Aut}(F)$ is generated by the Nielsen transformations.

(2) The fact that the group $\operatorname{Aut}(F)$ is generated by the Nielsen operations is similar in spirit to the fact, established in any self-respecting linear algebra course, that the group $\operatorname{Aut}(\mathbb{Z}^n) = \operatorname{GL}(n, \mathbb{Z})$ is generated by elementary matrices.

The following conjecture was formulated by James Andrews and Morton Curtis [**AC65**] in 1965.

Conjecture 21.18. (Andrews-Curtis Conjecture) *Let $\langle g_1, \ldots, g_n \,|\, r_1, \ldots, r_n \rangle$ be a finite presentation. If the corresponding group is the trivial group, then $(r_1, \ldots, r_n)$ can be obtained from $(g_1, \ldots, g_n)$ through a finite sequence of operations where we allow the Nielsen operations* (1), (2), (3) *and the following fourth operation:*

$$(4) \qquad (g_1, g_2, g_3, \ldots, g_n) \;\rightsquigarrow\; (hg_1h^{-1}, g_2, g_3, \ldots, g_n) \quad \textit{for some } h \in \pi.$$

The Andrews-Curtis Conjecture is notoriously difficult. It seems like most experts believe the conjecture to be false, but not much progress has been made over the last decades. Finally note that the Andrews-Curtis Conjecture will raise its head once more in Section 107.2.

21.5. The abelianization of a group.

The following definition should already be familiar from the algebra course.

Definition. Let π be a group. We refer to

$$[\pi, \pi] \;:=\; \langle\langle \{ \underbrace{[x, y]}_{= xyx^{-1}y^{-1}} \mid x, y \in \pi \} \rangle\rangle$$

as the *commutator subgroup* of π.

We start out with the following elementary lemma.

Lemma 21.19. *Let π be a group. Then the following hold:*

(1) *The group $\pi/[\pi, \pi]$ is abelian.*
(2) *We have*

$$[\pi, \pi] \;:=\; \Big\{ \prod_{i=1}^{n} [x_i, y_i] \,\Big|\, x_1, y_1, \ldots, x_n, y_n \in \pi \Big\}.$$

Remark. Note that in general not every element of the commutator subgroup is a commutator. For example, if $\pi = \langle a, b, c, d \rangle$ is the free group on four generators a, b, c, d, then we have $[a, b] \cdot [c, d] \in [\pi, \pi]$, but it is not a commutator.[376]

[376] Why not?

PROOF. Let π be a group.

(1) For any two elements $x, y \in \pi$ we have
$$xy\,[\pi,\pi] \;=\; yx\cdot x^{-1}y^{-1}xy\,[\pi,\pi] \;=\; yx\cdot \underbrace{[x^{-1},y^{-1}]}_{\in[\pi,\pi]}[\pi,\pi] \;=\; yx\,[\pi,\pi].$$
This shows that $\pi/[\pi,\pi]$ is indeed abelian.

(2) Note that the inverse of a commutator is again a commutator, i.e. given $x, y \in \pi$ we have
$$[x,y]^{-1} \;=\; (xyx^{-1}y^{-1})^{-1} \;=\; yxy^{-1}x^{-1} \;=\; [y,x].$$
Furthermore a conjugate of a commutator is again a commutator. Indeed, given $x, y, g \in \pi$ we have
$$\begin{aligned} g[x,y]g^{-1} = gxyx^{-1}y^{-1}g^{-1} &= gxg^{-1}gyg^{-1}gx^{-1}g^{-1}gy^{-1}g^{-1} \\ &= (gxg^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1} = [gxg^{-1}, gyg^{-1}]. \end{aligned}$$
It follows from this discussion and from Lemma 21.3 that each element of the commutator subgroup can be written as a product of commutators, i.e.
$$[\pi,\pi] \;:=\; \Big\{ \prod_{i=1}^{n}[x_i,y_i] \;\Big|\; x_1,y_1,\dots,x_n,y_n \in \pi \Big\}. \qquad \blacksquare$$

Definition. Let π be a group. We call
$$\pi_{\mathrm{ab}} \;:=\; \pi/[\pi,\pi]$$
the *abelianization* of π.[377]

The following proposition summarizes some of the key properties of the abelianization of groups.

Proposition 21.20.

(1) *If π is abelian, then $\pi_{\mathrm{ab}} = \pi$.*

(2) *Let π be a group and let $\alpha\colon \pi \to H$ be a homomorphism to an abelian group. Then there exists a unique homomorphism $\beta\colon \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \to H$ such that the following diagram commutes:*
$$\begin{tikzcd} \pi \arrow[r] \arrow[dr, "\alpha"'] & \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \arrow[d, "\beta"] \\ & H. \end{tikzcd}$$
In particular for any group π and any abelian group G the map
$$\begin{array}{rcl} \operatorname{Hom}(\pi_{\mathrm{ab}}, G) & \to & \operatorname{Hom}(\pi, G) \\ (\alpha\colon \pi_{\mathrm{ab}} \to G) & \mapsto & \big(\pi \to \pi_{\mathrm{ab}} \xrightarrow{\alpha} G\big) \end{array}$$
is a bijection.

[377]The abelianization of a group π is sometimes also written as $\mathrm{H}_1(\pi)$. The reason for that somewhat strange looking notation will become clear later on when we formulate the Hurewicz Theorem 52.5 and when we consider the homology groups, see Chapter **??**.

(3) *Let G and H be two groups. There exists a unique isomorphism*

$$(G*H)_{\mathrm{ab}} \xrightarrow{\cong} G_{\mathrm{ab}} \times H_{\mathrm{ab}}$$

which makes the following diagram commute:

$$\begin{tikzcd} G*H \arrow[r, two heads] \arrow[d, two heads] & G\times H \arrow[d, two heads] \\ (G*H)_{\mathrm{ab}} \arrow[r, "\cong"] & G_{\mathrm{ab}}\times H_{\mathrm{ab}}, \end{tikzcd}$$

where the other three maps are the obvious projection maps.

(4) *Let $F = \langle x_1, \dots, x_m\rangle$ be the free group on m generators $x_1, \dots, x_m$. Then the map*[378]

$$\begin{array}{rcl} \Phi\colon F_{\mathrm{ab}} = F/[F,F] & \to & \mathbb{Z}^m \\ [x_i] & \mapsto & e_i := \text{the } i\text{-th standard basis vector} \end{array}$$

is an isomorphism.

(5) *Let $\pi = \langle x_1, \dots, x_m \mid r_1, \dots, r_n\rangle$ be a finitely presented group. We denote by*

$$\Phi\colon \langle x_1, \dots, x_m\rangle \to \mathbb{Z}^m$$

the homomorphism that is given by $\Phi(x_i) = e_i$. There exists a unique isomorphism

$$\Psi\colon \pi/[\pi,\pi] \xrightarrow{\cong} \mathbb{Z}^m/\langle \Phi(r_1), \dots, \Phi(r_n)\rangle$$

such that the following diagram commutes

$$\begin{tikzcd} \langle x_1,\dots,x_m\rangle \arrow[rr, two heads, "x_i\mapsto e_i", "\Phi"'] \arrow[d, two heads] & & \mathbb{Z}^m \arrow[dd, two heads] \\ \pi = \langle x_1,\dots,x_m\rangle/\langle\langle r_1,\dots,r_n\rangle\rangle \arrow[d, two heads] & & \\ \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \arrow[rr, "\Psi", "\cong"'] & & \mathbb{Z}^m/\langle \Phi(r_1),\dots,\Phi(r_n)\rangle. \end{tikzcd}$$

(6) *The maps*

$$G \mapsto G_{\mathrm{ab}} = G/[G,G]$$

and

$$(\varphi\colon G\to H) \mapsto \left(\begin{array}{rcl} \varphi_*\colon G_{\mathrm{ab}} = G/[G,G] & \to & H_{\mathrm{ab}} = H/[H,H] \\ [g] & \mapsto & [\varphi(g)] \end{array}\right)$$

define a covariant functor from the category $\mathcal{Gr}$ of groups to the category $\mathcal{AbGr}$ of abelian groups.

(7) *Given a group G we denote by*

$$\begin{array}{rcl} \Phi_G\colon G & \to & G_{\mathrm{ab}} = G/[G,G] \\ g & \mapsto & [g] \end{array}$$

the projection map. These maps Φ_G define a natural transformation from the identity functor $\mathcal{Gr}\to\mathcal{Gr}$ to the abelianization functor $\mathcal{Gr}\to\mathcal{Gr}$ that we had introduced in (6).[379]

(8) *Isomorphic groups have isomorphic abelianizations.*

(9) *If* $\varphi\colon A \to B$ *is an epimorphism, then the induced map* $A_{\mathrm{ab}} \to B_{\mathrm{ab}}$ *is also an epimorphism.*

Examples.

(1) By Proposition 21.20 (1) and (3) we have

$$(\mathbb{Z}_3 * \mathbb{Z}_{35})_{\mathrm{ab}} \;\cong\; \mathbb{Z}_3 \times \mathbb{Z}_{35} \quad \text{and} \quad (\mathbb{Z}_5 * \mathbb{Z}_{21})_{\mathrm{ab}} \;\cong\; \mathbb{Z}_5 \times \mathbb{Z}_{21}.$$

It follows from the Chinese Remainder Theorem that the abelianizations are isomorphic. On the other hand on page 597 we had already observed that the Grushko Decomposition Theorem 19.21 implies that the groups $\mathbb{Z}_3 * \mathbb{Z}_{35}$ and $\mathbb{Z}_5 * \mathbb{Z}_{21}$ are not isomorphic. Thus we have seen that non-isomorphic groups can have isomorphic abelianizations.

(2) We give an explicit example for statement (5). Let

$$\pi \;=\; \langle x, y \mid x^2y^{-2}x, y^{-1}x^3yx\rangle.$$

Let $\Phi\colon \langle x,y\rangle \to \mathbb{Z}^2$ be the epimorphism given by $\Phi(x) = (1,0)$ and $\Phi(y) = (0,1)$. Then the abelianization of π is given by[380]

$$\pi_{\mathrm{ab}} \;=\; \mathbb{Z}^2/\langle \underbrace{\Phi(x^3y^{-2})}_{=\begin{pmatrix} 3 \\ -2\end{pmatrix}}, \underbrace{\Phi(y^{-1}x^3yx)}_{=\begin{pmatrix} 4 \\ 0\end{pmatrix}}\rangle \;=\; \mathbb{Z}^2/\begin{pmatrix} 3 & 4 \\ -2 & 0\end{pmatrix}\mathbb{Z}^2 \;\cong\; \mathbb{Z}_8.$$

(3) Let S be a set. It is straightforward to show that the abelianization of the free group $\langle S\rangle$ generated by S is isomorphic to the free abelian group $\mathbb{Z}^{(S)}$ generated by S.

(4) Given $k \in \mathbb{N}$ we denote by S_k the permutation group on k elements and we denote by $A_k \subset S_k$ the alternating group, i.e. the group of all permutations with positive sign. In the algebra course it was shown that for $k \geq 5$ the group A_k is simple, i.e. the only normal subgroups of A_k are the trivial group and the group A_k itself.[381] The commutator subgroup $[A_k, A_k]$ is normal. Since A_k is not abelian we cannot have $[A_k, A_k] = \{e\}$. Thus we see that $[A_k, A_k] = A_k$, which implies that the abelianization of A_k is the trivial group.

Now we provide the proof of Proposition 21.20.

Proof of Proposition 21.20.

(1) The first statement is obvious.[382]

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[378]By Lemma 19.14 there exists a unique homomorphism $F \to \mathbb{Z}^m$ with $x_i \mapsto e_i$. By (1) this map descends to a homomorphism $F/[F,F] \to \mathbb{Z}^m$. The way we write down the map is a little bit sloppy, since we only specify the map on the generators of $F/[F,F]$.

[379]Super-picky readers might argue that in (6) we introduced a functor from $\mathcal{G}r$ to $\mathcal{A}b\mathcal{G}r$, but of course we can also view it as a functor from $\mathcal{G}$ to $\mathcal{G}$.

[380]Why is the group isomorphic to $\mathbb{Z}_8$?

[381]A proof of this statement is also given in any self-respecting algebra book, see e.g. [**Rob95**, p. 69].

[382]Why?

(2) Let π be a group and let $\alpha\colon \pi \to H$ be a homomorphism to an abelian group. Then $[\pi,\pi]$ lies in the kernel of α. The desired statement now follows immediately from Lemma 21.1.

(3) The maps

$$\begin{array}{rcl}\Phi\colon (G*H)_{\mathrm{ab}} & \to & G_{\mathrm{ab}}\times H_{\mathrm{ab}} \\ [(x_1,\dots,x_k)] & \mapsto & \Big(\Big[\prod\limits_{x_i\in G} x_i\Big],\Big[\prod\limits_{x_i\in H} x_i\Big]\Big)\end{array} \quad \text{and} \quad \begin{array}{rcl}\Psi\colon G_{\mathrm{ab}}\times H_{\mathrm{ab}} & \to & (G*H)_{\mathrm{ab}} \\ ([g],[h]) & \mapsto & [g\cdot h]\end{array}$$

are easily seen to be well-defined, to be homomorphisms and to be inverses of one another.

(4) This statement follows immediately from the statements (1) and (3) together with the observation that $\langle x_1,\dots,x_m\rangle = \langle x_1\rangle * \dots * \langle x_m\rangle$, that $\langle x_i\rangle_{\mathrm{ab}} \cong \mathbb{Z}$ and that $\mathbb{Z}^m = \mathbb{Z}\times\dots\times\mathbb{Z}$.

(5) It is straightforward to see that there exists a unique homomorphism

$$\Psi\colon \pi/[\pi,\pi] \;\to\; \mathbb{Z}^m/\langle \Phi(r_1),\dots,\Phi(r_n)\rangle$$

that makes the diagram of statement (5) commute. It remains to show that Ψ is an isomorphism. We write $F=\langle x_1,\dots,x_m\rangle$ and we write $G=\langle y_1,\dots,y_n\rangle$. We consider the following diagram

$$\begin{tikzcd} G_{\mathrm{ab}} = G/[G,G] \arrow[r, "{[y_i]\mapsto e_i}", "\cong"'] \arrow[d, "{[y_i]\mapsto[r_i]}"'] & \mathbb{Z}^n \arrow[d, "{e_i\mapsto\Phi(r_i)}"] \\ F_{\mathrm{ab}} = F/[F,F] \arrow[r, "{\Phi([x_i])=e_i}", "\cong"'] \arrow[d, two heads] & \mathbb{Z}^m \arrow[d, two heads] \\ \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \arrow[r, "\Psi"] & \dfrac{\mathbb{Z}^m}{\langle \Phi(r_1),\dots,\Phi(r_n)\rangle}. \end{tikzcd}$$

Note that the diagram commutes. By (4) the top two horizontal maps are isomorphisms. The bottom vertical maps are clearly epimorphisms and it is clear that the images of the top vertical maps lie in the kernel of the bottom vertical maps. It now follows from an elementary "diagram chase" that the bottom horizontal map is also an isomorphism.[383] We leave the details to the reader.

[383]More precisely, we use the following fact: suppose we are given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\cong"] \arrow[d] & A' \arrow[d] \\ B \arrow[r, "\cong"] \arrow[d, two heads] & B' \arrow[d, two heads] \\ C \arrow[r] & C', \end{tikzcd}$$

of group homomorphisms where the bottom vertical maps are epimorphisms, where the top two horizontal maps are isomorphisms and where the kernels of the bottom vertical maps are the images of the top vertical maps. Then the bottom horizontal map is also an isomorphism. This general statement can be proved by a "diagram chase". This statement can also be viewed as a special case of the "five lemma".

(6) This statement follows easily from the definitions and the observation that the image of a commutator under a group homomorphism is once again a commutator.
(7) This statement is basically obvious once one has internalized the definition of a natural transformation.
(8) This statement is an immediate consequence of (6).
(9) We will prove this statement in Exercise 21.6. ■

21.6. **The amalgamated product of groups I.** In the formulation of the generalized version of the Seifert–van Kampen theorem we will need to replace the free product of two groups by the "amalgamated product" of two groups.

In Section 19.2 we had first defined the free product via an explicit construction. We had then explained that one can also characterize the free product via a universal property. In this section we will proceed the other way round, we will first state the desired universal property and then we will explicitly construct a group which satisfies the universal property.

Proposition 21.21. *Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. Then there exists a triple*

$$(A *_G B, \varphi\colon A \to A *_G B, \psi\colon B \to A *_G B)$$

where $A *_G B$ is a group and φ and ψ are homomorphisms with the following properties:

(a) *the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "\varphi"] \\ B \arrow[r, "\psi"'] & A *_G B, \end{tikzcd}$$

(b) *if $(H', \varphi'\colon A \to H', \psi'\colon B \to H')$ is another triple which satisfies property* (a), *then there exists a unique homomorphism $\Theta\colon A *_G B \to H'$ such that $\varphi' = \Theta \circ \varphi$ and $\psi' = \Theta \circ \psi$, i.e. such that the following diagram commutes*

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "\varphi"] \arrow[ddr, bend left, "\varphi'"] & \\ B \arrow[r, "\psi"] \arrow[drr, bend right, "\psi'"'] & A *_G B \arrow[dr, dashed, "\Theta"'] & \\ & & H'. \end{tikzcd}$$

Proof. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. As usual we view A and B as subgroups of $A * B$. We put

$$A *_G B \; := \; A * B / \langle\!\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\!\rangle$$

and we define $\varphi\colon A \to A *_G B$ as the composition of the obvious homomorphisms

$$A \;\hookrightarrow\; A * B \;\twoheadrightarrow\; A * B / \langle\!\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\!\rangle,$$

and precisely the same way we define the homomorphism $\psi\colon B \to A *_G B$. It follows from the construction of $A *_G B$ that $\varphi(\alpha(g)) = \psi(\beta(g))$ for all $g \in G$. Therefore the triple $(A *_G B, \varphi\colon A \to A *_G B, \psi\colon B \to A *_G B)$ satisfies property (a).

Now let $(H', \varphi'\colon A \to H', \psi'\colon B \to H')$ be another triple which satisfies property (a). We consider the following diagram

$$\begin{tikzcd}
G \arrow[rrr, hook, "\alpha"] \arrow[dd, "\beta"'] & & & A \arrow[dll, hook'] \arrow[dd, "\varphi"] \arrow[dddr, bend left, "\varphi'"] & \\
& A * B \arrow[dr, two heads] \arrow[ddrrr, bend right, "\Phi"] & & & \\
B \arrow[ur, hook] \arrow[rr, "\psi"'] \arrow[drrrr, bend right, "\psi'"'] & & A *_G B = & \dfrac{A * B}{\langle\langle \alpha(g)\beta(g)^{-1} \rangle\rangle} \arrow[dr, dashed, red] & \\
& & & & H'
\end{tikzcd}$$

where $\Phi\colon A * B \to H'$ is the unique homomorphism determined by ψ' and φ'. Since $\varphi' \circ \alpha = \psi' \circ \beta$ it follows that for any $g \in G$ we have

$$\Phi(\alpha(g) \cdot \beta(g)^{-1}) \;=\; \Phi(\alpha(g)) \cdot \Phi(\beta(g))^{-1} \;=\; \varphi'(\alpha(g)) \cdot \psi'(\beta(g))^{-1} \;=\; e.$$

Put differently, all elements of the form $\alpha(g)\beta(g)^{-1}$ in $A * B$ lie in the kernel of Φ. By Lemma 21.4 there exists a unique homomorphism $\Theta\colon A * B/\langle\langle\{\alpha(g)\beta(g)^{-1}\}_{g\in G}\rangle\rangle \to H'$ which makes the following diagram commute

$$\begin{tikzcd}
A * B \arrow[r] \arrow[dr, "\Phi"'] & A *_G B = A * B/\langle\langle\{\alpha(g)\beta(g)^{-1}\}_{g\in G}\rangle\rangle \arrow[d, "\Theta"] \\
& H'.
\end{tikzcd}$$

This proves the existence of Θ. The uniqueness follows from the observation that another choice of Θ' would give another choice for Φ, but this homomorphism is unique. ■

Definition. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. We write

$$A *_G B \;:=\; A * B/\langle\langle\{\alpha(g)\beta(g)^{-1}\}_{g\in G}\rangle\rangle$$

and we refer to $A *_G B$ as the *amalgamated product of A and B with amalgam G*.[384] We refer to the obvious homomorphisms $A \to A *_G B$ and $B \to A *_G B$ as the *natural homomorphisms*.[385]

Remark. A variation on the argument on page 582 shows that if

$$(H, \varphi\colon A \to H, \psi\colon B \to H)$$

[384] The notation is a little misleading since the homomorphisms α and β do not get mentioned in the notation, even though they play of course an important role. In particular different choices of α and β can lead to non-isomorphic amalgamated products.

[385] We leave it again to the reader to figure out in what sense these homomorphisms are "natural".

is a triple that has both the properties (a) and (b) of Proposition 21.21, then there exists a unique isomorphism $\Theta\colon A *_G B \xrightarrow{\cong} H$ such that $\varphi(a) = \Theta(a)$ for all $a \in A$ and $\psi(b) = \Theta(b)$ for all $b \in B$. Often we will use this uniquely defined isomorphism to identify H with the amalgamated product $A *_G B$.

Example. Let $G = \mathbb{Z}_2, A = \mathbb{Z}_4$ and $B = \mathbb{Z}_6$ and consider the homomorphisms

$$\begin{array}{rcl}\alpha\colon G & \to & A \\ [k] & \mapsto & [2k]\end{array} \quad \text{and} \quad \begin{array}{rcl}\beta\colon G & \to & B \\ [k] & \mapsto & [3k].\end{array}$$

Furthermore we consider the homomorphisms[386]

$$\begin{array}{rcl}\varphi\colon A = \mathbb{Z}_4 & \to & \mathrm{SL}(2,\mathbb{Z}) \\ k & \mapsto & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^k\end{array} \quad \text{and} \quad \begin{array}{rcl}\varphi\colon B = \mathbb{Z}_6 & \to & \mathrm{SL}(2,\mathbb{Z}) \\ k & \mapsto & \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^k.\end{array}$$

It is straightforward to verify that[387] $\varphi \circ \alpha = \psi \circ \beta$, therefore there exists by Proposition 21.21 a homomorphism

$$A *_G B \ = \ \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 \ \to \ \mathrm{SL}(2,\mathbb{Z}).$$

More surprisingly, this homomorphism is in fact an isomorphism. We refer to [**Rosb94**, p. 218] and [**Ser80**, p. 35] for a proof.

The following lemma summarizes some important special cases of the amalgamated product of groups.

Lemma 21.22.

(1) *For two groups A and B and $G = \{e\}$ the trivial group, the corresponding amalgamated product is the usual free product, i.e.*

$$A *_{\{e\}} B \ = A * B.$$

(2) *If $\alpha\colon G \to A$ is a group homomorphism and if $\beta\colon G \to \{e\}$ is the trivial homomorphism, then*

$$A *_G \{e\} \ = \ A/\langle\langle \alpha(G) \rangle\rangle.$$

(3) *Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be group homomorphisms. We consider the diagram*

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d] \\ B \arrow[r] & A *_G B. \end{tikzcd}$$

The following two statements hold:[388]

(a) *If the left vertical map β is an isomorphism, then the right vertical map is also an isomorphism and the inverse $A *_G B \to A$ is given by the identity on A and $\beta^{-1}\colon B \to G$.*

(b) *If the left vertical map β is an epimorphism, then the right vertical map is also an epimorphism.*

---

[386] Are these really homomorphisms?

[387] Indeed, the one non-trivial element of $G = \mathbb{Z}_2$ gets sent under both maps to $-\mathrm{id}$.

(4) *Let A, B, C and G, H be groups and let $\alpha\colon G \to A$, $\beta\colon G \to B$, $\widetilde{\beta}\colon H \to B$ and $\gamma\colon H \to C$ be group homomorphisms. Then the universal properties of the amalgamated products define an isomorphism*[389]

$$(A *_G B) *_H C \xrightarrow{\cong} A *_G (B *_H C).$$

(5) *Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be group homomorphisms and let $\varphi\colon H \to G$ be an epimorphism. The natural map $A *_G B \to A *_H B$ is an isomorphism.*

(6) *Let $\alpha\colon G * H \to A$ and $\beta\colon G * H$ to be group homomorphisms. If the restriction of β to H is trivial, then the natural map $A *_{G*H} B \to A/\langle\langle H\rangle\rangle *_G B$ is an isomorphism.*

Remark. Lemma 21.22 (4) is actually quite useful since it allows us to form iterated amalgamated products without having to worry about parentheses.

Sketch of proof. The first two statements follow immediately from the definition

$$A *_G B \;:=\; A * B/\langle\langle\{\alpha(g)\beta(g)^{-1}\}_{g\in G}\rangle\rangle.$$

The remaining statement follow either directly from the universal property of the amalgamated product or from the actual description. For some statements we will fill in the details in Exercise 21.12. ■

The following lemma gives us a convenient description of the amalgamated product of two finitely presented groups.

Lemma 21.23. *Let*

$$A \;=\; \langle x_1,\dots,x_k \mid r_1,\dots,r_p\rangle$$

and

$$B \;=\; \langle y_1,\dots,y_l \mid s_1,\dots,s_q\rangle$$

be two finitely presented groups. Furthermore let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms and let $g_1,\dots,g_m$ be generators of G. Then

$$A *_G B \;=\; \langle x_1,\dots,x_k,y_1,\dots,y_l \mid r_1,\dots,r_p,s_1,\dots,s_q \text{ and } \alpha(g_i)=\beta(g_i), i=1,\dots,m\rangle.$$

Examples.

(1) Let $A_1 = \langle x_1, y_1\rangle$ and $A_2 = \langle x_2, y_2\rangle$ be two free groups. Furthermore for $i = 1, 2$ let $\varphi_i\colon \langle t\rangle \to A_i$ be the homomorphism that is given by $\varphi_i(t^n) := [x_i, y_i]^n$. Then we obtain from Lemma 21.23 that

$$A_1 *_{\langle t\rangle} A_2 \;=\; \langle x_1,y_1,x_2,y_2 \mid [x_1,y_1]=[x_2,y_2]\rangle.$$

(2) Let $A = \langle a\rangle$ and let $B = \langle b\rangle$ be two infinite cyclic groups and let $G = \langle x, y \mid [x, y]\rangle$ be a group that is isomorphic to $\mathbb{Z}^2$. We consider the homomorphism $\alpha\colon G \to A$ that is given by $\alpha(x) = a$ and $\alpha(y) = e$ and we consider the homomorphism $\beta\colon G \to B$

[388] The statements are similar in spirit to the statement of Lemma 3.43 for the pushout of topological spaces.

[389] In an attempt to lessen the burden on the reader we refrained from writing down the super-precise statement.

that is given by $\beta(x) = e$ and $\beta(y) = b$. Then

$$A *_G B \underset{\substack{\uparrow \\ \text{Lemma 21.23}}}{=} \langle a, b \mid \alpha(x) = \beta(x), \alpha(y) = \beta(y) \rangle = \langle a, b \mid a = e, e = b \rangle \underset{\substack{\uparrow \\ \text{Tietze transformation (2), see page 632}}}{=} \langle | \rangle = \{e\}.$$

This shows in particular that the amalgamated product of two non-trivial groups can in fact be trivial.

Proof of Lemma 21.23. We need to show that

$$\underbrace{\frac{A*B}{\langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g\in G} \rangle\rangle}}_{\text{definition of } A*_G B} = \underbrace{\frac{A*B}{\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle}}_{\text{right-hand side of the proposition}}.$$

Put differently, we need to prove the following claim.

Claim. In the free product $A * B$ we have

$$\langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g\in G} \rangle\rangle = \langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle.$$

It is clear that the right-hand side is contained in the left-hand side. To show the reverse inclusion we need to show that for each $g \in G$ we have $\alpha(g)\beta(g)^{-1} \in \langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle$. We start out with two observations:

(1) We first consider the case $g = g_i^{-1}$ for some i. Then

$$\alpha(g)\beta(g)^{-1} = \alpha(g_i^{-1})\beta(g_i^{-1})^{-1} = \alpha(g_i)^{-1}\big(\alpha(g_i)\beta(g_i^{-1})\big)^{-1}\alpha(g_i),$$

thus $\alpha(g)\beta(g)^{-1}$ is a conjugate of the inverse of $\alpha(g_i)\beta(g_i^{-1})$, in particular it lies in $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle$.

(2) Note that if a, b and c are elements in a group π such that ac and b lie in a normal subgroup Γ, then $abc = ac \cdot c^{-1}bc$ also lies in Γ.

Now we turn to the proof of the claim. We define the *length* of $g \in G$ as

$$\ell(g) = \min\{n \mid \text{ we can write } g = g_{i_1}^{\epsilon_1} \cdot \dots \cdot g_{i_n}^{\epsilon_n} \text{ for } \epsilon_j \in \{-1, 1\} \text{ and } i_j \in \{1, \dots, m\}\}.$$

Since $g_1, \dots, g_m$ are generators of G any $g \in G$ has a length.

We prove the claim by induction on the length of $g \in G$. For $\ell(g) = 0$ there is nothing to prove. Now suppose we have shown the claim for all elements of length $n - 1$. Now let $g \in G$ be an element of length n. Thus we can write $g = g_{i_1}^{\epsilon_1} \cdot \dots \cdot g_{i_n}^{\epsilon_n}$ for $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$ and $i_1, \dots, i_n \in \{1, \dots, m\}$. We then have

$$\begin{aligned}
\alpha(g)\beta(g)^{-1} = \alpha(g)\beta(g^{-1}) &= \alpha\big(g_{i_1}^{\epsilon_1} \cdots g_{i_n}^{\epsilon_n}\big) \cdot \beta\big(g_{i_n}^{-\epsilon_n} \cdots g_{i_1}^{-\epsilon_1}\big) \\
&= \underbrace{\alpha\big(g_{i_1}^{\epsilon_1}\big) \cdots \alpha\big(g_{i_{n-1}}^{\epsilon_{n-1}}\big)}_{=:a} \cdot \underbrace{\alpha\big(g_{i_n}^{\epsilon_n}\big) \cdot \beta\big(g_{i_n}^{-\epsilon_n}\big)}_{=:b} \cdot \underbrace{\beta\big(g_{i_{n-1}}^{-\epsilon_{n-1}}\big) \cdots \beta\big(g_{i_1}^{-\epsilon_1}\big)}_{=:c}.
\end{aligned}$$

Note that b lies in the normal subgroup $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle$ by (1) and ac lies in the normal subgroup $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle$ by induction. Thus $abc = \alpha(g)\beta(g)^{-1}$ lies in $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1,\dots,m} \rangle\rangle$ by (2). ∎

21.7. **The amalgamated product of groups II** $(*)$**.** In this last section we will state and outline proofs of a few more advanced properties of amalgamated products.

Recall that on page 647 we saw that the amalgamated product of two non-trivial groups can be trivial. The following proposition gives a criterion for the amalgamated product to be non-trivial.

Proposition 21.24. *Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. If α and β are both monomorphisms, then the natural homomorphisms $A \to A *_G B$ and $B \to A *_G B$ are also both monomorphisms.*

Convention. If we are in the setting of Proposition 21.24, then we will use the natural monomorphisms $A \to A *_G B$ and $B \to A *_G B$ to identify A, B and G with the images in $A *_G B$ under these monomorphisms.

Remark. In Exercise 21.10 we will see that there exists an injective group homomorphism $\alpha\colon G \to A$ and a non-injective group homomorphism $\beta\colon G \to B$ such that the amalgamated product $A *_G B$ is trivial. This shows that there is no obvious way how one can weaken the hypothesis in Proposition 21.24 that both maps are monomorphisms.

The proof of Proposition 21.24 relies on the following proposition which is proved in [**Ser80**, p. 3], or alternatively [**vdW48**], [**LS77**, Theorem IV.2.6] and [**CgRR08**, Satz 5.2].

Theorem 21.25. (Normal Form Theorem) *Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group monomorphisms. We identify G with its image in A and B. Let $S \subset A$ be a set of coset representatives for $G\backslash A$ and let $T \subset B$ be a set of coset representatives for $G\backslash B$. Then any element h in $A *_G B$ we can be written* uniquely *as a product $h = g \cdot c_1 \cdots c_k$ where $g \in G$ and $c_1, \dots, c_k \in S \sqcup T$.*

Proof of Proposition 21.24. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group monomorphisms. Let S be a set of coset representatives for $G\backslash A$ and let T be a set of coset representatives for $G\backslash B$. We have the following modest commutative diagram

$$\begin{tikzcd} G \times S \arrow[rr, "(g,s)\mapsto g\cdot s"] \arrow[drr, "(g,s)\mapsto g\cdot s"'] & & A \arrow[d, "\text{natural homomorphism}"] \\ & & A *_G B. \end{tikzcd}$$

The top map is a bijection by definition of S. Furthermore the diagonal map is an injection by the Normal Form Theorem 21.25. Thus the vertical map is a monomorphism. ■

We conclude this chapter with another consequence of the Normal Form Theorem 21.25.

Proposition 21.26. *Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group monomorphisms. Furthermore let $\varphi\colon \pi \to A *_G B$ be a group epimorphism. We write $\widetilde{A} := \varphi^{-1}(A)$, $\widetilde{B} := \varphi^{-1}(B)$ and $\widetilde{G} := \varphi^{-1}(G)$. Then there exists a unique isomorphism $\psi\colon \widetilde{A} *_{\widetilde{G}} \widetilde{B} \to \pi$ which makes* *the following diagram commute:*

$$\begin{tikzcd}
\widetilde{A} *_{\widetilde{G}} \widetilde{B} \arrow[d, "\psi"'] \arrow[dr, "\text{induced by } \varphi\colon \widetilde{A} \to A \text{ and } \varphi\colon \widetilde{B} \to B"] & \\
\pi \arrow[r, "\varphi"'] & A *_G B.
\end{tikzcd}$$

PROOF. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group monomorphisms. It follows from the universal property of the amalgamated product that there exists a unique homomorphism $\psi\colon \widetilde{A} *_{\widetilde{G}} \widetilde{B} \to \pi$ which makes the above diagram commute. It remains to show that ψ is an isomorphism.

Let S be a set of coset representatives for $G\backslash A$ and let T be a set of coset representatives for $G\backslash B$. By the Normal Form Theorem 21.25 we know that any element h in $A *_G B$ we can be written *uniquely* as a product $h = g \cdot c_1 \cdots c_k$ where $g \in G$ and $c_1, \dots, c_k \in S \sqcup T$. We write $\ell(h) := k$.

Now we turn to the proof that ψ is an epimorphism. We need to show that every $g \in \pi$ lies in the image of ψ. We prove this statement by induction on $\ell(\varphi(g))$. If $\ell(\varphi(g)) \leq 1$, then $\varphi(g) \in A$ or $\varphi(g) \in B$. But this means that $g \in \psi(\widetilde{A})$ or $g \in \psi(\widetilde{B})$. Suppose we know that we have an $n \in \mathbb{N}$ such that the desired statement holds whenever $\ell(\varphi(g)) \leq n$. Let $g \in \pi$ with $\ell(\varphi(g)) = n + 1$. Since φ is an epimorphism we see that there exist $x, y \in \pi$ with $g = x \cdot y$ such that $\ell(\varphi(x)) = n$ and $\ell(\varphi(y)) = 1$. By induction we know that x and y lie in the image of ψ. But then so does $g = x \cdot y$.

It remains to show that ψ is also a monomorphism. Thus let $h \in \ker(\psi)$. Since φ restricts to a surjective map $\widetilde{G}\backslash\widetilde{A} \to G\backslash A$ we see that there exists a set $\widetilde{S}$ of coset representatives for $\widetilde{G}\backslash\widetilde{A}$ with $\varphi(\widetilde{S}) \subset S$. Similarly there exists a set $\widetilde{T}$ of coset representatives for $\widetilde{G}\backslash\widetilde{B}$ with $\varphi(\widetilde{T}) \subset T$. By the Normal Form Theorem 21.25 we can write $h = \widetilde{g} \cdot \widetilde{c}_1 \cdot \dots \cdot \widetilde{c}_k$ with $\widetilde{g} \in \widetilde{G}$ and $\widetilde{c}_1, \dots, \widetilde{c}_k \in \widetilde{S} \sqcup \widetilde{T}$. Note that

$$e \;=\; \varphi(e) \;=\; \varphi(\psi(h)) \underset{\substack{\uparrow \\ \text{since the diagram commutes}}}{=} \varphi(\widetilde{g}) \cdot \underbrace{\varphi(\widetilde{c}_1)}_{\in S\sqcup T} \cdot \dots \cdot \underbrace{\varphi(\widetilde{c}_k)}_{\in S\sqcup T} \;\in\; A *_G B.$$

It follows immediately from the uniqueness statement of the Normal Form Theorem 21.25 that $k = 0$, i.e. we see that $h = \widetilde{g} \in \widetilde{G}$. But the restriction of $\psi\colon \widetilde{A} *_{\widetilde{G}} \widetilde{B} \to \pi$ to $\widetilde{G}$ is of course just the inclusion $\widetilde{G} \to \pi$, in particular the restriction is a monomorphism. Thus we see that $h = e$. ■

Exercises for Chapter 21.

Exercise 21.1. Let π be a group and let $A, B \subset \pi$ be two subsets of π. Show that there exists a natural isomorphism between $(\pi/\langle\langle A\rangle\rangle)/\langle\langle B\rangle\rangle$ and $\pi/\langle\langle A \cup B\rangle\rangle$.

Exercise 21.2. Let π be a finitely generated group. Show that every finite-index subgroup is also finitely generated.

Remark. In Proposition 37.15 we will prove the more subtle result that every finite-index subgroup of a finitely presented group is finitely presented.

Exercise 21.3. Let π be a finitely generated group and let G be a finite group. Show that the set $\operatorname{Hom}(\pi, G)$ of homomorphisms from π to G is a finite set.

Exercise 21.4.

(a) Show that
$$\langle x_1, \dots, x_k \,|\, x_k \rangle \;\cong\; \langle x_1, \dots, x_{k-1} \rangle.$$

(b) Show that in the group $\langle x, y \,|\, xyx^{-1}y^{-1} \rangle$ we have the equalities $yx^{-1} = x^{-1}y$ and $x^{-1}y^{-1} = y^{-1}x^{-1}$.

Exercise 21.5.

(a) Let $k \geq 5$. Determine the abelianization of the permutation group S_k.

(b) Let G be an abelian group, let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be epimorphisms. Show that $A *_G B$ is also abelian.

Exercise 21.6. Given a group G we denote by G_{ab} its abelianization.

(a) Let $\varphi\colon A \to B$ be an epimorphism. Show that the induced map $A_{\text{ab}} \to B_{\text{ab}}$ is also an epimorphism.

(b) Let $\varphi\colon A \to B$ be a monomorphism. Is the induced map $A_{\text{ab}} \to B_{\text{ab}}$ also a monomorphism?

Exercise 21.7. Given any $l, m, n \in \mathbb{N}$. We consider the von Dyck group
$$T(l, m, n) \;:=\; \langle a, b \,|\, a^l, b^m, (ab)^n \rangle.$$
Show that $T(l, m, n)_{\text{ab}} \cong \mathbb{Z}_{\gcd(l,m,n)}$.
Remark. It follows in particular that if l, m, n are coprime, then the abelianization is trivial. If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, then it follows from Corollary 33.22 together with Lemma 33.21 that the von Dyck group $T(l, m, n)$ is nonetheless infinite.

Exercise 21.8. Let $\varphi\colon \langle x, y \rangle \to \mathbb{Z}^2$ be the epimorphism that is given by $\varphi(x) = (1, 0)$ and $\varphi(y) = (0, 1)$. Show that the kernel $\ker(\varphi)$ is normally generated by the commutator $[x, y] = xyx^{-1}y^{-1}$.

Exercise 21.9. Let $\varphi\colon G \to A$ and $\psi\colon G \to B$ group monomorphisms which we use to define the amalgamated product $A *_G B$. Suppose that the map $A \to A *_G B$ is an isomorphism. Show that ψ is an isomorphism.

Exercise 21.10. Let G be a group. Let $\alpha\colon G \to A$ be a monomorphism to a simple group A and let $\beta\colon G \to B$ be an epimorphism that is not an isomorphism. Show that the amalgamated product $A *_G B$ is trivial.

Exercise 21.11. Let A and G be groups and let $\varphi\colon A \to G$ be an epimorphism. Furthermore let G' be a copy of G. We consider the amalgamated product $G *_A G'$ given by using twice the epimorphism $\varphi\colon A \to G$ and $\varphi\colon A \to G' = G$. Show that the natural homomorphism $G \to G *_A G'$ is an isomorphism.

Exercise 21.12.

(a) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be group homomorphisms and let $\varphi\colon H \to G$ be an epimorphism. Show that the natural map $A *_G B \to A *_H B$ is an isomorphism.
(b) Let $\alpha\colon G * H \to A$ and $\beta\colon G * H$ to be group homomorphisms. We assume that the restriction of β to H is trivial. Show that the natural map $A *_{G*H} B \to A/\langle\langle H\rangle\rangle *_G B$ is an isomorphism.

Exercise 21.13. Let X be a path-connected topological space and let $x_0 \in X$. Furthermore let $C \subset X$ be a subset that is homeomorphic to S^1. We pick a homeomorphism $\varphi\colon S^1 \to C$ and a path $\gamma\colon [0,1] \to X$ from x_0 to $\varphi(1)$. We denote by $\widetilde{\varphi}\colon [0,1] \to X$ the map that is given by $t \mapsto \varphi(e^{2\pi i t})$. Show that the subgroup

$$\langle\langle C\rangle\rangle \;:=\; \langle\langle \gamma * \varphi * \widetilde{\gamma}\rangle\rangle \;\subset\; \pi_1(X, x_0)$$

neither depends on the choice of φ nor on the choice of γ.

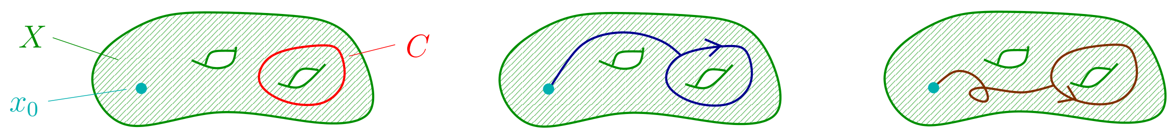


FIGURE 400. Illustration for Exercise 21.13.

Exercise 21.14. Let π be a group and let S be a generating set for π. We set $V := \pi$, we set $E := V \times S$ and we define the initial map $i\colon E = V \times S \to V$ by $i((v,s)) = v$ and we define the terminal map $t\colon E = V \times S \to V$ by $t((v,s)) = v \cdot s$. Note that (V, E, i, t) is an abstract graph. It is called the *Cayley graph* $\Gamma(\pi, S)$ of the group π with respect to the generating set S.

(a) We consider $\pi = \langle x, y \,|\, xyx^{-1}y^{-1}\rangle$ with the generating set $S = \{x, y\}$. Sketch $|\Gamma(\pi, S)|$ as a subset of $\mathbb{R}^2$.
(b) We consider $\pi = \langle x, y \,|\, \varnothing\rangle$ with the generating set $S = \{x, y\}$. Sketch $|\Gamma(\pi, S)|$ as a subset of $\mathbb{R}^2$.
(c) We consider $\pi = \langle x, y \,|\, x^2, xyx^{-1}y^{-1}\rangle$ with the generating set $S = \{x, y\}$. Sketch $|\Gamma(\pi, S|$ as a subset of $\mathbb{R}^3$.

Exercise 21.15. Let $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $n \in \mathbb{N}$.

(a) Show that the abelianization of $\mathrm{SL}(n, \mathbb{K})$ is the trivial group.
Hint. Try to write the elementary matrices as commutators.
(b) Determine the abelianization of $\mathrm{GL}(n, \mathbb{K})$.

Remark. A group with trivial abelianization is sometimes called *perfect*. We have thus shown that the groups $\mathrm{SL}(n, \mathbb{K})$ are perfect.

In this chapter we will generalize the "basic" Seifert–van Kampen Theorem 20.2 to the setting where we no longer need to assume that the intersection $U \cap V$ is simply connected. We will use the resulting Seifert–van Kampen Theorem 22.1 to determine the fundamental groups of the (non-) orientable surfaces Σ_g and N_g. In Chapter 23 we will use these calculations to give the classification of compact 2-dimensional smooth and topological manifolds.

Towards the end of this chapter we will use the Seifert–van Kampen Theorem 22.1 to prove Proposition 22.10 which says, in a nutshell, that every finitely presented group occurs as the fundamental group of a closed smooth manifold of dimension ≥ 4.

22.1. The formulation of the general Seifert–van Kampen Theorem.

With the group theoretic preparations from Chapter 21 we can now formulate and prove the general Seifert–van Kampen Theorem.

Theorem 22.1. (Seifert–van Kampen) *Let X be a topological space and let $X = U \cup V$ be a decomposition of X in two open subsets U and V such that $U \cap V$ is non-empty and path-connected. Let $x_0 \in U \cap V$. Then there exists an isomorphism*

$$\Phi\colon \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$$

such that the following diagram commutes:

$$\begin{tikzcd}
\pi_1(U \cap V, x_0) \arrow[r] \arrow[d] & \pi_1(U, x_0) \arrow[d] \arrow[ddr, bend left] & \\
\pi_1(V, x_0) \arrow[r] \arrow[drr, bend right] & \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \arrow[dr, "\Phi", "\cong"'] & \\
 & & \pi_1(X, x_0).
\end{tikzcd}$$

Here all the "undecorated" maps are the obvious inclusion induced homomorphisms.

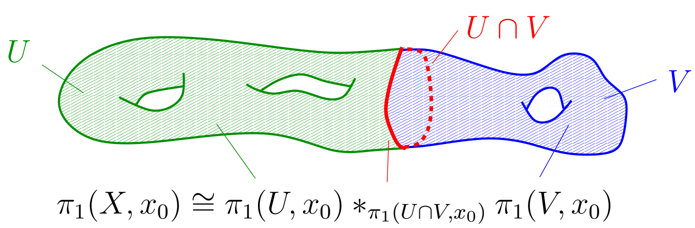


Figure 401. Illustration of Theorem 22.1.

The proof of the general Seifert–van Kampen Theorem is a modification of the proof of the more "basic" Seifert–van Kampen Theorem 20.2 where we had assumed that $U \cap V$ is in fact simply connected. We only sketch the proof of the general Seifert–van Kampen Theorem 22.1. The full details are for example given in Section 1.2 of [**Hat02**].

Sketch of the proof. The various inclusions of topological spaces give rise to the following commutative diagram:

$$\begin{tikzcd}
\pi_1(U\cap V,x_0) \arrow[r, "i"] \arrow[d, "j"'] & \pi_1(U,x_0) \arrow[dr] & \\
\pi_1(V,x_0) \arrow[rr] & & \pi_1(X,x_0).
\end{tikzcd}$$

By Proposition 21.21 there exists a unique homomorphism

$$\Phi\colon \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) \;\to\; \pi_1(X,x_0)$$

such that the following diagram commutes:

$$\begin{tikzcd}
\pi_1(U\cap V,x_0) \arrow[r, "i"] \arrow[d, "j"'] & \pi_1(U,x_0) \arrow[d, "k"] \arrow[dr, bend left] & \\
\pi_1(V,x_0) \arrow[r, "l"] \arrow[drr, bend right] & \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) \arrow[dr, "\Phi"] & \\
& & \pi_1(X,x_0).
\end{tikzcd}$$

It remains to show that Φ is surjective and that Φ is injective.

We first show surjectivity. We consider the following commutative diagram[390] of homomorphisms

$$\begin{tikzcd}
& \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) \arrow[dr, "\Phi"] & \\
\pi_1(U,x_0) * \pi_1(V,x_0) \arrow[ur] \arrow[rr] & & \pi_1(X,x_0).
\end{tikzcd}$$

By Lemma 20.1 the horizontal map is surjective. It follows that the diagonal map Φ on the right is also surjective.

Now we turn to the proof of the injectivity of Φ. We write $A=\pi_1(U,x_0)$, $B=\pi_1(V,x_0)$ and $G=\pi_1(U\cap V,x_0)$. As on page 594 we consider the set

$$S(A,B) \;:=\; \{\text{all finite sequences } (x_1,\dots,x_k) \text{ with each } x_i \text{ in } A \text{ or } B\}.$$

Furthermore we denote by $\sim_G$ the equivalence relation on $S(A,B)$ generated by the relation

$$(x_1,\dots,x_l,e,x_{l+2},\dots,x_k) \;\sim\; (x_1,\dots,x_l,x_{l+2},\dots,x_k)$$

where e is trivial in A or B, by the relation

$$(x_1,\dots,x_l,x_{l+1},\dots,x_k) \;\sim\; (x_1,\dots,x_lx_{l+1},\dots,x_k)$$

if both x_l and x_{l+1} lie in A or both lie in B, and also by the relation

$$(x_1,\dots,x_l,i(g),x_{l+2},\dots,x_k) \;\sim\; (x_1,\dots,x_l,j(g),x_{l+2},\dots,x_k)$$

[390] What are the maps, and why does the diagram commute?

for $g \in G$. Similar to Proposition 19.17 one can now show that the map

$$\begin{array}{rcl} S(A,B)/\sim_G & \to & A *_G B \\ [(x_1,\dots,x_k)] & \mapsto & x_1 \cdot \dots \cdot x_k \end{array}$$

is a well-defined bijection. We will use this isomorphism to identify $S(A,B)/\sim_G$ with $A *_G B$.

The proof of the present theorem is now almost identical to the proof of Theorem 20.2, one just needs to replace the discussion involving $S(A,B)/\sim$ by the corresponding objects in $S(A,B)/\sim_G$. More precisely, in Footnote 355 we pointed out where in the proof of Theorem 20.2 we used the hypothesis that $U \cap V$ is simply connected. At that point in the proof of Theorem 20.2 we need to replace $A * B = S(A,B)/\sim$ with $A *_G B = S(A,B)/\sim_G$. Full details for the proof can be found for example in Section 1.2 of [**Hat02**]. ■

The same way that we deduced Theorem 20.10 from Theorem 20.2 we can now obtain the following theorem from Theorem 22.1.

Theorem 22.2. (Seifert–van Kampen Theorem for Topological Manifolds) *Let M be an m-dimensional topological manifold and let $R, S \subset M$ be two m-dimensional submanifolds such that the following hold:*

(1) $M = R \cup S$,
(2) *$R \cap S$ is a component of ∂R and it is a component of ∂S,*
(3) *R and S are closed subsets of M.*

We pick a base point $x_0 \in R \cap S$. Then the inclusion induced map

$$\pi_1(R,x_0) *_{\pi_1(R\cap S,x_0)} \pi_1(S,x_0) \;\to\; \pi_1(M,x_0)$$

is an isomorphism.

Example. Given $n \in \mathbb{N}_0$ we denote as usual by $S^n_{\geq 0}$ the upper hemisphere of S^n and we denote by $S^n_{\leq 0}$ the lower hemisphere of S^n. We denote by $S^n_{=0} = S^n_{\geq 0} \cap S^n_{\leq 0}$ the "equator" of S^n, which is of course homeomorphic to S^{n-1}. Given any $n \in \mathbb{N}_{\geq 2}$ and any $P \in S^n_{=0}$ we can perform the following calculation:

$$\pi_1(S^n,P) \underset{\substack{\uparrow \\ \text{by the Seifert–van Kampen Theorem 22.2, note that we use} \\ \text{that our hypothesis } n\in\mathbb{N}_{\geq 2} \text{ implies that } S^{n-1} \text{ is path-connected}}}{\cong} \pi_1(S^n_{\leq 0},P) *_{\pi_1(S^n_{=0},P)} \pi_1(S^n_{\geq 0},P) \overset{\substack{\text{by page 116 the hemispheres are homeomorphic to } \overline{B}^n\text{, hence the fundamental groups are trivial} \\ \downarrow}}{\cong} \{e\} *_{\pi_1(S^{n-1},P)} \{e\} \underset{\substack{\uparrow \\ \text{by definition or by Lemma 21.22}}}{\cong} \{e\}.$$

This calculation gives a new, and arguably more conceptual proof of Proposition 14.14.

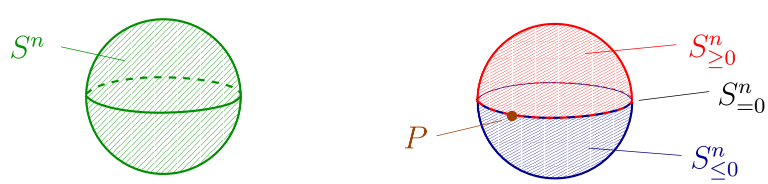


Figure 402

22.2. **The fundamental groups of surfaces.** Now we want to use the Seifert–van Kampen Theorem 22.1 to determine the fundamental groups of surfaces of genus $g \geq 2$. Before we do so we consider the torus, the Klein bottle and the real projective plane to gain some confidence in our newly acquired methods.[391]

(1) We first consider the torus $T = ([0,1] \times [0,1])/\sim$. In the following we will use the following notation, see also Figure 403 for an illustration:
   (a) We denote by Q the point on T corresponding to the four vertices of the square.
   (b) We write $T = U \cup V$ where
$$U = T \setminus \overline{B_{\frac{1}{8}}(\tfrac{1}{2},\tfrac{1}{2})} \subset T \quad \text{ and } V = B_{\frac{1}{4}}(\tfrac{1}{2},\tfrac{1}{2}) \subset T,$$
   i.e. U is the complement of a closed disk and V is an open disk and $U \cap V$ is an open annulus.
   (c) We let $P = (r,r) \in U \cap V$ and we denote by p the direct path from Q to P.
   (d) We pick a loop γ in $(U \cap V, P)$ that "goes counterclockwise around the annulus" and that represents a generator of the infinite cyclic group $\pi_1(U \cap V, P) \cong \mathbb{Z}$.[392]
   (e) We denote by x and y the two loops in (T,Q) corresponding to the horizontal and the vertical edge of the square. By a slight abuse of notation we denote the corresponding elements in $\pi_1(T,Q)$ by x and y as well.

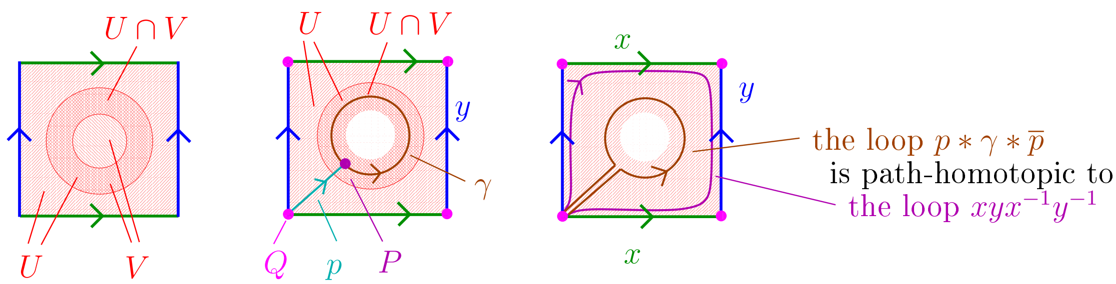


FIGURE 403

Now we have

$$\begin{array}{rclcl}
\pi_1(T,P) & \underset{\substack{\uparrow\\ \text{Seifert–van Kampen Theorem 22.1}}}{\cong} & \pi_1(U,P) *_{\pi_1(U\cap V,P)} \pi_1(V,P) & \underset{\substack{\uparrow\\ V \text{ is a disk, hence } \pi_1(V,P)=\{e\}}}{=} & \pi_1(U,P) *_{\pi_1(U\cap V,P)} \{e\} \\
& \underset{\substack{\uparrow\\ \text{by Lemma 21.22 we have } A *_G \{e\} = A/\langle\langle \alpha(G)\rangle\rangle}}{=} & \pi_1(U,P)/\langle\langle i_*(\pi_1(U\cap V,P))\rangle\rangle & \underset{\substack{\uparrow\\ \text{since } \pi_1(U\cap V,P)=\langle[\gamma]\rangle}}{=} & \pi_1(U,P)/\langle\langle[\gamma]\rangle\rangle \\
& \underset{\substack{\uparrow\\ \text{by Proposition 14.11 the path } p \text{ gives}\\ \text{rise to an isomorphism } p_*}}{\xrightarrow{\cong}} & \pi_1(U,Q)/\langle\langle[p*\gamma*\overline{p}]\rangle\rangle & \underset{\substack{\uparrow\\ \text{since } p*\gamma*\overline{p} \text{ is path-homotopic to } xyx^{-1}y^{-1}}}{=} & \pi_1(U,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle \;=\; (*).
\end{array}$$

[391] The following examples can also be handled using Theorem 22.2 instead of Theorem 22.1. The advantage of using Theorem 22.1 at this point is that in the illustrations one can draw $U \cap V$ better if U and V are open subsets.

[392] Since $U \cap V$ is an open annulus its fundamental group is isomorphic to $\mathbb{Z}$.

As in the proof of Lemma 20.9 we can show that $W := \partial([0,1]\times[0,1])/\sim$ is a deformation retract of U and that W is homeomorphic to the wedge of two circles. By Lemma 20.4 we can therefore make the identification $\pi_1(W,P) = \langle x, y\rangle$, see also Figure 404. Now we continue with the above calculation of $\pi_1(T,P)$. We have

$$\begin{aligned} (*) \;&=\; \pi_1(U,Q)/\langle\!\langle xyx^{-1}y^{-1}\rangle\!\rangle \;\underset{\substack{\uparrow\\ \text{by Lemma 18.14 and Proposition 18.16 (2)}}}{\xrightarrow{\cong}}\; \pi_1(W,P)/\langle\!\langle xyx^{-1}y^{-1}\rangle\!\rangle \\ &\underset{\substack{\uparrow\\ \text{since } W \text{ is the wedge of the circles } x \text{ and } y}}{=} \langle x,y\rangle/\langle\!\langle xyx^{-1}y^{-1}\rangle\!\rangle \;=\; \langle x,y \mid [x,y]\rangle \underset{\substack{\uparrow\\ \text{see Lemma 21.6 (1)}}}{\cong} \mathbb{Z}^2. \end{aligned}$$

Thus we have reproved that the fundamental group of the torus is isomorphic to the free abelian group $\mathbb{Z}^2$.

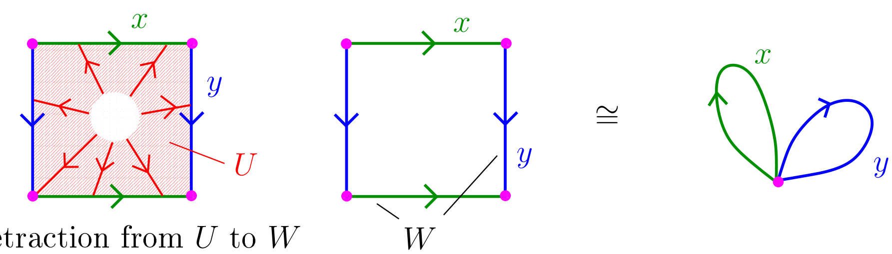


FIGURE 404

(2) We turn to the Klein bottle K. The approach to calculating the fundamental group of the Klein bottle is almost identical to the above approach to calculating the fundamental group of the torus. We just replace the situation illustrated in Figure 404 by the situation illustrated in Figure 405. The same argument as above shows that[393]

$$\pi_1(K) \;\cong\; \langle x, y \mid yxyx^{-1}\rangle.$$

Using this presentation and using Proposition 21.20 (5) we obtain in particular that the abelianization of $\pi_1(K)$ is given by

$$\pi_1(K)_{\mathrm{ab}} \;\cong\; \langle x,y \mid yxyx^{-1}\rangle_{\mathrm{ab}} \;\cong\; \mathbb{Z}^2/\binom{0}{2}\mathbb{Z} \;\cong\; \mathbb{Z}\oplus\mathbb{Z}_2.$$

On page 518 we already gave a very different calculation of $\pi_1(K)$. More precisely, we introduced the group G that is given by all self-homeomorphisms of $\mathbb{R}^2$ that can be written as concatenations of the two self-homeomorphisms

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s+1, 1-t) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (s,t) & \mapsto & (s, t+1) \end{array}$$

and their inverses and we showed that $\pi_1(K)\cong G$. On page 630 we already showed that there exists a homomorphism $\Phi\colon \langle x,y \mid yxyx^{-1}\rangle \to G$ with $\Phi(x) = B^{-1}A$ and $\Phi(y) = B$. This homomorphism is in fact an isomorphism. This can be seen as

[393] In particular this shows that $\pi_1(K)$ is finitely presented. This answers the question we had raised on page 631.

follows. Denote by H the group of all self-homeomorphisms of $\mathbb{R}^2$. We consider the diagram

$$\begin{tikzcd} \langle x,y \mid yxyx^{-1}\rangle \arrow[r, "\cong"] \arrow[rr, bend right, "\Phi", "x\mapsto B^{-1}A,\, y\mapsto B"'] & \pi_1(K) & \text{subgroup of } H \text{ generated by } A, B. \arrow[l, "\cong"'] \end{tikzcd}$$

Here the isomorphisms to $\pi_1(K)$ are the ones given by the two calculations of the fundamental group. If one goes carefully through the definitions of the two horizontal maps, then one can see that the diagram commutes. Since two of the three maps are isomorphisms and since the diagram commutes it follows that Φ is also an isomorphism.

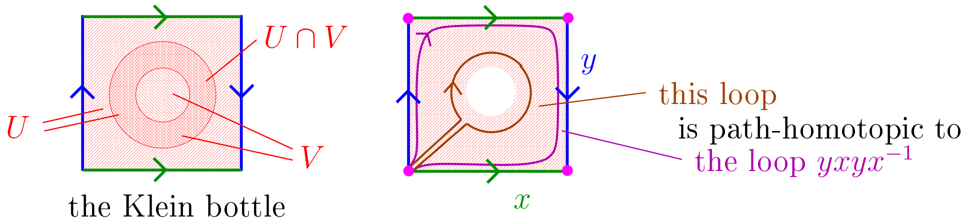


FIGURE 405

(3) Next we consider again the real projective space $\mathbb{R}P^2$. Here we use the description of $\mathbb{R}P^2$ from Lemma 3.42 as

$$\mathbb{R}P^2 \;=\; \overline{B}^2/\sim \qquad \text{where for } z \in S^1 \text{ we have } z \sim -z.$$

As in the previous two examples we decompose $\mathbb{R}P^2$ into an open disk V and the complement of a closed disk U such that $U \cap V$ is an annulus. We consider the base point $Q = [1] = [-1] \in \mathbb{R}P^2$ and we denote by α the loop

$$\begin{aligned} \alpha\colon [0,1] &\to \mathbb{R}P^2 \\ t &\mapsto [e^{i\pi t}]. \end{aligned}$$

By a slight abuse of notation we now denote by α also the corresponding element in $\pi_1(\mathbb{R}P^2, Q)$. Furthermore we denote by W the image of α in $\mathbb{R}P^2$. Evidently α is a generator of $\pi_1(W, Q)$. Then, with similar arguments as in the torus case, we see that

$$\pi_1(\mathbb{R}P^2) \underset{\substack{\uparrow\\ \text{Seifert–van Kampen Theorem 22.1}\\ \text{and Lemma 21.22 (2)}}}{\cong} \pi_1(U,Q)/\langle\langle p * \gamma * \overline{p}\rangle\rangle \underset{\substack{\uparrow\\ \text{since } W \text{ is a deformation}\\ \text{retract of } U}}{\cong} \pi_1(W,Q)/\langle\langle p * \gamma * \overline{p}\rangle\rangle \underset{\substack{\uparrow\\ \text{since } \pi_1(W,Q) = \langle\alpha\rangle\\ \text{and } [p*\gamma*\overline{p}] = \alpha^2}}{\cong} \langle\alpha\rangle/\langle\alpha^2\rangle \cong \mathbb{Z}_2.$$

In particular we obtain the same result, as we should, as in Corollary 16.18.

(4) Now that we have gained some confidence that the approach of determining fundamental groups using Seifert–van Kampen Theorem 22.1 does indeed give the right fundamental groups we turn to the surfaces of higher genus that we could not handle with our previous methods.

So let us consider the surface $F = E_8/\sim$ of genus 2. As in the torus case we write $F = U \cup V$ where V is an open disk and where U is the complement of a closed disk

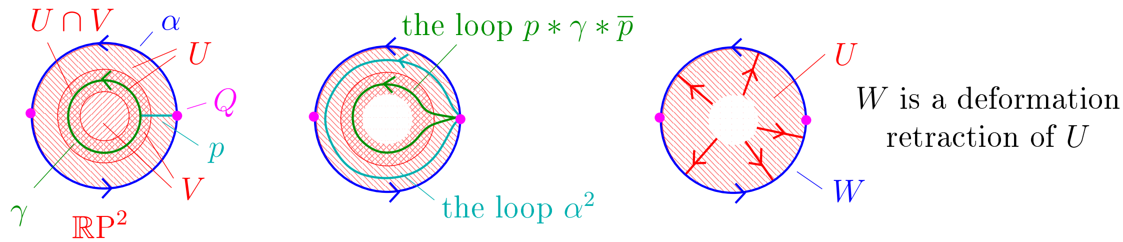


FIGURE 406

such that $U \cap V$ is an annulus. Similar to the torus case we write $W = \partial E_8/\sim$. By the proof of Proposition 20.8 we know that W is a deformation retraction of U and that W is homeomorphic to the wedge of four circles corresponding to x_1, y_1, x_2, y_2. See also Figures 407 and 408 for an illustration. Basically exactly the same argument

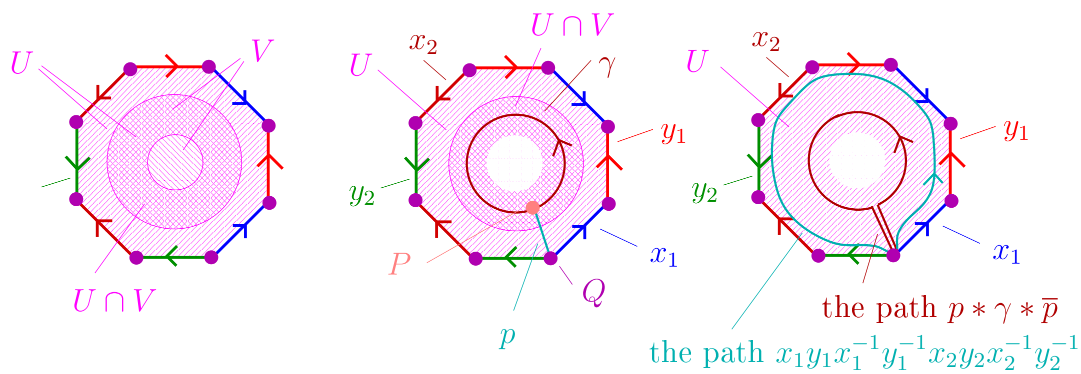


FIGURE 407

as in the torus case shows that

$$\begin{aligned}
\pi_1(E_8/\sim, P) \;&\cong\; \pi_1(U, Q)/\langle\langle \overbrace{\underbrace{x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1}}_{=[x_1,y_1][x_2,y_2]}}^{\text{path-homotopic to } p*\gamma*\overline{p}} \rangle\rangle \\
&\cong\; \pi_1(W, Q)/\langle\langle [x_1, y_1][x_2, y_2] \rangle\rangle \\
&\cong\; \langle x_1, y_1, x_2, y_2 \rangle / \langle\langle [x_1, y_1][x_2, y_2] \rangle\rangle \;=\; \langle x_1, y_1, x_2, y_2 \mid [x_1, y_1][x_2, y_2] \rangle.
\end{aligned}$$

Thus we have finally succeeded in calculating the fundamental group of the surface of genus 2.

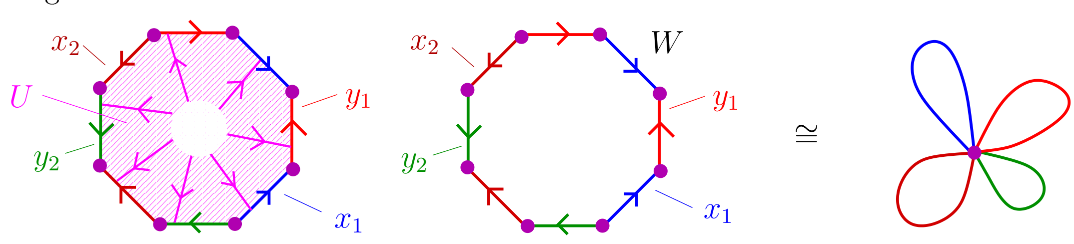


FIGURE 408

The following gives in particular a new proof for Proposition 20.8.

Proposition 22.3.

(1) *For any* $g \in \mathbb{N}_0$ *we have*

$$\pi_1(\textit{surface } \Sigma_g \textit{ of genus } g) \;\cong\; \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \cdot \dots \cdot [x_g, y_g] \rangle$$

and we have

$$\pi_1(\textit{surface of genus } g)_{\mathrm{ab}} \;\cong\; \mathbb{Z}^{2g}.$$

(2) *For* $g \neq h$ *the surfaces of genus* g *and* h *are not homeomorphic.*

Proof.

(1) Let $g \in \mathbb{N}_0$. For $g = 0$ we obtain the fundamental group from Proposition 14.14 and for $g = 1$ we obtain the presentation from the discussion on page 656. For $g \geq 2$ the presentation follows from a straightforward generalization of the calculation preceding the proposition. Finally the stated abelianization of the fundamental group is an immediate consequence of Proposition 21.20 (5).

(2) For $g \neq h$ we obtain from the above calculation and from the classification of finitely generated abelian groups that we gave in Theorem 19.4 that the abelianizations of the fundamental groups of the surfaces of genus g and h are not isomorphic. Hence the surfaces are not homeomorphic. ■

Remark.

(1) In Figure 409 we show loops that represent the generators x_1, y_1, x_2, y_2 of the fundamental group for the surface of genus 2.

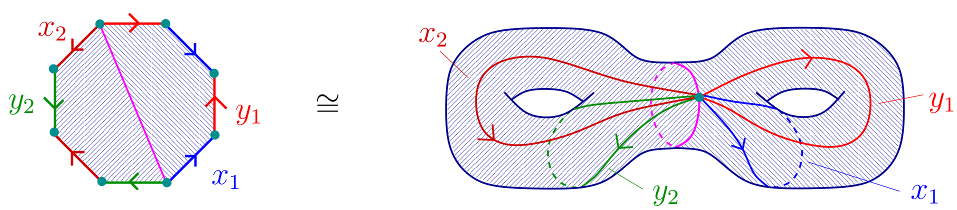


Figure 409

(2) As explained in Figure 220 we can view the surface Σ of genus 2 as the connected sum of two copies of the torus T. We can also use this point of view to determine the fundamental group of the surface Σ of genus 2. More precisely, we have $\Sigma = F_1 \cup F_1$ with $F_1 = T \setminus B^2$ and $F_2 = T \setminus B^2$ where $F_1 \cap F_2 = \partial F_1 = \partial F_2 \cong S^1$ and we obtain that[394]

$$\pi_1(\Sigma) \underset{\text{see page 646}}{\overset{\text{Seifert–van Kampen Theorem 22.2}}{\cong}} \pi_1(F_1) *_{\pi_1(F_1 \cap F_2)} \pi_1(F_2) \overset{\text{by Lemma 20.9 (2) where } t \mapsto [x_1,y_1] \text{ and } t \mapsto [x_2,y_2]}{\cong} \langle x_1, y_1 \rangle *_{\langle t \rangle} \langle x_2, y_2 \rangle$$
$$\cong \langle x_1, x_2, y_1, y_2 \mid [x_1, y_1] = [x_2, y_2] \rangle.$$

[394] Why is this group isomorphic to the above group $\langle x_1, y_1, x_2, y_2 \mid [x_1, y_1][x_2, y_2] \rangle$?

Let $g \geq 2$ and let π be the fundamental group of the surface of genus g. We have just shown that $\pi_{\mathrm{ab}} \cong \mathbb{Z}^{2g}$. This raises the question, whether perhaps π is isomorphic to $\mathbb{Z}^{2g}$ or to the free group F_{2g} on $2g$ generators. The proof of the following lemma is Exercise 22.1.

Lemma 22.4. *The group* $\pi = \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \cdot \dots \cdot [x_g, y_g] \rangle$ *is non-abelian.*

The lemma says in particular that π is not isomorphic to $\mathbb{Z}^{2g}$. It is much harder to determine whether or not π is a free group. We record this as a question for later.

Question 22.5. *Let π be the fundamental group of the surface of genus g. Is π isomorphic to the free group F_{2g} on $2g$ generators?*

We will give an answer to the question in Proposition 31.17 and also on page 1371.

Let π be the fundamental group of the surface of genus g. In Exercise 22.3 we will see that if $g \geq 2$, then π contains a free group on two generators as a subgroup. This raises the question, what other groups can occur as subgroups of π? Let us record this as a question.

Question 22.6. *Let π be the fundamental group of the surface of genus g. What isomorphism types of subgroups can appear? Is every subgroup again a free group or the fundamental group of a surface?*

We will provide the answer to Question 22.6 in Proposition 31.6.

22.3. **Non-orientable surfaces.** In this section we will discuss the fundamental groups of the non-orientable surfaces N_g of some genus $g \in \mathbb{N}$. It is also worth recalling that in Exercise 8.21 we showed that the connected sum of g copies of $\mathbb{RP}^2$ is diffeomorphic to the non-orientable surface of genus g that we introduced on page 206.

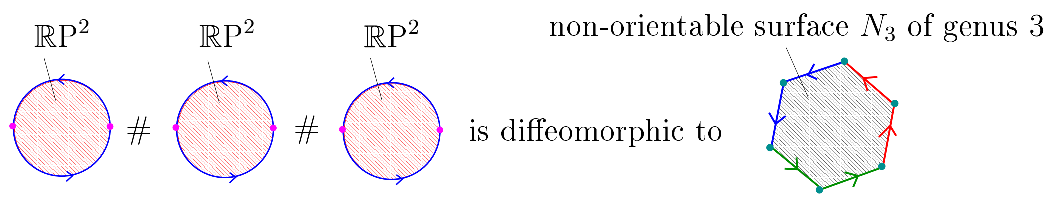


FIGURE 410

The following proposition is the analogue of Proposition 22.3.

Proposition 22.7.

(1) *For any $g \in \mathbb{N}$ we have*

$$\pi_1(\textit{non-orientable surface } N_g \textit{ of genus } g) \;\cong\; \langle x_1, \dots, x_g \mid x_1^2 \cdot \dots \cdot x_g^2 \rangle$$

and we have

$$\pi_1(\textit{non-orientable surface } N_g \textit{ of genus } g)_{\mathrm{ab}} \;\cong\; \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2.$$

(2) *For $g \neq h$ the non-orientable surfaces of genus g and h are not homeomorphic.*

PROOF.

(1) Let $g \in \mathbb{N}$. Recall that the non-orientable surface of genus one is by definition the real projective plane. In Corollary 16.18 we saw that $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2 \cong \langle x \mid x^2 \rangle$. The case $g \geq 2$ follows from a straightforward modification of the calculation of

the fundamental group of the surface of genus 2 that we gave on page 658. Using Figure 411 the reader should have no troubles with assembling a full argument. Finally note that the stated abelianization of the fundamental group is an immediate consequence of Proposition 21.20 (5).

(2) For $g \neq h$ we obtain from the above calculation and from the classification of finitely generated abelian groups that we gave in Theorem 19.4 that the abelianizations of the fundamental groups of the non-orientable surfaces of genus g and h are not isomorphic. Hence the non-orientable surfaces of genus g and h are not homeomorphic. ■

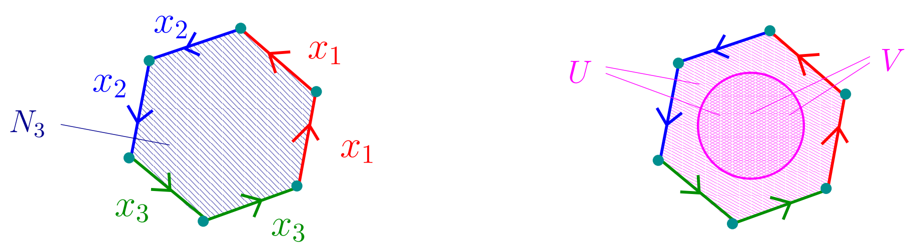


FIGURE 411. Illustration for the proof of Proposition 22.7.

As mentioned above, in Exercise 8.21 we showed that for any $g \in \mathbb{N}$ the non-orientable surface of genus g is diffeomorphic to the connected sum of g copies of $\mathbb{RP}^2$. In particular the Klein bottle is diffeomorphic to $\mathbb{RP}^2\#\mathbb{RP}^2$. Just for fun and for practice we provide the direct calculation of the fundamental group of $\mathbb{RP}^2\#\mathbb{RP}^2$ using the Seifert–van Kampen Theorem 22.2 for Smooth Manifolds. The calculation is somewhat interesting since it involves some interesting maps between fundamental groups.

Lemma 22.8. *We have*

$$\pi_1(\mathbb{RP}^2\#\mathbb{RP}^2) \;\cong\; \langle x, y \mid x^2 = y^2\rangle.$$

PROOF. In order to distinguish the two copies of $\mathbb{RP}^2 = \overline{B}^2/\sim$ we write $P = \mathbb{RP}^2$ and $Q = \mathbb{RP}^2$. The proof is of course a variation on the proof of Proposition 20.12. We have

$$\pi_1(\mathbb{RP}^2\#\mathbb{RP}^2) \;= \pi_1(P\#Q) \underset{\substack{\uparrow\\ \text{by the definition of } P\#Q \text{ and the Seifert–van Kampen Theorem 22.2}}}{\cong} \pi_1\big(P \setminus \tfrac12 B^2\big) *_{\pi_1(\frac12 S^1)} \pi_1\big(Q \setminus \tfrac12 B^2\big).$$

We pick a generator g for $\pi_1(\frac12 S^1)$. In the calculation of the fundamental group of the real projective plane on page 657 we saw that $\pi_1(P \setminus \frac12 B^2) \cong \langle x\rangle$ and that the inclusion induced map $\pi_1(\frac12 S^1) \to \pi_1(P \setminus \frac12 B^2)$ is given by $g \mapsto x^2$. Evidently the same holds for Q with $\pi_1(Q \setminus \frac12 B^2) \cong \langle y\rangle$. Summarizing we see that

$$\begin{aligned}
\pi_1(\mathbb{RP}^2\#\mathbb{RP}^2) \;&\cong\; \pi_1\big(P \setminus \tfrac12 B^2\big) *_{\pi_1(\frac12 S^1)} \pi_1\big(Q \setminus \tfrac12 B^2\big)\\
&\cong\; \text{amalgamated product of } \left(\begin{array}{ccc} \langle g\rangle & \to & \langle x\rangle \\ g & \mapsto & x^2\end{array} \text{ and } \begin{array}{ccc} \langle g\rangle & \to & \langle y\rangle \\ g & \mapsto & y^2\end{array}\right)\\
&\underset{\substack{\uparrow\\ \text{Lemma 21.23}}}{\cong}\; \langle x, y \mid x^2 = y^2\rangle \;=\; \langle x, y \mid x^2y^{-2}\rangle.
\end{aligned}$$

We have thus shown that $\pi_1(\mathbb{RP}^2\#\mathbb{RP}^2) \cong \langle x, y \mid x^2 = y^2\rangle$. ■

We conclude this section with a short discussion of the properties of fundamental groups of compact 2-dimensional smooth manifolds. We recall that the fundamental group of the real projective space $\mathbb{RP}^2$ is isomorphic to $\mathbb{Z}_2$, therefore it is torsion and in particular it is not torsion-free. It is less clear whether or not the fundamental groups of the other 2-dimensional smooth manifolds are torsion-free. More precisely, we have the following question.

Question 22.9.

(1) *Let $g \in \mathbb{N}$. Is the fundamental group*
$$\pi_1(\textit{surface of genus } g) \cong \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \cdot \dots \cdot [x_g, y_g]\rangle$$
torsion-free?

(2) *Let $k \geq 2$. Is the fundamental group*
$$\pi_1(k \cdot \mathbb{RP}^2) \cong \langle x_1, \dots, x_k \mid x_1^2 \cdot \dots \cdot x_k^2\rangle$$
torsion-free?

We will come back to this question in Exercise 25.10, Corollary 31.7 and Proposition **??**.

22.4. **Fundamental groups of high-dimensional smooth manifolds.** Later on, in Proposition 64.6 we will see that the fundamental group of a compact connected smooth manifold is necessarily finitely presented. In this section we want to study the converse: which finitely presented groups can occur as fundamental groups of smooth manifolds?

First note that it follows from Theorem 7.5 that only the trivial group and $\mathbb{Z}$ occur as the fundamental group of a compact connected 1-dimensional topological manifold. Similarly, we will see shortly in Theorem 23.4 that only "few" finitely presented groups occur as the fundamental group of compact connected 2-dimensional topological manifold. In contrast we will now see that once we go to dimension ≥ 4 any finitely presented group occurs as the fundamental group of a closed orientable smooth manifold

Proposition 22.10. *Let π be a finitely presented group. Given any $n \geq 4$ there exists a closed orientable connected non-empty n-dimensional smooth manifold M with $\pi_1(M) \cong \pi$.*

Remark.

(1) We will give an alternative proof of Proposition 22.10 on page 2550.
(2) Proposition 22.10 is a key ingredient of the proof of the fact, stated in Theorem 28.4, that closed smooth manifolds of a fixed dimension ≥ 4 cannot be classified.

The proof of Proposition 22.10 relies on two lemmas and one theorem. The first lemma is actually quite interesting in its own right.

Lemma 22.11. *Let W be a connected n-dimensional smooth manifold. Suppose we are given $k \in \mathbb{N}_0$ and a submanifold of the form $\overline{B}^{n-k} \times K$ where K is diffeomorphic to S^k.*

(1) *If $n \geq k+3$, then for every base point $w_0 \in W \setminus (B^{n-1} \times K)$ the inclusion induced map $\pi_1(W \setminus (B^{n-1} \times K), w_0) \to \pi_1(W, w_0)$ is an isomorphism.*

(2) *If $n = k+2$, then for every base point $w_0 \in W \setminus (B^{n-1} \times K)$ the inclusion induced map $\pi_1(W \setminus (B^{n-1} \times K), w_0) \to \pi_1(W, w_0)$ is an epimorphism.*

Proof of Lemma 22.11. We write $X = W \setminus (B^{n-k} \times K)$. Note that it follows from Proposition 15.11 that it suffices to prove the statement for some base point $w_0 \in S^{n-k-1} \times K$.

First we consider the case that $n \geq k+3$. Next note that we have the following two isomorphisms:

$$\pi_1(X, w_0) \underset{\substack{\uparrow\\ \text{the map is an isomorphism by Lemma 21.22 (3a) and since}\\ \text{the map } \pi_1(S^{n-k-1}\times K, w_0) \to \pi_1(\overline{B}^{n-k}\times K, w_0) \text{ is an isomorphism,}\\ \text{note that for the latter statement we used that } n \geq k+3}}{\xrightarrow{\cong}} \pi_1(X, w_0) *_{\pi_1(S^{n-k-1}\times K, w_0)} \pi_1\big(\overline{B}^{n-k} \times K, w_0\big) \underset{\substack{\uparrow\\ \text{by the Seifert–van Kampen}\\ \text{Theorem 22.2, which we can}\\ \text{apply by Proposition 8.2}}}{\xrightarrow{\cong}} \pi_1(W, w_0).$$

Since the horizontal maps are induced by the inclusions we see that the inclusion induced map $\pi_1(X, w_0) \to \pi_1(W, w_0)$ is indeed an isomorphism.

If $n = k+2$, then the inclusion induced map $\pi_1(S^{n-k-1} \times K) \to \pi_1(\overline{B}^{n-k} \times K)$ is an epimorphism but no longer an isomorphism. If we now use Lemma 21.22 (3b) instead of Lemma 21.22 (3a) we see that $\pi_1(X, w_0) \to \pi_1(W, w_0)$ is at the very least an epimorphism. ■

Lemma 22.12. *Let W be an n-dimensional smooth manifold and suppose we are given a boundary component of the form $S^{n-2} \times C$ where C admits a diffeomorphism $\varphi\colon S^1 \to C$. We set*

$$X \;:=\; (W \sqcup (S^{n-2} \times \overline{B}^2))/\sim \qquad \text{where } (x, \varphi(y)) \sim (x, y) \text{ for } (x,y) \in S^{n-2} \times S^1.$$

If $n \geq 4$, then for any $w_0 \in W$ the inclusion induced map $\pi_1(W, w_0) \to \pi_1(X, w_0)$ induces an isomorphism[395]

$$\pi_1(W, w_0)/\langle\langle C \rangle\rangle \xrightarrow{\cong} \pi_1(X, w_0).$$

Proof. Note that it follows from Proposition 15.11 that it suffices to prove the statement for some base point $w_0 \in S^{n-2} \times C$. Next note that we have the following two isomorphisms:

$$\pi_1(W, w_0)/\langle\langle C \rangle\rangle \underset{\substack{\uparrow\\ \text{the map is an isomorphism by Lemma 21.22 (2), here we use}\\ \text{that } \pi_1(S^{n-2}\times \overline{B}^2, w_0) = 0 \text{ and that } \pi_1(S^{n-2}\times C, w_0) = \langle C\rangle,\\ \text{note that for the latter statement we used that } n \geq 4}}{\xrightarrow{\cong}} \pi_1(W, w_0) *_{\pi_1(S^{n-2}\times C, w_0)} \pi_1\big(S^{n-2} \times \overline{B}^2, w_0\big) \underset{\substack{\uparrow\\ \text{by the Seifert–van Kampen}\\ \text{Theorem 22.2 which we can}\\ \text{apply by Proposition 8.15}}}{\xrightarrow{\cong}} \pi_1(X, w_0).$$

Since the horizontal maps are induced by the inclusions we see that the inclusion induced map $\pi_1(W, w_0)/\langle\langle C \rangle\rangle \to \pi_1(X, w_0)$ is indeed an isomorphism. ■

The proof of Proposition 22.10 also requires the following theorem, which is a special case of Theorem 9.15.

---

[395]Here we use that Exercise 21.13 tells us that $C = \{0\} \times C$ gives us a well-defined normal subgroup $\langle\langle C \rangle\rangle$.

Theorem 22.13. *Let M be a smooth manifold and let $\varphi\colon S^1\sqcup\cdots\sqcup S^1\to M$ be a map. If $\dim(M)\geq 3$, then φ is homotopic to a smooth embedding $\psi\colon S^1\sqcup\cdots\sqcup S^1\to M$.*

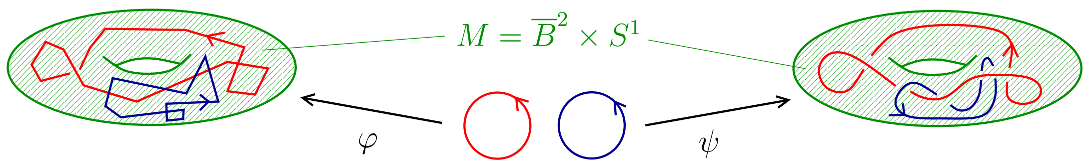


Figure 412. Illustration of Theorem 22.13.

Now we are fully equipped for proving Proposition 22.10.

Proof of Proposition 22.10. Throughout the proof we will mostly ignore base points. But at the end of the proof we will argue, why being a little sloppy with base points in between does not affect the outcome.

Let $\pi=\langle g_1,\dots,g_k\,|\,r_1,\dots,r_l\rangle$ be a finitely presented group. Furthermore let $n\geq 4$. We consider

$$X\;:=\;\underbrace{(S^1\times S^{n-1})\#\dots\#(S^1\times S^{n-1})}_{k\text{ copies of }S^1\times S^{n\text{-}1}}.$$

It follows basically from Proposition 8.35 that X is a closed orientable connected n-dimensional smooth manifold.

We see that

$$\pi_1(\overbrace{(S^1\times S^{n-1})\#\dots\#(S^1\times S^{n-1})}^{=X})\underset{\substack{\uparrow\\ \text{by Proposition 20.12 since } n\geq 3}}{\cong}\pi_1(S^1\times S^{n-1})*\dots*\pi_1(S^1\times S^{n-1})\underset{\substack{\uparrow\\ \text{by the discussion on page 520}\\ \text{we know that } \pi_1(S^1\times S^{n-1})\cong\mathbb{Z}}}{\cong}\overbrace{\langle g_1\rangle*\dots*\langle g_k\rangle}^{=\langle g_1,\dots,g_k\rangle}.$$

We use this isomorphism to make the identification $\pi_1(X,x_0)=\langle g_1,\dots,g_k\rangle$. Next we pick loops $b_1,\dots,b_l\colon(S^1,*)\to(X,x_0)$ that represent $r_1,\dots,r_l\in\pi_1(X,x_0)=\langle g_1,\dots,g_k\rangle$.

By hypothesis we have $\dim(X)=n\geq 4$. Thus we can apply Theorem 22.13 and we obtain that the map $b_1\sqcup\cdots\sqcup b_l\colon S^1\sqcup\cdots\sqcup S^1\to X$ is in fact homotopic to a smooth embedding $c_1\sqcup\cdots\sqcup c_l\colon S^1\sqcup\cdots\sqcup S^1\to X$. For $i=1,\dots,l$ we write $C_i:=c_i(S^1)$. It follows from Proposition 8.1 and the Tubular Neighborhood Theorem 8.24 that we can thicken up the C_i to disjoint submanifolds $\overline{B}^{n-1}\times C_1,\dots,\overline{B}^{n-1}\times C_l$. We consider[396]

$$Y:=X\setminus\bigcup_{i=1}^{l}(B^{n-1}\times C_i)\;\text{ and }\;Z:=\Big(Y\sqcup\coprod_{i=1}^{l}(S^{n-2}\times\overline{B}^2_i)\Big)\Big/\sim\quad\begin{array}{l}\text{where }(x,c_i(y))\sim(x,y)\text{ for}\\ x\in S^{n-2}\text{ and }y\in S^1_i=S^1.\end{array}$$

[396]The smooth manifold Y is effectively obtained from $(S^1\times S^{n-1})\#\dots\#(S^21\times S^{n-2})$ by removing copies of $\overline{B}^{n-1}\times S^1$ and gluing in copies of $S^{n-2}\times\overline{B}^2$. This operation is frequently called a surgery. We will discuss surgeries in greater detail in Chapter **??**.

It follows from Proposition 8.2 and Proposition 8.15 that Z is indeed a closed oriented connected n-dimensional smooth manifold. Finally we see that

$$\begin{aligned}
\pi_1(Z) \;&\overset{\substack{\text{Lemma 22.12 applied } l \text{ times}\\ \downarrow}}{\cong}\; \pi_1(Y)/\langle\!\langle C_1,\dots,C_l\rangle\!\rangle \;\overset{\substack{\text{Lemma 22.11 applied } k \text{ times}\\ \downarrow}}{\cong}\; \pi_1(X)/\langle\!\langle C_1,\dots,C_l\rangle\!\rangle \\
&\underset{\substack{\uparrow\\ \text{by the above, the } x_i \text{ are necessary since}\\ \text{we lost track of the base points}}}{\cong}\; \langle g_1,\dots,g_k\rangle/\langle\!\langle x_1r_1x_1^{-1},\dots,x_lr_lx_l^{-1}\rangle\!\rangle \;\underset{\substack{\uparrow\\ \text{by Lemma 21.2 (5)}}}{\cong}\; \underbrace{\langle g_1,\dots,g_k\rangle/\langle\!\langle r_1,\dots,r_l\rangle\!\rangle}_{=\langle g_1,\dots,g_k\,|\,r_1,\dots,r_l\rangle}.
\end{aligned}$$

We have thus found the promised smooth manifold. ■

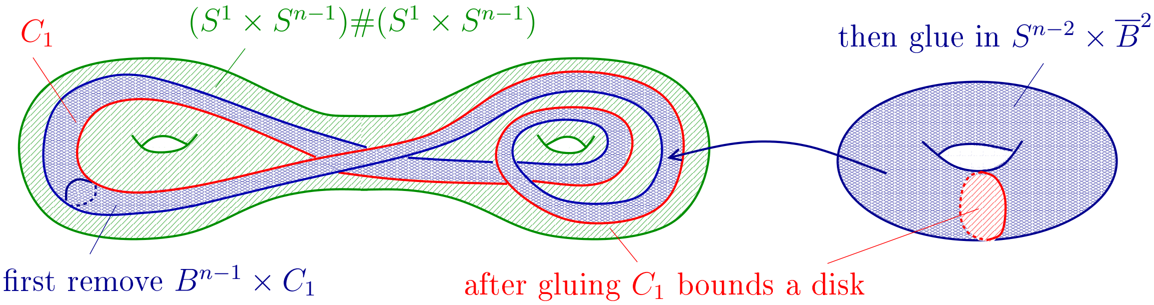


FIGURE 413. Illustration for the proof of Proposition 22.10.

In the proof of Proposition 22.10 we used on several occasions that we are dealing with dimensions ≥ 4. The following question arises naturally.

Question 22.14. *Is every finitely presented group the fundamental group of a closed* (*or at least compact*) *orientable connected 3-dimensional smooth manifold?*

We will answer Question 22.14 in Theorem **??**.

Exercises for Chapter 22.

Exercise 22.1. Let $g\in\mathbb{N}_{\geq 2}$. We consider the group

$$\pi \;:=\; \pi_1(\text{surface of genus } g) \;\cong\; \langle x_1,y_1,\dots,x_g,y_g\,|\,[x_1,y_1]\cdot\dots\cdot[x_g,y_g]\rangle.$$

(a) Show that π admits an epimorphism onto a free group of rank ≥ 2.
(b) What is the largest $k\in\mathbb{N}$ for which you manage to find an epimorphism onto a free group of rank k?

Exercise 22.2. We consider the curve C on the surface Σ_3 of genus three that is shown in Figure 434.

(a) Show that C represents the trivial element in $\pi_1(\Sigma_3,P)_{\text{ab}}$.
(b) Show that C represents a non-trivial element in $\pi_1(\Sigma_3,P)$.

Exercise 22.3. Let π be the fundamental group of the surface Σ of genus g. Show that π contains a free subgroup of rank 2.

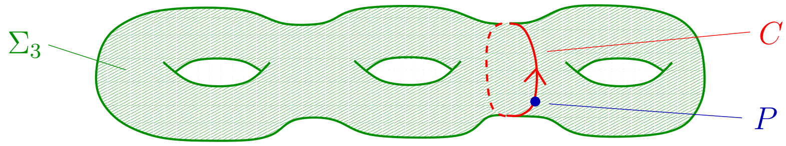


FIGURE 414. Illustration for Exercise 22.2.

Hint. You could make use of your solution to Exercise 22.1 or you could show that there exists an embedding $\varphi\colon S^1 \vee S^1 \to \Sigma$ such that $\varphi(S^1 \vee S^1)$ is a retract of Σ. We refer to Figure 415 for an illustration.

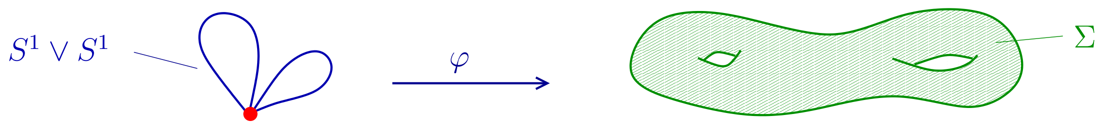


FIGURE 415. Illustration of Exercise 22.3.

Exercise 22.4. Let K be the Klein bottle. Recall that on page 656 we showed that $\pi_1(K) \cong \langle x, y \mid yxyx^{-1}\rangle$ and in Proposition 22.3 we showed that $\pi_1(K) \cong \langle x, y \,|\, x^2y^2\rangle$. Give a purely algebraic proof that the groups $\langle x, y \mid yxyx^{-1}\rangle$ and $\langle x, y \,|\, x^2y^2\rangle$ are isomorphic.

Exercise 22.5.

(a) Let M be an oriented connected m-dimensional smooth manifold and let N be a 1-dimensional submanifold *that is diffeomorphic to* S^1. Let $x_0 \in M \setminus N$. Suppose that $m \geq 4$. Show that the inclusion induced map $\pi_1(M \setminus N, x_0) \to \pi_1(M, x_0)$ is an isomorphism.
Hint. Use the Tubular Neighborhood Theorem 8.24.

(b) Show that the conclusion (a) does not hold for $m = 3$, i.e. give an example of a connected 3-dimensional smooth manifold M and a connected 1-dimensional submanifold N such that the inclusion induced map $\pi_1(M \setminus N, x_0) \to \pi_1(M, x_0)$ is not an isomorphism.
Remark. The smooth manifold M in the example does not have to be closed.

Exercise 22.6. Let X be the result of removing the interior of a smoothly embedded disk from a torus. Furthermore let Y be the result of removing the interiors of three disjointly smoothly embedded disks from the sphere S^2.

(a) Show that X and Y are homotopy equivalent.
(b) Show that X and Y are not homeomorphic.

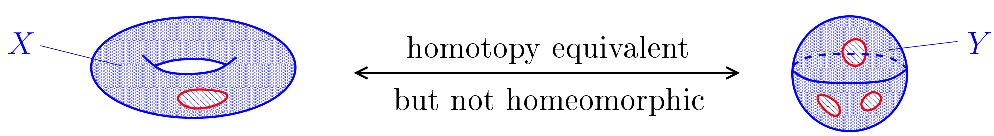


FIGURE 416. Illustration of Exercise 22.6.

Exercise 22.7. Let $g \in \mathbb{N}$ and let Σ be the surface of genus g minus one open disk. We pick a diffeomorphism $\varphi\colon S^1 \to \partial\Sigma$. Let $\psi\colon S^1 \times \partial\Sigma \to \partial\Sigma \times S^1$ be the diffeomorphism given by $(z,w) \mapsto (\varphi(z), \varphi^{-1}(w))$. We consider the inclusion map

$$\begin{array}{rcl} f\colon (S^1 \times \partial\Sigma) \cup_\psi (\partial\Sigma \times S^1) & \to & \Sigma \times \Sigma \\ [(z,w)] & \mapsto & \begin{cases} (z, \varphi(w)), & \text{if } (z,w) \in S^1 \times \partial\Sigma, \\ (\varphi(z), w), & \text{if } (z,w) \in \partial\Sigma \times S^1. \end{cases} \end{array}$$

Show that the induced map

$$f_*\colon \pi_1((S^1 \times \partial\Sigma) \cup_\varphi (\partial\Sigma \times S^1)) \;\to\; \pi_1(\Sigma \times \Sigma)$$

has non-trivial kernel.

Remark. You might want to use the Normal Form Theorem 21.25.

23. The classification of compact 2-dimensional manifolds I

One of the goals of mathematicians is to classify certain objects. For example one goal would be, given a dimension $n \in \mathbb{N}$, to "classify" all compact connected non-empty n-dimensional topological manifolds. But what does "classify" mean? One interpretation would be the following:

(I) Provide a collection $\{X_i\}_{i\in I}$ of compact connected non-empty n-dimensional topological manifolds with the following two properties:
  (a) Any compact connected non-empty n-dimensional topological manifold is homeomorphic to some X_i.
  (b) If $i \neq j$, then X_i and X_j are not homeomorphic.

(II) Furthermore provide an algorithm which determines, given a compact connected n-dimensional topological manifold M, to which X_i it is homeomorphic to.

With obvious modifications the same goal can also be formulated in the smooth setting.

For $n = 1$ we have achieved this goal with Theorems 7.1 and 7.5. In this setting a complete list is given by $\{S^1, [0,1]\}$ and given a compact connected non-empty 1-dimensional topological or smooth manifold it is trivial to determine to which of the two it corresponds.

In this chapter we turn our gaze to the 2-dimensional setting. We will state the classification theorem for topological and smooth manifolds and we will prove (I b) and (II) in the above setting. For the proof of (I a) we will need some patience though, we will only provide it in Chapter 108.

23.1. **The Surface Classification Theorem.** As mentioned above, one goal of any classification theorem is to provide a "complete" list of objects. We already have a good list of closed surfaces. Now we need to add surfaces with non-empty boundary to our arsenal. We start out with reminding the reader of the following subtle fact.[397]

Proposition 18.18.

(2) *Let M be a 2-dimensional smooth manifold. The boundary of M, viewed as a smooth manifold agrees with the boundary of M viewed as a topological manifold.*

Now let us move on to the first definition of this chapter,

Definition. Let Σ be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$. We pick smooth embeddings $\varphi_1, \dots, \varphi_n \colon \overline{B}^2 \to \Sigma$ with disjoint images. We refer to $\Sigma \setminus (\varphi_1(B^2) \cup \cdots \cup \varphi_n(B^2))$ as Σ *minus n open disks.*

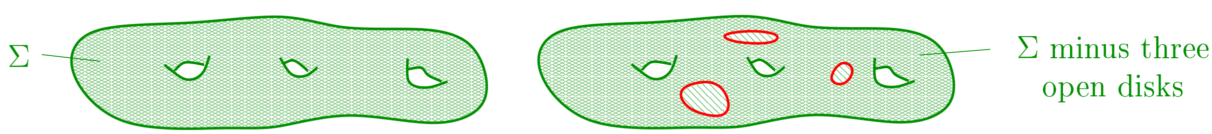


Figure 417

The following lemma captures some of the main features of the above definition.

[397]The statement is indeed subtle, for example we will only be able to prove the higher-dimensional analogue in Proposition 44.2 after a thorough study of homology groups.

Lemma 23.1. *Let Σ be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$.*

(1) *The topological space Σ minus n open disks is a compact connected topological submanifold of Σ with n boundary components, which are given by the boundaries of the closed disks whose interiors were removed.*
(2) *Σ minus n open disks is a submanifold of the smooth manifold Σ, in particular it is a smooth manifold in its own right.*
(3) *The diffeomorphism type of the smooth manifold Σ minus n open disks does not depend on the choice of the disks.*

SKETCH OF PROOF.

(1),(2) Let $\varphi_1, \ldots, \varphi_n \colon \overline{B}^2 \to \Sigma$ be smooth embeddings with disjoint images. Note that we obtain from applying Proposition 8.2 (2) iteratively altogether n times that $\Sigma \setminus (\varphi_1(B^2) \cup \cdots \cup \varphi_n(B^2))$ is a compact connected submanifold of Σ. In particular we see that it is a smooth and topological manifold in its own right. The fact that the boundary, as a topological manifold, is precisely given by $\varphi_1(S^1), \ldots, \varphi_n(S^1)$ follows from Propositions 6.27 and 18.18.
(3) This statement follows easily from Theorem 8.36. ■

This leads us to the following notation:

Notation. Given $g, n \in \mathbb{N}_0$ we write

$$\Sigma_{g,n} \;:=\; \text{the surface of genus } g \text{ minus } n \text{ open disks.}$$

Furthermore for $g \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we write

$$N_{g,n} \;:=\; \text{the non-orientable surface of genug } g \text{ minus } n \text{ open disks.}$$

Sometimes we refer to g as the *genus* of $\Sigma_{g,n}$ and also of $N_{g,n}$.

Examples.

(1) The surface $\Sigma_{0,2}$ is the result of removing two open disjoint disks from the 2-sphere. It is pretty elementary to show that the resulting smooth manifold is diffeomorphic to the "cylinder" $[0,1] \times S^1$. We refer to Figure 418 for an illustration of this statement.

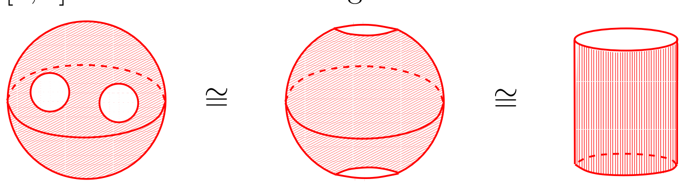


FIGURE 418

(2) A smooth manifold that is diffeomorphic to S^2 minus three open disks is often called a *pair of pants*. We refer to Figure 419 for an illustration which hopefully justifies the name. As we will see in Exercise 23.8, pairs of pants can be viewed as a "building block" for almost all compact connected 2-dimensional smooth manifolds.
(3) We saw in Figure 147 that the result of removing an open disk from $\mathbb{RP}^2$ is the Möbius band, i.e. $N_{1,1}$ is the Möbius band.

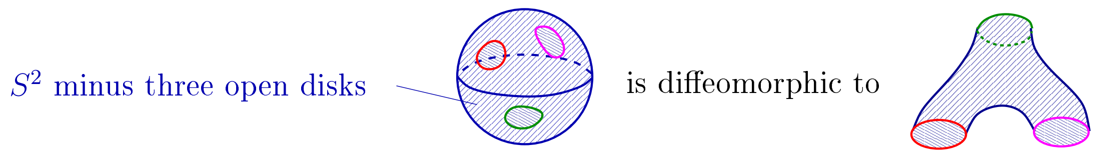


FIGURE 419

The following lemma shows that the fundamental groups of our new friends are rather dull and uninformative.

Lemma 23.2.

(1) *Given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}$ the surface $\Sigma_{g,n}$ admits a deformation retract to a topological graph of Euler characteristic $2-2g-n$. In particular $\pi_1(\Sigma_{g,n})$ is a free group on $2g+n-1$ generators.*

(2) *Given $k \in \mathbb{N}$ and $n \in \mathbb{N}$ the surface $N_{k,n}$ admits a deformation retract to a topological graph of Euler characteristic $2-k-n$. In particular $\pi_1(N_{k,n})$ is a free group on $k+n-1$ generators.*

Remark. Lemma 23.2 can be viewed as a generalization of the fact, essentially shown in Lemma 20.9, that $\Sigma_{g,1}$ admits a deformation retraction to a wedge Γ of $2g$ circles, which is of course a topological graph of Euler characteristic $1-2g$.

SKETCH OF PROOF.

(1) We denote by $C_1,\dots,C_n$ the boundary components of $\Sigma_{g,n}$. Let Γ be the topological space that is given by the wedge of $2g$-circles (which is obtained from identifying as usual the edges of the $4g$-gon pairwise), together with the boundary components $C_1,\dots,C_{n-1}$ and together with edges $e_1,\dots,e_{n-1}$ that connect the wedge point to $C_1,\dots,C_{n-1}$. The deformation retraction from $\Sigma_{g,n}$ to Γ is given by pushing the remaining boundary component C_n "inward". We illustrate this in Figure 420. (This is one of the rare occasions where we provide a "proof by picture". It is of course possible to give a completely rigorous proof, but one does not gain anything from doing so.) We equip Γ with the obvious structure of a topological graph which has $n = 1+(n-1)$ vertices and $2g+2(n-1)$ edges. Note that $\chi(\Gamma) = 2-2g-n$. Finally we see that

$$\pi_1(\Sigma_{g,n}) \underset{\substack{\uparrow \\ \text{by the above together with Lemma 18.14 and Proposition 18.16}}}{\cong} \pi_1(\Gamma) \cong \pi_1\left(\begin{array}{c}\text{connected topological graph } \Gamma \\ \text{with } \chi(\Gamma)=2-2g-n.\end{array}\right) \underset{\substack{\uparrow \\ \text{by Proposition 20.5}}}{\cong} \begin{array}{c}\text{free group on} \\ 2g+n-1 \text{ generators}\end{array}$$

(2) The proof of this statement is almost identical to the proof of the first statement. ■

In the following we will see that we can "recover" the closed surfaces Σ_g and N_g from any $\Sigma_{g,n}$ respectively any $N_{g,n}$. The key to doing so is the following definition.

Definition. Let M be a topological manifold. Let "$\sim$" be the equivalence relation on M that is generated by $x \sim y$ whenever x and y lie on the same component of ∂M. We write $\widehat{M} := M/\sim$.

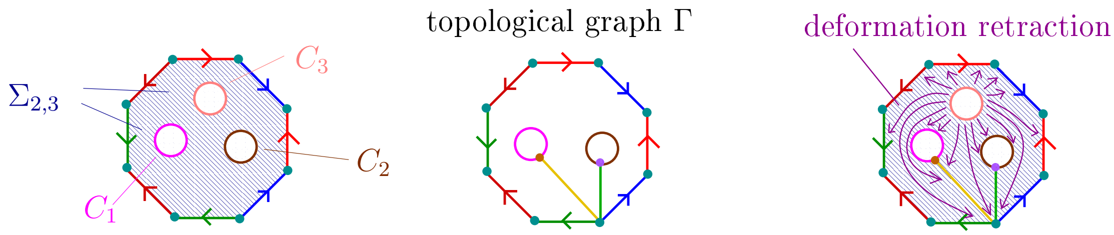


FIGURE 420. Illustration for the proof of Lemma 23.2.

The following lemma shows in particular that $\widehat{\Sigma_{g,n}}$ is homeomorphic to Σ_g and that $\widehat{N_{g,n}}$ is homeomorphic to N_g.

Lemma 23.3. *Let F be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$. If we set M to be F minus n open disks, then $\widehat{M}$ is homeomorphic to F.*

PROOF $(*)$. Let $\varphi_1, \dots, \varphi_n \colon \overline{B}^2 \to F$ be smooth embeddings with disjoint images. We need to show that for $M := F \setminus (\varphi_1(B^2) \cup \dots \cup \varphi_n(B^2))$ there exists a homeomorphism $\widehat{M} \to F$. By Theorem 8.32 we can extend the smooth embeddings $\varphi_1, \dots, \varphi_n$ to smooth embeddings $\overline{B}^2_2(0) \to F$ with disjoint images. We consider the following map:[398]

$$\begin{aligned} \Theta \colon \widehat{M} &\to F \\ P &\mapsto \begin{cases} \varphi_i(Q \cdot 2 \cdot (\|Q\| - 1)), & \text{if } P = [\varphi_i(Q)] \text{ for some } i \in \{1, \dots, n\} \text{ and } Q \in \overline{B}^2_2(0) \setminus B^2, \\ P, & \text{otherwise.} \end{cases} \end{aligned}$$

It follows from Lemmas 3.22 and 2.35 (2) that Θ is continuous. Note that by Lemma 2.40 we know that $\widehat{M}$ is compact. Now it follows almost immediately from Proposition 2.43 (3) that the given map $\Theta \colon \widehat{M} \to F$ is indeed a homeomorphism. ■

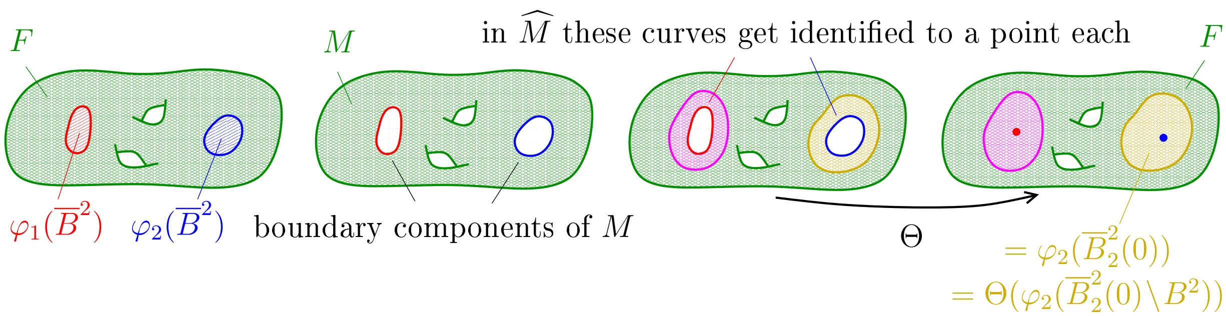


FIGURE 421. Illustration for the proof of Lemma 23.3.

23.2. **The classification theorem for surfaces.** The following theorem achieves Goal (I) that we set out on page 668.

Theorem 23.4. (Surface Classification Theorem) *Every compact connected non-*

[398]Note that discretely we are using Lemma 23.1 (1) which says that the boundary of M, viewed as a topological manifold, is indeed what we think it should be, namely it is given by $\varphi_1(S^1) \sqcup \dots \sqcup \varphi_n(S^1)$.

empty 2-dimensional topological manifold is homeomorphic to either

(1) *the surface* $\Sigma_{g,n}$ *for* unique $g, n \in \mathbb{N}_0$ *or*

(2) *to the surface* $N_{k,n}$ *for* unique $k \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$.

Furthermore, if we are given a smooth manifold, then we can upgrade "homeomorphic" to diffeomorphic.

Remark. The surfaces Σ_g, and thus also the surfaces $\Sigma_{g,n}$, come with a canonical orientation and each of these surfaces admits an orientation-reversing self-diffeomorphism given by a reflection. Thus, if we are given a compact *oriented* connected 2-dimensional smooth manifold, then we can in fact find an *orientation-preserving* diffeomorphism to the corresponding $\Sigma_{g,n}$.

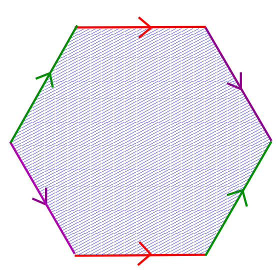
Theorem 23.4 leads us to the following definition.

Definition. Let M be a compact connected non-empty 2-dimensional topological manifold. By the Surface Classification Theorem 23.4 there exists a unique $g \in \mathbb{N}_0$ such that M is homeomorphic to some $\Sigma_{g,n}$ or to some $N_{g,n}$. We refer to g as the *genus* genus(M) *of* M.

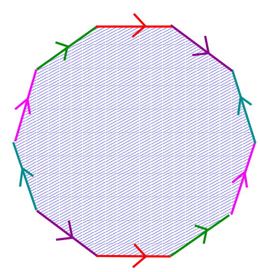
At the end of the section we will state a slightly more precise version of the Surface Classification Theorem 23.4 that gives us Goal (II) from page 668. Before we talk about the proof of the Surface Classification Theorem 23.4 let us discuss a few examples.

Examples.

(1) At least for closed orientable connected 2-dimensional manifolds it is perhaps tempting to dismiss the Surface Classification Theorem 23.4 (1) as "obvious". After all, what other closed orientable connected 2-dimensional smooth manifold could there possibly be? It is thus instructive to consider Figure 422. It shows a hexagon and a decagon where *opposite sides* are identified. The quotient spaces are closed orientable 2-dimensional smooth manifolds.[399] To which of the standard surfaces are they diffeomorphic to? We will come back to this question on page 1364.



hexagon with opposite sides identified



dodecagon with opposite sides identified

FIGURE 422

[399]Indeed, basically the same proof that showed that the $4g$-gon, with appropriate sides identified, is a closed orientable 2-dimensional smooth manifold also shows that the topological spaces shown in Figure 422 are closed orientable 2-dimensional smooth manifolds.

(2) Let $g \in \mathbb{N}$. We consider the surface Σ_g of genus g. The connected sum $\Sigma_g \# \mathbb{RP}^2$ contains a Möbius band as a submanifold of codimension 0,[400] hence $\Sigma_g \# \mathbb{RP}^2$ is non-orientable. It follows from the Surface Classification Theorem 23.4 that $\Sigma_g \# \mathbb{RP}^2$ is diffeomorphic to $k \cdot \mathbb{RP}^2$ for some unique $k \in \mathbb{N}$. In Exercise 23.2 we will determine k as a function of g.

The Surface Classification Theorem 23.4 is an immediate consequence of the following three theorems.

Theorem 108.7. *Let Σ be a compact connected non-empty 2-dimensional smooth manifold.*

(1) *If Σ is orientable, then Σ is diffeomorphic to the surface $\Sigma_{g,n}$ for some $g, n \in \mathbb{N}_0$.*
(2) *If Σ is non-orientable, then Σ is diffeomorphic to the surface $N_{k,n}$ for some $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.*

Theorem 23.5. *Every compact 2-dimensional topological manifold admits a smooth structure.*

Theorem 23.6. *The topological spaces $\Sigma_{g,m}$ and $N_{h,n}$ are pairwise non-homeomorphic.*

We now turn to the discussion of the proofs of the three theorems.

Proof of Theorem 108.7. We provide the proof of Theorem 108.7 in Chapter 108. Actually the proof is not as advanced as it may sound. The only ingredient one needs is the notion and existence of a handle decomposition which are given in Chapter 104. We will also sketch an alternative approach to proving Theorem 108.7 in Exercise **??**. Also note that there are several other approaches to proving Theorem 108.7. For example [**Mata02**, Chapter 5.2] and [**Hirs76**, Chapter 9.3] give a proof using “Morse theory”.[401] ■

Proof of Theorem 23.5. Let M be a compact 2-dimensional topological manifold. There are two approaches to showing that M admits a smooth structure.

(1) In 1922 Tibor Radó [**Rad26**] showed that M is homeomorphic to a finite 2-dimensional simplicial complex. Alternative proofs are given in [**Moi77**, p. 60], [**DM68**, Corollary 1], [**Tn92**, Theorem 4.1], [**MTh01**, Theorem 3.1.1], [**AS60**, p. 107] and [**GX13**, Appendix E]. A sketch of the proof is provided in [**Lee00**, Theorem 5.12].[402]

We will give the definition of a simplicial complex on page 1497. For the purpose of this proof it suffices to know that this means that M is homeomorphic to a topological space that is obtained from gluing finitely many triangles linearly along their sides.

Recall that in Proposition 6.21 we showed that the surface $E_{2g}/\sim$ of genus g admits a smooth structure. The proof can be modified without too many problems

[400]This follows from the observation that $\Sigma_g \# \mathbb{RP}^2$ contains $\mathbb{RP}^2 \setminus \frac{1}{2}B^2$, which by Figure 147 is a Möbius band.

[401]We will provide an introduction to Morse Theory in Chapter 104 but we will not discuss how Morse theory can be used directly to prove Theorem 108.7. We will only use Morse theory indirectly, since it allows us to study smooth manifolds via handle decompositions, which is the key to our proof of Theorem 108.7.

[402]The key ingredient to the proof is the Schönflies Theorem 50.3 which we will formulate later, but which we will not prove.

to show that any compact 2-dimensional topological manifold that admits a simplicial structure, also admits a smooth structure.

(2) In [**Hat13**, Theorem A] a direct proof is given for the fact that M admits a smooth structure. Note that this approach does not rely on the Schönflies Theorem 50.3. ■

Since we did not really prove the first two theorems we should at least provide full details for the third theorem. The idea is to find a complete set of invariants to distinguish the $\Sigma_{g,m}$ and $N_{h,n}$ pairwise. The first invariant we want to use is of course the (abelianization) of the fundamental group. But since this invariant is is not enough we have to look for further invariants. One idea would be to use the number of boundary components. But as we saw in Figure 396, even this extra information is not enough. To resolve this conundrum let us recall the following notation from page 670.

Notation. Let M be a topological manifold. Let "$\sim$" be the equivalence relation on M that is generated by $x \sim y$ whenever x and y lie on the same component of ∂M. We write $\widehat{M} := M/\sim$.

The following proposition contains all invariants that we will need.

Proposition 23.7. *For $g \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we have the following table of invariants:*

X	Σ_g	$\Sigma_{g,m}$ *with* $n \geq 1$	N_k	$N_{k,n}$ *with* $n \geq 1$
isomorphism type of $\pi_1(X)_{\text{ab}}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g+m-1}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{k+n-1}$
number of boundary components	0	n	0	n
isomorphism type of $\pi_1(\widehat{X})_{\text{ab}}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}_2.$

Proof. The proposition is an immediate consequence of Propositions 22.3 and 22.7, together with Lemmas 23.1, 23.2 and 23.3. ■

Now we can prove Theorem 23.6.

Proof of Theorem 23.6. It follows immediately from Proposition 23.7 that the $\Sigma_{g,m}$ and $N_{h,n}$ are pairwise non-homeomorphic. For completeness we point out that in Exercise 48.15 we will use (relative) homology groups to give an alternative proof of Theorem 23.6. ■

Proposition 23.7 also gives us a practical way to determine whether two surfaces are homeomorphic respectively diffeomorphic. This proposition also gives a hopefully satisfying answer to Goal (II) from page 668.

Proposition 23.8. *Let M and N be two compact connected non-empty 2-dimensional topological* (*respectively smooth*) *manifolds. The following are equivalent:*

(1) *M and N are homeomorphic* (*respectively diffeomorphic*),

(2) (a) $\pi_1(M)_{\text{ab}} \cong \pi_1(N)_{\text{ab}}$,

(b) *M and N have the same number of boundary components,*

(c) $\pi_1(\widehat{M})_{\text{ab}} \cong \pi_1(\widehat{N})_{\text{ab}}$.

In the smooth setting we can replace (c) *by*

(c') *M and N have the same orientability.*

Remark.

(1) Only much later, in Proposition 86.11, will we see that homeomorphisms between smooth manifolds preserve orientability. If we factor this proposition in, then we can replace (c) by (c') also in the topological setting.
(2) In Proposition 31.17 we will see that the fundamental group of the surface Σ_g is not isomorphic to the free group on $2g$ generators. Once we know this fact, one can replace (a) and (b) by $\pi_1(M) \cong \pi_1(N)$.

PROOF. In the topological setting this proposition follows immediately from Proposition 23.7 and the Surface Classification Theorem 23.4. In the smooth setting the proposition follows from the above, together with the fact that on page 300 we showed that the $\Sigma_{g,m}$ are orientable and that in Lemma 6.48 we showed that the $N_{k,n}$ are non-orientable. ■

We conclude this discussion of the classification theorem with a long remark on various other approaches to giving the topological classification of 2-dimensional topological manifolds.

Remark. The Surface Classification Theorem 23.4 says in particular that every compact connected non-empty 2-dimensional topological (respectively smooth) manifold is homeomorphic (respectively diffeomorphic) to one of the $\Sigma_{g,n}$ or one of the $N_{k,n}$. Basically all proofs in the literature that deal with the purely topological setting use the fact, mentioned on page 673, that a 2-dimensional topological manifold admits a simplicial structure. The idea then is to modify the simplicial structure to turn it into a simplicial structure of one of $\Sigma_{g,n}$ or $N_{k,m}$. The traditional account of this proof is given in [**ST34, ST80**] or alternatively in [**Mass91**, Theorem 5.1]. A variation on this proof, which is usually referred to as Conway's Zip Proof, is outlined in [**FW99, Wild10**], with full details given in [**Gle**] and [**SchubJ17**]. Another proof is given in [**Tn92**] and [**Meie**]. There is also a proof due to Eric Zeeman, see [**Ze66, Put2**]. The proof given in [**Arm88**, Chapter 7] is somewhat different, its basic idea is related to the approach taken in Exercise **??**. Finally we point to [**GX13**] for a book that is completely dedicated to the proof of the Surface Classification Theorem 23.4 in the topological setting.

By now we might have the impression that we have a solid understanding of surfaces. The following question provides a good test for our true understanding.

Question 23.9. *Let M be the surface of genus $g \in \mathbb{N}_0$ and let $p\colon \widetilde{M} \to M$ be a covering of finite degree d such that $\widetilde{M}$ itself is connected. By Proposition 17.1 and Lemma 16.3 we know that $\widetilde{M}$ is a closed orientable connected 2-dimensional smooth manifold. Can we determine the genus of $\widetilde{M}$ in terms of g and d?*

We will answer this question later on in Proposition 31.6. It is worth trying to come up with an answer with our current knowledge.

23.3. **Non-compact 2-dimensional manifolds.** Just for kicks we also want to consider *non-compact* 2-dimensional manifolds. A short moment's thought there is a plethora of examples:

(1) Given any connected 2-dimensional smooth manifold Σ and given any non-empty closed discrete subset $A \subset \Sigma$ one can easily show that the complement $\Sigma \setminus A$ is in fact a non-compact connected 2-dimensional smooth manifold. For example, this shows that $W = \mathbb{R}^2 \setminus \{(n,0) \mid n \in \mathbb{Z}\}$ is a non-compact connected 2-dimensional smooth manifold.

(2) In Figure 423 we show two non-compact 2-dimensional smooth manifolds X and Y for which the name "infinite genus" seems appropriate.

In Exercises 23.14 and Exercise 23.15 we will show, with some effort, that the non-compact 2-dimensional smooth manifolds W, X and Y are pairwise not diffeomorphic. This already shows that the classification of non-compact connected 2-dimensional smooth manifolds can get a little tricky. In fact the classification of non-compact orientable connected 2-dimensional smooth manifolds without boundary was first given by Ian Richards [**Ric63**] in 1963, building on earlier work of Béla Kerékjártó [**Kere23**] from 1923. In [**PM07**] this result was extended to the case of 2-dimensional smooth manifolds with non-empty boundary.

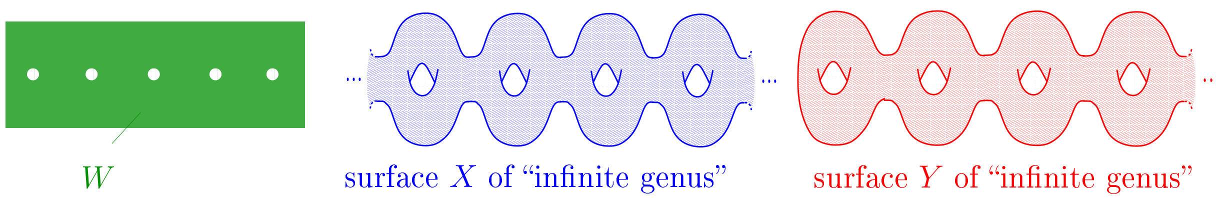


FIGURE 423

Remark. As an aside we point out that the surface Y of infinite genus shown in Figure 423 is sometimes called the *Loch Ness monster*. The reason is shown in Figure 424: the visible part of an infinite Loch Ness monster, together with the surface of the infinite Loch Ness form a 2-dimensional smooth manifold that is diffeomorphic to Y. More information on the mathematical Loch Ness monster and an explicit rigorous construction are given in [**ARM17**].

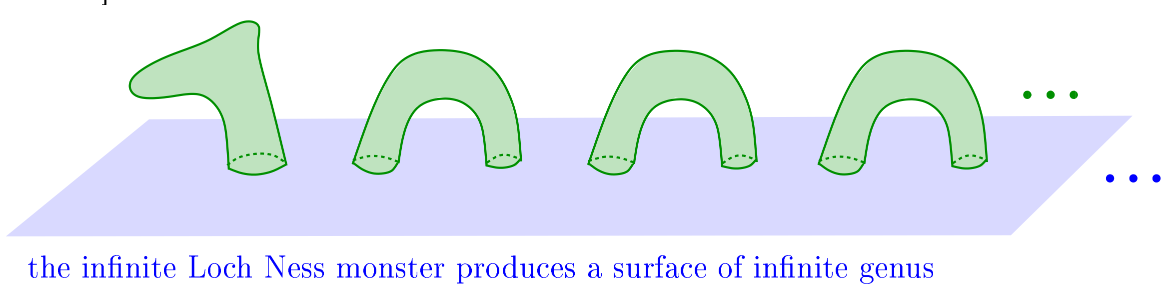


FIGURE 424

23.4. **Retractions onto boundary components of 2-dimensional manifolds.** $(*)$ In this section we will deal with the following question: for which 2-dimensional smooth manifolds is a given boundary component a retract? This discussion is useful but not essential and one can safely skip this section.

Without further apologies let us recall the following definition from page 484.

Definition. Let X be a topological space. We say that a subset A of X is a *retract of* X if there exists a retraction $r\colon X \to A$, i.e. a map with $r(a) = a$ for all $a \in A$.

For the reader's convenience we also recall the statement of Lemma 15.7.

Lemma 15.7. *The circle S^1 is not a retract of $\overline{B}^2$.*

We quickly recall the proof of Lemma 15.7.

Proof. We denote by $i\colon S^1 \to \overline{B}^2$ the inclusion map. Suppose there exists a retraction $r\colon \overline{B}^2 \to S^1$, i.e. a map with $r \circ i = \mathrm{id}_{S^1}$. We obtain the commutative diagram

$$\begin{tikzcd} & \overline{B}^2 \arrow[dr, "r"] & \\ S^1 \arrow[ur, "i"] \arrow[rr, "r\circ i = \mathrm{id}_{S^1}"'] & & S^1 \end{tikzcd}$$

which induces the commutative diagram

$$\begin{tikzcd} & \underbrace{\pi_1(\overline{B}^2,1)}_{=0} \arrow[dr, "r_*"] & \\ \underbrace{\pi_1(S^1,1)}_{\cong\mathbb{Z}} \arrow[ur, "i_*"] \arrow[rr, "(r\circ i)_* = (\mathrm{id}_{S^1})_*"'] & & \underbrace{\pi_1(S^1,1)}_{\cong\mathbb{Z}}. \end{tikzcd}$$

But a commutative diagram as given on the right-hand side cannot exist since an isomorphism of a non-trivial group cannot factor through the trivial group. ■

In light of Lemma 15.7 the following question arises:

Question 23.10. *Let Σ be a 2-dimensional smooth manifold and let C be a boundary component of Σ. Is C a retract of Σ?*

By the Surface Classification Theorem 23.4 it suffices to address Question 23.10 for the smooth manifolds $\Sigma_{g,m}$ and $N_{h,n}$. For notational convenience we will work throughout with $\Sigma = \Sigma_{g,m}$. We will deal with $\Sigma = N_{h,n}$ in Exercise 23.13.

So let us consider $\Sigma = \Sigma_{g,n}$. Lemma 15.7 says that the answer to Question 23.10 is no if $g = 0$ and $n = 1$. The proof of Lemma 15.7 suggests that if we want to answer Question 23.10, then given a boundary component C of $\Sigma_{g,n}$ we should try to understand the inclusion induced map $\pi_1(C) \to \pi_1(\Sigma_{g,n})$. Of course by Lemma 23.2 we know that $\pi_1(\Sigma_{g,n})$ is a free group on $2g + n - 1$ generators. But back then we mostly cared about the isomorphism type of $\pi_1(\Sigma_{g,n})$ and we did not worry about the relationship of $\Sigma_{g,n}$ and its boundary components.

The following lemma gives a new description of $\pi_1(\Sigma_{g,n})$. This description is more complicated in the sense that it is not totally immediate that $\pi_1(\Sigma_{g,n})$ is a free group. On the other hand it makes it clearer how the boundary components of $\Sigma_{g,n}$ interact with $\Sigma_{g,n}$.

Lemma 23.11. *Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. We denote the boundary components of $\Sigma_{g,n}$ by $C_1, \dots, C_n$. The following statements hold:*

(1) *Let $* \in \Sigma_{g,n}$ be the standard base point given by the "vertex" of $E_{2g}/\sim$. We pick an orientation for $\Sigma_{g,n}$. There exists a presentation*

$$\pi_1(\Sigma_{g,n}, *) \;=\; \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_n \mid [x_1,y_1]\cdot\dots\cdot[x_g,y_g]\cdot z_1\cdot\dots\cdot z_n\rangle$$

with the following properties:

(a) *The elements $x_1, y_1, \ldots, x_g, y_g$ are represented by the edges of the $2g$-gon in the "usual way".*

(b) *For any $i \in \{1, \ldots, n\}$ the element z_i is represented by a curve of the form $\alpha * \gamma * \overline{\alpha}$ where α is a path from $*$ to a point on C_i and γ is a path that goes once around C_i in an orientation-preserving way.*[403]

We refer to Figure 425 for an illustration.

(2) *For every $i \in \{1, \ldots, n\}$ the obvious map*

$$\underbrace{\langle x_1, y_1, .., x_g, y_g, z_1, .., \widehat{z_i}, .., z_n\rangle}_{\text{free group}} \to \langle x_1, y_1, .., x_g, y_g, z_1, .., z_n \mid \underbrace{[x_1, y_1] \ldots [x_g, y_g] \cdot z_1 .. z_n}_{\text{relation}}\rangle$$

is an isomorphism. The inverse is induced by the obvious assignments together with

$$z_i \;\mapsto\; z_{i-1}{}^{-1} \cdot \ldots \cdot z_1{}^{-1} \cdot [y_g, x_g] \cdot \ldots \cdot [y_1, x_g] \cdot z_n{}^{-1} \cdot \ldots z_{i+1}{}^{-1}.$$

In particular $\pi_1(\Sigma_{g,n})$ is a free group on $2g + n - 1$ generators.

(3) *If $g \geq 1$ or if $n \geq 2$, then for every $i \in \{1, \ldots, n\}$ the inclusion induced map $\pi_1(C_i) \to \pi_1(\Sigma_{g,n})$ is a monomorphism.*

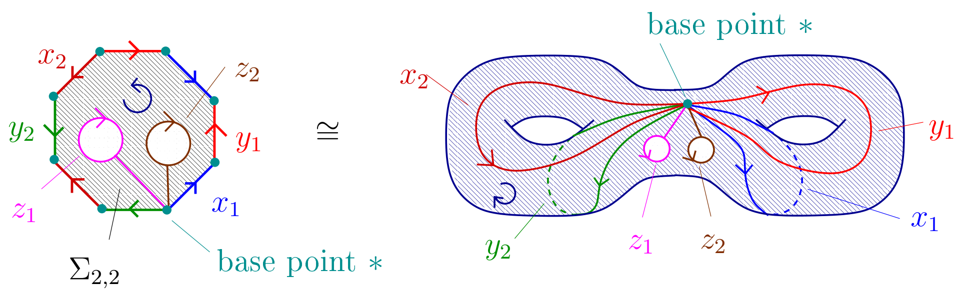


FIGURE 425. Illustration of the curves in Lemma 23.11.

SKETCH OF THE PROOF. Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$.

(1) First note that by Proposition 8.29 we can without loss of generality assume that the base point corresponds to any of the vertices of the $4g$-gon as shown in Figure 426. The following argument is very similar to the argument on the pages 656 and 658 and it relies heavily on the notation from Figure 426. We have

$$\begin{aligned}
\pi_1(\Sigma_{g,n}) &\overset{\substack{\text{Seifert–van Kampen Theorem 22.1}\\ \downarrow}}{=} \pi_1(U) *_{\pi_1(U \cap V)} \underbrace{\pi_1(V)}_{=\{e\}} \overset{\substack{\text{Lemma 21.22}\\ \downarrow}}{\cong} \pi_1(U)/\langle\langle \pi_1(U \cap V)\rangle\rangle \\
&\underset{\substack{\uparrow\\ \text{followings from the deformation retraction shown in Figure 426}\\ \text{together with Lemma 18.14 and Proposition 18.16 (2)}}}{=} \langle x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_n\rangle / \langle\langle [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \cdot z_1 \cdot \ldots \cdot z_n \rangle\rangle.
\end{aligned}$$

[403]Since we picked an orientation for $\Sigma_{g,n}$ each boundary component also inherits an orientation.

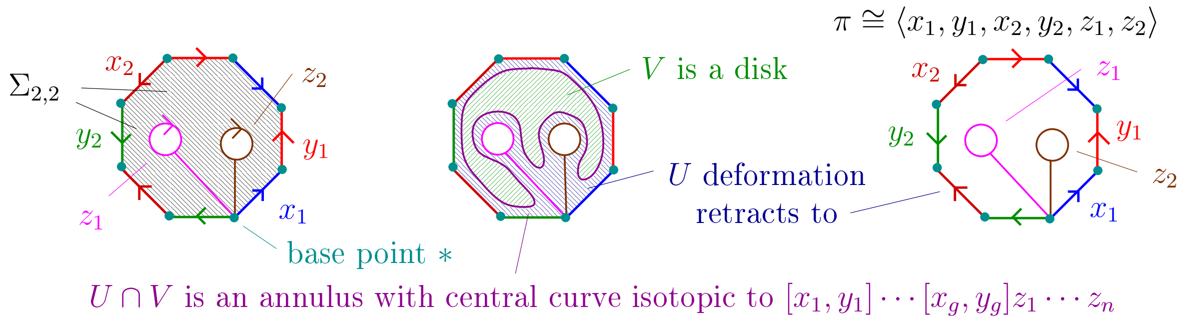


FIGURE 426. Illustration for the proof of Lemma 23.11.

(2) It follows immediately from Lemma **??** (2) that for any $i \in \{1, \dots, n\}$ the obvious map

$$\langle x_1, y_1, .., x_g, y_g, z_1, .., \widehat{z_i}, .., z_n\rangle \to \langle x_1, y_1, .., x_g, y_g, z_1, .., z_n \mid [x_1, y_1]\dots[x_g, y_g]\cdot z_1 .. z_n\rangle$$

is an isomorphism. The map that we give in the inverse direction is evidently a converse.

(3) Now suppose that $g \geq 1$ or that $n \geq 2$. Let $i \in \{1, \dots, n\}$. The image of $\pi_1(C_i)$ in $\pi_1(\Sigma_{g,n})$ is the subgroup generated by z_i. It suffices to show that z_i is an element of infinite order in $\pi_1(\Sigma_{g,n})$. We apply (2) to i. We obtain an isomorphism

$$\begin{array}{rcl} \pi_1(\Sigma_{g,n}) & \to & \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, \widehat{z_i}, \dots, z_n\rangle \\ z_i & \mapsto & {z_{i-1}}^{-1} \cdot \dots \cdot {z_1}^{-1} \cdot [y_g, x_g] \cdot \dots \cdot [y_1, x_1] \cdot \dots {z_n}^{-1} \cdot \dots \cdot {z_{i+1}}^{-1}. \end{array}$$

Since $g \geq 1$ or $n \geq 2$ we see that the element on the right-hand side is represented by a non-empty reduced word. Thus by definition it represents a non-trivial element in a free group. By Lemma 19.13 we know that every non-trivial element of a free group has infinite order. Thus we have shown that z_i is indeed an element of infinite order of $\pi_1(\Sigma_{g,n})$. ■

Finally the following technical lemma can be viewed as a refinement of Lemma 23.2 (1).

Lemma 23.12. *Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$.*

(1) *We denote the boundary components of $\Sigma_{g,n}$ by $C_1, \dots, C_n$. Given any $i \in \{1, \dots, n\}$ there exists a topological graph $G \subset \Sigma_{g,n}$ which contains all C_j with $i \neq j$ and which contains a point on C_j, a deformation retraction $r\colon \Sigma_{g,n} \to G$, a spanning tree $T \subset G$ and a homeomorphism*[404]

$$\Theta\colon G/T \to C_1 \vee \dots \vee \widehat{C_i} \vee \dots \vee C_n \vee \underbrace{S^1 \vee \dots \vee S^1}_{2g \text{ copies}}$$

such that the following diagram commutes:

$$\begin{tikzcd} & C_j \arrow[r] \arrow[dl] \arrow[d] & C_1 \vee \ldots \vee \widehat{C_i} \vee \ldots \vee C_n \vee S^1 \vee \ldots \vee S^1 \arrow[d, "\Theta"', "\cong"] \\ \Sigma_{g,n} \arrow[r, "r"] & G \arrow[r, "p"] & G/T. \end{tikzcd}$$

Note that $(p \circ r)_*\colon \pi_1(\Sigma_{g,n}) \to \pi_1(G/T)$ *is an isomorphism by Propositions 18.16 and 18.31 (1).*

(2) *The fundamental group* $\pi_1(\Sigma_{g,n})$ *is a free group on* $2g+n-1$ *generators.*

Given $k \in \mathbb{N}$ *and* $n \in \mathbb{N}$ *completely analogous statements also hold for* $N_{k,n}$*.*

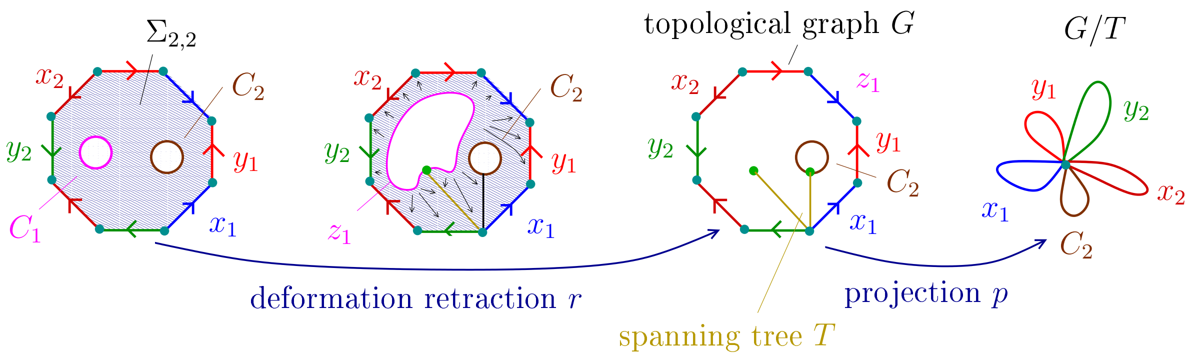


FIGURE 427. Illustration of Lemma 23.12.

SKETCH OF PROOF.

(1) In Figure 427 we show that, starting from any boundary component C_i of $\Sigma_{g,n}$, we can push “inward” to obtain a deformation retraction[405][406] to a subset that is a topological graph G that can be described as

$$G \;=\; \big(C_1 \sqcup \dots \sqcup \widehat{C}_i \sqcup \dots \sqcup C_n \sqcup \underbrace{S^1 \vee \dots \vee S^1}_{2g \text{ copies}}\big) \text{ together with a spanning tree } T.$$

The vertical map to the right is the homeomorphism given by the discussion on page 560. It follows immediately from the construction that the diagram commutes.

(2) This statement follows from (1) together with Lemma 20.4. ■

The proof for $N_{k,n}$ is almost identical to the above proof for $\Sigma_{g,n}$. We leave it to the reader to figure out the precise statements and to write down the proof.

The following lemma gives an answer to Question 23.10.

Lemma 23.13. *Let* $g \in \mathbb{N}_0$*.*

(1) *The unique boundary component of* $\Sigma_{g,1}$ *is not a retract of* $\Sigma_{g,1}$*.*

(2) *For* $k \geq 2$ *every boundary component of* $\Sigma_{g,k}$ *is a retract of* $\Sigma_{g,k}$*.*

PROOF. Let $g \in \mathbb{N}_0$.

(1) The case $g = 0$ was dealt with in Lemma 15.7. Now we assume that $g \geq 1$. We denote by C the unique boundary component of $\Sigma := \Sigma_{g,1}$. We had just seen in Lemma 23.11 that $\pi_1(C) \to \pi_1(\Sigma)$ is a monomorphism if $g \geq 1$. So we cannot apply the same argument as in Lemma 15.7 where we studied the induced map on the level

[404] Here $\widehat{C_i}$ means that we leave out C_i.

[405] We leave it to others with more time and energy to write down a rigorous argument. One possibility would be to apply several times the deformation retractions that we found in Exercise 18.24.

[406] This argument is evidently closely related to the proof of Lemma 20.9.

of fundamental groups. The idea now is to study instead the induced map on the *abelianizations* of the fundamental groups.

We denote by $i\colon C \to \Sigma$ the inclusion map and we suppose that there exists a retraction $r\colon \Sigma \to C$, i.e. a map with $r \circ i = \mathrm{id}_C$. We write $\pi_1(C) = \langle t \rangle$ and we use the isomorphism from Lemma 23.11 (1) to make the identification

$$\pi_1(\Sigma) \;=\; \langle x_1, y_1, \dots, x_g, y_g, z \mid [x_1, y_1] \cdot \dots \cdot [x_g, y_g] \cdot z \rangle$$

where z corresponds to t under the inclusion $C = \partial\Sigma \to \Sigma$. We obtain the commutative diagram

$$\begin{tikzcd} & \Sigma \arrow[dr, "r"] & \\ C \arrow[ur, "i"] \arrow[rr, "r\circ i=\mathrm{id}_C"'] & & C \end{tikzcd}$$

which induces by Proposition 21.20 (6) the commutative diagram

$$\begin{tikzcd} & \pi_1(\Sigma)_{\mathrm{ab}} \arrow[dr, "r_*"] & \\ \underbrace{\pi_1(C)_{\mathrm{ab}}}_{=\langle t\rangle} \arrow[ur, "i_*", "t\mapsto z"'] \arrow[rr, "(r\circ i)_*=\mathrm{id}"'] & & \underbrace{\pi_1(C)_{\mathrm{ab}}}_{=\langle t\rangle}. \end{tikzcd}$$

In $\pi_1(\Sigma)$ we have

$$z \;=\; ([x_1, y_1] \cdot \dots \cdot [x_g, y_g])^{-1} \;\in\; \pi_1(\Sigma).$$

This shows that z lies in the commutator subgroup of $\pi_1(\Sigma)$, i.e. z represents the trivial element in $\pi_1(\Sigma)_{\mathrm{ab}}$. Thus we have shown that the left diagonal map is the zero map. But the bottom horizontal map is an isomorphism of a non-trivial group and the diagram commutes. We have thus obtained a contradiction.

(2) This statement follows immediately from Lemma 23.12 since we can project the wedge of circles onto any circle. Alternatively, a retraction from $\Sigma_{2,n}$ to a boundary component is sketched in Figure 428. The general case is treated the same way.[407] ■

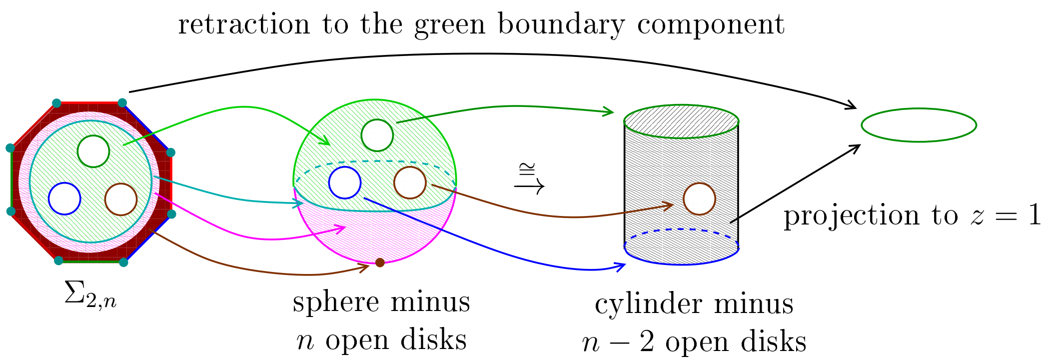


FIGURE 428

23.5. **Curves on surfaces.** In this section we will use the Surface Classification Theorem 23.4 to draw several hopefully interesting consequences on curves on surfaces. We start out with the following lemma.

Lemma 23.14. *Let Σ be a compact connected 2-dimensional smooth manifold. We denote the boundary components of Σ by $C_1, \dots, C_n$. Given any permutation $\sigma \in S_n$ there exists*

[407] The proof is somewhat similar to the proof of Lemma 15.10.

a diffeomorphism f of Σ with $f(C_i) = C_{\sigma(i)}$, $i = 1, \dots, n$. Furthermore, if Σ is oriented, then we can choose f to be orientation-preserving.

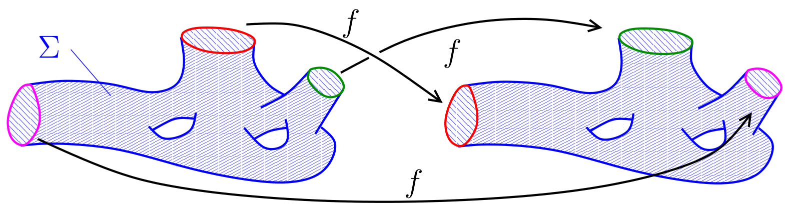


FIGURE 429. Illustration of Lemma 23.14.

PROOF. In the proof we deal with the setting that Σ is orientable. The case that Σ is non-orientable is dealt with in almost the same way. We pick an orientation for Σ and we equip the boundary components $C_1, \dots, C_n$ with the corresponding orientations. Furthermore we pick orientation-reversing diffeomorphisms $\varphi_1, \dots, \varphi_n \colon S^1 \to C_i$ and we consider

$$M \;:=\; (\Sigma \sqcup (\overline{B}^2_1 \sqcup \dots \sqcup \overline{B}^2_n))/\varphi_i(x) \sim x \qquad \text{with } x \in S^1 \text{ and } i \in \{1, \dots, n\}.$$

It follows from Proposition 6.30 that M is a closed oriented connected 2-dimensional smooth manifold with obvious orientation-preserving smooth embeddings $\psi_i \colon \overline{B}^2 \to M$. Now let $\sigma \in S_n$ be a permutation. Note that it follows from Theorem 8.36 that there exists a diffeotopy $F \colon M \times [0,1] \to M$ with $F_0 = \mathrm{id}$ such that for each $i \in \{1, \dots, n\}$ we have $F_1 \circ \varphi_i = \varphi_{\sigma(i)} \colon \overline{B}^2 \to M$. Note that F_1 is orientation-preserving by Lemma 8.6. The desired diffeomorphism is now given by the restriction of F_1 to Σ. ■

We move on to the following general definition.

Definition. Let M be a smooth manifold and furthermore let $N \subset M$ be a closed submanifold of codimension one. We say N is *non-separating* if $M \setminus N$ is connected, otherwise we see that N is *separating*.

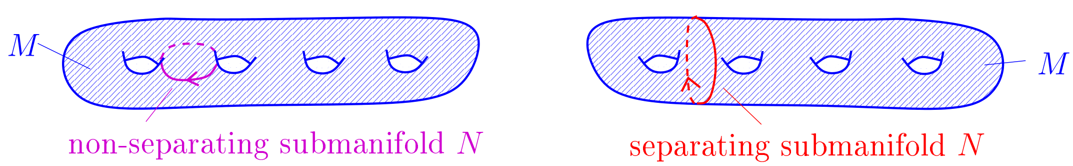


FIGURE 430

Next we recall that on page 360 we introduced the notion of cutting an orientable smooth manifold along a proper codimension-one submanifold. In Proposition 8.17 (4) and Proposition 8.18 we saw that this is operation is the inverse of the operation of gluing two boundary components that we introduced in Proposition 8.15. We will not recall the precise definitions but we hope that Figure 431 will bring back fond memories.

The following lemma explains how gluing and cutting affects surfaces.

Lemma 23.15. *Let Σ be a compact oriented connected 2-dimensional smooth manifold.*

(1) *Let C and D be two distinct boundary components of Σ. Gluing C to D via an orientation-reversing diffeomorphism increases the genus by one and it decreases the number of boundary components by two.*
(2) *Let F be a closed non-separating curve on Σ. Cutting along F decreases the genus by one and it increases the number of boundary components by two.* (*In particular this implies that the genus of Σ is ≥ 1.*)

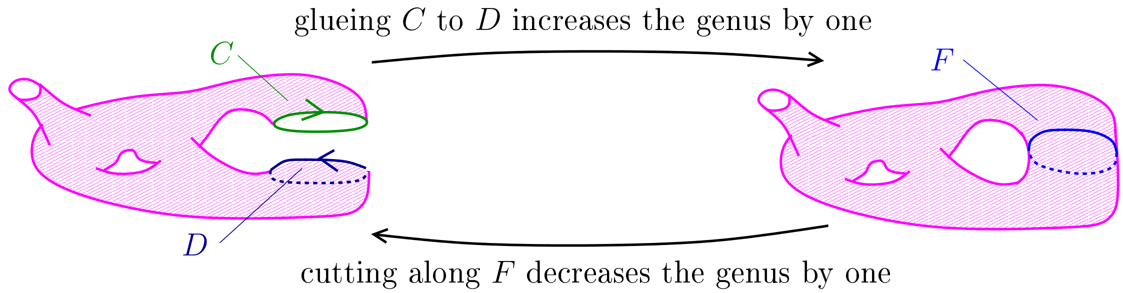


FIGURE 431. Illustration of Lemma 23.15.

SKETCH OF PROOF.

(1) By the Surface Classification Theorem 23.4 we can assume that $\Sigma = \Sigma_{g,n}$ for some $g \in \mathbb{N}_0$ and some $n \in \mathbb{N}_{\geq 2}$. We denote by M the result of gluing the boundary components C and D via an orientation-reversing diffeomorphism. Note that it follows from Proposition 8.15 that M is a compact oriented connected 2-dimensional smooth manifold. In particular it has some genus h. Also note that it follows from Lemma 23.14 that the diffeomorphism type of M does not depend on the choice of the two boundary components C and D.

   It is now elementary, albeit painful, to show clearly that gluing C to D via an orientation-reversing diffeomorphism increases the genus by one. We will deal with a special case in Exercise 23.12. The general case is not much harder. Finally note that evidently the number of boundary components goes down by two.

   For readers who are slightly unhappy about the above proof of the statement that the genus goes up by one we also give an alternative approach. Namely, the genus of M can be read off directly from suitable invariants. For example later on we will use the HNN-Seifert–van Kampen Theorem 26.3 to calculate $\pi_1(M)$ which, together with Proposition 23.7 implies that $h = g + 1$. Alternatively one can use the Euler characteristic, which we will introduce in Chapter 55, together with Lemmas 55.5 and 55.6 to see once again that $h = g + 1$.
(2) Note that if we cut along a non-separating curve, then the resulting 2-dimensional smooth manifold is again connected. Thus we see that Statement (2) is an immediate consequence of Statement (1) together with Proposition 8.18. ■

We move on to the following definition.

Definition. Let Σ be an oriented 2-dimensional smooth manifold and let C and D be two closed oriented curves in Σ. We say C and D are *equivalent* if there exists an orientation-preserving self-diffeomorphism f of Σ that restricts to an orientation-preserving diffeomorphism $f\colon C \to D$.

It is clear that if two curves are equivalent, then either both are separating or both are non-separating. Furthermore if they are both separating, then the results of cutting the 2-dimensional smooth manifold along the curves are diffeomorphic. The following proposition gives us the converse to this observation.

Proposition 23.16. *Let Σ be a compact oriented connected 2-dimensional smooth manifold and let C and D be two closed oriented curves in $\Sigma \setminus \partial\Sigma$.*

1. *If C and D are non-separating, then C and D are equivalent.*
2. *Suppose C and D are separating. If Σ cut along C is diffeomorphic to Σ cut along D, then C and D are equivalent.*

Example. It follows from Proposition 23.16 (1) that the two non-separating curves shown in Figure 432 are actually equivalent. Furthermore it follows easily from Proposition 23.16 (2) and the future Exercise 55.5 that any separating curve on the surface of genus four is equivalent to one of the separating curves shown in Figure 432.

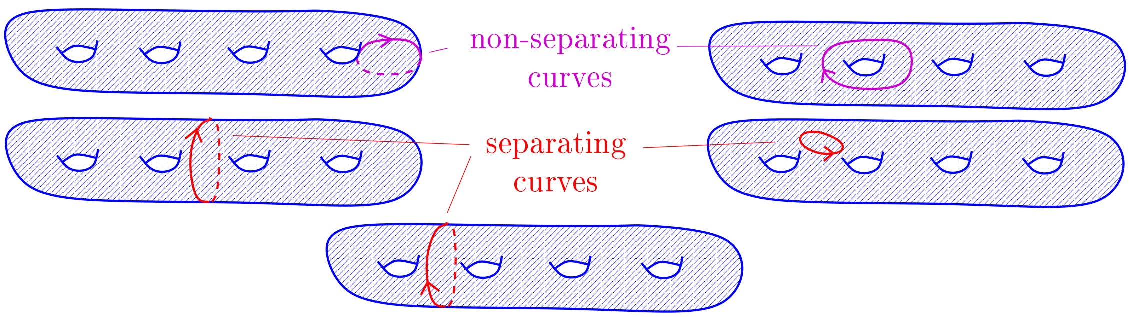


FIGURE 432

Remark. In Exercise 23.17 we will generalize Proposition 23.16 to certain pairs of curves.

SKETCH OF PROOF. Let Σ be a compact oriented connected 2-dimensional smooth manifold.

(1) Let C and D be two closed oriented non-separating curves on Σ. We denote by M respectively N the smooth manifold obtained from Σ by cutting along C respectively D. We denote by $C_\pm$ and $D_\pm$ the boundary components arising from C and D. Since C and D are non-separating it follows from Lemma 23.15 (2) and Lemma 23.14 that there exists an orientation-preserving diffeomorphism $\varphi\colon M \to N$ that restricts to orientation-preserving diffeomorphisms $\varphi\colon C_\pm \to D_\pm$. Using Lemma 18.9 and using a collar neighborhood one can arrange that under the obvious identifications $C_- = C_+$ and $D_- = D_+$ the maps $\varphi\colon C_- \to D_-$ and $\varphi\colon C_+ \to D_+$ agree. As in Proposition 8.15 (7) we see that $\varphi\colon M \to N$ descends to an orientation-preserving diffeomorphism $\overline{\varphi}\colon M/C_- = C_+ \to N/D_- = D_+$. If we combine this diffeomorphism

with the diffeomorphisms $\Sigma \to M/C_- = C_+$ and $\Sigma \to N/D_- = D_+$ from Proposition 8.18 we obtain the desired diffeomorphism.

(2) The argument is similar to the proof of Statement (1), except that we do not need to appeal to Lemma 23.15 (2), instead we can use our hypothesis directly. We leave it to the reader to fill in the details. ■

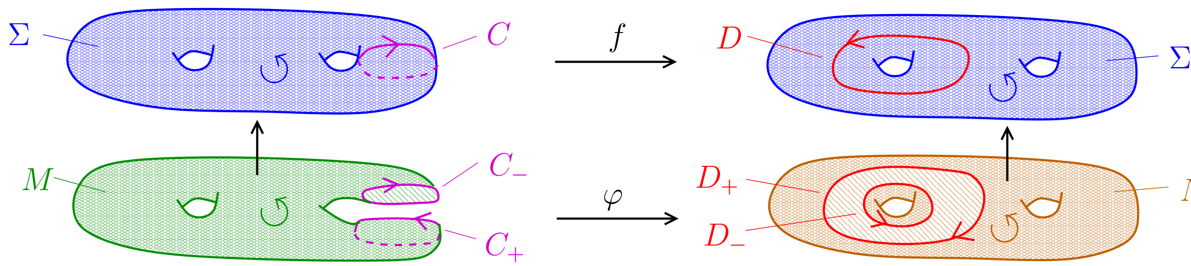


FIGURE 433. Illustration for the proof of Proposition 23.16.

In the remainder of this section we will state a few consequences of Proposition 23.16.

Proposition 23.17. *Let Σ be a compact orientable connected 2-dimensional smooth manifold and let C be a closed oriented curve on Σ.*

(1) *If C is non-separating, then C is not null-homotopic.*
(2) *If C is separating, and if the two components of Σ cut along C are not disks, then C is not null-homotopic.*

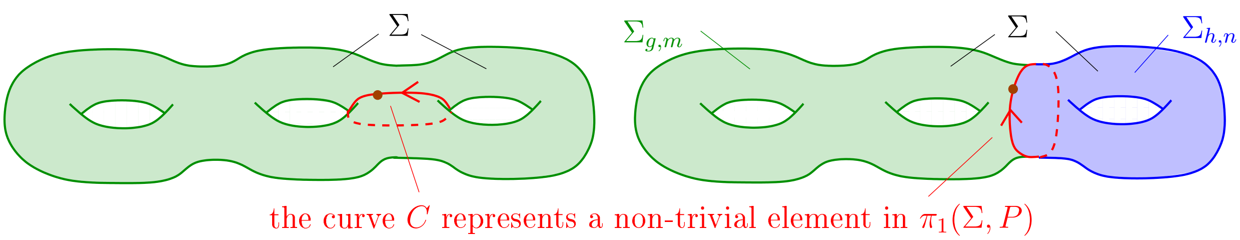


FIGURE 434. Illustration for (the proof of) Proposition 23.17.

PROOF. Let Σ be a compact 2-dimensional smooth manifold and let C be a closed oriented curve on Σ.

(1) First assume that C is non-separating. By the Surface Classification Theorem 23.4 we can assume that $\Sigma = \Sigma_{g,n}$ for some $g \in \mathbb{N}_0$ and some $n \in \mathbb{N}_0$. Since C is non-separating we know by Lemma 23.15 (2) that $g \geq 1$. Next note that by Proposition 23.16 we only need to show that there exists a single non-separating curve on $\Sigma_{g,n}$ that is not null-homotopic. By Lemma 23.11 we have a presentation
$$\pi_1(\Sigma_{g,n}, *) \;=\; \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_n \mid [x_1, y_1] \cdot \dots \cdot [x_g, y_g] \cdot z_1 \cdot \dots \cdot z_n \rangle$$
where each x_i is represented by a non-separating curve. But the x_i represent non-trivial elements in the abelianization of $\pi_1(\Sigma_{g,n})$, in particular the corresponding curves are not null-homotopic.

(2) Now we suppose that C is separating. We cut Σ along C and we obtain two connected surfaces. By the Surface Classification Theorem 23.4 we can assume that the two

surfaces are of the form $\Sigma_{g,m}$ and $\Sigma_{h,n}$. By hypothesis neither surface is a disk, so we have $g > 0$ or $m > 1$ and we have $h > 0$ or $n > 1$. To simplify the notation we now assume that $g = 1$ and $h = 1$ and $m = 2$ and $n = 2$. We perform the following calculation:

$$\begin{aligned}\pi_1(\Sigma) &\underset{\text{Seifert–van Kampen Theorem 22.2}}{\cong} \pi_1(\Sigma_{1,2}) *_{\pi_1(C)} \pi_1(\Sigma_{1,2}) \underset{\text{by Lemma 23.11, here } C \text{ corresponds to } c_1 \text{ and to } z_1^{-1}}{=} \langle a, b, c_1, c_2 \,|\, [a,b]\cdot c_1\cdot c_2\rangle *_{\pi_1(C)} \langle x, y, z_1, z_2 \,|\, [x,y]\cdot z_1\cdot z_2\rangle \\ &\underset{\text{by Lemma 21.23}}{=} \langle a, b, c_1, c_2, x, y, z_1, z_2 \,|\, [a,b]\cdot c_1\cdot c_2, [x,y]\cdot z_1\cdot z_2, c_1 = z_1^{-1}\rangle\end{aligned}$$

where the curve C corresponds to $c_1 = z_1^{-1}$. We need to show that c_1 represents a non-trivial element in $\pi_1(\Sigma)$. In our present setting we see that c_1 represents a non-trivial element in the abelianization, but this is *not* the case if $m = 1$. So we need to come up with an alternative argument to show that c_1 is non-trivial. Note that it follows from Lemma 21.13 that there exists a homomorphism

$$\varphi\colon \langle a, b, c_1, c_2, x, y, z_1, z_2 \mid [a,b]\cdot c_1\cdot c_2, [x,y]\cdot z_1\cdot z_2, c_1 = z_1^{-1}\rangle \;\to\; \langle u, v\rangle$$

with[408]

$$a \mapsto u, \quad b \mapsto v, \quad c_1 \mapsto [v,u], \quad c_2 \mapsto e, \quad x \mapsto v, \quad y \mapsto u, \quad z_1 \mapsto [u,v], \quad z_2 \mapsto e.$$

Finally note that $\varphi(c) = [u,v]$ is a non-trivial element of the free group $\langle u, v\rangle$. It follows that C is not null-homotopic. ■

Remark. What is the "right" proof for a statement is often a matter of taste. For example an alternative, arguably more conceptual, proof for Proposition 23.17 (1) can be obtained by the combination of the Surface Classification Theorem 23.4 together with the future HNN-Seifert–van Kampen Theorem 26.3, Lemma 23.11 (3) and Proposition 26.2. Similarly an alternative proof for Proposition 23.17 (2) can be obtained by the combination of the Surface Classification Theorem 23.4 with Seifert–van Kampen Theorem 22.2, Lemma 23.11 (3) and Proposition 21.24.

We conclude this section with the following consequence of Proposition 23.17.

Proposition 23.18. *Let Σ be a compact orientable 2-dimensional smooth manifold. Furthermore let $f\colon S^1 \to \Sigma$ be a smooth embedding. If f is null-homotopic, then there exists a smooth embedding $F\colon \overline{B}^2 \to \Sigma$ with $F|_{S^1} = f$.*

Proof. Let $f\colon S^1 \to \Sigma$ be a smooth embedding that is null-homotopic. By Proposition 23.17 (1) we know that $f(S^1)$ is separating. Let A and B be the two components of Σ cut along $f(S^1)$. By Proposition 23.17 (2) we know that at least one of A and B is diffeomorphic to a disk. Let us assume that A is diffeomorphic to $\overline{B}^2$. It follows easily from Lemma 18.9 (3), see also Proposition 30.1 (2), that there exists such a diffeomorphism $F\colon \overline{B}^2 \to A$ with $F|_{S^1} = f$. ■

[408] Recall that $[s,t]$ denotes the commutator of s and t, i.e. we have $[s,t] = sts^{-1}t^{-1}$. The verification that the relations are indeed satisfied relies on the trivial but useful observation that $[s,t]^{-1} = [t,s]$.

Exercises for Chapter 23.

Exercise 23.1. Determine the genus of the surface M shown in Figure 435.

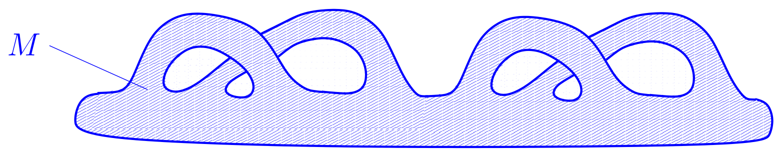


FIGURE 435. Illustration for Exercise 23.1.

Exercise 23.2. Let $g \in \mathbb{N}$. By the Surface Classification Theorem 23.4 we know that $\Sigma_g \# \mathbb{R}P^2$ is diffeomorphic to $k \cdot \mathbb{R}P^2$ for some unique $k \in \mathbb{N}$. Determine k as a function of g.

Exercise 23.3. Let X be the Möbius band minus two disks. We denote by A, B, C the three boundary components of X. We glue a torus minus one open disk and a Möbius band to the boundary components of X. By Proposition 8.15 we obtain a compact 2-dimensional smooth manifold.

(a) Do the resulting diffeomorphism types depend on the gluing?
(b) How do the resulting smooth manifolds fit into the classification scheme of the Surface Classification Theorem 23.4?

We refer to Figure 436 for an illustration.

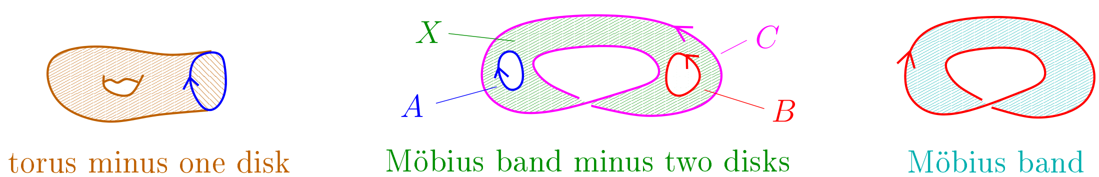


FIGURE 436. Illustration of Exercise 23.3.

Exercise 23.4. Recall that in Figure 147 we saw that $N_{1,1}$ is a Möbius band. In particular we can view $N_{0,1}$ as a submanifold of $\mathbb{R}^3$.

(a) Is $N_{2,1}$ diffeomorphic to a submanifold of $\mathbb{R}^3$?
(b) Show that $N_{3,1}$ is diffeomorphic to a submanifold of $\mathbb{R}^3$.
(c) For which $k \in \mathbb{N}$ is $N_{k,1}$ diffeomorphic to a submanifold of $\mathbb{R}^3$?

Exercise 23.5. Let F be the sphere minus k open disks. (Alternatively, let F be the disk $\overline{B}^2$ minus $k-1$ open disks.) We denote by $C_1, \ldots, C_k$ the boundary components of F. We pick a base point $x_0 \in F$. Furthermore for each $i \in \{1, \ldots, k\}$ we pick a path γ_i from a point on C_i to x_0. Finally let $\Gamma \subsetneq \pi_1(F, x_0)$ be a proper normal subgroup. Show that there exists an $i \in \{1, \ldots, k\}$ such that C_i, viewed as an element in $\pi_1(F, x_0)$ via the path γ_i, does not lie in Γ.
Hint. You might want to start out with your preferred base point x_0 and your preferred paths $\gamma_1, \ldots, \gamma_k$, e.g. as shown in Figure 437.

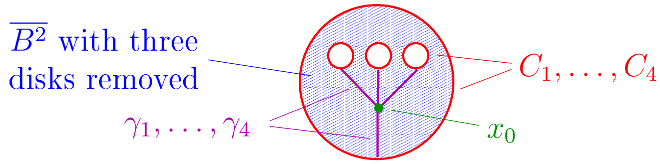


FIGURE 437. Illustration of Exercise 23.5.

Exercise 23.6. Let Σ be the torus minus one disk. We denote by $P \in \partial\Sigma$ a base point. Furthermore let x, y, z be the curves shown in Figure 438. We denote by x, y, z also the elements in $\pi_1(\Sigma, P)$ represented by these curves.

(a) Convince yourself of the fact that $\pi_1(\Sigma, P) = \langle x, y, z \,|\, [x, y] = z \rangle$.
Remark. This statement is closely related to the statement of Lemma 23.11, but it takes some effort to get the orientations of the curves right.

(b) Show that $z \in \pi_1(\Sigma, P)$ is not a proper power, i.e. show that there is no $w \in \pi_1(\Sigma, P)$ such that $z = w^d$ for some $d \in \mathbb{N}_{\geq 2}$.

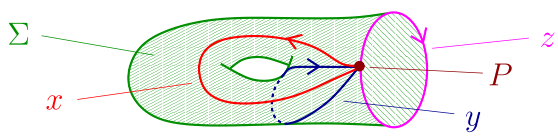


FIGURE 438. Illustration of Exercise 438.

Exercise 23.7. What is the genus of the surface shown in Figure 439?

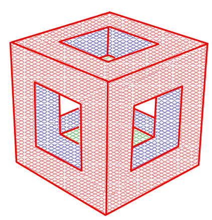


FIGURE 439. Illustration of Exercise 23.7.

Exercise 23.8. We say a 2-dimensional smooth manifold M admits a *pair of pants decomposition* if there exist submanifolds $P_1, \ldots, P_k$ such that each P_i is a pair of pants and such that for each $i \neq j$ the intersection $P_i \cap P_j$ is a (possibly empty) union of boundary components of P_i and also of P_j.

(a) Determine all $g, n \in \mathbb{N}_0$ such that $\Sigma_{g,n}$ admits a pair of pants decomposition.
(b) Determine all $g \in \mathbb{N}$ and $n \in \mathbb{N}_0$ for which $N_{k,n}$ admits a pair of pants decomposition.

Exercise 23.9. Let Σ be a compact connected 2-dimensional smooth manifold and let C and D be two closed oriented non-separating curves in $\Sigma \setminus \partial\Sigma$. In Proposition 23.16 we showed that if Σ is orientable, then there exists a self-diffeomorphism φ of Σ with $\varphi(C) = D$.

(a) Show that this conclusion does not necessarily hold if we drop the hypothesis that Σ is orientable.

(b) Where in the proof of Proposition 23.16 did we really make use of the hypothesis that Σ is orientable?

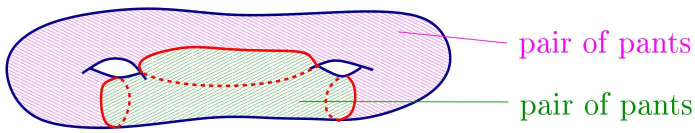


FIGURE 440. Illustration for Exercise 23.8.

Exercise 23.10. We consider the 2-dimensional smooth manifold M together with the map $\varphi\colon S^1 \to M$ that is shown in Figure 441. Show that φ is not homotopic to a smooth embedding.

Remark. At this stage it is a challenging, but by now means impossible, task to provide a completely rigorous proof.

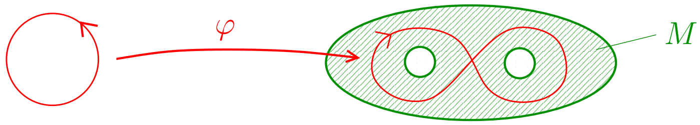


FIGURE 441. Illustration for Exercise 23.10.

Exercise 23.11.

(a) Let $k \in \mathbb{N}$. By Proposition 17.3 we know that the non-orientable surface N_k admits an essentially unique 2-fold covering $\Sigma \to N_k$ that is orientable. What is the genus of Σ?
Hint. First deal with the cases $N_1 \cong \mathbb{R}\mathrm{P}^2$ and $N_2 \cong \mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$. Given $k \in \mathbb{N}_{\geq 2}$ you can write N_k as the connected sum of an orientable smooth manifold with $\mathbb{R}\mathrm{P}^2$ or with $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$.

(b) Let $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. By Proposition 17.3 we now know that the non-orientable surface $N_{k,m}$ admits an essentially unique 2-fold covering $\Sigma \to N_{k,m}$ that is orientable. What is the genus of Σ and what is the number of boundary components of Σ?

Hint. In Exercise 55.2 we will give a different, perhaps less ad hoc approach to dealing with these two questions.

Exercise 23.12. Let F be the torus minus two open disks. We pick an orientation-reversing diffeomorphism $\varphi\colon C \to D$ between the two boundary components. Show that $F/c \sim \varphi(c)$ is diffeomorphic to the surface of genus wo.

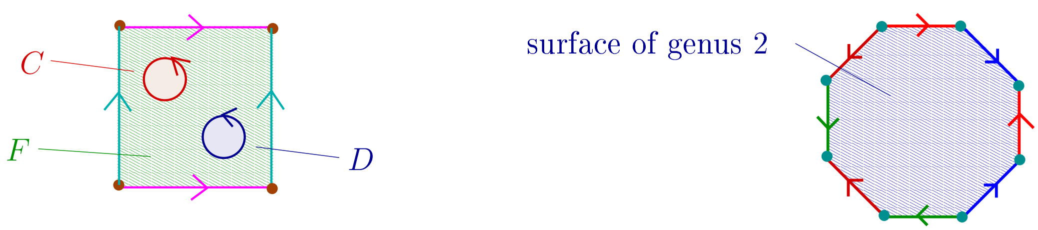


FIGURE 442. Illustration of Exercise 23.12.

Exercise 23.13. For which values of $g \in \mathbb{N}$ and $n \in \mathbb{N}$ is every boundary component of $N_{g,n}$ a retract of $N_{g,n}$?

FIGURE 443. Illustration for Exercise 443.

Exercise 23.14. Let $W = \mathbb{R}^2 \setminus \{(n,0) \,|\, n \in \mathbb{Z}\}$ and let X be the surface of infinite genus that is shown in Figure 444. Show that W and X are not diffeomorphic.
Hint. What compact submanifolds can occur?
Remark. The same argument also shows that W is not diffeomorphic to the other surfaces of "infinite genus" that are shown in Figure 445.
Remark. Later in Exercise 82.5 we will see that W and X are not homeomorphic. With the present knowledge it is at best quite challenging to give a watertight proof that W and X are not homeomorphic.

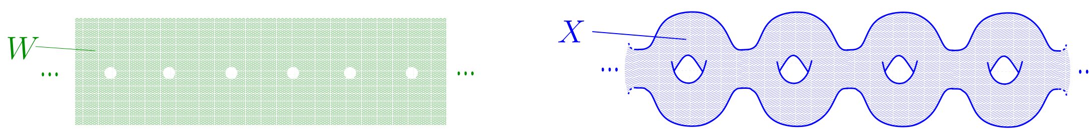


FIGURE 444. Illustration of Exercise 23.14.

Exercise 23.15. Let X be a topological space. Given a compact subset $K \subset X$ we define $m(K)$ to be the number of non-compact path-components of $X \setminus K$ that are *not* contained in a compact subset of X. As in Exercise 2.22 we refer to

$$\operatorname{End}(X) \;:=\; \max\{m(K) \,|\, K \text{ is a compact subset of } X\} \in \mathbb{N}_0 \cup \{\infty\}$$

as the *number of ends of X*.

(a) Determine the number of ends for each of the three non-compact surfaces shown in Figure 445.
(b) Which (if any) of the three "surfaces of infinite genus" shown in Figure 445 are homeomorphic?
(c) For each $n \in \mathbb{N}$ give an example of a non-compact connected 2-dimensional smooth manifolds such that the number of ends equals n.

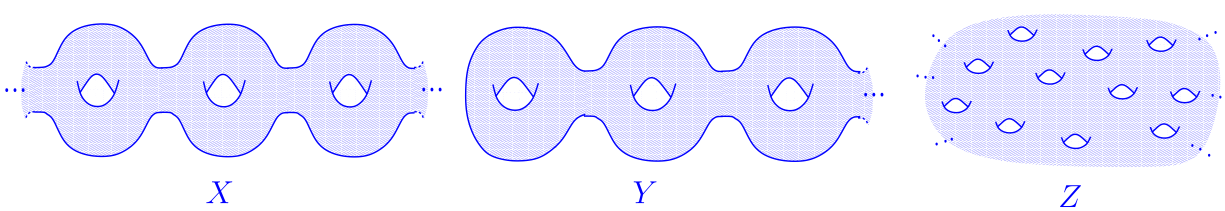


FIGURE 445. Illustration of Exercise 23.15.

Exercise 23.16. Let $n \in \mathbb{N}$, let M_n be S^2 minus n open disks and let C be a closed curve in $M_n \setminus \partial M_n$.

(a) Show that C is separating.
(b) Let $n = 3$. Show that C either bounds a disk or that it is isotopic to a boundary component of M_n.
(c) Show that the conclusion of (b) is in general not true for $n \geq 4$.

Exercise 23.17. Let Σ be a closed orientable connected 2-dimensional smooth manifold and let $\{A, B\}$ and $\{A', B'\}$ be pairs of closed curves on Σ such that A intersects B transversally in a single point and such that A' intersects B' transversally in a single point.

(a) Show that there exists a diffeomorphism $f\colon \Sigma \to \Sigma$ with $f(A) = A'$ and $f(B) = B'$.
Remark. The statement of this exercise, and of the closely related Proposition 23.16, are proved and extended, in [**FaM11**, Chapter 1.3]. They refer to statements of this type as the *change of coordinates principle*.
(b) Suppose that all the curves are oriented. Can we find f such that the maps $f\colon A \to A'$ and $f\colon B \to B'$ are both orientation-preserving?

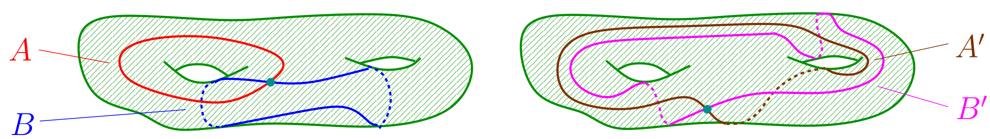


FIGURE 446. Illustration for Exercise 23.17.

24. Mapping cones, cylinders and tori

In this section we first introduce two systematic ways to construct new topological spaces out of a given topological space and we present three systematic ways to construct new topological spaces out of a map $f\colon X \to Y$ between given topological spaces. Initially it might not be entirely apparent why we should be interested, but all five examples will play an important role at various points. The main excuse for presenting these constructions at this very moment is that they allow us to play a little bit with our newly acquired abilities for computing fundamental groups.

24.1. The cone and suspension of a topological space.

In this section we define the cone and suspension of a topological space. The cone of a topological space is a particularly easy construction.

Definition. Given a non-empty topological space X the *cone on* X is defined to be the topological space

$$\operatorname{Cone}(X) := (X\times[0,1])/(X\times\{0\}) = (X\times[0,1])/\sim \text{ where } (x,0)\sim(x',0) \text{ for all } x,x'\in X.$$

Similarly we define the *open cone on* X to be

$$\operatorname{Cone}^o(X) := (X\times[0,1)])/(X\times\{0\}) = (X\times[0,1))/\sim \text{ where } (x,0)\sim(x',0) \text{ for all } x,x'\in X.$$

We refer to the point $X\times\{0\}$ as the *tip of the cone* $\operatorname{Cone}(X)$ *and of the open cone* $\operatorname{Cone}^o(X)$ or alternatively as the *cone point*. For completeness we define the cone and the open cone of the empty topological space to be the one point topological space $\{*\}$. In this setting we refer to $*$ as the tip of the cone.

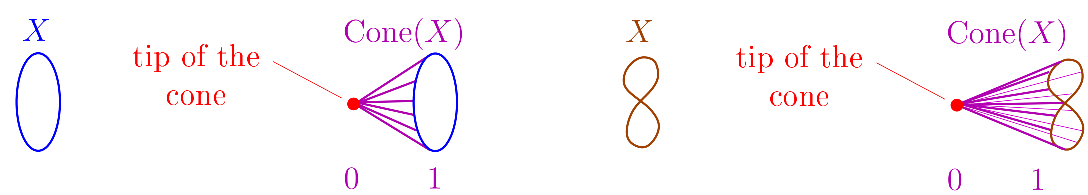


Figure 447

Examples.

(1) Let $n \in \mathbb{N}_0$. It follows from Lemma 3.22 that the map

$$\begin{array}{rcl}\operatorname{Cone}(S^n) & \to & \overline{B}^{n+1} \\ [(v,t)] & \mapsto & v\cdot t\end{array}$$

is continuous. One can easily verify that the map is a bijection. It follows immediately from Lemma 3.21 (4) and Proposition 3.12 that $\operatorname{Cone}(S^n)$ is compact. Since $\overline{B}^{n+1}$ is Hausdorff we obtain from Proposition 2.43 (3) that the given map is a homeomorphism. On a few occasions we will use this specific homeomorphism to make the identification $\operatorname{Cone}(S^n) = \overline{B}^{n+1}$.

(2) Let X be a subset of $\mathbb{R}^n$. We define its *physical cone* as

$$\widetilde{\mathrm{Cone}}(X) \;:=\; \text{union of all line segments from } \underbrace{(x,0)}_{\in\mathbb{R}^{n+1}} \text{ to } \underbrace{(0,1)}_{\in\mathbb{R}^{n+1}} \;\subset\; \mathbb{R}^{n+1}.$$

(a) In Exercise 24.1 we will see that if X is a compact subset of $\mathbb{R}^n$, then the map

$$\begin{array}{rrcl} f_X\colon & \mathrm{Cone}(X) & \to & \widetilde{\mathrm{Cone}}(X) \\ & [(x,t)] & \mapsto & t\cdot(x,0)+(1-t)\cdot(0,1) \end{array}$$

is a homeomorphism.

(b) After looking at Figure 447 and considering Examples (1) and (2a) one might walk away with the feeling that one has a pretty good understanding of the cone construction. But this optimism is potentially ill-founded. For example in Exercise 24.1 we see will see that for $X=(0,1)\subset\mathbb{R}$ the above map $f_X\colon \mathrm{Cone}(X)\to\widetilde{\mathrm{Cone}}(X)$ from (2a) is *not* a homeomorphism.

This definition does not seem to be particularly interesting, but we have one lemma whose proof is perhaps slightly more subtle than one might initially think.

Lemma 24.1. *Let X be a topological space. The tip of the cone* $\mathrm{Cone}(X)$ *is a deformation retract of* $\mathrm{Cone}(X)$.

Proof. If $X=\varnothing$, then it follows from the definition of $\mathrm{Cone}(\varnothing)=\{*\}$ that the cone equals the tip. Thus it remains to consider the case of a non-empty topological space. We consider the "obvious" map

$$\begin{array}{rrcl} \varphi\colon & \overbrace{(X\times[0,1])/(X\times\{0\})\;\times\;[0,1]}^{=\mathrm{Cone}(X)\times[0,1]} & \to & \overbrace{(X\times[0,1])/(X\times\{0\})}^{=\mathrm{Cone}(X)} \\ & ([(x,s)],t) & \mapsto & [(x,s\cdot(1-t))] \end{array}$$

One might think that this map "obviously" does the trick. The wary reader will not have failed to notice that here we are mixing the product and the quotient topologies, which, as we have learned the painful way in Section 5.2, can be quite dangerous and can lead to non-continuous maps.

So let us show that the map φ is indeed continuous.[409] Since there is an abundance of intervals we now write $S=[0,1]$ and $T=[0,1]$. We consider the following two maps

$$\begin{array}{rrcl} H\colon & (X\times S)\times T & \to & X\times S \\ & ((x,s),t) & \mapsto & (x,s\cdot(1-t)) \end{array} \quad\text{and}\quad \begin{array}{rrcl} p\colon & X\times S & \to & \mathrm{Cone}(X) \\ & (x,s) & \mapsto & [(x,s)]. \end{array}$$

[409] The fact that one actually needs to prove that the map φ is continuous seems to get swept under the carpet in every single book on (algebraic) topology that I have consulted.

Note that H and p are continuous.[410] We have the following commutative diagram

$$\begin{tikzcd} & (X \times S) \times T \arrow[dl, "p\times \mathrm{id}_T"'] \arrow[dr, "p\circ H"] & \\ \mathrm{Cone}(X) \times T \arrow[rr, "\varphi"] & & \mathrm{Cone}(X). \end{tikzcd}$$

By Lemma 5.15 the map p is a quotient map. Since T is regionally compact we obtain from Theorem 5.16 that $p \times \mathrm{id}_T$ is a quotient map. By the discussion above we know that $p \circ H$ is continuous. Thus it follows from Lemma 5.17 (2) that φ is continuous. ■

We move on to the next definition.

Definition. Let X be a non-empty topological space. We define the *suspension of* X to be the topological space

$$\Sigma(X) \ := \ (X \times [-1,1])/\sim$$

where we identify all points in $X \times \{-1\}$ to a single point and we identify all points in $X\times\{1\}$ to a single point. Furthermore, we define the suspension of the empty topological space to be the topological space $S^0 = \{\pm 1\}$, equipped with the discrete topology.

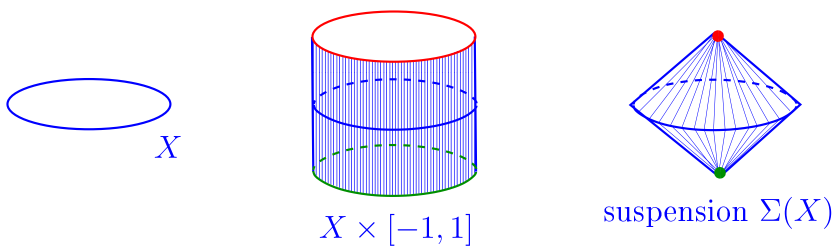


FIGURE 448

Before we give an example we state a basic lemma regarding suspensions.

Lemma 24.2. *Let X be a topological space.*

(1) *If X is compact, then* $\mathrm{Cone}(X)$ *and the suspension* $\Sigma(X)$ *are also compact.*
(2) *If X is Hausdorff, then* $\mathrm{Cone}(X)$ *and the suspension* $\Sigma(X)$ *are also Hausdorff.*
(3) $(*)$ *The maps*

$$\begin{array}{rcl} X & \to & \mathrm{Cone}(X) \\ x & \mapsto & [(x,1)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{Cone}(X) & \to & \Sigma(X) \\ [(x,t)] & \mapsto & [(x,1-t)] \end{array}$$

are closed embeddings.

(4) $(*)$ *The maps*

$$\begin{array}{rcl} X\times(0,1) & \to & \mathrm{Cone}^\circ(X) \\ (x,t) & \mapsto & [(x,t)] \end{array} \quad \begin{array}{rcl} \mathrm{Cone}^\circ(X) & \to & \mathrm{Cone}(X) \\ [(x,t)] & \mapsto & [(x,t)] \end{array} \quad \text{and} \quad \begin{array}{rcl} X\times(-1,1) & \to & \Sigma(X) \\ (x,t) & \mapsto & [(x,t)] \end{array}$$

are open embeddings.

[410] The map p is continuous by definition of the topology on $\mathrm{Cone}(X)$. But what is a super-formal argument for showing that H is continuous?

Sometimes we will use the embeddings in (3) and (4) to identify the domains with their images. For example we will use these maps to view X as a closed subset of $\mathrm{Cone}(X)$ and to view $\mathrm{Cone}(X)$ as a closed subset of $\Sigma(X)$. In particular we can and will at times view X as a closed subset of $\Sigma(X)$.

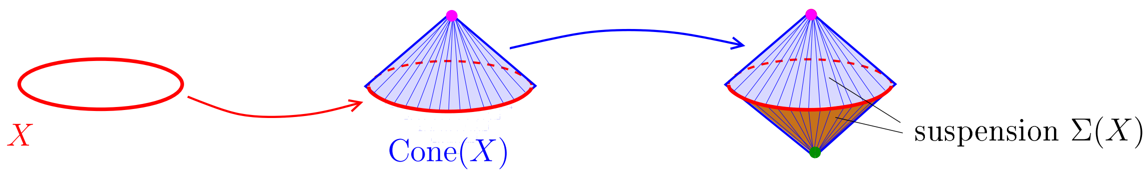


FIGURE 449. Illustration of Lemma 24.2 (3).

PROOF.

(1) This statement follows from Lemma 3.21 (4) and Proposition 3.12.
(2) In Exercise 24.4 we will show that if X is Hausdorff, then $\Sigma(X)$ is Hausdorff as well. The proof for $\mathrm{Cone}(X)$ is basically identical.
(3) We will prove this statement in Exercise 24.5.
(4) The proof of this statement is similar to the proof of (3). We leave it to the reader to fill in the details. ■

Example. It is pretty clear from Figure 448 that the suspension of S^n is homeomorphic to S^{n+1}. In fact throughout the course we will use the following homeomorphism[411] to identify $\Sigma(S^n)$ with S^{n+1}:

$$\begin{array}{rcl} \Sigma(S^n) & \xrightarrow{\cong} & S^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P,t)] & \mapsto & \big(\underbrace{P\cos\left(\frac{\pi}{2}t\right)}_{\in\mathbb{R}^{n+1}}, \underbrace{\sin\left(\frac{\pi}{2}t\right)}_{\in\mathbb{R}}\big). \end{array}$$

Note that this homeomorphism sends the "North Pole" of $\Sigma(S^n)$ to the "North Pole" of S^{n+1}.

For future reference we record the following lemma.

Lemma 24.3. *If X is a path-connected non-empty topological space, then its suspension $\Sigma(X)$ is simply connected.*

PROOF $(*)$. Let X be a path-connected non-empty topological space. In the following we denote by $p\colon X \times [-1,1] \to \Sigma(X)$ the obvious projection map. We consider the subsets $U = p(X \times [-1, \frac{1}{2}))$ and $V = p(X \times (-\frac{1}{2}, 1])$. It follows from the definition of the quotient topology on $\Sigma(X)$ that U and V are open subsets of $\Sigma(X)$. We pick a base point x on $X = X \times \{0\} \subset \Sigma(X)$. We calculate that

$$\pi_1(\Sigma(X),x) \underset{\substack{\uparrow \\ \text{Seifert–van Kampen} \\ \text{Theorem 22.1}}}{\cong} \pi_1(U,x) *_{\pi_1(U\cap V,x)} \pi_1(V,x) \underset{\substack{\uparrow \\ \text{as in Lemma 24.1 we see that } U \text{ and } V \text{ are contractible} \\ \text{hence their fundamental groups are trivial by Proposition 18.16 (3)}}}{=} \begin{array}{c}\text{amalgamated product} \\ \text{of trivial groups}\end{array} = \text{trivial group.}$$

[411]Using Proposition 2.43 (3) and Lemma 24.2 (1) one easily verifies that this map is indeed a homeomorphism.

We have thus shown that $\Sigma(X)$ is indeed simply connected. ■

We conclude this discussion of suspensions with the fairly obvious notion of the suspension of a map.

Definition. Given a map $f\colon X \to Y$ between topological spaces we denote by

$$\begin{array}{rcl} \Sigma(f)\colon \Sigma(X) & \to & \Sigma(Y) \\ [(x,t)] & \mapsto & [(f(x),t)] \end{array}$$

the *suspension of f*.

Most of the statements of the following lemma will not come as a surprise.

Lemma 24.4. $(*)$

(1) *If $f\colon X \to Y$ is a map between topological spaces, then $\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)$ is also continuous.*

(2) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are maps between topological spaces, then the maps $\Sigma(g \circ f)$ and $\Sigma(g) \circ \Sigma(f)$ from $\Sigma(X)$ to $\Sigma(Z)$ agree.*

(3) *The maps*

$$\begin{array}{rcl} X & \mapsto & \Sigma(X) \\ (f\colon X \to Y) & \mapsto & (\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)) \end{array}$$

define a covariant functor from the category of topological spaces to the category of topological spaces. We refer to this functor as the suspension functor.

(4) *Let $f, g\colon X \to Y$ be two maps between topological spaces. If f and g are homotopic rel a (possibly empty) subset A, then $\Sigma(f)$ and $\Sigma(g)$ are homotopic rel $\Sigma(A)$.*

(5) *If $f\colon X \to Y$ is a homotopy equivalence of topological spaces, then the suspension $\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)$ is also a homotopy equivalence.*

Proof $(*)$.

(1) We consider the following commutative diagram

$$\begin{tikzcd} X \times [-1,1] \arrow[r, "f \times \mathrm{id}_{[-1,1]}"] \arrow[d] \arrow[dr, dashed] & Y \times [-1,1] \arrow[d] \\ \Sigma(X) \arrow[r, "\Sigma(f)"'] & \Sigma(Y). \end{tikzcd}$$

By Lemma 3.8 (2b) we know that the top horizontal map is continuous. By Lemma 3.21 (3) the left and right vertical maps are continuous. It follows that the diagonal map is continuous. Finally we obtain from Lemma 3.22 that the bottom horizontal map is continuous.

(2) This statement follows immediately from the definitions.

(3) This statement follows immediately from (1) and (2) and the definitions.

(4) Clearly it suffices to prove the following claim.

Claim. If $F\colon X \times [0,1] \to Y$ is a map, then the map

$$\begin{array}{rcl} G\colon (\overbrace{(X \times [-1,1])/\sim}^{=\Sigma(X)}) \times [0,1] & \to & \overbrace{(Y \times [-1,1])/\sim}^{=\Sigma(Y)} \\ ([(x,t)],s) & \mapsto & [(F(x,s),t)] \end{array}$$

is continuous.

Verifying this claim is more subtle than one might have thought initially.[412] We denote by $\sim_\times$ the equivalence relation on $(X \times [-1,1]) \times [0,1]$ that is generated by $((x,t),s) \sim_\times ((x',t'),s)$ whenever $(x,t) \sim (x',t')$. We consider the following diagram:

$$\begin{tikzcd} ((X\times[-1,1])/\sim)\times[0,1] \arrow[ddrr, "G"'] & & ((X\times[-1,1])\times[0,1])/\sim_\times \arrow[ll, "{([(x,t)],s) \leftarrow [((x,t),s)]}"'] \arrow[d, "{[((x,t),s)]\mapsto(F(x,s),t)}"] \\ & & Y \times [-1,1] \arrow[d, "{(y,t)\mapsto[(y,t)]}"] \\ & & (Y \times [-1,1])/\sim . \end{tikzcd}$$

The top map is a homeomorphism by Lemma 5.18 and the fact that $[0,1]$ is regionally compact. The top vertical map is continuous by Lemma 3.22 and the bottom vertical map is continuous by Lemma 3.21 (3). It is clear that the diagram commutes. It now follows that G is continuous. ⊞

(5) The last statement is an immediate consequence of the proof of (4). ■

More interesting than the totally predictable Lemma 24.4 is perhaps the following scary example, which is dressed up as a lemma to make it more visible.

Lemma 24.5. *We consider the topological space $X = [-1,1]$, the subset $A = (-1,1)$ and the inclusion map $i\colon A \to X$. The induced map $\Sigma(i)\colon \Sigma(A) \to \Sigma(X)$ is* not *an embedding.*

PROOF. We will provide the proof in Exercise 24.10. ■

24.2. **Mapping cylinder and mapping cone.** In this section we associate to a map $f\colon X \to Y$ between two topological spaces two different topological spaces. The main interest in these topological spaces is that these topological spaces encode some of the information contained in the actual map. On numerous occasions this will allow us to translate questions about maps between topological spaces into questions about topological spaces.

We start out with the definition of a mapping cylinder.

Definition. Let $f\colon X \to Y$ be a map between topological spaces. We define the corresponding *mapping cylinder* to be the topological space

$$\mathrm{Cyl}(f) := \mathrm{Cyl}(f\colon X \to Y) := ((X \times [0,1]) \sqcup Y)/\sim \ \text{ where } (x,1) \sim f(x) \text{ for all } x \in X.$$

We illustrate the definition in Figure 450.

[412] At the very least it is more subtle than the author of these modest notes had been thinking for 20 years.

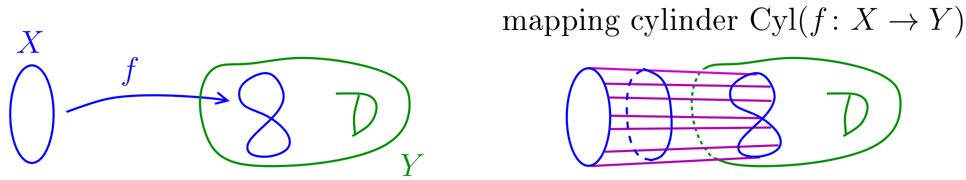


FIGURE 450

Remark. In many, arguably most, textbooks on algebraic topology the mapping cylinder is defined by identifying $(x, 0)$ with $f(x)$ instead of identifying $(x, 1)$ with $f(x)$. The definitions are evidently equivalent, but as the author knows from experience, the usage of two different definitions can cause some confusion. The totally non-mathematical advantage of our definition is that it leads to "horizontal" pictures which take up less space.

Example. If X is a topological space and if $f\colon X \to \{y\}$ is the constant map to the topological space consisting of a single element y, then the mapping cylinder $\operatorname{Cyl}(f)$ is homeomorphic to $\operatorname{Cone}(X)$. This statement is almost, but not completely[413] evident. We invite the general-topology-aficionado to fill in the details.

Definition. Let $f\colon X \to Y$ be a map between topological spaces. We refer to the maps

$$\begin{array}{rcl} X & \to & \operatorname{Cyl}(f\colon X \to Y) \\ x & \mapsto & [(x,0)] \end{array} \quad \text{and} \quad \begin{array}{rcl} Y & \to & \operatorname{Cyl}(f\colon X \to Y) \\ y & \mapsto & [y] \end{array}$$

as the *natural inclusions*. Sometimes we use these inclusions to identify X and Y with the corresponding subsets of the mapping cylinder.

The following lemma says that the natural inclusions are embeddings. This means that we can legitimately identify X and Y with their images in the mapping cylinder $\operatorname{Cyl}(f\colon X \to Y)$. On many occasions we will implicitly use the lemma without actually referring to it.

Lemma 24.6. $(*)$ *Let $f\colon X \to Y$ be a map between topological spaces.*

(1) *If X and Y are compact, then $\operatorname{Cyl}(f)$ is also compact.*
(2) *If X and Y are Hausdorff, then $\operatorname{Cyl}(f)$ is also Hausdorff.*
(3) *The natural inclusions $X \to \operatorname{Cyl}(f)$ and $Y \to \operatorname{Cyl}(f)$ are embeddings, in particular they are continuous.*
(4) *The images of the natural inclusions $X \to \operatorname{Cyl}(f)$ and $Y \to \operatorname{Cyl}(f)$ are closed subsets.*

PROOF. We leave the proofs of the first two and of the last statement as voluntary exercises to the reader. Finally we will provide the proof of the third statement in Exercise 24.6. ■

We continue with a (non-) example. We will not really make use of this lemma, but Lemma 24.7 should warn the reader that the topology on a mapping cylinder is not necessarily what one might think it is.

[413]Note that the statement is *not* true for the empty topological space. This observation shows that the statement requires at least a little bit of thought.

Lemma 24.7. (∗) *Let X be a topological space and let $A \subset X$ be a subset. We denote by $i\colon A \to X$ the inclusion map and we consider the map*

$$\begin{array}{rcl} \Theta\colon\ \mathrm{Cyl}(i) = ((A\times[0,1])\sqcup X)/(a,1)\sim i(a) & \to & \overbrace{(A\times[0,1])\cup(X\times\{0\})}^{\substack{\text{equipped with the subspace topology}\\ \text{coming from the product } X\times[0,1]}} \\ [P] & \mapsto & \begin{cases} P, & \text{if } P\in A\times[0,1], \\ (P,1), & \text{if } P\in X. \end{cases} \end{array}$$

The following two statements hold:

(1) *The map Θ is continuous and it is a bijection.*
(2) *If A is a closed subset of X, then Θ is a homeomorphism.*
(3) *There exist situations, e.g. $X = [0,\infty)$ and $A = [0,1)$, in which Θ is* not *a homeomorphism.*

PROOF.

(1) It follows easily from Lemma 3.22 that the given map Θ is continuous. Furthermore it is basically clear that Θ is a bijection.
(2) Suppose that A is a closed subset of X. Note that $A\times\{1\}$ is a closed subset of $A\times[0,1]$. It follows basically from Lemma 3.45 that Θ is a homeomorphism.
(3) We consider the case $X=[0,\infty)$ and $A=[0,1)$. We denote by
$$p\colon (A\times[0,1))\sqcup X \ \to\ \mathrm{Cyl}(i) = ((A\times[0,1))\sqcup X)/\sim$$
the projection map. We consider the open set $U := \{(x,t)\in[0,1)\times[0,1]\,|\,x>t\}$ of $A\times[0,1)$. Furthermore we consider the open subset $V := [0,2)$ of $X=[0,\infty)$. Note that $p^{-1}(p(U\sqcup V)) = U\sqcup V$. Thus it follows immediately from the definition of the quotient topology on $\mathrm{Cyl}(i) = ((A\times[0,1))\sqcup X)/\sim$ that $p(U\sqcup V)$ is an open subset of $\mathrm{Cyl}(i)$. On the other hand we have
$$\Theta(p(U\sqcup V)) \ = \ U\cup\{(x,1)\,|\,x\in V\},$$
but considering the point $(1,1)$ one can easily verify that this is *not* an open subset of $(A\times[0,1])\cup(X\times\{0\})$. ■

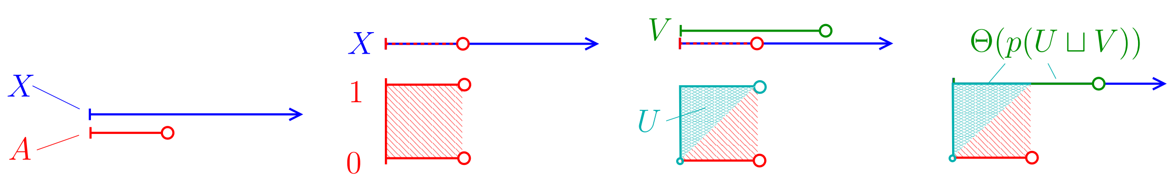


FIGURE 451. Illustration of the proof of Lemma 24.7 (3).

The following lemma summarizes some basic properties of the mapping cylinder.

Lemma 24.8. *Let $f\colon X \to Y$ be a map between topological spaces.*

(1) *The subspace Y is a deformation retract of* $\operatorname{Cyl}(f\colon X \to Y)$*, in fact an explicit deformation retraction is given by the following map*
$$\begin{array}{rcl} H\colon \operatorname{Cyl}(f\colon X\to Y)\times[0,1] & \to & \operatorname{Cyl}(f\colon X\to Y) \\ (P,t) & \mapsto & \begin{cases} P, & \text{if } P\in Y, \\ [(Q,s\cdot(1-t)+t)], & \text{if } P=[(Q,s)], \text{ where} \\ & \quad Q\in X \text{ and } s\in[0,1]. \end{cases} \end{array}$$

(2) *We denote by $r = H_1$ the retraction* $\operatorname{Cyl}(f\colon X\to Y)\to Y$ *given by (1). If we denote by $i\colon X \to \operatorname{Cyl}(f)$ and $j\colon Y \to \operatorname{Cyl}(f)$ the natural inclusions, then the following statements hold:*
   (a) *The map $j\colon Y \to \operatorname{Cyl}(f\colon X\to Y)$ and $r\colon \operatorname{Cyl}(f\colon X\to Y)\to Y$ are homotopy equivalences.*
   (b) *The map $i\colon X\to \operatorname{Cyl}(f\colon X\to Y)$ is a homotopy equivalence if and only if the map $f\colon X\to Y$ is a homotopy equivalence.*

Remark. On page 698 we saw that the mapping cylinder of a constant map $f\colon X \to \{y\}$ is homeomorphic to the cone on X. Thus we see that Lemma 24.8 (1) can be viewed as a generalization of Lemma 24.1.

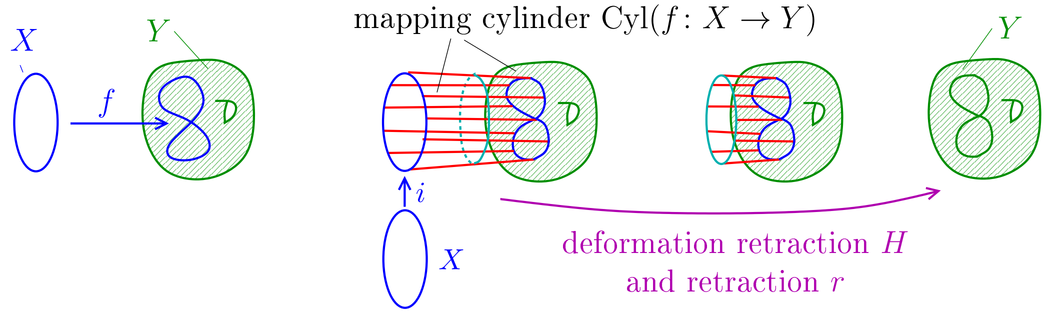


FIGURE 452. Illustration of Lemma 24.8 (1).

PROOF.

(1) We need to show that the map H is continuous. First note that the two maps
$$\begin{array}{rcl} (X\times[0,1])\times[0,1] & \to & \operatorname{Cyl}(f) \\ (Q,t) & \mapsto & [(Q,s\cdot(1-t)+t)] \end{array} \quad\text{and}\quad \begin{array}{rcl} Y\times[0,1] & \to & \operatorname{Cyl}(f) \\ (P,t) & \mapsto & P \end{array}$$
are continuous[414]. But this implies, by the somewhat delicate Lemma 18.23, that
$$H\colon ((X\times[0,1])\cup_{(x,1)\mapsto f(x)} Y)\times[0,1] \ \to\ \operatorname{Cyl}(f)$$
itself is continuous. Once continuity is out of the way it is basically clear that H is a deformation retraction. Therefore it follows from Lemma 18.14 that the inclusion $Y\to \operatorname{Cyl}(f\colon X\to Y)$ is a homotopy equivalence.

(2) (a) It follows from Lemma 18.14 that j and r are homotopy equivalences.

[414]For once this is basically obvious.

(b) We consider the map

$$\begin{array}{rcl} G\colon X\times[0,1] & \to & \operatorname{Cyl}(f\colon X\to Y) \\ (x,t) & \mapsto & [(x,t)]. \end{array}$$

This map is continuous by Lemma 3.22. We make the following observations:

$$\begin{array}{ccccc} \begin{array}{c} i=G_0\colon X\to\operatorname{Cyl}(f) \text{ is a} \\ \text{homotopy equivalence} \end{array} & \underset{\uparrow}{\Leftrightarrow} & \begin{array}{c} i=G_1\colon X\to\operatorname{Cyl}(f) \text{ is a} \\ \text{homotopy equivalence} \end{array} & \underset{\uparrow}{\Leftrightarrow} & \begin{array}{c} \overbrace{r\circ G_1}^{=f}\colon X\to Y \text{ is a} \\ \text{homotopy equivalence} \end{array} \\ & \begin{array}{c} \text{by Lemma 18.11 (3), since} \\ G_0 \text{ and } G_1 \text{ are homotopic} \end{array} & & \begin{array}{c} \text{by Lemma 18.11 (1), since by (2a) we} \\ \text{know that } r \text{ is a homotopy equivalence} \end{array} & \end{array}$$

We have thus shown that i is a homotopy equivalence if and only if $f=r\circ G_1$ is a homotopy equivalence. ■

In the above discussion we associated to a map $f\colon X\to Y$ between topological spaces a new topological space, namely the mapping cylinder $\operatorname{Cyl}(f\colon X\to Y)$. But we see in Lemma 24.8 (1) that the mapping cylinder, as a topological space, is in many ways the same as Y. Therefore we move on to the definition of the mapping cone which is a much more interesting construction.

Definition. Given a map $f\colon A\to X$ between topological spaces the corresponding *mapping cone* is defined as

$$\operatorname{Cone}(f\colon A\to X) \;:=\; (\operatorname{Cone}(A)\sqcup X)/\sim \;\text{ where } [(a,1)]\sim f(a) \text{ for all } a\in A.$$

We refer to the point $[A\times\{0\}]$ as the *cone point* of the mapping cone. We illustrate the definition in Figure 453.

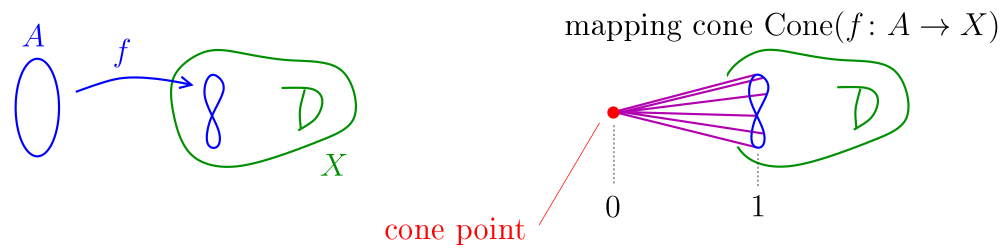


FIGURE 453

Remark. Let $f\colon A\to X$ be a map between topological spaces. If A is non-empty, then it follows almost immediately from the definitions that

$$\operatorname{Cone}(f\colon A\to X) \;=\; \operatorname{Cyl}(f\colon A\to X)/(A\times\{0\}).$$

The following lemma summarizes some properties of mapping cones.

Lemma 24.9. (∗) *Let $f\colon A\to X$ be a map between topological spaces.*

(1) *We denote by $i\colon A\to\operatorname{Cone}(A)$ the map that is given by $a\mapsto[(a,1)]$. The mapping cone $\operatorname{Cone}(f\colon A\to X)$ is the same*[415] *as the pushout of the maps $i\colon A\to\operatorname{Cone}(A)$*

and $f\colon A \to X$, i.e. we have the pushout diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & \mathrm{Cone}(A) \arrow[d] \\ X \arrow[r] & \mathrm{Cone}(f\colon A \to X). \end{tikzcd}$$

(2) *The natural inclusion*

$$\begin{array}{rcl} X & \to & \mathrm{Cone}(f\colon A\to X) \\ x & \mapsto & [x] \end{array} \quad \textit{respectively} \quad \begin{array}{rcl} \mathrm{Cone}^{\mathrm{o}}(A) & \to & \mathrm{Cone}(f\colon A\to X) \\ x & \mapsto & [x] \end{array}$$

is a closed respectively an open embedding. Often we will use these natural inclusions to identify X with its image in $\mathrm{Cone}(f\colon A \to X)$ *and to identify the open cone* $\mathrm{Cone}^{\mathrm{o}}(A)$ *with its image in* $\mathrm{Cone}(f\colon A\to X)$.

(3) *If the map $f\colon A \to X$ is a (closed or open) embedding, then the natural map* $\mathrm{Cone}(A) \to \mathrm{Cone}(f\colon A \to X)$ *is a (closed or open) embedding.*

(4) *If A is non-empty, then the map*

$$\begin{array}{rcl} \mathrm{Cone}(f\colon A\to X)/X & \to & \Sigma(A) \\ [(a,t)] & \mapsto & [(a,1-2t)] \end{array}$$

is a natural homeomorphism.

(5) *The cone point $*$ defines a closed subset of* $\mathrm{Cone}(f\colon A\to X)$.

(6) *There exists a natural deformation retraction from* $\mathrm{Cone}(f\colon A\to X)\setminus\{*\}$ *to X. In particular the subset X is a deformation retract of the complement of the cone point.*

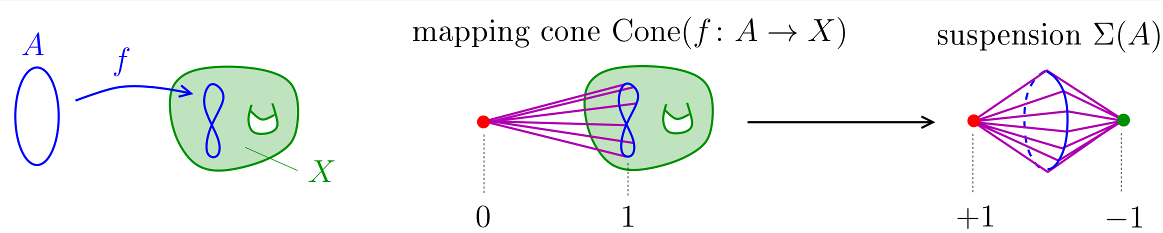


FIGURE 454. Illustration of Lemma 24.9 (4).

SKETCH OF PROOF.

(1) This statement follows basically immediately from the definitions.
(2) This statement follows immediately from (1) together with Lemma 3.43.
(3) This statement also follows immediately from (1) together with Lemma 3.43.
(4) It is clear that the given map is a bijection. One can easily verify, without any extra wizardry, that the given map is indeed a homeomorphism.
(5) This statement follows quite easily from the definition of the topologies involved.
(6) The proof of this statement is basically identical to the proof of Lemma 24.8 (1). ■

So what have we done so far? We have associated to a map $f\colon X \to Y$ between topological spaces the topological space $\mathrm{Cone}(f\colon X\to Y)$. This construction will allow us

[415] At this point one can now argue whether they are the same on the nose or whether the obvious map is just a natural homeomorphism.

to turn questions about maps between topological spaces into questions about topological spaces. As always in algebraic topology, not only do we want a map

$$\text{set of maps between topological spaces} \;\to\; \text{set of topological spaces},$$

but we really want a functor. It is clear what the category should be on the right. It is a little less clear what the category should be on the left. Here it is.

Definition. We refer to the category $\mathcal{MapTop}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{MapTop}) \;&:=\; \text{all maps between topological spaces}, \\ \mathrm{Mor}(f\colon X\to Y, \widetilde{f}\colon \widetilde{X}\to\widetilde{Y}) \;&:=\; \begin{array}{c}\text{all pairs of maps}\\ (\varphi\colon X\to\widetilde{X}, \psi\colon Y\to\widetilde{Y})\\ \text{such that the diagram}\end{array} \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y}\end{array} \text{ commutes}\end{aligned}$$

with the (hopefully) obvious composition of morphisms as the *category of maps between topological spaces.*

The following lemma says that we can view “taking the mapping cone” and “taking the mapping cylinder” as covariant functors.

Lemma 24.10. $(*)$

(1) *The maps*

$$(f\colon X\to Y) \mapsto \mathrm{Cyl}(f\colon X\to Y)$$

and

$$(\varphi\colon X\to\widetilde{X}, \psi\colon Y\to\widetilde{Y}) \mapsto \left(\begin{array}{rl} \mathrm{Cyl}(f\colon X\to Y) \to & \mathrm{Cyl}(\widetilde{f}\colon \widetilde{X}\to\widetilde{Y}) \\ [P] \mapsto & \begin{cases} [(\varphi(x),t)], & \text{if } P=(x,t) \text{ with,} \\ & \qquad x\in X,\ t\in[0,1] \\ [\psi(y)], & \text{if } P=y \text{ with } y\in Y\end{cases}\end{array}\right).$$

define a covariant functor from the category $\mathcal{MapTop}$ of maps between topological spaces to the category $\mathcal{Top}$ of topological spaces.

(2) *If, in the setting of* (1), *the maps φ and ψ are embeddings, then the corresponding map* $\mathrm{Cyl}(f\colon X\to Y)\to\mathrm{Cyl}(\widetilde{f}\colon\widetilde{X}\to\widetilde{Y})$ *is also an embedding.*

(3) *The analogues of* (1) *and* (2), *with obvious modifications, hold if we replace mapping cylinders by mapping cones.*

(4) *Given a morphism*[416]

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y}\end{array}$$

the map from (3) *for mapping cones fits into the following commutative diagram:*

$$\begin{array}{ccccc} Y & \hookrightarrow & \mathrm{Cone}(f\colon X\to Y) & \longrightarrow & \Sigma(X) \\ {\scriptstyle\psi}\downarrow & & \downarrow & & \downarrow{\scriptstyle\Sigma(\varphi)} \\ \widetilde{Y} & \hookrightarrow & \mathrm{Cone}(\widetilde{f}\colon\widetilde{X}\to\widetilde{Y}) & \longrightarrow & \Sigma(\widetilde{X})\end{array}$$

where the horizontal maps to the left are the inclusion maps that we introduced in Lemma 24.9 (2) and where the horizontal maps to the right are given by the projection and the homeomorphism from Lemma 24.9 (4).

Proof (∗). One needs to convince oneself that the map between the mapping cylinders and the mapping cones is in fact continuous. For the mapping cylinders that is an immediate consequence of Lemma 3.44 (5). For the mapping cones first note that the same argument as in the proof of Lemma 24.4 (1) shows that the map

$$\begin{array}{rcl} \mathrm{Cone}(X) & \to & \mathrm{Cone}(\widetilde{X}) \\ [(x,t)] & \mapsto & [(\varphi(x),t)] \end{array}$$

is continuous. The continuity of the map between the mapping cones now follows again from Lemma 3.44 (5).

The second statement will be proved in Exercise 24.9 (3). Finally statement (4) of the lemma follows basically from the definitions. ■

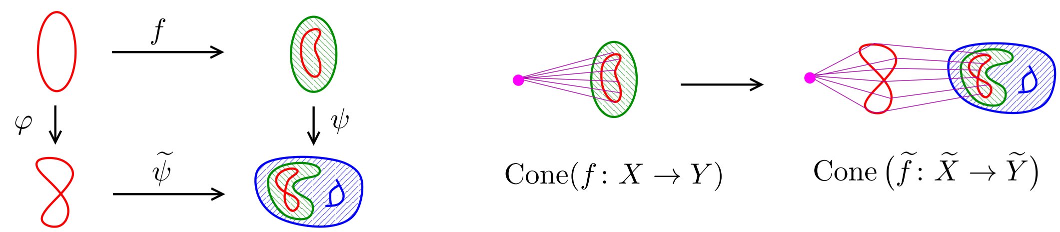


Figure 455. Illustration of Lemma 24.10.

The following lemma shows in particular that homotopic maps give rise to homotopy equivalent mapping cones. In particular at times this lemma allows us to translate the question whether two maps are homotopic to the question whether two topological spaces are homotopy equivalent.

Proposition 24.11. *Let $f_0, f_1\colon A \to X$ be maps between topological spaces. If f_0, f_1 are homotopic, then there exists a homotopy equivalence $\mathrm{Cone}(f_0) \to \mathrm{Cone}(f_1)$ rel the union of X and the cone point.*

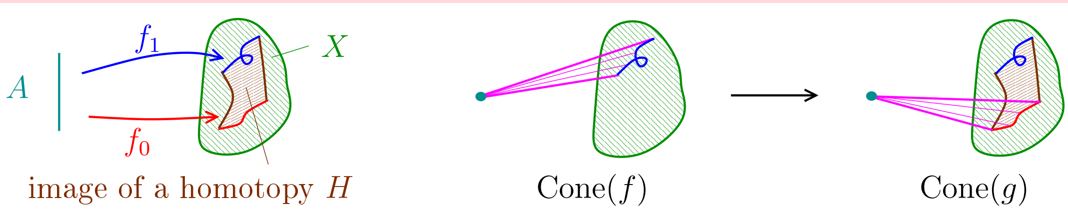


Figure 456. Illustration of Proposition 24.11.

It is not that difficult to write down explicit maps between the cones and to write down explicit homotopies. But we prefer to go take a more conceptual approach. Namely, instead of proving Proposition 24.11 directly we will prove the following more technical lemma.

[416] In fancy speak this commutative diagram says that the inclusion maps into the mapping cone and the projection from the mapping cone to the suspension are natural transformations between suitable functors.

Lemma 24.12. (∗) *Let $f_0, f_1 \colon A \to X$ be maps between topological spaces. Furthermore let $H \colon A \times [0,1] \to X$ be a homotopy between f_0 and f_1. Let $j \in \{0,1\}$. We denote by $i_j \colon A \to A \times \{j\}$ the obvious inclusion map.*

(1) *The inclusion map i_j induces by Lemma 24.10 (3) a map $\mathrm{Cone}(f_j) \to \mathrm{Cone}(H)$. This map is an embedding.*
(2) *If we use the map from (1) to view $\mathrm{Cone}(f_j)$ as a subset of $\mathrm{Cone}(H)$, then it is a deformation retract of $\mathrm{Cone}(H)$.*

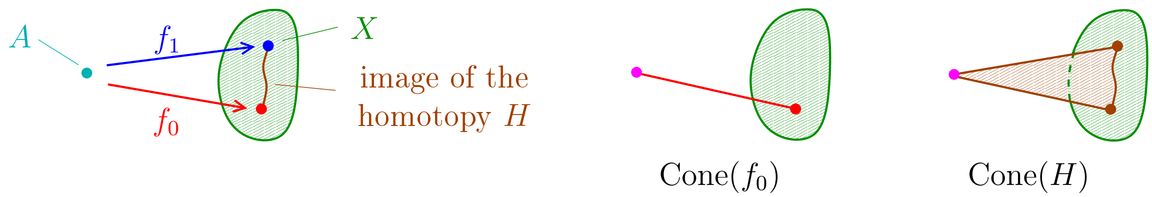


FIGURE 457. Illustration of Lemma 24.12.

PROOF OF PROPOSITION 24.11 ASSUMING LEMMA 24.12 (∗). Let H be a homotopy between f_0 and f_1. By Lemma 24.12 we can view $\mathrm{Cone}(f_0)$ and $\mathrm{Cone}(f_1)$ as deformation retracts of $\mathrm{Cone}(H)$. It follows from Lemma 18.14 and Lemma 18.11 that $\mathrm{Cone}(f_0)$ and $\mathrm{Cone}(f_1)$ are homotopy equivalent. A quick inspection of the maps involved shows that the two cones are homotopy equivalent rel the union of X and the cone point. ■

PROOF OF LEMMA 24.12 (∗). Let $f_0, f_1 \colon A \to X$ be maps between topological spaces. Furthermore let $H \colon A \times [0,1] \to X$ be a homotopy between f_0 and f_1. For notational convenience we only deal with the case $j = 0$. We denote by $i \colon A \to A \times \{0\}$ the obvious inclusion map. We get a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "i"'] & X \arrow[d, "="] \\ A \times [0,1] \arrow[r, "H"] & X. \end{tikzcd}$$

Thus we get from Lemma 24.10 (3) a map $\mathrm{Cone}(f_0) \to \mathrm{Cone}(H)$. It follows almost immediately from Lemma 24.10 (3) that this map is an embedding.

Since this proof suffers from an abundance of intervals playing different roles we rescale two of the intervals:

(1) We use the interval $[0,1]$ for the homotopy H.
(2) We use the interval $[0,2]$ for the definition of the mapping cone.
(3) We use the interval $[0,3]$ for the definition of the desired deformation retraction.

With this convention out of the way we pick a deformation retraction

$$\varphi \colon ([0,1] \times [0,2]) \times [0,3] \;\to\; [0,1] \times [0,2]$$

rel $([0,2] \times \{0\}) \cup (\{2\} \times [0,1])$ such that $\varphi(r,s,t) = (r, \frac{1}{3}st)$ whenever $r \in [0, \frac{1}{2}]$. It is straightforward to see that such a deformation retraction exists. We refer to Figure 458 for an illustration.

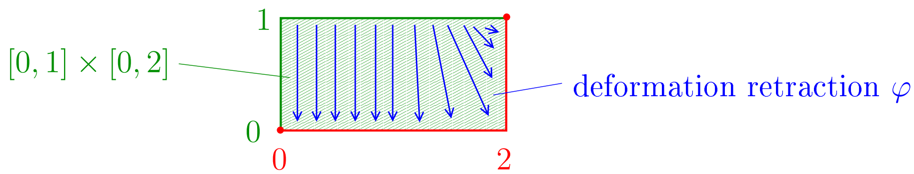


FIGURE 458

Now we consider the map

$$G\colon (\overbrace{((A\times[0,1]\times[0,2])\sqcup X)/\sim}^{=\mathrm{Cone}(H\colon A\times[0,1]\to X)})\times[0,3] \;\to\; \mathrm{Cone}(H)$$
$$([P],z) \;\mapsto\; \begin{cases} [(a,\varphi(x,y,z))], & \text{if } P=(a,x,y),\\ P, & \text{if } P\in X.\end{cases}$$

Using Lemma 3.21 (3) and Lemma 18.23 one can easily show that this map G is continuous. It is straightforward to verify that this map has the desired properties. ■

For completeness' sake we also state the following corollary.

Corollary 24.13. $(*)$ *Let $f\colon A\to X$ be a map between topological spaces and let $g\colon X\to Y$ be a homotopy equivalence. The map*

$$\begin{aligned}\mathrm{Cyl}(f\colon A\to X) &\;\to\; \mathrm{Cyl}(g\circ f\colon A\to Y)\\ [P] &\;\mapsto\; \begin{cases} [(a,t)], & \text{if } P=(a,t)\in A\times[0,1],\\ [g(P)], & \text{if } P\in X\end{cases}\end{aligned}$$

is a homotopy equivalence rel $A\times\{0\}$. The same way one sees that the mapping cones of f and $g\circ f$ are homotopy equivalent.

Example. Let $f\colon S^1\to\mathbb{R}^2$ be the inclusion and let $g\colon\mathbb{R}^2\to\{*\}$ be the retraction to the origin. In Figure 459 we show the corresponding mapping cones, which we now know are homotopy equivalent by Corollary 24.13.

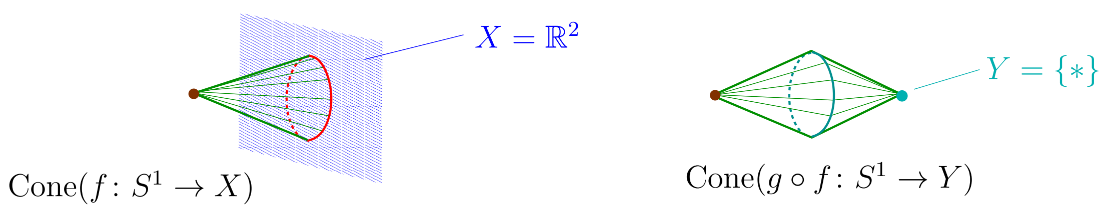


FIGURE 459

PROOF. The corollary is a straightforward consequence of Proposition 24.11. We leave it to the reader to fill in the details. ■

The following lemma calculates the fundamental groups of mapping cones.

Lemma 24.14. (∗) *Let $f\colon A \to X$ be a map between topological spaces and let $a_0 \in A$. We write $x_0 := f(a_0)$. If A is path-connected, then the inclusion induces an isomorphism*

$$\pi_1(X, x_0)/f_*(\pi_1(A, a_0)) \;\to\; \pi_1(\mathrm{Cone}(f\colon A \to X), [x_0]).$$

Example. Let $n \in \mathbb{N}$ and let $A = X = S^1$. We consider the map $f\colon A = S^1 \to X = S^1$ that is given by $f(z) = z^n$. It follows from Lemma 24.14 and Proposition 16.17, perhaps counter-intuitively, that $\pi_1(\mathrm{Cone}(f\colon A \to X)) \cong \mathbb{Z}_n$. We refer to Figure 460 for an attempt at an illustration of this example.

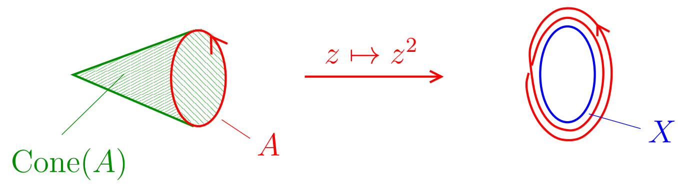


FIGURE 460

PROOF. We will provide the proof in Exercise 24.12. ■

We conclude our discussion of mapping cones with the following proposition.

Proposition 24.15. (∗) *Let X be a topological space and let A be a subset of X. We denote by $i\colon A \to X$ the inclusion map. We use Lemma 24.9 (2) to view X as a subset of* $\mathrm{Cone}(i\colon A \to X)$.

(1) *If A is contractible, then given any $a_0 \in A$ there exists a deformation retraction G from* $\mathrm{Cone}(i\colon A \to X)$ *to X such that G_1(cone point $[A \times \{0\}]) = a_0$.*
(2) *If $a_0 \in A$ is a deformation retract of A, then there exists a deformation retraction G from* $\mathrm{Cone}(i\colon A \to X)$ *to X such that for any $t \in [0,1]$ we have $G_1([(a_0, t)]) = a_0$.*

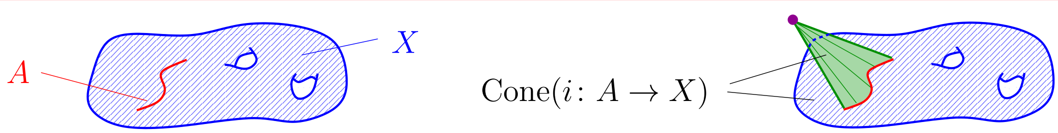


FIGURE 461. Illustration of Proposition 24.15.

PROOF. We first prove Statement (1). Thus let $a_0 \in A$. Since A is contractible it follows from Lemma 18.15 (2) that there exists a map $r\colon A \times [0,1] \to A$ such that $r_0 = \mathrm{id}$ and such that $r_1(a) = a_0$ for all $a \in A$. We pick continuous functions $x, y\colon [0,1] \times [0,1] \to [0,1]$ with the following properties:

(a) For any $t \in [0,1]$ we have $x(0,t) = 1$ or $y(0,t) = 0$.
(b) We have $x(s,0) = 0$ and $y(s,0) = s$ for all $s \in [0,1]$.
(c) We have $y(s,1) = 1$ for all $s \in [0,1]$.
(d) We have $x(1,t) = 0$ and $y(1,t) = 1$ for all $t \in [0,1]$.

We sketch graphs of such functions x and y in Figure 462. Now we consider the map

$$\begin{array}{rcl} F\colon \overbrace{((A \times [0,1])/(A \times \{0\}))}^{=\mathrm{Cone}(A)} \times [0,1] & \to & \overbrace{(A \times [0,1])/(A \times \{0\})}^{=\mathrm{Cone}(A)} \\ ([(a,s)], t) & \mapsto & [(r(a, x(s,t)), y(s,t))]. \end{array}$$

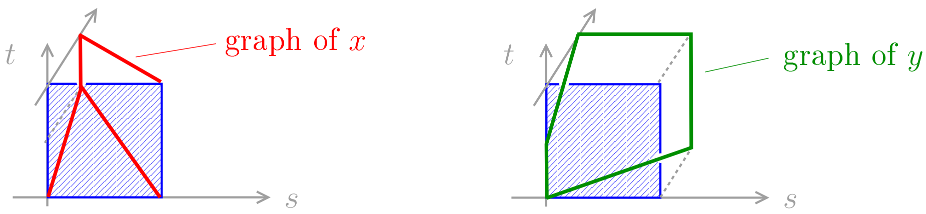


FIGURE 462

We make the following observations:

(0) Since x and y are continuous and since $[0,1]$ is regionally compact we obtain from Lemmas 3.22 and Lemma 5.18 that F is continuous.
(1) It follows from (a) that F is actually well-defined.
(2) It follows from (b) that $F_0 = \mathrm{id}$.
(3) It follows from (c) that F_1 takes values in $A \times \{1\}$.
(4) It follows from (d) that F is a homotopy rel $A \times \{1\}$.

In summary this shows that F is a deformation retraction from $\mathrm{Cone}(A)$ to $A \times \{1\}$. Now we consider the map

$$\begin{array}{rcl} G\colon \overbrace{(\mathrm{Cone}(A) \cup_{(a,1)=a} X)}^{=\mathrm{Cone}(i\colon A\to X)} \times [0,1] & \to & \mathrm{Cone}(i\colon A \to X) \\ ([P],t) & \mapsto & \begin{cases} [F(P,t)] & \text{if } P \in \mathrm{Cone}(A), \\ [P], & \text{if } P \in X. \end{cases} \end{array}$$

It follows from Lemma 18.23 that this map is continuous. Clearly G is a deformation retract from $\mathrm{Cone}(i\colon A \to X)$ to X. By construction it sends the cone point to a_0.

It remains to prove Statement (2). If we start out with a deformation retraction r from A to a_0, then the above homotopy F has the desired property. ■

24.3. **The mapping cone of a suspension** (∗)**.** In this section we show that under favorable circumstances the operation of taking a mapping cone (or mapping cylinder) commutes with the operation of taking the suspension of a map. This technical proposition will play a useful role later on. The proposition is also a good way to test whether we really understand the various topologies involved in the definition of mapping cones, mapping cylinders and suspensions.

Proposition 24.16. *If $f\colon X \to Y$ is a map between topological spaces, then the map*[417]

$$\begin{array}{rcl} \Phi\colon \Sigma(\mathrm{Cone}(f\colon X \to Y)) & \to & \mathrm{Cone}(\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)) \\ P & \mapsto & \begin{cases} [([(x,t)],s)], & \text{if } P = [([(x,s)],t)] \text{ with } x \in X,\ t \in [-1,1] \\ & \quad \text{and with } s \in [0,1], \\ [([y],t)], & \text{if } P = [([y],t)] \text{ with } y \in Y \text{ and } t \in [-1,1]. \end{cases} \end{array}$$

is a homeomorphism.

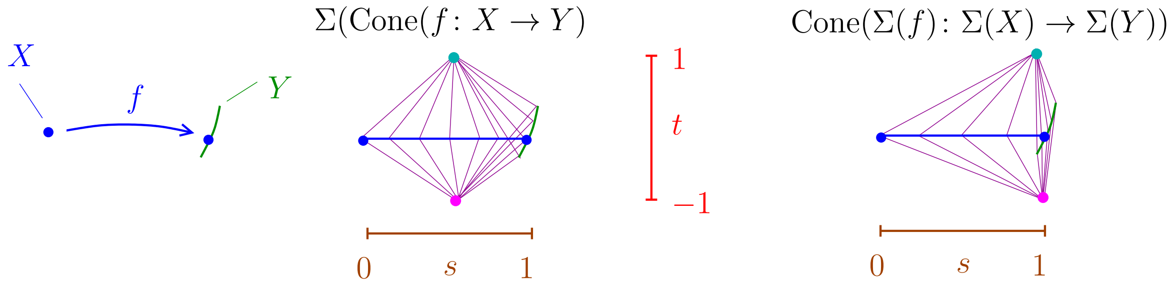


FIGURE 463. Illustration of Proposition 24.16.

In the proof of Proposition 24.16 we will need the following lemma which is quite similar to Proposition 5.20.

Lemma 24.17. *Let X be a topological space. If Y is a regionally compact topological space (e.g. if $Y = [0,1]$), then the map*

$$\begin{array}{rccc} \varphi\colon & \Sigma(X) \times Y & \to & \Sigma(X \times Y) \\ & ([(x,t)], y) & \mapsto & [((x,y),t)] \end{array}$$

is continuous.

In Figure 464 we illustrate Lemma 24.17. This cute picture is not meant to hide the grim reality that for arbitrary topological spaces X it actually requires a non-trivial input to prove Lemma 24.17.

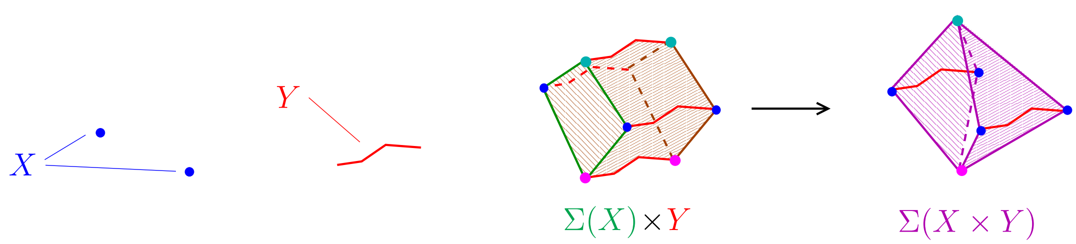


FIGURE 464. Illustration of Lemma 24.17.

PROOF. We denote by $p\colon (X \times Y) \times [-1,1] \to \Sigma(X \times Y)$ and $q\colon X \times [-1,1] \to \Sigma(X)$ the two obvious projection maps. We have the following commutative diagram

$$\begin{tikzcd} (X \times [-1,1]) \times Y \arrow[r, "(x,t,y)\mapsto(x,y,t)", "\cong"'] \arrow[d, "q\times \mathrm{id}_Y"'] & (X \times Y) \times [-1,1] \arrow[d, "p"] \\ \Sigma(X) \times Y \arrow[r, "\varphi"] & \Sigma(X \times Y). \end{tikzcd}$$

By Lemma 3.8 the top horizontal map is a homeomorphism. By Lemma 5.15 the maps p and q are quotient maps. Since Y is regionally compact we obtain from Theorem 5.16 that $q \times \mathrm{id}_Y$ is a quotient map. It follows from Lemma 5.17 that φ is also a quotient map. By Lemma 5.15 (2) this implies that φ is indeed continuous. ∎

[417]It takes a major effort to disentangle the notation. We write $[\,-\,]$ to indicate the equivalence class corresponding to a suspension and we write $[\,-\,]$ to indicate the equivalence class corresponding to the equivalence relation used to define mapping cones.

Proof of Proposition 24.16. Let $f\colon X \to Y$ be a map between topological spaces. One can easily verify that the given map Φ is a bijection. It remains to show that Φ and Φ^{-1} are continuous. We will do so in the following two claims. The domain and the target of Φ are both a scary mix of product and quotient topologies. Thus we need to tread super carefully.

Claim. The map Φ is continuous.

To prove the claim we introduce the following notation:

(a) We consider the map $\varphi\colon X \times \{1\} \to Y$ that is given by $\varphi(x, 1) = f(x)$.
(b) We denote by $p\colon \operatorname{Cone}(f\colon X \to Y) \times [-1, 1] \to \Sigma(\operatorname{Cone}(f\colon X \to Y))$ the projection map.

We consider the following diagram:

$$\begin{tikzcd}
(X\times[0,1])\times[-1,1] \arrow[d, "i"] & Y\times[-1,1] \arrow[dl, "j"] & \\
((X\times[0,1])\times[-1,1]) \cup_{\varphi\times \mathrm{id}_{[-1,1]}} (Y\times[-1,1]) \arrow[d, "{[(P,t)]\mapsto([P],t)}"] \arrow[ddddrr, bend left, "=:\Psi"] & & \\
((X\times[0,1]) \cup_\varphi Y)\times[-1,1] \arrow[d, "="] & & \\
\operatorname{Cone}(f\colon X\to Y)\times[-1,1] \arrow[d, "q"', "{([P],t)\mapsto[(P,t)]}"] & & \\
\Sigma(\operatorname{Cone}(f\colon X\to Y)) \arrow[rr, "\Phi"] & & \operatorname{Cone}(\Sigma(f)\colon \Sigma(X)\to\Sigma(Y)).
\end{tikzcd}$$

We make the following observations:

(1) By Lemma 5.19 we know that the vertical map between the second and the third row is a homeomorphism.
(2) It follows from (1) together with Lemma 3.22, applied to the projection map q, that Φ is continuous if Ψ is continuous.
(3) It follows from Lemma 3.44 (5) that Ψ is continuous if $\Psi \circ i$ and $\Psi \circ j$ are continuous.
(4) We have the following commutative diagram:

$$\begin{tikzcd}
Y\times[-1,1] \arrow[rr, "\Psi\circ i"] \arrow[dr, "{(P,t)\mapsto[(P,t)]}"'] & & \operatorname{Cone}(\Sigma(f)\colon \Sigma(X)\to\Sigma(Y)). \\
& \Sigma(Y) \arrow[ur, "{Q\mapsto[Q]}"'] &
\end{tikzcd}$$

The diagonal maps are continuous by Lemma 3.21 (3) and Lemma 3.44 (5). Thus we see that $\Psi \circ i$ is continuous. The same argument shows that $\Psi \circ j$ is continuous.

The combination of (2), (3) and (4) shows that Φ is indeed continuous. $\boxplus$

Claim. The map

$$\begin{array}{rcl} \overbrace{\mathrm{Cone}(\Sigma(f)\colon \Sigma(X)\to\Sigma(Y))}^{=((\Sigma(X)\times[0,1])\sqcup\Sigma(Y))/\sim} & \xrightarrow{\Phi^{-1}} & \Sigma(\mathrm{Cone}(f\colon X\to Y)) \\ P & \mapsto & \begin{cases} [([(x,s)],t)], & \text{if } P=[([(x,t)],s)] \text{ with } s\in[0,1] \text{ and} \\ & \text{with } [(x,t)]\in\Sigma(X), \text{ i.e. } x\in X,\ t\in[-1,1], \\ [([y],t)], & \text{if } P=[([y],t)] \text{ with } t\in Y,\ t\in[-1,1]. \end{cases} \end{array}$$

is continuous.

By Lemma 3.44 it suffices to show that the maps

$$\begin{array}{rcl} \Sigma(X)\times[0,1] & \to & \Sigma(\mathrm{Cone}(f\colon X\to Y)) \\ ([(x,t)],s) & \mapsto & [([(x,s)],t)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \Sigma(Y) & \to & \Sigma(\mathrm{Cone}(f\colon X\to Y)) \\ [(y,t)] & \mapsto & [([y],t)] \end{array}$$

are continuous. The second map is continuous by Lemma 24.4 (1). Furthermore it follows from the combination of Lemma 24.17 with Lemma 24.4 (1) that the first map is continuous. ■

24.4. **The mapping torus.** In this last section we discuss the mapping torus. This construction gives rise to many interesting topological spaces. We will also see that determining the fundamental group of a mapping torus is somewhat challenging, but not impossible.

Definition. Let X be a topological space and let $f\colon X\to X$ be a map. We refer to

$$\mathrm{Tor}(X,f) \;:=\; (X\times[0,1])\,/\,\sim \qquad \text{where } (x,0)\sim(f(x),1) \text{ for all } x\in X,$$

as the *mapping torus of* (X,f). We refer to the map

$$\begin{array}{rcl} \mathrm{Tor}(X,f)=(X\times[0,1])/(x,0)\sim(f(x),1) & \to & S^1 \\ [(x,t)] & \mapsto & e^{2\pi i t} \end{array}$$

as *the natural projection onto* S^1.[a] Sometimes X is called the *fiber* of the mapping torus and f is called the *monodromy* of the mapping torus.

[a] It follows from Lemma 3.22 that the natural projection is indeed continuous.

The following lemma shows that the mapping torus is a pushout. This is very useful since it allows us to use all the results from Lemma 3.43.

Lemma 24.18. *Let X be a topological space and let $f\colon X\to X$ be a map. The natural maps*

$$\begin{array}{rcl} \alpha\colon X\times[0,1] & \to & \mathrm{Tor}(X,f) \\ (x,t) & \mapsto & [(x,t)] \end{array} \quad \textit{and} \quad \begin{array}{rcl} \beta\colon X & \to & \mathrm{Tor}(X,f) \\ x & \mapsto & [(x,0)] \end{array}$$

induce the following natural homeomorphism:

$$\begin{tikzcd} X\times\{0,1\} \arrow[r, "(x,t)\mapsto x"] \arrow[d, hook, "(x,t)\mapsto(x,t)"] & X \arrow[d] \arrow[ddr, bend left, "\beta"] & \\ X\times[0,1] \arrow[r] \arrow[drr, bend right, "\alpha"'] & (X\times[0,1])\cup_{X\times\{0,1\}} X \arrow[dr, "\cong", "\Psi"'] & \\ & & \mathrm{Tor}(X,f). \end{tikzcd}$$

Proof. First note that it follows from Lemma 3.43 (3) that Ψ is continuous. Next we consider the natural map

$$\begin{array}{rcl} \Phi\colon (X\times[0,1])/\sim & \to & (X\times[0,1])\cup_{X\times\{0,1\}} X \\ [(x,t)] & \mapsto & [(x,t)]. \end{array}$$

It follows from Lemma 3.43 (1) and Lemma 3.22 that Φ is continuous. One can easily verify that Φ and Ψ are inverses of one another. Thus we have shown that both maps are homeomorphisms. ■

For completeness' sake we state the following almost self-evident lemma which gets used, often subconsciously, on numerous occasions.

Lemma 24.19. (∗) *Let X be a topological space and let $f\colon X\to X$ be a map.*

(1) *The map*

$$\begin{array}{rcl} \Theta\colon (X\times\mathbb{R})/(x,r) \overset{\text{with } x\in X \text{ and } r\in\mathbb{R}}{\overset{\downarrow}{\sim}} (f(x),r+1) & \to & \operatorname{Tor}(X,f)=(X\times[0,1])/(x,0)\sim(f(x),1) \\ [(x,t)] & \mapsto & [(x,t)] \end{array}$$

is a homeomorphism.

(2) *For every interval $I\subset\mathbb{R}$ of length less than one, the map*

$$\begin{array}{rcl} \varphi\colon X\times I & \to & (X\times\mathbb{R})/\sim \ \xrightarrow{\Theta}\ \operatorname{Tor}(X,f) \\ (x,t) & \mapsto & [(x,t)] \end{array}$$

is an embedding. Furthermore, if I is closed respectively open, then φ is a closed respectively open embedding.

Proof. First note that it is straightforward to write down an inverse to Θ. The second statement follows from elementary arguments. We leave it to the reader to fill in the details. ■

The following lemma takes care of our two favorite properties of topological spaces, namely compactness and being Hausdorff.

Lemma 24.20. *Let X be a topological space and let $f\colon X\to X$ be a homeomorphism.*

(1) *If X is Hausdorff, then the mapping torus $\operatorname{Tor}(X,f)$ is Hausdorff.*
(2) *If X is compact, then the mapping torus $\operatorname{Tor}(X,f)$ is compact.*

Proof.

(1) This statement will be proved in Exercise 24.13.
(2) This statement follows immediately from Proposition 3.12 (2) and Lemma 3.21 (4). ■

Next we will see that several well-established topological spaces are actually mapping tori.

Examples.

(1) We consider the topological space $[0,1]$ and let $f\colon [0,1] \to [0,1]$ be the self-homeomorphism of $[0,1]$ that is given by $f(w) = 1-w$. It follows immediately from the definitions that the corresponding mapping torus $\mathrm{Tor}([0,1],f)$ is precisely the Möbius band as defined on page 204. We refer to Figure 465 for an illustration.

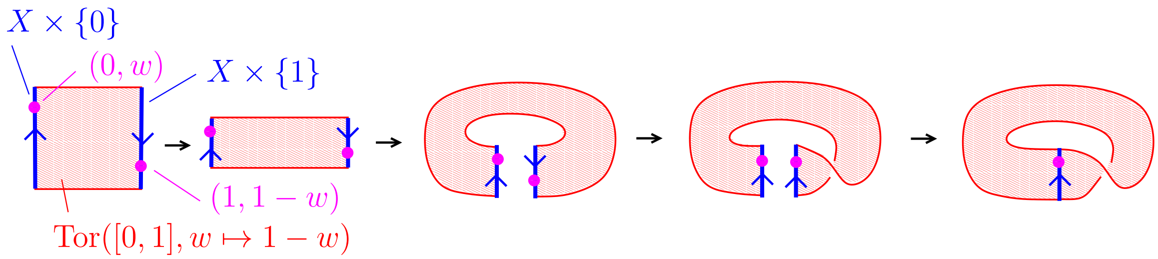


FIGURE 465. The Möbius band as a mapping torus.

(2) Let $X = S^1$ and let $f\colon X \to X$ be the homeomorphism that is given by $f(z) = \overline{z}$. It follows easily from Proposition 2.43 (3) and Lemma 24.20 (1) that the map

$$\begin{array}{rcl} \overbrace{([0,1]\times[0,1])\,/\,(x,0)\sim(x,1),(0,y)\sim(1,1-y)}^{\text{Klein bottle}} & \to & \mathrm{Tor}(S^1,f) \\ [(x,y)] & \mapsto & [(e^{2\pi \mathrm{i} y},x)] \end{array}$$

is a homeomorphism from the Klein bottle to the mapping torus $\mathrm{Tor}(S^1,f)$.[418] This homeomorphism is also illustrated in Figure 466.

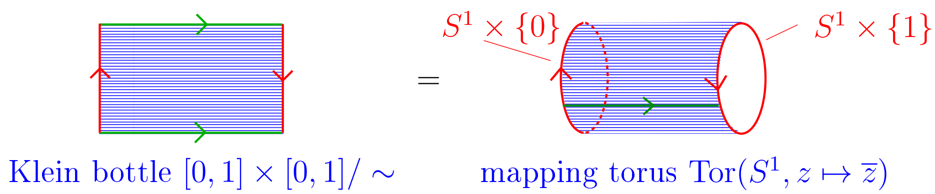


FIGURE 466

(3) As a final example we consider $X = \mathbb{Z}_3 = \{0,1,2\}$, equipped with the discrete topology. Let $f\colon X \to X$ be the homeomorphism that is given by $f(x) = x+1 \bmod 3$. As we illustrate in Figure 467, the mapping torus $\mathrm{Tor}(X,f)$ is homeomorphic to S^1.

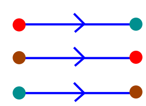
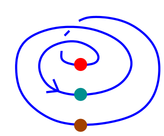
the mapping torus $\mathrm{Tor}(\mathbb{Z}_3, f(x) = x+1)$  is homeomorphic to 

FIGURE 467

We slow down a little to consider the case $\varphi = \mathrm{id}$ in more detail. On page 203 we already saw that $\mathrm{Tor}(S^1,\mathrm{id})$ is homeomorphic to the torus $S^1 \times S^1$. Thus the following lemma does not come as a surprise.

[418] What does the mapping torus $\mathrm{Tor}(S^1, z \mapsto -z)$ look like?

Lemma 24.21. *Given any topological space X the map*

$$\begin{array}{rcl}\Phi\colon \operatorname{Tor}(X,\mathrm{id}) & \to & X\times S^1\\ [(x,t)] & \mapsto & (x,e^{2\pi \mathrm{i} t})\end{array}$$

is a homeomorphism.

Convention. Let X be a topological space. We use the homeomorphism from Lemma 24.21 to make the identification $\operatorname{Tor}(X,\mathrm{id}) = X\times S^1$.

Proof (∗). It is easy to verify that Φ is a bijection. Using Remark 3.23 (1) one can easily show that Φ is continuous. It still remains to show that Φ is actually a homeomorphism.

First we provide an argument based on Proposition 5.23. First note that it follows from Proposition 2.43 (3) that the map $[0,1]/\{0,1\}\to S^1$ induced by $[t]\mapsto e^{2\pi\mathrm{i}}$ is a homeomorphism. We use this homeomorphism to replace S^1 by $[0,1]/\{0,1\}$. Next we consider the following diagram:

$$\begin{tikzcd} & X\times[0,1] \arrow[dl, "(x,t)\mapsto(x,{[t]})"', "p"] \arrow[dr, "(x,t)\mapsto{[(x,t)]}", "q"'] & \\ X\times([0,1]/\{0,1\}) \arrow[rr, "\Phi^{-1}", "(x,{[t]})\mapsto{[(x,t)]}"'] & & (X\times[0,1])/(x,0)\sim(x,1). \end{tikzcd}$$

By Proposition 5.23, applied to $Y=[0,1]$ and the compact subset $B=\{0,1\}$ we know that p is a quotient map. By Lemma 3.21 (3) we know that q is continuous. Since the diagram commutes we obtain from Lemma 5.17 that Φ^{-1} is continuous. We have thus shown that Φ is a homeomorphism.

In the rather unlikely event that the reader skipped Proposition 5.23 we also give a more direct argument. By Lemma 2.42 (2) it remains to show that Φ is an open map. So let U be an open subset of $\operatorname{Tor}(X,\mathrm{id}) = (X\times[0,1])/\sim$. Let $P:=[(x,t)]\in U$ be a point. We can assume that $t\in[0,1)$. We need to show that $Q:=\Phi([(x,t)])$ admits an open neighborhood Y that is contained in $V:=\Phi(U)$. In the following we denote by $p\colon X\times[0,1]\to\operatorname{Tor}(X,\mathrm{id})$ the projection map.

If $t\in(0,1)$, then by the definition of the topology on $\operatorname{Tor}(X,\mathrm{id})$ there exists an open subset $W\subset X$ and an open interval $(a,b)\subset(0,1)$ with $P\in W\times(a,b)\subset U$. We set $Y:=\Phi(W\times(a,b)) = W\times\{e^{2\pi\mathrm{i}t}\,|\,t\in(a,b)\}$. Note that $p^{-1}(Y)=Y$ is an open subset of $X\times[0,1]$. Thus Y has the desired property

Now suppose that $t=0$. In $X\times[0,1]$ we pick an open neighborhood $W_1\times[0,\epsilon_1)$ of the point $(x,0)$ that is contained in $p^{-1}(U)$. Similarly we pick an open neighborhood $W_2\times(1-\epsilon_2,1]$ of the point $(x,1)$ that is also contained in $p^{-1}(U)$. It is now straightforward to see that $Y:=(W_1\cap W_2)\times\{e^{2\pi\mathrm{i}s}\,|\,s\in(-\epsilon_2,\epsilon_1)\}$ has the desired property. ■

Now we continue the discussion of mapping tori with a few more examples.

Examples.

(1) In [**Rolf90**, Chapter 10.I.10] and [**BZH14**, Section 5.C] it is shown that the complement of the trefoil and the complement of the figure-8 knot are homeomorphic to a mapping torus, where the fiber is given by removing a point from the torus.

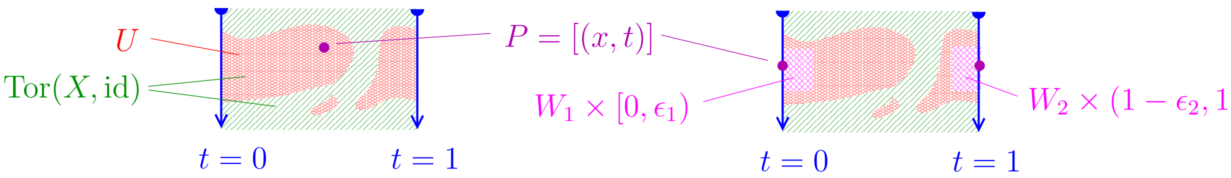


FIGURE 468

(2) If one wants to construct new interesting topological spaces out of a given topological space we need to find "non-trivial" self-homeomorphisms. For example let Σ be an oriented surface and let $C \subset \Sigma$ be a closed oriented curve on Σ, i.e. C is an oriented submanifold diffeomorphic to S^1. By the Tubular Neighborhood Theorem 8.24 there exists an orientation-preserving smooth embedding $\Phi\colon [-1,1] \times S^1 \to \Sigma$ such that $\Phi\colon \{0\} \times S^1 \to C$ is an orientation-preserving diffeomorphism. By Lemma 6.13 we can pick a smooth map $\gamma\colon [-1,1] \to [0, 2\pi]$ with $\gamma(-1) = 0$ and $\gamma(1) = 2\pi$ which is constant close to -1 and 1. We refer to the orientation-preserving diffeomorphism[419]

$$\begin{array}{rcl} \Sigma & \to & \Sigma \\ P & \mapsto & \begin{cases} P, & \text{if } P \notin \Phi([-1,1] \times S^1), \\ \Phi\big(t, z \cdot e^{\gamma(t)}\big) & \text{if } P = \Phi(t,z) \text{ for some } t \in [-1,1] \text{ and } z \in S^1, \end{cases} \end{array}$$

as a *Dehn*[420] *twist along the curve C*. The definition of a Dehn twist is illustrated in Figure 469.

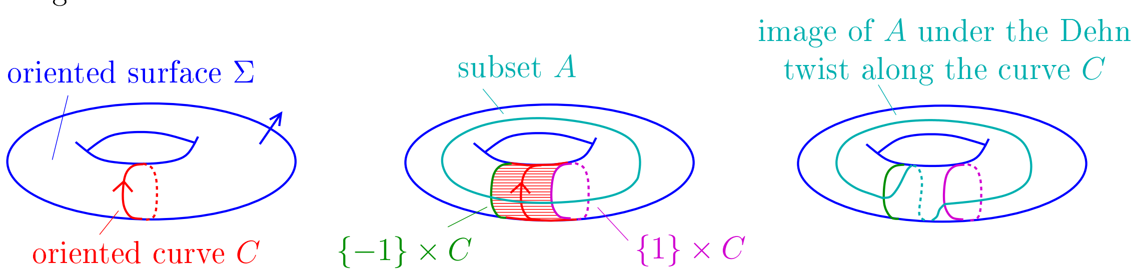


FIGURE 469

For completeness we also state the following lemma.

Lemma 24.22. *Let M be an n-dimensional smooth manifold and let $f\colon M \to M$ be a diffeomorphism. The following statements hold:*

(1) *The mapping torus* $\operatorname{Tor}(M, f)$ *is naturally an $(n+1)$-dimensional smooth manifold such that the obvious inclusion map $M \times (0,1) \to \operatorname{Tor}(M,f)$ is a smooth map.*
(2) *The boundary $\partial \operatorname{Tor}(M, f)$ is given by the mapping torus $\operatorname{Tor}(\partial M, f|_{\partial M})$, in particular, if M is closed, then $\operatorname{Tor}(M,f)$ is also closed.*

[419]One can easily verify that this map is a diffeomorphism. An inverse is given by a Dehn twist in the "opposite direction".

[420]Max Dehn (1878-1952) was a German–Jewish mathematician. In 1939 he fled to Norway. After the German occupation of Norway he fled to America via Siberia and Japan. He died in 1952 in Black Mountain, North Carolina.

(3) *If M is orientable and if f is orientation-preserving, then* $\operatorname{Tor}(M, f)$ *is also an orientable smooth manifold.*

Proof (∗). First consider the case that M has no boundary. By Proposition 6.51 (4) we know that $M \times [0,1]$ is an $(n+1)$-dimensional smooth manifold with boundary given by $M \times \{0,1\}$. Note that we obtain the mapping torus $\operatorname{Tor}(M, f)$ from the smooth manifold $M \times [0,1]$ by identifying the boundary component $M \times \{0\}$ with $M \times \{1\}$. Thus the statements are an immediate consequence of Proposition 8.15.

We will not make use of Lemma 24.22 when M has non-empty boundary. Thus we leave it to the curious reader to modify the above argument to the general case. ■

The lemma says in particular that self-diffeomorphisms of surfaces give rise to 3-dimensional smooth manifolds. This construction is an important source of examples of 3-dimensional smooth manifolds.

Definition. Let X and Y be topological spaces. Two homeomorphisms $f, g\colon X \to Y$ are called *isotopic*, if there exists an *isotopy* from f to g, i.e. a map

$$\begin{array}{rcl} H\colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & H(x,t) \end{array}$$

such that the following hold:

(1) We have $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$,
(2) for each $t \in [0,1]$ the map $X \to Y$ given by $x \mapsto H(x,t)$ is a homeomorphism.

Examples.

(1) We consider $X = [0,2]$ and the two self-homeomorphisms given by $f(x) = x$ and by $g(x) = \frac{1}{2} \cdot x^2$. Then[421]

$$\begin{array}{rcl} H\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & f(x) \cdot (1-t) + g(x) \cdot t \end{array}$$

is an isotopy from f to g.

(2) Lickorish [**Lic62**] showed that any self-homeomorphism of a compact orientable 2-dimensional smooth manifold is isotopic to the concatenation of finitely many Dehn twists.

The following lemma will be proved in Exercise 24.14.

Lemma 24.23. *Let X be a topological space. If $f, g\colon X \to X$ are two homeomorphisms that are isotopic, then the mapping tori* $\operatorname{Tor}(X, f)$ *and* $\operatorname{Tor}(X, g)$ *are homeomorphic.*

Example. We consider $X = S^1$ and the self-homeomorphism that is given by $g(z) = -z$. Note that the map $H(z,t) = e^{\pi i t} \cdot z$ defines an isotopy from $f = \mathrm{id}$ to g. It follows from Lemma 24.23 that the mapping torus $\operatorname{Tor}(S^1, g(z) = -z)$ is homeomorphic to the torus $\operatorname{Tor}(S^1, \mathrm{id}) = S^1 \times S^1$.

Now we can determine which surfaces are mapping tori.

[421] Let $t \in [0,1]$. Why is the map $[0,2] \to [0,2]$ given by $x \mapsto H(x,t)$ a homeomorphism?

Proposition 24.24. *The annulus, the Möbius band, the torus and the Klein bottle are, up to homeomorphism, the only connected, compact 2-dimensional topological manifolds that can be written as a mapping torus* $\operatorname{Tor}(X, f)$ *of a connected 1-dimensional topological manifold* X.

PROOF. By the examples (1), (2) on page 713 and by Lemma 24.21 we know that there exist the following types of homeomorphisms:

$$\begin{array}{rcl} \text{annulus } S^1 \times [0,1] & \cong & \operatorname{Tor}([0,1], \mathrm{id}) \\ \text{Möbius band} & \cong & \operatorname{Tor}([0,1], x \mapsto 1-x) \end{array} \quad \text{and} \quad \begin{array}{rcl} \text{torus } S^1 \times S^1 & \cong & \operatorname{Tor}(S^1, \mathrm{id}) \\ \text{Klein bottle} & \cong & \operatorname{Tor}(S^1, z \mapsto \overline{z}). \end{array}$$

In particular all these four smooth manifolds are homeomorphic to mapping tori. It remains to show that these are the only connected, compact 2-dimensional topological manifolds that can be written as a mapping torus.

It follows from Theorem 7.1 that any compact connected 1-dimensional topological manifold is homeomorphic to $[0,1]$ or to S^1. We know from Lemma 18.9 that any self-homeomorphism is isotopic to one of the self-homeomorphisms that we had written down above. But then it follows from Lemma 24.23 that any mapping torus is homeomorphic to one from the above list. ■

For proving statements about mapping tori the following lemma gives a useful alternative point of view. We leave the task of providing the elementary proof to the reader.

Lemma 24.25. *Let* X *be a topological space and let* $f\colon X \to X$ *be a homeomorphism. Then the following hold:*

(1) *The map*[422]
$$\begin{array}{rcl} \mathbb{Z} \times (X \times \mathbb{R}) & \to & X \times \mathbb{R} \\ (n, (x,t)) & \mapsto & (f^n(x), t+n) \end{array}$$
defines a continuous, proper and discrete action[423] *of the group* $\mathbb{Z}$ *on* $X \times \mathbb{R}$.

(2) *The map*
$$\begin{array}{rcl} \operatorname{Tor}(X,f) = (X \times [0,1])/\sim & \to & (X \times \mathbb{R})/\mathbb{Z} \\ [(x,t)] & \mapsto & [(x,t)] \end{array}$$
is a homeomorphism.

In the following, given a topological space X and a homeomorphism $f\colon X \to X$ we denote the corresponding quotient space by $(X \times \mathbb{R})/\mathbb{Z}_f$ and we will identify the mapping torus $\operatorname{Tor}(X, f)$ with $(X \times \mathbb{R})/\mathbb{Z}_f$ using the homeomorphism of Lemma 24.25.

Remark. It follows from Lemma 24.25 together with Proposition 6.32 that if X is a n-dimensional topological (respectively) smooth manifold and f is a homeomorphism, then the corresponding mapping torus $\operatorname{Tor}(X,f) = (X \times \mathbb{R})/\mathbb{Z}_f$ is a $(n+1)$-dimensional topological (respectively smooth) manifold. Furthermore, if X is in fact an n-dimensional smooth

---

[422] Here for $n \geq 1$ we write $f^n(x) = (\overbrace{f \circ \cdots \circ f}^{n\text{-times}})(x)$, for $n = 0$ we write $f^0(x) = x$ and for $n \leq -1$ we write $f^n(x) = (\underbrace{f^{-1} \circ \cdots \circ f^{-1}}_{|n|\text{-times}})(x)$.

[423] Why is it an action?

manifold and $f\colon X \to X$ is a diffeomorphism, then almost the same argument shows that the mapping torus $\mathrm{Tor}(X,f) = (X\times\mathbb{R})/\mathbb{Z}_f$ is an $(n+1)$-dimensional smooth manifold.

Lemma 24.26. *Let X be a topological space and let $f\colon X\to X$ be a homeomorphism. Then the map*

$$\begin{array}{rcl} X\times\mathbb{R} & \to & (X\times\mathbb{R})/\mathbb{Z}_f = \mathrm{Tor}(X,f) \\ (x,t) & \mapsto & [(x,t)] \end{array}$$

is a covering of infinite degree. Furthermore for $k\in\mathbb{N}$ the map

$$\begin{array}{rcl} \mathrm{Tor}(X,f^k) = (X\times\mathbb{R})/\mathbb{Z}_{f^k} & \to & (X\times\mathbb{R})/\mathbb{Z}_f = \mathrm{Tor}(X,f) \\ [(x,t)] & \mapsto & [(x,k\cdot t)] \end{array}$$

is a k-fold covering.

Proof. Let X be a topological space and let $f\colon X\to X$ be a homeomorphism. The first statement is an immediate consequence of Lemma 24.25 and Proposition 16.10.

Now let $k\in\mathbb{N}$. We consider the following commutative diagram of maps:

$$\begin{tikzcd} (X\times\mathbb{R})/\mathbb{Z}_{f^k} \arrow[rr, "{[(x,t)]\mapsto[(x,k\cdot t)]}"] \arrow[dr, "{[(x,t)]\mapsto[(x,k\cdot t)]}"'] & & (X\times\mathbb{R})/\mathbb{Z}_f. \\ & (X\times\mathbb{R})/(k\cdot\mathbb{Z})_f \arrow[ur, "{[(x,t)]\mapsto[(x,t)]}"'] & \end{tikzcd}$$

The left diagonal map is easily seen to be well-defined and to be a homeomorphism. By Proposition 16.10 the right diagonal map is a k-fold covering. It follows that the top horizontal map is also a k-fold covering. ■

Examples.

(1) We apply Lemma 24.26 to $X=[0,1]$ and the homeomorphism that is given by $f(x):=1-x$. Note that $f^2(x)=f(f(x))=f(1-x)=1-(1-x)=x$ for all x, i.e. $f^2=\mathrm{id}$. Thus we recover the statement from page 496 that the annulus $\mathrm{Tor}([0,1],\mathrm{id})$ is a 2-fold covering of the Möbius band $\mathrm{Tor}([0,1],f(x)=1-x)$.

(2) We apply Lemma 24.26 to $X=S^1$ and the self-homeomorphism of $X=S^1$ that is given by $f(z):=\overline{z}$. As in the previous example we see that $f^2=\mathrm{id}$. Thus we get, after the discussion on page 504, a new proof that the torus $\mathrm{Tor}(S^1,\mathrm{id})$ is a 2-fold covering of the Klein bottle $\mathrm{Tor}(S^1,f(z)=\overline{z})$.

Our next goal is to determine the fundamental groups of mapping tori. Before we can do so we need to introduce a new concept from group theory.

Definition. Let N be a group, let $k\in\mathbb{N}_0$ and let $\varphi\colon N\to N$ be an isomorphism such that $\varphi^k=\mathrm{id}$.[424] We define the *semidirect product of N and $\mathbb{Z}_k$ with respect to φ* as the group $N\rtimes_\varphi\mathbb{Z}_k$ where the underlying set is given by the direct product $N\times\mathbb{Z}_k$,[425] but where the group multiplication[426] is given by

$$(h,m)\cdot(h',m') := (h\cdot\varphi^m(h'),m+m').$$

If φ is understood from the context, then we drop it from the notation, i.e. we write $N\rtimes\mathbb{Z}_k$ instead of $N\rtimes_\varphi\mathbb{Z}_k$. If $\varphi\neq\mathrm{id}$, then we say that the semidirect product as *non-trivial.*

Examples.

(1) Let N be a group and $\varphi = \mathrm{id}_N$, then $N \rtimes_{\mathrm{id}} \mathbb{Z} = N \times \mathbb{Z}$ is the direct product of N and $\mathbb{Z}$.

(2) Let $G = \mathbb{Z}_2$ and let $N = \mathbb{Z}_n$ for some $n \in \mathbb{N}_0$. We denote by $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ the isomorphism that is given by multiplication by -1. Clearly this isomorphism satisfies $\varphi^2 = \mathrm{id}$, thus we can form the semidirect product $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$, which is called the *$n$-th dihedral group D_n*. It has $2n$ elements. As an example, the third dihedral group D_3 is isomorphic to the permutation group S_3 via the isomorphism [427]

$$\begin{array}{rcl} \Psi\colon \mathbb{Z}_3 \rtimes \mathbb{Z}_2 & \to & S_3 \\ (h,m) & \mapsto & (1\ 2\ 3)^h \cdot (1\ 2)^m. \end{array}$$

(3) Let $k \in \mathbb{N}_{\geq 2}$. In Exercise 31.17 we will see that we can write the direct product $\langle x_1, \dots, x_k\rangle \times \mathbb{Z}$ in many "inequivalent" ways as a semidirect product of a finitely generated free group with $\mathbb{Z}$.

Remark. In Exercise 24.16 we will see that $N = \{(h,0) \mid h \in N\} \subset N \rtimes \mathbb{Z}_k$ is a normal subgroup of $N \rtimes \mathbb{Z}_k$. Put differently, we have $N \triangleleft N \rtimes \mathbb{Z}_k$. This is the reason for the symbol "$\rtimes$" in the notation of the semidirect product.

We have the following elementary lemma.

Lemma 24.27. *Let G be a group and let $\gamma\colon G \to \mathbb{Z}$ be an epimorphism. We pick an element $t \in G$ with $\gamma(t) = 1$ and we write $N := \ker(\gamma)$. We denote by φ the isomorphism*

$$\begin{array}{rcl} \varphi\colon N & \to & N \\ h & \mapsto & tht^{-1}. \end{array}$$

Then the map[428]

$$\begin{array}{rcl} \Psi\colon G & \to & N \rtimes_\varphi \mathbb{Z} \\ g & \mapsto & \big(gt^{-\gamma(g)}, \gamma(g)\big) \end{array}$$

is an isomorphism of groups.

Proof. It is straightforward to verify that Ψ is injective and surjective. It remains to show that Ψ is a homomorphism: given $g, h \in G$ we have indeed

$$\begin{aligned} \Psi(g\cdot h) = (ght^{-\gamma(gh)}, \gamma(gh)) &= (gt^{-\gamma(g)} \cdot \underbrace{t^{\gamma(g)} \cdot ht^{-\gamma(h)} \cdot t^{-\gamma(g)}}_{=\varphi^{\gamma(g)}(ht^{-\gamma(h)})}, \gamma(g)+\gamma(h)) \\ &= (gt^{-\gamma(g)}, \gamma(g)) \cdot (ht^{-\gamma(h)}, \gamma(h)) = \Psi(g)\cdot\Psi(h). \end{aligned}$$

[424]Note that this condition is automatically satisfied if $k = 0$, i.e. for any isomorphism φ of any group we always have $\varphi^0 = \mathrm{id}$.

[425]In particular, if N is finite and $k \in \mathbb{N}$, then the cardinality of $N \rtimes \mathbb{Z}_k$ is given by $|N| \cdot k$.

[426]It is an elementary argument to show that $N \rtimes_\varphi \mathbb{Z}_k$ with the given multiplication is indeed a group.

[427]First note that a straightforward calculation shows that $(1\,2)^{-1} \cdot (1\,2\,3) \cdot (1\,2) = (1\,3\,2) = (1\,2\,3)^{-1}$. This implies that also for any $m \in \mathbb{Z}_2$ and $s \in \mathbb{Z}$ we have $(1\,2)^{-m} \cdot (1\,2\,3)^s \cdot (1\,2)^m = (1\,2\,3)^{s(-1)^m}$. Now for any $(h,m), (h',m') \in \mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$ we have

$$\begin{aligned} \Psi(h,m)\Psi(h',m') &= (1\,2\,3)^h \cdot (1\,2)^m \cdot (1\,2\,3)^{h'} \cdot (1\,2)^{m'} = (1\,2\,3)^h \cdot (1\,2)^m \cdot (1\,2)^{-m} \cdot (1\,2\,3)^{(-1)^m h'} \cdot (1\,2)^m \cdot (1\,2)^{m'} \\ &= (1\,2\,3)^{h+(-1)^m h'} \cdot (1\,2)^{m+m'} = \Psi(h+(-1)^m h', m+m') = \Psi((h,m)\cdot(h',m')). \end{aligned}$$

We have thus shown that Ψ is a homomorphism.

[428]Note that $\gamma(gt^{-\gamma(g)}) = \gamma(g) + \gamma(t)(-\gamma(g)) = 0$, i.e. $gt^{-\gamma(g)}$ does indeed lie in $N = \ker(\gamma)$.

■

The following proposition shows that the fundamental group of a mapping torus $\operatorname{Tor}(X, f)$ is isomorphic to a semidirect product of $\pi_1(X)$ with $\mathbb{Z}$.

Proposition 24.28. *Let X be some non-empty path-connected topological space and furthermore let $f\colon X \to X$ be some homeomorphism. Then there exists an isomorphism $\psi\colon \pi_1(\operatorname{Tor}(X, f)) \cong \pi_1(X) \rtimes \mathbb{Z}$ such that the following diagram commutes*

$$\begin{tikzcd} \pi_1(X) \arrow[r] \arrow[d] & \pi_1(\operatorname{Tor}(X,f)) \arrow[r] \arrow[d, "\cong"', "\psi"] & \pi_1(S^1) \arrow[d, "\cong"] \\ \pi_1(X) \arrow[r] & \pi_1(X) \rtimes \mathbb{Z} \arrow[r] & \mathbb{Z}. \end{tikzcd}$$

Here the top-left horizontal map is induced by the inclusion $X = X \times \{0\} \to \operatorname{Tor}(X, f)$ and the top-right horizontal map is induced by the canonical projection map $\operatorname{Tor}(X, f) \to S^1$. The bottom-left horizontal map is given by the natural inclusion into the first term of the semidirect product and the bottom-right horizontal map is given by the projection onto the second term of the semidirect product.[429]

Proof. We have

$$\pi_1(\operatorname{Tor}(X, f)) \underset{\substack{\uparrow \\ \text{Lemma 24.25}}}{=} \pi_1((X \times \mathbb{R})/\mathbb{Z}_f) \underset{\substack{\uparrow \\ \text{Theorem 16.16 (2) and 24.27}}}{\cong} \pi_1(X \times \mathbb{R}) \rtimes \mathbb{Z} \underset{\substack{\uparrow \\ \text{Proposition 18.16}}}{\cong} \pi_1(X) \rtimes \mathbb{Z}.$$

Thus we have obtained an isomorphism $\psi\colon \pi_1(\operatorname{Tor}(X, f)) \xrightarrow{\cong} \pi_1(X) \rtimes \mathbb{Z}$. We leave the verification that the given diagram commutes to the reader. ■

Remark. A careful reading of the proof of the proposition shows that we can identify the automorphism in the semidirect product. More precisely, let X be a path-connected topological space and let $f\colon X \to X$ be a homeomorphism. Let x be a base point and let γ be path from $f(x)$ to x. We denote by φ the automorphism of $\pi_1(X, x)$ that is given by

$$\begin{tikzcd} \pi_1(X,x) \arrow[r, "f_*"] \arrow[rr, bend right, "=:\varphi"'] & \pi_1(X, f(x)) \arrow[r, "\cong", "\text{Prop. 14.11}"'] & \pi_1(X,x). \end{tikzcd}$$

Then $\pi_1(\operatorname{Tor}(X, f)) \cong \pi_1(X, x) \rtimes_\varphi \mathbb{Z}$.

We conclude this section with a discussion of several examples.

Examples.

(1) Let $K = \operatorname{Tor}(S^1, f(z) := \overline{z})$ be the Klein bottle. Then the automorphism $f_* =: \varphi$ of $\pi_1(S^1) = \mathbb{Z}$ is multiplication by -1. It follows from the previous remark[430] that

$$\pi_1(\text{Klein bottle}) \;\cong\; \mathbb{Z} \rtimes_\varphi \mathbb{Z}.$$

[429] Note that both maps on the bottom are group homomorphisms.

[430] On page 656 we saw that $\pi_1(K) \cong \langle x, y \,|\, yxyx^{-1} \rangle$. What is an explicit isomorphism between the groups $\langle x, y \,|\, yxyx^{-1} \rangle$ and $\mathbb{Z} \rtimes_\varphi \mathbb{Z}$?

(2) Let $A \in \mathrm{GL}(n,\mathbb{Z})$ be a matrix. Recall that in Exercise 16.16 we showed that the map $f(A)\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ given by $[v] \mapsto [Av]$ is a homeomorphism. Furthermore we know that, under the usual identification $\pi_1(\mathbb{R}^n/\mathbb{Z}^n,0) = \mathbb{Z}^n$ as in Theorem 16.16, the induced map $f(A)_*$ on $\pi_1(\mathbb{R}^n/\mathbb{Z}^n,0) = \mathbb{Z}^n$ is given by multiplication by A. It follows from the above remark that $\pi_1(\mathrm{Tor}(\mathbb{R}^n/\mathbb{Z}^n, f(A))) \cong \mathbb{Z}^n \rtimes_A \mathbb{Z}$. Note that this implies that $\pi_1(\mathrm{Tor}(\mathbb{R}^n/\mathbb{Z}^n, f(A)))$ is solvable.[431]

(3) Let X be a connected topological space. We calculate that

$$\pi_1(X\times S^1) \underset{\substack{\uparrow\\ \text{Lemma 24.21}}}{=} \pi_1(\mathrm{Tor}(X,\mathrm{id}_X)) \underset{\substack{\uparrow\\ \text{by the above remark}}}{\cong} \pi_1(X)\rtimes_{\mathrm{id}}\mathbb{Z} = \pi_1(X)\times\mathbb{Z} = \pi_1(X)\times\pi_1(S^1).$$

So in this special case we obtain the same result as in Proposition 16.20.

Exercises for Chapter 24.

Exercise 24.1. Given a subset X of $\mathbb{R}^n$ we define its *physical cone* as

$$\widetilde{\mathrm{Cone}}(X) \;:=\; \text{union of all line segments from } \underbrace{(x,1)}_{\in\mathbb{R}^{n+1}} \text{ to } \underbrace{(0,0)}_{\in\mathbb{R}^{n+1}} \subset \mathbb{R}^{n+1}$$

and we consider the map

$$\begin{aligned} f_X\colon \mathrm{Cone}(X) &\to \widetilde{\mathrm{Cone}}(X)\\ [(x,t)] &\mapsto t\cdot(x,1) + (1-t)\cdot(0,1). \end{aligned}$$

(a) Let X be a compact subset of $\mathbb{R}^n$. Show that the map f_X is a homeomorphism.
(b) Let $X = (0,1) \subset \mathbb{R}$. Show that the map f_X is not a homeomorphism.
Remark. It is actually true that f_X is continuous and a bijection. To show that f_X is not a homeomorphism we need to show that there exists an open subset U of $\mathrm{Cone}(X)$ such that the image under f_X is not open. You could try your hand at

$$U \;:=\; \mathrm{Cone}(X) \setminus \{[(\tfrac{1}{n},\tfrac{1}{n})]\,|\,n\in\mathbb{N}_{\geq 2}\}.$$

We refer to Figure 470 for an illustration.

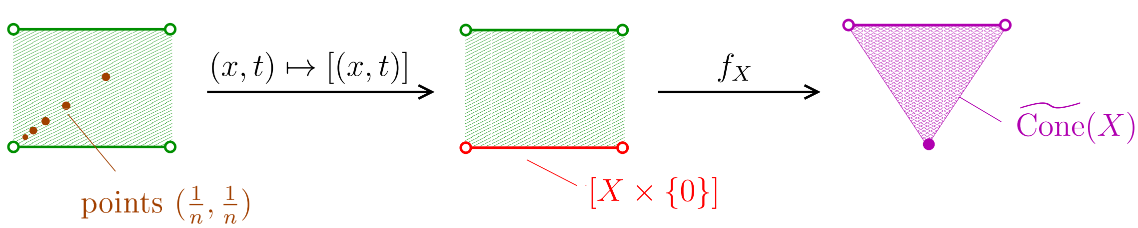


FIGURE 470

[431]Recall that a group G is solvable if there exist subgroups $G_0 = \{1\} \subset G_1 \subset G_2 \subset \cdots \subset G_k = G$ such that for $i = 0,\ldots,k-1$ the group G_i is normal in G_{i+1} and such that G_{i+1}/G_i is abelian. In our case the filtration is given by $\{0\} \subset \mathbb{Z}^n \subset \mathbb{Z}^n \rtimes_A \mathbb{Z}$.

Exercise 24.2. Let X be a topological space. Show that $\operatorname{Cone}(X)$ respectively the suspension $\Sigma(X)$ are homeomorphic to the pushout of the diagram

$$\begin{tikzcd} X \arrow[r, "x\mapsto(x,1)"] \arrow[d] & X\times[0,1] \\ \{*\} & \end{tikzcd} \quad \text{respectively} \quad \begin{tikzcd} X \arrow[r, "x\mapsto[(x,0)]"] \arrow[d, "x\mapsto[(x,0)]"'] & \operatorname{Cone}(X) \\ \operatorname{Cone}(X) & \end{tikzcd}$$

Remark. These descriptions of the cone and the suspension explain our definitions for the case that X is the empty topological space.

Exercise 24.3. Let X be a topological space.

(a) Show that the cone $\operatorname{Cone}(X)$ is naturally homeomorphic to the join $X * \{*\}$.
(b) Show that the suspension $\Sigma(X)$ is naturally homeomorphic to the join $X * S^0$.

Exercise 24.4. Let X be a topological space. Show that if X is Hausdorff, then the suspension $\Sigma(X)$ is also Hausdorff.

Exercise 24.5. Let X be a topological space. Show that the injections

$$\begin{array}{rcl} X & \to & \operatorname{Cone}(X) \\ x & \mapsto & [(x,1)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \operatorname{Cone}(X) & \to & \Sigma(X) \\ [(x,t)] & \mapsto & [(x,1-t)] \end{array}$$

are embeddings.

Exercise 24.6. Let $f\colon X\to Y$ be a map between topological spaces. Show that the natural inclusions

$$\begin{array}{rcl} X & \to & \operatorname{Cyl}(f\colon X\to Y) \\ x & \mapsto & (x,0) \end{array} \quad \text{and} \quad \begin{array}{rcl} Y & \to & \operatorname{Cyl}(f\colon X\to Y) \\ y & \mapsto & [y] \end{array}$$

are embeddings.

Exercise 24.7. Let $f\colon X\to Y$ be a map between topological spaces and let Z be a topological space.

(a) Show that the map

$$\Theta\colon \operatorname{Cyl}(f\times\mathrm{id}\colon X\times Z\to Y\times Z) \to \operatorname{Cyl}(f\colon X\to Y)\times Z$$
$$P \mapsto \begin{cases} ([(Q,t)],s), & \text{if } P=[(Q,t,s)] \text{ with } Q\in X, t,s\in[0,1], \\ ([Q],s), & \text{if } P=[(Q,s)] \text{ with } Q\in Y, s\in[0,1]. \end{cases}$$

is continuous.

(b) Suppose that Z is regionally compact. Show that the map Θ from (a) is a homeomorphism.
Hint. Use Lemma 5.18.

Exercise 24.8. Let X be a topological space and let A be a subset of X.

(a) We denote by $i\colon A\to X$ the inclusion. Show that the map

$$\begin{array}{rcl} A\times[0,1] & \to & \operatorname{Cyl}(i) = ((A\times[0,1])\sqcup X)/\sim \\ (a,t) & \mapsto & [(a,t)] \end{array}$$

is an embedding.

(b) Let $f\colon A \to X$ be an embedding. Is the map $A \times [0,1] \to \operatorname{Cyl}(f)$ necessarily an embedding?

Exercise 24.9. Give a proof of Lemma 24.10 (3).

Exercise 24.10. We consider the topological space $X = [-1,1]$, the subset $A = (-1,1)$ and the inclusion map $i\colon A \to X$. Show that the induced map $\Sigma(i)\colon \Sigma(A) \to \Sigma(X)$ is *not* an embedding.
Remark. It might help to have a look at the equally scary Exercise 24.1.

Exercise 24.11. Let X and Y be topological spaces. We consider the corresponding join $X * Y = (X \times [0,1] \times Y)/\sim$ as defined on page 207. Show that the map

$$\begin{array}{rcl} X * Y = (X \times [0,1] \times Y)/\sim & \cong & (\operatorname{Cone}(X) \times Y) \cup_{(X\times\{1\})\times Y = X \times (Y \times \{1\})} (X \times \operatorname{Cone}(Y)) \\ {[(x,t,y)]} & \mapsto & \begin{cases} (x, [(y,2t)]), & \text{if } t \in [0,\frac{1}{2}], \\ ([(x,2(t-1))], y), & \text{if } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

is a homeomorphism.
Remark. Evidently you need to verify that all the maps are actually continuous.

Exercise 24.12. Let $f\colon A \to X$ be a map between topological spaces and let $a_0 \in A$. We write $x_0 := f(a_0)$. We suppose that A is path-connected. Show that the natural inclusion map $X \to \operatorname{Cone}(f\colon A \to X))$ induces an isomorphism

$$\pi_1(X, x_0)/f_*(\pi_1(A, a_0)) \ \to \ \pi_1(\operatorname{Cone}(f\colon A \to X)).$$

Exercise 24.13. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. We assume that X is Hausdorff. Show that the mapping torus $\operatorname{Tor}(X, f)$ is also Hausdorff.

Exercise 24.14. Let X be a topological space. Show that if $f, g\colon X \to X$ are two homeomorphisms that are isotopic, then the mapping tori $\operatorname{Tor}(X, f)$ and $\operatorname{Tor}(X, g)$ are homeomorphic.

Exercise 24.15. Let

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{\varphi} & \pi & \xrightarrow{\psi} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 1 & \longrightarrow & \widetilde{K} & \xrightarrow{\widetilde{\varphi}} & \widetilde{\pi} & \xrightarrow{\widetilde{\psi}} & \widetilde{G} & \longrightarrow & 1 \end{array}$$

be a commutative diagram of group homomorphisms with the following properties:

(1) φ and $\widetilde{\varphi}$ are monomorphisms.
(2) ψ and $\widetilde{\psi}$ are epimorphisms.
(3) $\operatorname{im}(\varphi) = \ker(\psi)$ and $\operatorname{im}(\widetilde{\varphi}) = \ker(\widetilde{\psi})$.

Show that if two out of the three vertical maps α, β and γ are isomorphisms, then so is the third one.
Remark. We will generalize this statement later on in the Five Lemma 43.12.

Exercise 24.16. Let N be a group, let $k \in \mathbb{N}_0$ and let $\varphi\colon N \to N$ be an isomorphism such that $\varphi^k = \mathrm{id}$. As on page 718 we define the *semidirect product of N and $\mathbb{Z}_k$ with respect to*

φ as the group $N \rtimes_\varphi \mathbb{Z}_k$ where the underlying set is given by the direct product $N \times \mathbb{Z}_k$, and where the multiplication is given by

$$(h, m) \cdot (h', m') \ := \ (h \cdot \varphi^m(h'), m + m').$$

(a) Show that $N \rtimes_\varphi \mathbb{Z}_k$ is indeed a group.
(b) Show that $N = \{(h, 0) \in N \rtimes_\varphi \mathbb{Z}_k \,|\, h \in N\}$ is a normal subgroup of $N \rtimes \mathbb{Z}_k$.
(c) Is $\mathbb{Z}_k = \{(0, n) \in N \rtimes_\varphi \mathbb{Z}_k \,|\, n \in \mathbb{Z}_k\} \subset N \rtimes_\varphi \mathbb{Z}_k$ necessarily a normal subgroup?
(d) Suppose that $k = 0$. Let $n \in \mathbb{N}$ and let K be a subgroup of N of some finite index m. Show that if K is preserved by φ, then $K \rtimes n\mathbb{Z}$ is a subgroup of $N \rtimes \mathbb{Z}$ of index $m \cdot n$.

Exercise 24.17. Let π be a group and let $\phi\colon \pi \times \langle t \rangle \to \mathbb{Z}$ be an epimorphism. We suppose that $k := \phi(t) \neq 0$. We set $\Gamma := \ker(\pi \xrightarrow{\phi} \mathbb{Z} \to \mathbb{Z}_k)$. Since ϕ is an epimorphism there exists an $x \in \pi$ such that $\gcd(\phi(x), k) = 1$. Let $\mu\colon \Gamma \to \Gamma$ be the automorphism that is given by $g \mapsto xgx^{-1}$. Show that the map

$$\begin{array}{rcl} \Gamma \rtimes_\mu \mathbb{Z} & \to & \pi \times \mathbb{Z} \\ (g, n) & \mapsto & (gx^n, n + \phi(g)) \end{array}$$

is an isomorphism.
Hint. You could use Exercise 24.15 together with Exercise 24.16 (d).
Remark.

(1) By Exercise 21.2 and Proposition 37.15 we know that if π is finitely generated (presented), then so is Γ. The exercise thus says that if a finitely presented group π admits an epimorphism onto $\mathbb{Z}$, then we can write a product $\pi \times \mathbb{Z}$ in many interesting and perhaps surprising ways as a semidirect product.
(2) We will prove a manifold analogue in Proposition 79.12.

Exercise 24.18. Let (X, x_0) be a path-connected topological space and let $f\colon X \to X$ be a homeomorphism with $f(x_0) = x_0$. Show that there exists an isomorphism of the following form:

$$\pi_1(\operatorname{Tor}(X, f))_{\text{ab}} \ \cong \ \mathbb{Z} \oplus \operatorname{coker}\big(\pi_1(X, x_0)_{\text{ab}} \xrightarrow{f_* - \text{id}} \pi_1(X, x_0)_{\text{ab}}\big).$$

Exercise 24.19. Let $\varphi\colon \overline{B}^2 \to \overline{B}^2$ be the map that is given by $(x, y) \mapsto (x, -y)$. Is the corresponding mapping torus $\operatorname{Tor}(\overline{B}^2, \varphi)$ homeomorphic to the solid torus?

Exercise 24.20. Let G and π be groups and let $\varphi\colon \pi \to \operatorname{Aut}(G)$ be a homomorphism. We consider the set $G \times \pi$ with the multiplication

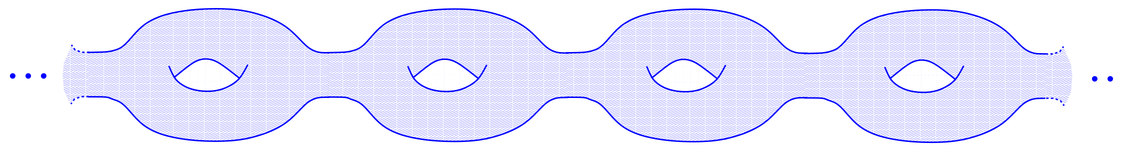
$$(g, a) \cdot (h, b) \ := \ (g \cdot \varphi(a)(h), a \cdot b).$$

(a) Show that $G \times \pi$ with the above multiplication is a group. This group is denoted by $G \rtimes_\varphi \pi$ and it is called the *semidirect product of G and π with respect to φ*.
(b) In what sense does this definition of the semidirect product of two groups generalize the concept introduced on page 718?
(c) Let K be a group and let $\alpha\colon K \to \pi$ be an epimorphism onto a group π. We suppose that α admits a right-inverse, i.e. we suppose that there exists a homomorphism $\varphi\colon \pi \to K$ with $\alpha \circ \varphi = \text{id}_\pi$. Show that K is isomorphic to a semidirect product of the form $\ker(\alpha) \rtimes \pi$.

(d) (i) Let $n \in \mathbb{N}$. Show that $\mathrm{GL}(n, \mathbb{R})$ is isomorphic to a semidirect product of the form $\mathrm{SL}(n, \mathbb{R}) \rtimes (\mathbb{R} \setminus \{0\})$.
(ii) For which $n \in \mathbb{N}$ is $\mathrm{GL}(n, \mathbb{R})$ isomorphic to a direct product $\mathrm{SL}(n, \mathbb{R}) \rtimes (\mathbb{R} \setminus \{0\})$?
(iii) Repeat (i) and (ii) with $\mathbb{C}$ instead of $\mathbb{R}$.

25. Limits

The goal of this section is to determine the fundamental group of "unbounded spaces", e.g. we would like to determine the fundamental group of the surface of "infinite genus" illustrated in Figure 471.[432] One idea might be to view the surface of "infinite genus" as the limit of a sequence given by compact surfaces. But then we need to make precise what we mean by a "limit of topological spaces" and "limit of the corresponding fundamental groups". We will do so in the following sections.



surface of "infinite genus"

Figure 471

25.1. **Preordered and directed sets.**

The following definition partly recalls some notions that we introduced on page 73.

Definition. Let X be a set and let $\leq$ be a relation on X.

(1) The relation is called *reflexive* if for every $x \in X$ we have $x \leq x$.
(2) The relation is called *transitive* if for every $x, y, z \in X$ we have
$$x \leq y \text{ and } y \leq z \implies x \leq z.$$
(3) A reflexive and transitive relation is called a *preorder*.
(4) A set together with a preorder is called a *preordered set*.
(5) A *directed set* is a preordered set such that for any $x, y \in X$ there exists a $z \in X$ with $x \leq z$ and $y \leq z$.

Examples.

(1) Let X be a subset of $\mathbb{R}$ and denote by $\leq$ the usual "less or equal" relation on the real numbers. Then $(X, \leq)$ is a directed set. In particular $(\mathbb{N}, \leq)$ is a directed set.
(2) Given any set X the relation "$\leq$" where $x \leq y$ only if $x = y$ is a preorder, which we call the *trivial preorder* on X.
(3) Let M be a set. We denote by $\mathcal{P}(M)$ the power set of M, i.e. the set of all subsets of M. Then "being a subset", i.e. the relation given by "$\subseteq$", defines a preorder on $\mathcal{P}(M)$. Thus $(\mathcal{P}(M), \subseteq)$ is a preordered set, in fact it is straightforward to see that it is even a directed set.
(4) The set $X = \{0, x, y\}$ with the trivial relations $0 \leq 0, x \leq x, y \leq y$ and the non-trivial relations $0 \leq x$ and $0 \leq y$ is a preordered set. We refer to $(\{0, x, y\}, \leq)$ as the *push-pull set*. Note that this is not a directed set, since there is no $z \in X$ with $x \leq z$ and $y \leq z$.

[432] Actually we also need to make precise what we mean by the surface of "infinite genus".

(5) We consider the set $\mathbb{N}$. The divisibility relation $k|l$ for $k,l \in \mathbb{N}$ is a preorder on $\mathbb{N}$ and with this preorder $\mathbb{N}$ is in fact a directed set.[433]

Exercise 25.1. We consider the following subset of $\mathbb{R}^2$:

$$X \;:=\; \bigcup_{n\in\mathbb{N}} \text{circle of radius } \tfrac{1}{n} \text{ around } (\tfrac{1}{n},0) \;=\; \bigcup_{n\in\mathbb{N}} \{(x,y)\in\mathbb{R}^2 \,|\, (x-\tfrac{1}{n})^2+y^2=\tfrac{1}{n^2}\}.$$

This subset, equipped with the subspace topology is colloquially referred to as the *Hawaiian earrings*, see Figure 472 for an illustration.

(a) Let $A=\{0\}\cup\{\frac{1}{n}\,|\,n\in\mathbb{N}\}$. Show that X is homeomorphic to $[0,1]/A$.
Remark. We encountered the topological space $[0,1]/A$ in Exercise 3.31.
(b) Show that X is compact.
(c) Is X homeomorphic to the wedge of countably infinitely many circles?
(d) Show that $\pi_1(X)$ is infinitely generated.
(e) Even better than (d): Show that $\pi_1(X)$ is uncountably generated, i.e. show that there is no countable generating set for $\pi_1(X)$.

The combination of (b) and (d) implies, perhaps somewhat surprisingly, that there exist compact topological spaces whose fundamental groups are infinitely generated.
Remark. The precise determination of $\pi_1(X)$ is actually rather delicate, we refer to **[MoM86]** and **[Smi92]** for details. In particular, despite appearances, it is shown in **[Smi92]** that $\pi_1(X)$ is not a free group.

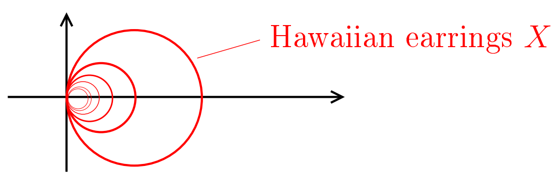


FIGURE 472. Illustration for Exercise 25.1.

Exercise 25.2. For $n\in\mathbb{N}$ let C_n be the circle in $\mathbb{R}^2\times\{0\}$ of radius $\frac{1}{2n}$ with center $(\frac{1}{n},0)$. The *harmonic archipelago* H is the topological space that obtained from $\mathbb{R}^2$ as follows: For each $n\in\mathbb{N}$ we delete an open disk between C_n and C_{n+1} and we glue on a cone such that the z-coordinate of the cone equals 1. We refer to Figure 473 for an illustration.

(a) Give a precise description of the harmonic archipelago H.
Remark. You might want to skip this exercise, unless you are inclined to super-rigorous mathematics.
(b) Show that the harmonic archipelago H is non-compact.
(c) Let $x_0=(0,0)$ be the origin. Show that for each n the inclusion induced map $\pi_1(C_n,x_0)\to\pi_1(H,x_0)$ is a monomorphism.
(d) As in Exercise 25.1 we refer to $X:=\bigcup_{n\in\mathbb{N}} C_n$ as the Hawaiian earrings. Show that the inclusion induced map $\pi_1(X,x_0)\to\pi_1(H,x_0)$ is not a monomorphism.
(e) Is $\pi_1(H,x_0)$ countable or uncountable?

[433]Indeed, given $k,l\in\mathbb{N}$ we have $k|kl$ and $l|kl$.

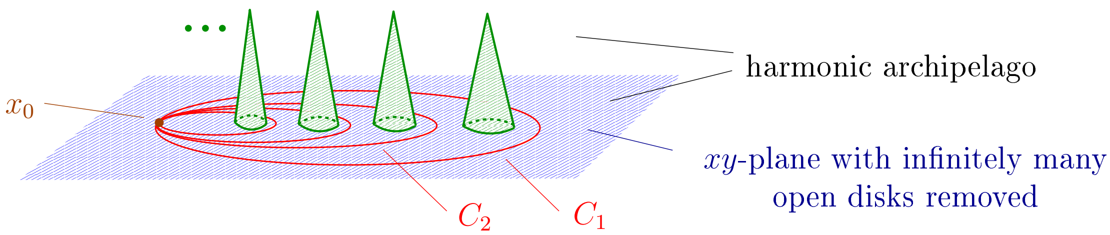


FIGURE 473. Illustration of Exercise 25.2.

25.2. **The direct limit of a direct system.**

Definition. Let $(I, \leq)$ be a preordered set[434] and let $\mathcal{C}$ be a category. A *direct system in the category* $\mathcal{C}$ *over* I is a family of objects $\{X_i\}_{i\in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ij}: X_i \to X_j\}$ for all $i, j \in I$ with $i \leq j$ such that the following two conditions are satisfied:

$$\begin{array}{llll} (1) & f_{ii} = \mathrm{id}_{X_i} & \text{for all } i \in I, \\ (2) & f_{ik} = f_{jk} \circ f_{ij} & \text{for all } i, j, k \in I \text{ with } i \leq j \leq k. \end{array}$$

Examples.

(A) We consider the directed set $I = (\mathbb{N}, \leq)$ and the category of groups. For $i \in \mathbb{N}$ we consider the group $\mathbb{Z}^i$ and for $i \leq j$ we denote by

$$\begin{array}{rcl} f_{ij}: \mathbb{Z}^i & \to & \mathbb{Z}^j \\ (x_1, \ldots, x_i) & \mapsto & (x_1, \ldots, x_i, 0, \ldots, 0) \end{array}$$

the inclusion map. Then $(\{\mathbb{Z}^i\}_{i\in\mathbb{N}}, \{f_{ij}\}_{i\leq j})$ forms clearly a direct system.

(B) Again we consider $I = (\mathbb{N}, \leq)$ and the category of groups. Given $i \in \mathbb{N}$ we define

$$F_i := \langle x_1, \ldots, x_i \rangle = \text{the free group on } i \text{ generators}$$

and given $i \leq j$ we set

$$\begin{array}{rcl} f_{ij}: F_i & \to & F_j \\ x_k & \mapsto & x_k, \text{ for } k = 1, \ldots, i. \end{array}$$

This is clearly a direct system of groups.

(C) Given $r \in \mathbb{N} \cup \{\infty\}$ we set

$$X_r := \big(0, r + \tfrac{1}{2}\big) \cup \bigcup_{j=1}^{r} \underbrace{\big\{z \in \mathbb{C} \,\big|\, \big|z - (j + \tfrac{1}{4}i)\big| = \tfrac{1}{4}\big\}}_{\text{circle of radius } \frac{1}{4} \text{ around } j + \frac{1}{4}i \in \mathbb{C}} \subset \mathbb{C}.$$

For example, for $r \in \mathbb{N}$ the set X_r is the subset of $\mathbb{C} = \mathbb{R}^2$ that is given by the interval $(0, r + \frac{1}{2})$ with r circles attached. We refer to Figure 474 for an illustration.

Now we consider the directed set $(\mathbb{N}, \leq)$ and for $r \leq s$ we denote by $f_{rs}: X_r \to X_s$ the obvious inclusion map. This is clearly a direct system of topological spaces.

[434]In the literature one often demands that I is a directed set, see e.g. [**Hat02**, p. 243], but this is not necessary.

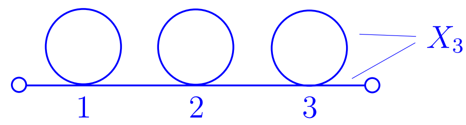


FIGURE 474

(D) We can generalize example (C) as follows. Let $(I, \leq)$ be a preordered set and let $\{X_i\}_{i\in I}$ be a family of topological spaces such that for any $i \leq j$ we have $X_i \subset X_j$. Given $i \leq j$ we denote by $\iota_{ij}\colon X_i \to X_j$ the inclusion map. Then $(\{X_i\}_{i\in I}, \{\iota_{ij}\}_{i\leq j})$ is a direct system in the category of topological spaces.

(E) Let $(\{0, x, y\}, \leq)$ be the push-pull set and let $\mathcal{C}$ be a category. A direct system in the category $\mathcal{C}$ over the preordered set $(\{0, x, y\}, \leq)$ consists of three objects A_0, A_x, A_y, the identity morphisms $\mathrm{id}_{A_0}, \mathrm{id}_{A_x}, \mathrm{id}_{A_y}$ and morphisms $f_x\colon A_0 \to A_x$ and $f_y\colon A_0 \to A_y$. We refer to such a system as a *pushout system* and often we arrange the morphisms in a diagram as follows

$$\begin{tikzcd} A_0 \arrow[r, "f_x"] \arrow[d, "f_y"'] & A_x \\ A_y. \end{tikzcd}$$

Definition. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be a category. Suppose we are given a direct system $(\{X_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ in the category $\mathcal{C}$. A *direct limit*[435] of the direct system is an object Y in $\mathcal{C}$ together with morphisms $g_i\colon X_i \to Y, i \in I$ in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i \leq j$ we have $g_j \circ f_{ij} = g_i\colon X_i \to Y$, i.e. for all $i \leq j$ the following diagram commutes

$$\begin{tikzcd} X_j \arrow[dr, "g_j"] & \\ & Y. \\ X_i \arrow[uu, "f_{ij}"] \arrow[ur, "g_i"'] & \end{tikzcd}$$

(2) If we are given another object Y' and morphisms $g'_i\colon X_i \to Y', i \in I$ that satisfy (1), then there exists a unique morphism $F\colon Y \to Y'$ such that for all $i \leq j$ the following diagram commutes[436]

$$\begin{tikzcd} X_j \arrow[dr, "g_j"] \arrow[drr, bend left, "g'_j"] & & \\ & Y \arrow[r, dashed, "F", "\exists!"'] & Y'. \\ X_i \arrow[uu, "f_{ij}"] \arrow[ur, "g_i"'] \arrow[urr, bend right, "g'_i"'] & & \end{tikzcd}$$

The "usual argument" for objects that satisfy a universal property[437] shows that if the direct limit exists, then it is unique in an appropriate sense. We denote the direct limit by $\varinjlim X_i$ or sometimes by $\varinjlim_{I} X_i$.[438]

Examples. Now we discuss the direct limits of the direct systems we had discussed above.

(A) We consider the above direct system in the category of groups that is given by the free abelian groups $\mathbb{Z}^i$, $i \in \mathbb{N}$, with the obvious inclusion homomorphisms $\mathbb{Z}^i \to \mathbb{Z}^j$ for $i \leq j$. We claim that[439]

$$\varinjlim \mathbb{Z}^i \;=\; \mathbb{Z}^{(\mathbb{N})} \;=\; \{(x_1, x_2, \dots) \mid x_i \in \mathbb{Z} \text{ but only finitely many } x_i\text{'s are non-zero}\},$$

where the maps $\mathbb{Z}^j \to \varinjlim \mathbb{Z}^i = \mathbb{Z}^{(\mathbb{N})}$, $j \in \mathbb{N}$ are the obvious inclusion maps. Indeed, given a group Y' and homomorphisms $g'_j \colon \mathbb{Z}^j \to Y'$ that satisfy condition (1) we define

$$\begin{aligned} F \colon \varinjlim \mathbb{Z}^i \;=\; \mathbb{Z}^{(\mathbb{N})} &\;\to\; Y' \\ (x_1, \dots, x_j, 0, \dots) &\;\mapsto\; g'_j(x_1, \dots, x_j). \end{aligned}$$

It is straightforward to see that this definition is well-defined, i.e. independent of the choice of x_j.[440] It is clear that the diagram (2) commutes. Furthermore, since $(x_1, \dots, x_j, 0, \dots) = g_j(x_1, \dots, x_j)$ it is clear that F is the unique homomorphism that makes the diagram (2) commute.

Thus we have now shown that the direct limit is the free abelian group on an infinite countable generating set.

(B) We consider the above direct system in the category of groups that is given by the free groups $F_i := \langle x_1, \dots, x_i \rangle$, $i \in \mathbb{N}$, with the obvious inclusion homomorphisms $F_i \to F_j$ for $i \leq j$. Similar to the proof of (A) one can show that

$$\varinjlim F_i \;=\; \langle x_1, x_2, x_3, \dots \rangle \;=\; \text{free group on the generators } x_1, x_2, x_3, \dots$$

where the maps $F_j \to \varinjlim F_i = \langle x_1, x_2, \dots \rangle$, $j \in \mathbb{N}$ are the obvious maps. Put differently, the direct limit is the free group on an infinite countable generating set.

(C) We continue with the direct system $(X_r, r \in \mathbb{N})$ in the category of topological spaces from above. Perhaps not surprisingly the direct limit is the interval $(0, \infty)$ with infinitely many circles attached, i.e.

$$\varinjlim X_r \;=\; X_\infty \;=\; \big(0, \infty\big) \;\cup\; \bigcup_{j=1}^{\infty} \big\{ z \in \mathbb{C} \,\big|\, \big|z - (j + \tfrac{1}{4} i)\big| = \tfrac{1}{4} \big\} \;\subset\; \mathbb{C},$$

---

[435] Somewhat confusingly a "direct limit" is often also called a "projective limit" or "colimit of a direct system".

[436] The notation $\exists!$ means that there exists a uniquely defined morphism that makes the diagram commute.

[437] See for instance the discussion on page 582 for an example of such an argument.

[438] Here we ignore the maps f_{ij} and g_i in the notation. Usually these maps are clear from the context.

[439] Why is the direct limit not given by $\mathbb{Z}^{\mathbb{N}}$, i.e. the group of all infinite sequences?

[440] Here we did not demand that $x_j \neq 0$.

where the maps $X_s \to \varinjlim X_r = X_\infty$, $s \in \mathbb{N}$ are the obvious inclusion maps. We leave the verification of this statement to the reader.

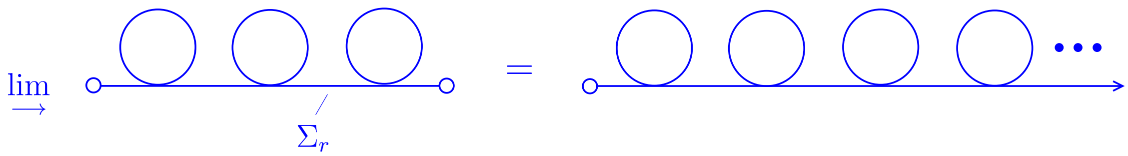


FIGURE 475

(D) Let $(I, \leq)$ be a preordered set and let $\{X_i\}_{i\in I}$ be a family of topological spaces such that for any $i \leq j$ we have $X_i \subset X_j$. In the category of topological spaces we have

$$\varinjlim X_i \;=\; X \;:=\; \bigcup_{i\in I} X_i,$$

where the topology on $X = \bigcup_{i\in I} X_i$ is given by the rule that $U \subset X$ is open if and only if $U \cap X_i$ is open for all $i \in I$. Furthermore the maps $g_j\colon X_j \to X = \bigcup_{i\in I} X_i$, $j \in \mathbb{N}$ are the obvious inclusion maps.[441] The verification of this statement is Exercise 25.3.

Definition. If the direct limit of a pushout system

$$\begin{tikzcd} A_0 \arrow[r, "f_x"] \arrow[d, "f_y"'] & A_x \\ A_y \end{tikzcd}$$

exists, then we refer to it as the *pushout* of the *pushout system*. The defining property of the pushout Z can be summarized in the following commutative diagram[442]

$$\begin{tikzcd} A_0 \arrow[r, "f_x"] \arrow[d, "f_y"'] & A_x \arrow[d, "g_x"] \arrow[ddr, bend left, "g_x'"] & \\ A_y \arrow[r, "g_y"] \arrow[drr, bend right, "g_y'"'] & \text{pushout } Z \arrow[dr, dashed, "\exists!"] & \\ & & Z'. \end{tikzcd}$$

Examples.

(1) We first consider the pushout in the category of groups. By Proposition 21.21 the pushout of a pushout system in the category of groups

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \\ B \end{tikzcd} \qquad \text{is given by} \qquad \begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d] \\ B \arrow[r] & A *_G B = \dfrac{A * B}{\langle\langle \{\alpha(g)\cdot\beta(g)^{-1}\}_{g\in G}\rangle\rangle}. \end{tikzcd}$$

---

[441] Why are the maps g_j continuous, i.e. why are they morphisms in the category of topological spaces.

[442] Here we suppress the morphism $g_0\colon A_0 \to Z$ from the diagram since it is given by $g_x \circ f_x = g_y \circ f_y$.

(2) Now we consider the pushout in the category of topological spaces. In Lemma 3.43 we already showed, in ever so slightly different words, that the pushout of the pushout system of topological spaces

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z & \end{tikzcd} \quad \text{is given by} \quad \begin{tikzcd} X \arrow[rr, "f"] \arrow[d, "g"'] & & Y \arrow[d] \\ Z \arrow[r] & Y \cup_X Z := (Y \sqcup Z)/f(x) \sim g(x). & \end{tikzcd}$$

Here we equip $Y \cup_X Z$ with the quotient topology induced from the obvious topology on $Y \sqcup Z$.

(3) The direct limit of a direct system does not always exist. For example, denote by $\mathcal{C}$ the category with two objects, namely the trivial group and the group $\mathbb{Z}_2$, and where the morphisms are the usual group homomorphisms. Then the pushout system

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \end{tikzcd} \quad \begin{array}{c}\text{does not have a direct limit}\\ \text{since there is no commutative}\\ \text{diagram of the form}\end{array} \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z}_2 \arrow[d, "\beta"] \\ \mathbb{Z}_2 \arrow[r, "\alpha"] & C \end{tikzcd}$$

with $C = 0$ or $C = \mathbb{Z}_2$ such that in the following two diagrams there exists the desired diagonal maps from C to $Y' = \mathbb{Z}_2$:[443]

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z}_2 \arrow[d, "\beta"] \arrow[ddr, bend left, "x\mapsto x"] & \\ \mathbb{Z}_2 \arrow[r, "\alpha"] \arrow[drr, bend right, "x\mapsto 0"'] & C \arrow[dr] & \\ & & Y' = \mathbb{Z}_2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z}_2 \arrow[d, "\beta"] \arrow[ddr, bend left, "x\mapsto 0"] & \\ \mathbb{Z}_2 \arrow[r, "\alpha"] \arrow[drr, bend right, "x\mapsto x"'] & C \arrow[dr] & \\ & & Y' = \mathbb{Z}_2. \end{tikzcd}$$

Proposition 25.1. *The direct limit of any direct system exists in the following categories:*

(0) *the category of sets,*
(1) *the category $\mathcal{Top}$ of topological spaces,*
(2) *the category $\mathcal{Gr}$ of groups,*
(3) *the category $\mathcal{AbGr}$ of abelian groups,*
(4) *the category $\mathcal{Ring}$ of rings,*
(5) *the category $R - \mathcal{Mod}$ of R-modules, where R is a commutative ring.*

SKETCH OF PROOF. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be one of the six given categories. Furthermore let $(\{X_i\}_{i\in I}, \{f_{ij}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in $\mathcal{C}$.

(0),(1) If we work in the category of sets, then we define the direct limit as the set

$$\varinjlim X_i \ := \ \Big(\bigsqcup_{i\in I} X_i\Big)/ \sim \ \text{ where } x \sim f_{ij}(x) \text{ for all } i \leq j,\ x \in X_i.$$

[443] Why is there no such diagram with $C = 0$ or $C = \mathbb{Z}_2$ and homomorphisms α and β?

If we work in the category of topological spaces, then $\bigsqcup_{i\in I} X_i$ is equipped with the topology where a subset U is open if and only if $U \cap X_i$ is open for all i.[444]

(2) If we work in the category of groups, then we cannot use the definition of (0) and (1), since the union of groups is not a group. Therefore we define the direct limit via the "smallest group that contains all X_i", i.e. we define it as the group

$$\varinjlim X_i \;:=\; \Big(\mathop{\ast}_{i\in I} X_i\Big) / \,\langle\langle \{ f_{ij}(x)\cdot x^{-1} \mid i\le j \text{ and } x\in X_i\}\rangle\rangle.$$

(3),(4),(5) If we work in the category of abelian groups, or in the category of rings or in the category of R-modules, then we define[445]

$$\varinjlim X_i \;:=\; \Big(\bigoplus_{i\in I} X_i\Big) / \,\{ f_{ij}(x) - x \mid \; i\le j \text{ and } x\in X_i\}.$$

Furthermore in all six cases we denote by $g_j\colon X_j \to \varinjlim X_i$, $j\in I$ the obvious maps. It is now straightforward to verify that in all five categories we provided a direct limit. We leave the details to the reader. ∎

Remark.

(1) Suppose we are given a direct system $(\{X_i\}_{i\in I}, \{f_{ij}\colon X_i\to X_j\}_{i\le j})$ in any of the four algebraic categories mentioned in the previous proposition. Now we assume that $(I,\le)$ is not only a preordered set, but that it is in fact a directed set. Then, as for direct limits of sets and topological spaces, we can also define

$$\varinjlim X_i \;:=\; \Big(\bigsqcup_{i\in I} X_i\Big)/\sim \quad \text{where } x\sim f_{ij}(x) \text{ for all } i\le j,\ x\in X_i$$

and we denote by $g_j\colon X_j\to \varinjlim X_i$, $j\in I$, the obvious maps. If we work in the category of groups, then $\varinjlim X_i$ is naturally a group as follows: if we are given $a\in X_i$ and $b\in X_j$, by definition of a directed set there exists $k\in I$ with $i\le k$ and $j\le k$, and we define

$$[a]\cdot[b] \;:=\; \big[\underbrace{f_{ik}(a)}_{\in X_k}\cdot \underbrace{f_{jk}(b)}_{\in X_k}\big] \;\in\; \varinjlim X_i.$$

It follows easily from the definition of a direct system that this definition of the product structure is independent of the choice of k and that this defines indeed a group structure on $\varinjlim X_i$. Similarly we deal with the case that we work in the other three algebraic categories.

(2) An important special case is given by $I=\mathbb{N}_0$ and where the maps $f_{ij}\colon X_i\to X_j$ are inclusion maps. In this case one can in fact take

$$\varinjlim X_i \;:=\; \bigcup_{i\in\mathbb{N}_0} X_i.$$

[444]Note that for the above example of the pushout we do *not* get the same topological space (why not?) but we get a topological space that is homeomorphic (why?) to the above pushout. Since we only care about the direct limit up to homeomorphism this makes no difference.

[445]Here we really mod out by the subgroup generated by the differences $f_{ij}(x)-x$.

The algebraic structure is defined as in the previous remark.

We continue our discussion with the following lemma that gives a particularly simple but commonly used calculation of a direct limit.

Lemma 25.2. *Let $f_n\colon X_n \to X_{n+1}$ be a sequence of morphisms in any category. Suppose there exists an $N \in \mathbb{N}$ such that all f_n for $n \geq N$ are isomorphisms. Then $\varinjlim X_n$ exists and it is naturally isomorphic to X_N, more precisely, a direct limit is given by $\varinjlim X_n := X_N$ and the maps*

$$\begin{array}{rcll} f_{N-1} \circ \cdots \circ f_{n+1} \circ f_n\colon X_n & \to & X_N & \text{if } n \leq N, \\ (f_n \circ \cdots \circ f_{N+1} \circ f_N)^{-1}\colon X_n & \to & X_n & \text{if } n > N \end{array}$$

define a direct limit.

Proof. We will provide the proof in Exercise 25.2. ■

25.3. **The direct limit in the topological category.** In this short section we discuss a few results about direct limits that are specific to the category of topological spaces. We start out with the discussion of an important example of direct limits in the topological category.

Lemma 25.3. *Let $(\{X_i\}_{i\in\mathbb{N}_0}, \{f_{ij}\}_{i\leq j})$ be a direct system in the category of topological spaces. We assume that all the maps $f_{ij}\colon X_i \to X_j$ are inclusion maps.*

(1) *In this case one can in fact take*

$$\varinjlim X_i \ := \ \bigcup_{i\in\mathbb{N}_0} X_i$$

where the topology is given, as in the proof of Proposition 25.1, by the condition that $U \subset X$ is open if and only if each $U \cap X_i$ is an open subset of X_i.

(2) *For each $i \in \mathbb{N}$ the obvious map $X_i \to \varinjlim X_i$ is an embedding.*

Proof (∗).

(1) The argument is the same as in the proof of Proposition 25.1 (1).
(2) Let $i \in \mathbb{N}$. The map $X_i \to \varinjlim X_i$ is evidently injective and continuous. By Lemma 2.42 (1) it remains to show that the map is an open map. Thus let $U_i \subset X_i$ be an open subset. By definition of the subspace topology there exists an open subset $U_{i+1} \subset X_{i+1}$ such that $U_{i+1} \cap X_i = U_i$. We iterate this procedure to obtain subsets $U_i, U_{i+1}, U_{i+2}, \ldots$. We set $U := \bigcup\limits_{j\geq n} U_j$. Evidently $U \cap X_i = U_i$. ■

Finally we conclude this section with two lemmas which show that in the category of topological spaces direct limits commute, under favorable circumstances, with direct products and quotients.

Lemma 25.4. (∗) *Let $(I, \leq)$ be a preordered set and let $(\{X_i\}_{i\in I}, \{f_{ij}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in the category of topological spaces. We denote by $g_i\colon X_i \to \varinjlim X_i$ the canonical maps. Furthermore let Y be a topological space. By definition of the direct limit* *the maps*

$$g_i \times \mathrm{id}_Y \colon X_i \times Y \;\to\; (\varinjlim X_i) \times Y$$

induce a map

$$\varinjlim (X_i \times Y) \;\to\; \big(\varinjlim X_i\big) \times Y.$$

If Y is regionally compact, then this map is a homeomorphism.

Example. Let $f\colon X \to Y$ and $g\colon X \to Z$ be maps between topological spaces. These maps define a pushout system, and taking the product with $[0,1]$ also defines a pushout system. We get the following two diagrams:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "i"] \\ Z \arrow[r, "j"] & X \cup_Y Z \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X\times[0,1] \arrow[r, "f\times\mathrm{id}"] \arrow[d, "g\times\mathrm{id}"'] & Y\times[0,1] \arrow[d, "I"] \arrow[ddr, bend left, "i\times\mathrm{id}"] & \\ Z\times[0,1] \arrow[r, "J"] \arrow[drr, bend right, "j\times\mathrm{id}"'] & X\times[0,1]\cup_{Y\times[0,1]} Z\times[0,1] \arrow[dr, dashed] & \\ & & (X\cup_Y Z)\times[0,1]. \end{tikzcd}$$

By the property of the pushout we have the diagonal map. Since $[0,1]$ is regionally compact we obtain from Lemma 25.4, or more directly from Lemma 5.18, that the diagonal map is a homeomorphism.

Sketch of proof. This lemma follows easily from the explicit description of the direct limit in the category of topological spaces that we gave in Proposition 25.1 together with Lemma 5.18. ■

Lemma 25.5. (∗) *Let $(I,\leq)$ be a preordered set and let $(\{X_i\}_{i\in I}, \{f_{ij}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in the category of topological spaces. Let G be a group. Suppose that for each $i \in I$ we are given an action of G on X_i such that given any $i \leq j$ the following diagram commutes:*

$$\begin{tikzcd} G \times X_i \arrow[r] \arrow[d, "\mathrm{id}_G \times f_{ij}"] & X_i \arrow[d, "f_{ij}"] \\ G \times X_j \arrow[r] & X_j. \end{tikzcd}$$

It follows almost immediately from this data that G also acts on $\varinjlim X_i$. We denote by $g_i\colon X_i \to \varinjlim X_i$ the canonical maps. By definition of the direct limit the corresponding maps

$$X_i/G \;\to\; (\varinjlim X_i)/G$$

induce a map

$$\varinjlim (X_i/G) \;\to\; \big(\varinjlim X_i\big)/G.$$

This map is a homeomorphism.

Sketch of proof. This lemma follows easily from the explicit description of the direct limit in the category of topological spaces that we gave in Proposition 25.1. ■

25.4. **Fundamental groups and direct limits.** In this section we want to address the following question:

Question 25.6. *Let $(\{X_i\}_{i\in I}, \{f_{ij}\}_{i\le j})$ be a direct system of topological spaces. Now we can form the direct limit $\varinjlim X_i$ of the topological spaces. The maps f_{ij} also induce homomorphisms on fundamental groups. The question is, when does "taking limits commute with taking fundamental groups", i.e. under what circumstances do we have*[446]

$$\pi_1\big(\varinjlim X_i\big) \;=\; \varinjlim \pi_1(X_i)\ ?$$

Example. We have already seen two instances where the equality of the question holds.

(1) Suppose X is a topological space that is the union of two open subsets U and V such that U, V and $U\cap V$ are path-connected. We fix a base point in $U\cap V$. Then

$$\pi_1\left(\varinjlim \begin{array}{c} U\cap V\to U \\ \downarrow \\ V \end{array}\right) \underset{\substack{\uparrow \\ \text{see page 732}}}{=} \pi_1(X) \underset{\substack{\uparrow \\ \text{Seifert–van Kampen} \\ \text{Theorem 22.1}}}{=} \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \underset{\substack{\uparrow \\ \text{see page 731}}}{=} \varinjlim \begin{array}{c} \pi_1(U\cap V)\to \pi_1(U) \\ \downarrow \\ \pi_1(V) \end{array}$$

(2) Let X be a path-connected topological space and let X_k, $k\in\mathbb{N}$ be a sequence of subsets of X such that the following hold:
   (a) each X_k is open,
   (b) each X_k is simply connected,
   (c) the sequence X_k is nested, i.e. for each k we have $X_k\subset X_{k+1}$,
   (d) we have $\bigcup\limits_{k\in\mathbb{N}} X_k = X$.

   Then for any $w\in X_1$ we have[447]

$$\pi_1\big(\varinjlim X_i, w\big) \underset{\substack{\uparrow \\ \text{see Lemma 25.3}}}{=} \pi_1\Big(\bigcup_{i\in\mathbb{N}} X_i, x_0\Big) \underset{\substack{\uparrow \\ \text{Lemma 20.6}}}{=} \{e\} \;=\; \varinjlim \{e\} \;=\; \varinjlim \pi_1(X_i, w).$$

The following proposition generalizes the second example.

Proposition 25.7. *Let $X_i, i\in\mathbb{N}$ be a sequence of topological spaces such that the following holds:*

(1) *the sequence is nested, i.e. for each $i\in\mathbb{N}$ we have $X_i\subset X_{i+1}$,*

and such that one of the following two conditions holds:

(2) *every compact subset of $\varinjlim X_i$ is already contained in one of the X_i, or*

(3) *each X_i is open in X_{i+1}.*

[446] As it is this question is not well-phrased, since fundamental groups depend on base points and we ignored them in the above discussion.

[447] Recall that by the discussion on page 729 the topological spaces $\{X_i\}_{i\in\mathbb{N}}$ together with the inclusion maps form a direct system in the category of topological spaces. The direct limit is given by the union of all X_i's, where the union is given a suitable topology. Furthermore note that the inclusion induced maps $\pi_1(X_i, w)\to\pi_1(X_j, w)$ turn the groups $\{\pi_1(X_i, w)\}_{i\in\mathbb{N}}$ into a direct system in the category of groups.

Then for any $w \in X_1$ we have

$$\pi_1\big(\varinjlim X_i, w\big) \;=\; \varinjlim \pi_1(X_i, w).$$

Remark. We will see that the proof of Proposition 25.7 is a generalization of the ideas behind the proof of Lemma 20.6.

The following lemma shows that the hypothesis (3) in Proposition 25.7 implies in fact hypothesis (2).

Lemma 25.8. *Let $X_i, i \in \mathbb{N}$ be a sequence of topological spaces such that each X_i is open in X_{i+1}. Then every compact subset of $\varinjlim X_i$ is already contained in one of the X_i.*

PROOF OF LEMMA 25.8. Let $X_i, i \in \mathbb{N}$ be a nested sequence of topological spaces such that each X_i is open in X_{i+1}.

Claim. For every $i \in \mathbb{N}$ the subset X_i is open in X.

By the definition of the topology on X we have to show that for each $j \in \mathbb{N}$ the intersection $X_i \cap X_j$ is open in X_j. Since the sequence is nested it follows that for $j \leq i$ we have $X_i \cap X_j = X_j$, in particular $X_i \cap X_j$ is open in X_j. Now we need to show that X_i is open in any X_{i+k} with $k \in \mathbb{N}_0$. We do so by induction on k. For $k = 0$ the statement is clear. Suppose we already know that X_i is open in X_{i+k}. Then X_i is an open subset in X_{i+k} and X_{i+k} is in turn an open subset of X_{i+k+1} by our hypothesis. It follows that X_i is open in X_{i+k+1}.[448] $\boxplus$

Now let $K \subset \varinjlim X_i$ be a compact subset. From the claim we know that each X_i is open in $X = \varinjlim X_i$ we obtain immediately from Lemma 2.41 and the hypothesis that the X_i's are nested that there exists a $k \in \mathbb{N}$ such that $K \subset X_i$. ■

The proof of Proposition 25.7 also relies on the following lemma that we will use implicitly and explicitly on numerous occasions.

Lemma 25.9. *Let (X, x_0) be a pointed topological space.*

(1) *Given any $g \in \pi_1(X, x_0)$ there exists a compact subset K of X with $x_0 \in K$ such that g lies in the image of the inclusion induced map $\pi_1(K, x_0) \to \pi_1(X, x_0)$.*
(2) *Let Y be a subset of X with x_0. If $g \in \pi_1(Y, x_0)$ lies in the kernel of the inclusion induced map $\pi_1(Y, x_0) \to \pi_1(X, x_0)$, then there exists a compact subset K of X such that g lies in the kernel of the inclusion induced map $\pi_1(Y, x_0) \to \pi_1(Y \cup K, x_0)$.*

PROOF.

(1) Let $g \in \pi_1(X, x_0)$. We represent g by a loop $\gamma\colon [0, 1] \to X$. By Lemma 2.40 the image $\gamma([0, 1])$ is compact. Thus $K := \gamma([0, 1])$ has the desired property.
(2) Now suppose that we are given $g = [\gamma] \in \pi_1(Y, x_0)$ that lies in the kernel of the inclusion induced map $\pi_1(Y, x_0) \to \pi_1(X, x_0)$. By definition this means that there exists a path-homotopy $H\colon [0, 1] \times [0, 1] \to X$ from the path γ to the constant path.

[448] Here we use the following general fact: if U is an open subset in V and V is an open subset of W, then U is an open subset of W. Why is that true?

We appeal again to Lemma 2.40 to deduce that $K := H([0,1] \times [0,1])$ is a compact subset of X. Evidently K has the desired property. ■

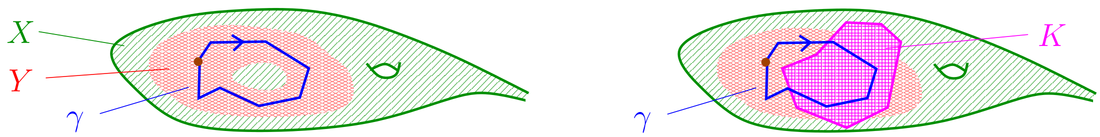


FIGURE 476. Illustration for Lemma 25.9.

Now we can provide the proof Proposition 25.7.

PROOF OF PROPOSITION 25.7. By Lemma 25.8 we only have to prove the proposition under the hypothesis (2). Thus let $X_i, i \in \mathbb{N}$ be a nested sequence of topological spaces such that every compact subset of $\varinjlim X_i$ is already contained in one of the X_i. Let $w \in X_1$. We write $X = \bigcup\limits_{i \in I} X_i = \varinjlim X_i$.

Given $k \leq l$ we denote by φ_{kl} the inclusion induced map $\pi_1(X_k, w) \to \pi_1(X_l, w)$. We denote by $\Phi\colon \varinjlim \pi_1(X_i, w) \to \pi_1(X, w)$ the unique homomorphism that makes the following diagram commute for any $k \leq l$:

$$\begin{tikzcd}
\pi_1(X_l,w) \arrow[dr, "\psi_l"] \arrow[drr, bend left] & & \\
& \varinjlim \pi_1(X_i,w) \arrow[r, dashed, "\Phi", "\exists!"'] & \pi_1(X,w). \\
\pi_1(X_k,w) \arrow[uu, "\varphi_{kl}"] \arrow[ur, "\psi_k"'] \arrow[urr, bend right] & &
\end{tikzcd}$$

We need to show that Φ is an isomorphism. First we show that the map Φ is surjective. Let $g \in \pi_1(X, w)$. By Lemma 25.9 (1) and our hypothesis (2) there exists a $k \in \mathbb{N}$ such that g lies in the image of the inclusion induced map $\pi_1(X_k, w) \to \pi_1(X, w)$. We consider the following commutative diagram

$$\begin{tikzcd}
& \varinjlim \pi_1(X_i,w) \arrow[r, "\Phi"] & \pi_1(X,w). \\
\pi_1(X_k,w) \arrow[ur] \arrow[urr, bend right, "{[\gamma]\mapsto g}"'] & &
\end{tikzcd}$$

By our choice of k we see that g lies in the image of the curved map, hence it also lies in the image of Φ. This concludes the proof that Φ is surjective.

Now we show that the map Φ is injective. So let $g \in \varinjlim \pi_1(X_i, w)$. Since $(\mathbb{N}, \leq)$ is a directed set we can use the description of the direct limit from page 733, i.e. we can write

$$\varinjlim \pi_1(X_i, w) \;=\; \Big(\bigsqcup_{i \in \mathbb{N}} \pi_1(X_i, w)\Big)/\sim \quad \text{where } g \sim \varphi_{kl}(g) \text{ for all } k \leq l,\ g \in \pi_1(X_k, w).$$

In particular we see that there exists a k such that g lies in the image of the map $\psi_k\colon \pi_1(X_k, w) \to \varinjlim \pi_1(X_k, w)$. It follows from Lemma 25.9 (1) and our hypothesis (2) that there exists an $l \geq k$ such that $\varphi_{kl}([\gamma])$ is trivial in $\pi_1(X_l, w)$. We obtain the following diagram

$$\begin{tikzcd} \pi_1(X_l,w) \arrow[dr] & & \\ & \varinjlim \pi_1(X_i,w) \arrow[r, "g\mapsto e"] & \pi_1(X,w). \\ \pi_1(X_k,w) \arrow[uu, "{[\gamma]\mapsto e}"] \arrow[ur, "{[\gamma]\mapsto g}"'] & & \end{tikzcd}$$

It follows immediately that g itself is trivial. This concludes the proof that Φ is injective. ■

Examples.

(1) As on page 728 we denote by X_∞ the interval $(0,\infty)$ with infinitely many circles attached. We pick the base point $w = \frac{1}{4}$. We then have [449]

$$\pi_1\big(X_\infty, w\big) \underset{\substack{\uparrow \\ \text{see page 730}}}{=} \pi_1\big(\varinjlim X_r, w\big) \underset{\substack{\uparrow \\ \text{Proposition 25.7}}}{=} \varinjlim \pi_1\big(X_r, w\big) \underset{\substack{\uparrow \\ \text{Proposition 20.5}}}{=} \varinjlim \langle x_1,\dots,x_r\rangle \underset{\substack{\uparrow \\ \text{see page 730}}}{=} \langle x_1, x_2, \dots\rangle.$$

Furthermore the proof of Proposition 20.5 and the above argument show that the generators $x_1, x_2, \dots$ of $\pi_1(X_\infty)$ can be chosen such that x_n, $n \in \mathbb{N}$, corresponds to a loop which goes "once around the n-th circle".

(2) Now we consider

$$\widetilde{X} \;:=\; \mathbb{R} \;\cup\; \bigcup_{j\in\mathbb{Z}} \underbrace{\big\{z \in \mathbb{C} \,\big|\, \big|z - (j + \tfrac{1}{4}i)\big| = \tfrac{1}{4}\big\}}_{\text{circle of radius } \frac{1}{4} \text{ around } j+\frac{1}{4}i\in\mathbb{C}} \;\subset\; \mathbb{C}.$$

Put differently, $\widetilde{X}$ is the real axis with infinitely many circles attached. The same argument as in (1) shows that $\pi_1(\widetilde{X})$ is the free group on infinitely many generators x_n, $n \in \mathbb{Z}$. In Figure 477 we illustrate a covering map $p\colon \widetilde{X} \to X = S^1 \vee S^1$. By Corollary 16.14 the map $\pi_1(\widetilde{X}) \to \pi_1(X)$ is a monomorphism. We had just seen that $\pi_1(\widetilde{X})$ is a free group on infinitely generators and by the discussion on page 606 we know that $\pi_1(X)$ is a free group on two generators. In fact, with appropriate orientations and conventions this shows that the homomorphism

$$\begin{array}{rcl} \pi_1(\widetilde{X}) = \langle \dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\rangle & \to & \langle s,t\rangle = \pi_1(X) \\ x_n & \mapsto & s^n t s^{-n} \end{array}$$

[449] The third equality follows from Proposition 20.5, to be more precise, it follows from the fact that for any $r \leq s$ the inclusion map $\iota_{rs}\colon X_r \to X_s$ and the isomorphisms of Proposition 20.5 give rise to a commutative diagram

$$\begin{tikzcd} \pi_1(X_r,w) \arrow[r, "\cong"] \arrow[d, "(\iota_{rs})_*"'] & \langle x_1,\dots,x_r\rangle \arrow[d, "x_i\mapsto x_i"] \\ \pi_1(X_s,w) \arrow[r, "\cong"] & \langle x_1,\dots,x_s\rangle. \end{tikzcd}$$

The fact that this diagram commutes follows easily from the proof of Proposition 20.5. We leave the details to the reader.

is a monomorphism. In particular we see that the free group $\langle s,t\rangle$ contains a subgroup that is "much larger" than the original free group itself.

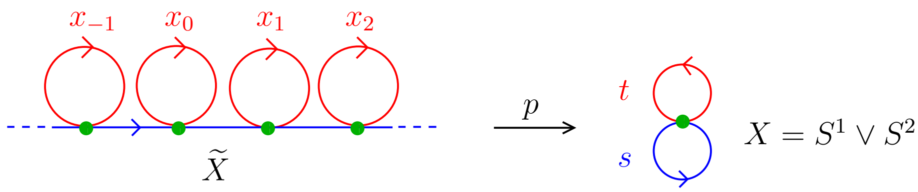


FIGURE 477

(3) An argument similar to (1) shows that $\pi_1(\mathbb{C}\setminus\mathbb{Z})$ is a free group generated by $x_n, n\in\mathbb{Z}$ where x_n is a loop which goes "once around the n-th hole". We leave the task of providing the details to the reader. We refer to Figure 478 for an illustration.

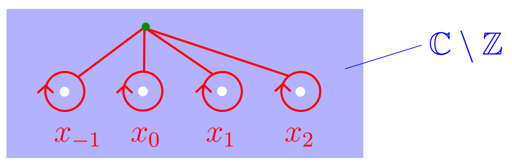


FIGURE 478

25.5. **The surface of "infinite genus" (∗).** In this section we return to the initial goal of this chapter, namely we want to determine the fundamental group of the surface of "infinite genus" that we illustrate in Figure 479. Before we can calculate its fundamental group we have to give a precise description of the surface of infinite genus.

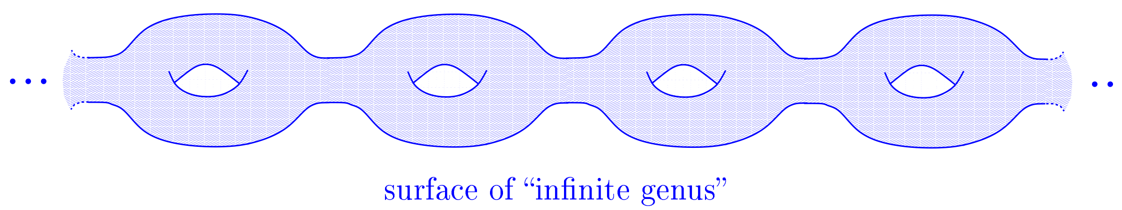


FIGURE 479

Definition. We write

$$F \;:=\; \text{the torus minus two open disks.}$$

We label the two boundary components by C_- and C_+. We pick orientation-preserving diffeomorphisms $\varphi_\pm\colon S^1\to C_\pm$. We define

$$\Sigma_\infty \;:=\; \Big(\bigcup_{i\in\mathbb{Z}}(F\times\{i\})\Big)\Big/\sim \qquad \text{where } (\varphi_+(x),i)\sim(\varphi_-(x),i+1) \text{ for } x\in S^1 \text{ and } i\in\mathbb{Z}.$$

We refer to Σ_∞ as the *surface of infinite genus*. It should be clear that this definition gives a rigorous description of the surface that we sketched in Figure 479.[450]

Lemma 25.10.

[450]One might be tempted to say that the surface of infinite genus is the connected sum of infinitely many tori. But it is not clear that "connected sum of infinitely many tori" is actually a well-defined concept.

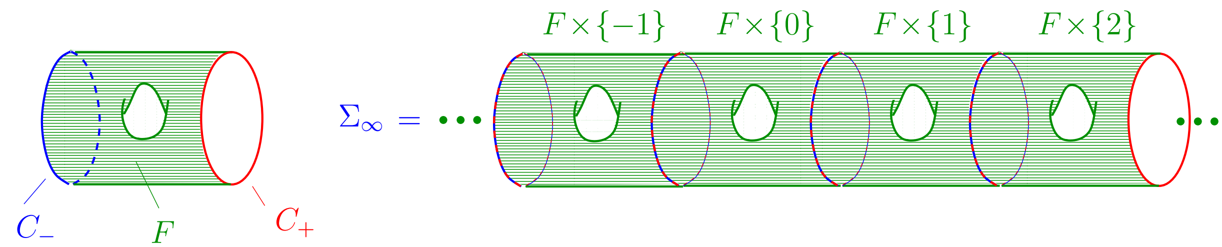


FIGURE 480

(1) *The surface Σ_∞ is a 2-dimensional orientable smooth manifold.*
(2) *Given $k \in \mathbb{N}$ we write*

$$X_k \;:=\; \Big(\bigcup_{i=-k+1}^{k}(F\times\{i\})\Big)/\sim \qquad \begin{array}{c}\textit{where } (\varphi_+(x),i)\sim(\varphi_-(x),i+1) \textit{ for } x\in S^1\\ \textit{and } i\in\{-k+1,\dots,k-1\}.\end{array}$$

The obvious maps $X_k \to \Sigma_\infty$ define a homeomorphism

$$\varinjlim X_i \xrightarrow{\cong} \Sigma_\infty.$$

SKETCH OF PROOF ($*$).

(1) This statement follows fairly easily from Proposition 8.15.
(2) It follows almost immediately from Proposition 2.43 (2) that for each $k \in \mathbb{Z}$ the two obvious maps $Y_k := F \times \{k\} \to \varinjlim X_i$ and $F \times \{k\} \to \Sigma_\infty$ are embeddings. Thus we can view each Y_k as a subset of $\varinjlim X_i$ and as a subset of Σ_∞. By unraveling the definitions of the topology one sees that in both cases a subset U is open if and only if the intersection with each Y_k is open. It now follows easily that the map $\varinjlim X_i \to \Sigma_\infty$ is a homeomorphism.

In both cases we leave it to the reader to fill in all the details. ■

In the following lemma we compute the fundamental group of the surface of infinite genus.

Lemma 25.11. *The fundamental group of the surface of infinite genus is the free group on infinitely many generators.*

Remark. In Proposition 31.5 we will see that the fundamental group of any non-compact orientable 2-dimensional smooth manifold is a free group.

PROOF. We continue with the notation from Lemma 25.10. By Lemma 25.10 (2) it suffices to show that the fundamental group of $\varinjlim X_i$ is a free group on infinitely many generators. By Lemma 23.11 we have[451]

$$\pi_1(F) \;=\; \langle a,b,c,c' \mid c=[a,b]\cdot c'\rangle \;=\; \langle x,y,z,z' \mid z=[x,y]\cdot z'\rangle$$

[451]This is indeed a consequence of Lemma 23.11, the only change is that we reversed the orientation of one of the boundary curves.

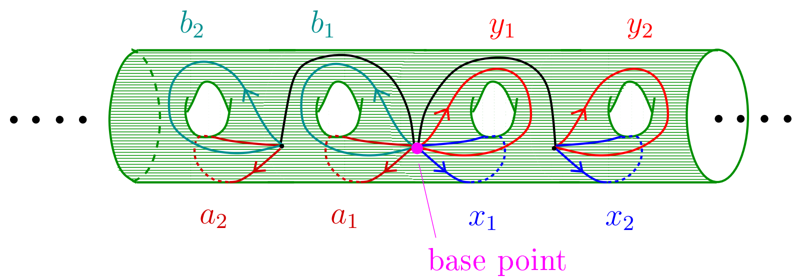


FIGURE 481. A set of generators for the fundamental group of a surface of infinite genus.

where the generators z, z' and c, c' correspond to the two boundary components. We refer to Figure 482 for an illustration. An induction argument, using the Seifert–van Kampen

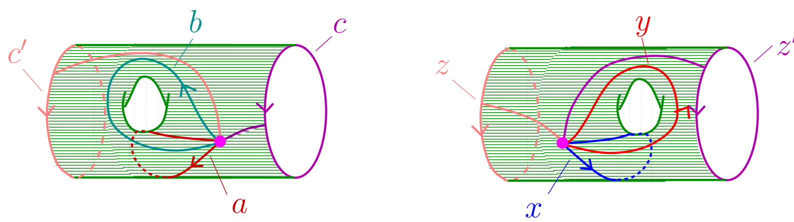


FIGURE 482

Theorem 22.2 for smooth manifolds together with Lemma 21.23 and the above presentations of $\pi_1(F)$ shows that for any $l \geq k$ we have a commutative diagram

$$\begin{array}{ccc}
\pi_1(X_k) & \xrightarrow{\cong} & \left\langle \begin{matrix} x_1, y_1, \ldots, x_k, y_k \\ a_1, b_1, \ldots, a_k, b_k \end{matrix} \, z_k, c_k \;\middle|\; \prod_{j=1}^{k}[a_j, b_j]c_k = \prod_{j=1}^{k}[x_j, y_j]z_k \right\rangle \\
\big\downarrow & & {\scriptstyle c_k \mapsto \prod_{j=k+1}^{l}[a_j,b_j]c_l} \big\downarrow {\scriptstyle z_k \mapsto \prod_{j=k+1}^{l}[x_j,y_j]z_l} \\
\pi_1(X_l) & \xrightarrow{\cong} & \left\langle \begin{matrix} x_1, y_1, \ldots, x_l, y_l \\ a_1, b_1, \ldots, a_l, b_l \end{matrix} \, z_l, c_l \;\middle|\; \prod_{j=1}^{l}[a_j, b_j]c_l = \prod_{j=1}^{l}[x_j, y_j]z_l \right\rangle
\end{array}$$

where the left vertical map is induced by the inclusion $X_k \to X_l$. We leave the details to the energetic reader. For any $l \geq k$ we also have the following commutative diagram of

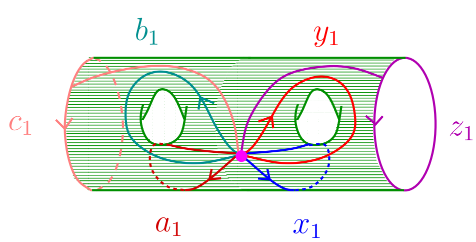


FIGURE 483

homomorphisms:

$$\begin{array}{ccc}
\left\langle \begin{array}{c} x_1,y_1,\dots,x_k,y_k \\ a_1,b_1,\dots,a_k,b_k \end{array} p \right\rangle & \xrightarrow[p\mapsto \prod_{j=1}^{k}[a_j,b_j]c_k]{\cong} & \left\langle \begin{array}{c} x_1,y_1,\dots,x_k,y_k \\ a_1,b_1,\dots,a_k,b_k \end{array} z_k,c_k \,\middle|\, \prod_{j=1}^{k}[a_j,b_j]c_k=\prod_{j=1}^{k}[x_j,y_j]z_k \right\rangle \\
\big\downarrow & & \Big\downarrow {\scriptstyle c_k\mapsto \prod_{j=k+1}^{l}[a_j,b_j]c_l \;\mid\; z_k\mapsto \prod_{j=k+1}^{l}[x_j,y_j]z_l} \\
\left\langle \begin{array}{c} x_1,y_1,\dots,x_l,y_l \\ a_1,b_1,\dots,a_l,b_l \end{array} p \right\rangle & \xrightarrow[p\mapsto \prod_{j=1}^{l}[a_j,b_j]c_l]{\cong} & \left\langle \begin{array}{c} x_1,y_1,\dots,x_l,y_l \\ a_1,b_1,\dots,a_l,b_l \end{array} z_l,c_l \,\middle|\, \prod_{j=1}^{l}[a_j,b_j]c_l=\prod_{j=1}^{l}[x_j,y_j]z_l \right\rangle
\end{array}$$

The horizontal homomorphisms are in fact isomorphisms, this is for example a consequence of the Tietze transformations from page 632.[452]

We denote by Σ_∞ the surface of infinite genus. Furthermore given any $i\in\mathbb{N}_0$ we write $U_i=X_i\setminus\partial X_i$. Note that each U_i is in fact an open subset of Σ_∞. We have

$$\pi_1(\Sigma_\infty) \underset{\substack{\uparrow \\ \text{since } \Sigma_\infty=\bigcup_{i\in\mathbb{N}_0}U_i}}{=} \pi_1\big(\varinjlim U_i\big) \overset{\substack{\text{Proposition 25.7} \\ \downarrow}}{=} \varinjlim \pi_1(U_i) \underset{\substack{\uparrow \\ \text{since } \pi_1(U_i)\xrightarrow{\cong}\pi_1(X_i)}}{=} \varinjlim \pi_1(X_i) \overset{\substack{\text{by the above two} \\ \text{commutative diagrams} \\ \downarrow}}{=} \varinjlim \left\langle \begin{array}{c} x_1,y_1,..,x_i,y_i \\ a_1,b_1,..,a_i,b_i \end{array} p \right\rangle \underset{\substack{\uparrow \\ \text{see example on} \\ \text{page 730}}}{=} \left\langle \begin{array}{c} x_1,y_1,.. \\ a_1,b_1,.. \end{array} p \right\rangle.$$

This concludes the proof of the lemma. ■

In the final part of this section we will see that we can actually use Lemma 25.11 to draw an interesting conclusion on fundamental groups of surfaces of *finite* genus. First we have the following lemma.

Lemma 25.12. *Let $g\geq 2$. We denote by Σ_∞ the surface of infinite genus and we denote by Σ_g the surface of genus g.*

(1) *There exists an action of $\mathbb{Z}$ on Σ_∞ that is discrete and continuous and such that the quotient $\Sigma_\infty/\mathbb{Z}$ is homeomorphic to Σ_g.*
(2) *There exists a covering $p\colon \Sigma_\infty\to\Sigma_g$ of infinite degree.*

Sketch of proof.

---

[452]Indeed, we write $X=\{x_1,y_1,\dots,x_m,y_m,a_1,b_1,\dots,a_m,b_m\}$, $u=\prod_{j=1}^{m}[a_j,b_j]$, $v=\prod_{j=1}^{m}[x_j,y_j]$, $c=c_m$ and $z=z_m$, then

$$\langle X,z,c\mid uc=vz\rangle=\langle X,z,c\mid z=v^{-1}uc\rangle \xleftarrow{\cong} \langle X,c\rangle \xrightarrow{\cong} \langle X,c,p\mid p=uc\rangle=\langle X,c,p\mid X,c=u^{-1}p\rangle \xleftarrow{\cong} \langle X,p\rangle$$

where all three isomorphisms are the isomorphisms of the Tietze transformation (2). It is straightforward to see that the concatenation equals for $m=k,l$ precisely the horizontal maps in the diagram.

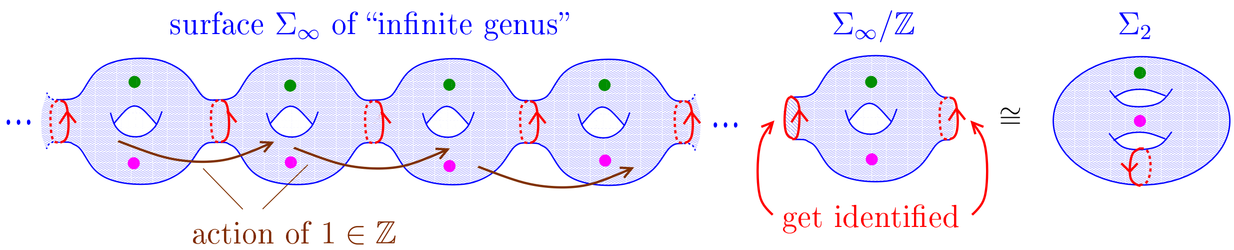


FIGURE 484. Illustration of Lemma 25.12.

(1) We continue with the notation from page 740 and Lemma 25.10. Let $g \in \mathbb{N}_{\geq 2}$. As before we consider

$$\Sigma_\infty := \Big(\bigcup_{i\in\mathbb{Z}} (F\times\{i\})\Big)/\sim \ \text{ where } (\varphi_+(x), i) \sim (\varphi_-(x), i+1) \text{ for } x \in S^1 \text{ and } i \in \mathbb{Z}.$$

Given $n \in \mathbb{Z}$ and $[(x,k)] \in \Sigma_\infty$ we define $n \cdot [(x,k)] := [(x, k+(g-1)\cdot n)]$. It follows from Exercise 25.14[453] that this action is continuous and proper. Furthermore we obtain from Exercise 25.14 that the quotient $\Sigma_\infty/\mathbb{Z}$ is homeomorphic to the topological space that is given by identifying the two boundary components of

$$\Big(\bigcup_{i=0}^{g-1} (F \times \{i\})\Big)/\sim \qquad \begin{array}{c}\text{where } (\varphi_+(x), i) \sim (\varphi_-(x), i+1) \text{ for } x \in S^1 \\ \text{and } i \in \{0, \dots, g-1\}.\end{array}$$

The experienced reader will surely be convinced that this topological space is homeomorphic to the surface of genus g. We leave it to the scrupulous reader to fill in the details.

(2) This statement follows immediately from (1) together with Proposition 16.9. ■

Using the results of this section we can prove the following proposition.

Proposition 25.13. *Let $g \geq 2$. We consider the automorphism*

$$\varphi\colon \underbrace{\langle \{x_{1,i}, y_{1,i}, \dots, x_{g-1,i}, y_{g-1,i}\}_{i\in\mathbb{Z}}\rangle}_{\text{free group of infinite rank}} \to \underbrace{\langle \{x_{1,i}, y_{1,i}, \dots, x_{g-1,i}, y_{g-1,i}\}_{i\in\mathbb{Z}}\rangle}_{\text{same free group of infinite rank}}$$

that is induced by

$$\begin{aligned} x_{j,i} &\mapsto x_{j,i+1} \\ y_{j,i} &\mapsto y_{j,i+1}. \end{aligned}$$

There exists an isomorphism

$$\pi_1(\Sigma_g) \cong \langle \{x_{1,i}, y_{1,i}, \dots, x_{g-1,i}, y_{g-1,i}\}_{i\in\mathbb{Z}}\rangle \rtimes_\varphi \mathbb{Z}.$$

SKETCH OF PROOF. We consider the surface Σ_∞ of infinite genus together with the $\mathbb{Z}$-action that is provided by Lemma 25.12 (1). By Proposition 16.9 (2) we know that the projection $p\colon \Sigma_\infty \to \Sigma_\infty/\mathbb{Z} = \Sigma_g$ is a covering.

We pick a base point $a_0 \in \Sigma_g$. Next let $\widetilde{a}_0 \in \Sigma_\infty$ be a point with $p(\widetilde{a}_0) = a_0$. It follows from Corollary 16.14 that $p_*\colon \pi_1(\Sigma_\infty, \widetilde{a}_0) \to \pi_1(\Sigma_g, a_0)$ is a monomorphism. We will use

[453]It follows from the Collar Neighborhood Theorem 8.12 together with Lemma 8.11 that there exist open disjoint neighborhoods of the two boundary components of F, thus the extra hypothesis for Exercise 25.14 (c) is in fact satisfied.

this monomorphism to view $\pi_1(\Sigma_\infty, \widetilde{a}_0)$ as a subgroup of $\pi_1(\Sigma_g, a_0)$. We pick a path $\widetilde{\mu} \in \Sigma_\infty$ from $a_0 = 0 \cdot a_0$ to $1 \cdot a_0$. We write $\mu := p \circ \widetilde{\mu}$ and we consider the isomorphism

$$\begin{array}{rcl} \psi\colon \pi_1(\Sigma_\infty, \widetilde{a}_0) & \to & \pi_1(\Sigma_\infty, \widetilde{a}_0) \\ g & \mapsto & [\mu] \cdot g \cdot [\mu]^{-1}. \end{array}$$

Now we have the following isomorphisms:

by Lemma 25.11 we have an explicit isomorphism $\pi_1(\Sigma_\infty, \widetilde{a}_0) \cong \langle \{x_{1,i}, y_{1,i}, \dots, x_{g-1,i}, y_{g-1,i}\}_{i\in\mathbb{Z}} \rangle$
we leave it to the reader to verify that ψ and φ match up

$$\pi_1(\Sigma_g, a_0) \xrightarrow{\cong} \pi_1(\Sigma_\infty, \widetilde{a}_0) \rtimes_\psi \mathbb{Z} \xrightarrow{\cong} \langle \{x_{1,i}, y_{1,i}, \dots, x_{g-1,i}, y_{g-1,i}\}_{i\in\mathbb{Z}} \rangle \rtimes_\varphi \mathbb{Z}.$$

by Theorem 16.16 (2) we have an epimorphism $\gamma\colon \pi_1(\Sigma_g, a_0) \to \mathbb{Z}$
with $\gamma([\mu]) = 1$ and with kernel $\pi_1(\Sigma_\infty, \widetilde{a}_0)$
thus we obtain the isomorphism from Lemma 24.27

We have thus found the promised isomorphism. ■

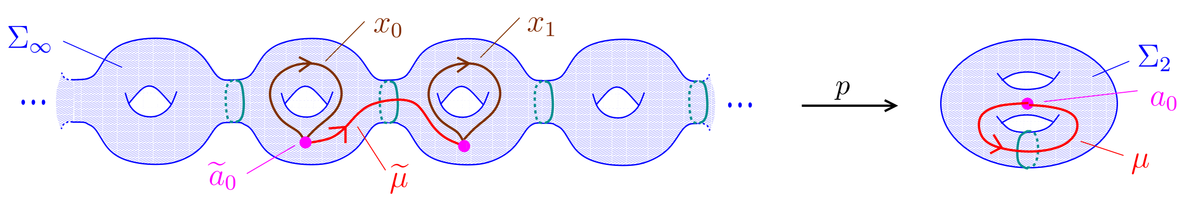


FIGURE 485. Illustration for the proof of Proposition 25.13.

Using Proposition 25.13 we can prove the following proposition.

Proposition 25.14. *Let $g \geq 2$. Every abelian subgroup of $\pi_1(\Sigma_g)$ is isomorphic to $\mathbb{Z}$.*

PROOF. This proposition is a pleasant application of Proposition 25.13 and Exercise 19.11. We will provide the details in Exercise 25.9. ■

25.6. **The inverse limit of an inverse system.** For completeness' sake we also introduce the inverse limit of an inverse system, even though it will not play a role in the intermediate future.

Definition. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be a category. An *inverse system in the category $\mathcal{C}$ over I* is a family of objects $\{X_i\}_{i\in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ji}\colon X_j \to X_i\}$ for all $i, j \in I$ with $i \leq j$[454] such that the following two conditions are satisfied:

(1)	$f_{ii} = \mathrm{id}_{X_i}$	for all $i \in I$,
(2)	$f_{ki} = f_{ji} \circ f_{kj}$	for all $i, j, k \in I$ with $i \leq j \leq k$.

Examples.

(A) For each $i \in \mathbb{N}$ we define $X_i := \mathbb{Z}^i$ and for $i \leq j$ we denote by

$$\begin{array}{rcl} f_{ji}\colon \mathbb{Z}^j & \to & \mathbb{Z}^i \\ (x_1, \dots, x_j) & \mapsto & (x_1, \dots, x_i) \end{array}$$

[454] So here for $i \leq j$ we have a morphism $X_j \to X_i$ instead of $X_i \to X_j$ in a direct system.

the projection onto the first i coordinates. Then $(\{\mathbb{Z}^i\}_{i\in\mathbb{N}}, \{f_{ji}\}_{i\leq j})$ forms an inverse system in the category of groups.

(B) Let p be a prime. For each $i \in \mathbb{N}$ we define $X_i := \mathbb{Z}_{p^i}$ and for $i \leq j$ we denote by

$$\begin{array}{rcl} f_{ji}\colon \mathbb{Z}_{p^j} & \to & \mathbb{Z}_{p^i} \\ n + p^j\mathbb{Z} & \mapsto & n + p^i\mathbb{Z} \end{array}$$

the obvious projection map. Then $(\{\mathbb{Z}_{p^i}\}_{i\in\mathbb{N}}, \{f_{ji}\}_{i\leq j})$ forms an inverse system in the category of rings.

(C) We let $I = \mathbb{N}$, but this time we use the preorder given by divisibility, i.e. we write $k \leq l$ if and only if $k|l$. For each $k \in \mathbb{N}$ we define $X_k := \mathbb{Z}_k$ and for $k|l$ we denote by

$$\begin{array}{rcl} f_{lk}\colon \mathbb{Z}_l & \to & \mathbb{Z}_k \\ n + l\mathbb{Z} & \mapsto & n + k\mathbb{Z} \end{array}$$

the obvious projection map. Then $(\{\mathbb{Z}_k\}_{k\in\mathbb{N}}, \{f_{lk}\}_{k|l})$ forms an inverse system in the category of rings.

The following definition is the "mirror" image of the definition of the direct limit of a direct system.

Definition. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be a category. Furthermore suppose we are given an inverse system $(\{X_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ in the category $\mathcal{C}$. An *inverse limit*[455] of the inverse system is an object Y in $\mathcal{C}$ together with morphisms $g_i\colon Y \to X_i$, $i \in I$, in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i \leq j$ we have $f_{ji} \circ g_j = g_i\colon Y \to X_i$, i.e. for all $i \leq j$ the following diagram commutes

$$\begin{tikzcd} & X_j \arrow[d, "f_{ji}"] \\ Y \arrow[ur, "g_j"] \arrow[r, "g_i"'] & X_i. \end{tikzcd}$$

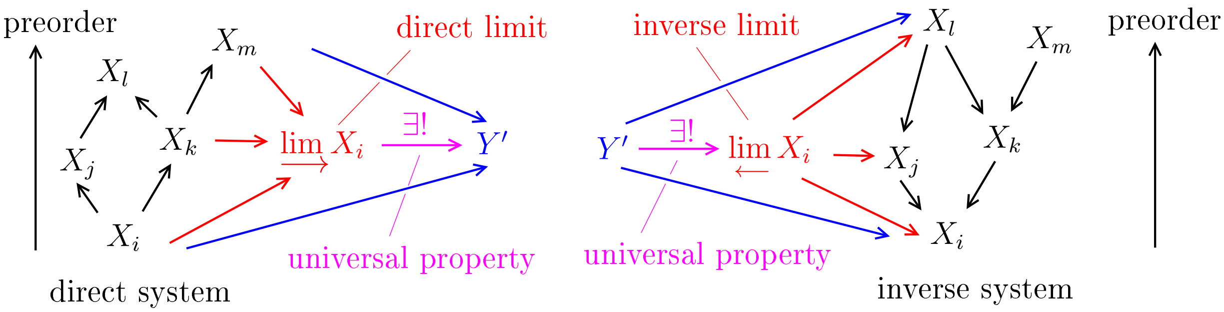
(2) If we are given another object Y' and morphisms $g_i'\colon Y' \to X_i$, $i \in I$, that satisfy (1), then there exists a unique morphism $F\colon Y' \to Y$ such that for all $i \leq j$ the following diagram commutes

$$\begin{tikzcd} & & X_j \arrow[dd, "f_{ji}"] \\ Y' \arrow[r, dashed, "F", "\exists!"'] \arrow[urr, bend left, "g_j'"] \arrow[drr, bend right, "g_i'"'] & Y \arrow[ur, "g_j"] \arrow[dr, "g_i"'] & \\ & & X_i. \end{tikzcd}$$

[455]Somewhat confusingly an "inverse limit" is often also called an "inductive limit" or "limit of an inverse system".

The "usual argument" for objects that satisfy a universal property shows that if the inverse limit exists, then it is unique in an appropriate sense. We denote the inverse limit by $\varprojlim X_i$ or sometimes by $\varprojlim_I X_i$.

In the following diagram we compare schematically the definition of the direct limit of a direct system to the inverse limit of an inverse system:[456]



Examples. In order to get an idea of how to construct inverse limits we first consider the special case that the preorder on I is trivial, in the sense that we have $i \leq j$ if and only if $i = j$.

(D) Let I be a set with the trivial preorder. Suppose that for every $i \in I$ we are given a set X_i. Together with the identity maps of the X_i this defines an inverse system over I in the category of sets. We claim that the inverse limit is given by[457]

$$\varprojlim X_i \;=\; \prod_{i\in I} X_i$$

together with the maps g_i that are given by the projection onto the i-component. Now we verify this statement. Since we only have $i \leq j$ if $i = j$ there is nothing to verify in part (1) of the definition of the inverse limit. Now suppose we are given a

---

[456] The difference between a direct limit and an inverse limit is not the direction of the arrows between the X_i. In fact given a preorder "$\leq$" one could define a preorder "$\widetilde{\leq}$" via $a \widetilde{\leq} b$ if and only if $b \leq a$, hence a direct system over $(I, \leq)$ is an inverse system over $(I, \widetilde{\leq})$. The real difference is whether the X_i map to the limit, or whether the limit maps to the X_i.

[457] Recall that the product $\prod_{i\in I} X_i$ is defined as the set of all maps $f\colon I \to \bigsqcup_{i\in I} X_i$ with the property that $f(i) \in X_i$ for all $i \in I$. Such a map f is often written as $(f(i))_{i\in I}$ and often we refer to $f(i)$ as the *i-component* of f. Furthermore we refer to the map

$$\begin{array}{rcl} \prod_{i\in I} X_i & \to & X_i \\ f & \mapsto & f(i) \end{array}$$

as the *projection onto the i-component*. Note that if the X_i's are groups, rings or modules, then the product inherits this algebraic structure by component-wise multiplication and addition.

set Y' and maps $g_i'\colon Y'\to X_i$. Then we define

$$\begin{array}{rcl} F\colon Y' & \to & \varprojlim X_i \\ y & \mapsto & \left(\begin{array}{rcl} I & \to & \prod\limits_{i\in I} X_i \\ i & \mapsto & g_i'(y)\end{array}\right).\end{array}$$

It is straightforward to verify that F is uniquely determined and that it has the desired property. We have thus determined the inverse limit $\varprojlim X_i$.

(E) Let I be a set with the trivial preorder and suppose that for any $i\in I$ we are given a topological space X_i. Together with the identity maps of the X_i this defines an inverse system over I in the category of topological spaces. As in (D) we define

$$\varprojlim X_i \;=\; \prod_{i\in I} X_i$$

and each map g_i is again given by the projection onto the i-component.

To conclude that we have found an inverse limit in the category of topological spaces we still have to equip $\prod\limits_{i\in I} X_i$ with a topology which has the following two properties:

(i) the maps $g_i\colon \prod\limits_{i\in I} X_i\to X_i$ are continuous,

(ii) given a topological space Y' and continuous maps $g_i'\colon Y'\to X_i$ the map F, defined as in (D), is continuous.[458]

We set

$$\mathcal{B} \;:=\; \left\{\prod_{i\in I} U_i \;\subset\; \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \text{ is open in } X_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i\neq X_i \end{array}\right\}.$$

It is straightforward to verify that $\mathcal{B}$ has the basis property that we introduced on page 102. We equip $\prod\limits_{i\in I} X_i$ with the topology generated by this basis and we refer to this topology as the *product topology*.[459] Now we need to verify that we found the correct topology on the direct limit.

(i) It follows immediately from the definitions that the maps g_i are continuous with respect to the product topology.

(ii) Suppose we are given Y and g_i' as above. We denote by F the map defined as above. We claim that F is continuous. By Proposition 2.37 it suffices to verify

[458] On the product $\prod\limits_{i\in I} X_i$ we could also consider the topology $\mathcal{T}$ that is generated by all sets of the form $\prod\limits_{i\in I} U_i$ with U_i open in X_i. This might be a more "natural" topology on $\prod\limits_{i\in I} X_i$. But with this topology $\mathcal{T}$ the map F does not necessarily have to be continuous. The problem is that $\mathcal{T}$ has "too many" open sets.

[459] If the index set I is finite, i.e. if we consider the product of finitely many topological spaces, then this definition of the product topology agrees with the definition given on page 162.

that the preimage of any set in $\mathcal{B}$ is open. So let $\prod_{i\in I} U_i$ be a set in $\mathcal{B}$. Then

$$F^{-1}\Big(\prod_{i\in I} U_i\Big) = \bigcap_{i\in I} (g_i')^{-1}(U_i) = \bigcap_{U_i\neq X_i} \underbrace{(g_i')^{-1}(U_i)}_{\text{open in } Y'} = \begin{array}{c}\text{intersection of finitely}\\ \text{many open subsets of } Y'.\end{array}$$

This concludes the proof that F is continuous.

Now we can prove fairly easily that inverse limits exist in many categories that are of interest to us.

Proposition 25.15. *The inverse limit of any inverse system exists in the following categories:*

(0) *the category $\mathcal{S}et$ of sets,*
(1) *the category $\mathcal{T}op$ of topological spaces,*
(2) *the category $\mathcal{G}r$ of groups,*
(3) *the category $\mathcal{T}op\mathcal{G}r$ of topological groups,*
(4) *the category $\mathcal{A}b\mathcal{G}r$ of abelian groups,*
(5) *the category $\mathcal{R}ing$ of rings,*
(6) *the category of R-modules, where R is a commutative ring.*

Furthermore, the direct limits of (0),...,(6) *are "coherent". For example if $\{X_i\}_{i\in I}$ is a direct system of topological groups, then the direct limit is the direct system of the corresponding topological spaces together together with an appropriate group structure.*

We will see that the proof of Proposition 25.15 is a relatively straightforward generalization of the examples (D) and (E) above.

Proof. Let $(I, \le)$ be a preordered set and let $\mathcal{C}$ be one of the seven given categories. Furthermore let $(\{X_i\}_{i\in I}, \{f_{ji} \colon X_j \to X_i\}_{i\le j})$ be an inverse system in the category $\mathcal{C}$. We define

$$\varprojlim X_i \;:=\; \Big\{(x_i)_{i\in I} \in \prod_{i\in I} X_i \,\Big|\, f_{ji}(x_j) = x_i \text{ for all } i \le j\Big\}.$$

Furthermore we denote by $g_i \colon \varprojlim X_i \to X_i$ the obvious projection maps. It follows immediately from the definitions that for any $i \le j$ we have $f_{ji} \circ g_j = g_i$. Furthermore, given an object Y' and morphisms $g_i' \colon Y' \to X_i$, $i \in I$ which satisfy the condition $f_{ji} \circ g_j' = g_i'$ for all $i \in I$ we define

$$\begin{array}{rcl} F\colon Y' & \to & \varprojlim X_i \\ y & \mapsto & \left(\begin{array}{ccc} I & \to & \prod_{i\in I} X_i \\ i & \mapsto & g_i'(y)\end{array}\right).\end{array}$$

It is straightforward to verify that $F(Y')$ lies in $\varprojlim X_i$ and that F is the unique map with the desired property. This shows that the inverse limit exists in the category of sets.[460]

[460]Now we will see that we can use the "same" inverse limit for all the categories, this is very different from the construction of the direct limit for the same categories. There, in the proof of Proposition 25.1, we had to use several different construction.

For the remaining six categories we will use the same definition of the direct limit as a set and the same maps, but we now have to verify that we have objects and morphisms in the given category.

(1) Suppose that we work in the category of topological spaces. We equip $\prod_{i\in I} X_i$ with the product topology that we defined in Example (E) above. We equip $\varprojlim X_i$ with the subspace topology. The same argument as in Example (E) shows that the maps g_i and F are continuous.

(2) In the category of groups we view $\prod_{i\in I} X_i$ as a group via component-wise multiplication. Since the f_{ij} are group homomorphisms we see that $\varprojlim X_i$ is a subgroup of $\prod_{i\in I} X_i$. It is straightforward to verify that the above maps are all morphisms in the category of groups.

(3) We equip the set $\varprojlim X_i$ with the topological structure and the group structure as in the above statements (1) and (2). It follows from Lemma 3.54 that this turns $\varprojlim X_i$ into a topological group, which turns out to be the inverse limit of the inverse system of topological groups.

(4),(5),(6) These categories are dealt with the same way as (2).

For all seven categories we have now provided an inverse limit. Since we always equipped the same set with different structures we see that the direct limits are coherent. ■

Remark. We consider the important special case that the preordered set $(I,\leq) = (\mathbb{N},\leq)$. Furthermore let

$$(\{X_n\}_{n\in\mathbb{N}}, \{f_{nm}\colon X_n \to X_m\}_{m\leq n})$$

be an inverse system over $(\mathbb{N},\leq)$ in any of the above seven categories. Given $n \in \mathbb{N}_{\geq 2}$ we write $g_n := f_{n,n-1}\colon X_n \to X_{n-1}$. We have

$$\begin{aligned}
\varprojlim X_i &= \big\{(x_1,x_2,x_3,\dots) \mid x_n \in X_n \text{ for all } n \in \mathbb{N} \text{ and } f_{nm}(x_n) = x_m \text{ for all } m \leq n\big\} \\
&\underset{\substack{\uparrow \\ \text{follows from } f_{nm} = f_{n,n-1}\circ\cdots\circ f_{m+1,m} = g_n\circ\cdots\circ g_{m+1}}}{=} \big\{(x_1,x_2,x_3,\dots) \mid x_n \in X_n \text{ for all } n \in \mathbb{N} \text{ and } g_n(x_n) = x_{n-1} \text{ for all } n \in \mathbb{N}_{\geq 2}\big\}
\end{aligned}$$

$$= \text{all infinite sequences of the form } \begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ x_3 & \in & X_3 \\ \downarrow & & \downarrow g_3 \\ x_2 & \in & X_2 \\ \downarrow & & \downarrow g_2 \\ x_1 & \in & X_1 \end{array}$$

This discussion shows that we can view of elements in the inverse limit as "upward going sequences".

Now we return to our initial examples of inverse systems and their inverse limits.

Examples.

(A) It follows quite easily from the previous remark that

$$\varprojlim_{n\in\mathbb{N}} \mathbb{Z}^n \;=\; \{(x_1,x_2,x_3,\dots)\mid x_n\in\mathbb{Z}\} \;\cong\; \mathbb{Z}^{\mathbb{N}}.$$

Note that this is an uncountable group. Recall that on page 730 we had determined that

$$\varinjlim \mathbb{Z}^i \;=\; \mathbb{Z}^{(\mathbb{N})} \;=\; \{(x_1,x_2,\dots)\mid x_i\in\mathbb{Z} \text{ but only finitely many } x_i \text{ are non-zero}\}$$

which is a countable group.[461] Thus we see that in this particular case the inverse limit is "much larger" than the direct limit.[462]

(B) Let p be a prime. The inverse limit $\widehat{\mathbb{Z}}_p := \varprojlim \mathbb{Z}_{p^i}$ of the rings $\mathbb{Z}_{p^i}=\mathbb{Z}/p^i\mathbb{Z}$ is called the *ring of the p-adic numbers.*[463] The p-adic numbers play a very prominent role in many branches of pure mathematics.

(C) The inverse limit $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}_n$ of the rings $\mathbb{Z}_n$, $n\in\mathbb{N}$, with the preorder on $\mathbb{N}$ defined by the divisibility relation $k|l$, is called the *ring of the profinite integers.* Using the Chinese Remainder Theorem one can show that

$$\widehat{\mathbb{Z}} \;\cong\; \prod_{\text{prime } p} \widehat{\mathbb{Z}}_p.$$

Now we continue with three examples that we had not considered yet.

(D) We let $(I,\le)=(\mathbb{N},\le)$. Given $i\in\mathbb{N}$ we set $X_i:=\mathbb{N}$ and for $i\le j$ we define

$$\begin{aligned} f_{ji}\colon X_j=\mathbb{N} &\;\to\; X_i=\mathbb{N}\\ n &\;\mapsto\; n+(j-i). \end{aligned}$$

Then $(\{X_i=\mathbb{N}\}_{i\in\mathbb{N}},\{f_{ji}\}_{i\le j})$ forms an inverse system in the category of sets. In Exercise 25.8 we will see that in this case the inverse limit is the empty set.

(E) Let X be a topological space. We denote by K the set of all compact subsets of X. We consider K with the preorder given by inclusion, i.e. we have $A\le B$ if $A\subseteq B$. Given $A\in K$ we define $X_A:=X\setminus A$. Furthermore given $A\le B$ we denote by[464] $f_{AB}\colon \pi_0(X_B)\to\pi_0(X_A)$ the map induced by the inclusion $X_B\to X_A$. It is straightforward to verify that the sets $\{\pi_0(X_A)\}_{A\in K}$, together with the inclusion induced maps f_{AB} for $A\le B$ define an inverse system in the category of sets. The elements of $\mathrm{ends}(X) := \varprojlim \pi_0(X_A)$ are called the *ends of X*. In Exercise 25.13 we will determine the number of ends of $X=\mathbb{R}^n$.

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[461] Why is $\mathbb{Z}^{(\mathbb{N})}$ countable?

[462] Note though that we are comparing limits of different system, first of a direct system and then of an inverse system.

[463] In number theory one often uses a different convention, namely one writes explicitly $\mathbb{Z}/p\mathbb{Z}$ for the quotient ring and one denotes the p-adic numbers by $\mathbb{Z}_p$. If the expression "$\mathbb{Z}_p$" appears in a paper or a book it is a good idea to check the convention used by the author.

[464] Recall that on page 135, given a topological space Y we defined $\pi_0(Y)$ as the set of path-components of Y. A map $f\colon Y\to Z$ between two topological spaces induces a map $f_*\colon \pi_0(Y)\to\pi_0(Z)$ in an obvious way.

(F) Given a topological space Y we denote by $C(Y,\mathbb{R})$ the abelian group of all continuous functions on Y. Note that for each subset $W\subset Y$ we have a homomorphism $C(Y,\mathbb{R})\to C(W,\mathbb{R})$ given by $f\mapsto f|_W$, i.e. given by restricting a function on Y to a function on W.

Let X be a topological space and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets such that for each $i\in\mathbb{N}$ we have $U_i\subset U_{i+1}$ and such that $\bigcup\limits_{i\in\mathbb{N}} U_i = X$. Then the groups $\{C(U_i,\mathbb{R})\}_{i\in\mathbb{N}}$ together with the restriction maps form an inverse system. We leave it to the reader to verify that the restriction maps from X to the U_i gives rise to an isomorphism

$$C(X,\mathbb{R}) \ \to\ \varprojlim C(U_i,\mathbb{R}).$$

Definition. Let $(\{0,x,y\},\leq)$ be the push-pull set as defined on page 726, i.e. the preordered set with the non-trivial relations $0\leq x$ and $0\leq y$, and let $\mathcal{C}$ be a category. An inverse system in the category $\mathcal{C}$ over $(\{0,x,y\},\leq)$ consists of three objects A_0,A_x,A_y, the identity morphisms $\mathrm{id}_{A_0},\mathrm{id}_{A_x},\mathrm{id}_{A_y}$ and morphisms $f_x\colon A_x\to A_0$ and $f_y\colon A_y\to A_0$. We refer to such a system as a *pullback system* and often we arrange the morphisms in a diagram as follows

$$\begin{tikzcd} & A_y \arrow[d, "f_y"] \\ A_x \arrow[r, "f_x"'] & A_0. \end{tikzcd}$$

If the inverse limit of such a pullback system exists, then we refer to it as the *pullback* of the *pullback system*. The defining property of the pullback Z can be summarized in the following diagram

$$\begin{tikzcd} Z' \arrow[dr, dashed, "\exists!"'] \arrow[drr, bend left, "g'_y"] \arrow[ddr, bend right, "g'_x"'] & & \\ & \text{pullback } Z \arrow[r, "g_x"'] \arrow[d, "g_y"] & A_y \arrow[d, "f_y"] \\ & A_x \arrow[r, "f_x"'] & A_0. \end{tikzcd}$$

The following lemma deals with the pullback in the category of topological spaces.

Lemma 25.16. *Let $p\colon X\to B$ be a map of topological spaces, let C be a topological space and let $f\colon C\to B$ be a map.*

(1) *In the category of topological spaces the pullback of the pullback system*

$$\begin{tikzcd} & X \arrow[d, "p"] \\ C \arrow[r, "f"'] & B \end{tikzcd} \quad \text{is given by} \quad \begin{tikzcd} f^*X := \{(c,x)\in C\times X \mid f(c)=p(x)\} \arrow[r, "(c{,}x)\mapsto x"] \arrow[d, "(c{,}x)\mapsto c"'] & X \arrow[d, "p"] \\ C \arrow[r, "f"'] & B. \end{tikzcd}$$

Often the pullback is denoted by f^*X.

(2) *If X and C are connected, then f^*X is also connected.*
(3) *Let $p\colon X \to B$ be a covering.*
 (a) *The pullback $q\colon f^*X \to C$ is also a covering.*
 (b) *If B is connected, then*
$$\deg(q\colon f^*X \to C) \;=\; \deg(p\colon X \to B).$$
 (c) *Let $c \in C$. We write $b = f(c)$ and we pick*[465] *$x \in X$ such that $p(x) = b$. The following equality holds:*
$$q_*(\pi_1(f^*X, (c,x))) \;=\; f_*^{-1}(p_*(\pi_1(X,x))) \;\subset\; \pi_1(C,c).$$

PROOF (∗).

(1) This statement follows quite easily from the definitions or alternatively from the proof of Proposition 25.15.
(2) We will prove this statement in Exercise 25.17.
(3) Let $p\colon X \to B$ be a covering. We write $Z := f^*X$ and $z := (c,x)$. Finally we denote by $q\colon f^*X \to C$ the map given by $(c,x) \mapsto c$.
 (a) Suppose $V \subset B$ is uniformly covered. In Exercise 25.16 we will verify that $f^{-1}(V)$ is uniformly covered. Since $f^{-1}(V)$ is open we see that $q\colon Z \to C$ is a covering.
 (b) Since B is connected it makes sense to speak of the degree of p. For any $c \in C$ we have a bijection
$$q^{-1}(c) \;=\; \{(c,x) \in C \times X \,|\, f(c) = p(x)\} \xrightarrow{\text{bijection}} p^{-1}(f(c)) \;=\; \deg(p).$$
 This shows that the degree of q equals the degree of p.
 (c) We need to show that $q_*(\pi_1(Z,z)) = f_*^{-1}(\pi_1(X,x))$. We show the two inclusions "$\subset$" and "$\supset$" separately.
 "$\subset$" Let $\widetilde{f}\colon Z \to X$ be the map given by $(c,x) \to x$. By construction we have $f \circ q = \widetilde{f} \circ p$. But this implies immediately that $f_*(q_*(\pi_1(Z,z))$ is contained in $p_*(\pi_1(X,x))$.
 "$\supset$" Let $\gamma\colon [0,1] \to C$ be a loop in (C,c) with $[f \circ \gamma] = f_*([\gamma]) \in p_*(\pi_1(X,x))$. We write $g := f \circ \gamma$. Let $\widetilde{g}\colon [0,1] \to X$ be the lift with $\widetilde{g}(0) = x$. Since $[g] \in p_*(\pi_1(X,x))$ we obtain from Lemma 16.15 (1) that $\widetilde{g}$ is actually a loop. Thus the map $\alpha\colon [0,1] \to C \times X$ that is given by $t \mapsto (\gamma(t), \widetilde{g}(t))$ is a loop. By construction the loop takes values in Z and evidently we have $q_*([\alpha]) = [\gamma]$. This shows that $[\gamma] \in q_*(\pi_1(Z,z))$. ■

We conclude this section on inverse limits with the following lemma.

Lemma 25.17. *Let $(I, \leq)$ be a preordered set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ be an inverse system of topological spaces.*
(1) *If all the X_i's are Hausdorff, then the inverse limit $\varprojlim X_i$ is a closed subset of $\prod_{i\in I} X_i$.*
(2) *If all the X_i's are compact and Hausdorff, then the inverse limit $\varprojlim X_i$ is also compact.*[466]

[465] Recall that covering maps are in particular surjective, in particular such x exists.

Example. Let p be a prime. The abelian group $\widehat{\mathbb{Z}}_p$ is the inverse limit of finite, hence compact groups. It follows from Proposition 25.18 that $\widehat{\mathbb{Z}}_p$ is also a compact topological group. On the other hand $\widehat{\mathbb{Z}}_p$ is infinite,[467] which shows that the topology on $\widehat{\mathbb{Z}}_p$ is *not* the discrete topology.

PROOF. Let $(I, \leq)$ be a preordered set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ be an inverse system of topological spaces.

(1) We suppose that all the X_i are Hausdorff. Given $i \in I$ we denote by $p_i\colon \prod_{j\in J} X_j \to X_i$ the projection map and we denote by $D_i = \{(x,x) \mid x \in X_i\} \subset X_i \times X_i$ the diagonal. By Exercise 3.21 the hypothesis that X_i is Hausdorff implies that the diagonal D_i is a closed subset of $X_i \times X_i$. By the proof of Proposition 25.15 we have

$$\begin{aligned}\varprojlim X_i \;&:=\; \Big\{(x_i)_{i\in I} \in \prod_{i\in I} X_i \,\Big|\, f_{lk}(x_l) = x_k \text{ for all } k \leq l\Big\}\\ &= \bigcap_{k\leq l}\Big\{x \in \prod_{i\in I} X_i \,\Big|\, \underbrace{((f_{lk}\circ p_l)\times p_k)}_{\substack{(f_{lk}\circ p_l)\times p_k \text{ is a map}\\ \prod_{i\in I} X_i \to X_k\times X_k}}(x) \in D_k\Big\} \;=\; \bigcap_{k\leq l}\underbrace{((f_{lk}\circ p_l)\times p_k)^{-1}(D_k)}_{\text{closed since } D_k \text{ closed}}.\end{aligned}$$

Thus we see that $\varprojlim X_i$ is the intersection of closed subsets of $\prod_{i\in I} X_i$, hence $\varprojlim X_i$ itself is a closed subset.

(2) Now we suppose that all the X_i are Hausdorff and compact. We obtain from Tychonoff's Theorem 3.17 that $\prod_{i\in I} X_i$ is compact. Together with (1) we see that $\varprojlim X_i$ is a closed subset of a compact topological space. Thus it follows from Lemma 2.17 (1) that $\varprojlim X_i$ is also compact. ■

25.7. **The profinite completion of a group** (∗)**.** Note that we can view any group as a topological group by equipping it with the discrete topology. With this topology a group is compact if and only if it is finite. Furthermore any homomorphism of groups equipped with the discrete topology is also a morphism in the category $\mathcal{TopGr}$ of topological groups.

Proposition 25.18. *Let $(I, \leq)$ be a preordered set and let $(\{G_i\}_{i\in I}, \{f_{ji}\colon G_j \to G_i\}_{i\leq j})$ be an inverse system of finite groups. Then the inverse limit $\varprojlim G_i$ exists in the category of topological groups and it is a compact topological group.*

PROOF. It follows from Proposition 25.15 that the inverse limit $\varprojlim G_i$ exists in the category of topological groups. Since all the G_i's are finite, in particular compact, it follows from Lemma 25.17 (2) that the inverse limit $\varprojlim G_i$ is also compact. ■

[466]Note that this is very different from the situation of the direct limit. The direct limit of topological spaces is not necessarily compact, for example the direct limit of the direct system $[0, n]$, $n \in \mathbb{N}$ is given by the non-compact space $\bigcup_{n\in\mathbb{N}} [0, n] = [0, \infty)$.

[467]In fact $\widehat{\mathbb{Z}}_p$ is uncountable (why?).

Now let π be a group. We consider the set of all finite index normal subgroups $\{\pi_i\}_{i\in I}$. We equip the index set I with the preorder defined by the convention that $i \leq j$ if and only if $\pi_j \subset \pi_i$.[468] Given $i \leq j$ we denote by $\varphi_{ji}\colon \pi/\pi_j \to \pi/\pi_i$ the obvious projection map. Then $(\{\pi/\pi_i\}_{i\in I}, \{\varphi_{ji}\colon \pi/\pi_j \to \pi/\pi_i\}_{i\leq j})$ forms an inverse system. We refer to

$$\widehat{\pi} \ := \ \varprojlim \pi/\pi_i$$

as the *profinite completion of* π. By Proposition 25.18 the profinite completion $\widehat{\pi}$ is a compact topological group. It follows immediately from the definition of an inverse limit that there exists a unique homomorphism $\pi \to \widehat{\pi}$ such that for any i the following diagram commutes:

$$\begin{tikzcd} \pi \arrow[r] \arrow[dr, bend right] & \widehat{\pi} \arrow[d] \\ & \pi/\pi_i. \end{tikzcd}$$

Examples.

(1) The profinite completion $\widehat{\mathbb{Z}}$ of the group $\mathbb{Z}$ equals precisely the profinite integers that we had encountered on page 751.
(2) The profinite completion of a finite group G is the group itself, i.e. $\widehat{G} = G$.[469]
(3) Let $G = \mathbb{Q}$. This group has no proper subgroups of finite index.[470] It follows immediately from the definitions that $\widehat{\mathbb{Q}} = \{0\}$.

Now let $\varphi\colon A \to B$ be a group homomorphism. We denote by $\{B_i\}_{i\in I}$ the finite index normal subgroups of B. For $B_j \subset B_i$ we denote again by $p_{ji}\colon B/B_j \to B/B_i$ the projection map. For every $i \in I$ the preimage $\varphi^{-1}(B_i)$ is a finite index normal subgroup of A.[471] Now we define $\psi_i\colon \widehat{A} \to A/\varphi^{-1}(B_i) \xrightarrow{\varphi} B/B_i$. Note that ψ_i is continuous since the first map is continuous by definition and the second map is a map between two topological spaces that are equipped with the discrete topology. It follows immediately from the definitions that for $B_j \subset B_i$ we have $p_{ji} \circ \psi_j = \psi_i$. It now follows from the definition of the inverse limit in the category of topological groups that there exists a unique continuous homomorphism $\widehat{\varphi}\colon \widehat{A} \to \widehat{B}$ such that for any i the following diagram commutes

$$\begin{tikzcd} \widehat{A} \arrow[r, "\widehat{\varphi}"] \arrow[dr, bend right, "\psi_i"'] & \widehat{B} \arrow[d] \\ & B/B_i. \end{tikzcd}$$

[468]Note that we wrote $\pi_j \subset \pi_i$ instead of $\pi_i \subset \pi_j$.
[469]Why is that?
[470]Why not?
[471]Why is it a finite index subgroup?

It follows easily from the definitions and Proposition 25.18 that the maps

$$A \mapsto \widehat{A}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}(A,B) & \to & \mathrm{Mor}(\widehat{A},\widehat{B}) \\ (\varphi\colon A\to B) & \mapsto & (\widehat{\varphi}\colon \widehat{A}\to\widehat{B}) \end{array}$$

define a functor from the category of groups to the category of compact topological groups.

Exercises for Chapter 25.

Exercise 25.3. Let $(I,\leq)$ be a preordered set and let $\{X_i\}_{i\in I}$ be a family of topological spaces such that for any $i\leq j$ we have $X_i\subset X_j$. We define

$$\varinjlim X_i \;=\; X \;:=\; \bigcup_{i\in I} X_i,$$

where the topology on $X=\bigcup_{i\in I} X_i$ is given by the rule that $U\subset X$ is open if and only if $U\cap X_i$ is open for all $i\in I$. Furthermore we denote by $g_j\colon X_j\to X=\bigcup_{i\in I} X_i$, $j\in\mathbb{N}$ the obvious inclusion maps. Show that $\varinjlim X_i$ together with the maps g_j is the direct limit of the direct system in the category of topological spaces.

Exercise 25.4. Let $\{X_i\}_{i\in\mathbb{N}}$ be a family of topological spaces. Suppose that for each $i\in\mathbb{N}$ we are given an injective map $f_i\colon X_i\to X_{i+1}$.

(a) Show the map $X_i\to\varinjlim X_i$ is not necessarily an embedding.
(b) Suppose that each $f_i\colon X_i\to X_{i+1}$ is an embedding. Show that in this case each map $X_i\to\varinjlim X_i$ is in fact an embedding.

Exercise 25.5.

(a) For $i\in\mathbb{N}$ we define $F_i=\langle x_i,x_{i+1}\rangle$. For $i\in\mathbb{N}$ we denote by $g_i\colon F_i\to F_{i+1}$ the homomorphism that is given by $g_i(x_i)=e, g_i(x_{i+1})=x_{i+1}$. Furthermore for $i<j$ we define $f_i\colon F_i\to F_j$ to be the homomorphism $g_{j-1}\circ\cdots\circ g_{i+1}\circ g_i$. What is the direct limit $\varinjlim F_i$ in the category of groups?
(b) Given $k\in\mathbb{N}$ we define $X_k=\mathbb{N}$ to be the topological space where the open sets are $\{1,\dots,k\},\{k+1\},\{k+2\},\dots$. For $k\leq l$ we denote by $f_{kl}\colon X_k\to X_l$ the identity map. (Note that with the choice of our topologies on X_k and X_l this map is in fact continuous). What is the direct limit $\varinjlim X_k$ in the category of topological spaces?
Remark. as a set the direct limit is again $\mathbb{N}$, but what is the topology?

Exercise 25.6. Suppose we are given two pushouts

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, dashed] \\ Z \arrow[r, dashed] & P \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \widetilde{Y} \arrow[d, dashed] \\ \widetilde{Z} \arrow[r, dashed] & \widetilde{P}. \end{tikzcd}$$

Furthermore suppose we are given a commutative diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] \arrow[dd] & & Y \arrow[dr] \arrow[dd, dashed] & \\ & \widetilde{X} \arrow[rr] \arrow[dd, dashed] & & \widetilde{Y} \arrow[dd, dashed] \\ Z \arrow[rr, dashed] \arrow[dr] & & P \arrow[dr, dashed, "\exists!"] & \\ & \widetilde{X} \arrow[rr, dashed] & & \widetilde{P}. \end{tikzcd}$$

An example for such a situation is sketched in Figure 486.

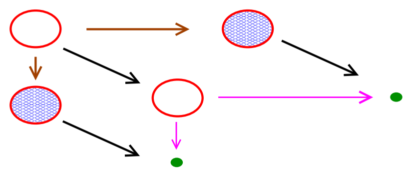

FIGURE 486

(a) Show that there exists a unique map $P \to \widetilde{P}$ which makes the cube commute.
(b) Suppose that the maps $X \to \widetilde{X}$, $Y \to \widetilde{Y}$ and $Z \to \widetilde{Z}$ are homotopy equivalences. Does it follows that the map $P \to \widetilde{P}$ is a homotopy equivalence?

Exercise 25.7. Proof Lemma 25.2. It is best to prove it using the definition of direct limits instead of using any of the above explicit descriptions of direct limits.

Exercise 25.8. We let $(I, \leq) = (\mathbb{N}, \leq)$. Given $i \in \mathbb{N}$ we set $X_i := \mathbb{N}$ and for $i \leq j$ we define

$$\begin{array}{rcl} f_{ji} \colon X_j = \mathbb{N} & \to & X_i = \mathbb{N} \\ n & \mapsto & n + (j - i). \end{array}$$

Then $(\{X_i = \mathbb{N}\}_{i \in \mathbb{N}}, \{f_{ji}\}_{i \leq j})$ forms an inverse system in the category of sets. Show that the inverse limit is the empty set.

Exercise 25.9. Let $g \geq 2$. We denote by Σ_g the surface of genus g. Show that every abelian subgroup of $\pi_1(\Sigma_g)$ is isomorphic to $\mathbb{Z}$.
Hint. Make use of Proposition 25.13 and Exercise 19.11.

Exercise 25.10. Let $g \in \mathbb{N}_{\geq 2}$. Use Proposition 25.13 to show that the fundamental group of the surface of genus g is torsion-free.

Exercise 25.11. Let

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \\ C & \end{tikzcd}$$

be a pushout system in some category which admits pushouts. (E.g. we could consider the category of groups, R-modules or topological spaces). We consider the direct limit, i.e. the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "r"] \\ C \arrow[r, "s"] & X. \end{tikzcd}$$

Show that if $f\colon A \to B$ is an isomorphism in the given category, then the opposite map $s\colon C \to X$ is also an isomorphism in the category.
Remark. This statement generalizes the statement of Lemma 3.43 (3c).

Exercise 25.12. Let $\mathcal{C}$ be some category. We consider the following commutative diagram of morphisms in $\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha"] & B \arrow[r, "\psi"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\ A' \arrow[r, "\varphi'"] & B' \arrow[r, "\psi'"] & C'. \end{tikzcd}$$

We suppose that the big rectangle forms a pushout square.

(a) We suppose that the square to the left is a pushout square. Show that the square to the right is also a pushout square.
(b) We suppose that the square to the right is a pushout square. Is the square to the left also a pushout square?

Exercise 25.13. Let X be a topological space. We denote by $\mathcal{K}$ the set of all compact subsets of X. We consider $\mathcal{K}$ with the preorder given by inclusion, i.e. given $A, B \in \mathcal{K}$ we have $A \leq B$ if $A \subseteq B$. Given $A \in \mathcal{K}$ we define[472] $X_A := \pi_0(X \setminus A)$. Furthermore given $A \leq B$ we denote by $f_{AB}\colon \pi_0(X_B) \to \pi_0(X_A)$ the map induced by the inclusion $X_B \to X_A$. It follows immediately from the discussion on page 480 that the sets $\{\pi_0(X_A)\}_{A\in\mathcal{K}}$, together with the inclusion induced maps f_{AB} for $A \leq B$ define an inverse system in the category of sets. The elements of $\varprojlim \pi_0(X_A)$ are called the *ends of* X.

(a) Let $\mathcal{T}op$ be the category of topological spaces and let $\mathcal{S}et$ be the category of sets. Show that
$$X \;\mapsto\; \varprojlim \pi_0(X_A)$$
gives rise to a covariant functor from $\mathcal{T}op$ to $\mathcal{S}et$.
(b) Show that for a topological space X the cardinality of $\varprojlim \pi_0(X_A)$, viewed as an element in $\mathbb{N}_0 \cup \{\infty\}$, equals the number of ends of M as defined in Exercise 2.22.
(c) Let $n \in \mathbb{N}_0$. Determine the number of ends of $X = \mathbb{R}^n$.
(d) For each $k \in \mathbb{N}_0$ give an example of a connected 1-dimensional CW-complex X with precisely k ends.

Exercise 25.14. Let X be a topological space, let A and B be two subsets of X and let $f\colon A \to B$ be a homeomorphism. We define

$$\widetilde{X} \;:=\; \Big(\bigcup_{i\in\mathbb{Z}} X \times \{i\}\Big) \,/\, (a,i) \sim (f(a), i-1) \qquad \text{for } a \in A \text{ and } i \in \mathbb{Z}.$$

We consider the action of $\mathbb{Z}$ on $\widetilde{X}$ that is given by $n\cdot[(x,i)] := [(x,i+n)]$ for $(x,i) \in X\times\{i\}$.

(a) Show that this action is continuous.

[472] Recall that by the definition on page 135, given a topological space Y we had denoted $\pi_0(Y)$ by the set of path-components of X.

(b) We write $X(f) = X/\sim$ where $a \sim f(a)$ for $a \in A$. Show that the maps

$$\begin{array}{rcl} \Phi\colon X(f) & \to & \widetilde{X}/\mathbb{Z} \\ [x] & \mapsto & [(x,0)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon \widetilde{X}/\mathbb{Z} & \to & X(f) \\ [(x,i)] & \mapsto & [x] \end{array}$$

are continuous and inverses of one another. (This implies of course that both maps are homeomorphisms.)

(c) Suppose that there exist disjoint open neighborhoods U of A and V of B. Show that the action is discrete.

Exercise 25.15. We consider the profinite integers, i.e. the inverse limit $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}_n$ of the rings $\mathbb{Z}_n$, $n \in \mathbb{N}$, with the preorder on $\mathbb{N}$ defined by the divisibility relation $k|l$. Does there exist an epimorphism $\widehat{\mathbb{Z}} \to \mathbb{Z}$ of groups?

Exercise 25.16. Show that the pullback $f^*X \to C$ of a covering $p\colon X \to B$ under a map $f\colon C \to B$ is again a covering.

Exercise 25.17. Let $f^*X \to C$ be the pullback of a map $p\colon X \to B$ und a map $f\colon C \to B$. Show that if X and C are connected, then f^*X is also connected.

Exercise 25.18. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a short exact sequence of groups. Furthermore let $g\colon Z \to C$ be a group homomorphism. Show that there exists a commutative diagram of group homomorphisms of the form below such that the top horizontal sequence is also exact:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "\widetilde{\alpha}"] \arrow[d, "="] & Y \arrow[r, "\widetilde{\beta}"] \arrow[d, "f"] & Z \arrow[r] \arrow[d, "g"] & 1 \\ 1 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 1. \end{tikzcd}$$

Exercise 25.19.

(a) We consider the inverse system $(X_i)_{i\in\mathbb{N}}$ where $X_i = \mathbb{Z}$ and where for each $i \geq j$ the map $X_i \to X_j$ is given by multiplication by 2^{i+j}. Determine $\varprojlim X_i$ in the category of abelian groups.

(b) Now we work in the category of sets. Show that there exists an inverse system $(\{X_i\}_{i\in\mathbb{N}}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ of non-empty sets such that $\varprojlim X_i = \varnothing$.

Remark. In Exercise 25.20 we will see that the inverse limit of an inverse system of *finite non-empty sets* is in fact non-empty.

Exercise 25.20. In this exercise we consider the category of topological space. Let $(\{X_i\}_{i\in\mathbb{N}}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ be an inverse system of topological spaces that are non-empty, Hausdorff and compact.

(a) For each $j \in \mathbb{N}$ we set

$$Z_j := \{(x_i)_{i\in\mathbb{N}} \mid \text{ for all } i \leq j \text{ we have } x_i = f_{ji}(x_j)\} \subset \prod_{i\in\mathbb{N}} X_i.$$

Show that Z_j is closed and non-empty.
Hint. You might want to consider Lemma 25.17 (1).

(b) Show that for any $i, j \in \mathbb{N}$ we have $Z_i \cap Z_j \neq \varnothing$.
(c) Apply Tychonoff's Theorem 3.17, together with Exercise 2.48, to conclude that $\varprojlim X_i \neq \varnothing$.

Remark. We can apply this statement in particular to an inverse system of finite non-empty sets, viewed as topological spaces by equipping each of the finite sets with the discrete topology. Evidently a finite discrete topological space is Hausdorff and compact.

Exercise 25.21. Let I be preordered set, let $\{f_i \colon A_i \to A_j\}_{i \leq j}$ be a direct system in the category of abelian groups and let $\{g_i \colon B_i \to B_j\}_{j \leq i}$ be an inverse system in the category of abelian groups. Suppose that for each $i \in I$ we are given a bilinear map $\varphi_i \colon A_i \times B_i \to \mathbb{Z}$ such that for any $i \leq j$, and $a \in A_i$ and $b \in B_j$ we have $\varphi_i(a_i, g_j(b_j)) = \varphi_j(f_i(a_i), b_j)$. Define an "interesting" bilinear map

$$\varinjlim A_i \ \times \ \varprojlim B_i \ \to \ \mathbb{Z}.$$

26. The HNN-Seifert–van Kampen Theorem

One of our main goals at the moment is to calculate fundamental groups of interesting topological spaces. As we have discovered, one of the most powerful tools is the Seifert–van Kampen Theorem 22.1.

In light of Exercise 3.32 (c) we can say, modulo technical details, that the Seifert–van Kampen Theorem 22.1 allows us to calculate the fundamental group of a topological space $X = Y \cup_f Z$ that is obtained from gluing two path-connected topological spaces Y and Z along path-connected open subsets $A \subset Y$ and $B \subset Z$. This situation is illustrated in Figure 487 to the left.

Sometimes one is given the situation that is shown in Figure 487 to the right. Namely we are given a path-connected topological space X and we glue two disjoint path-connected open subsets A and B together. In this chapter we will prove the HNN-Seifert–van Kampen Theorem which will allow us to calculate the fundamental group in the latter situation.

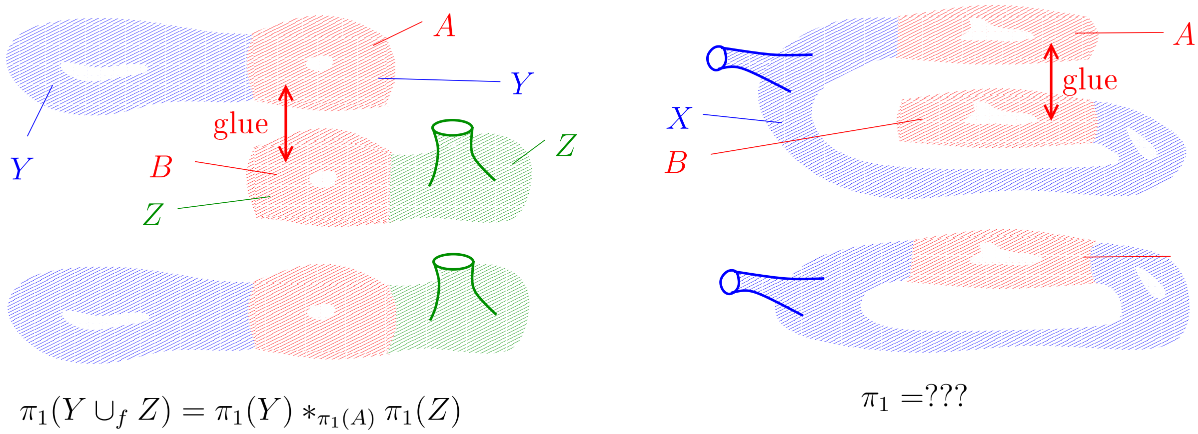

Figure 487

We start out with the following notation.

Notation. Let X be a topological space, let A and B be two subsets of X and let $f\colon A \to B$ be a homeomorphism. We write

$$X(f) \;:=\; X(f\colon A \to B) \;:=\; X/\sim \qquad \text{where } f(a) \sim a \text{ for } a \in A.$$

Examples.

(1) Let X be the torus with two open disks removed. We denote by A and B the two boundary components and we endow them with the orientation coming from a fixed orientation of the torus. Let $f\colon A \to B$ be an orientation-reversing diffeomorphism. Then one sees that $X(f\colon A \to B)$ is diffeomorphic[473] to the surface of genus 2. This statement is illustrated in Figure 488.[474]

(2) Let Y be a topological space and let $f\colon Y \to Y$ be a homeomorphism. We consider $X = Y \times [0,1]$. Furthermore, by an abuse of notation, we denote by $f\colon Y \times \{0\} \to Y \times \{1\}$ also the map defined by $f(y,0) = (f(y),1)$. It follows immediately from the

[473] Note that by Lemma 8.16 we know that $X(f\colon A \to B)$ is indeed a smooth manifold.

[474] What topological space do we obtain if f is chosen to be orientation-preserving?

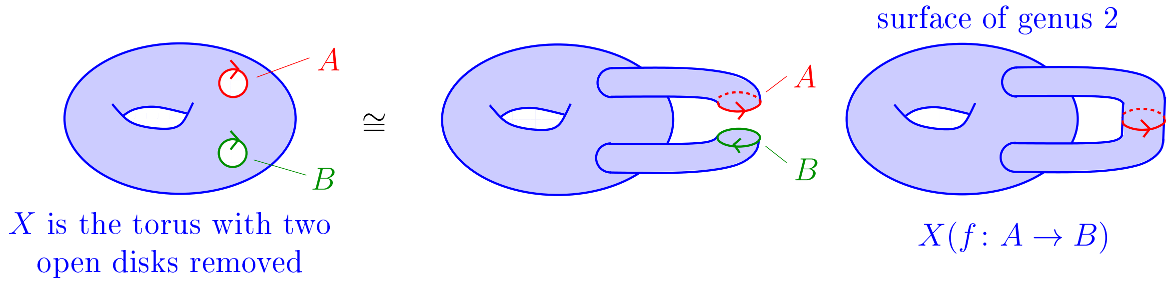

FIGURE 488

definitions that

$$X(f) \;=\; (Y \times [0,1])(f) \;=\; (Y \times [0,1])/\sim \;=\; \operatorname{Tor}(X, f),$$

where $\operatorname{Tor}(X, f)$ denotes the mapping torus that we introduced on page 711.

Our goal now is to determine $\pi_1(X(f))$ in terms of the fundamental groups of X, A and the induced map $f_*\colon \pi_1(A) \to \pi_1(B)$. To formulate the Seifert–van Kampen Theorem 22.1 we needed to introduce the notion of an amalgamated product of groups. So it does perhaps now come as a surprise that we need a new construction from group theory to formulate the desired result.

Definition. Let π and Γ be two groups and let $\alpha, \beta\colon \Gamma \to \pi$ be two homomorphisms. We refer to

$$\langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1}\rangle \;:=\; (\pi * \langle t\rangle) \,/\, \langle\!\langle \{\alpha(g)t\beta(g)^{-1}t^{-1}\}_{g\in\Gamma}\rangle\!\rangle$$

as the *HNN-extension corresponding to* (π, α, β).[475]

Example. Let π be a group and let Γ be the trivial group. It follows immediately from the definitions that the corresponding HNN-extension is the free product $\pi * \langle t\rangle$.

The following lemma summarizes some basic properties of HNN-extensions. It says in particular that a semidirect product is a special case of an HNN-extension.

Lemma 26.1.

(1) *Let $\pi = \Gamma$. Furthermore let $\alpha\colon \pi \to \pi$ and $\beta\colon \pi \to \pi$ be isomorphisms. We write $\varphi = \beta \circ \alpha^{-1}\colon \pi \to \pi$. In this setting the map*

$$\begin{array}{rcl} \Phi\colon \pi \rtimes_\varphi \mathbb{Z} & \to & \langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1}\rangle \\ (g, n) & \mapsto & g \cdot t^{-n} \end{array}$$

is an isomorphism.

(2) *Let $\pi = \langle x_1, \dots, x_k \mid r_1, \dots, r_l\rangle$ be a finitely presented group and let $g_1, \dots, g_m$ is a generating set for Γ. A presentation for the* HNN*-extension of (π, α, β) is given by*

$$\langle x_1, \dots, x_k, t \mid r_1, \dots, r_l, \alpha(g_i) = t\beta(g_i)t^{-1} \text{ for } i = 1, \dots, m\rangle.$$

[475]The concept of such a group was introduced by Graham Higman, Bernhard Neumann and Hannah Neumann [**HNN50**] in 1950, whence the name.

(3) *Let π and Γ be two groups and let $\alpha, \beta\colon \Gamma \to \pi$ be two homomorphisms. Then there exists an epimorphism*
$$\langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1}\rangle \;\to\; \mathbb{Z}$$
that is uniquely determined by $g \mapsto 0$ if $g \in \pi$ and $t^n \mapsto n$.

Sketch of proof.

(1) We first verify that the given map is indeed a homomorphism. Let (g, m) and (h, n) be two elements in $\pi \rtimes_\varphi \mathbb{Z}$. First we assume that $m \geq 0$. Then the following equalities hold in $\langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1}\rangle$:
$$\begin{array}{rcl} \Phi(g,m)\cdot\Phi(h,n) &=& gt^{-m}\cdot ht^{-n} \overset{\substack{\text{since } \alpha \text{ is an isomorphism we can write} \\ h=\alpha(\alpha^{-1}(h))=t\beta(\alpha^{-1}(h))t^{-1}=t\varphi(h)t^{-1}=t\cdot t\varphi(\varphi(h))t^{-1}\cdot t^{-1}=\dots=t^m\varphi^m(h)t^{-m} \\ \downarrow}}{=} gt^{-m}\cdot t^m\varphi^m(h)t^{-m}\cdot t^{-n} = g\varphi^m(h)t^{-m-n} \\ &=& \Phi(g\varphi^m(h), m+n) \underset{\substack{\uparrow \\ \text{definition of product structure on } \pi \rtimes_\varphi \mathbb{Z}}}{=} \Phi((g,m)(h,n)). \end{array}$$
If $m < 0$ then a very similar calculation, using that β is an isomorphism, shows again that $\Phi(g, m) \cdot \Phi(h, n) = \Phi((g, m)(h, n))$. This shows that Φ is indeed a homomorphism. By Lemma 19.12 there exists a unique homomorphism
$$\Psi\colon \pi * \langle t\rangle \;\to\; \pi \rtimes \mathbb{Z}$$
with $\Psi(g) = (g, 0)$ for every $g \in \pi$ and with $\Psi(t) = (0, 1)$. Using Lemma 21.4 and using an elementary calculation one can easily show that Ψ descends to a map $\langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1}\rangle \to \pi \rtimes \mathbb{Z}$. Since $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity on the generating sets π and t we see that both homomorphisms are the identity. In particular Φ is an isomorphism.

(2) This time we apply Lemma 21.4 twice to obtain the desired isomorphism and its inverse.

(3) This statement is once again an immediate consequence of Lemma 21.4. ■

The following proposition is an analogue of Proposition 21.24.

Proposition 26.2. (∗) *Let π and Γ be two groups and let $\alpha, \beta\colon \Gamma \to \pi$ be two homomorphisms. If α and β are monomorphisms, then the obvious inclusion map*
$$\begin{array}{rcl} \pi & \to & \langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1}\rangle := (\pi * \langle t\rangle)\,/\,\langle\!\langle \{\alpha(g)t\beta(g)^{-1}t^{-1}\}_{g\in\Gamma}\rangle\!\rangle \\ h & \mapsto & h \end{array}$$
is also a monomorphism.

Proof. The proposition is proved in [**Ser80**, p. 45]. Alternatively the proposition follows from the Normal Form Theorem for HNN-extensions which is formulated and proved in [**LS77**, Theorem IV.2.1] or alternatively in [**CgRR08**, Satz 6.3] and [**Sti93**, Chapter 9.2]. ■

Now we return to topological spaces. The following theorem can be considered a sibling of the Seifert–van Kampen Theorem 22.1.

Theorem 26.3. (HNN-Seifert–van Kampen Theorem) *Suppose we are in one of the following two settings:*

(1) *We are given a path-connected topological space X, two open disjoint subsets A and B of X and a homeomorphism $f\colon A \to B$.*
(2) *We are given a path-connected smooth manifold X, two distinct components A and B of ∂X and a homeomorphism $f\colon A \to B$.*

We pick a base point $x_0 \in A$ and we pick a path γ in X from x_0 to $f(x_0)$. Furthermore we write $\Gamma = \pi_1(A, x_0)$. Finally we denote by

$$\alpha\colon \Gamma = \pi_1(A, x_0) \;\to\; \pi_1(X, x_0)$$

the inclusion induced map and we denote by

$$\beta\colon \Gamma = \pi_1(A, x_0) \xrightarrow{f_*} \pi_1(B, f(x_0)) \to \pi_1(X, f(x_0)) \xrightarrow{\gamma_*} \pi_1(X, x_0)$$

the concatenation of the maps induced by f, the inclusion and the base point change isomorphism from Proposition 14.11, using the path γ. Then there exists an isomorphism

$$\Phi\colon \pi_1(X(f), x_0) \;\xrightarrow{\cong}\; \langle \pi_1(X, x_0), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1}\rangle$$

which has the following two properties:

(1) *The diagram*

$$\begin{tikzcd} & \pi_1(X, x_0) \arrow[dl] \arrow[dr] & \\ \pi_1(X(f), x_0) \arrow[rr, "\Phi"] & & \langle \pi_1(X, x_0), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1}\rangle \end{tikzcd}$$

with the obvious diagonal maps commutes.

(2) *The closed loop in $X(f)$ defined by γ corresponds under the isomorphism Φ to t.*

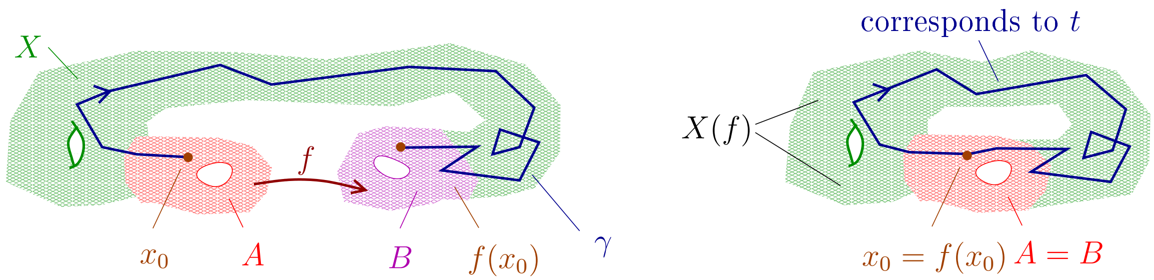

FIGURE 489. Illustration for the HNN-Seifert–van Kampen Theorem 26.3 (1).

Remark. The HNN-Seifert–van Kampen Theorem 26.3 is a frequently used tool in the calculation of fundamental groups of topological spaces. The popularity of the result is in contrast to the sparsity of proofs in the literature. A sketch of an outline of a proof is given in [**SW79**]. A much more general statement is proved in [**Bou07**, Proposition IV.5.5 on p. 431].

Example. As on page 761 we consider the surface X that is given by removing two open disks from a torus. We refer to the two boundary components as A and B. We equip A and B with the orientations given in Figure 490. We pick a diffeomorphism $f\colon A \to B$ that preserves the orientation. As discussed on page 761, the resulting topological space $X(f)$ is homeomorphic to the surface of genus two.

We pick a base point $x_0 \in A$. We pick a loop a in (A, x_0), we pick loops c, d in (X, x_0), a path γ in X from x to $f(x)$ and a loop b in $(B, f(x))$ as indicated in Figure 490. Finally we write $\widetilde{b} = \gamma * b * \overline{\gamma}$. By Lemma 23.11 and as in the proof of Lemma 25.11 we have an isomorphism

$$\pi_1(X, x_0) \xrightarrow{\cong} \langle c, d, a, \widetilde{b} \mid a = [c, d] \cdot \widetilde{b} \rangle.$$

We write $\Gamma = \pi_1(A, x_0)$. Note that $\Gamma = \langle a \rangle$ and $\alpha(a) = a$ and $\beta(a) = \widetilde{b}$. We see that

$$\begin{aligned}\pi_1(X(f), x_0) \underset{\substack{\uparrow \\ \text{Theorem 26.3 (2)}}}{\cong} \langle \pi_1(X), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle &\underset{\substack{\uparrow \\ \text{Lemma 26.1}}}{=} \langle c, d, a, \widetilde{b}, t \mid a = [c, d] \cdot \widetilde{b}, a = t\widetilde{b}t^{-1} \rangle \\ \underset{\substack{\uparrow \\ \text{Tietze transformations}}}{=} \langle c, d, \widetilde{b}, t \mid [c, d] \cdot \widetilde{b} = t\widetilde{b}t^{-1} \rangle &= \langle c, d, \widetilde{b}, t \mid \underbrace{[c, d] \cdot \widetilde{b}t\widetilde{b}^{-1}t^{-1}}_{=[c,d][\widetilde{b},t]} \rangle.\end{aligned}$$

Thus we have now obtained the same presentation that we had already obtained in Proposition 22.3.

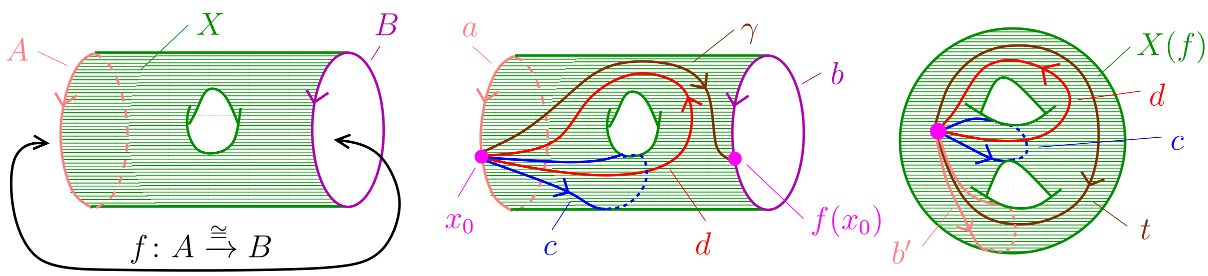

FIGURE 490

In the proof of Theorem 26.3 (1) we will need one extra piece of notation and one more lemma.

Notation. Let π and Γ be groups and let $\alpha, \beta\colon \Gamma \to \pi$ be homomorphisms. Given $i \in \mathbb{Z}$ we write $\pi_i := \pi$ and $\Gamma_i := \Gamma$. We write[476]

$$\dots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \dots := \varinjlim \big(\underbrace{\pi_{-i} \underset{\Gamma_{-i+1}}{*} \dots \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \dots \underset{\Gamma_i}{*} \pi_i}_{\text{iterated amalgamated product defined by } \alpha \text{ and } \beta} \big).$$

By a slight abuse of language we refer to this group as an infinite amalgamated product. By the universal property of the direct limit there exists an automorphism Ξ of this infinite amalgamated product that shifts all indices by one to the right.

[476]By Lemma 21.22 (4) we can form the iterated amalgamated product without having to worry about parentheses.

Lemma 26.4. *With the above notation there exists a well-defined map*

$$\begin{array}{rcl} \left(\dots\pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \dots\right) \rtimes_{\Xi} \mathbb{Z} & \xrightarrow{\cong} & \langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1}\rangle \\ (g_i, n) & \mapsto & gt^{-i}\cdot t^n, \quad \text{where } g_i \in \pi_i \end{array}$$

which is an isomorphism.

Example. If α and β are isomorphisms, then it follows easily from Lemma 21.22 (3) that the "infinite amalgamated product" is isomorphic to π. We see that in this special case the statement of Lemma 26.4 is basically the same as the statement of Lemma 26.1 (1).

Sketch of proof of Lemma 26.4. It follows easily from the universal property of the direct limit and of the amalgamated product that there exists a well-defined map

$$\begin{array}{rcl} \dots\pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \dots & \to & \langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1}\rangle \\ g_i & \mapsto & gt^{-i}, \quad \text{where } g_i = g \in \pi_i = \pi. \end{array}$$

A straightforward calculation as in the proof of Lemma 26.1 (1) shows that we get an induced map on the semidirect product. This shows that the map given in the lemma actually makes sense. It remains to show that the map is an isomorphism. Note that using Lemma 21.4 one can easily show that there exists a well-defined homomorphism from right to left which is an inverse to our given map. We leave it to the reader to fill in the details. ■

Proof of Theorem 26.3 (1). [477] Let X be a path-connected topological space, let A and B be two open disjoint path-connected subsets of X and let $f\colon A \to B$ be a homeomorphism. We pick a base point $x \in A$ and we pick a path γ in X from x to $f(x)$. Furthermore we define

$$\widetilde{X(f)} \;:=\; \Big(\bigcup_{i\in\mathbb{Z}} X \times \{i\}\Big) \,/\, (a,i) \sim (f(a), i-1) \quad \text{for } a \in A \text{ and } i \in \mathbb{Z}.$$

We equip $\widetilde{X(f)}$ with the base point $\widetilde{x}$ which is represented by $(x,0)$. There is an obvious action of $\mathbb{Z}$ on $\widetilde{X(f)}$ given by $n\cdot[(P,i)] := [(P,i+n)]$ for $(P,i) \in X \times \{i\}$. Since A and B are open and disjoint we can apply Exercise 25.14. We obtain that this action is discrete and continuous and that the map

$$\begin{array}{rcl} \Phi\colon X(f) & \to & \widetilde{X(f)}/\mathbb{Z} \\ [x] & \mapsto & [(x,0)] \end{array}$$

is a homeomorphism.

For $i \in \mathbb{Z}$ we now write $X_i := X \times \{i\}$, $A_i := A \times \{i\}$, $B_i := B \times \{i\}$ and $x_i := \{x\} \times \{i\}$. Furthermore we write $\pi_i := \pi_1(X_i, x_i)$ and $\Gamma_i = \pi_1(A_i, x_i)$. We refer to Figure 491 for an illustration.

[477] As we saw on pages 762 and in Lemma 26.1 (1), we can view Theorem 26.3 as a generalization of Proposition 24.28. In fact the subsequent proof is partly modelled on the proof of Proposition 24.28.

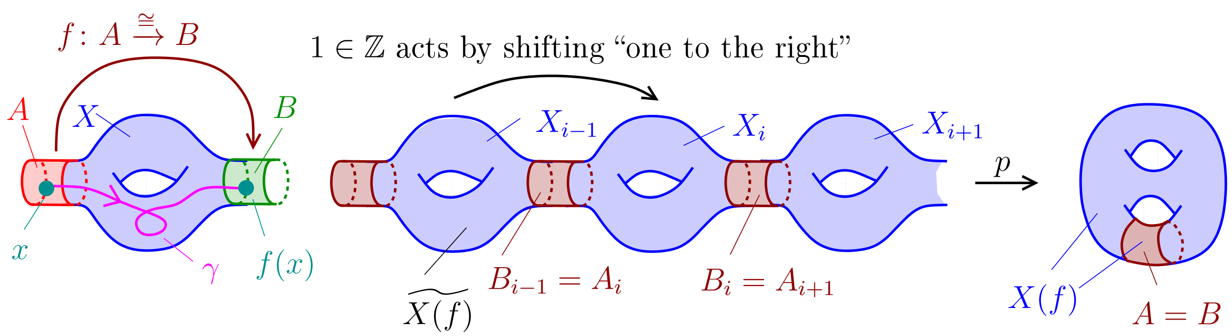

FIGURE 491. Illustration for the proof of Theorem 26.3 (1).

Claim. Given any $i \in \mathbb{Z}$ we consider the homomorphism

$$\pi_i \;=\; \pi_1(X_i, x_i) \underset{\substack{\uparrow \\ \text{induced by the} \\ \text{inclusion } X_i \to \widetilde{X(f)}}}{\longrightarrow} \pi_1(\widetilde{X(f)}, x_i) \underset{\substack{\uparrow \\ \text{induced by Proposition 14.11 (1) where we use} \\ i \text{ translates of the path } \gamma \text{ to connect } x_i \text{ to } \widetilde{x} = x_0}}{\longrightarrow} \pi_1(\widetilde{X(f)}, \widetilde{x}).$$

These homomorphisms induce an isomorphism

$$\dots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \dots \xrightarrow{\cong} \pi_1(\widetilde{X(f)}, \widetilde{x}).$$

Here given $i \in \mathbb{Z}$ the maps in the amalgamated product are defined as follows:

$$\overbrace{\pi_1(X_{i-1}, x_{i-1})}^{=\pi_{i-1}} \underset{\substack{\uparrow \\ \text{given by Proposition 14.11 (1)}}}{\xleftarrow{\gamma_*}} \pi_1(X_{i-1}, f(x_{i-1})) \underset{\substack{\uparrow \\ \text{induced by the} \\ \text{inclusion } B_{i-1} \to X_{i-1}}}{\leftarrow} \pi_1(B_{i-1}, f(x_i)) \xleftarrow{f_*} \overbrace{\pi_1(A_i, x_i)}^{=\Gamma_i} \underset{\substack{\uparrow \\ \text{induced by the} \\ \text{inclusion } A_i \to X_i}}{\rightarrow} \overbrace{\pi_i(X_i, x_i)}^{=\pi_i}.$$

To prove the claim we need to introduce one more gadget. Namely, given $n \in \mathbb{N}$ we write

$$\widetilde{Y}_n \;:=\; \Big(\bigcup_{i=-n}^{n} X_i \Big) \,/\, (a, i) \sim (f(a), i-1) \qquad \text{for } a \in A, i \in \{-n+1, \dots, n\}.$$

It follows from the Seifert–van Kampen Theorem 22.1, applied iteratively $2n$ times, that

$$\pi_1\big(\widetilde{Y}_n, \widetilde{x}\big) \xleftarrow{\cong} \pi_{-n} \underset{\Gamma_{-n+1}}{*} \pi_{-n+1} \dots \pi_{n-1} \underset{\Gamma_n}{*} \pi_n.$$

Now we see that

$$\pi_1\big(\widetilde{X(f)}, \widetilde{x}\big) \;=\; \pi_1\big(\varinjlim \widetilde{Y}_n, \widetilde{x}\big) \underset{\substack{\uparrow \\ \text{since } A,\, B \text{ are open in } X \text{ we can} \\ \text{apply Proposition 25.7 (3)}}}{\xleftarrow{\cong}} \varinjlim \pi_1\big(\widetilde{Y}_n, \widetilde{x}\big) \underset{\substack{\uparrow \\ \text{by the above calculation and the definition} \\ \text{of the right hand side}}}{\xleftarrow{\cong}} \dots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \dots$$

⊞

In summary we obtain isomorphisms

above homeomorphism (first $\cong$); by Theorem 16.16 (2) and Lemma 24.27, here the automorphism Θ is given by conjugation by $[\gamma] \in \pi_1(X(f), x)$ (second $\cong$)

$$\begin{aligned}
\pi_1(X(f), x) \;&\cong\; \pi_1\big(\widetilde{X(f)}/\mathbb{Z}, [\widetilde{x}]\big) \;\cong\; p_*(\pi_1(\widetilde{X(f)}, \widetilde{x})) \rtimes_\Theta \mathbb{Z} \\
&\cong\; \pi_1(\widetilde{X(f)}, \widetilde{x}) \rtimes \mathbb{Z} \;\cong\; \big(\ldots \pi_{-2} \underset{\Gamma_{-1}}{*} \pi_{-1} \underset{\Gamma_0}{*} \pi_0 \underset{\Gamma_1}{*} \pi_1 \underset{\Gamma_2}{*} \pi_2 \ldots\big) \rtimes_\Xi \mathbb{Z}
\end{aligned}$$

Corollary 16.14 (first $\cong$); here, as on page 765, we denote by Ξ the automorphism of the infinite amalgamated product that "shifts every term by one to the right", the isomorphism is then given by the above claim and the verification that under this isomorphism the automorphisms Θ and Ξ match up (second $\cong$)

$$\xrightarrow{\cong} \; \langle \pi_1(X(f), x), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1}\rangle.$$

Lemma 26.4

We leave it to the reader to verify that the given isomorphism has all the properties that were promised. ■

Sketch of the proof of Theorem 26.3 (2) (∗). The proof of Theorem 26.3 (2) is basically the same as the proof of Theorem 26.3 (1), we just need to replace the Seifert–van Kampen Theorem 22.1 by the Seifert–van Kampen Theorem 22.2 for smooth manifolds. We leave it to the reader to fill in the details and to make the necessary changes to the proof. ■

In many situations the following corollary to the HNN-Seifert–van Kampen Theorem 26.3 is quite useful.

Corollary 26.5. (∗) *Let X be a topological space. We suppose that it can be written as a union $X = Y \cup Z$ such that the following conditions are satisfied:*

(1) *Y and Z are open subsets,*
(2) *Y is path-connected,*
(3) *Z is simply connected,*
(4) *$Y \cap Z$ consists of two simply connected path components A and B, each of which is open in X.*

Let y be a point in Y and let γ be the concatenation of a path in Y from y to a point $a \in A$, a path in Z from a to a point $b \in B$ and a path in Y from b to y. Then the map

$$\pi_1(Y, y) * \langle t \rangle \;\to\; \pi_1(X, y)$$

that is given by the inclusion induced map $\pi_1(Y, y) \to \pi_1(X, y)$ and given by $t \mapsto [\gamma]$ is an isomorphism.

Example. We consider the topological space X that is obtained from "gluing an interval" to the torus. Using the notation from Figure 493 one can show quite easily, using Corollary 26.5 that $\pi_1(X, x_0) \cong \langle x, y, t \,|\, [x, y]\rangle \cong \mathbb{Z}^2 * \langle t \rangle$. We refer to Figure 493 for an illustration. We will work out the details of this example in Exercise 26.1.

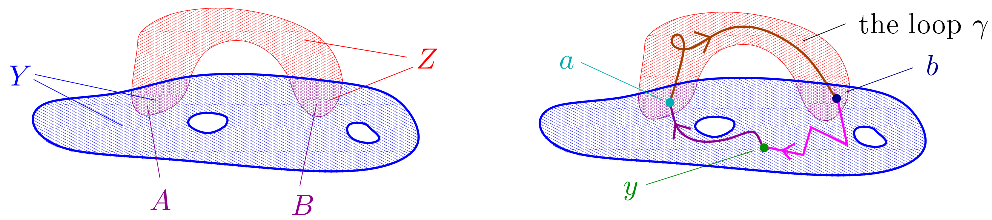

FIGURE 492. Illustration of Corollary 26.5.

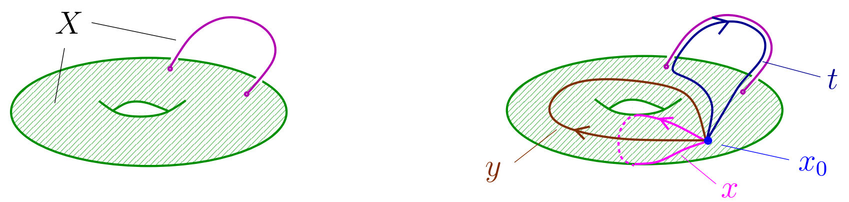

FIGURE 493

PROOF. Let Z' be a disjoint copy of Z (e.g. we could set $Z' = Z \times \{0\}$). We consider $W = (Y \sqcup Z')/a \sim a'$ where we identify each point $a \in A$ with the corresponding point in Z'. We denote by B' the copy of B in Z' and thus also the copy in W. Thus we see

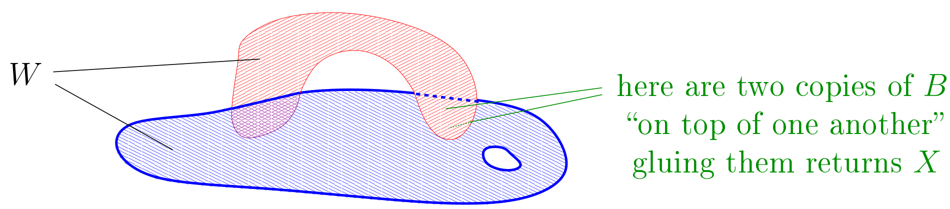

FIGURE 494. Illustration for the proof of Corollary 26.5.

that B and its doppelgänger B' are contained in W. We denote by $f\colon B' \to B$ the obvious homeomorphism. We obtain the following isomorphisms:[478]

by the HNN-Seifert–van Kampen Theorem 26.3 and since B is simply connected

$$\pi_1(X,y) \underset{\uparrow}{\cong} \pi_1(W(f),y) \overset{\downarrow}{\cong} \pi_1(W,y)*\langle t\rangle \underset{\uparrow}{\cong} (\pi_1(Y,y)*_{\pi_1(A,y)}\pi_1(Z',y))*\langle t\rangle \underset{\uparrow}{=} \pi_1(Y,y)*\langle t\rangle.$$

by Lemma 3.45 the obvious map $X \to W(f)$ is a homeomorphism | Seifert–van Kampen Theorem 20.2 | since A and Z' are simply connected

One can easily verify that this is precisely the map in the statement of the corollary. ■

We conclude this chapter with the following technical lemma which we will need once at a later occasion.

Lemma 26.6. $(*)$ *Let $n \in \mathbb{N}$ and let M be the 2-sphere with n open disks removed. We enumerate the boundary components by $C_1, \dots, C_n$. For $i = 1, \dots, n$ we pick a path*

[478] It follows from our hypothesis that A, Y, Z are open in X that in the settings of the HNN-Seifert–van Kampen Theorem 26.3 and the Seifert–van Kampen Theorem 20.2 we do in fact deal with open subsets.

$\beta_i\colon [0,1] \to C_i$ that goes once around C_i. We pick a base point $* \in M \setminus \partial M$. We pick paths $\alpha_1, \dots, \alpha_n\colon [0,1] \to M$ with the following properties:

1. *each α_i is a smooth embedding,*
2. *each α_i is a path from $*$ to the point $\beta_i(0)$,*
3. *for each i we have $\alpha_i([0,1)) \subset M \setminus \partial M$,*
4. *the images of the paths only intersect at $*$.*

For any $j \in \{1, \dots, n\}$ the homomorphism

$$\langle y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n \rangle \quad \to \quad \pi_1(M, *)$$

that is uniquely determined by

$$y_i \quad \mapsto \quad [\alpha_i * \beta_i * \overline{\alpha_i}]$$

is an isomorphism.

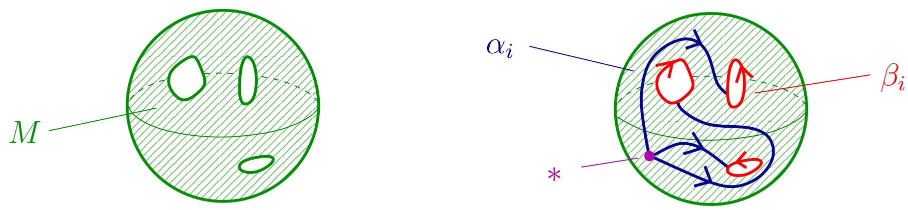

Figure 495. Illustration of Lemma 26.6.

Remark.

(1) This proposition should ring a bell, it sounds quite like Lemma 23.11. The difference is that in Lemma 23.11 we only said that there exist certain paths β_i such that the given map is an isomorphism. The above lemma gives a more precise statement what type of paths work.

(2) One might think that perhaps one can use any old paths α_i to obtain a basis for the free group $\pi_1(M, *)$. But as we will see in Exercise 26.6, this is not the case.

We conclude this chapter with a sketch of a proof of Lemma 26.6. Perhaps the most friendly words that can be uttered regarding the proof are that the proof shows that it can be quite hard to write down rigorous arguments once one leaves the friendly word of submanifolds and instead tries to deal with more general subsets.

Sketch of a proof. Without loss of generality we can assume that $j = n$. We consider $N := M \setminus \alpha_n([0,1])$. This is an open subset of the smooth manifold M, thus N itself is a smooth manifold. Furthermore by Exercise 10.2 we know that the inclusion $M \setminus \alpha_n([0,1]) \to M$ is a homotopy equivalence.[479] For $i = 1, \dots, n-1$ we write $X_i := \alpha_i((0,1]) \subset N$. It follows from (1) and Proposition 8.1 (1) that $X = X_1 \cup \dots \cup X_{n-1}$ is a submanifold of N. By the Tubular Neighborhood Theorem 8.24 we can pick a tubular map $[-1,1] \times X$. We

[479]The alert reader will not have failed to notice that we are referring to an exercise that is dealt with later in the notes. Fortunately we do not need to be afraid of circular logic. The exercise builds on an extension of the Tubular Neighborhood Theorem 8.24 to submanifolds with boundary. This formulation, see Theorem 10.5, is just a little awkward and was thus postponed to later on. But if one uses a naive extension of the Tubular Neighborhood Theorem 8.24 to the case of submanifolds with boundary, then one can easily prove Exercise 10.2 with our present knowledge.

write $W := N \setminus (-1,1) \times X$, we pick a base point $\star \in W$ and we pick path p in M from $*$ to $\star$.

Using Corollary 26.5 one can prove[480] the following:

(1) there exist loops $\eta_1, \dots, \eta_{n-1}$ in $(N, \star)$ such that the obvious map
$$\pi_1(W,\star) * \langle [\eta_1], \dots, [\eta_{n-1}] \rangle \;\to\; \pi_1(N,\star)$$
is an isomorphism,

(2) for each $i \in \{1, \dots, n-1\}$ the loop $p * \eta_i * \overline{p}$ is path-homotopic to $\alpha_i * \beta_i * \overline{\alpha_i}$.

Putting everything together we see that we have an isomorphism
$$\pi_1(W) * \langle [\alpha_1 * \beta_1 * \overline{\alpha_1}], \dots, [\alpha_{n-1} * \beta_{n-1} * \overline{\alpha_{n-1}}] \rangle \xrightarrow{\cong} \pi_1(M).$$
By Lemma 23.11 we know that the right-hand side is a free group of rank $n-1$. It follows from the Grushko-Neumann Theorem 19.19 that $\pi_1(W)$ is trivial. We have thus obtained the desired isomorphism.[481] ■

Exercises for Chapter 26.

Exercise 26.1. Give full details for the calculation of the fundamental group of the topological space shown in Figure 493.

Exercise 26.2. Let $\pi = \langle a \rangle$, let $\Gamma = \langle t \rangle$ and let $\alpha, \beta \colon \Gamma \to \pi$ be the homomorphisms given by $\alpha(t) = a$ and $\beta(t) = a^2$. Show that the corresponding HNN-extension is isomorphic to the semidirect product $\mathbb{Z}[\frac{1}{2}]_\varphi \rtimes \mathbb{Z}$ where $\varphi \colon \mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{2}]$ is the group homomorphism given by multiplication by 2.

Exercise 26.3. Let X be a path-connected topological space. Furthermore let $f \colon X \to X$ be a homeomorphism. In Proposition 24.28 we saw that there exists an isomorphism $\psi \colon \pi_1(\mathrm{Tor}(X, f)) \cong \pi_1(X) \rtimes \mathbb{Z}$. Give an alternative proof of this statement using the HNN-Seifert–van Kampen Theorem 26.3.

Exercise 26.4. Let X be the result of removing two open disks from the surface of genus g. We denote by C and D the resulting boundary components. We pick orientation-preserving diffeomorphisms $\varphi \colon S^1 \to C$ and $\psi \colon S^1 \to D$.

(a) We consider the homeomorphism
$$\begin{aligned} f \colon C \times S^1 &\to D \times S^1 \\ (c, z) &\mapsto (\psi(z), \varphi^{-1}(c)). \end{aligned}$$
What is the abelianization of the fundamental group of $(X \times S^1)(f)$?

(b) Does the answer to (a) change if we pick φ to be orientation-preserving and ψ to be orientation-reversing?

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[480] Here is the point where the demanding reader might feel that not enough details are provided.

[481] The esthetically-inclined reader might object to the use of the Grushko-Neumann Theorem 19.19. But any other approach also opens up a can of worms. For example one might be tempted to show that W is contractible.

Exercise 26.5. Let X be a topological space. In Lemma 24.3 we showed that if X is non-empty and path-connected, then the fundamental group of the suspension $\Sigma(X)$ is trivial.

(a) Suppose that X has two path-components. Show that $\pi_1(\Sigma(X)) \cong \mathbb{Z}$.
(b) Suppose that X has k path-components with $k \geq 1$. Show that $\pi_1(\Sigma(X))$ is isomorphic to a free group on $k-1$ generators.

Exercise 26.6. We consider the permutation group S_4 where given $\alpha, \beta \in S_4$ we write $\alpha \cdot \beta = \alpha \circ \beta$. We consider the cycles[482]

$$\sigma = (1\,2\,3\,4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \mu = (1\,2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \text{ and } \nu = (1\,3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

(a) Show that σ and μ form a generating set for S_4.
(b) Show that σ and ν do not form a generating set for S_4.
(c) Show that $(\mu \cdot \sigma) \cdot \mu \cdot (\mu \cdot \sigma)^{-1} = \nu$.
(d) Show that the endomorphism $\langle x, y \rangle \to \langle x, y \rangle$ that is given by $x \mapsto (yx)^{-1}x(yx)$ and $y \mapsto y$ is not an isomorphism.
(e) We consider M which is defined as the disk $\overline{B}^2$ minus two open disks[483] We consider the base point $*$ and the curves β_1 and β_2 as shown in Figure 496. Show that for $i = 1, 2$ there exist paths α_i from $*$ to the starting point of β_i such that $[\alpha_1 * \beta_1 * \overline{\alpha_1}]$ and $[\alpha_2 * \beta_2 * \overline{\alpha_2}]$ do *not* form a generating set for $\pi_1(M, *)$.

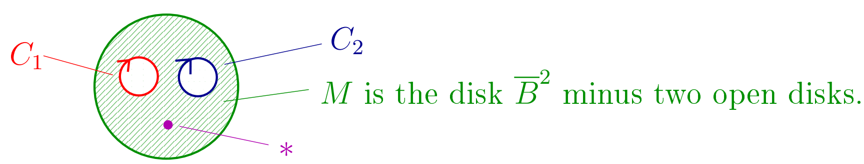


FIGURE 496

[482]Recall that for pairwise disjoint $a_1, \ldots, a_k \in \{1, \ldots, n\}$ we denote by $(a_1\ a_2\ \ldots\ a_k) \in S_n$ the permutation that is given by

$$\begin{aligned} \{1, \ldots, n\} &\to \{1, \ldots, n\} \\ m &\mapsto \begin{cases} a_{i+1}, & \text{if } m = a_i \text{ for some } i \in \{1, \ldots, k-1\}, \\ a_1 & \text{if } m = a_k, \\ m, & \text{otherwise.} \end{cases} \end{aligned}$$

Put differently, the numbers $(a_1, \ldots, a_k)$ get cyclically permuted and all other numbers stay the same.

[483]It follows easily from stereographic projection that M is diffeomorphic to S^2 minus three open disks.

27. An excursion into knot theory

In this chapter we will study the fundamental groups of the complements of knots. In particular we will use these calculations to show that the trivial knot, the trefoil and the figure-8 knot are pairwise different, i.e. we will show that they are pairwise not smoothly isotopic.

27.1. **One more avatar of spheres.** The spheres $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ are some of the most important topological spaces. We already saw that there are different ways to think about spheres, in particular on pages 119, 182 and 199 we made the following identifications

$$S^n = \mathbb{R}^n \cup \{\infty\}, \quad S^n = \overline{B}^n/S^{n-1}, \quad \text{and} \quad S^n = \overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_-.$$

Furthermore, on page 203 we had given a canonical homeomorphism

$$(S^{m-1}\times\overline{B}^n) \cup_{S^{m-1}\times S^{n-1}} (\overline{B}^m \times S^{n-1}) \xrightarrow{\cong} S^{m+n-1}.$$

Even though the homeomorphism is canonical, it is not particularly explicit. Furthermore it has the disadvantage that, if one looks up the definition, one sees that the restriction to the two solid tori $S^{m-1} \times \overline{B}^n$ and $\overline{B}^m \times S^{n-1}$ is not smooth. The following lemma rectifies these two problems.

Lemma 27.1. *Let $m, n \in \mathbb{N}_0$. We consider the following two maps:*

$$\begin{aligned} \Phi\colon S^{m-1}\times\overline{B}^n &\to A := \left\{(z,w) \in \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n} \,\middle|\, \|w\|^2 + \|z\|^2 = 1 \text{ and } \|z\|^2 \geq \tfrac{1}{2}\right\} \\ (a,b) &\mapsto \left(\frac{a}{\sqrt{\|a\|^2+\|b\|^2}}, \frac{b}{\sqrt{\|a\|^2+\|b\|^2}}\right) \end{aligned}$$

and

$$\begin{aligned} \Psi\colon \overline{B}^m\times S^{n-1} &\to B := \left\{(z,w) \in \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n} \,\middle|\, \|w\|^2 + \|z\|^2 = 1 \text{ and } \|w\|^2 \geq \tfrac{1}{2}\right\} \\ (a,b) &\mapsto \left(\frac{a}{\sqrt{\|a\|^2+\|b\|^2}}, \frac{b}{\sqrt{\|a\|^2+\|b\|^2}}\right). \end{aligned}$$

These maps have the following properties:

1. *The map $\Phi\colon S^{m-1}\times\overline{B}^n \to A$ is an orientation-preserving diffeomorphism.*
2. *The map $\Psi\colon \overline{B}^m\times S^{n-1} \to B$ is a diffeomorphism, it is orientation-preserving if and only if m is even.*
3. *The map*[484]

$$\begin{aligned} \Theta\colon (S^{m-1}\times\overline{B}^n) \cup_{S^{m-1}\times S^{n-1}} (\overline{B}^m\times S^{n-1}) &\to S^{m+n-1} \\ [P] &\mapsto \begin{cases} \Phi(P), & \text{if } P \in S^{m-1}\times\overline{B}^n, \\ \Psi(P), & \text{if } P \in \overline{B}^m\times S^{n-1} \end{cases} \end{aligned}$$

is well-defined, it is a diffeomorphism, and it is orientation-preserving if and only if m is even.

[484] We equip the boundaries of $S^{m-1}\times S^n$ and $\overline{B}^m\times S^{n-1}$ with the "obvious" collars. We use these collars and Proposition 8.15 (1b) to equip the left-hand side with the structure of a smooth manifold. Furthermore we use Proposition 8.15 (6) to equip the left-hand side with the orientation, where the orientation matches the orientation of $S^{m-1}\times\overline{B}^n$ and which matches the orientation of $\overline{B}^m\times S^{n-1}$ if and only if m is even.

Convention. We use the diffeomorphism Θ from Lemma 27.1 (2) to add the smooth manifold $(S^{m-1}\times \overline{B}^n)\cup_{S^{m-1}\times S^{n-1}}(\overline{B}^m\times S^{n-1})$ to our list of avatars of S^{m+n-1}.

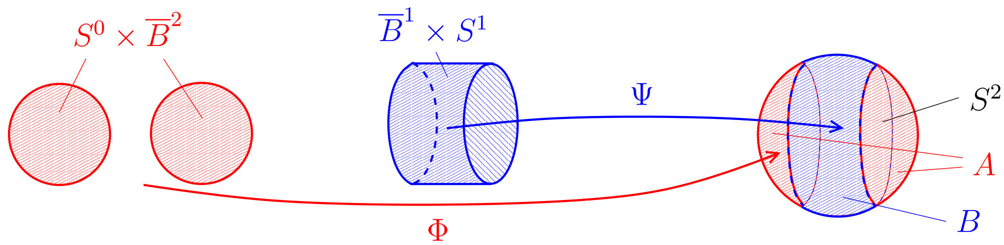


FIGURE 497. Illustration for Lemma 27.1.

SKETCH OF PROOF (*).

(2) Since the orientations are a little more interesting in the second case let us treat the second case in detail. We consider the map

$$\begin{aligned}\widetilde{\Psi}\colon B=\left\{(z,w)\in\mathbb{R}^m\times\mathbb{R}^n=\mathbb{R}^{m+n}\,\middle|\,\|w\|^2+\|z\|^2=1\ \&\ \|w\|^2\geq\tfrac{1}{2}\right\} &\to \overline{B}^m\times S^{n-1}\\ (z,w) &\mapsto \left(\tfrac{z}{\|w\|},\tfrac{w}{\|w\|}\right).\end{aligned}$$

One can easily verify that $\widetilde{\Psi}$ and Ψ are inverses of one another and that Ψ and $\widetilde{\Psi}$ are smooth. This observation implies that $\Psi\colon\overline{B}^m\times S^{n-1}\to B$ is a diffeomorphism. It remains to show that Ψ is orientation-preserving. By Lemma 6.46 it suffices to prove the following claim.

Claim. For $e_{m+1}=((0,\ldots,0),(1,0,\ldots,0))\in\overline{B}^m\times S^{n-1}$ the corresponding differential $\mathrm{D}\,\Psi_{e_{m+1}}\colon \mathrm{V}_{e_{m+1}}(\overline{B}^m\times S^{n-1})\to\mathrm{V}_{\Psi(e_{m+1})}B$ is orientation-preserving if and only if m is even.

By the orientation conventions from page 299 and Proposition 6.51 we know that $\{e_1,\ldots,e_m,e_{m+2},\ldots,e_{m+n}\}$ is a positive basis for $\mathrm{V}_{e_{m+1}}(\overline{B}^m\times S^{n-1})$. First note that a straightforward calculation shows that the differential of Ψ at the point e_{m+1} is actually the identity. Next note that $\Psi(e_{m+1})=e_{m+1}$. Finally note that by definition $\{v_1,\ldots,v_{m+n-1}\}$ is a positive basis for $\mathrm{V}_{\Psi(e_{m+1})}B=\mathrm{V}_{e_{m+1}}S^{m+n-1}$ if and only if $\det(e_{m+1},v_1,\ldots,v_{m+n-1})>0$. The claim follows almost immediately from these observations. ⊞

(1) The proof of this statement is almost identical to the above proof of (2).

(3) It is clear that Φ and Ψ agree on $S^{m-1}\times S^{n-1}$. Therefore it follows from Lemma 3.22 that the given map $\Theta\colon(S^{m-1}\times\overline{B}^n)\cup_{S^{m-1}\times S^{n-1}}(\overline{B}^m\times S^{n-1})\to S^{m+n-1}$ is well-defined and continuous. Using (1) one easily verifies that the map is a bijection. It follows from Proposition 2.43 (3) that the map Θ is a homeomorphism. Going through the definition of the smooth structure of the smooth manifold on the left, as defined in Proposition 8.15 (1b), shows that the map is actually a diffeomorphism. Finally it follows from (1) and (2) that the diffeomorphism is actually orientation-preserving. ■

Throughout this chapter we are mostly interested in the case $m = n = 2$. In this setting we have the following identifications which are given by definition and b Lemma 2.44:

$$S^3 \;=\; \left\{(w,x,y,z)\,\middle|\, w^2+x^2+y^2+z^2=1\right\} \;=\; \left\{(z,w)\in\mathbb{C}^2\,\middle|\,|w|^2+|z|^2=1\right\} \underset{\uparrow}{=} \mathbb{R}^3\cup\{\infty\}.$$

via the stereographic projection as defined in Lemma 2.44
note that in Exercise 6.17 we saw that this map
is (lucky us!) orientation-preserving

We will go back and forth between these models without mentioning these maps. In particular, as discussed on page 344, we will view $\mathbb{R}^3$ as a submanifold of the smooth manifold $S^3 = \mathbb{R}^3\cup\{\infty\}$. Furthermore we now view A and B as subsets of $S^3 = \mathbb{R}^3\cup\{\infty\}$. In this setting we have

$$\text{“central curve of } A\text{”} \;=\; \Phi(S^1\times\{0\}) \;=\; S^1\times\{0\} \;=\; \text{the trivial knot}$$

and we have

$$\text{“central curve of } B\text{”} \;=\; \Psi(\{0\}\times S^1) \;=\; \text{the } z\text{-axis}\cup\{\infty\}.$$

We visualize the solid torus A and the central curve of the solid torus B in Figure 498. The solid torus B is more difficult to illustrate. It consists of a closed disk attached to each point on the central curve. For example the closed disk attached to the origin is just the “obvious” closed disk in the xy-plane that touches the torus $\Phi(S^1\times S^1)$.[485]

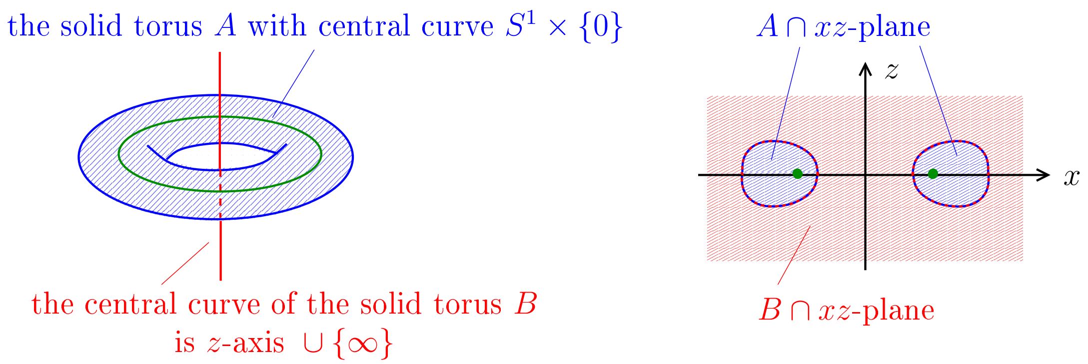


Figure 498

Lemma 3.13 and Lemma 27.1 give us two smooth embeddings of the 2-dimensional torus $S^1\times S^1$ into $\mathbb{R}^3$. The following lemma says that these two smooth embeddings are essentially the same.

Lemma 27.2. (∗) *We denote by $\Omega\colon S^3\setminus\{(0,1)\}\to\mathbb{R}^3$ the stereographic projection as defined in Lemma 2.44. We consider the two smooth embeddings*[486][487]

$$\begin{array}{ccc} \overline{B}^2\times S^1 \xrightarrow{\Theta} \mathbb{R}^3 & & \overline{B}^2\times S^1 \xrightarrow{\Xi} S^3\setminus\{(0,1)\}\xrightarrow{\Omega}\mathbb{R}^3 \\ ((x,y),e^{i\varphi}) \mapsto \begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1+\frac{x}{2}\\ 0\\ -\frac{y}{2}\end{pmatrix} & \text{and} & (z,w)\mapsto\left(\frac{w}{\sqrt{|z|^2+|w|^2}},\frac{z}{\sqrt{|z|^2+|w|^2}}\right)\end{array}$$

provided by Lemma 3.13 and provided by Lemma 27.1. There exists a smooth diffeotopy G of S^3 rel $\{(0,0)\}\times S^1$ with $G_0 = \mathrm{id}$ and $G_1\circ\Theta = \Omega\circ\Xi$.

[485]What is the disk attached to the point ∞?

Convention. In all the future examples we will not distinguish between the two smooth embeddings of the solid torus $\overline{B}^2 \times S^1 \to \mathbb{R}^3$ that are given in Lemma 27.2.

SKETCH OF PROOF (∗). By the Isotopy Extension Theorem 8.27 it suffices to show that there exists a smooth isotopy rel $\{(0,0)\} \times S^1$ from Θ to $\Omega \circ \Xi$.

We write $H = \{(x, 0, z) \in \mathbb{R}^3 \,|\, x > 0\}$. Let us first consider the restrictions of Θ and $\Omega \circ \Xi$ to the disk $\overline{B}^2 \times \{1\}$. In both cases the image lies in H. These maps are illustrated in Figure 499. We leave it to the dear reader to show that there exists a smooth isotopy $F\colon \overline{B}^2 \times \{1\} \times [0,1] \to H$ rel $\{(0,0,1)\}$ from Θ to $\Omega \circ \Xi$.

The desired isotopy is given by "rotating F" around the z-axis. Slightly more precisely it is given by the map

$$\begin{array}{rcl} \overline{B}^2 \times S^1 \times [0,1] & \to & \mathbb{R}^3 \\ ((x,y), e^{i\varphi}, t) & \mapsto & \text{rotation of } F((x,y),t) \in \mathbb{R}^3 \text{ by } \varphi \text{ around the } z\text{-axis.} \end{array}$$ ■

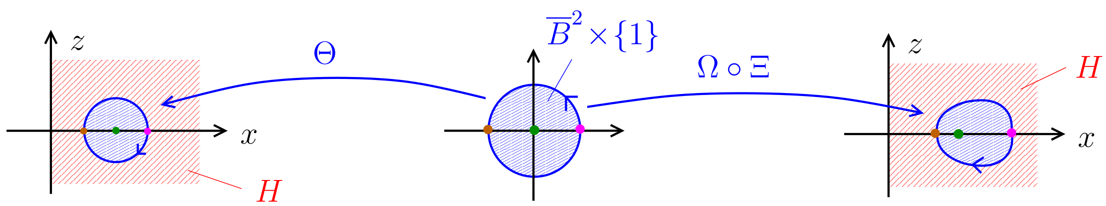


FIGURE 499. Illustration for the proof of Lemma 27.2.

27.2. **The fundamental group of the complement of the trefoil.** Now we want to finally turn to the study of knots. In particular we would like to study the knots that are illustrated in Figure 500. In Question 8.40 we had raised the question, how can one show that the trefoil is not trivial?

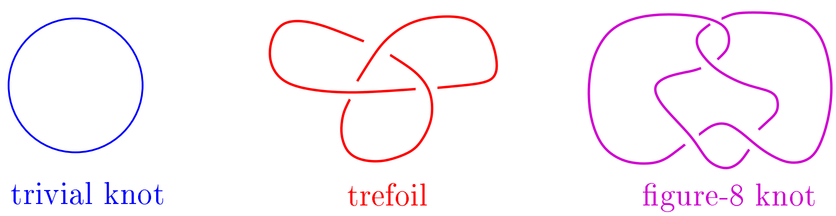


FIGURE 500. The trivial knot, the trefoil and the figure-8 knot.

Now we have some tools to attack this question. We saw in Proposition 8.43 that if K and J are smoothly isotopic knots, then the knot complements $S^3 \setminus K$ and $S^3 \setminus J$ are diffeomorphic. In particular if $\pi_1(S^3 \setminus K)$ and $\pi_1(S^3 \setminus J)$ are not isomorphic, then K and J are not smoothly isotopic. Thus we arrive at the following variation on Question 8.40.

Question 27.3. *Can the fundamental group be used to distinguish the trefoil from the trivial knot?*

[486]Note that $\Xi\colon \overline{B}^2 \times S^1 \to S^3$ is just the composition of the swap $\overline{B}^2 \times S^1 \to S^1 \times \overline{B}^2$ and the smooth embedding $\Phi\colon S^1 \times \overline{B}^2 \to S^3$.

[487]As a reality check note that it follows from Lemma 2.44 that Θ and $\Omega \circ \Xi$ agree on $\{(0,0)\} \times S^1$.

Before we continue we want to give a precise description of the trivial knot and the trefoil. In fact, we will use the occasion to introduce an interesting family of knots which contains the trefoil knot.

Definition.

(1) As on page 385 we define the *trivial knot* as the knot[488]
$$U := \{(e^{it},0) \mid t\in\mathbb{R}\} \quad\subset\quad \big\{(z,w)\in\mathbb{C}^2 \,\big|\, |w|^2+|z|^2=1\big\}$$
or equivalently
$$U := \{(\cos t,\sin t,0)\mid t\in\mathbb{R}\} \quad\subset\quad \mathbb{R}^3\cup\{\infty\}.$$

(2) Let $p,q\in\mathbb{N}$ be coprime. We consider the curve $C=\{(p\cdot t,q\cdot t)\mid t\in\mathbb{R}\}$ on the torus $T=[0,2\pi]\times[0,2\pi]/\sim$. We define the (p,q)-torus knot $T(p,q)$ as the image of C under the map
$$\begin{aligned}\Phi\colon T=[0,2\pi]\times[0,2\pi]/\sim \quad&\to\quad S^3\\ (s,t)\quad&\mapsto\quad \left(\tfrac{1}{\sqrt{2}}e^{is},\tfrac{1}{\sqrt{2}}e^{it}\right).\end{aligned}$$

(3) We define the *trefoil* to be the $(2,3)$-torus. This definition of the trefoil is illustrated in Figure 501.

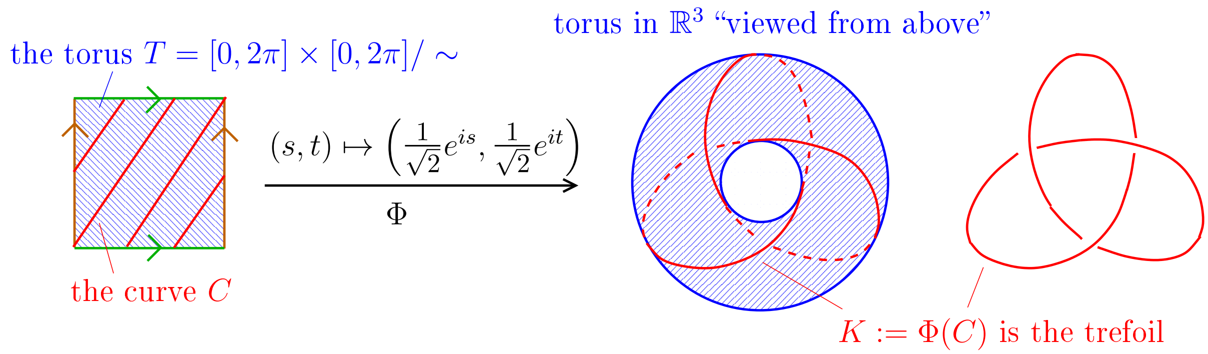


FIGURE 501

Now we turn to the calculation of the fundamental groups of the complements of the trivial knot and of the trefoil.

Proposition 27.4. *If U is the trivial knot, then $S^3\setminus U$ is diffeomorphic to $S^1\times\mathbb{C}$, in particular $\pi_1(S^3\setminus U)\cong\mathbb{Z}$.*

PROOF. Note that
$$\begin{aligned}S^3\setminus U \;&=\; \{(z,w)\in\mathbb{C}^2\mid |z|^2+|w|^2=1\}\setminus\{(z,0)\mid |z|=1\}\\ &=\; \{(z,w)\in\mathbb{C}^2\mid |z|^2+|w|^2=1 \text{ and } |w|^2>0\}.\end{aligned}$$
Now basically the same argument as in the proof of Lemma 27.1 shows that the map
$$\begin{aligned}\Phi\colon S^1\times\mathbb{C}\quad&\to\quad S^3\setminus U=\big\{(z,w)\in\mathbb{C}^2\,\big|\,|w|^2+|z|^2=1\text{ and }|w|^2>0\big\}\\ (a,b)\quad&\mapsto\quad\left(\frac{b}{\sqrt{|a|^2+|b|^2}},\frac{a}{\sqrt{|a|^2+|b|^2}}\right)\end{aligned}$$

[488]The name U comes from the fact that the trivial knot is often also called the *unknot*.

is a diffeomorphism. Since $S^1 \times \mathbb{C}$ is homotopy equivalent to S^1 we obtain from the above and from Proposition 18.16 (2) that $\pi_1(S^3 \setminus U) = \pi_1(S^1 \times \mathbb{C}) = \pi_1(S^1) \cong \mathbb{Z}$. ■

Remark. Let U be the trivial knot. A careful reading of the proof of Proposition 27.4 shows that a generator of $\pi_1(S^3 \setminus U)$ is given by a meridian that we defined on page 390. We refer to Figure 502 for an illustration of the trivial knot and a meridian.

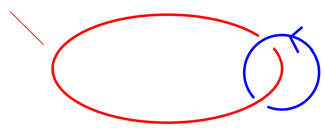


FIGURE 502

Proposition 27.5.

(1) *Let $p, q \in \mathbb{N}$ be coprime. Then $\pi_1(S^3 \setminus T(p,q)) \cong \langle x, y \mid x^p = y^q \rangle$.*

(2) *If K is the trefoil, then $\pi_1(S^3 \setminus K) \cong \langle x, y \mid x^2 = y^3 \rangle$.*

PROOF. By definition of the trefoil Statement (2) is just a special case of Statement (1). Thus we only need to prove Statement (1). Let $p, q \in \mathbb{N}$ be coprime. By Lemma 27.1 there exists a diffeomorphism

$$\Phi\colon S^3 \xrightarrow{\cong} \underbrace{\left(S^1 \times \overline{B}^2\right)}_{=:A} \cup_{S^1 \times S^1 = S^1 \times S^1} \underbrace{\left(\overline{B}^2 \times S^1\right)}_{=:B}$$

which is, up to scaling by a factor of $\sqrt{2}$, the identity on $T := S^1 \times S^1$. We identify the (p,q)-torus knot with its image under Φ, i.e. we have

$$K \;=\; \{(e^{psi}, e^{qsi}) \mid s \in \mathbb{R}\} \;\subset\; T.$$

We write $A_K = A \setminus K$, $B_K = B \setminus K$ and $T_K = T \setminus K$. Note that A_K and B_K are submanifolds of $X_K := S^3 \setminus K$ and with[489] $A_K \cap B_K = \partial A_k = \partial B_K = T_K$ and with $A_K \cup B_K = X_K$. Note that $x := S^1 \times \{0\}$ is a deformation retract of A_K and that $y := \{0\} \times S^1$ is a deformation retract of B_K. Finally note that T_K retracts onto a parallel copy of K. More precisely, for $\epsilon > 0$ sufficiently small, the subset T_K retracts onto the loop

$$z \;=\; \{(e^{psi+\epsilon}, e^{qsi}) \mid s \in \mathbb{R}\} \;\subset\; T_K.$$

By an abuse of notation we denote by x, y and z also the loops corresponding to the circles x, y and z with the obvious orientation. Then the inclusion induced maps

$$\begin{array}{ccccc} \pi_1(A_K) & \leftarrow & \pi_1(T_K) & \rightarrow & \pi_1(B_K) \\ i_*(g) & \leftarrow\!\shortmid & g & \mapsto & j_*(g) \end{array} \quad \text{become} \quad \begin{array}{ccccc} \langle x \rangle & \leftarrow & \langle z \rangle & \rightarrow & \langle y \rangle \\ x^p & \leftarrow\!\shortmid & z & \mapsto & y^q. \end{array}$$

Thus we see that

$$\pi_1(X_K) \underset{\substack{\uparrow \\ \text{Seifert–van Kampen Theorem 22.2}}}{=} \pi_1(A_K) *_{\pi_1(T_K)} \pi_1(B_K) \underset{\substack{\uparrow \\ \text{above discussion}}}{\cong} \langle x \rangle *_{x^p = y^q} \langle y \rangle \underset{\substack{\uparrow \\ \text{Lemma 21.23}}}{=} \langle x, y \mid x^p = y^q \rangle.$$

■

[489] Here we mean by ∂A_K and ∂B_K the boundary of A_K and B_K viewed as submanifolds of S^3.

Now the question is whether or not $\pi_1(S^3 \setminus K) = \langle x, y \mid x^2 = y^3 \rangle$ is isomorphic to $\pi_1(S^3 \setminus U) = \mathbb{Z}$. As usual we first consider the abelianization of $\pi_1(S^3 \setminus K)$, but it is straightforward to see, using Proposition 21.20 that the abelianization is isomorphic to $\mathbb{Z}$.[490][491] Despite this slight setback the following lemma does fortunately show that the group $\pi_1(S^3 \setminus K) = \langle x, y \mid x^2 = y^3 \rangle$ is not isomorphic to $\pi_1(S^3 \setminus U) = \mathbb{Z}$.

Proposition 27.6.

(1) *The group $\langle x, y \mid x^2 = y^3 \rangle$ admits an epimorphism onto the permutation group S_3, in particular the group $\langle x, y \mid x^2 = y^3 \rangle$ is not isomorphic to $\mathbb{Z}$.*
(2) *The trefoil is not smoothly isotopic to the trivial knot.*

Proof.

(1) It follows from Lemma 21.4 and a straightforward calculation that there exists a unique homomorphism
$$\Phi\colon \langle x, y \mid x^2 = y^3 \rangle \;\to\; S_3 = \text{permutation group on three elements}$$
with
$$\Phi(x) \;=\; \sigma \;:=\; (1\,2) \;=\; \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$
and
$$\Phi(y) \;=\; \tau \;:=\; (1\,2\,3) \;=\; \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$
The permutations σ and τ do not commute, hence the image of Φ is a non-abelian subgroup of S_3.[492] On the other hand the abelian group $\mathbb{Z}$ cannot admit an epimorphism onto a non-abelian group[493].
(2) It follows from (1) and above discussion that the trefoil is non-trivial. ■

The following question arises naturally.

Question 27.7. *Are the (p, q)-torus knots pairwise different, i.e. if (p, q) and (r, s) for coprime pairs of natural numbers, does it follow that $T(p, q)$ and $T(r, s)$ are not smoothly isotopic?*

The approach of the proof of Proposition 27.6 is most likely too pedestrian to deal with Question 27.7. We will return to Question 27.7 at a much later stage.

Remark. Let $p, q \in \mathbb{N}$ be coprime. We consider the map
$$\begin{array}{rcl} f\colon S^3 \;=\; \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} & \to & \mathbb{C} \\ (z, w) & \mapsto & z^q - w^p. \end{array}$$

---

[490] In fact that is not a coincidence, using homology groups we will soon see that for any knot J the abelianization of $\pi_1(S^3 \setminus J)$ is isomorphic to $\mathbb{Z}$.

[491] If one goes very carefully through the definitions and calculations then one can see that a meridian once again becomes a generator of the abelianization.

[492] In fact it is easy to see that the only non-abelian subgroup of S_3 is S_3 itself, i.e. Φ is an epimorphism onto S_3.

[493] Why not?

Note that by definition the (p,q)-torus knot $T(p,q)$ equals $f^{-1}(\{0\})$. It follows from work of Milnor, see [**Miln68**, Theorem 4.8] and [**Sav12**, Chapter 8] (see also [**EN85**, p. 125] and [**Rolf90**, Chapter 10.I]), that there exists a surface and a self-diffeomorphism of order pq together with a diffeomorphism $\Theta\colon S^3 \setminus T(p,q) \to \mathrm{Tor}(\Sigma,\varphi)$ such that the following diagram commutes

$$\begin{tikzcd} S^3 \setminus T(p,q) \arrow[rr, "\Theta", "\cong"'] \arrow[dr, "P\mapsto \frac{f(P)}{|f(P)|}"'] & & \mathrm{Tor}(\Sigma,\varphi) \arrow[dl, "{[(z,t)]\mapsto e^{2\pi i t}}"] \\ & S^1. & \end{tikzcd}$$

This generalizes the observation from 714 that the complement of the trefoil is diffeomorphic to a mapping torus.

The question arises, whether the fundamental group is strong enough to detect the smooth isotopy class of a knot. In the section we will see that this is perhaps a trifle to optimistic.

27.3. **The mirror image of a knot.** We continue with the following nicely geometric definition.

Definition. Given a knot $K \subset S^3$ the reflection of K in any hyperplane of $\mathbb{R}^4$ is called *mirror* of K and denoted by K^{mir}.

Lemma 27.8.

(1) *Let $K \subset S^3$ be a knot. The reflections in any two hyperplanes of $\mathbb{R}^4$ give rise to smoothly isotopic knots.*
(2) *Let $K \subset \mathbb{R}^3$ be a knot. We use the diffeomorphism $\mathbb{R}^3 \cup \{\infty\} \to S^3$ from Lemma 2.44 to view K as a knot in S^3. The image of K under a reflection of an affine hyperplane in $\mathbb{R}^3$ corresponds under this embedding to a mirror of K.*

Sketch of proof.

(1) We will prove this statement in Exercise 27.1, making use of Lemma 2.65 and Exercise 18.7.
(2) Let $K \subset \mathbb{R}^3$ be a knot. First note that the same argument as in (1) shows that reflection in any two affine hyperplanes of $\mathbb{R}^3$ lead to smoothly isotopic knots in $\mathbb{R}^3$. Thus it remains to show that there exists a single hyperplane H of $\mathbb{R}^3$ such that the reflection of K in H corresponds to the mirror image of $K \subset \mathbb{R}^3 \subset S^3$ as defined above. Next note that the stereographic projection, that we used in Lemma 2.44, commutes with the reflection in the $(x_1 = 0)$-hyperplane of $\mathbb{R}^4$ respectively $\mathbb{R}^3$. Thus we see that the hyperplane $(x_1 = 0)$ has the desired property. ■

Convention. Let $K \subset S^3$. Since usually we only care about knots up to smooth isotopy we use Lemma 27.8 to talk of "the mirror image" of K instead of the slightly more correct "a mirror image of K".

Example. Let $K \subset \mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} = S^3$ be a knot. Keeping Lemma 27.8 (2) we see in Figure 503 the trefoil and its mirror and we also see the figure-8 knot and its mirror.

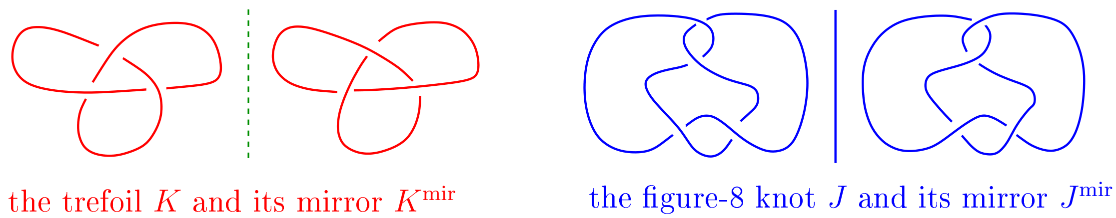


FIGURE 503

The question arises, whether the mirror image of a knot is "truly different" from the original knot. We are now naturally led to the following definition.

Definition. We say that a knot K in $\mathbb{R}^3$ is *amphichiral* if it is smoothly isotopic to its mirror image, otherwise we call the knot K *chiral*.

Examples.

(1) The trivial knot equals of course its mirror, in particular the trivial knot is amphichiral.
(2) In Figure 504 we show that the figure-8 knot J is smoothly isotopic to its mirror image J^*, i.e. the figure-8 knot is also amphichiral.

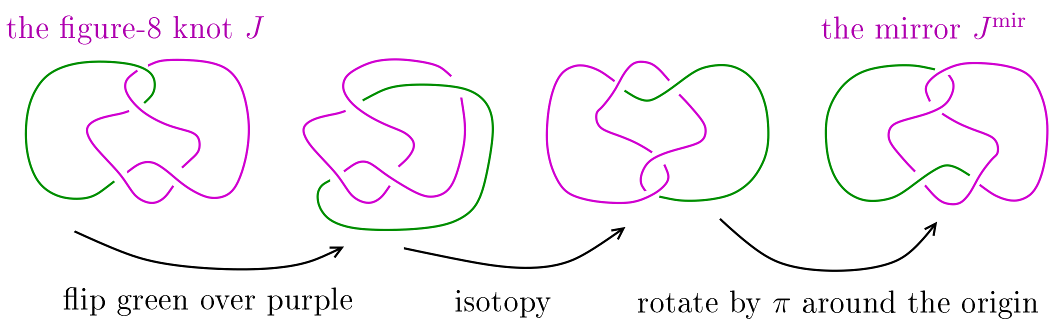


FIGURE 504

It is now natural to ask whether the trefoil is amphichiral. But endless hours of playing around with the trefoil do not lead to any success. So the suspicion arises that the following question should be answered in the negative.

Question 27.9. *Is the trefoil chiral?*

Much later, in Proposition 99.5 and **??** we will use two different approaches to give negative answer to this question. In the first approach we use the fundamental group together with the "longitude", in the latter approach we use linking pairings of the "2-fold branched covering of K".

We return to the study of fundamental groups. Let $K \subset S^3$ be a knot and let K^{mir} be its mirror image. Evidently $S^3 \setminus K$ is diffeomorphic to $S^3 \setminus K^{\text{mir}}$, so we cannot hope to distinguish K and K^{mir} using the fundamental groups of their complements. This leads us to the following question:

Question 27.10. *Let K and J be two knots such that the fundamental groups of $S^3 \setminus K$ and $S^3 \setminus J$ are isomorphic. Does this imply that K is smoothly isotopic to J or to its mirror J^*?*

27.4. **The connected sum of oriented submanifolds and knots** (∗)**.** In this section, given two submanifolds $K \subset M$ and $L \subset N$ we introduce the corresponding connected sum $K\#L \subset M\#N$. Eventually our main interest will be in the connected sum of knots $K, L \subset S^3$.

To introduce the connected sum of submanifolds and knots we need to make a few preparations.

Notation. Let $n \in \mathbb{N}$ and let $k \in \{1, \ldots, n-1\}$.

(1) We denote by $\overline{B}{}^n_2$ and $\overline{B}{}^k_2$ the closed balls of radius 2 in $\mathbb{R}^n$ and $\mathbb{R}^k$.
(2) We set $J^k := \{(x, 0) \in \overline{B}{}^n_2 \,|\, x \in \overline{B}{}^k_2\}$ and we view equip J^k with the obvious orientation.
(3) Let M be a closed oriented n-dimensional smooth manifold and let $K \subset M$ be a closed oriented k-dimensional submanifold. Furthermore let $\varphi\colon \overline{B}{}^n_2 \to M$ be a smooth embedding.
   (a) We say φ is *K-admissible* if $\varphi(J^k) = \varphi(\overline{B}{}^n_2) \cap K$, if $\varphi\colon \overline{B}{}^n_2 \to M$ is orientation-preserving and if $\varphi|_{J^k}\colon J^k \to \varphi(\overline{B}{}^n_2) \cap K$ is orientation-preserving.
   (b) We say φ is *K-anti-admissible* if the same conditions as in (a) hold, except that we now demand that both maps are orientation-reversing.

In Figure 505 we illustrate the definition of a K-admissible map in the special case that $M = S^3$ and that K is in fact a knot.

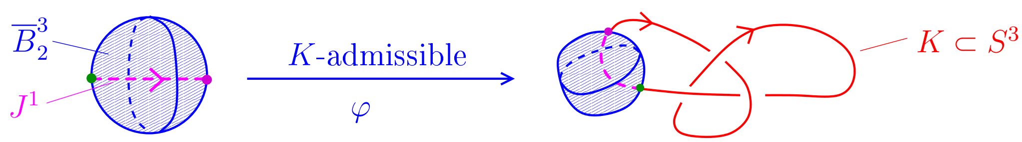


FIGURE 505

The following proposition says that K-admissible maps always exist and that in a sense they are unique.

Proposition 27.11. *Let M be a closed oriented n-dimensional smooth manifold. Furthermore let $K \subset M$ be a closed oriented k-dimensional submanifold.*

(1) *There exists a K-admissible map.*
(2) *If M is connected, then for any K-admissible maps $\varphi\colon \overline{B}{}^n_2 \to M$ and $\psi\colon \overline{B}{}^n_2 \to M$ there exists a diffeotopy $F\colon M \times [0,1] \to M$ such that $F_0 = \mathrm{id}$, such that $F_1 \circ \varphi = \psi$ and such that each $F_t \circ \varphi$ is K-admissible*

The obvious analogues of (1) *and* (2) *also hold for "K-anti-admissible".*

SKETCH OF PROOF. The first statement follows easily from the definition of a submanifold. The second statement is an analogue of Theorem 8.36. Unfortunately it seems like nobody ever bothered to write down a proof for Statement (2). Thus we muster all the authority we have and we claim that this statement can be proved in an analogous fashion. ■

Let us quickly move on to the key definition of this section which can be viewed as a variation on the definition on page 377.

Definition. Let M be a closed oriented n-dimensional smooth manifold and let $K \subset M$ be a closed oriented k-dimensional submanifold. Furthermore let N be a closed oriented n-dimensional smooth manifold and let $L \subset N$ be a closed oriented k-dimensional submanifold. We pick a K-admissible map $\varphi\colon \overline{B}^n_2 \to M$ and we pick an L-anti-admissible map $\psi\colon \overline{B}^n_2 \to N$. We use $\varphi\colon \overline{B}^n \to M$ and $\psi\colon \overline{B}^n \to N$ to define $M\#N$. Furthermore we use[494] $\overline{B}^k = \overline{B}^k_1 \xrightarrow{x \mapsto (x,0)} J^n_k \xrightarrow{\varphi} K$ and $\overline{B}^k = \overline{B}^k_1 \xrightarrow{x \mapsto (x,0)} J^n_k \xrightarrow{\psi} L$ to define the connected sum $K\#L$ as we did on page 377. We refer to the resulting pair $(M\#N, K\#L)$ as the *connected sum of* (M,K) *and* (N,L).

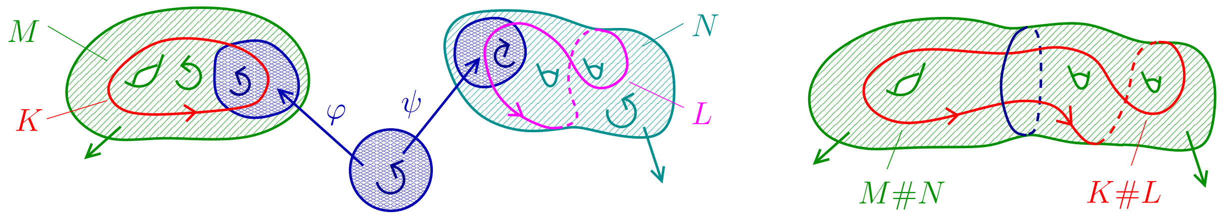


FIGURE 506

The following proposition can be viewed as an analogue of Proposition 8.35.

Proposition 27.12. *Let M be a closed oriented n-dimensional smooth manifold and let K be a closed oriented k-dimensional submanifold of M. Furthermore let N be a closed oriented n-dimensional smooth manifold and let L be a closed oriented k-dimensional submanifold of N.*

(1) *The subset $K\#L$ is a closed k-dimensional smooth submanifold of $M\#N$.*
(2) *If M and N are both connected, then the oriented diffeomorphism type of the pair $(M\#N, K\#L)$ does not depend on the choice of the K-admissible map and the L-anti-admissible map.*

SKETCH OF PROOF.

(1) We leave it to the reader to verify this statement. Note that here one needs to make use of the fact that the (anti-) admissible maps give us control on $\overline{B}^n_2$ and not just on $\overline{B}^n$.
(2) This statement follows fairly easily from Proposition 27.11. ■

Now we can finally specialize to our beloved setting of knots in S^3. Following most books in knot theory, see e.g. [**Rolf90**, p. 40] and [**Kaw96**, p. 30] we introduce the connected sum of two oriented knots as follows:

Definition.

(1) Let K and L be two oriented knots in S^3. We pick a K-admissible map $\varphi\colon \overline{B}^3_2 \to S^3$ and we pick an L-anti-admissible map $\psi\colon \overline{B}^3_2 \to S^3$. We perform the connected

[494] Note that here we deliberately restrict ourselves from balls of radius 2 to balls of radius 1.

sum $(S^3 \# S^3, K \# L)$ and we use the explicit orientation-preserving diffeomorphism $S^3 \# S^3 \to S^3$ from Lemma 8.37 to view $K \# L$ as an oriented knot in S^3.

(2) We say a knot K is *prime* if it is not smoothly isotopic to the connected sum of two non-trivial knots.

Examples.

(1) In Figure 507 we show the connected sum of the trefoil K and the figure-8 knot L. Admittedly it can take a minute to connect the picture to the actual definition. Just for fun we also show meridians of K, L and $K \# L$.

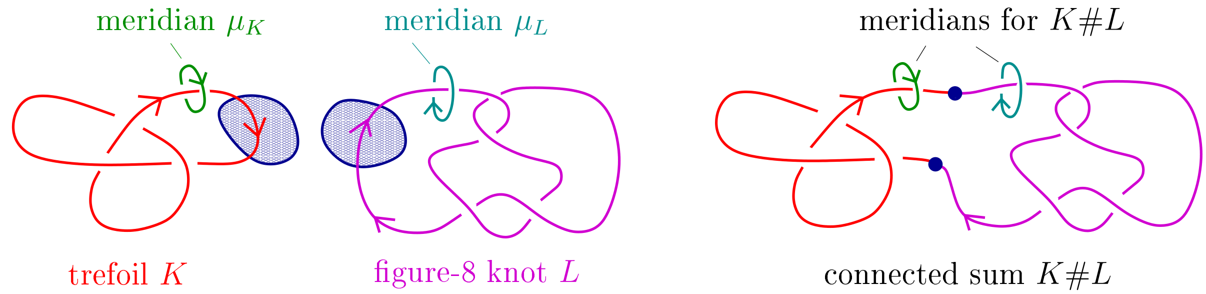


FIGURE 507

(2) In Figure 508 we show the trefoil and its "mirror image" K^{mir}, i.e. the reflection of K in the xy-hyperplane. The connected sum $K \# K$ is called the *square knot* and the connected sum $K \# K^{\text{mir}}$ is called the *granny knot*. It is notoriously tricky to distinguish the square knot and the granny knot, even up to taking "mirror images".

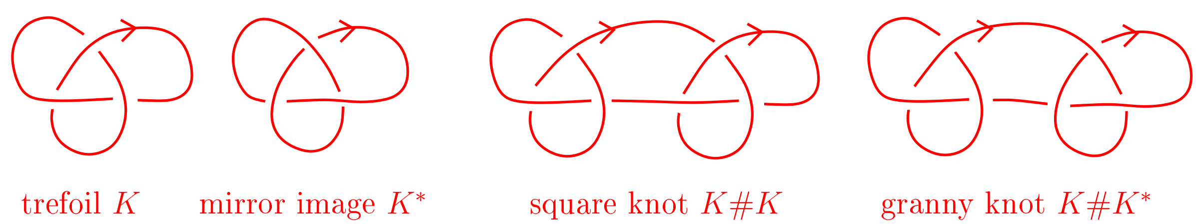


FIGURE 508

Proposition 27.13. *On the set of smooth isotopy classes of oriented knots the connected sum operation is well-defined and commutative.*

SKETCH OF PROOF. First note that it follows almost immediately from the Isotopy Extension Theorem 8.27 together with Proposition 27.12 that the connected sum operation is indeed well-defined.[495] A slight variation on the proof of Proposition 8.35 (5) shows that the connected sum of knots is commutative. ■

But now let us return to our main hobby these days, namely the calculation of fundamental groups. We have the following proposition.

[495]Note though that the statement does not follow just from Proposition 27.12 since we claim that $K \# L$ is well-defined up to isotopy in S^3. This is a priori not the same statement as saying that the oriented diffeomorphism type of $(S^3 \# S^3, K \# L)$ is well-defined.

Proposition 27.14. *Let K and L be two oriented knots in S^3. We pick meridians μ_K and μ_L and base points $x_0 \in \mu_K$ and $y_0 \in \mu_L$. There exists an isomorphism*
$$\pi_1(S^3 \setminus (K\#L)) \;\cong\; \pi_1(S^3 \setminus K, x_0) \underset{\substack{\uparrow \\ \textit{given by making the identification } \mu_K = \mu_J}}{*_{\langle \mu_K\rangle = \langle \mu_L \rangle}} \pi_1(S^3 \setminus L, y_0).$$

Sketch of proof. To preserve our sanity we only provide a sketch of the proof. We pick a K-admissible map $\varphi\colon \overline{B}^3_2 \to S^3$ and we pick an L-anti-admissible map $\psi\colon \overline{B}^3_2 \to S^3$. We write $D^3_+ := S^3 \setminus \varphi(B^3)$, $D^3_- := \varphi(\overline{B}^3)$ and $K^\pm := D^3_\pm \cap K$. Similarly we write $E^3_+ := S^3 \setminus \psi(B^3)$, $E^3_- := \psi(\overline{B}^3)$ and $L^\pm := E^3_\pm \cap K$. Since all meridians are equivalent we can arrange that $\mu_K \subset \varphi(S^2)$ and that $\mu_L \subset \psi(S^2)$.[496] Now we consider the following diagram

$$\begin{array}{ccccc}
 & & \text{isomorphism by the Seifert–van Kampen Theorem 22.2 for Smooth Manifolds} & & \\
 & & \downarrow & & \\
\pi_1(S^3 \setminus K\#L) & = & \pi_1(S^3\#S^3 \setminus K\#L) \;\cong\; \pi_1(D^3_+ \setminus K_+) & *_{\langle \mu_K\rangle = \langle \mu_L\rangle} & \pi_1(E^3_+ \setminus L_+) \\
 & & \downarrow\cong & & \downarrow\cong \\
 & & \pi_1(S^3 \setminus K) & *_{\langle \mu_K\rangle = \langle \mu_L\rangle} & \pi_1(S^3 \setminus L).
\end{array}$$

Here the vertical maps are isomorphisms by another two applications of the Seifert–van Kampen Theorem 22.2, applied this time to $S^3 \setminus K = (D^3_+ \setminus K_+) \cup (D^3_- \setminus K_-)$ and to $S^3 \setminus L = (E^3_+ \setminus L_+) \cup (E^3_- \setminus L_-)$. Note that here we secretly use Lemma 21.22 (3) twice. ■

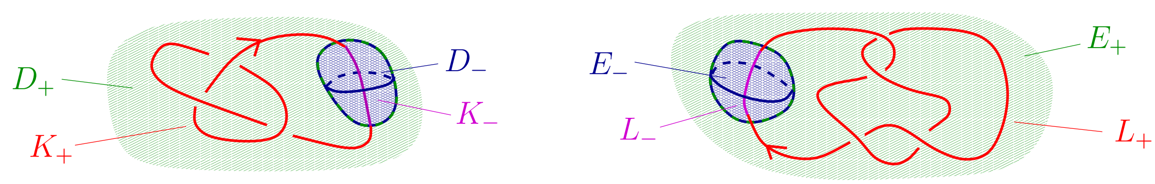


Figure 509. Illustration for the proof of Proposition 27.14.

Remark. In Section 70.7 we will introduce the "internal connected sum" of two oriented submanifolds. In particular we will introduce the "internal connected sum" of two oriented knots which, for most intents and purposes agrees with the connected sum of two oriented knots.

In Exercise 27.6 we will prove the following corollary to Proposition 27.14.

Corollary 27.15. *Let K and L be oriented knots. The fundamental groups of $S^3 \setminus (K\#L)$ and $S^3 \setminus (K\#L^*)$ are isomorphic.*

In particular we see that the square knot and the granny knot cannot be distinguished by the fundamental group of their complements. On the other hand we will see in Exercise **??** that the granny knot is neither smoothly isotopic to the square knot nor to its mirror image. In particular we see that in general the answer to Question 27.10 is negative.

[496]Readers who go through the proof with a fine logical comb will notice that the last sentence requires some extra thought.

Fortunately the following deep theorem shows that the situation is much better once we restrict to prime knots. Namely, under this extra hypothesis we actually do get a positive answer to Question 27.10.

Theorem 27.16. (Whitten Theorem) *Let K and L be two prime knots in $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} = S^3$. If $\pi_1(S^3 \setminus K)$ is isomorphic to $\pi_1(S^3 \setminus L)$, then K is smoothly isotopic to L or to its mirror image L^*.*

PROOF. Let K and L be two prime knots such that the fundamental groups $\pi_1(S^3 \setminus K)$ and $\pi_1(S^3 \setminus L)$ are isomorphic. Wilbur Whitten [**Whn87**, Corollary 2.1], building on the Gordon-Luecke Theorem 8.46, showed that there exists a diffeomorphism $F\colon S^3 \to S^3$ with $F(K) = L$. By possibly composing F with the reflection in a hyperplane we see that there exists an orientation-preserving diffeomorphism $F\colon S^3 \to S^3$ with $F(K) = L$ or with $F(K) = L^*$. The claimed statement now follows from Corollary 8.45. ■

27.5. **Knot diagrams.** In the previous section we gave a completely rigorous calculation of the isomorphism type of the fundamental group of the complement of a trefoil and we used this to show that the trefoil is not smoothly isotopic to the trivial knot. In the next section we will give a practical algorithm for determining the isomorphism type of the fundamental group of the complement of a given knot. The key to formulating the algorithm is the notion of a knot diagram that we will study in this section.

Before we start with knot theory we introduce one bit of notation.

Notation. Given a smooth map $f\colon S^1 \to \mathbb{R}^n$ and $z = e^{is} \in S^1$ we write

$$f'(z) \ := \ \text{derivative at the point } s \text{ of the function } \mathbb{R} \to \mathbb{R}^n \text{ given by } t \mapsto f(e^{it}).$$

Definition. A map $\gamma\colon S^1 \to \mathbb{R}^2$ is called *diagrammatic* if the following conditions are satisfied:

(a) the map is an immersion,
(b) if $z \neq w \in S^1$ satisfy $\gamma(z) = \gamma(w)$, then $\gamma'(z)$ and $\gamma'(w)$ are linearly independent,
(c) given any $P \in \gamma(S^1)$ the preimage $\gamma^{-1}(P)$ consists of either one or two points.

Furthermore any $z \in S^1$ for which there exists a $w \neq z \in S^1$ with $\gamma(z) = \gamma(w)$ is called a *double point of* γ. The definitions are illustrated in Figure 510.

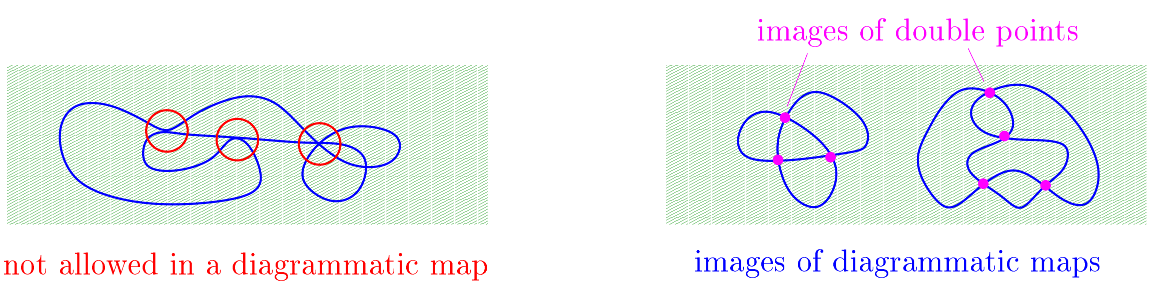


FIGURE 510

The following lemma will be proved in Exercise 27.4.

Lemma 27.17. *Every diagrammatic map has only finitely many double points.*

Definition.

(1) A *knot diagram* is a diagrammatic map $\gamma\colon S^1 \to \mathbb{R}^2$ together with a map
$$c\colon \{\text{double points of } \gamma\} \ \to \ \{\pm 1\}$$
which has the property that for $z \neq w \in S^1$ with $\gamma(z) = \gamma(w)$ we have $c(z) \neq c(w)$.

(2) Let (γ, c) be a knot diagram. Let z be a double point. If $c(z) = +1$, then we say that the double point is an *overcrossing*, otherwise we call it an *undercrossing*.

The next lemma shows that knot diagrams give rise to knots.

Lemma 27.18. *Let $(\gamma\colon S^1 \to \mathbb{R}^2, c)$ be a knot diagram. We can pick disjoint compact segments $I_1, \dots, I_k$ of S^1 and a smooth function $\eta\colon S^1 \to [-1, 0]$ with the following properties:*

(a) *Each segment contains a unique undercrossing and each undercrossing is contained in some segment.*

(b) *No segment contains an overcrossing.*

(c) *The value of η outside the segments is equal to 0.*

(d) *On each segment the function has a unique local minimum, namely at the corresponding undercrossing where the value of η is equal to -1.*

The following two statements hold:

(1) *The image of S^1 under the map $S^1 \to \mathbb{R}^3$ given by $z \mapsto (\gamma(z), \eta(z))$ is a knot.*

(2) *Any two choices of segments and η as above give rise to smoothly isotopic knots.*

If we equip S^1 with the standard orientation, then in the above statements we can also replace "knot" by "oriented knot".

Lemma 27.18 shows that a knot diagram gives rise to an essentially unique (oriented) knot. We refer to this knot as the *knot associated to the knot diagram*. At times we will rather blur the difference between a knot diagram and the associated knot.

Example. In Figure 511 we show to the left a knot diagram. To the right we show the associated knot. It is pretty clear that the resulting knot is smoothly isotopic to the trefoil.

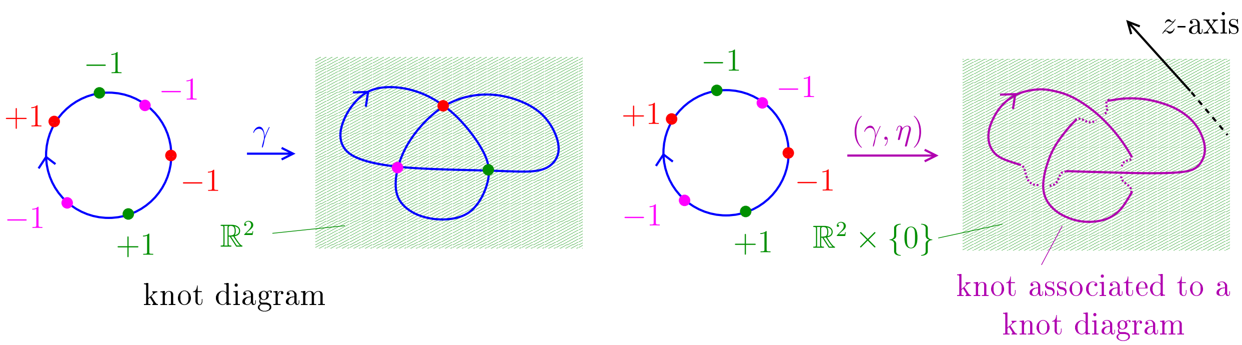


FIGURE 511

PROOF. It follows quite easily from Lemma 6.13 that such segments $I_1, \dots, I_k$ and that such a function η exist. Given such a function it is clear that the map $z \mapsto (\gamma(z), \eta(z))$ is injective. Since γ is an immersion it follows that the map $z \mapsto (\gamma(z), \eta(z))$ is in fact a smooth embedding. Thus it follows from Proposition 8.1 (1) that the image is a one-dimensional submanifold of $\mathbb{R}^3 \cup \{\infty\} = S^3$ that is diffeomorphic to S^1. In other words, the image is a knot.

Finally suppose we are given functions η and η' with the properties stated as above. We consider the map

$$\begin{array}{rcl} H\colon S^1 \times [0,1] & \to & \mathbb{R}^3 \cup \{\infty\} = S^3 \\ (z,t) & \mapsto & (\gamma(z), \eta(z)\cdot(1-t) + \eta'(z)\cdot t). \end{array}$$

This map is smooth. By construction we know that for any two distinct points $z, w \in S^1$ with $\gamma(z) = \gamma(w)$ we have $\eta(z) < \eta(w)$ if and only if $c(z) = -1$ if and only if $\eta'(z) < \eta'(w)$. Using this observation one can easily verify that each H_t is a smooth embedding. In other words, the map H_t is a smooth isotopy between the two (oriented) knots corresponding to η and η'. ■

In practice, i.e. in future examples, we do what every other sane topologist does, namely we draw a suggestive picture and we interpret it as a knot diagram and a corresponding knot in the obvious way. For example in Figure 512 we show the first knots in the standard table of knots. We refer to the standard textbooks on knot theory [**Rolf90, BZH14**] for a continuation of the table. But it should already be pretty clear from this table that knots form a straightforward way to produce an almost endless list of interesting topological problems.

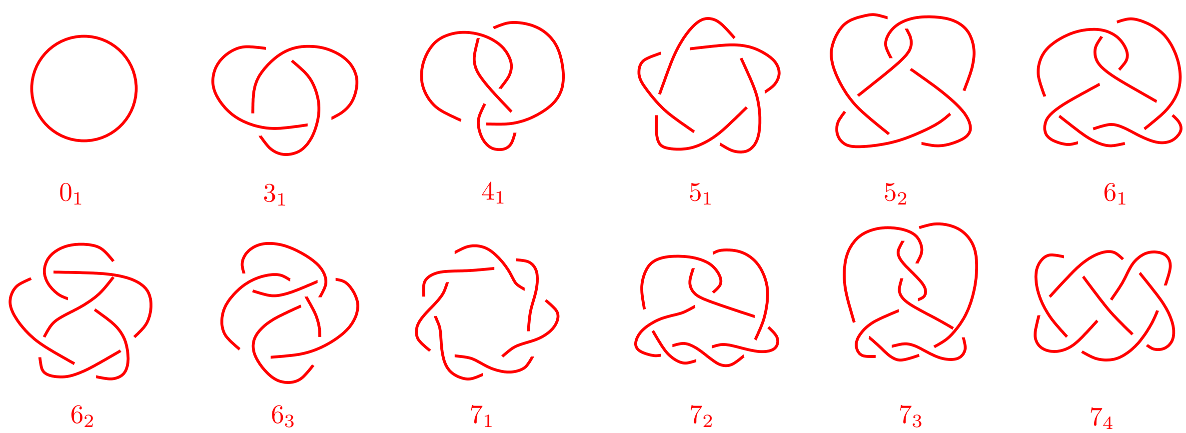


Figure 512

In Lemma 27.18 we saw that any knot diagram gives rise to a knot. The following proposition gives us the converse:

Proposition 27.19. *Given any knot $K \subset S^3$ there exists a knot diagram such that K is smoothly isotopic to the knot associated to the knot diagram.*

The proof of Proposition 27.19 mostly rests on the next lemma. The formulation of this lemma requires the following notation.

Notation. Given $v \in S^2$ we denote by

$$\begin{array}{rcl} \pi_v \colon \mathbb{R}^3 & \to & v^\perp := \{w \in \mathbb{R}^3 \,|\, \langle v, w\rangle = 0\} \\ w & \mapsto & w - \langle v, w\rangle v \end{array}$$

the projection onto the orthogonal complement of v.

Now we can formulate the lemma.

Lemma 27.20. *Let $K \subset \mathbb{R}^3$ be a knot. We pick a diffeomorphism $f \colon S^1 \to K$. We consider the following three properties of a vector $v \in S^2$:*

(a) *the map $\pi_v \circ f \colon S^1 \to v^\perp = \{w \in \mathbb{R}^3 \,|\, \langle v, w\rangle = 0\}$ is an immersion,*
(b) *whenever $z \neq w \in S^1$ satisfy $(\pi_v \circ f)(z) = (\pi_v \circ f)(w)$, then $(\pi_v \circ f)'(z)$ and $(\pi_v \circ f)'(w)$ are linearly independent,*
(c) *for every $P \in (\pi_v \circ f)(S^1)$ the preimage $(\pi_v \circ f)^{-1}(P)$ consists of either one or two points.*

The set

$$\{v \in S^2 \,|\, v \text{ satisfies (a), (b) and (c)}\}$$

has full measure in S^2.[497]

Remark. In Proposition 6.62 (2) we showed that a subset of full measure is dense. The chronically bored reader can try to show that the set $\{v \in S^2 \,|\, v \text{ satisfies (a), (b) and (c)}\}$ considered in Lemma 27.20 is open.

Proof. Let $K \subset \mathbb{R}^3$ be a knot. We pick a diffeomorphism $f \colon S^1 \to K$. We say that a subset of a smooth manifold W is *large* if it is open and if it has full measure in W. It follows from Proposition 6.62 (5b) that the intersection of two large subsets is again large. Furthermore, it follows from Proposition 6.62 (1) that any self-diffeomorphism of S^2 sends large sets to large sets.

It is convenient to consider an extra property of vectors in S^2 which is somewhat weaker than the above property (c):

(c′) given any $P \in (\pi_v \circ f)(S^1)$ the preimage $(\pi_v \circ f)^{-1}(P)$ consists of finitely many points.

We write

$$V(a) \;:=\; \{v \in S^2 \,|\, v \text{ satisfies (a)}\}.$$

Similarly we define $V(b), V(c), V(a, b), V(a, b, c')$ and so on. Note that with this language we need to show that $V(a, b, c)$ has full measure.

Let $v \in S^2$. We start out with three elementary observations regarding the projection maps π_v.

(1) Given $u \in \mathbb{R}^3 \setminus \{0\}$ we have

$$\pi_v(u) = 0 \quad\Longleftrightarrow\quad u \in \mathbb{R} \cdot v \quad\Longleftrightarrow\quad \tfrac{u}{\|u\|} \in \{v, -v\}.$$

(2) The projection map π_v is linear, hence for any $P \in \mathbb{R}^3$ we have $(\mathrm{D}\,\pi_v)_P = \pi_v$.
(3) It follows from (2) and the chain rule that given any $z \in S^1$ we have the equality $(\pi_v \circ f)'(z) = \pi_v(f'(z))$.

[497] We refer to page 316 for the definition of a subset of full measure.

Next we consider the three maps[498] [499]

$$\varphi\colon S^1 \to S^2 \qquad \psi\colon \overbrace{\{(z,w)\in S^1\times S^1 \,|\, z\neq w\}}^{=:M} \to S^2 \qquad \text{and} \qquad \rho\colon S^2 \to S^2$$
$$z \mapsto \frac{f'(z)}{\|f'(z)\|} \qquad (z,w) \mapsto \frac{f(z)-f(w)}{\|f(z)-f(w)\|} \qquad v \mapsto -v.$$

We make the following observations:

(4) If X is a subset of full measure (respectively large subset) of S^2, then it follows from the above discussion that $X \cap \rho(X)$ is also a subset of full measure (respectively large).

(5) Since f is smooth we see that the map φ is also smooth.

(6) Given $v \in S^2$ it follows easily from (1) and (3) that $\pi_v \circ f\colon S^1 \to \mathbb{R}^2$ is an immersion if and only if neither v nor $-v$ lies in $\varphi(S^1)$. In other words, we have the equality $V(a) = (S^2 \setminus \varphi(S^1)) \cap \rho(S^2 \setminus \varphi(S^1))$.

(7) Note that $M = \{(z,w) \in S^1 \times S^1 \,|\, z \neq w\}$ is an open subset of the smooth manifold $S^1 \times S^1$, thus it is a smooth manifold in an obvious way. With this smooth manifold structure the map $\psi\colon M \to S^2$ is easily seen to be smooth.

(8) For each $(x,y) \in S^1$ we make the identification[500] $\mathrm{T}_{(x,y)}\, S^1 = \mathbb{R}$ via the basis vector $(-y,x)$. Given $(z,w) \in M$ we use the above identification to make the identification $\mathrm{T}_{(z,w)}\, M = \mathrm{T}_{(z,w)}(S^1 \times S^1) = \mathbb{R}^2$. Furthermore given $v \in S^2$ we already saw on page 293 that we have the equality $\mathrm{T}_v\, S^2 = v^\perp = \{w \in \mathbb{R}^3 \,|\, \langle v,w\rangle = 0\} \subset \mathbb{R}^3$.

(9) Using the identifications from (8) an elementary calculation[501] shows that for a point $(z,w) \in M$ we can write the differential $\mathrm{D}\,\psi_{(z,w)}$, viewed as a homomorphism $\mathrm{T}_{(z,w)}\, M = \mathbb{R}^2 \to \mathrm{T}_{\psi(z,w)}\, S^2 \subset \mathbb{R}^3$, in the following way as a (3×2)–matrix:

$$\mathrm{D}\,\psi_{(z,w)} \;=\; \frac{1}{\|f(z)-f(w)\|} \cdot \underbrace{\big(\pi_{\psi(z,w)}(f'(z)) \quad -\pi_{\psi(z,w)}(f'(w))\big)}_{(3\times 2)\text{–matrix, in particular two columns}}.$$

(10) It follows from (9) that $(z,w) \in M$ is a regular point of ψ if and only if the vectors $\pi_{\psi(z,w)}(f'(z))$ and $\pi_{\psi(z,w)}(f'(w))$ are linearly independent.

(11) Let $v \in S^2$. It follows immediately from (1) and the linearity of π_v that we have $\pi_v(f(z)) = \pi_v(f(w))$ if and only if $\psi(z,w) = v$ or $\psi(z,w) = -v$.

---

[498] Since f is an immersion we see that the map φ is well-defined, i.e. we do not divide by zero. Similarly, since f is in particular injective we see that the map ψ is well-defined, i.e. once again we do not divide by zero.

[499] The astute reader will notice that throughout the argument it might be more reasonable to work with maps φ and ψ that take values in $\mathbb{RP}^2 = S^2/x \sim -x$ instead of taking values in S^2. We stick with the maps to S^2 since S^2 has the advantage that its tangent spaces can be described easily as vector subspaces of $\mathbb{R}^3$, which makes it easy to write down differentials. Also, the approach of working with maps to $\mathbb{RP}^2$ instead creates an extra layer of notation which is as annoying as our notation which requires the extra map ρ.

[500] As on many other occasions we use Proposition 6.39 to view the tangent space of a submanifold of some $\mathbb{R}^n$ as a vector subspace of $\mathbb{R}^n$.

[501] This calculation can be performed easily using the chain rule and by writing ψ as the composition of the map $(z,w) \mapsto f(z) - f(w)$ followed by the map $P \mapsto \frac{P}{\|P\|}$.

(12) We denote by $\sigma\colon S^1 \times S^1$ the diffeomorphism given by $(z, w) \mapsto (w, z)$. It follows immediately from the definitions that $\psi \circ \sigma = \rho \circ \psi$.

(13) It follows easily from (3), (10), (11) and (12) that
$$V(b) \;=\; \{\text{regular values of } \psi\}.$$

After these initial remarks we turn to our first claim.

Claim.

(α) The sets $V(a)$ and $V(a, b)$ are large.
(β) We have $V(a) = V(a, c')$ and $V(a, b) = V(a, b, c')$.

We prove the two statements of the claim in several steps.

(i) Since S^1 is compact we obtain from Lemma 2.40 and Lemma 2.17 that $S^2 \setminus \varphi(S^1)$ is an open subset of S^2. Furthermore, since φ is smooth and since $\dim(S^1) < \dim(S^2)$ we obtain from Proposition 6.62 (5) that $S^2 \setminus \varphi(S^1)$ is a subset of full measure. In summary, we have shown that $S^2 \setminus \varphi(S^1)$ is large. It follows from (5) and (6) that $V(a) = (S^2 \setminus \varphi(S^1)) \cap \rho(S^2 \setminus (\varphi(S^1)))$ is large.

(ii) It follows from Sard's Theorem 6.63 that the set of regular values of ψ is a subset of full measure. It follows from (i) and (13), together with Proposition 6.62 (1) that $V(a, b) = V(a) \cap V(b)$ has full measure.

(iii) Let $v \in V(a)$. It follows from (11), from the fact that S^1 is compact and from Exercise 8.3 that $\psi^{-1}(v)$ is finite. In particular we see that $V(a, c') = V(a)$ and thus also $V(a, b, c') = V(a, b)$.

(iv) It remains to show that $V(a, b)$ is open. Let $v \in V(a, b) = V(a) \cap V(b)$. From (13) and (iii), together with Exercise 6.29 (2), we obtain an open neighborhood U of $v \in S^2$ that is still contained in $V(b)$. Since $V(a)$ is open we see that $U \cap V(a)$ is an open neighborhood of $v \in S^2$ that is still contained in $V(a, b)$. It follows from Lemma 2.5 that $V(a, b)$ is indeed open. ⊞

Before we continue it is perhaps helpful to summarize what we have shown so far: In the claim we have seen that $V(a, b, c') = V(a, b)$ is large, in particular that it has full measure. But our actual goal is to show that $V(a, b, c)$ has full measure.

To do so we consider the map
$$\begin{array}{rcl}\Xi\colon \overbrace{\{(x, y, z) \in S^1 \times S^1 \times S^1 \,|\, x \neq y \text{ and } x \neq z\}}^{=:N} & \to & S^2 \times S^2 \\ (x, y, z) & \mapsto & (\psi(x, y), \psi(x, z)).\end{array}$$

We make the following simple observations:

(14) We view N as a smooth manifold in an obvious way. Evidently the map Ξ is smooth.

(15) It follows easily from (1) that we have $v \in V(c)$ if and only if none of the four vectors $(\pm v, \pm v)$ lies in $\Xi(N)$.

(16) If one of the four vectors $(\pm v, \pm v)$ lies in $\Xi(N)$, then so does (v, v). This can be seen as follows: If $\Xi(x, y, z) \in \{(\pm v, \pm v)\}$, then $f(x), f(y), f(z)$ lie on the line $\mathbb{R} \cdot v$. After possibly permuting x, y, z we can assume that $f(x) = f(y) + r \cdot v$ and that $f(x) = f(z) + s \cdot v$ with $r, s > 0$. But with this permutation we have $\Xi(x, y, z) = (v, v)$.

Claim. We consider the "partial diagonal"

$$\Delta := \{(v,v) \in S^2 \times S^2 \,|\, v \in V(a,b)\}.$$

The complement of $\Xi(N) \cap \Delta$ has full measure in Δ.

In the previous claim we saw that $V(a,b)$ is an open subset of S^2. Using this fact it is straightforward to see that Δ is a 2-dimensional submanifold of $S^2 \times S^2$. Thus, by Exercise 8.9 (b) it suffices to prove that if $(x,y,z) \in N$ satisfies $\Xi(x,y,z) = (v,v)$ for some $v \in V(a,b)$, then Ξ intersects Δ transversally, i.e. we have the equality

$$(\mathrm{D}\,\Xi_{(x,y,z)})(\mathrm{T}_{(x,y,z)}\,N) + \mathrm{T}_{(v,v)}\,\Delta \;=\; \mathrm{T}_{(v,v)}(S^2 \times S^2) \;\subset\; \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6.$$

We write $p = \pi_v(f'(x))$, $q = \pi_v(f'(y))$ and $r = \pi_v(f'(z))$ and we write $\mu = \frac{1}{\|f(x)-f(y)\|}$ and $\nu = \frac{1}{\|f(x)-f(z)\|}$. By (10) and (13), and by the fact that $v \in V(b)$, we know that p, q and r are pairwise linearly independent. In particular q and r form a basis for $\mathrm{T}_v\,S^2$. Note that this implies that $\binom{q}{q}$ and $\binom{r}{r}$ form a basis for $\mathrm{T}_{(v,v)}(S^2 \times S^2)$. Thus, using (9) we see that

$$(\mathrm{D}\,\Xi_{(x,y,z)})(\mathrm{T}_{(x,y,z)}\,N) + \mathrm{T}_{(v,v)}\,\Delta \;=\; \text{span of the columns of } \begin{pmatrix} \mu \cdot p & -\mu \cdot q & 0 & q & r \\ \nu \cdot p & 0 & -\nu \cdot r & q & r \end{pmatrix}.$$

Using that p, q and r are pairwise linearly independent it is not hard to see that the five columns of the matrix on the right span a 4-dimensional subspace of $\mathbb{R}^6$. Since this subspace is contained in the 4-dimensional subspace $\mathrm{T}_{(v,v)}(S^2 \times S^2)$ we see that it equals $\mathrm{T}_{(v,v)}(S^2 \times S^2)$. ⊞

We consider the "diagonal map" $d\colon V(a,b) \to \Delta \subset S^2 \times S^2$ that is given by $d(v) = (v,v)$. One can easily show that $d\colon V(a,b) \to \Delta$ is a diffeomorphism. It follows from (16) that $V(a,b,c) = d^{-1}(\Delta \setminus \Xi(N))$. By the claim $\Delta \setminus \Xi(N)$ is a subset of full measure of Δ. Since d is a diffeomorphism we obtain from Proposition 6.62 (5b) that $V(a,b,c) = d^{-1}(\Delta \setminus \Xi(N))$ has full measure in $V(a,b)$. We saw above that $V(a,b)$ has full measure in S^2. It follows easily from Proposition 6.62 (5a) that $\Delta \setminus \Xi(N)$ has full measure in S^2. ■

Now we can provide the proof of Proposition 27.19.

Proof of Proposition 27.19. Let $K \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be a knot. As we saw in Exercise 8.30 (a), it follows easily from Lemma 3.32 (4) and Lemma 8.5 (2) that, possibly after applying a smooth isotopy, we can assume that $K \subset \mathbb{R}^3$.

Next we pick $v \in S^2$ as in Lemma 27.20. It follows from Lemma 3.32 (4) and Lemma 8.5 (2) that, possibly after applying a smooth isotopy, we can assume that $v = e_3$. We define $\gamma := \pi_{e_3} \circ f\colon S^1 \to \mathbb{R}^2 \times \{0\} = \mathbb{R}^2$. It follows immediately from the fact that $v = e_3$ has the three properties (a), (b) and (c) stated in Lemma 27.20 that γ is diagrammatic. Next we consider the function

$$\begin{array}{rcl} \epsilon\colon S^1 & \to & \mathbb{R} \\ w & \mapsto & z\text{-coordinate of } f(w). \end{array}$$

Furthermore, given $z_1 \neq z_2 \in S^1$ with $\gamma(z_1) = \gamma(z_2)$ we define

$$c(z_i) := \begin{cases} +1, & \text{if } \epsilon(z_i) > \epsilon(z_{3-i}), \\ -1, & \text{if } \epsilon(z_i) < \epsilon(z_{3-i}). \end{cases}$$

It is clear that (γ, c) is a knot diagram. We pick a function $\eta\colon S^1 \to [-1, 0]$ as in Lemma 27.18. We consider the map

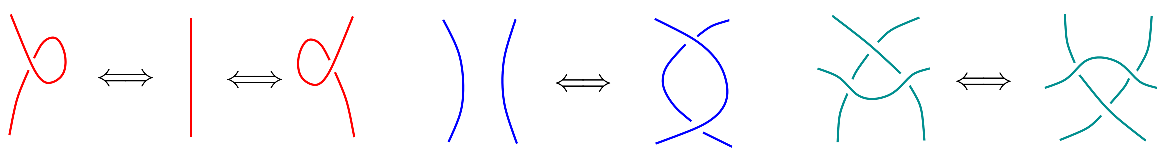
$$\begin{array}{rcl} H\colon S^1 \times [0,1] & \to & \mathbb{R}^3 \\ (z,t) & \mapsto & (\gamma(z), (1-t)\cdot\epsilon(t) + t\cdot\eta(t)). \end{array}$$

Clearly this map is smooth. Given any $z, w \in S^1$ with $\gamma(z) = \gamma(w)$ we have $\eta(z) < \eta(w)$ if and only if $c(z) = -1$ if and only if $\eta(z) < \eta(w)$. From this observation we deduce that each H_t is actually a smooth embedding. Thus we see that H is a smooth isotopy from K to the knot associated to the knot diagram (γ, η). ■

We have now seen that every knot, up to smooth isotopy, arises from a knot diagram. The question arises, when do two knot diagrams give rise to smoothly isotopic knots. The following theorem gives a complete answer. We formulate the theorem in a slightly informal way since we will not make use of it.

Theorem 27.21. *Two knot diagrams give rise to smoothly isotopic knots if and only if the two diagrams are related by a finite sequence of smooth isotopies of $\mathbb{R}^2$ and Reidemeister moves. The three Reidemeister moves are illustrated in Figure 513.*

Proof. The theorem, not surprisingly, goes back to work of Kurt Reidemeister [**Rei32**, Chapter 3] in 1932. The formulation and proof in [**Rei32**] is in terms of "polygonal knots". The statement of the theorem is well known in our setting, where knots are defined as submanifolds of S^3 and it appears in any textbook on knot theory. A satisfactory proof can be found in [**OSS15**, Theorem B.1.1]. ■



Reidemeister move 1 Reidemeister move 2 Reidemeister move 3

Figure 513

Remark. Throughout these lecture notes we will see on several occasions that the tools of algebraic topology, e.g. fundamental groups, linking pairings and Reidemeister torsion can be used fruitfully to study knots. Proposition 27.19 and Theorem 27.21 open up an alternative route to studying knots. More precisely, these two results show that there exists a bijection

$$\begin{array}{c}\text{knot diagrams up to smooth isotopy} \\ \text{and Reidemeister moves}\end{array} \xrightarrow{\cong} \text{knots up to smooth isotopy}$$

This bijection lies at the heart of the definition of several knot invariants, like the Jones polynomial [**Jon85**] that was introduced in 1984 and the HOMFLY-PT polynomial [**FYHLMO85,**

PT87] that was discovered in 1985. These polynomial invariants are quite different from any of the invariants that are obtained through algebraic-topological methods. We refer to [**Lic97a**, Chapters 3 and 15] and [**BZH14**, Chapter 17] for more information on these invariants.

27.6. **The Wirtinger presentation.** In the previous section we saw that we can associate a knot diagram to any given knot. In this section we will see that we give an explicit algorithm that, given a knot diagram, produces a presentation for the fundamental group of the complement $S^3 \setminus K$ of a knot K.

In this section we will take several liberties and we will not attempt to give full details in the proofs. The issue is that any attempt to make the proofs completely rigorous renders them totally unreadable.

Before we can state the promised algorithm we need to introduce a few more definitions.

Definition. Let $K \subset \mathbb{R}^3$ be an oriented knot that is associated to some knot diagram $(f\colon S^1 \to \mathbb{R}^2, c)$.

(1) We refer to the images of the double points as the *crossings of K*.
(2) Let $x \in f(S^1)$ be a crossing. Let $z, w \in S^1$ with $f(z) = f(w) = x$ and such that z is an undercrossing and w is an overcrossing. We say x is a *positive crossing* if the ordered basis $\{f'(w), f'(z)\}$ is a positive basis for $\mathbb{R}^2$. Otherwise we say that x is a *negative crossing*.
(3) We refer to the components of $K \cap (\mathbb{R}^2 \times \{0\})$ as the *strands of K*.

These definitions are illustrated in Figures 514 and 515.

Remark. In a picture of a diagram the strands are understood to correspond precisely to the segments drawn.

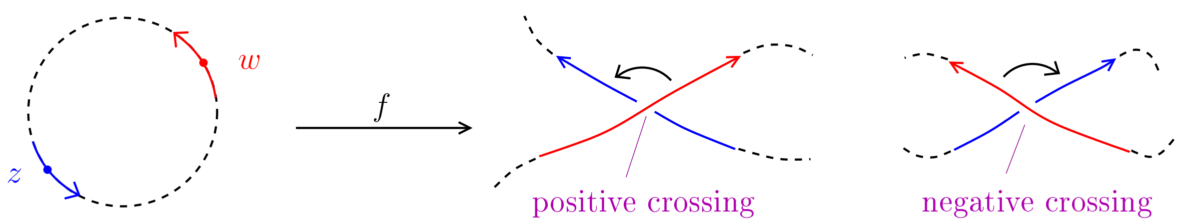


FIGURE 514

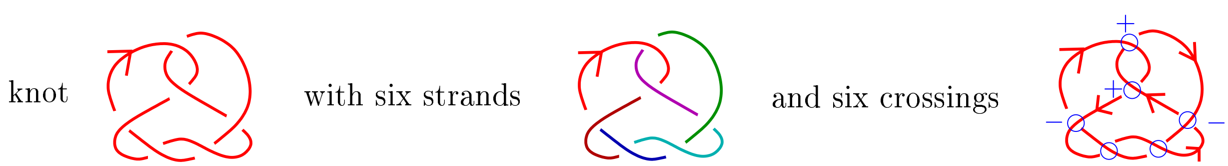


FIGURE 515

The following lemma will be proved in Exercise 27.8.

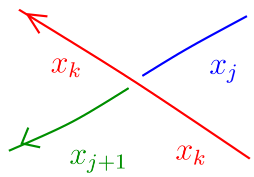
Lemma 27.22. *If a knot diagram has at least one crossing, then the number of crossings equals the number of strands.*

The following proposition shows how to determine the fundamental group of a knot that is associated to a knot diagram. We will sacrifice some rigor in the formulation in an attempt to make it more readable. We leave it to the reader to make the proposition totally rigorous.

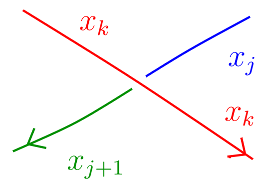
Proposition 27.23. *Let K be a knot that is associated to a given knot diagram that has at least one crossing. We enumerate the strands cyclically by $x_1, \dots, x_n$ where we make the identifications $x_0 = x_n$ and $x_{n+1} = x_1$. For the crossing between the strands x_j and x_{j+1} we define a relation r_j as shown in Figure 516. Then we have an isomorphism*

$$\pi_1(S^3 \setminus K) \;\cong\; \langle x_1, \dots, x_n \,|\, r_1, \dots, r_n \rangle.$$

Furthermore we can drop any one of the relations and we still obtain an isomorphism.



the positive crossing gives rise to the relation $r_j = x_k \cdot x_{j+1} \cdot x_k^{-1} \cdot x_j^{-1}$



the negative crossing gives rise to the relation $r_j = x_k^{-1} \cdot x_{j+1} \cdot x_k \cdot x_j^{-1}$

FIGURE 516

Remark. Given a knot K, any presentation with n generators and $n-1$ relators as obtained from Proposition 27.23 is called a *Wirtinger presentation of* $\pi_1(S^3 \setminus K)$.

In the following we first explain why Proposition 27.23 is plausible. Afterwards we will provide the technical proof of Proposition 27.23.

"PLAUSIBILITY ARGUMENT FOR PROPOSITION 27.23". Suppose we are given a knot diagram that has n crossings. We denote by K the corresponding knot. We work with the base point $* = (0,0,2) \in \mathbb{R}^3$ which lies "above" the knot K. We enumerate the strands cyclically. For the j-th strand we denote by x_j the oriented triangle that starts at $*$ and "circles once around the j-th strand" according to the "right-hand rule". We illustrate the definition in Figure 517.

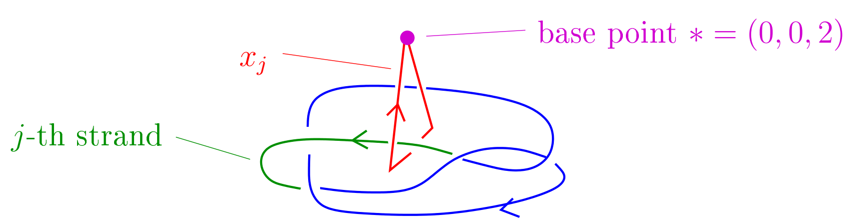


FIGURE 517

Next we consider the situation at a positive crossing. By "sliding" various x_j along the strands and by some further path-homotopies we can assume that we are in the situation in Figure 518 on the left. If one follows the loop $x_k * x_{j+1} * \overline{x_k} * \overline{x_j}$ on its journey one realizes that on three occasions one goes back and forth to the base point. Thus the loop is actually path-homotopic to the loop shown in Figure 518 on the right. But that loop is clearly null-homotopic. Almost the same logic applies to negative crossings.

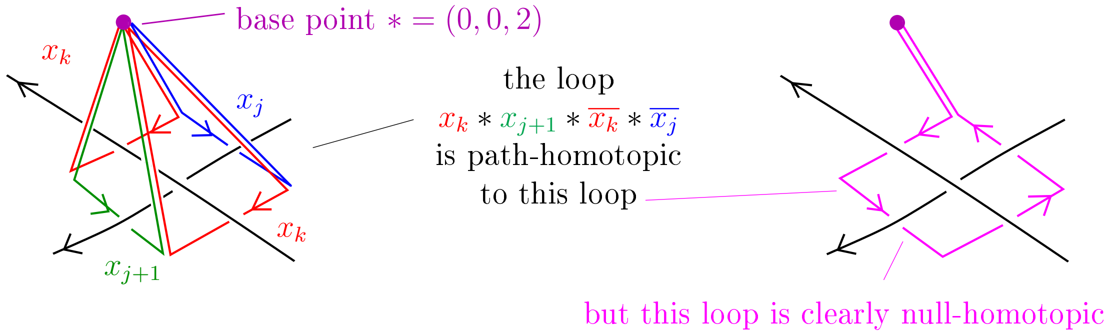


FIGURE 518

This discussion, together with Lemmas 19.14 and 21.4, shows that there exists a unique group homomorphism

$$\begin{array}{rcl}\langle x_1,\dots,x_n \,|\, \text{relations corresponding to crossings}\rangle & \to & \pi_1(S^3 \setminus K, *) \\ x_i & \mapsto & [x_i].\end{array}$$

with

It remains to show that this group homomorphism is in fact an isomorphism. We will do so next in the proper proof of Proposition 27.23. ■

SKETCH OF A PROOF FOR PROPOSITION 27.23. Suppose once again that we are given a knot diagram $(\gamma\colon S^1 \to \mathbb{R}^2, c)$ that has $n \geq 1$ crossings. We denote by $P_1,\dots,P_n$ the double points of γ. We pick a smooth function $\eta\colon S^1 \to [-1,0]$ as in the statement of Lemma 27.18. We denote by K the image of the map $S^1 \to \mathbb{R}^3$ given by $z \mapsto (\gamma(z),\eta(z))$. It follows from property (d) of Lemma 27.18 that

$$K \cap (\mathbb{R}^2 \times \mathbb{R}_{\leq -1}) \;=\; K \cap (\mathbb{R}^2 \times \{-1\}) \;=\; \{(P_1,-1),\dots,(P_n,-1)\}.$$

In contrast to the above plausibility argument we now actually work with the base point $\infty \in S^3 = \mathbb{R}^3 \cup \{\infty\}$. We set

$$A \;:=\; (\{(x,y,z) \in \mathbb{R}^3 \,|\, z \geq -1\} \cup \{\infty\}) \setminus K,$$

and

$$B \;:=\; (\{(x,y,z) \in \mathbb{R}^3 \,|\, z \leq -1\} \cup \{\infty\}) \setminus K.$$

We start out with the following claim.

Claim 1. The inclusion maps induce an isomorphism

$$\pi_1(A,\infty) *_{\pi_1(A\cap B,\infty)} \pi_1(B,\infty) \;\cong\; \pi_1(S^3 \setminus K).$$

Evidently the statement of the claim smells like it should be a consequence of the Seifert–van Kampen 22.1. The problem is that A and B are not open subsets of $S^3 \setminus K$.

We rectify this problem by replacing A and B by suitable open subsets. We define

$$U \;:=\; A \,\cup\, (\mathbb{R}^2 \setminus \{P_1, \dots, P_n\}) \times (-2, -1)$$

and

$$\begin{aligned} V \;:=\; B \,&\cup\, (\mathbb{R}^2 \setminus \gamma(S^1)) \times [-1, 0) \\ &\cup\, \{(z,t) \mid \text{ there exists } w \in S^1 \text{ with } z = \gamma(w) \text{ and } t \in [-1, \eta(z))\}. \end{aligned}$$

One can easily verify that U and V are open subsets of $S^3 \setminus K$. Furthermore the "obvious retractions" of the blue sets to the corresponding subsets of $\mathbb{R}^2 \times \{-1\}$ show that A is a deformation retract of U, that B is a deformation retract of V and that $A \cap B$ is a deformation retract of $U \cap V$. Now we see that the inclusion maps induce isomorphisms

$$\pi_1(S^3 \setminus K) \;\underset{\substack{\uparrow \\ \text{by the Seifert–van Kampen Theorem 22.1}}}{\overset{\cong}{\longleftarrow}}\; \pi_1(U, \infty) *_{\pi_1(U \cap V, \infty)} \pi_1(V, \infty) \;\underset{\substack{\uparrow \\ \text{by the above, Lemma 18.14} \\ \text{and Proposition 18.16 (2)}}}{\overset{\cong}{\longrightarrow}}\; \pi_1(A, \infty) *_{\pi_1(A \cap B, \infty)} \pi_1(B, \infty).$$

⊞

Our next goal is to understand the groups $\pi_1(A, \infty)$, $\pi_1(B, \infty)$ and $\pi_1(A \cap B, \infty)$. We handle these three groups consecutively. To better understand $\pi_1(A, \infty)$ we introduce the following objects.

(1) As in the plausibility argument we set $* = (0, 0, 2)$. We fix once and for all a path-connecting $*$ to ∞.
(2) We write $D := \{(\gamma(z), w) \mid z \in S^1 \text{ and } w \in [-1, \eta(z))\}$.
(3) We cyclically enumerate the strands of the knot.
(4) For $j \in \{1, \dots, n\}$ we denote by D_j the unique component of D that contains the j-th strand in its closure.
(5) For $j = 1, \dots, n$ we denote by x_j a loop in $*$ that is the boundary of a triangle that intersects D_j precisely once in a positive[502] direction. We use the path from (1) to view each x_j as a loop in ∞.

Some of the above objects are illustrated on the left of Figure 519.

Claim 2. The obvious map $\langle x_1, \dots, x_n \rangle \to \pi_1(A, \infty)$ is an isomorphism.

Note that each D_j is contractible and hence simply connected. Furthermore one can easily show that $A \setminus D$ is contractible. We use Corollary 26.5 iteratively n times to prove the claim. We leave it to the assiduous reader to fill in the details.[503] ⊞

Fortunately the next group $\pi_1(B, \infty)$ on our list is easy to handle.

Claim 3. The group $\pi_1(B, \infty)$ is trivial.

Using the stereographic projection from Lemma 2.44 and using the map defined on page 116 it is not hard to see that there exists a homeomorphism

$$f \colon (\{(x, y, z) \in \mathbb{R}^3 \mid z \leq -1\} \cup \{\infty\} \;\to\; \overline{B}^3$$

which sends the points in $\mathbb{R}^2 \times \{-1\}$ to S^2. Since $K \cap (\mathbb{R}^2 \times \mathbb{R}_{\leq -1}) = \{(P_1, -1), \dots, (P_n, -1)\}$ we see that B is homeomorphic to $\overline{B}^3$ with n points on its boundary removed. But this

[502]Here by "positive direction" we mean that the orientation vectors of the triangle followed by the orientation vector of the knot should form a positive basis for $\mathbb{R}^3$.

[503]The details involve replacing the set D_j by a suitable open subset to be able to apply Corollary 26.5.

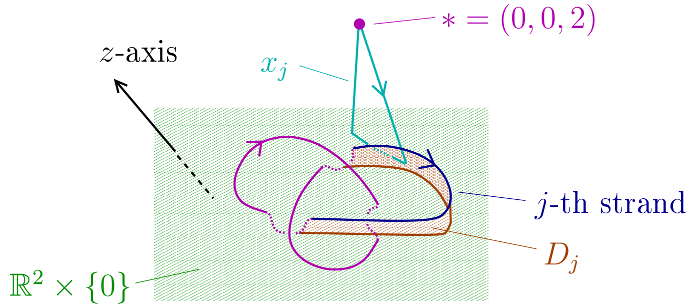


FIGURE 519

does not change the fact that the origin is a deformation retract. We have thus shown that $f(B)$, and hence also B, is contractible. But this implies by Proposition 18.16 (3) that $\pi_1(B,\infty)$ is indeed trivial. $\boxplus$

Finally we turn to the group $\pi_1(A\cap B,\infty)$. We pick orientation-preserving smooth embeddings $\varphi_i\colon \overline{B}^2 \to \mathbb{R}^2\times\{-1\}$, $i=1,\dots,n$ with disjoint images and such that for each i we have $\varphi_i(0)=(P_i,-1)$. It follows almost immediately[504] from Lemma 26.6 that we have an equality

$$\langle z_1,\dots,z_n \mid z_1\cdot\dots\cdot z_n\rangle \;=\; \pi_1(A\cap B,\infty)$$

where for any $i\in\{1,\dots,n\}$ the element z_i is represented by a curve of the form $\alpha_i*\gamma_i*\overline{\alpha}_i$ where α_i is a suitable path from ∞ to a point on $\varphi_i(S^1)$ and γ_i is a path that goes once around $\varphi_i(S^1)$ in an orientation-preserving way.

Now we can finally prove the desired statement. Indeed, we have

$$\begin{aligned}
\pi_1(S^3\setminus K) \underset{\substack{\uparrow\\ \text{by Claim 1}}}{\cong} \pi_1(A,\infty) *_{\pi_1(A\cap B,\infty)} \pi_1(B,\infty) &\underset{\substack{\uparrow\\ \text{by Claim 3 and Lemma 21.22 (2)}}}{\cong} \pi_1(A,\infty)/\langle\langle \pi_1(A\cap B,\infty)\rangle\rangle \\
\underset{\substack{\uparrow\\ \text{by Claim 2, Lemma 21.2 (4) and the}\\ \text{above fact that } \{z_1,\dots,z_n\} \text{ is a}\\ \text{generating set for } \pi_1(A\cap B,\infty)}}{\cong} \langle x_1,\dots,x_n\rangle/\langle\langle z_1,\dots,z_n\rangle\rangle &\underset{\substack{\uparrow\\ \text{the ``plausibility argument shows that each}\\ z_i \text{ is conjugate to } r_i, \text{ thus it follows from}\\ \text{Lemma 21.2 (5) that } \langle\langle z_1,\dots,z_n\rangle\rangle = \langle\langle r_1,\dots,r_n\rangle\rangle}}{\cong} \langle x_1,\dots,x_n\rangle/\langle\langle r_1,\dots,r_n\rangle\rangle.
\end{aligned}$$

It remains to show that we can drop any one of the relations. Thus let $j\in\{1,\dots,n\}$. It follows easily from $z_1\cdot\dots\cdot z_n=e$ that $z_j\in\langle z_1,\dots,z_{j-1},z_{j+1},\dots,z_n\rangle$. Now note that by Lemma 21.2 (4) this implies that $\langle\langle z_1,\dots,z_n\rangle\rangle = \langle\langle z_1,\dots,z_{j-1},z_{j+1},\dots,z_n\rangle\rangle$. Therefore we see that in the above equality we can drop the relation r_j. ■

As mentioned above, we continue with two examples, namely the trefoil and the figure-8 knot. First we consider the knot diagram for the trefoil K that is shown in Figure 520 to the left. We number the strands as shown. To the right we show how the three crossings give rise to three relations. As pointed out in Proposition 27.23, we can drop any one of the relations.

[504] The slightly cautious "almost" refers to the fact that we also make use of the fairly obvious fact that $((\mathbb{R}^2\times\{-1\})\cup\{\infty\})\setminus \bigcup\limits_{i=1}^{n}\varphi_i(B^2)$ is a deformation retract of $A\cap B=((\mathbb{R}^2\times\{-1\})\cup\{\infty\})\setminus\{P_1,\dots,P_n\}$.

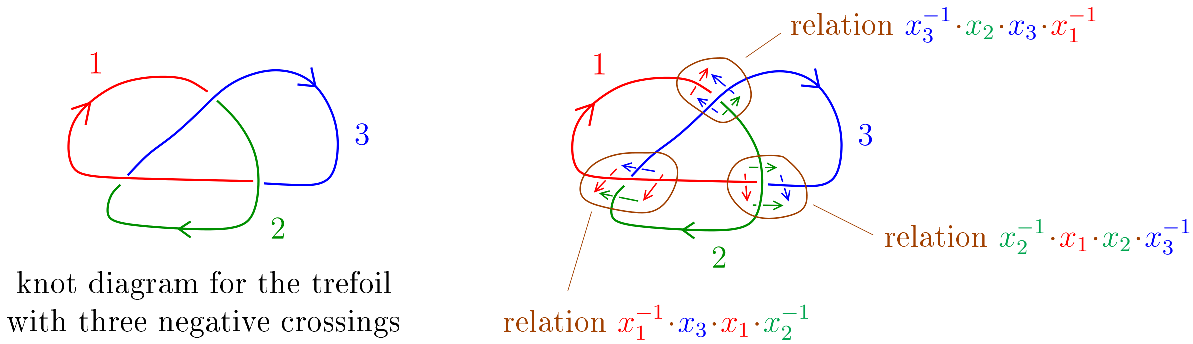


FIGURE 520

This discussion shows that we have proved the following lemma.

Lemma 27.24. *Let K be the trefoil. There exists an isomorphism*

$$\pi_1(S^3 \setminus K) \;\cong\; \langle x_1, x_2, x_3 \,|\, x_3^{-1}x_2x_3x_1^{-1}, x_2^{-1}x_1x_2x_3^{-1}\rangle.$$

Remark. Let K be the trefoil. In Proposition 27.5 we had also shown that there exists an isomorphism $\pi_1(S^3 \setminus K) \cong \langle x, y \mid x^2 = y^3\rangle$. In particular we have now shown that the two groups $\langle x_1, x_2, x_3 \,|\, x_3^{-1}x_2x_3x_1^{-1}, x_2^{-1}x_1x_2x_3^{-1}\rangle$ and $\langle x, y \mid x^2 = y^3\rangle$ are isomorphisms. In Exercise 27.9 we will give a purely algebraic proof of this statement.

The following corollary to Proposition 27.23 gives us the disappointing news that our old trick of using abelianizations of fundamental groups cannot be used to distinguish knots.

Corollary 27.25. *Given any knot K the abelianization of the group $\pi_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$.*

PROOF. We will prove the corollary in Exercise 27.5. Later, on page 1318, once we have introduced homology groups we will give a different, arguably more conceptual proof of the lemma. ■

After this sobering corollary we move on to the next example.

Lemma 27.26. *Let J be the figure-8 knot. There exists an isomorphism*

$$\pi_1(S^3 \setminus J) \;\cong\; \langle x_1, x_2, x_3, x_4 \,|\, x_3x_2x_3^{-1}x_1^{-1}, x_4^{-1}x_3x_4x_2^{-1}, x_1x_4x_1^{-1}x_3^{-1}\rangle.$$

PROOF. We leave it to the reader to use the knot diagram shown in Figure 521 to prove the lemma. ■

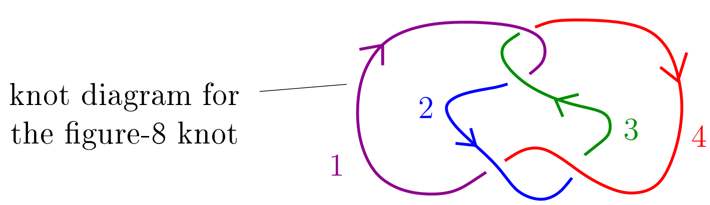


FIGURE 521

On its own Lemma 27.26 is not particularly enlightening. It is initially not clear whether this group is isomorphic to $\mathbb{Z}$ or to the fundamental group of the complement of the trefoil. By Corollary 27.25 we know that there is no hope that the abelianization will tell us something interesting.

So we need a different approach. Recall that in Proposition 27.6 we used the (non-) existence of an epimorphism onto the symmetric group S_3 to distinguish the trefoil from the trivial knot. The idea now is to repeat this trick with a different finite group. Recall that on page 719 we introduced the dihedral groups D_n which have $2n$ elements. It is encouraging that $D_3 \cong S_3$ thus we try our luck with D_5.[505]

Using Lemma 21.13 and enough time on our hands we obtain the following table:

	trivial knot	trefoil	figure-8 knot
epimorphism onto S_3	✗	✓	✗
epimorphism onto the dihedral group D_5	✗	✗	✓

This table shows that we have proved the following corollary.

Corollary 27.27. *The trivial knot, the trefoil and the figure-8 knot are pairwise not smoothly isotopic.*

This approach of trying to distinguish knots by the (non-) existence of epimorphisms onto finite groups is rather ad hoc, but not without its merits. For example the following proposition is proved in [**BF15, BR20**].

Proposition 27.28. *Let K be either the trivial knot, the trefoil or the figure-8 knot. Furthermore let J be another knot. If for all finite groups G the group $\pi_1(S^3 \setminus K)$ admits an epimorphism onto G if and only if $\pi_1(S^3 \setminus J)$ does, then J is smoothly isotopic to K or to the mirror image of K, i.e. the reflection of K in the xy-plane.*

We conclude this section with the following question which is still open.

Question 27.29. *Let K and J be two knots in $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} = S^3$. Suppose that the finite groups onto which $\pi_1(S^3 \setminus K)$ and $\pi_1(S^3 \setminus J)$ admit epimorphisms are the same. Does this imply that K and J are smoothly isotopic, up to possibly taking a reflection in the xy-plane?*

27.7. **The Reidemeister-Schreier process.** Even though Corollary 27.25 was quite disappointing, using a clever jiu-jitsu move we will see in this section that we can actually turn Corollary 27.25 to our advantage. The key to doing so is contained in the following definition and the subsequent lemma.

Definition. Let π be a group and let $n \in \mathbb{N}$. We say that a subgroup $\Gamma \subset \pi$ is *cocyclic of order n* if Γ is a normal subgroup such that π/Γ is isomorphic to $\mathbb{Z}_n$.

Lemma 27.30. *If π is a group whose abelianization is isomorphic to $\mathbb{Z}$, then given any $n \in \mathbb{N}$ there exists a unique subgroup that is cocyclic of order n. It is given by the kernel of any epimorphism $\pi \to \mathbb{Z}_n$.*

[505]The reader might ask, why we do not try our luck with D_4. But for reasons we cannot get into right now, given a knot K and an even number $2n$ there cannot be an epimorphism $\pi_1(S^3 \setminus K) \to D_{2n}$.

PROOF (*). Let π be a group whose abelianization is isomorphic to $\mathbb{Z}$ and let $n \in \mathbb{N}$. We denote by $\psi\colon \pi \to \pi_{\mathrm{ab}}$ the obvious projection and we pick an isomorphism $\varphi\colon \pi_{\mathrm{ab}} \xrightarrow{\cong} \mathbb{Z}$. It is clear that $(\varphi \circ \psi)^{-1}(n\mathbb{Z})$ is a subgroup of π that is cocyclic of order n. On the other hand, suppose that Γ is a subgroup of π that is cocyclic of order n. We pick an isomorphism $\theta\colon \pi/\Gamma \cong \mathbb{Z}_n$. It follows from Proposition 21.20 that there exists a unique epimorphism $\alpha\colon \mathbb{Z} \to \mathbb{Z}_n$ such that the following diagram commutes:

$$\begin{tikzcd}
\pi \arrow[r, "\psi", "\cong"'] \arrow[d, two heads, "g\mapsto g\Gamma"'] & \pi_{\mathrm{ab}} \arrow[r, "\varphi", "\cong"'] & \mathbb{Z} \arrow[d, two heads, "\alpha"] \\
\pi/\Gamma \arrow[rr, "\theta", "\cong"'] & & \mathbb{Z}_n.
\end{tikzcd}$$

It follows immediately that $\Gamma = (\varphi \circ \psi)^{-1}(n\mathbb{Z})$. ■

Corollary 27.25 and Lemma 27.30 allow us to make the following definition.

Definition. Given a knot K and $n \in \mathbb{N}$ we denote by $\pi(K, n)$ the unique subgroup of $\pi_1(S^3 \setminus K)$ that is cocyclic of order n.

Example. If K is the trivial knot, then we saw in Proposition 27.4 that we can make an identification $\pi_1(S^3 \setminus K) = \mathbb{Z}$. It follows easily that for any $n \in \mathbb{N}$ we have $\pi(K, n) = n \cdot \mathbb{Z}$, in particular $\pi(K, n)$ is isomorphic to $\mathbb{Z}$.

We have now a new tool for trying to distinguish knots:

Lemma 27.31. *Let K and J be two knots. If K and J are smoothly isotopic, then given any $n \in \mathbb{N}$ the groups $\pi(K, n)$ and $\pi(J, n)$ are isomorphic. In particular the abelianizations of these groups are isomorphic.*

The question arises, given a knot K and $n \in \mathbb{N}$, how can we determine $\pi(K, n)$ from a given presentation for $\pi_1(S^3 \setminus K)$? Fortunately the following proposition gives us an explicit algorithm. To facilitate the uptake we formulate the following proposition in a slightly informal way.

Proposition 27.32. *Let $\pi = \langle x_1, \dots, x_m, t \,|\, r_1, \dots, r_l \rangle$ be a group and let $\phi\colon \pi \to \mathbb{Z}_n$ be an epimorphism such that $\phi(x_1) = \cdots = \phi(x_m) = 0$ and $\phi(t) = 1$. Then a presentation for the subgroup $\ker(\phi\colon \pi \to \mathbb{Z}_n)$ is given by*

$$\langle y_{1,0}, \dots, y_{1,n-1}, \dots, y_{m,0}, \dots, y_{m,n-1}, u \,|\, s_1^0, \dots, s_1^{n-1}, \dots, s_l^0, \dots, s_l^{n-1} \rangle$$

where each s_i^j is obtained as follows:

(1) *For $i = 1, \dots, l$ and $j = 0, \dots, n-1$ we write $t^j r_i t^{-j}$ as an expression in*
$$x_1, tx_1t^{-1}, \dots, t^{n-1}x_1t^{-n+1}, \dots, x_m, tx_mt^{-1}, \dots, t^{n-1}x_mt^{-n+1} \text{ and } t^n.$$

(2) *Then we replace each term $t^j x_i t^{-j}$ by $y_{i,j}$ and we replace t^n by u. We obtain a word in the new generators and denote it by s_i^j.*

The inclusion map $\ker(\phi\colon \pi \to \mathbb{Z}_n) \to \pi$ is then given by sending each $y_{i,j}$ to $t^j x_i t^{-j}$ and by sending u to t^n.

PROOF. As we had mentioned above, the proposition follows fairly easily from the more general Reidemeister-Schreier process that is explained in [**LS77**, Chapter II.4], [**MKS76**, Chapter 2.3] or [**Bog08**, Chapter 2.9]. ■

Instead of studying the proof of Proposition 27.32 our time is better spent trying to understand two explicit examples.

Example. Let $K \subset S^3$ be the trefoil. We write $\pi = \pi_1(S^3 \setminus K)$. In Proposition 27.5 we showed that we can make the identification $\pi = \langle x, t \mid \underbrace{x^3 \cdot t^{-2}}_{=r} \rangle$. By Lemma 21.13 there exists a unique epimorphism $\phi\colon \pi \to \mathbb{Z}_2$ with $\phi(x) = 0$ and $\phi(t) = 1$. Evidently $\pi(K, 2) = \ker(\phi\colon \pi \to \mathbb{Z}_2)$. Now we will use Proposition 27.32 to find a presentation $\langle y_0, y_1, u \mid s_0, s_1 \rangle$ for $\ker(\phi\colon \pi \to \mathbb{Z}_2)$. We write

$t^0 r t^{-0} = \underbrace{x^3}_{=y_0^3} \cdot \underbrace{t^{-2}}_{=u^{-1}}$ and more interestingly $t^1 r t^{-1} = t x^3 t^{-3} = (txt^{-1})^3 \cdot t^{-2} = (\underbrace{txt^{-1}}_{y_1^3})^3 \cdot \underbrace{t^{-2}}_{=u^{-1}}$.

Thus we see that

$$\pi(K,2) \;=\; \ker(\phi\colon \pi \to \mathbb{Z}_2) \underset{\substack{\uparrow \\ \text{Proposition 27.32}}}{=} \langle y_0, y_1, u \mid y_0^3 u^{-1}, y_1^3 u^{-1} \rangle \underset{\substack{\uparrow \\ \text{by the Tietze transformations from Lemma ??}}}{=} \langle y_0, y_1 \mid y_0^3 = y_1^3 \rangle.$$

Using Proposition 21.20 we can now easily compute that the abelianization of the above group $\pi(K, 2) = \ker(\phi\colon \pi \to \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_3$.

This calculation, together with Lemma 27.31 and the above discussion of the trivial knot, gives a new proof of Proposition 27.6.

The previous example was fairly straightforward since we had started out with a presentation where one generator got sent to one, and the other ones got sent to zero. Unfortunately this is not the case for Wirtinger presentations, so we will need to work somewhat harder. In the following example we consider the figure-8 knot K and we carry out the first steps needed to calculate the isomorphism type of $\pi(K, 2)$.

Example. Let J be the figure-8 knot. In Lemma 27.26 we saw that we can make the identification

$$\pi_1(S^3 \setminus J) \;=\; \langle y_1, y_2, y_3, y_4 \mid y_1^{-1} y_4 y_1 y_3^{-1}, y_3^{-1} y_2 y_3 y_1^{-1}, y_2^{-1} y_1 y_2 y_4^{-1} \rangle.$$

We denote by $\phi\colon \pi_1(S^3 \setminus J) \to \mathbb{Z}_2$ the epimorphism that is given by sending each y_i to 1. We set $t := y_1$. The goal is to replace the generators y_1, y_2, y_3 by generators x_1, x_2, x_3 which satisfy $\phi(x_i) = 0$. In fact we have

$$\begin{aligned}
\pi_1(S^3 \setminus J) \;&=\; \langle t, y_2, y_3, y_4 \mid t^{-1} y_4 t y_3^{-1}, y_3^{-1} y_2 y_3 t^{-1}, y_2^{-1} t y_2 y_4^{-1} \rangle \\
&=\; \langle t, y_2, y_3, y_4 \mid \underbrace{t^{-1} y_4 t^{-1} t t y_3^{-1}}_{=t^{-1} y_4 t y_3^{-1}}, \underbrace{t^{-1} t y_3^{-1} y_2 t^{-1} t y_3 t^{-1}}_{=y_3^{-1} y_2 y_3 t^{-1}}, \underbrace{t^{-1} t y_2^{-1} t y_2 t^{-1} t y_4^{-1}}_{=y_2^{-1} t y_2 y_4^{-1}} \rangle \\
&\underset{\substack{\uparrow \\ \text{substitution } x_i = y_i t^{-1},\ x_i^{-1} = t y_i^{-1}, \text{ see Lemma 21.12}}}{=} \langle t, x_2, x_3, x_4 \mid t^{-1} x_4 t x_3^{-1}, t^{-1} x_3^{-1} x_2 t x_3, t^{-1} x_2^{-1} t x_2 x_4^{-1} \rangle.
\end{aligned}$$

At this point we have a presentation to which we can apply Proposition 27.32. A heroic calculation shows that $\pi(J,2)_{\mathrm{ab}} \cong \mathbb{Z} \oplus \mathbb{Z}_5$. Thus we obtain a new proof that the figure-8 knot and the trefoil are not smoothly isotopic.

The above procedure for distinguishing knots works fairly well in practice. For example, with enough time at one's hand, or alternatively a convenient computer implementation, one can use the above methods to show that all the knots shown in Figure 512 are in fact pairwise different.

Nonetheless, the above approach is not overly satisfying. It would be much better to have invariants of knots that are much quicker to calculate. We will come back to this challenge at a much later point.

The above discussion, in particular Proposition 27.32 raises the following purely algebraic question.

Question 27.33. *Let π be a group with a given finite presentation and let $\phi\colon \pi \to G$ be an epimorphism onto a finite group.*

(1) *Is the kernel* $\ker(\phi\colon \pi \to G)$ *a finitely presented group?*
(2) *If the answer to* (1) *is yes, how can we determine a finite presentation for the kernel in terms of the given data?*

It turns out that the answer to the first question is yes.

Proposition 27.34. *If π is a finitely presented group and $\phi\colon \pi \to G$ is an epimorphism onto a finite group, then* $\ker(\phi\colon \pi \to G)$ *is also finitely presented.*

The proof of the proposition is given in most books on combinatorial group theory, see e.g. [**LS77**, Chapter II.4], [**MKS76**, Chapter 2.3] or [**Bog08**, Chapter 2.9]. In fact the proof is algorithmic, in particular it also gives an affirmative answer to the second question. The algorithm is called the *Reidemeister-Schreier process*. The general Reidemeister-Schreier process is somewhat messy to formulate. Thus we refer to the above literature for details.

Remark. In the proof of Proposition 27.6 we showed that the group $\langle x, t \,|\, x^3 \cdot t^{-2} \rangle$ admits an epimorphism α onto the permutation group S_3. In [**Bog08**, Example 2.9.3] the general Reidemeister-Schreier process is used to determine a presentation for the kernel of $\alpha\colon \langle x, t \,|\, x^3 \cdot t^{-2} \rangle \to S_3$.

27.8. **High-dimensional knots.** Since the study of knots in S^3 was so much fun, it would be a shame not to consider high-dimensional knots.

Definition. An *n-dimensional knot in S^k* is a smooth submanifold of S^k that is diffeomorphic to S^n. For $n = 1$ and $k = 3$ we say that such a knot is *classical*, when we allow any dimension then sometimes we say the knot is *high-dimensional*.

We start out with a boring example.

Example. Let $n, k \in \mathbb{N}$ with $k \geq n$. We refer to $U := \{(x, 0) \in S^{n+2} \subset \mathbb{R}^{k+1} \,|\, x \in S^n\}$ as the *trivial n-dimensional knot in S^k*.

Definition. We say an n-dimensional knot in S^k is *trivial* if it is smoothly isotopic to the trivial n-dimensional knot in S^k.

In Exercise 11.9 we showed, building on Proposition 11.11, that every n-dimensional knot in a sphere of dimension $\geq 2n+3$ is trivial. The following theorem is a refinement of this result.

Theorem 27.35. (Haefliger's (Un-)knotting Theorem)

(1) *Let $n, k \in \mathbb{N}$. If $2k > 3(n+1)$, then every n-dimensional knot in S^k is trivial.*
(2) *Given any $m \in \mathbb{N}$ there exists a $(4m-1)$-dimensional knot in S^{6m} that is non-trivial.*

Example. It follows immediately from Haefliger's Unknotting Theorem 27.35, and an elementary calculation, that for any $n \geq 2$ every n-dimensional knot in S^{2n+1} is trivial. As we saw in Proposition 27.6, this conclusion does not hold for $n = 1$.

Proof. The first statement was proved by André Haefliger [**Hae61a**, p. 47] in 1961. (An alternative proof is given in [**Wall16**, p. 192].) Shortly afterwards André Haefliger [**Hae62b**] also proved the second statement. ■

Remark. In the remainder of this section we will only consider knots of codimension two, i.e. n-dimensional knots in S^{n+2}. In principle one could also study embedded spheres of other codimensions. But first note that the Generalized Smooth Schönflies Theorem 50.4 implies that, at least for $n \neq 4$, codimension one knots in S^n are trivial. Also note that André Haefliger [**Hae67, Hae62a, Hae61b, Hae62b**] showed that in codimensions ≥ 3 there can be knots that are non-trivial. But Eric Zeeman [**Ze63c**, p. 505] and John Stallings [**Stal63**, Corollary 9.3] showed that knots of codimension ≥ 3 are "unknotted" in the "PL-category" and in the "topological category". Thus usually topologists tend to stick to knots of codimension two. Nonetheless, we will make use of knots of larger codimension in page **??**.

We continue with the following very unsurprising lemma.

Lemma 27.36. *Let $n \in \mathbb{N}$.*

(1) *Let K and L be two n-dimensional knots in S^{n+2}. If K and L are smoothly isotopic, then $\pi_1(S^{n+2} \setminus K) \cong \pi_1(S^{n+2} \setminus L)$.*
(2) *If K is a trivial n-dimensional knot in S^{n+2}, then $\pi_1(S^{n+2} \setminus K) \cong \mathbb{Z}$.*

Proof.

(1) The proof of this statement is verbatim the same as the proof of Proposition 8.43.
(2) Let K be a trivial n-dimensional knot in S^{n+2}. We see that

$$\pi_1(S^{n+2} \setminus K) \underset{\substack{\uparrow \\ \text{by (1)}}}{\cong} \pi_1(S^{n+2} \setminus U) \overset{\substack{\text{follows from Exercise 27.15 together with Proposition 18.16} \\ \downarrow}}{\cong} \pi_1(S^1) \underset{\substack{\uparrow \\ \text{by Corollary 16.18}}}{\cong} \mathbb{Z}.$$

■

Evidently the question now is whether there exist non-trivial high-dimensional knots. Or perhaps better, the question is whether we will manage to explicitly construct such examples. The good news is, we can indeed do so.

In the following we will present a construction, which has its origins in the work of Emil Artin [**Art25**] and Eric Zeeman [**Ze65**] that turns an n-dimensional knot into a family of $(n+1)$-dimensional knots. We make the following preparations:

(1) As on page 199 we make the identification $S^{n+2} = \overline{B}^{n+2}_+ \cup_{S^{n+1}} \overline{B}^{n+2}_-$.
(2) By Lemma 27.1 we can make the identification $S^{n+3} = (S^1 \times \overline{B}^{n+2}) \cup_{S^1 \times S^{n+1}} (\overline{B}^2 \times S^{n+1})$.
(3) We make the identification $\mathbb{R}^{n+2} = \mathbb{C} \times \mathbb{R}^n$. In particular we view $\overline{B}^{n+2}$ as a subset of $\mathbb{C} \times \mathbb{R}^n$.
(4) Given $z \in S^1$ we denote by

$$\begin{array}{rccc} \rho_z \colon & \overline{B}^{n+2} & \to & \overline{B}^{n+2} \\ & (w,x) & \mapsto & (z \cdot w, x) \end{array}$$

the rotation of $\overline{B}^{n+2}$ by the "angle" $z \in S^1$ around "the $0 \times \overline{B}^n$-axis".

Now let $K \subset S^{n+2}$ be a knot. It follows fairly easily from Theorem 8.36 and the fact that K is a smooth submanifold that, after possibly applying a diffeotopy of S^{n+2}, we can assume that $J_- := K \cap \overline{B}^{n+2}_- = \{(0,w) \,|\, w \in \overline{B}^n\}$ is the "trivial disk knot" in $\overline{B}^{n+2}$. We denote by $J_+ := K \cap \overline{B}^{n+2}_+$ the "other disk knot". Next, given $m \in \mathbb{Z}$ we denote by Φ_m the diffeomorphism

$$\begin{array}{rccc} \Phi_m \colon & S^1 \times \overline{B}^{n+2} & \to & S^1 \times \overline{B}^{n+2} \\ & (z,x) & \mapsto & (z, \rho_{z^m}(x)). \end{array}$$

Informally speaking Φ_m spins $\overline{B}^{n+2}$ altogether m times as we go around the S^1-direction. Note that Φ_0 is the identity and also note that each Φ_m leaves the subset $S^1 \times S^{n-1}$ invariant as a subset. Now we define the *m-twist spin* $S_m(K)$ as follows:

$$S_m(K) \; := \; \underbrace{\Phi_m(S^1 \times J_+)}_{\subset S^1 \times \overline{B}^{n+2}} \cup_{S^1 \times S^{n-1}} \underbrace{(\overline{B}^2 \times S^{n-1})}_{\subset \overline{B}^2 \times S^{n+1}} \; \subset \; \underbrace{(S^1 \times \overline{B}^{n+2}) \cup_{S^1 \times S^{n+1}} (\overline{B}^n \times S^{n+1})}_{= S^{n+3}}.$$

We leave it to the reader to verify that $S_m(K)$ is indeed an $(n+1)$-dimensional knot. In Figure 522 we do our best to illustrate this construction for $n = 1$. Again informally speaking, $S_m(K)$ is given by spinning the disk knot J_+ around the S^1-direction, performing m spins around J_+ as you go around S^1, and then capping off the result by $\overline{B}^2 \times S^{n-1}$.

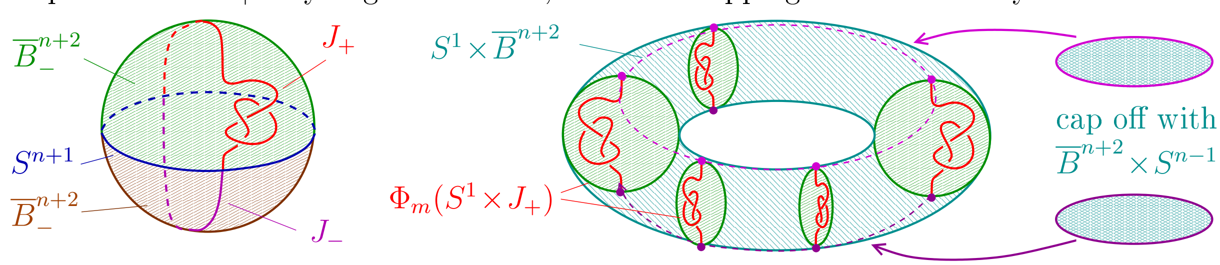


FIGURE 522

The following proposition will show shortly, perhaps unsurprisingly, that some twist spins are non-trivial. Perhaps more surprisingly the proposition also shows that (± 1)-twist spins are always trivial.

Proposition 27.37. *Let $K \subset S^{n+2}$ be an n-dimensional knot.*

(1) *For the 0-twist spin there exists an isomorphism $\pi_1(S^{n+3} \setminus S_0(K)) \cong \pi_1(S^{n+2} \setminus K)$.*

(2) *For every $\epsilon \in \{-1, 1\}$ the ϵ-twist spin knot $S_\epsilon(K)$ is trivial.*

Sketch of proof.

(1) We have the following isomorphisms:

$$\pi_1(S^{n+3} \setminus S_0(K)) \underset{\substack{\uparrow \\ \text{by the Seifert–van Kampen Theorem 22.2, we also use that } \Phi_0(S^1\times J_+) = S^1 \times J_+}}{\cong} \pi_1(S^1 \times (\overline{B}^{n+2} \setminus J_+)) *_{\pi_1(S^1\times(S^{n+1}\setminus S^{n-1}))} \pi_1(\overline{B}^2 \times (S^{n+1} \setminus S^{n-1}))$$

$$\downarrow \cong \quad \text{isomorphisms given by Proposition 16.20}$$

$$(\langle t\rangle \times \pi_1(\overline{B}^{n+2} \setminus J_+)) *_{\langle s\rangle \times \langle t\rangle} \langle s\rangle$$

$$\uparrow \cong \quad \text{isomorphism by the group theoretic fact that } \Gamma \xrightarrow{\cong} (\langle t\rangle \times \Gamma) *_{\langle s\rangle\times\langle t\rangle} \langle s\rangle$$

$$\pi_1(\overline{B}^{n+2} \setminus J_+)$$

$$\downarrow \cong \quad \text{isomorphism by Theorem 22.2}$$

$$\pi_1(S^{n+2} \setminus K).$$

By now the reader will be fully aware of the fact that it is supremely painful to completely keep track of all identifications, base points, induced maps and so on. Thus we also took some liberties to improve readability.

(2) This statement is proved implicitly in [**Ze65**, p. 486] and explicitly in [**FO15**, Corollary 2.1]. The reason for writing [**FO15**] was that it felt easier to prove the statement, than to try to read the proofs written down by others. So we warmly recommend to try to prove the statement on your own. ■

Corollary 27.38. *Given any $n \in \mathbb{N}$ there exists an n-dimensional knot in S^{n+2} that is non-trivial.*

Proof. Let $K \subset S^3$ be the trefoil and let $n \in \mathbb{N}$. We see that

$$\pi_1(S^{n+2} \setminus (n-1)\text{-st iterated 0-twist spin of } K) \underset{\substack{\uparrow \\ \text{Proposition 27.37}}}{\cong} \pi_1(S^3 \setminus K) \underset{\substack{\uparrow \\ \text{Proposition 27.5 and} \\ \text{Proposition 27.6}}}{\not\cong} \mathbb{Z}.$$

It now follows from Lemma 27.36 (2) that the $(n-1)$-st iterated 0-twist spin of the trefoil is a non-trivial knot. ■

Exercises for Chapter 27.

Exercise 27.1. Let $K \subset S^3$ be a knot and let H_1 and H_2 be two hyperplanes of $\mathbb{R}^4$. For $i = 1, 2$ we denote by K_i^* the reflection of K in the hyperplane H_i. Show that K_1^* and K_2^*

are smoothly isotopic.
Hint. Use Lemma 2.65 and Exercise 18.7.

Exercise 27.2. Given $k \in \mathbb{Z}$ we consider the map

$$\begin{array}{rcl} S^1 \times S^1 & \to & S^1 \times S^1 \\ (w,z) & \mapsto & (wz^k, z). \end{array}$$

Show that $(S^1 \times \overline{B}^2) \cup_{\varphi_k} (S^1 \times \overline{B}^2)$ is homeomorphic to S^3 if and only if $k \in \{-1, 1\}$.
Hint. It might be helpful to compute the fundamental group or the first homology group of this topological space.

Exercise 27.3. We consider the two knot diagrams shown in Figure 523. Use Reidemeister moves to show that the associated knots are smoothly isotopic.

FIGURE 523

Exercise 27.4. Show that every diagrammatic map has only finitely many double points.

Exercise 27.5. Let K be a knot. Use Proposition 27.23 to show that the abelianization of the group $\pi_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$.

Exercise 27.6. Let K and L be oriented knots. We denote by L^* the mirror image of L. Show that the fundamental groups of $S^3 \setminus (K\#L)$ and $S^3 \setminus (K\#L^*)$ are isomorphic.

Exercise 27.7. We defined the sign of a crossing of a diagram of an oriented knot. How does the sign change if we reverse the orientation of the knot?

Exercise 27.8. Show that if a knot diagram has at least one crossing, then the number of crossings equals the number of strands.

Exercise 27.9. Give a purely algebraic proof that the two groups $\langle x, y \mid x^2 = y^3 \rangle$ and $\langle a, b, c \,|\, c^{-1}bca^{-1}, b^{-1}abc^{-1} \rangle$ are isomorphic.
Remark. You could use the Tietze transformations introduced in Lemma **??** and you could use Lemma 21.12.

Exercise 27.10. Let $K \subset S^3$ be a knot. We consider the "cone"

$$C \;:=\; \{r \cdot P \,|\, P \in K \text{ and } r \in [0,1]\} \;\subset\; \overline{B}^4.$$

(a) Show that C is homeomorphic to the disk $\overline{B}^2$.
(b) We suppose that K is smoothly isotopic to the unknot. Show that C is a topological submanifold of the topological manifold $\overline{B}^4$.

Exercise 27.11. Let $K \subset S^3$ be the trefoil. As in Exercise 27.10 we consider the "cone"

$$C \;:=\; \{r \cdot P \,|\, P \in K \text{ and } r \in [0,1]\} \;\subset\; \overline{B}^4.$$

In Exercise 27.10 we shewed that C is homeomorphic to the disk $\overline{B}^2$. In the following we will show that C is not a submanifold of the topological manifold $\overline{B}^4$.

(a) Let $\alpha\colon \pi_1(S^3 \setminus K) \to \mathbb{Z}$ and $\beta\colon \mathbb{Z} \to \pi_1(S^3 \setminus K)$ be two maps. Show that the composition $\beta \circ \alpha\colon \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K)$ is not an isomorphism.

(b) Suppose that F is a proper 2-dimensional topological submanifold of $\overline{B}^4$. Show that given any point $P \in F \cap B^4$ there exists a map $\Phi\colon \overline{B}^4 \to \overline{B}^4$ that is an embedding such that $\Phi(0) = P$ and such that $\Phi(\overline{B}^2 \times \{0\}) = \Phi(\overline{B}^4) \cap C$.

(c) Show that C is not a *topological submanifold* of $\overline{B}^4$.
Hint. Evidently the idea is to apply (c) to the point $P = 0 \in C$. The difficulty is that the map Φ from (c) is not just a rescaling of the identity but it is potentially a completely different chart. To deal with the problem we introduce some notation.
  (i) Given $I \subset [0,1]$ we write $D_I := \{v \in \overline{B}^4 \,|\, \|v\| \in I\}$.
  (ii) Given $I \subset [0,1]$ we write $N_I := \Phi(D_I)$.
  (iii) Given $t \in (0,1]$ we write $D_t := D_{\{t\}}$ and $N_t := N_{\{t\}}$.
Show that the following statements hold:
  (1) Given any $t \in (0,1]$ there exists an $s \in (0,1)$ with $D_{[0,s]} \subset N_{[0,t)}$.
  (2) Given any $s \in (0,1]$ there exists a $t \in (0,1)$ with $N_{[0,t]} \subset D_{[0,s)}$.
  (3) Given any choice of $r < s$ in $(0,1]$ the two inclusion maps $D_r \backslash C \to D_{[r,s]} \backslash C$ and $D_s \backslash C \to D_{[r,s]} \backslash C$ are homotopy equivalences, in particular they induce isomorphism of fundamental groups. Each of the fundamental groups is isomorphic to $\pi_1(S^3 \backslash K)$.
  (4) Given any choice of $r < s$ in $(0,1]$ the two inclusion maps $N_r \backslash C \to N_{[r,s]} \backslash C$ and $N_s \backslash C \to N_{[r,s]} \backslash C$ are homotopy equivalences, in particular they induce isomorphism of fundamental groups. Each of the fundamental groups is isomorphic to $\mathbb{Z}$.
  (5) Use the above discussion together with (b) to show that such Φ cannot exist.

FIGURE 524. Illustration for Exercise 27.11.

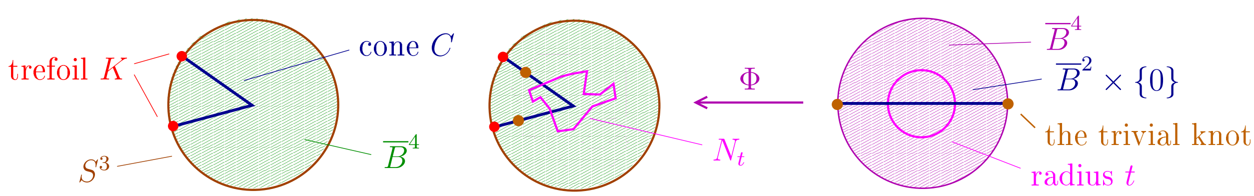


FIGURE 525. Second illustration for Exercise 27.11.

Exercise 27.12. Let K be the trefoil. Compute the abelianization of the group $\pi(K, 3)$.

Exercise 27.13. Let π be a group. We say that a subset $S \subset \pi$ *normally generates* π if $\langle\langle S \rangle\rangle = \pi$. We define the *weight of* π as

$$w(\pi) := \min\{\#S \mid S \text{ is a subset of } \pi \text{ that normally generates } \pi\} \in \mathbb{N}_0 \cup \{\infty\}.$$

(a) Determine the weight of $\mathbb{Z}^m$.

(b) Determine the weight of the free group on m generators.
(c) Let $K \subset S^3$ be a knot. Show that $w(\pi_1(S^3 \setminus K)) = 1$.

Exercise 27.14. We say two knots K and J in $\mathbb{R}^3$ are related by a *crossing change* if there exists a smooth map $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ and some $s \in (0,1)$ with the following properties:

(1) We have $F_0(S^1) = K$ and $F_1(S^1) = J$.
(2) For every $t \neq s$ the map $F_t\colon S^1 \to S^3$ is an embedding.
(3) The map $F_s\colon S^1 \to S^3$ is an immersion and there exist $w, z \in S^1$ with the following properties:
   (i) the restriction of F_s to $S^1 \setminus \{w, z\}$ is an injection,
   (ii) $F_s(w) = F_s(z)$ and the vectors $F_s'(w)$ and $F_s'(z)$ are linearly independent.

In Figure 526 we illustrate the definition of a crossing change. Given two knots K and

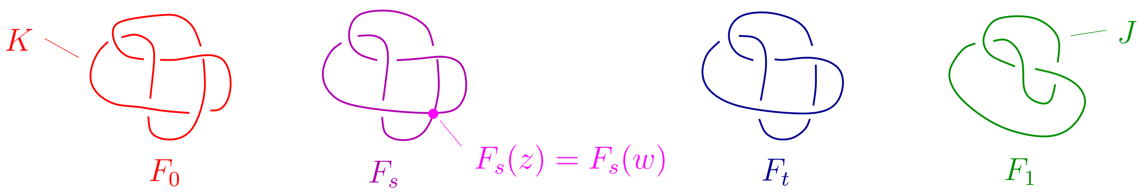


FIGURE 526. Illustration of Exercise 27.14.

J the minimal number of crossing changes needed to turn K into J is called the *Gordian distance between K and J*. The Gordian distance of a knot K to the trivial knot is called the *unknotting number of K*. Given explicit knots these numbers are notoriously difficult to calculate.

(a) Show that any two knots in $\mathbb{R}^3$ are related by a sequence of finitely many crossing changes. In other words, show that the Gordian distance between two knots is finite. *Hint.* You could use the fact that every knot admits a knot diagram.
(b) What are the best lower and upper bounds you can find on the Gordian distance between the trefoil and the figure-8 knot?

Remark. For readers who prefer completely formal proofs it can be a fun challenge to tweak the argument of the proof of Proposition 11.11 to obtain a proof of (a).

Exercise 27.15. Let $n \in \mathbb{N}$ and let $k \in \{1, \ldots, n-1\}$. We consider the submanifold $K := \{(x, 0) \in \mathbb{R}^{n+1} \mid x \in S^k\}$. Show that $S^n \setminus K$ is homotopy equivalent to S^{n-k-1}.

Exercise 27.16. Let $K \subset S^3$ be an oriented knot and let $m \in \mathbb{Z}$. As on page 805 we consider the m-twist spin $S_m(K) \subset S^4$. We view S^3 as a subset of S^4 via the embedding $z \mapsto (z, 0)$. The intersection $S_m(K) \cap S^3$ is a knot. What is the smooth isotopy type of this knot? A possible answer is that this knot is a connected sum of K with some suitable other knot.

28. Decision problems

In the last chapters we had developed several techniques for determining the fundamental groups of topological spaces. In particular we managed to find finite presentations for the fundamental groups of large classes of topological spaces, e.g. topological graphs, surfaces, knot complements and mapping tori. In fact we now have enough tools to determine, with enough effort and perseverance, a presentation for the fundamental groups of most "nice" spaces. Furthermore we will see later in Propositions 64.6 and 85.13 on that the fundamental group of every compact topological manifold admits a *finite* presentation.

The question that now arises is, what information can we extract from a finite presentation of a group? Here are a couple of natural questions.

Question 28.1.

(1) *Does there exist an algorithm that can decide whether or not a given finite presentation* $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ *represents the trivial group?*
(2) *Does there exist an algorithm that can decide whether or not two given finite presentations* $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ *and* $\Gamma = \langle h_1, \dots, h_m \mid s_1, \dots, s_n \rangle$ *represent isomorphic groups?*[506]
(3) *Does there exist an algorithm that can determine, given*
   (a) *a finite presentation* $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ *and*
   (b) *a word* w *in* $g_1, \dots, g_k$*, i.e. an element* $w \in \langle g_1, \dots, g_k \rangle$*,*
   whether or not w *represents the trivial element in the group* π*?*[507]
(4) *Does there exist an algorithm that can determine, given*
   (a) *a finite presentation* $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ *and*
   (b) *and two words* v *and* w *in* $g_1, \dots, g_k$*,*
   whether or not v *and* w *represent conjugate elements in the group* π*?*[508]

Remark.

(1) This question was first posed by Max Dehn [**Deh1911**] in 1911. In fact shortly afterwards, in 1912 Max Dehn [**Deh1912**] gave positive answers to the word problem and the conjugacy problem for the fundamental groups of compact 2-dimensional smooth manifolds.
(2) The above question is formulated in a slightly informal way. After all, what is an "algorithm" supposed to be? The questions are made more precise in [**CZ93**, Chapter 7], [**Sti82**] and [**MillC92**].

On page 632 we introduced the Tietze transformations on presentations. We pointed out that two finite presentations for a given group are in fact related by a finite sequence of Tietze transformations. In particular, if $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ is a presentation for the trivial group, then we can turn the presentation after finitely many Tietze transformations into the trivial presentation $\langle | \rangle$. The problem is that we do not know how many Tietze transformations are required to turn a presentation of a trivial group into a

[506]The question whether such an algorithm exists is often called the *isomorphism problem*.
[507]The question whether such an algorithm exists is often called the *word problem*.
[508]The question whether such an algorithm exists is often called the *conjugacy problem*.

trivial presentation. In particular at times it can be very hard to find the correct Tietze transformations.

Example. We consider the presentation

$$\pi \;=\; \langle a, b, c \mid a^3, b^3, c^4, ac = ca^{-1}, aba^{-1} = bcb^{-1} \rangle.$$

This is a presentation of the trivial group as can be seen as follows:[509]

(1) from $aba^{-1} \sim bcb^{-1}$ we get $(aba^{-1})^3 \sim (bcb^{-1})^3$,
(2) since $a^{-1}a \sim 1$ and $b^{-1}b \sim 1$ it follows that $(aba^{-1})^3 \sim ab^3a^{-1}$ and $(bcb^{-1})^3 \sim bc^3b^{-1}$,
(3) we have $b^3 \sim 1$, hence $ab^3a^{-1} \sim 1$, therefore we get from (1) and (2) that $bc^3b^{-1} \sim 1$ hence $c^3 \sim 1$,
(4) we have $c^3 \sim 1$ but we also have the relation $c^4 \sim 1$, so $c \sim 1$,
(5) since $aba^{-1} \sim bcb^{-1}$ it follows that $aba^{-1} \sim 1$ and so $b \sim 1$,
(6) from $ac \sim ca^{-1}$ and $c \sim 1$ it follows that $a \sim a^{-1}$, hence $a^2 \sim 1$,
(7) together with $a^3 \sim 1$ we obtain $a \sim 1$.
(8) we now showed that a, b, c are all equivalent to 1, thus we see that the group defined by the presentation is the trivial group.

This example shows that if a presentation corresponds to the trivial group, then it can be surprisingly complicated to verify this statement.

Now suppose we are given a finite presentation and we have to decide whether or not it is a presentation of the trivial group. We can modify the presentation using Tietze transformations. If at some point we end up with the trivial presentation we know that the given presentation corresponds to the trivial group. But what happens if after many attempts at modifying the presentation using Tietze transformations we still do not have the trivial presentation. Does that mean that the presentation corresponds to a non-trivial group? Or does it just mean that we did not try long enough?

So the real question is, does there exist an algorithm, that given a presentation which defines a non-trivial group, actually verifies that the group is non-trivial?

Unfortunately the answer is quite sobering.

Theorem 28.2. *The answer to all four questions raised in Question 28.1 is no.*

Remark. Here “no” means that one can show that no such algorithm exists. This is a much stronger statement than saying that we do not know of an example of such an algorithm.

Proof.

(1) Adyan [**Ady55**] and Michael Rabin [**Rabi58**] showed in 1955 that the answer to the first question is no. Alternatively see [**MillC92**, Theorem 3.3] for a proof.
(2) An algorithm that performs (2) would also give an algorithm for determining whether or not a given presentation is isomorphic to the trivial group with the empty presentation $\langle | \rangle$. But since the answer to (1) is no, it follows that the answer to (2) is also no. Historically, it was first proved by Sergey Novikov [**Nov65**] and William Boone [**Boo58**] that (2) has a negative answer before a negative answer to (1) was given.

[509] For elements g, h of $\langle a, b, c\rangle$ we write $g \sim h$ if they represent the same element in π.

(3) If there did exist an algorithm that dealt with the third problem, then we could apply it in particular to the generators. Since a presentation presents the trivial group if and only if all generators are trivial, it follows that an algorithm for (3) would also give an algorithm for (1). Since the latter cannot exist, the former cannot exist either.
(4) An element in a group is trivial if and only if it is conjugate to the trivial element. Hence the fact that (3) has a negative answer leads to a negative answer to (4). ■

Remark.

(1) There are many other non-existence results in the vein of Theorem 28.2. For example, [**MillC92**, Corollary 3.4], see also Exercise 28.2 says the following:
   (a) it is undecidable whether a finitely presented group is free,
   (b) it is undecidable whether a finitely presented group is abelian,
   (c) it is undecidable whether a finitely presented group is nilpotent,
   (d) it is undecidable whether a finitely presented group is solvable,
   (e) it is undecidable whether a finitely presented group is simple,
   (f) it is undecidable whether a finitely presented group is torsion-free.
(2) The statement of Theorem 28.2 that the answer to all four questions of Question 28.1 is no, is of course somewhat disappointing. But this raises the question whether there are interesting classes of groups for which the answer is actually yes. For example Mikhail Gromov [**Grom87**][510] introduced the notion of a "word hyperbolic group" for which the answer to all four questions turns out to be yes. Examples of word hyperbolic groups are given by
   (a) finite groups,
   (b) free groups, fundamental groups of surfaces of genus ≥ 2,
   (c) fundamental groups of hyperbolic smooth manifolds of any dimension,
   (d) fundamental groups of "Riemannian manifolds with negative sectional curvature".

   In fact in a precise sense a "random" finitely presented group is word hyperbolic, see e.g. [**Ol05**] for more details.

   The topic of word hyperbolic groups is a very important topic of mathematics in its own right. See e.g. [**Lö15**] for an introduction to word hyperbolic groups.

Of course our main goal is not to classify groups, but our goal is to understand "nice" topological spaces. For example ideally we would like to classify compact topological and smooth manifolds. The following theorem gives us some hope that this task is not completely hopeless.

Theorem 28.3. *Let $n \in \mathbb{N}_0$.*

(1) *There exist only countably many homeomorphism classes of compact n-dimensional topological manifolds.*

[510] Mikhail Gromov (1943-) is a French-Russian mathematician famous for his many contributions to geometry and group theory. He was the first person to view groups as "geometric objects", giving rise to the subject of "geometric group theory".

(2) *There exist only countably many diffeomorphism classes of compact n-dimensional smooth manifolds.*

PROOF. The first statement is proved in [**CK70**]. The second statement follows from [**Pets84**, p. 77] and the fact that every smooth manifold admits a Riemannian structure, see e.g. [**Wall16**, Theorem 1.3.1]. For $n \geq 6$ the second statement can also be deduced from the fact that every compact n-dimensional smooth manifold admits a finite simplicial structure, see Theorem 64.2, the fact that there are only countably many isomorphism classes of finite simplicial complexes, see Exercise 61.4, and the (very difficult) Theorem **??** which says that for $n \geq 6$ every compact topological manifold admits at most finitely many smooth structures. ■

Given $n \in \mathbb{N}$ we would like to classify closed orientable connected n-dimensional smooth manifolds up to diffeomorphism. For $n = 1$ we did so in Theorem 7.5 and for $n = 2$ we saw the classification in the Surface Classification Theorem 23.4. In particular we saw that in these dimensions only "few groups" appear as fundamental groups.

We also proved the following result which shows that in dimension ≥ 4 the situation is dramatically different.

Proposition 22.10. *Given any finitely presented group π and any $n \geq 4$ there exists a closed orientable connected n-dimensional smooth manifold with fundamental group π.*

Theorem 28.2 and Proposition 22.10 are at the heart of the proof of the following theorem.

Theorem 28.4. *Let $n \geq 4$.*

(1) *Then there is no algorithm that can decide whether or not two closed orientable n-dimensional smooth manifolds are homeomorphic.*
(2) *Then there is no algorithm that can decide whether or not two closed orientable n-dimensional smooth manifolds are diffeomorphic.*

PROOF. The first statement was first proved by Markov [**Mark58**]. We refer to [**BHP68**, Theorem 1] and [**Hak73**] for a detailed discussion of the proofs of both statements. ■

This leaves us with the case of 3-dimensional smooth manifolds. It turns out that in this dimension there does in fact exist such an algorithm.

Theorem 28.5. *There exists an algorithm that can decide whether or not two closed orientable connected 3-manifolds are diffeomorphic.*

In this case the algorithm is considerably more complicated than in the 1-dimensional and the 2-dimensional case. The proof that such an algorithm exists is due to the work of many mathematicians, first and foremost William Thurston [**Thu82**] and Grigori Perelman. The proof of the theorem was completed about 2003. It is impossible to mention here all the work that goes into proving Theorem 28.5. We refer to the survey paper [**AFW15**, Theorem 4.27] for details and precise references.

Exercises for Chapter 28.

Exercise 28.1.

(a) Show that every finite group is isomorphic to some permutation group S_n.
(b) Let $\pi = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ be a finitely presented group. Suppose that we know that π is finite. Show that there exists an algorithm which determines an injective homomorphism $\varphi \colon \pi \to S_n$ for some $n \in \mathbb{N}$.
(c) Let $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and $\Gamma = \langle h_1, \dots, h_m \mid s_1, \dots, s_n \rangle$ be two finitely presented groups. Suppose we know that π and Γ are finite. Show that there exists an algorithm that can determine whether or not π and Γ are isomorphic.

Exercise 28.2.

(a) Show that it is undecidable whether a finitely presented group is free.
(b) Show that it is undecidable whether a finitely presented group is abelian.
(c) Show that it is undecidable whether a finitely presented group is nilpotent.
(d) Show that it is undecidable whether a finitely presented group is solvable.

Somewhat more precisely, for each group property show that there is no algorithm that can decide whether or not a given finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ represents a group that has said property.

Exercise 28.3. A group π is called *residually finite* if given any non-trivial element $g \in \pi$ there exists a homomorphism $\alpha \colon \pi \to G$ to a finite group G such that $\alpha(g)$ is non-trivial. Show that the word problem is solvable in a finitely generated group that is residually finite.
Remark. A solution to this exercise can be found in [**MillC92**, Theorem 5.2].

Part IV

The universal covering

After spending a lot of time on determining the fundamental groups of topological spaces we now want to return to the study of covering maps. One key goal of this section is to address the following question.

Question 29.1. *Does every path-connected topological space B admit a path-connected covering $p\colon X \to B$ such that X is simply connected?*

We know of course that the answer is yes for the torus, the Klein bottle and the Möbius band. But does such a covering exist for the surface of genus 2? Or does it exist for the complement $S^3 \setminus K$ of a knot $K \subset S^3$? How about the wedge $S^1 \vee S^1$ of two circles?

29.1. **Local properties of topological spaces.** Before we can address Question 29.1 we need to recall the definition of a local property of topological spaces that we already gave in Section 2.11.

Definition. Let P be a property of topological spaces. We say a topological space X is *locally* P if given any $Q \in X$ and any neighborhood U of Q there exists an open neighborhood V of Q that is contained in U that has the property P.

Examples.

(1) By Proposition 18.16 we know that every contractible topological space is simply connected. Thus every locally contractible topological space is also locally simply connected. Furthermore every simply connected topological space is by definition path-connected, thus every locally simply connected topological space is also locally path-connected.
(2) On page 139 we saw that every open subset of $\mathbb{R}^n$ is locally simply connected.
(3) By Lemma 6.9 (1) every topological manifold is locally homeomorphic to some non-empty convex subset of $\mathbb{R}^n$. Since non-empty convex subsets are contractible we see that every topological manifold is locally contractible.
(4) In Exercise 29.1 we will show that every topological graph is locally contractible and we will show that the wedge of locally contractible topological spaces is again locally contractible.
(5) We consider
$$X \;:=\; \mathbb{R}^2 \;\setminus\; \bigcup_{n=1}^{\infty} \tfrac{1}{n} \times [-1,1],$$
with the usual subspace topology coming from $\mathbb{R}^2$. Then X is path-connected but not locally path-connected. Indeed, we consider the point $(0,0) \in X$ and the neighborhood $U = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2}) \cap X$. It is straightforward to show that there does not exist a path-connected neighborhood V of $(0,0)$ which is contained in U. We refer to Figure 527 for an illustration.
(6) We consider
$$X \;:=\; \mathbb{R}^2 \;\setminus\; \bigcup_{n=1}^{\infty} \big(\tfrac{1}{n}, 0\big),$$
with the subspace topology coming from $\mathbb{R}^2$. Then X is path-connected, locally path-connected but it is not locally simply connected. Indeed the origin admits no

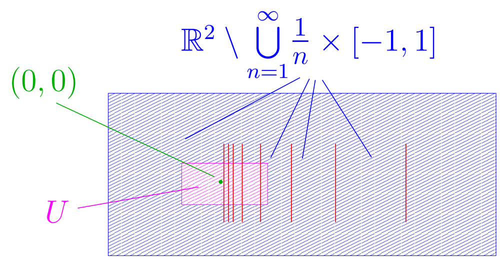


FIGURE 527

neighborhood of the desired form, since any neighborhood of the origin contains "infinitely many holes". We made this statement precise in Exercise 20.1.

29.2. **Lifting maps to coverings.** Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Furthermore let $f\colon (Z, z_0) \to (B, b_0)$ be a map. Recall that on page 507 we said that f *lifts*, if there exists a *lift* $\widetilde{f}\colon (Z, z_0) \to (X, x_0)$, i.e. a map $\widetilde{f}\colon Z \to X$ with $\widetilde{f}(z_0) = x_0$ and which makes the following diagram commute

$$\begin{tikzcd} & (X, x_0) \arrow[d, "p"] \\ (Z, z_0) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (B, b_0). \end{tikzcd}$$

For example, if $f\colon ([0,1], 0) \to (B, b_0)$ is a map from the interval $[0,1]$ to B, i.e. if f is a path with starting point b_0, then we saw in Proposition 16.11 that there exists a unique lift of f to a map $\widetilde{f}\colon ([0,1], 0) \to (X, x_0)$.

Example. Consider the following maps:

$$\begin{tikzcd} & (S^1, 1) \arrow[d, "p"', "z \mapsto z^3"] \\ (S^1, 1) \arrow[ur, dashed, "???\ \text{lift}\ \widetilde{f}\,???"] \arrow[r, "z \mapsto z^2", "f"'] & (S^1, 1). \end{tikzcd}$$

Here the map p is a 3-fold covering. Does the map f lift to a map $\widetilde{f}$?

To answer the question we need to study lifts of maps a little more carefully. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces and let $f\colon (Z, z_0) \to (B, b_0)$ be a map. Suppose that there exists a lift $\widetilde{f}\colon (Z, z_0) \to (X, x_0)$. We then obtain the commutative diagram

$$\begin{tikzcd} & (X, x_0) \arrow[d, "p"] \\ (Z, z_0) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (B, b_0) \end{tikzcd}$$

from the functoriality of the fundamental group it follows that $f_* = \big(p \circ \widetilde{f}\big)_* = p_* \circ \widetilde{f}_*$, i.e. we get the commutative diagram

$$\begin{tikzcd} & \pi_1(X, x_0) \arrow[d, "p_*"] \\ \pi_1(Z, z_0) \arrow[ur, "\widetilde{f}_*"] \arrow[r, "f_*"'] & \pi_1(B, b_0). \end{tikzcd}$$

Thus it follows that if f lifts, then

$$\operatorname{im}(f_*) \;=\; \operatorname{im}\left(\left(p\circ\widetilde{f}\right)_*\right) \;=\; \operatorname{im}\left(p_*\circ\widetilde{f}_*\right) \;\subset\; \operatorname{im}(p_*).$$

So summarizing, if there exists a lift of f, then

$$\operatorname{im}(f_*\colon \pi_1(Z,z_0)\to\pi_1(B,b_0)) \;\subset\; \operatorname{im}(p_*\colon \pi_1(X,x_0)\to\pi_1(B,b_0)).$$

Example. In the above example we have $\pi_1(S^1,1)=\mathbb{Z}$ and $\operatorname{im}(f_*)=2\mathbb{Z}$ whereas we have $\operatorname{im}(p_*)=3\mathbb{Z}$. So we see that f does not lift.

The following proposition says, that the above conclusion has a converse, for “reasonably” topological spaces. More precisely, we have the following proposition which, under a modest technical hypothesis, gives a complete criterion for when a map lifts.

Proposition 29.2. *Let $p\colon (X,x_0)\to(B,b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected topological space and let $f\colon (Z,z_0)\to(B,b_0)$ be a map.*

(1) *If Z is locally path-connected, then we have the following equivalence of statements:*

$$\begin{tikzcd} & (X,x_0) \arrow[d, "p"] \\ (Z,z_0) \arrow[r, "f"'] \arrow[ur, dashed, "\text{there exists a lift } \widetilde{f}"] & (B,b_0) \end{tikzcd} \quad\Longleftrightarrow\quad \operatorname{im}(f_*)\subset\operatorname{im}(p_*).$$

(2) *There exist at most one such lift, in other words, any two such lifts agree.*[511]

Examples. Let $p\colon (X,x_0)\to(B,b_0)$ be a covering of pointed topological spaces.

(1) The condition on the fundamental groups in Proposition 29.2 is automatically satisfied if Z is simply connected.

(2) Now let $f\colon (S^n,s_0)\to(B,b_0)$ be a map. The topological manifold S^n is path-connected and locally path-connected. By Proposition 14.14 the sphere S^n is simply connected if $n\ge 2$. Thus it follows from (1) that there exists a uniquely determined lift $\widetilde{f}\colon (S^n,s_0)\to(X,x_0)$ of the map $f\colon (S^n,s_0)\to(B,b_0)$.

In the proof of Proposition 29.2 we will need the following lemma.

Lemma 29.3. *Let $p\colon (X,x_0)\to(B,b_0)$ be a covering of pointed spaces, let γ be a path in B with starting point b_0 and let δ be another path in B with $\delta(0)=\gamma(1)$. We denote by $\widetilde{\gamma}$ the lift of γ with starting point x_0. Then*

$$\begin{array}{c}\textit{endpoint of the lift of } \gamma*\delta\\ \textit{to the starting point } x_0\end{array} \;=\; \begin{array}{c}\textit{endpoint of the lift of } \delta\\ \textit{to the starting point } \widetilde{\gamma}(1).\end{array}$$

Proof. Let $p\colon (X,x_0)\to(B,b_0)$ be a covering of pointed spaces, let γ be a path in B with starting point b_0 and let δ be another path in B with $\delta(0)=\gamma(1)$. We denote by $\widetilde{\gamma}$ the lift of γ with starting point x_0 and we denote by $\widetilde{\delta}$ the lift of δ to the starting point $\widetilde{\gamma}(1)$. Finally we denote by $\widetilde{\gamma*\delta}$ the lift of $\gamma*\delta$ to the starting point x_0. For $t\in[0,1]$ we write $\beta(t)=(\gamma*\delta)(\frac{1}{2}(t+1))$ and $\widetilde{\beta}(t)=(\widetilde{\gamma*\delta})(\frac{1}{2}(t+1))$. Note that by definition we

[511] Note that for (2) we do not need the annoying hypothesis that Z is locally path-connected.

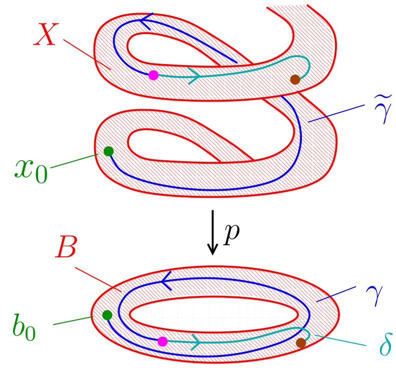


FIGURE 528. Illustration of Lemma 29.3

have $\beta(t) = \delta(t)$ for $t \in [0,1]$ and that $\widetilde{\beta}$ and $\widetilde{\delta}$ are both lifts of $\beta = \delta$ to the starting point $\beta(0) = \gamma(1) = \delta(0)$. Thus it follows that

$$\text{endpoint of } \widetilde{\gamma * \delta} \;=\; (\widetilde{\gamma * \delta})(1) \;=\; \widetilde{\beta}(1) \underset{\substack{\uparrow \\ \text{by Proposition 16.11}}}{=} \widetilde{\delta}(1) \;=\; \text{endpoint of } \widetilde{\delta}.$$

■

Now we can give the proof of Proposition 29.2. To make the proof slightly more readable we prove the two statements (1) and (2) separately.

PROOF OF PROPOSITION 29.2 (1). Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected and locally path-connected topological space and let $f\colon (Z, z_0) \to (B, b_0)$ be a map.

By the discussion on page 818 we only have to prove the "$\Longleftarrow$"-direction of the proposition. So suppose that we have

$$\operatorname{im}(f_*\colon \pi_1(Z, z_0) \to \pi_1(B, b_0)) \;\subset\; \operatorname{im}(p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)).$$

We have to construct a lift $\widetilde{f}\colon (Z, z_0) \to (X, x_0)$. We will do so as follows. Let $z \in Z$ be a point. Since Z is path-connected we can find a path $\alpha\colon [0,1] \to Z$ with $\alpha(0) = z_0$ and $\alpha(1) = z$. According to Proposition 16.11 we can lift the path $f \circ \alpha\colon [0,1] \to B$ to a path $\widetilde{(f \circ \alpha)}\colon [0,1] \to X$ with starting point x_0. We define

$$\widetilde{f}(z) \;:=\; \widetilde{(f \circ \alpha)}(1).$$

This definition is illustrated in Figure 529.

We have to verify the following statements:

(1) the map $\widetilde{f}\colon Z \to X$ is well-defined,
(2) the map $\widetilde{f}$ is continuous,
(3) the map $\widetilde{f}$ is a lift.
(4) and the map $\widetilde{f}$ is the unique lift.

We will do so in the following:

(1) We have to show that for $z \in Z$ the point $\widetilde{f}(z) \in X$ is well-defined, i.e. independent of the choice of the path α. Thus let $\beta\colon [0,1] \to Z$ be another path with $\beta(0) = z_0$

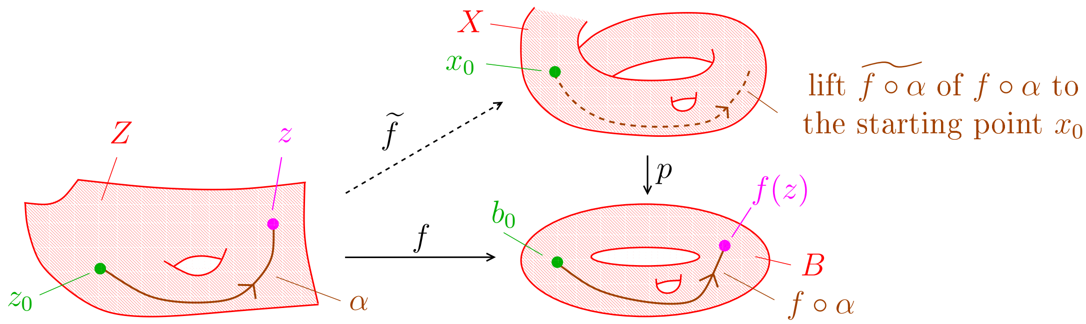


FIGURE 529

and $\beta(1) = z$. (This situation is illustrated in Figure 530.) It follows that

$$
\begin{aligned}
(\widetilde{f \circ \alpha})(t=1) &= (\text{lift of } f \circ \alpha \text{ to the starting point } x_0)(t=1) \\
&\underset{\uparrow}{=} (\text{lift of } (f \circ \alpha) * \overline{(f \circ \beta)} * (f \circ \beta) \text{ to the starting point } x_0)(t=1)
\end{aligned}
$$

by Corollary 16.13 since $f \circ \alpha$ and $(f \circ \alpha) * \overline{(f \circ \beta)} * (f \circ \beta)$ are path-homotopic

$$
\begin{aligned}
&= (\text{lift of } (f \circ (\underbrace{\alpha * \overline{\beta}}_{=:\gamma, \text{ a loop in } (Z, z_0)})) * (f \circ \beta) \text{ to the starting point } x_0)(t=1) \\
&\underset{\uparrow}{=} (\text{lift of } f \circ \beta \text{ to the starting point } (\widetilde{f \circ \gamma})(1))(t=1)
\end{aligned}
$$

Lemma 29.3, where $\widetilde{f \circ \gamma}$ denotes the lift of $f \circ \gamma$ to the starting point x_0

$$
\underset{\uparrow}{=} (\text{lift of } f \circ \beta \text{ to starting point } x_0)(t=1) \;=\; (\widetilde{f \circ \beta})(t=1).
$$

by our hypothesis we have $[f \circ \gamma] = f_*([\gamma]) \in \operatorname{im}(f_*) \subset p_*(\pi_1(X, x_0))$,
hence $\widetilde{f \circ \gamma}$ is a loop in x_0 by Lemma 16.15[512]

Thus we have shown that the map $\widetilde{f}$ is well-defined.

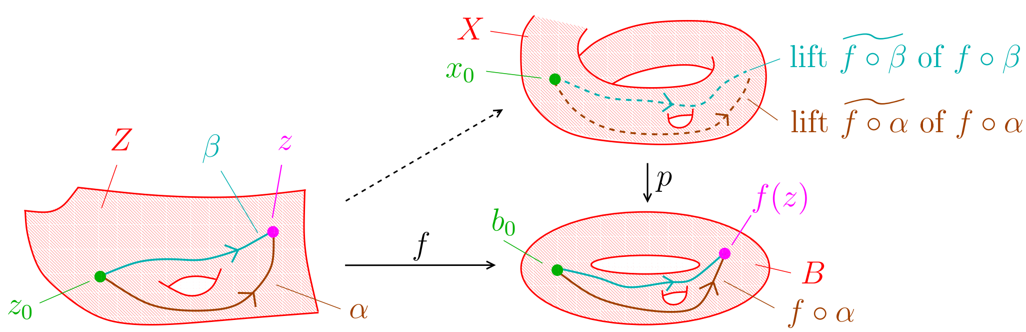


FIGURE 530

[512]Lemma 16.15 (1) says the following: if δ is a loop in (B, b_0) and if $\widetilde{\delta}$ denotes the lift of δ to the starting point x_0, then the following holds:

$$\widetilde{\delta} \text{ is a loop in } (X, x_0) \quad \Longleftrightarrow \quad [\delta] \text{ lies in } p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0).$$

We apply the lemma to $\delta = f \circ \gamma$.

(2) Now we want to show that $\widetilde{f}$ is continuous. So let $z \in Z$. We pick a path α from z_0 to z. Since p is a covering there exists an open neighborhood V of $\widetilde{f}(z)$ such that the restriction of p to V is an embedding. We denote by $q\colon p(V) \to V$ the inverse map. Since Z is locally path-connected there exists an open neighborhood U of z that is contained in $f^{-1}(p(V))$ and that is path-connected. Since f and q are continuous it suffices to prove the following claim.

Claim. On the open neighborhood U of z the map $\widetilde{f}$ agrees with $q \circ f$.

So let $w \in U$. Since U is path-connected there exists a path δ from z to w that lies in U. Then

$$\begin{aligned}
\widetilde{f}(w) &= \text{endpoint of the lift of } f\circ(\alpha*\delta) = (f\circ\alpha)*(f\circ\delta) \text{ to the starting point } x_0\\
&\underset{\substack{\uparrow\\ \text{by Lemma 29.3 and the definition of } \widetilde{f}(z) \text{ as the endpoint of } \widetilde{f\circ\alpha}}}{=} \text{endpoint of the lift of } f\circ\delta \text{ to the starting point } \widetilde{f}(z)\\
&\underset{\substack{\uparrow\\ \text{since } q=p^{-1} \text{ we know that } q\circ(f\circ\delta) \text{ is the lift of } f\circ\delta \text{ to the starting point } f(z)}}{=} \text{endpoint of } q\circ(f\circ\delta) = (q\circ f)(\delta(1)) = (q\circ f)(w).
\end{aligned}$$

(3) We need to show that $\widetilde{f}$ is a lift, i.e. we need to show that $p\circ\widetilde{f} = f$. So let $z \in Z$. Then

$$p\big(\widetilde{f}(z)\big) \underset{\substack{\uparrow\\ \text{pick path } \alpha \text{ from } z_0 \text{ to } z \text{ and denote by } \widetilde{f\circ\alpha} \text{ the lift of } f\circ\alpha}}{=} p\big(\big(\widetilde{f\circ\alpha}\big)(1)\big) = (f\circ\alpha)(1) = f(\alpha(1)) = z.$$

■

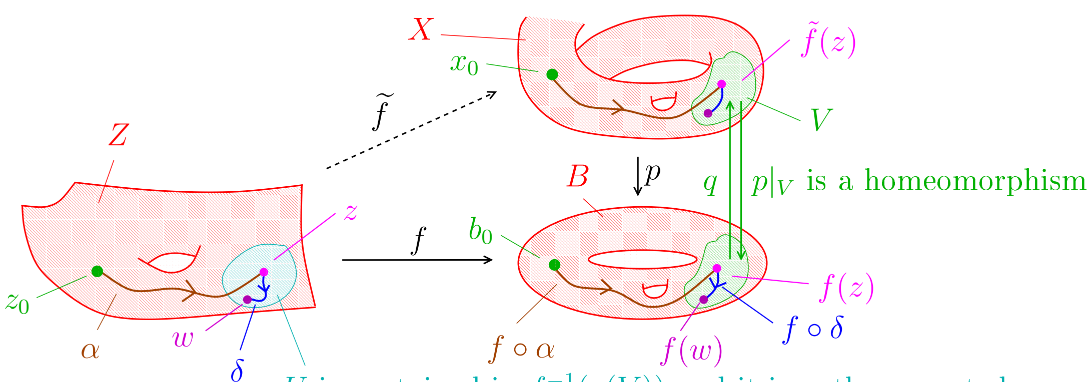


FIGURE 531. Illustration for the proof of Proposition 29.2 (1).

PROOF OF PROPOSITION 29.2 (2). Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected topological space. Finally suppose we are given a map $f\colon (Z,z_0) \to (B,b_0)$.

We will now see, using the uniqueness of the lift of paths established in Proposition 16.11, that any two lifts $\widetilde{f},\widetilde{f}'\colon (Z,z_0) \to (X,x_0)$ agree. Thus let $z \in Z$. Since Z is path-connected there exists a path $\alpha\colon [0,1] \to Z$ from z_0 to z. Then $f\circ\alpha\colon [0,1] \to B$

is a path and $\widetilde{f} \circ \alpha$ and $\widetilde{f'} \circ \alpha$ are both lifts of $f \circ \alpha$ to the starting point x_0. By Proposition 16.11 we have $\widetilde{f} \circ \alpha = \widetilde{f'} \circ \alpha$. In particular the endpoints agree. But this implies the desired statement that $\widetilde{f}(z) = (\widetilde{f} \circ \alpha)(1) = (\widetilde{f'} \circ \alpha)(1) = \widetilde{f'}(z)$. ■

In Exercise 29.3 we will prove the following corollary.

Corollary 29.4. *Let $\pi \colon X \to B$ be a covering of topological spaces. Then every open simply connected and locally path-connected subset $U \subset B$ is uniformly covered.*

29.3. **Semi-locally simply connected topological spaces.** Our next goal will be to show that every "reasonable" path-connected non-empty topological spaces admits a covering that is simply connected. To make precise what we mean by "reasonable" we need to introduce the following definition.

Definition. A topological space X is called *semi-locally simply connected*[513]if for every point $x \in X$ there exists an open[514] neighborhood U of x such that the inclusion induced map $\pi_1(U, x) \to \pi_1(X, x)$ is the trivial map.

Examples.

(1) Every simply connected topological space is semi-locally simply connected.
(2) Let H be the Hawaiian earring that we introduced on page 727. In Exercise 29.2 we will see that H is not semi-locally simply connected.
(3) Evidently every topological space that is locally simply connected is also semi-locally simply connected. But the converse does not hold. For example let us consider the cone(H) on the Hawaiian earring.
   (a) By Lemma 24.1 we know that the tip of cone(H) is a deformation retract. This implies that cone(H) is simply connected, in particular it is semi-locally simply connected.
   (b) On the other hand we will see in Exercise 29.2 that cone(H) is not locally simply connected.

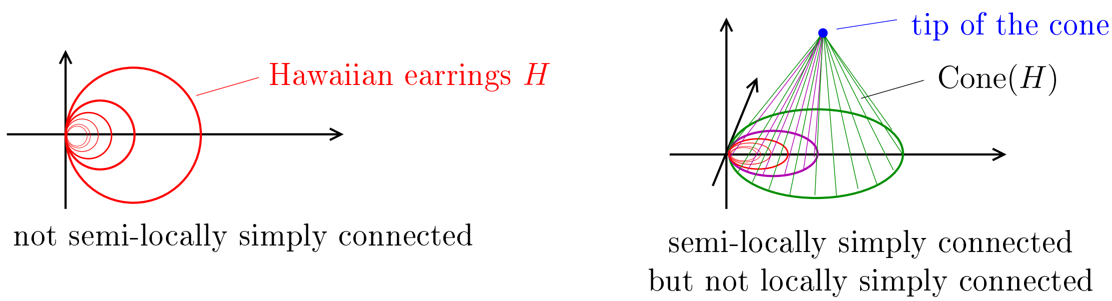


FIGURE 532

[513]Note that in contrast to "locally simply connected" there is no general notion of what "semi-locally" means. In other words, we define "semi-locally simply connected" as a single object.

[514]If there exists a neighborhood W of x such that the inclusion induced map $\pi_1(W, x) \to \pi_1(X, x)$ is the trivial map, then the interior U of W is an open neighborhood of x and the inclusion induced map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial, since it factors through the trivial map $\pi_1(W, x) \to \pi_1(X, x)$. Thus in the definition it makes no real difference whether we demand a "neighborhood" or an "open neighborhood".

29.4. **Existence of covering spaces.** Let $p\colon (X,x_0)\to(B,b_0)$ be a covering of pointed topological spaces. In Corollary 16.14 we saw that

$$p_*\colon \pi_1(X,x_0) \;\to\; \pi_1(B,b_0)$$

is an injective map. Thus we can view $\pi_1(X,x_0)$ as a subgroup of $\pi_1(B,b_0)$.

Generalizing Question 29.1 one can now ask, whether to each subgroup Γ of $\pi_1(B,b_0)$ there exists a corresponding covering with $p_*(\pi_1(X,x_0))=\Gamma$. We will see that this is indeed the case, at least under the rather technical assumption that the topological space B is locally path-connected and semi-locally simply connected.

The following proposition gives in particular an affirmative answer to Question 29.1 under rather modest hypotheses.

Proposition 29.5. *Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected, let $y_0\in Y$ and let $\Gamma\subset\pi=\pi_1(Y,y_0)$ be a subgroup.*

(1) *There exists a path-connected covering*[515] $p\colon (X,x_0)\to(Y,y_0)$ *of pointed topological spaces such that* $p_*(\pi_1(X,x_0))=\Gamma$.
(2) *For the covering p from (1) we have*
$$[X:Y] \;=[\pi_1(Y,y_0):\Gamma].$$

Examples.

(1) We consider $(Y,y_0)=(S^1,1)$. We pick an identification[516] $\pi_1(S^1,1)=\mathbb{Z}$. Given $n\in\mathbb{N}$ we consider $\Gamma=n\mathbb{Z}\subset\mathbb{Z}=\pi_1(S^1,1)$. Then the covering
$$\begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$
has the property that $p_*(\pi_1(S^1,1))=n\mathbb{Z}$. Another example is given by the covering
$$\begin{array}{rcl} q\colon \mathbb{R}/n\mathbb{Z} & \to & S^1 \\ t+n\mathbb{Z} & \mapsto & e^{2\pi i t}. \end{array}$$
Here we also have $\operatorname{im}(q_*)=n\mathbb{Z}$.
(2) Let Z be a simply connected topological space and let $z_0\in Z$. Furthermore let π be a group that acts on Z continuously and discretely. We consider $Y=Z/\pi$, we denote by $q\colon Z\to Y=Z/\pi$ the projection map and we write $y_0=q(z_0)$. Using the isomorphism of Theorem 16.16 we can identify π with $\pi_1(Y,y_0)$.

Now let $\Gamma\subset\pi=\pi_1(Y,y_0)$ be a subgroup. Then the group Γ also acts on Z and we can consider the projection map
$$p\colon X:=Z/\Gamma \;\to\; Y:=Z/\pi$$
We denote by x_0 the image of z_0 under the projection map $Z\to X=Z/\Gamma$. One can show easily that $p\colon X=Z/\Gamma\to Y=Z/\pi$ is a covering map with $p_*(\pi_1(X,x_0))=\Gamma$.

[515] Here we say that a covering $p\colon X\to Y$ is path-connected if X is path-connected.

[516] This means that we pick an isomorphism $\pi_1(S^1,1)\xrightarrow{\cong}\mathbb{Z}$ and then ignore it in the notation.

We summarize the situation in the following diagram

$$\begin{tikzcd}
Z \arrow[dr] \arrow[dd, "q"'] & \\
& X = Z/\Gamma \arrow[dl, "p"'] \\
Y = Z/\pi &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\pi_1(Z, z_0) = \{e\} \arrow[d] \\
\pi_1(X, x_0) = \Gamma \arrow[d, hook, "p_*"'] \\
\pi_1(Y, y_0) = \pi.
\end{tikzcd}$$

Now we turn to the proof of Proposition 29.5. Note that we only need show the existence of such a covering, the statement about the degree of the covering is an immediate consequence of Lemma 16.15 (3).

We start out with a general discussion of the construction of the covering. Only afterwards we will turn to the nitty-gritty of the proof. In particular for the time being let Y be any path-connected topological space, without any other adjectives. Furthermore let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup.

To get an idea for the construction, let us first consider the case that Γ is the trivial group. In this case we want to construct a covering $p\colon X \to Y$ such that X is simply connected. Let us assume for a second, that we have such an X. We pick $x_0 \in X$ with $p(x_0) = y_0$. Then, somewhat similar to the statement of Theorem 16.16, we have a map

$$\begin{array}{rcl} \Phi\colon X & \to & \{\text{path-homotopy classes of paths starting in } y_0\} \\ x & \mapsto & p \circ (\text{any path from } x_0 \text{ to } x). \end{array}$$

In fact the proof of Theorem 16.16 comes close to showing that the map Φ is a bijection. Hence Φ is in fact a homeomorphism, provided we equip the right-hand side with an appropriate topology.

Now that we are not given X, the idea is, that given (Y, y_0) we can define the right-hand side of the above bijection, and with a suitable topology we will have found our X.

As on page 233 we denote by $Y^{[0,1]}$ the set of all maps $[0,1] \to Y$. Next we consider the path space[517]

$$P_{y_0} \;:=\; \{\gamma \in Y^{[0,1]} \,|\, \gamma(0) = y_0\} \;=\; \{\text{all paths in } Y \text{ with starting point } y_0\}.$$

For $u, v \in P_{y_0}$ we write

$$u \sim v \quad :\Longleftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [u * \overline{v}] \in \Gamma.$$

Using the fact that $\Gamma \subset \pi_1(Y, y_0)$ is a subgroup one can show easily that $\sim$ is indeed an equivalence relation on P_{y_0}. The definitions of P_{y_0} and $\sim$ are illustrated in Figure 533. We write $X_{y_0,\Gamma} := P_{y_0}/\sim$ and $x_0 := [e_{y_0}]$, i.e. x_0 is the equivalence class of the constant path.

Now we consider the evaluation map

$$\begin{array}{rcl} p\colon X_{y_0,\Gamma} = P_{y_0}/\sim & \to & Y \\ {[v]} & \mapsto & v(1). \end{array}$$

[517]On page 2660 the path space will be called $P(Y, y_0)$. Later on it will play an important role, but for the purpose of this discussion it is easier to work with a shorter notation.

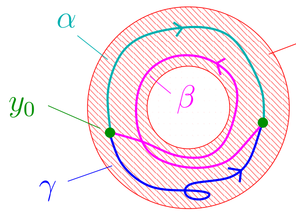


FIGURE 533

Note that this map is well-defined, i.e. it is independent of the choice of the representative v. It remains to equip $X_{y_0,\Gamma} = P_{y_0}/\sim$ with a sensible topology and to show that the map $p\colon X_{y_0,\Gamma} = P_{y_0}/\sim \to Y$ has all the desired properties.

In fact in the following we provide two approaches to defining such a topology:

(1) The first one is is "by hand", namely we construct a topology on $X_{y_0,\Gamma} = P_{y_0}/\sim$ from scratch.
(2) The second approach is "off-the-shelves", meaning we make use of the compact-open topology that we introduced on page 234.

Put differently, both Lemma 29.6 and Lemma 29.7 can be used to complete the proof of Proposition 29.5.

We start out with the "by hand" approach.

Lemma 29.6. *Let Y be a path-connected topological space, let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup. We write $P := P_{y_0}$ and we write $X := X_{y_0,\Gamma}$. Given $x = [f] \in X = P/\sim$ and given an open neighborhood V of $p(x) = f(1) \in Y$ we consider*

$$U(x, V) \;:=\; \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \subset X = P/\sim .$$

With this notation the following statement holds:

(0) *The family of all such subsets $U(x, V)$ of X satisfy the "basis property" that we introduced on page 102.*

The topology that is generated on $X = X_{Y_0,\Gamma}$ via the sets $U(x, V)$ is called the whisker topology*. If Y is locally path-connected and semi-locally simply connected, and if we equip $X = X_{Y_0,\Gamma}$ with the whisker topology, then the following statements hold:*

(1) *The above evaluation map $p\colon X \to Y$ is continuous and open.*
(2) *The map $p\colon X \to Y$ is a covering.*
(3) *The topological space X is path-connected,*
(4) *We have $p_*(\pi_1(X, x_0)) = \Gamma$.*

PROOF. Let Y be a path-connected, let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup. We write $P := P_{y_0}$ and we write $X := X_{y_0,\Gamma}$. Given $x = [f] \in X = P/\sim$ and given an open path-connected neighborhood V of $p(x) = f(1) \in Y$ we consider

$$U(x, V) \;:=\; \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \subset X = P/\sim .$$

We denote by $\mathcal{B}$ the family of all such subsets of X.

(0) We need to show that $\mathcal{B}$ has the basis property.
   (B1) For each $x \in X$ we have $x \in U(x, X)$. This shows that $\mathcal{B}$ satisfies (B1).

(B2) Now let $x = [f]$ be a point in the intersection of $U(a, A)$ and $U(b, B)$. Let V be any open neighborhood V of $f(1)$ with $V \subset A \cap B$, e.g. we could take $V = A \cap B$. It is straightforward to see that $U(x, V) \subset U(a, A) \cap U(b, B)$. We refer to Figure 534 for an illustration.

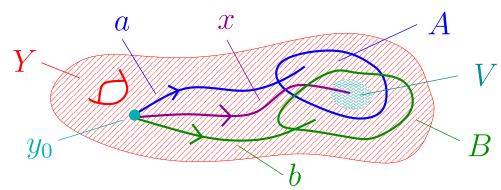


FIGURE 534

In the following we assume that Y is locally path-connected and that Y is semi-locally simply connected. We equip $X = X_{Y_0,\Gamma}$ with the *whisker topology*, i.e. we equip it with the topology generated by $\mathcal{B}$. Since Y is locally path-connected it follows easily that the whisker topology is also generated by

$$\mathcal{C} \;:=\; \{U(x, V) \in \mathcal{B} \,|\, x \in X \text{ and } V \text{ is a } \textit{path-connected} \text{ open neighborhood of } p(x)\}.$$

(1) We need to show that the map $p\colon X = P/\sim \to Y$ is continuous and open. We start with the following observation: for every $x \in X$ and every open path-connected neighborhood V of $p(x)$ we have $p(U(x, V)) = V$.[518]

Now we turn to the actual proof of continuity. So let $A \subset Y$ be an open subset and let $x = [f] \in p^{-1}(A)$. It suffices to show that there exists an open subset B of Y such that $U(x, B) \subset p^{-1}(A)$. Since Y is locally path-connected, there exists an open neighborhood B of x that is contained in A and that is path-connected. By the above observation we have $p(U(x, B)) = B \subset A$, i.e. $U(x, B) \subset p^{-1}(A)$.

Finally we show that p is also open. By Lemma 2.39 it suffices to verify that images of the basis $\mathcal{C}$ of the topology of X are open. But this statement we had just verified in the above observation.

(2) Now we want to prove that $p\colon X = P/\sim \to Y$ is a covering map. So let $y \in Y$ be a point. We need to show that there exists a connected open neighborhood V of y such that the restriction of p to each component of $p^{-1}(V)$ is a homeomorphism.

Since Y is semi-locally simply connected, there exists an open neighborhood U of y such that the inclusion induced map $\pi_1(U, x) \to \pi_1(Y, y)$ is the trivial map. We claim that V already has the desired property.

Claim.

$$p^{-1}(V) \;=\; \bigsqcup_{\widetilde{y} \in p^{-1}(y)} U(\widetilde{y}, V).$$

It is clear that the right-hand side is contained in the left-hand side. Next we want to show that the left-hand side is contained in the right-hand side. So let $[g] \in p^{-1}(V)$. Here g is a path in Y from y_0 to a point $v \in V$. Since V is in particular path-connected

[518]By definition we have $p(U(x, V)) \subset V$. But since V is path-connected we also have $V \subset p(U(x, V))$.

there exists a path h in V from v to y. We then have

$$[g] \;=\; [g*h*\overline{h}] \;\in\; U(\underbrace{[g*h]}_{\in p^{-1}(y)},V).$$

We refer to Figure 535 for an illustration.

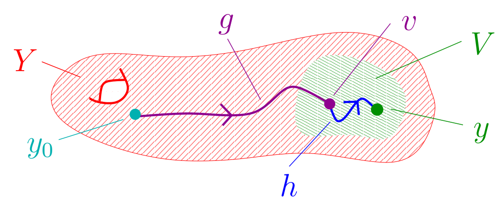


FIGURE 535

It remains to show that the sets on the right-hand side of the claim are disjoint. So let $[v'],[v'']\in p^{-1}(y)$ and let $[g]\in U([v'],V)\cap U([v''],V)$. Then there exist paths u' and u'' in V with $[v'*u']=[v''*u'']$. It follows that

$$[v'] \;=\; [v'*u'*\overline{u}'] \;=\; [v''*u''*\overline{u}'] \;\underset{\substack{\uparrow\\ u''*\overline{u}' \text{ is null-homotopic since the map } \pi_1(V,y)\to\pi_1(Y,y) \text{ is the trivial map}}}{=}\; [v''].$$

Thus $U([v'],V)\cap U([v''],V)$ intersect only if $[v']=[v'']$. We refer to Figure 536 for an illustration. ⊞

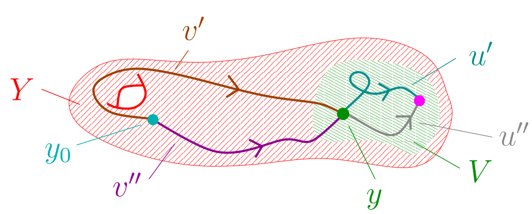


FIGURE 536

The sets $U(\widetilde{y},V)$ are by definition open. By the above claim the sets are also disjoint. Thus it suffices to prove the following claim.

Claim. For each $\widetilde{y}\in p^{-1}(y)$ the restriction of the projection map $p\colon X=P/\sim\,\to Y$ to the map $p\colon U(\widetilde{y},V)\to V$ is a homeomorphism.

So let $\widetilde{y}\in p^{-1}(y)$. As we had just pointed out in (1), the map $p\colon U(\widetilde{y},V)\to V$ is surjective. From the fact that the inclusion induced map $\pi_1(V,y)\to\pi_1(Y,y)$ is trivial it also follows immediately that p is injective. Furthermore we had just verified in (1) that p is continuous and open. Therefore $p\colon U(\widetilde{y},V)\to V$ is a homeomorphism. ⊞

(3) Now we want to show that $X=P/\sim$ is path-connected. We start out with the following claim.

Claim. Let $\alpha\colon[0,1]\to Y$ be a path. Then

$$\begin{array}{rcl}\widetilde{\alpha}\colon[0,1] & \to & X=P/\sim\\ t & \mapsto & \left[\begin{array}{rcl}[0,1] & \to & Y\\ s & \mapsto & \alpha(ts)\end{array}\right].\end{array}$$

is a continuous map.

By Proposition 2.37 it suffices to show that for each $U(x,V)$ in the basis $\mathcal{C}$ of the topology on X the preimage $\widetilde{\alpha}^{-1}(U(x,V))$ is open in $[0,1]$. So suppose we are given $U(x,V)$. We denote by $\{I_j\}_{j\in J}$ the path-components of $\alpha^{-1}(V)$. Since α is continuous each I_j is an open subset of $[0,1]$. Note that for each $j \in J$ we either have[519] $\widetilde{\alpha}(I_j) \subset U(x,V)$ or $\widetilde{\alpha}(I_j) \cap U(x,V) = \varnothing$. It follows that

$$\widetilde{\alpha}^{-1}(U(x,V)) \;=\; \text{union of all the } I_j \text{ with the property that } \widetilde{\alpha}(I_j) \subset U(x,V).$$

Thus we see that $\widetilde{\alpha}^{-1}(U(x,V))$ is a union of open subsets of $[0,1]$, hence the preimage itself is open. ⊞

To show that X is path-connected it suffices to show that for every point $x \in X$ there exists a path in X from $x_0 = [e_{y_0}]$ to x. So let $x = [\alpha] \in X$. Then such a path is given by $\widetilde{\alpha}\colon [0,1] \to X$, where $\widetilde{\alpha}$ is defined as in the claim. By the claim $\widetilde{\alpha}$ is indeed continuous.

(4) It remains to show that $p_*(\pi_1(X,x_0)) = \Gamma$. Let $g \in \pi_1(Y,y_0)$. We choose a representatives $\alpha\colon [0,1] \to Y$ of g. We consider the following path[520] in $X = P/\sim$

$$\begin{array}{rcl} \widetilde{\alpha}\colon [0,1] & \to & X = P/\sim \\ t & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & Y \\ s & \mapsto & \alpha(ts) \end{array}\right]. \end{array}$$

Note that $\widetilde{\alpha}$ is evidently the lift of α to the starting point x_0. Then we have

$$g \in p_*(\pi_1(X,x_0)) \underset{\substack{\uparrow \\ \text{Lemma 16.15 (1)}}}{\Leftrightarrow} \widetilde{\alpha} \text{ is a loop} \Leftrightarrow \text{ in } X = P/\sim \text{ we have } \widetilde{\alpha}(1) = x_0 = [e_{y_0}] \underset{\substack{\uparrow \\ \text{definition of } \sim \text{ on } P}}{\Leftrightarrow} \alpha \in \Gamma.$$ ■

Next we use the "off-the-shelves"-topology, namely we will make use of the compact-open topology. For the reader's convenience we recall the definition from page 233.

Definition. Let Y and A be topological spaces. The *compact-open topology* on the set Y^A, i.e. on the set of all maps from A to Y, is the topology generated, in the sense of the definition on page 105, by all the sets of the form

$$M(K,U) \;:=\; \{f \in Y^A \,|\, f(K) \subset U\}$$

where $K \subset A$ is compact and where $U \subset Y$ is open.

Lemma 29.7. *Let Y be a path-connected topological space and let $y_0 \in Y$. Furthermore let $\Gamma \subset \pi_1(Y,y_0)$ be a subgroup.*

(i) *We equip $Y^{[0,1]}$ = set of all maps $[0,1] \to Y$ with the compact-open topology.*
(ii) *We equip $P_{y_0} = \{\gamma \in Y^{[0,1]} \,|\, \gamma(0) = y_0\}$ with the subspace topology.*
(iii) *We equip $X_{y_0,\Gamma} = P_{y_0}/\sim$ with the quotient topology.*

[519] This might require a few minutes of thought.
[520] Note that this map is continuous by the claim in (3).

We refer to this topology on $X_{y_0,\Gamma} = P_{y_0}/\sim$ as the CO-topology *on $X_{y_0,\Gamma}$. We write $X := X_{y_0,\Gamma}$. If Y is locally path-connected and semi-locally simply connected, then the following statements hold:*

(1) *The above evaluation map $p\colon X \to Y$ is continuous and open.*
(2) *The map $p\colon X \to Y$ is a covering.*
(3) *The topological space X is path-connected,*
(4) *We have $p_*(\pi_1(X, x_0)) = \Gamma$.*

Proof. Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected, let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup. We write $X := X_{y_0,\Gamma}$ and we equip $X := P/\sim$ with the CO-topology. We make a few preparations:

(i) We consider the maps

$$\begin{tikzcd} P \arrow[r, hook] \arrow[d, "q"'] & Y^{[0,1]} \arrow[d, "\mathrm{ev}"', "f\mapsto f(1)"] \\ X = P/\sim \arrow[r, "p"] & Y \end{tikzcd}$$

where q is the projection map and $\mathrm{ev}\colon Y^{[0,1]} \to Y$ is given by the evaluation at 1. By Lemma 3.21 (3) we know that q is continuous. Furthermore by Proposition 5.4 we know that the map $\mathrm{ev}\colon Y^{[0,1]} \to Y$ is continuous. Thus it follows from Lemma 2.30 (3) and Lemma 3.22 (2) that the induced map $p\colon X = P/\sim \to Y$ is continuous.

(ii) Since Y is locally path-connected it follows from Lemma 2.71 that the open path-connected subsets of Y form a basis for the topology of Y. Since $[0,1]$ is Hausdorff we obtain from Lemma 5.8 that the sets $M(K,U)$, with $K \subset [0,1]$ compact and U open and path-connected form a basis for the topology of the compact-open topology on $Y^{[0,1]}$.

(iii) Let $x = [f] \in X = P/\sim$ and let V be an open path-connected neighborhood of $p(x) = f(1) \in Y$. As in Lemma 29.6 we consider

$$U(x,V) \;:=\; \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \subset X = P/\sim .$$

We want to show that $U(x,V)$ is an open subset of X. By definition of the quotient topology we need to show that $q^{-1}(U(x,V))$ is an open subset of P. This statement is easily seen to be a consequence of the following claim.

Claim. Let $g\colon [0,1] \to Y$ be a path with $g(0) = y_0$ and $g(1) \in V$. The set

$$\left\{ h \in P \;\middle|\; \begin{array}{c} \text{there exists a path } u\colon [0,1] \to V \text{ with } u(0) = g(1) \\ \text{such that } h \text{ is path-homotopic to } g * u \end{array} \right\}$$

is an open subset of P.

Let $h \in P$ be a path that is path-homotopic to $g * u$ for some path $u\colon [0,1] \to V$. Note that it follows from the fact that Y is semi-locally simply connected, together with Corollary 2.76, that there exists an $n \in \mathbb{N}$ and open subsets $D_1, \dots, D_n$ of Y with the following properties:

(a) For each $i \in \{0, \dots, n-1\}$ we have $h([\frac{i}{n}, \frac{i+1}{n}]) \subset D_i$.

(b) For each $i \in \{0,\dots,n-1\}$ the inclusion induced map $\pi_1(D_i, h(\frac{i}{n})) \to \pi_1(Y, h(\frac{i}{n}))$ is the trivial map.

Since Y is locally path-connected we can furthermore find open path-connected neighborhood $E_1,\dots,E_{n-1}$ of $h(\frac{1}{n}),\dots,h(\frac{n-1}{n})$. Now let

$$Z := \bigcap_{i=0}^{n-1} M([\tfrac{i}{n},\tfrac{i+1}{n}], D_i) \;\cap\; \bigcap_{i=1}^{n-1} M(\{\tfrac{i}{n}\}, E_i) \;\cap\; M(\{1\}, V).$$

We make the following observations:

(α) By definition of the compact-open topology we know that Z is an open subset of $Y^{[0,1]}$. Furthermore it follows from (a) and the choice of $E_1,\dots,E_{n-1}$ that $h \in Z$.

(β) By Lemma 2.5 it remains to show that $Z \cap P$ is contained in the set that we described in the statement of the claim.

Now let $k \in Z \cap P$.

(γ) Since $k(1) \in V$ and since V is path-connected there exists a path $v\colon [0,1] \to V$ from $h(1)$ to $k(1)$.

(δ) Since the $E_1,\dots,E_{n-1}$ are path-connected we can arrange, after a path-homotopy, that $k(\frac{i}{n}) = (h * v)(\frac{i}{n})$ for $i = 0,\dots,n-1$.

(η) It follows from (b) together with a slight variation on Lemma 14.13 that we can show that k is path-homotopic to $h * v$.

Since k is path-homotopic to $h * v$ and since h is path-homotopic to $g * u$ we have now shown that k is path-homotopic to $g * (u * v)$. But this means that k lies again in the set defined in the statement of the claim. ⊞

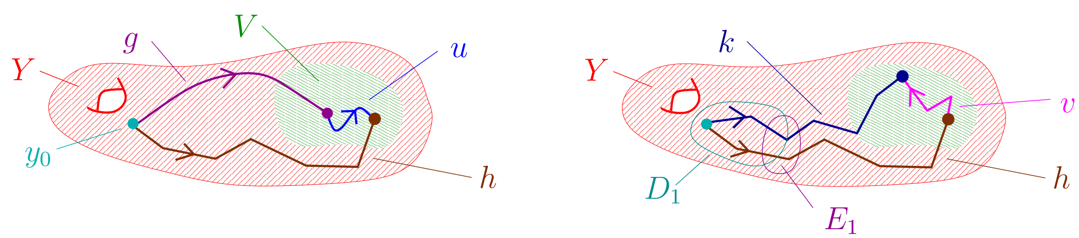


FIGURE 537. Illustration for the proof of Lemma 29.7.

(iv) Let $\alpha\colon [0,1] \to Y$ be a path with $\alpha(0) = y_0$. It follows easily from Proposition 5.6 and Lemma 2.30 (4) that the map

$$\begin{array}{rcl} [0,1] & \to & P \subset Y^{[0,1]} \\ t & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & Y \\ s & \mapsto & \alpha(ts) \end{array} \right). \end{array}$$

is continuous.

Now we turn to the actual proofs of the four statements.

(1) In (i) we already showed that $p\colon X = P/\sim \to Y$ is continuous. We still need to show that p is open. It follows from an elementary argument, see Exercise 3.23, that it suffices to show that the evaluation map $\mathrm{ev} = p \circ q\colon P \to Y$ is open. In fact it follows from (ii), together with Lemmas 2.28 and Lemma 2.39 that it suffices to prove the following claim.

Claim. For each compact subset $K \subset [0,1]$ and each path-connected open subset $V \subset Y$ we have $\operatorname{ev}(M(K,V) \cap P) = V$.

A straightforward argument, that uses that Y and V are path-connected, shows that

$$\operatorname{ev}(M(K,V) \cap P) \;=\; \begin{cases} \varnothing, & \text{if } M(K,V) = \varnothing, \\ Y, & \text{if } 1 \notin K \text{ and } M(K,V) \neq \varnothing, \\ V, & \text{if } 1 \in K. \end{cases}$$

Thus we see that in any case the image $\operatorname{ev}(M(K,V))$ is an open subset of Y. ⊞

(2) Now we want to prove that $p\colon X = P/\sim \to Y$ is a covering map. So let $y \in Y$ be a point. We need to show that there exists a connected open neighborhood V of y such that the restriction of p to each component of $p^{-1}(V)$ is a homeomorphism.

Since Y is semi-locally simply connected, there exists an open neighborhood V of y such that the inclusion induced map $\pi_1(V,y) \to \pi_1(Y,y)$ is the trivial map. We claim that V already has the desired property.

Claim.

$$p^{-1}(V) \;=\; \bigsqcup_{\widetilde{y} \in p^{-1}(y)} U(\widetilde{y}, V).$$

For the reader's convenience we repeat word-by-word the proof from page 827. First note that it is clear that the right-hand side is contained in the left-hand side. Next we want to show that the left-hand side is contained in the right-hand side. So let $[g] \in p^{-1}(V)$. Here g is a path in Y from y_0 to a point $v \in V$. Since V is in particular path-connected there exists a path h in V from v to y. We then have

$$[g] \;=\; [g * h * \overline{h}] \;\in\; U(\underbrace{[g*h]}_{\in p^{-1}(y)}, V).$$

It remains to show that the sets on the right-hand side of the claim are disjoint. So let $[v'], [v''] \in p^{-1}(y)$ and let $[g] \in U([v'],V) \cap U([v''],V)$. Then there exist paths u' and u'' in V with $[v' * u'] = [v'' * u'']$. It follows that

$$[v'] \;=\; [v' * u' * \overline{u}'] \;=\; [v'' * u'' * \overline{u}'] \;\underset{\uparrow}{=}\; [v''].$$

$u'' * \overline{u}'$ is null-homotopic since the map $\pi_1(V,y) \to \pi_1(Y,y)$ is the trivial map

Thus $U([v'],V) \cap U([v''],V)$ intersect only if $[v'] = [v'']$. ⊞

By (iii) we know that the sets $U(\widetilde{y}, V)$ are open. By the above claim the sets are also disjoint. Thus it suffices to prove the following claim.

Claim. For each $\widetilde{y} \in p^{-1}(y)$ the restriction of the projection map $p\colon X = P/\sim \to Y$ to the map $p\colon U(\widetilde{y}, V) \to V$ is a homeomorphism.

The proof of this claim is again almost verbatim the same as on page 827. So let $\widetilde{y} \in p^{-1}(y)$. It follows easily from the fact that V that the map $p\colon U(\widetilde{y}, V) \to V$ is surjective. From the fact that the inclusion induced map $\pi_1(V,y) \to \pi_1(Y,y)$ is trivial it also follows immediately that p is injective. Furthermore we had just verified in

(1) that p is continuous and open. Therefore $p\colon U(\widetilde{y},V)\to V$ is a homeomorphism. $\boxplus$

(3) It follows from from the fact that Y is path-connected together with (iv) that P is path-connected. By Lemma 3.21 (5) this implies that $X=P/\sim$ is path-connected.

(4) It remains to show that $p_*(\pi_1(X,x_0))=\Gamma$. The proof we now give is basically identical to the proof on page 828. Thus let $g\in\pi_1(Y,y_0)$. We choose a representatives $\alpha\colon[0,1]\to Y$ of g. We consider the following map:

$$\begin{array}{rcl}\widetilde{\alpha}\colon[0,1] & \to & X=P/\sim \\ t & \mapsto & \left[\begin{array}{rcl}[0,1] & \to & Y \\ s & \mapsto & \alpha(ts)\end{array}\right].\end{array}$$

By (iv) we know that $\widetilde{\alpha}$ is in fact continuous. It is now clear that $\widetilde{\alpha}$ is the lift of α to the starting point x_0. Then we have

$$g\in p_*(\pi_1(X,x_0)) \underset{\substack{\uparrow\\ \text{Lemma 16.15 (1)}}}{\Leftrightarrow} \widetilde{\alpha}\text{ is a loop} \Leftrightarrow \text{in } X=P/\sim \text{ we have } \widetilde{\alpha}(1)=x_0=[e_{y_0}] \underset{\substack{\uparrow\\ \text{definition of }\sim\text{ on }P}}{\Leftrightarrow} \alpha\in\Gamma.$$ ■

Remark.

(1) The construction of the universal covering of a topological space has its origin in the work of Hermann Weyl from 1913, see [**Wey1913**][**Wey64**, p. 58].

(2) Let Y be a path-connected topological space, let $y_0\in Y$ and let $\Gamma\subset\pi=\pi_1(Y,y_0)$ be a subgroup. Recall that in Lemma 29.6 we introduced the whisker topology and that in Lemma 29.7 we introduced the CO-topology on $X_{y_0,\Gamma}:=P_{y_0}/\sim$.
   (a) In [**FZ07**, Lemma 2.1] it is shown that the CO-topology is contained in the whisker topology.
   (b) In [**FZ07**, Example 2.2] it is shown that the two topologies are in general not equal, for example they disagree for the Hawaiian earrings that we introduced on page 727.
   (c) It follows from Proposition 29.8 (see also [**FZ07**, Lemma 2.1]) that the two topologies agree for every topological space that is locally path-connected and semi-locally simply connected.

(3) The various choices of topologies on $X_{y_0,\Gamma}=P_{y_0}/\sim$ are discussed in greater detail in [**FZ07, VZ14**] and [**Pav20**]. The history of the construction of covering spaces is also outlined [**Pav20**].

29.5. **Uniqueness of covering spaces.** Now the question arises, to what degree is the covering corresponding to the subgroup Γ unique. To answer that question we need an appropriate notion of two coverings being equivalent.

Definition. Let $p\colon X\to B$ and $q\colon Y\to B$ be two coverings of a topological space B. We say p and q are *equivalent*, if there exists a homeomorphism $\Phi\colon X\to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "\Phi", "\cong"'] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B. & \end{tikzcd}$$

Similarly we define the equivalence of coverings of pointed topological spaces.

Example. Given $n \in \mathbb{N}$ the two coverings

$$\begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon \mathbb{R}/n\mathbb{Z} & \to & S^1 \\ t+n\mathbb{Z} & \mapsto & e^{\mathrm{i}t} \end{array}$$

are equivalent. In fact the desired map $\Phi\colon S^1 \to \mathbb{R}/n\mathbb{Z}$ is given by $e^{2\pi \mathrm{i} t} \mapsto [nt]$.

Now we can formulate the following proposition:

Proposition 29.8. *Let B be a topological space, let $b_0 \in B$ and let $\Gamma \subset \pi_1(B, b_0)$ be a subgroup.*

(1) *Let $p\colon (X, x_0) \to (B, b_0)$ be a path-connected and locally path-connected covering such that $p_*(\pi_1(X, x_0)) = \Gamma$. This covering has the following universal property: for any other path-connected covering $q\colon (Y, y_0) \to (B, b_0)$ with $\Gamma \subset q_*(\pi_1(Y, y_0))$ there exists a unique covering $r\colon (X, x_0) \to (Y, y_0)$ such that the following diagram commutes:*

$$\begin{tikzcd} (X, x_0) \arrow[dr, dashed, "\exists!\, r"] \arrow[dd, "p"'] & \\ & (Y, y_0) \arrow[dl, "q"] \\ (B, b_0). & \end{tikzcd}$$

(2) *If B is path-connected, locally path-connected and semi-locally simply connected, then there exists, up to equivalence, a unique path-connected covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$.*

Remark. Let B be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $b_0 \in B$ and let $\Gamma \subset \pi_1(B, b_0)$ be a subgroup. It follows from Proposition 29.8 that the whisker topology and the CO-topology a on the set $X = \{\gamma \in B^{[0,1]} \,|\, \gamma(0) = b_0\}/\sim$, which we introduced in Lemmas 29.6 and 29.7, are actually the same.

PROOF. Let B be a topological space and let $b_0 \in B$.

(1) Let $p\colon (X, x_0) \to (B, b_0)$ and $q\colon (Y, y_0) \to (B, b_0)$ be two path-connected coverings such that $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$. We assume furthermore that X is locally path-connected. It follows from Proposition 29.2 that there exists a unique map $r\colon (X, x_0) \to (Y, y_0)$ which makes the following diagram commute

$$\begin{tikzcd} & (Y, y_0) \arrow[d, "q"] \\ (X, x_0) \arrow[ur, "r"] \arrow[r, "p"'] & (B, b_0). \end{tikzcd}$$

It remains to show that r is a covering. This will be done in Exercise 29.4.

(2) The existence of such a covering is just the statement of Proposition 29.5. The uniqueness statement is an immediate consequence of (1)[521] and the usual proof of uniqueness for an object satisfying a universal property. We leave the details to the reader. ■

Propositions 29.5 and 29.8 (2) lead us to the following definition.

Definition. Let B be a topological space that is path-connected, locally path-connected and semi-locally simply connected, let $b_0 \in B$ and let $\Gamma \subset \pi_1(B, b_0)$ be a subgroup.

(1) As in the proof of Proposition 29.5 we consider the set

$$\widetilde{B}^{\Gamma}_{b_0} \;:=\; \{\text{all paths in } B \text{ with starting point } b_0\}/\sim$$

where $u \sim v$ if and only u and v have the same endpoint and if $[u * \overline{v}] \in \Gamma$. We equip the set $\widetilde{B}^{\Gamma}_{b_0}$ with the topology introduced in the proof of Proposition 29.5. We refer to $\widetilde{B}^{\Gamma}_{b_0}$ together with map $p\colon \widetilde{B}^{\Gamma}_{b_0} \to B$ given by $[u] \mapsto u(1)$ as the *canonical covering corresponding to* (B, b_0, Γ). Furthermore we refer to the point in $\widetilde{B}^{\Gamma}_{b_0}$ that is represented by the constant path as the *canonical base point of* $\widetilde{B}^{\Gamma}_{b_0}$.

(2) We refer to any path-connected covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces with $p_*(\pi_1(X, x_0)) = \Gamma$ as *the covering of B corresponding to* Γ. By Proposition 29.8 (2) any two such coverings are equivalent, in particular any such covering is equivalent to the above canonical covering. In practice usually we do not care about this indeterminacy.

Proposition 29.8 can be viewed as a "machine" that constructs covering spaces of a given topological space X. In the next section we discuss the case that X is a surface in somewhat greater detail to get a better feeling for what is happening.

29.6. **The universal covering.** In Proposition 29.8 we saw that subgroups of fundamental groups give rise to covering spaces. An obvious choice of a subgroup is the trivial group and we immediately obtain the following corollary to Proposition 29.8.

Corollary 29.9. *Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $x_0 \in X$. Then the following holds:*

(1) *There exists, up to equivalence, a unique path-connected covering*

$$p\colon (\widetilde{X}, \widetilde{x}_0) \;\to\; (X, x_0)$$

such that $\widetilde{X}$ is simply connected.

[521]If one does the argument carefully, then one sees, that here we use the following fact: if B is locally path-connected, then also every covering space of B is locally locally path-connected.

(2) *Let* $q\colon (Y,y_0)\to(X,x_0)$ *be another path-connected covering. Then there exists a unique covering* $r\colon (\widetilde{X},\widetilde{x}_0)\to(Y,y_0)$ *such that the following diagram commutes:*

$$
\begin{tikzcd}
(\widetilde{X},\widetilde{x}_0) \arrow[dd, "p"'] \arrow[dr, "r"] & \\
& (Y,y_0) \arrow[dl, "q"] \\
(X,x_0). &
\end{tikzcd}
$$

Proof. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $x_0\in X$.

(1) We apply Proposition 29.8 to the trivial subgroup of $\pi_1(X,x_0)$ and we obtain, up to equivalence, a unique path-connected covering $p\colon (\widetilde{X},\widetilde{x}_0)\to(X,x_0)$ such that $p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ is the trivial group. But then it follows from Corollary 16.14 that $\widetilde{X}$ is simply connected.
(2) The second statement is also an immediate consequence of Proposition 29.8. ■

Definition. Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $x_0\in X$.

(1) We refer to the canonical covering $p\colon \widetilde{X}_{x_0}:=\widetilde{X}^{\{e\}}_{x_0}\to X$, as defined on page 834, as the canonical universal covering of (X,x_0).
(2) We refer to any covering $p\colon (\widetilde{X},\widetilde{x}_0)\to(X,x_0)$ such that $\widetilde{X}$ is simply connected as a *universal covering* of (X,x_0). As we remarked in the definition on page 834, very often we suppress the fact that the universal covering is well-defined only up to equivalence.[522]
(3) Sometimes we ignore base points and we say that a covering $p\colon \widetilde{X}\to X$ is the universal covering if $\widetilde{X}$ is simply connected.

Remark. Note that most "reasonable" topological spaces are locally path-connected and semi-locally simply connected. For example we saw on page 816 that every topological manifold is locally simply connected, which implies in particular that it is locally path-connected and semi-locally simply connected. Thus, given a path-connected topological manifold we can talk of is universal covering.

Examples.

(1) The map

$$
\begin{array}{rcl}
p\colon \mathbb{R} & \to & \mathbb{R}/\mathbb{Z}=S^1\\
t & \mapsto & e^{2\pi i t}
\end{array}
$$

is the universal covering of S^1.

[522] In the literature the universal covering of a topological space is usually indicated by the same symbol but decorated with a tilde, e.g. the universal covering of X is usually denoted by $\widetilde{X}$.

(2) More generally, the map

$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & (S^1)^n \\ (t_1,\dots,t_n) & \mapsto & \left(e^{2\pi \mathrm{i} t_1},\dots,e^{2\pi \mathrm{i} t_n}\right) \end{array}$$

is the universal covering of the n-dimensional torus $(S^1)^n$.

(3) The projection map $p\colon S^3 \to S^3/\mathbb{Z}_p = L(p,q)$ is the universal covering of the lens space $L(p,q)$.

Remark. The universal covering is often also just called the *universal cover*. Also very often one suppresses the actual map p from the notation. For example, often one says that "$\mathbb{R}^n$ is the universal covering of $(S^1)^n$". The name "universal cover(ing)" stems from the fact, proved in Corollary 29.9, that the universal cover(ing) of a topological space X "covers" every other covering of X.

We have now shown that for every topological space X that is locally path-connected and semi-locally simply connected there exists a universal covering $p\colon \widetilde{X} \to X$. So far we have given an abstract construction, but if possible we would like to visualize it. We have just given an explicit description of the universal covering of the n-dimensional torus $(S^1)^n$ and the lens spaces. Now we will describe the universal covers of $X = S^1 \vee S^2$ and $X = S^1 \vee S^1$.

First we consider $X = S^1 \vee S^2$ where we identify $i \in S^1$ with $(0,0,-1) \in S^2$. As on page 609 we consider the topological space

$$\widetilde{X} \;=\; \left(\mathbb{R} \cup (\mathbb{Z}\times S^2)\right) / \, n \sim (n,(0,0,-1)), \quad n \in \mathbb{Z}.$$

It is straightforward to see that the map

$$\begin{array}{rl} p\colon \left(\mathbb{R} \cup (\mathbb{Z}\times S^2)\right) / \, n \sim (n,(0,0,-1)) \to S^1 \vee S^2 & \\ P \mapsto \begin{cases} e^{\mathrm{i}(2\pi P + \frac{\pi}{2})}, & \text{if } P \in \mathbb{R} \\ Q, & \text{if } P = (n,Q) \text{ with } n \in \mathbb{Z}, Q \in S^2 \end{cases} \end{array}$$

is well-defined and in fact a covering. We saw on page 609 that $\widetilde{X}$ is simply connected. Thus $p\colon \widetilde{X} \to X$ is the universal covering of X.

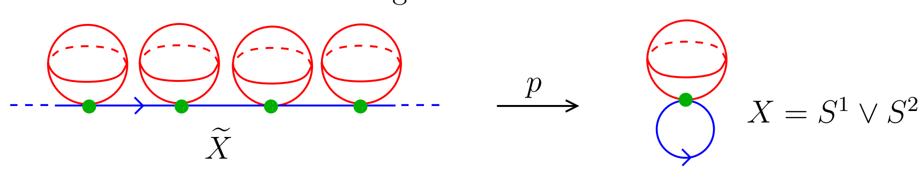


Figure 538

Now we turn to $X = S^1 \vee S^1$. In the following we give a semi-rigorous description of the universal covering of $S^1 \vee S^1$ as a subset of $\mathbb{R}^2$. Iteratively we build a sequence of topological spaces Y_k, $k \in \mathbb{N}_0$ as follows:

(1) The space Y_0 is the cross centered at the origin where each segment has length 1.

(2) Given Y_k we attach to each endpoint of Y_k three segments of length $\frac{1}{4^k}$[523] such that each endpoint of Y_k now has a neighborhood that is a cross. We denote the resulting space by Y_{k+1}.

Finally we put $Y = \bigcup_{k\in\mathbb{N}_0} Y_k$. This construction is illustrated in Figure 539. We summarize

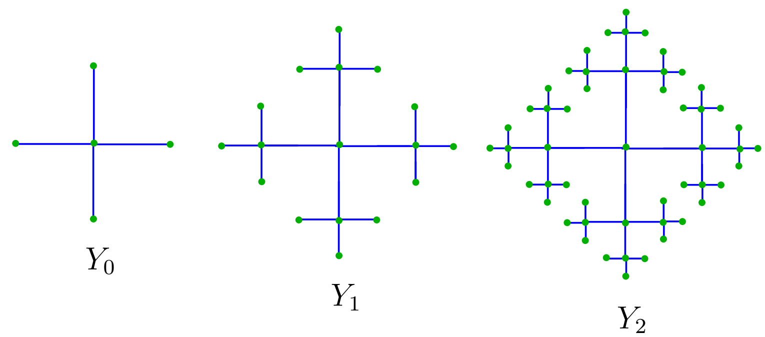


FIGURE 539

some of the key properties of the construction in the following claim.

Claim.

(1) Each Y_k is a tree.[524]
(2) Y is simply connected.

SKETCH OF PROOF. First note that $\chi(Y_0) = 5 - 4 = 1$. For $k \geq 1$ we obtain Y_k from Y_{k-1} by adding three vertices and three edges, hence $\chi(Y_k) = \chi(Y_{k-1}) = \cdots = \chi(Y_0) = 1$. This shows that each Y_k is a tree. It follows from Proposition 18.29 that each Y_k is simply connected.

Given $k \in \mathbb{N}$ we denote by U_k the result of removing the 4^{k+1} endpoints of Y_k. Note that each U_k is open in Y_k, hence each U_k is open in Y. Furthermore each U_k deformation retracts to Y_{k-1}. Hence it follows from the fact that each Y_k is simply connected that also each U_k is simply connected.

It follows as in Lemma 20.6 that $Y = \bigcup_{k\in\mathbb{N}} Y_k = \bigcup_{k\in\mathbb{N}} U_k$ itself is simply connected. ■

Now we construct a covering $p\colon Y \to S^1 \vee S^1$. To distinguish the two copies of S^1 we name them A and B, which are wedged together along $a \in A$ and $b \in B$. We give both A and B the usual counterclockwise orientation. Furthermore we orient each edge of the infinite topological graph Y as indicated in Figure 540. We consider the map $p\colon Y \to A \vee B$ that is given as follows:

(1) each vertex of the infinite topological graph Y gets sent to the point $a = b \in A \vee B$,
(2) each horizontal edge of Y gets sent in an orientation-preserving way to $A \setminus \{a\}$,
(3) each vertical edge of Y gets sent in an orientation-preserving way to $B \setminus \{b\}$.

[523] In Figure 539 and 540 the lengths of the segments is $\frac{1}{2^k}$ since these are easier to draw.

[524] Recall that by definition a tree is a topological graph with finitely many vertices V and edges E, such that the Euler characteristic $\chi(G) = \#V - \#E = 1$.

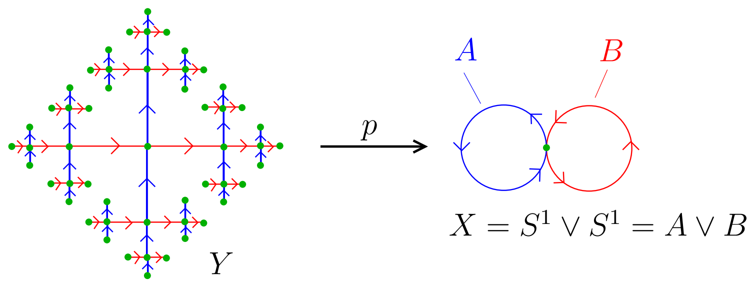


FIGURE 540

This map is illustrated in Figure 540. It is relatively straightforward to convince oneself that $p\colon Y \to X$ is in fact a covering.[525] Since Y is simply connected, this is the universal covering of $X = A \vee B = S^1 \vee S^1$.

We have now determined explicitly the universal covers of several topological spaces. The surfaces of genus $g \geq 2$ are arguably one of the most interesting types of topological manifolds. Therefore it is natural to ask the following question.

Question 29.10. *What does the universal covering of the surface of genus $g \geq 2$ look like?*

We continue with the following lemma which will prove helpful at times.

Lemma 29.11. *Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected with universal covering $p\colon \widetilde{X} \to X$. Given any map $f\colon X \to X$, given any $x_0 \in X$, given any $\widetilde{x}_0 \in \widetilde{X}$ and given any $\widetilde{y}_0 \in \widetilde{X}$ with $p(\widetilde{y}_0) = f(x_0)$ there exists a unique map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ with $\widetilde{f}(\widetilde{x}_0) = \widetilde{y}_0$ such that the following diagram commutes:*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X. \end{array}$$

Furthermore, if f is a homeomorphism, then $\widetilde{f}$ is also a homeomorphism.

Remark. Lemma 29.11 is a generalization of Exercise 17.3 (a).

PROOF. We consider the map $f \circ g := p\colon \widetilde{X} \to X$. Since $\widetilde{X}_0$ is the universal covering, since $\pi_1(\widetilde{X}, \widetilde{x}_0)$ and since X is locally path-connected and semi-locally simply connected we obtain from Proposition 29.2 that there exists a unique lift $\widetilde{g}\colon \widetilde{X} \to \widetilde{X}$ of g with $\widetilde{g}(\widetilde{x}_0) = \widetilde{y}_0$. This map is precisely the map we were trying to find. We will show in Exercise 29.10 that $\widetilde{f}$ is a homeomorphism. ■

---

[525] We have to check that each point in $X = A \vee B$ admits an open neighborhood that is uniformly covered. It is clear that such a point exists for all points that are not the wedge point. For the wedge point itself the desired open neighborhood is given by small open intervals on A and B. Then each component of the preimage consists of a little cross with center P, with one horizontal edge going into P, one horizontal edge going out of P, one vertical edge going into P and one vertical edge going out of P.

We conclude this section with a more general statement. Recall that in Theorem 16.16 we saw that if X is a simply connected topological space and if G is a group that acts continuously and discretely on X, then the fundamental group of the quotient space X/G is isomorphic to G. Now we will show that, under modest extra hypotheses, the converse holds. More precisely, we have the following proposition.

Proposition 29.12. $(*)$ *Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $y_0 \in Y$. We write $\pi = \pi_1(Y, y_0)$. Then there exists a simply connected topological space X, namely the universal covering of Y, and a continuous and discrete action of π on X such that Y is homeomorphic to X/π.*

Proof $(*)$. Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $y_0 \in Y$ be a base point. In Proposition 29.5 we already showed that Y admits a simply connected covering. We recall the explicit construction given in the proof of Proposition 29.5. Thus we consider

$$P \;:=\; \{f \in Y^{[0,1]} \mid f(0) = y_0\} \;=\; \{\text{all paths in } Y \text{ with starting point } y_0\}.$$

For $u, v \in P$ we write

$$u \sim v \quad :\Leftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [u * \overline{v}] = e \in \pi_1(Y, y_0)$$

and we put $X = P/\sim$. As in Lemma 29.6 given $x = [f] \in X = P/\sim$ and an open path-connected neighborhood V of $f(1)$ we define

$$U(x, V) \;:=\; \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \;\subset\; X = P/\sim .$$

Furthermore, as in Lemma 29.6 we endow X with the whisker topology, i.e. with the topology generated by these sets $U(x, V)$.

Claim. The map

$$\begin{array}{rcl} \pi_1(Y, y_0) \times X & \to & X \\ ([g], [u]) & \mapsto & [g * u] \end{array}$$

is a well-defined action.

We first show that the map is independent of the choice of u. So suppose that $u, v \in P$ are equivalent. Then $g * u$ and $g * v$ still have the same endpoint. Now we need to consider the second condition in the definition of $\sim$. We have

$$[(g * u) * (\overline{g * v})] \;=\; [g * u * \overline{v} * \overline{g}] \;=\; [g] * \underbrace{[u * \overline{v}]}_{=e \in \pi_1(Y, y_0)} * [g]^{-1} \;=\; e \;\in \pi_1(Y, y_0).$$

It is straightforward to see that the map does not depend on the choice of representative g. Finally we want to show that it is indeed an action. So let $[g], [h] \in \pi_1(Y, y_0)$. Then

$$[g] \cdot ([h] \cdot [u]) \;=\; [g] \cdot [h * u] \;=\; [g * (h * u)] \;=\; [(g * h) * u] \;=\; [g * h] \cdot [u] \;=\; ([g] \cdot [h]) \cdot [u].$$

⊞

Now we need to show the following:

(1) the group $\pi_1(Y, y_0)$ acts continuously and discretely on $X = P/\sim$,
(2) there exists a homeomorphism $f\colon X/\pi_1(Y, y_0) \to Y$.

We do this in the following two steps:

(1) It follows immediately from the definition of the whisker topology on $X = P/\sim$ and from Proposition 2.37 that the action is continuous.

Now we need to show that the action is discrete. So let $x = [\gamma] \in P/\sim$. We set $z := \gamma(1) \in Y$. Since Y is semi-locally simply connected there exists an open neighborhood V of $z = \gamma(1)$ such that the inclusion induced map $\pi_1(V, z) \to \pi_1(Y, z)$ is trivial. It suffices to show that for $[g] \in \pi_1(Y, y_0)$ we have

$$[g] \cdot U(x, V) \cap U(x, V) \neq \varnothing \quad \Longrightarrow \quad [g] = e \in \pi_1(Y, y_0).$$

So suppose that $[g] \cdot U(x, V) \cap U(x, V) \neq \varnothing$. This means that there exist paths a and b in U, starting from P such that $g * \gamma * a$ and $\gamma * b$ have the same endpoint and they are path-homotopic in Y. But then

$$[g] \;=\; [g] \cdot \underbrace{[\gamma] \cdot \overbrace{[a * \overline{b}]}^{=e \in \pi_1(Y, y_0) = \{e\}} * [\gamma]^{-1}}_{=e \in \pi_1(Y, y_0)} \;=\; [(g * \gamma * a) * \overline{(\gamma * b)}] \;=\; e \;\in\; \pi_1(Y, y_0).$$

We refer to Figure 541 for an illustration.

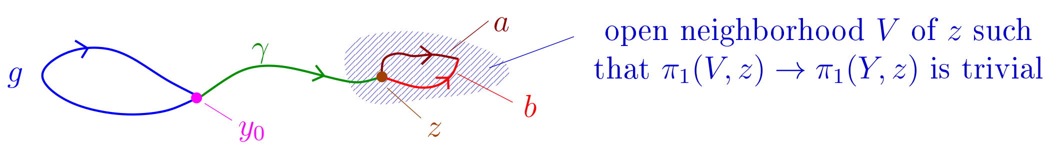


Figure 541

(2) We consider the map

$$\begin{aligned} f \colon X/\pi_1(Y, y_0) &\to Y \\ [\gamma] &\mapsto \gamma(1). \end{aligned}$$

It is straightforward to see that this map is a bijection. It follows immediately from Lemma 3.22 that the map is continuous and it follows from Lemma 2.39 that the map is open. Thus it is a homeomorphism. ■

Exercises for Chapter 29.

Exercise 29.1.

(a) Show that every topological graph is locally contractible.

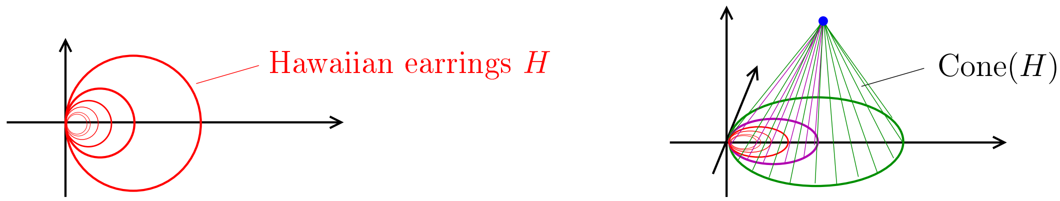
(b) Let $\{X_i\}_{i \in I}$ be a family of locally contractible topological spaces. Show that any wedge $\bigvee_{i \in I} X_i$ is also locally contractible.

Remark. For a rigorous argument you will want to use Proposition 18.20.

Exercise 29.2. Let H be the Hawaiian earring that we introduced on page 727 and that we show in Figure 542 to the left.

(a) Show that H is not semi-locally simply connected.

(b) Show that cone(H) is not locally simply connected.



not semi-locally simply connected not locally simply connected

FIGURE 542

Exercise 29.3. Let $\pi\colon X \to B$ be a covering of topological spaces. Show that every simply connected and locally path-connected subset $U \subset B$ is uniformly covered.

Exercise 29.4. Let $p\colon (X, x_0) \to (B, b_0)$ and $q\colon (Y, y_0) \to (B, b_0)$ be two path-connected coverings of topological spaces such that $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$. It follows from Proposition 29.2 that there exists a unique map $r\colon (X, x_0) \to (Y, y_0)$ which makes the following diagram commute

$$\begin{tikzcd} & (Y, y_0) \arrow[d, "q"] \\ (X, x_0) \arrow[ur, "r"] \arrow[r, "p"'] & (B, b_0). \end{tikzcd}$$

Show that r is a covering.

Exercise 29.5. We consider Figure 543.

(a) Let X be the wedge of two circles and let $p\colon Y \to X$ be the universal covering that is illustrated in the figure. Denote the usual generators of $\pi_1(X)$ by a and b. By an abuse of notation we denote by a and b also representing loops. What does the lift of the loop $a * b * \overline{a} * \overline{b}$ to Y to the starting point y_0 look like?

(b) We continue with the notation from (1). Now consider the loop

$$\gamma := a * a * b * a * a * \overline{b} * \overline{a} * b * b * \overline{a} * \overline{a} * \overline{b} * \overline{a} * \overline{b}.$$

We denote by $\widetilde{\gamma}$ the lift of γ to the starting point y_0. Is y_0 the endpoint of $\widetilde{\gamma}$?

(c) Does there exist a homeomorphism $f\colon Y \to Y$ with $f(y_0) = z_0$?

Exercise 29.6. Let $X = S^1 \cup \{*\}$ be $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ with two "1", where the topology is defined as for the line with two zeros, see page 95. Does X admit a universal cover? If yes, can you describe it explicitly?

Exercise 29.7.

(a) Let M be the Möbius band. We pick an identification $\pi_1(M) = \mathbb{Z}$. Given $k \in \mathbb{N}$ we denote by Γ_k the kernel of the epimorphism $\pi_1(M) = \mathbb{Z} \to \mathbb{Z}_k$. Let $M_k \to M$ be the covering corresponding to Γ_k. What well-known topological space is M_k? You do not have to justify your answer.

(b) Draw the universal covering of $\mathbb{R}\mathrm{P}^2 \vee S^1$ as a subset of $\mathbb{R}^3$.

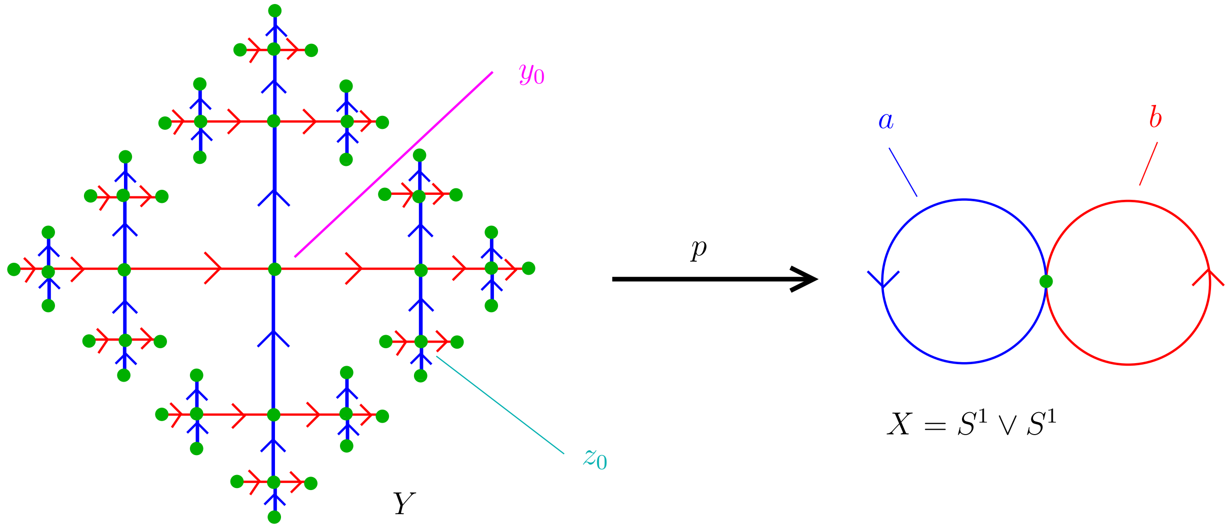


FIGURE 543

(c) Let $X = A \vee B$ be the wedge of two circles. Let $\Gamma \subset \pi_1(X) = \langle a, b \rangle$ be the commutator subgroup. What does the covering $p\colon \widetilde{X} \to X$ corresponding to Γ look like? You can draw it as a subset of $\mathbb{R}^2$.
Hint. You have seen $\widetilde{X}$ many times in your life.

Exercise 29.8. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. We denote by $M := \operatorname{Tor}(X, f) := (X \times [0,1])/(x,0) \sim (f(x),1)$ the corresponding mapping torus (see page 711.) Let $p\colon \widetilde{M} \to M$ be a finite covering. Show that $\widetilde{M}$ is homeomorphic to a mapping torus.

Exercise 29.9. We say two smooth manifolds M and N are *commensurable* if M and N admit finite covers $\widetilde{M}$ and $\widetilde{N}$ that are diffeomorphic.

(a) Show that the torus and the surface of genus two are not commensurable.
(b) Let $g, h \in \mathbb{N}_{n \geq 2}$. Let M be the surface of genus g and let N be the surface of genus h. Show that M and N are commensurable.
(c) Determine which compact connected 2-dimensional smooth manifolds are commensurable.

Exercise 29.10. Let X be a topological space and let $p\colon \widetilde{X} \to X$ be a covering. Let $f\colon X \to X$ be a homeomorphism and let $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ be a map such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \widetilde{X} \arrow[d, "p"] \\ X \arrow[r, "f"] & X. \end{tikzcd}$$

Show that $\widetilde{f}$ is also a homeomorphism.
Remark. This is a generalization of Exercise 17.3 (b).

Exercise 29.11. Give an example of a topological space X, a finite covering $p\colon \widetilde{X} \to X$ and a homeomorphism $f\colon X \to X$ which does *not* lift to a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$, i.e. such there is no map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X. \end{array}$$

Exercise 29.12. Let X be a path-connected topological space that is locally path-connected and let $p\colon \widetilde{X} \to X$ be a finite covering of degree d.

(a) Let $f\colon X \to X$ be a homeomorphism. Show that there exists a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ such that $p \circ \widetilde{f} = f^d \circ p$.
(b) Does the conclusion of (a) also hold, if we do not assume that f is a homeomorphism?

Exercise 29.13. Let X be a path-connected topological space that is locally path-connected and semi-locally simply connected. Furthermore let $x_0 \in X$. By Proposition 29.8 there exists an, essentially unique, covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. We refer this covering as the *universal abelian covering of* (X, x_0).

(a) Let X be torus minus one open disk. Give an explicit description of the universal abelian cover of X.
(b) Let X be the sphere minus three open disks. Give an explicit description of the universal abelian cover of X.

Evidently the expression "explicit" is a little vague, you could interpret it as an explicit description as a submanifold of say $\mathbb{R}^3$.

Exercise 29.14. Let $f\colon (X, x_0) \to (Y, y_0)$ be a map between pointed topological spaces such that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a monomorphism. We assume that both X and Y are locally path-connected and semi-locally simply connected. Let Γ be a subgroup of $\pi_1(Y, y_0)$ Let $q\colon \widetilde{Y} \to Y$ be the covering corresponding to Γ. Show that the pullback $p\colon f^*\widetilde{Y} \to X$, as defined in Lemma 25.16, is the covering corresponding to the subgroup $f_*^{-1}(\Gamma)$ of $\pi_1(X, x_0)$.

Exercise 29.15. Let X be a path-connected topological space that is locally path-connected and that is semi-locally simply connected. Furthermore let $p\colon \widetilde{X} \to X$ be the universal covering. Let $x_0 \in X$ be a base point and let $\varphi\colon \pi_1(X, x_0) \to \Gamma$ be a group homomorphism. We consider

$$\begin{array}{rcl} q\colon \widetilde{X}_\varphi := (\widetilde{X} \times \Gamma)/\overbrace{(x,g) \sim (h \cdot x, \varphi(h) \cdot g)}^{x \in X, g \in \Gamma \text{ and } h \in \pi_1(X,x_0)} & \to & X \\ {[(x,g)]} & \mapsto & p(x) \end{array}$$

where we equip Γ with the discrete topology, $\widetilde{X} \times \Gamma$ with the product topology and $\widetilde{X}_\varphi$ with the quotient topology.

(a) Suppose that φ is an epimorphism. Show that $q\colon \widetilde{X}_\varphi \to X$ is the covering corresponding to $\ker(\varphi)$.

(b) In general, show that the number of components of $\widetilde{X}_\varphi$ has the same cardinality as $\operatorname{coker}(\varphi)$ and show that the restriction of $q\colon \widetilde{X}_\varphi \to X$ to each component of $\widetilde{X}_\varphi$ is the covering corresponding to $\ker(\varphi)$.

30. Topics in covering theory (∗)

In this chapter we will discuss several applications and examples of the existence of (universal) coverings of suitable topological spaces.

30.1. **Self-diffeomorphisms of S^1.** Using covering theorem we can now prove the following proposition on self-diffeomorphisms of S^1.

Proposition 30.1. *Let $f\colon S^1 \to S^1$ be a diffeomorphism.*

(1) *If f is orientation-preserving, then f is diffeotopic to the identity, otherwise f is diffeotopic to the reflection $r\colon S^1 \to S^1$ given by $(x, y) \mapsto (x, -y)$.*[526]

(2) *The diffeomorphism f extends to a diffeomorphism of $\overline{B}^2$.*

Remark. In Section **??**, given an orientable smooth manifold M, we will introduce the "mapping class group" $\mathrm{MCG}^+(M)$ which basically measures "how many interesting" orientation-preserving self-diffeomorphisms a smooth manifold has. In this language Proposition 30.1 says that $\mathrm{MCG}^+(S^1)$ is trivial. In Section **??** we will also discuss mapping class groups of higher dimensional spheres. In particular we will see that in general the naive analogues in higher dimensions do *not* hold.

Proof.

(1) As usual we identify S^1 with $\mathbb{R}/\mathbb{Z}$. Thus let $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a diffeomorphism. First we consider the case that f is orientation-preserving. We denote by $p\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ the usual covering map. By Proposition 29.2 applied to the map $f \circ p\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ there exists a map $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \mathbb{R} \arrow[d, "p"] \\ \mathbb{R}/\mathbb{Z} \arrow[r, "f"] & \mathbb{R}/\mathbb{Z}. \end{tikzcd}$$

We make the following observations regarding $\widetilde{f}$:

(a) Since p is a local diffeomorphism and since f is a smooth map we see that $\widetilde{f}$ is also smooth.

(b) Since f is orientation-preserving and since p is a local diffeomorphism we see that $\widetilde{f}$ is also orientation-preserving. This implies[527] that $\widetilde{f}$ is a strictly monotonously increasing function.

(c) Since f is a diffeomorphism and since the above diagram commutes we see that for any $x \in \mathbb{R}$ the restriction of $\widetilde{f}$ to the interval $[x, x+1)$ defines a bijection with $[f(x), f(x+1))$.

[526] Note that in Lemma 18.9 we proved an analogous statement for self-homeomorphisms of S^1. The case of self-homeomorphisms is technically easier since it is often easier to write down a continuous map than to write down a smooth map.

[527] Why?

(d) For any $x \in \mathbb{R}$ we have, by the commutativity of the diagram, the equality $f(x+1) - f(x) \in \mathbb{Z}$.

(e) It follows from (b), (c) and (d) that for any $x \in \mathbb{R}$ we have $f(x+1) = f(x) + 1$.

Now we consider the map

$$\begin{array}{rcl} \widetilde{F}\colon \mathbb{R} \times [0,1] & \to & \mathbb{R} \\ (x,t) & \mapsto & \widetilde{f}(x) \cdot (1-t) + x \cdot t. \end{array}$$

This map is evidently smooth. Since $x \mapsto x$ is a strictly monotonously increasing diffeomorphism of $\mathbb{R}$ one can see easily that for any $t \in [0,1]$ the map $\widetilde{F}_t\colon \mathbb{R} \to \mathbb{R}$ has the above Properties (a) to (e). This implies in particular that the map $\widetilde{F}$ descends to a map

$$\begin{array}{rcl} F\colon (\mathbb{R}/\mathbb{Z}) \times [0,1] & \to & \mathbb{R}/\mathbb{Z} \\ (x,t) & \mapsto & \widetilde{f}(x) \cdot (1-t) + x \cdot t \end{array}$$

where each $F_t\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a homeomorphism. Since each $\widetilde{F}_t$ is a local diffeomorphism and since p is a local diffeomorphism we obtain from Lemma 6.23 (3) that each F_t is in fact a diffeomorphism. We have thus shown that F is indeed a diffeotopy from f to the identity.

Finally we consider the case that f is orientation-reversing. In this case $r \circ g$ is orientation-preserving. We apply the above result to $r \circ g$ and we easily obtain the desired diffeotopy from f to r.

(2) Let $f\colon S^1 \to S^1$ be a diffeomorphism. By (1) we know that there exists a diffeotopy $F\colon S^1 \times [0,1] \to S^1$ from f to $g = \mathrm{id}$ or to $g = r$. By Lemma 6.13 there exists a smooth map $f\colon [\frac{1}{2}, 1] \to [0,1]$ with $f(t) = 0$ for $t \in [\frac{1}{2}, \frac{5}{8}]$ and $f(t) = 1$ for $t \in [\frac{7}{8}, 1]$. The promised extension is now given by

$$\begin{array}{rcl} \overline{B}^2 & \to & \overline{B}^2 \\ P = (x,y) & \mapsto & \begin{cases} P = (x,y), & \text{if } \|P\| \le \frac{1}{2} \text{ and } f \text{ is orientation-preserving}, \\ (x,-y), & \text{if } \|P\| \le \frac{1}{2} \text{ and } f \text{ is orientation-reversing}, \\ F(\frac{P}{\|P\|}, 2 - 2\cdot\|x\|), & \text{if } \|P\| \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

■

The following corollary is an immediate consequence of Proposition 30.1.

Corollary 30.2. *If C and D are two oriented smooth manifolds that are diffeomorphic to S^1, then any two orientation-preserving diffeomorphisms $C \to D$ are diffeotopic.*

We conclude this section with the following corollary.

Corollary 30.3. *Let M be a 2-dimensional smooth manifold (possibly disconnected) and let A and B be two distinct boundary components of M. Suppose A and B are equipped with orientations. Given any two orientation-preserving diffeomorphisms $f, g\colon A \to B$ the resulting smooth manifolds*

$$M/f(a) \sim a \quad \text{and} \quad M/g(a) \sim a \quad (\text{in both cases with } a \in A)$$

are diffeomorphic.

Loosely speaking, the corollary says that if we glue two oriented boundary components of a surface together in an orientation-preserving way, then the diffeomorphism type of the resulting smooth manifold does not depend on the specific choice of the orientation-preserving diffeomorphism.

Proof. It follows from Proposition 6.27 and Theorem 7.5 that A and B are both diffeomorphic to S^1. The corollary is now an immediate consequence of Lemma 8.16 and Corollary 30.2. ■

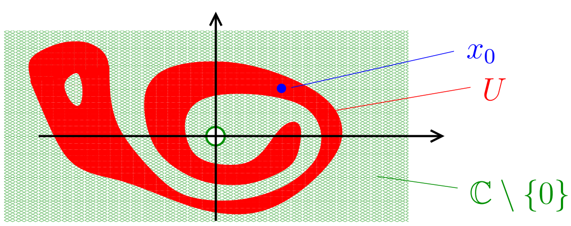
30.2. **Complex logarithms.** In this short section we will discuss complex logarithms. This discussion does not play a role in the remainder of these notes, but it is a fun application of covering theory.

Definition. Let $U \subset \mathbb{C} \setminus \{0\}$ be a subset. We say *the complex logarithm exists on U* if there exists a map[528] $\ln \colon U \to \mathbb{C}$ such that $\exp(\ln(z)) = z$ for all $z \in U$.

The following proposition gives a very pleasing sufficient and necessary criterion for complex logarithms to exist.

Proposition 30.4. (*) *Let $U \subset \mathbb{C} \setminus \{0\}$ be an open path-connected subset. The following statement holds:*[529]

the complex logarithm exists on U $\iff$ *for some $x_0 \in U$ the inclusion induced map $\pi_1(U, x_0) \to \pi_1(\mathbb{C} \setminus \{0\}, x_0)$ is the trivial map.*



the map $\pi_1(U, x_0) \to \pi_1(\mathbb{C} \setminus \{0\}, x_0)$
is the trivial map, hence
the complex logarithm exists on U

Figure 544. Illustration of Proposition 30.4.

Proof (*). Let $U \subset \mathbb{C} \setminus \{0\}$ be an open path-connected subset and let $x_0 \in U$. We consider the following rather simple minded diagram

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "z \mapsto \exp(z)"] \\ U \arrow[ur, dashed] \arrow[r, hook, "i"'] & \mathbb{C} \setminus \{0\}. \end{tikzcd}$$

We make the following observations:

(1) As on page 493 one can show that the vertical map exp is actually a covering map.
(2) The question whether the complex logarithm exists on U is equivalent to the question whether the inclusion map $i \colon U \to \mathbb{C} \setminus \{0\}$ lifts to a map $U \to \mathbb{C}$.

[528] As always, all maps are understood to be continuous.

[529] It follows from Proposition 15.11 that the condition on the right-hand side does not depend on the choice of the base point.

(3) Since $\exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is surjective we can pick a $y_0 \in \mathbb{C}$ with $\exp(y_0) = x_0$.
(4) Note that $\mathbb{C}$ is simply connected. The image $\operatorname{im}(\exp_*\colon \pi_1(\mathbb{C}, y_0) \to \pi_1(\mathbb{C} \setminus \{0\}, x_0))$ is therefore the trivial group.
(5) Since U is an open subset of $\mathbb{C} \setminus \{0\}$ it is in particular locally path-connected. Thus we can apply Proposition 29.2. Together with (4) we obtain from Proposition 29.2 the statement

$$\begin{array}{c}\text{there exists a lift of}\\ \text{the map } i\colon U \to \mathbb{C}\setminus\{0\}\end{array} \iff \begin{array}{c}\text{the map } i_*\colon \pi_1(U, x_0) \to \pi_1(\mathbb{C}\setminus\{0\}, x_0)\\ \text{is the trivial map.}\end{array}$$

(6) The combination of (2) and (5) is precisely the statement we had set out to prove. ■

30.3. **Preimages of subsets under coverings.** Let (B, b_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. In Proposition 29.5 (2) we saw that subgroups of $\pi_1(B, b_0)$ give rise to coverings of B. Arguably the most common source of subgroups is given by considering kernels of group homomorphisms $\pi_1(B, b_0) \to G$.

Example. Let B be the 2-dimensional sphere S^2 minus three open disks. We pick a base point b_0 for B and we pick loops u, v, w as shown in Figure 545. From Lemma 23.11 we know that $\pi_1(B, b_0) = \langle u, v, w \,|\, uvw \rangle$. By Lemma 21.4 there exists a unique homomorphism $\varphi\colon \pi_1(B, b_0) \to \mathbb{Z}_6$ with $\varphi(u) = 2$, $\varphi(v) = 3$ and $\varphi(w) = -5$. Let $p\colon X \to B$ be the covering corresponding to $\ker(\varphi)$. By Proposition 17.1 we know that X is again a compact orientable 2-dimensional smooth manifold, where the boundary is given by $\partial X = p^{-1}(\partial B)$. Thus we know from the Surface Classification Theorem 23.4 that there exist unique $g \in \mathbb{N}_0$ and $n \in \mathbb{N}$ such that X is homeomorphic to $\Sigma_{g,n}$, i.e. such that X is homeomorphic to the surface of genus g minus n open disks.

FIGURE 545

Therefore the following question arises:

Question 30.5. *In the previous example, what are g and n?*

The final proposition of this chapter will allow us to answer one half of Question 30.5, namely we can determine n. We will determine g in Exercise 31.3.

Proposition 30.6. *Let (B, b_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. Let $\varphi\colon \pi_1(B, b_0) \to G$ be a group epimorphism and let $p\colon (X, x_0) \to (B, b_0)$ be the covering with $p_*(\pi_1(X, x_0)) = \ker(\varphi)$. Finally let A be a path-connected subset of B and let $a_0 \in A$. We denote by $i\colon A \to B$ the* *inclusion map and we pick a path γ from b_0 to a_0. We consider the homomorphism*

$$\psi := \varphi \circ \gamma_* \circ i_* \colon \pi_1(A, a_0) \xrightarrow{i_*} \pi_1(B, a_0) \xrightarrow{\gamma_*} \pi_1(B, b_0) \xrightarrow{\varphi} G.$$

With these preparations the following two statements hold:

(1) *Each path-component of $p^{-1}(A)$ is the covering of A corresponding to the subgroup* $\ker(\psi \colon \pi_1(A, a_0) \to G)$.

(2) *If G is a finite group, then*

$$\text{number of path-components of } p^{-1}(A) \;=\; [G : \operatorname{im}(\psi \colon \pi_1(A, a_0) \to G)].$$

Example. As promised we can now determine the number n of boundary components of the covering X in the previous example. More precisely, denote by U, V, W the three boundary components of B. We make the following three observations:

(1) The image of $\pi_1(V)$ in $\mathbb{Z}_6$ is the subgroup generated by $\varphi(u) = 2$, thus the subgroup has index two, which by Proposition 30.6 implies that $p^{-1}(U)$ consists of two components.
(2) The image of $\pi_1(W)$ in $\mathbb{Z}_6$ is the subgroup generated by $\varphi(v) = 3$, thus $p^{-1}(V)$ consists of three components.
(3) The image of $\pi_1(U)$ in $\mathbb{Z}_6$ is the subgroup generated by $\varphi(w) = -5$. Since -5 is coprime to 6 we see that this subgroup is all of $\mathbb{Z}_6$, thus $p^{-1}(W)$ consists of precisely one component.

Altogether we see that X has $2 + 3 + 1 = 6$ boundary components.

Remark. The reader who is in need of a brain teaser might want to come up with an appropriate generalization of Proposition 30.6 to the case where we consider coverings of B coming from a non-normal subgroup.

Proof. To simplify the notation a little bit we assume that $a_0 = b_0$, in particular we ignore the role of γ. Throughout the proof we refer to Figure 546 for an illustration.

(1) Let W be a path-component of $p^{-1}(A)$. We pick a point $w_0 \in W$ with $p(w_0) = a_0$. We start out with the following slightly subtle claim.

Claim. We have the following equality:

$$p_*(\pi_1(X, w_0)) \;=\; \ker(\varphi \colon \pi_1(B, a_0) \to G).$$

We pick a path γ in X from x_0 to w_0. Since $p \circ \gamma$ is a loop in a_0 we can write $h := [p \circ \gamma] \in \pi_1(B, a_0)$. We consider the following diagram

$$\begin{tikzcd}
\pi_1(X, w_0) \arrow[r, "\gamma_*", "\cong"'] \arrow[d, "p_*"] & \pi_1(X, x_0) \arrow[d, "p_*"] \\
\pi_1(B, a_0) \arrow[r, "g \mapsto hgh^{-1}", "\cong"'] \arrow[d, "\varphi"] & \pi_1(B, a_0) \arrow[d, "\varphi"] \\
G \arrow[r, "g \mapsto \varphi(h) g \varphi(h)^{-1}", "\cong"'] & G.
\end{tikzcd}$$

The top square commutes by Proposition 14.11 (2) and (5). Evidently the bottom square also commutes. By definition of (X, x_0) we have

$$p_*(\pi_1(X, x_0)) \;=\; \ker(\varphi\colon \pi_1(B, a_0) \to G).$$

The claim now follows from the fact that the diagram commutes and that the horizontal maps are isomorphisms.

In the following we denote by $j\colon W \to X$ the inclusion map. We consider the following diagram

$$\begin{tikzcd}
\pi_1(W, w_0) \arrow[r, "j_*"] \arrow[d, "p_*"] & \pi_1(X, w_0) \arrow[d, "p_*"] \\
\pi_1(A, a_0) \arrow[r, "i_*"] \arrow[dr, "\psi"'] & \pi_1(B, a_0) \arrow[d, "\varphi"] \\
& G.
\end{tikzcd}$$

Recall that we need to show the equality $p_*(\pi_1(W, w_0)) = \ker(\psi)$. We show the two inclusions separately.

"$\subset$" This inclusion follows immediately from the definitions, from the commutativity of the above diagram and from the claim.

"$\supset$" Thus let $g \in \ker(\psi)$. We represent it by a loop γ in (A, a_0). By the above claim and Lemma 16.15 the loop $i \circ \gamma$ in (B, a_0) lifts to a loop $\widetilde{\gamma}$ in (X, w_0). By definition the image of $\widetilde{\gamma}$ lies in $p^{-1}(A)$ and it contains w_0. In particular the image of $\widetilde{\gamma}$ lies in the path-component of $p^{-1}(A)$ that contains w_0, in other words, $\widetilde{\gamma}$ is actually a loop in (W, w_0). Since $g = p_*([\widetilde{\gamma}])$ we obtain that $g \in p_*(\pi_1(W, w_0))$.

(2) We write $n = |G|$ and $k = |\operatorname{im}(\psi\colon \pi_1(A, a_0) \to G)|$. It follows from Lemma 16.15 and (1) that each path-component of $p^{-1}(A)$ is a k-fold covering of A. By Lemma 16.15 the map $p\colon X \to B$ is an n-fold covering. Evidently $p\colon p^{-1}(A) \to A$ is also an n-fold covering. It follows almost immediately that $p^{-1}(A)$ has necessarily n/k path-components. But $n/k = |G|/|\operatorname{im}(\psi)|$ is by definition the degree of $\operatorname{im}(\psi)$ in G. ■

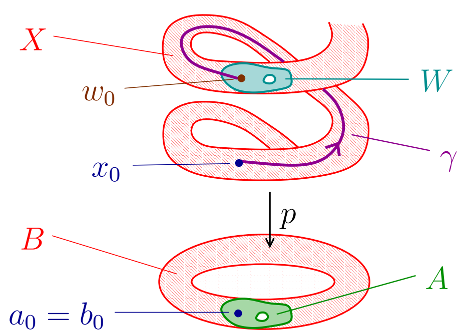


Figure 546. Illustration for the proof of Proposition 30.6.

30.4. **Covering spaces of surfaces.** Suppose we start out with a surface F of genus g and a finite index subgroup Γ of $\pi_1(F)$, then by Proposition 29.5 we obtain a corresponding finite covering $p\colon \widetilde{F} \to F$. By Proposition 17.1 and Lemma 16.3 (4) the topological space

$\widetilde{F}$ is again a closed orientable connected 2-dimensional smooth manifold. From the Surface Classification Theorem 23.4 we know that $\widetilde{F}$ is again a surface of some genus h. How can we determine h from g and Γ?

For future reference we record the question.

Question 30.7. *Given a surface F of genus g and a finite index subgroup Γ of $\pi_1(F)$, what is the genus of the corresponding finite cover?*

It will take a while till we can give a satisfactory answer. In fact we will obtain a complete answer only once we have introduced the Euler characteristic.

But we want to discuss one example in detail. We consider the surface F of genus two that we view as the connected sum of two tori $S = S^1 \times S^1$ and $T = S^1 \times S^1$. We denote by u, v the standard generators for $\pi_1(S)$ given by the two copies of S^1 in each torus. Similarly we denote by x, y the standard generators for $\pi_1(T)$. We pick an embedded closed disk in S and an embedded closed disk in T which both lie in the complement of the standard generators. We denote the corresponding open disks by D and E and we write $S^* = S \setminus D$ and $T^* \setminus E$. We build the connected sum $S\#T$ by gluing S^* to T^* along the boundary.[530] Furthermore, by a slight abuse of notation, we denote by u, v and x, y also the generators of $\pi_1(S^*) = \langle u, v\rangle$ and $\pi_1(T^*) = \langle x, y\rangle$.

By the discussion on page 659 the abelianization of $\pi_1(S\#T) = \pi_1(S^* \cup_{\partial S^* = \partial T^*} T^*)$ is the free abelian group on u, v, x, y. Now we consider the epimorphism

$$\varphi\colon \pi_1(S\#T) \;\to\; \pi_1(S\#T)/[\pi_1(S\#T), \pi_1(S\#T)] \;\to\; \mathbb{Z}_3$$

given by $u \mapsto 1, v \mapsto 0, x \mapsto 0$ and $y \mapsto 0$. Our goal in the following discussion is to visualize the covering $p\colon \widetilde{F} \to F = S\#T$ corresponding to $\ker(\varphi)$. Since φ is an epimorphism onto a group of order 3 we see that $\ker(\varphi)$ is a subgroup of index 3. By Lemma 16.15 this implies that p is a 3-fold covering.

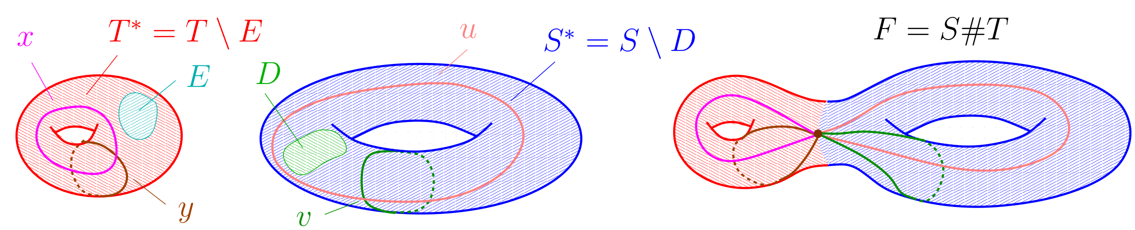


FIGURE 547

[530]It follows from Corollary 30.3 that we can be a bit casual in our language, i.e. we can talk about "gluing surfaces" without specifying explicit diffeomorphisms of boundary components, as long as we take into account orientations.

We first consider the torus $S = S^1 \times S^1$ and furthermore we consider the epimorphism $\psi\colon \pi_1(S) = \langle u, v \mid [u, v]\rangle \to \mathbb{Z}_3$ given by $\psi(u) = 1$ and $\psi(v) = 0$. The covering corresponding to $\ker(\psi) \subset \pi_1(S)$ is equivalent to the covering[531]

$$\begin{array}{rcl} \widetilde{S} = S^1 \times S^1 & \to & S = S^1 \times S^1 \\ (a, b) & \mapsto & (a^3, b). \end{array}$$

Now we consider the epimorphism $\psi^*\colon \pi_1(S^*) \to \mathbb{Z}_3$ given by $\psi^*(u) = 1$ and $\psi^*(v) = 0$. By the argument of Footnote 531 the covering corresponding to $\ker(\psi^*)$ is then given by $p\colon \widetilde{S}^* := \widetilde{S} \setminus p^{-1}(D) \to S^* = S \setminus D$. Recall that we perform the connected sum operation

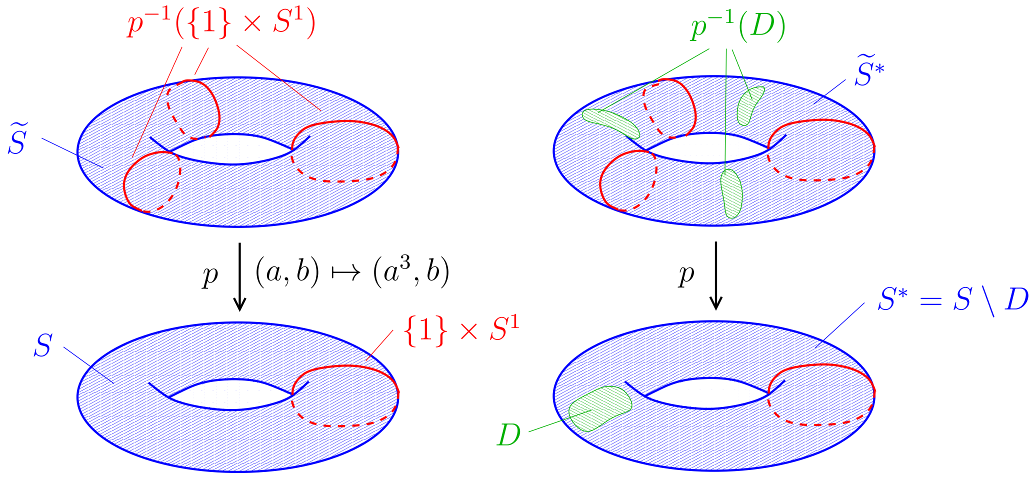


FIGURE 548

$S\#T$ by gluing $T^* := T \setminus E$ to $S^* = S \setminus D$ to obtain $F = S\#T$. The same way we can also glue three copies of T^*, i.e. $T^* \times \{1, 2, 3\}$, to the three boundary components of $\widetilde{S}^*$ to obtain a closed smooth manifold $\widetilde{F}$. It follows from Lemma 2.35 (2') that the map

$$\begin{array}{rcl} q\colon \widetilde{F} & \to & F \\ P & \mapsto & \begin{cases} p(P), & \text{if } P \in \widetilde{S}^*, \\ Q, & \text{if } P = Q \times \{n\} \text{ with } Q \in T^* \text{ and } n \in \{1, 2, 3\} \end{cases} \end{array}$$

is continuous. Similar to Footnote 531 one can now see that $q\colon \widetilde{F} \to F$ is indeed the covering of F corresponding to $\ker(\varphi)$. This construction is illustrated in Figure 549 which also shows that $\widetilde{F}$ is a surface of genus 4.

Exercise for Chapter 30.

Exercise 30.1. We write $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

[531]Indeed, since $\widetilde{S} \to S$ is a connected 3-fold covering we know from Lemma 16.15 that $p_*(\pi_1(\widetilde{S}))$ is a subgroup of $\pi_1(S)$ of index three. Furthermore it is contained in the subgroup $\ker\big(\pi_1(S) \xrightarrow{\psi} \mathbb{Z}_3\big)$ which also has index three, thus $p_*(\pi_1(\widetilde{S})) = \ker(\psi\colon \pi_1(S) \to \mathbb{Z}_3)$. But by Proposition 29.8 all path-connected coverings that correspond to the same subgroup are equivalent.

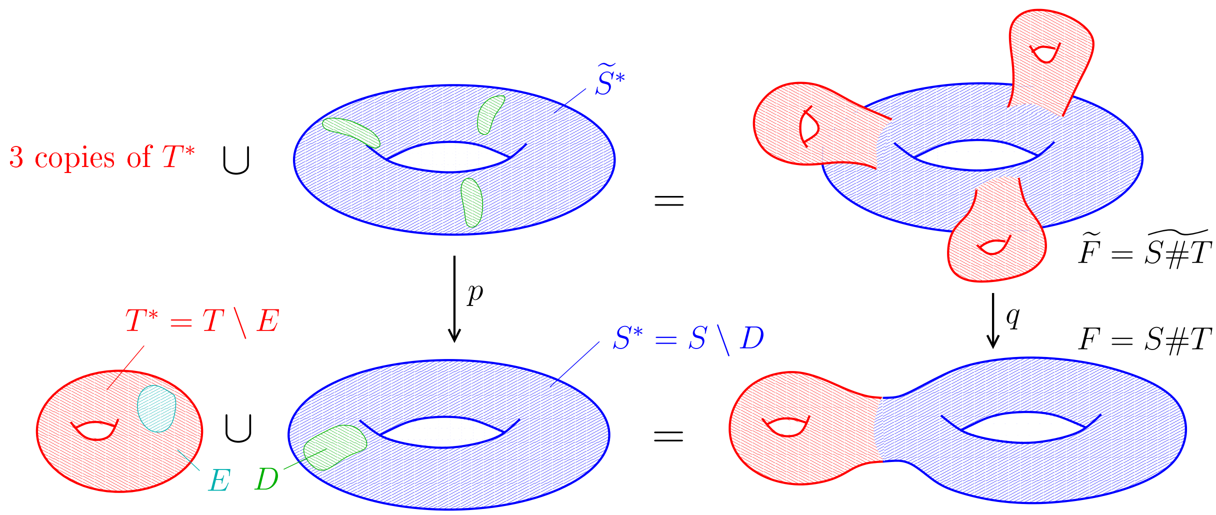


FIGURE 549

(a) Let $m \in \mathbb{N}$. Show that the map

$$\begin{array}{ccc} \mathbb{C}^* & \to & \mathbb{C}^* \\ z & \mapsto & z^m \end{array}$$

is a holomorphic map between complex manifolds and show that this map is a covering map with degree $= m$.

(b) Show that the exponential map

$$\begin{array}{rcl} \exp\colon \mathbb{C} & \to & \mathbb{C}^* := \mathbb{C} \setminus \{0\} \\ z & \mapsto & \exp(z) \end{array}$$

is a holomorphic map between complex manifolds and show that this map is a covering map of infinite degree.

(c) A *complex logarithm* on an open subset $U \subset \mathbb{C}^*$ is a map $f\colon U \to \mathbb{C}$ such that for all $z \in U$ we have $f(\exp(z)) = z$.
   (i) Show that every complex logarithm is holomorphic.
   (ii) Show that on any simply connected open subset $U \subset \mathbb{C}^*$ one can define a complex logarithm.
   (iii) Show that it is not possible to define a complex logarithm on all of $\mathbb{C}^*$.

Exercise 30.2. Let Σ be a compact oriented connected 2-dimensional smooth manifold and let $f\colon \partial\Sigma \to \partial\Sigma$ be a diffeomorphism. Show that there exists a diffeomorphism $\Sigma \to \Sigma$ with $F|_{\partial\Sigma} = f$ if and only if f is either orientation-preserving or orientation-reversing.
Hint. Use Lemma 23.14, Proposition 30.1 and the Collar Neighborhood Theorem 8.12.

Exercise 30.3. Let $n \in \mathbb{N}_{\geq 3}$.

(a) We consider the connected sum $M = \mathbb{RP}^n \# \mathbb{RP}^n$. We consider the following homomorphism

$$\begin{array}{ccccccc} \varphi\colon \pi_1(\mathbb{RP}^n\#\mathbb{RP}^n) & \xrightarrow{\cong} & \pi_1(\mathbb{RP}^n) * \pi_1(\mathbb{RP}^n) & \xrightarrow{\cong} & \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a \mid a^2\rangle * \langle b \mid b^2\rangle & \to & \mathbb{Z}_2. \\ & \uparrow & & \uparrow & a & \mapsto & 1 \\ & \text{Proposition 20.12} & & \text{Corollary 16.18} & b & \mapsto & 1 \end{array}$$

We denote by $\widetilde{M}$ the cover corresponding to $\ker(\varphi\colon \pi_1(M) \to \mathbb{Z}_2)$. Show that $\widetilde{M}$ is diffeomorphic to $S^1 \times S^{n-1}$.
Hint. Use Proposition 30.6.

(b) Let $k \in \mathbb{N}$. We consider the connected sum M of k copies of $\mathbb{RP}^n$. As in (a) we have an obvious epimorphism $\pi_1(M) \to \mathbb{Z}_2$. It turns out that the corresponding covering space $\widetilde{M}$ is diffeomorphic to the connected sum of m copies of $S^1 \times S^{n-1}$. Determine m in terms of k.

31. Coverings of topological graphs and applications to group theory

In this chapter we will apply the results obtained in Chapter 29 to prove several results in group theory.

31.1. Coverings of topological graphs.

In our proofs of the promised group-theoretic results we will need the following proposition which is also of interest in its own right.

Proposition 31.1. *Let $G = (V, E, i, t)$ be a connected abstract graph and let $p\colon \widetilde{X} \to |G|$ be a covering of the topological realization $X := |G|$ of G. Then there exists a natural*[532] *abstract graph $\widetilde{G} = (\widetilde{V}, \widetilde{E}, i, t)$, a natural homeomorphism $f\colon |\widetilde{G}| \to \widetilde{X}$ and a natural map $q\colon \widetilde{G} \to G$ of abstract graphs with the following properties:*

1. *We have $\#\widetilde{V} = [\widetilde{X} : X] \cdot \#V$ and $\#\widetilde{E} = [\widetilde{X} : X] \cdot \#E$.*
2. *The following diagram commutes:*

$$\begin{tikzcd} |\widetilde{G}| \arrow[r, "f"] \arrow[d, "|q|"'] & \widetilde{X} \arrow[d, "p"] \\ |G| \arrow[r, "="] & X. \end{tikzcd}$$

Here $|q|$ denotes the topological realization of the map q, as defined in Lemma 15.2.

Remark. In Proposition 36.37 we will generalize the above proposition to the setting of "CW-complexes".

Proof. Let $G = (V, E, i, t)$ be a connected abstract graph. Recall that on page 222 we defined the topological realization of G as follows:

$$|G| \ := \ \big(V \ \sqcup \ (E \times [0, 1])\big)/\sim \quad \text{where for } e \in E \text{ we have } (e, 0) \sim i(e) \text{ and } (e, 1) \sim t(e).$$

Note that by Lemma 4.1 we know that $|G|$ is connected. Given $e \in E$ we set $\mu(e) := [(e, \frac{1}{2})]$, i.e. $\mu(e)$ is the "midpoint" of e. Furthermore, given $e \in E$ we denote by $\gamma_e\colon [0, 1] \to X$ the path given by $t \mapsto [(e, t)]$.

Note that in Exercise 18.38 we showed that $X = |G|$ is locally contractible, in particular it is locally path-connected. Thus in the following discussion we will have no troubles with applying the results from Chapter 29.

Now let $p\colon \widetilde{X} \to |G|$ be a covering of the topological realization $X = |G|$ of G. We set $\widetilde{V} := p^{-1}(V) \subset \widetilde{X}$ and we set $\widetilde{E} := p^{-1}(\{\mu(e) \,|\, e \in E\}) \subset \widetilde{X}$. Since $X = |G|$ is connected we know that the degree $[\widetilde{X} : X]$ is defined. By definition, or if you want, by Lemma 16.1, we have $\#\widetilde{V} = [\widetilde{X} : X] \cdot \#V$ and $\#\widetilde{E} = [\widetilde{X} : X] \cdot \#E$.

Now let $\widetilde{e} \in \widetilde{E}$. We denote by $\varphi(\widetilde{e})$ the unique edge in E with $p(\widetilde{e}) = \mu(\varphi(\widetilde{e}))$. By Proposition 29.2 there exists a unique lift $\widetilde{\gamma}_{\widetilde{e}}\colon [0, 1] \to \widetilde{X}$ of $\gamma_{\varphi(\widetilde{e})}\colon [0, 1] \to X$ with $\widetilde{\gamma}(\frac{1}{2}) = \widetilde{e}$. We set $i(\widetilde{e}) = \widetilde{\gamma}(0)$ and $t(\widetilde{e}) = \widetilde{\gamma}(1)$. Since $\widetilde{\gamma}_{\widetilde{e}}$ is a lift of γ_e we see that $i(\widetilde{e})$ and $t(\widetilde{e})$ actually lie in $\widetilde{V}$. In other words, $(\widetilde{V}, \widetilde{E}, i, t)$ is actually an abstract graph. Furthermore

[532] We leave the pleasure of figuring out what "natural" means to the reader.

note that the two maps $p\colon \widetilde{V} \to V$ and $\varphi\colon \widetilde{E} \to E$ define by, basically by construction, a map $q\colon \widetilde{G} = (\widetilde{V}, \widetilde{E}, i, t) \to (V, E, i, t) = G$ of abstract graphs.

Next we define $f\colon |\widetilde{G}| \to \widetilde{X}$ via the inclusion $\widetilde{V} \to \widetilde{X}$ and by sending $(\widetilde{e}, t)$ to $\widetilde{\gamma}_{\widetilde{e}}(t)$. It follows immediately from Lemma 3.22 that f is continuous. Furthermore, using the existence and uniqueness statement of Proposition 29.2 it is not difficult to see that the map f is actually a bijection. Next note that, with a little bit of an effort, one can also show that f is an open map. Thus we obtain from Lemma 2.42 that f is actually a homeomorphism.

Finally note that it is straightforward to see that the given diagram in the proposition does indeed commute. ■

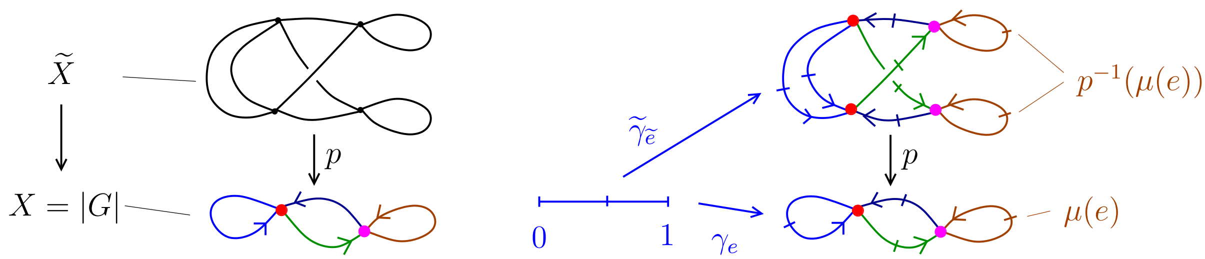


FIGURE 550. Illustration for the proof of Proposition 31.1.

31.2. **Fundamental group of countable topological graphs.** In Proposition 20.5 we showed that the fundamental group of any finite topological graph is a free group. We generalize this result to countable topological graphs.

Proposition 31.2. *If X is a countable connected non-empty topological graph, then $\pi_1(X)$ is a free group.*

Remark. In Theorem 37.9 we will show that the fundamental of any (not necessarily countable) connected topological graph is a free group. Since the proof of this more general statement is technically rather demanding we feel that it is worth to prove this intermediate statement.

The proof of Proposition 31.2 builds on the following reasonably straightforward lemma.

Lemma 31.3. *Let H be a finite connected graph and let G be a connected non-empty subgraph of H. Let v be a vertex of G. We denote by $i\colon |G| \to |H|$ the inclusion. We set $s := \chi(H) - \chi(G)$. Then there exists a homomorphism $\varphi\colon \langle x_1, \dots, x_s\rangle \to \pi_1(|H|, v)$ such that the map*

$$i_* * \varphi\colon \pi_1(|G|, v) * \langle x_1, \dots, x_s\rangle \;\to\; \pi_1(|H|, v)$$

is an isomorphism.

PROOF OF LEMMA 31.3. Let $H = (W, F, i, t)$ be a finite connected graph. Furthermore let $G = (V, E, i, t)$ be a connected non-empty subgraph of H. By Proposition 4.8 (1) there exists a spanning tree S for G. By a rather elementary argument, which we outsourced to Exercise 31.1, there exists a spanning tree T of H such that $T \cap G = S$. Let $e_1, \dots, e_m$ be the edges of G not contained in S and let $e_{m+1}, \dots, e_n$ be the edges of H not contained in T.

For $j = 1, \ldots, m$ we pick a path α_j in $|S|$ from v to $i(e_j)$, we denote by β_j the obvious path form $i(e_j)$ to $t(e_j)$ and we pick a path γ_j in $|S|$ from $t(e_j)$ to v. We proceed analogously for $j = m+1, \ldots, n$, except that now can no longer demand that the paths α_j and γ_j lie in $|S|$, but we will demand that they lie in $|T|$. Finally we consider the following diagram:

$$\begin{tikzcd}
& \pi_1(|G|,v) * \langle y_{m+1}, \ldots, y_{m+n} \rangle \arrow[dr] & \\
\langle y_1, \ldots, y_m \rangle * \langle y_{m+1}, \ldots, y_{m+n} \rangle \arrow[ur, "\Theta_G * \mathrm{id}"] \arrow[rr, "\Theta_H"] & & \pi_1(|H|,v).
\end{tikzcd}$$

Here the homomorphisms Θ_G and Θ_H are the maps induced by $y_j \mapsto [\alpha_j * \beta_j * \gamma_j]$. By Proposition 20.5 we know that both Θ_G and Θ_H are isomorphisms. This shows that the homomorphism $\langle x_1, \ldots, x_{n-m} \rangle \to \pi_1(|H|,v)$ given by $x_j \mapsto \Theta_H(y_{m+j})$ has the desired properties. Finally one can easily verify that $\chi(G) = 1+m$ and $\chi(H) = 1+n$. Thus we see that $n - m = \chi(H) - \chi(G)$. ∎

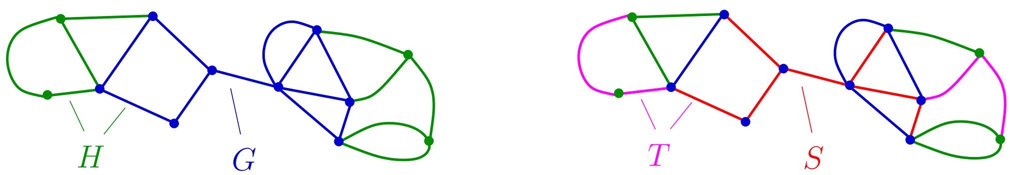


FIGURE 551. Illustration for the proof of Lemma 31.3.

Now we can provide the proof of Proposition 31.2.

PROOF OF PROPOSITION 31.2. Let $G = (V, E, i, t)$ be a countable connected non-empty abstract graph. We pick a vertex $v \in V$. We need to show that $\pi_1(G, v)$ is a free group. By Exercise 4.7 we know that there exists a sequence $\{G_j\}_{j \in \mathbb{N}}$ of subgraphs with the following properties:

(1) Each G_j is finite connected.
(2) Each G_j contains v.
(3) We have $G = \bigcup\limits_{i \in \mathbb{N}} G_j$ in the sense that $V = \bigcup\limits_{j \in \mathbb{N}} V_j$ and $E = \bigcup\limits_{j \in \mathbb{N}} G_j$.

We see that

$$\pi_1(|G|,v) \;=\; \pi_1\Big(\bigcup_{j \in \mathbb{N}} |G_j|, v\Big) \underset{\substack{\uparrow \\ \text{by Exercise 4.6 and Proposition 25.7}}}{=} \varinjlim \pi_1(|G_j|,v) \underset{\substack{\uparrow \\ \text{follows from Lemma 31.3 and} \\ \text{the discussion on page 730}}}{=} \text{free group.}$$

∎

31.3. **Subgroups of free groups.** In this section we will apply our topological results to the study of free groups. First we introduce the following definition.

Definition. Given a free group F, i.e. given a group F that is isomorphic to the free group $\langle S \rangle$ generated by some set S, we refer to the cardinality of S as the *rank* of F.[533]

[533]It follows from the discussions on the pages 641 and 582 that the rank of F is well-defined.

The following proposition is arguably the most interesting result of this chapter. The statement is purely group-theoretical, but as we will see, a topological point of view leads to a very neat proof.

Proposition 31.4. *Let F be a free group of countable rank.*

(1) *Every subgroup of F is again a free group.*
(2) *If F is finitely generated and if G is a finite-index subgroup of F, then*[534]

$$\operatorname{rank}(G) - 1 \ = \ [F : G] \cdot (\operatorname{rank}(F) - 1).$$

Examples.

(1) If $F = \mathbb{Z}$ then the formula from Proposition 31.4 does indeed show that every finite-index subgroup of $\mathbb{Z}$ is again a free group of rank one, i.e. every finite-index subgroup of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$.
(2) Let $\alpha\colon F := \langle a, b\rangle \to \mathbb{Z}_3$ be the epimorphism defined given by $\alpha(a) = 1$ and $\alpha(b) = 0$. It follows from Proposition 31.4 that the kernel $G := \ker(\alpha)$ is a free group of rank $[F : G] \cdot (\operatorname{rank}(F) - 1) + 1 = 3 \cdot 1 + 1 = 4$. In Exercise 31.2 we will determine four elements of $G = \ker(\alpha)$ that are generators for this free group.

Remark.

(1) In Corollary 37.10 we will extend the statement to all free groups, i.e. we will drop the condition that the rank should be countable. The idea of the proof is fairly similar to the one used below, but the proof of the more general statement is overall "technologically" somewhat more fancy.
(2) There are purely group theoretic proofs of Proposition 31.4 and of Corollary 37.10, see e.g. [**Schre27**], [**HallM76**, Theorem 7.2.1] or [**Bau93**, Chapter III.3], but the topological proof that we had just provided is arguably conceptually the clearest proof.
(3) Corollary 37.10 and Proposition 31.4 are a special case of the *Kurosh Subgroup Theorem*. This theorem says that if G is the product of groups $A_i, i \in I$ and if H is a subgroup of G, then H is a free product of a free group and groups $B_i, i \in I$ where each B_i is a (potentially trivial) group that is conjugate to a subgroup of A_i. We refer to [**LS77**, Theorem IV.1.10] for details and a proof.

Proof. Let $F = \langle E\rangle$ be a free group on a countable generating set E and let G be a subgroup of F. Note that it follows easily from Lemma 1.7 that G is a countable set. Let $\Gamma = (\{x_0\}, E, i, t)$ be the abstract graph with one vertex and one edge for each element of E. By the discussion on page 560 we know that the topological realization $X := |G|$ is homeomorphic to the wedge $\bigvee_{e\in E} S^1_e$. It follows from Lemma 20.4 that there exists an isomorphism $\varphi\colon F = \langle E\rangle \to \pi_1(X, x_0)$. By Proposition 36.10 (6) and Proposition 29.5 there exists a path-connected covering $p\colon (Y, y_0) \to (X, x_0)$ of degree $[F : G]$ such that $p_*(\pi_1(Y, y_0)) = G$. By Corollary 16.14 the map p_* is injective. Thus we see that $\pi_1(Y, y_0)$ itself is isomorphic to G. By Proposition 31.1 we can equip Y with the structure of a

[534]Does the equality hold if G is infinite-index and if we interpret $[F : G] \cdot (\operatorname{rank}(F) - 1)$, as usual, as $\infty \cdot (\operatorname{rank}(F) - 1) = \infty$ for $F \ncong \mathbb{Z}$?

topological graph. Since E and G are countable we see that Y is a countable topological graph. We deduce from Proposition 31.2 that $G \cong \pi_1(Y, y_0)$ is a free group. This completes the proof of (1).

Now suppose that F is finitely generated and that G is a finite-index subgroup of F. We see that we have the following equalities:

$$\underbrace{\operatorname{rank}(\pi_1(Y,y_0))}_{=\operatorname{rank}(G)} - 1 \underset{\text{Proposition 20.5 (2)}}{\overset{\uparrow}{=}} \chi(Y) \underset{\text{Proposition 31.1}}{\overset{\uparrow}{=}} [Y:X]\cdot\chi(X) \underset{\text{Proposition 20.5 (2)}}{\overset{\uparrow}{=}} [F:G]\cdot(\underbrace{\operatorname{rank}(\pi_1(X,x_0))}_{=\operatorname{rank}(F)} - 1).$$

This completes the proof of (2). ■

31.4. **Subgroups of surface groups.** We start out with the following proposition that is similar in spirit to Proposition 31.2, but potentially it is more surprising.

Proposition 31.5. *If M is a connected 2-dimensional smooth manifold that is non-compact, then $\pi_1(M)$ is a free group.*

Remark. A somewhat different proof of Proposition 31.5 is given in [**Sti93**, p. 142].

Example. The 2-dimensional smooth manifolds shown in Figure 552 are non-compact and thus their fundamental groups are free. For the example on the left we saw this already in Lemma 25.11.

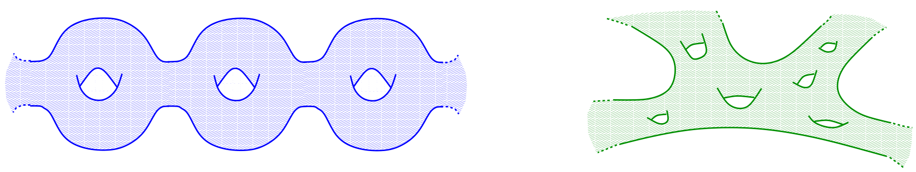


FIGURE 552

PROOF. Let M be a connected 2-dimensional smooth manifold that is non-compact. In Corollary 44.7 we will show, using the Collar Neighborhood Theorem 8.12, that M is homotopy equivalent to a smooth manifold with empty boundary. So we might as well assume that M has no boundary.

By Proposition 6.64 we know that there exists a sequence $X_1, X_2, \ldots$ of n-dimensional smooth submanifolds of M with the following four properties:

(1) The sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$.
(2) Each X_i is compact and connected.
(3) The X_i are submanifolds in the sense of the definition on page 270.
(4) We have $\bigcup_{i\in\mathbb{N}} \mathring{X}_i = M$.

We pick an embedding $\varphi\colon \overline{B}^2 \to X_1 \setminus \partial X_1$ and given any $i \in \mathbb{N}$ we set $X_i' := X_i \setminus \varphi(B^2)$. With a moderate effort we see, by iteratively applying the Surface Classification Theorem 23.4 and Lemma 23.12, that there exists a sequence of finite topological graphs $\{G_i\}_{i\in I}$ with the following properties:

(5) G_1 contains a point $v \in \varphi(S^1)$.
(6) For each $i \in \mathbb{N}$ we have $G_i \subset G_{i+1}$.

(7) Each G_i is a deformation retract of X'_i.

We see that

$$\begin{aligned}\pi_1(M\setminus\varphi(B^2),v) &= \pi_1\Big(\bigcup_{j\in\mathbb{N}} X'_j,v\Big) \overset{\substack{\text{follows from (4) and Proposition 25.7}\\ \downarrow}}{=} \varinjlim \pi_1(X'_j,v)\\ &\underset{\substack{\uparrow\\ \text{by (7) and Proposition 18.16}}}{=} \varinjlim \pi_1(|G_j|,v) \underset{\substack{\uparrow\\ \text{follows from Lemma 31.3 and the discussion on page 730}}}{=} \pi_1(G_1,v)*\text{free group}.\end{aligned}$$

Now we can perform the following calculation:

$$\begin{aligned}\pi_1(M,v) &= \pi_1((M\setminus\varphi(B^2))\cup_C\varphi(\overline{B}^2),v) \overset{\substack{\text{Seifert–van Kampen Theorem 22.2}\\ \downarrow}}{\cong} \pi_1(M\setminus\varphi(B^2),v) *_{\pi_1(C,v)} \overbrace{\pi_1(\varphi(\overline{B}^2),v)}^{=\{e\}}\\ &\underset{\substack{\uparrow\\ \text{by the above calculation of } \pi_1(M\setminus\varphi(B^2),v)\\ \text{together with Lemma 21.22 (2)}}}{\cong} (\pi_1(G_1,v)*\text{free group})/\langle\!\langle \pi_1(C,v)\rangle\!\rangle \underset{\substack{\uparrow\\ \text{by Lemma 23.12 we know that we can}\\ \text{write } \pi_1(G_1,v)=\pi_1(C,v)*\text{free group}}}{=} \text{free group.}\end{aligned}$$

■

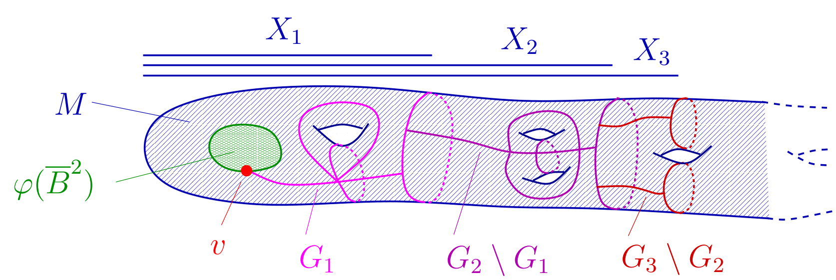


FIGURE 553. Illustration for the proof of Proposition 31.5.

Next we address the following question.

Question 22.6. *Let π be the fundamental group of the surface of genus $g\geq 2$. What isomorphism types of subgroups can appear? Is every subgroup again a free group or the fundamental group of a surface?*

The following proposition gives an affirmative answer.

Proposition 31.6. *Let M be the surface of genus g and let $\Gamma\subset\pi_1(M)$ be a subgroup.*

(1) *If the index $[\pi:\Gamma]$ is finite, then Γ is the fundamental group of the surface of genus $[\pi:\Gamma]\cdot(g-1)+1$.*
(2) *If the index $[\pi:\Gamma]$ is infinite, then Γ is a free group.*

Remark. In Exercise 31.6 we will consider the case of subgroups of the fundamental group of a *non-orientable* surface of some genus $g\in\mathbb{N}$.

PROOF. Let $g\in\mathbb{N}_{\geq 2}$ and let Σ be the surface of genus g. We pick a base point $x_0\in\Sigma$. Let $\Gamma\subset\pi_1(\Sigma,x_0)$ be a subgroup. We obtain from Proposition 36.10 (6) and Proposition 29.5 that there exists a path-connected covering $p\colon(\widetilde{\Sigma},\widetilde{x}_0)\to(\Sigma,x_0)$ of index $[\pi_1(M,x_0):\Gamma]$

with $p_*(\pi_1(\Sigma,\widetilde{x}_0)) = \Gamma$. By Proposition 17.1 we know that $\widetilde{\Sigma}$ is an orientable 2-dimensional smooth manifold with empty boundary.

(1) If the index $[\pi : \Gamma]$ is finite, then we know by Lemma 16.3 (4) that $\widetilde{\Sigma}$ is again compact. This implies that we now know that $\widetilde{\Sigma}$ is a closed 2-dimensional smooth manifold. In Exercise 31.6 we will show that the genus is given by the slightly weird formula
(2) By Lemma 16.3 (5) we know that $\widetilde{\Sigma}$ is non-compact. Thus it follows from Proposition 31.5 that $\Gamma \cong \pi_1(\widetilde{\Sigma})$ is a free group. ■

We will prove the following corollary to Proposition 31.6 in Exercise 31.4. Note that it gives an affirmative answer to Question 22.9 (1).

Corollary 31.7. *For every $g \in \mathbb{N}$ the fundamental group of the surface of genus g is torsion-free.*

We conclude this section with the following proposition which is closely related to Proposition 21.14. We will not prove the proposition, but we will it use at one occasion later on.

Proposition 31.8. *Let M be a surface of genus ≥ 2. If $\Gamma \subset \pi_1(M)$ is a* normal *non-trivial subgroup of infinite-index, then $\pi_1(M)$ is a free group of* infinite *rank.*

PROOF. Let M be a surface of genus ≥ 2 and let $\Gamma \subset \pi_1(M)$ be a *normal* non-trivial subgroup of infinite-index. By Proposition 31.6 it remains to show that Γ is infinitely generated. A proof for this statement is given in [**Cat03**, Lemma 3.4], alternatively one can prove the second statement using [**Lü97**, Theorem 0.7]. A significantly stronger statement is proved in [**BrH07**, Theorem 1]. ■

31.5. **Residually finite groups and Hopfian groups.** Let (X, x_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. In Proposition 29.8 we saw that there exists a bijection between the set of subgroups of $\pi_1(X, x_0)$ and the set of equivalence classes of path-connected coverings of (X, x_0).

Let (X, x_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected that is compact. It follows from the above, that given a (finite-index) subgroup of $\pi_1(X, x_0)$ we can associate to (X, x_0) a (finite) covering $p\colon \widetilde{X} \to X$. For the most part we like to stick to the world of compact topological spaces X. By Lemma 16.3 this means that we are interested in finite-index subgroups of $\pi_1(X, x_0)$. This leads us to the following, purposefully vague question.

Question 31.9. *Which (finitely generated) groups have lots of finite-index subgroups?*

In this section we will try to address Question 31.9. The following definition, which we already stated in Exercise 19.14, gives one popular way to turn Question 31.9 into a proper mathematical question.

Definition. A group π is called *residually finite* if given any non-trivial element $g \in \pi$ there exists a homomorphism $\alpha\colon \pi \to G$ to a finite group G such that $\alpha(g)$ is non-trivial.

Examples.

(1) The group $\mathbb{Z}$ is residually finite, indeed given $g \in \mathbb{Z}$ we can consider the epimorphism $\alpha\colon \mathbb{Z} \to \mathbb{Z}/(|g|+1)\cdot\mathbb{Z}$ onto a finite group, which evidently satisfies $\alpha(g) \neq 0$.
(2) It follows easily from the classification of finitely generated abelian groups, see 19.4, a slight generalization of (1), that every finitely generated abelian group is residually finite.
(3) In Exercise 31.12 we will see that the group $(\mathbb{Q}, +)$ is not residually finite.

The following is now a meaningful interpretation of Question 31.9.

Question 31.10. *Which (finitely generated) groups are residually finite?*

Before we address Question 31.10 let us heighten the interest in Question 31.10 by stating a surprising property of residually finite groups.

Definition. A group π is called *Hopfian*[535] if every epimorphism $\varphi\colon \pi \to \pi$ is in fact a monomorphism.

Example.

(1) It follows immediately from Lemma 19.8 (5) that every finitely generated abelian group is Hopfian.
(2) Note that in contrast to (1), the *infinitely generated* free abelian group $\mathbb{Z}^{(\mathbb{N})}$ is actually not Hopfian. Indeed, the homomorphism $\varphi\colon \mathbb{Z}^{(\mathbb{N})} \to \mathbb{Z}^{(\mathbb{N})}$ which is given by $\varphi(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$ is an epimorphism, but evidently it is not a monomorphism.

It is self-evident that being Hopfian is a convenient property. As we will see later in the Hopf Conjecture 69.14, the notion of a Hopfian group also naturally arises in topology. The following proposition relates the two concepts that we just introduced.

Proposition 31.11. *Let π be a finitely generated group. If π is residually finite, then it is Hopfian.*

Remark. Shortly we will see that the converse to Proposition 31.11 does not hold.

Proof. Let π be a finitely generated group that is residually finite. Let $\varphi\colon \pi \to \pi$ be an epimorphism. Now we assume that φ is not a monomorphism. We pick a non-trivial element $g \neq e \in \ker(\varphi)$. Since π is residually finite there exists a homomorphism $\alpha\colon \pi \to G$ onto a finite group G such that $\alpha(g) \neq e$.

Since π is finitely generated we know by the rather elementary Exercise 21.3 that the set $\operatorname{Hom}(\pi, G)$ of homomorphisms from π to G is finite. We set $n := \#\operatorname{Hom}(\pi, G)$ and we denote by $\beta_1, \dots, \beta_n\colon \pi \to G$ the distinct homomorphisms.

Next we consider the homomorphisms $\beta_1 \circ \varphi, \dots, \beta_n \circ \varphi\colon \pi \to G$. Since φ is an epimorphism we see that these homomorphisms are also distinct. But note they all satisfy $(\beta_i \circ \varphi)(g) = \beta_i(\varphi(g)) = \beta_i(e) = e$. So these n homomorphisms are different from our initial homomorphism $\alpha\colon \pi \to G$. This means that we have found $n+1$ distinct homomorphisms from $\pi \to G$. This is a contradiction to the definition of n. ■

[535] This notion is once again named after Heinz Hopf.

Now that we are fully convinced of the value of residually finite groups, let us explore this class of groups in greater detail. The following proposition gives a convenient characterization of residually finite groups.

Proposition 31.12. *Let π be a group. The following two statements are equivalent.*

(1) *The group is residually finite.*
(2) *Given any non-trivial element $g \in \pi$ there exists a finite-index subgroup Γ with $g \notin \Gamma$.*

If π is countable, then the above are also equivalent to the following statement:

(3) *There exists a sequence $\pi \supset \pi_0 \supset \pi_1 \supset \pi_2 \supset \dots$ of finite index normal subgroups of π such that $\bigcap\limits_{i\in\mathbb{N}} \pi_i = \{e\}$.*

The proof of Proposition 31.12 rests on the following group-theoretic concept.

Definition. Let G be a group and let H be a subgroup. We refer to

$$H^c \;:=\; \bigcap_{g\in G} gHg^{-1}$$

as the *normal core of G*.

The following lemma is the key ingredient for the proof of Proposition 31.12. But note that the lemma is also interesting in its own right.

Lemma 31.13. *Let G be a group and let H be a subgroup.*

(1) *The normal core of H is a normal subgroup of G.*
(2) *If X is a subgroup of G that is contained in H and that is normal in G, then $X \subset H^c$.*
(3) *If H is of finite-index d in G, then H^c is a subgroup of G of index at most $d!$, in particular H^c is also of finite-index in G.*

Proof of Lemma 31.13.

(1) Let $k \in G$. We have

$$kH^ck^{-1} \;=\; \bigcap_{g\in G} k\cdot gHg^{-1}\cdot k^{-1} \;=\; \bigcap_{g\in G} (kg)H(kg)^{-1} \;=\; \bigcap_{g\in G} gHg^{-1} \;=\; H^c.$$

(2) Let X be a subgroup of G that is contained in H and that is normal in G. Then

$$X \;\underset{\substack{\uparrow\\ \text{since } X \text{ normal in } G}}{=}\; \bigcap_{g\in G} gXg^{-1} \;\underset{\substack{\uparrow\\ \text{since } X\subset H}}{\subset}\; \bigcap_{g\in G} gHg^{-1} \;=\; H^c.$$

(3) The proof of the last statement is arguably more interesting. Thus let H be a subgroup of index $d \in \mathbb{N}$ of G. We consider the set G/H of right coset of G with respect to H. We denote by $\mathrm{Perm}(G/H)$ the group of permutations of G/H, i.e. it is the group of all bijections of the set G/H. Note that G/H and thus also $\mathrm{Perm}(G/H)$

are finite. We consider the map

$$\begin{array}{rcl} \varphi\colon G & \mapsto & \operatorname{Perm}(G/H) \\ g & \mapsto & \left(\begin{array}{rcl} G/H & \to & G/H \\ xH & \mapsto & gxH \end{array}\right). \end{array}$$

This map is easily seen to be a homomorphism. Since $\operatorname{Perm}(G/H)$ is finite we see that $\ker(\varphi)$ is a finite-index normal subgroup of G. We claim that $\ker(\varphi) \subset H$. Thus let $g \in \ker(\varphi)$. This implies in particular that $\varphi(g)(H) = H$, but that means that $gH = H$, i.e. that $g \in H$. We have thus shown that indeed $\ker(\varphi) \subset H$. From (2) and the above observation that $\ker(\varphi)$ is normal in G we obtain that $\ker(\varphi) \subset H^c$. But since the index of $\ker(\varphi)$ in G is at most $d!$ we see that the index of the normal H^c in G is also at most $d!$. ■

Proof of Proposition 31.12. Let π be a group.

(1)⇒(2) Let $g \in \pi$ be a non-trivial element. Since π is residually finite there exists a homomorphism $\alpha\colon \pi \to G$ to some finite group G such that $\alpha(g)$ is non-trivial. The kernel $\Gamma := \ker(\alpha)$ is a finite-index subgroup of π with $g \notin \Gamma$.

(2)⇒(1) Let $g \in \pi$ be a non-trivial element and let Γ be a finite-index subgroup of π with $g \notin \Gamma$. By Lemma 31.13 we know that the normal core Γ^c is a finite-index normal subgroup of π. Thus we see that the homomorphism $\alpha\colon \pi \to \pi/\Gamma^c$ is an epimorphism onto a finite group with $\gamma(g) \neq e$.

(3)⇒(1) This statement is trivial. Note that here we do not need that π is countable.

(1)⇒(3) We outsource the straightforward proof to Exercise 31.9. ■

The following lemma gives a topological criterion for a (fundamental) group to be residually finite.

Lemma 31.14. *Let (X, x_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. The following two statements are equivalent:*

1. *The group $\pi_1(X, x_0)$ is residually finite.*
2. *Given any loop $\gamma\colon [0,1] \to X$ in (X, x_0) that represents a non-trivial element in $\pi_1(X, x_0)$ there exists a finite covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that the corresponding lift $\widetilde{\gamma}\colon [0,1] \to \widetilde{X}$ is not a loop.*

Proof. Let (X, x_0) be a topological space that is path-connected, locally path-connected and semi-locally simply connected. First we prove the (1)⇒(2)-implication. Thus we assume that the group $\pi_1(X, x_0)$ is residually finite. Let $\gamma\colon [0,1] \to X$ be a loop in (X, x_0) that represents a non-trivial element in $\pi_1(X, x_0)$. By Proposition 31.12 there exists a finite-index subgroup Γ of $\pi_1(X, x_0)$ such that $[\gamma] \notin \Gamma$. Since X is locally path-connected and semi-locally simply connected we know by Proposition 29.5 that there exists a covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ of X of degree $[\pi : \Gamma]$ with $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \Gamma$. By Lemma 16.15 we know that the corresponding lift $\widetilde{\gamma}\colon [0,1] \to \widetilde{X}$ is not a loop.

Now we prove the (2)⇒(1)-implication. Let $g \in \pi_1(X, x_0)$ be a non-trivial element. We pick a representative loop $\gamma\colon [0,1] \to (X, x_0)$. By hypothesis there exists a finite covering

$p\colon (\widetilde{X},\widetilde{x}_0) \to (X,x_0)$ such that the corresponding lift $\widetilde{\gamma}\colon [0,1] \to \widetilde{X}$ is not a loop. By Lemma 16.15 we know that $g = [\gamma]$ does not lie in $p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ and that this is a finite-index subgroup of $\pi_1(X,x_0)$. It follows from Proposition 31.12 that $\pi_1(X,x_0)$ is residually finite. ∎

Proposition 31.15. *Every free group is residually finite.*

In the following we will provide two proofs of Proposition 31.15. The first is algebraic, very short and elegant. The other proof is topological in nature and great fun. The eagle-eyed reader will notice that the basic ideas of the two proofs are actually not that dissimilar.

Remark. In Exercise 19.14 we gave a third proof for Proposition 31.15. Furthermore, a completely different proof of Proposition 31.15, based again on topology, is given in [**Hem72**].

First proof of Proposition 31.15. Let X be a set and let $g \in \langle X\rangle$ be a non-trivial element in the free group generated by the set X. We can and will write g as a product $g = y_1^{\epsilon_1} \cdot \dots \cdot y_k^{\epsilon_k}$ with $y_1,\dots,y_k \in X$ and $\epsilon_1,\dots,\epsilon_k \in \{-1,1\}$ such that the following condition is satisfied:

$(*)$ For every $i \in \{1,\dots,k-1\}$ we have $(y_i,\epsilon_i) \neq (y_{i+1},-\epsilon_{i+1})$.

It follows easily from $(*)$ that for every $x \in X$ we can pick a permutation $\sigma_x \in S_{k+1}$ with the following two properties:

(1) For every $i \in \{1,\dots,k\}$ with $x = y_i$ and $\epsilon_i = 1$ we have $\sigma_x(i+1) = i$.
(2) For every $i \in \{1,\dots,k\}$ with $x = y_i$ and $\epsilon_i = -1$ we have $\sigma_x(i) = i+1$. In other words, we have $\sigma_x^{-1}(i+1) = i$.

By Lemma 19.14 there exists a unique homomorphism $\varphi\colon \langle X\rangle \to S_{k+1}$ such that $\varphi(x) = \sigma_x$ for every $x \in X$. We claim that $\varphi(g) \in S_{k+1}$ is a non-trivial permutation. In fact we have

$$\varphi(g)(k+1) = \varphi(y_1^{\epsilon_1}\cdot\ldots\cdot y_k^{\epsilon_k})(k+1) = \varphi(y_1)^{\epsilon_1}(\ldots(\underbrace{\varphi(y_k)^{\epsilon_k}(k+1)}_{=k})\ldots) = \ldots = \varphi(y_1)^{\epsilon_1}(2) = 1.$$

This shows that the permutation $\varphi(g) \in S_{k+1}$ sends $k+1$ to 1, in other words, it is a non-trivial permutation. ∎

Sketch of second proof of Proposition 31.15. For simplicity we only provide the proof that the free group on two generators is residually finite. Recall that by Lemma 20.4 we can make an identification $\langle a,b\rangle = \pi_1(S^1_a \vee S^1_b,*)$ where a and b correspond to loops around S^1_a and S^1_b. It follows fairly easily from from Lemma 16.15 and Lemma 31.14 that it remains to show that for any loop $\gamma = a_1^{m_1} * b_1^{n_1} * \cdots * a_k^{m_k} * b_k^{n_k}\colon [0,1] \to X = S^1_a \vee S^1_b$ there exists a finite-covering $p\colon \widetilde{X} \to X$ such that the lift of γ is not a loop. In Figure 554 we show the following two-step construction of such a covering for the loop $\gamma = a * b * \overline{a} * \overline{b}$.[536]

(1) First we build an “upward turning lift $\widetilde{\gamma}$ of γ”.
(2) Afterward we complete the “image of $\widetilde{\gamma}$” to a finite covering $\widetilde{X}$.

[536] The reader will notice that this is basically the solution to Exercise 16.10.

It is an extraordinary charming exercise to turn the above idea into a rigorous argument for the general case. We would not dream of robbing the reader from this experience. ■

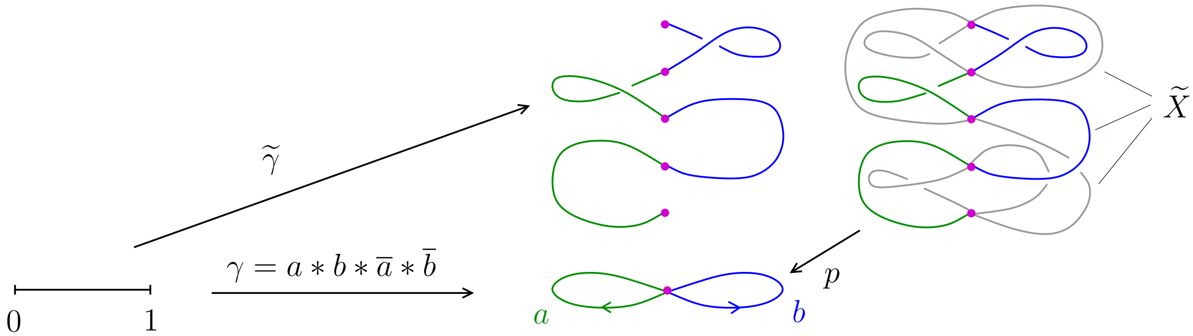


FIGURE 554. Illustration for the proof of Proposition 31.15.

Basically for free we now get the following result.

Corollary 31.16. *Every finitely generated free group is Hopfian.*

PROOF. This statement follows immediately from Proposition 31.15 together with Proposition 31.11. An alternative proof of the corollary is given in [**MKS76**, p. 109]. ■

In Question 22.5 we had asked whether the fundamental group of the surface of genus g is isomorphic to the free group F_{2g} on $2g$ generators. Now we can answer this question in the negative.

Proposition 31.17. *If $g \geq 1$, then the group*

$$\pi_1(\textit{surface of genus } g) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \,|\, [x_1, y_1] \cdot \cdots \cdot [x_g, y_g] \rangle$$

is not isomorphic to the free group F_{2g} on $2g$ generators.

In fact Proposition 31.17 is an immediate consequence of the following more general statement.

Proposition 31.18. *Let $X = \{x_1, \ldots, x_k\}$ be a finite set with k elements. If r is a non-trivial element in the free group $\langle x_1, \ldots, x_k \rangle$, then the group $\langle x_1, \ldots, x_k \mid r \rangle$ is not isomorphic to $\langle x_1, \ldots, x_k \rangle$.*

PROOF. Let X is a finite set and let r be an element in $\langle X \rangle$. Suppose there exists an isomorphism $\varphi \colon \langle X \,|\, r \rangle \to \langle X \rangle$. We denote by $\psi \colon \langle X \rangle \to \langle X \,|\, r \rangle$ the obvious epimorphism. Then $\varphi \circ \psi \colon \langle X \rangle \to \langle X \rangle$ is an epimorphism. It follows from Corollary 31.16 that $\varphi \circ \psi$ is actually an isomorphism. This implies in particular that ψ is a monomorphism. But r lies in the kernel of ψ, this is only possible if $r = e \in \langle X \rangle$. ■

Our next goal is to give a useful criterion for a group to be residually finite. To do so we need to introduce the following definition.

Definition. Let R be a ring. We say that a group G is linear over R if there exists an $n \in \mathbb{N}$ and a monomorphism $G \to \mathrm{GL}(n, R)$.

Examples.

(1) In Exercise 19.13 we showed that the free group on two generators is linear over $\mathbb{Z}$.
(2) Let G be a group and let R be a ring. In Exercise 31.15 we will show that if G admits a finite-index subgroup which is linear over R, then G itself is linear over R.

Now we can formulate the promised proposition.

Proposition 31.19. *Let G be a finitely generated group. If G is linear over $\mathbb{C}$, then G is residually finite.*

Sketch of proof. We only provide a full proof of the following easier claim which is the content of Exercise 19.14 (a).

Claim. The group $\mathrm{GL}(n,\mathbb{Z})$ is residually finite.

Let $A = (a_{ij}) \in \mathrm{GL}(n,\mathbb{Z})$ be a matrix with $A \neq \mathrm{id}$. We pick some $m \in \mathbb{N}$ such that $m > \max\{|a_{ij} - \delta_{ij}| \,|\, i,j = 1,\dots,n\}$. The ring homomorphism $\mathbb{Z} \to \mathbb{Z}_m$ induces a group homomorphism $\alpha\colon\ \mathrm{GL}(n,\mathbb{Z}) \to \mathrm{GL}(n,\mathbb{Z}_m)$. Basically by construction we have $\alpha(A) \neq \mathrm{id}$. Since $\mathrm{GL}(n,\mathbb{Z}_m)$ is a finite group we are done. ⊞

Now let G be a finitely generated subgroup of $\mathrm{GL}(n,\mathbb{C})$. The proof that G is residually finite consists of two steps:

(1) First, since G is finitely generated one sees, almost immediately, that there exists a finitely generated subring S of $\mathbb{C}$ such that $G \subset \mathrm{GL}(n,S)$.
(2) Next one needs to show that S is a "residually finite ring", i.e. one needs to show that given any non-zero $s \in S$ there exists a ring homomorphism $\alpha\colon S \to T$ to some finite ring T with $\alpha(s) \neq 0$. This part requires some input from ring theory.

Once one has proved (1) and (2) it is straightforward to modify the proof of the claim to conclude that G is residually finite.

The full details for the above proof are given in [**Mal40, ?**] and also in [**Weh73**, Theorems 4.1, 4.3] and in [**Hem87**]. ■

We already mentioned that $(\mathbb{Q},+)$ is not residually finite and that $\mathbb{Z}^{(\mathbb{N})}$ is not Hopfian. These are amusing observations, but most group theorists and geometric topologists restrict their interest to finitely generated groups. Thus the question arises, whether perhaps every finitely generated group is residually finite or Hopfian.

Proposition 31.20. *Given $m,n \in \mathbb{N}$ we consider the Baumslag-Solitar group*

$$\mathrm{BS}(m,n) \;=\; \langle x,y \,|\, x^{-1}y^m x \;=\; y^n\rangle.$$

The following statements hold:

(1) *The group $\mathrm{BS}(m,n)$ is Hopfian if and only if m and n are "meshed", i.e. if and only if m or n divides the other or if m and n have precisely the same prime divisors.*
(2) *The group $\mathrm{BS}(m,n)$ is residually finite if and only if $|m| = 1$, or if $|n| = 1$ or if $|m| = |n|$.*

Example. By Proposition 31.20 the group $\mathrm{BS}(2,6)$ is Hopfian but not residually finite. This shows, as promised, that the converse to Proposition 31.11 does not hold.

PROOF. The first statement was proved in 1962 by Gilbert Baumslag and Solitar [**BaS62**]. The second statement is proved in [**Mes72**, Theorem C] and also [**Cam90**, p. 61]. ■

By Proposition 31.20 we now know that there exist finitely generated groups which do not have "many" finite quotients. The question arises, does every non-trivial group admit at least some finite index subgroups? It is now time to recall the following standard definition from group theory.

Definition. A group G is called *simple* if the only normal subgroups are the trivial group $\{e\}$ and the whole group G.

Example. One of the most basic statements in the theory of finite groups is the fact that for $n \geq 5$ the alternating group A_n is actually simple, we refer to [**Bog08**, Chapter 1.12] for a proof.

It is now natural to ask whether there exist infinite simple groups. The answer is yes, but one needs to be rather creative when it comes to producing such groups. This discussion leads us to the following groups which were introduced by Richard Thompson in 1965.

Definition.

(1) A continuous self-homeomorphism $f\colon [0,1] \to [0,1]$ is called *dyadic* if there exists an $m \in \mathbb{N}$ such that the restriction of f to each interval $[\frac{i}{2^m}, \frac{i+1}{2^m}]$ is linear and the slope on each such interval is a (possibly negative) power of 2.
(2) We say a map $g\colon S^1 \to S^1$ is *dyadic* if there exists a dyadic map $f\colon [0,1] \to [0,1]$ such that $g(e^{2\pi i t}) = e^{2\pi i f(t)}$.

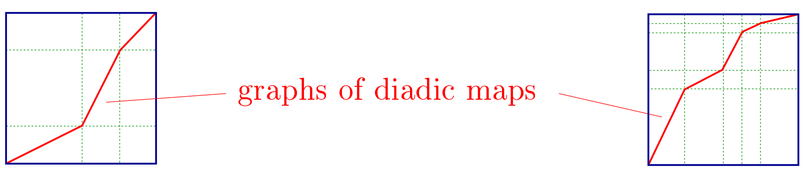


FIGURE 555

We leave it to the reader to verify that the set of dyadic self-homeomorphisms of $[0,1]$ and of S^1 actually form groups. This observation leads us to the key definition.

Definition.

(1) *Thompson's group F* is defined as the group of dyadic self-homeomorphisms of $[0,1]$.
(2) *Thompson's group T* is defined as the subgroup of the self-homeomorphisms of S^1 that is generated by the dyadic self-homeomorphisms together with the homeomorphism given by $z \mapsto -z$.

The following theorem summarizes some of the many interesting properties of Thompson's groups.

Theorem 31.21.

(1) *Thompson's group F and Thompson's group T are both finitely presented.*
(2) *The commutator subgroup of Thompson's group F is simple, in particular Thompson's group F is not residually finite.*
(3) *Thompson's group T is simple, in particular it is not residually finite.*

PROOF.

(1) The statements are proved in [**CFP96**] and [**Geo08**, Chapters 9.2].
(2) By [**CFP96**] and [**Geo08**, Chapters 9.2] the commutator subgroup of Thompson's group F is simple. One can easily verify that Thompson's group F is non-abelian and torsion-free. It now follows from an elementary argument that it is not residually finite.
(3) It is shown in [**CFP96**] and [**Geo08**, Theorem 9.4.3] that Thompson's group T is simple. Clearly T is infinite, hence it cannot be residually finite. ■

In this section we have shown, among many other statements, that free groups are residually finite. It is natural to ask which of the other friendly groups which we have encountered are actually residually finite. Let us pose two questions:

Question 31.22.

(1) *Are the fundamental groups of closed 2-dimensional smooth manifolds residually finite?*
(2) *Is the fundamental group of every knot complement residually finite?*

We will return to these questions later on in Proposition 34.10.

Exercises for Chapter 31.

Exercise 31.1. Let H be a finite connected graph and let G be a connected non-empty subgraph of H. Let S be a spanning tree of G. Show that there exists a spanning tree T of H such that $T \cap G = S$.
Remark. This exercise is a rather straightforward generalization of Proposition 4.8 (1).

Exercise 31.2. Let $\alpha\colon \langle a, b\rangle \to \mathbb{Z}_3$ be the epimorphism defined given by $\alpha(a) = 1$ and $\alpha(b) = 0$. Furthermore let X be the wedge of two circles A and B. As usual we equip X with the CW-structure with one 0-cell P and two 1-cells a and b. Furthermore we make the identification $\pi_1(X) = \langle a, b\rangle$.

(a) "Construct" the covering $p\colon \widetilde{X} \to X$ corresponding to $\ker(\alpha)$, i.e. construct the covering p such that $p_*(\pi_1(\widetilde{X})) = \ker(\alpha)$. More precisely, give the CW-structure on $\widetilde{X}$ and indicate, e.g. using colors, the cellular map p.
(b) We know from Proposition 31.4 (2) that $\ker(\alpha)$ is a free group on four generators. Determine four elements of $\ker(\alpha)$ that generate $\ker(\alpha)$ as a free group. You do not have to justify your answer.

Exercise 31.3. In Question 30.5 we had considered an explicit 6-fold covering $X \to B$ where B is the 2-dimensional sphere minus three open disks. On page 849 we saw that X has six boundary components. Determine the genus of X.

Exercise 31.4. Let $g \in \mathbb{N}_{\geq 2}$. Use Proposition 31.5 to show that the fundamental group of the surface of genus g is torsion-free.
Remark. We posed this already as Exercise 25.10 but with different instructions.

Exercise 31.5. Let P be a property of groups. We say a group G is *virtually* P if G admits a finite-index subgroup that has property P. Let G be a finitely generated group. Show that if G is virtually abelian, then there exists a short exact sequence of the form $0 \to \mathbb{Z}^d \to G \to A \to 1$ where $d \in \mathbb{N}_0$ and A is a finite group.

Exercise 31.6. Let $p\colon \widetilde{\Sigma} \to \Sigma$ be a covering map of degree d between two closed *non-orientable* connected 2-dimensional smooth manifolds. Show that the following equality holds:

$$\operatorname{genus}(\widetilde{\Sigma}) \;=\; d \cdot (\operatorname{genus}(\Sigma) - 1) + 1.$$

Hint. Let $\varphi\colon \overline{B}^2 \to \Sigma$ be an embedding. Consider the compact connected 2-dimensional smooth manifolds $M := \Sigma \setminus \varphi(B^2)$ and $\widetilde{M} := p^{-1}(M)$. What can you say about the fundamental group and the number of boundary components of $\widetilde{M}$?
Remark. This exercise is meant as the non-orientable analogue of Proposition 31.6. In Lemma 55.9 we will use "Euler characteristics" to give an alternative proof of this equality.

Exercise 31.7. We consider the non-compact 2-dimensional smooth manifold M shown in Figure 556. Does there exists a closed 2-dimensional smooth manifold N such that M is a covering space of N?

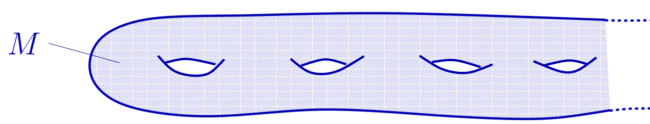


FIGURE 556. Illustration for Exercise 31.7.

Exercise 31.8. In Exercise 19.14 we introduced the notion of a residually finite group and we showed that the free group on two generators is residually finite.

(a) Show that every finitely generated free group is residually finite.
Remark. Use Proposition 31.4 (2).
(b) Show that every free group is residually finite.

Exercise 31.9. Let π be a countable residually finite group. Show that there exists a descending sequence $\pi \supset \pi_0 \supset \pi_1 \supset \pi_2 \supset \dots$ of finite index normal subgroups of π such that we have $\bigcap\limits_{i \in \mathbb{N}} \pi_i = \{e\}$.

Exercise 31.10. Let G be a finitely generated group. Show that given any $n \in \mathbb{N}$ there exist only finitely many subgroups of G of index n.

Exercise 31.11. Let G be a group. A subgroup Γ of G is called *characteristic* if for every automorphism $\varphi\colon G \to G$ we have $\varphi(\Gamma) = \Gamma$.

(a) Show that if subgroup is characteristic, then it is also normal.
(b) Let H be a subgroup of G. Suppose that G is finitely generated. Show that there exists a characteristic subgroup $K \subset G$ with $K \subset H$ and such that K has finite index in G.
(c) Show that in (b) we cannot drop the hypothesis that G is finitely generated.

Exercise 31.12. Show that the group $(\mathbb{Q}, +)$ is not residually finite.

Exercise 31.13. Let $\varphi\colon A \to B$ be a homomorphism between free groups of the same finite rank. Suppose that $\varphi_*\colon A_{\text{ab}} \to B_{\text{ab}}$ is an isomorphism. Show that φ is a monomorphism.
Hint. Use Proposition 31.4 and Corollary 31.16.

Exercise 31.14. Let G be a group and let $\varphi\colon G \to G$ be an automorphism. We suppose that G is residually finite and finitely generated. Show that the semidirect product $G \rtimes_\varphi \mathbb{Z}$, that we defined on page 718, is also residually finite.
Hint. Make good use of Exercise 31.11.

Exercise 31.15. Let R be a ring, let G be a group and let H be a subgroup of finite index $d \in \mathbb{N}$. Suppose there exists a monomorphism $\varphi\colon H \to \mathrm{GL}(n, R)$. Show that there exists a monomorphism $\varphi\colon G \to \mathrm{GL}(n \cdot d, R)$.

Exercise 31.16.

(a) Provide an example of an infinitely generated group that is Hopfian.
(b) Show that every infinite simple group is Hopfian.

Exercise 31.17. Let $k \in \mathbb{N}_{\geq 2}$. We consider the direct product $\langle x_1, \ldots, x_k \rangle \times \mathbb{Z}$. Show that given any $N \in \mathbb{N}$ there exists a finitely generated free group F of rank $\geq N$ and an automorphism φ of F such that $\langle x_1, \ldots, x_k \rangle \times \mathbb{Z}$ is isomorphic to the semidirect product $F \rtimes_\varphi \mathbb{Z}$.
Remark. Most of the hard work was already done in Exercise 24.17.

32. Hyperbolic geometry

In this chapter we will switch gears and we will introduce some basics on hyperbolic geometry. The reason for doing so is that in Chapter 33 we will see that every surface of genus ≥ 2 admits a hyperbolic structure, and this fact will be extremely helpful in getting a better understanding of surfaces.

32.1. Hyperbolic space.

32.1. Hyperbolic space. We start out with the following definition.

Definition. Let V be a real vector space. A *symmetric form on V* is a map[537]

$$g\colon V \times V \rightarrow \mathbb{R}$$

that is bilinear, i.e. for all $u, v, w \in V$ and $\mu, \lambda \in \mathbb{R}$ we have

$$\begin{aligned} g(\lambda u + \mu v, w) &= \lambda g(u, w) + \mu g(v, w) \\ g(u, \lambda v + \mu w) &= \lambda g(u, v) + \mu g(u, w) \end{aligned}$$

and that is symmetric, i.e. for all $v, w \in V$ we have

$$g(v, w) = g(w, v).$$

We say g is *positive-definite* if for all $v \in V \setminus \{0\}$ we have

$$g(v, v) > 0.$$

Now we can recall the following key definition from the theory of smooth manifolds, see e.g. [**Lee02**, Chapter 17].

Definition.

(1) A *Riemannian structure* on a smooth manifold M is a smooth[538] map g which assigns to each point P a positive-definite symmetric form g_P on the tangent space $\mathrm{T}_P\, M$.
(2) A *Riemannian manifold* is a pair (M, g) consisting of a smooth manifold M together with a Riemannian structure g.

As so often in mathematics, given some structure one is very much interested in maps that preserve the given structure. This leads us to the following definition.

Definition.

(1) Given a smooth map $f\colon M \rightarrow N$ between smooth manifolds and given $P \in M$ we denote by $\mathrm{D}\, f_P = f_*\colon \mathrm{T}_P\, M \rightarrow \mathrm{T}_{f(P)}\, N$ the differential, i.e. the induced homomorphism from the tangent space $\mathrm{T}_P\, M$ to the tangent space $\mathrm{T}_{f(P)}\, N$.
(2) Let $f\colon M \rightarrow N$ be a local diffeomorphism. Given a Riemannian structure h on N we denote by f^*h the Riemannian structure on M that at each point $P \in M$ is given by $(f^*h)(v, w) = h(f_*v, f_*w)$.[539]
(3) We say that a map $f\colon (M, g) \rightarrow (N, h)$ between Riemannian manifolds is a (*local*) *isometry* if f is a (local) diffeomorphism and if $g = f^*h$.

[537]Sometimes such a structure is called a *bilinear symmetric form*, but we abbreviate this to *symmetric form*.

[538]The notion of "smooth" in this context is made precise in [**Lee02**, Chapter 17]. Once one has digested the definition of a "smooth map" it is straightforward to see that all examples below satisfy this condition.

(4) Given a Riemannian manifold (M, g) we denote by $\operatorname{Isom}(M, g)$ the group of isometries of (M, g). If M is orientable, then we denote by $\operatorname{Isom}^+(M, g)$ the group of orientation-preserving isometries of (M, g).
(5) We say that Riemannian manifolds (M, g) and (N, h) are *isometric* if there exists an isometry from (M, g) to (N, h).

In this chapter and the following chapter we are mostly interested in Riemannian structures on surfaces, and more specifically, in hyperbolic structures. Now we give the definition of hyperbolic space.

Definition. The *hyperbolic half-plane* is the smooth manifold

$$\mathbb{H} \;=\; \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \;=\; \{z = x + \mathrm{i}y \in \mathbb{C} \mid y > 0\}$$

together with the Riemannian structure that assigns to the tangent space $\mathrm{T}_z\,\mathbb{H} = \mathbb{R}^2$ at a point $z = x + \mathrm{i}y \in \mathbb{H}$ the positive-definite symmetric form

$$\begin{aligned} g\colon \mathbb{R}^2 \times \mathbb{R}^2 &\to \mathbb{R} \\ (v, w) &\mapsto \frac{1}{y^2} \cdot \langle v, w \rangle \end{aligned}$$

where $\langle v, w \rangle$ denotes the usual scalar product on $\mathbb{R}^2$.

Example. In Figure 557 we see a decomposition of $\mathbb{H}$ into angels and devils. With respect to the hyperbolic metric all the angels have the same size and all the devils have the same size, even though they differ in the usual Euclidean sense.

FIGURE 557

As we mentioned before, we are very much interested in the group $\operatorname{Isom}^+(\mathbb{H})$ of orientation-preserving isometries of the Riemannian manifold $\mathbb{H}$. This leads us to the following definition.

Definition. A *Möbius transformation* of $\mathbb{H}$ is a map of the form[540]

$$\begin{aligned} \mathbb{H} &\to \mathbb{H} \\ z &\mapsto \frac{az + b}{cz + d} \end{aligned}$$

where $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$.

[539] It follows from [**Lee02**, Proposition 13.9] that f^*h is indeed a Riemannian structure on M. Note that in the proof, to show that f^*h is indeed non-degenerate, one needs that f is a local diffeomorphism.

[540] An elementary calculation shows that the image of a Möbius transformation $\mathbb{H} \to \mathbb{C}$ does indeed lie again in $\mathbb{H}$.

Examples. We give three special types of Möbius transformations of $\mathbb{H}$:

(1) given $r>0$ the scalar multiplication $z\mapsto rz=\frac{\sqrt{r}\cdot z+0}{0\cdot z+\frac{1}{\sqrt{r}}}$ is a Möbius transformation,

(2) given $d\in\mathbb{R}$ the horizontal translation $z\mapsto z+d=\frac{1\cdot z+d}{0\cdot z+1}$ is a Möbius transformation,

(3) the map $z\mapsto -\frac{1}{z}=\frac{0\cdot z+1}{(-1)\cdot z+0}$ is a Möbius transformation.

The third Möbius transformation is the composition of taking the inverse $z\mapsto\frac{1}{z}$ and the reflection $z\mapsto -z$ in the origin. We illustrate this map in Figure 558. For simplicity we refer to the map $z\mapsto -\frac{1}{z}$ as an inversion.

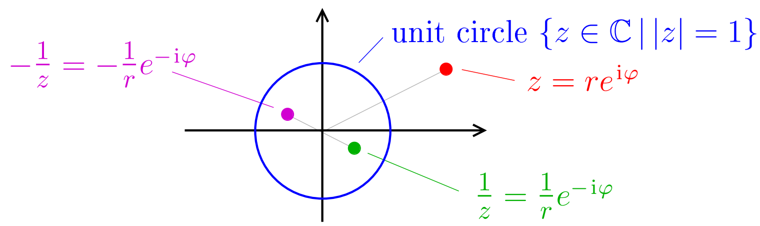


FIGURE 558

Rather amazingly we are already done with listing all elements in $\mathrm{Isom}^+(\mathbb{H})$. More precisely, we have the following proposition.

Proposition 32.1. *The map*

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{R}) & \to & \mathrm{Isom}^+(\mathbb{H}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \left(\begin{array}{ccc} \mathbb{H} & \to & \mathbb{H} \\ z & \mapsto & \frac{az+b}{cz+d} \end{array}\right) \end{array}$$

is a group isomorphism.

SKETCH OF PROOF. One can easily calculate by hand that each Möbius transformation is an orientation-preserving isometry of $\mathbb{H}$. Furthermore, it is an amusing calculation to verify that the given map is indeed a group homomorphism. It is elementary to see that the map is a monomorphism. The only bit of the proposition which requires some thought is the statement that every orientation-preserving isometry is already a Möbius transformation. This statement is for example shown in [**And05**, Theorem 3.19]. ■

Now that we have seen that every orientation-preserving isometry of $\mathbb{H}$ is a Möbius transformation it makes sense to study Möbius transformations in greater detail.

Proposition 32.2. *Every Möbius transformation of $\mathbb{H}$ is the composition of scalar multiplications, horizontal translations and inversions.*

PROOF. Let $a,b,c,d\in\mathbb{R}$ with $ad-bc=1$ and denote by

$$\Phi(z) \;=\; \frac{az+b}{cz+d}$$

the corresponding Möbius transformation. If $c=0$, then

$$\Phi(z) \;=\; \frac{a}{d}z+\frac{b}{d}$$

is the composition of the scalar multiplication[541] $z \mapsto \frac{a}{d}z$ with the horizontal translation $z \mapsto \frac{b}{d} + z$. If $c \neq 0$, then

$$\Phi(z) = \frac{az+b}{cz+d} = \frac{acz+bc}{c^2z+dc} \underset{\substack{\uparrow \\ \text{since } ad-bc=1}}{=} \frac{acz+ad-1}{c^2z+cd} = \frac{-1+a(cz+d)}{c^2z+cd} = \frac{-1}{c^2z+dc} + \frac{a}{c}.$$

Thus we see that Φ is the composition of the following maps

$$z \mapsto c^2 z, \quad z \mapsto z + dc, \quad z \mapsto \frac{-1}{z} \quad \text{and } z \mapsto z + \frac{a}{c}.$$ ■

Definition.

(1) The *hyperbolic disk*[542] is the smooth manifold

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z|^2 < 1\}$$

together with the Riemannian structure which assigns to the tangent space $\mathrm{T}_z\,\mathbb{D} = \mathbb{R}^2$ at a point $z \in \mathbb{D}$ the positive-definite symmetric form

$$\begin{aligned} g_z \colon \mathbb{R}^2 \times \mathbb{R}^2 &\to \mathbb{R} \\ (v,w) &\mapsto \frac{2}{(1-|z|^2)^2}\langle v,w\rangle \end{aligned}$$

where, as above, $\langle v,w\rangle$ denotes the usual scalar product on $\mathbb{R}^2$.

(2) A *Möbius transformation* of $\mathbb{D}$ is a map of the form[543]

$$\begin{aligned} \mathbb{D} &\to \mathbb{D} \\ z &\mapsto e^{\mathrm{i}\theta} \cdot \tfrac{z-a}{1-\overline{a}z} \end{aligned}$$

for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$.

Examples.

(1) In Figure 559 we see three different decompositions of $\mathbb{D}$ into subsets. In each of the three cases these subsets have the same hyperbolic size, even though they differ in the usual Euclidean sense.

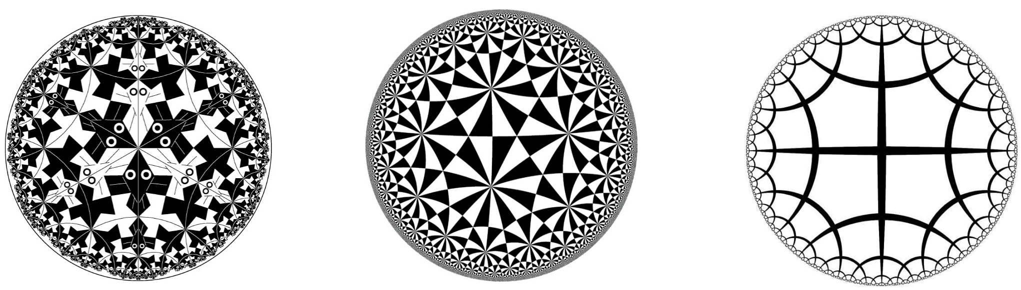


FIGURE 559

[541] Note that $c = 0$ implies that $1 = ad - bc = ad$, in particular $ad > 0$, hence $\frac{a}{d} > 0$.

[542] The hyperbolic disk is often also called the *Poincaré model for hyperbolic space.*

[543] An elementary calculation shows that the image of a Möbius transformation $\mathbb{D} \to \mathbb{C}$ does indeed lie again in $\mathbb{D}$.

(2) If we set $a = 0$ in the definition of a Möbius transformation of $\mathbb{D}$, then we obtain precisely the rotation by the angle θ. Put differently, the Möbius transformations preserving 0 are precisely given by rotations around the origin.

The following proposition will allow us to go back and forth between the hyperbolic half-plane and the hyperbolic disk.

Proposition 32.3. *The maps*

$$\begin{array}{rcl} \Phi\colon \mathbb{H} & \to & \mathbb{D} \\ z & \mapsto & \frac{z-\mathrm{i}}{z+\mathrm{i}} \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon \mathbb{D} & \to & \mathbb{H} \\ w & \mapsto & \frac{\mathrm{i}+\mathrm{i}w}{1-w} \end{array}$$

have the following properties:

(1) *Φ and Ψ are inverses to one another, in particular they are biholomorphisms,*
(2) *they extend to homeomorphisms between $\overline{\mathbb{H}} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\} \cup \{\infty\}$[544] and $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$,*
(3) *they are orientation-preserving,*[545]
(4) *they are isometries,*
(5) *they give a bijection between the Möbius transformations of $\mathbb{D}$ and $\mathbb{H}$.*[546]

Proof. All five statements follow from a straightforward calculation. The statements can also be deduced, more or less directly, from any book that deals with Möbius transformations, see e.g. [**Schwe79**, Chapter 6] and [**And05**, Chapter 2]. ■

Remark. Proposition 32.3 says that $\mathbb{D}$ and $\mathbb{H}$ are two equivalent ways to describe hyperbolic space. In fact there are several other ways to describe hyperbolic space, for example we could also use the Beltrami-Klein model

`https://en.wikipedia.org/wiki/Beltrami-Klein_model`

or the hyperboloid model

`https://en.wikipedia.org/wiki/Hyperboloid_model`.

Some of the connections between these models are explained in the following video of Henry Segerman and Saul Schleimer:

`https://www.youtube.com/watch?v=eGEQ_UuQtYs`.

The following proposition summarizes some of the key properties of Möbius transformations.

Proposition 32.4. [547]

(1) *Compositions of Möbius transformations are again Möbius transformations and the inverse of a Möbius transformation is again a Möbius transformation.*

---

[544]Here $\overline{\mathbb{H}}$ has the same topology as defined on page 95.

[545]The hyperbolic half-plane $\mathbb{H}$ and the hyperbolic disk $\mathbb{D}$ are both subsets of $\mathbb{C} = \mathbb{R}^2$ and inherit from $\mathbb{C} = \mathbb{R}^2$ the canonical orientation.

[546]More precisely, if α is a Möbius transformation of $\mathbb{H}$, then $\Phi \circ \alpha \circ \Psi$ is a Möbius transformation of $\mathbb{D}$ and conversely, if α is a Möbius transformation of $\mathbb{D}$, then $\Psi \circ \alpha \circ \Phi$ is a Möbius transformation of $\mathbb{H}$.

(2) *Möbius transformations are biholomorphisms, they are isometries and they are orientation-preserving.*
(3) *Möbius transformations act transitively, i.e. given any two points P and Q there exists a Möbius transformation which sends P to Q.*
(4) *Given $P, Q \in \mathbb{D}$ and non-zero vectors $v \in \mathrm{T}_P\,\mathbb{D}$ and $w \in \mathrm{T}_Q\,\mathbb{D}$ with $\|v\|_g = \|w\|_g$ there exists a Möbius transformation ϕ with $\phi(P) = Q$ and $d\phi_P(v) = w$.*
(5) *Let ϕ and ψ be two Möbius transformations. Suppose there exists a $P \in \mathbb{D}$ such that $\phi(P) = \psi(P)$ and such that $d\phi_P = d\psi_P$. Then $\phi = \psi$.*

Proof. By Proposition 32.3 it suffices to prove the statements for $\mathbb{D}$ or for $\mathbb{H}$, whichever is more convenient.

(1),(2) The first two statements follow from a straightforward calculation, see e.g. [**Schwe79**, p. 43].

(3) We prove this statement for $\mathbb{D}$. Note that by (1) it suffices to show that given any $P \in \mathbb{D}$ there exists a Möbius transformation ψ with $\psi(P) = 0$. But such a Möbius transformation is given by

$$\psi(z) \;=\; \tfrac{z-P}{1-\overline{P}z}.$$

(4) Using (3) we see that we only have to deal with the case that $P = Q = 0$. But then the desired Möbius transformation is given by a rotation.

(5) By (3) there exists a Möbius transformation θ such that $\theta(0) = P$ and there exists a Möbius transformation ω such that $\omega(\phi(P)) = \omega(\psi(P)) = 0$. We consider the Möbius transformations $\phi' = \omega \circ \phi \circ \theta$ and $\psi' = \omega \circ \psi \circ \theta$. It follows from our hypothesis that both ϕ' and ψ' fix 0. As we pointed out above, the only Möbius transformations of $\mathbb{D}$ that fix 0 are rotations.

By our hypothesis we have $d\phi'_0 = d\psi'_0$, which means that the rotations ϕ' and ψ' agree. But this implies that ϕ and ψ also agree. ■

Note that Möbius transformations extend in an obvious way uniquely to homeomorphisms of $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and $\overline{\mathbb{H}} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\} \cup \{\infty\}$. We will denote the extension of a Möbius transformation with the same symbol.

Lemma 32.5. *Let $\{P, Q, R\}$ and let $\{S, T, U\}$ be two sets of three distinct points on the boundary $\partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. Then there exists a Möbius transformation ψ of $\mathbb{H}$ with $\psi(P) = S$ and $\psi(\{Q, R\}) = \{T, U\}$.*

Proof. Let P, Q, R be three distinct points on $\partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. It suffices to show that there exists a Möbius transformation ψ with $\psi(P) = \infty$ and with $\psi(\{Q, R\}) = \{0, 1\}$. We proceed as follows:

(1) We first claim that there exists a Möbius transformation α with $\alpha(P) = \infty$. Indeed, if $P = \infty$ then we just take $\alpha = \mathrm{id}$. Otherwise we can combine a horizontal translation with the inversion to obtain a Möbius transformation α with $\alpha(P) = \infty$. Given such

[547] The statement of the proposition applies to Möbius transformations of $\mathbb{D}$ and to Möbius transformations of $\mathbb{H}$.

α we write $Q' = \alpha(Q)$ and $R' = \alpha(R)$. Note that the points Q', R' lie on $\mathbb{R} = \overline{\mathbb{H}} \setminus \{\infty\}$. Without loss of generality we can assume that $Q' < R'$.

(2) Next we pick a horizontal translation such that $\beta(Q') = 0$. Note that we still have $(\beta \circ \alpha)(P) = \beta(\infty) = \infty$. Also note that it follows from $Q' < R'$ that $\beta(R') > 0$.

(3) Finally we pick a scalar multiplication γ such that $(\gamma \circ \beta)(R') = 1$. Note that γ fixes 0 and ∞. Thus we see that $\psi = \gamma \circ \beta \circ \alpha$ has the desired property. ■

Definition. A *hyperbolic line* is a subset of $\mathbb{D}$ of one of the following two types:

(1) it is the intersection of $\mathbb{D}$ with a Euclidean line through the origin,

(2) it is the intersection of $\mathbb{D}$ with a Euclidean circle that intersects the circle $S^1 = \partial\overline{\mathbb{D}}$ orthogonally.

In each of the two cases we refer to the intersection of the Euclidean object with $S^1 = \partial\overline{\mathbb{D}}$ as the *endpoints* of the hyperbolic line. We refer to Figure 560 for an illustration.

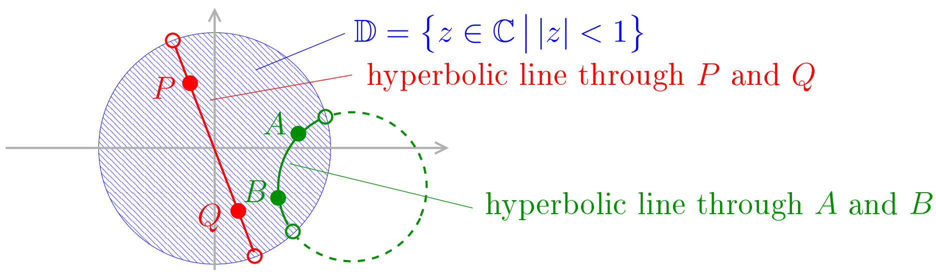


FIGURE 560

Lemma 32.6.

(1) *For any two distinct points P and Q in $\mathbb{D}$ there exists a unique hyperbolic line that contains P and Q.*

(2) *For any two distinct points P and Q in $\partial\overline{\mathbb{D}}$ there exists a unique hyperbolic line with P and Q as endpoints.*

(3) *Let P and Q be two distinct points in $\partial\overline{\mathbb{D}}$. Every Möbius transformation ϕ sends the hyperbolic line with endpoints P and Q to the hyperbolic line with endpoints $\phi(P)$ and $\phi(Q)$.*

(4) *Möbius transformations act transitively on hyperbolic lines, i.e. given any two hyperbolic lines g and h there exists a Möbius transformation ϕ with $\phi(g) = h$.*

PROOF. In principle one can prove the lemma directly in $\mathbb{D}$. But it is easier to prove it using $\mathbb{H}$. In $\mathbb{H}$ the hyperbolic lines are defined to be vertical lines $\{x + iy \mid y > 0\}$ and Euclidean half-circles such that the origin lies on $\mathbb{R}$. It is straightforward to see that the diffeomorphisms of Proposition 32.3 define a bijection between the hyperbolic lines in $\mathbb{D}$ and in $\mathbb{H}$.

The corresponding statements for hyperbolic lines in $\mathbb{H}$ are straightforward to prove. They can also be obtained from any textbook on hyperbolic geometry, see e.g. [**And05**] and [**Bon09**, Chapter 2]. ■

32.2. **Angles in Riemannian manifolds.** We continue our quick introduction to Riemannian manifolds and hyperbolic space with a short discussion of angles.

Definition.

(1) Let g be a positive-definite symmetric form on a real vector space V. Given two non-zero vectors $v, w \in V \setminus \{0\}$ we define[548]

$$\text{angle between } v \text{ and } w \ := \ \text{unique } \alpha \in [0, \pi] \text{ with } g(v, w) = \cos(\alpha) \cdot \|v\| \cdot \|w\|.$$

(2) Given a Riemannian manifold (M, g) and non-zero vectors $v, w \in \mathrm{T}_P M$ we use the form g_P to define the angle between v and w.

Examples.

(1) Let U be an open subset of $\mathbb{R}^n$ and let $f\colon U \to \mathbb{R}_{>0}$ be a smooth function. For any $P \in U$ we can define a positive-definite symmetric form g_P on $\mathrm{T}_P U = \mathbb{R}^n$ via

$$g_P(v, w) \ = \ f(P) \cdot \langle v, w \rangle,$$

where $\langle\, ,\, \rangle$ denotes the usual scalar product. A straightforward calculation shows that in this Riemannian structure the angle at any point is the usual Euclidean angle in $\mathbb{R}^n$. This argument implies in particular to the Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$, in both cases the hyperbolic angle, defined by the hyperbolic Riemannian structure, is the same as the Euclidean angle.

(2) A hyperbolic triangle is defined to be a compact subset of $\mathbb{H}$ whose boundary is given by segments of three hyperbolic lines. One of the many elegant theorems in hyperbolic geometry, see [**And05**, Theorem 5.16], says that given such a hyperbolic triangle its angle sum equals π minus its "area". Even without the definition of "area" it is pretty clear that this result implies that the angle sum of a hyperbolic triangle is always less than π. This statement is illustrated in Figure 561.

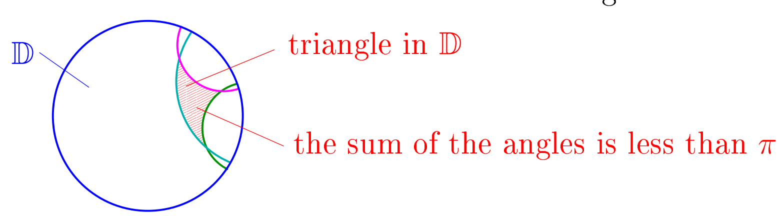


FIGURE 561

The following lemma follows easily from the definitions and from Proposition 32.4 (2).

Lemma 32.7. *Local isometries, in particular Möbius transformations, preserve angles.*

Definition.

(1) Let $P \in \mathbb{D}$ and let g be a hyperbolic line through P. Let Q be one of the two endpoints of g. We refer to the set of points between P and Q as a *ray emanating from P.*

(2) Let $P \in \mathbb{D}$ and let r, s be two rays emanating from P. We refer to the angle between the tangent vectors of r and s at P as the *angle between the rays r and s.*

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[548] We have $|g(v, w)| \le \big|\|v\| \cdot \|w\|\big|$ by the Cauchy-Schwarz inequality, see [**Zh11**, p. 27].

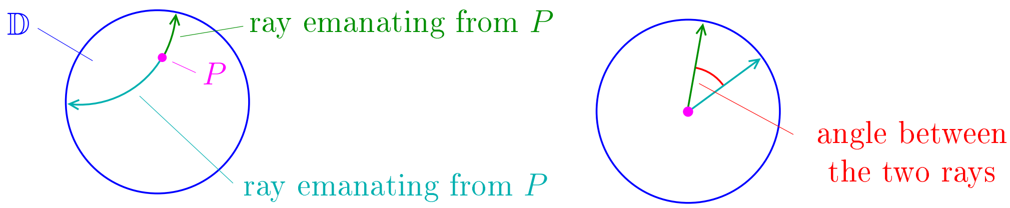


Figure 562

Examples.

(1) Given $\varphi \in [0, 2\pi)$ the set $S_\varphi := \{re^{i\varphi} \mid r \in [0,1)\}$ is a ray emanating from 0.
(2) It follows from the above discussion of angles in $\mathbb{D}$, that given $\varphi, \theta \in [0, 2\pi)$ the angle between the two rays S_φ and S_θ in $\mathbb{D}$ is the usual Euclidean angle.

Lemma 32.8. *Let r, s be two rays emanating from $P \in \mathbb{D}$ and let r', s' be two rays emanating from $P' \in \mathbb{D}$. Then the following holds:*

$$\begin{array}{c}\textit{there is a Möbius transformation } \phi \\ \textit{with } \phi(r \cup s) = r' \cup s'\end{array} \iff \textit{angle between } r, s = \textit{angle between } r', s'.$$

Furthermore, if such a Möbius transformation exists, then it is unique.

Proof. By Lemma 32.7 Möbius transformations preserve angles and by Proposition 32.4 Möbius transformations act transitively on $\mathbb{D}$. Therefore we can without loss of generality assume that $P = P' = 0$. The lemma follows from the observation, made on page 876, that Möbius transformations preserving 0 are precisely given by rotations around the origin. ■

32.3. **The distance metric of a Riemannian manifold.**

Definition. Let (M, g) be a connected Riemannian manifold.

(1) Given $v \in T_P M$ we write $\|v\|_g = \sqrt{g_P(v,v)}$. If g is understood from the context, then we just write $\|v\|$.
(2) Let $\gamma\colon [a,b] \to M$ be a piecewise smooth path. By definition we can pick a subdivision $a = s_0 < s_1 < \cdots < s_m = b$ such that for $i = 0, \ldots, m-1$ the map $\gamma|_{[s_i, s_{i+1}]}$ is smooth. We define the *length of* γ as[549]
$$\ell_{(M,g)}(\gamma) := \sum_{i=0}^{m-1} \int_{t=s_i}^{t=s_{i+1}} \|\gamma'(t)\|_g \, dt.$$
If (M, g) is understood from the context, then we write $\ell(\gamma)$ instead of $\ell_{(M,g)}(\gamma)$.
(3) Let P and Q be two points on M.
$$d_{(M,g)}(P,Q) := \inf\{\ell(\gamma) \mid \underbrace{\gamma \text{ is a smooth path in } M \text{ from } P \text{ to } Q}_{\text{exists by Lemma 8.30}}\}.$$
If (M, g) is understood from the context, then we just write $d(P, Q)$.

[549] Recall that given a smooth path $\delta\colon [a,b] \to M$ and given any $t \in [a,b]$ the path defines, basically by definition of the tangent space $T_{\delta(t)} M$, an element in $T_{\delta(t)} M$. We denote this vector by $\delta'(t)$.

We start our discussion of the distance function d on Riemannian manifolds with the following somewhat technical proposition.

Proposition 32.9. *Let (M,g) be a connected Riemannian manifold. Then the following hold:*

(1) *the pair $(M, d_{(M,g)})$ is a metric space,*
(2) *the topology defined by the metric $d_{(M,g)}$ agrees with the topology of M,*
(3) *given any $P \in M$ and given any open neighborhood U of P there exists an $r > 0$ such that*

$$B_r^{(M,g)}(P) := \{Q \in M \mid d_{(M,g)}(P,Q) < r\} \subset U.$$

PROOF. Let (M,g) be a connected n-dimensional Riemannian manifold. We first show that $(M, d_{(M,g)})$ is a metric space.

It is clear that $d = d_{(M,g)}$ is symmetric. It follows easily from the definitions and the fact that the concatenation of two piecewise smooth paths is again piecewise smooth that d satisfies the triangle inequality.

Now let $P \neq Q$ be two distinct points in M. We need to show that $d(P,Q) > 0$. Since M is Hausdorff and since it is locally diffeomorphic to open subsets of $\mathbb{R}^n$ we can find a chart $\Phi\colon U \to B_{2r}(0)$ for P with $\Phi(P) = 0$ and such that $Q \notin U$. We write $\Psi = \Phi^{-1}$. Before we continue we introduce the following definitions:

(1) We write $U_r = \Phi^{-1}(\{x \in \mathbb{R}^n \mid \|x\| = r\})$ and given a subset $I \subset [0, 2r)$ we write $U_I = \Phi^{-1}(\{x \in \mathbb{R}^n \mid \|x\| \in I\})$.
(2) For a positive-definite bilinear form h on $\mathbb{R}^n$ we set
$$\|h\| := \min\left\{\sqrt{h(v,v)}\,\middle|\, v \in S^{n-1}\right\}.$$
Note that h is positive-definite, that $h\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous and that S^{n-1} is compact. It follows from Lemma 2.40 that the minimum exists and that $\|h\| > 0$. It is straightforward to see that for any $v \in \mathbb{R}^n$ we have
$$h(v,v) \;\geq\; \|h\| \cdot \|v\|_{\text{eucl}}.$$
(3) We set
$$C := \min\left\{\|(\Psi^* g)_P\| \,\middle|\, P \in \overline{B_r(0)}\right\}.$$
Since $P \mapsto \|(\Psi^* g)_P\|$ is continuous and since $\overline{B_r(0)}$ is compact it follows from Lemma 2.40 that $C > 0$.

Claim. Let $\gamma\colon [0,1] \to M$ be a piecewise smooth path from P to Q. There exists a $t \in [0,1]$ with $\gamma(t) \in U_r$.

Suppose there is no $t \in (0,1)$ with $\gamma(t) \in U_r$. We consider the sets $A = U_{[0,r)}$ and $B = M \setminus U_{[0,r]}$. The set A is clearly open. Furthermore it follows from Lemma 6.10 that $U_{[0,r]}$ is closed, hence B is open. Since $\gamma([0,1]) \cap U_r = \emptyset$ we have $[0,1] = \gamma^{-1}(A) \sqcup \gamma^{-1}(B)$. But both sets $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$ are open by the continuity of γ and both are non-empty since $0 \in \gamma^{-1}(A)$ and $1 \in \gamma^{-1}(B)$. But this contradicts the fact, established with some effort in Proposition 2.55, that $[0,1]$ is connected. ⊞

Now it suffices to prove the following claim.

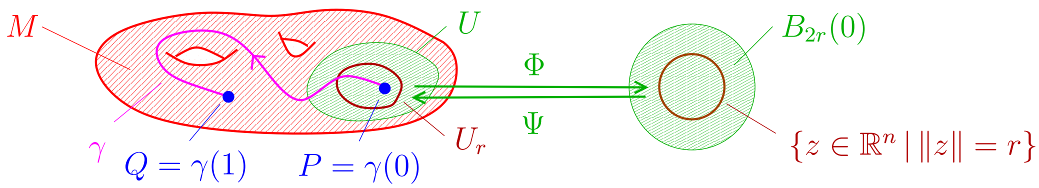


FIGURE 563. Illustration of the proof of Proposition 32.9

Claim. Let $\gamma\colon [0,1] \to M$ be a piecewise smooth path from P to Q. Then $\ell_g(\gamma) \geq C \cdot r$.

Let $T := \inf\{t \in (0,1) \,|\, \gamma(t) \in U_r\}$. By the above claim this definition makes sense since we take the infimum of a non-empty set. By Lemma 6.10 the set U_r is a closed subset of M. Since γ is continuous we also have $\gamma(T) \in U_r$. Now we have

$$\begin{aligned}
\ell_g(\gamma) \;\geq\; \ell_g(\gamma|_{[0,T]}) \;&=\; \ell_{\Psi^* g}(\Phi \circ \gamma|_{[0,T]}) && = \int\limits_{t=0}^{t=T} \|(\Phi\circ\gamma)'(t)\|_{\Psi^* g}\, dt \\
&\underset{\substack{\uparrow \\ \text{for any } v \in \mathrm{T}_P\,\overline{B_r(0)} = \mathbb{R}^n \text{ we have} \\ \|v\|_{(\Psi^* g)_P} \,\geq\, \|(\Psi^* g)_P\| \cdot \|v\|_{\mathrm{eucl}}}}{\geq} \int\limits_{t=0}^{t=T} C \cdot \|(\Phi\circ\gamma)'(t)\|_{\mathrm{eucl}}\, dt && \underset{\substack{\uparrow \\ \text{[RudiW76, Theorem 6.13]}}}{\geq} C \cdot \Bigg\| \int\limits_{t=0}^{t=T} (\Phi\circ\gamma)'(t)\, dt \Bigg\|_{\mathrm{eucl}} \\
&=\; C \cdot \|\underbrace{(\Phi\circ\gamma)(T)}_{\|-\|_{\mathrm{eucl}}=r} - \underbrace{(\Phi\circ\gamma(0))}_{=0}\|_{\mathrm{eucl}} && = C \cdot r.
\end{aligned}$$

This concludes the proof of the claim and therefore the proof of statement (1) of the proposition.

Next we turn to the proof of statement (2). This means that we need to show that the topology defined by d is the same as the original topology of M. The proof is again somewhat technical and uses ideas similar to the proof of (1). We refer to **[Lee97**, Lemma 6.2] for the proof.

Finally statement (3) is just an immediate consequence of the definitions and of statement (2). ■

Lemma 32.10. *Let (M,g) and (N,h) be two connected Riemannian manifolds. Furthermore let $f\colon (M,g) \to (N,h)$ be a local isometry. Then the following hold:*

(1) *For any piecewise smooth path $\gamma\colon [a,b] \to M$ we have*
$$\ell_{(N,h)}(f \circ \gamma) \;=\; \ell_{(M,g)}(\gamma).$$

(2) *For any $P, Q \in M$ we have*
$$d_{(N,h)}(f(P), f(Q)) \;\leq\; d_{(M,g)}(P,Q).$$

PROOF. Let $f\colon (M,g) \to (N,h)$ be a local isometry. We start out with the following claim.

Claim.

(1) For a smooth path $\gamma\colon [c,d] \to M$ and $t \in (c,d)$ we have $\|(f\circ\gamma)'(t)\|_{(N,h)} = \|\gamma'(t)\|_{(M,g)}$.

(2) For any piecewise smooth path $\gamma\colon [a,b] \to M$ we have $\ell_{(N,h)}(f\circ\gamma) = \ell_{(M,g)}(\gamma)$.

Now we provide the proof of the two statements.

(1) The first statement is an immediate consequence of the definitions and the hypothesis that f is a local isometry.

(2) By the additivity of the lengths of piecewise smooth paths it suffices to consider the case that $\gamma\colon [a,b] \to M$ is a smooth path. But in that case the equality follows immediately from the definitions and from (1). ⊞

Now let P and Q be two points on M. For any piecewise smooth path p in M from P to Q we have

$$d_{(N,h)}(f(P), f(Q)) \underset{\substack{\uparrow \\ \text{by definition of } d_{(N,h)}}}{\leq} \ell_{(N,h)}(f\circ p) \underset{\substack{\uparrow \\ \text{by the claim}}}{=} \ell_{(M,g)}(p).$$

It follows that the left-hand side is also less or equal than the infimum over all paths on the right-hand side. Thus $d_{(N,h)}(f(P), f(Q)) \leq d_{(M,g)}(P,Q)$. ■

32.4. **The hyperbolic distance function.** We start out with the following definition.

Definition. Let (M,g) be a Riemannian manifold. A path $\gamma\colon [0,a] \to M$ is called a *geodesic* if the following conditions are satisfied:

(1) γ is smooth,
(2) $\|\gamma'(t)\|_g = 1$ for every $t \in [0,a]$ and
(3) $\ell(\gamma) = d_{(M,g)}(\gamma(0), \gamma(a))$.

The following proposition summarizes the key facts about the distance metric on $\mathbb{D}$.

Proposition 32.11.

(1) *Let $r \in (0,1)$ and $\varphi \in \mathbb{R}$. In the Riemannian manifold $\mathbb{D}$ the path*

$$\begin{array}{rcl} \gamma\colon [0,r] & \to & \mathbb{D} \\ t & \mapsto & te^{\mathrm{i}\varphi} \end{array}$$

is, up to reparametrization, the unique shortest path from 0 to $re^{\mathrm{i}\varphi}$.

(2) *We have*

$$d_{(\mathbb{D},g)}\big(0, re^{\mathrm{i}\varphi}\big) = \ell(\gamma) = \int_{t=0}^{t=r} \frac{\sqrt{2}}{1-|\gamma(t)|^2}\|\gamma'(t)\|_{\text{eucl}}\,dt = \int_{t=0}^{t=r} \frac{\sqrt{2}}{1-t^2}\,dt.$$

(3) *Given $\varphi \in \mathbb{R}$ we have*

$$\lim_{r\to 1} d_{(\mathbb{D},g)}\big(0, re^{\mathrm{i}\varphi}\big) = \infty.$$

(4) *Given any P and Q in $\mathbb{D}$ there is a unique geodesic from P to Q.*

Proof. The first statement can be deduced easily from [**Bon09**, Lemma 2.4 and Proposition 2.22]. The second statement follows immediately from the first statement, and the third statement is a straightforward consequence of the second statement. For the fourth statement, note that it suffices to prove it for $P = 0$. In that case it is a straightforward consequence of (1). ■

Lemma 32.12. *Let (P,Q) and (S,T) be two pairs of distinct points on $\mathbb{D}$ such that $d(P,Q) = d(S,T)$. Then there exists a unique Möbius transformation ϕ with $\phi(P) = S$ and $\phi(Q) = T$.*

Example. Let $r \in (0,1)$, $\varphi \in [0,2\pi]$, $\psi \in [0,2\pi]$ and $\gamma \in (0,\pi)$. We consider the pairs of points $P = re^{i\varphi}, Q = re^{i(\varphi+\gamma)}$ and $S = re^{i\psi}, T = re^{i(\psi-\gamma)}$. We then have

$$d(S,T) \;=\; d(T,S) \;=\; d(re^{i(\psi-\gamma)}, re^{i\psi}) \underset{\substack{\uparrow \\ \text{since rotation by } \varphi+\gamma-\psi \text{ is an isometry}}}{=} d(re^{i\varphi}, re^{i(\varphi+\gamma)}).$$

By Lemma 32.12 there exists a Möbius transformation ϕ with $\phi(P) = S$ and $\phi(Q) = T$. This Möbius transformation is illustrated in Figure 564. Recall that from Proposition 32.4 and Lemma 32.6 we know that Möbius transformations preserve hyperbolic lines and that they are orientation-preserving. In particular ϕ sends the hyperbolic line through P and Q to the hyperbolic line through S and T. Since ϕ is orientation-preserving it "swaps" the two sides of the hyperbolic line. We refer to Figure 564 for an illustration.

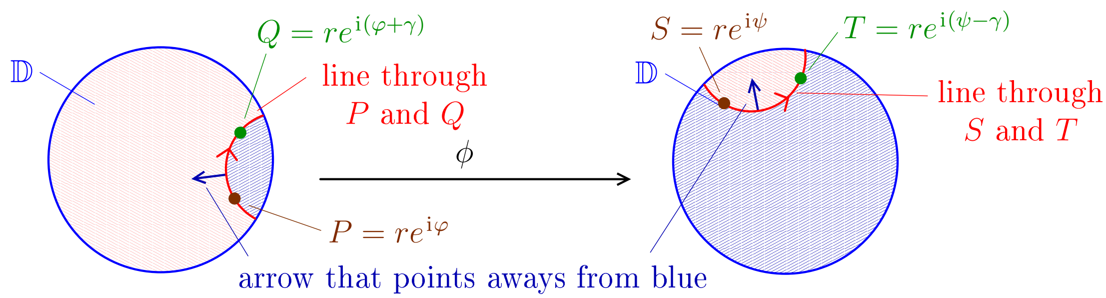


FIGURE 564

PROOF OF LEMMA 32.12. It follows from Proposition 32.4 (3) that, without loss of generality, we can assume that $P = 0$ and $S = 0$. We write $Q = re^{i\varphi}$ and $T = se^{i\psi}$. It is a direct consequence of Proposition 32.11 (1) and our hypothesis that

$$d(0, re^{i\varphi}) \;=\; d(P,Q) \;=\; d(S,T) \;=\; d(0, se^{i\psi})$$

that we have $r = s$. But this means that we can apply the rotation by the angle $\varphi - \psi$, which is a Möbius transformation, to turn (P,Q) into (S,T). This proves the existence of ϕ. The uniqueness follows from the observation, that we had made already on page 876, that the only Möbius transformations that fix 0 are precisely rotations. ■

Lemma 32.13. *Let $P \in \mathbb{D}$ and $r > 0$. Let $S, T \in B_r(P)$. Then the image of the unique geodesic in $\mathbb{D}$ from S to T lies in $B_r(P)$.*

PROOF. As usual, this time by Propositions 32.4, 32.6 and 32.11, it suffices to prove the lemma for $P = 0$. In this special case the reader will have no troubles providing the elementary argument. We also refer to Figure 565 for an illustration. ■

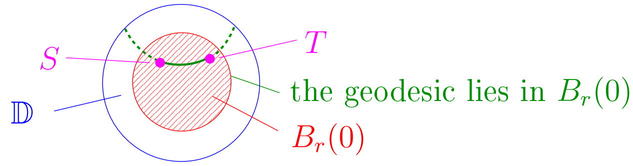


FIGURE 565

32.5. **Complete metric spaces.** We conclude this chapter with a short discussion of complete metric spaces.

Definition.

(1) We say that a metric space (X, d) is *complete* if every Cauchy sequence in (X, d) converges.
(2) We say that a connected Riemannian manifold (M, g) is *complete* if the corresponding metric space $(M, d_{(M,g)})$ is complete.

Examples.

(1) The metric space given by $\mathbb{R}^n$ and the Euclidean metric is complete.
(2) The metric space given by the open ball B^n and the Euclidean metric is not complete. Indeed the sequence $a_n = 1 - \frac{1}{n}$ is a Cauchy sequence, but it does not converge in the open ball B^n. Similarly we see that for example the metric space $\mathbb{C} \setminus \{0, 1\}$ with the usual Euclidean metric is not complete.

The following proposition gives a useful criterion for a metric space to be complete.

Proposition 32.14. *Every compact metric space is complete.*

PROOF. Let (X, d) be a compact metric space and let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X, d). We want to show that the sequence converges, i.e. we want to show that

$$\underset{x\in X}{\exists}\ \underset{\epsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n\geq N}{\forall}\ d(a_n, x) < \epsilon.$$

Let us assume that no such x exists. Thus we negate the above statement and we obtain that

$$\underset{x\in X}{\forall}\ \underset{\epsilon>0}{\exists}\ \underset{N\in\mathbb{N}}{\forall}\ \underset{n\geq N}{\exists}\ d(a_n, x) \geq \epsilon.$$

Given $x \in X$ we pick $\epsilon_x > 0$ as above and we write

$$U_x \ := \ B_{\epsilon_x}(x) \quad \text{and} \quad V_x \ := \ B_{\epsilon_x/2}(x).$$

Note each U_x has the property that it does not contain all but finitely many of the a_n.

The family of open sets $\{V_x\}_{x\in X}$ covers X. Since X is compact there exist $x_1, \ldots, x_k$ such that $X = V_{x_1} \cup \cdots \cup V_{x_k}$. Now let $\epsilon := \frac{1}{2}\min\{\epsilon_{x_1}, \ldots, \epsilon_{x_k}\}$. For this $\epsilon > 0$ we pick $N \in \mathbb{N}$ as in the definition of a Cauchy sequence. This means in particular that for all $m \geq N$ we have $d(a_N, a_m) < \epsilon$.

Since $X = V_{x_1} \cup \cdots \cup V_{x_k}$ there exists an $i \in \{1, \ldots, k\}$ with $a_N \in V_{x_i}$. It suffices to prove the following claim to obtain the desired contradiction.

Claim. For any $m \geq N$ we have $a_m \in U_{x_i}$.

Let $m \geq N$. We see that

$$d(x_i, a_m) \underset{\substack{\uparrow \\ \text{triangle inequality}}}{\leq} d(x_i, a_N) + d(a_N, a_m) \underset{\substack{\uparrow \\ \text{since } a_N \in V_{x_i} \text{ and } d(a_N, a_m) < \epsilon}}{<} \tfrac{1}{2}\epsilon_{x_i} + \epsilon \underset{\substack{\uparrow \\ \text{since } \epsilon \leq \frac{1}{2}\epsilon_{x_i}}}{\leq} \tfrac{1}{2}\epsilon_{x_i} + \tfrac{1}{2}\epsilon_{x_i} = \epsilon_{x_i}.$$

This implies that $a_m \in B_{\epsilon_{x_i}}(x_i) = U_{x_i}$. ■

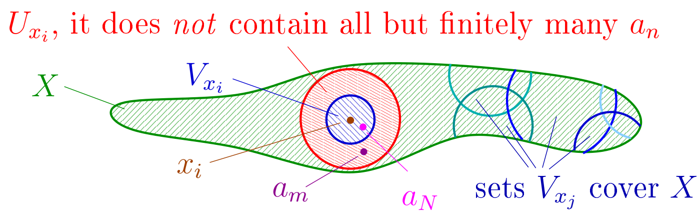


FIGURE 566. Illustration of the proof of Proposition 32.14.

The Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$ are of course not compact, but nonetheless, as we will now see, we can use Proposition 32.14 to show that $\mathbb{D}$ and $\mathbb{H}$ are complete.

Lemma 32.15. *The Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$ are complete.*

PROOF. By Proposition 32.3 the two Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$ are isometric. Therefore it suffices to show that $\mathbb{D}$ is complete. We denote by $d = d_{(\mathbb{D},g)}$ the metric on $\mathbb{D}$ corresponding to the Riemannian structure. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to d. An elementary argument using the triangle inequality and the definition of a Cauchy sequence shows that there exists a $C \in \mathbb{R}$ such that $d(0, a_n) \leq C$ for all $n \in \mathbb{N}$.

It follows from Proposition 32.11 that there exists an $r \in (0, 1)$ such that all a_n are contained in the closed Euclidean ball $\overline{B}_r^{eucl}(0)$. By the Heine-Borel Theorem 2.20 this ball is a compact subset of $\mathbb{R}^2$, viewed with the usual topology. It follows from Proposition 32.9 that $\overline{B}_r^{eucl}(0)$ is also compact in the topology defined by d. Now we see that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the compact metric space $(\overline{B}_r^{eucl}(0), d)$. It follows from Proposition 32.14 that the sequence converges. ■

Exercises for Chapter 32.

Exercise 32.1. Does the smooth manifold $B^2 \setminus \{0\}$ admit a complete Riemannian metric?

33. The universal covering of surfaces

Throughout this chapter we mean by a *surface* a connected 2-dimensional smooth manifold. For example the surface of genus g, $g \geq 0$, but also $\mathbb{C}$, $\mathbb{C} \setminus \{0,1\}$ and the result of removing finitely many points from Σ_g are surfaces.

Definition. Let M be a surface. We say that a Riemannian structure g on M is *hyperbolic* if the following two conditions are satisfied:

(1) (M,g) is locally isometric to $\mathbb{D}$, i.e. if given any $P \in M$ there exists an isometry $\Phi\colon U \to V$ from an open neighborhood U of P to an open subset V of $\mathbb{D}$,
(2) (M,g) is complete.

If g is hyperbolic, then we refer to (M,g) as a hyperbolic surface.

Remark. By Proposition 32.14 the condition (2) of the definition is automatically satisfied if M is a closed surface.

In this chapter we will pursue the following three goals:

(1) we want to show that for any $g \geq 2$ the surface of genus g is a complex manifold,
(2) we intend to prove that for any $g \geq 2$ the surface of genus g is hyperbolic,
(3) given $g \geq 2$ we want to determine the universal covering of the surface of genus g.

33.1. **Hyperbolic surfaces.** In this section we will show that the surfaces of genus ≥ 2 are complex manifolds and that they are hyperbolic. In the subsequent subsection we will then consider more general types of surfaces.

Definition. Let M be a 2-dimensional topological manifold without boundary. A *Möbius structure* for M is a family of homeomorphisms $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ from open subsets of M to open subsets of $\mathbb{D}$ such that $\bigcup\limits_{i\in I} U_i = M$ and such that for any $i,j \in I$ the transition map

$$\underbrace{\Phi_i(U_i\cap U_j)}_{\subset\mathbb{D}} \xrightarrow{\left(\Phi_i|_{U_i\cap U_j}\right)^{-1}} U_i\cap U_j \xrightarrow{\Phi_j|_{U_i\cap U_j}} \underbrace{\Phi_j(U_i\cap U_j)}_{\subset\mathbb{D}}$$

is given by a Möbius transformation. Sometimes we refer to a smooth manifold together with a Möbius structure as a *Möbius manifold.*

Example.

(1) We will always view $\mathbb{D}$ as equipped with the Möbius structure that is given by the chart $\mathrm{id}\colon \mathbb{D} \to \mathbb{D}$.
(2) Any open subset of a Möbius manifold is also a Möbius manifold.

The following lemma says that Möbius structures are useful for solving two problems at once: we can use them to show that a smooth manifold is a complex manifold and we can use them to show that a smooth manifold has a hyperbolic structure.

Lemma 33.1. *Let M be a surface and let $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ be a Möbius structure for M. Then the following hold:*

(1) *The charts form a holomorphic atlas for M, in particular M is a complex 1-dimensional smooth manifold.*
(2) (a) *The smooth manifold M admits a unique Riemannian structure g such that all the charts in the atlas $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ are isometries.*
  (b) *If the Riemannian structure from* (a) *is complete, then (M, g) is hyperbolic.*

In the following we will always view a Möbius manifold as a 1-dimensional complex manifold and as a Riemannian manifold with the structures coming from Lemma 33.1.

Proof. Let M be a surface and let $\{\Phi_i\colon U_i \to V_i\}_{i\in I}$ be a Möbius structure for M.

(1) The first statement is an immediate consequence of the definitions and of the fact, proved in Proposition 32.4 (2), that Möbius transformations are biholomorphisms.
(2) (a) We denote by g the Riemannian structure on $\mathbb{D}$. We define a Riemannian structure on M as follows. Let $P \in M$. We pick a chart $\Phi_i\colon U_i \to V_i$ from our Möbius structure and we define $h_P := \Phi_i^*(g_{\Phi_i(P)})$. Now suppose that we had picked a different chart Φ_j from the Möbius structure. The transition map from Φ_i to Φ_j is by definition a Möbius transformation, which is an isometry by Proposition 32.4 (2). It follows easily that Φ_j gives rise to the same definition of h_P. It is now straightforward to verify that this defines a Riemannian structure on M and that it has the desired properties.
  (b) This statement is an immediate consequence of (a) and of the definition of a hyperbolic surface. ■

The following proposition gives in particular an affirmative answer to Question 12.10.

Proposition 33.2. *Let $g \geq 2$. Then the following hold:*

(1) *The surface of genus g admits a Möbius structure.*
(2) *The surface of genus g admits the structure of a 1-dimensional complex manifold.*
(3) *The surface of genus g admits a hyperbolic Riemannian structure.*

Proof. We prove the proposition for the case that $g = 2$. The general case is proved by almost the same argument.

We first show that the surface of genus 2 admits a Möbius structure. By [**Bon09**, Lemma 5.3] (see also [**Frie16a**, Lemma 21.5]) there exists an $r > 0$ such that the interior angle at any vertex of the hyperbolic octagon H_8 in $\mathbb{D}$, defined by $Q_k = re^{2\pi i k/16}$, $k = 1, 3, \ldots, 15$, equals $\frac{\pi}{4}$.[550] This hyperbolic octagon is illustrated in Figure 567.

Let $k \in \{0, 1, 4, 5\}$. By Lemma 32.12 and the example on page 884 there exists a unique Möbius transformation Φ_k with $\Phi_k(Q_{2k-1}) = Q_{2k+5}$ and $\Phi_k(Q_{2k+1}) = Q_{2k+3}$. The Möbius transformations Φ_0 and Φ_1 are illustrated in Figure 568.

[550] Here is a sketch for the proof: For $r \to 0$ the interior angle converges to the interior angle of a Euclidean regular octagon, namely $\frac{3\pi}{4}$, whereas for $r \to \infty$ the interior angle converges to 0. The interior angle changes continuously with r (that is sort of clear, except that it is somewhat painful to provide a rigorous proof), so by the Intermediate Value Theorem there exists an r such that the interior angle equals $\frac{\pi}{4}$.

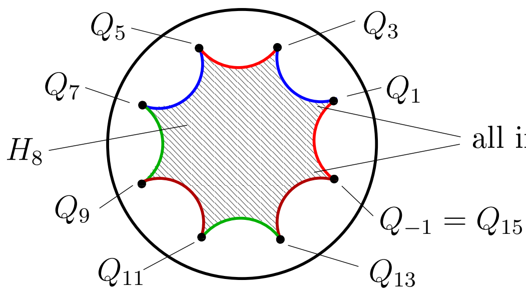


FIGURE 567

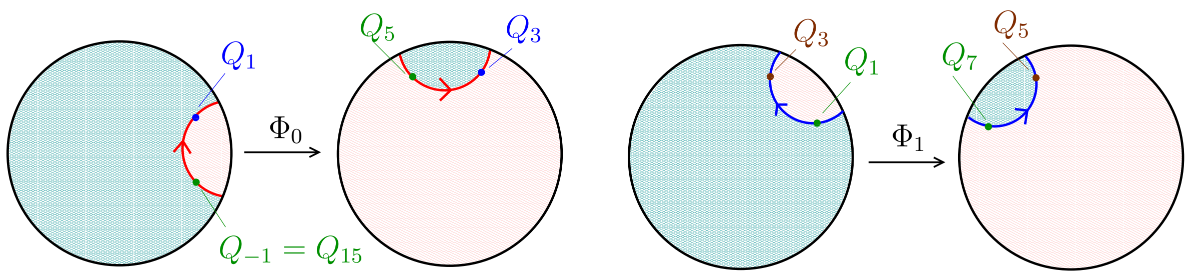


FIGURE 568

Given $k \in \{0, 1, 4, 5\}$ we declare any point P on the edge connecting Q_{2k-1} and Q_{2k+1} to be equivalent to $\Phi_k(P)$. We denote by $\sim$ the equivalence relation that is generated by these equivalences. It is straightforward to see that $H_8/\sim$ is homeomorphic to the surface of genus 2 that we defined on page 205 as $E_8/\sim$. In the following we denote by $p\colon H_8 \to H_8/\sim$ the obvious projection map.

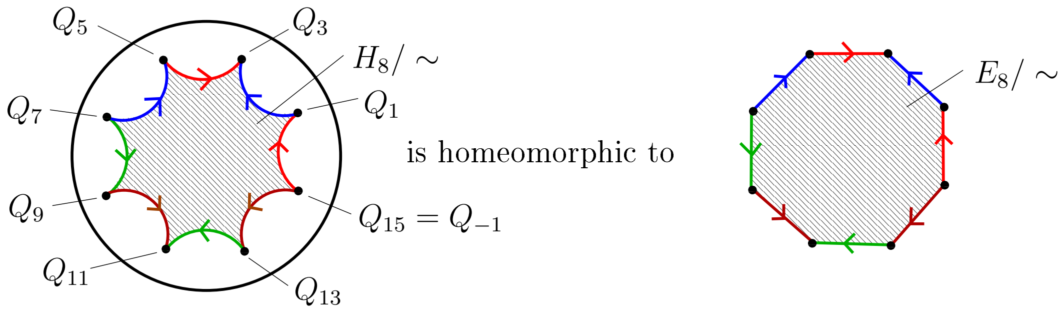


FIGURE 569

Now we will give an explicit Möbius structure for $H_8/\sim$. More precisely, given any $P \in H_8/\sim$ we will give a chart ψ_P such that the transitions maps are given by Möbius transformations. So let $P \in H_8/\sim$.

(1) If $P = p(z)$ for some $z \in \mathring{H}_8$, then the map

$$\begin{array}{rcl} \psi_P\colon p(\mathring{H}_8) & \to & \mathring{H}_8 \\ p(w) & \mapsto & w \end{array}$$

is a chart around $P = p(z)$.

(2) Now suppose that $P = p(z)$ where z lies in the interior of one of the eight edges of ∂H_8. In the following we deal with the case that z lies on the edge from Q_{-1} to Q_1. We pick an $r > 0$ such that $B_{2r}(z)$ does not contain any of the vertices $Q_1, Q_3, \dots, Q_{15}$ and does not hit any of the other edges. Then we consider the map

$$\begin{array}{rcl} \psi_P\colon p(B_r(z) \cap H_8) \cup p(\Phi_0(B_r(z)) \cap H_8) & \to & B_r(z) \\ p(w) & \mapsto & \begin{cases} w, & \text{if } w \in B_r(z) \cap H_8 \\ \Phi_0^{-1}(w), & \text{if } w \in \Phi_0(B_r(z)) \cap H_8. \end{cases} \end{array}$$

This map is a chart around $P = p(z)$. We refer to Figure 570 for an illustration of the definition of the map ψ_P.

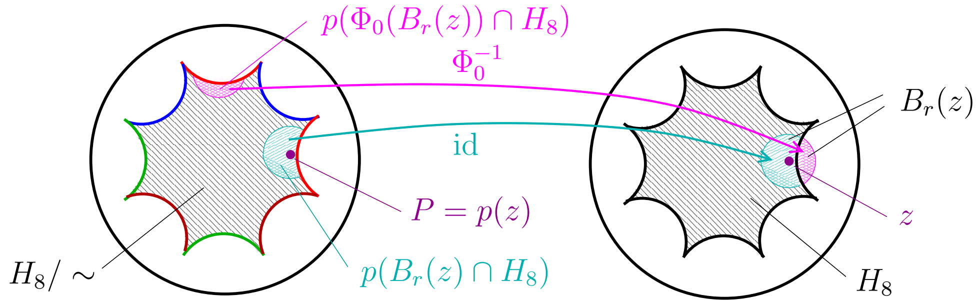


FIGURE 570

(3) Finally suppose that $P = p(Q_1) = \dots = p(Q_{15})$. We choose $r \in (0, \frac{1}{2}d(Q_{-1}, Q_1))$. Given $k \in \{1, 3, \dots, 15\}$ we define $\gamma(k)$ via the following table:

k	1	3	5	7	9	11	13	15
$\gamma(k)$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{1}{2}\pi$	$\frac{7}{4}\pi$	0	$\frac{1}{4}\pi$	$\frac{3}{2}\pi$.

Recall that we had chosen the octagon H_8 such that the angle at any vertex equals $\frac{\pi}{4}$. For $k = 1, 3, \dots, 15$ we can therefore apply Lemma 32.8 to find a unique Möbius transformation θ_k with the following two properties:

(a) $\theta_k(Q_k) = 0$,

(b) The map θ_k applied to the two rays emanating from Q_k equals the two rays $\{se^{i\gamma(k)} \mid s \geq 0\}$ and $\{se^{i(\gamma(k)+\frac{\pi}{4})} \mid s \geq 0\}$ emanating from 0.

We consider the map

$$\begin{array}{rcll} \psi_P\colon p(B_r(Q_1) \cap H_8) \cup \dots \cup p(B_r(Q_{15}) \cap H_8) & \to & B_r(0) & \\ p(z) & \mapsto & \theta_k(z), & \text{if } z \in B_r(Q_k) \cap H_8. \end{array}$$

We leave it to the reader to verify that ψ_P is well-defined and that it is indeed a homeomorphism. This map is a chart around $P = p(z)$. We refer to Figure 571 for an illustration of the definition of the map ψ_P.

It is straightforward to verify that all the transition maps between the above maps are given by Möbius transformations. We have thus shown that the maps $\{\psi_P\}_{P \in H_8}$ form a Möbius structure for $H_8/\sim$. This concludes the proof of (1).

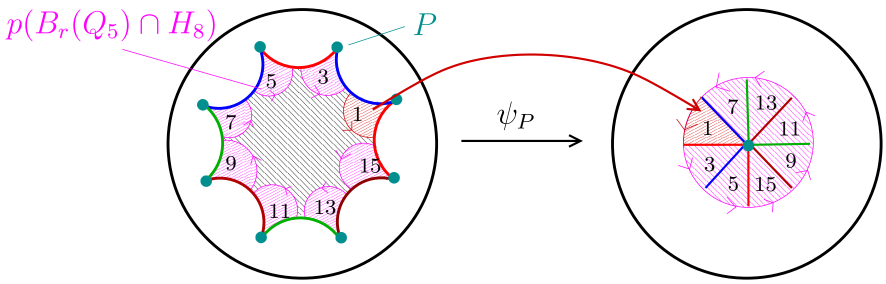


FIGURE 571

Now we turn to the proof of (2). The existence of a Möbius structure together with Lemma 33.1 (2) shows immediately that the surface of genus 2 is a 1-dimensional complex manifold.

Finally we provide the proof of (3). In light of (1) and Lemma 33.1 (2) it suffices to show that the Riemannian manifold $(H_8/\sim, g)$ is complete. By Proposition 32.9 the topology on M defined by the metric $d_{(M,g)}$ agrees with the given topology of M. Since M is compact it now follows from Proposition 32.14 that (M, g) is complete. ■

33.2. **More hyperbolic structures on the surfaces of genus $g \geq 2$ (∗).** Now we want to study the question, how many hyperbolic metrics does a surface of genus g admit. To make this question more precise, we introduce the following definition.

Definition. Let M be a surface of genus $g \geq 2$ and let h, h' be two hyperbolic metrics on M. We say h and h' are *equivalent* if there exists an isometry from (M, h) to (M, h').

This raises the question, whether a surface of genus $g \geq 2$ admits non-equivalent hyperbolic metrics. We return to the construction of a hyperbolic metric in Proposition 33.2. In our construction we started out with a regular $4g$-gon with interior angle $\frac{2\pi}{4g}$ and we identified pairs of sides. Two facts were crucial to obtain a hyperbolic metric:

(1) the sides were identified via an isometry,
(2) the sum of the interior angles equals 2π.

Or put differently, any choice of a $4g$-gon with side lengths $k_1, \ldots, k_{4g}$ and interior angles $\alpha_1, \ldots, \alpha_{4g}$ that satisfy the equations

$$k_1 = k_3,\ k_2 = k_4,\ \ldots,\ k_{4g-3} = k_{4g-1},\ k_{4g-2} = k_{4g} \quad \text{and} \quad \alpha_1 + \cdots + \alpha_{4g} = 2\pi$$

gives rise to a hyperbolic metric on a topological space that is homeomorphic to the surface of genus g. It is of course not clear whether different parameters give rise to non-equivalent hyperbolic structures. One way to distinguish them, in principle at least, is to look at the shortest length of a closed curve that is not null-homotopic. We will not elaborate on this procedure.

How many such $4g$-gons are there? On the one hand we have parameters $k_1, k_3, \ldots, k_{4g-1}$ and $\alpha_1, \ldots, \alpha_{4g}$. On the other hand we have the condition that $\alpha_1 + \cdots + \alpha_{4g} = 2\pi$. Furthermore we want a $4g$-gon, so the last angle and the length of the last edge are determined by the others. Thus we obtain two other conditions. Summarizing we have $2g + 4g = 6g$ parameters that need to satisfy 3 conditions. Of course it is not clear which of these

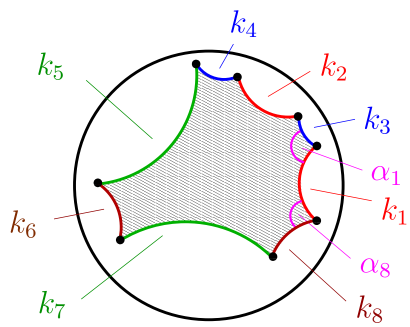


FIGURE 572

Riemannian structures are isometric. This naive discussion suggests that the space of hyperbolic metrics is $(6g-3)$-dimensional. Surprisingly this is not too far off the correct answer, in fact in [**BP92**, Chapter B.4] the following proposition is proved.

Proposition 33.3. *The space of hyperbolic metrics on a surface of genus $g \geq 2$ can be viewed as a smooth manifold of dimension $6g-6$.*

We will not make use of this proposition. The space of hyperbolic metric is called the *Teichmüller space*, more information can be found at

`https://en.wikipedia.org/wiki/Teichmueller_space`.

The theory is named after Oswald Teichmüller (1913-1943) who was a brilliant mathematician and an ardent Nazi, see

`https://en.wikipedia.org/wiki/Oswald_Teichmueller`

`http://www-history.mcs.st-andrews.ac.uk/Biographies/Teichmuller.html`

and

`https://de.wikipedia.org/wiki/Oswald_Teichmueller`.

33.3. **More examples of hyperbolic surfaces.** In this section we will generalize the discussion from the previous section to surfaces that are non-compact.

In the following let $Q_1 = 1, Q_2 = \mathrm{i}, Q_3 = -1$ and $Q_0 = Q_4 = -\mathrm{i}$. We denote by H_4 the closed, non-compact subset of $\mathbb{D}$ that is bounded by the four hyperbolic lines with endpoints Q_k, Q_{k+1} where $k = 0, 1, 2, 3$. We refer to Figure 574 for an illustration.

We consider the Möbius transformations[551]

$$\Phi_1(z) := (-\mathrm{i}) \cdot \frac{z - \frac{-1+\mathrm{i}}{2}}{1 - \frac{-1-\mathrm{i}}{2} z} \quad \text{and} \quad \Phi_2(z) := (-\mathrm{i}) \cdot \frac{z + \frac{-1-\mathrm{i}}{2}}{1 + \frac{-1+\mathrm{i}}{2} z}.$$

We record a few key properties of Φ_1 and Φ_2 in a lemma.

Lemma 33.4.

[551] Recall that a Möbius transformation of $\mathbb{D}$ is a map of the form $\psi(z) = e^{\mathrm{i}\theta} \cdot \frac{z-a}{1-\bar{a}z}$ for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$.

(1) *We have* $\Phi_1(Q_2) = Q_4$ *and* $\Phi_1(Q_3) = Q_3$ *and we also have* $\Phi_2(Q_2) = Q_4$ *and* $\Phi_1(Q_1) = Q_1$.
(2) *The restriction of* Φ_1 *to the hyperbolic line with endpoints* Q_3 *and* Q_2 *is given by reflection in the* x*-axis.*[552]
(3) *The restriction of* Φ_2 *to the hyperbolic line with endpoints* Q_1 *and* Q_2 *is given by reflection in the* x*-axis.*

Proof. The first statement follows immediately from plugging in the points. A straightforward, albeit slightly painful calculation shows that for any $\alpha \in \mathbb{R}$ we have

$$\Phi_1\big(-1 + \mathrm{i} + e^{-\mathrm{i}\alpha}\big) \;=\; -1 - \mathrm{i} + e^{\mathrm{i}\alpha}.$$

This proves the second statement. The third statement is proved by a similarly tedious calculation. ■

Remark. We consider the Möbius transformation Φ_1 in more detail. It sends the hyperbolic line with endpoints Q_3 and Q_2 to the hyperbolic line with endpoints Q_3 and Q_4. In particular it sends the two halves determined by the hyperbolic lines to one another. Which half gets sent to which half is determined by the fact that Φ_1 is orientation-preserving. We refer to Figure 573 for an illustration of the Möbius transformations Φ_1 and Φ_2.

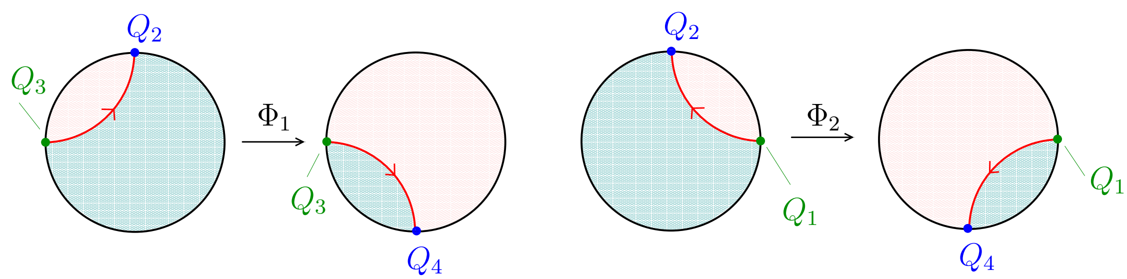


Figure 573

Given $k \in \{1,2\}$ we declare any point P on the line with the endpoints Q_k and Q_{k+1} to be equivalent to $\Phi_{3-k}(P)$. We denote by $\sim$ the equivalence relation that is generated by these equivalences.

Proposition 33.5.

(1) *The topological space* $H_4/\sim$ *admits a Möbius structure. In particular it is a 1-dimensional complex manifold.*
(2) *The Riemannian structure on* $H_4/\sim$ *coming from* (1) *and Lemma 33.1 is complete.*
(3) *The topological space* $H_4/\sim$ *admits a hyperbolic metric.*

Proof. We start out with the proof of (1). We have to show that $H_4/\sim$ admits a Möbius structure. The following argument is almost identical to the proof of Proposition 33.2 (1).

[552]The Möbius transformation Φ_1 sends the hyperbolic line with endpoints Q_3 and Q_2 to the hyperbolic line with endpoints Q_3 and Q_4. There are many such Möbius transformations, but the Möbius transformation we picked has the extra property, that on the hyperbolic line it is the "obvious map".

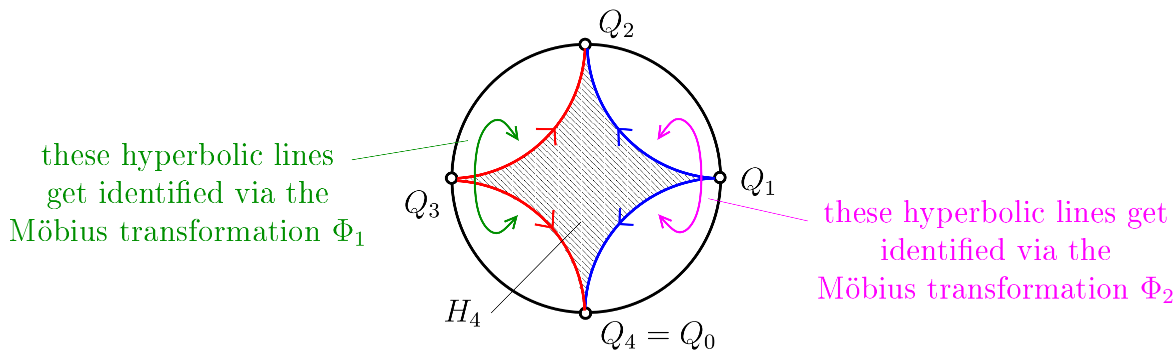


FIGURE 574

We denote by $p\colon H_4 \to H_4/\sim$ the projection map. In the following let $P \in H_4/\sim$.

(1) If $P = p(z)$ for some $z \in \mathring{H}_4$, then

$$\begin{array}{rcl} \psi_P\colon p\big(\mathring{H}_4\big) & \to & \mathring{H}_4 \\ p(w) & \mapsto & w \end{array}$$

is a chart around $P = p(z)$.

(2) Now suppose that $P = p(z)$ where z lies on ∂H_4. In the following we deal with the case that z lies on the edge from Q_1 to Q_2. All other cases are dealt with almost the same way. We pick an $r > 0$ such that $B_{2r}(z)$ intersects no other component of ∂H_4. We consider the map

$$\begin{array}{rcl} \psi_P\colon p(B_r(z) \cap H_4) \cup p(\Phi_2(B_r(z)) \cap H_4) & \to & B_r(z) \\ p(w) & \mapsto & \begin{cases} w, & \text{if } w \in B_r(z) \cap H_4 \\ \Phi_2^{-1}(w), & \text{if } w \in \Phi_1^{-1}(B_r(z)) \cap H_4. \end{cases} \end{array}$$

This map is a chart around $P = p(z)$.

The charts that we just constructed form an atlas for $H_4/\sim$. It follows immediately from the definitions that all transition maps are given by Möbius transformations. It follows that the maps ψ_P form a Möbius structure for $H_4/\sim$. As in the proof of Proposition 33.2 it follows that $H_4/\sim$ is a 1-dimensional complex manifold and that these charts satisfy the conditions of Lemma 33.1. This concludes the proof of (1).

We continue with the proof of (2). We have to show that the Riemannian structure on $H_4/\sim$ coming from (1) and Lemma 33.1 is complete. The subsequent argument is a variation on the proof of Proposition 32.14.

We denote by d the metric corresponding to the Riemannian structure on $H_4/\sim$. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $H_4/\sim$ with respect to d. As in the proof of Proposition 32.14 we see that there exists a $C \in \mathbb{R}$ such that $d(0, a_n) \leq C$ for all $n \in \mathbb{N}$.

Claim. There exists an $r \in [0, 1)$ such that for any $z \in H_4$ with $d(0, p(z)) \leq C$ we have $|z| \leq r$.

Given $s \in [0,1)$ we define

$$g(s) \ := \int\limits_{t=0}^{t=s} \frac{\sqrt{2}}{1-t^2}\, dt.$$

Clearly we have $\lim\limits_{s\to 1} g(s) = \infty$. In particular there exists an $r \in [0,1)$ such that $g(s) > C+1$ for all $s \geq \frac{1}{2}r$.

We claim that r has the desired property. So let $z = x + iy \in H_4$ with $|z| > r$. Since H_4 is a subset of the square spanned by ± 1 and $\pm i$ we see that we have $|z| \leq |x| + |y|$. Put differently, we have $|x| \geq \frac{1}{2}r$ or $|y| \geq \frac{1}{2}r$. Without loss of generality we can assume that $|x| \geq \frac{1}{2}r$.

We need to show that $d(0, p(z)) \geq C+1$. It suffices to show that for any piecewise smooth path $\gamma\colon [0,a] \to H_4/\sim$ with $\gamma(0) = 0$ and $\gamma(a) = p(z)$ we have $\ell(\gamma) \geq C+1$.

We denote by

$$\begin{array}{rcl} q\colon H_4/\sim & \to & (-1,1) \\ {[x+iy]} & \mapsto & x \end{array}$$

the projection onto the real part. It follows from Lemma 33.4 that this map is indeed well-defined. A straightforward calculation shows that for all $t \in [0,a]$ we have

$$\big\|\gamma'(t)\big\|_{H_4/\sim} \ \geq \ \big\|(q\circ\gamma)'(t)\big\|_{\mathbb{D}}.$$

It follows that

$$\ell(\gamma) \ = \int\limits_0^a \big\|\gamma'(t)\big\|_{H_4/\sim}\, dt \underset{\substack{\uparrow \\ \text{above inequality}}}{\geq} \int\limits_0^a \big\|(q\circ\gamma)'(t)\big\|_{\mathbb{D}}\, dt \underset{\substack{\uparrow \\ \text{definition of } d_{(\mathbb{D},g)}}}{\geq} d_{(\mathbb{D},g)}(0, \overbrace{q(\gamma(a))}^{=x}) \underset{\substack{\uparrow \\ \text{Proposition 32.11}}}{=} g(|x|) \overset{\substack{\text{since } |x| \geq \frac{1}{2}r \\ \downarrow}}{\geq} C+1.$$

⊞

By the above claim the sequence $\{a_n\}_{n\in\mathbb{N}}$ lies in the compact set $p(\{z \in H_4 \mid |z| \leq r\})$. But by Proposition 32.14 the metric space $p(\{z \in H_4 \mid |z| \leq r\})$ is complete, so the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges in $p(\{z \in H_4 \mid |z| \leq r\})$, in particular it converges in $p(H_4) = H_4/\sim$. Thus we have proved that $H_4/\sim$ is complete. This concludes the proof of (2).

Finally statement (3) is an immediate consequence of the definition of a hyperbolic metric and of statement (2). ■

We had just seen in Proposition 33.5 that $H_4/\sim$ admits a Möbius structure, in particular it is a 1-dimensional complex manifold. It is relatively straightforward to show that $H_4/\sim$ is homeomorphic to the sphere minus three points, or equivalently, to $\mathbb{C}$ minus two points. Now we show the much stronger statement that $H_4/\sim$ is actually biholomorphic to $\mathbb{C} \setminus \{0,1\}$.

Proposition 33.6. *There exists a biholomorphism $H_4/\sim \to \mathbb{C}\setminus\{0,1\}$.*

Remark. We equip $H_4/\sim$ with the Riemannian metric g coming from the Möbius structure and we equip $\mathbb{C}\setminus\{0,1\}$ with the usual Euclidean Riemannian metric that we denote by h.

By Proposition 33.5 the metric space $(H_4/\sim, g)$ is complete whereas $(\mathbb{C}\setminus\{0,1\}, h)$ is evidently not complete. Thus we see that the biholomorphism $H_4/\sim\,\to \mathbb{C}\setminus\{0,1\}$ from Proposition 33.6 is *not* an isometry.

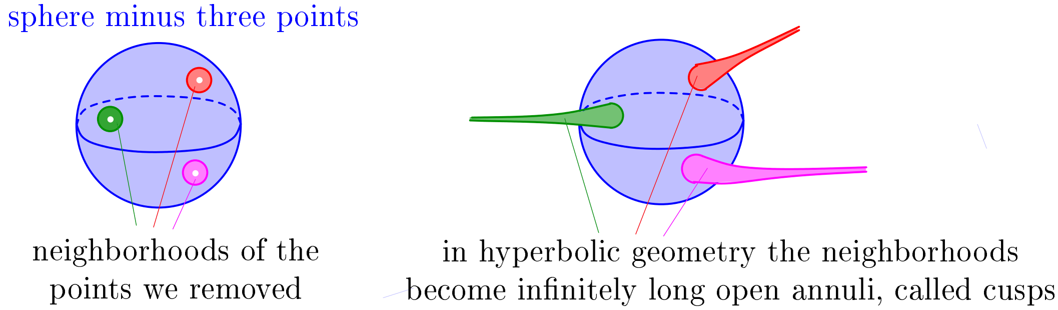


FIGURE 575

PROOF OF PROPOSITION 33.6. We denote by Δ the open subset of $\mathbb{D}$ bounded by the three hyperbolic lines with endpoints $-1, 1$ and i. (We refer to Figure 576 for an illustration of Δ.) It follows from applying the Riemann Mapping Theorem 12.11 together with Carathéodory's Theorem 12.12 that there exists a biholomorphism $\Psi\colon \Delta \to \mathbb{D}$ that extends to a homeomorphism $\Psi\colon \overline{\Delta} \to \overline{\mathbb{D}}$. We combine Ψ with the biholomorphism from $\mathbb{D}$ to $\mathbb{H}$ coming from Proposition 32.3. This way we obtain a biholomorphism $\Phi\colon \Delta \to \mathbb{H}$ that extends to a homeomorphism $\Phi\colon \overline{\Delta} \to \overline{\mathbb{H}}$.

Using Lemma 32.5 we can, if necessary, compose Φ with a Möbius transformation of $\mathbb{H}$, to arrange that $\Phi(i) = \infty$ and that $\Phi(\{-1,1\}) = \{0,1\}$.

Note that Φ restricts to a homeomorphism $\Phi\colon \partial\overline{\Delta} \to \partial\overline{\mathbb{H}} = \mathbb{R}\cup\{\infty\}$. By the above we have $\Phi(\partial\overline{\Delta}) = \mathbb{R}\setminus\{0,1\}$. Now we consider the map

$$\begin{array}{rcl} \Psi\colon H_4 & \to & \mathbb{C}\setminus\{0,1\} \\ z & \mapsto & \begin{cases} \Phi(z), & \text{if } z \in \overline{\Delta}, \\ \overline{\Phi(\overline{z})}, & \text{if } \overline{z} \in \overline{\Delta}, \end{cases} \end{array}$$

Note that by the above we have $f((-1,1)) = (0,1)$. Thus it follows from the Schwarz Reflection Principle, see Proposition 12.3, that the restriction of Ψ to the interior of H_4 is holomorphic. It follows from Lemma 33.4 that if P and Q are two equivalent points on ∂H_4, then $\Psi(P) = \Psi(Q)$. Therefore the map Ψ factors through a map $H_4/\sim\,\to \mathbb{C}\setminus\{0,1\}$. Using once again the Schwarz Reflection Principle one can show that this map is in fact a biholomorphism. We leave the verification of this step to the reader. ■

Definition. Given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ we now denote by $F_{g,n}$ the surface that is obtained from the surface of genus g by removing n points. We refer to $F_{g,n}$ as the *n-punctured surface of genus g.*

For example the three-punctured sphere $F_{0,3}$ is the sphere with three points removed. As we pointed out above, $F_{0,3}$ is diffeomorphic to $\mathbb{C}\setminus\{0,1\}$. In particular, as we have just seen, the three-punctured sphere admits a hyperbolic structure. The following theorem says that "most" surfaces admit a hyperbolic structure.

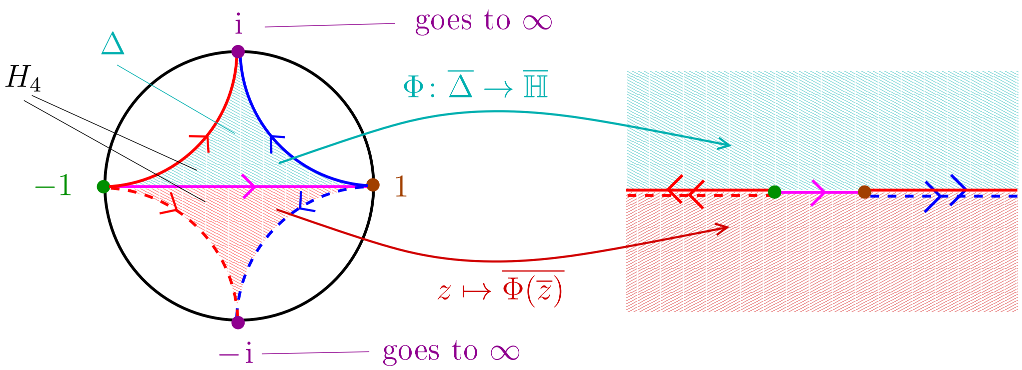
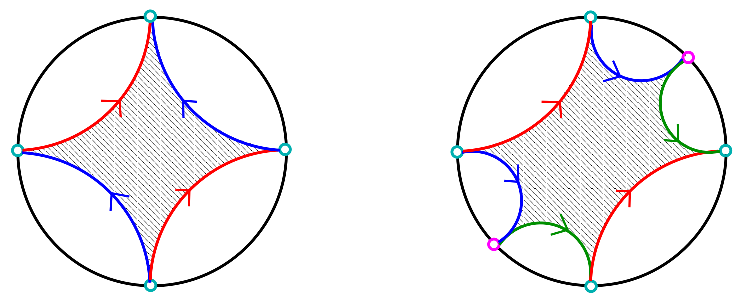


FIGURE 576. Illustration of the proof of Proposition 33.6

Theorem 33.7. *Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}_0$. If $2g+n-2 > 0$, then $F_{g,n}$ admits a hyperbolic structure.*

SKETCH OF A PROOF. Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}_0$. If $2g + n - 2 > 0$ then variations on the argument of Propositions 33.2 and 33.5 show that $F_{g,n}$ admits a hyperbolic metric. In Figure 577 we sketch the constructions needed to show that the once-punctured and the twice-punctured torus have a Möbius structure. The proof that the corresponding Riemannian manifolds are complete is similar to the proof of Proposition 33.5. ■



once punctured torus twice punctured torus

FIGURE 577

33.4. **The universal covering of surfaces.**

Definition. Let $f\colon M \to N$ be a map between two Möbius manifolds.

(1) We say that f is a *Möbius map* if for every two chart $\Phi\colon U \to V$ in the Möbius structure of M and any chart $\Psi\colon X \to Y$ in the Möbius structure of N the restriction of the map

$$\underbrace{\Phi(U \cap f^{-1}(X))}_{\subset \mathbb{D}} \xrightarrow{\Psi\circ f\circ\Phi^{-1}} \underbrace{\Psi(X \cap f(U))}_{\subset \mathbb{D}}$$

to any component of $\Phi(U \cap f^{-1}(X))$ is given by a Möbius transformation.

(2) We say that f is a *Möbius isomorphism*, if f is a *Möbius map*, if f is a bijection, and if f^{-1} is also a *Möbius map*.

The following lemma is an immediate consequence of the definitions and the fact, proved in Proposition 32.4, that Möbius transformations are biholomorphisms and isometries.

Lemma 33.8. *Let M and N be two Möbius manifolds and let $f\colon M \to N$ be a Möbius map. Then the following two statements hold:*

(1) *the map f is a local biholomorphism,*
(2) *the map f is a local isometry.*

The following theorem is the main result of this chapter.

Theorem 33.9. (Hadamard's Theorem) *Let $g \geq 2$, and let $M = H_{4g}/\sim$ be the surface of genus g with the Möbius structure constructed in Proposition 33.2. Then there exists a covering map $p\colon \mathbb{D} \to M$ with the following properties:*

(1) *p is a Möbius map,*
(2) *p is a local isometry,*
(3) *p is a local biholomorphism.*

The hyperbolic disk $\mathbb{D}$ is of course simply connected. The theorem thus says that for any $g \geq 2$ the hyperbolic disk $\mathbb{D}$ is the universal covering of the surface of genus g. Thus we have now answered Question 29.10.

The proof of Theorem 33.9 will require the following two sections.

33.5. **Proof of Theorem 33.9 I.** In this section we will lay the technical foundations for the proof of Theorem 33.9.

In the following lemma we first summarize a few properties of Möbius maps.

Lemma 33.10.

(1) *If $U \subset \mathbb{D}$ is a connected subset and if $f\colon U \to \mathbb{D}$ is a Möbius map, then f is given by a Möbius transformation.*
(2) *Let $f\colon M \to N$ be a map between two Möbius manifolds. If f is locally a Möbius map, then f is a Möbius map.*
(3) *The composition of two Möbius maps is again a Möbius map.*
(4) *A bijective Möbius map is a Möbius isomorphism.*

Proof.

(1) Let $U \subset \mathbb{D}$ be a connected subset and let $f\colon U \to \mathbb{D}$ be a Möbius map. It follows from Proposition 32.4 (4) that given $P \in U$ there exists a unique Möbius transformation ϕ_P such that $f = \phi_P$ in an open neighborhood of P. We consider the map

$$\begin{array}{rcl}\phi\colon U & \to & \text{set of all Möbius transformations} \\ P & \mapsto & \phi_P.\end{array}$$

We equip the right-hand side of the map ϕ with the discrete topology. It follows from the definitions that this map is continuous. But then it follows from Lemma 2.61 that this map is constant. But that is exactly what we had tried to show.

(2) This statement follows immediately from (1) and the definitions.

(3) This statement is an immediate consequence of definitions and the fact, proved in Proposition 32.4, that the compositions of two Möbius transformations is again a Möbius transformation.
(4) Let $f\colon M \to N$ be a bijective Möbius map. In Proposition 32.4 we saw that the inverse of a Möbius transformation is again a Möbius transformation. It follows that f^{-1} is locally a Möbius map. But then it follows from (2), that f^{-1} is in fact a Möbius map. ■

Lemma 33.11. *Let U be an open connected subset of $\mathbb{D}$ and let M be a Möbius manifold. Furthermore let $\Phi, \Psi\colon U \to M$ be two Möbius maps.*[553] *Suppose one of the following two conditions is satisfied:*

(1) *There exists an open non-empty subset of U on which Φ and Ψ agree, or*
(2) *there exists a point $Q \in U$ with $\Phi(Q) = \Psi(Q)$ and a non-zero vector $v \in \mathrm{T}_Q\,\mathbb{D}$ such that $\mathrm{D}\,\Phi_Q(v) = \mathrm{D}\,\Psi_Q(v)$,*

then the Möbius maps Φ and Ψ agree on all of U.

Proof. Let U be an open connected subset of $\mathbb{D}$, let M be a Möbius manifold and let $\Phi, \Psi\colon U \to M$ be two Möbius maps. We consider one more statement:

(3) There exists a point $Q \in U$ such that $\Phi(Q) = \Psi(Q)$ and such that $\mathrm{D}\,\Phi_Q = \mathrm{D}\,\Psi_Q$.

Clearly we have (1) ⇒ (3). We also have (2) ⇒ (3). Indeed, the differentials $\mathrm{D}\,\Phi_Q$ and $\mathrm{D}\,\Psi_Q$ are homomorphisms between two 1-dimensional complex vector spaces. So if they are agree on one non-zero vector, then they agree on all of the vector space $\mathrm{T}_Q\,\mathbb{D}$.

So it suffices to show that if (3) is satisfied, then the maps Φ and Ψ agree on all of U. So suppose that there exists a point $Q \in U$ such that $\Phi(Q) = \Psi(Q)$ and such that $\mathrm{D}\,\Phi_Q = \mathrm{D}\,\Psi_Q$. We consider

$$Y \;:=\; \{P \in U \mid \Phi(P) = \Psi(P) \text{ and } \mathrm{D}\,\Phi_P = \mathrm{D}\,\Psi_P\}.$$

By our hypothesis we know that Y contains Q, in particular Y is non-empty. Since Φ and Ψ are smooth it follows that Y is closed. It remains to show that Y is open.

So let $P \in Y$. We pick a map $\Theta\colon W \to \mathbb{D}$ from the Möbius structure of M such that W is an open neighborhood of $\Phi(P) = \Psi(P)$. We denote by V the component of $\Phi^{-1}(W) \cap \Psi^{-1}(W)$ that contains P. Since Φ and Ψ are Möbius maps it follows from Lemma 33.10 (3) that the maps $\Theta \circ \Phi\colon V \to \mathbb{D}$ and $\Theta \circ \Psi\colon V \to \mathbb{D}$ are also Möbius maps. Since V is connected it follows from Lemma 33.10 (1) that both maps are given by Möbius transformations. These two Möbius transformations agree at P and they have the same differential at P. It follows from Proposition 32.4 (5) that Φ and Ψ agree on an open neighborhood of P. This shows that Y is also open. ■

Definition. Let M be a Möbius manifold and let $Q \in M$ and $r > 0$. We say $B_r^M(Q)$ is *small* if there exists a Möbius isomorphism $\Omega\colon B_r^{\mathbb{D}}(0) \to B_r^M(Q)$ with $\Omega(0) = Q$.

[553] Here we view $U \subset \mathbb{D}$ as equipped with the obvious Möbius structure.

Proposition 33.12. *Let M be a Möbius manifold and let $Q \in M$. Then there exists an $r > 0$ such that $B_r^M(Q)$ is small.*

In the proof of Proposition 33.12 we will make use of the following lemma.

Lemma 33.13. *Let (X, g) and (Y, h) be Riemannian manifolds. Let $Q \in X$ and $r > 0$. Let $f\colon B_{3r}^X(Q) \to (Y, h)$ be a local isometry and let $S, T \in B_r^X(Q)$. Then we have*

$$d_Y(f(S), f(T)) \;\le\; d_X(S, T).$$

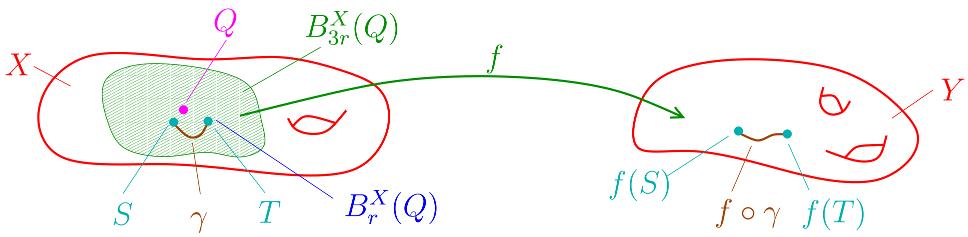


Figure 578. Illustration for Lemma 33.13.

Proof. Let $S, T \in B_r^X(Q)$. Furthermore let $\epsilon > 0$. By the triangle inequality we have $d_X(S, T) < 2r$. In particular there exists a piecewise smooth path γ in X from S to T such that $\ell_X(\gamma) < 2r$ and with $\ell_X(\gamma) < d_X(S, T) + \epsilon$. Given any t we have

$$d_X(Q, \gamma(t)) \;\le\; d_X(Q, S) + d_X(S, \gamma(t)) \;<\; r + \ell_X(\gamma) \;<\; 3r.$$

Thus we have shown that the image of γ lies in $B_{3r}^X(Q)$. In particular the image of γ lies in the domain of f. Thus we see that

$$d_Y(f(S), f(T)) \underset{\substack{\uparrow \\ \text{since } f\circ\gamma \text{ is a path from} \\ f(S) \text{ to } f(T)}}{\le} \ell_Y(f \circ \gamma) \underset{\substack{\uparrow \\ \text{by Lemma 32.10, since} \\ f \text{ is a local isometry}}}{=} \ell_X(\gamma) \;<\; d_X(S, T) + \epsilon.$$

Since this inequality holds for all $\epsilon > 0$ we see that $d_Y(f(S), f(T)) \le d_X(S, T)$. ■

Now we can provide the proof of Proposition 33.12.

Proof of Proposition 33.12. Let M be a Möbius manifold and let $Q \in M$. We pick a map $\Phi\colon U \to V$ from the Möbius structure of M such that U is an open neighborhood of Q. By Proposition 32.4 (3), after possibly composing Φ with a Möbius transformation, we can furthermore assume that $\Phi(Q) = 0$. We write $\Omega := \Phi^{-1}$. By Proposition 32.9 (3) and by the openness of V there exists an $r > 0$ such that $B_{3r}^M(Q) \subset U$ and such that $B_{3r}^{\mathbb{D}}(0) \subset V$.[554]

It follows immediately from Lemmas 33.8 and 33.13 and from our choice of r that the following holds:

(1) For any $S, T \in B_r^M(Q)$ we have $d_{\mathbb{D}}(\Phi(S), \Phi(T)) \le d_M(S, T)$.
(2) For any $S, T \in B_r^{\mathbb{D}}(Q)$ we have $d_M(\Omega(S), \Omega(T)) \le d_{\mathbb{D}}(S, T)$.

It follows immediately from these two inequalities, from $\Phi(Q) = 0$ and from the fact that $\Phi = \Omega^{-1}$ that Ω restricts to an isometry $\Omega\colon (B_r^{\mathbb{D}}(0), d_{\mathbb{D}}) \to (B_r^M(Q), d_M)$ of metric spaces which is furthermore a bijection. Furthermore, since Φ is a Möbius isomorphism

[554]Note the discrete factor 3 in the radius of the ball.

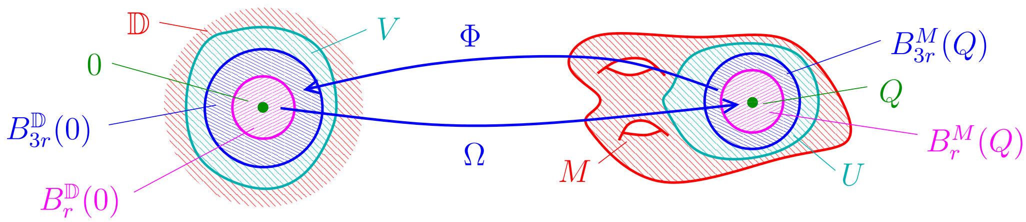


FIGURE 579. Illustration for the proof of Proposition 33.12.

it follows that the inverse map $\Omega = \Phi^{-1}\colon (B^{\mathbb{D}}_r(0), d_{\mathbb{D}}) \to (B^M_r(Q), d_M)$ is also a Möbius isomorphism. ■

Lemma 33.14. *Let M be a Möbius manifold. Let $Q \in M$ and let $r > 0$ such that $B^M_r(Q)$ is small. Let $\Phi\colon \mathbb{D} \to M$ be a Möbius map such that $B^M_r(Q) \subset \Phi(\mathbb{D})$. Then the following hold:*

(1) *For any $X \in \Phi^{-1}(Q)$ the map Φ restricts to a Möbius isomorphism $B^{\mathbb{D}}_r(X) \to B^M_r(Q)$.*
(2) *Let $s \leq \frac{r}{2}$ and let $Y \in \mathbb{D}$ such that $\Phi(Y) \in B^M_s(Q)$. Then there exists an $X \in \Phi^{-1}(Q)$ with $Y \in B^{\mathbb{D}}_s(X)$.*

PROOF. Let M be a Möbius manifold. Let $Q \in M$ and let $r > 0$ such that $B^M_r(Q)$ is small. We pick a Möbius isomorphism $\Omega\colon B^{\mathbb{D}}_r(0) \to B^M_r(Q)$ as in the definition of "small". Let $\Phi\colon \mathbb{D} \to M$ be a Möbius map such that $B^M_r(Q) \subset \Phi(\mathbb{D})$.

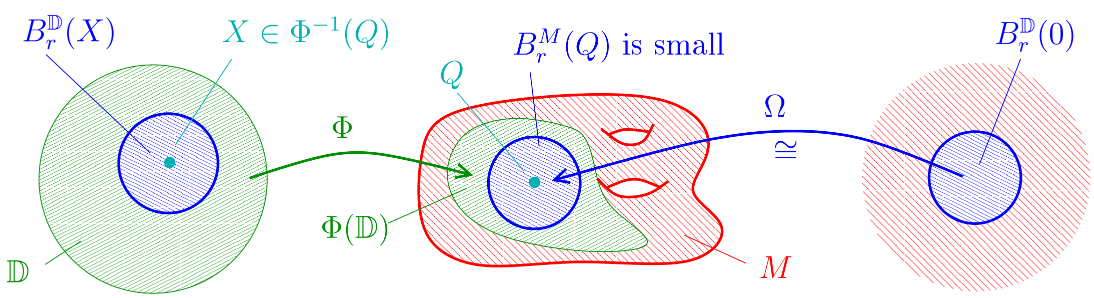


FIGURE 580. Illustration for the proof of Lemma 33.14 (1).

(1) Let $X \in \Phi^{-1}(Q)$. It follows immediately from the fact that Möbius maps are local isometries and from Lemma 33.13 that $\Phi(B^{\mathbb{D}}_r(X)) \subset B^M_r(Q)$. Therefore we can consider the Möbius map $\Omega^{-1} \circ \Phi\colon B^{\mathbb{D}}_r(X) \to B^{\mathbb{D}}_r(0)$. By Lemma 33.10 this Möbius map is the restriction of a Möbius transformation. In particular it is a Möbius isomorphism. But since Ω is a Möbius isomorphism as well we obtain that $\Phi\colon B^{\mathbb{D}}_r(X) \to B^M_r(Q)$ is also a Möbius isomorphism.
(2) Let $s \leq \frac{r}{2}$ and let $Y \in \mathbb{D}$ such that $\Phi(Y) \in B^M_s(Q)$, i.e. such that $d_M(Q, \Phi(Y)) < s$. By the triangle inequality we have
$$B^M_s(\Phi(Y)) \ \subset\ B^M_{2s}(Q) \ \subset\ B^M_r(Q).$$
The same argument as in (1) shows that the map Φ restricts to a Möbius isomorphism $B^{\mathbb{D}}_s(Y) \to B^M_s(\Phi(Y))$. We have $d_M(Q, \Phi(Y)) < s$, i.e. we have $Q \in B^M_s(\Phi(Y))$.

Hence there exists an $X \in B^{\mathbb{D}}_s(Y)$ with $\Phi(X) = Q$. We refer to Figure 581 for an illustration. ■

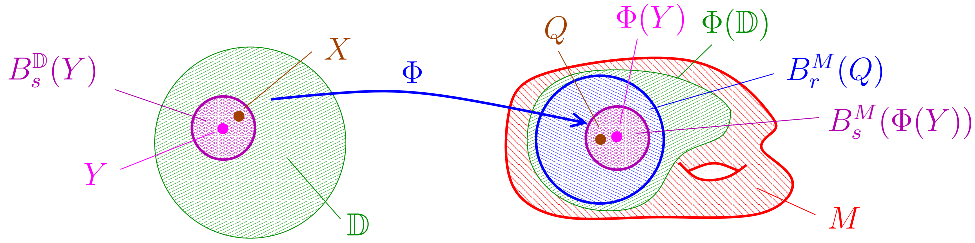


FIGURE 581. Illustration for the proof of Lemma 33.14 (2).

33.6. **Proof of Theorem 33.9 II.** In this section we will now finally provide the proof of Theorem 33.9. We will only cover the case $g = 2$, the proof of the general case is almost identical.

Throughout this section let $M = H_8/\sim$ be the surface of genus 2 with the Möbius structure constructed in Proposition 33.2. In the following we will show that there exists a Möbius map $p\colon \mathbb{D} \to M$ that is in fact a covering map. It then follows from Lemma 33.8 that p is also a local isometry and a local biholomorphism.

For the remainder of the proof we adopt the following notation and conventions.

(1) We view $\mathring{H}_8$ as a subset of $\mathbb{D}$ and also of $H_8/\sim$.
(2) Given $Z \in \mathbb{D}$ we denote by

$$\begin{array}{rcl}\gamma_Z\colon [0,1] & \to & \mathbb{D} \\ t & \mapsto & t\cdot Z\end{array}$$

the radial path from the origin $0 \in \mathbb{D}$ to Z.
(3) We say that $U \subset \mathbb{D}$ is *convex* if it is convex in the usual Euclidean sense, i.e. if given any $P, Q \in U$ the points $t\cdot P + (1-t)\cdot Q$, $t \in [0,1]$ also lie in U. On several occasions throughout the proof we will make use of the following elementary observations:
(a) convex sets are path-connected,
(b) the intersection of two convex sets is also convex,
in particular, combining (a) and (b) we get
(c) the intersection of two convex sets is path-connected.

Our first goal is to construct a suitable map $p\colon \mathbb{D} \to M = H_8/\sim$. So let $Z \in \mathbb{D}$. We write $\gamma = \gamma_Z$. We say $t \in [0,1]$ is *good* if there exists a convex open neighborhood U of $\gamma([0,t])$ and a map $\Psi\colon U \to M$ which has the following properties:

(i) Ψ is a Möbius map,
(ii) it restricts to the identity on $\mathring{H}_8 \cap U \subset M$.

We start out with the following lemma.

Lemma 33.15. *The point $t = 1$ is good.*

PROOF. We set

$$T := \{t \in [0,1] \mid t \text{ is good}\}.$$

We start out with the following observations:

(1) We have that $0 \in T$ since the identity on the open neighborhood $\overset{\circ}{H}_8$ of 0 has the required properties.[555]
(2) Being good is clearly an open condition, hence $T \subset [0,1]$ is open.

We want to show that $T = [0,1]$. By the above, and since $[0,1]$ is connected, it suffices to show that $s := \sup(T)$ is good.

Since $0 \in S$ and since T is open we see that $s > 0$. Therefore there exists an increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ of good numbers $t_n \in [0,t)$ that converges to s. Since each t_n is good we can pick for each n a convex open neighborhood U_n of $\gamma([0,t_n])$ and a map $\Psi_n \colon U_n \to M$ that has the desired properties (i) and (ii).

Let $m,n \in \mathbb{N}$. The Möbius maps Ψ_m and Ψ_n agree on the non-empty open connected subset $\overset{\circ}{H}_8 \cap U_n \cap U_m$. It follows from Lemma 33.11 that the maps Ψ_m and Ψ_n agree on $U_n \cap U_m$.

By the continuity of γ the sequence $\{\gamma(t_n)\}_{n\in\mathbb{N}}$ converges to $\gamma(t)$, in particular it is a Cauchy sequence with respect to $d_{\mathbb{D}}$. It follows from Lemma 32.10 that $\{\Psi_n(\gamma(t_n))\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to d_M.[556] It follows from Proposition 33.2 that the metric space $(M = H_8/\sim, d_M)$ is complete. Therefore the Cauchy sequence $\{\Psi_n(\gamma(t_n))\}_{n\in\mathbb{N}}$ converges to a point $S \in M$.

It follows from Proposition 33.12 that there exists an $r > 0$ and a Möbius isomorphism $\Omega \colon B_r^{\mathbb{D}}(0) \to B_r^M(S)$ such that $\Omega(0) = S$. We pick an n such that $\Psi_n(\gamma(t_n)) \in B_r^M(S)$ and such that $d_{\mathbb{D}}(\gamma(t), \gamma(s)) < r$. We write $t = t_n, \Psi = \Psi_n, U = U_n$ and we write $X = \gamma(t)$. Let $v \in \mathrm{T}_X \mathbb{D}$ be a non-zero vector. We let $w := d(\Omega^{-1} \circ \Psi)_X(v)$. The Möbius maps Ω and Ψ are in particular local isometries, therefore we have $\|w\| = \|v\|$. It follows from Proposition 32.4 (4) that there exists a Möbius transformation Θ with $\Theta(X) = \Omega^{-1}(\Psi(X))$ and such that $d\Theta_X(v) = w$.

The situation is illustrated in Figure 582. The Möbius maps $\Omega \circ \Theta \colon B_r^{\mathbb{D}}(X) \to M$ and $\Psi \colon U \to M$ agree at $X = \gamma(t)$ and we have $\mathrm{D}(\Omega \circ \Theta)_X(v) = \mathrm{D}\,\Psi_X(v)$. It follows from Lemma 33.11 that $\Omega \circ \Theta$ and Ψ agree on the path-connected set $B_r^{\mathbb{D}}(X) \cap U$.

Now we consider the map

$$\begin{array}{rcl} \Phi \colon U \cup B_r^{\mathbb{D}}(\gamma(t)) & \to & M \\ Q & \mapsto & \begin{cases} \Psi(Q), & \text{if } Q \in U, \\ \Omega^{-1}(\Theta(Q)), & \text{if } Q \in B_r^{\mathbb{D}}(\gamma(t)). \end{cases} \end{array}$$

By the above discussion this map is well-defined and it is locally a Möbius map. It follows from Lemma 33.10 (3) that this map is a Möbius map. It follows from $d_{\mathbb{D}}(\gamma(t), \gamma(s)) < r$

[555] Here we use that for each point $Z \in \overset{\circ}{H}_8$ the identity map $\mathrm{id} \colon \overset{\circ}{H}_8 \to \overset{\circ}{H}_8$ is part of the Möbius structure of M.

[556] Indeed, suppose we have $\epsilon > 0$. Since $\{\gamma(t_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence there exists an N such that $d_{\mathbb{D}}(\gamma(t_m), \gamma(t_n)) < \epsilon$ for all $m,n \geq N$. Now let $m,n \geq N$. Without loss of generality we have $m > n$. Then

$$d_M(\Psi_m(\gamma(t_m)), \Psi_n(\gamma(t_n))) \underset{\substack{\uparrow \\ \text{since } \Psi_m \text{ and } \Psi_n \text{ agree on } \gamma|_{[0,t_n]}}}{=} d_M(\Psi_m(\gamma(t_m)), \Psi_m(\gamma(t_n))) \underset{\substack{\uparrow \\ \text{Lemma 32.10}}}{\leq} d_{\mathbb{D}}(\gamma(t_m), \gamma(t_n)) < \epsilon.$$

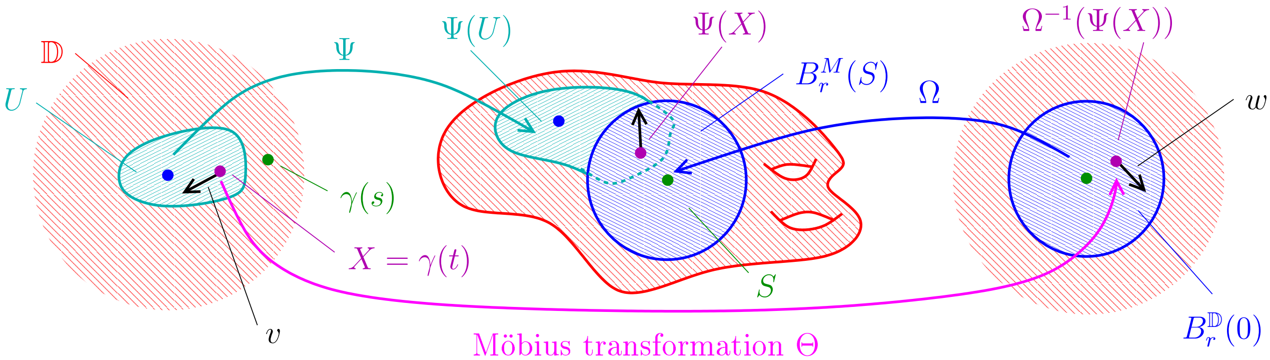


FIGURE 582

and the fact that γ is a geodesic in $\mathbb{D}$ that $\gamma([t,s]) \subset B_r^{\mathbb{D}}(\gamma(t))$, in particular $U \cup B_r^{\mathbb{D}}(\gamma(t))$ is an open subset of $\mathbb{D}$ that contains $\gamma([0,t]) \cup \gamma([t,s]) = \gamma([0,s])$. It is now straightforward to see that we can find a *convex* open subset W of $U \cup B_r^{\mathbb{D}}(\gamma(t))$ that contains $\gamma([0,s])$. The Möbius map $\Phi\colon W \to M$ now certifies that s is also good. ■

Thus we have now shown that there exists a convex open neighborhood U of $\gamma([0,1])$ and a local isometry $\Psi\colon U \to H_8/\sim$ which agrees with the identity map on $U \cap \mathring{H}_8$. We define $p(Z) := \Psi(Z)$. By Lemma 33.11 this definition does not depend on the choice of U and Ψ.

Lemma 33.16. *The map $p\colon \mathbb{D} \to M$ is a Möbius map.*

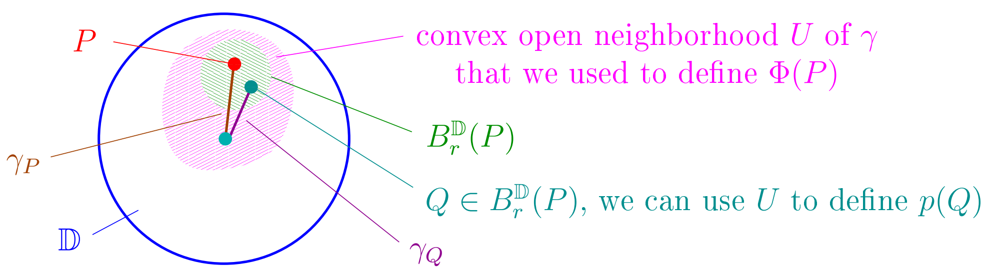


FIGURE 583. Illustration of the proof of Lemma 33.16.

PROOF. By Lemma 33.10 (2) it suffices to show that p is locally a Möbius map. This means that it suffices to show that every point $P \in \mathbb{D}$ admits an open neighborhood W such that the restriction of p to W is a Möbius map. Thus let $P \in \mathbb{D}$. By the above claim there exists a convex open neighborhood U of $\gamma_P([0,1])$ and a Möbius map $\Psi\colon U \to H_8/\sim$ which restricts to the identity on $U \cap \mathring{H}_8$. Since U is open we can find an $r > 0$ such that for all $Q \in B_r^{\mathbb{D}}(P)$ the image of the radial path γ_Q from 0 to Q also lies in U. (We refer to Figure 583 for an illustration.) This implies that we can use the Möbius map $\Psi\colon U \to M$ to define p for any $Q \in B_r^{\mathbb{D}}(P)$. Put differently, $p = \Psi$ on $B_r^{\mathbb{D}}(P)$. We have thus shown that the restriction of p to the open neighborhood $W = B_r^{\mathbb{D}}(P)$ of P is a Möbius map. ■

The following lemma now concludes the proof of Theorem 33.9

Lemma 33.17. *The map $p\colon \mathbb{D} \to M = H_8/\sim$ is a covering map.*

PROOF. First we show that the map $p\colon \mathbb{D} \to M = H_8/\sim$ is surjective. Recall that by construction p is the identity on $\mathring{H}_8$. Since p is continuous[557] it follows that the restriction of p to $H_8 \to M = H_8/\sim$ is surjective.[558]

It remains to show that every $Q \in M$ admits a uniformly covered neighborhood. Let $Q \in M$. By Proposition 33.12 there exists an $r > 0$ such that $B^M_{2r}(Q)$ is small.[559] We claim that $B^M_r(Q)$ is uniformly covered. More precisely, we make the following claims:

(1) $p^{-1}(B^M_r(Q)) = \bigcup\limits_{X\in p^{-1}(Q)} B^{\mathbb{D}}_r(X)$,
(2) for $X \neq X' \in p^{-1}(Q)$ we have $B^{\mathbb{D}}_r(X) \cap B^{\mathbb{D}}_r(X') = \varnothing$,
(3) given any $X \in p^{-1}(Q)$ the restriction of p to $B^{\mathbb{D}}_r(X) \to B^M_r(Q)$ is a homeomorphism.

We apply Lemma 33.14 to the map $p\colon \mathbb{D} \to M$. It gives us the following three facts:

(a) If $Y \in \mathbb{D}$ satisfies $p(Y) \in B^M_r(Q)$, then there exists an $X \in p^{-1}(Q)$ with $Y \in B^{\mathbb{D}}_r(X)$.
(b) For any $X \in p^{-1}(Q)$ the restriction of p to $B^{\mathbb{D}}_r(X) \to B^M_r(Q)$ is a Möbius isomorphism, in particular it is a homeomorphism.
(c) For any $X \in p^{-1}(Q)$ the restriction of p to $B^{\mathbb{D}}_{2r}(X)$ is injective.

Now the desired properties (1) and (3) follow immediately from (a) and (b). It remains to prove (2). Suppose that we have $X \neq X' \in p^{-1}(Q)$ with $B^{\mathbb{D}}_r(X) \cap B^{\mathbb{D}}_r(X') \neq \varnothing$. We need to show $X = X'$. We pick $Y \in B^{\mathbb{D}}_r(X) \cap B^{\mathbb{D}}_r(X')$. The triangle inequality implies that $X' \in B^{\mathbb{D}}_{2r}(X)$. But p restricted to $B^{\mathbb{D}}_{2r}(X)$ is injective by (c). Hence $X = X'$. ■

33.7. **Picard's Theorem ❀.** We also have the following variation on Theorem 33.9.

Proposition 33.18. *Let $H_4/\sim$ be the three-punctured sphere with the Möbius structure constructed in Proposition 33.5. Then there exists a covering map $p\colon \mathbb{D} \to H_4/\sim$ with the following properties:*

(1) *p is a Möbius map,*
(2) *p is a local isometry,*
(3) *p is a local biholomorphism.*

PROOF. The proof of Proposition 33.18 is almost the same as the proof of Theorem 33.9. In the proof of Theorem 33.9 we used that $H_8/\sim$ is complete. In the present case it follows from Proposition 33.5 (3) that $H_4/\sim$ is complete. The construction of the Möbius map $\Phi\colon \mathbb{D} \to H_4/\sim$ is verbatim the same as the construction of the Möbius map $\Phi\colon \mathbb{D} \to H_8/\sim$. The same argument as before shows that $\Phi\colon \mathbb{D} \to H_4/\sim$ is surjective and that it is a covering map. ■

[557]The map p is a Möbius map, in particular it is a local biholomorphism which implies that it is a local homeomorphism. Thus it follows from Lemma 2.74 that p itself is continuous.

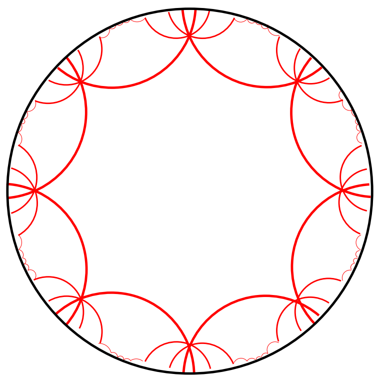
[558]This is the only part of the proof where we actually use that we work with $H_8/\sim$, the remainder of the proof works for any smooth manifold M with a Möbius structure such that the corresponding Riemannian manifold is complete.

[559]Note the factor 2.

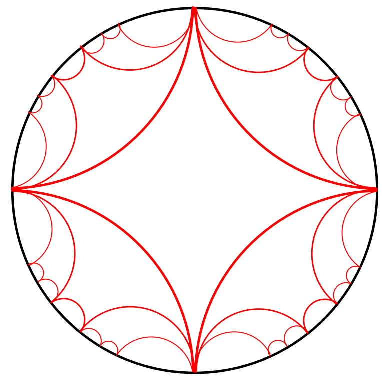
Remark. We consider again the covering $p\colon \mathbb{D} \to H_8/\sim$ that we had just constructed. It follows from Corollary 29.4 that the interior of H_8 is uniformly covered. In particular we see that

$$\mathbb{D} \setminus \underbrace{p^{-1}(\partial H_8)}_{\text{“1-dimensional”}} = \bigsqcup \text{copies of } \mathring{H}_8.$$

Put differently, up to the “one-dimensional” subset $p^{-1}(\partial H_8)$ we can cover $\mathbb{D}$ by infinitely many disjoint copies of the open hyperbolic octagon. Such a decomposition is often called a *tessellation*. This tessellation is shown in Figure 584 on the left. The same argument also applies to the covering $\mathbb{D} \to H_4/\sim$ that we also had just constructed. The corresponding tessellation of $\mathbb{D}$ is shown in Figure 584 on the right.



preimages of $\mathring{H}_8$ under the universal cover $p\colon \mathbb{D} \to H_8/\sim$



preimages of $\mathring{H}_4$ under the universal cover $p\colon \mathbb{D} \to H_4/\sim$

FIGURE 584

At the end of this chapter we return to complex analysis. There are different types of holomorphic functions defined on $\mathbb{C}$.

(1) There are the (rather dull) constant functions.
(2) There are holomorphic functions $f\colon \mathbb{C} \to \mathbb{C}$ that are surjective. For example it is an easy consequence of the fundamental theorem of algebra that all maps defined by non-constant polynomials are surjective.
(3) There are also non-constant maps that are not surjective, for example the exponential function $z \mapsto \exp(z)$ is never zero[560], in fact its image is $\mathbb{C} \setminus \{0\}$.

On the other hand it is fairly easy prove, using Liouville's Theorem 12.1, that the image of a non-constant holomorphic function on $\mathbb{C}$ is dense.

Now we can prove Picard's Theorem[561] which is a significantly stronger statement than the above statement that the image is dense.

Theorem 33.19. (Picard's Theorem) *Let $f\colon \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function. Then there exists at most one $z \in \mathbb{C}$ which does not lie in the image of f.*

[560] Why not?

[561] Emile Picard (1856-1941) was a French mathematician. Many readers will have encountered him already in real analysis, as one of the two namesakes of the Picard-Lindelöf Theorem.

PROOF. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that there exist two different complex numbers a, b that do not lie in the image of f. We need to show that f is constant. We consider the biholomorphism

$$\begin{aligned} \alpha\colon \mathbb{C} \setminus \{a, b\} &\to \mathbb{C} \setminus \{0, 1\} \\ z &\mapsto \frac{z-a}{b-a}. \end{aligned}$$

Furthermore we denote by $\beta\colon \mathbb{C} \setminus \{0, 1\} \to H_4/\sim$ the biholomorphism from Proposition 33.6. Now we consider the following diagram of maps

$$\begin{tikzcd} & & & \mathbb{D} \arrow[d, "\Phi"] \\ \mathbb{C} \arrow[r, "f"] & \mathbb{C} \setminus \{a,b\} \arrow[r, "\alpha", "\cong"'] & \mathbb{C} \setminus \{0,1\} \arrow[r, "\beta", "\cong"'] & H_4/\sim \end{tikzcd}$$

where $\Phi\colon \mathbb{D} \to H_4/\sim$ is the covering map from Proposition 33.18. Recall that Φ is a Möbius map which implies by Lemma 33.8 that Φ is a local biholomorphism.

Since $\mathbb{C}$ is simply connected we can appeal to Proposition 29.2 to obtain a lift of the map $\beta \circ \alpha \circ f\colon \mathbb{C} \to H_4/\sim$ to the universal covering. More precisely, there exists a map $\widetilde{\beta \circ \alpha \circ f}\colon \mathbb{C} \to \mathbb{D}$ such that the following diagram commutes

$$\begin{tikzcd} & & & \mathbb{D} \arrow[d, "\Phi"] \\ \mathbb{C} \arrow[rrru, bend left, "\widetilde{\beta\circ\alpha\circ f}"] \arrow[r, "f"'] & \mathbb{C} \setminus \{a,b\} \arrow[r, "\cong", "\alpha"'] & \mathbb{C} \setminus \{0,1\} \arrow[r, "\cong", "\beta"'] & H_4/\sim . \end{tikzcd}$$

Note that it follows from the fact that $\beta \circ \alpha \circ f$ is holomorphic and the fact that Φ is a local biholomorphism, that the map $\widetilde{\beta \circ \alpha \circ f}\colon \mathbb{C} \to \mathbb{D}$ is also holomorphic. But $\mathbb{D}$ is of course bounded. Therefore it follows from Liouville's Theorem 12.1 that $\widetilde{\beta \circ \alpha \circ f}$ is constant. But then $\beta \circ \alpha \circ f$ is also constant. Since α and β are biholomorphisms this implies that f itself is already constant. ■

Remark. In the literature Theorem 33.19 is sometimes referred to as Picard's Little Theorem, to distinguish it from Picard's Great Theorem which says that if z_0 is an essential singularity of a holomorphic function, then in any neighborhood of z_0 the function f assumes all, but possibly one, value *infinitely* many times. We refer to [**Con78**, Section 12] for details.

33.8. **Triangle groups** $(*)$**.** In this short bonus section we will discuss the triangle groups. But before we can do so we need to introduce the notion of the reflection in a hyperbolic plane.

Definition. We define the reflection in the hyperbolic line $l := \{x + 0\mathrm{i} \,|\, x \in (-1, 1)\} \subset \mathbb{D}$ to be the orientation-reversing isometry

$$\begin{aligned} \rho_l\colon \mathbb{D} &\to \mathbb{D} \\ z = x + \mathrm{i}y &\mapsto \overline{z} = x - \mathrm{i}y. \end{aligned}$$

Now let k be any hyperbolic line in $\mathbb{D}$. By Lemma 32.6 there exists a Möbius transformation Φ with $\Phi(k) = l$. We define the *reflection in k* to be the map

$$\begin{array}{rcl} \rho_k\colon \mathbb{D} & \to & \mathbb{D} \\ x & \mapsto & \Phi^{-1} \circ \rho_l \circ \Phi \end{array}$$

In Exercise 33.1 we will see that the definition of ρ_k does not depend on the choice of Φ.

Next recall that given any two distinct points $P, Q \in \mathbb{D}$ there exists by Lemma 32.6 a unique hyperbolic line $g(P, Q)$ through P and Q. Now we can formulate the key proposition of this section.

Proposition 33.20. *Let $l, m, n \in \mathbb{N}$ with $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.*

(1) *There exists a hyperbolic triangle Δ_{ABC} with interior angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$ at the vertices A, B and C.*

Let G be the subgroup of that is generated by the reflections in the hyperbolic lines $g(C, A)$, $g(A, B)$ and $g(B, C)$. The following statements hold:

(2) (a) *Given any $g \neq h \in G$ the intersection $g \cdot \Delta_{ABC} \cap h \cdot \Delta_{ABC}$ is either empty or a vertex or a side.*
  (b) $\bigcup\limits_{g \in G} g \cdot \Delta_{ABC} = \mathbb{D}$.

(3) *The map*

$$\begin{array}{rcl} \langle x, y, z \,|\, x^2, y^2, z^2, (xy)^l = (yz)^m = (zx)^n \rangle & \to & G \\ x & \mapsto & \textit{reflection in } g(C, A) \\ y & \mapsto & \textit{reflection in } g(A, B) \\ z & \mapsto & \textit{reflection in } g(B, C). \end{array}$$

is an isomorphism.

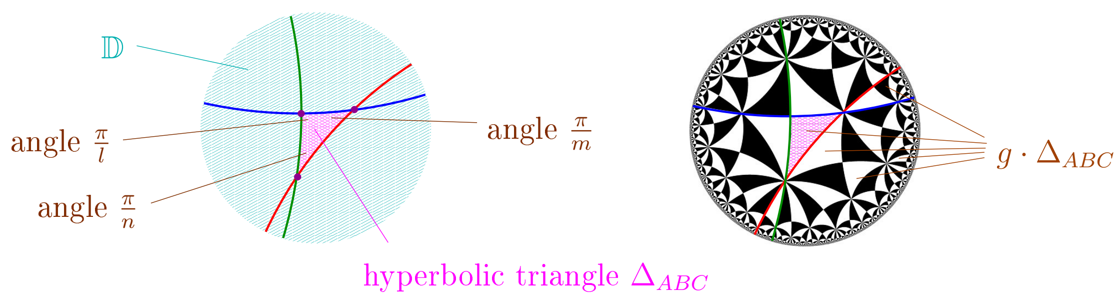


FIGURE 585. Illustration of Proposition 33.20.

PROOF.

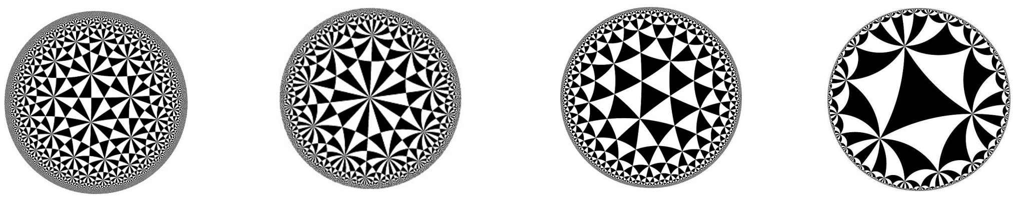
(1) This statement is shown in [**Rat19**, Theorem 3.5.6]. In fact in that reference it is also shown that the triangle is unique up to an isometry.
(2) The statements (2a) and (2b) are both shown in [**Cara54**, p. 177-182] or alternatively in [**Mask71**].
(3) (a) First note that the map is actually well-defined. In fact this follows from the observation that every reflection has order 2 and that xy, yz and zx get sent to a rotation around a vertex with rotation angles $\frac{2\pi}{l}$, $\frac{2\pi}{m}$ and $\frac{2\pi}{n}$.

(b) The fact that the map is in fact an isomorphism is shown in [**Mag74**, Theorem 2.8]. ■

Remark.

(1) A decomposition of the euclidean plane $\mathbb{R}^2$ or the hyperbolic disk $\mathbb{D}$ into isometric geometric shapes that only overlap on the boundary is called a *tessellation*. Proposition 33.20 thus shows that the hyperbolic disk $\mathbb{D}$ can be tessellated by any hyperbolic triangle with angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$ for $l, m, n \in \mathbb{N}$ with $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} < 1$. Tessellations for various values of (l, m, n) are shown in Figure 586.

(2) Note that, as illustrated in Figure 584, Hadamard's Theorem 33.9 and Proposition 33.18 also provide tessellations of $\mathbb{D}^2$.



$(2,3,7)$ $(2,3,9)$ $(3,3,4)$ $(6,6,6)$

FIGURE 586

Proposition 33.20 partly motivates the following definition.

Definition. Let $l, m, n \in \mathbb{N}$.

(1) The group

$$T^*(l,m,n) := \langle x, y, z \mid x^2, y^2, z^2, (xy)^l = (yz)^m = (zx)^n \rangle$$

is called the (l, m, n)-triangle group.

(2) The group

$$T(l,m,n) := \ker \left\{ \begin{array}{rcl} T^*(l,m,n) & \to & \mathbb{Z}_2 \\ x, y, z & \mapsto & 1 \end{array} \right\}$$

is called the (l, m, n)-von Dyck group.

Remark. Let $l, m, n \in \mathbb{N}$.

(1) (a) If $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} < 1$, then we saw in Proposition 33.20 that $T^*(l, m, n)$ is isomorphic to the subgroup of $\mathrm{Isom}(\mathbb{D})$ generated by reflections in the hyperbolic lines defined by a hyperbolic triangle with interior angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$.

(b) If $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} = 1$, then it is shown in [**Mag74**, Section II.4] that $T^*(l, m, n)$ is isomorphic to the subgroup of $\mathrm{Isom}(\mathbb{R}^2)$ generated by reflections in the euclidean lines defined by a *euclidean* triangle with interior angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$.

(c) Finally, if $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} > 1$, then it is shown in [**Mag74**, Section II.4] that $T^*(l, m, n)$ is isomorphic to the subgroup of $\mathrm{Isom}(S^2)$ generated by reflections in the great circles defined by a *spherical* triangle with interior angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$.

(2) In each of the three cases above the (l, m, n)-von Dyck group is the intersection of the (l, m, n)-triangle group with the subgroup of orientation-preserving isometries.

The following lemma gives a particularly simple presentation of the von Dyck groups.

Lemma 33.21. *Given any $l, m, n \in \mathbb{N}$ the map*

$$\begin{array}{rcl} \langle a, b \,|\, a^l, b^m, (ab)^n \rangle & \to & T(l, m, n) \\ a & \mapsto & xy \\ b & \mapsto & yz \end{array}$$

is an isomorphism.

Proof. We will prove Lemma 33.21 in Exercise 33.2. ■

We conclude this chapter with the following corollary to Proposition 33.20.

Corollary 33.22. *Let $l, m, n \in \mathbb{N}$. If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, then the triangle group $T^*(l, m, n)$ and the von Dyck group $T(l, m, n)$ are infinite.*

Proof. Let $l, m, n \in \mathbb{N}$ with $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. We denote by G the group introduced in Proposition 33.20. Note that it follows from Proposition 33.20 (2), the fact that Δ_{ABC} is compact and the fact that $\mathbb{D}$ is non-compact that G is infinite. By Proposition 33.20 (3) this implies that the triangle group $T^*(l, m, n)$ is infinite. The index two subgroup $T(l, m, n)$ is evidently also infinite. ■

Exercises for Chapter 33.

Exercise 33.1. We define reflection in the hyperbolic line $l := \{x \,|\, x \in (-1, 1)\} \subset \mathbb{D}$ to be the map

$$\begin{array}{rcl} \rho_l \colon \mathbb{D} & \to & \mathbb{D} \\ z = x + \mathrm{i}y & \mapsto & \overline{z} = x - \mathrm{i}y. \end{array}$$

Now let k be any hyperbolic line in $\mathbb{D}$. By Lemma 32.6 there exists a Möbius transformation Φ with $\Phi(k) = l$. We define the *reflection in k* to be the map

$$\begin{array}{rcl} \rho_k \colon \mathbb{D} & \to & \mathbb{D} \\ x & \mapsto & \Phi^{-1} \circ \rho_l \circ \Phi \end{array}$$

Show that the definition of ρ_k does not depend on the choice of Φ.

Exercise 33.2. Let $l, m, n \in \mathbb{N}$. Show that the map

$$\begin{array}{rcl} \langle a, b \,|\, a^l, b^m, (ab)^n \rangle & \to & T(l, m, n) \\ a & \mapsto & xy \\ b & \mapsto & yz \end{array}$$

is an isomorphism.
Hint. First use Proposition 27.32 to find an explicit presentation for $T(l, m, n)$.

34. The deck transformation group

34.1. **The deck transformation group and its relation to fundamental groups.** The following definition introduces the main actor of this chapter.

Definition. Let $p\colon X \to B$ be a covering.

(1) A *deck transformation* is a homeomorphism $d\colon X \to X$ such that $p(d(x)) = p(x)$ for all x, i.e. such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "d", "\cong"'] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & B. & \end{tikzcd}$$

The group of deck transformations[562] is called the *deck transformation group* of the covering $p\colon X \to B$ and we denote it by $D(p\colon X \to B)$.

(2) We say that the covering is *regular* if for every $b \in B$ the deck transformation group D acts transitively on $p^{-1}(b)$, i.e. if given any two points x, x' with $p(x) = p(x') = b$ there exists a deck transformation d with $d(x) = x'$. Otherwise we say that the covering is *irregular*.

For completeness' sake we also mention the following definition even though we will barely make use of it:

(3) Let P be a property of isomorphism types of groups, e.g. cyclic, free, nilpotent, solvable etc.We say the covering $p\colon X \to B$ is a P-covering if the covering is regular and if the deck transformation group has the property P.

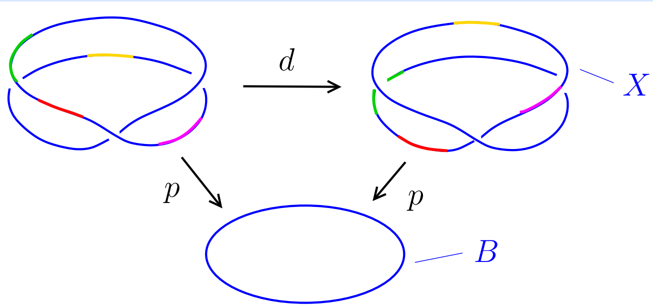


Figure 587. Schematic image of a deck transformation.

Example.

(1) We consider the covering

$$\begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^2. \end{array}$$

Then the map $d\colon S^1 \to S^1$ given by $d(z) := -z$ is a deck transformation. It is now clear that the covering is regular.

[562] Why do deck transformations form a group?

(2) Let $p\colon \widetilde{X} \to X$ be a 2-fold covering. We define

$$\begin{array}{rcl} f\colon \widetilde{X} & \to & \widetilde{X} \\ Q & \mapsto & \text{the unique other point in } p^{-1}(p(Q)). \end{array}$$

In Lemma 16.4 we saw that f is a deck transformation. It follows immediately that any 2-fold covering is regular.

(3) As an important special case of (2) consider a non-orientable path-connected smooth manifold M. As in the proof of Proposition 17.3 we consider the 2-fold covering

$$\begin{array}{rcl} \widetilde{M} := \{(Q,O) \mid Q \in M \text{ and } O \text{ an orientation of } \mathrm{T}_Q\, M\} & \to & M \\ (Q,O) & \mapsto & Q. \end{array}$$

It is easy to show that the map $d\colon \widetilde{M} \to \widetilde{M}$ given by $d((Q,O)) := (Q,-O)$ is a deck transformation of p.

(4) We consider again the first two coverings of topological graphs from Figure 313 which are also shown again in Figure 588. The covering maps are the ones suggested by the colors and the directions. We first make the following observations. A deck transformation d does the following:
  (a) it sends a vertex again to a vertex,
  (b) it sends a blue edge to a blue edge, preserving the orientation,
  (c) it sends a red edge to a red edge, preserving the orientation,
  (d) if e is an edge with endpoints P and Q, then $d(e)$ is an edge with endpoints $d(P)$ and $d(Q)$, in particular if e is a loop, then $d(e)$ is also a loop.

Now we consider the two examples.
  (a) The example on the left admits one non-trivial deck transformation, namely the map that is given by "rotation by π" around the center of the topological graph. The covering is now easily seen to be regular.
  (b) The loop a_3 in the topological graph on the right is the only red loop, so any deck transformation has to send it to itself. It follows that there can be no deck transformation that sends the vertex to the right to any of the other vertices.[563] Thus the covering on the right is irregular.

(5) Finally we consider the 4-fold coverings illustrated in Figure 589. They are both regular coverings. For the topological graph on the left the deck transformation group is isomorphic to $\mathbb{Z}_4$, a generator is given by "rotation by $\frac{\pi}{2}$". The deck transformation group on the right is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, here one $\mathbb{Z}_2$ summand is given by "rotation by π" and the other one corresponds to "swapping the inner and the outer circle".

Lemma 34.1. *Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed path-connected topological spaces and let d_1 and d_2 be two deck transformations of p. Then the following holds*

$$d_1 \;=\; d_2 \quad\Longleftrightarrow\quad d_1(x_0) \;=\; d_2(x_0).$$

PROOF. Evidently we only need to show the "$\Leftarrow$"-direction. Thus let d be a deck transformation of p. One way of stating that d is a deck transformation is to state that the

[563] Does this covering admit any deck transformation?

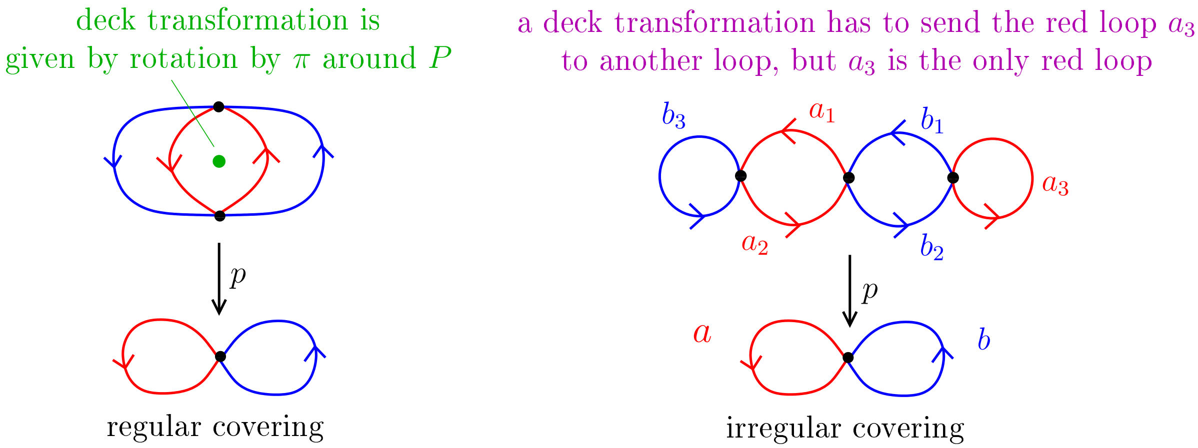


FIGURE 588

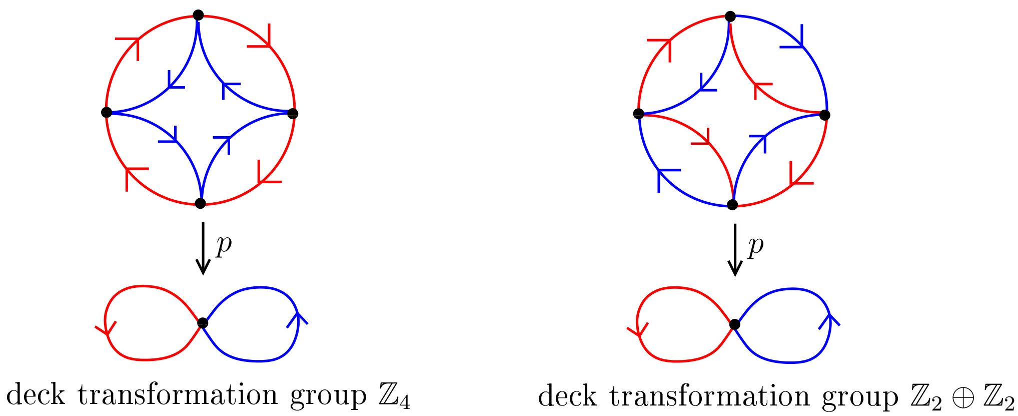


FIGURE 589

following diagram commutes:

$$\begin{tikzcd}
& (X, d(x_0)) \arrow[d, "p"] \\
(X, x_0) \arrow[ur, "d"] \arrow[r, "p"'] & (B, b_0).
\end{tikzcd}$$

Put differently, the map d can be viewed as a lift of the map $(X, x_0) \to (B, b_0)$ to the covering $(X, d(x_0)) \to (B, b_0)$. By the uniqueness of lifts[564], that we showed in Proposition 29.2 (2), we now see that two deck transformations d_1 and d_2 of p with $d_1(x_0) = d_2(x_0)$ agree. ■

Example. We let $B = (S^1 \times \{a\} \sqcup S^1 \times \{b\})/(1, a) \sim (1, b)$, i.e. B is the wedge of two circles and we consider

$$X = (S^1 \sqcup (S^1 \times \mathbb{Z}_5))/(e^{2\pi i n/5} \sim (1, n)) \text{ for } n \in \mathbb{Z}_5,$$

[564]Here we use that X is path-connected.

i.e. X is a circle with five circles attached at the points $e^{2\pi i k/5}$, $k = 0, \ldots, 4$. Then the map

$$p\colon (S^1 \sqcup (S^1 \times \mathbb{Z}_5))/\sim \;\to\; G = (S^1 \times \{a\} \sqcup S^1 \times \{b\})/(1,a) \sim (1,b)$$

that is given by $p(z) = (z^5, a)$ for $z \in S^1$ and $p((z,n)) = (z,b)$ for $(z,n) \in S^1 \times \mathbb{Z}_5$, is a covering map of degree five. This covering illustrated in Figure 590.

We consider the map

$$\begin{array}{rcl} \mathbb{Z}_5 & \to & D(p\colon X \to B) \\ m & \mapsto & \left(\begin{array}{rclc} X & \to & X & \\ z & \mapsto & e^{2\pi i m/5} z, & \text{for } z \in S^1, \\ (z,n) & \mapsto & (z, m+n) & \text{for } (z,n) \in S^1 \times \mathbb{Z}_5 \end{array} \right) \end{array}$$

In Figure 590 this corresponds to "rotation of X by the angle $\frac{2\pi m}{5}$". It is straightforward to show that this map is well-defined, i.e. the maps on the right are deck transformations, and using Lemma 34.1 one can easily see that this map is an isomorphism. Also one can now convince oneself easily that the covering $p\colon X \to B$ is regular.

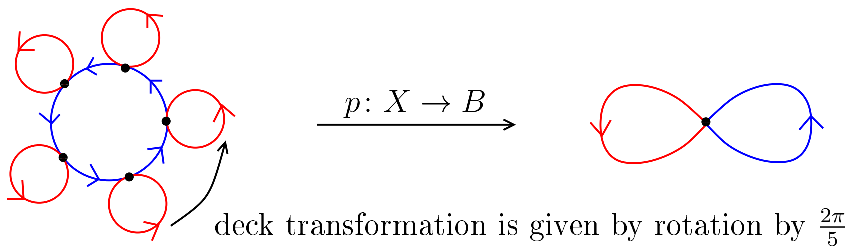


FIGURE 590

Lemma 34.2. *Let X be a path-connected topological space on which a group G acts continuously and discretely. We consider the corresponding covering $p\colon X \to X/G$. Then $p\colon X \to X/G$ is regular and the map*

$$\begin{array}{rcl} \Phi\colon G & \to & D(p\colon X \to X/G) \\ g & \mapsto & \left(\begin{array}{rcl} X & \to & X \\ x & \mapsto & gx \end{array} \right) \end{array}$$

is an isomorphism of groups.

PROOF. It is straightforward to see that the maps $x \mapsto gx$ are indeed deck transformations. It follows almost immediately that the covering $p\colon X \to X/G$ is regular. Also it follows easily from the axioms of a group action that Φ is a homomorphism of groups. Furthermore it is clear that Φ is a monomorphism.

It remains to show that Φ is an epimorphism. So let $d \in D(p\colon X \to X/G)$ be a deck transformation. We pick $x \in X$. Since $p(d(x)) = p(x)$ we have $d(x) = gx$ for some $g \in G$. Since X is path-connected it follows from Lemma 34.1 that $d(y) = gy$ for all $y \in X$. This shows that Φ is an epimorphism. ■

Lemma 34.3. *Any 2-fold covering of path-connected topological spaces is regular and the deck transformation group contains precisely two elements.*

PROOF. Let $p\colon X \to B$ be a 2-fold covering of path-connected topological spaces. By Lemma 16.4 we know that

$$\begin{array}{rcl} d\colon X & \to & X \\ x & \mapsto & \text{the unique other element of } p^{-1}(p(x)) \end{array}$$

is a deck transformation. Evidently it is regular. It follows easily from Lemma 34.1 and our hypothesis that X, B are path-connected that any deck transformation of p is either the identity or it equals the above deck transformation d. ■

We recall the following important, albeit somewhat technical, definition.

Definition. Let G be a group and let $H \subset G$ be a subgroup. The *normalizer* $N(H)$ of the subgroup $H \subset G$ is defined as

$$N(H) \;:=\; \{g \in G \mid g^{-1}Hg = H\}.$$

The following lemma summarizes a few elementary properties of the normalizer of a subgroup.

Lemma 34.4. *Let G be a group and let $H \subset G$ be a subgroup.*

(1) *The subgroup H of G is normal if and only if $N(H) = G$.*
(2) *The normalizer $N(H)$ of $H \subset G$ is a subgroup of G.*
(3) *The normalizer $N(H)$ of $H \subset G$ is the largest subgroup of G which contains H as a normal subgroup, in the sense that if A is a subgroup of G which contains H as a normal subgroup, then $A \subset N(H)$.*

PROOF. Both statements are easily verified. Alternatively see [**Isa94**, p. 26] for the elementary proof. ■

We return to topology and we consider one more example in detail.

Example. Let Y be a topological space that is path-connected, locally path-connected and semi-locally simply connected and let $y_0 \in Y$. Furthermore let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup. By Proposition 29.5 there exists a path-connected covering $p\colon (X, x_0) \to (Y, y_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$. We recall the construction of X. We first considered

$$P \;:=\; \{\text{all paths in } Y \text{ with starting point } y_0\}$$

and for $u, v \in W$ we wrote

$$u \sim v \quad :\Leftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [u * \overline{v}] \in \Gamma.$$

We defined $X = P/\sim$ and the projection map $p\colon P/\sim \to Y$ that is given by $[w] \mapsto w(1)$. We then showed that, with an appropriate topology on $P/\sim$, the map $p\colon P/\sim \to Y$ is a path-connected covering with $p_*(\pi_1(P/\sim, x_0)) = \Gamma$.

As in the proof of Proposition 29.12 we now would like to construct deck transformations by precomposing elements in P by a loop in y_0. More precisely, let g be a loop in (Y, y_0). We consider the map

$$\begin{array}{rcl} P & \to & P \\ w & \mapsto & g * w \end{array}$$

We need to check that the map respects our equivalence relation $\sim$ on P. So suppose that $u, v \in P$ are equivalent. Then $g * u$ and $g * v$ still have the same endpoint. Now we need to consider the second condition. We have

$$[(g * u) * (\overline{g * v})] \;=\; [g * u * \overline{v} * \overline{g}] \;=\; [g] * \underbrace{[u * \overline{v}]}_{\in \Gamma} * [g]^{-1}.$$

In general it is not the case that the right-hand side lies again in Γ. But it is true if $[g]$ lies in the normalizer of Γ. As in the proof of Proposition 29.12 it is now straightforward to see that the map

$$\begin{array}{rcl} \Psi\colon N(\Gamma)/\Gamma & \to & D(p\colon X \to Y) \\ {[g]} & \mapsto & \begin{pmatrix} P/\sim & \to & P/\sim \\ {[u]} & \mapsto & [g * u] \end{pmatrix} \end{array}$$

is a well-defined homomorphism. We will see in Proposition 34.5 that this in fact an isomorphism of groups.

Now we want to consider the general case of a covering. So let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed path-connected topological spaces. We write $\pi := \pi_1(B, b_0)$ and $\Gamma := p_*(\pi_1(X, x_0))$.[565] Now the question arises whether there is a connection between the groups π, Γ and $D(p\colon X \to B)$.

Similar to the proof of Theorem 16.16 we can consider the following map[566]

$$\begin{array}{rcl} \Phi\colon D(p\colon X \to B) & \to & \pi/\Gamma \\ d & \mapsto & \big[p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))\big]. \end{array}$$

It is straightforward to see that this map is well-defined, i.e. it does not depend on the choice of the path.

The question now arises, whether this map Φ is a group isomorphism. But in general this is not the case, after all, if Γ is not normal in π, then π/Γ is not even a group. But we have the following proposition:

Proposition 34.5. *Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. We suppose that X and B are both path-connected and locally path-connected. We write $\pi := \pi_1(B, b_0)$ and $\Gamma := p_*(\pi_1(X, x_0))$. The above map*

$$\begin{array}{rcl} \Phi\colon D(p\colon X \to B) & \to & \pi/\Gamma \\ d & \mapsto & \big[p \circ (\textit{path in } X \textit{ from } x_0 \textit{ to } d(x_0))\big]. \end{array}$$

restricts to an isomorphism of groups

$$\Phi\colon D(p\colon X \to B) \xrightarrow{\cong} N(\Gamma)/\Gamma.$$

Proof. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed path-connected topological spaces. We write $\pi := \pi_1(B, b_0)$, $\Gamma := p_*(\pi_1(X, x_0))$ and $D := D(p\colon X \to B)$.

(A) First we have to show that $\Phi(D)$ lies indeed in $N(\Gamma)/\Gamma$. Thus let $d \in D$. We need to show that $\Phi(d)$ lies in $N(\Gamma)/\Gamma$. Thus let f be a loop in (X, x_0) and and let g be a

[565] Recall that by Corollary 16.14 the induced map $p_*\colon \Gamma \to \pi$ is injective, thus $\Gamma \cong \pi_1(X, x_0)$.

[566] We denote hereby with $[p \circ f]$ not only the equivalence class of the loop $p \circ f$ in π, but also its equivalence class π/Γ, since the notation $[p \circ f]\Gamma$ is too cumbersome.

path in X from x_0 to $d(x_0)$. We have to verify that $[\overline{p \circ f}] \cdot [p \circ g] \cdot [p \circ f] \in \Gamma$. Indeed the following holds

$$[\overline{p \circ f}] \cdot [p \circ g] \cdot [p \circ f] \;=\; [\overline{p \circ f} * p \circ g * p \circ f] \;=\; \big[p \circ \big(\underbrace{\overline{f} * g * f}_{\text{loop in } (X, x_0)}\big)\big] \;\in\; p_*(\pi_1(X, x_0)) = \Gamma.$$

(B) Now we have to show that Φ is a group homomorphism. So let d and e be two deck transformations. We pick a path g from x_0 to $d(x_0)$ and we pick a path h from x_0 to $e(x_0)$. Then

$$\Phi(d) \cdot \Phi(e) \underset{\substack{\uparrow \\ \text{definition of } \Phi(e) \text{ and } \Phi(d)}}{=} [p \circ g] \cdot [p \circ h] \underset{\substack{\uparrow \\ \text{since } p \circ d = p}}{=} [p \circ g] \cdot [p \circ (d \circ h)] \overset{\substack{\text{endpoint of } g \text{ equals starting point of } d \circ h \\ \downarrow}}{=} [g * (d \circ h)] \underset{\substack{\uparrow \\ \text{since } g * (d \circ h) \text{ is a path} \\ \text{from } x_0 \text{ to } d(e(x_0)), \text{ see Figure 591}}}{=} \Phi(e \circ d).$$

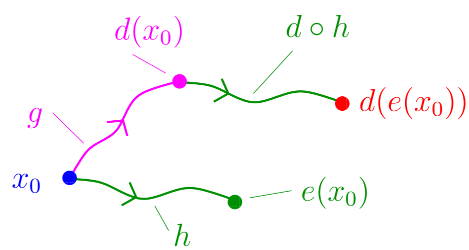


FIGURE 591

(C) Now we will construct a map $\Psi \colon N(\Gamma)/\Gamma \to D$ in the inverse direction. Thus let $[f] \in N(\Gamma) \subset \pi = \pi_1(B, b_0)$ where f is a loop in (B, b_0). Let x_1 be the endpoint of the lift $\widetilde{f}$ of f to the starting point x_0.[567] We have $p(x_1) = p\big(\widetilde{f}(1)\big) = f(1) = x_0$. Thus we obtain the following diagram of maps of pointed topological spaces:

$$\begin{tikzcd} & (X, x_1) \arrow[d, "p"] \\ (X, x_0) \arrow[r, "p"] & (B, b_0). \end{tikzcd}$$

Then

$$\begin{aligned} p_*(\pi_1(X, x_1)) \underset{\substack{\uparrow \\ \text{by Proposition 14.11 since } \widetilde{f} \text{ is a path from } x_0 \text{ to } x_1}}{=} \big\{ \big[p \circ \big(\overline{\widetilde{f}} * g * \widetilde{f}\big)\big] \,\big|\, [g] \in \pi_1(X, x_0) \big\} &= [f]\, p_*(\pi_1(X, x_0))\, [f]^{-1} \\ &\underset{\substack{\uparrow \\ \text{since } [f] \in N(\Gamma)}}{=} p_*(\pi_1(X, x_0)). \end{aligned}$$

[567] According to Corollary 16.13 the definition of x_1 does not depend on the choice of the representative of $[f] \in N(\Gamma)$.

It follows from Proposition 29.2 that there exists a lift $\widetilde{p}\colon (X,x_0)\to(X,x_1)$ of p, i.e. there exists a map $\widetilde{p}$ such that the following diagram commutes:

$$\begin{tikzcd} & (X,x_1) \arrow[d, "p"] \\ (X,x_0) \arrow[ur, "\widetilde{p}"] \arrow[r, "p"] & (B,b_0). \end{tikzcd}$$

The map is $\widetilde{p}$ is now by definition a deck transformation.

We have thus constructed a map $N(\Gamma)\to D$. We still need to show that this map descends to a map $N(\Gamma)/\Gamma\to D$. So let $[f]\in N(\Gamma)$ and let $[g]\in\Gamma$. We denote by $\widetilde{f}$ the lift of f to the starting point x_0. Similarly we define $\widetilde{g*f}$. By the above we only need to show that the endpoints of $\widetilde{f}$ and $\widetilde{g*f}$ agree. In fact we have

$$\widetilde{g*f}(1) \underset{\substack{\uparrow \\ \text{Lemma 29.3}}}{=} \begin{array}{c}\text{endpoint of the lift of } f\\ \text{to the starting point } \widetilde{g}(1)\end{array} \underset{\substack{\uparrow \\ \text{by Lemma 16.15 (2) (1) since } [g]\in p_*(\pi_1(X,x_0))}}{=} \begin{array}{c}\text{endpoint of the lift of } f\\ \text{to the starting point } x_0\end{array} = \widetilde{f}(1)$$

(D) Now we want to show that Φ is a bijection. It suffices to show that $\Psi\circ\Phi=\mathrm{id}$ and $\Phi\circ\Psi=\mathrm{id}$.

First let $d\in D$. Then

$$\begin{aligned}\Psi(\Phi(d)) &= \Psi(\text{loop } f \text{ in } (B,b_0) \text{ that lifts to a path from } x_0 \text{ to } d(x_0))\\ &= \text{deck transformation that sends } x_0 \text{ to } d(x_0) \underset{\substack{\uparrow \\ \text{Lemma 34.1}}}{=} d\end{aligned}$$

This shows that $\Psi\circ\Phi$ is the identity.

Now let $g\in N(\Gamma)/\Gamma$. We pick a representative loop s in b_0 and we denote by $\widetilde{s}$ the lift of s to the starting point x_0. We denote by x_1 the endpoint of $\widetilde{s}$. Then

$$\Phi(\Psi(g)) = \big[p\circ\big(\text{path in } X \text{ from } x_0 \text{ to } \overbrace{\Psi(g)(x_0)}^{=x_1}\big)\big] \underset{\substack{\uparrow \\ \widetilde{s} \text{ is a path from } x_0 \text{ to } x_1}}{=} \big[p\circ\widetilde{s}\big] = [s] = g.$$

This shows that $\Phi\circ\Psi$ is the identity. ■

In most applications we will use the following special case of Proposition 34.5.

Proposition 34.6. *Let (X,x_0) be a pointed topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $\gamma\colon\pi_1(X,x_0)\to G$ be an epimorphism onto a group G. We denote by $p\colon(\widetilde{X},\widetilde{x}_0)\to(X,x_0)$ the covering corresponding to $\ker(\gamma)$ that is given by Proposition 29.5 and that is unique up to covering equivalence by Proposition 29.8. Then the map*

$$\begin{aligned}\Phi\colon D(p\colon\widetilde{X}\to X) &\to \pi_1(X,x_0)/p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) \xrightarrow{\gamma} G.\\ d &\mapsto \big[p\circ(\textit{path from } x_0 \textit{ to } d(x_0))\big]\end{aligned}$$

is an isomorphism.

PROOF. First note that $p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) = \ker(\gamma)$ is a normal subgroup of $\pi_1(X,x_0)$. Therefore the normalizer of $\Gamma := p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ equals $\pi_1(X,x_0)$. It now follows immediately from Proposition 34.5 that the homomorphism on the left is an isomorphism. The homomorphism on the right is an isomorphism since $p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) = \ker(\gamma)$. ■

The following corollary now gives a complete characterization of regular coverings.

Corollary 34.7. *Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces that are path-connected and locally path-connected. Then the following statements are equivalent:*

(1) *The covering is regular.*
(2) *The deck transformation group $D(p\colon X \to B)$ acts transitively on $p^{-1}(b_0) \subset X$.*
(3) *The subgroup $p_*(\pi_1(X,x_0))$ is a normal subgroup of $\pi_1(B,b_0)$.*
(4) *There exists a group G which acts discretely on X and there exists a homeomorphism $f\colon X/G \to B$ such that the following diagram commutes:*

$$\begin{tikzcd} & X \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X/G \arrow[rr, "f"'] & & B \end{tikzcd}$$

where $q\colon X \to X/G$ is the canonical projection map.

If p is a finite covering then the above statements are furthermore equivalent to

(5) $\#D(p\colon X \to B) = [X : B]$.

Remark. Let $p\colon (X,x_0) \to (B,b_0)$ be a 2-fold covering of path-connected pointed topological spaces. In Lemma 34.3 we saw that p is regular, which by Corollary 34.7 means that the index two subgroup $p_*(\pi_1(X,x_0))$ of $\pi_1(B,b_0)$ is normal. In fact we already showed in Exercise 17.1 that any index 2 subgroup Γ of a group π is normal.[568]

PROOF. Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed path-connected topological spaces. We write $\pi := \pi_1(B,b_0)$, $\Gamma := p_*(\pi_1(X,x_0))$ and $D = D(p\colon X \to B)$. The following four steps prove the equivalence of (1), (2) (3) and (4).

(1) $\Leftrightarrow$ (2) Recall that a covering is regular if for every $b \in B$ the group D acts transitively on $p^{-1}(b)$. Thus (1) $\Rightarrow$ (2) is trivial. Now suppose that d acts transitive on $p^{-1}(b_0)$. Let b be some other point in B. Since B is path-connected there exists a path s from b to b_0.

Now let x, x' be two points in $p^{-1}(b)$. We denote by $\widetilde{s}$ the lift of s to the starting point x and similarly we define $\widetilde{s}'$. We denote by x_0 and x_0' the endpoints of $\widetilde{s}$ and $\widetilde{s}'$. Since D acts transitively on $p^{-1}(b_0)$ there exists a deck transformation d with $d(x_0) = x_0'$. By the uniqueness of lifts to a starting point we then also have $d \circ \widetilde{s} = \widetilde{s}'$. But then we also have that d turns the starting point of $\widetilde{s}$ into the starting point of $\widetilde{s}'$, i.e. we have $d(x) = x'$.

[568] We already made use of this fact in the proof of Corollary 17.4.

(2) ⇔ (3) We consider again the map

$$\begin{aligned} \Phi\colon D(p\colon X \to B) &\mapsto \pi/\Gamma \\ d &\mapsto \big[p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))\big]. \end{aligned}$$

We first prove the following claim.

Claim. We have the following equivalence of statements:

$$D \text{ acts transitively on } p^{-1}(b_0) \subset X \quad \Leftrightarrow \quad \Phi\colon D \to \pi/\Gamma \text{ is surjective.}$$

First suppose that D acts transitively on $p^{-1}(b_0) \subset X$. Let $g \in \pi/\Gamma$ and pick a loop f that represents g. We denote by $\widetilde{f}$ the unique lift of f to the starting point x_0 and we denote by $\widetilde{x}_0$ the endpoint of $\widetilde{f}$. Since D acts transitively there exists a deck transformation d with $d(x_0) = \widetilde{x}_0$. Now

$$\Phi(d) = [p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))] \underset{\substack{\uparrow \\ \text{since } \widetilde{f} \text{ is a path from } x_0 \text{ to } d(x_0)}}{=} [p \circ (d \circ \widetilde{f})] \underset{\substack{\uparrow \\ \text{since } p \circ d = p}}{=} [p \circ \widetilde{f}] = [f] = g.$$

Now suppose that $\Phi\colon D \to \pi/\Gamma$ is surjective. Let $\widetilde{x}_0 \in p^{-1}(b_0)$. We pick a path $\widetilde{f}$ from x_0 to $\widetilde{x}_0$ and we denote by f the loop $p \circ \widetilde{f}$. Since Φ is surjective there exists a deck transformation d with $\Phi(d) = [f]$. But then $d(x_0) = \widetilde{x}_0$. ⊞

Now we have the following equivalences

$$\begin{aligned} D \text{ acts transitively on } p^{-1}(b_0) \subset X &\iff \Phi\colon D \to \pi/\Gamma \text{ is surjective} \\ &\underset{\substack{\uparrow \\ \text{Proposition 34.5}}}{\iff} \pi = N(\Gamma) \underset{\substack{\uparrow \\ \text{by the definitions}}}{\iff} \Gamma \text{ is normal in } \pi. \end{aligned}$$

(1) ⇒ (4) We will show that the deck transformation group D has the desired properties.

We first show that the deck transformation group acts discretely on X. So let $x \in X$. We write $b = p(x)$. Since $p\colon X \to B$ is a covering there exists a connected open neighborhood U of b that is uniformly covered. This means that the restriction of p to each component of $p^{-1}(U)$ is a homeomorphism. Now let V be the component of $p^{-1}(U)$ that contains x. Then $gV \cap V = \varnothing$ for each non-trivial g. This shows that the action of D is discrete.

The map $p\colon X \to B$ has the property that $p(gx) = p(x)$ for all $g \in D$. Therefore the map p factors through a map $f\colon X/D \to B$. The map f is clearly surjective. Furthermore it is injective since we suppose that D acts transitively on each $p^{-1}(b)$. Finally the map f is not only continuous but, as follows easily from Lemma 16.3 (1), it is also open. Thus f is a homeomorphism.

(4) ⇒ (1) Now suppose that such a homeomorphism $f\colon X/G \to B$ exists. First note that for each $g \in G$ the map $x \mapsto gx$ not only defines a deck transformation of the covering $q\colon X \to X/G$ but also of the covering $p\colon X \to B$. Now let $b \in B$ and let $x, x' \in X$ with $p(x) = p(x')$. Since f is a homeomorphism we also have $q(x) = q(x')$. Hence there exists a $g \in G$ with $gx = x'$.

Now we suppose that p is a finite covering. We show the following equivalence:

(5) ⇔ (3) We have the following equivalences:[569]

$$\begin{aligned} \#D = [X:B] &\overset{\text{Lemma 16.15 (3)}}{\iff} \#D = [\pi:\Gamma] \overset{\text{Proposition 34.5}}{\iff} [N(\Gamma):\Gamma] = [\pi:\Gamma] \\ &\iff N(\Gamma) = \pi \qquad\quad \iff \Gamma \subset \pi \text{ is a normal subgroup.} \end{aligned}$$ ■

We conclude this chapter with the following proposition which often, rather conveniently, allows us to replace an irregular covering by a regular covering.

Proposition 34.8. *Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces that are path-connected, locally path-connected and semi-locally simply connected. If p is a finite-degree covering, then there exists a covering $q\colon (Y,y_0) \to (X,x_0)$ such that $p \circ q\colon (Y,y_0) \to (B,b_0)$ is a finite-degree* regular *covering. For convenience this statement is summarized in the following diagram:*

$$\begin{tikzcd}
(Y,y_0) \arrow[d, dashed, "\text{(2) there exists a covering } q"'] \arrow[dd, bend left, "\text{(3) such that } p\circ q \text{ is a finite-degree regular covering}"] \\
(X,x_0) \arrow[d, "\text{(1) given a finite-degree covering } p"'] \\
(B,b_0).
\end{tikzcd}$$

Now we can provide the proof of Proposition 34.8.

Proof. Let $p\colon (X,x_0) \to (B,b_0)$ be a finite-degree covering of pointed topological spaces that are path-connected, that are locally path-connected and that are semi-locally simply connected. In the following we write $G := \pi_1(X,x_0)$ and $H := p_*(\pi_1(B,b_0))$. By Lemma 16.15 we know that H is a finite-index subgroup of G. It follows from Lemma 31.13 that the normal core H^c of H in G is a finite-index normal subgroup of G that is contained in H. Since B is locally path-connected and semi-locally simply connected we obtain from Proposition 29.5 a finite-degree path-connected covering $r\colon (Y,y_0) \to (B,b_0)$ of pointed topological spaces such that $r_*(\pi_1(Y,y_0)) = H^c$.

The existence of the desired covering map $q\colon (Y,y_0) \to (X,x_0)$ with $r = p \circ q$ is an immediate consequence of Proposition 29.8 and the fact that H^c is contained in H. ■

34.2. **Fundamental groups of surfaces.** In this section we combine the results from the previous section with the main result from Chapter 33 to draw some interesting consequences on the structure of fundamental groups of surfaces. First we have the following proposition.

Proposition 34.9. *Let $g \in \mathbb{N}_{\geq 2}$. We denote by Σ_g the surface of genus g. We pick a base point $x_0 \in \Sigma_g$. There exists a monomorphism $\pi_1(\Sigma_g,x_0) \to \mathrm{SL}(2,\mathbb{R})$.*

Proof. Throughout the proof we will make use of the notion of a Möbius structure, of a Möbius manifold and of a Möbius map between Möbius manifolds. We refer to pages 887 and 897 for the definitions, in case the reader needs a refresher.

[569]Where do we actually use in the argument that the covering is finite?

Now let $g \geq 2$. As in the proof of Proposition 33.2 we can and will make the identification $\Sigma_g = H_{4g}/\sim$ where H_{4g} is a regular hyperbolic $4g$-gon in $\mathbb{D}$ with interior angle $\frac{\pi}{2g}$. Furthermore, as in the proof of Proposition 33.2 we equip $\Sigma_g = H_{4g}/\sim$ with the canonical Möbius structure. By Hadamard's Theorem 33.9 there exists a covering map $p\colon \mathbb{D} \to \Sigma_g$ such that p is a Möbius map.

Claim. Every deck transformation of the covering map $p\colon \mathbb{D} \to \Sigma_g$ is a Möbius transformation.

Let $f\colon \mathbb{D} \to \mathbb{D}$ be a deck transformation. Recall that this means that the following diagram commutes:

$$\begin{tikzcd} \mathbb{D} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathbb{D} \arrow[dl, "p"] \\ & \Sigma_g. & \end{tikzcd}$$

Since p is a Möbius map and since p is a local diffeomorphism we see that f is locally a Möbius map. By Lemma 33.11 (2) this implies that f is in fact a Möbius map. It now follows from Lemma 33.11 (1) that f is in fact a Möbius transformation. $\boxplus$

Now we can easily conclude the proof of the proposition, indeed we have the following isomorphisms:

$$\pi_1(\Sigma_g, x_0) \underset{\substack{\uparrow \\ \text{by Proposition 34.5}}}{\cong} D(p\colon \mathbb{D} \to \Sigma_g) \underset{\substack{\uparrow \\ \text{by the claim}}}{\subset} \text{group of Möbius transformations of } \mathbb{D} \underset{\substack{\uparrow \\ \text{by Proposition 32.1}}}{\cong} \mathrm{SL}(2, \mathbb{R}).$$

■

The following proposition gives in particular a positive answer to Question 31.22.

Proposition 34.10. *Let Σ be a closed connected 2-dimensional smooth manifold. We pick a base point $x_0 \in \Sigma$. The following two statements hold:*

(1) *The group $\pi_1(\Sigma, x_0)$ is linear over $\mathbb{R}$.*
(2) *The group $\pi_1(\Sigma, x_0)$ is residually finite.*

Remark.

(1) In [**NewM85**, Lemma 1], see also [**Scot78**, Section 3] together with [**Bou81**, Chapitre V, §4, Section 4], it shown that $\pi_1(\Sigma, x_0)$ is in fact linear over $\mathbb{Z}$.
(2) There are many other proofs that $\pi_1(\Sigma, x_0)$ is residually finite, see e.g. [**Chis99**, Corollary 4.9] and [**Hem72**] for alternative arguments.

Proof. Let Σ be a closed connected 2-dimensional smooth manifold. By Proposition 17.3 we know that there exists a finite covering $p\colon (\widetilde{\Sigma}, \widetilde{x}_0) \to (\Sigma, x_0)$ such that $\widetilde{\Sigma}$ is a closed orientable connected 2-dimensional smooth manifold. By Proposition 34.9 we know that $\pi_1(\widetilde{\Sigma}, \widetilde{x}_0)$ is isomorphic to a subgroup of $\mathrm{SL}(2, \mathbb{R})$. In particular we see that $\pi_1(\widetilde{\Sigma}, \widetilde{x}_0)$ is linear over $\mathbb{R}$.

By Corollary 16.14 and Lemma 16.15 we know that $\pi_1(\widetilde{\Sigma}, \widetilde{x}_0)$ is isomorphic to a finite-index subgroup of $\pi_1(\Sigma, x_0)$. Thus it follows from the above together with Exercise 31.15 that $\pi_1(\Sigma, x_0)$ is also linear over $\mathbb{R}$. By Proposition 31.19 this implies that $\pi_1(\Sigma, x_0)$ is residually finite. ■

Exercises for Chapter 34.

Exercise 34.1. In Figure 592 below we show two coverings $p\colon Y \to X = S^1 \vee S^1$ and $q\colon Z \to X = S^1 \vee S^1$.

(a) Determine which coverings are regular.
(b) For the regular covering(s) in (a), determine the isomorphism type of the deck transformation group.

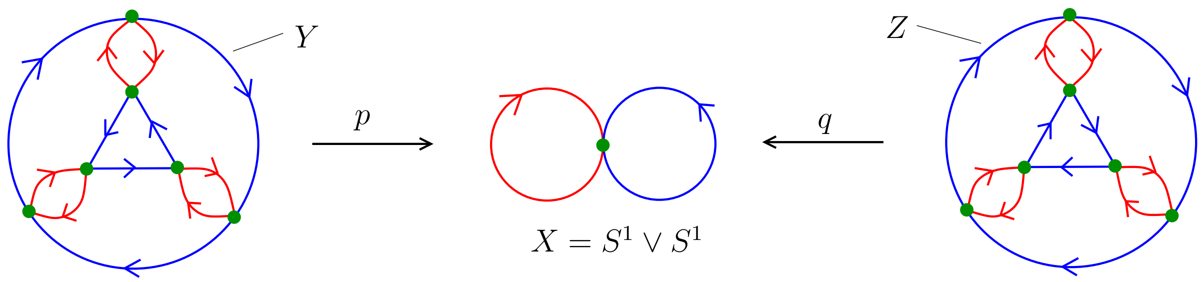


FIGURE 592. Illustration for Exercise 34.1.

Exercise 34.2. Let Σ be the surface of genus 2. Does there exists a regular covering $p\colon \widetilde{\Sigma} \to \Sigma$ such that the deck transformation group is isomorphic to the permutation group S_3?

Exercise 34.3. Let G be a finite group. Show that there exists a closed orientable 2-dimensional smooth manifold that admits a free and transitive action by G.

Exercise 34.4. Let $p\colon \widetilde{X} \to X$ be a covering with deck transformation group G and let $K \subset \widetilde{X}$ be a compact subset. We suppose that X is Hausdorff and locally compact. Show that the set $\{g \in G \,\|\, g \cdot K \cap K \neq \varnothing\}$ is finite.

Remark. In [**Scot78**, Lemma 1.2] claims that the same conclusion holds without the hypothesis that X is locally compact. Try to prove this more general statement.

35. Related constructions in algebraic geometry and Galois theory (∗)

In this short chapter we indicate some connections between the topics that we have covered so far to algebraic geometry and Galois theory. Since the author of these notes does not know much about either field the following discussion should be taken with a grain of salt. In particular for an expert the subsequent discussion will certainly look too simplistic.

35.1. **The fundamental group of an algebraic variety** (∗).

Definition. Let $\mathbb{K}$ be a field.[570]

(1) Given a set of multivariable polynomials S, i.e. given a subset $S \subset \mathbb{K}[t_1, \dots, t_n]$ we denote by
$$V(S) \;=\; \{(x_1, \dots, x_n) \in \mathbb{K}^n \mid f(x_1, \dots, x_n) = 0 \text{ for all } f \in S\}$$
the *vanishing set of* S, i.e. the set of all points in $\mathbb{K}^n$ where all the polynomials in S, viewed as maps $\mathbb{K}^n \to \mathbb{K}$, vanish.

(2) We say $U \subset \mathbb{K}^n$ is *Zariski open*,[571] if there exists a subset S of $\mathbb{K}[t_1, \dots, t_n]$ such that $U = \mathbb{K}^n \setminus V(S)$.

Remark. Let S be a subset of $\mathbb{K}[t_1, \dots, t_n]$. Since the ring $\mathbb{K}[t_1, \dots, t_n]$ is Noetherian there exists a finite set $\{f_1, \dots, f_k\} \subset \mathbb{K}[t_1, \dots, t_n]$ such that $V(S) = V(f_1, \dots, f_k)$.

Lemma 35.1. *Let $\mathbb{K}$ be a field. The Zariski open subsets form a topology on $\mathbb{K}^n$.*

Remark.

(1) The topology defined by the Zariski open subsets is of course called the *Zariski topology on* $\mathbb{K}^n$.

(2) If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, then every Zariski open subset is also open with respect to the usual topology on $\mathbb{R}^n$ or $\mathbb{C}^n = \mathbb{R}^{2n}$. But for $n \geq 1$ the converse does not hold, in fact "most" subsets that are open with respect to the usual topology are not open with respect to the Zariski topology.

Proof. Let $\mathbb{K}$ be a field.

(1) The vanishing set of the empty set, i.e. $V(\varnothing)$ is all of $\mathbb{K}^n$, hence the empty set $\varnothing = \mathbb{K}^n \setminus V(\varnothing)$ is open.

(2) The vanishing set of the constant polynomial $f = 1$ is the empty set, therefore $\mathbb{K}^n = \mathbb{K}^n \setminus V(1)$ is open.

(3) We need to show that the intersection of finitely many Zariski open sets is again Zariski open. By induction it suffices to consider the case of two Zariski open sets. So suppose we are given Zariski open sets $X = \mathbb{K}^n \setminus V(S)$ and $Y = \mathbb{K}^n \setminus V(T)$. Then
$$\mathbb{K}^n \setminus (X \cap Y) \;=\; (\mathbb{K}^n \setminus X) \cup (\mathbb{K}^n \setminus Y) \;=\; V(S) \cup V(T) \;=\; V\big(\{s \cdot t\}_{s \in S, t \in T}\big).$$

---

[570] For example we could take $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ but $\mathbb{K}$ could also be a field of non-zero characteristic, e.g. for a prime p we could take $\mathbb{K} = \mathbb{F}_p$ or $\mathbb{K}$ could be the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$.

[571] Oscar Zariski (1899-1986) was a Russian-American mathematician working in algebraic geometry.

(4) We need to show that the union of Zariski open sets is again Zariski open. So suppose we are given a family of Zariski open sets $X_i = \mathbb{K}^n \setminus V(S_i)$, $i \in I$. Then

$$\mathbb{K}^n \setminus \bigcup_{i\in I} X_i \;=\; \bigcap_{i\in I}(\mathbb{K}^n \setminus X_i) \;=\; \bigcap_{i\in I} V(S_i) \;=\; V\Big(\bigcup_{i\in I} S_i\Big).$$

Thus we have verified that the Zariski open sets have all the properties of a topology. ■

Definition. Let $\mathbb{K}$ be a field.

(1) A *variety over* $\mathbb{K}$ is defined as the vanishing set of a set of multivariable polynomials.
(2) The *Zariski topology on a variety* $V \subset \mathbb{K}^n$ is defined as the subspace topology on V induced from the Zariski topology on $\mathbb{K}^n$.

Example. Let $\mathbb{K}$ be a field and let and $f(s,t) = s^2 + t^2 - 1$. We define

$$S^1_{\mathbb{K}} \;:=\; V(f) \;=\; \{(x,y) \in \mathbb{K}^2 \mid f(x,y) = x^2 + y^2 - 1 = 0\} \;=\; \{(x,y) \in \mathbb{K}^2 \mid x^2 + y^2 = 1\}.$$

For $\mathbb{K} = \mathbb{R}$ we obtain the usual circle S^1.

Any variety V is again a topological space, so in principle one could study the usual fundamental group. But in many cases, e.g. if the characteristic of $\mathbb{K}$ is non-zero, one will see that all continuous maps $S^1 \to V$ are constant, hence the usual fundamental group is trivial.

The correct analogue of a finite covering $p\colon X \to B$ between topological spaces for varieties is a finite étale morphism. The precise definition of a finite étale morphism is irrelevant for our purpose.[572] A heuristic fact is that varieties tend to have many finite étale morphisms, but there is no well-behaved analogue of an infinite covering in the context of varieties. In particular usually there is no concept of a "universal" étale morphism. The question is now, whether one can recover the notion of a fundamental group from the knowledge of finite covers.

Given a variety V we can consider all finite regular étale morphisms $p_i\colon W_i \to V$, $i \in I$. The corresponding deck transformations groups $D(p_i\colon W_i \to V)$, $i \in I$ are finite groups that form an inverse system. Now we can define the étale fundamental group of V as

$$\pi_1^{\text{ét}}(V) \;:=\; \varprojlim_i D(p_i\colon W_i \to V).$$

By Proposition 25.18 the étale fundamental group $\pi_1^{\text{ét}}(V)$ is naturally a compact topological group. In algebraic geometry the étale fundamental group of a variety plays a role similar to the role of the fundamental group of a topological space in topology.

If V is a complex variety without singularities, then the étale fundamental group is isomorphic to the profinite completion[573] of the usual topological fundamental group $\pi_1(V)$, i.e. we have

$$\pi_1^{\text{ét}}(V) \;\cong\; \widehat{\pi_1(V)}.$$

We refer to [**Groth71**] for a proof of this statement.

[572] Furthermore it is unknown to the author of these lecture notes.

[573] Recall that on page 755 we introduced the profinite-completion $\widehat{\pi}$ of a group π.

More information on this topic can be found at

`https://en.wikipedia.org/wiki/etale_fundamental_group`

35.2. **Galois theory** $(*)$**.** The theory of coverings of topological spaces has some similarities to the Galois theory of field extensions.

Throughout this section let K be a field of characteristic zero[574] and let L/K be a field extension of K, i.e. L is a field which contains K as a subfield. The dimension of L as a K vector space is denoted by $[L:K]$. We consider the group

$$\mathrm{Gal}(L/K) = \text{ all field automorphisms of } L \text{ which are the identity on } K.$$

Put differently, a field automorphism $f\colon L \to L$ lies in $\mathrm{Gal}(L/K)$ if and only if the following diagram commutes:

$$\begin{tikzcd} & K \arrow[dl] \arrow[dr] & \\ L \arrow[rr, "f"] & & L. \end{tikzcd}$$

We consider two examples:

(1) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is a field extension with $[\mathbb{Q}(\sqrt{2})\colon \mathbb{Q}] = 2$ and $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ consists of the identity and the homomorphism $a + b\sqrt{2} \mapsto a - b\sqrt{2}$.
(2) $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is a field extension with $[\mathbb{Q}(\sqrt[3]{2})\colon \mathbb{Q}] = 3$ and the group $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$ consists only of the identity.

Given a field K we denote by $\overline{K}$ the algebraic closure of K. A field extension L/K is called *Galois*, if for each $f \in \mathrm{Gal}(\overline{K}/K)$ we have $f(L) = L$.[575] In the usual course on Galois theory one shows that L/K is Galois if and only if each polynomial in $K[x]$ which admits a zero in L splits over L, i.e. all other zeros of f lies also in L. For a Galois extension L/K we refer to $\mathrm{Gal}(L/K)$ as the *Galois group* of the field extension.

We return to the above two examples:

(1) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is a Galois field extension.
(2) $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not a Galois field extension, since $\sqrt[3]{2}$ is a zero of x^3-2, but the complex zero $\sqrt[3]{2}e^{2\pi \mathrm{i}/3}$ of f does not lie in $\mathbb{Q}(\sqrt[3]{2})$.[576]

If L/K is a finite extension, then the main theorem of Galois theory says that the following two statements are equivalent:

(1) The field extension L/K is Galois.
(2) $\#\,\mathrm{Gal}(L/K) = [L:K]$.

The statement is formally very similar to the statement of Corollary 34.7 which says that for some finite covering $p\colon X \to B$ of topological spaces the following statements are equivalent:

(1) The covering $p\colon X \to B$ is regular.

[574] This restriction to characteristic zero is not necessary, but it simplifies the discussion.

[575] Strictly speaking this is the definition of a normal extension, but since K is a field of characteristic zero, a normal extension is already Galois.

[576] We put $z = \sqrt[3]{2}$ and $w = e^{2\pi \mathrm{i}/3}$. It is a good exercise to show that $\mathbb{Q}(z,w)/\mathbb{Q}$ is a normal field extension with $[\mathbb{Q}(z,w)\colon \mathbb{Q}] = 6$. The group $\mathrm{Gal}(\mathbb{Q}(z,w)/\mathbb{Q})$ is isomorphic to the permutation group S_3.

(2) $\#D(p\colon X \to B) = [X : B]$.

The analogy between Galois theory and covering theory goes considerably further. Many statements about the Galois group of a Galois extension have a corresponding statement for the deck transformation group of a normal covering. For example, given a finite extension L/K the (normal) subgroups of Galois group $\mathrm{Gal}(L/K)$ correspond to (normal) intermediate fields and (normal) subgroups of the deck transformation group correspond to (regular) intermediate coverings.

Now let L/K be a Galois extension of infinite degree. All finite intermediate Galois extensions of K, i.e. all finite Galois extensions F/K with $F \subset L$, form a preordered set via the inclusion. The corresponding Galois groups form an inverse set. Indeed, given two finite Galois extensions F/K and G/K with $F \subseteq G$ the restriction of an automorphism of G to an automorphism of F defines a group homomorphisms $\mathrm{Gal}(G/K) \to \mathrm{Gal}(F/K)$. Therefore we can form the inverse limit of these Galois groups. By [**FJ08**, Section 1.3] there exists a canonical isomorphism of groups

$$\mathrm{Gal}(L/K) \;=\; \varprojlim_{\substack{F/K \text{ finite} \\ \text{Galois extension}}} \mathrm{Gal}(F/K).$$

On the right-hand side we consider the inverse limit of an inverse system of finite groups. It follows from Proposition 25.18 that the right-hand side, and thus also the Galois group $\mathrm{Gal}(L/K)$, is naturally a compact topological group. This topology on $\mathrm{Gal}(L/K)$ is called the *Krull topology*.

Example. Let p be a prime. We denote by $\mathbb{F}_p$ the finite field with p elements and we denote by $\overline{\mathbb{F}}_p$ its algebraic closure. It is shown in [**FJ08**, Section 1.5] that there exists a canonical isomorphism

$$\mathrm{Gal}\left(\overline{\mathbb{F}}_p/\mathbb{F}_p\right) \;\cong\; \widehat{\mathbb{Z}}$$

where $\widehat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$, or equivalently the profinite integers that we introduced on page 751.

Part V

CW-complexes and Cofibrations

36. CW-complexes I: Definitions, basic properties and constructions

In this chapter we will introduce CW-complexes. Initially the definition of a CW-complex might look rather confusing and artificial. But in this and in Chapters 37 and 38 we will see that the notion of a CW-complex strikes the right balance:

(1) CW-complexes are topological space that have many desirable properties,
(2) the notion is flexible enough so that most of the topological spaces that we are interested in are actually CW-complexes.

36.1. Definition of finite-dimensional CW-complexes and examples.

We start out with the definition of a finite-dimensional CW-complex and its topological realization.

Definition.

(a) A CW-*complex* X *of dimension* ≤ -1 is defined to be the empty set. Its *topological realization* $|X|$ is also defined to be the empty set.
(b) We define inductively higher dimensional CW-complexes. Let $n \in \mathbb{N}_0$. A CW-*complex* X^n *of dimension* $\leq n$ is a pair $(X^{n-1}, \{\{\varphi_i\colon S^{n-1} \to |X^{n-1}|\}_{i\in I_n}\})$ where the following holds:
   (i) X^{n-1} is a CW-complex X^{n-1} of dimension $\leq n-1$, with topological realization denoted by $|X^{n-1}|$,
   (ii) $\{\varphi_i\colon S^{n-1} \to |X^{n-1}|\}_{i\in I_n}$ is a set of maps.

   The *topological realization* $|X^n|$ *of* X^n is defined as the topological space
$$|X^n| \;:=\; \Big(|X^{n-1}| \;\sqcup\; \bigsqcup_{i\in I_n} \overline{B}^n\Big)/\sim$$
   where for each $i \in I_n$ we use the map φ_i to identify the points on the corresponding copy of $S^{n-1} = \partial\overline{B}^n$ with the corresponding points in $|X^{n-1}|$.
(c) A CW-*structure* for a topological space Y is a CW-complex X together with a homeomorphism $f\colon |X| \to Y$.

Remark.

(1) Let X be a CW-complex as above and let $n \in \mathbb{N}_0$. It follows from Lemma 3.44 (4) that the obvious map $|X^{n-1}| \to |X^n|$ is an embedding. Thus we can and will view $|X^{n-1}|$ as a subspace of $|X^n|$.
(2) Note that $\overline{B}^0$ consists of a point and that $S^{-1} = \varnothing$. It follows from the above definition that a CW-complex of dimension ≤ 0 is exactly the same as set of points with the discrete topology.
(3) CW-complexes were introduced by John H. C. Whitehead[577]. The name comes from "C" for "closure-finite", and the "W" for "weak topology". We will explain on pages 941 and 954 what "closure-finite" and "weak topology" mean. Some people also say it goes back to the initials C. W. of Whitehead. Some authors, see e.g. [**Lee00**, p. 132] and [**Mun84**, p. 214] require "closure-finiteness" as part of the definition, but as we will see, this property follows from the definition as given.

---

[577] John Henry Constantine Whitehead (1904-1960) was a British mathematician.

(4) In the notation we usually do not distinguish between the CW-complex and its topological realization, i.e. we usually denote the topological realization of a CW-complex X again by X.

Before we discuss a long list of examples of CW-complexes we want to introduce a few more definitions.

Definition. Let X be a CW-complex as above and let $k \in \mathbb{N}_0$. We continue with the notation used in the previous definition

(1) The images of the $\overline{B}_i^k$, $i \in I_k$, in X are called the *k-dimensional cells*, or often short *k-cells*, of X.
(2) The images of the B_i^k, $i \in I_k$, in X are called the *open k-cells* of X.
(3) The maps $\varphi_i \colon S_i^{k-1} \to X$[578] are called the *attaching maps* of the k-cells and the obvious maps $\overline{B}_i^k \to X$ are referred to as the *characteristic maps* of the k-cells.
(4) We refer to the CW-complex X^k as the *k-skeleton* of X.
(5) If X is empty, then we define its dimension to be -1. If X is non-empty, then we define the *dimension* $\dim(X) \in \mathbb{N}_0 \cup \{\infty\}$ as the maximal dimension of a cell.

Remark. At times, when we say that "a CW-complex X is k-dimensional", we allow the possibility, that the highest dimension of a cell is actually less than k. This slightly fuzzy language should not pose a problem in practice.

Example. We consider the topological space which is sketched in Figure 593, namely

$$Y \;:=\; ([-1,1] \times [-1,1]) \cup \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + (y-1)^2 = 1 \text{ and } y \geq 1\}.$$

The topological space X thus consists of a square together together with a semicircle.

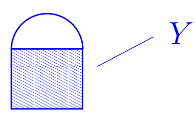


Figure 593

Our goal is to find a CW-structure for X. First we consider the 0-dimensional CW-complex which consists of the 4 vertices of the square. Next we add a 1-cell. To do so we consider the map φ which sends $\partial \overline{B}^1 = \{\pm 1\}$ to the two points $\pm 1 \times -1$. The topological space $X^0 \cup_\varphi \overline{B}^1$ consists of the four points $\{\pm 1, \pm 1\}$ together with an edge which connects the lower two points. This topological space is sketched in the center of Figure 594. The same way we attach four more 1-cells and we obtain the 1-dimensional CW-complex which is sketched on the right of Figure 594. Finally we attach a 2-cell to X^1 with the attaching map shown in Figure 595 and we obtain a CW-complex X^2 whose topological realization is homeomorphic to Y.

Now we will see that most of the topological spaces that we have encountered so far can be described as CW-complexes.

[578]Here we already use the convention that in our notation we do not distinguish between a CW-complex and its topological realization.

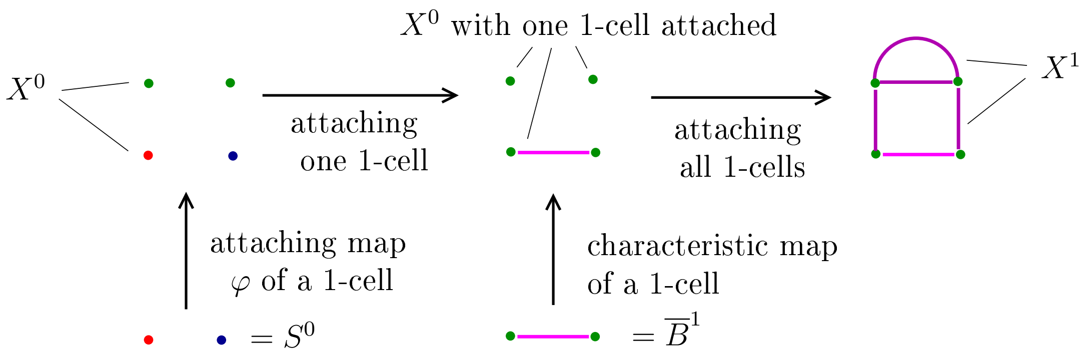


FIGURE 594. The CW-complex X^1.

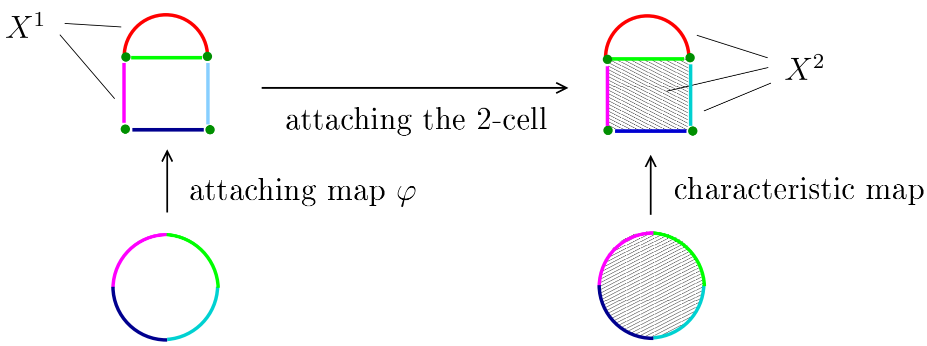


FIGURE 595. Attaching the 2-cell to the 1-skeleton X^1.

(1) It follows almost immediately from the definitions that every topological graph, as defined on page 223, admits naturally the structure of a one-dimensional CW-complex where the vertices correspond precisely to the 0-cells and the edges correspond precisely to the 1-cells. Conversely, every 1-dimensional CW-complex admits naturally the structure of a topological graph. Therefore in the following we will freely go back and forth between these objects.

(2) As a special case of (1) we see that S^1 can be viewed as a CW-complex with one 0-cell and one 1-cell. Furthermore we can view the wedge $S^1 \vee S^1$ as a CW-complex with one 0-cell and two 1-cells. Finally $\mathbb{R}$ can be viewed as a CW-complex where the 0-cells are given by the integers $\mathbb{Z}$ and the 1-cells are given by the intervals $[k, k+1]$ with $k \in \mathbb{Z}$. We refer to Figure 596 for an illustration.

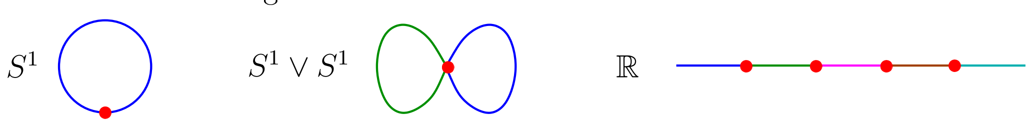


FIGURE 596

(3) Now we consider the torus $T = ([0,1] \times [0,1])/\sim$. We denote by $p\colon [0,1] \times [0,1] \to T$ the obvious projection map. We can view T as a CW-complex with one 0-cell corresponding to the point $p(0,0) = p(0,1) = p(1,0) = p(1,1)$, two 1-cells corresponding to

$p(0\times[0,1]) = p(1\times[0,1])$ and to $p([0,1]\times 0) = p([0,1]\times 1)$ and one 2-cell corresponding to $p([0,1]\times[0,1])$. Note that since we have only one 0-cell there is a unique choice of attaching map for the 1-cells. Finally the attaching map for the 2-cell is illustrated in Figure 597.[579] In Figure 597 we sketch the CW-structure of the torus and also the corresponding CW-structure of the torus viewed as a subset of $\mathbb{R}^3$. We refer to it as the *standard* CW-*structure* of the torus.

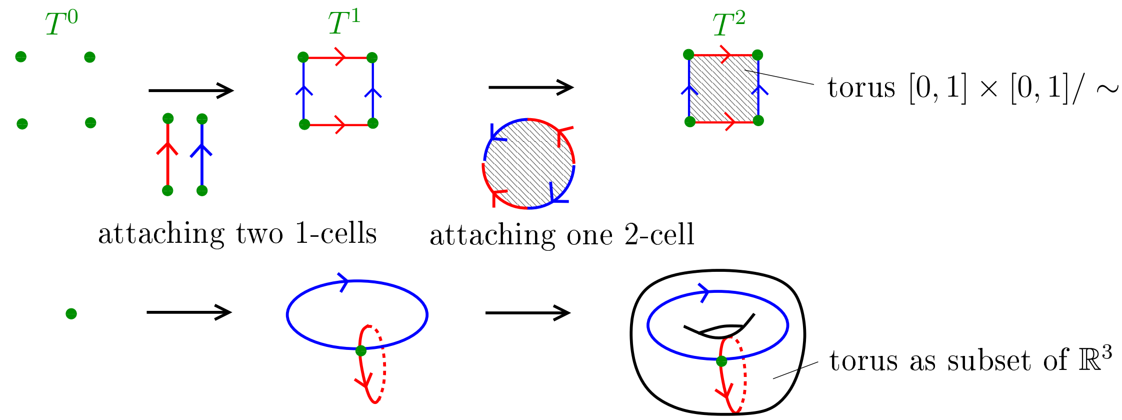


FIGURE 597. The torus as CW-complex.

(4) Next we consider the Klein bottle $K = ([0,1]\times[0,1])/\sim$. As usual we denote by $p\colon [0,1]\times[0,1] \to K$ the projection map. Similar to the torus we can view K as CW-complex with one 0-cell corresponding to $p(0,0) = p(0,1) = p(1,0) = p(1,1)$, two 1-cells corresponding to $p(0\times[0,1]) = p(1\times[0,1])$ and to $p([0,1]\times 0) = p([0,1]\times 1)$ and one 2-cell corresponding to $p([0,1]\times[0,1])$ with the attaching map sketched in Figure 598.[580]
The 1-skeleta of the torus and the Klein bottle are both wedges of two circles. Furthermore both CW-complexes have precisely one 2-cell. The difference between the torus and the Klein bottle is that the attaching maps for the 2-cells are different.
(5) In Figure 599 we show that we can endow the torus with different CW-structures.
(6) In Figure 600 we sketch how we can view the surface of genus 2, the projective plane $\mathbb{RP}^2$, the non-orientable surface of genus 3 and the Möbius band as CW-complexes.[581]

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[579]Strictly speaking the attaching map is given by

$$\begin{array}{rcl} S^1 & \mapsto & T^1 \\ e^{\,it} & \mapsto & \begin{cases} p(\frac{2}{\pi}t, 0), & \text{if } t \in [0, \frac{\pi}{2}] \\ p(1, \frac{2}{\pi}(t-\frac{\pi}{2})), & \text{if } t \in [\frac{\pi}{2}, \pi] \\ p(1-\frac{2}{\pi}(t-\pi), 1), & \text{if } t \in [\pi, \frac{3\pi}{2}] \\ p(0, 1-\frac{2}{\pi}(t-\frac{3}{2}\pi)), & \text{if } t \in [\frac{3\pi}{2}, 2\pi]. \end{cases} \end{array}$$

But writing out the attaching map with explicit formulas is clearly not very instructive.

[580]Evidently, as in Footnote 579, one could spell out the attaching map with precise formulas, but there is no point in doing it.

[581]What does the 1-skeleton of the given CW-structure for the surface of genus two look like, if we view the surface Σ as a "physical object" in $\mathbb{R}^3$?

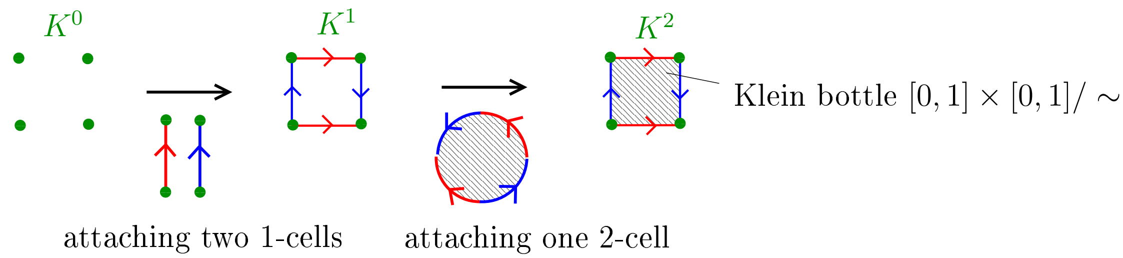

FIGURE 598. The Klein bottle as CW-complex.

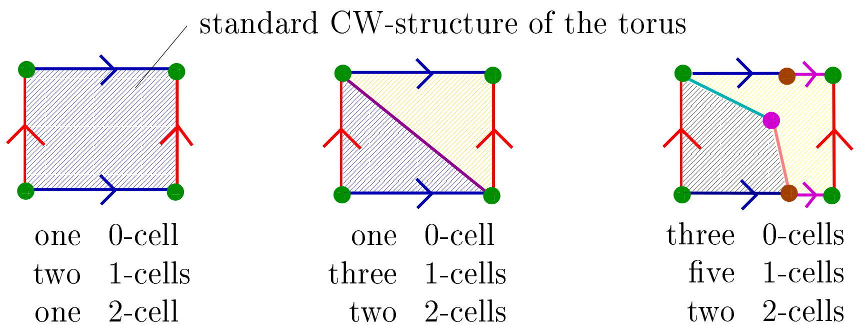

FIGURE 599. The torus with three different CW-structures.

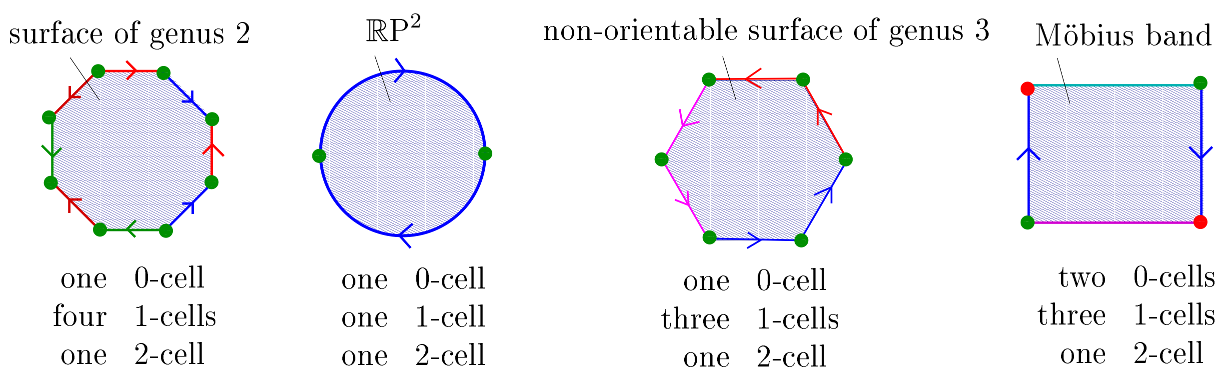

FIGURE 600

(7) We conclude this discussion of surfaces with the 2-sphere. Since it makes no difference, we will immediately discuss the case of any sphere. We can describe the n-dimensional sphere S^n as a CW-complex X as follows: We take a single 0-cell given by a point P and one n-cell where the attaching map $\varphi\colon S^{n-1} \to X^0 = \{P\}$ sends the whole boundary S^{n-1} to the point P. The topological realization of X is thus given by

$$X^0 \cup_\varphi \overline{B}^n \;=\; \{P\} \cup_{P=\partial \overline{B}^n} \overline{B}^n \;=\; \overline{B}^n/S^{n-1} \;\underset{\substack{\uparrow \\ \text{identification from page 182}}}{\xrightarrow{\cong}}\; S^n.$$

We refer to this CW-structure on S^n as the *canonical* CW-*structure on* S^n. Note that the identification on page 182 identifies unique 0-cell to the left with the North

Pole $(0,\dots,0,1)$. Unless we say something else we will equip S^n with the above CW-structure.

We continue the discussion of CW-complexes with two more explicit examples, namely the real projective spaces and the complex projective spaces which we had already defined on page 194. In the following let $\mathbb{A}=\mathbb{R}$ or $\mathbb{A}=\mathbb{C}$. We denote by $\sim$ the equivalence relation on $\mathbb{A}^{n+1}\setminus\{0\}$ given by multiplication by an element in $\mathbb{A}\setminus\{0\}$ and we define

$$\mathbb{A}\mathrm{P}^n \;:=\; (\mathbb{A}^{n+1}\setminus\{0\})/\sim\,.$$

On page 287 and Lemma 12.5 we saw that $\mathbb{A}\mathrm{P}^n$ is a smooth manifold. Given a point $(v_0,\dots,v_n)\in\mathbb{A}^{n+1}\setminus\{0\}$ we denote by $[v_0:\ldots:v_n]$ its equivalence class. For $k\leq n$ the map

$$\begin{array}{rcl}\mathbb{A}\mathrm{P}^k & \to & \mathbb{A}\mathrm{P}^n\\ {[v_0:\ldots:v_k]} & \mapsto & [v_0:\ldots:v_k:0:\ldots:0]\end{array}$$

is easily seen to be a smooth embedding and we use it to view $\mathbb{A}\mathrm{P}^k$ as a submanifold of $\mathbb{A}\mathrm{P}^n$.

We can give an explicit description of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ as CW-complexes.

Lemma 36.1. *Let $n\in\mathbb{N}_0$.*

(1) *We can view $\mathbb{R}\mathrm{P}^n$ as a* CW*-complex with the following properties:*
   (a) *the* CW*-structure has exactly one cell in the dimensions $0,1,\dots,n$ and no other cells,*
   (b) *for each $k\leq n$ the k-skeleton of $\mathbb{R}\mathrm{P}^n$ is given by $\mathbb{R}\mathrm{P}^k$.*

(2) *We can view $\mathbb{C}\mathrm{P}^n$ as a* CW*-complex with the following properties:*
   (a) *the* CW*-structure has exactly one cell in the dimensions $0,2,\dots,2n$ and no other cells,*
   (b) *for each $k\leq n$ the $2k$-skeleton of $\mathbb{C}\mathrm{P}^n$ is given by $\mathbb{C}\mathrm{P}^k$.*

PROOF.

(1) We prove the statement by induction on n. For $n=0$ the topological space $\mathbb{R}\mathrm{P}^0$ consists of a single point which means that it is a CW-complex with a single 0-cell. Now suppose that we have equipped $\mathbb{R}\mathrm{P}^{n-1}$ with a CW-structure with the properties stated in (a) and (b). We view $\mathbb{R}\mathrm{P}^{n-1}$ as the subspace of $\mathbb{R}\mathrm{P}^n$ that is given by setting the last coordinate to zero. We consider the attaching map

$$\begin{array}{rcl}\varphi\colon S^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1}\\ (x_1,\dots,x_n) & \mapsto & [x_1:\ldots:x_n].\end{array}$$

It remains to show that there exists a homeomorphism $\mathbb{R}\mathrm{P}^{n-1}\cup_\varphi\overline{B}^n\to\mathbb{R}\mathrm{P}^n$ that restricts to the identity on $\mathbb{R}\mathrm{P}^{n-1}$. We consider the map

$$\begin{array}{rcl}\Theta\colon \mathbb{R}\mathrm{P}^{n-1}\cup_\varphi\overline{B}^n & \to & \mathbb{R}\mathrm{P}^n\\ {[P]} & \mapsto & \begin{cases}[x_1:\cdots:x_n:0], & \text{if } P=[x_1:\cdots:x_n]\in\mathbb{R}\mathrm{P}^{n-1},\\ [P:1-\|P\|], & \text{if } P\in\overline{B}^n.\end{cases}\end{array}$$

It follows immediately from the definition of φ that the map Θ is well-defined. Furthermore it follows easily from Lemma 3.22 that the map Θ is continuous. It is basically clear that the map Θ is a surjection and it is straightforward to verify that the map Θ is an injection. Using Proposition 3.40, Lemma 3.22 and Proposition 2.43 (3) one can easily verify that the map is a homeomorphism. Finally it is clear that the restriction of the map Θ to $\mathbb{R}\mathrm{P}^{n-1}$ is the identity.

(2) The proof for $\mathbb{C}\mathrm{P}^n$ is basically identical to the proof of (1). We just need to view S^{2n-1} and $\overline{B}^{2n}$ as subsets of $\mathbb{C}^n$. We leave it to the reader to make the necessary minuscule modifications. ■

To enhance the reader's interest in CW-complexes we now give a sneak preview of a later result.

Theorem 64.5. *Every compact connected n-dimensional smooth manifold M admits a finite* CW*-structure with the following properties:*

(1) *the maximal dimension of a cell is n,*
(2) *the boundary ∂M is a subcomplex.*

Furthermore, if $M = A \cup B$ is a decomposition into two n-dimensional submanifolds such that $A \cap B$ is a union of boundary components of A and a union of boundary components of B, then we can find a CW*-structure on M which has all the above properties and such that A and B are subcomplexes.*

In the above discussion we have seen that many of the topological spaces we are interested in can be viewed as CW-complexes. But note that not every topological space can be described as a CW-complex. For example consider the topological space X which consists of precisely two points P and Q and which is endowed with the trivial topology. (Recall that this means that the only open sets are the empty set and all of X.) Then we cannot view X as CW-complex since any CW-complex with finitely many points consists only of 0-cells together with the *discrete* topology. For the same reason $\{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ is not a CW-complex.

36.2. **Two topologies on $\mathbb{R}^\infty$.** In the next section we want to introduce infinite-dimensional CW-complexes. Before we do so we want to get more acquainted with $\mathbb{R}^\infty$ and topologies on the set $\mathbb{R}^\infty$. First of all we write

$$\mathbb{R}^\infty \;:=\; \mathbb{R}^{(\mathbb{N})} \;:=\; \{(x_1, x_2, \dots) \,|\, \text{only finitely many } x_i \text{ are non-zero}\}.$$

As in the finite-dimensional case we can equip $\mathbb{R}^\infty$ with the Euclidean metric[582]

$$d((x_1, x_2, \dots), (y_1, y_2, \dots)) \;:=\; \sqrt{\sum_{i=1}^{\infty} |x_i - y_i|^2}.$$

By Lemma 2.1 the metric d defines a topology on $\mathbb{R}^\infty$, more precisely

$$\mathcal{S} \;:=\; \{U \subset \mathbb{R}^\infty \,|\, \text{ for every } P \in U \text{ there exists an } \epsilon > 0 \text{ such that } B_\epsilon(P) \subset U\}$$

is a topology on $\mathbb{R}^\infty$.

This topology might feel like it is the "right topology" on $\mathbb{R}^\infty$. But the next two examples might sow some doubts.

[582]Note that the series on the right is in fact a finite sum, since all but finitely many terms are zero. It is again straightforward to verify that d is a metric.

Examples.

(1) We claim that the function

$$\begin{aligned} \alpha\colon \mathbb{R}^\infty &\to \mathbb{R} \\ (x_1, x_2, \dots) &\mapsto \sum_{i=1}^{\infty} 2^i \cdot x_i \end{aligned}$$

is not continuous with respect to $\mathcal{S}$. To show this claim we consider the open subset $(-1,1) \subset \mathbb{R}$, then $\alpha^{-1}((-1,1))$ contains the origin 0. We claim that there is no $\epsilon > 0$ with $B_\epsilon(0) \subset \alpha^{-1}((-1,1))$. Indeed, suppose such $\epsilon > 0$ exists. We pick $i \in \mathbb{N}$ with $\frac{1}{2^i} < \epsilon$ and we consider the point

$$Q = (0, \dots, 0, \tfrac{1}{2^i}, 0, \dots).$$

Then $Q \in B_\epsilon(0)$ but $\alpha(Q) = 1$, i.e. $Q \notin \alpha^{-1}((-1,1))$.

(2) Let B be the infinite diagonal matrix where the diagonal entries are given by 2^i for $i = 1, 2, \dots$, i.e. let

$$B := \begin{pmatrix} 2 & 0 & 0 & \dots \\ 0 & 4 & 0 & \dots \\ 0 & 0 & 8 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We claim that the map

$$\begin{aligned} \beta\colon \mathbb{R}^\infty &\to \mathbb{R}^\infty \\ x &\mapsto B \cdot x \end{aligned}$$

is not continuous with respect to $\mathcal{S}$. Indeed, an argument similar to (1) shows that the preimage of the open set $U = B_1(0)$ is not open.

The maps α and β feel quite natural, so we might want to look around for a different topology on $\mathbb{R}^\infty$ for which these two maps are in fact continuous. Of course that would be achieved by equipping $\mathbb{R}^\infty$ with the discrete topology, but clearly that's not what we want. Another reasonable condition on the desired topology is that for any $n \in \mathbb{N}_0$ the inclusion map

$$\begin{aligned} \iota_n\colon \mathbb{R}^n &\hookrightarrow \mathbb{R}^\infty \\ (x_1, \dots, x_n) &\mapsto (x_1, \dots, x_n, 0, \dots) \end{aligned}$$

should also be continuous. This immediately eliminates the discrete topology as a contender for the "right topology".

In fact this last condition leads us almost immediately to an alternative topology on $\mathbb{R}^\infty$. We saw in the proof of Proposition 25.1 that in the category of topological spaces we have

$$\varinjlim \mathbb{R}^n = \mathbb{R}^\infty,$$

where $\mathbb{R}^\infty$ is equipped with the "weak topology", i.e. with the topology

$$\mathcal{T} := \{U \subset \mathbb{R}^\infty \,|\, \text{for every } n \in \mathbb{N} \text{ the set } U \cap \mathbb{R}^n \text{ is open in } \mathbb{R}^n\}.$$

Put differently, a map $f\colon \mathbb{R}^\infty \to X$ to a topological space X is continuous with respect to $\mathcal{T}$ if and only if for each $n \in \mathbb{N}$ the map $\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty \xrightarrow{f} X$ is continuous.

Examples.

(1) By definition of $\mathcal{T}$ the inclusion map $\mathbb{R}^n \to \mathbb{R}^\infty$ is continuous with respect to $\mathcal{T}$.
(2) The above map $\alpha\colon \mathbb{R}^\infty \to \mathbb{R}$ is continuous with respect to the topology $\mathcal{T}$. Indeed, for any $n \in \mathbb{N}$ the restriction $\mathbb{R}^n \to \mathbb{R}^\infty \to \mathbb{R}$ is a linear map, hence continuous.
(3) In Exercise 36.2 we will see that $\mathcal{S} \subset \mathcal{T}$. This implies in particular that the identity map defines a continuous map $(\mathbb{R}^\infty, \mathcal{T}) \to (\mathbb{R}^\infty, \mathcal{S})$.
(4) Let a_i, b_i, $i \in \mathbb{N}$ be real numbers. The infinite hyperrectangle
$$(a_1, b_1) \times (a_2, b_2) \times (a_3, b_3) \times \ldots$$
is easily seen to be an open subset of $(\mathbb{R}^\infty, \mathcal{T})$.[583]

The following lemma shows in particular that the above map $\beta\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ is continuous with respect to the topology $\mathcal{T}$.

Lemma 36.2. *Let $C = (c_{ij})_{i,j\in\mathbb{N}}$ be an infinite diagonal matrix. Then the map*[584]
$$\begin{array}{rcl} \gamma\colon \mathbb{R}^\infty & \to & \mathbb{R}^\infty \\ x & \mapsto & C \cdot x \end{array}$$
is continuous with respect to the topology $\mathcal{T}$.

Proof. Let $U \in \mathcal{T}$. We need to show that $\gamma^{-1}(U)$ lies in $\mathcal{T}$. So given any $n \in \mathbb{N}$ we need to show that $\gamma^{-1}(U) \cap \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$. So let $n \in \mathbb{N}$. Given $m \in \mathbb{N}$ we denote by $C_{m,n}$ the submatrix of C given by the top left $m \times n$-submatrix of C and we denote by $\gamma_{m,n}\colon \mathbb{R}^n \to \mathbb{R}^m$ the map given by $x \mapsto C_{m,n} \cdot x$. We have[585]
$$\begin{aligned} \gamma^{-1}(U) \cap \mathbb{R}^n &= \left\{x \in \mathbb{R}^n \,\middle|\, C \cdot x \in U\right\} &&= \left\{x \in \mathbb{R}^n \,\middle|\, C \cdot x \in \bigcup_{m\in\mathbb{N}} (U \cap \mathbb{R}^m)\right\} \\ &= \bigcup_{m\in\mathbb{N}} \left\{x \in \mathbb{R}^n \,\middle|\, C \cdot x \in U \cap \mathbb{R}^m\right\} &&= \bigcup_{m\in\mathbb{N}} \left\{x \in \mathbb{R}^n \,\middle|\, C_{m,n} \cdot x \in U \cap \mathbb{R}^m\right\}. \\ &= \bigcup_{m\in\mathbb{N}} \underbrace{\gamma_{m,n}^{-1}(\overbrace{U \cap \mathbb{R}^m}^{\text{open in } \mathbb{R}^m})}_{\substack{\text{open in } \mathbb{R}^n \text{ since} \\ \gamma_{m,n} \text{ is continuous}}}. \end{aligned}$$
This shows that $\gamma^{-1}(U) \cap \mathbb{R}^n$ is a union of open subsets of $\mathbb{R}^n$, hence itself is open. ■

This discussion shows hopefully that the following convention is reasonable.

Convention. Unless we say something else we will view $\mathbb{R}^\infty$ as a topological space with the topology $\mathcal{T}$. Put differently, we think of $\mathbb{R}^\infty$ as $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$. Similarly we define $\mathbb{C}^\infty = \varinjlim \mathbb{C}^n$ as topological spaces.

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[583] This example is less obvious. The topology $\mathcal{T}$ has more open sets than $\mathcal{S}$ which made it easier in (2) to argue that the map α is continuous. But here in (3) it also makes it harder, since we have to check the continuity for more open sets on the right-hand side.

[584] Does this also work for any infinite-square matrix?

[585] For each $k \in \mathbb{N}$ we view $\mathbb{R}^k$ as a subset of $\mathbb{R}^\infty$ via the inclusion ι_k.

Remark. In Exercise 36.3 we will verify that the map

$$\begin{array}{rcl} \mathbb{C}^\infty & \to & \mathbb{R}^\infty \\ (x_1+iy_1, x_2+iy_2, \dots) & \to & (x_1, y_1, x_2, y_2, \dots) \end{array}$$

is a homeomorphism. At times we will use this homeomorphism to make the identification $\mathbb{C}^\infty = \mathbb{R}^\infty$.

In the future, the more we work with $\mathbb{R}^\infty$ and $\mathbb{C}^\infty$, the more we will see that the "direct limit topology" is in fact the "right" topology to work with.

To get acquainted with this new topological space we want to prove the following lemma.

Lemma 36.3. *Given any compact subset K of $\mathbb{R}^\infty$ there exists an $n \in \mathbb{N}$ with $K \subset \mathbb{R}^n$.*

Proof. So let $K \subset \mathbb{R}^\infty$ be a compact set. Suppose that there is no $n \in \mathbb{N}$ such that $K \subset \mathbb{R}^n$. Then there exists an infinite increasing sequence $n_1, n_2, n_3, \dots$ of natural numbers and points $x_1, x_2, x_3, \dots$ in K such that $x_i \in \mathbb{R}^{n_i} \setminus \mathbb{R}^{n_i - 1}$. We denote by X the set whose elements are precisely the x_i.

Claim 1. The set X is a discrete subset of $\mathbb{R}^\infty$.

To simplify the notation we assume that $n_i = i$ for all $i \in \mathbb{N}$. For each i we denote by z_i the i-coefficient of x_i. Put differently, z_i is the highest non-zero coefficient of x_i. Furthermore for each $i \in \mathbb{N}$ we pick $\epsilon_i \in (0, \frac{\|z_i|}{2})$. Finally, given $x \in \mathbb{R}$ and $\epsilon > 0$ we write $I_\epsilon(x) = (x-\epsilon, x+\epsilon)$. We define

$$\begin{array}{rcccccccccl} U_1 & = & I_{\epsilon_1}(z_1) & \times & I_{\epsilon_2}(0) & \times & I_{\epsilon_3}(0) & \times & I_{\epsilon_4}(0) & \times & \dots \\ U_2 & = & \mathbb{R} & \times & I_{\epsilon_2}(z_2) & \times & I_{\epsilon_3}(0) & \times & I_{\epsilon_4}(z_4) & \times & \dots \\ U_3 & = & \mathbb{R} & \times & \mathbb{R} & \times & I_{\epsilon_3}(z_3) & \times & I_{\epsilon_4}(0) & \times & \dots \\ \vdots & & & & & & & & & & \end{array}$$

As we remarked on page 938, each U_i is an open subset of $\mathbb{R}^\infty$. Evidently U_i contains x_i and thus each U_i is an open neighborhood of x_i. Furthermore the sets U_i, $i \in \mathbb{N}$ are easily seen to be disjoint. But this shows, by definition of the subspace topology, that X is discrete. ⊞

Claim 2. The set X is a closed subset of K.

We need to show that $X \setminus K$ is open. By definition of the subspace topology it suffices to show that $\mathbb{R}^\infty \setminus X$ is open. In the following we continue with the notation from the previous claim. Thus let $y \in \mathbb{R}^\infty$. We pick $n \in \mathbb{N}$ such that $y \in \mathbb{R}^n$. In $\mathbb{R}^n$ we can and will pick an open neighborhood U of $(y_1, \dots, y_n)$ that does not contain $x_1, \dots, x_n \in \mathbb{R}^n$. One can now verify easily that

$$V := U \times I_{\epsilon_{n+1}}(0) \times I_{\epsilon_{n+2}}(0) \times \dots$$

is an open neighborhood of $y \in \mathbb{R}^\infty$ that does not intersect X. ⊞

It follows from Claim 1, Claim 2 and Lemma 2.18 (2) that X is indeed finite. But that is a contradiction to our assumption that there is no $n \in \mathbb{N}$ such that K is contained in $\mathbb{R}^n$. ■

We conclude this section with a short discussion of the infinite-dimensional sphere S^∞.

Definition. We refer to

$$S^\infty \ := \ \{x \in \mathbb{R}^\infty \,|\, d(x, \text{origin}) = 1\} \ = \ \Big\{(x_1, x_2, \dots) \in \mathbb{R}^\infty \,\Big|\, \sum_{i=1}^{\infty} x_i^2 = 1\Big\},$$

equip with the subspace topology coming from $\mathbb{R}^\infty$, as the *infinite-dimensional sphere.*

Even though we did not show it yet, we strongly expect that the finite-dimensional spheres S^n are not contractible. In contrast to this expectation we have the following mind-bending lemma.

Lemma 36.4. *The infinite-dimensional sphere S^∞ is contractible.*

Proof. We want to show that S^∞ is contractible. We recall that by Lemma 18.13 it suffices to show that there exists a homotopy H from id_{S^∞} to a constant map. Our goal is to find such H. We set

$$X \ := \ \{(0, p_2, p_3, \dots) \ \in \ \mathbb{R}^\infty \,|\, p_i \in \mathbb{R}\}.$$

Next we consider the maps

$$\begin{array}{rcl} F\colon \mathbb{R}^\infty \times [0,1] & \to & \mathbb{R}^\infty \\ (P,t) & \mapsto & \underbrace{P\cdot(1-t) + (0,P)\cdot t}_{\neq 0 \text{ if } P \neq 0} \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon X \times [0,1] & \to & \mathbb{R}^\infty \\ (P,t) & \mapsto & \underbrace{P\cdot(1-t) + (1,0,\dots)\cdot t}_{\neq 0 \text{ if } P \neq 0}. \end{array}$$

For non-zero $P \in \mathbb{R}^\infty$ and $P \in X$ we define $\widetilde{F}(P,t)$ and $\widetilde{G}(P,t)$ to be the normalization of $F(P,t)$ and $G(P,t)$ to length one. Next we consider the map

$$\begin{array}{rcl} H\colon S^\infty \times [0,1] & \to & S^\infty \\ (P,t) & \mapsto & \begin{cases} \widetilde{F}(P, 2t), & \text{if } t \in [0, \frac{1}{2}], \\ \widetilde{G}(\widetilde{F}(P,1), 2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

Clearly $H_0 = \text{id}$ and it is straightforward to see that H_1 is a constant map. It is tempting to conclude that we are done. But in fact we still need to show the following claim.

Claim. The map H is continuous.

By Lemma 14.3 it suffices to show that the restriction of H to the subsets $S^\infty \times [0, \frac{1}{2}]$ and $S^\infty \times [\frac{1}{2}, 1]$ are continuous. Since intervals are regionally compact it follows from Lemma 25.4 that it suffices to show for each $n \in \mathbb{N}$ the restrictions of H to the subsets $S^n \times [0, \frac{1}{2}]$ and $S^n \times [\frac{1}{2}, 1]$ are continuous. But now we are in familiar territory and the continuity is a consequence of elementary arguments. ■

36.3. **Infinite-dimensional CW-complexes.** The following definition now generalizes the notion of a CW-complex.

Definition.

(a) A CW-*complex* X is defined as a sequence

$$X^{-1} \ := \ \varnothing \subset X^0 \subset X^1 \subset X^2 \subset \dots$$

of CW-complexes (in the sense of the definition of page 930) such that for each $m \le n$ the CW-complex X^m is the m-skeleton of X^n. Let $n \in \mathbb{N}_0$. We adopt the following language:

(i) The (open) n-cells of X^n are called *the (open) n-cells of* X.

(ii) We refer to X^n as the *n-skeleton of* X

(b) We define the *topological realization* $|X|$ of such a CW-complex X as

$$|X| \;:=\; \varinjlim |X^n|.$$

Recall that on page 930 we argued that for each $n \in \mathbb{N}_0$ we can view $|X^n|$ as a subset of $|X^{n+1}|$. Thus it follows from Lemma 25.3 that we can and will write

$$|X| \;:=\; \bigcup_{n\in\mathbb{N}_0} |X^n|$$

where a subset $U \subset |X|$ is open if and only if for every $n \in \mathbb{N}_0$ the intersection $U \cap |X^n|$ is an open subset of $|X^n|$. As before, in the notation we usually do not distinguish between the notation for a CW-complex and its topological realization.

(c) We define the dimension $\dim(X) \in \mathbb{N}_0 \cup \{\infty\}$ of a non-empty CW-complex X as the maximal dimension of a cell.[586] We define the dimension of the empty CW-complex to be -1.

(d) We say a CW-complex is *finite* if it has only finitely many cells and we say it is *countable* if it has only countably many cells.

(e) A CW-*structure* for a topological space Y is a CW-complex X together with a homeomorphism $f\colon |X| \to Y$.

Remark.

(1) Let X be a CW-complex and let $n \in \mathbb{N}_0$. By Lemma 25.3 we know that the inclusion map $|X^n| \to |X|$ is an embedding. Thus we can and will view the n-skeleton as a subset of X.

(2) The topology given in (b) on a union of topological spaces is sometimes called the the *weak topology*. This explains the "W" of the name CW-complex.

(3) It follows almost immediately from the definitions that the topological realization $|X|$ of a CW-complex is, *as a set*, the disjoint union of all the open cells.

(4) On a few occasions we will refer to $X^\infty := X$ as the ∞-skeleton.

Examples.

(1) We consider the infinite sphere

$$S^\infty \;=\; \{x \in \mathbb{R}^\infty \,|\, d(x,\text{origin}) = 1\} \;=\; \Big\{(x_1, x_2, \dots) \in \mathbb{R}^\infty \,\Big|\, \sum_{i=1}^{\infty} x_i^2 = 1\Big\}$$

which equip with the subspace topology coming from $\mathbb{R}^\infty$. It follows easily from the fact that $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ that we have the equality $S^\infty = \varinjlim S^n$. We consider iteratively the spheres S^n as CW-complexes, where S^0 consists of two 0-cells and where S^n is built starting from S^{n-1} by attaching two n-cells ("the upper hemisphere

[586] Here we define $\dim(X) := \infty$ if X admits cells of arbitrarily large dimensions.

and the lower hemisphere") to S^{n-1}. We then obtain an ascending sequence

$$S^0 \subset S^1 \subset S^2 \subset \dots$$

of CW-complexes which satisfies precisely the required properties of a CW-complex as defined above. This shows that the sphere S^∞ is an infinite-dimensional CW-complex. We refer to Figure 601 for an illustration.[587]

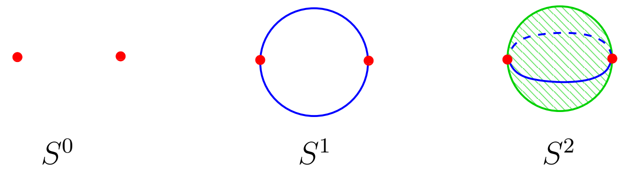


FIGURE 601

FIGURE 602. An ascending sequence of CW-complexes $S^0 \subset S^1 \subset S^2 \dots$.

(2) Let I be a set and for each $i \in I$ let $n_i \in \mathbb{N}$. In Exercise 36.10 we will see that the wedge $\bigvee_{i\in I} S_i^{n_i}$ has a natural CW-structure with one 0-cell and where for each $n \in \mathbb{N}_0$ the n-cells correspond precisely to the n-dimensional spheres in our family of spheres $\{S_i^{n_i}\}_{i\in I}$.

The following definition and notation is the obvious infinite-dimensional analogue of the finite-dimensional definitions introduced on page 194.

Definition. Let $\mathbb{A} = \mathbb{R}$ or $\mathbb{A} = \mathbb{C}$.

(1) We define the *infinite-dimensional projective space* $\mathbb{A}\mathrm{P}^\infty$ as the set

$$\mathbb{A}\mathrm{P}^\infty = (\mathbb{A}^\infty \setminus \{0\})/(\mathbb{A} \setminus \{0\})$$

together with the quotient topology.

(2) Given a point $(x_1, x_2, \dots) \in \mathbb{A}^\infty \setminus \{0\}$ we denote its equivalence class in $\mathbb{A}\mathrm{P}^\infty$ by $[x_1 : x_2 : \dots]$.

(3) Given $n \in \mathbb{N}$ we view $\mathbb{A}\mathrm{P}^n$ as a subset of $\mathbb{A}\mathrm{P}^\infty$ via the inclusion map

$$\begin{array}{rcl} \mathbb{A}\mathrm{P}^n & \to & \mathbb{A}\mathrm{P}^\infty \\ [x_0 : \dots : x_n] & \mapsto & [x_0 : \dots : x_n : 0 : \dots]. \end{array}$$

The following lemma shows in particular that we can view the real infinite projective space $\mathbb{R}\mathrm{P}^\infty$ as an infinite-dimensional CW-complex.

Lemma 36.5.

(1) *For each $n \in \mathbb{N}$ the obvious map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^\infty$ is an embedding. Furthermore these maps define a homeomorphism $\varinjlim \mathbb{R}\mathrm{P}^n \xrightarrow{\cong} \mathbb{R}\mathrm{P}^\infty$.*

(2) *The topological space $\mathbb{R}\mathrm{P}^\infty$ admits a CW-structure such that for every $n \in \mathbb{N}$ the n-skeleton is given by $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^\infty$ where $\mathbb{R}\mathrm{P}^n$ is equipped with the CW-structure coming from Lemma 36.1.*

[587] If we equip S^∞ with the topology $\mathcal{S}$ coming from the Euclidean metric on $\mathbb{R}^\infty$, then this argument does not work and it is not clear (and presumably wrong), that $(S^\infty, \mathcal{S})$ is a CW-complex.

(3) *The infinite real projective space* $\mathbb{RP}^\infty$ *admits a* CW*-structure which has precisely one* n*-cell for every* $n \in \mathbb{N}_0$.
(4) (a) *The action* $\{\pm 1\} \times S^\infty \to S^\infty$ *given by* $(\epsilon, v) \mapsto \epsilon \cdot v$ *is continuous and discrete.*
 (b) *The obvious projection* $S^\infty \to S^\infty/x \sim -x$ *is a 2-fold covering.*
 (c) *The map*

$$\begin{array}{rcl} S^\infty/x \sim -x & \to & \mathbb{RP}^\infty \\ [(x_1, x_2, \dots)] & \mapsto & [x_1 : x_2 : \dots] \end{array}$$

is a homeomorphism.

Proof (*).

(1) We leave it as a very mildly interesting exercise to the reader to show that for each $n \in \mathbb{N}$ the obvious map $\mathbb{RP}^n \to \mathbb{RP}^\infty$ is an embedding. Next we note that we have the following equalities:

$$\begin{array}{rcl} \varinjlim \mathbb{RP}^n \;=\; \varinjlim (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\}) & \xrightarrow[\cong]{\text{homeomorphism by Lemma 25.5}} & \big(\varinjlim (\mathbb{R}^{n+1} \setminus \{0\})\big)/(\mathbb{R} \setminus \{0\}) \\ & = & (\mathbb{R}^\infty \setminus \{0\})/(\mathbb{R} \setminus \{0\}) \;=\; \mathbb{RP}^\infty. \end{array}$$

It follows easily from the definitions that this homeomorphism is the inclusion induced map.
(2) This statement follows immediately from (1) and the definition of a CW-complex on page 940.
(3) This statement is an immediate consequence of (2) and the definition of the CW-structure on each $\mathbb{RP}^n$.
(4) (a) The statement is not difficult to prove, but one needs to argue carefully with the given topology on S^∞. We will fill in the details in Exercise 36.4.
 (b) This statement follows from (a) and Proposition 16.9.
 (c) It is straightforward to verify that the given map is a bijection. We leave it to the reader to verify that the map is continuous and that its inverse is continuous. ■

The following lemma is the complex sibling of Lemma 36.5.

Lemma 36.6.
(1) *For each* $n \in \mathbb{N}$ *the obvious map* $\mathbb{CP}^n \to \mathbb{CP}^\infty$ *is an embedding. Furthermore the maps* $\mathbb{CP}^n \to \mathbb{CP}^\infty$ *define a homeomorphism* $\varinjlim \mathbb{CP}^n \xrightarrow{\cong} \mathbb{CP}^\infty$.
(2) *The topological space* $\mathbb{CP}^\infty$ *admits a* CW*-structure such that for every* $n \in \mathbb{N}$ *the* $2n$*-skeleton is given by* $\mathbb{CP}^n \to \mathbb{CP}^\infty$ *where* $\mathbb{CP}^n$ *is equipped with the* CW*-structure coming from Lemma 36.1.*
(3) *The infinite complex projective space* $\mathbb{CP}^\infty$ *admits a* CW*-structure which has precisely one* n*-cell for every even* $n \in \mathbb{N}_0$.

Proof. The proof of this lemma is basically verbatim the same as the proof of the corresponding parts of Lemma 36.5. ■

Convention. We will use Lemmas 36.5 (1) and 36.6 as an excuse to view each $\mathbb{R}\mathrm{P}^n$ as a subset of $\mathbb{R}\mathrm{P}^\infty$ and to view each $\mathbb{C}\mathrm{P}^n$ as a subset of $\mathbb{C}\mathrm{P}^\infty$.

The following lemma says that characteristic maps are continuous and it gives a convenient approach to showing that a map starting at a CW-complex is continuous.

Lemma 36.7. *Let X be a* CW*-complex. We denote by $\{\Phi_i\colon \overline{B}^{n_i} \to X\}_{i\in I}$ the characteristic maps of all the cells of X. The following four statements hold:*

(1) *Every characteristic map Φ_i is continuous.*
(2) *A subset $U \subset X$ is open if and only if for each $i \in I$ the preimage of U under the characteristic map Φ_i is open, i.e. if each $\Phi_i^{-1}(U) \subset \overline{B}^{n_i}$ is an open subset of $\overline{B}^{n_i}$.*
(3) *The statement in (2) also holds if we replace "open" by "closed".*
(4) *A map $f\colon X \to Y$ to a topological space Y is continuous if and only if each map $f \circ \Phi_i\colon \overline{B}^{n_i} \to Y$ is continuous.*

Proof ($*$). Given $n \in \mathbb{N}_0$ we denote by $I_n \subset I$ the indices corresponding to the characteristic maps of the n-dimensional cells.

(1) Let $i \in I_n$. The characteristic map $\Phi\colon \overline{B}^n \to X$ is the composition of the maps

$$\overline{B}^n = \overline{B}^n_i \to |X^{n-1}| \sqcup \coprod_{j\in I_n} \overline{B}^n_j \to |X^n| = \Big(|X^{n-1}| \sqcup \coprod_{j\in I_n} \overline{B}^n_j\Big)/\sim\ \to |X| = \varinjlim |X^k|.$$

All these maps are obviously continuous, hence the composition is also continuous.

(2) Once one has unraveled the definition of the topology on a CW-complex one sees that the first statement of the lemma is obvious. In the following we will try to convince the reader that this glib statement is actually correct.

Thus let $U \subset X$ be a subset. By the definition on page 941 we know that U is open if and only if for each $n \in \mathbb{N}_0$ the intersection $U \cap |X_n|$ is an open subset of $|X_n|$. The statement follows from this observation together with the following claim.

Claim. Let $n \in \mathbb{N}$ and let U be a subset of $|X_n|$. The following two statements are equivalent:

(a) U is an open subset of $|X_n|$.
(b) $U \cap |X_{n-1}|$ is an open subset of $|X_{n-1}|$ and for every $i \in I_n$ the preimage $\Phi_i^{-1}(U)$ is an open subset of $\overline{B}^n$.

By the definition on page 930 we have

$$|X^n| \;=\; \Big(|X^{n-1}| \;\sqcup\; \coprod_{i\in I_n} \overline{B}^n_i\Big)/\sim \;\text{ where } \Phi_i(x) \sim x \text{ for } x \in \partial\overline{B}^n_i.$$

The statement now follows easily from the definition of the quotient topology and the disjoint union topology Lemma 3.44 (4). We leave it to the reader to weave these slogans into a coherent argument. ⊞

(3) The "only if"-direction follows from Lemma 36.7 (1). The "if"-direction follows from (1) and a slight generalization of Lemma 2.34 which one can easily prove using Lemma 1.3 (7). We leave it to the reader to make sense of this cryptic remark.

(4) Let $f\colon X \to Y$ be a map to a topological space. If f is continuous, then it follows from Lemma 36.7 (1) that each map $f \circ \Phi_i\colon \overline{B}^{n_i} \to Y$ is continuous. Now suppose conversely that each map $f \circ \Phi_i\colon \overline{B}^{n_i} \to Y$ is continuous. Let $U \subset Y$ be an open subset. We need to show that $f^{-1}(U)$ is an open subset of X. By (1) it suffices to show that each preimage $\Phi_i^{-1}(f^{-1}(U))$ is an open subset of $\overline{B}^{n_i}$. But this follows from the observation that $\Phi_i^{-1}(f^{-1}(U)) = (f \circ \Phi_i)^{-1}(U)$ and the hypothesis that each $f \circ \Phi_i$ is continuous. ■

36.4. **Homotopies on infinite CW-complexes** $(*)$**.** On several occasions throughout these notes, in particular in the next section, we will want to construct a homotopy on an infinite-dimensional CW-complex by working our way up through the skeleta. In the end one needs to combine infinitely many homotopies which obviously leads to continuity issues. This rather technical section deals with these issues. It is best to initially skip this section and to go back to it if later on one really wants to know how an argument works.

For the undaunted reader we start out with the following subtle lemma.

Lemma 36.8. *Let X be a* CW*-complex. The following three topologies on $X \times [0,1]$ agree:*
(1) *The product topology on the product $X \times [0,1]$.*
(2) *The topology on $X \times [0,1]$ viewed as the direct limit $\varinjlim (X^k \times [0,1])$.*[588]
(3) *The topology which is defined on the set $X \times [0,1]$ by the condition that a given subset $U \subset X \times [0,1]$ is open if and only if for each characteristic map $\Phi\colon \overline{B}^n \to X$ the preimage $(\Phi \times \mathrm{id}_{[0,1]})^{-1}(U)$ is an open subset of $\overline{B}^n \times [0,1]$.*

PROOF. Let X be a CW-complex. First note that it follows immediately from the fact that $X = \varinjlim X_k$ together with Lemma 3.8 (1c) that any subset that is open with respect to (1) is also open with respect to (2). Furthermore it follows from Lemma 36.7 (1) together with Lemma 3.8 (2b) that any subset that is open with respect to (2) is also open with respect to (3). Thus it suffices to show that any set that is open with respect to (3) is also open with respect to (1).

Let $W \subset X \times [0,1]$ be a subset that is open with respect to (3). We need to show that W is also open with respect to the product topology on $X \times [0,1]$. Let $(x_0, t_0) \in W$ be a point. By Lemma 3.11 it suffices to show the following statement:

$(*)$ There exists a neighborhood V of $x_0 \in X$ and a neighborhood K of $t_0 \in [0,1]$ such that $V \times K$ is contained in W.

To prove $(*)$ we start out with the following claim.

Claim. There exists a compact neighborhood K of $t_0 \in [0,1]$ such that $\{x_0\} \times K \subset W$.

Let $\Phi\colon \overline{B}^n \to X$ be the characteristic map of the open cell that contains x_0. Note that $\Phi\colon B^n \to X$ is injective. Thus there exists a unique $y_0 \in B^n$ with $\Phi(y_0) = x_0$. We write $\Psi := \Phi \times \mathrm{id}_{[0,1]}$. Since W is open with respect to (3) we know that $\Psi^{-1}(W)$ is an open

[588]Recall that we showed in Lemma 25.3 that this is the topology where a given subset W of $X \times [0,1] = \varinjlim (X^k \times [0,1])$ is open if and only each intersection $W \cap (X^k \times [0,1])$ is an open subset of $X^k \times [0,1]$.

subset of $\overline{B}^n \times [0,1]$. Since $(y_0,t_0) \in \Psi^{-1}(W)$ we obtain from the definition of the product topology on $\overline{B}^n \times [0,1]$ there exists in particular an open neighborhood J of $t_0 \in [0,1]$ such that $\{y_0\} \times J \subset \Psi^{-1}(W)$. Since $[0,1]$ is regionally compact we see that there exists a compact neighborhood K of t_0 that is contained in J. Now we see that

$$\{x_0\} \times K \;=\; \{\Phi(y_0)\} \times K \underset{\substack{\uparrow \\ \text{definition of } \Psi}}{=} \Psi(\{y_0\} \times K) \;\subset\; \Psi(\{y_0\} \times J) \underset{\substack{\uparrow \\ \text{since } \{y_0\} \times J \subset \Psi^{-1}(W)}}{\subset} \Psi(\Psi^{-1}(W)) \underset{\substack{\uparrow \\ \text{Lemma 1.3 (2)}}}{\subset} W.$$

Thus K has the desired property. ⊞

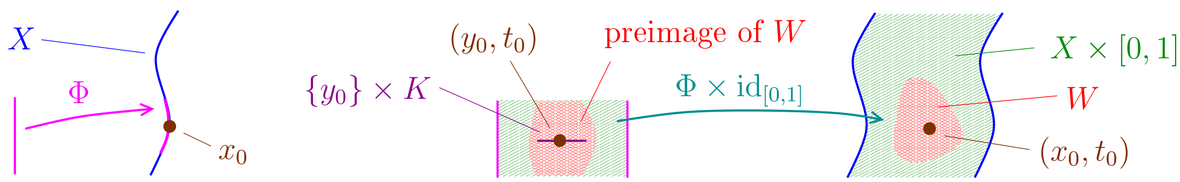


FIGURE 603. Illustration for the proof of Lemma 36.8.

Next we consider $V := \{x \in X \,|\, \{x\} \times K \subset W\}$. By the choice of K we have $x_0 \in V$. Thus to prove $(*)$ it remains to prove the following claim.

Claim. The subset V is open in X.

We use Lemma 36.7 (2) to show that V is open in X. Thus let $\Phi\colon \overline{B}^n \to X$ be the characteristic map of an n-cell of X. We need to show that the preimage $\Phi^{-1}(V)$ is open in $\overline{B}^n$. We write $U := (\Phi \times \mathrm{id}_{[0,1]})^{-1}(W)$. Since W is open with respect to (3) we see that U is an open subset of $\overline{B}^n \times [0,1]$. Basically by definition we have the equality

$$\Phi^{-1}(V) \;=\; \{y \in \overline{B}^n \,|\, \{y\} \times K \subset U\}.$$

It follows from elementary point-set topology, see Exercise 3.10, that the set on the right-hand side is indeed open. This concludes the proof of the second claim and thus of the lemma. ■

Later on we will frequently need the following corollary.

Corollary 36.9. *Let X be a* CW*-complex and let $\Phi\colon X \times [0,1] \to Y$ be a map to a topological space Y. The following statements are equivalent:*

(1) *The map f is continuous.*
(2) *For every $k \in \mathbb{N}_0$ the restriction of f to $X^k \times [0,1]$ is continuous.*
(3) *For every characteristic map $\Phi\colon \overline{B}^n \to X$ the map*

$$\overline{B}^n \times [0,1] \xrightarrow{(v,t)\mapsto(\Phi(v),t)} X \times [0,1] \xrightarrow{f} Y$$

is continuous.

PROOF. The corollary follows immediately from the equivalence of the three topologies in Lemma 36.8. ■

36.5. **Properties of CW-complexes.** We have seen, as promised, that many topological spaces that we are interested in are indeed CW-complexes. In this section we will prove a somewhat technical proposition that shows that CW-complexes have, again as promised, many nice topological properties.

To formulate the key result of this section we need the following definition which we will explore in much greater detail in the subsequent section.

Definition. A *subcomplex* of a CW-complex X is a subset of X that is the union of arbitrarily many cells of X. We say a subcomplex is *finite* it contains only finitely many cells of X.

Now we can state the main result of this section.

Proposition 36.10.

(1) *Every point on a* CW*-complex is closed.*[589]
(2) (a) CW*-complexes are Hausdorff.*
  (b) CW*-complexes are normal.*[590]
(3) (a) *Every cell of a* CW*-complex* X *is compact and it is a closed subset of* X*.*
  (b) *Every finite* CW*-complex is compact.*
  (c) *The closure of an open cell is the corresponding cell. More precisely, given a characteristic map* $\Phi\colon \overline{B}^n \to X$ *the closure of* $\Phi(B^n)$ *is indeed* $\Phi(\overline{B}^n)$*.*
  (d) *If a closed subset of* X *contains an open cell, then it also contains the corresponding cell.*
(4) *Let* X *be a* CW*-complex, let* $x \in X$ *and let* U *be an open neighborhood* U *of* x*.*
  (a) *There exists an open neighborhood* V *of* x *that is contained in* U *and that admits a deformation retraction to* $\{x\}$*.*
  (b) *There exists a closed neighborhood* V *of* x *that is contained in* U *and that admits a deformation retraction to* $\{x\}$*.*
(5) *Every point on a* CW*-complex is good in the sense of the definition on page 604.*
(6) CW*-complexes are locally contractible, locally path-connected, locally simply-connected and semi-locally simply connected.*
(7) *Let* X *be a* CW*-complex. The following statements are equivalent:*
  (a) X *is connected.*
  (b) X *is path-connected.*
  (c) *The 1-skeleton* X^1 *is connected.*
  (d) *The 1-skeleton* X^1 *is path-connected.*
  (e) *Given any two distinct points* P, Q *in the 0-skeleton of* X *there exists an embedding* $\gamma\colon [0,1] \to X^1$ *with* $\gamma(0) = P$, $\gamma(1) = Q$ *and such that* $\gamma([0,1])$ *is a subcomplex of* X^1*.*
(8) *There exists a map* Φ *which assigns to each subcomplex* A *a subset* $\Phi(A) \subset X$ *such that the following statements hold:*
  (a) *for each subcomplex* A *the set* $\Phi(A)$ *is an open neighborhood of* A *such that* A *is a deformation retract of* $\Phi(A)$,

(b) *for any two subcomplexes* A_1 *and* A_2 *we have*

$$\Phi(A_1 \cap A_2) = \Phi(A_1) \cap \Phi(A_2) \quad \text{and} \quad \Phi(A_1 \cup A_2) = \Phi(A_1) \cup \Phi(A_2),$$

(c) *for any two subcomplexes* A *and* B *with* $A \subset B$ *we have* $\Phi(A) \subset \Phi(B)$.

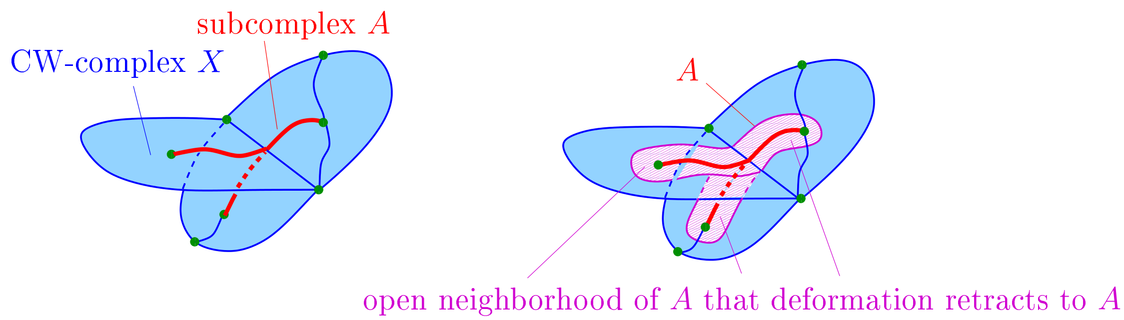


FIGURE 604. Illustration of Proposition 36.10 (8).

Remark.

(1) In a slightly simplified form Proposition 36.10 (8) says that we can assign to each subcomplex A in a "systematic way" an open neighborhood $\Phi(A)$ that deformation retracts to A.
(2) In Proposition 36.10 (8) it is essential that A is a subcomplex. For example, if we consider $X = \mathbb{R}$ and $A = \{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$, then A does not admit an open neighborhood U such that A is a deformation retract of U.[591]
(3) Proposition 36.10 (6) sounds somewhat technical, but it ensures that later on we can apply Proposition 29.5 to obtain coverings of CW-complexes.
(4) Many other useful properties of CW-complexes are proved in [**LW69**, Section II.3].
(5) There is a limit to what degree CW-complexes are nice, for example in Exercise 18.35 we saw that the wedge $\bigvee_{i \in \mathbb{N}_0} [0,1]_i$, which is easily seen to be a CW-complex, is not metrizable.

PROOF (*). We start out with the following notation. Given a subset A of $\overline{B}^n$ and $\epsilon > 0$ we denote by

$$N_\epsilon(A) \ := \ \left\{ x \in \overline{B}^n \,\middle|\, \text{there exists an } y \in A \text{ with } \|x - y\| < \epsilon \right\}$$

the ϵ-neighborhood around A. One easily verifies that $N_\epsilon(A)$ is an open subset of $\overline{B}^n$. Furthermore given two non-empty subsets X and Y of $\overline{B}^n$ we follow the notation introduced on page 175 and we write

$$d(X, Y) \ := \ \inf\{\|x - y\| \,|\, x \in X \text{ and } y \in Y\}.$$

[589]Note that we need (1) in the proof of (2), thus we cannot just use Lemma 2.13 to deduce (2) from (1).

[590]Recall that on page 120 we introduced the definition that a topological space X is called *normal* if any two disjoint closed subsets A and B of X are separated by neighborhoods, i.e. if there exist disjoint neighborhoods U of A and V of B.

[591]Why not?

Furthermore, if X or Y is empty, then we define $d(X,Y)=\infty$ and evidently we write $\infty>0$.

Now let X be a CW-complex. We denote by Γ the set of cells of X and given $n\in\mathbb{N}_0$ we denote by Γ_n the set of n-cells. Given a cell γ we denote by Φ_γ the corresponding characteristic map.

Let $A\subset X$ be a subset and let ϵ be a map that assigns to each $\gamma\in\Gamma$ a positive number ϵ_γ. Given $n\in\mathbb{N}_0$ we define a subset $N_\epsilon^n(A)$ inductively as follows:

(i) For $n=0$ we define $N_\epsilon^0(A):=A\cap X^0$.

(ii) Suppose that $N_\epsilon(A)^n$ has already been defined. We define

$$N_\epsilon^{n+1}(A) \;:=\; N_\epsilon^n(A)\cup\bigcup_{\gamma\in\Gamma_{n+1}}\Phi_\gamma\Big(N_{\epsilon_\gamma}(\overbrace{\Phi_\gamma^{-1}(A)}^{\subset\overline{B}^{n+1}})\cup\underbrace{(1-\epsilon_\gamma,1]\times\overbrace{\Phi_\gamma^{-1}(N_\epsilon^n(A))}^{\subset S^n}}_{\text{radial coordinates on }\overline{B}^{n+1}}\Big).$$

It is straightforward to verify that each $N_\epsilon^n(A)$ is an open neighborhood of $A\cap X^n$ in X^n. The definition of $N_\epsilon^{n+1}(A)$ is illustrated in Figure 605. Finally we set $N_\epsilon(A):=\bigcup\limits_{n\in\mathbb{N}}N_\epsilon^n(A)$.

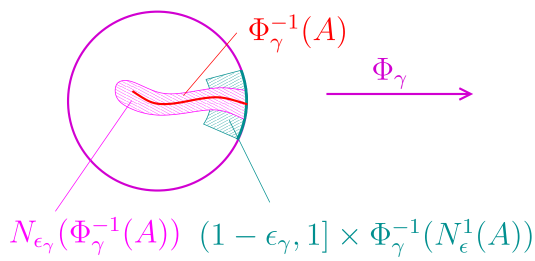


FIGURE 605

It follows easily from the definition of the topology of a CW-complex that $N_\epsilon(A)$ is an open neighborhood of A in X. The definition of $N_\epsilon(A)$ is illustrated in Figure 606.

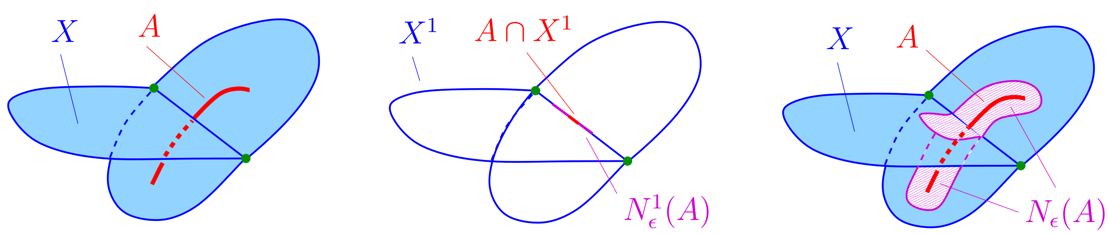


FIGURE 606

Now we turn to the proofs of the eight statements:

(1) This statement follows fairly easily from Lemma 3.44. We will fill in the details in Exercise 36.6.

(2) Since by (1) we know that points are closed it suffices to show that CW-complexes are normal. Thus let A and B of X be disjoint closed subsets of X. We need to show that there exist open disjoint neighborhoods of A and B.

So let A and B be two disjoint closed subsets of X. We will find ϵ as above such that $N_\epsilon(A)$ and $N_\epsilon(B)$ are disjoint. We will define the values of ϵ inductively on the dimension of the cells. For 0-cells we take $\epsilon = 1$. Now suppose that we have already found disjoint open sets $N^n_\epsilon(A)$ and $N^n_\epsilon(B)$. Let γ be an $(n+1)$-cell with characteristic map $\Phi_\gamma\colon \overline{B}^{n+1} \to X$.

Claim. We have the following inequalities:

$$\begin{array}{lrcl}
(1) & \mu_1 := d(\Phi_\gamma^{-1}(N^n_\epsilon(A)), \Phi_\gamma^{-1}(B)) & > & 0, \\
(2) & \mu_2 := d(\Phi_\gamma^{-1}(N^n_\epsilon(B)), \Phi_\gamma^{-1}(A)) & > & 0, \\
(3) & \mu_3 := d(\Phi_\gamma^{-1}(B), \Phi_\gamma^{-1}(A)) & > & 0.
\end{array}$$

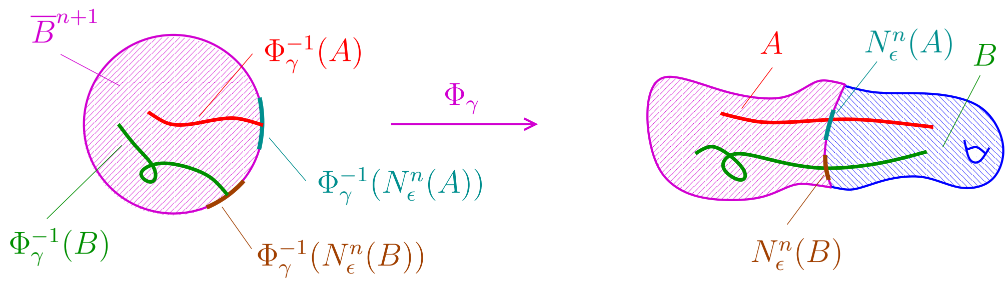


FIGURE 607

We deal with the three inequalities separately:

(a) We first consider μ_1. Suppose that $\mu_1 = 0$. If this was the case then there would exist a sequence of points in $\Phi_\gamma^{-1}(B)$ that converges to a point of $\Phi_\gamma^{-1}(B)$ in $\partial\overline{B}^{n+1}$ of distance zero from $\Phi_\gamma^{-1}(N^n_\epsilon(A))$. But this is impossible since $\Phi_\gamma^{-1}(N^n_\epsilon(B))$ is a neighborhood of $\Phi_\gamma^{-1}(B) \cap \partial\overline{B}^{n+1}$ in $\partial\overline{B}^{n+1}$ which is disjoint from $\Phi_\gamma^{-1}(N^n_\epsilon(A))$.

(b) The proof of the second inequality $\mu_2 > 0$ is proved exactly the same as the proof of the inequality $\mu_1 > 0$.

(c) It follows from Lemma 36.7 (1) and our hypothesis that A and B are disjoint closed subsets that the preimages $\Phi_\gamma^{-1}(A)$ and $\Phi_\gamma^{-1}(B)$ are disjoint closed subsets of $\overline{B}^{n+1}$. By Lemma 2.17 (1) we know that they are in fact disjoint compact subsets of $\overline{B}^n$. Thus it follows from Corollary 3.19 (2) that $\mu_3 = d(\Phi_\gamma^{-1}(B), \Phi_\gamma^{-1}(A)) > 0$. ⊞

We set $\epsilon_\gamma := \frac{1}{2}\min\{\mu_1, \mu_2, \mu_3\}$. We do this for all $(n+1)$-cells γ. It then follows immediately from the definitions that $N^{n+1}_\epsilon(A)$ and $N^{n+1}_\epsilon(B)$ are disjoint.

(3) (a) Every k-cell is the image of the compact set $\overline{B}^k$, hence it is compact by Lemma 2.40. It follows from Lemma 2.17 (2) and the fact, proved in (2), that X is Hausdorff that every k-cell is also a closed subset of X.

(b) A finite CW-complex is the union of finitely many cells. Thus it follows from (3a) together with Lemma 2.16 that a finite CW-complex is compact.

(c) Let $\Phi\colon \overline{B}^n \to X$ be a characteristic map. We have the following inclusions

$$\Phi(\overline{B}^n) \underset{\substack{\uparrow \\ \text{Lemma 2.33}}}{\subset} \overline{\Phi(B^n)} \underset{\substack{\uparrow \\ \text{definition of the closure}}}{=} \bigcap \text{closed subsets containing } \Phi(B^n) \underset{\substack{\uparrow \\ \text{by (3a)}}}{\subset} \Phi(\overline{B}^n).$$

We have thus shown that we indeed have $\overline{\Phi(B^n)} = \Phi(\overline{B}^n)$.

(d) This statement follows immediately from (3b) and the definition of the closure of a set.

(4) Let $x \in X$ and let U be an open neighborhood of x. We need to prove the following two statements;
 (a) There exists an open neighborhood V of x that is contained in U and that admits a deformation retraction to $\{x\}$.
 (b) There exists a closed neighborhood V of x that is contained in U and that admits a deformation retraction to $\{x\}$.

In the following we will prove Statement (a). We leave it to the reader to make the slight modifications to obtain Statement (b).

So let us start with the proof of Statement (a). We leave it to the long-suffering reader to show that there exists an ϵ such that $N_\epsilon(x)$ is contained in U.

It remains to show that $N_\epsilon(x)$ admits a deformation retraction to $\{x\}$. We denote by m the unique natural number with $x \in X^m \setminus X^{m-1}$. It follows easily from the definitions that $N_\epsilon^m(x)$ is an open ball around x, in particular $N_\epsilon^m(x)$ deformation retracts onto $\{x\}$. Therefore it suffices to find a deformation retraction from $N_\epsilon(x)$ to $N_\epsilon^m(x)$.

First note that for $n > m$ "sliding radially outward" we obtain a deformation retraction

$$R(n)\colon N_\epsilon^n(x) \times [0,1] \;\to\; N_\epsilon^n(x)$$

from $N_\epsilon^n(x)$ to $N_\epsilon^{n-1}(x)$. We denote by $r_n := R(n)_1\colon N_\epsilon^n(x) \to N_\epsilon^{n-1}(x)$ the corresponding retraction. For each $k \in \mathbb{N}_{\geq m+1}$ we consider the map

$$\begin{array}{rcl} f_k\colon N_\epsilon(x) & \to & N_\epsilon^{k-1}(x) \\ y & \mapsto & \begin{cases} (r_k \circ \cdots \circ r_l)(y), & \text{if } y \in N_\epsilon^l(x) \text{ and } l \geq k, \\ y, & \text{if } y \in N_\epsilon^{k-1}(x). \end{cases} \end{array}$$

It follows from Corollary 36.9 that this map f_k is continuous. Now we consider the following map

$$\begin{array}{rcl} F\colon N_\epsilon(x) \times [0,1] & \to & N_\epsilon(x) \\ (y,s) & \mapsto & \begin{cases} y, & \text{if } x \in N_\epsilon^k(x) \text{ and } s \in [0, \frac{1}{2^{k+1}}], \\ r_k\big(y, 2^{k+1} \cdot (s - \frac{1}{2^{k+1}})\big), & \text{if } x \in N_\epsilon^k(x) \text{ and } s \in [\frac{1}{2^{k+1}}, \frac{1}{2^k}], \\ f_k(y), & \text{if } x \in N_\epsilon^k(x) \text{ and } s \in [\frac{1}{2^k}, 1]. \end{cases} \end{array}$$

We refer to Figure 613 for an illustration. It follows from Lemma 14.3 that the restriction of F to each $N_\epsilon^k(x) \times [0,1]$ is continuous. Thus it follows from Corollary 36.9 that $F\colon N_\epsilon(x) \times [0,1] \to N_\epsilon(x)$ is continuous. This map is the desired deformation retraction.

(5) This statement follows immediately from (2) and (4).

(6) Note that this statement follows immediately from (4) since every topological space that admits a deformation retraction to a point is contractible, and since every contractible topological space is path-connected and simply connected. Finally note that a topological space that is locally simply connected is in particular semi-locally simply connected.

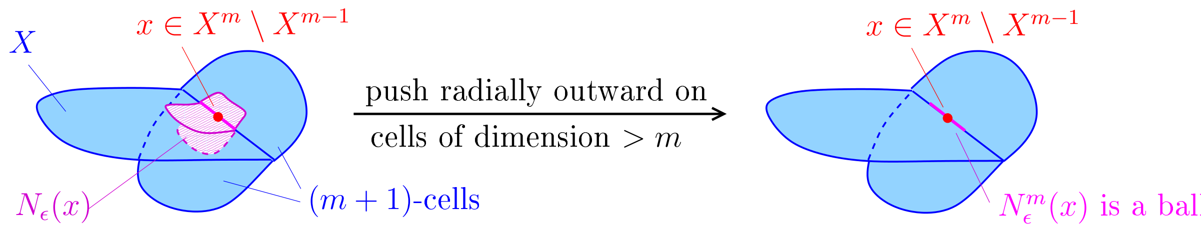


FIGURE 608

(7) By Lemma 2.72 an locally path-connected topological space is connected if and only if it is path-connected. This shows the equivalence (a) $\Leftrightarrow$ (b) and (c) $\Leftrightarrow$ (d). It follows from Lemma 4.1 (5) and the discussion on page 932 that we also have (d) $\Leftrightarrow$ (e). We will prove (d) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) in Exercise 36.7.

(8) We pick an $\epsilon \in (0,1)$. Given a subcomplex A of X we set $\Phi(A) := N_\epsilon(A)$. In each cell of $X \setminus A$ the set $N_\epsilon(A)$ is a product neighborhood of the boundary of this cell. So a deformation retraction of $N_\epsilon(A)$ onto A can be constructed as in (4). We leave the details to the reader. It is clear that Φ has all the other desired properties. ■

Our next goal is to formulate an addendum to Proposition 36.10. To do so we need to introduce the following definition.

Definition. We say that a CW-complex is *locally finite* if every cell is disjoint from all but finitely many cells of X.

Examples. The CW-complex $\mathbb{C}\mathrm{P}^\infty$ is not locally finite, whereas the CW-complexes illustrated in Figure 609 are locally finite.

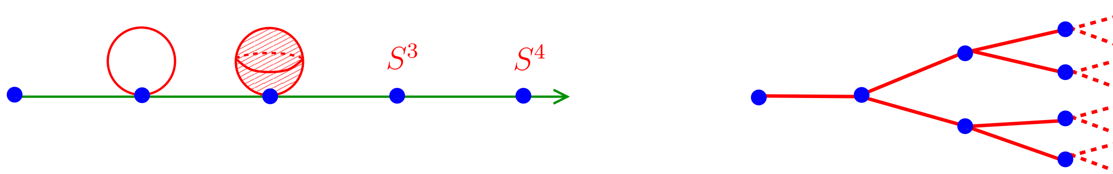


FIGURE 609. Examples of locally finite CW-complexes.

The following proposition is the promised addendum to Proposition 36.10.

Proposition 36.11. *A* CW*-complex is regionally compact if and only if it is locally finite.*

PROOF. We will not make use of Proposition 36.11, except that in the negative sense, that now we know that many CW-complexes we are interested in are unfortunately not regionally compact. The proposition is proved in [**Geo08**, Proposition 10.1.8]. ■

One of the curses of the working topologist is the unfortunate fact that a bijection continuous map between topological spaces does not need to be a homeomorphism. The following lemma gives a nice criterion for a map between CW-complexes to be a homeomorphism.

Lemma 36.12. *Let $f\colon X \to Y$ be a map between* CW*-complexes. Suppose the following three conditions are satisfied:*

(1) *f is a bijection,*
(2) *f is continuous,*
(3) *the images of the cells of X are precisely the cells of Y.*

Then $f\colon X \to Y$ is a homeomorphism.

Proof $(*)$. We denote by $g = f^{-1}\colon Y \to X$ the inverse of f. It remains to show that g is continuous. Let $\{\Phi_i\colon \overline{B}^{m_i} \to X\}_{i\in I}$ be the characteristic maps of the CW-complex X and let $\{\Phi_j\colon \overline{B}^{n_j} \to Y\}_{j\in J}$ be the characteristic maps of the CW-complex Y. By Lemma 36.7 it suffices to show that for each $j \in J$ the map $g \circ \Psi_j\colon \overline{B}^{n_j} \to X$ is continuous. We consider the cell $\Psi_j(\overline{B}^{n_j})$. By hypothesis (3) there exists an $i \in I$ such that $f(\Phi_i(\overline{B}^{m_i})) = \Psi_j(\overline{B}^{n_j})$. Since $\Phi_i(\overline{B}^{m_i})$ is compact and since we know by Proposition 36.10 that $\Psi_j(\overline{B}^{n_j}) \subset Y$ is Hausdorff it follows from Proposition 2.43 (3) that $f\colon \Phi_i(\overline{B}^{m_i}) \to \Psi_j(\overline{B}^{n_j})$ is a homeomorphism. But this implies that $f^{-1} = g$ is continuous on $\Psi_j(\overline{B}^{n_j})$. It follows that, as desired, the map $g \circ \Psi_j$ is continuous. ■

We conclude this section with the following proposition.

Proposition 36.13. *Every* CW*-complex is paracompact.*

Proof. As always, when it comes to CW-complexes, the proof is slightly delicate. Since we will not really make use of the proposition we refer instead to [**FrPi90a**, Theorem 1.3.5], or alternatively to [**Hat2**, Proposition 1.20], [**Lee00**, Theorem 5.22], [**LW69**, Theorem II.4.2] and [**Miy52**], for a proof. ■

36.6. **The Finiteness Theorem for CW-complexes.** The following proposition is one of the key technical results about infinite CW-complexes.

Theorem 36.14. (Finiteness Theorem for CW-complexes) *Every compact* (*connected*) *subset of a* CW*-complex X is contained in a finite* (*connected*) *subcomplex.*

Remark. The statement of the proposition and also its proof have more than a passing resemblance with Lemma 36.3.

Example. We consider the wedge $X := \bigvee\limits_{i\in\mathbb{N}_0} S^1_i$. As discussed on page 942, we can view X as a CW-complex with one 0-cell, given by the wedge point, and 1-cells indexed by $i \in \mathbb{N}_0$. It follows from Theorem 36.14 that each compact subset of X is contained in the union of finitely many S^1_i. In fact this statement already follows from Exercise 18.33.

Before we start out with the proof of Theorem 36.14 we formulate the following pretty corollary.

Corollary 36.15. *A* CW*-complex is compact if and only if it is finite.*

Proof. In Proposition 36.10 (3) we showed that every finite CW-complex is compact. Now we want to prove the converse. Thus let X be a compact CW-complex. It follows from

Theorem 36.14 that X is contained in a finite subcomplex Y of X. But from $X \subset Y \subset X$ it follows that $X = Y$, i.e. X is a finite CW-complex. ■

The proof of Theorem 36.14 will be a fairly straightforward consequence of the following rather technical lemma.

Lemma 36.16. *Let X be a* CW*-complex.*

(1) *Let $A \subset X$ be a subset. If every open cell of X contains at most one point of A, then A is closed and discrete.*
(2) *If $C \subset X$ is a compact subset, then C is contained in a finite union of open cells.*
(3) *Every cell of X is contained in a finite subcomplex of X.*[592]

Remark. Lemma 36.16 (3) together with Proposition 36.10 (3a) and (3c) shows in particular that the *closure* of each open cell is contained in the union of *finitely* many open cells. This property is sometimes referred to as X being "closure finite" which explains the "C" in the name CW-complex.

Proof of Lemma 36.16 ($*$). We start out with the following claim.

Claim. Let Y be a CW-complex. Then the statements (1), (2) and (3) are equivalent.

Let Y be a CW-complex. First we show (1) $\Rightarrow$ (2). Let $C \subset Y$ be a compact subset. For each cell that intersects C we pick a single point. We denote the resulting set of points by A. By (1) the set A is closed and discrete. Since A is a closed subset of compact set we obtain from Lemma 2.17 (1) that A is compact. Since A is compact and discrete we obtain from Lemma 2.18 (1) that A is in fact finite. But that means that C intersects only finitely many cells, i.e. C is contained in a finite union of open cells as claimed.

Now we will prove (2) $\Rightarrow$ (3). So let $C_\sigma \subset Y$ be an n-cell. We denote by U_σ the corresponding open cell. By Proposition 36.10 (3) and by definition of a cell we know that $C_\sigma \setminus U_\sigma$ is a compact subset of the $(n-1)$-skeleton Y^{n-1}. It follows from (2) that the subset $C_\sigma \setminus U_\sigma$ is contained in a finite union of open cells of dimension $\leq n-1$. By the same argument each of these lower-dimensional closed cells is again contained in a finite union of open cells of smaller dimension, except again for the cell itself. A straightforward induction argument now shows that we end up with finitely many cells which form a subcomplex of Y.

Next we will show (3) $\Rightarrow$ (1). Let $A \subset Y$ be a subset such that no two points lie in the same open cell. We consider the intersection of A with a closed cell. By (3) this intersection is contained in a finite subcomplex Y'. Since A has at most one point in common with any open cell this intersection $A \cap Y'$ must be finite and hence closed in $A \cap Y'$ by Proposition 36.10 (2). By Proposition 36.10 (3) the finite subcomplex Y' is a closed subset of X. Hence $A \cap Y'$ is a closed subset of X. For any point $x \in A$ the set $A \setminus \{x\}$ also satisfies the hypothesis for A, and so it must also be a closed subset of X. Hence $\{x\}$ is open in A, so A is discrete. ⊞

Now let X be a CW-complex. First note that statement (1) holds for X^0 by definition of the topology on X^0. Now suppose we know (1) for the n-skeleton X^n for some $n \in \mathbb{N}_0$.

[592] Recall that by definition, see page 947, a subcomplex is *finite* if it contains only finitely many cells of X. So even a subcomplex that is given by a single cell might a priori contain many other cells.

By claim we know that (2) and (3) also for X^n. But the proof of (2) $\Rightarrow$ (3) for a given n-cell only used (2) for subsets of X^{n-1}. Thus we actually have (3) for X^{n+1}. Thus we also have (1) for X^{n+1}. Summarizing we have all three statements for X^n for all n. But any cell lies in some X^n, and so we know (3) for X itself, and by the claim this implies that (1) and (2) also hold for X. ■

Now we can provide the proof of Theorem 36.14.

Proof of Theorem 36.14. Let X be a CW-complex and let $C \subset X$ be a compact subset. By Lemma 36.16 (2) the subset C is contained in a union of finitely many open cells. By Lemma 36.16 (3) each of these open cells is contained in a finite subcomplex. The union of these finitely many finite subcomplexes is a finite subcomplex Z which contains C.
If C is connected, then we take the component of Z that contains C. By Lemma 36.21 (3) this is again a subcomplex. ■

36.7. **Cellular maps.** Before we move on to considering subcomplexes and constructions of CW-complexes let us quickly introduce the type of maps between CW-complexes that are reasonable.

Definition.
(1) A map $g\colon X \to Y$ between two CW-complexes is called *cellular* if for each $n \in \mathbb{N}_0$ we have $g(X^n) \subset Y^n$.
(2) A map $g\colon X \to Y$ between two CW-complexes is called a *cellular isomorphism* if f is cellular, if f is a bijection and continuous, and if f^{-1} is also cellular.

Example. In general maps between CW-complexes are not cellular. For example consider the CW-complex $X = [0,1]$ with two 0-cells and one 1-cell and we consider the CW-complex $Y = [0,1] \times [0,1]$ with four 0-cells, two 1-cells and one 2-cell. Then the diagonal map

$$\begin{array}{rcl} g\colon X = [0,1] & \to & Y = [0,1] \times [0,1] \\ t & \mapsto & (t,t) \end{array},$$

which is illustrated in Figure 610 is not a cellular map.

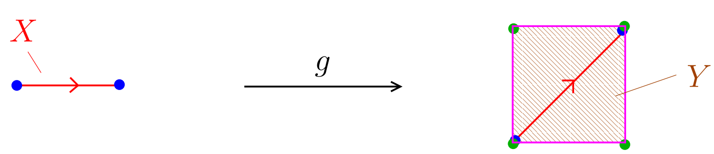


Figure 610

It is clear that the composition of two cellular maps is again cellular. This leads us to the following definition.

Definition. We call the category $\mathcal{CW}$ with

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{CW}) & := & \text{all CW-complexes}, \\ \mathrm{Mor}(X,Y) & := & \text{all } \textit{cellular} \text{ maps from } X \text{ to } Y, \end{array}$$

together with the usual composition of maps the *category of* CW-*complexes*. This category should not be confused with the *full category of* CW-*complexes* that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{FullCW}) &:= \text{all CW-complexes}, \\ \mathrm{Mor}(X,Y) &:= \text{all } \textit{continuous} \text{ maps from } X \text{ to } Y. \end{aligned}$$

Now is as good as any other moment to introduce a little more categorical language.

Definition. A covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ between two categories is called *faithful* if for every two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ the map $\mathrm{Mor}(X,Y) \to \mathrm{Mor}(F(X), F(Y))$ is an injection. If the map is in fact a bijection, then we call the functor *fully faithful*. The same definition applies, with the obvious modifications, to contravariant functors.

To practice the definitions we formulate the following lemma.

Lemma 36.17. *The maps*

$$\begin{aligned} \text{CW-}\textit{complex } X &\mapsto |X| \\ (f\colon X \to Y) &\mapsto (f\colon |X| \to |Y|) \end{aligned}$$

define a covariant functor from the category $\mathcal{CW}$ of CW-*complexes to the category of topological spaces. The functor is faithful but not fully faithful. The maps also define a covariant functor from the full category $\mathcal{FullCW}$ of* CW-*complexes to the category of topological spaces, this time the functor is fully faithful.*

Proof. The statements are basically obvious. The fact that we just found a continuous map which is not cellular gives us that the functor $\mathcal{CW} \to \mathcal{Top}$ is not fully faithful. ■

36.8. **Subcomplexes.** In this section we will discuss in great deal the concept of a subcomplex. We defined the notion on a subcomplex in a perfunctory way on page 947, but to give the concept its due prominence let us repeat the definition.

Definition. A *subcomplex* of a CW-complex X is a subset of X that is the union of arbitrarily many cells of X.

Examples.

(1) In Figure 611 we consider the smooth manifold X that is given by removing an open disk from a torus. We see that X has the structure of a CW-structure such that the boundary ∂X is a CW-subcomplex.

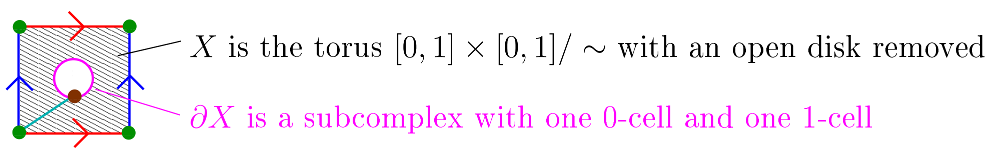


Figure 611

(2) Let X be a CW-complex. For each $n \in \mathbb{N}_0$ the n-skeleton X^n is a subcomplex.

The name "subcomplex" suggests that a subcomplex should again be a CW-complex. That is indeed the case.

Lemma 36.18. *Let X be a* CW*-complex and let A be a subcomplex of X.*

(1) *The subset A admits a* CW*-structure such that for each $n \in \mathbb{N}_0$ we have the equality $|A^n| = A \cap |X^n|$ and such that the characteristic maps of A correspond precisely to the characteristic maps of X that take values in A.*
(2) *The inclusion map $A \to X$ is a closed embedding, in particular A is a closed subset of X.*

Proof. We start out with the proof of Statement (1). We set $Y^{-1} = |Y^{-1}| := \varnothing$. For $n = 0, 1, 2, 3, \ldots$ we iteratively do the following:

(1) Let $\{\varphi_i \colon S^{n-1} \to |Y^{n-1}|\}_{i \in I}$ be the attaching maps from the CW-structure of X that correspond to n-cells that lie in A. We define $Y^n = (Y^{n-1}, \{\varphi_i \colon S^{n-1} \to |Y^{n-1}|\}_{i \in I})$.
(2) We set $|Y^n| := \Big(|Y^{n-1}| \sqcup \coprod_{i \in I} \overline{B}^n\Big)/\sim$.

Finally we define Y to be the CW-complex that is given by the sequence of CW-complexes $Y_{-1} \subset Y_0 \subset Y_1 \subset \ldots$.

Since A is by definition the union of cells we see that the obvious map $\varphi \colon |Y| \to A$ is a bijection. It remains to show that the map φ is actually a homeomorphism. Note that by definition, if $\Phi \colon \overline{B}^n \to |Y|$ is a characteristic map of the CW-complex Y, then $\varphi \circ \Phi \colon \overline{B}^n \to |X|$ is a characteristic map of the CW-complex X. Thus it follows from Lemma 36.7 (1) and (4) that φ is continuous.

By Lemma 2.42 (1) it remains to show that the map $\varphi \colon |Y| \to X$ is a closed map.

Claim. The map $\varphi \colon |Y| \to X$ is a closed map.

Lt $C \subset |Y|$ be a closed subset. We need to show that $\varphi(C)$ is also a closed subset of A. By Lemma 36.7 (1) and (3) it suffices to show that for each cell $e \subset X$ the set $\varphi(C) \cap e$ is closed. By Lemma 36.16 we know that e intersects only finitely many cells. In particular there are only finitely many cells contained in $A \cap e$. We denote these by $a_1, \ldots, a_m$. Now we see that

$$\begin{aligned}
\varphi(C) \cap e \underset{\substack{\uparrow \\ \text{since } \varphi(C) \subset A}}{=} \varphi(C) \cap (A \cap e) &\underset{\substack{\uparrow \\ \text{since } A \text{ is a subcomplex}}}{=} \varphi(C) \cap \bigcup_{i=1}^{m} a_i = \bigcup_{i=1}^{m} (\varphi(C) \cap a_i) \\
&\underset{\substack{\uparrow \\ \text{since } C \text{ is closed in } |Y| \text{ and since} \\ a_1, \ldots, a_m \in S \text{ are cells of } Y}}{=} \bigcup_{i=1}^{m} \text{closed subset of } a_i \underset{\substack{\uparrow \\ \text{by Lemma 2.38 (2) since the } a_i \\ \text{are closed subsets of } e}}{=} \bigcup_{i=1}^{m} \text{closed subset of } e = \text{closed subset of } e.
\end{aligned}$$

⊞

Now we turn to the proof of Statement (2). In fact this follows immediately from the claim. Indeed, evidently $|Y|$ is a closed subset of $|Y|$ and from the claim we obtain that $\varphi(|Y|)$ is a closed subset of X. But $\varphi(|Y|) = A$. Thus we have shown that A is a closed subset of X. ■

Definition. Let A and X be CW-complex. A map $f \colon A \to X$ is called a *cellular embedding* if f is injective and if the image of a cell of A is a cell of X.

Lemma 36.19. (∗) *Let $f\colon A \to X$ be a cellular embedding between two* CW*-complexes. The following statements hold:*

(1) *The image $f(A)$ is a subcomplex of A and the map $f\colon A \to f(A)$ is a cellular isomorphism.*
(2) *The map $f\colon A \to X$ is a closed embedding.*

Proof (∗). It follows immediately from the above definition of a subcomplex and of a cellular embedding that the image of a cellular embedding $f\colon A \to X$ is a subcomplex. Next we consider the map $f^{-1}\colon f(A) \to A$. We deduce from Lemma 36.7 together with Proposition 2.43 (3) that the map $f^{-1}\colon f(A) \to A$ is continuous. It follows almost immediately that $f\colon A \to f(A)$ is a cellular isomorphism. Finally we obtain from the above together with Lemma 36.18 that $f\colon A \to X$ is a closed embedding. ■

Lemma 36.20. *Let X be a* CW*-complex and let $\{X_i\}_{i\in I}$ be a family of subcomplexes such that $\bigcup\limits_{i\in I} X_i = X$. The following three statements hold:*

(1) *A subset $U \subset X$ is open if and only if each $X_i \cap U$ is an open subset of X_i.*
(2) *A subset $A \subset X$ is closed if and only if each $X_i \cap A$ is a closed subset of X_i.*
(3) *A map $f\colon X \to Y$ to a topological space Y is continuous if and only if each map $f|_{X_i}\colon X_i \to Y$ is continuous.*

Proof.

(1) This statement is an easy consequence of Lemma 36.7 (2).
(2) The "only if"-direction is an immediate consequence of Lemma 2.4 (2). The "if"-direction follows from (1) and Lemma 2.34 (1) ⇒ (2).
(3) The "only if"-direction is an immediate consequence of Lemma 2.30 (3). The "if"-direction follows from (1) and Lemma 2.34 (1) ⇒ (3). ■

The following lemma shows in particular that some straightforward constructions provide examples of subcomplexes.

Lemma 36.21. *Let X be a* CW*-complex.*

(1) *For any $n \in \mathbb{N}_0$ the n-skeleton X^n is a subcomplex of X.*
(2) *The union and also the intersection of arbitrarily many subcomplexes is again a subcomplex.*
(3) *Every path-component of a complex is again a subcomplex.*

Proof.

(1) This statement is clear.
(2) This statement is also clear.
(3) This statement is basically obvious since each closed ball is path-connected. ■

We conclude this section with the following proposition.

Proposition 36.22. *Let X be a CW-complex and let $Y = Y_0 \subset Y_1 \subset Y_2 \subset \ldots$ be a sequence of subcomplexes such that the following two conditions are satisfied:*

(1) *we have $X = \bigcup\limits_{i \in \mathbb{N}} Y_i$,*

(2) *each Y_i is a deformation retract of Y_{i+1}.*

Then Y_0 is a deformation retract of X.

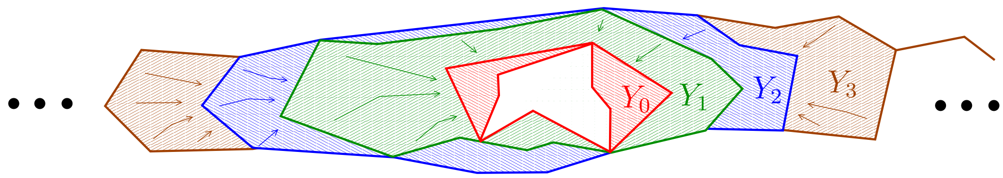


FIGURE 612. Illustration of Proposition 36.22.

PROOF (∗). By hypothesis there exists for each $k \in \mathbb{N}_0$ a deformation retraction

$$R(k)\colon Y_k \times [0,1] \ \to \ Y_{k-1}.$$

We denote by $r_k := R(k)_1\colon Y_k \to Y_{k-1}$ the corresponding retraction. For each $k \in \mathbb{N}_0$ we consider the map

$$\begin{aligned} f_k\colon X \times [0,1] &\ \to \ Y_{k-1} \times [0,1] \\ (x,t) &\ \mapsto \ \begin{cases} (r_k \circ \cdots \circ r_l)(x), & \text{if } x \in Y_l \text{ and } l \geq k, \\ x, & \text{if } x \in Y_{k-1}. \end{cases} \end{aligned}$$

It follows from Corollary 36.9 that this map f_k is continuous. Now we consider the following map

$$\begin{aligned} F\colon X \times [0,1] &\ \to \ X \\ (x,t) &\ \mapsto \ \begin{cases} x, & \text{if } x \in Y_k \text{ and } t \in [0, \frac{1}{2^{k+1}}], \\ r_k\big(x, 2^{k+1} \cdot (t - \frac{1}{2^{k+1}})\big), & \text{if } x \in Y_k \text{ and } t \in [\frac{1}{2^{k+1}}, \frac{1}{2^k}], \\ f_k(x), & \text{if } x \in Y_k \text{ and } t \in [\frac{1}{2^k}, 1]. \end{cases} \end{aligned}$$

We refer to Figure 613 for an illustration. It follows from Lemma 14.3 that the restriction of F to each $Y_k \times [0,1]$ is continuous. Thus it follows from Corollary 36.9 that $F\colon X \times [0,1] \to X$ is continuous. This map is the desired deformation retraction. ■

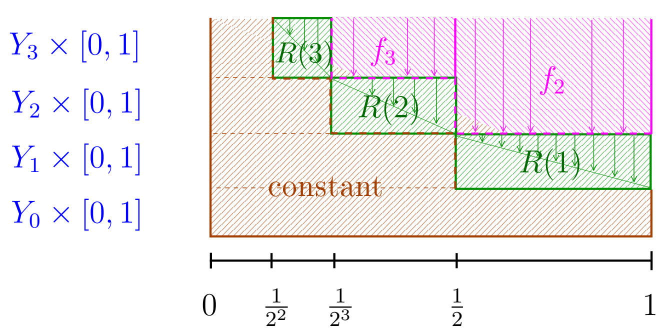


FIGURE 613. Illustration for the proof of Proposition 36.22.

36.9. **Products of CW-complexes I.** Let X and Y be two CW-complexes. We can form the product topological space $X \times Y$. Ideally one would like to equip $X \times Y$ with a CW-structure. Throughout this section we use the homeomorphism from Proposition 2.53 (2) to identify the n-dimensional closed ball $\overline{B}^n$ with the n-dimensional cube $I^n = [0,1]^n$ and we use it to make the identification $S^{n-1} = \partial I^n$. For $p + q = n$ we evidently have $I^n = I^p \times I^q$ and we have

$$\partial(I^n) \;=\; \partial I^p \times I^q \;\cup\; I^p \times \partial I^q.$$

The idea is now to define a CW-structure on $X \times Y$ such that the product of a p-cell in X with a q-cell in Y corresponds to a $(p+q)$-cell in $X \times Y$.

In the following we want to execute this idea rigorously. First, to be on the safe side, we consider the case of two finite complexes.

Definition. Let X and Y be two CW-complexes. Given $p \in \mathbb{N}_0$ let $\{\varphi_j \colon S^{p-1} \to X^{p-1}\}_{j \in J_p}$ and $K_p := \{\psi_k \colon S^{p-1} \to Y^{p-1}\}_{k \in K_p}$ be the set of attaching maps of p-dimensional cells of X and Y. Furthermore we denote by $\{\Phi_j \colon \overline{B}^p \to X^p\}_{j \in J_p}$ and $\{\Psi_k \colon \overline{B}^p \to Y^p\}_{k \in K_p}$ the corresponding characteristic maps of the cells. We define a new CW-complex Z as follows:

(1) We put $Z^{-1} = \varnothing$.
(2) Suppose Z^{n-1} is already defined for some $n \in \mathbb{N}_0$. Given $j \in J_p$ and $k \in K_{n-p}$ we define[593]

$$\begin{array}{rcl} \theta_{(j,k)} \colon \partial I^n = \partial I^p \times I^{n-p} \;\cup\; I^p \times \partial I^{n-p} & \to & Z^{n-1} \\ (x,y) & \mapsto & (\Phi_j(x), \Psi_k(y)). \end{array}$$

We set

$$Z^n \;:=\; \Big(Z^{n-1} \;\sqcup\; \bigsqcup_{p+q=n} \; \bigsqcup_{j \in J_p, k \in K_{n-p}} \big(\underbrace{I^p \times I^{n-p}}_{=\overline{B}^n}\big)_{(j,k)}\Big) / \sim$$

where for each $j \in J_p$, $k \in K_{n-p}$ we define $(x,y) \in (I^p \times I^{n-p})_{(j,k)}$ with the point $\theta_{(i,j)}(x,y) \in Z^{n-1}$.
(3) Given $j \in J_p$ and $k \in K_{n-p}$ we denote by $\Theta_{(j,k)} \colon I^n = I^p \times I^{n-p} \to Z^n$ the corresponding characteristic map.
(4) We set $Z := \varinjlim Z^n$.

As the discussion suggests, the above construction does indeed (often) define a CW-structure on the topological space $X \times Y$. To simplify the initial discussion we first consider the case that X and Y are two finite CW-complexes.

Proposition 36.23. *Let X and Y be two* CW*-complexes. We use the notation from the above definition.*

(1) *The above data turns Z into a* CW*-complex.*
(2) *The characteristic maps $\Theta_{(i,j)}$ define a continuous bijection $\Theta \colon Z \to X \times Y$.*
(3) *Suppose we are in one of the following two settings:*
   (a) *X and X are both finite* CW*-complexes, or

[593] Why does the image of this attaching map lie in $(X \times Y)^{n-1}$?

(b) *$Y = [0,1]$ where Y is equipped with the usual* CW*-structure with two 0-cells and one 1-cell,*

then the map $\Theta\colon Z \to X \times Y$ from (2) is a homeomorphism. In particular the topological space $X \times Y$ has a CW*-structure such that for each $n \in \mathbb{N}_0$ we have*

$$\underbrace{(X \times Y)^n}_{n\text{-skeleton of } X\times Y} = \bigcup_{p+q=n} X^p \times Y^q.$$

Definition. Let X and Y be finite CW-complexes. Suppose we are in one of the two settings of Proposition 36.23 (3).

(1) We refer to the above CW-structure on $X \times Y$ as the *product* CW*-structure*.
(2) Given a p-cell e for X and given a q-cell for Y we denote by $e \times f$ the corresponding $(p+q)$-cell for $X \times Y$.

Remark. The second case of Proposition 36.23 (3) might appear somewhat random, but it will actually prove important for showing that several constructions, like mapping cylinders and mapping cones, give rise to CW-complexes.

Proof (∗). Let X and Y be two CW-complexes.

(1) This statement follows basically immediately from the definitions.
(2) It follows from Lemma 36.7 (4) that the map $\Theta\colon Z \to X \times Y$, which is defined via the characteristic maps $\Theta_{(i,j)}$, is continuous.
(3) (a) Suppose that X and X are both finite CW-complexes. In this case Z has only finitely many cells, hence Z is compact by Corollary 36.15. Furthermore note that $X \times Y$ is Hausdorff by Proposition 36.10 (2) and Proposition 3.12. It follows from Proposition 2.43 (3) that $\Theta\colon Z \to X \times Y$ is indeed a homeomorphism.
(b) Suppose we have $Y = [0,1]$ with the obvious CW-structure. It follows basically immediately from the equivalence of the topologies (1) and (3) of Lemma 36.8 that the map $\Theta\colon Z \to X \times Y$ is a homeomorphism.

Since $\Theta\colon Z \to X \times Y$ is a homeomorphism and since Z is a CW-complex we have endowed $X \times Y$ with a CW-structure. The statement about the skeleta follows immediately from the definition of the CW-structure on Z. ■

Examples.

(1) If we equip $S^1 = [0,1]/0 \sim 1$ with the usual CW-structure with one 0-cell and one 1-cell, then the product CW-structure on the torus $S^1 \times S^1 = ([0,1] \times [0,1])/\sim$ is precisely the CW-structure that we had already encountered on page 933. We refer to Figure 614 for an illustration.

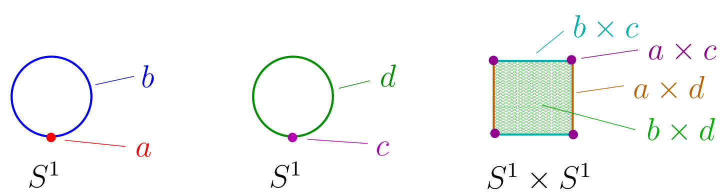


Figure 614

(2) Given $a = s_0 < s_1 < \cdots < s_k = b$ and $c = t_0 < t_1 < \cdots < t_l = d$ we can view the intervals $[a, b]$ and $[c, d]$ as CW-complexes where the 1-cells are given precisely by the intervals $[s_i, s_{i+1}]$, $i = 0, \dots, k-1$ and $[t_j, t_{j+1}]$, $j = 0, \dots, l-1$. The product CW-complex is the topological space $[a, b] \times [c, d]$ where the 2-cells are given by the rectangles $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$ with $i \in \{0, \dots, k-1\}$ and $j \in \{0, \dots, l-1\}$. The analogous statement holds also for higher-dimensional hyperrectangles. For example, given $k \in \mathbb{N}$ and $n \in \mathbb{N}$ we can equip the cube $[0, 1]^n$ with a CW-structure where the n-cells are precisely given by the cubes $[\frac{s_1}{k}, \frac{s_1+1}{k}] \times \cdots \times [\frac{s_n}{k}, \frac{s_n+1}{k}]$ with $s_1, \dots, s_n \in \{0, \dots, k-1\}$.

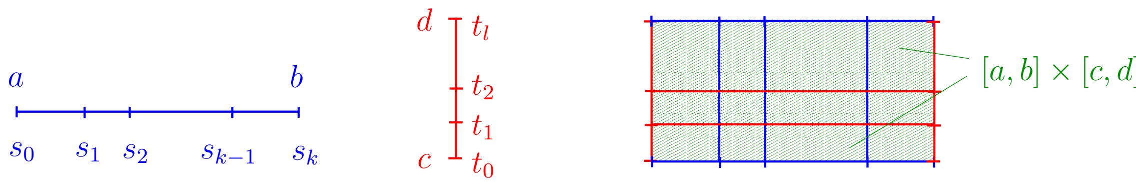


FIGURE 615

The formulation of Proposition 36.23 already suggests that the conclusion does not hold without some assumptions on X and Y. In fact, the following lemma, which goes back to [**Dow52**], shows that there are indeed examples of CW-complexes X and Y such that the map $\Theta\colon Z \to X \times Y$ is *not* a homeomorphism. In other words, in general the "naive" attempt to endow a product of CW-complexes with a CW-structure fails.

Lemma 36.24. *Let $S := \{(s_1, s_2, \dots) \,|\, s_i \in \mathbb{N}\}$ be the set of all infinite sequences of positive integers. Let $X = \bigvee\limits_{i \in [0,1]} [0, 1]_s$ be the wedge of uncountably many copies of $[0, 1]$, obtained by wedging all the $0 \in [0, 1]_s$. Similarly we form the wedge $Y = \bigvee\limits_{j \in \mathbb{N}} [0, 1]_j$ of countably many copies of $[0, 1]$. We equip X and Y with the "obvious CW-structure", see e.g. Lemma 36.32 (4). In this setting the map $\Theta\colon Z \to X \times Y$ from Proposition 36.23 is* not *a homeomorphism.*

PROOF. For $s \in S$ and $j \in \mathbb{N}$ we consider $(\frac{1}{s_j}, \frac{1}{s_j}) \in [0, 1]_s \times [0, 1]_j \in X \times Y$. We denote by $P \subset X \times Y$ the subset defined by all these points. Since each 2-cell of the CW-complex Z contains a single point of P we obtain from Lemma 36.7 (3) that P is a closed subset of Z. To show that Θ is not a homeomorphism it remains to prove the following claim.

Claim. The set P is not closed with respect to the product topology on $X \times Y$.

We denote by $x_0 \in X$ and $y_0 \in Y$ the wedge points. Note that $(x_0, y_0) \notin P$. Thus it suffices to show that every neighborhood of $(x_0, y_0) \in X \times Y$ contains a point of P. In fact, by definition of the product topology, see page 162, it suffices to show that any subset $U \times V$ of $X \times Y$ of the form $U = \bigvee\limits_{s \in S} [0, a_s] \subset X$ and $V = \bigvee\limits_{j \in \mathbb{N}} [0, b_j] \subset Y$ contains a point of P. We pick a sequence $t = (t_1, t_2, \dots) \in S$ such that for every $j \in \mathbb{N}$ we have $t_j > j$ and such that $t_j > \frac{1}{b_j}$. Furthermore we pick a $k \in \mathbb{N}$ with $k > \frac{1}{a_t}$. It follows that $t_k > k > \frac{1}{a_t}$, which implies that $\frac{1}{t_k} < a_t$. Recall that by construction we also have $\frac{1}{t_k} < b_k$. Thus we see that $(\frac{1}{t_k}, \frac{1}{t_k})$ is a point of P that lies in $[0, a_t] \times [0, b_k)$. Hence it also lies in $U \times V$. ■

36.10. **Products of CW-complexes II** $(*)$**.** In this section we will discuss the product of two infinite CW-complexes in somewhat greater detail. We will not make use of the results of this and the coming section, thus these two sections can be skipped safely.

First we mention the following proposition which can be viewed as a generalization of Proposition 36.23.

Proposition 36.25. *Let X and Y be two* CW*-complexes. If at least one of the two* CW*-complexes has only countably many cells, then the map $\Theta\colon Z \to X \times Y$ from Proposition 36.23 (2) is a homeomorphism, in particular $X \times Y$ has a natural* CW*-structure.*

Proof. This statement is proved in [**LW69**, Theorem II.5.2]. If one of X or Y is a finite CW-complex, then the statement is also proved in [**Hat02**, Theorem A.6]. If both CW-complexes have countably many cells, then the statement is also proved in [**FrPi90a**, Proposition 2.2.3] or [**Miln56b**, Lemma 2.1]. ■

Definition. Let X and Y be two CW-complexes.

(1) If at least one of the two CW-complexes has only countably many cells, then we refer to the above CW-structure on $X \times Y$ as the *product* CW*-structure*.
(2) Following [**LW69**, p. 56] we denote by $X \otimes Y$ the topological space whose underlying set is given by $X \times Y$ but whose topology is the unique topology that turns the map $\Theta\colon Z \to X \otimes Y$ from Proposition 36.23 into a homeomorphism.

Remark. Note that [**Brooke17**, Theorem 1] gives a necessary and sufficient condition for the product of two CW-complexes to be a CW-complex.

We conclude this section with a short discussion on the relationship between $X \otimes Y$ and $X \times Y$.

Proposition 36.26. $(*)$ *Let X and Y be two* CW*-complexes.*

(1) *The identity map* $\mathrm{id}\colon X \otimes Y \to X \times Y$ *is a continuous bijection.*
(2) *A subset $K \subset X \otimes Y$ is compact if and only if it is compact in $X \times Y$.*
(3) *Given any $(x, y) \in X \times Y$ the induced map $\pi_1(X \otimes Y, (x, y)) \to \pi_1(X \times Y, (x, y))$ is an isomorphism.*
(4) *If at least one of the two* CW*-complexes has only countably many cells, then the map $X \otimes Y \to X \times Y$ is a homeomorphism.*

The proof of Proposition 36.26 (3) is mostly outsourced to the following lemma.

Lemma 36.27. $(*)$ *Let $\Theta\colon A \to B$ be a continuous bijection between two topological spaces which has the property that a subset $K \subset A$ is compact if and only if $\Theta(K) \subset B$ is compact. If A is Hausdorff, then for any $a_0 \in A$ the map $\Theta_*\colon \pi_1(A, a_0) \to \pi_1(B, f(a_0))$ is an isomorphism.*

Proof $(*)$. We write $b_0 := f(a_0)$. The proof of the lemma naturally breaks up into showing that $\Theta_*\colon \pi_1(A, a_0) \to \pi_1(B, b_0)$ is an epimorphism and that it is a monomorphism.

Let us first show that the map Θ_* is an epimorphism. Suppose we are given an element $[f\colon (S^1,*) \to (B,b_0)] \in \pi_1(B,b_0)$. By Lemma 2.40 we know that $L := f(S^1)$ is a compact subset of B. We set $K := \Theta^{-1}(L)$. Since Θ is a bijection we have $\Theta(K) = L$. It follows from our hypothesis that K is a compact subset of A. Since Θ is a bijection, since K is compact and since L is Hausdorff we obtain from Proposition 2.43 (3) that $\Psi := \Theta|_K\colon K \to L$ is a homeomorphism. In particular the inverse $\Psi^{-1}\colon L \to K$ is continuous. Next we set $g := \Psi^{-1} \circ f\colon (S^1,*) \to (A,a_0)$. We have $\Theta_*([g]) = [\Theta \circ g] = [\Theta|_K \circ \Psi^{-1} \circ f] = [f]$. Thus we have shown that Θ_* is an epimorphism.

The proof that Θ_* is a monomorphism is quite similar. The key observation is that the image of a homotopy in B between two loops is a compact subset. We leave it to the reader to fill in the few remaining details. ■

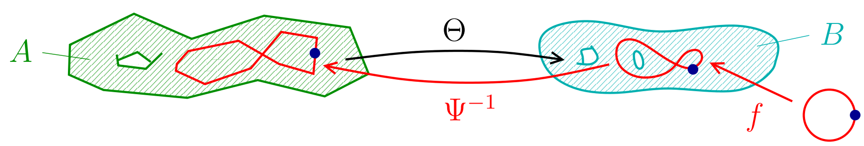


FIGURE 616. Illustration for the proof of Lemma 36.27.

PROOF OF PROPOSITION 36.26 (∗). Let X and Y be two CW-complexes.

(1) It follows immediately from Proposition 36.23 (2) and the definition of the topological space $X \otimes Y$ that the identity map $\mathrm{id}\colon X \otimes Y \to X \times Y$ is a continuous bijection.
(2) Let $K \subset X \otimes Y$. If K is compact in $X \otimes Y$. It follows from (1) and Lemma 2.40 that K is also compact in $X \times Y$. Now suppose conversely that K is compact in the product topological space $X \times Y$.
   (a) Note that $X \times Y$ is Hausdorff by Proposition 36.10 (2) and Proposition 3.12. Thus it follows from Lemma 2.17 (2) that K is a closed subset of $X \times Y$. It follows from (1) that K is closed in $X \otimes Y$.
   (b) We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the obvious projections. It follows from Lemma 2.40 that $p(K)$ and $q(K)$ are compact. By Theorem 36.14 we know that there exist finite subcomplexes $X' \subset X$ and $Y' \subset Y$ such that $p(K) \subset X'$ and $q(K) \subset Y'$. But this implies that K is contained in the finite subcomplex $X' \otimes Y'$ of $X \otimes Y$. By Theorem 36.14 we know that $X' \otimes Y'$ is compact. Together with (a) we see that K is a closed subset of $X \otimes Y$ that is contained in a compact subset $X' \otimes Y'$. It follows from Lemma 2.17 (1) that K itself is compact.
(3) This statement follows immediately from (1) and (2), the fact that $X \times Y$ is Hausdorff and Lemma 36.27.
(4) This statement is just a reformulation of Proposition 36.23 (3). ■

36.11. **A convenient category of topological spaces** (∗)**.** Let X and Y be two CW-complexes. We can equip the set $X \times Y := \{(x,y) \,|\, x \in X \text{ and } y \in Y\}$ with two different topologies, namely the familiar product topology from page 162 and the less familiar topology $X \otimes Y$ from the previous section. As we saw in Lemma 36.24, in general these two topologies are different.

This is quite a nuisance. In this section we will discuss one approach to addressing this issue. We will not make much use of the ideas discussed in this section, thus the reader can safely move on to the next section.

Let us first reflect about what "product" is supposed mean in the first place. After contemplating on the concept of "product" for a while one will surely arrive at the following definition.

Definition. Let $\mathcal{C}$ be a category and let $X, Y \in \mathrm{Ob}(\mathcal{C})$ be two objects. A *product of X and Y in the category $\mathcal{C}$* is some object $P \in \mathrm{Ob}(\mathcal{C})$ together with two morphisms $p\colon P \to X$ and $q\colon P \to Y$ such that the following property is satisfied: Whenever we are given two morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ there exists a unique morphism $\Phi\colon Z \to P$ such that $f = p \circ \Phi$ and $g = q \circ \Phi$. The situation is summarized in the following diagram

$$\begin{tikzcd} & & X \\ Z \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] \arrow[r, dashed, "\Phi"] & P \arrow[ur, "p"] \arrow[dr, "q"] & \\ & & Y. \end{tikzcd}$$

Sometimes we refer to P also as the *categorical product* of X and Y if we want to distinguish it from the usual set-theoretic product.

Remark. The definition of the product can be formulated in terms of inverse limits. More precisely, we consider the set $I = \{*, \star\}$ with the trivial preorder. An inverse system over I in a category $\mathcal{C}$ is just the data of two objects X and Y in $\mathcal{C}$. The corresponding inverse limit, if it exists, is *by definition*[594] precisely a product of X and Y in $\mathcal{C}$.

The usual "formal nonsense" shows that the product, if it exists, is unique up to a canonical isomorphism. Thus we will use usual parlance and we will speak of *the product*, if it exists.

Examples.

(1) Let $\mathcal{S}et$ be the category of sets and let $X, Y \in \mathrm{Ob}(\mathcal{S}et)$. One can easily verify that the product is given by the set $X \times Y := \{(x, y) \,|\, x \in X \text{ and } y \in Y\}$ together with the obvious maps $X \times Y \to X$ and $X \times Y \to Y$.
(2) Let $\mathcal{T}op$ be the category of topological spaces and let $(X, \mathcal{A}), (Y, \mathcal{B}) \in \mathrm{Ob}(\mathcal{T}op)$. Let $\mathcal{P}$ be the product topology on the set $X \times Y$ that we defined on page 162. It follows almost immediately from Lemma 3.6 (2) that in the category of topological space the product is given by the topological space $(X \times Y, \mathcal{P})$ together with the obvious maps $X \times Y \to X$ and $X \times Y \to Y$.
(3) Let $\mathcal{PT}op$ be the category of pointed topological spaces that we introduced on page 476. Furthermore let $(X, x_0), (Y, y_0) \in \mathrm{Ob}(\mathcal{PT}op)$ together with the obvious maps. In Exercise 36.12 we will see that the product in $\mathcal{PT}op$ is given by the smash product $(X, x_0) \wedge (Y, y_0)$ that we introduced on page 577, together with the obvious morphisms $(X, x_0) \wedge (Y, y_0) \to (X, x_0)$ and $(X, x_0) \wedge (Y, y_0) \to (Y, y_0)$.

[594] We refer to page 745 for the definition of an inverse limit.

The following theorem shows that if we move from considering all topological spaces to a slightly more constrained setup, then we can combine the two concepts of products that we introduced for CW-complexes.

Theorem 36.28. *There exists a full subcategory $\mathcal{C}$ of the category of topological spaces with the following properties:*

(1) *Every regionally compact Hausdorff space is an object of $\mathcal{C}$.*
(2) *Every* CW*-complex is an object of $\mathcal{C}$.*
(3) *Every direct system in $\mathcal{C}$ has a direct limit and every inverse system in $\mathcal{C}$ has an inverse limit.*
(4) *If X and Y are two* CW*-complexes, then the* CW*-complex $X \otimes Y$ is in fact the product in $\mathcal{C}$.*

Remark. A category as in Theorem 36.28 is often called a *convenient category*. Note though that the notion of a convenient category might be used somewhat differently by different authors. In fact there are many "convenient categories", possibly with somewhat different properties from the ones we stated. We refer to [**Stee67, Miln59, Spa63, VogR71, McCor67, Stri, Pre02, BrownR63**] and [**Stro11**, Chapter 3.4] for more examples and for the discussion of various notions of a "convenient category".

We will outline the proof of Theorem 36.28 by giving an explicit example of a convenient category. We will need the following definitions.

Definition.

(1) A subset A of a topological space X is called *compactly closed* if for every map $\varphi\colon K \to X$, where K is a compact Hausdorff space, the preimage $\varphi^{-1}(A)$ is a closed subset of K.[595]
(2) A topological space X is called *compactly generated* if every compactly closed subset of X is actually closed.[596]
(3) A topological space X is called *weakly Hausdorff* if for every map $\varphi\colon K \to X$, where K is a compact Hausdorff space, the image $\varphi(K)$ is closed.[597]
(4) A topological space X is called a CGWH-*space*[598] if it is compactly generated and if it is weakly Hausdorff.

First, for the readers who like weird topological spaces we give some non-examples.

Examples.

(1) We equip the set $X = (\mathbb{N} \times \mathbb{N}) \cup \{*\}$ with the topology where $U \subset X$ is open if one of the following holds:
   (a) the point $*$ is contained in U, or
   (b) for all but finitely many $n \in \mathbb{N}$ the set U contains all but finitely many points in $\{n\} \times \mathbb{N}$.

[595] It follows from Lemma 2.32 that every closed subset of X is actually compactly closed.

[596] Sometimes in the literature a compactly generated topological space is also called a *k-space*.

[597] By Lemma 2.17 (2) we know that a topological space that is Hausdorff is also, as it should be, weakly Hausdorff.

[598] Evidently the acronym CGWH stands for Compactly Generated and Weakly Hausdorff.

In Exercise 36.13 we will see that X is not compactly generated.

(2) We consider $X = \mathbb{R} \setminus \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ and we consider $Y = \mathbb{R}/Z$ where $Z \subset \mathbb{R}$ is the subset given by integers.[599] In Exercise 36.16 we will see that the product topological space $X \times Y$ is not compactly generated.

(3) In Exercise 36.17 we will give an example of a topological space that is weakly Hausdorff but that is not Hausdorff.

We proceed with an actual example of a CGWH-space.

Proposition 36.29. *Every* CW*-complex is a* CGWH*-space.*

Proof. Let X be a CW-complex.

(1) Let A be a compactly closed subset of X. It follows easily from Lemma 36.7 (3) that A is in fact a closed subset of X.

(2) By Proposition 36.10 (2) we know that X is Hausdorff. It follows from Lemma 2.17 (2) that X is in fact weakly Hausdorff. ■

We continue with the following simple-minded definition.

Definition. The *category $\mathcal{CGWH}$ of* CGWH*-spaces* is the full subcategory of the category of topological spaces whose objects are precisely the CGWH-spaces.

Remark. The category of CGWH-spaces was introduced by Michael McCord [**McCor69**], building on work by Norman Steenrod [**Stee67**].

The following theorem shows that the category of CGWH-spaces has many charming properties. The theorem provides us in particular with a proof of Theorem 36.28.

Theorem 36.30. *The category $\mathcal{CGWH}$ of* CGWH*-spaces has the following properties:*

(0) *The category $\mathcal{CGWH}$ is a full subcategory of the category of topological spaces.*

(1) *Every regionally compact Hausdorff space is an object of $\mathcal{CGWH}$.*

(2) *Every* CW*-complex is an object of $\mathcal{CGWH}$.*

(3) *Every direct system in $\mathcal{CGWH}$ has a direct limit and every inverse system in $\mathcal{CGWH}$ has an inverse limit.*

(4) *Given any two objects X and Y in $\mathcal{CGWH}$ there exists a product in the category $\mathcal{CGWH}$.*

(5) *Let X and Y be two objects in $\mathcal{CGWH}$. The following statements hold:*
   (a) *The product of X and Y in $\mathcal{CGWH}$ is given by the set $X \times Y$ together with a topology $\mathcal{T}$ that contains the usual product topology $\mathcal{P}$. In particular the identity map*
   $$(X \times Y, \mathcal{T}) \ \to \ (X \times Y, \mathcal{P})$$
   is a bijection and continuous.
   (b) *If X and Y are compact and Hausdorff, then $\mathcal{T} = \mathcal{P}$, i.e. the map in* (a) *is a homeomorphism.*

[599] We use this cumbersome notation to make sure that we do not confuse the quotient space $\mathbb{R}/Z$ with the quotient of $\mathbb{R}$ by the obvious action of the group $(\mathbb{Z}, +)$ on the topological space $\mathbb{R}$.

(c) *If X and Y are two* CW*-complexes, then the identity map $(X \times Y, \mathcal{T}) \to X \otimes Y$ is a homeomorphism.*

Remark. Let X and Y be two CGWH-spaces. We denote by $X \widetilde{\times} Y$ the categorical product of X and Y in the category of CGWH-spaces. Theorem 36.30 (4) can be reformulated as saying that we have a continuous bijection $X \widetilde{\times} Y \to X \times Y$. Furthermore, if X and Y are two CW-complexes, then the identity map $X \widetilde{\times} Y \to X \otimes Y$ is a homeomorphism.

Sketch of proof. Most of the statements below were first proved by Michael McCord [**McCor69**], building on work by Norman Steenrod [**Stee67**]. Arguably the best account of the proof is given in [**Stri**]. In the following we show that our theorem, as formulated, can be deduced from the statements in [**Stri**].

(0) This statement holds by definition.
(1) [**Stri**, Proposition 1.7] says that every regionally compact Hausdorff space is in fact a CGWH-space.
(2) In Proposition 36.29 we just saw that every CW-complex is a CGWH-space.
(3) In [**Stri**, Propositions 2.23 and 2.30] it is shown that every direct system and every inverse system in the category of CGWH-spaces actually has a limit in the category of CGWH-spaces.
(4) As we discussed on page 965, this is just a special case of (3).
(5) Let X and Y be two topological spaces. Let $\mathcal{T}$ be the usual product topology on $X \times Y$. Furthermore let $\mathcal{P}$ be the topology on $X \times Y$ that is generated by all compactly closed subsets of $(X \times Y, \mathcal{T})$. By Lemma 2.32 we know that every closed subset is compactly closed, thus we see that $\mathcal{T} \subset \mathcal{P}$. It follows from [**Stri**, Proposition 2.4 and Corollary 2.16] that $(X \times Y, \mathcal{P})$ is the product in the category of CGWH-spaces.
   (a) This statement follows immediately from the above discussion.
   (b) Now suppose that X and Y are compact and Hausdorff. By Proposition 3.12 we know that the product topological space $X \times Y$ is also compact and Hausdorff. But it follows almost immediately from the definitions that in a compact Hausdorff space a subset is compactly closed if and only if it is closed. Thus we see that $\mathcal{P} = \mathcal{T}$.
   (c) Now let X and Y be CW-complexes. We denote by $\mathcal{X}$ the topology of $X \widetilde{\times} Y$ that we introduced on page 963. We need to show that $\mathcal{P} = \mathcal{X}$. We make the following two observations:
      (i) By definition a subset $A \subset X \times Y$ is closed in $\mathcal{P}$ if and only if the intersection of A with each compact subset of $(X \times Y, \mathcal{T})$ is closed.
      (ii) By definition a subset $A \subset X \times Y$ is closed in $\mathcal{X}$ if and only if the intersection of A with each cell of the CW-complex $X \otimes Y$ is closed.

   Using Lemma 36.7 (3), Theorem 36.14 and Exercise 3.14 one can show that $\mathcal{P}$ and $\mathcal{X}$ have the same closed subsets. We leave it to the reader to fill in the details. ■

Remark. Given CGWH-spaces X and Y it is also convenient to equip X^Y with a slight modification of the usual compact-open topology. We refer to [**Stri**, Proposition 2.24] for details.

We conclude this section with the following proposition that gives us three other convenient properties of the category of CGWH-spaces. The reader who has struggled with the intricacies of Theorem 5.16 and Proposition 5.23 will surely appreciate these statements.

Proposition 36.31.

(1) *Let $f\colon X \to Y$ be a bijection between two topological spaces. If f is proper, i.e. if the preimage of every compact subset of Y is a compact subset of X, and if X and Y are* CGWH*-spaces, then f is actually a homeomorphism.*
(2) *Let $f\colon W \to X$ and $g\colon Y \to Z$ be quotient maps. If W, X, Y and Z are* CGWH*-spaces, then $f \times g\colon W \times Y \to X \times Z$ is also a quotient map.*
(3) *Let X and Y be topological spaces and let $B \subset Y$ be a closed subset. If X and Y are* CGWH*-spaces, then the obvious map $(X \times Y)/(X \times B) \to X \times (Y/B)$ is a homeomorphism.*

PROOF. This proposition is proved in [**Stri**, Propositions 2.20, 2.40 and 3.17]. ∎

Remark. The category of CGWH-spaces admits a fully faithfully embedding into the category of “quasiseparated condensed sets”. The category of “quasiseparated condensed sets” is particularly well-suited for dealing with technical problems that arise in topology. We refer to [**Scho19**, Theorem 2.16] and [**Scho19b**, Proposition 1.2] for details.

36.12. **Constructions of more CW-complexes.** In this section we will see how one can build new CW-complexes out of given CW-complexes. We start out with the following lemma.

Lemma 36.32.

(1) *Let $\{X_i\}_{i\in I}$ be a family of* CW*-complexes. The disjoint union $\coprod_{i\in I} X_i$ admits a natural* CW*-structure such that each X_i is a subcomplex.*
(2) *Let X and Y be* CW*-complexes, let A be a subcomplex of X and let $f\colon A \to Y$ be a cellular map. As on page 197 we consider the topological pushout:*
$$X \cup_A Y \ := \ (X \sqcup Y)/\sim \qquad \text{where } a \sim f(a) \text{ for } a \in A$$
Furthermore we consider the following pushout diagram:
$$\begin{array}{ccc} A & \hookrightarrow & X \\ \downarrow f & & \downarrow x\mapsto[x] \\ Y & \xrightarrow{y\mapsto[y]} & X \cup_A Y. \end{array}$$
The following statements hold:[600]
  (a) *The map pushout $X \cup_A Y$ admits a natural* CW*-structure.*
  (b) *The map $Y \to X \cup_A Y$ is a cellular embedding and for any subcomplex B of X that is disjoint from A, the map $B \to X \cup_A Y$ is also a cellular embedding.*
  (c) *If $f\colon A \to Y$ is cellular embedding, then the opposite map $X \to X \cup_A Y$ is also a cellular embedding.*
(3) *Let X be a* CW*-complex and let A be a subcomplex. The quotient X/A admits a natural* CW*-structure such that the projection map $X \to X/A$ is cellular and where*

the cells of X/A are in one-to-one correspondence to the cells of X that do not entirely lie in A together with one 0-cell given by the point $[A]$.[601]

(4) *Let $\{X_i\}_{i\in I}$ be a family of* CW*-complexes and suppose that for each $i \in I$ we are given a point $a_i \in X_i^0$. If we use these points to form the wedge $\bigvee_{i\in I} X_i$, then the wedge admits a natural* CW*-complex structure such that each X_i is a subcomplex.*

(5) (∗) *Let (X, x_0) and (Y, y_0) be pointed* CW*-complexes. If at least one of X or Y has only countably many cells, then the smash product*

$$X \wedge Y \;=\; (X \times Y)/((X \times \{y_0\}) \cup (\{x_0\} \times Y)),$$

as defined in Exercise 18.30, has a natural CW*-structure.*

(6) (∗) *Let X be a* CW*-complex, let A be a complex of X and let $f\colon A \to X$ be a cellular map with the following properties:*

(a) *For each n-cell σ of A we have either $f(\sigma) \subset X^{n-1}$ or $f|_\sigma$ is a homeomorphism.*

(b) *The image of f is contained in a subcomplex of X that is disjoint from A.*

Then $X/\sim$, where $a \sim f(a)$ for all $a \in A$, admits a natural CW*-structure such that for any subcomplex B that is disjoint from $f(A)$ the natural map $B \to X/\sim$ is a cellular embedding.*

(*We point out that by Lemma 36.19 we know that the image of a cellular embedding is in fact a cellular homeomorphism onto a subcomplex and we will often use this cellular isomorphism to identify the domain with its image.*)

Convention. When we form the wedge of a family of a CW-complexes, then we always use a point that lies in the 0-skeleton, so that we can view the wedge in a natural way as a CW-complex.

Proof (∗).

(1) The first statement is basically obvious.

(2) First note that it is pretty clear what the CW-structure on $X \cup_A Y$ should be. The problem that arises is that one needs to show that the topology of the new CW-complex agrees in fact with the intended topology. If X (and Y) are finite CW-complexes then this follows easily from Proposition 2.43 (3). The general case gets more technical. We refer to [**LW69**, p. 60 and Theorem 5.11] or alternatively to [**FrPi90a**, Theorem 2.31 on p. 62] for complete proofs. The remaining two statements follow easily from the actual definition of the CW-structure on $X \cup_A Y$.

(3) Note that this statement is the special case of (2) where we consider the constant map $f\colon A \to Y = \{*\}$ where $Y = \{*\}$ is a topological space consisting of a single point.

(4) This follows easily from (1) and (2).

[600] The statements below should be viewed as an extension of the discussion of the properties of the pushout that we gave in Lemma 3.43.

[601] It is is worth considering the case that $A = \varnothing$ is the empty subcomplex. By the definition on page 179 we know that $X/A = X/\varnothing$ equals the disjoint union of X with the topological space $\{\varnothing\}$ consisting of a single point. The above definition of the CW-structure on X/A still makes, namely it is given by the disjoint union of the CW-complex with $\{\varnothing\}$ viewed as a CW-complex in the only possible way.

(5) This statement follows immediately from Proposition 36.23 (3) and (2).
(6) This statement follows from [**LW69**, Proposition 5.8]. We can apply [**LW69**, Proposition 5.8] since our hypothesis on f ensures that f defines a "cellular equivalence relation" in the sense of [**LW69**, p. 32]. ■

Later on we will make good use of the following lemma.

Lemma 36.33. *Let X be a* CW*-complex and let $n \in \mathbb{N}$. Let $\{\varphi_i \colon \overline{B}^{n+1} \to X^n\}_{i \in I}$ be the characteristic maps of the $(n+1)$-cells of X. For each $i \in I$ let $\overline{B}_i^{n+1}$ be a copy of $\overline{B}^{n+1}$ and let S_i^n be a copy of S^n. We write $S_i^n = \partial \overline{B}_i^{n+1}$.*

(1) *The map*
$$\begin{array}{rcll} f \colon \bigvee\limits_{i \in I} \overbrace{(\overline{B}_i^{n+1}/S_i^n)}^{\text{homeomorphic to } S^{n+1}\text{, see page 182}} & \to & X^{n+1}/X^n & \\ {[x]} & \mapsto & [\varphi_i(x)] & \text{if } x \in \overline{B}_i^{n+1} \end{array}$$
is a homeomorphism.

(2) *If X has only one 0-cell $\{x_0\}$ and no cells in dimensions $1, \dots, n$, then the homeomorphism f, together with the homeomorphism $\overline{B}^{n+1}/S^n \to S^{n+1}$ from page 182 induces a homeomorphism*
$$\bigvee_{i \in I} S_i^{n+1} \to X^{n+1}.$$

(3) *For every $i \in I$ the subset $X^{n+1} \setminus \Phi_i(B_i^{n+1})$ is a subcomplex of X^{n+1} and the map*
$$\begin{array}{rcl} \overline{B}_i^{n+1}/S_i^n & \to & X^{n+1}/(X^{n+1} \setminus \Phi_i(B_i^{n+1})) \\ {[x]} & \mapsto & [\Phi_i(x)] \end{array}$$
is a homeomorphism.

Example. In Figure 617 we see the torus together with a CW-structure X that has two 2-dimensional cells. It follows from Lemma 36.33 (1) that $X/X^1 = X^2/X^1$ is homeomorphic to the wedge of two spheres.

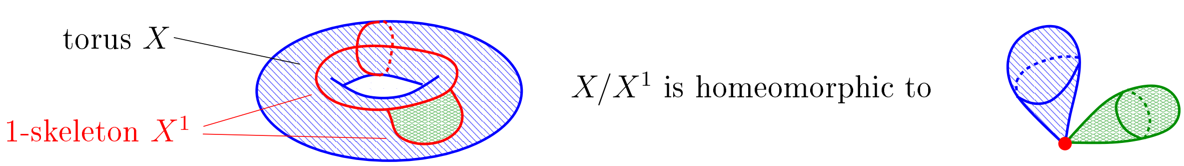


FIGURE 617

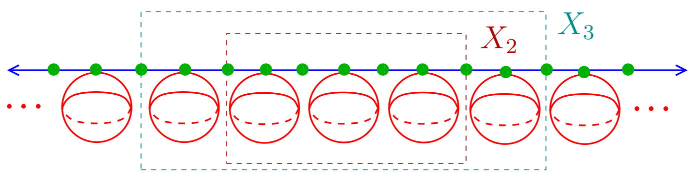
PROOF (∗).

(1) First note that it is basically clear that the given map f is a bijection. Furthermore it follows immediately from Lemma 18.25 that f is continuous. We use Lemma 36.32 to view both sides of Lemma 36.33 as CW-complexes. The map f gives a bijection between the cells of the two sides. Thus it follows from Lemma 36.12 that f is a homeomorphism.
(2) This statement follows immediately from (1).

(3) Since we remove a "top-dimensional" open cell from X^{n+1} it follows immediately from the definitions that $X^{n+1} \setminus \Phi_i(B_i^{n+1})$ is a subcomplex of X^{n+1}. Similar to the argument in (1) we now see that the map is a homeomorphism. ∎

Lemma 36.34. *Let $f_i\colon X_i \to X_{i+1}$, $i \in \mathbb{N}$ be a sequence of cellular inclusion maps between* CW*-complexes. Then the direct limit*[602] $\varinjlim X_i = \bigcup_{i\in I} X_i$ *admits a* CW*-structure such that each X_i is a subcomplex of* $\varinjlim X_i = \bigcup_{i\in I} X_i$.

Example. Lemma 36.34 allows us to view the line with infinitely many spheres attached as a CW-complex.[603] We refer to Figure 618 for an illustration.



the topological space $X = \bigcup_{i\in\mathbb{N}} X_i$

FIGURE 618

PROOF. We leave it to the reader to write down the, notationally messy but mathematically elementary, proof. ∎

We conclude this section with the following corollary that shows that several constructions that we are familiar with can also be performed in the category of CW-complexes.

Corollary 36.35. *Let X be a* CW*-complex.*[604]

(1) *The cone*

$$\operatorname{Cone}(X) \;=\; (X \times [0,1])/\sim \qquad \textit{where } (x,0) \sim (y,0) \textit{ for every } x,y \in X$$

admits a natural CW*-structure such that the inclusion map* $X \to \operatorname{Cone}(X)$ *given by* $x \mapsto [(x,1)]$ *is a cellular embedding.*

(2) *The suspension*

$$\Sigma(X) \;=\; (X \times [-1,1])/\sim \qquad \textit{where for every } x,y \in X \textit{ we have } (x,-1) \sim (y,-1) \textit{ and } (x,1) \sim (y,1)$$

admits a natural CW*-structure such that the map* $X \to \Sigma(X)$ *given by* $x \mapsto [(x,0)]$ *is a cellular embedding.*

(3) *If A is a* CW*-complex and if $f\colon A \to X$ is a cellular map, then the mapping cone*

$$\operatorname{Cone}(f\colon A \to X) \;=\; (\quad)/\sim \quad \textit{where } (a,1) \sim f(a) \textit{ for all } a \in A$$

[602] We write explicitly "direct limit" to remind the reader that, according to the discussion on page 731, the topology on $\varinjlim X_i = \bigcup_{i\in I} X_i$ is given by the condition that a subset U is open if and only if each intersection $U \cap X_i$ is an open subset of X_i.

[603] It follows from Proposition 2.45 that the topology on the "line with infinitely many spheres" coming from $\mathbb{R}^3$ agrees with the topology coming from the CW-structure.

and the mapping cylinder

$$\mathrm{Cyl}(f\colon A\to X) \;=\; (A\times[0,1]\sqcup X)/\sim \;\; \textit{where } (a,1)\sim f(a) \textit{ for all } a\in A$$

admit natural CW*-structures. Furthermore the obvious maps* $X\to \mathrm{Cone}(f\colon A\to X)$ *and* $\mathrm{Cyl}(f\colon A\to X)$ *are cellular embeddings and the map* $A\mapsto \mathrm{Cyl}(f\colon A\to X)$ *given by* $a\mapsto [(a,0)]$ *is a cellular embedding.*

(4) *If* $f\colon X\to X$ *is a cellular map, then the mapping torus*

$$\mathrm{Tor}(X,f) \;:=\; (X\times[0,1])\,/\sim \qquad \textit{where } (x,0)\sim(f(x),1) \textit{ for all } x\in X$$

admits a natural CW*-structure such that the map* $X\to\mathrm{Tor}(X,f)$ *given by* $x\mapsto (x,0)$ *is a cellular embedding.*

(*As in the statement of Lemma 36.32 we point out that often we will use a cellular embedding to identify the domain with its image.*)

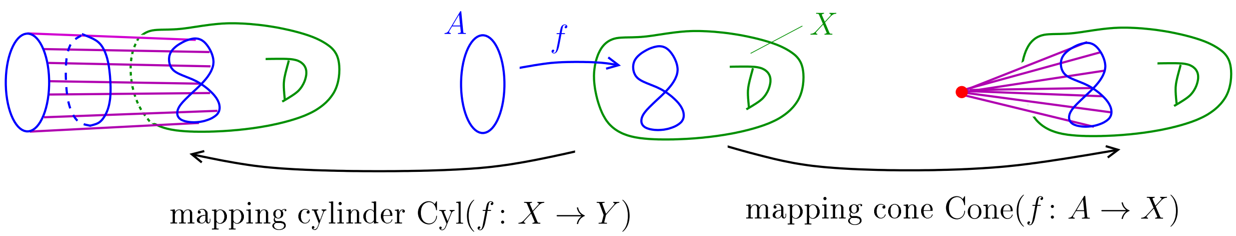


FIGURE 619

PROOF. All four statements can be proved fairly easily using Proposition 36.23 (3b) and Lemma 36.32. We only point out that the CW-structure on the mapping torus $\mathrm{Tor}(X,f)$ is obtained in a 2-stage process, first we equip the mapping cylinder $\mathrm{Cyl}(f\colon X\to X)$ with the natural CW-structure given by (3). Afterwards we apply Lemma 36.32 (6). We leave it to the reader to fill all the other details. ■

We conclude this discussion of constructions of CW-structures with the following slightly sobering proposition.

Proposition 36.36. *There exists a finite 3-dimensional CW-complex that admits an open subset that does not admit a* CW*-structure.*

PROOF. This statement is proved in [**Cau92**, Exemple 2]. ■

36.13. **Coverings of CW-complexes.** In this section we will see that a covering of a CW-complex inherits a natural CW-structure.

Proposition 36.37. *Let* $p\colon \widetilde{X}\to X$ *be a covering of a connected* CW*-complex* X*. We can equip* $\widetilde{X}$ *with a natural structure of a* CW*-complex such that the following statements hold:*

(1) *The map* $p\colon \widetilde{X}\to X$ *is a cellular map.*

[604] Strictly speaking, or Bourbakily speaking if that's an adverb, we should really specify in each case what the "natural CW-structure" is. But this should hopefully be clear in each case, and we do not encumber the reader with unnecessary extra notation.

(2) *For any $k \in \mathbb{N}_0$ we have*[605]

$$\#k\text{-cells of } \widetilde{X} \;=\; \big[\widetilde{X} : X\big] \cdot \#k\text{-cells of } X.$$

(3) *The dimension of $\widetilde{X}$ is the same as the dimension of X.*

Remark. Proposition 36.37 is a generalization of Proposition 31.1 where we showed that any covering of a topological graph is again a topology graph. In fact, as we will see, the proofs are structurally very similar.

PROOF. Let $p\colon \widetilde{X} \to X$ be a covering of a connected CW-complex X of degree $n := [\widetilde{X} : X]$. Let $\psi\colon \overline{B}^k \to X$ be a characteristic map of a k-cell of X. We write $x = \psi(0)$. We consider the following diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ \overline{B}^k \arrow[ur, dashed, "\widetilde{\psi}"] \arrow[r, "\psi"'] & X. \end{tikzcd}$$

Since $\overline{B}^k$ is simply connected and locally path-connected we can apply Proposition 29.2 which says in this context that for each $\widetilde{x} \in p^{-1}(x)$ there exists a unique lift $\overline{B}^k \to \widetilde{X}$ with $\widetilde{\psi}(0) = \widetilde{x}$. By the definition on page 495, the degree $n = [\widetilde{X} : X]$ of the covering equals the cardinality of $p^{-1}(x)$. Thus we see that ψ gives rise to n maps $\overline{B}^k \to \widetilde{X}$. It is straightforward to verify that all these lifts of the characteristic maps for the cells of X define a CW-structure for $\widetilde{X}$. We leave the joyful task of verifying the details to the reader. Alternatively we refer to [**Bre93**, Theorem IV.8.10] for details.

It follows immediately from the above construction and the definitions that the CW-structure we had just defined on $\widetilde{X}$ has the desired properties (1) to (3). ■

Examples.

(1) We equip S^1 with the CW-structure consisting of one 0-cell and one 1-cell. We denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$ the universal covering of S^1. If we apply the construction of the CW-structure on $\mathbb{R}$ given in the proof of Proposition 36.37, then we end up equipping $\mathbb{R}$ with the CW-structure where the 0-cells correspond to $\mathbb{Z}$ and the 1-cells are precisely the intervals $[k, k+1]$, with $k \in \mathbb{Z}$. We illustrate the CW-structure of $\mathbb{R}$ in Figure 620.

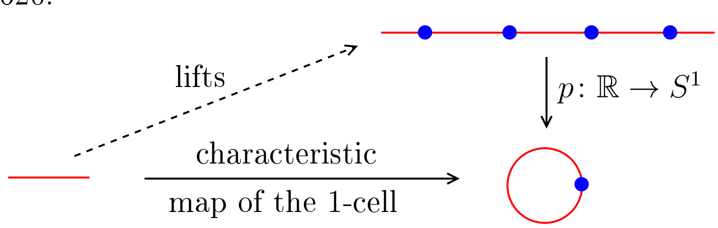


FIGURE 620

[605] Note that this is an equality of cardinalities. For example, if $[\widetilde{X} : X]$ is infinite countable and if X has finitely many k-cells, then $\widetilde{X}$ has infinitely many, but countably many, k-cells.

(2) Almost the same approach as in (1), applied to the universal covering $\mathbb{R}^2 \to S^1 \times S^1$, endows $\mathbb{R}^2$ with a CW-structure that is illustrated in Figure 621.

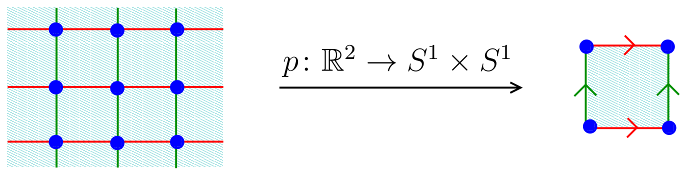


FIGURE 621

(3) In the proof of Hadamard's Theorem 33.9 we constructed an explicit covering map $\mathbb{D} \to H_8/\sim \cong E_8/\sim$. If we equip $E_8/\sim$ with the "usual" CW-structure then we obtain from Proposition 36.37 the CW-structure on $\mathbb{D}$ that is shown in Figure 622.

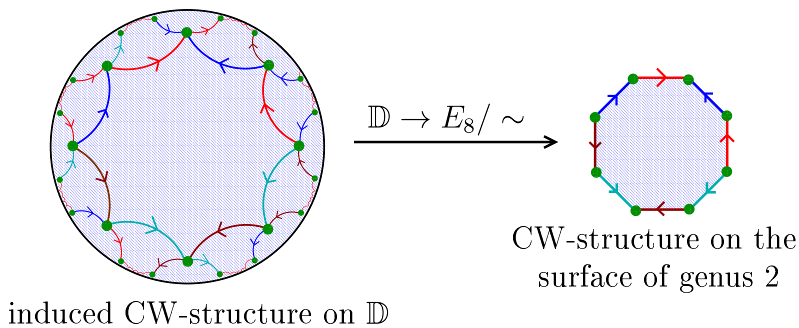


FIGURE 622

Exercises for Chapter 36.

Exercise 36.1.

(a) Does every compact subset of $\mathbb{R}^2$ admit a CW-structure?
(b) Let $n \in \mathbb{N}$. Does the open ball $B^n_1(0)$ admit a CW-structure?
(c) Let C be the Cantor set as defined on page 100. Does C admit a CW-structure?

Exercise 36.2.

(a) We equip $\mathbb{R}^\infty$ with the metric that is given by

$$d((x_1, x_2, \dots), (y_1, y_2, \dots)) \ := \ \sqrt{\sum_{i=1}^{\infty} |x_i - y_i|^2}.$$

We consider the following two topologies on $\mathbb{R}^\infty$:

$$\mathcal{S} \ := \ \{U \subset \mathbb{R}^\infty \,|\, \text{for every } P \in U \text{ there exists } \epsilon > 0 \text{ such that } B_\epsilon(P) \subset U\}$$

and

$$\mathcal{T} \ := \ \{U \subset \mathbb{R}^\infty \,|\, \text{for every } n \in \mathbb{N} \text{ the set } U \cap \mathbb{R}^n \text{ is open in } \mathbb{R}^n\}.$$

Show that $\mathcal{S} \subset \mathcal{T}$.

(b) We view $\mathbb{R}^\infty$ as equipped with the topology $\mathcal{S}$. Does the Heine-Borel Theorem hold in $\mathbb{R}^\infty$? More precisely, is every bounded closed subset of $\mathbb{R}^\infty$ compact?

(c) We view $\mathbb{R}^\infty$ as equipped with the topology $\mathcal{T}$. Show that the projection map $p\colon \mathbb{R}^\infty \to \mathbb{R}^n$ is continuous and open.

Exercise 36.3. Show that the map

$$\begin{array}{rcl} \mathbb{C}^\infty & \to & \mathbb{R}^\infty \\ (x_1 + iy_1, x_2 + iy_2, \dots) & \to & (x_1, y_1, x_2, y_2, \dots) \end{array}$$

is a homeomorphism.

Exercise 36.4. Show that the action

$$\begin{array}{rcl} \{\pm 1\} \times S^\infty & \to & S^\infty \\ (\epsilon, v) & \mapsto & \epsilon \cdot v \end{array}$$

of the group $\{\pm 1\}$ on the topological space S^∞ is continuous and discrete.

Exercise 36.5. Let $P, Q \in S^\infty$. Show that there exists a homeomorphism h of S^∞ with $h(P) = Q$.

Exercise 36.6. Let X be a CW-complex and let $x \in X$. Show that $\{x\}$ is a closed subset of $|X|$.
Hint. Make good use of Lemma 3.44.

Exercise 36.7. Let X be a CW-complex.

(a) Show that if the 1-skeleton X^1 is path-connected, then X itself is path-connected.
(b) Show that if X is connected, then X^1 is also connected.

Hint. Make good use of Lemmas 2.57, 2.58 and 2.62.

Exercise 36.8. Show that a CW-complex is locally finite if and only if it is regionally compact.

Exercise 36.9. We equip $\mathbb{R}\mathrm{P}^2$ with the CW-structure with one cell in dimensions $0, 1, 2$. Let $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/\pm 1$ be the usual 2-fold covering. The CW-structure on $\mathbb{R}\mathrm{P}^2$ induces a CW-structure on S^2. What does this CW-structure on S^2 look like?

Exercise 36.10. Let I be a set and for each $i \in I$ let $n_i \in \mathbb{N}$. Show that the wedge $\bigvee_{i\in I} S_i^{n_i}$ has a natural CW-structure with one 0-cell and where for each $n \in \mathbb{N}_0$ the n-cells correspond precisely to the n-dimensional spheres in the family of spheres $\{S_i^{n_i}\}_{i\in I}$.
Hint. Use Lemma 3.28 and use the fact, shown on page 182, that for each $n \in \mathbb{N}$ the sphere S^n is homeomorphic to $\overline{B}^n/S^{n-1}$.

Exercise 36.11. Let X be a finite CW-complex. Show that here exists an $n \in \mathbb{N}_0$ and a map $\varphi\colon |X| \to \mathbb{R}^n$ that is an embedding.

Exercise 36.12. Recall that we denote by $\mathcal{PTop}$ the category of pointed topological spaces. Suppose we are given $(X, x_0), (Y, y_0) \in \mathrm{Ob}(\mathcal{PTop})$. Show that the direct product of (X, x_0) and (Y, y_0) in $\mathcal{PTop}$ is given by the smash product $(X, x_0) \wedge (Y, y_0)$ that we introduced on page 577.

Exercise 36.13. We consider the set $X = (\mathbb{N} \times \mathbb{N}) \cup \{*\}$. We say $U \subset X$ is open if $* \notin U$ or if for all but finitely many $n \in \mathbb{N}$ the set U contains all but finitely many points in $\{n\} \times \mathbb{N}$.

(a) Show that these open sets do indeed define a topology on X.
(b) Show that X is Hausdorff.
(c) Show that every infinite subset of X contains an infinite closed and discrete subset.
(d) Show that every compact subset of X is finite.
(e) Show that the compactly generated topology on X is the discrete topology.

These steps show that X is not compactly generated.

Exercise 36.14. Let $\mathcal{C}$ be a convenient category of topological spaces, i.e. let $\mathcal{C}$ be a full subcategory $\mathcal{C}$ of the category of topological spaces that has the properties stated in Theorem 36.28. Let X and Y be CW-complexes. Show that if he usual product $X \times Y$ lies in $\mathcal{C}$, then $X \times Y$ is homeomorphic to the product CW-complex $X \otimes Y$.

Exercise 36.15. Let X and Y be CGWH-spaces. We denote by $X \widetilde{\times} Y$ the product in the category of CGWH-spaces, which exists by Theorem 36.30 (4). Show that for each $n \in \mathbb{N}$, each $x_0 \in X$ and $y_0 \in Y$ the natural maps $p\colon X \widetilde{\times} Y \to X$ and $q\colon X \widetilde{\times} Y \to Y$ induce an isomorphism

$$p_* \times q_* \colon \pi_n(X \widetilde{\times} Y, (x_0, y_0)) \xrightarrow{\cong} \pi_n(X, x_0) \times \pi_n(Y, y_0).$$

Remark. The exercise is to verify that the proofs of Propositions 16.20 and 40.8 also apply to $X \widetilde{\times} Y$ instead of the usual product topological space $X \times Y$.

Exercise 36.16. We consider $X = \mathbb{R} \setminus \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ and we consider $Y = \mathbb{R}/Z$ where $Z \subset \mathbb{R}$ is the subset given by integers. We consider the subset

$$A \;=\; \left\{ \left(\tfrac{1}{i} + \tfrac{a_i}{j}, i + \tfrac{1}{2j}\right) \in X \times Y \,\middle|\, i, j \in \mathbb{N} \right\} \qquad \text{where } a_i = \tfrac{1}{2}\left(\tfrac{1}{i} - \tfrac{1}{i+1}\right).$$

(a) Show that the closure of A contains $(0, [0])$.
(b) Show that for any compact subset $K \subseteq X \times Y$, the set $K \cap A$ has only finitely many points.
(c) Show that X is not compactly generated.

Exercise 36.17.

(a) We consider

$$X \;:=\; \{0\} \cup \bigcup_{n,m \in \mathbb{N}} \left\{\tfrac{1}{n} + \tfrac{1}{m+n}\right\} \;\subset\; \mathbb{R}.$$

We equip X with the usual metric coming from $\mathbb{R}$. We set $x := 0$.
(i) Show that each point in $X \setminus \{x\}$ is open.
(ii) Show that X is not regionally compact at x.

(b) Let X be a countable metric space which admits a point $x \in X$ such that the following conditions are satisfied:
(i) Each point in $X \setminus \{x\}$ is open.
(ii) The topological space X is not regionally compact at x.

Let $Y := X \cup \{\infty\}$ be the one-point compactification as defined in Exercise 2.66.

(α) Show that compact sets of Y are closed. (This implies that Y is weakly Hausdorff.)

(β) Show, by considering the points x and ∞, that Y is not Hausdorff.

Remark. The combination of (a) and (b) shows that there exist topological spaces that are weakly Hausdorff but not Hausdorff.

Exercise 36.18. Let X be a CW-complex with countably many cells.

(a) Show that there exists a sequence $A_1 \subset A_2 \subset \ldots$ of *finite* subcomplexes A_i of X such that $X = \bigcup_{i \in \mathbb{N}_0} A_i$.

(b) If X is connected, then show that we can arrange in (a) that every A_i is connected.

37. CW-complexes II: Fundamental groups

In this chapter we will consider fundamental groups of CW-complexes. First we collect a few general statements. Then we will show that fundamental groups of 1-dimensional CW-complexes are free groups. Finally we will see how can can calculate fundamental groups of more general CW-complexes in practice.

37.1. **Fundamental groups of CW-complexes.** Using Proposition 36.10 we can now prove the following very useful version of the Seifert–van Kampen Theorem 22.1. In view of Theorem 64.5, which we mentioned already on page 936, this can be viewed as a generalization of the Seifert–van Kampen Theorem 22.2 for smooth manifolds.

Theorem 37.1. (Seifert–van Kampen Theorem for CW-complexes) *Let X be a* CW*-complex and let $X = A \cup B$ be a decomposition of X in two subcomplexes such that $A \cap B$ is path-connected. Let $x_0 \in A \cap B$. Then*

$$\pi_1(X, x_0) \;\cong\; \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0).$$

Examples.

(1) Let X be the wedge of two circles. We can view X as a CW-complex with one 0-cell x_0 and two 1-cells a and b. Then $A = \{x_0, a\}$ and $B = \{x_0, b\}$ are subcomplexes, each of which is homeomorphic to S^1. We refer to Figure 623 for an illustration. It follows, as on page 606, that

$$\pi_1(X, x_0) \;\underset{\substack{\uparrow\\ \text{Theorem 37.1}}}{\cong}\; \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0) \;\underset{\substack{\uparrow\\ \text{since } A \text{ and } B \text{ are homeomorphic to } S^1}}{\cong}\; \langle a\rangle * \langle b\rangle \;=\; \langle a, b\rangle.$$

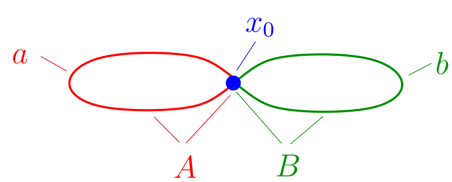


Figure 623

(2) We consider

$$K \;:=\; \underbrace{(S^1 \times S^1)}_{=:A} \cup \underbrace{(\overline{B}^2 \times \{1\})}_{=:B},$$

i.e. K is given by a torus where the "central hole" is filled in with a disk. We refer to Figure 624 for an illustration.[606] In the above definition of K we already wrote K as the union of two subsets. It is quite straightforward to see that K can be given a CW-structure such that A and B are subcomplexes.[607] We pick the base point

[606]Here the name K stands for "kiacherl".

[607]How would you define the CW-structures?

$x_0 = (1,1)$. We obtain that

$$\pi_1(K,x_0) \underset{\substack{\uparrow \\ \text{Theorem 37.1}}}{\cong} \pi_1(A,x_0) *_{\pi_1(A\cap B,x_0)} \pi_1(B,x_0) \underset{\substack{\uparrow \\ \text{since } A\cap B = S^1\times\{1\}}}{\cong} \underbrace{\langle x,y|[x,y]\rangle *_{\langle s\rangle} \{e\}}_{\text{with } x \leftrightarrow s} \underset{\substack{\uparrow \\ \text{Lemma 21.22}}}{\cong} \langle y\rangle.$$

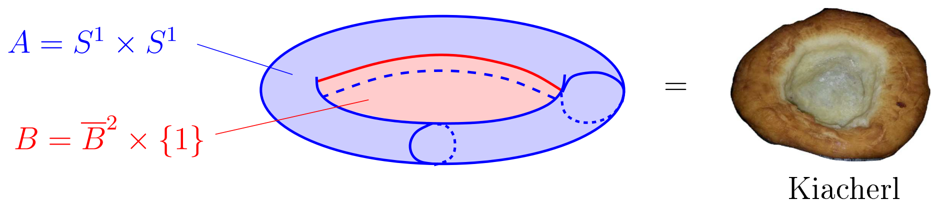


FIGURE 624

PROOF OF THEOREM 37.1 (*). [608] Let X be a CW-complex and let $X = A \cup B$ be a decomposition of X in two subcomplexes such that $A \cap B$ is path-connected. Note that $A \cap B$ is also a subcomplex. Let $x_0 \in A \cap B$.

We use the notation of Proposition 36.10 (8). We write $U = \Phi(A)$ and $V = \Phi(B)$. Note that by Proposition 36.10 (8b) we have $U \cap V = \Phi(A \cap B)$. Since $A \cap B$ is path-connected and since $A \cap B$ is a deformation retract of $\Phi(A \cap B)$ we see that $U \cap V = \Phi(A \cap B)$ is also path-connected. Therefore we can apply the Seifert–van Kampen Theorem 22.1 to the decomposition $X = U \cup V$. We obtain the following commutative diagram

$$\begin{tikzcd}
\pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) \arrow[r, "\cong"] & \pi_1(X,x_0) \\
\pi_1(A,x_0) *_{\pi_1(A\cap B,x_0)} \pi_1(B,x_0) \arrow[r] \arrow[u] & \pi_1(X,x_0), \arrow[u, "="]
\end{tikzcd}$$

where the two vertical maps are induced by the inclusions. By Proposition 36.10 (8a) and Proposition 18.16 (2) the three inclusion induced maps $\pi_1(A,x_0) \to \pi_1(U,x_0)$, $\pi_1(B,x_0) \to \pi_1(V,x_0)$ and $\pi_1(A\cap B,x_0) \to \pi_1(U \cap V,x_0)$ are isomorphisms. The top horizontal map is an isomorphism by the Seifert–van Kampen Theorem 22.1. It follows that the bottom horizontal map is also an isomorphism. ■

Next we want to compute the fundamental group of a CW-complex that is obtained by gluing two subcomplexes of a connected CW-complex. We recall and introduce the following two definitions.

Definition. Let X be a topological space, let A be a subset of X and let $f\colon A \to X$ be an embedding. Following the discussion on page 761 we define

$$X(f) := X(f\colon A\to X) := X/\sim \qquad \text{where } f(a)\sim a \text{ for every } a \in A.$$

Furthermore we define the *partial mapping torus*[609]

$$\operatorname{Tor}(f) := \operatorname{Tor}(f\colon A\to X) := (X \sqcup (A\times[0,1]))/\sim \qquad \text{where } (a,0)\sim a \text{ and } (a,1)\sim f(a) \text{ for every } a\in A.$$

[608]The proof is very similar to the proof of the Seifert–van Kampen Theorem for Smooth Manifolds 22.2.

[609]If $A = X$, then one can easily show that the partial mapping torus is naturally homeomorphic to the mapping torus that we introduced on page 711.

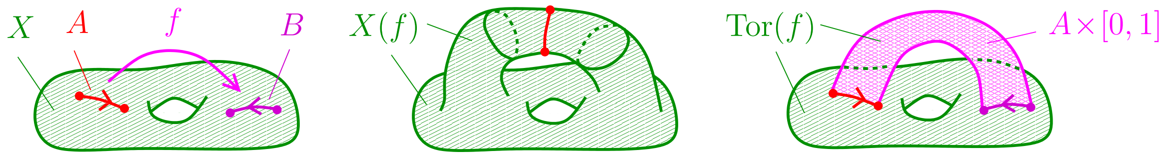


FIGURE 625. partial-mapping-torus

Continuing with the above notation, note that the projection $A \times [0,1] \to A$ induces a map $p\colon \operatorname{Tor}(f) \to X(f)$. In Proposition 38.5 we will see that under reasonably mild hypotheses the map $p\colon \operatorname{Tor}(f) \to X(f)$ is a homotopy equivalence. In particular by Proposition 18.16 we see that p induces an isomorphism of fundamental groups.

Now we can formulate, prove and discuss the following theorem.

Theorem 37.2. (HNN-Seifert–van Kampen Theorem for CW-complexes) *Suppose we are given a path-connected* CW*-complex X and two path-connected disjoint subcomplexes A and B. Let $f\colon A \to B$ be a cellular isomorphism. We pick a base point $x_0 \in A$. Let $\gamma\colon [0,1] \to X$ be an embedding with $\gamma(0) = x_0$ and with $\gamma(1) = f(x_0)$ such that $\gamma([0,1])$ is a subcomplex of the 1-skeleton X^1.*[610] *We write $\Gamma = \pi_1(A, x_0)$. Finally we denote by*

$$\alpha\colon \Gamma = \pi_1(A, x_0) \to \pi_1(X, x_0)$$

the inclusion induced map and we denote by

$$\beta\colon \Gamma = \pi_1(A, x_0) \xrightarrow{f_*} \pi_1(B, f(x_0)) \to \pi_1(X, f(x_0)) \xrightarrow{\gamma_*} \pi_1(X, x_0)$$

the concatenation of the maps induced by f, the inclusion and the base point change isomorphism from Proposition 14.11, using the path γ. Then there exists an isomorphism

$$\Phi\colon \pi_1(\operatorname{Tor}(f), x_0) \xrightarrow{\cong} \langle \pi_1(X, x_0), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle$$

which has the following two properties:

(1) *The diagram*

$$\begin{tikzcd}
 & \pi_1(X, x_0) \arrow[dl] \arrow[dr] & \\
\pi_1(\operatorname{Tor}(f), x_0) \arrow[rr, "\Phi"] & & \langle \pi_1(X, x_0), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle
\end{tikzcd}$$

with the obvious diagonal maps commutes.

(2) *The closed loop in $\operatorname{Tor}(f)$ given by the composition of γ with the path $s \mapsto [(x_0, 1-s)]$ corresponds under the isomorphism Φ to t.*

Remark.

(1) Basically the same way that we obtained the HNN-Seifert–van Kampen Theorem 26.3 from the Seifert–van Kampen Theorem 22.1 one can also obtain the above HNN-Seifert–van Kampen Theorem 37.2 for CW-complexes from the Seifert–van Kampen Theorem 37.1 for CW-complexes. Below we will give a different proof of the

[610] Note that it follows from our hypothesis that X is path-connected together with Proposition 36.10 (7) that such γ exists.

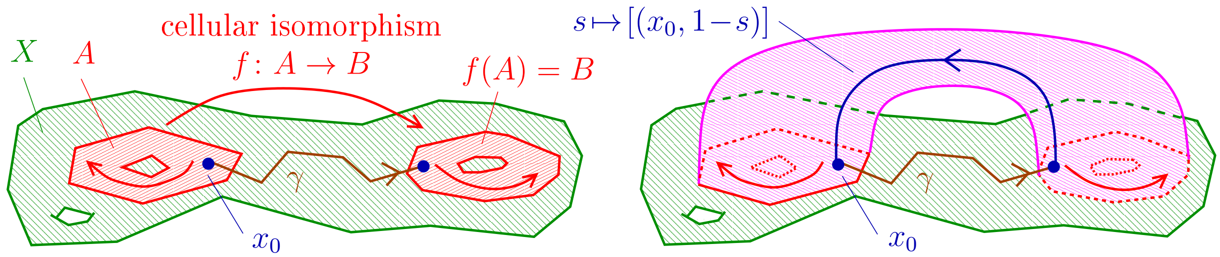


FIGURE 626. Illustration of the HNN-Seifert–van Kampen Theorem 37.2.

HNN-Seifert–van Kampen Theorem 26.3 for CW-complexes which is easier than the argument provided in the proof of the HNN-Seifert–van Kampen Theorem 37.2.

(2) The original HNN-Seifert–van Kampen Theorem 26.3 is more general than the above HNN-Seifert–van Kampen Theorem 26.3 for CW-complexes, but for most applications the latter result will be just fine.

(3) Using Proposition 38.5 one can also deal with certain situations where the map $f\colon A \to B$ is not a cellular isomorphism. We leave it to the reader to figure out what these generalized statements might be.

SKETCH OF A PROOF OF THEOREM 37.2. Let X be a path-connected CW-complex X and let $f\colon A \to B$ be a cellular isomorphism between two path-connected disjoint subcomplexes of X. Let $x_0 \in A$ and let $\gamma\colon [0,1] \to X$ be an embedding with $\gamma(0) = x_0$ and with $\gamma(1) = f(x_0)$ such that $C := \gamma([0,1])$ is a subcomplex.

It follows from Proposition 36.23 (3b) and Lemma 36.32 that $\operatorname{Tor}(f)$ is equipped with a natural CW-structure such that we can view $A \times [0,1]$ and X as subcomplexes of $\operatorname{Tor}(f)$.

We consider $Y := (X \sqcup (\{x_0\} \times [0,1]))/\sim$ where $(x_0, 0) \sim x_0$ and $(x_0, 1) \sim f(x_0)$. We view Y as a CW-complex given by attaching a single 1-cell to X. We denote by D the image of $\{x_0\} \times [0,1]$ in Y. Furthermore we denote by $\delta\colon [0,1] \to D$ the path given by $s \mapsto (x_0, 1-s)$. Now note that we have the following two isomorphisms:

$$\pi_1(Y, x_0) \underset{\substack{\uparrow \\ \text{Seifert–van Kampen Theorem 37.1} \\ \text{for CW-complexes}}}{\cong} \pi_1(X, x_0) *_{\pi_1(C, x_0)} \pi_1(C \cup D, x_0) \underset{\substack{\uparrow \\ \text{since } C \text{ contractible and since} \\ C \cup D \text{ is homeomorphic to } S^1}}{\cong} \pi_1(X, x_0) * \langle t = [\gamma * \delta] \rangle$$

and

$$\pi_1(A, x_0) * \pi_1(B, f(x_0)) \underset{\uparrow}{\cong} \pi_1(A, x_0) * \pi_1(D \cup B, x_0) \underset{\uparrow}{\cong} \pi_1(A \cup D \cup B, x_0).$$

by the Seifert–van Kampen Theorem 37.1 and the base point change $f(x_0) \mapsto x_0$ induced by δ

Finally we also have the isomorphism:

$$\pi_1(\operatorname{Tor}(f), x_0) \underset{\uparrow}{\cong} \pi_1(Y, x_0) *_{\pi_1(A \cup D \cup B, x_0)} \underbrace{*\pi_1(A \times [0,1], x_0)}_{\cong \pi_1(A, x_0)}.$$

by the Seifert–van Kampen Theorem 37.1, here we use that $Y \cap (A \times [0,1]) = A \cup D \cup B$ is connected

Note that in the above discussion basically all maps are induced by the obvious inclusion maps We leave it to the intrepid reader to go through the explicit descriptions of the various homomorphisms and the description of the amalgamated product in Proposition 21.21 to show that we do indeed end up with the promised description of $\pi_1(\mathrm{Tor}(f), x_0)$. ∎

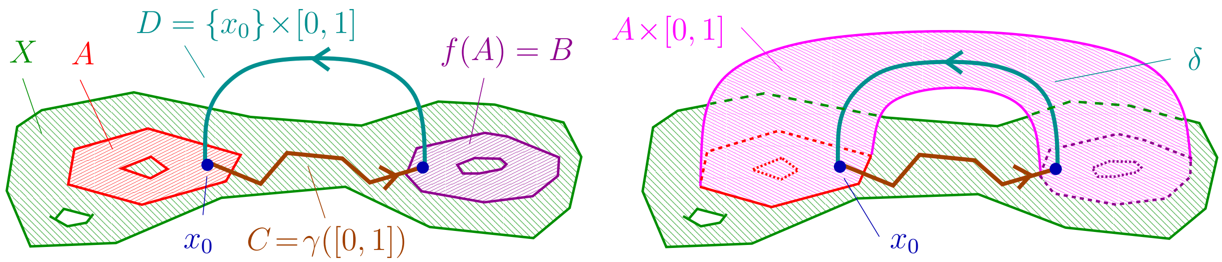


FIGURE 627. Illustration for the proof of Theorem 37.2.

We conclude this section with the following proposition that is an immediate consequence of Proposition 25.7 and the Finiteness Theorem 36.14

Proposition 37.3. *Let X be a* CW*-complex and let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of subcomplexes such that $X = \bigcup_{i \in \mathbb{N}} X_i$. Then for any $w \in X_1$ we have*

$$\pi_1(X, w) \;=\; \varinjlim \pi_1(X_i, w).$$

Examples.

(1) We have[611]

$$\begin{array}{ccccccccc} & & & & \uparrow & & \uparrow\cong & & \\ & & & & \pi_1(\mathbb{RP}^3) & & \mathbb{Z}_2 & & \\ & & & & \uparrow & & \uparrow\cong & & \\ \pi_1(\mathbb{RP}^\infty) & \underset{\substack{\uparrow \\ \text{see page 942}}}{=} & \pi_1\Big(\bigcup_{n\in\mathbb{N}} \mathbb{RP}^n\Big) & \underset{\substack{\uparrow \\ \text{Proposition 37.3}}}{=} & \varinjlim \; \pi_1(\mathbb{RP}^2) & = & \varinjlim \; \mathbb{Z}_2 & \cong & \mathbb{Z}_2. \\ & & & & \uparrow & & \uparrow & & \\ & & & & \pi_1(\mathbb{RP}^1) & & \mathbb{Z} & & \end{array}$$

(2) In Exercise 37.1 we will use Proposition 37.3 to show that there exists a CW-complex X with $\pi_1(X) \cong (\mathbb{Q}, +)$.

37.2. **The Euler characteristic of a CW-complex.** In the following sections we will give ea complete calculation of the fundamental group of a 1-dimensional CW-complex. In the subsequent discussion it will be convenient to have the following definition at our disposal.

Definition. Given a finite CW-complex X we define its Euler characteristic $\chi(X)$ as follows:

$$\chi(X) \;:=\; \sum_{n \in \mathbb{N}_0} (-1)^n \cdot \text{number of } n\text{-cells}.$$

Examples.

(1) If G is a finite topological graph, then, according to the discussion on page 932, we can view G as a 1-dimensional CW-complex where the number of 0-cells equals

[611] By Corollary 16.18 we know that $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$ for $n \geq 2$. But why is $\pi_1(\mathbb{RP}^1) \cong \mathbb{Z}$?

the number of vertices and the number of 1-cells equals the number of edges of G. Thus it follows immediately from the definitions that in this case the above Euler characteristic agrees with the definition of the Euler characteristic of a topological graph that we gave on page 228.

(2) For the CW-structures provided by Lemma 36.1 we have $\chi(\mathbb{R}\mathrm{P}^n) = \frac{1}{2}(1+(-1)^n)$ and $\chi(\mathbb{C}\mathrm{P}^n) = n+1$.

(3) A short glance at the three examples in Figure 599 shows that these three different CW-structures for the torus have the same Euler characteristic. Later on, in Chapter 55, we will find out whether or not that is a coincidence.

The following three propositions give us several convenient statements about the Euler characteristic of a finite CW-complex.

Proposition 37.4. *Let $p\colon \widetilde{X} \to X$ be a finite covering of a finite* CW*-complex X. We equip $\widetilde{X}$ with the structure of a* CW*-complex given by Proposition 36.37. Then*

$$\chi(\widetilde{X}) \;=\; [\widetilde{X} : X] \cdot \chi(X).$$

Proof. Let $d := [\widetilde{X} : X]$. As we mentioned in Proposition 36.37 (2), it follows immediately from the construction of the CW-structure on $\widetilde{X}$ that for each k we have

$$\#k\text{-cells of } \widetilde{X} \;=\; \big[\widetilde{X} : X\big] \cdot \#k\text{-cells of } X.$$

The proposition is an immediate consequence of this equality. ■

The following lemma can be viewed as a generalization of Lemma 18.32.

Lemma 37.5. *Let X be a finite* CW*-complex and let A be a subcomplex. We equip the quotient X/A with the* CW*-structure given by Lemma 36.32 (3). Then*

$$\chi(X/A) \;=\; \chi(X) - \chi(A) + 1.$$

Proof. It follows immediately from the construction of the CW-structure on X/A given in Lemma 36.32 (3) that for each $k \in \mathbb{N}_0$ we have the following equalities:

$$\#k\text{-cells of } X/A \;=\; \begin{cases} \#k\text{-cells of } X \text{ not in } A, & \text{if } k > 0, \\ \#k\text{-cells of } X \text{ not in } A \;+\; 1, & \text{if } k = 0. \end{cases}$$

Evidently these equalities implies the equality $\chi(X/A) = \chi(X) - \chi(A) + 1$. ■

Proposition 37.6. *Let X be a non-empty finite connected 1-dimensional* CW*-complex. Then $\chi(X) = 0$ if and only if X is contractible.*

Proof. This statement is an immediate consequence of Proposition 20.5 and the fact that we can view any 1-dimensional CW-complex as a topological graph of the same Euler characteristic. ■

We have now assembled enough information to get us through this chapter. Later on in Chapter 55 we will give a much more elaborate discussion of the Euler characteristic of a CW-complex.

37.3. **The fundamental group of 1-dimensional CW-complexes.** The goal in this section is to show that the fundamental group of a non-empty connected 1-dimensional CW-complex X is a free group. The idea is quite simple:

(1) If the CW-complex X has a single 0-cell, then it follows basically from Proposition 20.7 that $\pi_1(X)$ is a free group.
(2) In the general case we show that X is homotopy equivalent to a 1-dimensional CW-complex with a single 0-cell.

The following definition is the key to proving Statement (2).

Definition.

(1) A *tree* is a 1-dimensional CW-complex that admits a deformation retraction to a 0-cell.
(2) Let G be a 1-dimensional CW-complex. A *spanning tree* is a subcomplex T that is a tree and that contains all vertices of G.

Remark.

(1) On page 228 we defined a *tree* to be a finite connected non-empty topological graph with Euler characteristic 1. It follows from Proposition 37.6 that the above definition of a tree is consistent with the definition that we had given on page 228.
(2) We leave it to the reader to show that for a finite topological graph the above definition of a spanning tree is equivalent to the one we had given on page 229.
(3) A spanning tree is often also called a *maximal tree*.

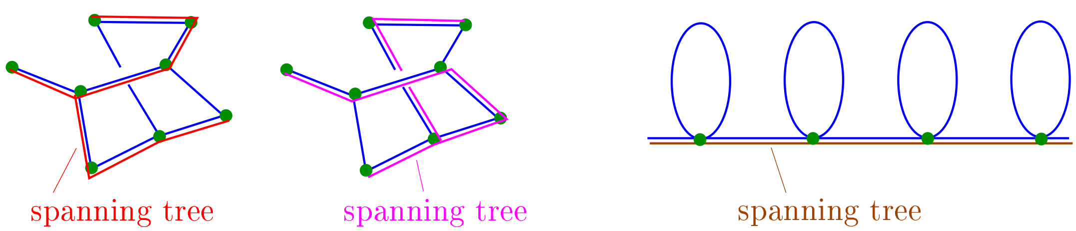


FIGURE 628

In Figure 628 we give examples of spanning trees in topological graphs. We see in particular that in general there is no unique spanning tree.

Proposition 37.7.

(1) *Given any connected 1-dimensional* CW*-complex G and given any tree $S \subset G$ there exists a spanning tree T that contains S and such that S is a deformation retract of T.*
(2) *Every connected non-empty 1-dimensional* CW*-complex admits a spanning tree.*

Remark. A somewhat different proof of Proposition 37.7 is given in [**FrPi90a**, Lemma 2.6.3].

PROOF. Statement (2) follows from Statement (1) by applying Statement (1) to a tree that is given by a single vertex. Thus it remains to prove Statement (2). Now let G be a connected 1-dimensional CW-complex and let $S \subset G$ be a tree.

Claim. There exists a sequence $S = Y_0 \subset Y_1 \subset Y_2 \subset \ldots$ of subcomplexes of G with the following properties:

(1) The union $Y := \bigcup_{i\in\mathbb{N}} Y_i$ contains all 0-simplices of G.
(2) Each Y_i is a deformation retract of Y_{i+1}.

To prove the claim we first construct a sequence $S = X_0 \subset X_1 \subset X_2 \subset \ldots$ of subcomplexes of G as follows.

(a) We set $X_0 := S$.
(b) Given X_i we define X_{i+1} to be the union of X_i with all 1-dimensional cells which contain at least one 0-cell of X_i.

We make the following observations:

(i) It follows from Lemma 36.21 (2) that X_{i+1} is a subcomplex of G.
(ii) We claim that $G = \bigcup_{i\in\mathbb{N}} X_i =: X$. This can be seen as follows. Basically by construction we have for each characteristic map[612] $\Phi\colon \overline{B}^n \to G$, $n = 0, 1$ that $\Phi^{-1}(X) = \varnothing$ or $\Phi^{-1}(X) = \overline{B}^n$. Since G is 1-dimensional it follows from Lemma 36.7 that X is a subset of G that is closed and that is open. Since G is connected and since X is non-empty we see that $X = G$.

Next we do the following iterative construction:

(a) We set $Y_0 := X_0 = S$.
(b) Suppose we are given Y_i. For each 0-cell v in Y_{i+1} that is not contained in Y_i we pick precisely one 1-cell e_v of Y_{i+1} that contains v. We define Y_{i+1} to be the union of Y_i with all these 1-cells. By Lemma 36.21 (2) we know that Y_{i+1} is a subcomplex of Y_i.

By construction the 0-cells of each Y_i are precisely the 0-cells of X_i. It follows from (ii) that $Y := \bigcup_{i\in\mathbb{N}} Y_i$ contains all 0-cells of G. It remains to show that each Y_i is a deformation retract of Y_{i+1}. As we will see, this is almost obvious, except that we need to worry a little bit about continuity issues. Let V be the set of 0-cells in Y_{i+1} that are not contained in Y_i. Note that the corresponding 1-cell e_v contains two 0-cells, namely v and one 0-cell w in Y_i. Let $\varphi_v\colon e_v \times [0,1] \to e_v$ be the obvious deformation retraction to w. Finally we consider the map

$$\begin{array}{rcl} F\colon \Big(\overbrace{Y_i \cup \bigsqcup_{v\in V} e_v}^{=Y_{i+1}}\Big) \times [0,1] & \to & Y_{i+1} = Y_i \cup \bigsqcup_{v\in V} e_v \\ ([P],t) & \mapsto & \begin{cases} P, & \text{if } P \in Y_i, \\ \varphi_v(P,t), & \text{if } P \in e_v. \end{cases} \end{array}$$

It follows almost immediately from Corollary 36.9 that the map is continuous. It is now basically clear that F defines a deformation retraction from Y_{i+1} to Y_i. ⊞

[612]For $n = 0$ the statement is a tautology. For $n = 1$ note that if $\Phi^{-1}(X) \neq \varnothing$, then there exists an i with $\Phi^{-1}(X_i) \neq \varnothing$. Since X_i is a subcomplex we see that $\Phi^{-1}(X_i)$ contains at least one 0-cell. But by construction this means that $\Phi^{-1}(X_{i+1}) = \overline{B}^1$.

We set $Y := \bigcup\limits_{i\in\mathbb{N}} Y_i$ By Lemma 36.21 (2) we know that Y is a subcomplex of G. We need to show that Y is contractible. Since S is contractible it follows from Lemma 18.11 (2) and Lemma 18.14 that it suffices to that S is a deformation retract of Y. But this is an immediate consequence of Statement (2) of the claim together with Proposition 36.22. ■

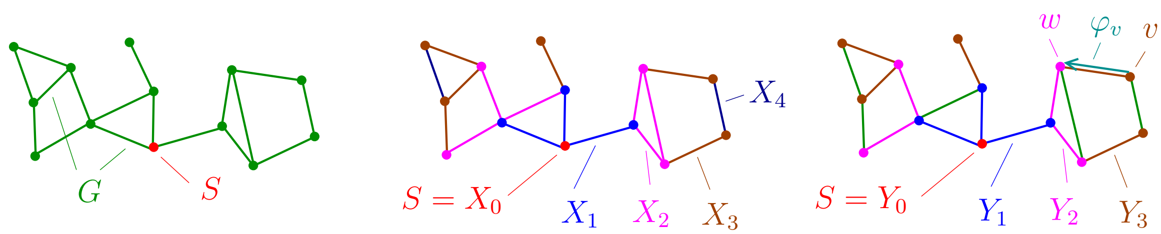


FIGURE 629. Illustration for the proof of Proposition 37.7.

Proposition 37.8. *Let X be a 1-dimensional* CW*-complex.*

(1) *For every spanning tree T of X the following holds:*
    (a) *The projection $p\colon X \to X/T$ is a homotopy equivalence.*
    (b) *The quotient X/T X is homeomorphic to a wedge of circles. If X is finite, then X/T is homeomorphic to the wedge of $1-\chi(X)$ circles.*
(2) *If X is connected and non-empty, then X is homotopy equivalent to the wedge of circles. Furthermore if X is finite, then X is homotopy equivalent to the wedge of $1-\chi(X)$ circles.*

Remark.

(1) Proposition 37.8 is a fairly straightforward generalization of Proposition 18.31. In fact the proof below is at times verbatim the same as the proof of Proposition 18.31.
(2) Later on, in Corollary 39.10, we will generalize Proposition 37.8 in a suitable way to the case of arbitrary CW-complexes.

PROOF. Let X be a 1-dimensional CW-complex.

(1a) Let T be a spanning tree of X. By hypothesis and by definition there exists a 0-cell t_0 of T such that there exists a deformation retraction from T to $\{t_0\}$, i.e. there exists a homotopy $R\colon T\times[0,1] \to T$ such that $R_1 = \mathrm{id}$ and such that $R_0 = c_{t_0}$ is the constant map. For notational convenience we make the obvious identification $\overline{B}^1 = [-2,2]$. Now let $\{\Phi_i\colon [-2,2] \to X\}_{i\in I}$ be the characteristic maps of the 1-cells of X that are not contained in T. We write $a_i = \Phi_i(-2)$ and $b_i = \Phi_i(2)$.

By Lemma 36.32 (3) we can view X/T as a 1-dimensional CW-complex with a single 0-cell $*$ and one 1-cell for each $i \in I$. We denote by $\{\Psi_i\colon [-2,2] \to X/T\}_{i\in I}$ the characteristic maps of these 1-cells. We consider the map

$$\begin{array}{rcl} q\colon X/T & \to & X \\ x & \mapsto & \begin{cases} R(a_i, -y-1), & \text{if } x = \Psi_i(y) \text{ for some } y \in [-2,-1], \\ \Psi_i(2\cdot y), & \text{if } x = \Psi_i(y) \text{ for some } y \in [-1,1], \\ R(b_i, y-1), & \text{if } x = \Psi_i(y) \text{ for some } y \in [1,2]. \end{cases} \end{array}$$

It follows from Lemma 36.7 (1) and (4) that q is continuous. (We refer to Figure 362 from a long time ago for an illustration of q.) It remains to prove the following claim.

Claim. The maps p and q are homotopy inverses of one another.

The proof of the claim is fairly elementary, just notationally a little messy. First we consider the following map

$$\begin{array}{rcl} F\colon (X/T)\times[0,1] & \to & X/T \\ (x,t) & \mapsto & \begin{cases} *, & \text{if } x=\Psi_i(y) \text{ for some } y\in[-2,-1-t], \\ \Psi_i(\frac{2y}{1+t}), & \text{if } x=\Psi_i(y) \text{ for some } y\in[-1-t,1+t], \\ *, & \text{if } x=\Psi_i(y) \text{ for some } y\in[1+t,2]. \end{cases} \end{array}$$

Using Corollary 36.9 (3) one can easily show that F is continuous. It follows immediately from the definitions that $F_0=p\circ q$ and $F_1=\mathrm{id}_{X/T}$. Next we consider the following map

$$\begin{array}{rcl} G\colon X\times[0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} R(x,t), & \text{if } x\in T, \\ R(a_i,y+2+t), & \text{if } x=\Phi_i(y) \text{ for some } y\in[-2,-1-t], \\ \Phi_i(\frac{2y}{1+t}), & \text{if } x=\Phi_i(y) \text{ for some } y\in[-1-t,1+t], \\ R(b_i,-y+2+t), & \text{if } x=\Phi_i(y) \text{ for some } y\in[1+t,2]. \end{cases} \end{array}$$

Again, using Corollary 36.9 (3) one can easily show that G is continuous. Furthermore it follows again immediately from the definitions that $G_0=q\circ p$ and $G_1=\mathrm{id}_X$. ⊞

(1b) Let T be a spanning tree of X. We denote by I the set of 1-cells of X that are not contained in T. As mentioned in (1), it follows from Lemma 36.32 (3) that we can view X/T as a 1-dimensional CW-complex with a single 0-cell $*$ and one 1-cell for each $i\in I$. By Lemma 36.33 we know that X/T is homeomorphic to the wedge $\bigvee_{i\in I} S^1_i$.

Finally we suppose that X is finite. At this point it suffices to show that X/T is homeomorphic to the wedge of $1-\chi(X)$ circles, in other words, we want to show that $|I|=1-\chi(X)$. This in turn follows from the following calculation:

$$1-|I| \underset{\substack{\uparrow \\ \text{see page 942}}}{=} \chi\Big(\bigvee_{i\in I} S^1_i\Big) = \chi(X/T) \underset{\substack{\uparrow \\ \text{by Lemma 37.5}}}{=} \chi(X)-\chi(T) \underset{\substack{\uparrow \\ \text{by Proposition 37.6}}}{=} \chi(X).$$

(2) Now we assume that X is connected and non-empty. By Proposition 37.7 we know that X admits a spanning tree T. The desired conclusion now follows from (1) together with (2). ■

Now we no longer have any troubles with computing the fundamental group of 1-dimensional CW-complexes. More precisely, we have the following theorem which generalizes Proposition 20.5 and 31.2

Theorem 37.9. *Let X be a non-empty connected 1-dimensional* CW*-complex. Furthermore let $x_0 \in X$. Then the following holds:*

(1) *The group $\pi_1(X, x_0)$ is isomorphic to a free group. In fact, if T is a spanning tree for X, then the cardinality of the generating set of the free group equals the cardinality of the set of edges in $X \setminus T$.*
(2) *If X is a finite* CW*-complex, then $\pi_1(X, x_0)$ is a free group with $1-\chi(X)$ generators.*
(3) *An explicit description of the isomorphism in (1) is given as follows. Pick a base point x_0 in the spanning tree and for each 1-cell e that is not contained in T we pick a path α_e from p to one of the endpoints and pick a path β_e from the other endpoint back to p[613] and let t_e be the loop given by first going along α_e, then going through e and then going back to the base point via β_e. Then the map*

$$\begin{array}{rcl} \big\langle \{x_e\}_{e \text{ cell not in } T} \big\rangle & \to & \pi_1(X, x_0) \\ x_e & \mapsto & [t_e] \end{array}$$

is an isomorphism.

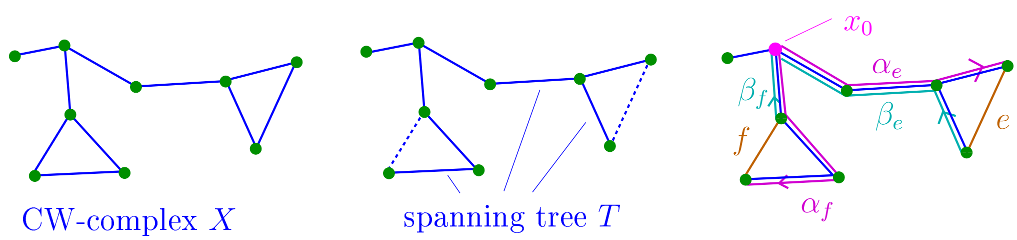


FIGURE 630

PROOF. Let X be a non-empty connected 1-dimensional CW-complex. By Proposition 37.7 we know that X admits a spanning tree T. We denote by E the set of 1-cells of X that are not contained in T. Let x_0 be a point in T. We denote by $p\colon X \to X/T$ the projection map. We write $* = p(x_0)$. We have the following isomorphisms:

$$\pi_1(X, x_0) \xrightarrow{p_*} \pi_1(X/T, *) \xrightarrow{\cong} \pi_1\Big(\bigvee_{e \in E} S^1_e, *\Big) \xleftarrow{\cong} \langle \{x_e\}_{e \in E} \rangle.$$

by Proposition 37.8 we know that the map $p\colon X \to X/T$ is a homotopy equivalence, from Proposition 18.16 (1a) we obtain that p_* is an isomorphism

Proposition 18.16 (1b)

isomorphism given by Proposition 20.7

If one goes through the explicit isomorphism, then one sees that the map given in (3) is indeed an isomorphism.

Finally suppose that X is a finite 1-dimensional CW-complex. It is not terribly hard to show directly that E has cardinality $1 - \chi(X)$. But since we proved the desired statement regarding the rank of $\pi_1(X, x_0)$ already in Proposition 20.5 we will not work out the details of the argument. ■

We conclude this discussion of CW-complexes with an application to group theory, namely to the study of free groups. Recall that in Proposition 31.4 we used topological

[613] The two paths exist since a tree is in particular connected.

graphs to show that if F is a free group of countable rank, then every subgroup is also a free group. Now that we have developed the theory of CW-complexes we can remove the slightly artificial restriction on the cardinality of the generating set. More precisely, we have the following corollary to our previous results.

Corollary 37.10. *Every subgroup of a free group (of any rank) is again a free group.*

Remark. Even for free groups of countable rank the proof of Corollary 37.10 might be of interest, since it differs somewhat from the proof of Proposition 31.4.

PROOF. Let $F = \langle S\rangle$ be a free group on a set S and let G be a subgroup of F.

(1) By attaching one 1-cell for each $s \in S$ to a one-point set $\{P\}$ we obtain a 1-dimensional CW-complex X with one 0-cell P and with one 1-cell for each $s \in S$. By Theorem 37.9 we have $\pi_1(X, P) \cong F$. By Proposition 36.10 (6) and Proposition 29.5 there exists a covering $p\colon (Y, Q) \to (X, P)$ of degree $[F : G]$ such that $p_*(\pi_1(Y, Q)) = G$. By Corollary 16.14 the map p_* is injective. Thus we see that $\pi_1(Y, Q)$ itself is isomorphic to G. By Proposition 36.37 we can equip Y with the structure of a 1-dimensional CW-structure. We deduce from Theorem 37.9 that $G \cong \pi_1(Y, Q)$ is a free group.

(2) Now suppose that F is a free group of finite rank m and suppose that G is a finite-index subgroup of F. By Proposition 37.4 we know that $\chi(Y) = [F : G] \cdot \chi(X)$. It is now a consequence of Theorem 37.9 that $\operatorname{rank}(G) - 1 = [F : G] \cdot (\operatorname{rank}(F) - 1)$. ■

Example. Let G be the CW-complex consisting of one 0-cell v and two 1-cells which we denote by s and t. By a slight abuse of notation we denote the corresponding elements in $\pi_1(G, v)$ by s and t as well. By Proposition 20.5 we can identify $\pi_1(G, v)$ with the free group $\langle s, t\rangle$ on the generators s and t. Now let $\phi\colon \pi_1(G, v) = \langle s, t\rangle \to \mathbb{Z}$ be the homomorphism that is given by $\phi(s) = 0$ and $\phi(t) = 1$. By Proposition 29.5 there exists a covering $q\colon (\widehat{G}, \widehat{v}) \to (G, v)$ with $q_*(\pi_1(\widehat{G}, \widehat{v})) = \ker(\phi)$, furthermore by Proposition 29.8 this covering is unique up to equivalence, in the sense of the definition on page 832.

We consider the infinite covering illustrated in Figure 313, which is very similar to the finite covering studied on pages 914.[614] For $i \in \mathbb{Z}$ we denote by x_i the loop in $(\widetilde{G}, \widetilde{v})$ as illustrated in Figure 632. We denote by x_i also the corresponding element in $\pi_1(\widetilde{G}, \widetilde{v})$. It follows from Theorem 37.9 that $\pi_1(\widetilde{G}, \widetilde{v})$ equals the free group $\langle \dots x_{-2}, x_{-1}, x_0, x_1, x_2 \dots\rangle$. It is straightforward to see that

$$p_*(x_i) = t^i s t^{-i}.$$

By Corollary 16.14 the homomorphism p_* is a monomorphism. Thus we see that $t^i s t^{-i}$, $i \in \mathbb{Z}$, generate an infinitely generated free subgroup of the finitely generated generated

[614] This covering can also be described rigorously. More precisely, we view G as the union of two copies A and B of S^1 glued along the common point 1. We define

$$\widetilde{G} = (\mathbb{R} \sqcup (S^1 \times \mathbb{Z}))/(n \sim (1, n)) \text{ for } n \in \mathbb{Z}.$$

Then the projection map

$$p\colon (\mathbb{R} \sqcup (S^1 \times \mathbb{Z}))/\sim \; \to \; G = (S^1 \times \{A\} \sqcup S^1 \times \{B\})/(1, A) \sim (1, B)$$

is given by $p(x) = (e^{2\pi i x}, A)$ for $x \in \mathbb{R}$ and $p((z, n)) = (z, B)$ for $(z, n) \in S^1 \times \mathbb{Z}$.

free group $\langle s,t\rangle$. Furthermore we see that $p_*(\pi_1(\widetilde{G},\widetilde{v})) \subset \ker(\phi\colon \pi_1(G,v)\to\mathbb{Z})$. It is easy to see that the deck transformation group is isomorphic to $\mathbb{Z}$.

Now we claim that the coverings $p\colon \widetilde{G}\to G$ and $q\colon \widehat{G}\to G$ are equivalent. By Proposition 29.8 it suffices to show that $p_*(\pi_1(\widetilde{G},\widetilde{v})) = q_*(\pi_1(\widehat{G},\widehat{v}))$. We consider the following sequence of maps

$$\mathbb{Z} \underset{\substack{\uparrow\\ \text{see above}}}{\cong} D(p\colon \widetilde{G}\to G) \underset{\substack{\uparrow\\ \text{isomorphism of Proposition 34.6}}}{\cong} \pi_1(G,v)/p_*(\pi_1(\widetilde{G},\widetilde{v})) \underset{\substack{\uparrow\\ \text{since } p_*(\pi_1(\widetilde{G},\widetilde{v}))\subset\ker(\phi\colon \pi_1(G,v)\to\mathbb{Z})}}{\twoheadrightarrow} \pi_1(G,v)/\ker(\phi\colon \pi_1(G,v)\to\mathbb{Z}) \xrightarrow[\cong]{\phi} \mathbb{Z}$$

Thus, reading from left to right, we get an epimorphism from $\mathbb{Z}$ onto itself. But any such epimorphism has to be an isomorphism. It follows that the third map is also an isomorphism, i.e. $p_*(\pi_1(\widetilde{G},\widetilde{v})) = \pi_1(G,v)/\ker(\phi\colon \pi_1(G,v)\to\mathbb{Z})$, but the latter group is precisely $q_*(\pi_1(\widehat{G},\widehat{v}))$. So we get the desired equality of subgroups.

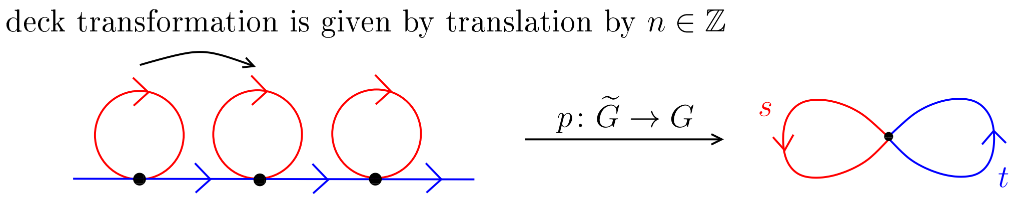


FIGURE 631

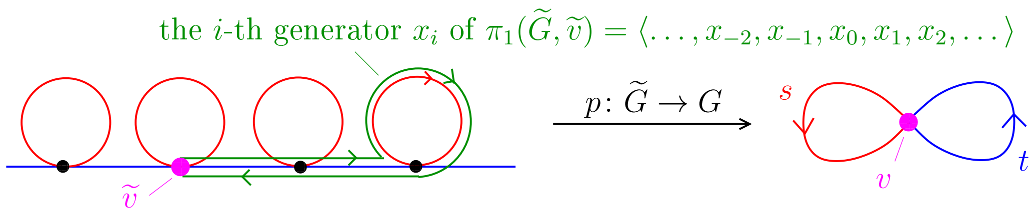


FIGURE 632

37.4. **Fundamental groups of CW-complexes.** In this section we will discuss fundamental groups of CW-complexes. For the statement of our first proposition it is useful to introduce the following definition.

The following definition is evidently inspired by the definition of a CW-complex.

Definition. Let X be a topological space and let $\varphi\colon S^{n-1}\to X$ be a map. We refer to $X\cup_\varphi \overline{B}^n$ as the *topological space X with an n-cell attached via the attaching map φ*. We refer to the image of $\overline{B}^n$ in $X\cup_\varphi\overline{B}^n$ as an n-cell.

Remark. On page 692 we gave an explicit homeomorphism $\operatorname{Cone}(S^{n-1})\xrightarrow{\cong}\overline{B}^n$. Using this homeomorphism one sees almost immediately that, using the notation from the above definition, we have a homeomorphism $\operatorname{Cone}(\varphi\colon S^{n-1}\to X)\xrightarrow{\cong}X\cup_\varphi\overline{B}^n$ which is the identity on X. In other words, attaching a cell is just a special case of taking a mapping cone.

The following proposition now tells us how attaching a cell affects the fundamental group of a given topological space.

Proposition 37.11. *Let X be a topological space, let $n \geq 1$, let $\varphi\colon S^{n-1} \to X$ be a map and let x_0 be a point in $\varphi(S^{n-1})$. Then the following hold:*

$(n > 2)$ *The inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^n, x_0)$ is an isomorphism.*

$(n = 2)$ *The map $\varphi\colon S^1 \to X$ defines an element in $\pi_1(X, x_0)$ that we denote by $[\varphi]$, and the inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^n, x_0)$ descends to an isomorphism*

$$\pi_1(X, x_0)/\langle\langle[\varphi]\rangle\rangle \xrightarrow{\cong} \pi_1\big(X \cup_\varphi \overline{B}^2, x_0\big).$$

$(n = 1)$ *Let $\epsilon \in \{-1, 1\}$ such that $x_0 = \varphi(\epsilon)$. We set $y_0 := \varphi(-\epsilon)$.*

(a) *Suppose that x_0 and y_0 lie in the same path-component of X. Let γ be a loop in $X \cup_\varphi \overline{B}^1$ that is given by the concatenation of the obvious path in the 1-cell from $x_0 = \varphi(\epsilon)$ to $y_0 = \varphi(-\epsilon)$ and some path in X from $y_0 = \varphi(-\epsilon)$ to $x_0 = \varphi(\epsilon)$. With this notation the map*

$$\pi_1(X, x_0) * \langle t\rangle \ \to \ \pi_1(X \cup_\varphi \overline{B}^1, x_0)$$

that is given by the inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^1, x_0)$ and given by $t \mapsto [\gamma]$ is an isomorphism.

(b) *Suppose that x_0 and y_0 lie in two different path components of X. Then we get an isomorphism*

$$\pi_1(X, x_0) * \pi_1(X, y_0) \ \to \ \pi_1(X \cup_\varphi \overline{B}^1, x_0).$$

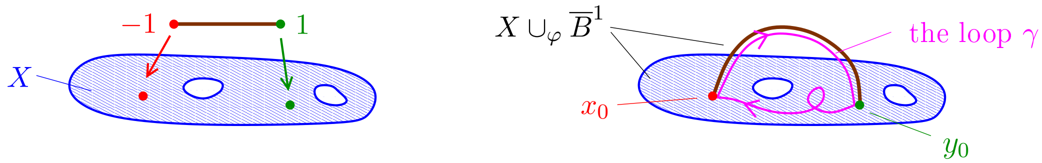


FIGURE 633. Illustration of Proposition 37.11 $(n-1)$ (a).

Examples.

(1) Let $X = S^1$ and let $n \in \mathbb{N}$. We attach to S^1 a 2-cell via the attaching map $\varphi\colon S^1 \to S^1$ that is given by $\varphi(z) = z^n$. Then[615]

$$\pi_1\big(S^1 \cup_\varphi \overline{B}^2, 1\big) \underset{\substack{\uparrow \\ \text{by Proposition 37.11}}}{=} \pi_1(S^1, 1)/\langle[\varphi]\rangle \underset{\substack{\uparrow \\ \text{see Proposition 16.17}}}{=} \langle x\rangle/\langle x^n\rangle \ \cong \ \mathbb{Z}_n.$$

(2) We attach a 2-cell to $X = \mathbb{R}^2$ via the attaching map $\mathrm{id} = \varphi\colon S^1 \to S^1$ that is given by the identity. The resulting topological space is illustrated in Figure 634. This topological space is again simply connected by Proposition 37.11.

(3) Using Theorem 37.9 and Proposition 37.11 we can now easily compute fundamental groups of finite CW-complexes. For example, as in Figure 635 we view the surface Σ of genus 2 as a CW-complex with one 0-cell P, four 1-cells x_1, y_1, x_2, y_2 and one 2-cell. We denote by X the 1-skeleton of Σ and we denote by $\varphi\colon S^1 \to X$ the attaching map

[615]We leave it to the reader to show that for $n = 2$ the topological space $S^1 \cup_\varphi \overline{B}^2$ is actually homeomorphic to $\mathbb{RP}^2$.

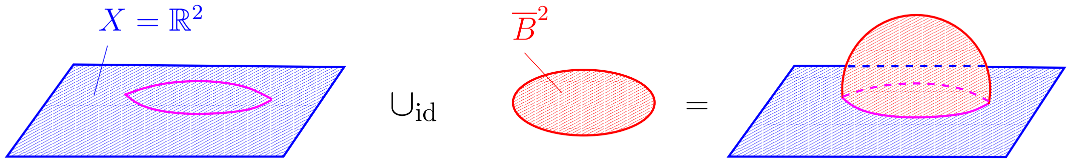


FIGURE 634

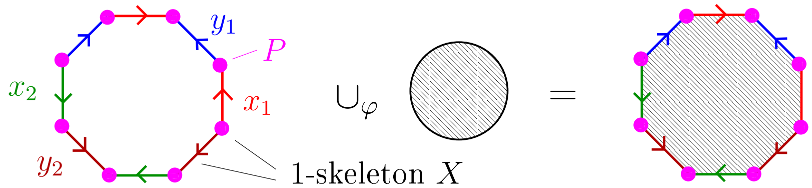


FIGURE 635

of the 2-cell. We have

$$\pi_1(\Sigma,P) \;=\; \pi_1\big(X\cup_\varphi \overline{B}^2,P\big) \underset{\substack{\uparrow\\ \text{by Proposition 37.11}}}{=} \pi_1(X,P)/\langle\langle[\varphi]\rangle\rangle \underset{\substack{\uparrow\\ \text{by Theorem 37.9 and since } [\varphi]=[x_1,y_1][x_2,y_2]}}{=} \langle x_1,y_1,x_2,y_2\rangle/\langle\langle [x_1,y_1][x_2,y_2]\rangle\rangle.$$

Of course we had obtained the same result much earlier. The advantage of CW-complexes is that Proposition 37.11 absorbs many of the technical arguments we had to deal with in the proof of Proposition 22.3, i.e. in our initial calculation of $\pi_1(\Sigma,P)$.

PROOF OF PROPOSITION 37.11 FOR $n\geq 2$ $(*)$. Let X be a topological space, let $n\geq 2$, let $\varphi\colon S^{n-1}\to X$ be a map and let x_0 be a point in $\varphi(S^{n-1})$. The key idea of the proof is to apply the Seifert–van Kampen Theorem 22.1 to the following two open[616] subsets of $X\cup_\varphi \overline{B}^n$

$$U \;=\; X\cup_\varphi \big\{z\in \overline{B}^n \,\big|\, \|z\|>\tfrac14\big\} \quad\text{ and }\quad V \;=\; \big\{z\in \overline{B}^n \,\big|\, \|z\|<\tfrac34\big\}.$$

The remainder of the proof consists in showing that the resulting isomorphism implies the desired, a priori slightly different, isomorphism.

We pick a point $x_1\in U\cap V$ and we pick a path in U from x_0 to x_1. We refer to Figure 636 for an illustration. Note that V is a ball, hence $\pi_1(V,x_1)$ is the trivial group.

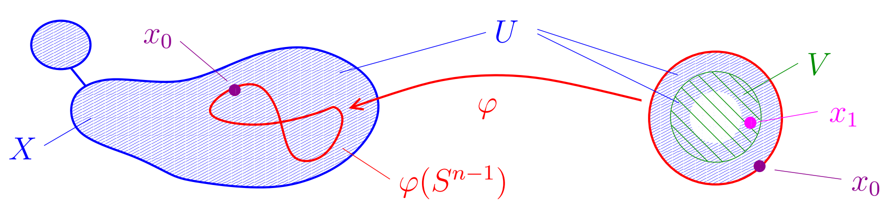


FIGURE 636

[616] Why are they open subsets?

We consider the following diagram of maps

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r] \arrow[d, "\cong"'] & \pi_1(X\cup_\varphi \overline{B}^n,x_0) \arrow[d, "="'] \\
\pi_1(U,x_0) \arrow[r] \arrow[d, "\cong"'] & \pi_1(X\cup_\varphi \overline{B}^n,x_0) \arrow[d, "\cong"] \\
\pi_1(U,x_1) \arrow[r] \arrow[d] & \pi_1(X\cup_\varphi \overline{B}^n,x_1) \arrow[d, "="] \\
\pi_1(U,x_1)/\langle\langle \pi_1(U\cap V,x_1)\rangle\rangle \arrow[r] \arrow[d, "\cong"', "\text{Lemma 21.22}"] & \pi_1(X\cup_\varphi \overline{B}^n,x_1) \arrow[d, "="] \\
\pi_1(U,x_1) *_{\pi_1(U\cap V,x_1)} \underbrace{\pi_1(V,x_1)}_{=\{e\}} \arrow[r, "\cong", "\text{Seifert–van Kampen}"'] & \pi_1(X\cup_\varphi \overline{B}^n,x_1).
\end{tikzcd}$$

We want to prove the following three statements:

(i) the diagram commutes,

(ii) all vertical maps except possibly the third vertical map on the left are isomorphisms,

(iii) the bottom horizontal map is an isomorphism.

The two vertical maps on the second level are the isomorphisms induced by the choice of base point coming from the chosen path from x_0 to x_1. All the other vertical and horizontal maps are induced by inclusions or they are the obvious algebraic map. It follows easily that the diagram commutes.

Most of the maps are obviously isomorphisms. Using Lemma 18.23 one can easily show that X is a deformation retract of U. It follows from Lemma 18.14 and Proposition 18.16 (2) that the top left vertical map is an isomorphism. The bottom vertical map on the left is an isomorphism by Lemma 21.22.

Finally the bottom horizontal map is an isomorphism by the Seifert–van Kampen Theorem 22.1.

Now we consider the case $n \geq 3$. In this case $\pi_1(U\cap V, x_1)$ is trivial since $U\cap V$ is homotopy equivalent to S^{n-1} which has trivial fundamental group by Proposition 14.14. It is now clear that the one missing vertical map on the left is also an isomorphism. Since all vertical maps are now isomorphisms and since the bottom horizontal map is an isomorphism it follows that the top horizontal map is an isomorphism.

Finally we consider the case $n = 2$. In this case $U \cap V$ is homotopy equivalent to S^1 which implies that $\pi_1(U\cap V,x_1) \cong \mathbb{Z}$. Under the isomorphism $\Phi\colon \pi_1(X,x_0)\to\pi_1(U,x_1)$ this infinite cyclic subgroup corresponds to $\langle[\varphi]\rangle$. It is straightforward to see that we get a commutative diagram

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[rr] \arrow[dr] \arrow[dd] & & \pi_1\big(X\cup_\varphi \overline{B}^n,x_0\big) \arrow[dd] \\
& \pi_1(X,x_0)/\langle\langle[\varphi]\rangle\rangle \arrow[ur] \arrow[dl, "\cong"'] & \\
\pi_1(U,x_1)/\langle\langle \pi_1(U\cap V,x_1)\rangle\rangle \arrow[rr, "\cong"'] & & \pi_1\big(X\cup_\varphi \overline{B}^n,x_1\big).
\end{tikzcd}$$

This concludes the proof of the proposition. ■

Proof of Proposition 37.11 for $n = 1$ $(*)$. The proof of the case $n = 1$ is quite similar to the proof of the case $n \geq 2$. The only serious difference is that we need to replace the Seifert–van Kampen Theorem 22.1 by Corollary 26.5, which in turn is a corollary to the HNN-Seifert–van Kampen Theorem 26.3. We leave it to the reader to fill in the details. ■

In the following corollary we will see that every finitely presented group is the fundamental group of a finite CW-complex. The key to proving the corollary is the following construction, which associates to any finite presentation a 2-dimensional CW-complex. The construction is illustrated in Figure 637. If the reader feels comfortable with that illustration it might be best to skip the official definition.

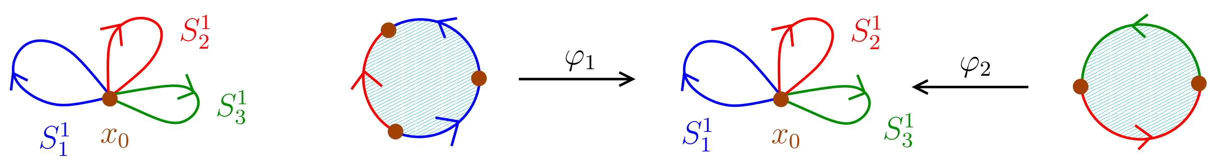
Construction. Let $\langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m\rangle$ be a finite presentation. For $i = 1, \ldots, k$ let S^1_i be a copy of S^1. For $j = 1, \ldots, m$ we do the following:

(1) We write the relation r_j as a reduced word $g_{s(1)}^{\epsilon_1} \cdot \cdots \cdot g_{s(n)}^{\epsilon_n}$ with $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$.
(2) We denote by $\varphi_j \colon S^1 = [0,1]/0 \sim 1 \to \bigvee\limits_{i=1}^{k} S^1_i$ the (hopefully obvious) path which on the interval $[\frac{i-1}{n}, \frac{i}{n}]$ travels through $S^1_{s(i)}$ at constant speed in the direction determined by ϵ_i.

We denote by X the CW-complex that is given by attaching m 2-cells to $\bigvee\limits_{i=1}^{k} S^1_i$ along the attaching maps $\varphi_1, \ldots, \varphi_m$. We refer to X as the CW-*complex associated to the presentation.* (We refer to Figure 637 for an illustration.)

2-dimensional CW-complex for the presentation

$\langle g_1, g_2, g_3 \,|\, g_1 \cdot g_2^{-1} \cdot g_1, g_3 \cdot g_2\rangle$



CW-complex is given by attaching two 2-cells to $S^1_1 \vee S^1_2 \vee S^1_3$

Figure 637

Now we can state the corollary that we promised above.

Corollary 37.12. *Let $\langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m\rangle$ be a finite presentation for a group π. If Y is the associated 2-dimensional* CW-*complex, then $\pi_1(Y) \cong \pi$.*

Proof. Let $\pi = \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m\rangle$ be a finitely presented group. We use the notation introduced in the above construction of the CW-complex associated to π. We denote by X the wedge of the k circles and we denote by x_0 the wedge point. Following the discussion on page 606 we can make the identification $\pi_1(X, x_0) = \langle g_1, \ldots, g_k\rangle$. This identification has the property that for $j = 1, \ldots, m$ the corresponding loop $\varphi_j \colon S^1 \to X$ represents $r_j \in \pi_1(X, x_0) = \langle g_1, \ldots, g_k\rangle$. We write $Y_0 = X$ and for $j = 1, \ldots, m$ we iteratively define Y_j to be the 2-dimensional CW-complex that is given by attaching a 2-cell to Y_{j-1} with the attaching map φ_j. We obtain that

$$\begin{array}{rcl} & \text{Proposition 37.11, applied } m \text{ times} & \text{since } [\varphi_j]=r_j \\ & \downarrow & \downarrow \\ \pi_1(Y_m,x_0) & \cong \langle g_1,\dots,g_k\rangle/\langle\!\langle[\varphi_1]\rangle\!\rangle\dots/\langle\!\langle[\varphi_m]\rangle\!\rangle & = \langle g_1,\dots,g_k\rangle/\langle\!\langle r_1\rangle\!\rangle\dots/\langle\!\langle r_m\rangle\!\rangle \\ & = \langle g_1,\dots,g_k\rangle/\langle\!\langle r_1,\dots,r_m\rangle\!\rangle & = \langle g_1,\dots,g_k\,|\,r_1,\dots,r_m\rangle. \\ & \uparrow & \uparrow \\ & \text{by Exercise 21.1} & \text{by definition} \end{array}$$

■

The following proposition says in particular that for many CW-complexes the fundamental group of a CW-complex is the same as the fundamental group of its 2-skeleton.

Proposition 37.13. *Let X be a connected* CW*-complex and let x_0 be a point in the 0-skeleton of X.*

(1) *If X has finitely many cells in each dimension, then the following statements hold:*
   (a) *The inclusion induced map*[617] *$\pi_1(X^1,x_0)\to\pi_1(X,x_0)$ is an epimorphism.*
   (b) *For any $k\geq 2$ the inclusion induced map $\pi_1(X^k,x_0)\to\pi_1(X,x_0)$ is an isomorphism.*
(2) *If X has only finitely many cells in dimensions 0 and 1, then $\pi_1(X,x_0)$ is finitely generated.*
(3) *If X has only finitely many cells in dimensions 0, 1 and 2, then $\pi_1(X,x_0)$ is finitely presented.*

Remark. In Proposition 40.9 we will generalize Proposition 37.13 to all connected CW-complexes.

Example. Let $X=\mathbb{RP}^\infty$ be the infinite real projective space with the CW-structure from page 942. For this CW-complex the 1-skeleton is $X^1=\mathbb{RP}^1\cong S^1$ and the 2-skeleton is $X^2=\mathbb{RP}^2$. It follows from Proposition 37.13 (1) that $\mathbb{Z}\cong\pi_1(S^1)\to\pi_1(\mathbb{RP}^\infty)$ is an epimorphism and it follows from Proposition 37.13 (1) that $\mathbb{Z}_2\cong\pi_1(\mathbb{RP}^2)\to\pi_1(\mathbb{RP}^\infty)$ is an isomorphism. This calculation of $\pi_1(\mathbb{RP}^\infty)$ is certainly easier than the argument from page 983.

Proof.

(1) We first suppose that X is a finite CW-complex. Then the two statements follow immediately from iteratively applying Proposition 37.11 finitely many times. If X is an infinite CW-complex, but with finitely many cells in each dimension, then one can reduce the two statements to the finite case using Propositions 37.3. We leave the details to the reader.
(2) Now we turn to the proof of (2). By (1) it suffices to show that the fundamental group of a finite connected 1-dimensional CW-complex is finitely generated, but we showed this in Theorem 37.9.
(3) Finally we turn to the proof of (3). By (1) it suffices to show that a finite connected 2-dimensional CW-complex X has a finitely presented fundamental group. We pick $x_0\in X$. We denote by $\varphi_1,\dots,\varphi_k$ the attaching maps of the 2-cells. For $i=1,\dots,k$

[617] Recall that for $k\in\mathbb{N}$ we denote by X^k the k-skeleton of X, i.e. the union of all cells of dimension $\leq k$.

we pick a path p_i from x_0 to a point on $\varphi_i(S^1)$ and we write $\psi_i = p_i * \varphi_i * \overline{p}_i$. We then have

$$\begin{aligned} \pi_1(X) &\underset{\substack{\uparrow \\ \text{Propositions 14.11 and 37.11}}}{\cong} \pi_1(X^1)/\langle\langle[\psi_1]\rangle\rangle \ldots /\langle\langle[\psi_k]\rangle\rangle \underset{\substack{\uparrow \\ \text{by Exercise 21.1}}}{=} \pi_1(X^1)/\langle\langle[\psi_1], \ldots, [\psi_k]\rangle\rangle \\ &\underset{\substack{\uparrow \\ \text{Statement (2)}}}{=} \{\text{finitely generated free group}\}/\langle\langle[\psi_1], \ldots, [\psi_k]\rangle\rangle. \end{aligned}$$

We have thus shown that $\pi_1(X)$ is a finitely presented group. ■

Remark. On page 936 we already mentioned Theorem 64.5 which says in particular that every compact smooth manifold admits the structure of a finite CW-complex. Together with Proposition 37.13 (3) we now obtain that the fundamental group of any connected compact smooth manifold M is finitely presented.

The following corollary is an immediate but rather convenient consequence of Proposition 37.13 (1).

Corollary 37.14. *Let X be a connected* CW*-complex with countably many cells. If X has no 1-cells, then X is simply connected, i.e. $\pi_1(X) = 0$.*

Examples.

(1) On page 935 we pointed out that for $n \geq 2$ the sphere S^n admits a CW-structure with one 0-cell and one n-cell. From Corollary 37.14 we now recover the result from Proposition 14.14 that S^n is simply connected.

(2) It follows from Lemma 36.1 and Corollary 37.14 that for any n we have $\pi_1(\mathbb{C}\mathrm{P}^n) = 0$. Similarly it follows, using Lemma 36.6, that $\pi_1(\mathbb{C}\mathrm{P}^\infty) = 0$.

We conclude this chapter with a purely group theoretic result. To put the result in context, first recall that in Exercise 21.2 we saw that every finite-index subgroup of a finitely generated group is finitely generated. Now we can prove the following analogue.

Proposition 37.15. *Every finite-index subgroup of a finitely presented group is finitely presented.*

Remark. We will provide a proof of Proposition 37.15 that makes heavy use of topological methods. Not surprisingly one can also give a purely group theoretic proof. In fact such a proof can be given by the Reidemeister-Schreier process that we alluded to in Section 27.7. We refer to [**LS77**, p. 103] and [**MKS76**, p. 90] for details.

PROOF. Let $\pi = \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle$ be a finitely presented group. Let Y be the finite 2-complex associated to this presentation that we constructed on page 995. By Corollary 37.12 we know that $\pi_1(Y) \cong \pi$. Now let Γ be a finite-index subgroup of π. By Proposition 36.10 together with Proposition 29.5 there exists a finite covering $p\colon \widetilde{Y} \to Y$ such that $\pi_1(\widetilde{Y}) \cong \Gamma$. By Proposition 36.37 (2) the 2-complex $\widetilde{Y}$ has finitely many 1-cells and 2-cells. Thus it follows from Proposition 37.13 (3) that $\Gamma \cong \pi_1(\widetilde{Y})$ is a finitely presented group. ■

Exercises for Chapter 37.

Exercise 37.1. Let $A = S^1 \times [0,1]$ be an annulus.

(a) (i) We consider

$$X \ := \ \Big(\bigcup_{k\in\mathbb{N}_0} A \times \{k\}\Big)/(z,0,k) \sim (z^2,1,k-1) \qquad \text{for } z \in S^1, k \in \mathbb{N}.$$

Determine $\pi_1(X)$.

(ii) We consider

$$Y \ := \ \Big(\bigcup_{k\in\mathbb{N}_0} A \times \{-k\}\Big)/(z,0,-k) \sim (z^2,1,-k-1) \qquad \text{for } z \in S^1, k \in \mathbb{N}.$$

Determine $\pi_1(Y)$.

In both cases the group you obtain is an abelian group that is isomorphic to a subgroup of $(\mathbb{R},+)$.

(b) Give an example of topological space Z such that $\pi_1(Z) \cong (\mathbb{Q},+)$.

Exercise 37.2. Next let X and Y be two finite CW-complexes. On page 961 we introduced the product CW-structure which by Proposition 36.23 is indeed a CW-structure for the topological space $X \times Y$. Is $\chi(X \times Y)$ determined by $\chi(X)$ and $\chi(Y)$?

38. CW-complexes III: Advanced properties

In this chapter we will prove several key technical theorems on CW-complexes, in particular we will prove the Homotopy Extension Theorem 38.1 and the Cellular Approximation Theorem 38.13. The reader who is in a hurry could just have a look at the Cellular Approximation Theorem 38.13 and then move on.

38.1. **The Homotopy Extension Theorem.** On several occasions we will need the following somewhat technical theorem.

Theorem 38.1. (Homotopy Extension Theorem) *Let X be a* CW*-complex, let Y be a topological space and let $f_0\colon X \to Y$ be map. Furthermore, suppose that $A \subset X$ is a subcomplex and suppose we are given a homotopy $G\colon A \times [0,1] \to Y$ from $g_0 := f_0|_A$ to some map $g_1\colon A \to Y$. Then there exists a homotopy $F\colon X \times [0,1] \to Y$ starting from f_0 such that the restriction $F|_{A\times[0,1]}$ equals G.*

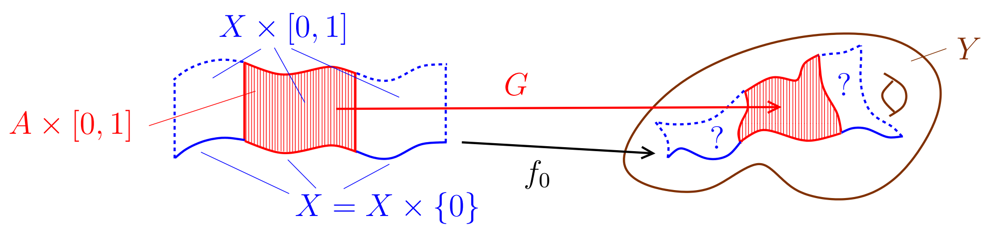


FIGURE 638. Illustration of the Homotopy Extension Theorem 38.1.

Remark. The statement of the Homotopy Extension Theorem does not hold for arbitrary subsets of X. For example, let $X = \mathbb{R}$ and $A = (0,1)$. We write $Y = \mathbb{R}\times\{0\}\cup(0,1)\times[0,1]$ and let $f\colon X \to Y$ be the obvious inclusion map and let $G\colon (0,1)\times[0,1] \to Y$ be the obvious inclusion map. (See Figure 639 for an illustration.) It is an amusing exercise to show that G cannot be extended to a homotopy on all of $\mathbb{R}$.[618] It is an equally amusing exercise to show that the homotopy can be extended if throughout we replace the open interval $A = (0,1)$ by the closed interval $A = [0,1]$.

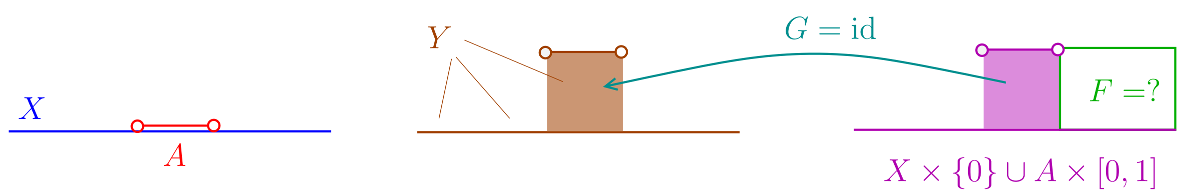


FIGURE 639

Before we delve into the long and strenuous proof of the Homotopy Extension Theorem 38.1 let us prove the following neat corollary.

[618] Here is a hint: if there was an extension of G to a homotopy $F\colon X\times[0,1] \to Y$, what would/should $F(1,1)$ be?

Corollary 38.2. *Let X be connected* CW*-complex, let P be a point in the 0-skeleton of X and let Q be some other point on Y.*

(1) *Given any path $\gamma\colon [0,1] \to X$ with $\gamma(0) = P$ and $\gamma(1) = Q$ there exists a homotopy $F\colon X \times [0,1] \to X$ with $F(P,t) = \gamma(t)$ for all $t \in [0,1]$ and such that $F_0 = \mathrm{id}$.*
(2) *There exists a map $h\colon X \to X$ with $h(P) = Q$ that is homotopic to the identity.*

Example. We consider the CW-complex X and the two points P and Q shown in Figure 640. It should be pretty clear that there is no homeomorphism f of X with $f(P) = Q$. But Corollary 38.2 gives us a decent replacement, namely a map $h\colon X \to X$ with $h(P) = Q$ which is homotopic to the identity.

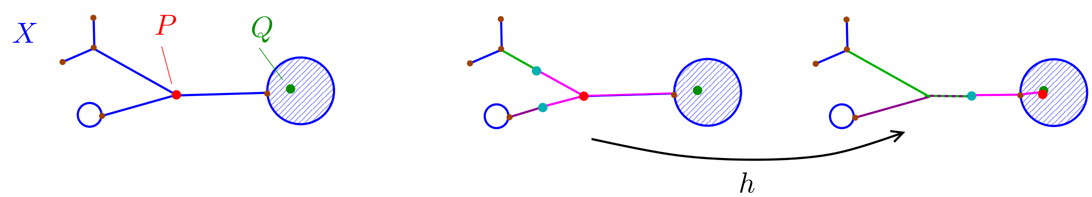


FIGURE 640

PROOF. Let X be connected CW-complex, let P be a point in the 0-skeleton of X and let Q be some other point on Y.

(1) Let $\gamma\colon [0,1] \to X$ be a path with $\gamma(0) = P$ and $\gamma(1) = Q$. We write $A := \{P\}$. Since P is a point in the 0-skeleton of X we see that A is a subcomplex of A. We apply the Homotopy Extension Theorem 38.1 to $f_0 = \mathrm{id}\colon X \to X$, $g_0(P) := P$, $g_1(P) = Q$ and $G(P,t) = \gamma(t)$. The resulting homotopy $F\colon X \times [0,1] \to X$ has all the desired properties.
(2) Since X is connected we obtain from Proposition 36.10 that there exists a map $\gamma\colon [0,1] \to X$ with $\gamma(0) = P$ and $\gamma(1) = Q$. From (1) we obtain a homotopy $F\colon X \times [0,1] \to X$ with $F_0 = \mathrm{id}$ and $F_1(P) = Q$. The map $h := F_1$ evidently has all the promised properties. ■

Before we can provide the proof of Theorem 38.1 we will need to formulate and prove several lemmas.

Lemma 38.3. *For any $n \in \mathbb{N}_0$ there exists a deformation retraction from the solid cylinder $\overline{B}^n \times [0,1]$ to the "cup" $\overline{B}^n \times \{0\} \cup S^{n-1} \times [0,1]$.*

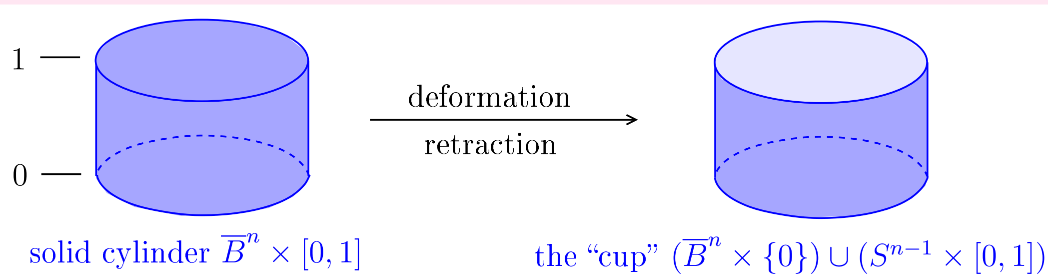


FIGURE 641

Proof. It is straightforward to verify that the map

$$\begin{array}{rcl} r\colon \overline{B}^n \times [0,1] & \to & (\overline{B}^n \times \{0\}) \cup (S^{n-1} \times [0,1]) \\ P & \mapsto & \begin{array}{c}\text{the unique point on } \overline{B}^n \times \{0\} \cup S^{n-1} \times [0,1] \text{ that} \\ \text{gets hit by the ray from } (0,2) \in \mathbb{R}^n \times \mathbb{R} \text{ through } P.\end{array} \end{array}$$

is a retraction. We refer to Figure 642 for an illustration. Since $\overline{B}^n \times [0,1]$ is convex we can define the corresponding deformation retraction as follows:

$$\begin{array}{rcl} (\overline{B}^n \times [0,1]) \times [0,1] & \to & \overline{B}^n \times [0,1] \\ (P,t) & \mapsto & P \cdot (1-t) + r(P) \cdot t. \end{array}$$

■

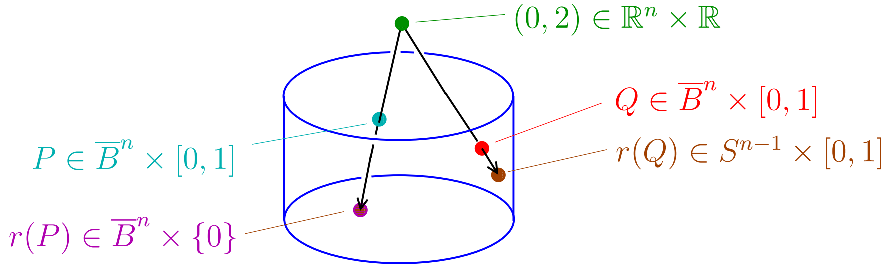


Figure 642

Lemma 38.4. *If X is a* CW*-complex and if A is a subcomplex of X, then there exists a deformation retraction from $X \times [0,1]$ to the subcomplex $(X \times \{0\}) \cup (A \times [0,1])$.*

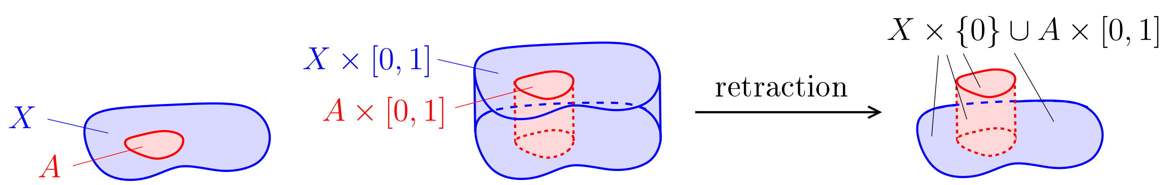


Figure 643. Illustration of Lemma 38.4.

Proof. Let X be a CW-complex and let A be a subcomplex. For each $n \in \mathbb{N}_0$ we set

$$Z_n := X \times \{0\} \cup (X^{n-1} \cup A) \times [0,1].$$

Note that each Z_n is a subcomplex of the product CW-complex $X \times [0,1]$. Evidently we have a sequence

$$(X \times \{0\}) \cup (A \times [0,1]) = Z_0 \subset Z_1 \subset Z_2 \subset \ldots \quad \text{with } X \times [0,1] = \bigcup_{n \in \mathbb{N}_0} Z_n.$$

Thus it follows from Proposition 36.22 that it suffices to prove the following claim.

Claim. For every $n \in \mathbb{N}_0$ there exists a deformation retraction from Z_{n+1} to Z_n.

Let $n \in \mathbb{N}_0$. We denote by $\Phi_i\colon \overline{B}^n \to X$, $i \in I$, the characteristic maps of all the n-cells of X that do not lie in A. By Lemma 38.3 there exists a deformation retraction $r\colon \overline{B}^n \times [0,1] \times [0,1] \to \overline{B}^n \times [0,1]$ from $\overline{B}^n \times [0,1]$ to $\overline{B}^n \times \{0\} \cup S^{n-1} \times [0,1]$. The idea is to apply this retraction to all k-cells of X to obtain the desired deformation retraction. More precisely we consider the map

$$\begin{array}{rcl} r_n\colon \overbrace{(X\times\{0\}\cup(X^n\cup A)\times[0,1])}^{=Z_{n+1}}\times[0,1] & \to & \overbrace{X\times\{0\}\cup(X^n\cup A)\times[0,1]}^{=Z_{n+1}} \\ ((P,t),s) & \mapsto & \begin{cases} \Phi_i(r((Q,t),s)), & \text{if } P=\Phi_i(Q) \text{ with } \\ & \qquad Q\in\overline{B}^n \text{ and } i\in I, \\ (P,t), & \text{otherwise.} \end{cases} \end{array}$$

This map is easily seen to be a deformation retraction from Z_{n+1} to Z_n.[619] ■

Now we can provide the proof of Theorem 38.1.

Proof of Theorem 38.1. Let X be a CW-complex, let Y be a topological space and let $f_0\colon X\to Y$ be map. Furthermore, suppose that $A\subset X$ is a subcomplex and suppose we are given a homotopy $G\colon A\times[0,1]\to Y$ from $g_0:=f_0|_A$ to some map $g_1\colon A\to Y$. By Lemma 38.4 there exists a retraction

$$H\colon X\times[0,1] \;\to\; (X\times\{0\})\cup(A\times[0,1]).$$

The theorem follows from the following claim.

Claim. The map

$$\begin{array}{rcll} F\colon X\times[0,1] \xrightarrow{H} (X\times\{0\})\cup(A\times[0,1]) & \to & Y & \\ (x,0) & \mapsto & f_0(x), & \text{for } x\in X, \\ (a,t) & \mapsto & G(a,t), & \text{for } (a,t)\in A\times[0,1]. \end{array}$$

has all the desired properties.

First we have to verify that the map is continuous. In fact, for continuity we do have to worry a little bit about the right-hand map. But note that Lemma 36.18 (2) says that the subcomplex A is a closed subset of X. Thus we see that the map on the right-hand side is continuous when restricted to the *closed* subsets $X\times\{0\}$ and $A\times[0,1]$. Thus it is continuous by Lemma 2.35 (2). Note it is clear that F has the desired properties, i.e. that it is a homotopy $F\colon X\times[0,1]\to Y$ starting from f_0 such that the restriction $F|_{A\times[0,1]}$ equals G. ■

The following proposition gives a typical application of the Homotopy Extension Theorem 38.1.

Proposition 38.5. *Let $f\colon A\to B$ be a cellular map between* CW*-complexes that has the following property:*

(∗) *For each n-cell σ of A we have either $f(\sigma)\subset B^{n-1}$ or $f|_\sigma$ is a homeomorphism.*

We consider the corresponding mapping cylinder

$$\mathrm{Cyl}(f) \;:=\; \mathrm{Cyl}(f\colon A\to B) \;:=\; ((A\times[0,1])\sqcup B)/\sim \;\; \textit{where } (a,1)\sim f(a) \textit{ for all } a\in A.$$

[619] It follows from Lemma 36.7 (4) that the map is in fact continuous.

that we defined on page 697. Finally let Y be a CW*-complex together with a cellular embedding $i \sqcup j\colon A \sqcup B \to Y$. The map*

$$\begin{array}{rcl} \Phi\colon (Y \sqcup \mathrm{Cyl}(f\colon A \to B))/\sim & \to & Y/i(a) \sim j(f(a)) \\ {[P]} & \mapsto & \begin{cases} [P], & \text{if } P \in Y, \\ [i(a)], & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1], \\ [j(b)], & \text{if } P = [b] \text{ with } b \in B \end{cases} \end{array}$$

is a homotopy equivalence.

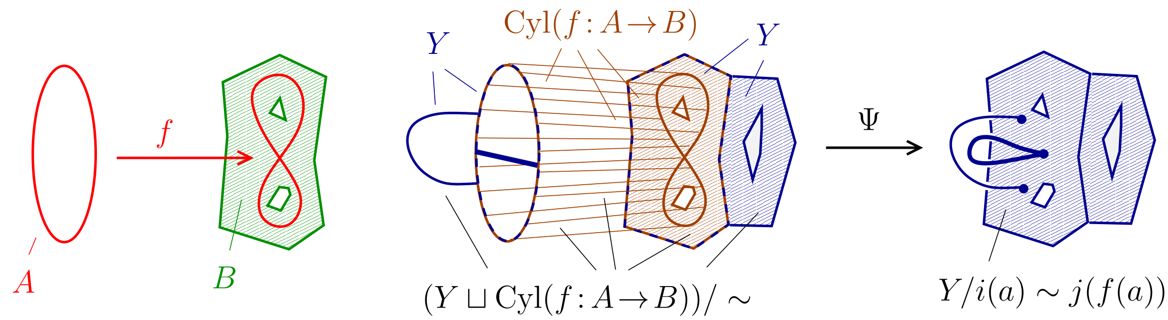


FIGURE 644. Illustration of Proposition 38.5.

Example. In many applications the map f will actually be a cellular isomorphism. In this special case Proposition 38.5 basically says that one can compress a product $A \times [0,1]$ to a single copy of A without changing the homotopy type. Two such examples are shown in Figure 645.

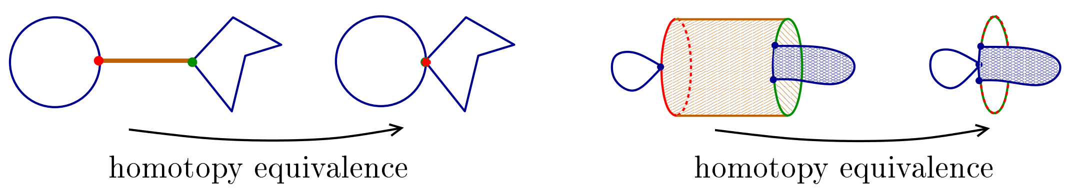


FIGURE 645

Remark. Note that the statement of Proposition 38.5 is related in spirit to the statement of Lemma 39.9 (2).

PROOF. First note that by Lemma 36.19 we can and will view $A \sqcup B$ as a subcomplex of Y. In the following let $R\colon \mathrm{Cyl}(f\colon A \to B) \times [0,1] \to B$ be the natural deformation retraction given by Lemma 24.8. Furthermore note that it follows quite easily from Corollary 36.35 (3) and Lemma 36.32 (2) that we can and will equip $\mathrm{Cyl}(f)$ and $(Y \sqcup \mathrm{Cyl}(f))/\sim$ with natural CW-structures. Finally, it follows from our slightly technical hypothesis on f together with Lemma 36.32 (6) that $Y/\sim$ also admits a natural CW-structure. Next we consider the homotopy

$$\begin{array}{rcl} G\colon (A \sqcup B) \times [0,1] & \to & \mathrm{Cyl}(f) \subset (Y \sqcup \mathrm{Cyl}(f))/\sim \\ {[(Q,t)]} & \mapsto & \begin{cases} R([(Q,0)],t), & \text{if } Q \in A, \\ Q, & \text{if } Q \in B. \end{cases} \end{array}$$

We obtain the following diagram:

$$\begin{tikzcd}
(A \sqcup B) \times \{0\} \arrow[rr, hook] \arrow[dd, hook, "i \sqcup j"'] & & (A \sqcup B) \times [0,1] \arrow[dl, "G"'] \arrow[dd, hook, "(i \sqcup j) \times \mathrm{id}"] \\
& (Y \sqcup \mathrm{Cyl}(f))/\sim & \\
Y \times \{0\} \arrow[rr, hook] \arrow[ur, hook] & & Y \times [0,1] \arrow[ul, dashed, "F"'] \\
& Y/a \sim f(a) \arrow[uu, dashed, "\Psi"] & Y. \arrow[l, two heads] \arrow[u, "y \mapsto (t,1)"']
\end{tikzcd}$$

We make the following observations and clarifications:

(1) The map $i \sqcup j\colon A \sqcup B \to Y$ is the inclusion map.

(2) It follows from the Homotopy Extension Theorem 38.1 and the fact that $A \sqcup B$ is a subcomplex of Y that there exists a map $F\colon Y \times [0,1] \to (Y \sqcup \mathrm{Cyl}(f))/\sim$ that makes the two triangles involving F commute.

(3) For any $a \in A$ we have

$$F_1(a) \;=\; F(a,1) \;=\; G(a,1) \;=\; \underbrace{f(a)}_{\in B} \;=\; G(f(a),1) \;=\; F(f(a),1) \;=\; F_1(f(a)).$$

It follows from this calculation that the map $F_1\colon Y \to (Y \sqcup \mathrm{Cyl}(f))/\sim$ descends to a map $\Psi\colon Y/a \sim f(a) \to (Y \sqcup \mathrm{Cyl}(f))/\sim$.

It remains to show the following claim.

Claim. The maps $\Phi\colon (Y \sqcup \mathrm{Cyl}(f))/\sim \to Y/a \sim f(a)$ and $\Psi\colon Y/a \sim f(a) \to (Y \sqcup \mathrm{Cyl}(f))/\sim$ are homotopy inverses of one another.

First we consider the map

$$\begin{array}{rcl} H\colon (Y/a \sim f(a)) \times [0,1] & \to & Y/a \sim f(a) \\ ([\underbrace{P}_{\in Y}],t) & \mapsto & \Phi(F(P,t)). \end{array}$$

One can easily verify that the map is, despite first appearances, actually well-defined. Using the fact that $Y/a \sim f(a)$ is a CW-complex and using Corollary 36.9 one can show quite easily that H is continuous. Basically by construction we have $H_0 = \mathrm{id}$ and $H_1 = \Phi \circ \Psi$. Thus we have found our first homotopy. Finally we consider the map

$$\begin{array}{rcl} H\colon (Y \sqcup \mathrm{Cyl}(f))/\sim & \to & (Y \sqcup \mathrm{Cyl}(f))/\sim \\ ([P],t) & \mapsto & \begin{cases} R(P,t), & \text{if } P \in \mathrm{Cyl}(f), \\ F(P,t), & \text{if } P \in Y. \end{cases} \end{array}$$

Similar to the above one sees that H is continuous, that $H_0 = \mathrm{id}$ and that $H_1 = \Psi \circ \Phi$. ■

38.2. **Dropping base points.** Even though base points are necessary and valuable, in practice they are frequently a major nuisance. Fortunately, using the Homotopy Extension Theorem 38.1, we can prove a proposition that often allows us to stop worrying about base points.

We start out with a few definitions. Some of them we have encountered before.

Definition.

(1) A *pointed topological space* is a pair (X, x_0) where X is a topological space and $x_0 \in X$ is a point. We refer to x_0 as the *base point* of the pointed topological space.
(2) A map $f\colon (X, x_0) \to (Y, y_0)$ between pointed topological spaces is a map $f\colon X \to Y$ with $f(x_0) = y_0$.
(3) We say that two maps $f, g\colon (X, x_0) \to (Y, y_0)$ between pointed topological spaces are *homotopic* if there exists a homotopy $F\colon X \times [0,1] \to Y$ rel $\{x_0\}$ from f to g.
(4) Given two pointed topological spaces (X, x_0) and (Y, y_0) we write[620]
$$\langle (X, x_0), (Y, y_0) \rangle := \{\text{maps } (X, x_0) \to (Y, y_0)\}/\sim \quad \begin{array}{c}\text{where } f \sim g \text{ if there exists a} \\ \text{homotopy in the above sense} \\ \text{between } f \text{ and } g.\end{array}$$

Example. Let (X, x_0) be a pointed topological space, let $n \in \mathbb{N}$ and let $*$ be the usual base point of S^n. By definition we have $\langle (S^n, *), (X, x_0) \rangle = \pi_n(X, x_0)$.

We have a natural map $\langle (X, x_0), (Y, y_0) \rangle \to [X, Y]$. It is worth spending a minute on trying to find examples that show that in general this map is neither an injection nor a surjection. In this section we want to give some criteria that give us some control over this map. We start out with the following easy, albeit fairly restrictive, example.

Proposition 38.6. *Let (Y, y_0) be a path-connected topological space. If $\pi_1(Y, y_0)$ is abelian, then the map*
$$\begin{array}{rcl} \langle (S^1, *), (Y, y_0) \rangle & \to & [S^1, Y] \\ [f] & \mapsto & [f] \end{array}$$
is a bijection.

Proof. It follows from the fact that Y is path-connected together with Proposition 18.33 (1) that the map is a surjection. Furthermore it follows from the hypothesis that $\pi_1(Y, y_0)$ is abelian together with Proposition 18.33 (2) that the map is an injection. ■

Now we want to move away from the fairly simple topological space S^1. In the proof of Proposition 18.33 we had to construct fairly explicitly maps on S^1 and $S^1 \times [0,1]$. In the following we will now replace S^1 by CW-complexes, we have now developed several tools for building maps *out* of a CW-complex. This leads us to the following definition.

Definition. A *pointed* CW-*complex* is a pair (X, x_0) where X is a CW-complex and x_0 is a point in the 0-skeleton X^0 of X.

Proposition 38.7. *Let (X, x_0) be pointed a* CW-*complex and let (Y, y_0) be a pointed topological space. We consider the map*
$$\begin{array}{rcl} \langle (X, x_0), (Y, y_0) \rangle & \to & [X, Y] \\ [f] & \mapsto & [f]. \end{array}$$

[620] The notation $\langle X, Y \rangle$ goes back to [**Hat02**, p. 357], where it says "the notation $\langle \ldots \rangle$ is intended to suggest pointed homotopy classes".

The following two statements hold:

(1) *If Y is path-connected, then the map $\langle (X,x_0),(Y,y_0)\rangle \to [X,Y]$ is a surjection.*
(2) *If Y is simply connected, then the map $\langle (X,x_0),(Y,y_0)\rangle \to [X,Y]$ is a bijection.*

Remark. In Proposition 121.25 we will formulate a variation on Proposition 38.7 for CW-complexes with abelian fundamental groups.

For the record we state the following corollary to our results.

Corollary 38.8.

(1) *Let (X,x_0) be a path-connected pointed topological space and let $n \in \mathbb{N}$. If $n=1$, then we assume that $\pi_1(X,x_0)$ is abelian and if $n \geq 2$ we assume that $\pi_1(X,x_0)$ is trivial. In this setting the map*

$$\underbrace{\langle (S^n,x_0),(X,x_0)\rangle}_{=\pi_n(X,x_0)} \xrightarrow{[f]\mapsto[f]} [S^n,X]$$

is a bijection.

(2) *Given any $n \in \mathbb{N}$ and given any base point $x_0 \in S^n$ the map*

$$\langle (S^n,x_0),(S^n,x_0)\rangle \xrightarrow{[f]\mapsto[f]} [S^n,S^n]$$

is a bijection.

Proof of Corollary 38.8.

(1) (a) The case $n=1$ is precisely the content of Proposition 38.6.
 (b) Now let $n \in \mathbb{N}_{\geq 2}$. We can and will equip S^n with a CW-structure such that x_0 is a 0-cell. The desired statement now follows from Proposition 38.7 (2).
(2) This statement follows immediately from (1) and the fact, established in Corollary 16.18 and Proposition 14.14, that for $n \geq 2$ the fundamental group $\pi_1(S^n)$ is trivial. ■

Now we turn to the proof of Proposition 38.7. This proof relies very much on the following lemma.

Lemma 38.9. *Let (X,x_0) and (Y,y_0) be pointed CW-complexes and let $F\colon X\times[0,1]\to Y$ be a homotopy with $F_0(x_0)=F_1(x_0)=y_0$. We consider the loop $\gamma\colon [0,1]\to Y$ that is given by $\gamma(t):=F(x_0,t)$. If the loop γ is null-homotopic, then there exists a homotopy $J\colon X\times[0,1]\to Y$ with $J_0=F_0$ and $J_1=F_1$ and such that $J(x_0,t)=y_0$ for all $t\in[0,1]$.*

Proof of Lemma 38.9. We prove the lemma in three easily digestible steps:

(1) We consider the loop $\gamma\colon [0,1]\to Y$ that is given by $\gamma(t):=F(x_0,t)$. By our hypothesis the loop γ is null-homotopic. By definition this means that there exists a map $\varphi\colon [0,1]\times[0,1]\to Y$ with $\varphi(t,0)=\gamma(t)$ for all $t\in[0,1]$ and such that $\varphi(x,y)=y_0$ for all other points in $\partial([0,1]\times[0,1])$.

(2) We consider the map[621]

$$G\colon \overbrace{(X\times[0,1])\times\{0\}\cup\underbrace{(X\times\{0,1\}\cup\{x_0\}\times[0,1])}_{\text{subcomplex of } X\times[0,1]}\times[0,1]}^{\text{subcomplex of }(X\times[0,1])\times[0,1]} \to Y$$
$$(x,s,t) \mapsto \begin{cases} F(x,s), & \text{if } t=0,\\ F(x,s), & \text{if } s=0,1,\\ \varphi(s,t), & \text{if } x=x_0.\end{cases}$$

It follows from Lemma 36.20 that this map G is continuous.

(3) By the Homotopy Extension Theorem 38.1 the map G from (2) extends to a map $H\colon (X\times[0,1])\times[0,1]\to Y$. It follows basically immediately from the definitions that $J := H_1\colon X\times[0,1]\to Y$ has the desired properties. ■

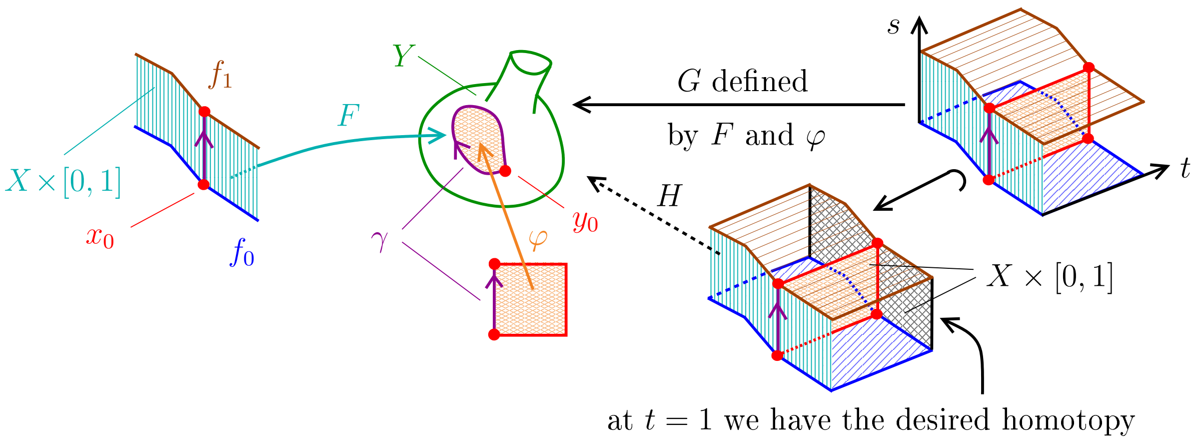


Figure 646. Illustration for the proof of Lemma 38.9.

Now we turn to the actual proof of Proposition 38.7.

Proof of Proposition 38.7. Let (X,x_0) and (Y,y_0) be pointed CW-complexes.

(1) We suppose that Y is path-connected. Let $f\colon X\to Y$ be map. We set $y_1=f(x_0)$. Since Y is path-connected there exists a path $\gamma\colon[0,1]\to Y$ with $\gamma(0)=y_1$ and $\gamma(1)=y_0$. Since $\{x_0\}$ is a subcomplex of X we obtain from Corollary 38.2 that we can extend γ, viewed as a homotopy $\{x_0\}\times[0,1]\to Y$, to a homotopy $F\colon X\times[0,1]\to Y$. This shows that $f=F_0$ is homotopic to a map $g:=F_1$ with $g(x_0)=y_0$. This implies that the map $\langle (X,x_0),(Y,y_0)\rangle\to[X,Y]$ is a surjection.

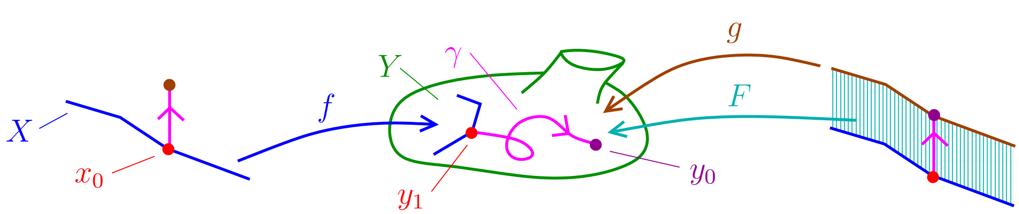


Figure 647. Illustration for the proof of Proposition 38.7 (2).

[621] Secretly we use Proposition 36.23 which tells us in this instance that the product CW-structure $X\times[0,1]\times[0,1]$ is indeed a CW-structure for the topological space $X\times[0,1]\times[0,1]$.

(2) By (1) it remains to show that the map $\langle (X,x_0),(Y,y_0)\rangle \to [X,Y]$ is injective. Thus let $f_0,f_1\colon (X,x_0)\to (Y,y_0)$ be two maps which represent the same element in $[X,Y]$. This means that there exists a homotopy $F\colon X\times[0,1]\to Y$ with $F_0=f_0$ and $F_1=f_1$. We need to show that there exists a homotopy $J\colon X\times[0,1]\to Y$ with $J_0=f_0$ and $J_1=f_1$ and such that $J(x_0,t)=y_0$ for all $t\in[0,1]$. We consider the loop $\gamma\colon [0,1]\to Y$ that is given by $\gamma(t):=F(x_0,t)$. Since Y is simply connected we know that γ is null-homotopic. Thus we can apply Lemma 38.9 and we obtain the desired homotopy J. ■

We conclude this section with the following proposition.

Proposition 38.10. $(*)$ *Let X be a CW-complex and let x_0 and x_1 be two points in the 0-skeleton of X. If X is connected, then there exists a map $f\colon X\to X$ with the following properties:*

(1) *The map f is homotopic to the identity.*
(2) *We have $f(x_0)=x_1$.*
(3) *The map f defines a homotopy equivalence $(X,x_0)\to(X,x_1)$ of pointed topological spaces.*

Proof $(*)$. Since X is connected we obtain from Proposition 36.10 (7) that there exists a path $\gamma\colon[0,1]\to X$ with $\gamma(0)=x_0$ and $\gamma(1)=x_1$. Since x_0,x_1 lie in the 0-skeleton of X we can apply Corollary 38.2 and we obtain a homotopy $F\colon X\times[0,1]\to X$ with $F(x_0,t)=\gamma(t)$ for all $t\in[0,1]$ and such that $F_0=\mathrm{id}$. We set $f:=F_1$. Evidently f is homotopic to the identity and $f(x_0)=x_1$. It remains to show that $f\colon (X,x_0)\to(X,x_1)$ is a homotopy equivalence of pointed topological spaces.

We need to find a homotopy inverse to f. Once again we appeal to Corollary 38.2 to obtain a homotopy $G\colon X\times[0,1]\to X$ with $G(x_1,t)=\overline{\gamma}(t):=\gamma(1-t)$ for all $t\in[0,1]$ and such that $G_0=\mathrm{id}$. We set $g:=G_1$. We need to show that $g\circ f\colon (X,x_0)\to(X,x_0)$ and $f\circ g\colon (X,x_1)\to(X,x_1)$ are homotopic[622] to the identity. We consider the homotopy[623]

$$\begin{array}{rcl} F\colon X\times[0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} F(x,2t), & \text{if } t\in[0,\frac12], \\ G(f(x),2t-1), & \text{if } t\in[\frac12,1]. \end{cases} \end{array}$$

Note that this is a homotopy from id_X to $g\circ f\colon X\to X$. Furthermore note that the loop given by $\gamma(t)=F(x_0,t)$ is precisely the loop $\gamma*\overline{\gamma}$. By Proposition 14.6 (3) we know that this loop is null-homotopic. Thus we can apply Lemma 38.9. We obtain a homotopy $J\colon X\times[0,1]\to X$ with $J_0=F_0=\mathrm{id}_X$, $J_1=F_1=g\circ f$ and $J(x_0,t)=x_0$ for all $t\in[0,1]$. But that is precisely what we needed. ■

38.3. **Combining infinitely many homotopies and the telescope construction** $(*)$.

In Lemma 18.3 we saw that one can combine finitely many homotopies. Later on we will want to combine infinitely many homotopies of CW-complexes. This slightly delicate procedure can be performed using the following lemma.

[622] Here, as a reminder, "homotopic" means "homotopic as maps of pointed topological spaces".

[623] It follows from Lemma 14.3 that F is indeed continuous.

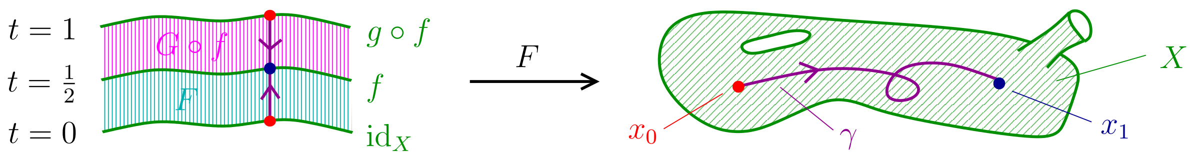


FIGURE 648. Illustration for the proof of Proposition 38.10.

Lemma 38.11. *Let X be a CW-complex, let A be a subset of X and let Y be a topological space. Finally let $W_0 \subset W_1 \subset W_2 \subset \ldots$ be a sequence of subcomplexes of X such that for each $k \in \mathbb{N}_0$ there exists an $l \in \mathbb{N}_0$ such that the k-skeleton X^k is contained in W_l. (In many applications we have $W_k = X^k$.) Let $f_0, f_1, \ldots$ be maps from X to Y. Suppose that for each $k \in \mathbb{N}_0$ we are given a homotopy $H(k)\colon X \times [0,1] \to Y$ rel $A \cup W_k$ from f_k to f_{k+1}. (Since it is easy to overlook this bit of the hypothesis we stress that $H(k)$ is in particular a homotopy rel W_k). We define*[624]

$$\begin{aligned} g\colon X &\to Y \\ x &\mapsto f_k(x) \quad \text{where } k \in \mathbb{N}_0 \text{ such that } x \in W_k. \end{aligned}$$

For $k = 0, 1, 2, \ldots$ we write $s_k = \sum_{j=1}^{k} \frac{1}{2^j}$. Then the map

$$\begin{aligned} H\colon X \times [0,1] &\to Y \\ (x,t) &\mapsto \begin{cases} H(k)(x, 2^{k+1} \cdot (t - s_k)), & \text{if } t \in [s_k, s_{k+1}), \\ g(x), & \text{if } t = 1 \end{cases} \end{aligned}$$

that is illustrated in Figure 649 is a homotopy rel A from f_0 to g. We refer to the homotopy H as the combination *of $\{H(k)\}_{k \in \mathbb{N}_0}$.*

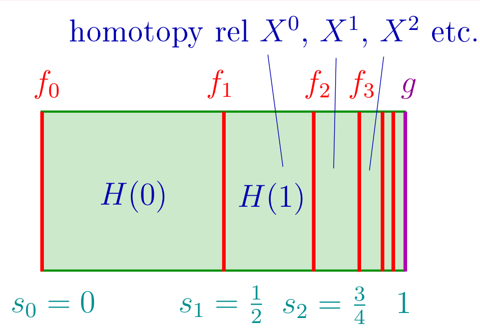


FIGURE 649. Illustration of Lemma 38.11.

PROOF. We only need to show that H is continuous. Note that for each $l \in \mathbb{N}_0$ it follows from our choice of maps f_l and Lemma 14.3 that the restriction of H to $W_l \times [0,1]$ is continuous. Since each X^k is contained in some W_l we see that the restriction of H to each $X^k \times [0,1]$ is continuous. But now it follows from Corollary 36.9 that $H\colon X \times [0,1] \to Y$ is also continuous. ■

[624]It follows easily from the setup that the map g is well-defined, i.e. given $x \in X$ the point $g(x)$ does not depend on the choice of k.

We conclude this technical section with an application that we will need once, namely in the proof Proposition **??**. But the application doubles also as a nice illustration for how to apply the previous lemma.

Definition. Let Z be a CW-complex. We equip $[0,\infty)$ with the CW-structure where the 0-skeleton is given by $\mathbb{N}_0$. We equip $Z \times [0,\infty)$ with the product CW-structure. We refer to the subcomplex

$$T(Z) \;:=\; \bigcup_{i\in\mathbb{N}_0} Z^i \times [i,i+1] \quad\subset\; Z\times[0,\infty)$$

as the *telescope of* Z. .

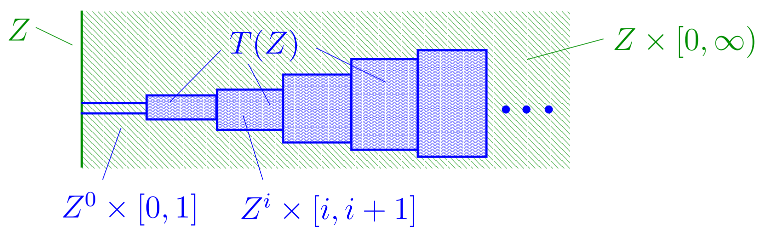


FIGURE 650

The following lemma shows that the telescope $T(Z)$ is homotopy equivalent to Z.

Lemma 38.12. *Let Z be a* CW*-complex. The following statements hold:*

(1) *$T(Z)$ is a deformation retract of $Z\times[0,\infty)$.*
(2) *The telescope $T(Z)$ is homotopy equivalent to Z.*

Remark. One can also show that Lemma 38.12 is a consequence of the much more general Proposition 119.10.

PROOF. First note that $Z = Z\times\{0\}$ is clearly a deformation retract of $Z\times[0,\infty)$. Thus it follows from Lemmas 18.14 and 18.11 that (2) is a consequence of (1). Therefore it remains to prove (1).

As in the definition of the telescope $T(Z)$ we view $[0,\infty)$, $X := Y := Z\times[0,\infty)$ and $T(Z)$ as CW-complexes. It follows from Lemma 38.4 that for each $k\in\mathbb{N}_0$ there exists a deformation retraction

$$r_k\colon (Z\times[0,1])\times[0,1] \;\to\; Z\times[0,1]$$

from $Z\times[0,1]$ to $(Z^k\times[0,1])\cup(Z\times\{1\})$. We denote by

$$\begin{array}{rcl}\rho_k\colon Z\times[k,k+1] & \to & (Z^{k-1}\times[k,k+1])\cup(Z^k\times\{k+1\})\\ (z,t) & \mapsto & (r_k(z,t-k)+(0,k),1)\end{array}$$

the corresponding retraction, with the intervals shifted by k. We make the following preparations:

(1) Given $k\in\mathbb{N}_0$ we define $W_k := Z^k\times[0,\infty)$. Note that it follows from the definition of the product CW-structure on $Z\times[0,\infty)$ that the k-skeleton of $X = Z\times[0,\infty)$ is contained in W_{k+1}.

(2) For each $k \in \mathbb{N}_0$ we consider the map

$$\begin{array}{rcl} f_k\colon X = Z \times [0,\infty) & \to & Y = Z \times [0,\infty) \\ (z,t) & \mapsto & \begin{cases} (z,t), & \text{if } x \in Z^k \text{ and } t \geq k+1, \\ (\rho_k \circ \cdots \circ \rho_1 \circ \rho_0)(x), & \text{if } z \in Z^k \text{ and } t < k+1. \end{cases} \end{array}$$

(3) We consider the map

$$\begin{array}{rcl} H(k)\colon X \times [0,1] & \to & Y = Z \times [0,\infty) \\ ((z,t),s) & \to & \begin{cases} (z,t), & \text{if } t \geq k+1, \\ r_{k+1}(\rho_k(z,t),s), & \text{if } t < k+1. \end{cases} \end{array}$$

This defines a homotopy from f_k to f_{k+1} rel W_k. Note that $H(0)_1 = \mathrm{id}_X$.

As in the statement of Lemma 38.11 we consider the map

$$\begin{array}{rcll} g\colon X & \to & Y & \\ x & \mapsto & f_k(x) & \text{where } k \in \mathbb{N}_0 \text{ such that } x \in W_k. \end{array}$$

One can easily verify that this is a retraction of $X = Z \times [0,\infty)$ to the telescope $T(Z)$. It follows from Lemma 38.11 that there exists a homotopy from $H(0)_1 = \mathrm{id}$ to g, i.e. $T(Z)$ is in fact a deformation retract of $X = Z \times [0,\infty)$. ■

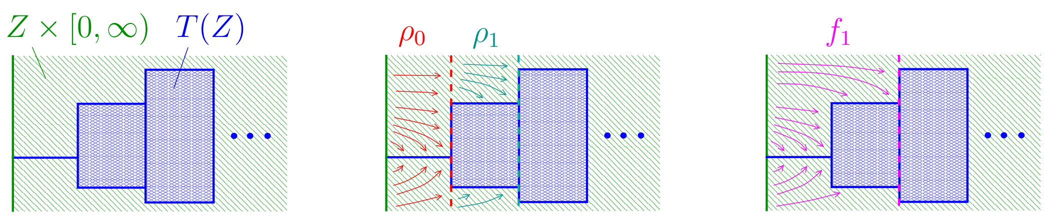


FIGURE 651. Illustration for the proof of Lemma 38.12.

38.4. **The Cellular Approximation Theorem.** We recall the following definition from page 955.

Definition. A map $g\colon X \to Y$ between CW-complexes is called *cellular* if for each $n \in \mathbb{N}_0$ we have $g(X^n) \subset Y^n$

It will be instructive to return to the example of a non-cellular map that we gave on page 955.

Example. In general maps between CW-complexes are not cellular. For example consider the CW-complex $X = [0,1]$ with two 0-cells and one 1-cell and we consider the CW-complex $Y = [0,1] \times [0,1]$ with four 0-cells, two 1-cells and one 2-cell. Then the diagonal map

$$\begin{array}{rcl} g\colon X = [0,1] & \to & Y = [0,1] \times [0,1] \\ t & \mapsto & (t,t) \end{array},$$

which is illustrated in Figure 652 on the left, is not a cellular map.

The following theorem is the main result of this section.

Theorem 38.13. (Cellular Approximation Theorem) *Let $g\colon X \to Y$ be a map between* CW*-complexes and let A be a subcomplex of X such that the restriction of g to A is a cellular map.*[625] *The map g is homotopic rel A to a cellular map $\widetilde{g}\colon X \to Y$.*

Remark. Let $g\colon X \to Y$ be a *homeomorphism* between CW-complexes. By the Cellular Approximation Theorem 38.13 we know that g is homotopic to a cellular map h. But in general we cannot arrange that h is again a *homeomorphism*. For example consider the torus $T = ([0,1]\times[0,1])/\sim = \mathbb{R}^2/\mathbb{R}^2$ with the standard CW-structure from page 933 which has precisely one 0-cell $*$ that corresponds to the origin. As in Theorem 16.16 we make the identification $\pi_1(\mathbb{R}^n/\mathbb{Z}^2,*)$ with $\mathbb{Z}^2$. Let $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ v & \mapsto & P\cdot v. \end{array}$$

In Lemma 6.35 we saw that this map is a homeomorphism. Furthermore, in Exercise 16.16 we saw that the induced map on $\pi_1(\mathbb{R}^2/\mathbb{Z}^2,*) = \mathbb{Z}^2$ is given by multiplication by P. On the other hand the map on $\pi_1(\mathbb{R}^2/\mathbb{Z}^2,*) = \mathbb{Z}^2$ induced by any *cellular homeomorphism* $\widetilde{g}$ on $\mathbb{R}^2/\mathbb{Z}^2$ is represented by a matrix of the form[626] $\begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix}$ or $\begin{pmatrix} 0 & \pm1 \\ \pm1 & 0 \end{pmatrix}$. It follows from Exercise 18.36 (a) that g cannot be homotopic, rel $*$, to any such $\widetilde{g}$.

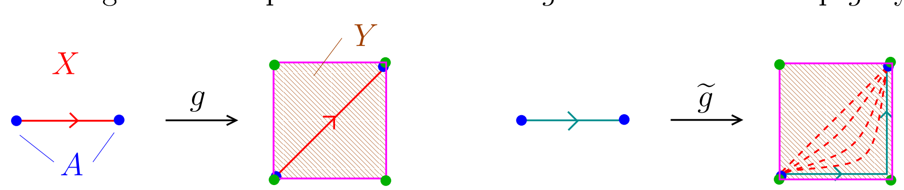
Example. We consider again the above example of a non-cellular map. More precisely, we consider again the above diagonal map

$$\begin{array}{rcl} g\colon X = [0,1] & \to & Y = [0,1]\times[0,1] \\ t & \mapsto & (t,t). \end{array}$$

We want to deform g into a cellular map, while keeping the endpoints fixed. Put differently, we want to find a homotopy rel $A = \{(0,0),(1,1)\}$ from g to a cellular map $\widetilde{g}$. Such a map $\widetilde{g}$ is given by

$$\begin{array}{rcl} \widetilde{g}\colon X = [0,1] & \to & Y = [0,1]\times[0,1] \\ t & \mapsto & \begin{cases} (2t,0), & \text{if } t\in[0,\frac{1}{2}] \\ (1,2t-1), & \text{if } t\in[\frac{1}{2},1] \end{cases} \end{array}$$

One way of stating this example is that we turn g into a cellular map $\widetilde{g}$ by "pushing g



g is not a cellular map $\qquad\qquad$ $\widetilde{g}$ is a cellular map

FIGURE 652

off the high-dimensional cell". The proof of Cellular Approximation Theorem 38.13 is built

[625]For example $f\colon X \to Y$ could be any map between CW-complexes, then the hypothesis is satisfied for A the empty set.

[626]Why is that?

on the same concept. But somewhat similar to the proof of Proposition 14.14, we have to worry about space-filling curves that exist by Proposition 2.60.

Before we attack the proof of the Cellular Approximation Theorem 38.13 we want to formulate a relative version thereof. To formulate this relative version we need the following semi-obvious definition.

Definition. A *pair of* CW*-complexes* is a pair (X, A) where X is a CW-complex and A is a subcomplex of X.

Now we can formulate the promised theorem.

Theorem 38.14. (Cellular Approximation Theorem for Pairs) *Suppose we are given a map $g\colon (X,A) \to (Y,B)$ between pairs of* CW*-complexes. Let Z be a subcomplex of A such that the restriction of g to Z is a cellular map. Then g is homotopic rel Z to a cellular map $\widetilde{g}\colon (X,A) \to (Y,B)$.*[627]

Proof assuming the Cellular Approximation Theorem 38.13. The desired homotopy is constructed as follows:

(1) By the Cellular Approximation Theorem 38.13 applied to $g|_A\colon A \to B$ there exists a homotopy F rel Z from $g|_A\colon A \to B$ to a cellular map.
(2) By the Homotopy Extension Theorem 38.1 we can extend the homotopy F from (1) to a homotopy $G\colon X \times [0,1] \to Y$. Note that G is still a homotopy rel Z. We write $h = g_1\colon X \to Y$. Note that $h|_A$ is cellular and that $h(A) \subset B$.
(3) We apply the Cellular Approximation Theorem 38.13 to h and we obtain a homotopy H rel A from h to a cellular map.
(4) The combination of the homotopies G and H has all the desired properties. ■

Now we turn to the proof of Cellular Approximation Theorem 38.13. The following proposition is the key step towards the desired proof.

Proposition 38.15. *Let V be a* CW*-complex and W be a topological space. Furthermore let $\psi\colon S^{n-1} \to V^{n-1}$ be a map to the $(n-1)$-skeleton of V and let $\varphi\colon S^{k-1} \to W$ be another map. Finally let*

$$h\colon V \cup_\psi \overline{B}^n \;\to\; W \cup_\varphi \overline{B}^k$$

be a map with $h(\psi(S^{n-1})) \subset W$. If $n < k$, then h is homotopic rel V to a map h' such that $h'(B^n)$ does not intersect B^k.

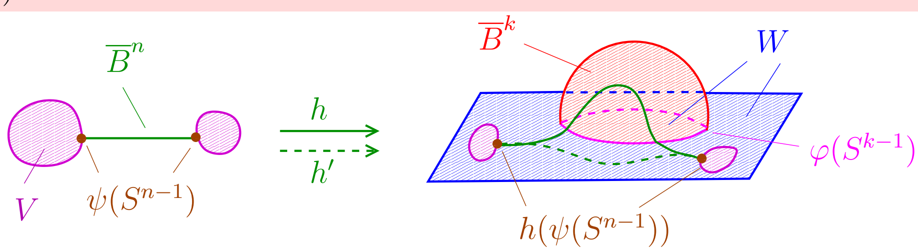


Figure 653. Illustration of Proposition 38.15.

[627]Here we refer to page 572 for the discussion on the notion of homotopies of pairs of topological spaces rel a subset.

The statement of Proposition 38.15 is fairly obvious, if there exists a point $P \in B^k$ that does not get hit by $h(B^n)$. Indeed, if such a point P exists, then we can use the deformation retraction $\overline{B}^n \setminus \{p\} \to S^{n-1}$ to push h on B^n off the ball B^k. Put in general such a point P does not exist, since even for $n < k$ there exist continuous surjective maps $B^n \to B^k$. To get around this problem requires a fairly hard, not particularly inspiring, argument which we will give later on in Section 38.5.

In the following we will use Proposition 38.15 to prove the Cellular Approximation Theorem 38.13.

Corollary 38.16. *Let Z be a finite n-dimensional* CW*-complex.*

(1) *Let $g\colon Z \to Y$ be a map to a k-dimensional* CW*-complex Y with $g(Z^{n-1}) \subset Y^{n-1}$. Let B be a subcomplex of Z with $g(B) \subset Y^n$. If $n < k$, then g is homotopic rel $Z^{n-1} \cup B$ to a map g' with $g'(Z) \subset Y^{k-1}$.*
(2) *Let $g\colon Z \to Y$ be a map to a* CW*-complex Y such that $g(Z^{n-1}) \subset Y^{n-1}$. Let B be a subcomplex of Z with $g(B) \subset Y^n$. The map g is homotopic rel $Z^{n-1} \cup B$ to a map g' with $g'(Z) \subset Y^n$.*

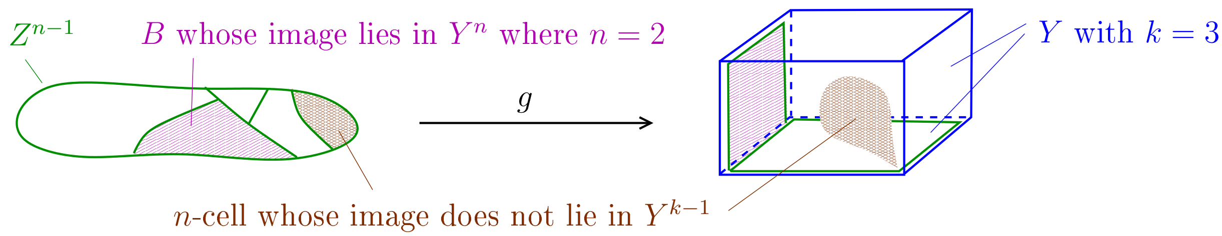


Figure 654. Illustration for Corollary 38.16 (1).

Proof. Let Z be a finite n-dimensional CW-complex. By Proposition 36.10 (3) the CW-complex Z is compact. It follows from Theorem 36.14 that the image of the compact set Z in a CW-complex is contained in a finite subcomplex. Therefore we can assume in both (1) and (2) that Y is in fact a finite CW-complex.

(1) Let $g\colon Z \to Y$ be a map to a finite k-dimensional CW-complex Y with $g(Z^{n-1}) \subset Y^{n-1}$. Let B be a subcomplex of Z with $g(B) \subset Y^n$. We suppose that $n < k$. We denote by $\widehat{Z}$ the union of Z^{n-1} with all the n-cells of Z whose image is contained in Y^{k-1}. In other words we can write $Z = \widehat{Z} \cup e_1 \cup \cdots \cup e_r$ where $e_1, \ldots, e_r$ are the n-cells such that the image is not contained in Y^{k-1}. We denote by $f_1, \ldots, f_s$ the k-cells of Y. Note that by hypothesis on B the interiors of the cells $e_1, \ldots, e_r$ are disjoint from B.

In the following we apply Proposition 38.15 precisely $r \cdot s$ times to get the desired homotopy. More precisely, we do the following:

(a) We apply Proposition 38.15 to the map

$$h\colon (\widehat{Z} \cup e_1 \cup \cdots \cup e_{r-1}) \cup e_r \quad \to \quad (Y \cup f_1 \cup \cdots \cup f_{s-1}) \cup f_s$$

to obtain a homotopy rel $\widehat{Z} \supset Z^{n-1} \cup B$ to a map h_{rs} such that the image of the interior of e_r under the map h_{rs} does not intersect the interior of f_s. Next we apply Proposition 38.15 to the map

$$h_{rs}\colon (\widehat{Z} \cup e_1 \cup \cdots \cup e_{r-2} \cup e_r) \cup e_{r-1} \quad \to \quad (Y \cup f_1 \cup \cdots \cup f_{s-1}) \cup f_s.$$

Iterating this process we obtain a homotopy rel $\widehat{Z}$ to a map such that the image of the interiors of the cells $e_1, \dots, e_r$ does not intersect the interior of f_s.

(b) Again iterating the process above we obtain a homotopy rel $\widehat{Z}$ to a map such that image of the interiors of the cells $e_1, \dots, e_r$ does not intersect the interior of any of the f_i. But that means that the image of Z lies in Y^{k-1}.

(2) Let $g\colon Z \to Y$ be a map to a CW-complex Y such that $g(Z^{n-1}) \subset Y^{n-1}$ and let B be a subcomplex of Z with $g(B) \subset Y^n$. As we pointed out in the beginning of the proof, we can assume that Y is finite-dimensional, we denote the dimension by k. But then statement (2) is just an immediate consequence of applying (1) iteratively $k - n$ times. ■

Now we turn to the proof of the Cellular Approximation Theorem 38.13.

PROOF. Let X be a CW-complex and let A be a (possibly empty) subcomplex of X. Furthermore let Y be a CW-complex. We prove the Cellular Approximation Theorem 38.13 only in the special case that X has finitely many cells. The general case is dealt with in [**Hat02**, Theorem 4.8].

Let $g\colon X \to Y$ be map.

(1) We say an n-cell e of X is *g-bad* if $g(e)$ is not contained in Y^n.
(2) If g is non-cellular, then we define

$$n(g) \;:=\; \text{minimal dimension of a cell that is } g\text{-bad}.$$

If g is cellular, then we define $n(g) := \infty$.

Claim. Let $g\colon X \to Y$ be a non-cellular map such that the restriction of g to A is a cellular map. Then there exists a homotopy from g to a map g' rel A such that $n(g') > n(g)$.

We write $n = n(g)$. We apply Corollary 38.16 (2) to $g\colon Z := X^n \to Y$ and $B = A^n$.[628] We obtain a homotopy G rel $X^{n-1} \cup A^n$ from $g\colon Z = X^n \to Y$ to a map $g'\colon Z = X^n \to Y$ with $g'(X^n) \subset Y^n$. We can extend this homotopy to a homotopy G on $(X^n \cup A) \times [0,1]$ via the trivial homotopy on $A \times [0,1]$. Now we apply the Homotopy Extension Theorem 38.1 to obtain a homotopy rel $X^n \cup A$ from $g\colon X \to Y$ to a map $g'\colon X \to Y$ such that $g'(X^n) \subset Y^n$. Note that by the choice of our homotopy we have $g'|_{X^{n-1}} = g|_{X^{n-1}}$. By definition of $n(g)$ the map $g'|_{X^{n-1}} = g|_{X^{n-1}}$ is cellular. Thus we see that $n(g') \geq n+1 = n(g)+1$. ⊞

[628]Note that by definition of n we have $g(X^{n-1}) \subset Y^{n-1}$.

Finally let $g\colon X\to Y$ be a non-cellular map such that the restriction of g to A is a cellular map. We apply the claim iteratively as many times as possible to obtain a sequence of maps $g=g_1,g_2,g_3,\ldots\colon X\to Y$, each of which is homotopic to g rel A, such that $n(g)=n(g_1)<n(g_2)<\ldots$. Note that n can take on only finitely many values, namely $\{0,\ldots,\dim(X),\infty\}$. This means that the sequence is finite which implies that the last map is cellular. ■

38.5. **Proof of Proposition 38.15** $(*)$**.** [629] Before we can even think of proving Proposition 38.15 we need to introduce a few more definitions and we need to prove a rather technical lemma.

Definition.

(1) A *closed half-space in* $\mathbb{R}^n$ is a set of the form $\{v\in\mathbb{R}^n\,|\,\langle v,w\rangle\geq 0\}$, for some fixed non-zero vector $w\in\mathbb{R}^n$.
(2) A *convex polyhedron in* $\mathbb{R}^n$ is a bounded subset of $\mathbb{R}^n$ that is the intersection of finitely many closed half-spaces.
(3) A *polyhedron in* $\mathbb{R}^n$ is the union of a finite number of convex polyhedra.[630]
(4) Given a polyhedron P a map $f\colon P\to\mathbb{R}^k$ is called *piecewise linear* if there exists a subdivision of P into finitely many convex polyhedra $Q_1,\ldots,Q_m$ such that the restriction of f to each Q_i is affine linear.[631]

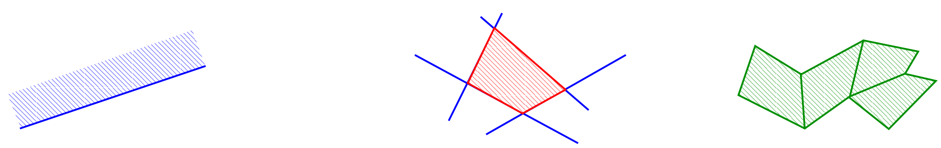


FIGURE 655

The following, very technical lemma, will get us around the problem of space filling curves, i.e. it will get us around the problem that we saw in Proposition 2.60 that there exist surjective continuous maps $\overline{B}^n\to\overline{B}^k$ even if $n<k$.

Lemma 38.17. *Let W be a topological space and let $\varphi\colon S^{k-1}\to W$ be map. Furthermore let $g\colon[0,1]^n\to Z:=W\cup_\varphi\overline{B}^k$ be a map. Then there exists a (possibly empty) polyhedron $K\subset[0,1]^n$ and a map $\widetilde{g}\colon[0,1]^n\to Z$ with the following properties:*

(1) *The map $\widetilde{g}$ is homotopic to g rel $g^{-1}(W)$.*
(2) *We have $\widetilde{g}(K)\subset B^k$.*
(3) *The restriction $\widetilde{g}|_K\colon K\to B^k$ is piecewise linear.*
(4) *There exists a non-empty open subset $U\subset B^k$ such that $\widetilde{g}^{-1}(U)\subset K$.*

[629] Here the $(*)$ indicates once again that we do not provide the proof in the lecture. You are reading this section at your own risk.

[630] Note that the union of zero convex polyhedra is the empty set.

[631] A map $f\colon V\to\mathbb{R}^k$ from a subset V of $\mathbb{R}^n$ is called *affine linear* if there exists a $P\in\mathbb{R}^k$ and a $k\times n$–matrix A such that $f(v)=P+Av$ for all $v\in V$.

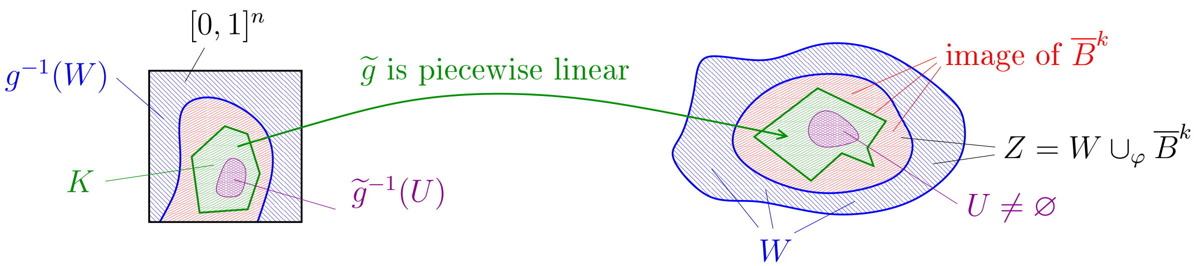


FIGURE 656. Illustration of Lemma 38.17.

Example. Suppose that in the setting of Lemma 38.17 there exists an open subset $U \subset B^k$ such that $g([0,1]^n) \cap U = \varnothing$. Then the empty polyhedron $K = \varnothing$ and the map $\widetilde{g} = g$ already have the desired property.

PROOF. Let W be a topological space and let $\varphi\colon S^{k-1} \to W$ be map. Furthermore let $g\colon [0,1]^n \to Z := W \cup_\varphi \overline{B}^k$ be a map. We write $D_1 := \overline{B}_{\frac{1}{4}}(0) \subset B^k$ and $D_2 := \overline{B}_{\frac{1}{2}}(0) \subset B^k$. The set $g^{-1}(D_2) \subset [0,1]^n$ is closed and therefore compact by the Heine-Borel Theorem 2.20. We obtain from Proposition 2.77 that g is uniformly continuous on $g^{-1}(D_2)$.[632] This implies that we can find an $\epsilon > 0$ with the following two properties:

(i) For any $x, y \in g^{-1}(D_2)$ with $\|x - y\| < \epsilon$ we have $\|g(x) - g(y)\| < \frac{1}{8}$.
(ii) We have[633]

$$\epsilon \;<\; \text{distance between } g^{-1}(D_1) \text{ and } [0,1]^n \setminus g^{-1}\big(\mathring{D}_2\big).$$

We divide $[0,1]^n$ into cubes such that the diameter of each cube is less than ϵ. We denote by K_1 the union of all the cubes that meet $g^{-1}(D_1)$ and we denote by K_2 the union of all the cubes meeting K_1. By the second property of ϵ we have $K_2 \subset g^{-1}(D_2)$. We refer to Figure 657 for an illustration.

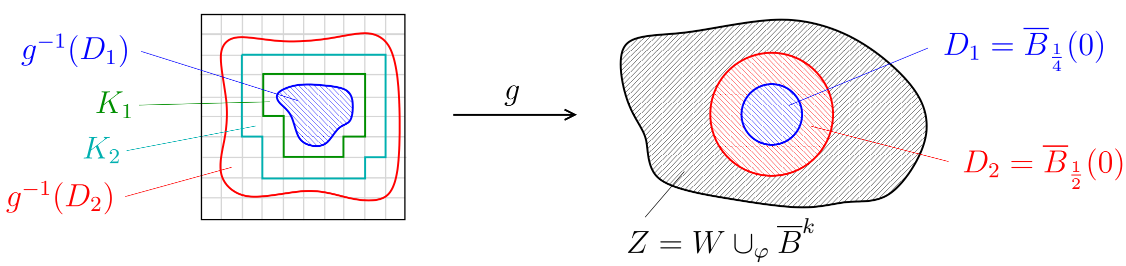


FIGURE 657

[632] Why can't we conclude that g is uniformly continuous on all of $[0,1]^n$?

[633] Here recall that, according to the definition on page 175, for two non-empty subsets A and B of $\mathbb{R}^n$ we refer to

$$d(A,B) \;:=\; \inf\{\|P - Q\| \,|\, P \in A \text{ and } Q \in B\}$$

as the distance between A and B. If A and B are compact and disjoint, then it follows from Corollary 3.19 (2) that $d(A,B) > 0$.

We need one more definition: Suppose we are given points $P_1,\dots,P_{n+1}\in\mathbb{R}^n$ such that the vectors $P_2-P_1,\dots,P_{n+1}-P_1$ are linearly independent. We refer to

$$\text{convex hull of } P_1,\dots,P_{n+1} \;=\; \Big\{\sum_{i=1}^{n+1} t_iP_i\,\Big|\,\sum_{i=1}^{n+1} t_i=1\Big\}$$

as a *simplex* in $\mathbb{R}^n$.[634] We will make use of the following trivial observation: if we are given $w_1,\dots,w_{n+1}\in\mathbb{R}^k$, then there exists a unique affine linear map Φ with $\Phi(P_i)=w_i$, namely the map that is given by

$$\Phi\Big(\sum_{i=1}^{n+1} t_iP_i\Big) \;:=\; \sum_{i=1}^{n+1} t_iw_i.$$

We leave it to the reader to show that each cube can be subdivided[635] into simplices. We also refer to Figure 658 for an illustration.

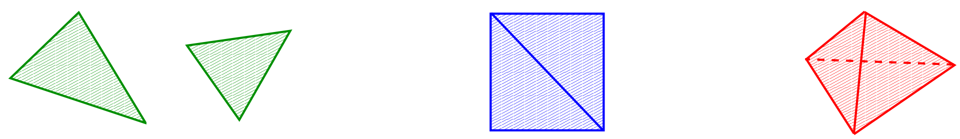


FIGURE 658

Now we subdivide all the cubes in K_2 into simplices. By the above discussion there exists a map $h\colon K_2\to\mathbb{R}^k$ that equals g on all the vertices of the simplices in the subdivision of the cubes.[636] Furthermore we choose a map $\psi\colon K_2\to[0,1]$ with $\psi|_{\partial K_2}\equiv 0$ and $\varphi|_{K_1}\equiv 1$.[637] Now we consider the homotopy[638]

$$\begin{array}{rcl} F\colon [0,1]^n\times[0,1] & \to & Z=W\cup_\varphi \overline{B}^k \\ (P,t) & \mapsto & \begin{cases} (1-t\cdot\psi(P))\cdot g(P)+t\cdot\psi(P)\cdot h(P), & \text{if } P\in K_2 \\ g(P), & \text{otherwise.}\end{cases}\end{array}$$

This is a homotopy from g to a map $\widetilde{g}\colon[0,1]\to Z=W\cup_\varphi\overline{B}^k$ that restricts to the piecewise linear map h on K_1.

Claim. The point $0\in D_1$ does not lie in $\widetilde{g}\big(\overline{[0,1]^n\setminus K_1}\big)$.

---

[634]Given a subset S of $\mathbb{R}^k$ we defined on page 123 the *convex hull* of S as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. As we saw in Exercise 2.36, if S is a finite subset of $\mathbb{R}^k$, then the *convex hull* of S is given by

$$\Big\{\sum_{s\in S} a_s\cdot s\,\Big|\,a_s\in[0,1] \text{ and } \sum_{s\in S}a_s=1\Big\}.$$

[635]Here, by "subdivide" we mean that given any cube K there exist simplices $S_1,\dots,S_k$ such that the interiors of the S_i are disjoint and such that $K=S_1\cup\dots\cup S_k$.

[636]Note that it follows from the uniqueness statement above that the map g is in fact continuous.

[637]Such a map can be constructed as follows: we assign the value 1 to all vertices in K_1 and we assign the value 0 to all other vertices in K_2. By the above "trivial observation" this defines a map on K_2 with the desired property.

[638]Since $\varphi|_{\partial K_2}\equiv 0$ we see that F is indeed continuous.

The claim is proved in two steps:

(1) On $[0,1]^n \setminus K_2$ we have $\widetilde{g} = g$. By Property (ii) of ϵ we have $g^{-1}(D_1) \subset K_2$. It follows that the image of $[0,1]^n \setminus K_2$ lies outside of D_1.

(2) Now let σ be a simplex of K_2 that does not lie in K_1. Note that $K_2 \subset g^{-1}(D_2)$. It follows from Property (i) of ϵ that $g(\sigma)$ is contained in some ball B_σ of radius $\frac{1}{8}$. Since $g(\sigma) \subset B_\sigma$ and since B_σ is convex we also have $h(\sigma) \subset B_\sigma$. Again by the convexity of B_σ we have $\widetilde{g}(\sigma) \subset B_\sigma$. We know that B_σ is not contained in D_1 since σ contains points outside K_1, hence outside $g^{-1}(D_1)$. The radius of B_σ is half that of D_1, so it follows that 0 is not in B_σ, and hence 0 does not lie in $\widetilde{g}(\sigma)$. ⊞

Since $\widetilde{g}\big(\overline{[0,1]^n \setminus K_1}\big)$ is compact it follows from the claim that there exists in fact an open neighborhood U of 0 that does not intersect the image of $[0,1]^n \setminus K_1$. Put differently, we have $\widetilde{g}^{-1}(U) \subset K_1$. It is now straightforward to verify that $K = K_1$, $\widetilde{g}$ and U have the desired properties. ■

Lemma 38.18. *Let W be a topological space and let $\varphi\colon S^{k-1} \to W$ be the attaching map of a k-cell. Furthermore let $h\colon \overline{B}^n \to Z := W \cup_\varphi \overline{B}^k$ be a map with $h(S^{n-1}) \subset W$. If $n < k$, then h is homotopic rel S^{n-1} to a map h' that misses B^k.*

Proof. Let W be a topological space and let $\varphi\colon S^{k-1} \to W$ be the attaching map of a k-cell. Furthermore let $h\colon \overline{B}^n \to Z := W \cup_\varphi \overline{B}^k$ be a map with $h(S^{n-1}) \subset W$. We suppose that $n < k$. We have to show that h is homotopic rel S^{n-1} to a map h' that misses B^k.

By Proposition 2.52 there exists a homeomorphism $\Phi\colon [0,1]^n \to \overline{B}^n$. So after precomposing h with Φ we might as well assume that the domain of h is $[0,1]^n$, i.e. that h is a map $h\colon [0,1]^n \to Z := W \cup_\varphi \overline{B}^k$ with $h(\partial[0,1]^n) \subset W$.

We pick a polyhedron K, a map $\widetilde{h}\colon [0,1]^n \to Z$ and a non-empty open subset $U \subset B^n$ with $\widetilde{h}^{-1}(U) \subset K$ as in the statement of Lemma 38.17. Since $n < k$ and since $\widetilde{h}|_K\colon K \to B^k$ is piecewise linear we deduce that $\widetilde{h}(K)$ cannot contain the open subset U of B^n.[639] We pick a point $P \in U \setminus \widetilde{h}(K)$. By statement (4) of Lemma 38.17 we know that P does not lie in the image of $\widetilde{h}\colon [0,1]^n \to Z$.

As we pointed out on page 549, there exists a deformation retraction from $\overline{B}^k \setminus \{P\}$ to S^{k-1}. We can extend this deformation retraction via the identity map to a deformation retraction from $Z = W \cup_\varphi (\overline{B}^k \setminus \{P\})$ to W. If we compose $\widetilde{h}$ with this deformation retraction we obtain a homotopy, rel S^{n-1}, from $\widetilde{h}$ to a map h' that misses B^k. ■

Now we can finally provide the proof of Proposition 38.15.

Proof of Proposition 38.15. Let V be a CW-complex and let W be some topological space. Furthermore let $\psi\colon S^{n-1} \to V^{n-1}$ be a map to the $(n-1)$-skeleton of V and let $\varphi\colon S^{k-1} \to W$ be another map. Finally let

$$h\colon V \cup_\psi \overline{B}^n \ \to\ W \cup_\varphi \overline{B}^k$$

[639] Indeed, since $\widetilde{h}|_K\colon K \to B^k$ is piecewise linear and since $n < k$ we know that the image $\widetilde{h}(K)$ is set of Lebesgue-measure zero in B^n whereas a non-empty open subset has positive Lebesgue-measure.

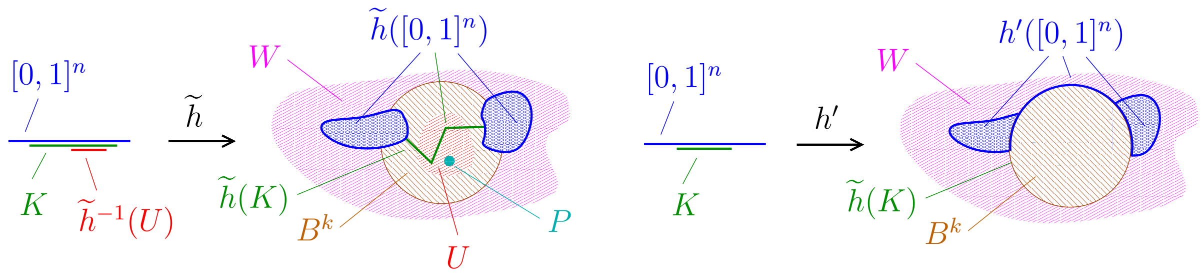


FIGURE 659. Illustration for the proof of Proposition 38.15.

be a map with $h(\psi(S^{n-1})) \subset W$. We suppose that $n < k$.

We denote by θ the obvious map $\overline{B}^n \to V \cup_\psi \overline{B}^n$, furthermore we write $h = g \circ \theta$. By our hypothesis we have $h(S^{n-1}) \subset W$. By Lemma 38.18 there exists a homotopy G rel S^{n-1} from h to a map $h' \colon \overline{B}^n \to W$ that misses B^k. The desired homotopy is now given by[640][641]

$$\begin{aligned} F\colon (V \cup_\psi \overline{B}^n) \times [0,1] &\to W \cup_\varphi \overline{B}^k \\ (P,t) &\mapsto \begin{cases} G(Q,t), & \text{if } P = \theta(Q), \text{ for some } Q \in \overline{B}^k, \\ g(P), & \text{otherwise.} \end{cases} \end{aligned}$$ ■

Exercises for Chapter 38.

Exercise 38.1. Let X be a contractible CW-complex and let $x_0 \in X$. Show that X admits a deformation retraction to the point x_0.
Hint. Use the Homotopy Extension Theorem 38.1.

Exercise 38.2. We consider the map

$$\begin{aligned} \varphi\colon S^1 &\to S^1 \\ e^{it} &\mapsto \begin{cases} e^{3it}, & \text{if } t \in [0, \frac{2\pi}{3}], \\ e^{-3it}, & \text{if } t \in [\frac{2\pi}{3}, \frac{4\pi}{3}], \\ e^{3it}, & \text{if } t \in [\frac{4\pi}{3}, 2\pi]. \end{cases} \end{aligned}$$

Put differently, φ wraps S^1 first around S^1 clockwise, then counter clockwise and then once again clockwise. We write $Y = S^1$ and we consider

$$X \;:=\; Y \cup_\varphi \overline{B^2},$$

[640]We view $V \cup_\psi \overline{B}^n$ as a CW-complex where V is a subcomplex and the image of $\overline{B}^n$ is an n-cell. Furthermore we equip $V \cup_\psi \overline{B}^n$ with the product CW-structure. It follows from Lemma 36.8 and Lemma 36.20, applied to the subcomplexes given by $V \times [0,1]$ and $\theta(\overline{B}^n) \times [0,1]$, that the map F is continuous.

[641]One might be tempted to use a different approach to show that F is continuous: Namely one might want to write $(V \cup_\psi \overline{B}^n) \times [0,1] = (V \times [0,1]) \cup_{\psi \times \mathrm{id}_{[0,1]}} (\overline{B}^n \times [0,1])$ and then argue using Lemma 3.44 (5) that the given map is continuous. But this approach is problematic since it is not clear that the two topologies on $(V \cup_\psi \overline{B}^n) \times [0,1] = (V \times [0,1]) \cup_{\psi \times \mathrm{id}_{[0,1]}} (\overline{B}^n \times [0,1])$ agree.

where we identify $z \in \partial\overline{B}^2 = S^1$ with $\varphi(z) \in S^1 = Y$.

(a) Show that X is homotopy equivalent to $\overline{B}^2$.

(b) Is X homeomorphic to $\overline{B}^2$?

Exercise 38.3. Let X be a CW-complex which admits a single 0-cell. Show that every subcomplex Y of X is path-connected.

Exercise 38.4. Let X be a CW-complex. Use the Cellular Approximation Theorem 38.13 to give a direct proof that if X is path-connected, then its 1-skeleton X^1 is also path-connected.

Remark. We gave a roundabout proof of this statement in Proposition 36.10 (7).

39. Cofibrations (∗)

One of the most useful properties of CW-complexes is the Homotopy Extension Theorem 38.1 which says that any homotopy on a subcomplex can be extended to a homotopy of the total CW-complex. In this chapter we will discuss several consequences of this property. In fact, since the "homotopy extension property" arises in many other settings we first formalize this concept by introducing the notion of a "cofibration". Afterwards we will prove properties of cofibrations, which will in particular apply to CW-complexes. For the most part the results in this chapter are not essential for the understanding of the subsequent chapters, thus not much harm is done by initially skipping this chapter.

Throughout this chapter we will on numerous occasions make use of the notion of a mapping cylinder and of a mapping cone. Thus we start out by recalling the definition of a mapping cylinder and a mapping cone that we had initially given on pages 697 and page 701.

Definition. Let $f\colon X \to Y$ be a map between topological spaces. We define the corresponding *mapping cylinder* to be the topological space

$$\mathrm{Cyl}(f) \;:=\; ((X\times[0,1])\sqcup Y)/\sim \;\text{ where } (x,1)\sim f(x) \text{ for all } x\in X.$$

We refer to the maps

$$\begin{array}{c} X \xrightarrow{i} \mathrm{Cyl}(f) \\ x \mapsto (x,0) \end{array} \quad \text{respectively} \quad \begin{array}{c} \mathrm{Cyl}(f) \xrightarrow{r} Y \\ P \mapsto \begin{cases} P, & \text{if } P\in Y, \\ f(Q), & \text{if } P=[(Q,t)] \text{ with } Q\in X, t\in[0,1]. \end{cases} \end{array}$$

as the *natural inclusion* respectively the *natural retraction*.[642]

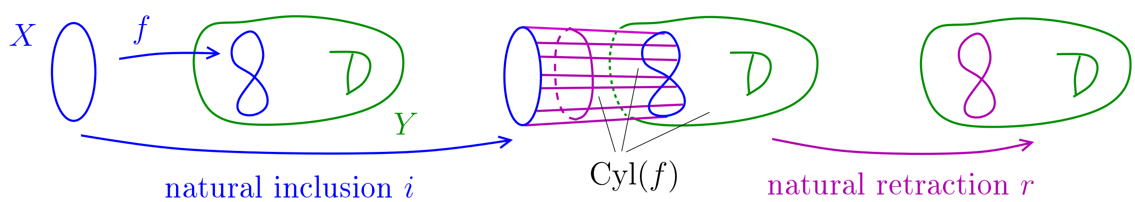


Figure 660

Definition. Given a map $f\colon A \to X$ between topological spaces the corresponding *mapping cone* is defined as

$$\mathrm{Cone}(f\colon A\to X) \;:=\; (A\times[0,1])\sqcup X/\sim \quad \text{where} \quad \begin{array}{l} (a,0)\sim(b,0) \text{ for all } a,b\in A \text{ and} \\ (a,1)\sim f(a) \text{ for all } a\in A. \end{array}$$

We refer to the point $[A\times\{0\}]\in\mathrm{Cone}(f)$ as the *cone point*. We illustrate the definition for an inclusion map $i\colon A\to X$ in Figure 661.

39.1. Cofibrations: definition and examples.

39.1. Cofibrations: definition and examples. Now we introduce the leading contender for our attention in this chapter.

[642] Note that by Lemma 24.8 we know that there exists a deformation retraction $R\colon \mathrm{Cyl}(f)\times[0,1]\to \mathrm{Cyl}(f)$ from $\mathrm{Cyl}(f)$ to Y, viewed as a subspace of $\mathrm{Cyl}(f)$, such that $R_1 = r$.

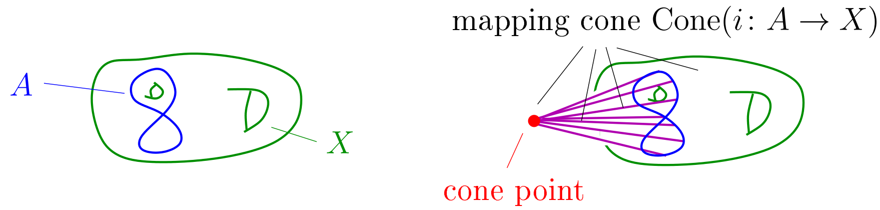


FIGURE 661

Definition.

(1) A map $i\colon A \to X$ between topological spaces is called a *cofibration* if it has the homotopy extension property, i.e. if given any topological space Y and any two maps $f\colon X \times \{0\} \to Y$ and $G\colon A \times [0,1] \to Y$ with $f(i(a),0) = G(a,0)$ for all $a \in A$ there exists a map $F\colon X \times [0,1] \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} A \times \{0\} \arrow[rr, hook] \arrow[d, "i"'] & & A \times [0,1] \arrow[dl, "G"'] \arrow[d, "i\times \mathrm{id}"] \\ & Y & \\ X \times \{0\} \arrow[ur, "f"] \arrow[rr, hook] & & X \times [0,1]. \arrow[ul, dashed, "F"'] \end{tikzcd}$$

We illustrate the definition in Figure 662.

(2) If $i\colon A \to X$ is a cofibration such that $i(A)$ is a closed subset of X, then we refer to i as a *closed cofibration*.

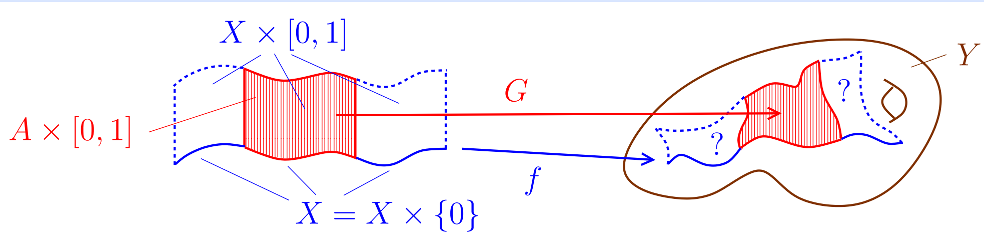


FIGURE 662

Remark.

(1) As we will see in Lemma 39.8, in most cases of interest the map $i\colon A \to X$ is for all intents and purposes an inclusion. For such situations the definition of the homotopy extension property is illustrated in Figure 662.
(2) In some books what we call a cofibration is called a "Hurewicz cofibration". Also, in some books, see e.g. **[Pic92]**, a cofibration is defined as what we call a closed cofibration.

The following theorem gives us one of the most important examples of a cofibration.

Theorem 39.1. *Let X be a* CW*-complex. Given any subcomplex A the corresponding inclusion map $i\colon A \to X$ is a closed cofibration.*

Proof. The statement of the Homotopy Extension Theorem 38.1 is precisely that the inclusion $A \to X$ is a cofibration. By Lemma 36.18 (2) we know that the cofibration is in fact a closed cofibration. ■

We continue with the following fairly elementary lemma which will be proved in Exercise 39.2.

Lemma 39.2. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are two (closed) cofibrations, then the composition $g \circ f\colon X \to Z$ is also a (closed) cofibration.*

Before we give more examples of cofibrations it is convenient to add a characterization of cofibrations to our repertoire.

Lemma 39.3. *If $i\colon A \to X$ is a map between two topological spaces, then the following two statements are equivalent:*

(1) *The map $i\colon A \to X$ is a cofibration.*
(2) *The map*[643]

$$\begin{array}{rcl} s\colon \mathrm{Cyl}(i) & \to & X \times [0,1] \\ P & \mapsto & \begin{cases} (i(a), 1-t), & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1], \\ (P, 0), & \text{if } P \in X \end{cases} \end{array}$$

admits a left-inverse, in other words, there exists a map $r\colon X \times [0,1] \to \mathrm{Cyl}(i)$ such that $r \circ s = \mathrm{id}_{\mathrm{Cyl}(i)}$.

If A is a subset of X and if $i\colon A \to X$ is the inclusion map, then (1) and (2) are equivalent to

(3) *The set $(X \times \{0\}) \cup (A \times [0,1])$ is a retract of $X \times [0,1]$.*

Example. Let I be an interval in $\mathbb{R}$ and let $P_1, \ldots, P_k$ be finitely many distinct points in I. In Figure 663 we show a retraction from $I \times [0,1]$ to $(I \times \{0\}) \cup (\{P_1, \ldots, P_k\} \times [0,1])$. Together with Lemma 39.3 this shows that the inclusion $\{P_1, \ldots, P_k\} \to I$ is a cofibration.

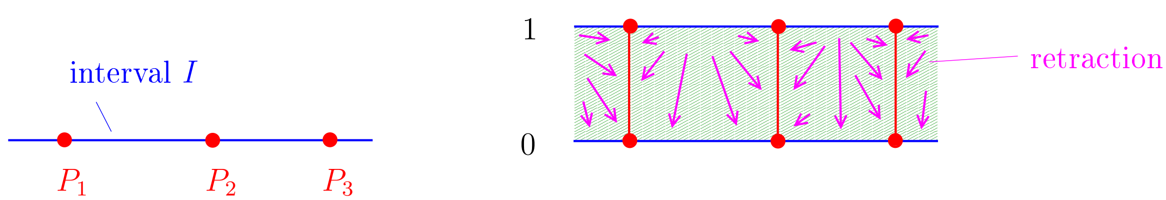


Figure 663

Proof (∗). First we prove the "(1) ⇒ (2)"-direction. Thus we now suppose that the map $i\colon A \to X$ is a cofibration. We consider the inclusion map $j\colon X \times \{0\} \to \mathrm{Cyl}(i)$ which is given by $j(x,0) = x$ and we consider the map

$$\begin{array}{rcl} G\colon A \times [0,1] & \to & \mathrm{Cyl}(i) \\ (a,s) & \mapsto & [(a, 1-s)]. \end{array}$$

[643] It follows from Lemma 3.44 (5) that this map is continuous.

We obtain the following diagram:

$$\begin{tikzcd}
A \times \{0\} \arrow[rr, hook] \arrow[dd, hook, "i"'] & & A \times [0,1] \arrow[dl, "G"'] \arrow[dd, hook, "i \times \mathrm{id}"] \\
& \mathrm{Cyl}(i) & \\
X \times \{0\} \arrow[rr, hook] \arrow[ur, hook, "j"] & & X \times [0,1]. \arrow[ul, dashed, "r"]
\end{tikzcd}$$

One easily verifies that the black diagram commutes. Since $i\colon A \to X$ is a cofibration we obtain a map $r\colon X \times [0,1] \to \mathrm{Cyl}(i)$ which makes the diagram commute. It is straightforward to verify that $r \circ s = \mathrm{id}_{\mathrm{Cyl}(i)}$.

The proof of the converse, i.e. the proof of the "(2) ⇒ (1)"-direction is also quite straightforward. We will provide the proof in Exercise 39.3.

Now we assume that A is a subset of X and that $i\colon A \to X$ is the inclusion map. We need to provide the proof for the equivalence of (2) and (3). This turns out to be quite subtle.

(a) First we assume that A is a closed subset of X.[644] Note that the image of $\mathrm{Cyl}(i)$ in $X \times [0,1]$ under the map s is precisely $(X \times \{0\}) \cup (A \times [0,1])$. Thus we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Cyl}(i) \arrow[rr, "s"] \arrow[dr, "s"'] & & (X \times \{0\}) \cup (A \times [0,1]) \arrow[dl, hook] \\
& X \times [0,1]. &
\end{tikzcd}$$

To show that (2) and (3) are equivalent it suffices to show that the top horizontal map is a homeomorphism. It is basically clear that the top horizontal map is a bijection. Furthermore it follows easily from Lemma 3.44 (5) that the map is continuous. Finally it follows from our hypothesis that A is a *closed* subset, together with Lemma 24.7 (2), that the top horizontal map is indeed a homeomorphism.

(b) The logic of (a) breaks down if A is not a closed subset. More precisely, in Lemma 24.7 (3) we saw that in general the top horizontal map is *not* a homeomorphism. Rather surprisingly it is nonetheless true that (2) and (3) are equivalent. This delicate argument is given in [**Strø68**, Lemma 4] or alternatively in [**Hat02**, Proposition A.18]. ■

Lemma 39.3 allows us to prove the following proposition which gives us many more examples of cofibrations.

Proposition 39.4. *Let M be a smooth manifold.*

(1) *For any a union W of components of ∂M the inclusion $i\colon W \to M$ is a closed cofibration.*
(2) *For any compact proper submanifold N of M the inclusion $i\colon N \to M$ is a closed cofibration.*

[644] This is also the only case we will ever need in practice.

The proof of Proposition 39.4 relies on the following lemma.

Lemma 39.5. *There exists a retraction*

$$\rho\colon [0,1]\times[0,1] \;\to\; (\{0\}\times[0,1])\cup([0,1]\times\{0\})$$

that has the property that $\rho(\{1\}\times[0,1]) = \{(1,0)\}$.

Proof. In Figure 664 we show how such a retraction ρ can be constructed. ∎

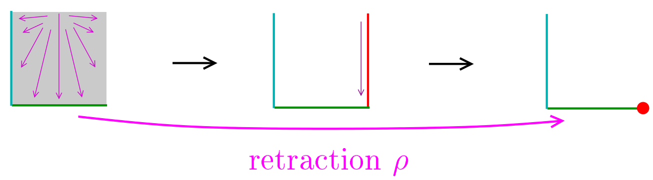


Figure 664. Illustration for the proof of Lemma 39.5.

Sketch of a proof of Proposition 39.4. Let M be an m-dimensional smooth manifold. Recall that in Lemma 39.5 we saw that there exists a retraction

$$\varphi = (\varphi_1,\varphi_2)\colon [0,1]\times[0,1] \;\to\; (\{0\}\times[0,1])\cup([0,1]\times\{0\})$$

that has the property that $\varphi(\{1\}\times[0,1]) = \{(1,0)\}$.

(1) Let W be a union of components of ∂M. Our goal is to show that the inclusion $i\colon W\to M$ is a cofibration. By the Collar Neighborhood Theorem 8.12 there exists a collar neighborhood $[0,1]\times W$. We consider the map[645]

$$\begin{aligned} r\colon M\times[0,1] \;&\to\; \mathrm{Cyl}(i) = ((W\times[0,1])\sqcup M)/\sim \;\text{ where } (x,1)\sim x \text{ for } x\in W \\ (P,t) \;&\mapsto\; \begin{cases} (\underbrace{(z\cdot\varphi_1(s,t),Q)}_{\in[0,1]\times W},\varphi_2(s,t)), & \text{if } P=(s,Q) \text{ with } s\in[0,1] \text{ and } Q\in W, \\ P, & \text{if } P\notin[0,1]\times W. \end{cases}\end{aligned}$$

(We refer to Figure 665 for an illustration of the map r.) One can easily verify that r is a left-inverse to the map $s\colon \mathrm{Cyl}(i)\to M\times[0,1]$ that we had considered in Lemma 39.3 (2). Thus it follows from Lemma 39.3 that $i\colon W\to M$ is a cofibration. Finally it follows from Proposition 6.27 (3a) that i is in fact a closed cofibration.

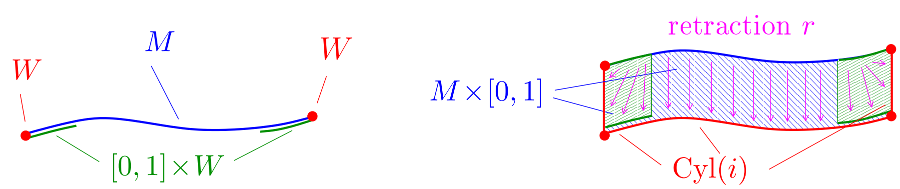


Figure 665. Illustration of the proof of Proposition 39.4 (1).

[645]Recall that by definition of a collar neighborhood the set $[0,1]\times\partial M$ is actually a closed subset of M. It now follows from this observation together with Lemma 8.11 and Lemma 2.35 (2) that the map r is indeed continuous.

(2) Let N be a compact proper n-dimensional submanifold of M. We want to show that the inclusion $j\colon N \to M$ is a cofibration. By the General Tubular Neighborhood Theorem 10.5 the submanifold N admits a tubular neighborhood. To simplify the discussion we will only treat the case that N admits a trivial tubular neighborhood $\overline{B}^{m-n} \times N$. The general case is notationally more messy since we have to work with a trivialization of the tubular neighborhood, similar to the argument in the proof of Proposition 10.11. We leave it to the doubting Thomases among the readership to deal with the general case.

We consider the map[646]

$$r\colon M\times[0,1] \to \operatorname{Cyl}(j) = ((N\times[0,1])\sqcup M)/\sim \ \text{ where } (x,1)\sim x \text{ for } x\in N$$
$$(P,t) \mapsto \begin{cases} (\underbrace{(z\cdot\varphi_1(|z|,t),Q)}_{\in \overline{B}^{m-n}\times N},\varphi_2(|z|,t)), & \text{if } P=(z,Q) \text{ with } z\in\overline{B}^{m-n} \text{ and } Q\in N,\\ P, & \text{if } P\notin \overline{B}^{m-n}\times N. \end{cases}$$

(We refer to Figure 666 for an illustration of the map r.) As in the proof of (1) we conclude from Lemma 39.3 that $j\colon N \to M$ is a cofibration. Since N is assumed to be compact it is a closed subset by Lemma 2.17 (2). ■

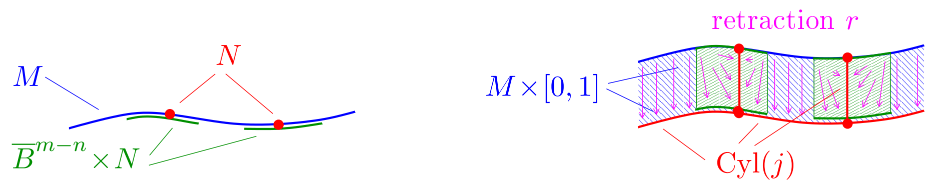


FIGURE 666. Illustration of the proof of Proposition 39.4 (2).

The next proposition gives another rich source of examples of cofibrations.

Proposition 39.6.
(1) *If $f\colon X \to Y$ is a map between two topological spaces, then the natural inclusion $i\colon X \to \operatorname{Cyl}(f)$ is a closed cofibration.*
(2) *Let X be a non-empty topological space. The natural inclusion $X \to \operatorname{Cone}(X)$ is a closed cofibration.*

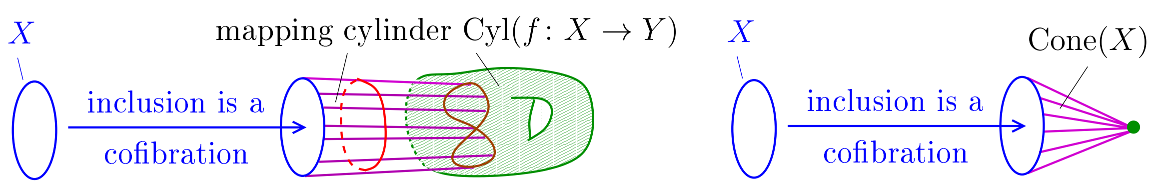


FIGURE 667

PROOF. Note that if X is a non-empty topological space and $f\colon X \to \{*\}$ is the constant map, then, as we pointed out on page 698, the mapping cylinder $\operatorname{Cyl}(f)$ is basically the cone of X. Thus we see that Statement (2) is a special case of Statement (1).

[646]It follows from the definition of a tubular neighborhood, see page 418, together with Lemma 2.35 (2) and Lemma 89.1 that the map r is indeed continuous.

It remains to prove Statement (1). Let $f\colon X \to Y$ be a map between topological spaces. By Lemma 24.6 (4) we know that the image of X in $\mathrm{Cyl}(f)$ is a closed subset. Thus it remains to show that $i\colon X \to \mathrm{Cyl}(f)$ is a cofibration. By Lemma 24.6 (3) the map i is an embedding, thus we can identify X with its image in $\mathrm{Cyl}(f)$.

By Lemma 39.3 (3)⇒(1) it suffices to prove the following claim:

Claim. There is a retraction r from $\mathrm{Cyl}(f)\times[0,1]$ to its subset $(\mathrm{Cyl}(f)\times\{0\})\cup(X\times[0,1])$.

In Figure 668 we do our best to illustrate the objects involved. To construct the desired

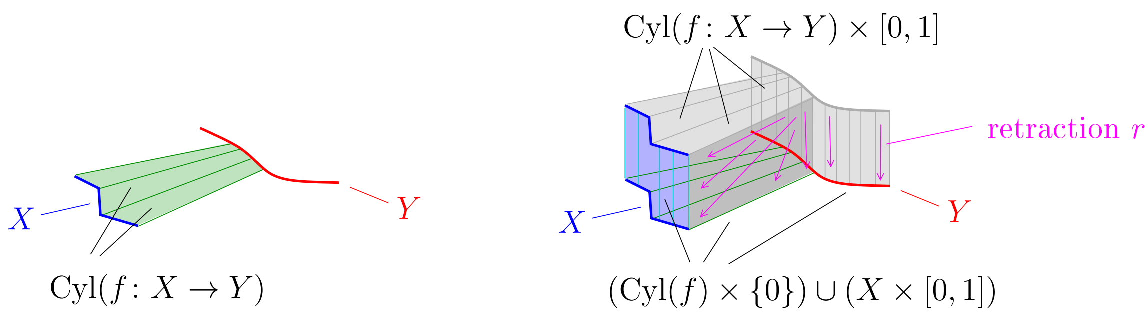


FIGURE 668. Illustration for the proof of Proposition 39.6.

retraction r we recall that in Lemma 39.5 we saw that there exists a retraction

$$s\colon [0,1]\times[0,1] \;\to\; (\{0\}\times[0,1])\cup([0,1]\times\{0\})$$

that has the property that $s(\{1\}\times[0,1]) = \{(1,0)\}$. Now the desired retraction is given by the map[647]

$$\begin{array}{rcl} r\colon\ \mathrm{Cyl}(f)\times[0,1] & \to & (\mathrm{Cyl}(f)\times\{0\})\cup(X\times[0,1]) \\ P & \mapsto & \begin{cases} [(x,s(t,u))], & \text{if } P=[(x,t,u)] \text{ with } x\in X \text{ and } t,u\in[0,1], \\ [(y,0)], & \text{if } P=[(y,u)] \text{ with } y\in Y \text{ and } u\in[0,1]. \end{cases}\end{array}$$ ■

We conclude this section with a proposition which says that any map $f\colon X \to Y$ between topological spaces can be "replaced" by a cofibration $X \to \mathrm{Cyl}(f)$.

Proposition 39.7. *Let $f\colon X\to Y$ be a map between topological spaces. We denote by $i\colon X\to\mathrm{Cyl}(f)$ the natural inclusion. The natural retraction $r\colon \mathrm{Cyl}(f)\to Y$ is a homotopy equivalence and the following diagram commutes:*

$$\begin{tikzcd} & X \arrow[dl, "i"'] \arrow[dr, "f"] & \\ \mathrm{Cyl}(f) \arrow[rr, "r"] & & Y. \end{tikzcd}$$

By Proposition 39.6 (1) we know that the left diagonal map $i\colon X\to\mathrm{Cyl}(f)$ is a cofibration.

PROOF. We showed in Lemma 24.8 (2a) that the natural retraction $r\colon \mathrm{Cyl}(f)\to Y$ is a homotopy equivalence. It is clear that the given diagram commutes. Thus we are done by Proposition 39.6 (1). That was quick. ■

[647] This map is continuous by Lemma 2.35 (2).

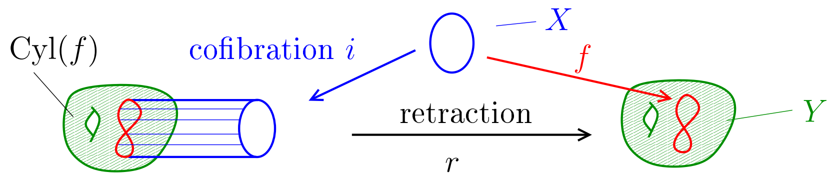


FIGURE 669. Illustration for Proposition 39.7.

39.2. **Properties of cofibrations.** The following lemma summarizes a few key properties of cofibrations.

Lemma 39.8. *If $i\colon A \to X$ is a cofibration, then the following three statements hold:*

(1) *the map $i\colon A \to X$ is injective,*
(2) *the map $i\colon A \to X$ is an embedding,*
(3) *if X is Hausdorff, then $i(A)$ is a closed subset of X, i.e. i is a closed cofibration.*

Remark. In Exercise 39.5 we will see that in Lemma 39.8 (3) we cannot drop the hypothesis that X is Hausdorff.

PROOF. Let $i\colon A \to X$ be a cofibration.

(1) Similar to the proof of Lemma 39.3 we consider the inclusion map $j\colon X \times \{0\} \to \mathrm{Cyl}(i)$ which is given by $j(x, 0) = x$ and we consider the map

$$\begin{array}{rcl} G\colon A \times [0,1] & \to & \mathrm{Cyl}(i) \\ (a, s) & \mapsto & [(a, 1-s)]. \end{array}$$

We obtain the following diagram:

$$\begin{tikzcd}
A \times \{0\} \arrow[rr, hook] \arrow[dd, hook, "i"'] & & A \times [0,1] \arrow[dl, "G"'] \arrow[dd, hook, "i\times\mathrm{id}"] \\
& \mathrm{Cyl}(i) & \\
X \times \{0\} \arrow[ur, hook, "j"] \arrow[rr, hook] & & X \times [0,1]. \arrow[ul, dashed, "F"']
\end{tikzcd}$$

One easily verifies that the black diagram commutes. Since i is a cofibration we obtain a map $F\colon X \times [0,1] \to \mathrm{Cyl}(i)$ which makes the diagram commute. Note that by design for any $t \in (0,1]$ the map $G\colon A \times \{t\} \to \mathrm{Cyl}(i)$ is injective. Since the diagram commutes this implies that $i \times \mathrm{id}_{\{t\}}\colon A \times \{t\} \to X \times \{t\}$ is also injective, i.e. $i\colon A \to X$ is injective.

(2) We continue with the argument and the notation in (1). Let $t \in (0,1]$. The map $G|_{A\times\{t\}}\colon A \times \{t\} \to A \times \{1-t\}$ is evidently a homeomorphism. Thus we see that

$$(G|_{A\times\{t\}})^{-1} \circ F|_{X\times\{t\}}\colon i(A) \times \{t\} \ \to \ A \times \{t\}$$

is an inverse to the map $i \times \mathrm{id}_{\{t\}}\colon A \times \{t\} \to X \times \{t\}$. This implies that the map $i \times \mathrm{id}_{\{t\}}\colon A \times \{t\} \to X \times \{t\}$ is an embedding, but this also implies that the map $i\colon A \to X$ is an embedding.

(3) Note that

$$\begin{array}{rcl} & \text{for } x\in A \text{ we have } F(x,0)=G(x,0)=[(x,1)]=[x] \text{ by construction of } F, \text{ furthermore} & \\ & \text{for } x\notin A \text{ the point } [x] \text{ does not lie in } \mathrm{Cyl}(i), \text{ thus it does not lie in the image of } F & \\ A & \overset{\downarrow}{=} & \{x\in X \mid F(x,0)=[x]\} \\ & = & \left\{ \begin{array}{c} \text{the points in } X \text{ on} \\ \text{which the maps} \end{array} \begin{array}{rcl} X & \to & \mathrm{Cyl}(i) \\ x & \mapsto & F(x,0) \end{array} \text{ and } \begin{array}{rcl} X & \to & \mathrm{Cyl}(i) \\ x & \mapsto & [x] \end{array} \text{ agree} \right\}. \end{array}$$

It follows from this description of A, our hypothesis that X is Hausdorff (which implies by (2) and Lemma 24.6 (2) that $\mathrm{Cyl}(i)$ is also Hausdorff) and Exercise 2.30 that A is indeed a closed subset of X. ■

The following lemma shows that inclusion maps that are cofibrations are particularly nicely behaved.

Lemma 39.9. *Let X be a topological space and let $A \subset X$ be a non-empty subset. We denote by $i\colon A \to X$ the inclusion map. If i is a closed cofibration, then the following three statements hold:*

(1) *The map*[648]

$$\begin{array}{rcl} \varphi\colon \mathrm{Cone}(i\colon A\to X) & \to & X/A \\ P & \mapsto & \left\{ \begin{array}{ll} [P], & \text{if } P\in X, \\ [A], & \text{if } P=[(a,t)] \text{ with } a\in A \text{ and } t\in[0,1] \end{array} \right. \end{array}$$

is a homotopy equivalence.

(2) *If A is contractible, then the projection map $X \to X/A$ is a homotopy equivalence.*

(3) *If $a_0 \in A$ is a point such that $\{a_0\}$ is a deformation retract of A, then the projection map $X \to X/A$ is a homotopy equivalence between the pointed topological spaces (X, a_0) and $(X/A, [A])$.*

Remark. The statement of Lemma 39.9 (2) is related in spirit to the statement of Proposition 38.5.

PROOF. Let X be a topological space and let $A \subset X$ be a closed subset such that the inclusion $i\colon A \to X$ is a cofibration.

(1) Similar to the proof of Lemma 39.8 (1) we consider the following two maps:

(a) First of all we consider the obvious inclusion map $j\colon X \times \{0\} \to \mathrm{Cone}(i)$ which is given by $j(x, 0) = x$.

(b) We consider the map

$$\begin{array}{rcl} G\colon A\times[0,1] & \to & \mathrm{Cone}(i) \\ (a,s) & \mapsto & [(a,1-s)]. \end{array}$$

Note that G_1 sends all points in A to the cone point.

[648]It follows from the ever popular Lemma 2.35 (2) and the fact that A is a closed subset of X that the map φ is continuous.

We obtain the following diagram:

$$\begin{tikzcd}
A \times \{0\} \arrow[rr, hook] \arrow[dd, hook, "i"'] & & A \times [0,1] \arrow[dl, "G"'] \arrow[dd, hook, "i\times\mathrm{id}"] \\
& \mathrm{Cone}(i) & \\
X \times \{0\} \arrow[rr, hook] \arrow[ur, hook, "j"] & & X \times [0,1]. \arrow[ul, dashed, "F"']
\end{tikzcd}$$

One easily verifies that the diagram commutes. Since i is a cofibration we obtain a map $F\colon X \times [0,1] \to \mathrm{Cone}(i)$ which makes the diagram commute. Note that the map $F_1\colon X \to \mathrm{Cone}(i)$ sends all points in A to the cone point. Thus F_1 descends to a map $\psi\colon X/A \to \mathrm{Cone}(i)$. It remains to prove the following claim.

Claim. The maps $\varphi\colon \mathrm{Cone}(i) \to X/A$ and $\psi\colon X/A \to \mathrm{Cone}(i)$ are homotopy inverses of one another.

Note that $F(A\times[0,1]) = G(A\times[0,1]) \subset A$. Therefore it follows from Lemma 18.21 that the map

$$H\colon (X/A) \times [0,1] \xrightarrow{([x],t)\mapsto[(x,t)]} (X \times [0,1])/(A \times [0,1]) \xrightarrow{[(x,t)]\mapsto[F(x,t)]} X/A$$

is continuous. It follows immediately from the definitions that H is a homotopy between $\mathrm{id}_{X/A}$ and $\varphi \circ \psi$.

Conversely we consider the map

$$\begin{array}{rcl} H\colon \mathrm{Cone}(i) \times [0,1] & \to & \mathrm{Cone}(i) \\ (P,s) & \mapsto & \begin{cases} F(P,s), & \text{if } P \in X, \\ [a,(1-s)\cdot t], & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1]. \end{cases} \end{array}$$

One easily verifies that this is a homotopy from $\mathrm{id}_{\mathrm{Cone}(i)}$ to $\psi \circ \varphi$. ⊞

(2) We consider the following diagram

$$\begin{tikzcd}
X \arrow[rr, hook] \arrow[dr] & & \mathrm{Cone}(i\colon A \to X) \arrow[dl, "\varphi"] \\
& X/A. &
\end{tikzcd}$$

It follows immediately from the definition of φ that this diagram commutes. Now suppose that A is contractible. By Proposition 24.15 (1) there exists a deformation retraction G from $\mathrm{Cone}(i\colon A \to X)$ to X. By statement (1) we know that φ is a homotopy equivalence. Thus it follows from Lemma 18.11 (1) that the map $X \to X/A$ is a homotopy equivalence.

(3) Now assume that $a_0 \in A$ is a point which has the property that $\{a_0\}$ is a deformation retract of A. By Proposition 24.15 (2) there exists a deformation retraction G from $\mathrm{Cone}(i\colon A \to X)$ to X such that for any $t \in [0,1]$ we have $G_1([(a_0,t)]) = a_0$. Also note that the homotopy equivalence H from (1) has the property that $H(a_0,t) = [(a_0, 1-t)]$. If one combines these observations with the argument in (2) it is not

difficult to see that the projection map is in fact a homotopy equivalence of the pointed topological spaces (X, a_0) and $(X/A, [a_0])$. ∎

The following corollary is quite useful.

Corollary 39.10. *Let X be a CW-complex and let $A \subset X$ be a subcomplex.*

(1) *If A is contractible, then the projection map $X \to X/A$ is a homotopy equivalence.*
(2) *If $a_0 \in A$ is a point such that $\{a_0\}$ is a deformation retract of A, then the projection map $X \to X/A$ is a homotopy equivalence between the pointed topological spaces (X, a_0) and $(X/A, [A])$.*

Remark. Corollary 39.10 (1) is a generalization of Proposition 37.8.

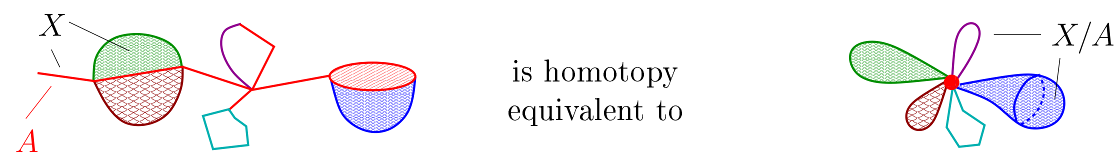


FIGURE 670. Illustration of Corollary 39.10.

PROOF. By Theorem 39.1 we know that the inclusion $A \to X$ is a closed cofibration. The two statements now follow immediately from the corresponding statements in Lemma 39.9. ∎

The following proposition is a neat corollary to Corollary 39.10.

Proposition 39.11. *Let X be a 0-connected CW-complex.*

(1) *There exists a contractible 1-dimensional subcomplex T which contains all 0-cells.*[649]
(2) *Given any T as in (1) the quotient X/T is a CW-complex with precisely one 0-cell. Furthermore the projection $X \to X/T$ is a homotopy equivalence.*

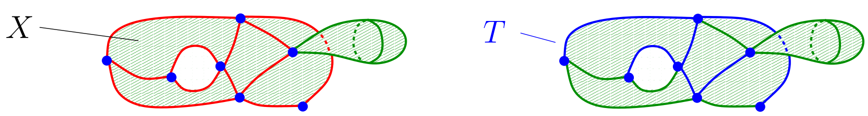


FIGURE 671. Illustration of Proposition 39.11.

Remark.

(1) Using Proposition 37.11 it is very easy to calculate the fundamental group of a finite 0-connected CW-complex with a single 0-cell. Proposition 39.11 says in particular that given a finite 0-connected CW-complex we can always reduce the calculation of the fundamental group to the case of a finite CW-complex with a single 0-cell.
(2) Let X be a 0-connected CW-complex with countably many 1-cells. Let $x_0 \in X$ be a 0-cell. By Proposition 40.9 the inclusion $X^1 \to X$ of the 1-skeleton induces an epimorphism $\pi_1(X^1, x_0) \to \pi_1(X, x_0)$. Furthermore it follows from Proposition 39.11 and 20.7 that $\pi_1(X^1, x_0)$ is a free group on a countable generating set. It follows from this discussion, together with Lemma 1.7 (2), that $\pi_1(X, x_0)$ is countable.

[649]Such a subcomplex is basically what we called a spanning tree on page 229.

PROOF. Let X be a 0-connected CW-complex.

(1) In Proposition 37.7 (1) we showed that the 1-skeleton X^1 admits a subcomplex T that contains all 0-cells of X and that admits a deformation retraction to x_0.
(2) By Lemma 36.32 the quotient X/T admits a CW-structure with one 0-cell. Furthermore note that it follows from Corollary 39.10 (2) that the projection map $X \to X/T$ is a homotopy equivalence between. ■

39.3. **Products and cofibrations** (∗)**.** In this and the following two sections we collect several technical useful results on cofibrations. As stand-alone sections these are arguably not much fun to read. Thus it is better to move on and to come back to these three sections when the results are needed.

Proposition 39.12.

(1) *If $f\colon A \to X$ and $g\colon B \to Y$ are two (closed) cofibrations, then the product map $f \times g\colon A \times B \to X \times Y$ is also a (closed) cofibration.*
(2) *If $f\colon A \to X$ is a (closed) cofibration and if Z is a topological space, then the resulting map $f \times \mathrm{id}_Z\colon A \times Z \to X \times Z$ is also a (closed) cofibration.*

Example. Let I be an interval and let $P \in I$. On page 1024 we saw that the inclusion $\{P\} \to I$ is a cofibration. It follows from Proposition 39.12 that given any topological space Z the inclusion $\{P\} \times Z \to I \times Z$ is a cofibration.

PROOF. It is clear that Statement (2) is an immediate consequence of Statement (1). Thus it remains to prove Statement (1). Now let $f\colon A \to X$ and $g\colon B \to Y$ be two cofibrations,

By Lemma 39.8 we can assume that $f\colon A \to X$ and $g\colon B \to Y$ are both inclusions. Note that by Lemma 39.3 (1) ⇒ (3) there exist retractions $r\colon X \times [0,1] \to (X \times \{0\}) \cup (A \times [0,1])$ and $s\colon Y \times [0,1] \to (Y \times \{0\}) \cup (B \times [0,1])$. We denote by $r =: (r_X, r_{[0,1]})$ the coordinate functions of r viewed as a map with values in $X \times [0,1]$. Similarly we define $s =: (s_Y, s_{[0,1]})$. Next we consider the map

$$\begin{array}{rcl} (X \times Y) \times [0,1] & \to & ((X \times Y) \times \{0\}) \cup ((A \times B) \times [0,1]) \\ ((x,y),t) & \mapsto & \Big(\underbrace{(r_X(x,t), s_Y(y,t))}_{\in X \times Y}, \underbrace{\tfrac{1}{2}(r_{[0,1]}(x,t) + s_{[0,1]}(y,t))}_{\in [0,1]}\Big). \end{array}$$

One can easily verify that this map is a retraction. It follows from Lemma 39.3 (3) ⇒ (1) that $f \times g\colon A \times B \to X \times Y$ is indeed a cofibration.

If f and g are in fact closed cofibrations, then it follows from Lemma 3.9 (1) and the above that $f \times g$ is also a closed cofibration. ■

We continue with the following characterisation of closed cofibrations [**Strø66, Strø68**].

Theorem 39.13. *Let X be a topological space and let A be a closed subset. The following two statements are equivalent:*

(1) *The inclusion $i\colon A \to X$ is a cofibration.*
(2) *There exists a map $\phi\colon X \to [0,1]$ such that $A = \phi^{-1}(\{0\})$ and there exists a homotopy $H\colon X \times [0,1] \to X$ rel A such that $H_0 = \mathrm{id}_X$ and such that $H(x,t) \in A$ whenever $t > \phi(x)$.*

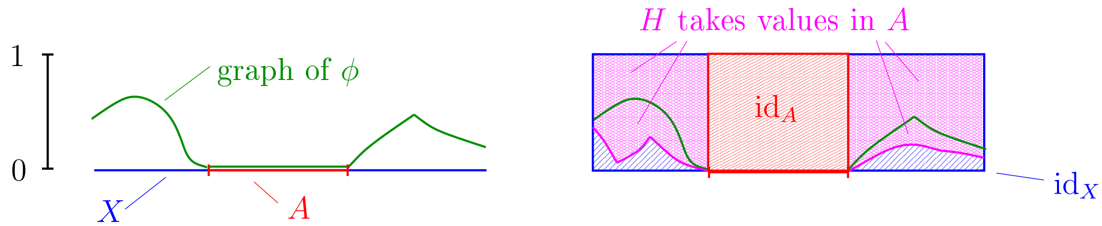


FIGURE 672. Illustration for Theorem 39.13.

Remark.

(1) We continue with the notation of the above Theorem 39.13. We consider the open subset $U := \phi^{-1}([0,1))$. The homotopy H restricts to a homotopy $K\colon U\times[0,1]\to X$ rel A with $K_0 = \mathrm{id}$ and with $K_1(U)\subset A$. This almost sounds like A is a deformation retract of U, but the given situation is somewhat looser, since K can take values outside of U.

(2) Sometimes a topological space X together with a closed subset A that admit H and ϕ as in Theorem 39.13 (2) is called a *neighborhood deformation retract pair*, or short, *NDR-pair*.

(3) Let X be a topological space and let $x_0\in X$. The pointed space (X,x_0) is called *well-pointed* if the inclusion $\{x_0\}\to X$ is a cofibration. If (X,x_0) is well-pointed, then in most practical situations the point $x_0\in X$ is good in the sense of the definition on page 604. Nonetheless, as discussed in

`https://mathoverflow.net/questions/87387/`

it is not necessarily the case that x_0 is good.

Example. Let M be a manifold. By the Collar Neighborhood Theorem 8.12 we can pick a collar neighborhood $[0,2]\times\partial M$. We consider the maps

$$\begin{array}{rcl}\phi\colon M & \to & [0,1] \\ P & \mapsto & \begin{cases} s, & \text{if } P=(Q,s) \text{ with } Q\in\partial M \text{ and } t\in[0,1], \\ 1, & \text{otherwise}\end{cases}\end{array}$$

and

$$\begin{array}{rcl}F\colon M\times[0,1] & \to & M \\ (P,t) & \mapsto & \begin{cases} (Q,\frac{2(s-t)}{2-t}), & \text{if } P=(Q,s) \text{ with } Q\in\partial M \text{ and } s\in[t,2], \\ (Q,0), & \text{if } P=(Q,s) \text{ with } Q\in\partial M \text{ and } s\in[0,t], \\ P, & \text{otherwise.}\end{cases}\end{array}$$

(Both maps are illustrated, to the best of the author's abilities, in Figure 673.) One can easily verify that the conditions of Theorem 39.13 are satisfied. This gives basically a new proof of Proposition 39.4 (1).

PROOF (∗). First we deal with the "(1)⇒(2)"-direction. Thus we suppose that $i\colon A\to X$ is a cofibration. Note that by Lemma 39.3 (1)⇒(3) we know that there exists a retraction $r\colon X\times[0,1]\to (X\times\{0\})\cup(A\times[0,1])$. We denote the two coordinate functions of r, viewed

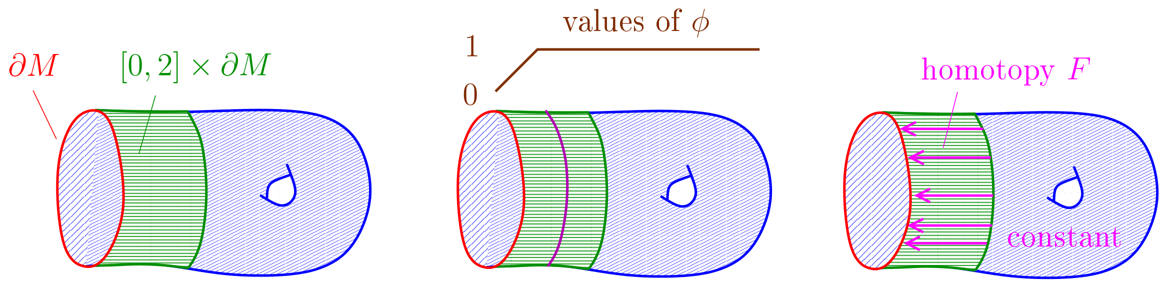


FIGURE 673

as a map with values in $X \times [0,1]$, by $r_X\colon X \times [0,1] \to X$ and $r_{[0,1]}\colon X \times [0,1] \to [0,1]$. We consider the following two maps:

$$\begin{array}{rcl} \phi\colon X & \to & [0,1] \\ x & \mapsto & \max\limits_{t\in[0,1]} |t - r_{[0,1]}(x,t)| \end{array} \quad \text{and} \quad \begin{array}{rcl} H\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & r_X(x,t). \end{array}$$

Note that by definition of a retraction we have $H_0 = \mathrm{id}_X$.

Claim. The map ϕ is well-defined and continuous.

First we consider the map $\theta\colon X \times [0,1] \to [0,1]$ that is given by $\theta(x,t) := |t - r_{[0,1]}(x,t)|$. Since $[0,1]$ is compact and since θ is continuous we obtain from Lemma 2.40 (2) that θ assumes a maximum, hence ϕ is well-defined.

It remains to show that $\phi\colon X \to [0,1]$ is continuous. Let V be an open subset of $[0,1]$. By Lemma 2.5 it suffices to show that given any $x_0 \in \phi^{-1}(V)$ there exists an open neighborhood U of x_0 with $U \subset \phi^{-1}(V)$. First we consider the case that $\phi(x_0) \in (0,1)$. This implies that we can pick $\epsilon > 0$ such that $(\phi(x_0) - \epsilon, \phi(x_0) + \epsilon) \subset V$. Note that by definition of ϕ and θ we have $\theta(\{x_0\} \times [0,1]) \subset [0, \phi(x_0)]$. Since $[0,1]$ is compact and since $[0, \phi(x_0) + \epsilon)$ is an open subset of $[0,1]$ we obtain from Exercise 3.6 that there exists an open neighborhood U_1 of $x_0 \in X$ such that

$$\{x_0\} \times [0,1] \ \subset\ U_1 \times [0,1] \ \subset\ \theta^{-1}([0, \phi(x_0) + \epsilon)).$$

Hence, for every $x \in U_1$ we have $\phi(x) \le \phi(x_0) + \epsilon$. Next suppose we are given $t_0 \in [0,1]$ such that $\theta(x_0, t_0) \in (\phi(x_0) - \epsilon, \phi(x_0) + \epsilon)$. By definition of the product topology there exists an open subset $U_2 \subset X$ such that

$$(x_0, t_0) \ \in\ U_2 \times \{t_0\} \ \subset\ \theta^{-1}((\phi(x_0) - \epsilon, \phi(x_0) + \epsilon)).$$

Note that this implies that for every $x \in U_2$ we have $\phi(x) \ge \phi(x_0) - \epsilon$. In particular we see that $U_1 \cap U_2 \subset \phi^{-1}((\phi(x_0) - \epsilon, \phi(x_0) + \epsilon)) \subset V$. Basically the same argument also works if $\phi(x_0) = 0$ or $\phi(x_0) = 1$. ⊞

Claim. We have $A = \phi^{-1}(0)$.

First let $x \in A$. Since r is a retraction we know that $r(x,t) = (x,t)$ for all $t \in [0,1]$. This means that $r_{[0,1]}(x,t) = t$ for all $t \in [0,1]$. Thus we see that $\phi(x) = 0$. Conversely, suppose that we have $x \in X$ with $\phi(x) = 0$. One easily verifies that this implies that for every $t > 0$ we have $r(x,t) \in A \times [0,1]$, in other words, we have $\{x\} \times (0,1] \subset r^{-1}(A \times [0,1])$.

Since $A \times [0,1]$ is a closed subset of $X \times [0,1]$ it follows from elementary arguments, see e.g. Exercise 2.11 (a), that $\{x\} \times [0,1]$ also belongs to $r^{-1}(A \times [0,1])$. But this implies that $x \in A$. ⊞

It remains to prove the following claim.

Claim. Given $(x,t) \in X \times [0,1]$ with $t > \phi(x)$ we have $H(x,t) \in A$.

So suppose we are given $(x,t) \in X \times [0,1]$ with $t > \phi(x)$. Note that this implies that $t > \phi(x) \geq |t - r_{[0,1]}(x,t)|$. This is only possible if $r_{[0,1]}(x,t) > 0$. Thus we see that $r(x,t) \in A \times [0,1]$. This implies that $H(x,t) \in A$. This concludes the proof of the claim and thus completes the proof of the "(2)⇒(1)"-direction.

Finally we provide the proof for the "(1)⇒(2)"-direction. Given $\phi\colon X \to [0,1]$ and the homotopy $H\colon X \times [0,1] \to X$ we consider the map

$$\begin{array}{rcl} r\colon X \times [0,1] & \to & (X \times \{0\}) \cup (A \times [0,1]) \\ (x,t) & \mapsto & (H(x,t), \max\{t - \phi(x), 0\}). \end{array}$$

One can easily verify that this map is indeed a retraction. Thus it follows from Lemma 39.3 (3)⇒(1) that the inclusion $i\colon A \to X$ is a cofibration. ■

Proposition 39.14. *Let $A \subset X$ and $B \subset Y$ be subsets of topological spaces. If the inclusion maps $i\colon A \to X$ and $j\colon B \to Y$ are closed cofibrations, then the inclusion map*

$$(A \times Y) \cup (X \times B) \;\to\; X \times Y$$

is also a closed cofibration.

Remark. In Exercise 39.6 we will see that the conclusion of Proposition 39.14 does not hold, if neither A nor B are closed subsets.

Proof. Since i and j are closed cofibrations we can pick maps $\phi\colon X \to [0,1]$, $\psi\colon Y \to [0,1]$ and homotopies $G\colon X \times [0,1] \to X$ and $H\colon Y \times [0,1] \to Y$ as in Theorem 39.13. We consider the maps

$$\begin{array}{rcl} \gamma\colon X \times Y & \to & [0,1] \\ (x,y) & \mapsto & \min\{\phi(x), \psi(y)\} \end{array}$$

and

$$\begin{array}{rcl} K\colon (X \times Y) \times [0,1] & \to & X \times Y \\ ((x,y),t) & \mapsto & (G(x, \min\{t, \psi(y)\}), H(y, \min\{t, \phi(x)\})). \end{array}$$

It is easy to see that $\gamma^{-1}(\{0\}) = (X \times B) \cup (A \times Y)$. Furthermore it is basically clear that $K_0 = \mathrm{id}$. Finally note that by construction we have $K((x,y),t) \in (A \times Y) \cup (X \times B)$, whenever $t > \gamma(x,y)$. Therefore it follows from Theorem 39.13 that the inclusion map $(A \times X) \cup (X \times B) \to X \times Y$ is a closed cofibration. ■

Corollary 39.15. *Let X be a topological space and let $A \subset X$ be a subset. If the inclusion $i\colon A \to X$ is a closed cofibration, then the inclusion $(X \times \{0,1\}) \cup (A \times [0,1]) \to X \times [0,1]$ is also a closed cofibration.*

PROOF. On page 1024 we saw that the inclusion map $\{0,1\} \to [0,1]$ is a cofibration. It follows from this observation together with Proposition 39.14 that the given inclusion map is a closed cofibration. ∎

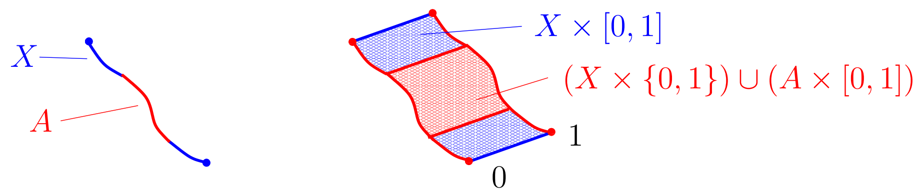


FIGURE 674. Illustration for Corollary 39.15.

Lemma 39.16. *Let $i\colon A \to X$ be a closed cofibration. If i is a homotopy equivalence, then $i(A)$ is a deformation retract of X.*

PROOF. Let $i\colon A \to X$ be a cofibration. By Lemma 39.8 we can assume that A is a closed subset of X and that $i\colon A \to X$ is the inclusion. Let $q\colon X \to A$ be a homotopy inverse of i. Recall that this means that $q \circ i = q\colon A \to A$ is homotopic to id_A and $i \circ q$ is homotopic to id_X.

We pick a homotopy $F\colon A \times [0,1] \to A$ from $q \circ i = q\colon A \to A$ to id_A. Since $i\colon A \to X$ is a cofibration we can extend the map $q\colon X \to A$ on X and the homotopy $F\colon A \times [0,1] \to A$ to a homotopy $\widetilde{F}\colon X \times [0,1] \to A$. Note that $\widetilde{F}_1|_A = F_1|_A = \mathrm{id}_A$, in other words, $\widetilde{F}_1\colon X \to A$ is a retraction. We write $r := \widetilde{F}_1\colon X \to A \subset X$.

Note that $r\colon X \to X$ is homotopic to $q = i \circ q\colon X \to X$ which in turn is homotopic to id_X. Thus it follows from Lemma 18.4 (1) that there exists a homotopy G from id_X to r.

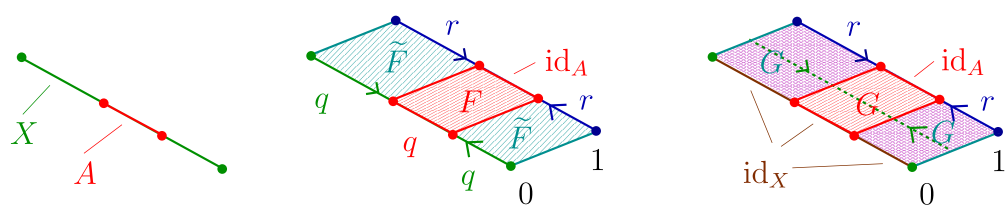


FIGURE 675. First illustration for the proof of Lemma 39.16.

Now we consider the following map

$$\begin{aligned} K\colon ((X \times \{0,1\}) \cup (A \times [0,1])) \times [0,1] &\to X \\ (x,s,t) &\mapsto \begin{cases} x, & \text{if } s = 0, \\ G(r(x), 1-t), & \text{if } s = 1, \\ G(x, (1-t)s), & \text{if } x \in A, \\ G(x,s), & \text{if } t = 0. \end{cases} \end{aligned}$$

It follows easily from the hypothesis that A is a closed subset together with Lemma 2.35 (2) that the map is continuous.

By Corollary 39.15 we know that $(X \times \{0,1\}) \cup (A \times [0,1]) \to X \times [0,1]$ is a cofibration. Thus we can extend the homotopy K to a homotopy $\widetilde{K}\colon (X \times [0,1]) \times [0,1] \to X$. The map $\widetilde{K}_1 \colon X \times [0,1] \to X$ is now a homotopy rel A from id_X to the retraction r. We have thus shown that A is a deformation retract of X. ■

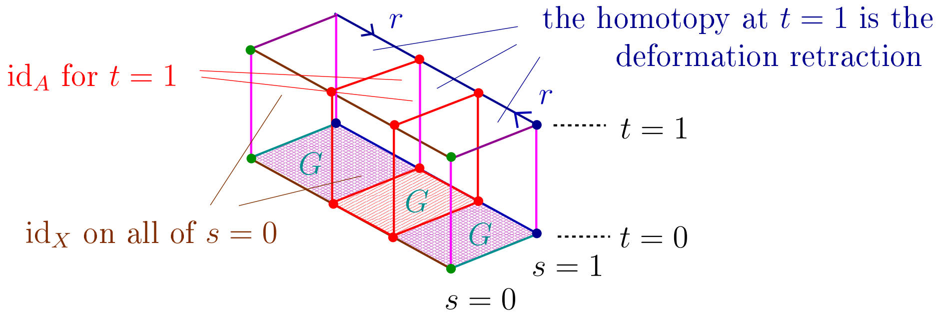


FIGURE 676. Second illustration for the proof of Lemma 39.16.

39.4. **Push-outs and cofibrations** $(*)$**.** In this section we will consider the relationship between cofibrations and pushouts. We recall the following notation from page 732.

Notation. Let $f\colon A \to X$ and $g\colon A \to Y$ be two maps between topological spaces. We write
$$X \cup_A Y \;:=\; (X \sqcup Y)/\sim \qquad \text{where } f(a) \sim g(a) \text{ for all } a \in A.$$
We refer to $X \cup_A Y$ as the *pushout of f and g.*

The next proposition continues a theme that we already encountered in Lemma 3.43 and Exercise 25.11.

Proposition 39.17. *Let $f\colon A \to X$ and $g\colon A \to Y$ be two maps between topological spaces. We consider the corresponding pushout diagram*
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "i"] \\ Y \arrow[r, "j"] & X \cup_A Y. \end{tikzcd}$$
The following two statements hold:

1. *If f is a (closed) cofibration, then the "opposite map" $j\colon Y \to X \cup_A Y$ is also a (closed) cofibration.*
2. *If f is a closed cofibration and if f is a homotopy equivalence, then the "opposite map" $j\colon Y \to X \cup_A Y$ is also a homotopy equivalence.*

The same statements hold, by symmetry, for g and the "opposite map" i.

Example. Let $f\colon A \to Y$ be a map between topological spaces. We consider the pushout diagram
$$\begin{tikzcd} A \arrow[r, "a\mapsto(a{,}0)"] \arrow[d, "f"'] & A \times [0,1] \arrow[d, "i"] \\ Y \arrow[r, "j"] & \underbrace{(A \times [0,1]) \cup_A Y}_{=\mathrm{Cyl}(f)}. \end{tikzcd}$$

By the discussion on page 1033 we know that the top horizontal map is a cofibration. Thus it follows from Proposition 39.17 that the natural inclusion $Y \to \mathrm{Cyl}(f\colon A \to Y)$ is a cofibration. Thus, together with Proposition 39.6 (1) we see that both the natural inclusions $A \to \mathrm{Cyl}(f\colon A \to Y)$ and $Y \to \mathrm{Cyl}(f\colon A \to Y)$ are cofibrations.

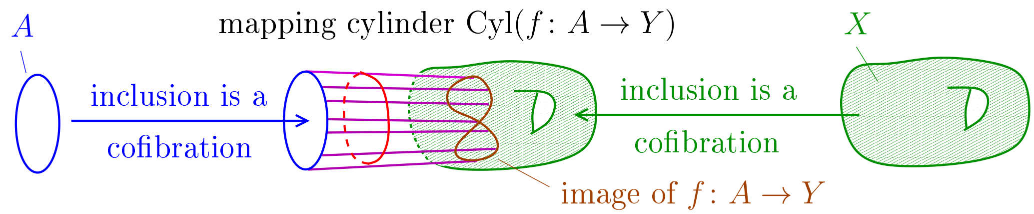


Figure 677

Proof. Throughout the proof we assume that $f\colon A \to X$ is a cofibration. By Lemma 39.8 we can assume that A is a subset of X and that $f\colon A \to X$ is the inclusion.

First we want to show that $j\colon Y \to X \cup_A Y$ is also a cofibration. Since f is a cofibration we know by Lemma 39.3 (1)⇒(3) that there exists a retraction

$$r\colon X \times [0,1] \;\to\; (X \times \{0\}) \cup (A \times [0,1]).$$

We introduce the following extra maps:

(1) We denote by $i\colon Y \to X \cup_A Y$ and $j\colon X \to X \cup_A Y$ the two obvious maps.
(2) We consider the pushout diagram

$$\begin{tikzcd}
Y \times \{0\} \arrow[r] \arrow[d, "g\times\mathrm{id}"'] & Y \times [0,1] \arrow[d, "k"] \\
((X \cup_A Y) \times \{0\}) \arrow[r] & ((X \cup_A Y) \times \{0\}) \cup (Y \times [0,1]).
\end{tikzcd}$$

We denote by $k\colon Y \times [0,1] \to ((X \cup_A Y) \times \{0\}) \cup (Y \times [0,1])$ the natural map.

Now we consider the following diagram:

$$\begin{tikzcd}
A \times [0,1] \arrow[r, "f\times\mathrm{id}_{[0,1]}"] \arrow[d, "g\times\mathrm{id}_{[0,1]}"'] & X \times [0,1] \arrow[d, "i\times\mathrm{id}_{[0,1]}"] \arrow[drr, "r"] & & \\
Y \times [0,1] \arrow[r, "j\times\mathrm{id}_{[0,1]}"] \arrow[drrr, bend right, "k"'] & (X \cup_A Y) \times [0,1] \arrow[drr, dashed, red, "s"] & & (X \times \{0\}) \cup (A \times [0,1]) \arrow[d, "(\overline{f}\times\mathrm{id}_{[0,1]})\cup(f\times\mathrm{id}_{[0,1]})"] \\
& & & ((X \cup_A Y) \times \{0\}) \cup (Y \times [0,1]).
\end{tikzcd}$$

We make the following observations:

(1) It follows immediately from Lemma 3.43 (3b) that we can view $Y \times [0,1]$ as a subset of $(X \cup_A Y) \times [0,1]$.

(2) The diagonal dashed map s comes from the fact, discussed on page 735, that the pushout commutes with taking products with the regionally compact topological space $[0,1]$.
(3) It follows from (1) and (2) that we can view the map s as a retraction. Thus it follows from Lemma 39.3 that the map $Y \to X \cup_A Y$ is indeed a cofibration.

Finally note that if f is a closed cofibration, then it follows from the above together with Lemma 3.43 (3d) that the map $Y \to X \cup_A Y$ is also a closed cofibration.

Now we assume that f is a closed cofibration and that it is a homotopy equivalence. We need to show that $j\colon Y \to X \cup_A Y$ is a homotopy equivalence. By Lemma 3.43 (3b) we know that j is an embedding. By Lemma 18.14 it now suffices to show that $j(Y)$ is a deformation retract of $X \cup_A Y$. By Lemma 39.16 we know that there exists a deformation retraction $G\colon X \times [0,1] \to X$ from X to A. We consider the map

$$\begin{array}{rcl} H\colon (X \cup_A Y) \times [0,1] & \to & X \cup_A Y \\ ([P],t) & \mapsto & \begin{cases} i(G(Q),t), & \text{if } P = i(Q) \text{ with } Q \in X, \\ j(Q), & \text{if } P = j(Q) \text{ with } Q \in Y. \end{cases} \end{array}$$

It follows from Lemma 18.23 that this map is actually continuous. Now that we put our worries regarding continuity to rest it is clear that this map is a deformation retraction from $X \cup_A Y$ to $j(Y)$. ■

The following theorem is the main result of this section.

Theorem 39.18. (Homotopy Pushout Theorem) *Suppose we are given the following commutative diagram of maps between topological spaces:*

$$\begin{tikzcd} X \arrow[d, "\varphi_X"] & A \arrow[l, hook', "i"'] \arrow[r, "f"] \arrow[d, "\varphi_A"] & Y \arrow[d, "\varphi_Y"] \\ X' & A' \arrow[l, hook', "i'"'] \arrow[r, "f'"] & Y'. \end{tikzcd}$$

If the vertical maps are homotopy equivalences and if the maps i and i' to the left are closed cofibrations, then the induced map

$$\begin{array}{rcl} X \cup_A Y & \to & X' \cup_{A'} Y' \\ {[P]} & \mapsto & \begin{cases} \varphi_X(P), & \text{if } P \in X, \\ \varphi_Y(P), & \text{if } P \in Y \end{cases} \end{array}$$

between the pushouts is a homotopy equivalence.

Remark. We outsource the proof, which follows quite closely the argument in [**Rap16**], to the next section. Alternative proofs are given in [**BrownR68**, p. 249], [**BrownR06**, Proposition 7.5.7] and [**tD08**, Proposition 5.3.4].

Proposition 39.19. *Let $\varphi\colon Y \to Y'$ be a homotopy equivalence between two topological spaces. Furthermore let A be a topological space and let $f\colon A \to Y$ and $f'\colon A \to Y'$ be two maps such that $\varphi \circ f$ is homotopic to f'. Then there is a homotopy equivalence*

$$\mathrm{Cone}(f\colon A \to Y) \ \to \ \mathrm{Cone}(f'\colon A \to Y')$$

Example. Let $f_0, f_1 \colon A \to Y$ be two maps between topological spaces. If f_0 and f_1 are homotopic, then it follows immediately from Proposition 39.19, applied to the map $\varphi = \mathrm{id}\colon Y \to Y' = Y$, that the mapping cones $\mathrm{Cone}(f\colon A \to Y)$ and $\mathrm{Cone}(f'\colon A \to Y)$ are homotopy equivalent. Note that this recovers for the most part the statement of Proposition 24.11.

PROOF. Let $F\colon A \times [0,1] \to Y'$ be a homotopy between $\varphi \circ f$ and f'. We consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Cone}(A) \arrow[d, "\mathrm{id}"] & A \arrow[l, hook'] \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & Y \arrow[d, "\varphi"] \\
\mathrm{Cone}(A) \arrow[d, "x\mapsto(x,0)"] & A \arrow[l, hook'] \arrow[r, "\varphi\circ f"] \arrow[d, "a\mapsto(a,0)"] & Y' \arrow[d, "y\mapsto(y,0)"] \\
\mathrm{Cone}(A)\times[0,1] & A\times[0,1] \arrow[l, hook'] \arrow[r, "(a{,}t)\mapsto(F(a{,}t){,}t)"] & Y'\times[0,1] \\
\mathrm{Cone}(A) \arrow[u, "x\mapsto(x,1)"] & A \arrow[l, hook'] \arrow[r, "f'"] \arrow[u, "a\mapsto(a,1)"] & Y'. \arrow[u, "y\mapsto(y,1)"]
\end{tikzcd}$$

The vertical maps are evidently homotopy equivalences. Furthermore we know by Proposition 39.6 (2) and Proposition 39.12 that the left horizontal maps are closed cofibrations. It follows from the Homotopy Pushout Theorem 39.18 together with Lemma 18.11 (1) that the pushout on top is homotopy equivalence to the pushout at the bottom. But the pushout on top is precisely $\mathrm{Cone}(f\colon A \to Y)$ and the pushout at the bottom is precisely $\mathrm{Cone}(f'\colon A \to Y')$. ■

Remark. Another typical application of the Homotopy Pushout Theorem 39.18 is given in the proof of Proposition 103.18 where we show that "handle attachments" and "cell attachments" lead to homotopy equivalent topological spaces.

39.5. **Proof of Theorem 39.18** (*). In this section we provide a proof of Theorem 39.18, following the argument of [**Rap16**]. The least that can be said is that the proof is not for the faint of heart.

The following lemma is a key step towards the proof of Theorem 39.18.

Lemma 39.20. *Let $f\colon A \to X$ and $g\colon A \to Y$ be two maps between topological spaces. We consider the corresponding pushout diagram*

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[d, "g"'] & B \arrow[l, bend right, "s"'] \arrow[d, "i"] \\
X \arrow[r, "j"] & X \cup_A B.
\end{tikzcd}$$

Let f be a homotopy equivalence. If g is a closed cofibration, and if there exists a map $s\colon B \to A$ with $f \circ s = \mathrm{id}_B$ and which is a closed cofibration, then the map "opposite to f", i.e. the map j is a homotopy equivalence.

Remark. Sometimes it takes a while to spot the differences to earlier results. In the similar looking Proposition 39.17 we had assumed that f is not only a homotopy equivalence but also a closed cofibration. Now we assume g is a closed cofibration and that we have the extra map s to play with.

PROOF OF LEMMA 39.20. First note that it follows from the hypotheses that $f \circ s = \mathrm{id}_B$ and that f is a homotopy equivalence, together with Lemma 18.11 (1), that s is also a homotopy equivalence. We consider the following diagram:

$$\begin{tikzcd}
B \arrow[r, "s", "\simeq"'] \arrow[d, "g\circ s"'] & A \arrow[rr, "f", "\simeq"'] \arrow[d, "u"'] \arrow[ddr, "g"] & & B \arrow[d, "g\circ s"'] \arrow[ddr, "i"] & \\
X \arrow[r, "v"] \arrow[drr, bend right, "="'] & P := X \cup_B A \arrow[rr, "k"] \arrow[dr, "l"] & & X \arrow[dr] & \\
& & X \arrow[rr, "j"] & & X \cup_A B.
\end{tikzcd}$$

We make the following clarifications and observations:

(1) To the top left we have a pushout square. In particular the maps $u\colon A \to X \cup_B A$ and $v\colon X \to X \cup_B A$ are the corresponding natural maps.
(2) The map $k\colon X \cup_B A \to X$ is defined via the identity on X and the map $g \circ s \circ f$ on A.
(3) It follows easily from Lemma 3.43 (2) and (4) that the square to the top right is a pushout diagram.
(4) The map $l\colon X \cup_B A \to X$ is defined via the identity on X and the map g on A.
(5) It follows easily from the definitions that the diagram commutes.
(6) Since s is a closed cofibration and a homotopy equivalence it follows from Proposition 39.17 (2) that the map $v\colon X \to X \cup_B A$ is also a closed cofibration and a homotopy equivalence.
(7) Note that $k \circ v = \mathrm{id}_X$ and $l \circ v = \mathrm{id}_X$. Since v and id_X are homotopy equivalences we obtain from Lemma 18.11 (1) that k and l are also homotopy equivalences.
(8) Since $g \circ s$ is a closed cofibration we obtain from Proposition 39.17 (1) that the map u is also a closed cofibration.

Next we consider the following two pushout diagrams:

$$\begin{tikzcd}
P \arrow[r, "l", "\simeq"'] \arrow[d, "p\mapsto(p,1)"] & X \arrow[d, "t"'] \\
(P\times[0,1]) \cup_{A\times[0,1]} A \arrow[r] & \underbrace{((P\times[0,1])\cup_{A\times[0,1]} A)\cup_P X}_{=:\mathrm{Cyl}(l)_A}
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
P = X\cup_B A \arrow[r, "k", "\simeq"'] \arrow[d, "\varphi(p)=(p,1)"] & X \arrow[d, "m"] \\
\mathrm{Cyl}(l)_A \arrow[r, "w"] & \mathrm{Cyl}(l)_A \cup_P X.
\end{tikzcd}$$

Providing the proof of the following claim is the main difficulty in the proof of the lemma.

Claim. The map $w\colon\ \mathrm{Cyl}(l)_A \to \mathrm{Cyl}(l)_A \cup_P X$ is a homotopy equivalence.

To prove the claim we first consider the following pushout diagram:

$$\begin{tikzcd}
A\times[0,1] \arrow[r, "(a,t)\mapsto a"] \arrow[d, "u\times\mathrm{id}"] & A \arrow[d] \arrow[ddr, bend left, "g"] & \\
P\times[0,1] \arrow[r, "z"] \arrow[drr, bend right, "(p,t)\mapsto l(p)"'] & (P\times[0,1])\cup_{A\times[0,1]} A \arrow[dr, dashed, "\Theta"] & \\
 & & X.
\end{tikzcd}$$

We take the resulting map Θ and use it to produce the following even more confusing pushout diagram

$$\text{(I)}\qquad \begin{tikzcd}
P \arrow[r, "l"] \arrow[d, "\simeq"', "p\mapsto z(p,0)"] & X \arrow[d, "t"', "\simeq"] \arrow[ddr, bend left, "\mathrm{id}"] & \\
(P\times[0,1])\cup_{A\times[0,1]} A \arrow[r] \arrow[drr, bend right, "\Theta"'] & \mathrm{Cyl}(l)_A=(P\times[0,1]\cup_{A\times[0,1]}A)\cup_P X \arrow[dr, dashed, "\Xi", "\simeq"'] & \\
 & & X.
\end{tikzcd}$$

We continue with the following comments:

(9) We claim that the left-hand vertical map is a closed cofibration. To verify this statement we consider the following commutative diagram

$$\begin{tikzcd}
(A\times[0,1])\cup_{A\times\{0,1\}}(P\times\{0,1\}) \arrow[r] \arrow[d, hook] & A\cup_{A\times\{0,1\}}(P\times\{0,1\}) \arrow[d] & P\cup_A P \arrow[l, "\cong"'] & P. \arrow[l] \arrow[dll, bend left, "p\mapsto z(p,0)"] \\
P\times[0,1] \arrow[r] & (P\times[0,1])\cup_{A\times[0,1]} A & &
\end{tikzcd}$$

We make a few comments:

(a) The left top horizontal map is induced by the projection $A\times[0,1]\to A$ and the identity on $P\times\{0,1\}$.
(b) The left vertical map is the "obvious" map. It is a closed cofibration by the combination of Corollary 39.15 with Lemma 3.45.
(c) The middle vertical map and the bottom horizontal map are the obvious maps.
(d) We leave it to the reader to verify that the square is in fact a pushout diagram.
(e) It follows from (b) and (d) together with Proposition 39.17 that the middle vertical map is a closed cofibration.
(f) The map $P\cup_A P\to A\cup_{A\times\{0,1\}}(P\times\{0,1\})$ is given by sending the "left copy" of P to $P\times\{0\}$ and the "right copy" of P to $P\times\{1\}$. There is an obvious inverse which shows that this map is actually a homeomorphism.
(g) The map $P\to P\cup_A P$ is given by sending P to the "left copy" of P. Since $A\to P$ is a closed cofibration we obtain again from Proposition 39.17 that the map $P\to P\cup_A P$ is a closed cofibration.
(h) It follows immediately from the definitions that the triangle to the right commutes. Thus we obtain from (e), (f) and (g) together with Lemma 39.2 that the

diagonal map is a closed cofibration. But that is precisely what we wanted to show.

(10) We claim that the left-hand vertical map $P \to (P \times [0,1]) \cup_{A\times[0,1]} A$ of (I) is a homotopy equivalence. This can be seen as follows. By (9) and Lemma 39.8 we can identify P with its image under the inclusion. It follows from Lemma 18.23 that the obvious deformation retraction on $P \times [0,1]$ and the constant map on A give rise to a deformation retraction from $(P \times [0,1]) \cup_{A\times[0,1]} A$ to the image of P. Thus it follows from the elementary Lemma 18.14 that the inclusion is indeed a homotopy equivalence.

(11) We consider the pushout square in (I). It follows from (9), (10) and Proposition 39.17 (2) that the right-hand vertical map t in (I) is also closed cofibration and a homotopy equivalence.

(12) We deduce from (11) and Lemma 18.11 (1) that Ξ is a homotopy equivalence.

Next we consider the following diagram:

(II)
$$\begin{tikzcd}
P = X \cup_B A \arrow[r, "k", "\simeq"'] \arrow[d, "\varphi(p)=(p,0)", "\simeq"'] \arrow[dd, bend right=60, "l"'] & X \arrow[d, "m"] \\
\mathrm{Cyl}(l)_A \arrow[r, "w"] \arrow[d, "\Xi", "\simeq"'] & \mathrm{Cyl}(l)_A \cup_P X \\
X. &
\end{tikzcd}$$

We make the following clarifications and observations:

(13) $\mathrm{Cyl}(l)_A \cup_P X$ is the pushout of the pushout diagram to the to top.

(14) The composition $P \xrightarrow{\varphi} \mathrm{Cyl}(l)_A \xrightarrow{\Xi} X$ equals the map l.

(15) Since l is a homotopy equivalence and since we just saw that $\Xi\colon \mathrm{Cyl}(l)_A \to X$ is a homotopy equivalence we obtain again from Lemma 18.11 (1) that $\varphi\colon P \to \mathrm{Cyl}(l)_A$ is a homotopy equivalence.

(16) We will now argue that the map $\varphi\colon P \to \mathrm{Cyl}(l)_A$ is a closed cofibration. To do so we consider the following diagram:

$$\begin{tikzcd}
 & P \arrow[d] \arrow[dd, bend left=60, "\varphi"] \\
(A \times [0,1]) \cup_{A\times\{0,1\}} (P \times \{0,1\}) \arrow[r] \arrow[d] & X \cup_A P \arrow[d] \\
P \times [0,1] \arrow[r] & \mathrm{Cyl}(l)_A.
\end{tikzcd}$$

We make the following observations:

(a) The top vertical map is a closed cofibration by Proposition 39.17 and our hypothesis that $A \to X$ is a closed cofibration.

(b) The map $X \cup_A P \to \mathrm{Cyl}(l)_A$ is defined by t on X and by φ on P.

(c) The vertical map to the left is induced by the inclusions.

(d) The horizontal map on the top is induced by the projection $A \times [0,1] \to A$, the obvious identification $P \times \{0\} \to P$ and the map l on $P = P \times \{1\} \to X$.

(e) We consider the maps

$$(P \times [0,1]) \cup_{(A\times[0,1])\cup_{A\times\{0,1\}}(P\times\{0,1\})} (X \cup_A P) \;\leftrightarrow\; \underbrace{((P\times[0,1])\cup_{A\times[0,1]} A)\cup_P X}_{=\mathrm{Cyl}(l)_A}$$

which are the obvious maps on $P \times [0,1]$, the obvious map on X and which going left-to-right is given by the obvious map $P \to P \times \{0\}$ and which right-to-left is given by the natural map $A \to X \cup_B A = P$. These maps are well-defined and inverses of one another, hence they are isomorphisms. These maps show that the square is a pushout diagram.

(f) It follows from Corollary 39.15 together with Lemma 3.45 that the left vertical map in the square is a closed cofibration.

(g) It follows (f) together with Proposition 39.17 (1) that the right vertical map in the square is also a closed cofibration.

(h) By (a) and (g) the map of our desire is a composition of two closed cofibrations, hence it is itself a closed cofibration by Lemma 39.2.

(17) It follows from (15) and (16) together with Proposition 39.17 (2), applied the diagram (II), that the map $m\colon X \to \mathrm{Cyl}(l)_A \cup_P X$ is a homotopy equivalence.

(18) By (7), (15) and (17) we know that m, k, φ are homotopy equivalences. By the diagram (II) we have $w \circ \varphi = m \circ k$. Thus we obtain from Lemma 18.11 (1) that w is also a homotopy equivalence. ⊞

Finally we consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "t"'] \arrow[d, "j"] \arrow[rr, bend left, "\mathrm{id}"] & \mathrm{Cyl}(l)_A \arrow[r, "\simeq", "\Xi"'] \arrow[d, "w", "\simeq"'] & X \arrow[d, "j"] \\
X \cup_A B \arrow[r, "t'"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathrm{Cyl}(l)_A \cup_P X \arrow[r, "n"] & X \cup_A B.
\end{tikzcd}$$

Here are the customary clarifications and observations:

(19) Recall that the map $t\colon X \to \mathrm{Cyl}(l)_A$ is the natural inclusion.

(20) By definition of t and Ξ we have $\Xi \circ t = \mathrm{id}_X$.

(21) The map $t'\colon X \cup_A B \to \mathrm{Cyl}(l)_A \cup_P X$ is defined by $w \circ t\colon X \to \mathrm{Cyl}(l)_A$ and it is defined by $m \circ g \circ s\colon B \to X$. Furthermore the map $n\colon \mathrm{Cyl}(l)_A \cup_P X \to X \cup_A B$ is defined by Ξ on $\mathrm{Cyl}(l)_A$ and by the natural map $j\colon X \to X \cup_A B$ on X.

(22) One can easily verify that $n \circ t' = \mathrm{id}_{X\cup_A B}$.

(23) It follows immediately from the definitions that the diagram commutes.

It follows from (20), (22) and (23) and the above commutative diagram that j is a retract of the map w, in the sense of the definition in Exercise 18.19. Since w is a homotopy equivalence we obtain from Exercise 18.19 that j is also a homotopy equivalence. We are done! ■

The next lemma is the same statement as Lemma 39.20 except that we can drop the hypothesis on the existence of s.

Lemma 39.21. *Let $f\colon A \to X$ and $g\colon A \to Y$ be two maps between topological spaces. We consider the corresponding pushout diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "i"] \\ X \arrow[r, "j"] & X \cup_A B. \end{tikzcd}$$

Let f be a homotopy equivalence. If g is a closed cofibration, then the map "opposite to f", i.e. the map j is a homotopy equivalence.

Proof of Lemma 39.21. We consider the maps

$$\begin{tikzcd} A \arrow[r, "i_f", "a\mapsto(a,0)"'] \arrow[rr, bend left, "f"] & \mathrm{Cyl}(f) \arrow[r, "r"] & B \arrow[l, bend left, "s"] \end{tikzcd}$$

where $i_f\colon A \to \mathrm{Cyl}(f)$ and $s\colon B \to \mathrm{Cyl}(f)$ are the natural inclusions and $r\colon \mathrm{Cyl}(f) \to B$ is the natural retraction. We make the following observations:

(1) It follows from Proposition 39.6 and the discussion on page 1039 that i_f and s are closed cofibrations.
(2) It follows immediately from the definitions that $f = r \circ i_f$ and $r \circ s = \mathrm{id}_B$.
(3) By Lemma 24.6 (3) and Lemma 24.8 we know that r is a homotopy equivalence.
(4) It follows from the hypothesis that f is a homotopy equivalence, together with (2) and (3) and together with Lemma 18.11 (1), that i_f is a homotopy equivalence.

Next we consider the following diagram

$$\begin{tikzcd} A \arrow[r, "i_f"'] \arrow[rrr, bend left, "f"] \arrow[dd, "g"'] & \mathrm{Cyl}(f) \arrow[rr, "r"] \arrow[d, "\mu"] & & B \arrow[ll, bend left, "s"] \arrow[dl, "\nu"] \arrow[dd, "i"] \\ & X \cup_A \mathrm{Cyl}(f) \arrow[r, "\beta"] & (X \cup_A \mathrm{Cyl}(f)) \cup_{\mathrm{Cyl}(f)} B \arrow[dr, "\gamma", "\cong"'] & \\ X \arrow[ur, "\alpha"] \arrow[rrr, "j"] & & & X \cup_A B. \end{tikzcd}$$

We continue with the following clarifications and observations:

(5) The quadrilateral to the left is a pushout diagram. In particular α and μ are the natural maps of a pushout.
(6) The quadrilateral in the top center is also a pushout diagram. In particular β and ν are the natural maps of a pushout.
(7) The map γ is the natural homeomorphism given by Lemma 3.43 (5).
(8) It follows immediately from the definitions that the diagram commutes.

(9) We consider the quadrilateral to the left.
  (a) By (1) and (4) we know that i_f is a homotopy equivalence and that it is a closed cofibration. Thus we obtain from (5) together with Proposition 39.17 (2) that the opposite map $\alpha\colon X \to X \cup_A \mathrm{Cyl}(f)$ is a homotopy equivalence.
  (b) By hypothesis g is a closed cofibration, thus we obtain from (5) together with Proposition 39.17 (1) that μ is also a closed cofibration.

(10) Next we obtain the quadrilateral in the top center. By (1), (3), (6) and (9b) we can apply Lemma 39.20 and we obtain that the map β is a homotopy equivalence.

(11) By (7), (9a) and (10) we know that α, β and γ are homotopy equivalences. Since the diagram commutes we see that the bottom map $j\colon X \to X \cup_A B$ is indeed a homotopy equivalence. ■

Now we can provide the proof of Theorem 39.18.

Proof of Theorem 39.18. We are given the following commutative diagram

$$\begin{tikzcd} X \arrow[d, "\varphi_X"] & A \arrow[l, hook', "i"'] \arrow[r, "f"] \arrow[d, "\varphi_A"] & Y \arrow[d, "\varphi_Y"] \\ X' & A' \arrow[l, hook', "i'"'] \arrow[r, "f'"] & Y' \end{tikzcd}$$

where all of the vertical maps are homotopy equivalences and where the maps i and i' to the left are closed cofibrations. Recall that we need to show that the induced map

$$\Phi\colon X \cup_A Y \;\to\; X' \cup_{A'} Y'$$

is a homotopy equivalence.

Case 1. First we assume that f and f' are homotopy equivalences. We consider the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\varphi_X", "\simeq"'] & X \cup_A Y \arrow[d, "\Phi"] \\ X' \arrow[r, "u'"] & X' \cup_{A'} Y' \end{tikzcd}.$$

We make the following comments:

(1) The horizontal maps u and u' are the obvious natural maps coming from the following pushout diagrams

$$\begin{tikzcd} A \arrow[r, "f", "\simeq"'] \arrow[d, "i"'] & Y \arrow[d, "v"] \\ X \arrow[r, "u"] & X \cup_A Y \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} A' \arrow[r, "f'", "\simeq"'] \arrow[d, "i'"'] & Y' \arrow[d, "v'"] \\ X' \arrow[r, "u'"] & X' \cup_{A'} Y'. \end{tikzcd}$$

Since f and f' are homotopy equivalences and since i and i' are closed cofibrations we obtain from Lemma 39.21, applied to these two pushout diagrams, that the horizontal maps u and u' are homotopy equivalences.

(2) The left vertical map of the diagram of the claim is a homotopy equivalence by hypothesis.

(3) It follows from (1) and (2), together with Lemma 18.11 (1), that Φ is a homotopy equivalence.

Case 2. Next we assume that f and f' are cofibrations. We consider the following diagram:

$$\begin{tikzcd}
 & A \arrow[rr, "i"] \arrow[dl, "f"'] \arrow[dd, "\varphi_A"', "\simeq"] & & X \arrow[dl] \arrow[dd, "k"] \arrow[ddr, "\varphi_X"', "\simeq"] & \\
Y \arrow[rr, "j"] \arrow[dd, "\varphi_Y"', "\simeq"] & & X \cup_A Y \arrow[dd, "l"] \arrow[ddr, color=blue, "\Phi"] & & \\
 & A' \arrow[rr] \arrow[dl, "f'"] & & X \cup_A A' \arrow[r, "q"] \arrow[dl, "l'"] & X'. \arrow[dl, "s"] \\
Y' \arrow[rr] & & X \cup_A Y' \arrow[r, "r"'] & X' \cup_{A'} Y' &
\end{tikzcd}$$

We make the following observations and clarifications:

(1) The top square and bottom left square are the obvious pushout diagrams.
(2) The vertical map $l\colon X \cup_A Y \to X \cup_A Y'$ is the map induced by φ_Y. The square in the front is a pushout diagram by Lemma 3.43 (4).
(3) The square in the back is defined to be a pushout diagram.
(4) Since i is a closed cofibration and φ_A is a homotopy equivalence it follows from Lemma 39.21, applied to the pushout diagram in the back, that k is a homotopy equivalence.
(5) Since i is a closed cofibration we obtain, by applying Proposition 39.17 to the pushout diagram on the top, that the "opposite map" $j\colon Y \to X \cup_A Y$ is a closed cofibration.
(6) Since j is a cofibration and since φ_Y is homotopy equivalence we obtain again from Lemma 39.21, this time applied to the pushout diagram in front, that l is a homotopy equivalence.
(7) Since f' is a closed cofibration we obtain again from Proposition 39.17 that the "opposite map" l' is a closed cofibration.
(8) We turn to the right part of the diagram. The map $q\colon X \cup_A A' \to X'$ is defined by φ_X on X and i' on A. Furthermore the map $r\colon X \cup_A Y' \to X' \cup_{A'} Y'$ is defined by φ_X on X' and id on Y'. Finally $s\colon X' \to X' \cup_{A'} Y'$ is the natural map. One can easily verify that the right hand part of the diagram commutes.
(9) Since φ_X and k are homotopy equivalences and since $q \circ k = \varphi_X$ we obtain from Lemma 18.11 (1) that q is a homotopy equivalence.
(10) As we mentioned in (2), the bottom left square is a pushout square. Furthermore it follows immediately from the definitions that the big square formed by the two squares at the bottom is also a pushout square. It follows from a fairly elementary argument, see e.g. Exercise 25.12, that the bottom right square is also a pushout square.
(11) It follows from (7), (9) and (10) together with the hard-fought Lemma 39.21 that r is a homotopy equivalence.
(12) Since $\Phi = r \circ l$ we obtain from (6) and (11) that Φ is indeed a homotopy equivalence.

Case 3. Finally we deal with the case that f and f' are arbitrary maps. We consider the following diagram

$$\begin{tikzcd}
X \arrow[dd, "\varphi_X"', "\simeq"] & A \arrow[l, "i"'] \arrow[rr, "f"] \arrow[dr, "j"] \arrow[dd, "\varphi_A"', "\simeq"] & & Y \arrow[dd, "\varphi_Y"', "\simeq"] \\
& & \mathrm{Cyl}(f) \arrow[ur, "\simeq", "r"'] \arrow[dd, "t"] & \\
X' & A' \arrow[l, "i'"'] \arrow[rr, "f'"] \arrow[dr, "j'"'] & & Y' \\
& & \mathrm{Cyl}(f) \cup_A A'. \arrow[ur, "q"'] &
\end{tikzcd}$$

We make the following clarifications and observations:

(1) The black part of the diagram is precisely the commutative diagram of our setup.
(2) The top triangle is given by the usual factorization of the map $f\colon A \to Y$ into the natural inclusion $j\colon X \to \mathrm{Cyl}(f)$ and the natural retraction $r\colon \mathrm{Cyl}(f) \to Y$. By Proposition 39.6 we know that j is a closed cofibration. Furthermore, by Lemma 24.8 (2a) we know that r is a homotopy equivalence.
(3) The left parallelogram is defined to be the pushout of the maps $j\colon A \to \mathrm{Cyl}(f)$ and $\varphi_A\colon A \to A'$.
(4) Since j is a closed cofibration we obtain from (3) and Proposition 39.17 that the map $j'\colon A' \to \mathrm{Cyl}(f) \cup_A A'$ is also a closed cofibration.
(5) Since φ_A is a homotopy equivalence and since j is a closed cofibration we obtain from Lemma 39.21 that t is a homotopy equivalence.
(6) The map $q\colon \mathrm{Cyl}(f) \cup_A A' \to Y'$ is defined via t on $\mathrm{Cyl}(f)$ and f' on A'.
(7) It follows easily from the definitions that the right parallelogram commutes, i.e. we have $q \circ t = \varphi_Y \circ r$. Since r, φ_Y and t are homotopy equivalences and since $q \circ t = \varphi_Y \circ r$ we obtain from Lemma 18.11 (1) that q is a homotopy equivalence.

Next we consider the commutative diagram

$$\begin{tikzcd}
X \arrow[d, "\varphi_X"] & A \arrow[l, hook', "i"'] \arrow[r, "j"] \arrow[d, "\varphi_A"] & \mathrm{Cyl}(f) \arrow[d, "t"] \\
X' & A' \arrow[l, hook', "i'"'] \arrow[r, "j'"] & \mathrm{Cyl}(f) \cup_A A'
\end{tikzcd}$$

Here we just make a single observations:

(8) By (2) and (4) and by hypothesis we know that all horizontal maps are closed cofibrations. Furthermore by hypothesis and (5) we know that all vertical maps are homotopy equivalences. Thus we obtain from Case 2 that the induced map

$$\Xi\colon X \cup_A \mathrm{Cyl}(f) \ \to\ X' \cup_{A'} (\mathrm{Cyl}(f) \cup_A A')$$

is a homotopy equivalence.

We move on to consider the following diagram:

$$(*)\qquad \begin{tikzcd}
X \cup_A \mathrm{Cyl}(f) \arrow[dd, "\simeq"', "\nu"] \arrow[dr, "\Xi", "\simeq"'] & & \mathrm{Cyl}(f) \arrow[ll, "\alpha"'] \arrow[r, "r", "\simeq"'] \arrow[dd, "\simeq"', "t"] & Y \arrow[dd, "\simeq"', "\varphi_Y"] \\
& X' \cup_{A'} (\mathrm{Cyl}(f) \cup_A A') \arrow[dl, "\mu"', "\cong"] & & \\
X' \cup_A \mathrm{Cyl}(f) & & \mathrm{Cyl}(f) \cup_A A' \arrow[ll, "\gamma"] \arrow[ul, "\beta"'] \arrow[r, "\simeq", "q"'] & Y'.
\end{tikzcd}$$

We make the following clarifications and observations:

(9) The lower diagonal map μ is the natural homeomorphism that is provided by Lemma 3.43 (5).

(10) The horizontal map α at the top left and the diagonal map β are the natural maps coming from pushouts. The vertical map ν to the left is induced by φ_X and the horizontal map γ at the bottom left map is induced by i'. One can easily verify that the diagram commutes.

(11) Recall that by hypothesis $i\colon A \to X$ and $i'\colon A' \to X'$ are closed cofibrations. Thus we obtain from Proposition 39.17 that the maps α and β are closed cofibrations. Since $\gamma = \mu \circ \beta$ and since μ is a homeomorphism we see that γ is also a closed cofibrations.

Finally we consider the following diagram:

$$\begin{tikzcd}
(X \cup_A \mathrm{Cyl}(f)) \cup_{\mathrm{Cyl}(f)} Y \arrow[r] \arrow[d, "\cong"] & (X' \cup_A \mathrm{Cyl}(f)) \cup_{\mathrm{Cyl}(f) \cup_A A'} Y' \arrow[d, "\cong"] \\
X \cup_A Y \arrow[r, "\Phi"] & X' \cup_{A'} Y'.
\end{tikzcd}$$

As usual we cannot be refrained from making comments.

(12) The top horizontal map is induced by ν and φ_Y.

(13) By (5), (8), (9) and our hypothesis we know that all the vertical maps in the diagram $(*)$ are homotopy equivalences. Furthermore by (2) and (7) we know that the right horizontal maps of $(*)$ are homotopy equivalences. Thus we obtain from Case 1 that the top horizontal map in our present diagram is a homotopy equivalence.

(14) The vertical maps are the natural homeomorphism provided by Lemma 3.43 (5).

(15) One can easily verify that the diagram commutes. Since the top horizontal map is a homotopy equivalence and since the vertical maps are homeomorphisms we see that the bottom horizontal map Φ is also a homotopy equivalence. And that had been our goal since page 1040. ■

This concludes our discussion of closed cofibrations. More information on closed cofibrations can be found in [**tD08**, Chapter 5], [**BrownR06**, Chapter 7], [**May99a**, Chapter 6] and [**Pic92**, Chapter 2.3].

Exercises for Chapter 39.

Exercise 39.1. Solve the following problems "by hand", i.e. without making use of any of above results. For example you should not use Lemma 39.8.

(a) Let $A = [a, b]$ be a compact interval and let $X = \mathbb{R}$. Show that the inclusion $i\colon A \to X$ is a cofibration.
(b) Let $A = \mathbb{R} \setminus \{0\}$ and $X = \mathbb{R}$. Show that the inclusion $i\colon A \to X$ is not a cofibration.
(c) Let $A = (-1, 1)$ and $X = \mathbb{R}$. Show that the inclusion $i\colon A \to X$ is not a cofibration.

Exercise 39.2. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two (closed) cofibrations. Show that $g \circ f\colon X \to Z$ is a (closed) cofibration.

Exercise 39.3. Prove the "(2) $\Rightarrow$ (1)"-direction of Lemma 39.3.

Exercise 39.4. Given an example of a topological space X and a non-empty subset A such that the map $\mathrm{Cone}(i\colon A \to X) \to X/A$ given in Lemma 39.9 (1) is not a homotopy equivalence.

Exercise 39.5. Let $X = \{x, y\}$ be a set with two elements and the trivial topology furthermore let $A = \{x\}$. Show that the inclusion $A \to X$ is a cofibration.
Remark. This is an example of a cofibration that is not closed.

Exercise 39.6. Let $X = \{a, b\}$ be the topological space with two elements a and b and where the open sets are precisely $\varnothing, \{a\}, X$. We set $A := \{a\}$.

(a) Show that the inclusion $A \to X$ is a cofibration.
(b) Show that the inclusion $(X \times A) \cup (A \times X) \to X \times X$ is not a cofibration.

Remark. This exercise shows that the conclusion of Proposition 39.14 does not hold if neither A nor B are closed subsets.

Exercise 39.7. Let $i\colon A \to X$ be a closed cofibration. Show that the induced map $\mathrm{Cone}(A) \to \mathrm{Cone}(X)$ is also a cofibration.
Hint. Make sense of the following diagram and use it is as a guideline:

$$\begin{tikzcd}
A \times \{0\} \arrow[r] \arrow[d] & X \times \{0\} \arrow[r] \arrow[d] & \{*\} \arrow[d] \\
A \times [0,1] \arrow[r] & (X \times \{0\}) \cup_{A \times \{0\}} (A \times [0,1]) \arrow[r] \arrow[d] & \mathrm{cone}(A) = * \cup_{A \times \{0\}} (A \times [0,1]) \arrow[d] \\
 & X \times [0,1] \arrow[r] & \mathrm{Cone}(X).
\end{tikzcd}$$

Part VI

Singular Homology

40. Higher homotopy groups: a primer

After these rather technical chapters on CW-complexes we now return to our original goal, namely trying to distinguish topological spaces. Using the fundamental group we had established several non-trivial results, for example we proved that

(1) S^1 and S^k with $k \geq 2$ are not homeomorphic,
(2) $\mathbb{R}^2$ and $\mathbb{R}^n$ with $n \geq 3$ are not homeomorphic,
(3) surfaces of different genus are not homeomorphic,
(4) we showed that S^1 is not a deformation retract of $\overline{B}^2$,
(5) the trefoil is not isotopic to the trivial knot.

On the other hand there are still many even rather elementary questions we cannot answer. For example at the moment we cannot show that for $k \neq l$ with $k, l \geq 2$ the spheres S^k and S^l are not homeomorphic and we cannot show that for $k \neq l$ with $k, l \geq 3$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic. Also, the somewhat mysterious lens spaces $L(p, q)$ still evaded our attempts at distinguishing them for a given p.

The problem with fundamental groups is that we only probe a topological space X with loops. The natural idea is now to go from using maps $S^1 \to X$ to studying maps $S^n \to X$ for $n \geq 2$. As we will see, this idea leads us naturally to the definition of the "higher homotopy groups" of a topological space.

40.1. Definition of the higher homotopy groups.

Notation. Throughout this chapter we write $I = [0, 1]$ for the unit interval. Given $n \in \mathbb{N}$ we denote by $I^n \subset \mathbb{R}^n$ the cube in $\mathbb{R}^n$ and we denote, as usual, by ∂I^n its boundary as a subset of $\mathbb{R}^n$.

Now let X be a topological space and let $x_0 \in X$. With the definitions from page 571 we can write

$$\pi_1(X, x_0) \;=\; \text{homotopy classes of maps } (I, \partial I) \to (X, x_0).$$

This definition can be easily generalized to higher dimensions:

Definition. Given a pointed topological space (X, x_0) and $n \in \mathbb{N}_0$ we define

$$\pi_n(X, x_0) \;=\; \text{homotopy classes of maps } (I^n, \partial I^n) \to (X, x_0).$$

For $n \geq 1$ we refer to $\pi_n(X, x_0)$ as the *n-th homotopy group.*

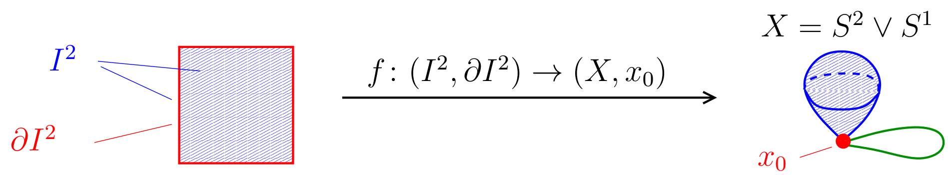


Figure 678. Illustration of an element in $\pi_2(S^2 \vee S^1, x_0)$.

Remark. This definition makes in particular sense for $n = 0$, we can thus consider $\pi_0(X, x_0)$. In Section 2.9 we had already defined $\pi_0(X)$ as the set of path-components of X. In fact it is straightforward to see that the map

$$\begin{array}{rcl} \pi_0(X,x_0) & \to & \pi_0(X) \\ [\gamma\colon I^0=\{0\}\to X] & \mapsto & \text{path-component of } X \text{ that contains } \gamma(0) \end{array}$$

is a bijection.

Note that $\pi_n(X,x_0)$ is a priori only a set. So to justify the name "homotopy group" we need to define a group structure on $\pi_n(X,x_0)$ which for $n=1$ agrees with our previous definition of the group structure. Here is what we do:

Definition. Let (X,x_0) be a pointed topological space and let $n\geq 1$. Furthermore let $f,g\colon (I^n,\partial I^n)\to(X,x_0)$ be two maps. We define

$$\begin{array}{rcl} f*g\colon (I^n,\partial I^n) & \to & (X,x_0) \\ (x_1,\dots,x_n) & \mapsto & \begin{cases} f(2x_1,\dots,2x_n), & \text{if } (x_1,\dots,x_n)\in[0,\frac{1}{2}]^n, \\ g(2x_1-1,\dots,2x_n-1), & \text{if } (x_1,\dots,x_n)\in[\frac{1}{2},1]^n, \\ x_0, & \text{otherwise.} \end{cases} \end{array}$$

This map is illustrated in Figure 679.[650]Applying Lemma 14.3 several times one sees that this map is indeed continuous.

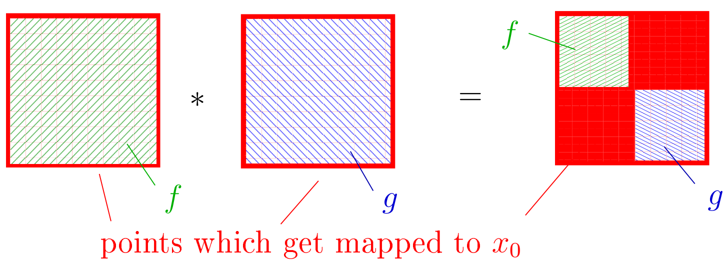


FIGURE 679

It is straightforward to see that the homotopy class of $f*g$ only depends on the homotopy classes of f and g. This leads us to the following definition:

Definition. Let (X,x_0) be a pointed topological space and let $n\geq 1$. We define

$$\begin{array}{rcl} \pi_n(X,x_0)\times\pi_n(X,x_0) & \to & \pi_n(X,x_0) \\ ([f],[g]) & \mapsto & [f]\cdot[g]:=[f*g]. \end{array}$$

Note that for $n=1$ this is precisely the same definition of the product structure that we had we already introduced in Chapter 14.2.

Now we can formulate the following proposition:

[650]For mysterious reasons in all figures of this chapter the origin $(0,0)$ is in the top left corner and the point $(1,1)$ is the vertex at the bottom right.

Proposition 40.1. *Let (X, x_0) be a pointed topological space and let $n \geq 1$. The following two statements hold:*

(1) *The map*

$$\begin{array}{rcl} \pi_n(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([f], [g]) & \mapsto & [f] \cdot [g] := [f * g] \end{array}$$

defines a group structure on $\pi_n(X, x_0)$.

(2) *Given a map $f\colon (I^n, \partial I^n) \to (X, x_0)$ the inverse of $[f] \in \pi_n(X, x_0)$ is represented by the map $f \circ \Phi\colon (I^n, \partial I^n) \to (X, x_0)$ where $\Phi\colon I^n \to I^n$ is the homeomorphism given by $\Phi(x_1, x_2, \ldots, x_n) = (1 - x_1, x_2, \ldots, x_n)$.*

As we will see, the proof of Proposition 40.1 is very similar to the proof of Proposition 14.7 and the proof of Proposition 14.6. To deal with the technical differences between the case $n = 1$ and the general case we need a few more definitions and one lemma.

Definition.

(1) A *cuboid* is any subset of $\mathbb{R}^n$ of the form $[c_1, d_1] \times \cdots \times [c_n, d_n]$.

(2) We say a cuboid $[c_1, d_1] \times \cdots \times [c_n, d_n]$ is *non-degenerate* if for all $i = 1, \ldots, n$ we have $d_i > c_i$.

Let X be a topological space, let $x_0 \in X$ and let $f\colon I^n \to X$ be a map.

(3) We say a cuboid $A \subset I^n$ is *boundary-trivial* if f sends all points on the boundary ∂A of A to x_0.

(4) Let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset I^n$ be a boundary-trivial cuboid and furthermore let $C = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be a non-degenerate cuboid that is contained in A. We define

$$\begin{array}{rcl} f(A, C)\colon I^n & \to & X \\ x = (x_1, \ldots, x_n) & \mapsto & \begin{cases} f(x), & \text{if } x \notin A, \\ x_0, & \text{if } x \in A \setminus C, \\ f\Big(a_1 + (x_1 - c_1)\frac{b_1 - a_1}{d_1 - c_1}, \ldots, a_n + (x_n - c_n)\frac{b_n - a_n}{d_n - c_n}\Big) & \text{if } x \in C. \end{cases} \end{array}$$

By applying Lemma 14.3 several times one can easily verify that this map is continuous.

The definition of $f(A, C)$ is sketched in Figure 680.

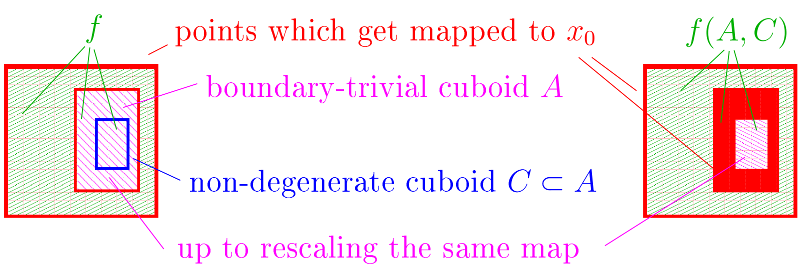


Figure 680

Lemma 40.2. *Let (X, x_0) be a pointed topological space and let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a map. Furthermore let $A \subset I^n$ be a boundary-trivial cuboid and let C be a non-degenerate* *cuboid that is contained in A. Then the maps f and $f(A, C)$ from $(I^n, \partial I^n)$ to (X, x_0) are homotopic.*

Proof $(*)$. Let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a map, let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset I^n$ be a boundary-trivial cuboid and let $C = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be a non-degenerate cuboid that is contained in A. For $t \in [0, 1]$ we write

$$o_t \;=\; \big[a_1(1-t) + c_1 t, b_1(1-t) + d_1 t\big] \times \cdots \times \big[a_n(1-t) + c_n t, b_n(1-t) + d_n t\big].$$

Note that $C_0 = A$ and $C_1 = C$. It is now straightforward to see that the map

$$\begin{array}{rcl} I^n \times [0,1] & \to & X \\ (x,t) & \mapsto & f(A, C_t)(x) \end{array}$$

is a homotopy between f and $f(A, C)$. ■

Now we can provide the proof of Proposition 40.1

Proof of Proposition 40.1. The neutral element of the product structure is given by the equivalence class of the constant map

$$\begin{array}{rcl} c_{x_0}\colon I^n & \to & X \\ x & \mapsto & x_0. \end{array}$$

Indeed, it follows immediately from Lemma 40.2, applied to $A = [0, 1]^n$ and $C = [0, \frac{1}{2}]^n$, that given any map $f\colon (I^n, \partial I^n) \to (X, x_0)$ we have $f * c_{x_0} \simeq f$. Basically the same argument shows also that $c_{x_0} * f \simeq f$.

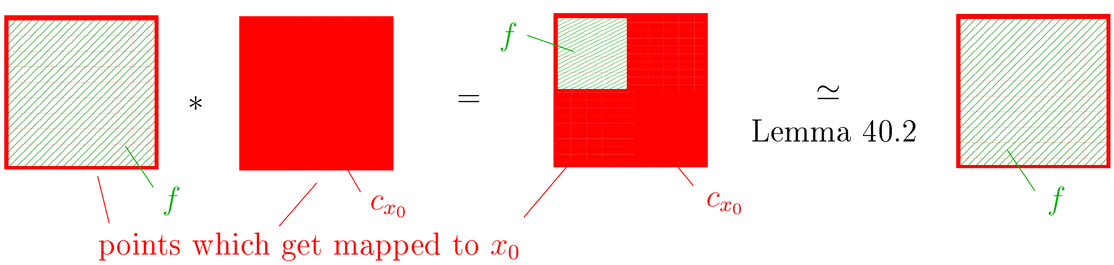


Figure 681

Now consider three maps $f, g, h\colon (I^n, \partial I^n) \to (X, x_0)$. By applying Lemma 40.2 several times one can easily show that $(f * g) * h \simeq f * (g * h)$, see e.g. Figure 682. This proves that the multiplication on $\pi_n(X, x_0)$ is associative.

Now we turn to the most interesting question, namely finding the inverse of a given element in $\pi_n(X, x_0)$. Thus suppose we are given a map $f\colon (I^n, \partial I^n) \to (X, x_0)$. We consider the map

$$\begin{array}{rcl} \overline{f}\colon I^n & \to & X \\ (x_1, \ldots, x_n) & \mapsto & f(1 - x_1, x_2, \ldots, x_n). \end{array}$$

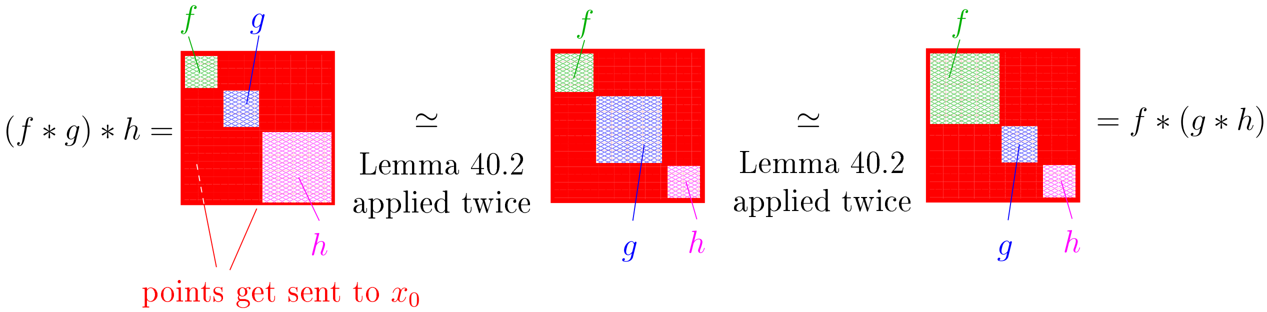


FIGURE 682

We claim that $[f] \cdot [\overline{f}] = [f * \overline{f}] = [c_{x_0}]$. Thus we need to show that $f * \overline{f}$ is homotopic to c_{x_0}. It follows from Lemma 40.2 that $f * \overline{f}$ is homotopic to the map

$$\begin{array}{rcl} h\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (x_1, \dots, x_n) & \mapsto & \begin{cases} f(2x_1, x_2, \dots, x_n), & \text{if } x_1 \in [0, \frac{1}{2}], \\ \underbrace{\overline{f}(2x_1 - 1, x_2, \dots, x_n)}_{= f(2-2x_1, x_2, \dots, x_n)}, & \text{if } x_1 \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

Thus it suffices to show that h is homotopic to c_{x_0}. It is in fact straightforward to see that the map[651][652]

$$\begin{array}{rcl} H\colon I^n \times [0,1] & \to & X \\ (x_1, \dots, x_n, t) & \mapsto & \begin{cases} f(2(x_1 - \frac{t}{2}), x_2, \dots, x_n), & \text{if } x_1 \in [\frac{t}{2}, \frac{1}{2}], \\ f(2 - 2(x_1 + \frac{t}{2}), x_2, \dots, x_n), & \text{if } x_1 \in (\frac{1}{2}, 1 - \frac{t}{2}], \\ x_0, & \text{otherwise} \end{cases} \end{array}$$

is a homotopy from h to c_{x_0}. We refer to Figure 683 for an illustration of the argument. Basically the same argument also shows that $[\overline{f}] \cdot [f] = [\overline{f} * f] = [c_{x_0}]$. We have thus verified both statements of the proposition. ■

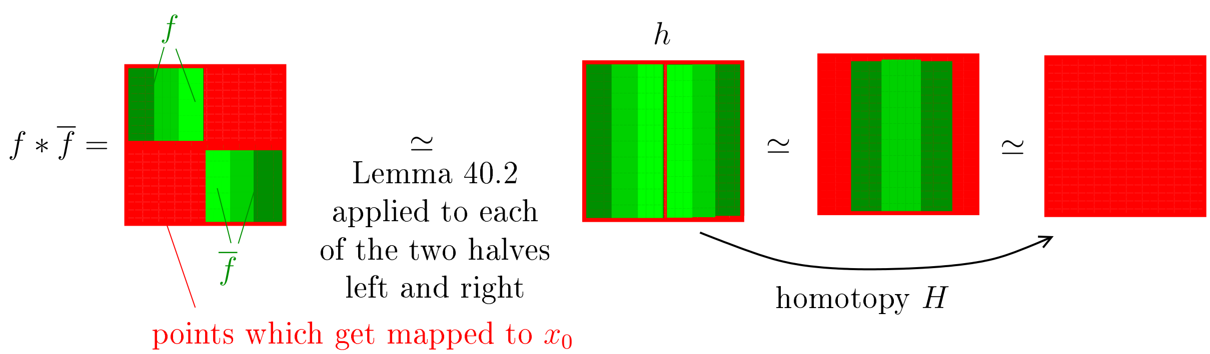


FIGURE 683

[651]This map is continuous by Lemma 14.3.

[652]Even for $n = 1$ this homotopy is different from the homotopy used in Proposition 14.6 (3). Why is that?

Example. There is one non-empty topological space for which we can immediately compute the higher homotopy groups, namely the topological space consisting of a single point P. In this case the only map $I^n \to \{P\}$ is the constant map, hence we see that $\pi_n(\{P\}) = 0$ for any $n \geq 1$.

The higher homotopy groups $\pi_n(X, x_0)$ seem even more complicated than the fundamental group. Surprisingly at least their algebraic structure is much simpler.

Proposition 40.3. *For any pointed topological space (X, x_0) and any $n \geq 2$ the homotopy group $\pi_n(X, x_0)$ is abelian.*

Remark. The higher homotopy groups were initially introduced by Eduard Čech in 1932 [**Čec32**]. Since it was quickly noticed that they are abelian it was initially assumed that higher homotopy groups are of little interest. This perception changed soon afterwards.

PROOF. Suppose we are given two maps $f, g\colon (I^n, \partial I^n) \to (X, x_0)$. We have to show that $f * g$ is homotopic to $g * f$. The homotopy is described in Figure 684 for $n = 2$. It is straightforward to write down an explicit homotopy between $f * g$ and $g * f$ by composing four instances of the homotopy provided by Lemma 40.2. We leave the details to the reader. ∎

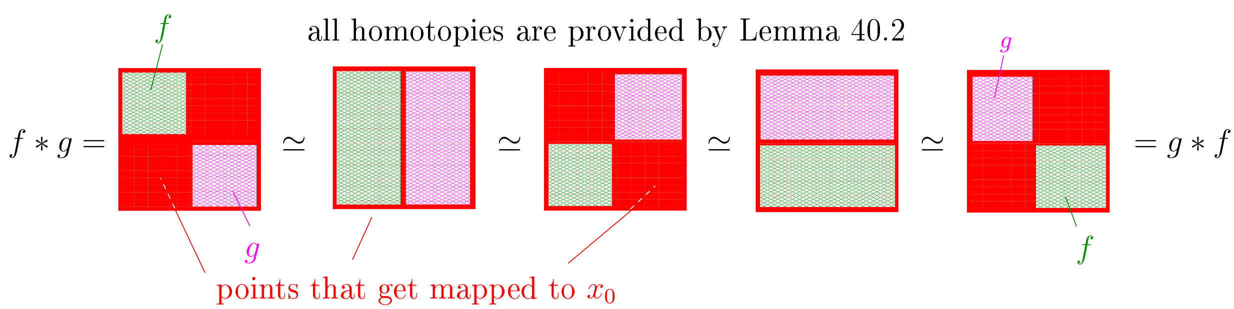


FIGURE 684. Illustration of the proof of Proposition 40.3.

40.2. **Alternative descriptions of homotopy groups.** In this short section we present several alternative points of views regarding elements in homotopy groups. In the following let $n \in \mathbb{N}$.

(1) We denote by $\Phi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ the homeomorphism from page 126.
(2) We write $* = (0, \ldots, 0, 1) \in S^n$. We denote by $\Psi\colon (\overline{B}^n/S^{n-1}, [S^{n-1}]) \to (S^n, *)$ the homeomorphism from page 182.

Now let X be a topological space and let $x_0 \in X$. The above homeomorphisms Φ and Ψ, together with Lemma 3.24, define bijections

$$\begin{array}{cl} & \text{(homotopy classes) of maps } (I^n, \partial^n) \to (X, x_0) \\ \Longleftrightarrow & \text{(homotopy classes) of maps } (\overline{B}^n, S^{n-1}) \to (X, x_0) \\ \Longleftrightarrow & \text{(homotopy classes) of maps } (\overline{B}^n/S^{n-1}, [S^{n-1}]) \to (X, x_0) \\ \Longleftrightarrow & \text{(homotopy classes) of maps } (S^n, *) \to (X, x_0). \end{array}$$

The four equivalences combined say that we can view elements in $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, *) \to (X, x_0)$. This point of view is perhaps closest to our visual approach to thinking about topological spaces. But as we have seen above, for technical purposes it is often much easier to work with the maps $(I^n, \partial I^n) \to (X, x_0)$.

Remark. It is visually most natural to work with the last point of view, namely to think of elements in $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, *) \to (X, x_0)$. Unfortunately with this definition it is a little less clear how the group structure on $\pi_n(X, x_0)$ is defined. To visualize the group structure we need to work a little:

(1) First note that the homeomorphism $\Phi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ descends to a homeomorphism $\overline{\Phi}\colon I/\partial I^n \to \overline{B}^n/S^{n-1}$.
(2) We consider the homeomorphism $\Theta := \overline{\Phi}^{-1} \circ \Psi^{-1}\colon S^n \to I^n/\partial I^n$. Note that it identifies the North Pole $(0, \dots, 0, 1)$ with the point $[\partial I^n]$.
(3) We write $K := \{(x_1, \dots, x_n) \in I^n \mid x_1 = \frac{1}{2}\}$.

In Figure 685 we show K and its journey through the maps Φ and Ψ.

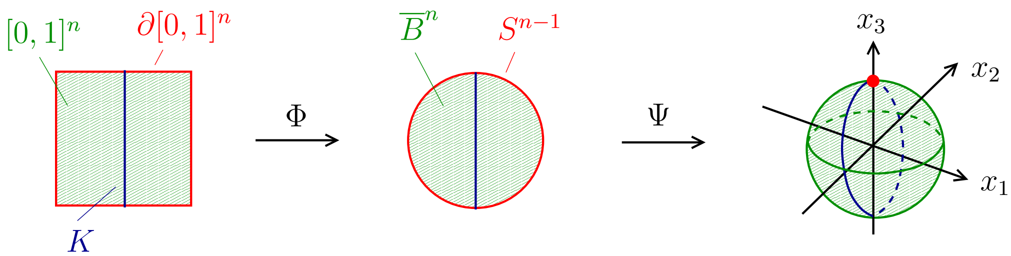


Figure 685

(4) We denote by $S^n_1 \vee S^n_2$ the wedge of two copies of S^n along the point $* \in S^n$.
(5) We refer to the map[653]

$$\begin{array}{rcl} S^n \xrightarrow{\Theta} I^n/\partial I^n \to I^n/(\partial I^n \cup K) & \to & S^n_1 \vee S^n_2 \\ [x_1, \dots, x_n] & \mapsto & \begin{cases} \Theta^{-1}([2x_1, x_2, \dots, x_n]) \in S^n_1, & \text{if } x_1 \in [0, \frac{1}{2}), \\ \Theta^{-1}([2x_1 - 1, x_2, \dots, x_n]), & \text{if } x_1 \in [\frac{1}{2}, 1] \end{cases} \end{array}$$

as the *pinching map*.
(6) It follows basically immediately from the definitions that the group structure on $\pi_n(X, x_0)$ is given as follows:

$$\begin{array}{rcl} \pi_n(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([f_1\colon S^n \to X], [f_2\colon S^n \to X]) & \mapsto & \big[S^n \to S^n_1 \vee S^n_2 \xrightarrow{f_1 \vee f_2} X\big]. \end{array}$$

In Figure 686 we sketch the pinching map and we sketch the corresponding definition of the group structure on $\pi_n(X, x_0)$.[654]

We conclude this section with the following lemma that can be viewed as a generalization of Lemma 14.1.

[653] It follows easily from Lemma 2.35 (2') that this map is continuous.

[654] The meticulous reader will notice that it is perhaps not completely clear how the maps f and g are defined on the two smaller spheres.

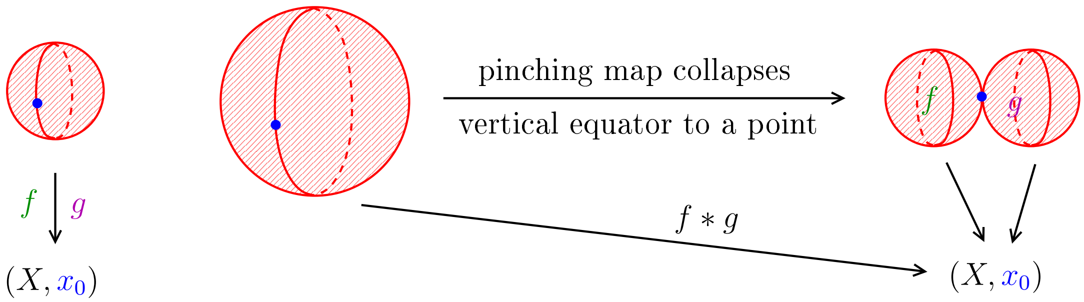


FIGURE 686

Lemma 40.4. *Let X be a topological space, let $x_0 \in X$ and let $n \geq 1$. Furthermore let $f\colon (S^n, *) \to (X, x_0)$ be a map. Then the following statements are equivalent:*

(1) *the map $f\colon S^n \to X$ is homotopic to a constant map,*
(2) *there exists a map $F\colon \overline{B}^{n+1} \to X$ so that $F|_{S^n} = f$,*
(3) *f represents the trivial element in $\pi_n(X, x_0)$.*

PROOF (∗). The equivalence of (2) and (3) is proved in a similar way as Lemma 14.1. We leave it to the reader to make the fairly straightforward modifications.

We turn to the proof that (3) implies (2). This is basically obvious. Indeed, the fact that f represents the trivial element in $\pi_n(X, x_0)$ means that there exists a homotopy of maps of pairs $(S^n, *) \to (X, x_0)$ from f to the constant map. But this means in particular that f is homotopic to the constant map c_{x_0}.

Finally suppose that (1) holds. This means that we assume that there exists a homotopy $H\colon S^n \times [0,1] \to X$ with $H_0 = f$ and $H_1 = c_{x_0}$. This time we do not assume that $H(*, t) = *$ for all $\in [0,1]$. We consider the map

$$\begin{array}{rcl} F\colon \overline{B}^{n+1} & \to & X \\ P & \mapsto & \begin{cases} H(Q, 2-2t), & \text{if } P = Q \cdot t \text{ with } Q \in S^n \text{ and } t \in [1, \frac{1}{2}], \\ x_0, & \text{otherwise.} \end{cases} \end{array}$$

It follows from Lemma 2.35 (2) that this map is indeed continuous. By construction we have $F|_{S^n} = f$. Thus we see that (2) is satisfied. This concludes the proof that all three statements are equivalent. ■

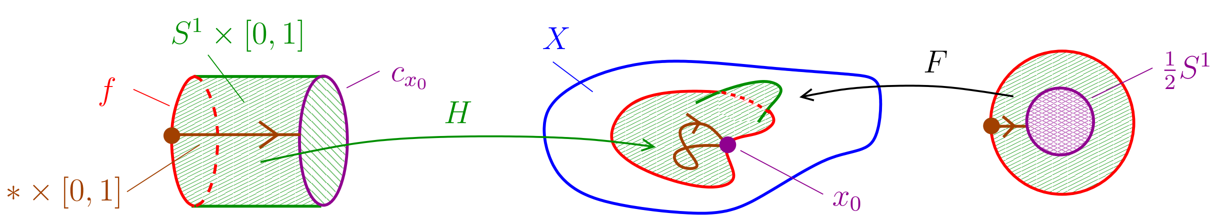


FIGURE 687. Illustration for the proof of Lemma 40.4.

40.3. **Properties and basic calculations of higher homotopy groups.** Now we want to understand some of the basic properties of higher homotopy groups. Perhaps not surprisingly, many of the basic statements about the fundamental group can be easily generalized to the setup of higher homotopy groups.

For example the following proposition can be viewed as a generalization of Proposition 14.11.

Proposition 40.5. *Let X be a topological space, let x_0 and x_1 be two points in X and let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1. Given a map $f\colon (I^n, \partial I^n) \to (X, x_1)$ we consider the map*[655][656]

$$\begin{array}{rcl} f^\gamma\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (t_1, \ldots, t_n) & \mapsto & \begin{cases} f\big(2t_1 - \frac{1}{2}, \ldots, 2t_n - \frac{1}{2}\big), & \text{if } (t_1, \ldots, t_n) \in \big[\frac{1}{4}, \frac{3}{4}\big]^n, \\ \gamma(t), & \text{if } (t_1, \ldots, t_n) \in \partial\big(\big[\frac{t}{4}, \frac{4-t}{4}\big]^n\big) \text{ for } t \in [0,1]. \end{cases} \end{array}$$

The definition of f^γ is illustrated in Figure 688. The map

$$\begin{array}{rcl} \gamma_*\colon \pi_n(X, x_1) & \to & \pi_n(X, x_0) \\ {[f]} & \mapsto & [f^\gamma] \end{array}$$

is well-defined and it is a group isomorphism.

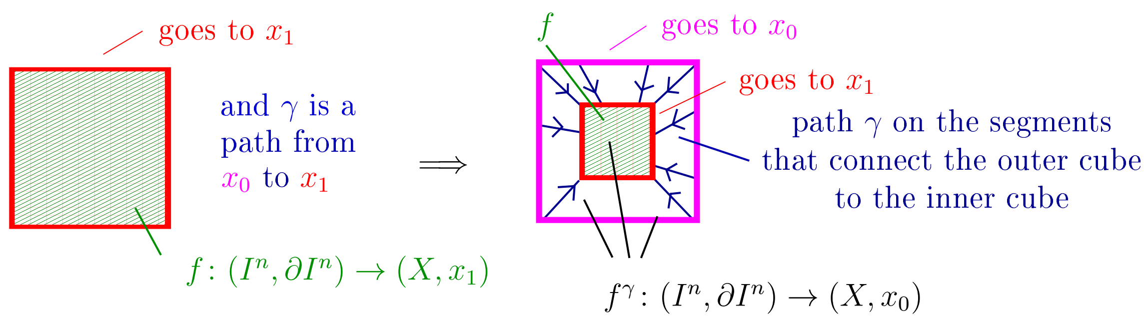


FIGURE 688. Illustration of Proposition 40.5.

Remark. If X is a path-connected topological space and if we are only interested in the isomorphism type of the n-th homotopy group of X, then sometimes we use this proposition as an excuse to just write $\pi_n(X)$, i.e. we suppress the base point from the notation.

PROOF. Let X be a topological space, let x_0 and x_1 be two points in X and let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1. We need to prove the following claim.

Claim.

(1) The map $\Phi\colon \pi_n(X, x_1) \to \pi_n(X, x_0)$ given by $[f] \mapsto [f^\gamma]$ is well-defined,
(2) the map Φ is a bijection, and
(3) the map Φ is a group homomorphism.

[655] If we view elements in $\pi_1(X, x_i)$ as represented by maps $(S^1, *) \to (X, x_i)$, what does the corresponding map $\pi_1(X, x_1) \to \pi_1(X, x_0)$ look like?

[656] Using Lemma 2.35 (2) one can easily verify that the map is continuous.

We quickly discuss the three statements of the claim:

(1) This statement can be verified easily by hand.
(2) As usual we denote by $\overline{\gamma}$ the path given by $\overline{\gamma}(t) = \gamma(1-t)$. It is straightforward to verify that the map $\Psi\colon \pi_n(X, x_0) \to \pi_n(X, x_1)$ given by $[g] \mapsto [f^{\overline{\gamma}}]$ is an inverse to Φ.
(3) Now let $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ be maps. We need to show that $f^\gamma * g^\gamma$ is homotopic to $(f * g)^\gamma$. (These two maps are illustrated in Figure 689.) It should be clear that verifying this statement is elementary, in the sense that one can write down an explicit homotopy. On the other hand it should also be clear that writing down an explicit homotopy is supremely painful and highly non-instructive. Thus we will skip this step and leave it to the eager reader to fill in the details. ■

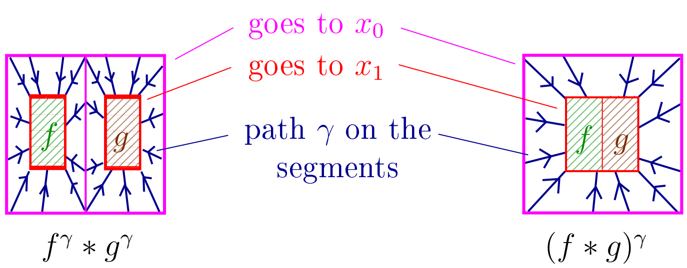


FIGURE 689. Illustration of the proof of Proposition 40.5.

Next we will see that higher homotopy groups are functorial.

Definition. Let $f\colon X \to Y$ be a map between two topological spaces, let $n \in \mathbb{N}$ and let $x_0 \in X$. We write $y_0 = f(x_0)$. The map

$$\begin{array}{rcl} f_*\colon \pi_n(X, x_0) & \to & \pi_n(Y, y_0) \\ [\sigma\colon (I^n, \partial I^n) \to (X, x_0)] & \mapsto & [f \circ \sigma\colon (I^n, \partial I^n) \xrightarrow{\sigma} (X, x_0) \xrightarrow{f} (Y, y_0)] \end{array}$$

is well-defined map and it is a group homomorphism for every $n \geq 1$.[657] As usual we say that f_* is the map induced by f.

The following proposition is straightforward generalization of Proposition 15.6.

Proposition 40.6. *Let $n \geq 1$. Recall that we denote by $\mathcal{PTop}$ the category of pointed topological spaces and that we denote by $\mathcal{Gr}$ be the category of groups. Then*

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PTop}) & \to & \mathrm{Ob}(\mathcal{Gr}) \\ (X, x_0) & \mapsto & \pi_n(X, x_0) \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}(\mathcal{PTop}) & \to & \mathrm{Mor}(\mathcal{Gr}) \\ (f\colon (X, x_0) \to (Y, y_0)) & \mapsto & (f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)) \end{array}$$

is a covariant functor.

[657]Both statements follow basically immediately from the definitions.

Proof. It follows immediately from the definitions that

$$\mathrm{id}_* = \mathrm{id} \qquad \text{for all pointed pairs } (X, x_0),$$

and that

$$(g \circ f)_* = g_* \circ f_* \qquad \text{for all maps } f\colon (X, x_0) \to (Y, y_0) \text{ and } g\colon (Y, y_0) \to (Z, z_0).$$

But this is exactly what we need to show. ■

We continue with the following fairly elementary but useful proposition.

Proposition 40.7. *Let $n \geq 1$.*

(1) *Let $f, g\colon X \to Y$ be two maps between topological spaces and let $x_0 \in X$. If f and g are homotopic, then*

$$\begin{array}{c} f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0)) \\ \text{is an isomorphism} \end{array} \iff \begin{array}{c} f_*\colon \pi_n(X, x_0) \to \pi_n(Y, g(x_0)) \\ \text{is an isomorphism.} \end{array}$$

The analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism".

(2) *Let $f\colon X \to Y$ be a map between topological spaces and let $x_0 \in X$ be a point. If f is a homotopy equivalence, then*

$$f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

is an isomorphism.

(3) *If X is a contractible topological space, then the following two statements hold:*
   (a) *for any base point $x_0 \in X$ we have $\pi_n(X, x_0) = 0$,*
   (b) *any map $f\colon S^n \to X$ can be extended to a map $\overline{B}^{n+1} \to X$.*

(4) *Let (X, x_0) be a pointed topological space. If we denote by Y the component of X that contains x_0 and if we denote by $i\colon Y \to X$ the inclusion map, then the map*

$$i_*\colon \pi_n(Y, x_0) \to \pi_n(X, x_0)$$

is an isomorphism.

Proof.

(1) This statement is proved basically the same way as Proposition 18.16 (1). We leave it to the reader to make the necessary changes to the argument.
(2) As in the case of Proposition 18.16 one can easily deduce Statement (2) from Statement (1).
(3) Let X be a contractible topological space.
   (a) By definition of "contractible" the topological space X is homotopy equivalent to a topological space consisting of a single point $\{*\}$. Evidently $\pi_n(\{*\}, *) = 0$, which by (2) implies that $\pi_n(X, x_0) = 0$ for some base point x_0. In Lemma 18.13 we saw that a contractible topological space is path-connected. Thus it follows from Proposition 40.5 that $\pi_n(X, x_0) = 0$ for any base point x_0.
   (b) This statement is proved basically the same way as Lemma 14.1. For completeness' sake we carry out the argument. So let $f\colon S^n \to X$ be a map. We pick a base point $*$ for S^n and we set $x_0 := f(*)$. By (a) we know that $\pi_n(X, x_0) = 0$. We take the point of view from page 1060. Thus we can view $[f]$ as an element

in $\pi_n(X, x_0) = 0$. By definition $[f] = 0$ means that there exists a homotopy $F\colon S^n \times [0,1] \to X$ such that $F(p,0) = f(p)$ for all $p \in S^n$, that $F(p,1) = x_0$ for all $p \in S^n$ and that $F(*,t) = x_0$ for all $t \in [0,1]$. We consider the map

$$\begin{array}{rcl} \overline{B}^{n+1} & \to & X \\ r \cdot v & \mapsto & F(v,(1-r)) \end{array}$$

where we write a point in $\overline{B}^{n+1}$ as $r \cdot v$ with $r \in [0,1]$ and $v \in S^n$. It follows from Lemma 3.25 (2) that this map is indeed continuous. Clearly this is the desired extension.

(c) This statement is an easy consequence of Lemma 2.70 and the fact that I^n is path-connected. ■

Example. The topological space $\mathbb{R}^n$ is homotopy equivalent to a point P. Thus it follows from Proposition 40.7 (2) and the discussion on page 1059 that $\pi_k(\mathbb{R}^n) \cong \pi_k(P) \cong 0$. Exactly the same way, using the discussion on page 548, one can show that all the higher homotopy groups of convex subsets of $\mathbb{R}^n$ vanish, e.g. we have $\pi_k\big(\overline{B}^n\big) = \pi_k(B^n) = 0$.

The following proposition is the high-dimensional analogue of Proposition 16.20.

Proposition 40.8. *Let A and B be two topological spaces and let $a_0 \in A$ and $b_0 \in B$. We consider the inclusion maps*

$$\begin{array}{rcl} i\colon A & \to & A \times B \\ a & \mapsto & (a, b_0) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon B & \to & A \times B \\ b & \mapsto & (a_0, b) \end{array}$$

and we consider the projection maps

$$\begin{array}{rcl} p\colon A \times B & \to & A \\ (a,b) & \mapsto & a \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon A \times B & \to & B \\ (a,b) & \mapsto & b. \end{array}$$

(1) *The maps*

$$\begin{array}{rcl} \Phi\colon \pi_n(A, a_0) \times \pi_n(B, b_0) & \to & \pi_n(A \times B, (a_0, b_0)) \\ (x,y) & \mapsto & i_*(x) \cdot j_*(y) \end{array}$$

and

$$\begin{array}{rcl} \Psi\colon \pi_n(A \times B, (a_0, b_0)) & \to & \pi_n(A, a_0) \times \pi_n(B, b_0) \\ z & \mapsto & (p_*(z), q_*(z)) \end{array}$$

are both homomorphisms[658] *and they are inverses of one another, in particular both are isomorphisms.*

(2) *The map Φ agrees with the map*

$$\begin{array}{rcl} \Theta\colon \pi_n(A, a_0) \times \pi_n(B, b_0) & \to & \pi_n(A \times B, (a_0, b_0)) \\ ([f\colon (S^n, *) \to (A, a_0)], [g\colon (S^n, *) \to (B, b_0)]) & \mapsto & \left[\begin{array}{rcl} (S^n, *) & \to & (A \times B, (a_0, b_0)) \\ z & \mapsto & (f(z), g(z)) \end{array} \right]. \end{array}$$

In particular we obtain from (1) that Θ is a homomorphism.

[658] In fact in this case it follows immediately from Proposition 40.3 and Lemma 16.19 that Φ is a homomorphism.

PROOF. The proof of this proposition is almost verbatim the same as the proof of Proposition 16.20. We leave it to the reader to make the minute modifications. ■

We continue with the following very general and actually very convenient proposition regarding homotopy groups of CW-complexes. The proposition greatly generalizes Proposition 37.13 (1).

Proposition 40.9.

(1) *Let X be a* CW*-complex and let $k \in \mathbb{N}_0$. As usual we denote by X^k the k-skeleton of X. Let $x_0 \in X^k$. The following two statements hold:*
   (a) *For $j = 1, \dots, k-1$ the inclusion induced map $\pi_j(X^k, x_0) \to \pi_j(X, x_0)$ is an isomorphism.*
   (b) *The inclusion induced map $\pi_k(X^k, x_0) \to \pi_k(X, x_0)$ is an epimorphism.*

(2) *Let X be a* CW*-complex with a single 0-cell x_0 and let $n \in \mathbb{N}$. If X has no cells in dimensions $1, \dots, n$, then $\pi_n(X, x_0) = 0$ for $k = 1, \dots, n$.*

Note that the above proposition is a generalization of Corollary 37.14.

PROOF.

(1) Let X be a CW-complex and let $x_0 \in X^k$. We denote by $i\colon X^k \to X$ the inclusion. We take the point of view from page 1060, i.e. we view elements of $\pi_j(X, x_0)$ as homotopy classes of maps $(S^j, *) \to (X, x_0)$. As on page 935 we consider the j-sphere S^j as a CW-complex with one 0-cell $*$ and one j-cell.
   (a) First we show that for $j = 1, \dots, k$ the map $i_*\colon \pi_j(X^k, x_0) \to \pi_j(X, x_0)$ is an epimorphism.
   Let $[g] \in \pi_j(X, x_0)$ be some element. In other words, we are given a map $g\colon (S^j, *) \to (X, x_0)$ be a map. By the Cellular Approximation Theorem 38.13, applied to $g\colon S^j \to X$ and the subcomplex $\{*\} \subset S^j$, there exists a homotopy F rel $\{*\}$ from the map g to a cellular map $\widetilde{g}$. Since F is a homotopy rel $\{*\}$ we see that F also defines a homotopy of maps $(S^j, *) \to (X, x_0)$ of pairs of topological spaces. Thus we see that $[g] = [\widetilde{g}] \in \pi_j(X, x_0)$.
   Since $\widetilde{g}$ is cellular we know that $\widetilde{g}(S^j)$ lies in the j-skeleton of X. Since $j \leq k$ we see in particular that $\widetilde{g}(S^j)$ is contained in X^k. In other words, $\widetilde{g}$ is a map $(S^k, *) \to (X^k, x_0)$. This shows that $[g] = [\widetilde{g}] = [i \circ \widetilde{g}] = i_*([\widetilde{g}])$ lies in the image of $\pi_j(X^k, x_0)$.
   (b) Now we show that for $j = 1, \dots, k-1$ the map $i_*\colon \pi_j(X^k, x_0) \to \pi_j(X, x_0)$ is a monomorphism. So suppose that we are given $[f_0], [f_1] \in \pi_j(X^k, x_0)$ with $i_*([f_0]) = i_*([f_1])$. By the Cellular Approximation Theorem 38.13 we can assume that the maps $f_0, f_1\colon (S^j, *) \to (X^k, x_0)$ are cellular. The fact that $i_*([f_0]) = i_*([f_1])$ means that there exists a homotopy $H\colon S^j \times [0,1] \to X$ with $H_0 = f_0$ and $H_1 = f_1$. This time we apply the Cellular Approximation Theorem 38.13 to the map $H\colon S^j \times [0,1] \to X$ and the subcomplex $(S^j \times \{0\}) \cup (S^j \times \{1\})$ and we obtain a cellular map $\widetilde{H}\colon S^j \times [0,1] \to X$ such that $\widetilde{\mathrm{H}}_0 = H_0 = f_0$ and $\widetilde{H}_1 = H_1 = f_1$. Since $\widetilde{H}$ is cellular it takes values in the $(j+1)$-skeleton of $X \times [0,1]$. Since $j < k$

we see in particular that $\widetilde{H}$ takes values in $X^k \times [0,1]$. But this means that we can use $\widetilde{H}$ to show that $[f_0] = [f_1] \in \pi_j(X^k, x_0)$.

(2) Let $n \in \mathbb{N}$. Let X be a CW-complex with a single 0-cell x_0 and which has no cells in dimensions $1, \ldots, n$. Let $k \in \{1, \ldots, n\}$. By hypothesis the k-skeleton of X equals $\{x_0\}$. By (1a) we know that the map $\pi_k(\{x_0\}, x_0) \to \pi_k(X, x_0)$ is an epimorphism. But the former group is evidently trivial, hence $\pi_k(X, x_0)$ is trivial as well. ■

In Proposition 14.14 we saw that $\pi_1(S^n) = 0$ for $n \geq 2$. Now we can prove the analogous statement for higher homotopy groups. Now we obtain, basically for free, the following much more general proposition.

Proposition 40.10. *Let $n \in \mathbb{N}$. For any $k \in \{1, \ldots, n-1\}$ we have*

$$\pi_k(S^n) \;=\; 0.$$

PROOF.

(1) Again, as on page 935 we consider the n-sphere S^n as a CW-complex with one 0-cell and one n-cell. With this CW-structure it follows immediately from Proposition 40.9 that $\pi_k(S^n) = 0$ for $k = 1, \ldots, n-1$.
(2) An alternative approach to proving Proposition 40.10 is given by the point of view taken in Exercise 9.1 where we used the Whitney Approximation Theorem 9.3 and Proposition 6.62 to show that for an $n \geq 2$ we have $\pi_1(S^n) = 0$. ■.

The following proposition is also a straightforward generalization from the setting of fundamental groups to the setting of higher homotopy groups. The proof is almost verbatim the same as the proof of Proposition 37.3 and of Proposition 25.7.

Proposition 40.11. *Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of subsets such that $X = \bigcup_{i \in \mathbb{N}} X_i$ and such that of one the following three conditions holds:*

(1) *every compact subset of X is already contained in one of the X_i, or*
(2) *each X_i is open in X, or*
(3) *X is a CW-complex and each X_i is a subcomplex.*

Then for any $w \in X_1$ and any $n \in \mathbb{N}$ the inclusion induced maps $\pi_n(X_i, w) \to \pi_n(X, w)$ induce an isomorphism

$$\pi_n(X, w) \;=\; \varinjlim \pi_n(X_i, w).$$

In Lemma 36.4 we already saw that S^∞ is contractible, thus we know by that all homotopy groups are trivial. But it is also illuminating to apply Proposition 40.11 to obtain this result.

Corollary 40.12. *For every $n \in \mathbb{N}$ we have $\pi_n(S^\infty) = 0$.*

Proof. We equip S^∞ and all spheres S^k, $k \in \mathbb{N}$ with the CW-complex structure that we defined on page 942. We have

$$\pi_n(S^\infty) \;=\; \pi_n\Big(\bigcup_{k\in\mathbb{N}} S^k\Big) \;\underset{\substack{\uparrow \\ \text{by Proposition 40.11 and by definition} \\ \text{of the topology of } S^\infty}}{=}\; \varinjlim \pi_n(S^k) \;\underset{\substack{\uparrow \\ \text{by Proposition 40.10 we have} \\ \pi_n(S^k)=0 \text{ for } k>n}}{=}\; 0. \qquad \blacksquare$$

40.4. **Covering spaces and higher homotopy groups.** One of the main tools in studying fundamental groups was to study the relationship between the fundamental group of a topological space and coverings of the given topological space. Perhaps somewhat surprisingly, for $n \geq 2$ we have a completely different relationship between the n-th homotopy group and coverings.

Proposition 40.13. *Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Let $n \geq 2$. Then the following two statements hold:*

(1) *The map $p_*\colon \pi_n(X, x_0) \to \pi_n(B, b_0)$ is an isomorphism.*
(2) *The map*[659]

$$\begin{array}{rcl} \pi_n(B, b_0) & \to & \pi_n(X, x_0) \\ [f\colon (S^n, *) \to (B, b_0)] & \mapsto & [\text{unique lift of } f \text{ to a map } (S^n, *) \to (X, x_0)] \end{array}$$

is well-defined and it is the inverse to p_*.

Before we provide the proof of Proposition 40.13 let us first state an immediate, and very useful, corollary to Proposition 40.13 and let us give a few examples.

Corollary 40.14. *Let X be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Given any $n \geq 2$ the universal covering $p\colon \widetilde{X} \to X$ induces an isomorphism*[660]

$$p_*\colon \pi_n(\text{universal covering } \widetilde{X} \text{ of } X) \xrightarrow{\cong} \pi_n(X).$$

Now we present some examples for Corollary 40.14. We will consider a few more examples after the proof of Proposition 40.13.

Examples.

(1) Let $k \in \mathbb{N}$ and $n \geq 2$. We have

$$\pi_n((S^1)^k) \;=\; \pi_n(\mathbb{R}^k/\mathbb{Z}^k) \;\underset{\substack{\uparrow \\ \text{Proposition 40.13}}}{\xleftarrow{\cong}}\; \pi_n(\mathbb{R}^k) \;\underset{\substack{\uparrow \\ \text{page 1065}}}{=}\; 0.$$

In particular we see that all of the higher homotopy groups of S^1 and the torus $S^1 \times S^1$ vanish.

---

[659]Here we take the point of view from page 1060 where we saw that we view the elements of $\pi_n(B, b_0)$ as homotopy classes of maps $(S^n, *) \to (B, b_0)$.

[660]In Corollary 29.9 we showed that every topological space X that is path-connected, locally path-connected and semi-locally simply connected admits a universal covering $p\colon \widetilde{X} \to X$, i.e. a covering such that $\widetilde{X}$ is simply connected.

(2) We consider $X = S^1 \vee S^2$. On page 836 we had determined its universal covering $\widetilde{X}$. For convenience we picture it again in Figure 690. It follows from Proposition 40.13 that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\pi_2(\widetilde{X})$. At the moment we cannot determine $\pi_2(\widetilde{X})$, but since $\widetilde{X}$ has "infinitely many spheres" one might expect $\pi_2(\widetilde{X})$ to be pretty big.

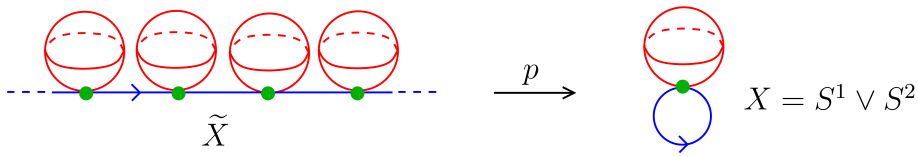


FIGURE 690

(3) One of our goals is still to complete the classification of lens spaces, in particular we would like to answer Question 16.7. So let $L(p,q)$ and $L(p,r)$ be two lens spaces. Then for any $n \geq 2$ we have

$$\pi_n(L(p,q)) \underset{\substack{\uparrow \\ \text{Proposition 40.13}}}{\xleftarrow{\cong}} \pi_n(S^3) \xrightarrow{\cong} \pi_n(L(p,r)).$$

We do not know the higher homotopy groups of S^3, but this argument shows that higher homotopy groups are useless for distinguishing lens spaces.

PROOF OF PROPOSITION 40.13. Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces and furthermore let $n \geq 2$. Our goal is to construct an inverse to the map $p_*\colon \pi_n(X,x_0) \to \pi_n(B,b_0)$.

We saw on page 1060 that we can think of elements in $\pi_n(B,b_0)$ as homotopy classes of maps $(S^n,*) \to (B,b_0)$. Thus let $f\colon (S^n,*) \to (B,b_0)$ be given. We are in the following setting[661]

$$\begin{tikzcd} & (X,x_0) \arrow[d, "p"] \\ (S^n,*) \arrow[r, "f"'] & (B,b_0). \end{tikzcd} \quad \xrightsquigarrow[\text{since } \pi_1(S^n)=0]{\text{by Proposition 29.2}} \quad \begin{tikzcd} & (X,x_0) \arrow[d, "p"] \\ (S^n,*) \arrow[r, "f"'] \arrow[ur, dashed, "\text{exists unique } \widetilde{f}"] & (B,b_0). \end{tikzcd}$$

Furthermore it follows again from Proposition 29.2, applied to $Z = S^n \times [0,1]$, that the lifts of two homotopic maps $(S^n,*) \to (B,b_0)$ are two homotopic maps $(S^n,*) \to (X,x_0)$.[662] Summarizing we just constructed a map

$$\begin{array}{rcl} \lambda\colon \pi_n(B,b_0) & \to & \pi_n(X,x_0) \\ [f] & \mapsto & [\widetilde{f}]. \end{array}$$

[661] In Proposition 29.2 we need that the sphere S^n is of path-connected and locally path-connected, but there are no hypotheses on B or X.

[662] Here there is a little hidden subtlety: if $F\colon S^n \times [0,1] \to B$ is a homotopy between two maps $f,g\colon (S^n,*) \to (B,b_0)$, then we can lift f,g to maps $\widetilde{f},\widetilde{g}\colon (S^n,*) \to (X,x_0)$. Furthermore we can apply Proposition 29.2 to lift F to a map $\widetilde{F}\colon S^n \times [0,1] \to X$ such that $\widetilde{F}(*,0) = x_0$. We still need to show that $\widetilde{F}(*,t) = x_0$ for all $t \in [0,1]$. We know that for every $t \in [0,1]$ we have $\widetilde{F}(*,t) \in p^{-1}(b_0)$. By Lemma 16.3 the preimage $p^{-1}(b_0)$ is discrete. Since $[0,1]$ is connected it follows that the map $p\colon [0,1] \to p^{-1}(b_0)$ is constant by Lemma 2.61. But since it takes the value x_0 for $t = 0$ it takes the value x_0 for all $t \in [0,1]$.

Now we have to show that λ is an inverse to p_*. First of all, given any $[f] \in \pi_n(B, b_0)$ we have

$$(p_* \circ \lambda)([f]) \;=\; p_*\big([\widetilde{f}]\big) \;=\; \big[p \circ \widetilde{f}\big] \;=\; [f].$$

Now let $[g] \in \pi_n(X, x_0)$. We then have

$$(\lambda \circ p_*)([g]) \;=\; \lambda([p \circ g]) \;\underset{\substack{\uparrow \\ \text{since } g \text{ is the unique lift of } p \circ g \text{ to } x_0.}}{=}\; [g].$$

■

Now we give a few more applications of Proposition 40.13.

Examples.

(1) Let Σ be a surface of genus ≥ 2. In Theorem 33.9 we have seen that there exists a covering $p\colon \mathbb{D} \to \Sigma$ where $\mathbb{D} = B_1(0) \subset \mathbb{C}$. Since $\mathbb{D}$ is convex it follows from Proposition 40.13 and the discussion on page 1065 that $\pi_n(\Sigma) \cong \pi_n(\mathbb{D}) = 0$ for all $n \geq 2$.

Together with the example on page 1068 we have now shown that the higher homotopy groups of every surface of genus ≥ 1 vanish.

(2) Let Σ be a surface of genus ≥ 1 and let $f\colon \Sigma \to \Sigma$ be a self-homeomorphism. We denote by $\mathrm{Tor}(\Sigma, f)$ the mapping torus that we introduced on page 711. For $n \geq 2$ we have

$$\pi_n(\mathrm{Tor}(\Sigma, f)) \underset{\substack{\uparrow \\ \text{by Lemma 24.26} \\ \text{and Proposition 40.13}}}{\cong} \pi_n(\Sigma \times \mathbb{R}) \underset{\substack{\uparrow \\ \text{Proposition 40.7}}}{\cong} \pi_n(\Sigma) \underset{\substack{\uparrow \\ \text{previous example}}}{=} 0.$$

Lemma 40.15. *Let X be a 1-dimensional* CW*-complex and let $x_0 \in X$ be base point. Then $\pi_n(X, x_0) = 0$ for every $n \geq 2$.*

Sketch of proof. Let X be a 1-dimensional CW-complex, let $x_0 \in X$ and let $n \geq 2$. We first assume that X is finite and connected. We write $k = 1 - \chi(X)$. We have

$$\pi_n(X) \underset{\substack{\uparrow \\ \text{Proposition 37.8 (2) and} \\ \text{Proposition 40.7}}}{\cong} \pi_n\Big(\bigvee_{i=1}^{k} S^1\Big) \underset{\substack{\uparrow \\ \text{Proposition 40.13}}}{\cong} \pi_n\Big(\text{universal covering of } \bigvee_{i=1}^{k} S^1\Big) \underset{\substack{\uparrow \\ \text{Proposition 40.11 and Figure 691}}}{\cong} \varinjlim \pi_n(\text{trees } T_i) \underset{\substack{\uparrow \\ \text{Proposition 18.29} \\ \text{and Proposition 40.7}}}{=} \varinjlim 0 = 0.$$

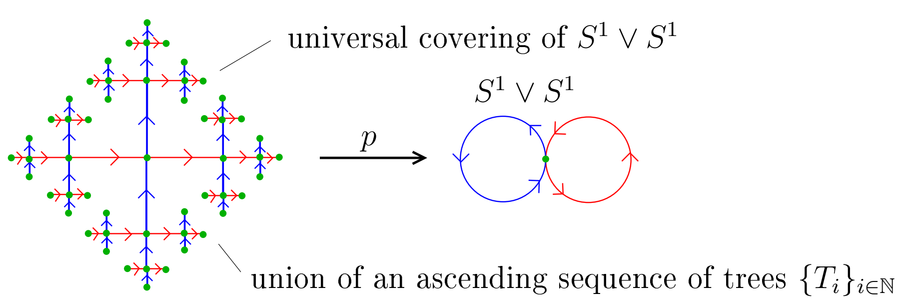


Figure 691

Now suppose that X is an arbitrary 1-dimensional CW-complex. Let $[f] \in \pi_n(X, x_0)$. It follows from Theorem 36.14 that the image of f lies in a finite connected subcomplex Y of X. But by the above discussion $[f] = 0 \in \pi_n(Y, x_0)$, hence, by functoriality of π_n, we also have $[f] = 0 \in \pi_n(X, x_0)$. ■

The discussion of $\pi_2(S^1 \vee S^2)$ on page 1069 strongly suggests that even simple topological spaces can have surprisingly large higher homotopy groups. This raises the following anxious question, which after these examples is no longer totally absurd.

Question 40.16. *Are the homotopy groups of a countable* CW*-complex countable?*

Remark. On page 1032 we saw that at least the fundamental of a countable CW-complex is countable.

We will answer this question in Proposition 85.19.

40.5. **Are there any higher homotopy groups that are non-trivial?** We have now computed lots of higher homotopy groups, but so far the results have been rather dull, all the groups we have computed so far are trivial.

So are there are any higher homotopy groups that are non-trivial? A natural candidate for a non-trivial group is surely $\pi_n(S^n)$. The following question naturally arises:

Question 40.17. *Let $n \geq 2$. Does the identity map* $\mathrm{id}\colon S^n \to S^n$ *define a non-trivial element in the homotopy group $\pi_n(S^n, *)$?*

We can also ask, what are the higher homotopy groups of spheres, i.e. what can we say about $\pi_k(S^n)$ for $k > n$? On page 1068 we saw that all these groups are trivial for $n = 1$. But what happens for $n > 1$? Can the higher homotopy groups of spheres be non-trivial?

To construct our first example of a potentially interesting map from a higher-dimensional to a lower-dimensional sphere we recall that on page 197 we gave an identification $\mathbb{C}\mathrm{P}^1 = S^2$. We consider the following map which is usually referred to as the *Hopf*[663] *map*:

$$\begin{array}{rcl} H\colon S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 = \mathbb{C}\mathrm{P}^1 \\ (z_1, z_2) & \mapsto & [z_1 : z_2]. \end{array}$$

The homotopy class of H defines an element in $\pi_3(S^2)$.

Question 40.18. *Does the Hopf map $H\colon S^3 \to S^2$ represent a non-trivial element in the homotopy group $\pi_3(S^2)$?*

Remark. The definition of the Hopf map is very elegant, but unfortunately we pay for the elegance by loosing any intuition for what the map looks like. So how are supposed to think about the Hopf map? A straightforward calculation shows that for each $P \in S^2$ the preimage $H^{-1}(\{P\})$ is a circle. Furthermore, if we use the explicit homeomorphism from Lemma 2.44 to make the identification of S^3 with $\mathbb{R}^3 \cup \{\infty\}$ then we can make the following observations:

(1) the circle $S^1 \times \{0\}$ corresponds to the points $(z, 0) \in S^3$, in particular the circle gets sent to the point $[1 : 0] \in \mathbb{C}\mathrm{P}^1$,

[663] Heinz Hopf (1894-1971) was a German mathematician.

(2) the circle that is given by the z-axis together with ∞ corresponds to the points $(0,z) \in S^3$, in particular the circle gets sent to the point $[0:1] \in \mathbb{C}\mathrm{P}^1$.

Next recall that Lemma 27.1 we gave an explicit identification[664]

$$\big(S^1 \times \overline{B}^2\big) \cup_{S^1 \times S^1} \big(\overline{B}^2 \times S^1\big) \xrightarrow{\cong} S^3.$$

It follows from the definition of this identification that on the torus $S^1 \times S^1$ the "diagonal circle" $\{(z,z) \,|\, z \in S^1\}$ is the preimage of the point $[1:1] \in \mathbb{C}\mathrm{P}^1$. In Figure 692 we show the preimages of H for some points in $\mathbb{C}\mathrm{P}^2 = S^2$.

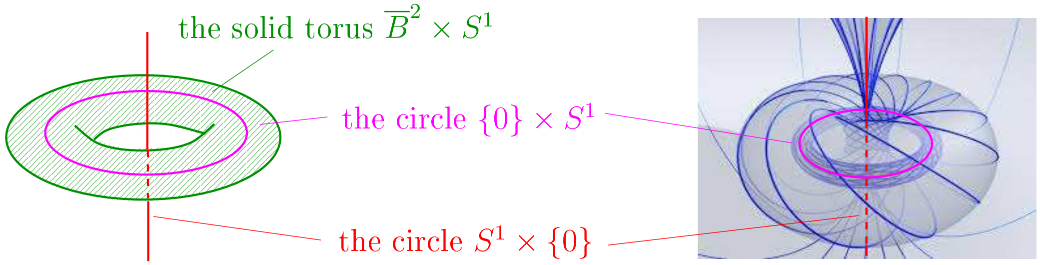


FIGURE 692

We introduce one more construction of maps that represent potentially non-trivial elements in homotopy groups of spheres.

Definition.

(1) Let X be a topological space. Recall that on page 694 we defined the *suspension of* X to be the topological space

$$\Sigma(X) \;:=\; (X \times [-1,1])/\sim$$

where we identify all points in $X \times \{-1\}$ to a single point and we identify all points in $X \times \{1\}$ to a single point. We refer to the point $[X \times \{1\}]$ as the *North Pole* of $\Sigma(X)$.

(2) Given a map $f\colon X \to Y$ between topological spaces we denote by

$$\begin{aligned} \Sigma(f)\colon \Sigma(X) &\to \Sigma(Y) \\ [(x,t)] &\mapsto [(f(x),t)] \end{aligned}$$

the *suspension of f*.

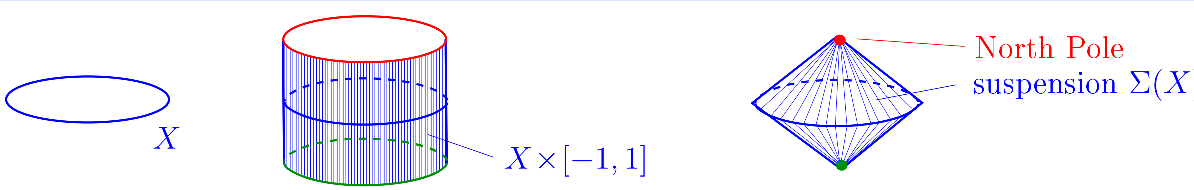


FIGURE 693

[664]Note that in Lemma 27.2 we saw that the corresponding smooth embedding $S^1 \times \overline{B}^2 \to S^3$ is essential given by the standard smooth embedding.

Remark. Let $n \in \mathbb{N}_0$. Recall that on page 695 we wrote down an explicit homeomorphism $\Sigma(S^n) \to S^{n+1}$ that sends the North Pole of $\Sigma(S^n)$ to the North Pole $(0,\dots,0,1)$ of S^{n+1}. We will use this homeomorphism to identify $\Sigma(S^n)$ with S^{n+1}.

Now we have the following elementary lemma.

Lemma 40.19. *Let $k \in \mathbb{N}_0$.*

(1) *Let (X, x_0) be a pointed topological space. We denote by $*$ the North Pole of $\Sigma(X)$. The map*

$$\begin{array}{rcl} \pi_k(X, x_0) & \to & \pi_{k+1}(\Sigma(X), *) \\ [f\colon S^k \to X] & \mapsto & \big[\Sigma(f)\colon \Sigma(S^k) = S^{k+1} \to \Sigma(X)\big] \end{array}$$

is a well-defined group homomorphism. We refer to it as the suspension homomorphism*.*

(2) *Let $n \in \mathbb{N}_0$. We denote by $*$ the North Pole of S^n and S^{n+1}. The map*

$$\begin{array}{rcl} \pi_k(S^n, *) & \to & \pi_{k+1}(S^{n+1}, *) \\ [f\colon S^k \to S^n] & \mapsto & \big[\Sigma(f)\colon \Sigma(S^k) = S^{k+1} \to \Sigma(S^n) = S^{n+1}\big] \end{array}$$

is a well-defined group homomorphism. We refer to it also as the suspension homomorphism*.*

Proof.

(1) It follows from Lemma 24.4 (4) that this map is well-defined.[665] We leave it the task of verifying that the map is in fact a group homomorphism as a somewhat challenging exercise to the reader. Alternatively we refer to [**Hilt53**, Theorem 2.1] for a proof.
(2) This statement follows immediately from (1). ■

We conclude this chapter with the following question:

Question 40.20. *Let $H\colon S^3 \to S^2$ be the Hopf map. Is the image of $[H] \in \pi_3(S^2)$ under the iterated suspension homomorphisms $\pi_3(S^2, x_0) \to \pi_{3+k}(S^{2+k}, x_1)$ non-trivial?*

Unfortunately at the moment we do not have the tools to answer any of the above questions. In general it is very hard to compute higher homotopy groups as there are few computational techniques that can be used to show that higher homotopy groups are non-zero. One of the key problems is that there is no analogue to the Seifert–van Kampen Theorem 22.1. We will get to know a few techniques to compute higher homotopy groups of spheres, but we will cover these only in much later chapter, since the calculations will require the use of singular homology groups. But even then we will only be able to determine some homotopy groups of spheres. The results we obtain are summarized in Chapter 118. But note that the full computation of homotopy groups of spheres is still an open problem.

This sorry state of affairs means that as of right now we cannot use higher homotopy groups to distinguish any of the topological spaces that we could not distinguish already

[665] In an earlier version of these it was written that "it is straightforward to see that the map is well-defined". If one traces through the ancestry of Lemma 24.4 (4) one realizes that this lemma is, despite appearances, by no means obvious.

using the fundamental group. In particular we still cannot show that for $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic. We will only be able to do so after developing a new tool, namely the singular homology groups, in the coming sections.

Exercises for Chapter 40.

Exercise 40.1. Let X be a topological space, let $x_0 \in X$ and let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a map with $n \geq 1$. We consider the map

$$\begin{array}{rcl} \overline{f}\colon I^n & \to & X \\ (x_1, \dots, x_n) & \mapsto & f(1-x_1, 1-x_2, \dots, 1-x_n). \end{array}$$

Is $[\overline{f}] \in \pi_n(X, x_0)$ the inverse of $[f] \in \pi_n(X, x_0)$?

Exercise 40.2. Let $k \leq l$ be natural numbers, let $f\colon S^k \to S^l$ be a map and let $x_0 \in S^k$. Show that for any $i \in \mathbb{N}$ the map $f_*\colon \pi_i(S^k, x_0) \to \pi_i(S^l, f(x_0))$ is the trivial map.
Hint. Use Proposition 40.10.

Exercise 40.3. In this exercise you can assume that we already know that $\pi_3(S^2) \cong \mathbb{Z}$. Show that there exists an epimorphism $\pi_3(S^1 \vee S^2) \to \mathbb{Z}^3 \oplus \mathbb{Z}_4$.

Exercise 40.4. Let $n \in \mathbb{N}$ and let K be a non-empty compact subset of $\mathbb{R}^n$. Let $r \in \mathbb{R}$ such that $r > \sup\{\|P\| \,|\, P \in K\}$. Show that for any $k \in \mathbb{N}$ the inclusion induced map $\pi_k(S^{n-1}_r(0)) \to \pi_k(\mathbb{R}^n \setminus K)$ is a monomorphism.

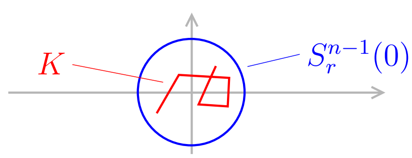


FIGURE 694. Illustration of Exercise 40.4.

Exercise 40.5. We consider the pseudocircle, i.e. we consider the slightly weird topological space $X = \{A, B, C, D\}$ from page 107. Show that $\pi_n(X, A) = 0$ for any $n \geq 2$.
Hint. Find a convenient covering of X.

Exercise 40.6. Let (X, x_0) be a topological space. We denote by $\Sigma(X)$ the suspension of X and we denote by $*$ the North Pole of $\Sigma(X)$. Is the suspension homomorphism

$$\pi_1(X_0, x_0) \;\to\; \pi_2(\Sigma(X), *),$$

that we introduced in Lemma 40.19, necessarily a monomorphism?

41. The homology groups of a topological space

One of our main goals is to show that for $m \neq n$ the spheres S^m and S^n are not homeomorphic. We have seen that for most cases fundamental groups are of no use. Furthermore, it might be that in principle higher homotopy groups can distinguish such spheres, but since we cannot compute higher homotopy groups, those are of no use either. So we need a new approach to finding invariants that can distinguish spheres.

In this chapter we want to introduce a new functor from the category of topological spaces to an algebraic category, more precisely, we will introduce a functor from the category of topological spaces to the category of abelian groups.

41.1. Singular chains.

Definition. Let $n \in \mathbb{N}_0$.

(1) Given k points $P_1, \dots, P_k$ in $\mathbb{R}^n$ such that $P_2 - P_1, \dots, P_k - P_1$ are linearly independent we refer to the convex hull of $P_1, \dots, P_k$ as a *k-simplex.*

(2) Given $n \in \mathbb{N}_0$ we refer to the convex hull of the standard basis vectors $e_1, \dots, e_{n+1}$ of $\mathbb{R}^{n+1}$, i.e. to

$$\Delta^n \;:=\; \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \,|\, x_0 + \dots + x_n = 1 \text{ and } x_i \geq 0 \text{ for all } i = 0, \dots, n\}$$

as the *standard n-simplex.*[666]

(3) Given the standard n-simplex Δ^n we define its boundary as[667]

$$\partial \Delta^n \;:=\; \bigcup_{i=0}^{n} \{(t_0, \dots, t_n) \in \Delta^n \,|\, t_i = 0\}.$$

(4) Finally we write

$$\mathring{\Delta}^n \;:=\; \Delta^n \setminus \partial \Delta^n.$$

Sometimes it will be helpful to view $\mathring{\Delta}^n$ as a smooth manifold where a smooth atlas for $\mathring{\Delta}^n$ is given by projection to the first n coordinates. We equip the smooth manifold $\mathring{\Delta}^n$ with the orientation where for each $P \in \mathring{\Delta}^n$ a basis $v_1, \dots, v_n$ of $\mathrm{V}_P \mathring{\Delta}^n$ is positive if the vectors $\{(1, \dots, 1), v_1, \dots, v_n\}$ are a positive basis for $\mathbb{R}^{n+1}$.

Example. The standard 0-simplex Δ^0 is just the point $\{1\} \in \mathbb{R}$, the standard 1-simplex Δ^1 equals the segment in $\mathbb{R}^2$ from $(1, 0)$ to $(0, 1)$ and the standard 2-simplex Δ^2 is the triangle in $\mathbb{R}^3$ spanned by the vertices $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$. We refer to Figure 695 for an illustration of the standard simplices in dimension 0, 1 and 2 and for an illustration of the boundary of the standard 2-simplex.

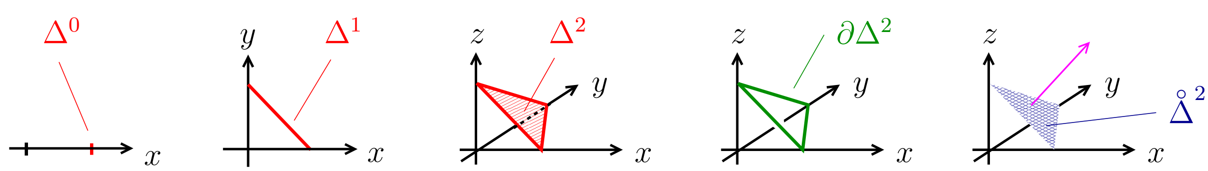


FIGURE 695

For the record we state the following elementary lemma.

[666] In Exercise 2.36 (a) we showed that the standard n-simplex Δ^n is the convex hull of $e_i \in \mathbb{R}^{n+1}$, $i = 1, \dots, n+1$. In particular the standard n-simplex is an n-simplex.

[667] The definition of the "boundary" of the standard n-simplex Δ^n is the "obvious one". Nonetheless note that if we had taken the original definition of the boundary of a subset A in some subspace X as introduced on page 92 and applied it to $\Delta^n \subset \mathbb{R}^{n+1}$, then the boundary of Δ^n in $\mathbb{R}^{n+1}$ would be all of Δ^n.

Lemma 41.1. *Let $n \in \mathbb{N}$.*

(1) *Every n-simplex, in particular the standard n-simplex Δ^n, is compact.*

We set

$$◺_n := \left\{ (x_1, \ldots, x_n) \in [0,1]^n \,\middle|\, \sum_{i=1}^{n} x_i \in [0,1] \right\}$$

and refer to it as the planar n-simplex. *The following statements hold:*

(2) *The maps*

$$\begin{array}{rcl} \Phi\colon \Delta^n & \to & ◺_n \\ (t_0, \ldots, t_n) & \mapsto & (t_0, \ldots, t_{n-1}) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon ◺_n & \to & \Delta^n \\ (s_0, \ldots, s_{n-1}) & \mapsto & \left(x_1, \ldots, x_n, 1 - \sum_{i=1}^{n} x_i\right) \end{array}$$

are homeomorphisms that are inverses of one another.

(3) *The map*

$$\Delta^n \xrightarrow{\Phi} ◺_n \xrightarrow[\text{applied to } ◺_n \text{ and } Q = (\frac{1}{n+1}, \ldots, \frac{1}{n+1})]{\text{the homeomorphism } \Xi \text{ obtained from Proposition 2.53 (2)}} \overline{B}^n$$

is a homeomorphism that restricts to a homeomorphism from $\partial\Delta^n$ to the sphere S^{n-1}.

Proof.

(1) We had proved this statement in Exercise 2.36 (b). Of course it is not difficult to prove the statement "from scratch".
(2) Note that Φ extends "by the same formula" to a map $\mathbb{R}^{n+1} \to \mathbb{R}^n$ which is evidently continuous. Hence the restriction to Δ^n is also continuous. Now that this is settled we see that the remaining statements are basically trivial.
(3) This statement follows easily from (1) and Proposition 2.53 (2). ■

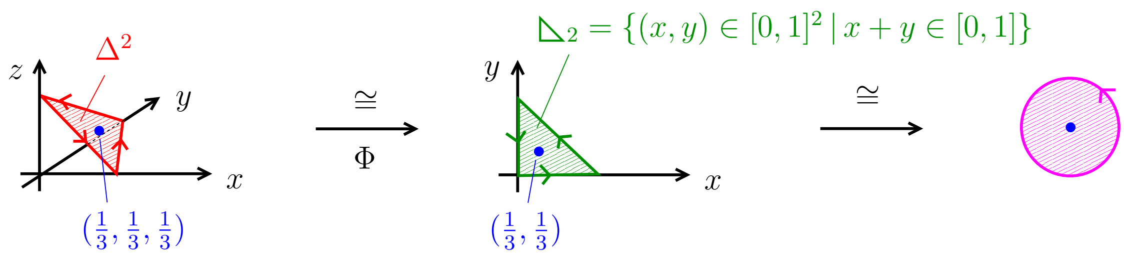


Figure 696. Illustration of the homeomorphism from Lemma 41.1.

Convention. Let $n \in \mathbb{N}$. We use the homeomorphism from Lemma 41.1 to make the identifications $\Delta^n = \overline{B}^n$ and $\partial\Delta^n = S^{n-1}$.

Definition. An *r-dimensional face* of the standard n-simplex is defined as a subset of the form

$$\{(t_0, \ldots, t_n) \in \Delta^n \mid t_{d_1} = t_{d_2} = \cdots = t_{d_{n-r}} = 0\},$$

where $d_1 < d_2 < \cdots < d_{n-r} \in \{0, \ldots, n\}$ are fixed. We illustrate the definition in Figure 697 on the left.

Remark.

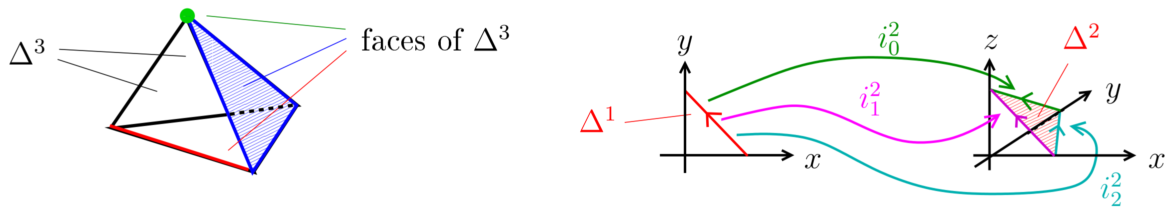


FIGURE 697

(1) The boundary $\partial\Delta^n$ of the standard n-simplex is the union of all $(n-1)$-dimensional faces of Δ^n.

(2) Every r-dimensional face of Δ^r is homeomorphic to the standard r-simplex. For later we fix homeomorphism in the case of $(n-1)$-dimensional faces. More precisely, for $j \in \{0, \ldots, n\}$ we consider the map

$$\begin{array}{rcl} i_j^n \colon \Delta^{n-1} & \to & \Delta^n \\ (t_0, \ldots, t_{n-1}) & \mapsto & (t_0, \ldots, t_{j-1}, 0, t_j, \ldots, t_{n-1}). \end{array}$$

This map is sometimes called the *j-th face map*. It is straightforward to see that these face maps are homeomorphisms. The face maps $i_0^2, i_1^2, i_2^2 \colon \Delta^1 \to \Delta^2$ are illustrated in Figure 697 on the right.

The first idea for the definition of homology groups is now the following. In the definition of the fundamental group $\pi_1(X, x_0)$ we had considered all maps $f \colon [0,1] \to X$ with the extra condition $f(0) = f(1) = x_0$. Using this extra condition we could concatenate loops to eventually end up with a group structure. Now given $n \in \mathbb{N}_0$ we want to consider all maps from Δ^n to X with no extra condition on the boundary. This time there is no reasonable way how two such maps can be composed to give a new map. Since we nonetheless want the notion of a group we turn the set of all maps $\Delta^n \to X$ into a group by "brute force", i.e. by considering the free abelian group generated by these maps.

This leads us to the following definition.

Definition. Let X be a topological space and let $n \in \mathbb{N}_0$.

(1) A *singular n-simplex* is a map $\sigma \colon \Delta^n \to X$.

(2) We define

$$\mathrm{C}_n(X) \ := \ \text{free abelian group generated by the set of singular } n\text{-simplices in } X.$$

For $n \in \mathbb{Z}_{<0}$ we set $\mathrm{C}_n(X) = 0$.

(3) We refer to an element in $\mathrm{C}_n(X)$ as a *singular n-chain in X*.

Remark. In principle, given a map $\sigma \colon \Delta^n \to X$ to some topological space X we need to show that this map is actually continuous. If one really feels compelled to give an argument, then often it is more convenient to show the equivalent statement that the map $\sigma \circ \Psi \colon \text{◺}_n \to X$ is continuous, where $\Psi \colon \text{◺}_n \to \Delta^n$ is the homeomorphism given in Lemma 41.1 (1).

Examples.

(1) For $n = 0$ a singular n-simplex is a map from Δ^0 to X. Such a map is uniquely determined by the corresponding point in X. Therefore frequently we will identify singular 0-simplices with points in X.

(2) For points $z_0, \ldots, z_k \in \mathbb{R}^n$ we consider the map[668]

$$\begin{array}{rcl} [z_0, \ldots, z_k] \colon \Delta^k & \to & \mathbb{R}^n \\ (t_0, \ldots, t_k) & \mapsto & \sum\limits_{j=0}^{k} t_j \cdot z_j. \end{array}$$

The map $[z_0, \ldots, z_k]$ thus sends the standard k-simplex Δ^k to the simplex spanned by $z_0, \ldots, z_k$ in $\mathbb{R}^n$. Put differently, $[z_0, \ldots, z_k]$ is a singular k-simplex in $\mathbb{R}^n$.

Remark. Let X be a topological space. By the discussion on page 580 we can view every element in $\mathrm{C}_n(X)$, i.e. every singular n-chain, as a linear combination $a_1\sigma_1 + \cdots + a_k\sigma_k$ where $a_1, \ldots, a_k \in \mathbb{Z}$ and where $\sigma_1, \ldots, \sigma_k$ are singular n-simplices.

Now let $f \colon X \to Y$ be a map between topological spaces. For each n we consider

$$\begin{array}{rcl} f_* \colon \mathrm{C}_n(X) & \to & \mathrm{C}_n(Y) \\ \sum\limits_{i=1}^{k} a_i \cdot \sigma_i & \mapsto & \sum\limits_{i=1}^{k} a_i \cdot (f \circ \sigma_i). \end{array}$$

It is easy to verify that $X \mapsto \mathrm{C}_n(X)$ and $f \mapsto f_*$ defines a functor from the category of topological spaces to the category of abelian groups.

Have we made any progress towards our goal of distinguishing topological spaces? Not really. For example, if X is a smooth manifold of dimension ≥ 1, then given any $n \in \mathbb{N}_0$ there exist uncountably many maps $\Delta^n \to X$, i.e. the group $\mathrm{C}_n(X)$ is a free abelian group with uncountably many generators. Such groups are totally useless for distinguishing non-homeomorphic topological spaces.

41.2. **Definition of the homology groups of a topological space.** Before we give the definition of homology groups it is perhaps helpful to recall the definition of de Rham cohomology.

Let M be a smooth manifold. We write

$$\Omega^n(M) \ := \ \{\text{smooth } n\text{-forms on } M\}.$$

The isomorphism type of this vector space is an invariant of the diffeomorphism type of the smooth manifold M, but it is rather useless, since for $1 \leq n \leq \dim(M)$ this is a vector space of uncountable dimension. To cut $\Omega^n(M)$ into something much smaller we defined linear maps $d_n \colon \Omega^n(M) \to \Omega^{n+1}(M)$ with $d_{n+1} \circ d_n = 0$. Finally we defined the n–th de Rham cohomology group of M as

$$\mathrm{H}^n_{\mathrm{dR}}(M) \ := \ \ker(d_n \colon \Omega^n(M) \to \Omega^{n+1}(M)) / \operatorname{im}(d_{n-1} \colon \Omega^{n-1}(M) \to \Omega^n(M)).$$

We are now in a similar situation: to a topological space X we associated the free abelian group $\mathrm{C}_n(X)$ which is “too big”. The idea now is to introduce a “differential” as for the case of differential forms. But in this case there is no good way to associate to a map $\Delta^n \to X$ a new map $\Delta^{n+1} \to X$. But by restricting the map $\Delta^n \to X$ to each

[668] Using Lemma 41.1 (2) one can easily show that this map is continuous.

of the $(n-1)$-dimensional faces of Δ^n which are, as we have seen, homeomorphic to the standard $(n-1)$-simplex, we can associate to a singular n-simplex $\sigma\colon \Delta^n \to X$ a sum of singular $(n-1)$-simplices, i.e. a singular $(n-1)$-chain. To do this we will use the maps $i_k^n\colon \Delta^{n-1} \to \Delta^n$, $k = 0, \dots, n$ from page 1077 to identify the standard $(n-1)$-simplex with the $(n-1)$-dimensional faces of Δ^n. Now we can define

$$\widetilde{\partial}_n\sigma \;:=\; \sum_{j=0}^{n} \sigma \circ i_j^n.$$

By Lemma 19.1 this defines a map $\widetilde{\partial}_n\colon\ \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$. But it is quite straightforward to see that for this definition the composition of the maps $\widetilde{\partial}_n\colon\ \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$ and $\widetilde{\partial}_{n-1}\colon\ \mathrm{C}_{n-1}(X) \to \mathrm{C}_{n-2}(X)$ is not zero[669]. But we can rectify this problem by cleverly introducing signs. This leads us to the following definition.

Definition. Let X be a topological space and let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. The *boundary* $\partial_n\sigma \in \mathrm{C}_{n-1}(X)$ of σ is defined as

$$\partial_n\sigma \;:=\; \sum_{j=0}^{n} (-1)^j \cdot \sigma \circ i_j^n.$$

By Lemma 19.1 the map $\sigma \mapsto \partial_n\sigma$ extends uniquely to a linear map

$$\begin{aligned} \partial_n\colon\ \mathrm{C}_n(X) &\to \mathrm{C}_{n-1}(X) \\ \sum_{i=1}^{k} a_i\sigma_i &\mapsto \sum_{i=1}^{k} a_i \cdot \partial\sigma_i, \end{aligned}$$

which we refer to as the *n-th boundary map.* If it is clear with what dimension we are working with, then we just write ∂ instead of ∂_n.

Examples.

(1) For any singular 1-simplex $\sigma\colon \Delta^1 \to X$ we have[670]

$$\partial\sigma \;=\; (-1)^0 \cdot \sigma \circ i_0^1 + (-1)^1 \cdot \sigma \circ i_1^1 \;=\; \sigma(0,1) - \sigma(1,0).$$

Put differently, a singular 1-simplex can be viewed as a map σ from the "interval" Δ^1 to X. The boundary of σ is thus the "endpoint" $\sigma(0,1)$ "minus" the "starting point" $\sigma(1,0)$. Here we view points in X again as singular 0-simplices. We sketch the definition of a singular 1-simplex in Figure 698.

(2) In Figure 699 we sketch the boundary of a singular 2-simplex.

As promised, these boundary maps have the following fundamental property:

Proposition 41.2. *Let X be a topological space. For any $n \in \mathbb{N}_0$ the composition*

$$\partial_{n-1} \circ \partial_n\colon\ \mathrm{C}_n(X) \;\to\; \mathrm{C}_{n-2}(X)$$

is the zero map.

---

[669] Indeed, let S be a set and let $\mathbb{Z}^{(S)}$ be the free abelian group generated by S. We define $\epsilon\colon \mathbb{Z}^{(S)} \to \mathbb{Z}$ to be the linear map that assigns to each $s \in S$ the value 1. It follows easily from the definitions that for any $\sigma\colon \Delta^n \to X$ we have $\epsilon(\widetilde{\partial}_n\sigma) = n+1$ and $\epsilon((\widetilde{\partial}_{n-1} \circ \widetilde{\partial}_n)(\sigma)) = n \cdot (n+1)$.

[670] Here we again use the convention that given a point $x \in X$ we denote by x also the corresponding singular 0-simplex.

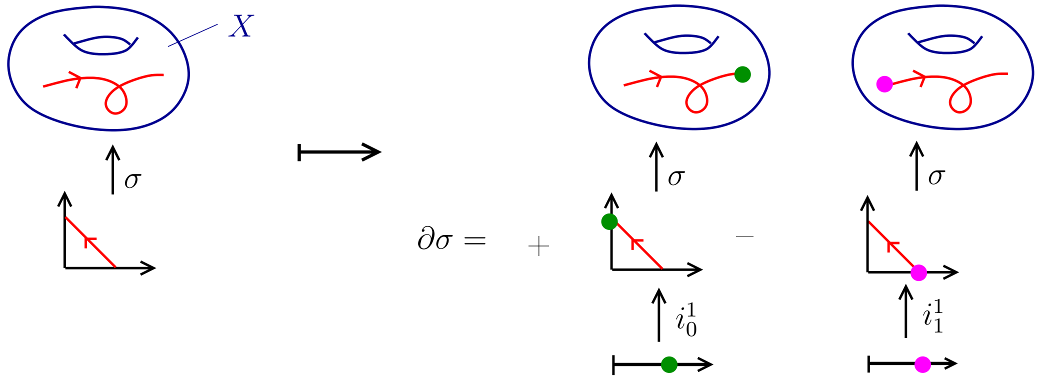


FIGURE 698. Boundary of a singular 1-simplex.

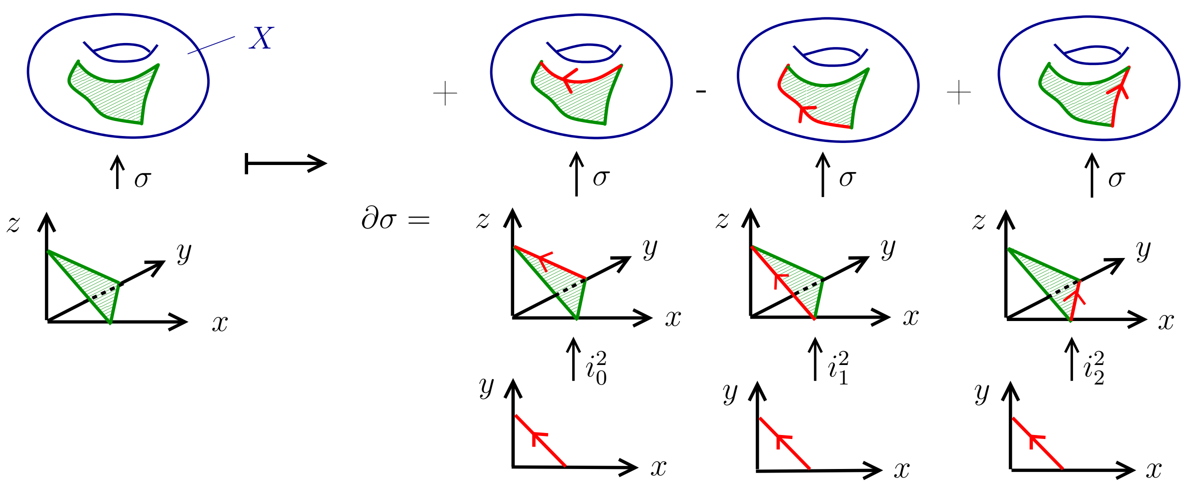


FIGURE 699. Boundary of a singular 2-simplex.

PROOF. It suffices to show that $\partial_{n-1}(\partial_n(\sigma)) = 0$ for every singular n-simplex $\sigma\colon \Delta^n \to X$. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. Then it follows immediately from the definitions that

$$\begin{aligned}
\partial_{n-1}(\partial_n(\sigma)) &= \partial_{n-1}\left(\sum_{j=0}^{n}(-1)^j \cdot \sigma \circ i_j^n\right) = \sum_{k=0}^{n-1}(-1)^k \cdot \sum_{j=0}^{n}(-1)^j \cdot (\sigma \circ i_j^n) \circ i_k^{n-1} \\
&= \sum_{0=k<j\le n}(-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) + \sum_{0=j\le k\le n-1}(-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) =: (*).
\end{aligned}$$

Now we want to show that each expression $\sigma \circ (i_j^n \circ i_k^{n-1})$ appears precisely twice, but with opposite signs. From the definitions it follows easily that for $j \le k$ we have[671]

$$i_j^n \circ i_k^{n-1} = i_{k+1}^n \circ i_j^{n-1}.$$

[671]Indeed, if $j \le k$, then the zero-entries of $(i_j^n \circ i_k^{n-1})(t_0, \ldots, t_{n-1})$ are the entries j and $k+1$.

Now we continue with the above calculation:

$$(*) \underset{\substack{\uparrow \\ \text{above calculation of } i_j^n \circ i_k^{n-1}}}{=} \sum_{0=k<j\le n} (-1)^{k+j}\cdot\sigma\circ\left(i_j^n\circ i_k^{n-1}\right) + \sum_{0=j\le k\le n-1}(-1)^{k+j}\cdot\sigma\circ\left(i_{k+1}^n\circ i_j^{n-1}\right)$$

$$\underset{\substack{\uparrow \\ \text{substitution } l=k+1}}{=} \sum_{0=k<j\le n} (-1)^{k+j}\cdot\sigma\circ\left(i_j^n\circ i_k^{n-1}\right) + \sum_{0=j<l\le n}(-1)^{l+j-1}\cdot\sigma\circ\left(i_l^n\circ i_j^{n-1}\right) \underset{\substack{\uparrow \\ \text{all summands left and right cancel}}}{=} 0.$$

■

The following is arguably the most important definition of algebraic topology.

Definition. Let X be a topological space and let $n\in\mathbb{N}_0$. We define the *n-th singular homology group of X* as the quotient group[672]

$$\mathrm{H}_n(X) := \frac{\ker(\partial_n\colon \mathrm{C}_n(X)\to\mathrm{C}_{n-1}(X))}{\operatorname{im}(\partial_{n+1}\colon \mathrm{C}_{n+1}(X)\to\mathrm{C}_n(X))}.$$

For $n\in\mathbb{Z}_{<0}$ we extend this definition to $\mathrm{H}_n(X):=0$. Throughout the lecture notes we will mostly shorten the name "singular homology" to "homology".[673]

One of the key goals of this course is to develop tools for computing homology groups of topological spaces. In particular we want to compute the homology groups for all spheres. In the subsequent discussions it is helpful to introduce a few more definitions.

Definition. Let X be a topological space.

(1) We say $c\in\mathrm{C}_n(X)$ is a *singular n-cycle*, or short *n-cycle*, if $\partial c=0$.
(2) If c is an n-cycle, then we refer to $[c]\in\mathrm{H}_n(X)$ as the *homology class of c*.
(3) We say two singular n-chains c and d are *homologous* if there exists a singular $(n+1)$-chain e with $\partial e=c-d$. In particular, if c and d are cycles, then they are homologous if they represent the same homology class.
(4) An n-cycle c is called *null-homologous*, if $[c]=0\in\mathrm{H}_n(X)$.
(5) Given a singular 1-simplex $\sigma\colon\Delta^1\to X$ we denote by $\overline{\sigma}$ the singular 1-simplex that is given by $\overline{\sigma}(t,1-t)=\sigma(1-t,t)$. Put differently, $\overline{\sigma}$ is given by reversing the orientation of σ.

Example. In Figure 700 we show that torus $T=([0,1]\times[0,1])/\sim$ together with two singular 1-simplices c and d and together with two singular 2-simplices α and β. It follows easily from the definitions that c and d are 1-cycles. It is a good exercise to verify that $\alpha-\beta$ is a 2-cycle. With our present knowledge it is impossible though to determine whether these cycles represent non-trivial homology classes.

The following lemma will be proved in Exercises 41.2 and 41.4.

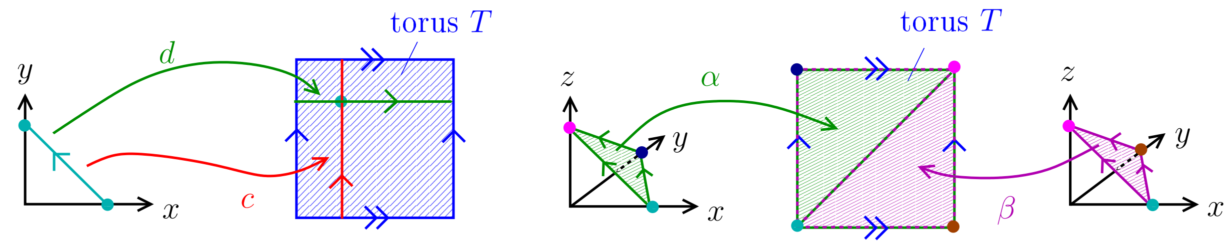
---

[672]It follows from Proposition 41.2 that we have the following inclusion:

$$\operatorname{im}(\partial_{n+1}\colon \mathrm{C}_{n+1}(X)\to\mathrm{C}_n(X)) \subset \ker(\partial_n\colon\mathrm{C}_n(X)\to\mathrm{C}_{n-1}(X)).$$

Furthermore the groups $\mathrm{C}_*(X)$ are abelian, in particular any subgroup is also abelian and normal. Therefore it does indeed make sense to consider the quotient group.

[673]Later on, in Chapters 48 and 63, we will also introduce "cellular homology" and "simplicial homology". At that point it will be useful to remember that our present homology groups have a more precise name, namely singular homology groups.



c and d are 1-cycles

$\alpha - \beta$ is a 2-cycle

FIGURE 700

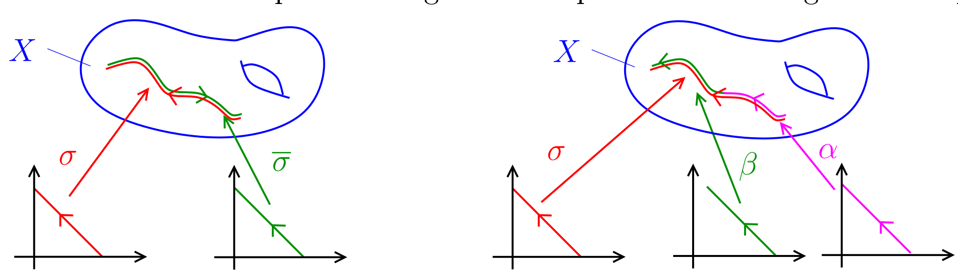
Lemma 41.3. *Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex.*

(1) *If σ is a constant map, then σ is null-homologous.*
(2) *The singular 1-chains $-\sigma$ and $\overline{\sigma}$ are homologous.*
(3) *Let $s \in (0,1)$ and consider the singular 1-simplices*

$$\begin{array}{rcl} \alpha\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(st, 1-st) \end{array} \quad \textit{and} \quad \begin{array}{rcl} \beta\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(s+t(1-s), 1-(s+t(1-s))). \end{array}$$

Then the singular 1-chains σ and $\alpha + \beta$ are homologous.

The second part of the lemma says that for a singular 1-simplex the geometric operation of reversing the orientation is (up to the boundary of a simplicial 2-chain) the same as algebraically inverting the sign. Furthermore, the third part of the lemma says that for a singular 1-simplex (again up to the boundary of a simplicial 2-chain) it does not make a difference whether or not we split the singular 1-simplex into two singular 1-simplices.



σ and $-\overline{\sigma}$ are homologous

σ and $\alpha + \beta$ are homologous

FIGURE 701. Illustration of Lemma 41.3.

Examples.

(1) We consider Figure 702. We see three singular chains on the torus, and it is straightforward to show that all three singular chains are cycles. Now the question arises, which of these cycles are in fact null-homologous. In Figure 703 we see that the first cycle $a - b + c$ in Figure 702 is the boundary of a singular 2-simplex D. One can easily see that the second cycle, $a + \overline{b} + c$ in Figure 702 is *not* the boundary of a singular 2-simplex. On the other hand by Lemma 41.3 we know that $b + \overline{b}$ is a null-homologous cycle, i.e. there exists an $E \in C_2(X)$ such that $b + \overline{b} = \partial E \in C_1(X)$.

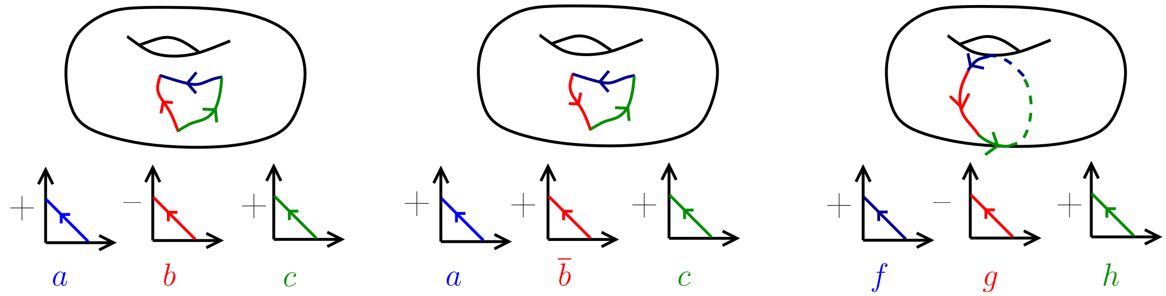


FIGURE 702. Three examples of 1-cycles on the torus.

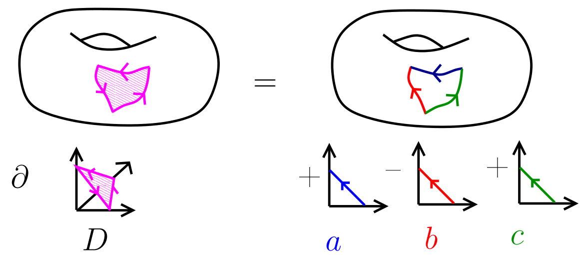


FIGURE 703. The first cycle in Figure 702 is the boundary of a 2-dimensional singular chain.

Thus it follows that

$$a+\overline{b}+c \;=\; a+\partial E-b+c \;=\; a-b+c+\partial E=\partial D+\partial E \;=\; \partial(D+E),$$

i.e. $a+\overline{b}+c$ is null-homologous.

It is not clear whether the third cycle in Figure 702 is the boundary of a 2-dimensional singular chain. We will see later that this not the case. But it will take us quite some time to develop the techniques which will allow us to prove this statement.

(2) In Figure 704 we show the torus $T=([0,1]\times[0,1])/\sim$ and two singular simplices $\alpha\colon \Delta^2\to T$ and $\beta\colon \Delta^2\to T$ that are given by affine linear maps.[674] It is straightforward to verify that $\alpha-\beta$ is a cycle, hence it represents an element in $\mathrm{H}_2(T)$. We will see later on page 1083 whether or not this element is zero in $\mathrm{H}_2(T)$.

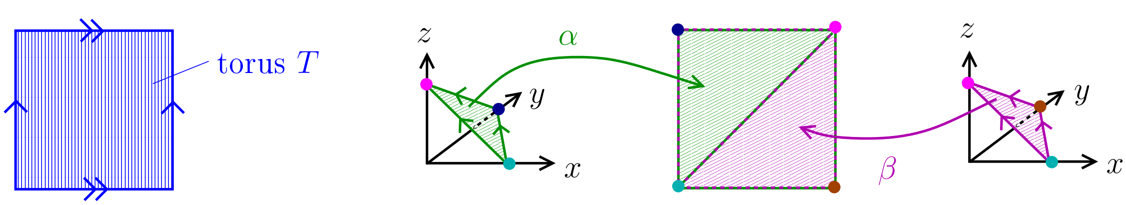


FIGURE 704

[674]More precisely, α is the unique affine linear map with $\alpha(1,0,0)=[(0,0)]$, $\alpha(0,1,0)=[(1,0)]$ and $\alpha(0,0,1)=[(1,1)]$ whereas β is the unique affine linear map with $\beta(1,0,0)=[(0,0)]$, $\beta(0,1,0)=[(0,1)]$ and $\beta(0,0,1)=[(1,1)]$.

(3) In Exercise 41.6 we will try to generalize the approach taken for the torus to other 2-dimensional smooth manifolds, namely the sphere, the Klein bottle and the projective space $\mathbb{R}P^2$.

41.3. **First calculations of homology groups.** Given a topological space the corresponding homology groups are defined as the quotient of two abelian groups which in general will be infinitely generated. Somewhat surprisingly we will see in the coming chapters that the homology groups are nonetheless very frequently finitely generated abelian groups.

We start out our computations of homology groups with the 0-th homology of a path-connected topological space. The key to the calculation is the following definition.

Definition. Given a topological space X we refer to the map

$$\begin{array}{rcl}\epsilon_X\colon\ \mathrm{C}_0(X) & \to & \mathbb{Z} \\ \sum\limits_{i=1}^{k} a_i\sigma_i & \mapsto & \sum\limits_{i=1}^{k} a_i\end{array}$$

as the *augmentation map.*

The following lemma shows that the augmentation map is a neat addition to the singular chain complex of a topological space.

Lemma 41.4. *Let X be a topological space.*

(1) *The map $\epsilon_X\circ\partial_1\colon\ \mathrm{C}_1(X)\to\mathbb{Z}$ is the zero map.*
(2) *The map*

$$\begin{array}{rcl}\epsilon_X\colon\ \mathrm{H}_0(X) & \to & \mathbb{Z} \\ [\sigma] & \mapsto & [\epsilon_X(\sigma)]\end{array}$$

is well-defined. In the following we refer to this map also as the augmentation map.

Proof. Let X be a topological space.

(1) Let $c=\sum\limits_{i=1}^{n} b_i\sigma_i\in\mathrm{C}_1(X)$. We calculate that

$$\epsilon(\partial_1(c)) \;=\; \epsilon\Big(\sum_{i=1}^{n} b_i\cdot\partial_1(\sigma_i\Big) \underset{\substack{\uparrow \\ \text{for every singular 1-simplex } \sigma \text{ we} \\ \text{have } \partial\sigma=\sigma(0,1)-\sigma(1,0)\text{, see page 1079}}}{=} \epsilon\Big(\sum_{i=1}^{n} b_i\cdot\sigma_i(0,1)-b_i\cdot\sigma_i(1,0)\Big) \underset{\substack{\uparrow \\ \text{by definition of } \epsilon}}{=} \sum_{i=1}^{n} b_i-\sum_{i=1}^{n} b_i \;=\; 0.$$

(2) Note that

$$\begin{array}{rcl}\mathrm{H}_0(X) & = & \ker(\partial_0\colon\ \mathrm{C}_0(X)\to\mathrm{C}_{-1}(X))\,/\,\mathrm{im}(\partial_1\colon\ \mathrm{C}_1(X)\to\mathrm{C}_0(X)) \\ & \underset{\substack{\uparrow \\ \text{since } \mathrm{C}_{-1}(X)=0}}{=} & \mathrm{C}_0(X)\,/\,\mathrm{im}(\partial_1\colon\ \mathrm{C}_1(X)\to\mathrm{C}_0(X)).\end{array}$$

Thus we see that Statement (2) is an immediate consequence of Statement (1). ■

Now we can formulate the following proposition.

Proposition 41.5. *Let X be a path-connected non-empty topological space and let P be a point in X. Then the augmentation map*

$$\epsilon = \epsilon_X\colon \mathrm{H}_0(X) \;\to\; \mathbb{Z}$$

is a natural isomorphism where the inverse is given by the map

$$\begin{aligned} \iota\colon \mathbb{Z} &\to \mathrm{H}_0(X) \\ n &\mapsto n\cdot[P]. \end{aligned}$$

Proof. Let X be a non-empty path-connected topological space. Let P be a point in X. It is clear that $\epsilon\circ\iota = \mathrm{id}_{\mathbb{Z}}$, in particular the homomorphism ϵ is surjective. It remains to show that ϵ is injective.

Thus let $c \in \mathrm{C}_0(X)$ with $\epsilon(c) = 0$. By definition of $\mathrm{C}_0(X)$ we can find points $Q_1,\dots,Q_k$ in X and $a_1,\dots,a_k \in \mathbb{Z}$ such that $c = \sum\limits_{i=1}^{k} a_i\cdot Q_i$. We choose a point P in X. Since X is path-connected we can find, for $i = 1,\dots,k$, a path from P to Q_i. Put differently, for each $i = 1,\dots,k$ we can find a singular 1-simplex σ_i with $\partial\sigma_i = Q_i - P$. Then we have

$$\partial\left(\sum_{i=1}^{k} a_i\sigma_i\right) \;=\; \sum_{i=1}^{k} a_i\partial\sigma_i \;=\; \sum_{i=1}^{k} a_i(Q_i - P) \;=\; \sum_{i=1}^{k} a_iQ_i - \underbrace{\left(\sum_{i=1}^{k} a_i\right)}_{=\epsilon(c)=0}\cdot P \;=\; \sum_{i=1}^{k} a_iQ_i.$$

This implies that $[c] = 0 \in \mathrm{H}_0(X)$. Thus we have shown that ϵ is injective. ■

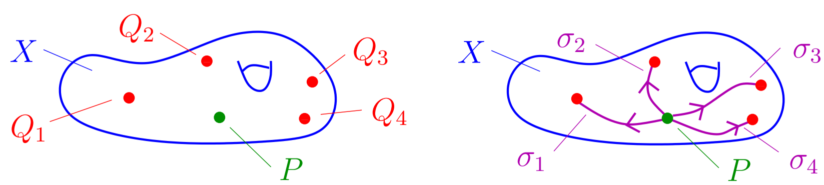


Figure 705. Illustration of the proof of Proposition 41.5.

The proof of the following lemma is Exercise 41.3.

Lemma 41.6. *Let X be a topological space which consists of a single point. Then the following holds*

$$\mathrm{H}_n(X) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

The calculation of $\mathrm{H}_0(X)$ in Proposition 41.5 and the calculation of the homology of a point in Lemma 41.6 are the only calculations of homology groups of path-connected topological spaces that one can do "by hand". For all other topological spaces we will need to work significantly harder.

41.4. **Algebraic chain complexes.** Before we continue with the discussion of homology groups it is convenient to introduce several algebraic definitions. The following definitions are just the obvious algebraic abstractions of the definitions that we had already introduced.

Definition.

(1) An *(algebraic) chain complex* $(\mathrm{C}_*, \partial_*)$ is a sequence

$$\ldots \to \mathrm{C}_n \xrightarrow{\partial_n} \mathrm{C}_{n-1} \xrightarrow{\partial_{n-1}} \ldots \mathrm{C}_1 \xrightarrow{\partial_1} \mathrm{C}_0 \to 0$$

of maps between abelian groups[675] such that for every $i \in \mathbb{N}$ we have $\partial_{i-1} \circ \partial_i = 0$. As before we introduce the convention that $\mathrm{C}_n = 0$ for $n \in \mathbb{Z}_{<0}$.

(2) We define the *n-th homology group of the chain complex* precisely the same way as we defined the homology of a topological space, namely we set

$$\mathrm{H}_n(C) \ := \ \ker(\partial_n\colon \mathrm{C}_n \to \mathrm{C}_{n-1}) \,/\, \operatorname{im}(\partial_{n+1}\colon \mathrm{C}_{n+1} \to \mathrm{C}_n).$$

(3) As before we refer to elements in the kernel of $\partial_n\colon \mathrm{C}_n \to \mathrm{C}_{n-1}$ as *cycles*. Furthermore we say that a cycle $z \in C_n$ is *null-homologous* if $[z] = 0 \in \mathrm{H}_n(C)$.

On some occasions we will need the following generalization:

(4) A *generalized (algebraic) chain complex* $(\mathrm{C}_*, \partial_*)$ is a sequence

$$\ldots \to \mathrm{C}_3 \xrightarrow{\partial_3} \mathrm{C}_2 \xrightarrow{\partial_2} \mathrm{C}_1 \xrightarrow{\partial_1} \mathrm{C}_0 \xrightarrow{\partial_0} \mathrm{C}_{-1} \xrightarrow{\partial_{-1}} \mathrm{C}_{-2} \to \ldots$$

of maps between abelian groups such that for every $i \in \mathbb{Z}$ we have $\partial_{i-1} \circ \partial_i = 0$. Basically all present and future definitions and results for chain complexes also make sense for generalized chain complexes. In most cases we will not explicitly state the generalization to generalized chain complex.

Now we introduce a new notion, namely the notion of a chain map between chain complexes.

Definition. A *chain map* $f\colon C_* \to D_*$ between two chain complexes[676] consists of a family $\{f_n\colon \mathrm{C}_n \to D_n\}_{n \in \mathbb{N}_0}$ of maps such that for each n the following equality holds:[677]

$$f_{n-1} \circ \partial_n \ = \ \partial_n \circ f_n.$$

Put differently the maps f_n have the property that the following diagram commutes:

$$\begin{tikzcd}
\ldots \arrow[r, "\partial"] & \mathrm{C}_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & \mathrm{C}_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & \mathrm{C}_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & \ldots \\
\ldots \arrow[r, "\partial"] & D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r, "\partial_{n-1}"] & \ldots
\end{tikzcd}$$

Remark. On many occasions we will use the elementary observation that the composition of two chain maps $f\colon C_* \to D_*$ and $g\colon D_* \to E_*$ defines a chain map $g \circ f\colon C_* \to E_*$.

Now we can prove the following lemma:

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[675] It is perhaps worth pointing out that we do not assume that the chain groups C_* are free abelian groups.

[676] As usual we suppress bits from the notation which are understood from the context. In this instance we suppress the notation for the boundary maps in the chain complexes. Put differently, instead of saying "let $(\mathrm{C}_*, \partial_*)$ be a chain complex" we say simply "let C_* be a chain complex".

[677] More precisely, for any $c \in C$ we have $f_{n-1}(\partial_n(c)) = \partial_n(f_n(c))$ where on the left-hand side we consider the boundary map $\mathrm{C}_n \to \mathrm{C}_{n-1}$, and on the right-hand side we consider the boundary map $D_n \to D_{n-1}$ which we also denote by ∂_n. Usually it is obvious from the context which boundary map we consider.

Lemma 41.7. *Let $f\colon C_* \to D_*$ be a chain map between chain complexes C_* and D_*. Then the map*

$$\begin{array}{rcl} f_*\colon\ \mathrm{H}_n(C_*) & \to & \mathrm{H}_n(D_*) \\ {[c]} & \mapsto & [f_n(c)] \end{array}$$

is well-defined.

In the following we refer to the map $f_*\colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ as the *induced map on homology*.

Proof. Let $c \in C_n$ be a cycle. First we have to show that $f_n(c)$ is again a cycle. Indeed we have

$$\partial_n(f_n(c)) \underset{\substack{\uparrow \\ \text{definition of a chain map}}}{=} f_{n-1}(\partial_n c) \underset{\substack{\uparrow \\ \text{since } c \text{ is a cycle}}}{=} f_n(0) = 0.$$

Now let $c, d \in C_n$ be two cycles which represent the same element in $\mathrm{H}_n(C)$. We have to show that $[f_n(c)] = [f_n(d)]$. Indeed we have

$$[f_n(c)] - [f_n(d)] = [f_n(c-d)] \underset{\substack{\uparrow \\ \text{since } c,d \text{ are homologous there} \\ \text{exists an } e \in \mathrm{C}_{n+1} \text{ with } \partial_{n+1}e = c-d}}{=} [f_n(\partial_{n+1}(e))] \underset{\substack{\uparrow \\ \text{definition of a chain map}}}{=} [\partial_{n+1}(f_{n+1}(e))] = 0 \in \mathrm{H}_n(D).$$

■

We can state the above results more formally. To do so it is convenient to introduce the following definition.

Definition. We refer to category $\mathcal{ChCplx}$ that is given by

$$\mathrm{Ob}(\mathcal{ChCplx}) := \text{all chain complexes,}$$

and with morphisms

$$\mathrm{Mor}(\mathrm{C}_*, D_*) := \text{all chain maps from } C_* \text{ to } D_*.$$

with the obvious composition of morphisms as the *category of chain complexes.*

The following lemma follows easily from Lemma 41.7 and from the definitions:

Lemma 41.8. *For each $n \in \mathbb{N}_0$ the maps*

$$\mathrm{C}_* \mapsto \mathrm{H}_n(C_*)$$

and

$$(f\colon C_* \to D_*) \mapsto (f_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*))$$

define a functor from the category $\mathcal{ChCplx}$ of chain complexes to the category $\mathcal{AbGr}$ of abelian groups.

41.5. **The functoriality of homology groups.** Let $f\colon X \to Y$ be a map between topological spaces. We already saw on page 1078 that for each $n \in \mathbb{N}_0$ the map f induces a map

$$\begin{array}{rcl} f_*\colon\ \mathrm{C}_n(X) & \to & \mathrm{C}_n(Y) \\ \sum\limits_{i=1}^{k} a_i \cdot \sigma_i & \mapsto & \sum\limits_{i=1}^{k} a_i \cdot f \circ \sigma_i. \end{array}$$

Now we can formulate the following lemma:

Lemma 41.9.

(1) *Let $f\colon X \to Y$ be a map between topological spaces and let $n \in \mathbb{N}_0$. For every $c \in \mathrm{C}_n(X)$ we have*
$$f_*(\partial_n c) \;=\; \partial_n(f_*(c)) \;\in\; \mathrm{C}_n(Y).$$
In particular $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ is a chain map.

(2) *The maps*
$$\text{topological space } X \;\mapsto\; \mathrm{C}_*(X)$$
and
$$(f\colon X \to Y) \;\mapsto\; (f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y))$$
define a functor from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{ChCplx}$ of chain complexes.

Proof.

(1) Let $f\colon X \to Y$ be a map between topological spaces and let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. We have
$$f_*(\partial\sigma) = f_*\left(\sum_{j=0}^{n}(-1)^j\cdot\sigma\circ i_j^n\right) = \sum_{j=0}^{n}(-1)^j\cdot f\circ(\sigma\circ i_j^n) = \sum_{j=0}^{n}(-1)^j\cdot\underbrace{(f\circ\sigma)}_{=f_*\sigma}\circ i_j^n = \partial(f_*\sigma).$$
Since the singular n-simplices generate $\mathrm{C}_n(X)$ we obtain the desired equality for all singular n-chains. By definition this means that $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ is a chain map.

(2) The second statement is an almost immediate consequence of the first statement. ■

In particular, using Lemma 41.7 we immediately obtain the following lemma.

Lemma 41.10. *Let $f\colon X \to Y$ be a map between topological spaces. Then the map*
$$\begin{aligned} f_*\colon \mathrm{H}_n(X) &\to \mathrm{H}_n(Y) \\ \left[\sum_{i=1}^{k} a_i\cdot\sigma_i\right] &\mapsto \left[\sum_{i=1}^{k} a_i\cdot(f\circ\sigma_i)\right] \end{aligned}$$
is well-defined.

Example. Let $f\colon X \to Y$ be a map between two topological spaces. It follows immediately from the definitions that the following diagram commutes
$$\begin{tikzcd} \mathrm{H}_0(X) \arrow[rr, "f_*"] \arrow[dr, "\epsilon_X"'] & & \mathrm{H}_0(Y). \arrow[dl, "\epsilon_Y"] \\ & \mathbb{Z} & \end{tikzcd}$$
where the maps ϵ_X and ϵ_Y denote the augmentation maps. In particular, if X and Y are path-connected topological spaces, then it follows easily from Proposition 41.5 that the map $f_*\colon \mathrm{H}_0(X) \to \mathrm{H}_0(Y)$ is an isomorphism. Furthermore, if $X = Y$, i.e. if f is a self-map on a path-connected topological space X, then f_* is in fact the identity on $\mathrm{H}_0(X)$.

The combination of Lemmas 41.9 and 41.8 now gives us the following lemma.

Lemma 41.11. *For each $n \in \mathbb{N}_0$ the map*

$$X \quad \mapsto \quad \mathrm{H}_n(X)$$

together with the map

$$(f\colon X \to Y) \quad \mapsto \quad \left(\begin{array}{rcl} f_*\colon \mathrm{H}_n(X) & \to & \mathrm{H}_n(Y) \\ \left[\sum\limits_{i=1}^{k} a_i \cdot \sigma_i\right] & \mapsto & \left[\sum\limits_{i=1}^{k} a_i \cdot (f \circ \sigma_i)\right] \end{array} \right)$$

defines a functor from the category of topological spaces to the category of abelian groups.

Example. Let X and Y be non-empty topological spaces. Let $f\colon X \to Y$ be a map. We suppose that f is constant, i.e. we suppose that there exists a $y \in Y$ such that $f(X) = \{y\}$. We denote by $i\colon \{y\} \to Y$ the inclusion. Let $n \in \mathbb{N}$. We consider the commutative diagram

$$\begin{tikzcd} & \{y\} \arrow[dr, hook, "i"] & \\ X \arrow[ur, "f"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

which gives rise to the commutative diagram

$$\begin{tikzcd} & \mathrm{H}_n(\{y\}) \arrow[dr, "i_*"] & \\ \mathrm{H}_n(X) \arrow[ur, "f_*"] \arrow[rr, "f_*"'] & & \mathrm{H}_n(Y). \end{tikzcd}$$

By Lemma 41.6 we know that $\mathrm{H}_n(\{y\}) = 0$. Thus we see that $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ factors through the trivial group, i.e. the map is the trivial map. On many occasions we will make use of this observation without referring to this discussion.

41.6. **Direct products and direct sums** $(*)$**.** In this short section we recall the definition of the direct product and the direct sum of a family of abelian groups.

Definition. Let $G_i, i \in I$ be a family of abelian groups. We define the *direct product* as

$$\prod_{i \in I} G_i \ := \ \Big\{\text{all maps } g\colon I \to \bigcup_{i \in I} G_i \text{ such that for all } i \in I \text{ we have } g(i) \in G_i\Big\}.$$

Given such a map we usually write $g_i := g(i)$ for $i \in I$ and we denote such a map by $\{g_i\}_{i \in I}$. The direct product is again a group with respect to the group structure

$$\begin{array}{rcl} \prod\limits_{i \in I} G_i \times \prod\limits_{i \in I} G_i & \to & \prod\limits_{i \in I} G_i \\ \big((a_i)_{i \in I}, (b_i)_{i \in I}\big) & \mapsto & (a_i + b_i)_{i \in I}. \end{array}$$

Furthermore we consider the *direct sum*

$$\bigoplus_{i \in I} G_i \ := \ \Big\{(g_i)_{i \in I} \in \prod_{i \in I} G_i \,\Big|\, \text{there exists only finitely many } i \in I \text{ with } g_i \neq e\Big\}.$$

The same way one can also define the direct product respectively the direct sum of rings, algebras, vector spaces, modules and so on.

Example. Let $I = \mathbb{N}$ and for each $i \in I = \mathbb{N}$ we pick $G_i = \mathbb{R}$. Then

$$\prod_{i \in \mathbb{N}} \mathbb{R} \quad = \quad \text{all real-valued sequences } i \mapsto a_i$$

and

$$\bigoplus_{i \in \mathbb{N}} \mathbb{R} \quad = \quad \text{all real-valued sequences with only finitely many non-zero terms.}$$

In general the direct product is "significantly larger" than the direct sum. For example the direct product $\prod_{i\in\mathbb{N}}\mathbb{Z}$ is uncountable while the direct sum $\bigoplus_{i\in\mathbb{N}}\mathbb{Z}$ is countable.[678]

Convention. Let $\{G_i\}_{i\in I}$ be family of abelian groups.

(1) For some element $g_j \in G_j$, following the discussion on page 580, we sometimes denote by g_j also the element in $\bigoplus_{i\in I} G_i$ that is given by $i \mapsto 0$ if $i \neq j$ and $j \mapsto g_j$. In particular we can view each G_j as a subgroup of $\bigoplus_{i\in I} G_i$.

(2) Given $i_1,\dots,i_l \in I$ and $g_{i_j} \in G_{i_j}$ we can use (1) to consider the corresponding element

$$g_{i_1}+\dots+g_{i_l} \in \bigoplus_{i\in I} G_i.$$

It follows immediately from the definitions that any element of $\bigoplus_{i\in I} G_i$ is of that form.

Finally we introduce the following notation.

Notation.

(I) Let $\{G_i\}_{i\in I}$ be a family of abelian groups and let H be an abelian group. Furthermore let $\{\varphi_i\colon G_i \to H\}_{i\in I}$ be a family of homomorphisms. We refer to

$$\begin{array}{rcl} \bigoplus\limits_{i\in I}\varphi_i\colon \bigoplus\limits_{i\in I} G_i & \to & H \\ g_{i_1}+\dots+g_{i_l} & \mapsto & \underbrace{\varphi_{i_1}(g_{i_1})}_{\in H}+\dots+\underbrace{\varphi_{i_l}(g_{i_l})}_{\in H} \end{array}$$

as the *direct sum of the homomorphisms* $\{\varphi_i\}_{i\in I}$.

(II) Let G be an abelian group, let $\{\mathrm{H}_i\}_{i\in I}$ be a *finite* family of abelian groups and let $\{\varphi_i\colon G \to \mathrm{H}_i\}_{i\in I}$ be a family of homomorphisms. We refer to

$$\begin{array}{rcl} \bigoplus\limits_{i\in I}\varphi_i\colon G & \to & \bigoplus\limits_{i\in I} \mathrm{H}_i \\ g & \mapsto & \sum\limits_{i\in I}\underbrace{\varphi_i(g)}_{\in \mathrm{H}_i} \end{array}$$

also as the *direct sum of the family of homomorphisms* $\{\varphi\}_{i\in I}$.

(III) Let $\{\varphi_i\colon G_i \to \mathrm{H}_i\}_{i\in I}$ be a *finite* family of homomorphisms between abelian groups. We refer to

$$\begin{array}{rcl} \bigoplus\limits_{i\in I}\varphi_i\colon \bigoplus\limits_{i\in I} G_i & \to & \bigoplus\limits_{i\in I} \mathrm{H}_i \\ \sum\limits_{i\in I} g_i & \mapsto & \sum\limits_{i\in I}\underbrace{\varphi_i(g_i)}_{\in \mathrm{H}_i} \end{array}$$

also as the *direct sum of the homomorphisms* $\{\varphi_i\}_{i\in I}$.

The notation is arguably somewhat ambiguous. But usually it should be clear from the context what we precisely mean when we write $\bigoplus_{i\in I}\varphi_i$. Sometimes, in desperation, we will refer to $\bigoplus_{i\in I}\varphi_i$ a direct sum of type (I), (II) or (III).

[678] Why is that the case?

For reference we jot down the following lemma which follows immediately from the definitions.

Lemma 41.12. $(*)$ *Let I be a finite set and let $\{\varphi_i\colon G_i \to \mathrm{H}_i\}_{i\in I}$ be a family of homomorphisms between abelian groups.*

(1) *Let F be an abelian group and let $\{\alpha\colon F \to G_i\}_{i\in I}$ be a family of homomorphisms. Then*
$$\Big(\bigoplus_{i\in I}\varphi_i\Big)\circ\Big(\bigoplus_{i\in I}\alpha_i\Big) \;=\; \bigoplus_{i\in I}\alpha_i\circ\varphi_i\colon F \;\to\; \bigoplus_{i\in I}G_i.$$
(2) *Let $\{\alpha_i\colon F_i \to G_i\}_{i\in I}$ be another family of homomorphisms between abelian groups. Then*
$$\Big(\bigoplus_{i\in I}\varphi_i\Big)\circ\Big(\bigoplus_{i\in I}\alpha_i\Big) \;=\; \bigoplus_{i\in I}\alpha_i\circ\varphi_i\colon \bigoplus_{i\in I}F_i \;\to\; \bigoplus_{i\in I}G_i.$$

41.7. **The homology groups of a direct sum.** Now we return to the study of chain complexes. Let C_* and D_* be two chain complexes. The same way as we introduced the direct sum of abelian groups we can also define the direct sum of chain complexes. The following lemma says that homology commutes with direct sums and direct products. The proof is a standard exercise in homological algebra, which oddly enough nobody ever feels like doing, see [**Weib94**, p. 5].

Lemma 41.13. *If $\{\mathrm{C}_a\}_{a\in A}$ is a family of chain complexes, then the obvious inclusion maps $\mathrm{C}_a \to \bigoplus_{a\in A}\mathrm{C}_a$ and $\mathrm{C}_a \to \prod_{a\in A}\mathrm{C}_a$ induce isomorphisms*
$$\bigoplus_{a\in A}\mathrm{H}_n(\mathrm{C}_a) \;\xrightarrow{\cong}\; \mathrm{H}_n\Big(\bigoplus_{a\in A}\mathrm{C}_a\Big) \quad \textit{and} \quad \prod_{a\in A}\mathrm{H}_n(\mathrm{C}_a) \;\xrightarrow{\cong}\; \mathrm{H}_n\Big(\prod_{a\in A}\mathrm{C}_a\Big).$$

Now we can formulate and prove the following lemma.

Lemma 41.14. *Let X be a topological space with path-components $\{X_a\}_{a\in A}$. Given $a\in A$ we denote by $i_a\colon X_a \to X$ the inclusion map. Then the map*
$$\bigoplus_{a\in A}i_a\colon \quad \bigoplus_{a\in A}\mathrm{C}_*(X_a) \;\to\; \mathrm{C}_*(X)$$
is an isomorphism of chain complexes. In particular for each $n\in\mathbb{N}_0$ the map
$$\bigoplus_{a\in A}i_{a*}\colon \quad \bigoplus_{a\in A}\mathrm{H}_n(X_a) \;\to\; \mathrm{H}_n(X)$$
is an isomorphism.

The lemma says in particular that the homology groups of a topological space X are the direct sum of the homology groups of the path-components of X. Put differently, this shows that in most cases it suffices to consider the homology groups of path-connected topological spaces.

Examples.

(1) One of our goals is to calculate the homology groups of the spheres. Now we can calculate the homology groups of the 0-dimensional sphere $S^0 = \{\pm 1\}$. More precisely,

by Lemmas 41.6 and 41.14 we have

$$\mathrm{H}_n(S^0) \cong \begin{cases} \mathbb{Z}^2, & \text{if } n=0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Later in the proof of Proposition 43.4 we will cleverly reduce the computation of the homology groups of all other spheres to this calculation.

(2) Let X be a topological space and let $\{U_a\}_{a\in A}$ be a family of disjoint open subsets of X. It follows fairly easily from Lemma 41.14 together with Lemma 2.68 (4) and (5) that the map

$$\bigoplus_{a\in A} i_{a*} \colon \bigoplus_{a\in A} \mathrm{H}_n(U_a) \to \mathrm{H}_n\Big(\bigcup_{a\in A} U_a\Big)$$

is an isomorphism.

Proof. Evidently the standard simplex Δ^n is path-connected. Thus it follows from Lemma 2.70 that the image of a map $\sigma\colon \Delta^n \to X$ is contained in a path-component X_a. This shows that

$$\bigsqcup_{a\in A} \text{singular } n\text{-simplices in } X_a \;=\; \text{singular } n\text{-simplices in } X.$$

It follows that the map

$$\begin{aligned} \bigoplus_{a\in A} i_a \colon \bigoplus_{a\in A} \mathrm{C}_n(X_a) &\to \mathrm{C}_n(X) \\ \Big(\sum_{i=1}^k c_i(\sigma\colon \Delta^n \to X_{a_i})\Big) &\mapsto \Big(\sum_{i=1}^k c_i(\sigma\colon \Delta^n \to X_{a_i})\Big) \end{aligned}$$

is an isomorphism of free abelian groups. From the definitions it follows immediately that this a chain map. The second statement of the lemma now follows from Lemma 41.13. ■

The following corollary is an immediate consequence of Proposition 41.5 and Lemma 41.13.

Corollary 41.15. *Let X be a topological space. As on page 135 we denote by $\pi_0(X)$ the set of path-components of X. Then*

$$\mathrm{H}_0(X) \cong \mathbb{Z}^{(\pi_0(X))}.$$

In particular, if X consists of n path-components with $n \in \mathbb{N}0$, then

$$\mathrm{H}_0(X) \cong \mathbb{Z}^n.$$

Exercises for Chapter 41.

Exercise 41.1. Let $n \in \mathbb{N}$. As in Lemma 41.1 we consider

$$◺_n := \Big\{(x_1,\dots,x_n) \in [0,1]^n \,\Big|\, \sum_{i=1}^n x_i \in [0,1]\Big\} \subset \mathbb{R}^n.$$

We denote by $\Phi\colon \Delta^n \to ◺_n$ the obvious projection map. We pick any $P \in \mathring{◺}_n$. Next we consider the maps

$$\begin{aligned} f_P\colon \mathring{◺}_n &\to \mathbb{R}^n \\ (x_1,\dots,x_n) &\mapsto \tfrac{((x_1,\dots,x_n)-P)}{x_1\cdot\dots\cdot x_n\cdot(1-x_1-\dots-x_n)} \end{aligned} \quad\text{and}\quad \begin{aligned} g\colon \mathbb{R}^n &\to B^n \\ (x_1,\dots,x_n) &\mapsto \tfrac{x_1^2+\dots+x_n^2}{1+x_1^2+\dots+x_n^2}\cdot(x_1,\dots,x_n) \end{aligned}$$

Note that f_P is defined in such a way that it sends rays emanating from P to rays emanating from the origin.

(a) Show that $f_P\colon \mathring{\triangle}_n \to \mathbb{R}^n$ is an orientation-preserving diffeomorphism.
(b) Show that $g\colon \mathbb{R}^n \to B^n$ is an orientation-preserving diffeomorphism.
Note that for the most part we had solved this problem on page 295.
(c) Show that $g \circ f_P\colon \mathring{\triangle}_n \to B^n$ extends to a homeomorphism $\Theta\colon \triangle_n \to \overline{B}^n$.
(d) Let $\Xi\colon \triangle_n \to \overline{B}^n$ be the homeomorphism obtained from Proposition 2.53 (2) applied to P. Show that Θ and Ξ are homotopic rel P.

Remark. The composition $\Psi \circ \Phi\colon \Delta^n \to \overline{B}^n$ is now a homeomorphism such that the restriction $\mathring{\Delta}^n \to B^n$ is a diffeomorphism. Furthermore this diffeomorphism is orientation-preserving if and only if n is even. Here we use the definitions on page 1075 to view $\mathring{\Delta}^n$ as an oriented smooth manifold.

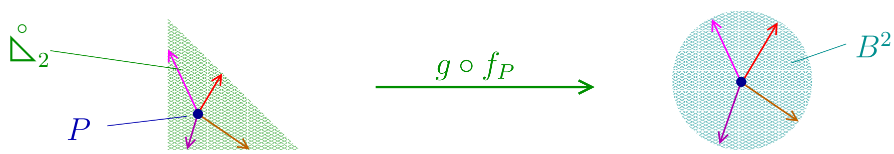


FIGURE 706. Illustration for Exercise 41.1.

Exercise 41.2.

(a) Let X be a topological space and let $x \in X$. For each $n \in \mathbb{N}_0$ denote by σ_n the singular n-simplex $\sigma_n\colon \Delta^n \to X$ which is defined by $\sigma_n(P) = x$ for all $P \in \Delta^n$. Determine $\partial\sigma_n \in \mathrm{C}_{n-1}(X)$.
(b) Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex. Show that if σ is a constant map, then σ is null-homologous.

Exercise 41.3. Let X be a topological space which consists of a single point. Show that the following holds:

$$\mathrm{H}_n(X) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Exercise 41.4.

(a) Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex. We consider the singular 1-simplices

$$\begin{array}{rcl} \sigma_1\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(\tfrac{1}{2}t, 1-\tfrac{1}{2}t) \end{array} \quad \text{and} \quad \begin{array}{rcl} \sigma_2\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & (\tfrac{1}{2}+\tfrac{1}{2}t, \tfrac{1}{2}-\tfrac{1}{2}t). \end{array}$$

Show that σ and $\sigma_1 + \sigma_2$ are homologous.
(b) Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex. We consider the singular 1-simplex

$$\begin{array}{rcl} \overline{\sigma}\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(1-t, t). \end{array}$$

Show that $\sigma + \overline{\sigma}$ is a 1-cycle and that this 1-cycle is null-homologous.

We refer to Figure 701 for an illustration of the definitions.

Exercise 41.5. Let $\Sigma = E_8/\sim$ be the surface of genus 2. Provide an example of a 2-cycle $\sum_{i=1}^{m} a_i \cdot \sigma_i \in \mathrm{C}_2(\Sigma)$ with the following three properties:

(1) At least one a_i is non-zero.
(2) The images of the singular 2-simplices cover the whole surface.
(3) The images of the singular 2-simplices overlap only in "1-dimensional" subsets.

Remark. Note that on page 1083 we provided an example of such a 2-cycle for the torus $\Sigma = ([0,1] \times [0,1])/\sim$.

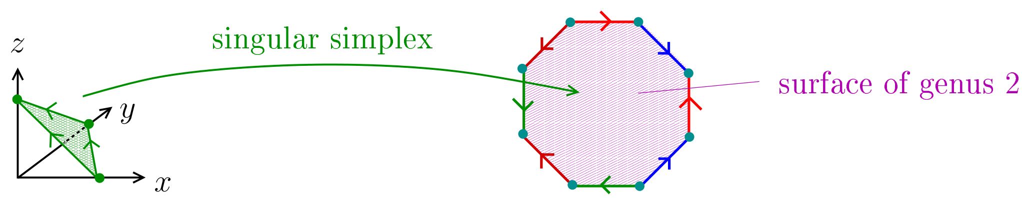


FIGURE 707. Illustration of Exercise 41.5.

Exercise 41.6.

(a) Try to find "interesting" elements in the following homology groups:
 (i) $\mathrm{H}_2(S^2)$,
 (ii) H_2(surface Σ of genus 2),
 (iii) H_2(Klein bottle K) and
 (iv) $\mathrm{H}_2(\mathbb{R}\mathrm{P}^2)$.
 More precisely, try to find 2-cycles which have the properties (1), (2) and (3) from Exercise 41.5.
(b) Make a guess for which surfaces you expect H_2 to be zero and for which it is non-zero.

Exercise 41.7. Let $n \in \mathbb{N}$.

(a) Provide an example of an n-cycle $\sum_{i=1}^{m} a_i \cdot \sigma_i \in \mathrm{C}_n(S^n)$ which has the properties (1), (2) and (3) from Exercise 41.5.
(b) Make an educated guess what the homology groups of S^n might be.

Exercise 41.8. Provide an example of a chain complex $(\mathrm{C}_*, \partial_*)$ such that all C_* are finitely generated free abelian groups and such that $\mathrm{H}_i(C_*) = 0$ for all even $i \in \mathbb{N}_0$ and such that $\mathrm{H}_i(C_*) = \mathbb{Z}_2$ for all odd $i \in \mathbb{N}_0$.

Exercise 41.9.

(a) Determine the isomorphism types of the homology groups of the chain complex

$$0 \to \mathrm{C}_1 = \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 3 & 7 \\ 4 & 2 \end{pmatrix}} \mathrm{C}_0 = \mathbb{Z}^2 \to 0.$$

(b) Determine the isomorphism types of the homology groups of the chain complex

$$0 \to \mathrm{C}_2 = \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 2 & 5 \\ 4 & 10 \end{pmatrix}} \mathrm{C}_1 = \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} -2 & 1 \end{pmatrix}} \mathrm{C}_0 = \mathbb{Z} \to 0.$$

(Why is this a chain complex?)

Exercise 41.10. Given a topological space X and $n \in \mathbb{N}_0$ we consider the map

$$\begin{array}{rcl} \epsilon\colon \mathrm{C}_n(X) & \to & \mathbb{Z} \\ \sum\limits_{i=1}^{k} a_i\sigma_i & \mapsto & \sum\limits_{i=1}^{k} a_i. \end{array}$$

(a) Show that the following diagram commutes

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \mathrm{C}_3(X) & \xrightarrow{\partial_3} & \mathrm{C}_2(X) & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_2(X) & \xrightarrow{\partial_0} & 0 \\ & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon} & & \\ \dots & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

(b) Show that if n is even, then for every cycle $\sigma \in \mathrm{C}_n(X)$ we have $\epsilon(\sigma) = 0$.

Exercise 41.11. Let $f\colon C \to B$ be a homomorphism between abelian groups and let A be a subgroup of B with the following two properties:

(1) We have $f(C) \cap A = 0$.
(2) The subgroups $f(C)$ and A generate B.

Show that the map

$$\begin{array}{rcl} A & \to & \operatorname{coker}(f) = B/\operatorname{im}(f) \\ a & \mapsto & [a] \end{array}$$

is an isomorphism.

Exercise 41.12. Suppose we are chain complex of the following form

$$D_* \;=\; \dots \to C_{k+2} \xrightarrow{\partial_{k+2}} A_{k+1} \oplus B_{k+1} \xrightarrow{\partial_{k+1}} A_k \oplus B_k \xrightarrow{\partial_k} C_{k-1} \to \dots$$

Given $m \in \{k, k+1\}$ we denote by $i_m\colon A_m \to A_m \oplus B_m$ and $j_m\colon B_m \to A_m \oplus B_m$ the natural inclusions and we denote by $p_m\colon A_m \oplus B_m \to A_m$ and $q_m\colon A_m \oplus B_m \to B_m$ the natural projections. We suppose that $p_k \circ \partial_{k+1} \circ i_{k+1}\colon A_{k+1} \to A_k$ is an isomorphism. Show that we can "cancel" A_{k+1} and A_k from the exact sequence without changing the isomorphism type of the homology groups. More precisely, consider the following chain complex:

$$\widetilde{D}_* \;=\; \dots \to C_{k+2} \xrightarrow{p_{k+1}\circ\partial_{k+2}} B_{k+1} \xrightarrow{q_k\circ\partial_{k+1}\circ j_{k+1}} B_k \xrightarrow{\partial_k\circ j_k} C_{k-1} \to \dots$$

Show that for each $i \in \mathbb{N}_0$ there exists an isomorphism $\mathrm{H}_i(D_*) \to \mathrm{H}_i(\widetilde{D}_*)$.

Remark. This implies in particular that D_* is acyclic if and only if $\widetilde{D}_*$ is acyclic.

42. Homology and homotopies

It was not overly difficult to introduce the homology groups of a topological space. We also proved several straightforward properties of homology groups. But so far we are unable to calculate any interesting homology group. In this section we will show in particular that homotopy equivalent topological spaces have isomorphic homology groups. That does not yet lead to the calculation of non-trivial homology groups, but this result will play an important role in the calculation of the homology groups of spheres that we will give in Proposition 43.4.

42.1. **Chain homotopies.**

In this section we introduce a criterion that ensures that two chain maps $f, g\colon C_* \to D_*$ induce the same maps on the corresponding homology groups.

Definition. Let $f, g\colon C_* \to D_*$ be chain maps between chain complexes C_* and D_*. A *chain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between f and g* consists of family $\{P_n\colon \mathrm{C}_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ such that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; f_n - g_n.$$

If there exists a chain homotopy between chain maps f and g, then we say that f and g are *chain homotopic* and we write $f \simeq g$.

For the definition of a chain homotopy and the subsequent discussion of chain homotopies it is sometimes helpful to keep in mind the following diagram that combines all objects involved in the definition:

$$\begin{tikzcd}
{} \arrow[r, "\partial_{n+2}"] & \mathrm{C}_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_*{,}g_*"] & \mathrm{C}_n \arrow[r, "\partial_n"] \arrow[d, "f_*{,}g_*"] \arrow[dl, "P_n"] & \mathrm{C}_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_*{,}g_*"] \arrow[dl, "P_{n-1}"] & {} \\
{} \arrow[r, "\partial_{n+2}"] & D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r, "\partial_{n-1}"] & {}
\end{tikzcd}$$

The following basic lemma will be proved in Exercise 42.1

Lemma 42.1. *Let C_* and D_* be two chain complexes.*

(1) *Chain homotopy defines an equivalence relation on the set of all chain maps from C_* to D_*. In particular, if $f, g, h\colon C_* \to D_*$ are chain maps with $f \simeq g$ and $g \simeq h$, then $f \simeq h$.*
(2) *Let $f, g\colon C_* \to D_*$ be chain maps, and let $e\colon B_* \to \mathrm{C}_*$ and $h\colon D_* \to E_*$ be two other chain maps. Then*

$$f \simeq g \quad\Longrightarrow\quad f \circ e \simeq g \circ e \quad \text{and} \quad h \circ f \simeq h \circ g.$$

Now we can formulate the following lemma.

Lemma 42.2. *Let $f, g\colon C_* \to D_*$ be two chain maps between chain complexes. If f and g are chain homotopic, then $f_* = g_*\colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ for all $n \in \mathbb{N}_0$.*

Proof. Let $P = \{P_n\}_{n\in\mathbb{N}_0}$ be a chain homotopy between f and g. Then for any cycle $z \in C_n$ we get the equalities

$$f(z) - g(z) \;=\; (f-g)(z) \;=\; (\partial_{n+1}P_n + P_{n-1}\partial_n)(z) \;=\; (\partial_{n+1}P_n)(z) + P_{n-1}\underbrace{\partial_n z}_{=0} \;=\; \partial P_n z,$$

i.e. $f(z)$ and $g(z)$ are homologous. ■

Definition. Let C_* and D_* be two chain complexes.

(1) A chain map $f\colon \mathrm{C}_* \to D_*$ is called a *chain homotopy equivalence between C_* and D_**, if it admits a chain homotopy inverse, i.e. a chain map $g\colon D_* \to \mathrm{C}_*$ with $g \circ f \simeq \mathrm{id}_{C_*}$ and $f \circ g \simeq \mathrm{id}_{D_*}$.

(2) If there exists a chain homotopy equivalence between C_* and D_*, then we say that the chain complexes C_* and D_* are *chain homotopy equivalent* and we write $C_* \simeq D_*$.

Remark. It is again straightforward to show that "homotopy equivalence" of chain complexes is indeed an equivalence relation.

For later we record the following straightforward corollary to Lemma 42.2.

Corollary 42.3. *If $f\colon C_* \to D_*$ is a chain homotopy equivalence between chain complexes, then the induced maps $f_*\colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ are isomorphisms.*

We also record the following useful little lemma.

Lemma 42.4. $(*)$ *Let $(\mathrm{C}_*, \partial_*)$ be a chain complex. If there exist "diagonal" maps P_n, $n \in \mathbb{Z}_{\geq -1}$, to obtain a diagram*[679]

$$\begin{tikzcd}
\dots \arrow[r] & C_3 \arrow[r, "\partial_3"] \arrow[d, "\mathrm{id}"] \arrow[dl, "P_3"'] & C_2 \arrow[r, "\partial_2"] \arrow[d, "\mathrm{id}"] \arrow[dl, "P_2"'] & C_1 \arrow[r, "\partial_1"] \arrow[d, "\mathrm{id}"] \arrow[dl, "P_1"'] & C_0 \arrow[r, "\partial_0"] \arrow[d, "\mathrm{id}"] \arrow[dl, "P_0"'] & 0 \\
\dots \arrow[r] & C_3 \arrow[r, "\partial_3"'] & C_2 \arrow[r, "\partial_2"'] & C_1 \arrow[r, "\partial_1"'] & C_0 \arrow[r, "\partial_0"'] & 0
\end{tikzcd}$$

such that for every $n \in \mathbb{N}_0$ we have the equality

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; \mathrm{id}_{\mathrm{C}_n},$$

then all homology groups of $(\mathrm{C}_*, \partial_*)$ vanish.

Proof $(*)$. Let D_* be the zero chain complex and for each $n \in \mathbb{N}_0$ let $f_n\colon \mathrm{C}_n \to D_n = 0$ be the trivial map. Since the homology groups of D_* it suffices, by Corollary 42.3, to show that f_* is a chain homotopy equivalence. For each $n \in \mathbb{N}_0$ let $g_n\colon D_n \to C_n$ be the trivial map. Evidently $f_n \circ g_n = \mathrm{id}_{D_*}$. Thus it remains to show that $g_* \circ f_* \simeq \mathrm{id}_{C_*}$. In other words we need to show that $0 \simeq \mathrm{id}_{C_*}$. But that is exactly what the maps P_n do. ■

42.2. **Homology and homotopic maps.** The name *chain homotopy* introduced in the previous section suggests that this concept is related to the notion of a homotopy between two topological spaces. As we will see shortly, this is indeed the case. For the reader's

[679]The diagram has no particular role, except that it shows the domain and the target of the maps P_n and that it is convenient for keeping track of the various maps.

convenience we recall that if $f, g\colon X \to Y$ are two maps between topological spaces, then a *homotopy between the maps f and g* is a map

$$F\colon X \times [0,1] \to Y,$$

such that

$$F(x,0) \;=\; f(x) \text{ and } F(x,1) \;=\; g(x) \text{ for all } x \in X.$$

If there exists a homotopy between f and g, then we say that f and g are *homotopic* and we write $f \simeq g$.

Proposition 42.5. *Let X and Y be two topological spaces and let $f, g\colon X \to Y$ be two maps. If f and g are homotopic, then the following hold:*

(1) *the induced maps f_* and g_* from $\mathrm{C}_*(X)$ to $\mathrm{C}_*(Y)$ are chain homotopic,*
(2) *for each $n \in \mathbb{N}_0$ we have $f_* = g_*\colon\ \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$.*

Remark. The proposition can be stated more succinctly as follows:

$$f \simeq g\colon X \to Y \quad\Longrightarrow\quad f_* \simeq g_*\colon\ \mathrm{C}_*(X) \to \mathrm{C}_*(Y) \quad\Longrightarrow\quad f_* = g_*\colon\ \mathrm{H}_n(X) \to \mathrm{H}_n(Y).$$

We start out with a discussion of the key idea behind the proof of Proposition 42.5. Afterwards we will fill in the details. So let $f, g\colon X \to Y$ be two maps between topological spaces and let $F\colon X \times [0,1] \to Y$ be a homotopy between the maps f and g. Our goal is to find a chain homotopy between f_* and g_*. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. We need to find $P(\sigma) \in \mathrm{C}_{n+1}(Y)$ and $P(\partial\sigma) \in \mathrm{C}_n(Y)$ such that

$$\partial P(\sigma) \;=\; -P(\partial\sigma) + f_*(\sigma) - g_*(\sigma).$$

Note that $F \circ (\sigma \times \mathrm{id}_{[0,1]})$ defines a map $\Delta^n \times [0,1] \to Y$. Now we can rewrite the above desired equality as follows:

$$\partial P(\sigma) \;=\; -P(\partial\sigma) + \begin{array}{c}\text{map } F\circ(\sigma\times \mathrm{id}_{[0,1]})\\ \text{restricted to } \underbrace{\Delta^n \times \{0\}}_{=\Delta^n}\end{array} - \begin{array}{c}\text{map } F\circ(\sigma\times \mathrm{id}_{[0,1]})\\ \text{restricted to } \underbrace{\Delta^n \times \{1\}}_{=\Delta^n}\end{array}.$$

To get an idea we consider Figure 708. We make the following trivial observation: we have

$$\partial(\Delta^n \times [0,1]) \;=\; \partial\Delta^n \times [0,1] \;\cup\; \Delta^n \times \{0\} \;\cup\; \Delta^n \times \{1\}.$$

Comparing the desired equality of maps and the equality of sets suggests that each map $P_k\colon\ \mathrm{C}_k(X) \to \mathrm{C}_{k+1}(Y)$ should take the role of replacing a singular k-simplex $\mu\colon \Delta^k \to X$ by the map $F \circ (\mu \times \mathrm{id}_{[0,1]})\colon \Delta^k \times [0,1] \to Y$. This is a good idea, but the problem is that $\Delta^k \times [0,1]$ is not a $(k+1)$-simplex. The idea now is to write $\Delta^k \times [0,1]$ as the union of $(k+1)$-simplices. In Figure 709 we see for example that we can view $\Delta^1 \times [0,1]$ as the union of the triangles with the vertices v_0, w_0, w_1 respectively v_0, v_1, w_1.

To generalize this observation we have to introduce some more notation.

Notation. Let $n \in \mathbb{N}_0$.

(1) For $i = 0, \ldots, n$ we write

$$v_i \;:=\; (0,\ldots,1,0,\ldots,0) \times \{0\} \;\in\; \mathbb{R}^{n+1} \times \{0\},$$

and

$$w_i \;:=\; (0,\ldots,1,0,\ldots,0) \times \{1\} \;\in\; \mathbb{R}^{n+1} \times \{1\}.$$

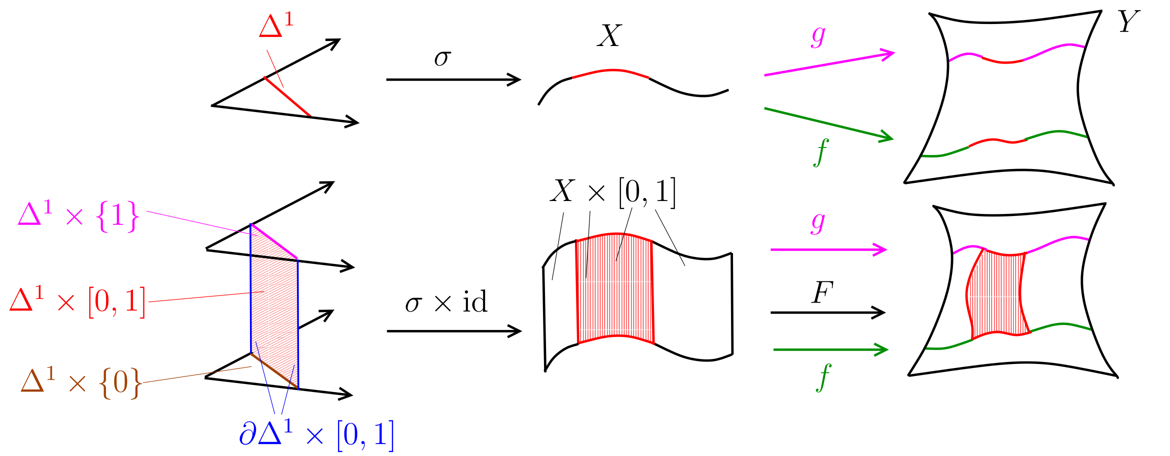


FIGURE 708

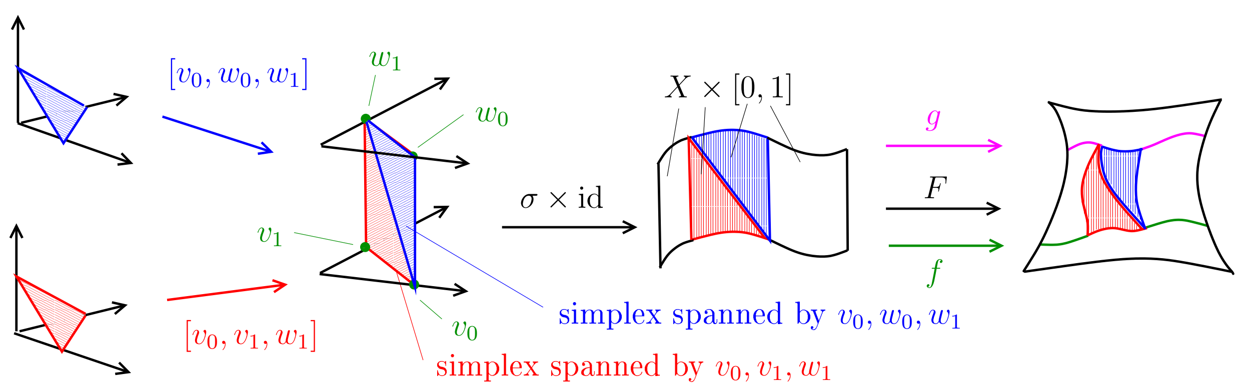


FIGURE 709

The points $v_0, \ldots, v_n$ thus span the simplex $\Delta^n \times 0$ and the points $w_0, \ldots, w_n$ span the simplex $\Delta^n \times 1$. Furthermore for each i the points $v_0, \ldots, v_i, w_i, \ldots, w_n$ span an $(n+1)$-simplex that is a subset of $\Delta^n \times [0,1]$.[680]

(2) For $a_0, \ldots, a_k \in \Delta^n \times [0,1]$ we consider, as on page 1078, the map

$$\begin{array}{rcl} [a_0, \ldots, a_k] \colon \Delta^k & \to & \Delta^n \times [0,1] \\ (\lambda_0, \ldots, \lambda_k) & \mapsto & \sum\limits_{j=0}^{k} \lambda_j a_j. \end{array}$$

The map $[a_0, \ldots, a_k]$ thus sends the standard k-simplex Δ^k to the simplex spanned by $a_0, \ldots, a_k$ in $\Delta^n \times [0,1]$.

(3) We define

$$\Omega_n \;:=\; \sum_{j=0}^{n} (-1)^j \cdot \underbrace{[v_0, \ldots, v_j, w_j, \ldots, w_n]}_{\Delta^{n+1} \to \Delta^n \times [0,1]} \;\in\; \mathrm{C}_{n+1}(\Delta^n \times [0,1]).$$

[680]It is a straightforward exercise to show that the union of these $(n+1)$-dimensional simplices is precisely $\Delta^n \times [0,1]$. We will not make use of this fact.

In the following lemma we compute the boundary of $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0,1])$. The statement might not be particularly visually appealing, but it will turn out to be very useful.

Lemma 42.6. *Let $n \in \mathbb{N}_0$. Given $t \in \{0,1\}$ we write*

$$\begin{array}{rcl} \eta_t\colon \Delta^n & \to & \Delta^n \times [0,1] \\ x & \mapsto & (x,t). \end{array}$$

With this notation we have

$$\partial_{n+1}(\Omega_n) + \sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*\big(\underbrace{\Omega_{n-1}}_{\in \mathrm{C}_n(\Delta^{n-1}\times[0,1])} \big) \;=\; \eta_1 - \eta_0 \;\in\; \mathrm{C}_n(\Delta^n \times [0,1]).$$

Proof. The proof of the lemma is basically an elementary, albeit slightly confusing calculation. First note that

$$\begin{array}{rcl} \partial\Omega_n & = & \partial\Big(\sum\limits_{j=0}^{n} (-1)^j \cdot [v_0,\dots,v_j,w_j,\dots,w_n] \Big) \\ & = & \sum\limits_{k=0}^{n+1} (-1)^k \cdot \sum\limits_{j=0}^{n} (-1)^j \cdot [v_0,\dots,v_j,w_j,\dots,w_n] \circ i_k^{n+1}. \end{array}$$

It is straightforward to see that for any $k \in \{0,\dots,n+1\}$ we have[681]

$$[v_0,\dots,v_j,w_j,\dots,w_n] \circ i_k^{n+1} \;=\; \begin{cases} [v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n], & \text{if } k \le j, \\ [v_0,\dots,v_j,w_j,\dots,\widehat{w_{k-1}},\dots,w_n], & \text{if } k > j. \end{cases}$$

Thus it follows that[682]

$$\begin{array}{rcl} \partial\Omega_n & = & \sum\limits_{0\le k\le j\le n} (-1)^k \cdot (-1)^j \cdot [v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n] \\ & & + \sum\limits_{n+1\ge k>j\ge 0} (-1)^k \cdot (-1)^j \cdot [v_0,\dots,v_j,w_j,\dots,\widehat{w_{k-1}},\dots,w_n]. \end{array}$$

[681] The first case, $k \le j$ is slightly easier: one has to figure out, where the h-th vertex e_h of the standard n-simplex gets sent to. If $h < k$, then i_k^{n+1} sends the h-th vertex again to the h-th vertex, and $[v_0,\dots,v_j,w_j,\dots,w_n]$ sends this vertex to v_h. If $h \ge k$, then i_k^{n+1} sends the h-th vertex again to the $(h+1)$-th vertex, and $[v_0,\dots,v_j,w_j,\dots,w_n]$ sends this vertex to v_{h+1}. If $k \le j$ this corresponds just to the map $[v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n]$.
The second case, $k > j$ is proved precisely the same way.

[682] Here we use the usual notation that $\widehat{v_k}$ means that we do not plug in $\widehat{v_k}$, the same applies to $\widehat{w_{k-1}}$.

Now we split the first sum into the cases $k=j$ and $k<j$ and we split the second sum into the cases $k=j+1$ and $k>j+1$. This leads us to the following equality:

$$\begin{aligned}\partial\Omega_n \; = \;& \sum_{k=0}^{n}(-1)^{2k}\cdot[v_0,\ldots,v_{k-1},w_k,\ldots,w_n] \\ &+\sum_{k=1}^{n+1}(-1)^{2k+1}\cdot[v_0,\ldots,v_{k-1},w_k,\ldots,w_n] \\ &+\sum_{0\le k<j\le n}(-1)^k\cdot(-1)^j\cdot[v_0,\ldots,\widehat{v_k},\ldots,v_j,w_j,\ldots,w_n] \\ &+\sum_{n+1\ge k>j+1\ge 1}(-1)^k\cdot(-1)^j\cdot[v_0,\ldots,v_j,w_j,\ldots,\widehat{w_{k-1}},\ldots,w_n].\end{aligned}$$

Next we consider the first two sums. The summands cancel each other, except for the summands for $k=0$ and $k=n+1$. Thus we obtain that

$$(*)\qquad\begin{aligned}\partial\Omega_n \; = \;& [w_0,\ldots,w_n]-[v_0,\ldots,v_n] \\ &+\sum_{0\le k<j\le n}(-1)^k\cdot(-1)^j\cdot[v_0,\ldots,\widehat{v_k},\ldots,v_j,w_j,\ldots,w_n] \\ &+\sum_{n+1\ge k>j+1\ge 1}(-1)^k\cdot(-1)^j\cdot[v_0,\ldots,v_j,w_j,\ldots,\widehat{w_k},\ldots,w_n].\end{aligned}$$

The first summand is just η_1 and the second is $-\eta_0$. It remains to show that the remaining summand corresponds to the remaining term in the formula. Thus we continue our proof with the discussion of the one term in the statement of the lemma that we have not studied yet. We have[683]

$$\sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times\mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \underset{\substack{\uparrow\\ \text{definition of } (i_j^n\times\mathrm{id}_{[0,1]})_* \text{ and substitution } r=j}}{=} \sum_{r=0}^{n}\sum_{s=0}^{n-1}(-1)^r\cdot(-1)^s\cdot(i_r^n\times\mathrm{id}_{[0,1]})\circ[v_0,\ldots,v_s,w_s,\ldots,w_{n-1}].$$

It follows from an elementary (but arguably confusing) argument, using only the definitions, that we have the following equalities of maps $\Delta^n\to\Delta^n\times[0,1]$:

$$(i_r^n\times\mathrm{id}_{[0,1]})\circ[v_0,\ldots,v_s,w_s,\ldots,w_{n-1}] \;=\; \begin{cases}[v_0,\ldots,\widehat{v_r},\ldots,v_{s+1},w_{s+1},\ldots,w_n], & \text{if } r\le s,\\[1ex] [v_0,\ldots,v_s,w_s,\ldots,\widehat{w_{r+1}},\ldots,w_n], & \text{if } r>s.\end{cases}$$

Thus we see that

$$\begin{aligned}\sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times\mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \;=\;& \sum_{0\le r\le s\le n-1}(-1)^s\cdot(-1)^r\cdot[v_0,\ldots,\widehat{v_r},\ldots,v_{s+1},w_{s+1},\ldots,w_n] \\ &+\sum_{n\ge r>s\ge 0}(-1)^s\cdot(-1)^r\cdot[v_0,\ldots,v_s,w_s,\ldots,\widehat{w_{r+1}},\ldots,w_n].\end{aligned}$$

[683] Here $v_0,\ldots,v_{n-1},w_0,\ldots,w_{n-1}$ are now, unfortunately somewhat confusingly, the corresponding vertices of $\Delta^{n-1}\times[0,1]$.

In the first term we perform the substitutions $r = k$ and $s = j - 1$ and in the second term we perform the substitutions $r = k - 1$, $s = j$. We obtain the following equality:

$$\begin{aligned}\sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times \mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \;&=\; \sum_{0\le k\le j-1\le n-1}(-1)^{j-1}\cdot(-1)^k\cdot[v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n]\\ &\quad+\sum_{n\ge k-1>j\ge 0}(-1)^j\cdot(-1)^{k-1}\cdot[v_0,\dots,v_j,w_j,\dots,\widehat{w_k},\dots,w_n].\end{aligned}$$

We see immediately that these terms are just the negatives of the corresponding terms in the above equation $(*)$. ■

The following lemma implies in particular the first statement of Proposition 42.5. The second statement of Proposition 42.5 regarding the induced maps on homology follows immediately from the first statement together with Lemma 42.2. In other words, the following lemma completes the proof of Proposition 42.5.

Lemma 42.7. *Let X be a topological space and let $F\colon X\times[0,1]\to X$ be a homotopy from a map f to a map g. Given any $n\in\mathbb{N}_0$ we define*

$$P_n\colon\ \mathrm{C}_n(X)\ \to\ \mathrm{C}_{n+1}(Y)$$

to be the map that is induced by

$$(\sigma\colon\Delta^n\to X)\ \mapsto\ (F\circ(\sigma\times\mathrm{id}_{[0,1]}))_*(\Omega_n).$$

Then for any $n\in\mathbb{N}_0$ we have

$$\partial P_n + P_{n-1}\partial \;=\; g_* - f_*$$

as maps from $\mathrm{C}_n(X)$ to $\mathrm{C}_n(Y)$.

Proof. Let X be a topological space and let $F\colon X\times[0,1]\to X$ be a homotopy from a map f to a map g. Furthermore let $\sigma\colon\Delta^n\to X$ be a singular k-simplex. Then

$$\begin{aligned}(\partial P_n+P_{n-1}\partial)(\sigma) \;&=\; \partial(P_n(\sigma))+P_{n-1}\Big(\sum_{j=0}^{n}(-1)^j\cdot\sigma\circ i_j^n\Big) \;=\; \partial(P_n(\sigma))+\sum_{j=0}^{n}(-1)^j\cdot P_{n-1}(\sigma\circ i_j^n)\\ &=\; \partial((F\circ(\sigma\times\mathrm{id}_{[0,1]}))_*(\Omega_n))+\sum_{j=0}^{n}(-1)^j\cdot(F\circ(\sigma\circ_j^n\times\mathrm{id}_{[0,1]}))_*(\Omega_{n-1})\\ &\underset{\substack{\uparrow\\ \text{by Lemma 41.9 (1) and since } (\sigma\times\mathrm{id})\circ(i_j^n\times\mathrm{id})=(\sigma\circ i_j^n)\times\mathrm{id}}}{=}\; (F\circ(\sigma\times\mathrm{id}_{[0,1]}))_*\Big(\partial(\Omega_n))+\sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times\mathrm{id}_{[0,1]})_*(\Omega_{n-1})\Big)\\ &\underset{\substack{\uparrow\\ \text{Lemma 42.6}}}{=}\; (F\circ(\sigma\times\mathrm{id}_{[0,1]}))_*(\eta_1-\eta_0) \underset{\substack{\uparrow\\ \text{since } F \text{ is a homotopy from } f \text{ to } g}}{=} f\circ\sigma-g\circ\sigma \;=\; g_*(\sigma)-f_*(\sigma).\end{aligned}$$

■

We recall the following definition from page 546.

Definition. Let X and Y be topological spaces.

(1) A map $f\colon X\to Y$ is a *homotopy equivalence between X and Y*, if there exists a *homotopy inverse* to f, i.e. if there exists a continuous map $g\colon Y\to X$ with $g\circ f\simeq \mathrm{id}_X$ and $f\circ g\simeq\mathrm{id}_Y$.

(2) If there exists a homotopy equivalence between X and Y, then we say that the topological spaces X and Y are *homotopy equivalent* and we write $X \simeq Y$.
(3) We say X is *contractible*, if X is homotopy equivalent to $Y = \{*\}$, i.e. if X is homotopy equivalent to a topological space which consists of precisely one point.

Now we obtain the following corollary to Proposition 42.5. This corollary mirrors the fact, proved in Propositions 18.16 (2) and 40.7 (2), that homotopy equivalences induce isomorphisms of fundamental groups and higher homotopy groups.

Corollary 42.8. *Let X and be Y topological spaces and let $n \in \mathbb{N}_0$.*

(1) *If $f\colon X \to Y$ is a homotopy equivalence (e.g. a homeomorphism), then the induced map $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism.*
(2) *If X and Y are homotopy equivalent, then $\mathrm{H}_n(X) \cong \mathrm{H}_n(Y)$.*
(3) *If X is contractible, then $\mathrm{H}_0(X) \cong \mathbb{Z}$ and $\mathrm{H}_n(X) = 0$ for $n \geq 1$.*
(4) *If X is a deformation retract of Y, then the inclusion induced map $\mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism.*

Proof.

(1) Let $f\colon X \to Y$ be a homotopy equivalence and let $g\colon Y \to X$ be a homotopy inverse of f, i.e. g is a map such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. It follows from Proposition 42.5 and the functoriality of homology that $f_* \circ g_* = (f \circ g)_* = \mathrm{id}_{\mathrm{H}_n(Y)}$ and similarly $g_* \circ f_* = \mathrm{id}_{\mathrm{H}_n(X)}$. But that means precisely that $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism.
(2) This statement follows immediately from (1).
(3) This statement follows immediately from (2) and Lemma 41.6.
(4) This statement follows immediately from (1) and Lemma 18.14. ■

Examples.

(1) On page 548 we saw that every convex subset of $\mathbb{R}^n$, e.g. the open disk B^n, the closed disk $\overline{B}^n$, any simplex or $\mathbb{R}^n$, is contractible. Thus it follows from Corollary 42.8, that
$$\mathrm{H}_i(\text{convex subset of } \mathbb{R}^n) \;\cong\; \mathrm{H}_i(B^n) \;\cong\; \mathrm{H}_i(\overline{B}^n) \;\cong\; \mathrm{H}_i(\mathbb{R}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$
(2) In Exercise 42.2 we will use Corollary 42.8 to show that if A is a compact convex subset of $\mathbb{R}^n$ and if x is a point in the interior of A, then the inclusion $\mathbb{R}^n \setminus A \to \mathbb{R}^n \setminus \{x\}$ induces isomorphisms $\mathrm{H}_i(\mathbb{R}^n \setminus A) \to \mathrm{H}_i(\mathbb{R}^n \setminus \{x\})$ for all i.

Remark. Given a map $f\colon X \to Y$ between topological spaces we denote by $[f]$ the homotopy equivalence class of f, and we denote by $[X, Y]$ the set of equivalence classes of maps from X to Y. As we pointed out on page 544,
$$\begin{aligned} \mathrm{Ob}(\mathcal{HomTop}) &:= \text{ all topological spaces}, \\ \mathrm{Mor}(X, Y) &:= [X, Y] \end{aligned}$$
together with the composition
$$\begin{aligned} \mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) &\to \mathrm{Mor}(X, Z) \\ ([f], [g]) &\mapsto [g \circ f] \end{aligned}$$

defines a category $\mathcal{HomTop}$, called the *homotopy category*. Proposition 42.5 thus says that for each $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} X & \mapsto & \mathrm{H}_n(X) \\ \text{and} & & \\ [f] & \mapsto & f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y) \end{array}$$

define a functor from the category $\mathcal{HomTop}$ to the category $\mathcal{AbGr}$ of abelian groups. In particular we now have the following commutative diagram of categories and functors:

$$\begin{tikzcd}
\begin{array}{c}\text{category } \mathcal{Top} \text{ of} \\ \text{topological spaces}\end{array} \arrow[rr, "X \mapsto X"] \arrow[dr, "X \mapsto \mathrm{H}_n(X)"'] & & \begin{array}{c}\text{homotopy} \\ \text{category } \mathcal{HomTop}\end{array} \arrow[dl, "X \mapsto \mathrm{H}_n(X)"] \\
& \begin{array}{c}\text{category } \mathcal{AbGr} \text{ of abelian} \\ \text{groups.}\end{array} &
\end{tikzcd}$$

Exercises for Chapter 42.

Exercise 42.1. Let C_* and D_* be two chain complexes.

(a) Show that chain homotopy defines an equivalence relation on the set of all chain maps from C_* to D_*.
(b) Let $f, g\colon C_* \to D_*$ be chain maps, and let $e\colon B_* \to \mathrm{C}_*$ and $h\colon D_* \to E_*$ be two other chain maps. Show that if $f \simeq g$, then $f \circ e \simeq g \circ e$ and $h \circ f \simeq h \circ g$.

Exercise 42.2. Let A be a compact convex subset of $\mathbb{R}^n$ and let $x \in A$ be a point. Show that the inclusion $\mathbb{R}^n \setminus A \to \mathbb{R}^n \setminus \{x\}$ induces isomorphisms $\mathrm{H}_i(\mathbb{R}^n \setminus A) \to \mathrm{H}_i(\mathbb{R}^n \setminus \{x\})$ for all i.
Hint. Use Exercise 18.13.

Exercise 42.3. Let $X = S^1$ and let $Y = \overline{B}^2 \times S^1$. We consider the map $f\colon X \to Y$ shown in Figure 710. Show that for every $i \in \mathbb{N}$ the induced map $f_*\colon \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ is the zero map.

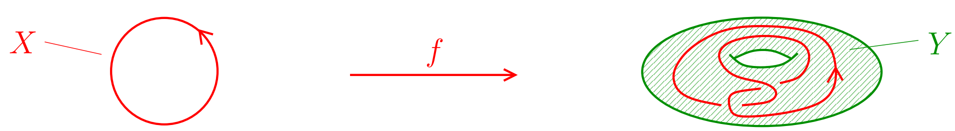


FIGURE 710. Illustration of Exercise 42.3.

43. The Excision Theorem

In this section, given a topological space X and a "good" subset A we will relate the homology groups of X, A and the quotient space X/A via a long exact sequence. We will use this long exact sequence to fulfill our dream of calculating the homology groups of all spheres. Unfortunately it will take a very long time to prove the existence of this long exact sequence.

43.1. **Reduced homology groups.** In this section we introduce the reduced homology groups of a topological space. These are a minute but at times surprisingly useful variation on the usual homology groups.

Definition. Let X be a topological space. By Lemma 41.4 (1) we know that the following is a generalized[684] chain complex

$$\begin{array}{ccccccccccc} & & & & & & \text{degree } 0 & & \text{degree } -1 & & \\ & & & & & & \downarrow & & \downarrow & & \\ \dots & \xrightarrow{\partial_3} & \mathrm{C}_2(X) & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \xrightarrow{\epsilon} & \mathbb{Z} & \to & 0. \\ & & & & & & \sum\limits_{i=1}^k r_i\cdot\sigma_i & \mapsto & \sum\limits_{i=1}^k r_i & & \\ & & & & & & & \uparrow & & & \\ & & & & & & \multicolumn{3}{c}{\text{augmentation map from page 1084}} & & \end{array}$$

We introduce the following three definitions.

(1) We refer to this chain complex as the *augmented chain complex* $\widetilde{\mathrm{C}}_*(X)$ *of* X.
(2) Given $n \in \mathbb{Z}_{\geq -1}$ we define the *n-th reduced homology group* $\widetilde{\mathrm{H}}_n(X)$ *of* X to be the n-th homology group of the augmented chain complex.
(3) If $f\colon X \to Y$ is a map between topological spaces we denote by $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ the map that is the usual map on $\widetilde{\mathrm{C}}_n(X) = \mathrm{C}_n(X)$ for $n \geq 0$ and that is the identity on the final $\mathbb{Z}$-term.

The following elementary lemma summarizes a few properties of reduced homology groups.

Lemma 43.1.

(0) *Given a topological space X we have*[685]

$$\widetilde{\mathrm{H}}_{-1}(X) \;=\; \begin{cases} 0, & \text{if } X \neq \varnothing, \\ \mathbb{Z}, & \text{if } X = \varnothing. \end{cases}$$

(1) *If $f\colon X \to Y$ is a map between two topological spaces, then the induced maps $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ are chain maps. In particular we see that for each $n \in \mathbb{Z}_{\geq -1}$ we get a well-defined induced map $f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$.*
(2) *For every $n \in \mathbb{Z}_{\geq -1}$ the maps*

$$\begin{array}{rcl} X & \mapsto & \widetilde{\mathrm{H}}_n(X) \\ (f\colon X \to Y) & \mapsto & (f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)) \end{array}$$

[684]Recall that a generalized chain complex is like a chain complex, but we also allow non-zero groups in negative degrees.

define a functor from the category of topological spaces to the category of abelian groups.

(3) *For each $n \in \mathbb{Z}_{\geq -1}$ the maps*

$$\begin{array}{rcl} ж\colon \widetilde{\mathrm{H}}_n(X) & \to & \mathrm{H}_n(X) \\ [\sigma] & \mapsto & [\sigma] \end{array}$$

are well-defined. Furthermore they define a natural transformation from the functor $X \mapsto \widetilde{\mathrm{H}}_n(X)$ to the functor $X \mapsto \mathrm{H}_n(X)$.

(4) *For every topological space X the following statements hold:*

(a) *Given any point $P \in X$ the map*

$$\begin{array}{rcl} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} & \to & \mathrm{H}_0(X) \\ [\sigma] \oplus n & \mapsto & [\sigma] + [n \cdot P] \end{array}$$

is an isomorphism.

(b) *For every $n \geq 1$ the natural map $ж\colon \widetilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X)$ is the identity.*

(5) *If X is a topological space with $n \in \mathbb{N}$ path-components, then $\widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}^{n-1}$. In particular X is path-connected if and only if $\widetilde{\mathrm{H}}_0(X) = 0$.*

(6) *The statements of Corollary 42.8 (1), (2) and (4) also hold the same way for reduced homology groups and the corresponding induced maps. For example, if X is a deformation retract of some topological space Y, then the inclusion induced maps $\widetilde{\mathrm{H}}_k(X) \to \widetilde{\mathrm{H}}_k(Y)$ are isomorphisms for all $k \in \mathbb{Z}_{\geq -1}$.*

(7) *If X is a contractible topological space, then $\widetilde{\mathrm{H}}_k(X) = 0$ for all $k \in \mathbb{Z}_{\geq -1}$.*

The lemma shows in particular that reduced homology groups are almost the same as the usual homology groups. So it might sound slightly silly to introduce them. But we will see that on several occasions reduced homology groups make it easier to formulate some statements.

Remark.

(1) If X is a non-empty topological space, then it follows from Lemma 43.1 (4) that we have isomorphisms

$$\mathrm{H}_n(X) \;\cong\; \begin{cases} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z}, & \text{if } n = 0, \\ \widetilde{\mathrm{H}}_n(X), & \text{if } n \neq 0. \end{cases}$$

Furthermore, by Lemma 43.1 (3) we know that for $n \geq 1$ the isomorphism is natural. On other hand for $n = 0$ the isomorphism depends on the choice of a point.

(2) In Lemma 43.2 we will formulate a useful variation on Lemma 43.1 (3).

(3) By Lemma 43.1 (7) we know for any $k \in \mathbb{N}_0$ and any $n \in \mathbb{Z}_{\geq -1}$ we have

$$\widetilde{\mathrm{H}}_n(\Delta^k) \;=\; \widetilde{\mathrm{H}}_n(B^k) \;=\; \widetilde{\mathrm{H}}_n(\overline{B}^k) \;=\; \widetilde{\mathrm{H}}_n(\mathbb{R}^k) \;=\; 0.$$

[685]This calculation shows that in almost all cases we can ignore $\widetilde{\mathrm{H}}_{-1}(X)$. Nonetheless, some formulas later on become cleaner if we take into account that $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$.

(4) We will often use Lemma 43.1 without explicitly mentioning it. For example we will refer to earlier results about homology groups and state the corresponding results for reduced homology groups.

PROOF.

(0) First let X be a non-empty topological space. It follows immediately from the definitions that the augmentation map ϵ is an epimorphism which entails that $\widetilde{\mathrm{H}}_{n-1}(X) = 0$. The fact that $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$ is clear.

(1) Let $f\colon X \to Y$ be a map between two topological spaces. It is a straightforward consequence of the definition of the augmentation maps that the following diagram commutes:

$$\begin{tikzcd} \mathrm{C}_0(X) \arrow[rr, "f_*"] \arrow[dr, "\epsilon"'] & & \mathrm{C}_0(Y) \arrow[dl, "\epsilon"] \\ & \mathbb{Z}. & \end{tikzcd}$$

It follows almost immediately from the fact that this diagram commutes that the induced maps $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ are chain maps. Furthermore we obtain from Lemma 41.7 that we get induced maps on reduced homology.

(2) This statement is basically obvious.

(3) This statement follows easily from the definitions.

(4) Let X be a topological space. By the definition of $\widetilde{\mathrm{H}}_n(X)$ we only have to deal with the case $n = 0$. Let $P \in X$ be a point. It is straightforward to show that the maps

$$\begin{array}{rcl} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} & \to & \mathrm{H}_0(X) \\ {[\sigma]} \oplus n & \mapsto & [\sigma] + [n \cdot P] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{H}_0(X) & \to & \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} \\ {[\sigma]} & \mapsto & [\sigma - \epsilon_X(\sigma) \cdot P] + \epsilon_X(\sigma) \end{array}$$

are inverses of one another. In particular both maps are isomorphisms.

(5) This statement follows immediately from (1), Corollary 41.15 and Lemma 19.8 (4).

(6) This statement is proved exactly the same way as Corollary 42.8.

(7) This final statement follows immediately from (5) and (6). ■

43.2. **Long exact sequences.** We need two more definitions before we can formulate one of the key theorems of homology theory.

Definition. Let X be a topological space and let $A \subset X$ be a subset. We say A is *good* if A is closed and if there exists an open neighborhood U of A in X such that A is a deformation retract of U.

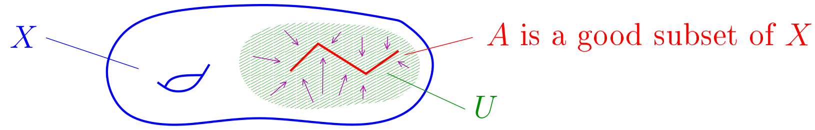


FIGURE 711. Illustration of a good subset.

Remark. Our definition of a good subset is the same as on [**Hat02**, p. 114], except that we do not demand that a good subset is non-empty.

Examples.

(1) Let $x \in X$ be a point. Basically by definition the subset $\{x\} \subset X$ is good in the above sense if and only the point is good in the sense of the definition on page 604.

(2) The sphere S^{n-1} is a good subset of the closed ball $\overline{B}^n$. Indeed, it is clear that S^{n-1} is a closed subset. Furthermore, if we consider the open neighborhood $U = \overline{B}^n \setminus \{0\}$, then a deformation retraction is given by the map

$$\begin{array}{rcl} (\overline{B}^n \setminus \{0\}) \times [0,1] & \to & \overline{B}^n \setminus \{0\} \\ (x,t) & \mapsto & \left(1 - t \cdot \frac{1-\|x\|}{\|x\|}\right) \cdot x. \end{array}$$

(3) We can generalize the second example: It follows easily from Proposition 6.27 (3) and the Collar Neighborhood Theorem 8.12 that every boundary component of a smooth manifold is good.

(4) Finally it follows from Lemma 36.18 (2) and Proposition 36.10 (8a) that every subcomplex A of a CW-complex X is good.

The following definition might be familiar from earlier algebra courses.

Definition. A sequence of group homomorphisms

$$\dots\ A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \dots$$

is called *exact*, if for each n we have

$$\ker(f_n) = \operatorname{im}(f_{n+1}).$$

Example. Let A and B be abelian groups. The sequence

$$0 \to A \xrightarrow{a \mapsto (a,0)} A \oplus B \xrightarrow{(a,b) \mapsto b} B \to 0$$

is easily seen to be exact.

Remark. We collect a few basic statements about exact sequences which we will use frequently.

(1) In any exact sequence the composition of two consecutive maps is necessarily zero.

(2) We have the following equivalences of statements:

$$\begin{array}{lcl} \text{the sequence } 0 \to A \xrightarrow{f} B \quad \text{is exact} & \iff & f \text{ is injective} \\ \text{the sequence } \quad A \xrightarrow{f} B \to 0 \text{ is exact} & \iff & f \text{ is surjective} \\ \text{the sequence } 0 \to A \xrightarrow{f} B \to 0 \text{ is exact} & \iff & f \text{ is an isomorphism.} \end{array}$$

(3) Sometimes we also use the following observations:

$$\begin{array}{lcl} \text{the sequence } 0 \to A \to B \xrightarrow{f} C \text{ is exact} & \implies & A \cong \ker(f) \\ \text{the sequence } A \xrightarrow{f} B \to C \to 0 \text{ is exact} & \implies & C \cong \operatorname{coker}(f). \end{array}$$

The following lemma gives us the first slightly interesting example of an exact sequence.

Lemma 43.2. *If X is a non-empty topological space, then the natural sequence*

$$0 \to \widetilde{\mathrm{H}}_0(X) \xrightarrow{\text{ж}} \mathrm{H}_0(X) \xrightarrow[\text{from Lemma 41.4}]{\text{augmentation map } \epsilon_X} \mathbb{Z} \to 0$$

is exact.

Proof. Let $P \in X$ be a point. We consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathrm{H}}_0(X) \arrow[r, "\text{ж}"] \arrow[dr, "v\mapsto(v,0)"'] & \mathrm{H}_0(X) \arrow[r, "\epsilon_X"] & \mathbb{Z} \arrow[r] & 0 \\
& & \widetilde{\mathrm{H}}_0(X)\oplus\mathbb{Z} \arrow[u, "\cong"', "(v,n)\mapsto(\text{ж}(v),[n\cdot P]"] \arrow[ur, bend right, "(v,n)\mapsto n"'] & &
\end{tikzcd}$$

We make the following observations:

(1) It follows immediately from the definitions that the diagram commutes.
(2) By Lemma 43.1 (4) we know that the vertical map is an isomorphism.
(3) The "bottom sequence" via $\widetilde{\mathrm{H}}_0(X)\oplus\mathbb{Z}$ is precisely of the form of the above example, i.e. this sequence is exact.
(4) It follows easily from the above that the original sequence is also exact. ■

Now we can formulate the following theorem.

Theorem 43.3. *Let X be a topological space and let $A \subset X$ be a good subset.*[686] *We denote by $i\colon A \to X$ the inclusion map and we denote by $p\colon X \to X/A$ the projection map. For each $n \in \mathbb{N}_0$ there exists a natural*[687] *homomorphism $\partial\colon \widetilde{\mathrm{H}}_n(X/A) \to \widetilde{\mathrm{H}}_{n-1}(A)$ such that the sequence*

$$\begin{array}{cccccccc}
 & & & \dots & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_{n+1}(X/A) & \xrightarrow{\partial} \\
\to & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_n(X/A) & \xrightarrow{\partial} \\
\to & \widetilde{\mathrm{H}}_{n-1}(A) & \xrightarrow{i_*} & \dots & & &
\end{array}$$

is exact.

Example. In Figure 712 we show a topological space X together with a good subset A and we show the inclusion map $i\colon A \to X$ and the projection $p\colon X \to X/A$. Theorem 43.3 relates the (reduced) homology groups of A, X and X/A in a mysterious way.

Remark.

(1) In Exercise 43.18 we will see that in general the conclusion of Theorem 43.3 does not hold, if we do not assume that A is a good subset.

---

[686] The most common situation is that X is a CW-complex and that A is a subcomplex.

[687] Here natural means the following: If $f\colon (X,A) \to (Y,B)$ is a map between pairs of topological spaces and if A and B are good subsets, then the following diagram commutes:

$$\begin{tikzcd}
\widetilde{\mathrm{H}}_n(X/A) \arrow[r, "\partial"] \arrow[d, "f_*"'] & \widetilde{\mathrm{H}}_{n-1}(A) \arrow[d, "f_*"] \\
\widetilde{\mathrm{H}}_n(Y/B) \arrow[r, "\partial"] & \widetilde{\mathrm{H}}_{n-1}(B).
\end{tikzcd}$$

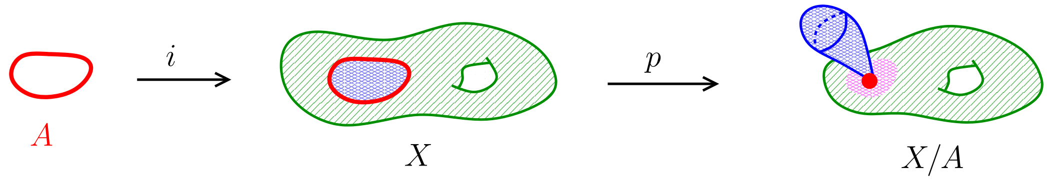


FIGURE 712

(2) If A is non-empty, then it follows from Lemma 43.1 that the last term that is possibly non-zero is $\widetilde{\mathrm{H}}_0(X/A)$.

It will take a while to prove Theorem 43.3. Therefore we will first consider several applications which will convince us that proving Theorem 43.3 is worth the effort.

43.3. **The homology groups of spheres.** Using Theorem 43.3 we can now determine the homology groups of spheres and we get a nice and satisfactory result.

Proposition 43.4. *For each $n \in \mathbb{Z}_{\geq -1}$ we have*
$$\widetilde{\mathrm{H}}_k(S^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Remark.

(1) Put differently, the combination of Lemma 43.1 and Proposition 43.4 says that for any $n > 0$ we have
$$\mathrm{H}_k(S^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0 \text{ or } k = n, \text{ and} \\ 0, & \text{if } k \neq 0, n. \end{cases}$$
Furthermore we have
$$\mathrm{H}_k(S^0) \;\cong\; \begin{cases} \mathbb{Z}^2, & \text{if } k = 0, \text{ and} \\ 0, & \text{if } k \neq 0. \end{cases}$$

(2) Note that the statement of Proposition 43.4 is in marked contrast to the situation for higher homotopy groups. As formulated in Question 40.18 we have the strong suspicion that $\pi_3(S^2) \neq 0$.

By Proposition 6.26 we know that for $k \neq l$ the spheres S^k and S^l are not diffeomorphic. The following corollary is a serious upgrade of this fact.

Corollary 43.5. *For $k \neq l$ the spheres S^k and S^l are not homeomorphic.*

So we have finally managed to distinguish spheres of different dimensions. Our initial idea had been to use higher homotopy groups to prove that result. Even though the definition of higher homotopy groups is arguably much more natural than the definition of homology groups, it turns out that the latter groups are significantly easier to compute.

PROOF OF PROPOSITION 43.4 ASSUMING THEOREM 43.3. We prove the proposition by induction on n. First note that for $n = -1$ we have $S^n = S^{-1} = \varnothing$, thus the statement follows immediately from Lemma 43.1 (0).[688]

[688]To readers who prefer S^0 as a starting point we point out that S^0 consists of precisely two points. Thus the promised isomorphisms follow from Lemmas 41.6, 41.14 and 43.1.

Now we suppose that the statement of the proposition holds for $n-1$. We want to determine the reduced homology of S^n. On page 182 we remarked that S^n is homeomorphic to $\overline{B}^n/S^{n-1}$. We had just seen on page 1108 that S^{n-1} is a good subset of $\overline{B}^n$. Therefore we can apply Theorem 43.3 to $X=\overline{B}^n$ and the closed subset $A=S^{n-1}$ and we obtain the following exact sequence

$$\begin{array}{cccccccc} & \dots & \longrightarrow & \widetilde{\mathrm{H}}_{k+1}(\overline{B}^n) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_{k+1}(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial} \\ \to & \widetilde{\mathrm{H}}_k(S^{n-1}) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_k(\overline{B}^n) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_k(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial} \\ \to & \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) & \xrightarrow{i_*} & \dots \end{array}$$

From the induction hypothesis, from the homeomorphism $S^n\cong\overline{B}^n/S^{n-1}$ and from the fact that $\widetilde{\mathrm{H}}_i(\overline{B}^n)=0$ for all i (obtained on page 1106) we now see that the above long exact sequence gives rise to the following exact sequence:

$$\begin{array}{cccccccc} & \dots & \longrightarrow & 0 & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_{k+1}(S^n) & \xrightarrow{\partial} \\ \to & \widetilde{\mathrm{H}}_k(S^{n-1}) & \xrightarrow{i_*} & 0 & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_k(S^n) & \xrightarrow{\partial} \\ \to & \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) & \xrightarrow{i_*} & \dots \end{array}$$

In particular we obtain for each k an exact sequence

$$0 \to \widetilde{\mathrm{H}}_{k+1}(S^n) \xrightarrow{\partial} \widetilde{\mathrm{H}}_k(S^{n-1}) \to 0.$$

But this means that the maps ∂ are isomorphisms, and using our induction hypothesis we obtain the desired result that

$$\widetilde{\mathrm{H}}_k(S^n) = \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) \cong \begin{cases} \mathbb{Z}, & \text{if } k-1=n-1, \text{ i.e. if } k=n \text{ and} \\ 0, & \text{if } k\neq n. \end{cases}$$ ■

In Lemma 2.59 we had used elementary arguments to show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. Furthermore, in Lemma 18.17 we had used the fundamental group to prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n\geq 3$. Now we can finally show that for any $k\neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic.

Proposition 43.6. *For any $k\neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic.*

PROOF. Clearly we only have to deal with the case $k,l\geq 1$. Let $k,l\in\mathbb{N}$ and suppose there exists a homeomorphism $f\colon\mathbb{R}^k\to\mathbb{R}^l$. Then we obtain that

$$\mathbb{Z} \underset{\substack{\uparrow \\ \text{Proposition 43.4}}}{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{k-1}) \overset{\substack{\text{Corollary 42.8} \\ \downarrow}}{\cong} \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^k\setminus\{0\}) \underset{\substack{\uparrow \\ \text{isomorphism since } f \text{ is a homeomorphism}}}{\xrightarrow{f_*}} \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^l\setminus\{f(0)\}) \overset{\substack{\text{Corollary 42.8} \\ \downarrow}}{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{l-1}).$$

So we have $\widetilde{\mathrm{H}}_{k-1}(S^{l-1})\cong\mathbb{Z}$. But by Proposition 43.4 that is only possible if $k=l$. ■

In Lemma 15.7 we had used the fundamental group to show that S^1 is not a retract of $\overline{B}^2$. Precisely the same argument, together with Proposition 43.4, now gives us the following proposition.

Proposition 43.7. *For every $n \in \mathbb{N}$ the sphere $S^{n-1} = \partial \overline{B}^n$ is not a retract of the closed ball $\overline{B}^n$.*

Remark. I found the following amusing description of Proposition 43.7 online:
"Suppose a shark jumps into a shoal of fish (a kind of big ball). The small fishes start escaping in all directions towards the border of the shoal, where the fishes stand still. Yet they escape with a certain disposition to follow a continuous flow, as they usually do, since everybody tends to follow its neighborhoods. But since there is no continuous retraction to the boundary, somebody doesn't know where to go, and stay there for a moment, much to the shark's satisfaction. There is also a 2D version, with a wolf entering into a herd of sheep."

Proof. Suppose there exists a retraction $r\colon \overline{B}^n \to S^{n-1}$. We denote by $i\colon S^{n-1} \to \overline{B}^n$ the inclusion map. We consider the following diagram

$$\begin{tikzcd} & \widetilde{\mathrm{H}}_{n-1}(\overline{B}^n) \arrow[dr, "r_*"] & \\ \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \arrow[ur, "i_*"] \arrow[rr, "(r\circ i)_*"] & & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}). \end{tikzcd}$$

This diagram commutes since it follows from the functoriality of the homology groups that $r_* \circ i_* = (r \circ i)_*$. Since r is a retraction it follows that $r \circ i = \mathrm{id}_{S^{n-1}}$, i.e. $(r \circ i)_* = (\mathrm{id}_{S^{n-1}})_*$ is the identity map on $\widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ which, by Proposition 43.4, is isomorphic to $\mathbb{Z}$. The lower map is thus an isomorphism, but the upper map factorizes through the trivial group, i.e. the composition of i_* and r_* cannot be an isomorphism of $\mathbb{Z}$. Thus we have obtained a contradiction. ■

Theorem 43.8. (Brouwer[689] Fixed Point Theorem) *Every map $f\colon \overline{B}^n \to \overline{B}^n$ admits a fixed point, i.e. for every map $f\colon \overline{B}^n \to \overline{B}^n$ there exists a point $x \in \overline{B}^n$ such that $f(x) = x$.*[690]

Proof. Suppose there exists a map $f\colon \overline{B}^n \to \overline{B}^n$ without a fixed point. Since f admits no fixed points there exists for each x precisely one ray which starts at $f(x)$ and which goes through x. We consider the following map which is illustrated in Figure 713:

$$\begin{array}{rcl} \Phi\colon \overline{B}^n & \to & \overline{B}^n \\ x & \mapsto & \begin{array}{c}\text{the intersection point of } S^{n-1} \text{ with the} \\ \text{uniquely determined ray from } f(x) \text{ to } x.\end{array} \end{array}$$

[689] Luitzen Brouwer (1881-1966) was a Dutch mathematician.

[690] How can one prove the statement for $n = 1$ using elementary methods from real analysis?

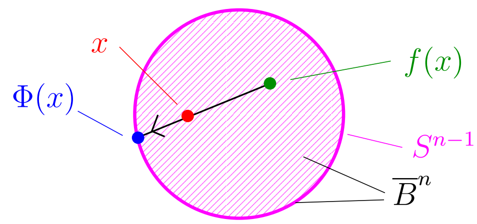


FIGURE 713

It is elementary to show that Φ is continuous.[691] It is evident that $\Phi(x) = x$ for all $x \in S^{n-1}$ and that $\Phi(x) \in S^{n-1}$ for all $x \in S^{n-1}$. The map Φ is thus a retraction from $\overline{B}^n$ to S^{n-1}. But according to Proposition 43.7 this not possible. Thus we have obtained a contradiction. ■

The Brouwer Fixed Point Theorem pops up many times in mathematics. For example we can use it to prove the following theorem in linear algebra.

Theorem 43.9. (Perron-Frobenius Theorem) *Let $A = (a_{ij})$ be an $n \times n$-real matrix. If all entries are positive, then A has a positive eigenvalue λ with an eigenvector v such that all entries of v are non-negative.*

Remark. There are more precise statements one can prove about matrices with positive entries we refer to [**Meyr00**, Section 8.2] and also to

`https://en.wikipedia.org/wiki/Perron-Frobenius_theorem`

for details. Supposedly the Perron-Frobenius Theorem and some of the more refined results mentioned in [**Meyr00**, Section 8.2] play a major role in Google's algorithm, see e.g.

`http://www.math.pku.edu.cn/teachers/yaoy/Fall2011/lecture07.pdf`

for more information.

PROOF. We consider the convex set

$$T \;:=\; \Big\{(t_1,\dots,t_n) \in [0,1]^n \,\Big|\, \sum_{i=1}^{n} t_i = 1\Big\}.$$

This set is just the standard $(n-1)$-simplex endowed with a different name. We saw in Lemma 41.1 that it is homeomorphic to $\overline{B}^{n-1}$. It follows immediately from this observation and from the Brouwer Fixed Point Theorem 43.8 that any self-map $f\colon T \to T$ has a fixed point.

[691]Here is the precise argument: given $x \in \overline{B}^n$ the ray r_x emanating from x and through $f(x)$ is given by $r + x(t) = x + t(f(x) - x)$ for $t \in \mathbb{R}_{\geq 0}$. For any x there exists a unique $t_x \geq 0$ with $\|r_x(t_x)\| = 1$, and $\Phi(x) = r_x(t_x)$. It suffices to show that $x \mapsto t_x$ is continuous. One can determine t_x through the quadratic equation

$$1 = \|r_x(t_x)\|^2 = \langle r_x(t_x), r_x(t_x)\rangle = 2\langle x, f(x) - x\rangle + \|f(x) - x\| = \langle x + t_x(f(x) - x), x + t_x(f(x) - x)\rangle$$

and the quadratic formula for polynomials of degree two which is continuous in the coefficients.

Let $A = (a_{ij})$ be an $n \times n$-real matrix such that all entries are positive. We consider the map

$$\begin{array}{rcl} f\colon T & \to & T \\ v = (v_1, \ldots, v_n) & \mapsto & \frac{Av}{|Av|}. \end{array}$$

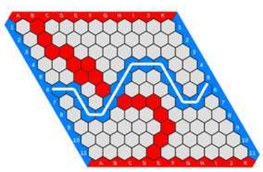
Note that this map is defined since our hypothesis that all a_{ij} are positive and the fact that all entries of v are non-negative and that at least one entry of v is positive imply that $Av \neq 0$. As we had just discussed above, the map $f\colon T \to T$ has a fixed point $v \in T$. But this means that $Av = |Av| \cdot v$. Put differently, we have just shown that v is an eigenvector to the positive eigenvalue $|Av|$. ■

We conclude this discussion of the Brouwer Fixed Point Theorem with two applications outside of mathematics:

(1) The board game "Hex" was invented in 1942 by Piet Hein[692] and it was re-invented in 1948 by John Nash[693] The game is illustrated in Figure 714 and an online version can be found at

`http://www.lutanho.net/play/hex.html`

David Gale [**Gal79**] showed in 1979 that the statement that the board game "Hex" cannot end in a draw, is equivalent to the Brouwer Fixed Point Theorem. We also refer to [**Gh14**, p. 102] for a discussion of Hex and the Brouwer Fixed Point Theorem.



11 × 11 Hex gameboard showing a winning configuration for Blue

Figure 714

(2) The Brouwer Fixed Point Theorem can also be used to show the existence of a "Nash equilibrium" in game theory. We refer to the following website for more information:

`https://en.wikipedia.org/wiki/Nash_equilibrium`.

43.4. **Basic homological algebra.** Now that we have made use of Theorem 43.3 we also need to provide a proof of Theorem 43.3. In this chapter we will first develop several algebraic methods which later on will be used in the proof of the existence of the exact sequence of Theorem 43.3.

[692]Piet Hein (1905-1996) was a Danish mathematician.

[693]John Nash (1928-2015) was an American mathematician who was awarded the Nobel prize in economics in 1994. The movie "a beautiful mind" was based on his struggles with schizophrenia.

Definition. A *short exact sequence* is an exact sequence of the form

$$0 \to A \to B \to C \to 0,$$

which consists of at most three non-trivial groups. A *short exact sequence of chain complexes*

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} \mathrm{C}_* \to 0$$

consists of chain complexes A_*, B_*, C_* and chain maps $i\colon A_* \to B_*$ and $p\colon B_* \to \mathrm{C}_*$ such that for each n the maps

$$0 \to A_n \xrightarrow{i} B_n \xrightarrow{p} \mathrm{C}_n \to 0$$

form a short exact sequence.

Remark. Throughout the subsequent discussion in the following pages it is helpful to keep in mind the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r, "i"] \arrow[d, "\partial_{n+1}"] & B_{n+1} \arrow[r, "p"] \arrow[d, "\partial_{n+1}"] & \mathrm{C}_{n+1} \arrow[r] \arrow[d, "\partial_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "i"] \arrow[d, "\partial_n"] & B_n \arrow[r, "p"] \arrow[d, "\partial_n"] & \mathrm{C}_n \arrow[r] \arrow[d, "\partial_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i"] \arrow[d, "\partial_{n-1}"] & B_{n-1} \arrow[r, "p"] \arrow[d, "\partial_{n-1}"] & \mathrm{C}_{n-1} \arrow[r] \arrow[d, "\partial_{n-1}"] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r, "i"] & B_{n-2} \arrow[r, "p"] & \mathrm{C}_{n-2} \arrow[r] & 0.
\end{tikzcd}$$

Now suppose that we are given such a short exact sequence of chain complexes. We will define a map

$$\partial_n\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$$

as follows:

(1) Let $z \in \mathrm{H}_n(C)$.
(2) We pick $c \in C_n$ with $z = [c]$.
(3) By the surjectivity of p we have $c = p(b)$ for some $b \in B_n$.
(4) We have

$$p(\partial b) \underset{\substack{\uparrow \\ \text{since the diagram commutes}}}{=} \partial p(b) = \partial c \underset{\substack{\uparrow \\ \text{since } c \text{ is a cycle}}}{=} 0.$$

(5) Since the second horizontal row in the above diagram is exact it follows immediately from (4) that $\partial b = i(a)$ for some uniquely determined $a \in A_{n-1}$.
(6) We put $\partial_n(z) := [a]$.

We refer to this map $\partial_n\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ as the *connecting homomorphism.*

The following lemma summarizes two of the key properties of the connecting homomorphisms.

Lemma 43.10.

(1) *Let $0 \to A_* \to B_* \to C_* \to 0$ be a short exact sequence of chain complexes. The connecting homomorphism $\partial_n \colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ is well-defined and it is indeed a homomorphism.*

(2) *The connecting homomorphism is natural, i.e.*

$$\begin{tikzcd} 0 \arrow[r] & A_* \arrow[r, "i"] \arrow[d, "a"] & B_* \arrow[r, "p"] \arrow[d, "b"] & C_* \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A'_* \arrow[r, "i"] & B'_* \arrow[r, "p"] & C'_* \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram of chain maps where the horizontal sequences are exact, then the following diagram commutes for every $n \in \mathbb{N}$:

$$\begin{tikzcd} \mathrm{H}_n(C) \arrow[r, "\partial_n"] \arrow[d, "c_*"'] & \mathrm{H}_{n-1}(A) \arrow[d, "a_*"] \\ \mathrm{H}_n(C') \arrow[r, "\partial_n"] & \mathrm{H}_{n-1}(A') \end{tikzcd}$$

Remark. Let us explain why we say that the connecting homomorphism is "natural". Let $\mathcal{S}$ be the category of short exact sequences of chain complexes where the morphisms are chain maps. Lemma 43.10 (2) says given any $n \in \mathbb{N}_0$ the connecting homomorphism $\partial_n \colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ is a natural transformation from the functor

$$\begin{array}{rcl} \mathcal{S} & \to & \mathcal{AbGr} \\ (A_* \to B_* \to \mathrm{C}_*) & \mapsto & \mathrm{H}_n(C) \end{array} \quad \text{to the functor} \quad \begin{array}{rcl} \mathcal{S} & \to & \mathcal{AbGr} \\ (A_* \to B_* \to \mathrm{C}_*) & \mapsto & \mathrm{H}_{n-1}(A). \end{array}$$

The proof of this lemma and the following proposition are done by a "diagram chase". This means that one needs to keep the commutative diagram on page 1115 in mind and one has to use on several occasions that the vertical maps are chain maps and that the horizontal sequences are exact. In both cases the proofs are quite elementary.

Proof.

> The proof below is elementary but not very readable. It is much easier to prove the lemma on your own than to try to read the proof.

For the undeterred reader we now provide the fully gory details.

(1) We start out with the following claim.

Claim. The map

$$\partial_n \colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$$

is well-defined.

Let $z \in \mathrm{H}_n(C)$. We denote by c, b and a the elements as in the definition of ∂_n. We need to show that the element a is indeed a cycle and that the definition of $\partial_n([c])$ does not depend on the choices of c and b.

The proofs of these two statements is not particularly difficult and it is done via "diagram chasing". First we note that

$$i(\partial a) \underset{\substack{\uparrow \\ \text{since the diagram commutes}}}{=} \partial i(a) = \partial(\partial b) = 0.$$

Since i is injective we obtain that $\partial a = 0$, i.e. a is indeed a cycle. Now we show that the homology class $[a]$ depends only on the homology class $[c]$.

(a) If we chose a different element in B_n, then it follows from the fact that the horizontal sequences are exact that this element would be of the form $b + i(a')$ for some $a' \in A_n$. But then we obtain the following equality:

$$\partial(b + i(a')) = \partial b + \partial(i(a')) \overset{\substack{\text{since the diagram commutes} \\ \downarrow}}{=} \underbrace{\partial b}_{=i(a)} + i(\partial a') = i(a + \partial a').$$

Since $[a] = [a + \partial a'] \in \mathrm{H}_{n-1}(A)$ this does not change the homology class.

(b) If we picked an other representative for $[c]$, then this representative would be of the form $c + \partial c'$ for some $c' \in \mathrm{C}_{n+1}$. Since the map $p\colon B_n \to C_n$ is an epimorphism there exists a $b' \in B_{n+1}$ with $p(b') = c'$. It follows that

$$p(b + \partial b') = p(b) + p(\partial b') \underset{\substack{\uparrow \\ \text{since the diagram commutes}}}{=} c + \partial p(b') = c + \partial c'.$$

Since $\partial(b + \partial b') = \partial b$ we see that we have ended up with the same element in B_{n-1} as before. ⊞

Claim. The map

$$\partial_n\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$$

is a homomorphism.

Thus let $[c_1]$ and $[c_2]$ be given. For $j = 1, 2$ we choose b_j as in this definition of ∂_n, and we denote by a_j the uniquely determined element with $i(a_j) = \partial b_j$. Then we also have $p(b_1 + b_2) = c_1 + c_2$ and $i(a_1 + a_2) = i(a_1) + i(a_2) = \partial b_1 + \partial b_2 = \partial(b_1 + b_2)$. It follows that

$$\partial_n([c_1 + c_2]) = [a_1 + a_2] = [a_1] + [a_2] = \partial_n([c_1]) + \partial_n([c_2]).$$ ⊞

(2) The second statement follows easily from the definitions. We leave the verification of the details to the reader. ■

Now we can formulate the following proposition which explains the name "connecting homomorphism".

Proposition 43.11.

(1) *Let*

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$

be a short exact sequence of chain complexes. Then the sequence

$$\dots \to \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B) \xrightarrow{p_*} \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \xrightarrow{i_*} \mathrm{H}_{n-1}(B) \xrightarrow{p_*} \dots$$

is exact.

(2) *The connecting homomorphism is natural. More precisely, let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & A'_* & \xrightarrow{i} & B'_* & \xrightarrow{p} & C'_* & \longrightarrow & 0 \end{array}$$

be a commutative diagram of chain maps where the horizontal sequences are exact. Then the following diagram commutes:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \mathrm{H}_n(A) & \xrightarrow{i_*} & \mathrm{H}_n(B) & \xrightarrow{p_*} & \mathrm{H}_n(C) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(A) & \longrightarrow & \dots \\ & & \downarrow a_* & & \downarrow b_* & & \downarrow c_* & & \downarrow a_* & & \\ \dots & \longrightarrow & \mathrm{H}_n(A') & \xrightarrow{i_*} & \mathrm{H}_n(B') & \xrightarrow{p_*} & \mathrm{H}_n(C') & \xrightarrow{\partial} & \mathrm{H}_{n-1}(A') & \longrightarrow & \dots \end{array}$$

The fact that the square to the right commutes is just in plain words the statement that the connecting homomorphism is natural.

In the following we refer to the exact sequence of Proposition 43.11 as the as *long exact sequence in homology* corresponding to the short exact sequence of chain complexes.

Proof of Proposition 43.11. First note that Statement (2) is an immediate consequence of Lemma 43.10. We included the statement for the convenience of the reader. Thus let us turn to the actual meat, namely Statement (1).

The proof below is elementary but not very readable. It is much easier to prove the proposition on your own than to try to read the proof.

Let

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$

be a short exact sequence of chain complexes. Then the sequence

$$\dots \to \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B) \xrightarrow{p_*} \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \xrightarrow{i_*} \mathrm{H}_{n-1}(B) \xrightarrow{p_*} \dots$$

is exact. We examine the exactness at the three groups $\mathrm{H}_{n-1}(A), \mathrm{H}_n(B)$ and $\mathrm{H}_n(C)$. Put differently, at each of the three groups we want to show the following two statements:

(1) the image of the map to the left is contained in the image of the map to the right, or equivalently, the composition of the map to the left and the map to the right is zero,
(2) the kernel of the map to the right is contained in the image of the map to the left.

Now we perform these steps:

(A) First let $[c] \in \mathrm{H}_n(C)$. We want to show that $i(\partial[c]) = 0$. We choose b with $p(b) = c$ and denote by $a \in A_{n-1}$ the element with $i(a) = \partial b$. Then we have

$$i(\partial[c]) \underset{\substack{\uparrow \\ \text{definition of } \partial\colon\ \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)}}{=} i([a]) = [\partial b] = 0 \in \mathrm{H}_{n-1}(B).$$

Now let $[a] \in \ker(i_*)$. Then $i(a) = \partial b$ for some $b \in B_n$. It follows immediately from the definitions that $\partial[p(b)] = [a]$.

(B) From $p \circ i = 0$ it follows that $p_* \circ i_* = 0$, i.e. $\operatorname{im}(i_*) \subset \ker(p_*)$.
Conversely let $b \in \ker(p_*)$. Then $p(b) = \partial c'$ for some $c' \in \mathrm{C}_{n+1}$. Since p is surjective, there exists a $b' \in B_{n+1}$ with $p(b') = c'$. From the commutativity it follows that $p(\partial b') = \partial p(b') = \partial c' = p(b)$. Thus it follows that $p(b - \partial b') = 0$, i.e. $b - \partial b' = i(a)$ for some $a \in A_n$. But then it follows that $[b] = [b - \partial b'] = [i(a)] \in \operatorname{im}(i_*)$.

(C) Let $[b] \in \mathrm{H}_n(B)$. We have to show that $\partial(p_*([b])) = 0$. In the definition of $\partial p_*([b])$ we choose some $b \in B$. Since $\partial b = 0$ it follows immediately that $\partial p_*([b]) = 0$.
Finally let $[c] \in \ker(\partial \colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A))$. We choose $b \in B_n$ with $p(b) = c$. Then there exists a uniquely determined $a \in A_{n-1}$ with $i(a) = \partial b$. Since we have $[a] = \partial[c] = 0 \in \mathrm{H}_{n-1}(A)$ there exists an $a' \in A_n$ with $\partial a' = a$. Then it follows that

$$\partial(b - i(a')) \;=\; \partial b - \partial i(a') \;=\; \partial b - i(\partial a') \;=\; \partial b - i(a) \;=\; 0,$$

i.e. $b - i(a')$ is a cycle. On the other hand we have $p(b - i(a')) = p(b) - (p \circ i)(a') = p(b)$. Thus we see that $p_*([b - i(a')]) = [c]$. We have thus shown that $\ker(\partial) \subset \operatorname{im}(p_*)$.

Thus we have proved the claim and therefore we have also completed the proof of the proposition. ■

The proof of the following lemma is also done via "diagram chasing". We will provide the proof in Exercise 43.6.

Lemma 43.12. (Five Lemma) *Let*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, two heads, "a"] & B \arrow[r, "j"] \arrow[d, "b", "\cong"'] & C \arrow[r, "k"] \arrow[d, "c"] & D \arrow[r, "l"] \arrow[d, "d", "\cong"'] & E \arrow[d, hook, "e"] \\ A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E' \end{tikzcd}$$

be a diagram of groups (here for once we do not suppose that the groups are abelian) such that the horizontal sequences are exact and such that every squares commute (if all the groups are abelian, then it suffices it the squares commute up to a sign). If a is surjective, if b and d are isomorphisms, and if e is injective, then c is an isomorphism.[694]

43.5. **Relative homology groups.** In this section we will introduce the relative homology groups which will play an important role in the proof of Theorem 43.3 and also in the remainder of this course.

Definition. Let (X, A) be a pair of topological spaces.[695] We view each chain group $\mathrm{C}_n(A)$ as a subgroup of $\mathrm{C}_n(X)$. Thus we can define

$$\mathrm{C}_n(X, A) \;:=\; \mathrm{C}_n(X)/\,\mathrm{C}_n(A)$$

[694] In many applications it suffices to remember the following slightly weaker statement: if the two vertical maps on the left and right are isomorphisms, then the middle map is also an isomorphism.

and we consider the boundary map[696]

$$\begin{array}{rrcl} \partial_n\colon & \mathrm{C}_n(X,A) & \to & \mathrm{C}_{n-1}(X,A) \\ & [c] & \mapsto & [\partial_n c]. \end{array}$$

We refer to

$$\dots \mathrm{C}_{n+1}(X,A) \xrightarrow{\partial_{n+1}} \mathrm{C}_n(X,A) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(X,A) \xrightarrow{\partial_{n-1}} \dots$$

as the chain complex of the pair (X,A).[697] Given any $n \in \mathbb{N}_0$ we define the *n-th relative homology group of* (X,A) as follows:

$$\mathrm{H}_n(X,A) \;:=\; \mathrm{H}_n(\mathrm{C}_*(X,A)).$$

Remark. Let (X,A) be a pair of topological spaces and let $n \in \mathbb{N}_0$. We write

$$S \;:=\; \{\sigma\colon \Delta^n \to X \mid \mathrm{im}(\sigma) \cap (X \setminus A) \neq \varnothing\},$$

i.e. S denotes the set of all singular n-simplices of X that are not completely contained in A. Then it is straightforward to show that the map

$$\begin{array}{rcl} \mathbb{Z}^{(S)} & \to & \mathrm{C}_n(X,A) \\ \sum\limits_{i=1}^{n} n_i\sigma_i & \mapsto & \left[\sum\limits_{i=1}^{n} n_i\sigma_i\right] \end{array}$$

is an isomorphism. In particular we see that $\mathrm{C}_n(X,A)$ is a free abelian group.

Examples.

(1) If $A = \varnothing$, then for all $n \in \mathbb{N}_0$ we have $\mathrm{C}_n(\varnothing) = 0$ and $\mathrm{C}_n(X,\varnothing) = \mathrm{C}_n(X)$, i.e. we have $\mathrm{H}_n(X,\varnothing) = \mathrm{H}_n(X)$. In the following we will freely go back and forth between the notations $\mathrm{H}_n(X,\varnothing)$ and $\mathrm{H}_n(X)$.

(2) If X is a path-connected topological space and if A is non-empty, then $\mathrm{H}_0(X,A) = 0$. Indeed, let $P \in X$ be a point. We pick a point $Q \in A$. Since X is path-connected we can pick a map $\sigma\colon \Delta^1 \to X$ such that $\sigma(0,1) = P$ and $\sigma(1,0) = Q$. As discussed on page 1079, in $\mathrm{C}_*(X,A)$ we have $\partial\sigma = P - Q$, but since $Q = 0 \in \mathrm{C}_0(X,A)$ we have $\partial\sigma = P$, which shows that $P = 0 \in \mathrm{H}_0(X,A)$. But this clearly implies that $\mathrm{H}_0(X,A) = 0$. We refer to Figure 715 for an illustration.

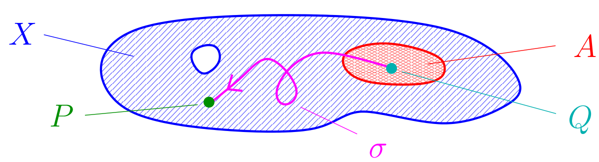


FIGURE 715

[695]Recall that according to the definition on page 477 this means that X is a topological space and $A \subset X$ is a subset.

[696]The map ∂_n is well-defined, since for any $c \in \mathrm{C}_n(A) \subset \mathrm{C}_n(X)$ the boundary ∂c lies obviously in $\mathrm{C}_{n-1}(A) \subset \mathrm{C}_{n-1}(X)$.

[697]Since $\partial_{n-1} \circ \partial_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n-2}(X)$ is the zero map it follows immediately that the composition $\partial_{n-1} \circ \partial_n\colon \mathrm{C}_n(X,A) \to \mathrm{C}_{n-2}(X,A)$ is also the zero map.

(3) Whereas it is often difficult to write down singular cycles in absolute chain complexes $\mathrm{C}_*(X)$, it can be surprisingly easy to write down singular cycles in a relative chain complex $\mathrm{C}_*(X,A)$. Indeed, given any map $\sigma\colon \Delta^n \to X$ with $\sigma(\partial\Delta^n) \subset A$ we have

$$\partial_n(\sigma) \;=\; \sum_{j=0}^{n}(-1)^j\cdot\sigma\circ i_j^n \;\underset{\substack{\uparrow \\ \text{since } \sigma\circ i_j^n\colon \Delta^{n-1} \text{ takes values in } A \text{ we see that } \sigma\circ i_j^n \in \mathrm{C}_{n-1}(A)}}{=}\; 0 \;\in\; \mathrm{C}_{n-1}(X,A)=\mathrm{C}_{n-1}(X)/\,\mathrm{C}_{n-1}(A).$$

Thus σ defines a cycle in the chain complex $\mathrm{C}_n(X,A)$. As an example we consider $X=\mathbb{R}^2$ and $A=B_1(-2)\cup B_1(2)$. Let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex with $\sigma(0,1)\in B_1(2)$ and $\sigma(1,0)\in B_1(-2)$. (We refer to Figure 716 for an illustration.) It follows from the above discussion that σ is a cycle in $\mathrm{C}_*(X,A)$. Later, on page 1126, we will see whether or not $[\sigma]\in \mathrm{H}_1(X,A)$ is non-trivial.

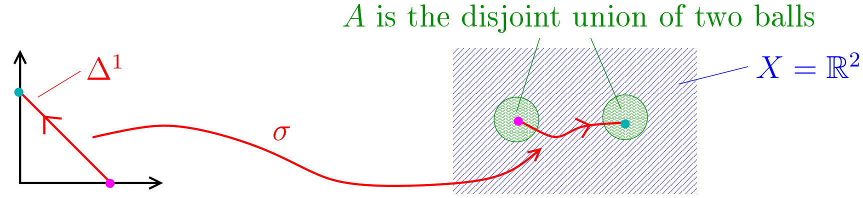


FIGURE 716

Definition.

(1) Recall that we refer to the category that is given by

$$\begin{aligned}\mathrm{Ob}(\mathcal{PairTop}) &:= \text{ all pairs of topological spaces,}\\ \mathrm{Mor}((X,A),(Y,B)) &:= \text{ all continuous maps } f\colon X\to A \text{ with } f(A)\subset B\end{aligned}$$

with the usual composition of maps as the *category of pairs of topological spaces.*

(2) Let $f\colon (X,A)\to(Y,B)$ be a map of pairs of topological spaces, i.e. a map $f\colon X\to Y$ with $f(A)\subset B$. Then f induces maps

$$\begin{aligned}f_*\colon\ \mathrm{H}_n(X,A) &\to \mathrm{H}_n(Y,B)\\ \Big[\sum_{j=1}^{n}a_j\sigma_j\Big] &\mapsto \Big[\sum_{j=1}^{n}a_j(f\circ\sigma_j)\Big].\end{aligned}$$

It is straightforward to show that these maps are well-defined.

The following lemma follows almost immediately from the definitions.

Lemma 43.13. *For each $n\in\mathbb{N}_0$ the maps*

$$\begin{aligned}(X,A) &\to \mathrm{H}_n(X,A)\\ (f\colon (X,A)\to(Y,B)) &\mapsto (f_*\colon\ \mathrm{H}_n(X,A)\to\mathrm{H}_n(Y,B))\end{aligned}$$

define a covariant functor from the category $\mathcal{PairTop}$ of pairs of topological spaces to the category $\mathcal{AbGr}$ of abelian groups.

Remark. If (Y,B) is a pair of topological spaces and if $X\subset Y$ and $A\subset B$, then we have the obvious inclusion map $(X,A)\to(Y,B)$ and we therefore obtain the induced maps $\mathrm{H}_n(X,A)\to\mathrm{H}_n(Y,B)$ in relative homology. If it is clear from the context that we are dealing with such an inclusion induced map, then on many occasions we will not explicitly point out that we are working with the inclusion induced maps.

Next we study the interaction between ordinary homology, relative homology and reduced homology. We introduce the following notation.

Notation. Let X be a topological space, let $A \subset X$ and let $n \in \mathbb{N}_0$. We introduce the following maps:

$$\begin{tikzcd} \mathrm{H}_n(X) \arrow[r, "="] \arrow[rr, bend right, "=:\text{я}=:\text{я}_{X,A}"'] & \mathrm{H}_n(X, \varnothing) \arrow[r] & \mathrm{H}_n(X, A) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widetilde{\mathrm{H}}_n(X) \arrow[r, "\text{ж}"] \arrow[rr, bend right, "=:\text{ж}_{X,A}"'] & \mathrm{H}_n(X) \arrow[r, "\text{я}"] & \mathrm{H}_n(X, A). \end{tikzcd}$$

Lemma 43.14. *Let X be a topological space and let $n \in \mathbb{N}_0$. The following statements hold:*

(1) *Let $A \subset X$. The maps $\text{я}_{X,A}$ and $\text{ж}_{X,A}$ are natural in the sense that if we are given a map $f\colon (X, A) \to (Y, B)$ of pairs of topological spaces then the following diagrams commute:*

$$\begin{tikzcd} \mathrm{H}_n(X) \arrow[r, "f_*"] \arrow[d, "\text{я}_{X,A}"'] & \mathrm{H}_n(Y) \arrow[d, "\text{я}_{Y,B}"] \\ \mathrm{H}_n(X, A) \arrow[r, "f_*"] & \mathrm{H}_n(Y, B) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widetilde{\mathrm{H}}_n(X) \arrow[r, "f_*"] \arrow[d, "\text{ж}_{X,A}"'] & \widetilde{\mathrm{H}}_n(Y) \arrow[d, "\text{ж}_{Y,B}"] \\ \mathrm{H}_n(X, A) \arrow[r, "f_*"] & \mathrm{H}_n(Y, B). \end{tikzcd}$$

(2) *For each $x_0 \in X$ the map*

$$\text{ж}_{X,\{x_0\}}\colon \widetilde{\mathrm{H}}_n(X) \ \to \ \mathrm{H}_n(X, \{x_0\})$$

is a natural isomorphism.

Proof.

(1) The first statement follows immediately from the definitions.
(2) We will provide the proof of this statement in Exercise 43.13. ■

Before we can state the next proposition we need to introduce the notion of a triple of topological spaces which is defined in almost the same fashion as a pair of topological spaces.

Definition.

(1) A *triple of topological spaces* is a triple (X, B, A) where X is a topological space and $A \subset B \subset X$ are subsets.
(2) A map $f\colon (X, B, A) \to (X', B', A')$ between triples of topological spaces is a continuous map $f\colon X \to X'$ such that $f(B) \subset B'$ and $f(A) \subset A'$.

Proposition 43.15. *Let X be a topological space and let $A \subset B \subset X$ be two subsets. We denote by $i\colon (B,A) \to (X,A)$ and by $p\colon (X,A) \to (X,B)$ the obvious maps.*

(1) *For each $n \in \mathbb{N}_0$ the map*[698][699]

$$\begin{array}{rcl} \partial_n\colon\ \mathrm{H}_n(X,B) & \to & \mathrm{H}_{n-1}(B,A) \\ \Big[\sum\limits_{j=1}^{k} a_j \cdot (\sigma_j\colon \Delta^n \to X)\Big] & \mapsto & \Big[\sum\limits_{j=1}^{k} a_j \cdot \partial\sigma_j\Big] \end{array}$$

is well-defined.

(2) *We consider the category of triples of topological spaces.*[700] *The maps ∂_n from (1) define a natural transformation from the functor $(X,B,A) \to \mathrm{H}_n(X,B)$ to the functor $(X,B,A) \to \mathrm{H}_{n-1}(B,A)$. In down-to-earth language this means that for a map $f\colon (X,A,B) \to (\widetilde{X},\widetilde{A},\widetilde{B})$ of triples of topological spaces the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}_n(X,B) & \xrightarrow{\quad\partial_n\quad} & \mathrm{H}_{n-1}(B,A) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{H}_n(\widetilde{X},\widetilde{B}) & \xrightarrow{\quad\partial_n\quad} & \mathrm{H}_{n-1}(\widetilde{B},\widetilde{A}). \end{array}$$

In particular the following diagram commutes:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \mathrm{H}_n(B,A) & \longrightarrow & \mathrm{H}_n(X,A) & \longrightarrow & \mathrm{H}_n(X,B) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(B,A) & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ \dots & \longrightarrow & \mathrm{H}_n(\widetilde{B},\widetilde{A}) & \longrightarrow & \mathrm{H}_n(\widetilde{X},\widetilde{A}) & \longrightarrow & \mathrm{H}_n(\widetilde{X},\widetilde{B}) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(\widetilde{B},\widetilde{A}) & \longrightarrow & \dots. \end{array}$$

(3) *The maps*

$$\dots \to \mathrm{H}_n(B,A) \xrightarrow{i_*} \mathrm{H}_n(X,A) \xrightarrow{p_*} \mathrm{H}_n(X,B) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(B,A) \to \dots$$

form an exact sequence.

The long exact sequence of Proposition 43.15 (2) is sometimes called the *long exact sequence of the triple* (X,B,A) and we refer to ∂_n as the *connecting homomorphism* of the triple (X,B,A).

PROOF. It follows immediately from the definitions and the third isomorphism theorem[701] that

$$0 \to \mathrm{C}_*(B,A) \xrightarrow{i_*} \mathrm{C}_*(X,A) \xrightarrow{p_*} \mathrm{C}_*(X,B) \to 0$$

is a short exact sequence of chain complexes.

[698]The fact that here the symbol "∂_n" denotes the boundary map of singular chains and the desired map $\mathrm{H}_n(X,B) \to \mathrm{H}_{n-1}(B,A)$ is an unfortunate problem in our choice of notation.

[699]Here we use that every homology class in $\mathrm{H}_n(X,B)$ admits by definition a representative in the quotient $\mathrm{C}_n(X,B) = \mathrm{C}_n(X)/\mathrm{C}_n(B)$, so it can be represented by a singular n-chain in X.

[700]It should be clear from the definitions preceding the definition what the objects and morphisms of this category are.

[701]The third isomorphism theorem in group theory implies that we have a natural isomorphism $(\mathrm{C}_*(X)/\mathrm{C}_*(A))/(\mathrm{C}_*(B)/\mathrm{C}_*(A)) \cong \mathrm{C}_*(X)/\mathrm{C}_*(B)$.

Claim. The maps ∂_n introduced in Statement (1) of the propositions are precisely the connecting homomorphisms, as defined on page 1115, of the above short exact sequence of chain complexes. In particular the maps ∂_n are well-defined.

The claim follows basically immediately from the definitions. But for completeness' sake we give some details. We consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_n(B,A) & \longrightarrow & \mathrm{C}_n(X,A) & \longrightarrow & \mathrm{C}_n(X,B) & \longrightarrow & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \longrightarrow & \mathrm{C}_{n-1}(B,A) & \longrightarrow & \mathrm{C}_{n-1}(X,A) & \longrightarrow & \mathrm{C}_{n-1}(X,B) & \longrightarrow & 0. \end{array}$$

Let $z \in \mathrm{H}_n(X,A)$. Now we will determine the image of z under the connecting homomorphism $\mathrm{H}_n(X,B) \to \mathrm{H}_{n-1}(B,A)$ which during the proof of the claim we denote by ∂^c.

We pick a representative $\sum\limits_{j=1}^{k} a_j \cdot (\sigma_j \colon \Delta^n \to X)$ in $\mathrm{C}_n(X)$. Note that $\sum\limits_{j=1}^{k} a_j \cdot (\sigma_j \colon \Delta^n \to X)$ also defines an element in $\mathrm{C}_n(X,A)$. We then take the boundary of the above singular n-chain under the boundary map $\partial_n \colon \mathrm{C}_n(X,A) \to \mathrm{C}_{n-1}(X,A)$ and we end up with the singular n-chain $\sum\limits_{j=1}^{k} a_j \cdot \partial\sigma_j \in \mathrm{C}_{n-1}(X,A)$. As we have seen in the proof of Lemma 43.10 this singular n-chain lies in fact in $\mathrm{C}_{n-1}(B,A) \subset \mathrm{C}_{n-1}(X,A)$. The homology class thereof is by definition precisely $\partial^c(z) \in \mathrm{H}_{n-1}(B,A)$. Summarizing we have shown that

$$\partial^c(z) \;=\; \left[\sum_{j=1}^{k} a_j \cdot \partial\sigma_j\right].$$

This shows that the map defined in (1) agrees with ∂^c. ⊞

Statement (2) of the lemma is an easy consequence of Lemma 43.10. The fact that the stated sequence of homology groups is exact is now an immediate consequence of Proposition 43.11. ■

For convenience we record two important special cases of Proposition 43.15 in the following corollary.

Corollary 43.16.

(1) *Let X be a topological space and let $B \subset X$ be a subset. The connecting homomorphism of Proposition 43.15 applied to $A = \varnothing$ gives rise to along exact sequence*

$$\dots \longrightarrow \mathrm{H}_n(B) \xrightarrow{i_*} \mathrm{H}_n(X) \xrightarrow{p_*} \mathrm{H}_n(X,B) \xrightarrow{\partial} \mathrm{H}_{n-1}(B) \longrightarrow \dots$$

Furthermore, for $n=1$ the connecting homomorphism $\partial \colon \mathrm{H}_1(X,B) \to \mathrm{H}_0(B)$ takes values in $\widetilde{\mathrm{H}}_0(B)$ and we obtain a long exact sequence[702]

$$\dots \longrightarrow \widetilde{\mathrm{H}}_n(B) \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{\text{ж}_{X,B*}} \mathrm{H}_n(X,B) \xrightarrow{\partial} \widetilde{\mathrm{H}}_{n-1}(B) \longrightarrow \dots$$

(2) *The connecting homomorphisms in* (1) *are natural. In particular, given a map $f \colon (X,A) \to (Y,B)$ between pairs of topological spaces we obtain a commutative*

diagram

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & \mathrm{H}_n(X) \arrow[r, "\text{я}"] \arrow[d, "f_*"] & \mathrm{H}_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & \mathrm{H}_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \dots \\
\dots \arrow[r] & \mathrm{H}_n(B) \arrow[r, "i_*"] & \mathrm{H}_n(Y) \arrow[r, "\text{я}"] & \mathrm{H}_n(Y,B) \arrow[r, "\partial"] & \mathrm{H}_{n-1}(B) \arrow[r] & \dots
\end{tikzcd}$$

The analogous statement also holds for reduced homology groups.

The long exact sequences of Corollary 43.16 (1) are sometimes called the *long exact sequence (in reduced homology) of the pair* (X,B).

Proof. First note that the statement regarding unreduced homology follows from Proposition 43.15 applied to $A=\emptyset$. Now we turn to the statement regarding reduced homology. This statement follows basically immediately from the unreduced case together with Lemma 43.1 (4) and Lemma 43.2. We leave it to the reader to fill in the dull details for the reduced case. ■

Remark. Let (X,B,A) be a triple of topological spaces. Then $(X,B,\emptyset)\to(X,B,A)$ is a map between triples of topological spaces and by the naturality of the long exact sequence we obtain the following commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_n(B) \arrow[r] \arrow[d] & \mathrm{H}_n(X) \arrow[r] \arrow[d] & \mathrm{H}_n(X,B) \arrow[r, "\partial"] \arrow[d, "="] & \mathrm{H}_{n-1}(B) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \mathrm{H}_n(B,A) \arrow[r] & \mathrm{H}_n(X,A) \arrow[r] & \mathrm{H}_n(X,B) \arrow[r, "\partial"] & \mathrm{H}_{n-1}(B,A) \arrow[r] & \dots .
\end{tikzcd}$$

Example. Let X be a topological space. Furthermore let $x_0\in X$ be a point. We denote by $i\colon\{x_0\}\to X$ the inclusion map. By Corollary 43.16 we have a long exact sequence

$$\to \underbrace{\mathrm{H}_1(\{x_0\})}_{=0} \xrightarrow{i_*} \mathrm{H}_1(X)\to\mathrm{H}_1(X,\{x_0\})\xrightarrow{i_*}\mathrm{H}_0(\{x_0\})\to\mathrm{H}_0(X)\to\mathrm{H}_0(X,\{x_0\})\to 0.$$

We obtain from that the map $i_*\colon \mathrm{H}_0(\{x_0\})\to\mathrm{H}_0(X)$ is a monomorphism. Together with Lemma 41.6 we see that for each $n\in\mathbb{N}$ the map $\mathrm{H}_n(X)\to\mathrm{H}_n(X,\{x_0\})$ is an isomorphism.

We move on to a more elaborate example.

Example. We let $X=\mathbb{R}^2$ and $A=B_1^2(-2)\cup B_1^2(2)$. We want to determine the relative homology groups of (X,A). We denote by $i\colon A\to X$ the inclusion map. By Corollary 43.16 we have a long exact sequence

$$\mathrm{H}_2(X)\to\mathrm{H}_2(X,A)\to\mathrm{H}_1(A)\xrightarrow{i_*}\mathrm{H}_1(X)\to\mathrm{H}_1(X,A)\to\mathrm{H}_0(A)\to\mathrm{H}_0(X)\to\mathrm{H}_0(X,A)\to 0.$$

By Corollary 42.8 (3) together with Lemma 41.14 we have $\mathrm{H}_i(X)=\mathrm{H}_i(\mathbb{R}^2)=0$ and also $\mathrm{H}_i(A)=\mathrm{H}_i(B_1(-2))\oplus\mathrm{H}_i(B_1(2))=0$ for $i\geq 1$. Thus we see from the above long exact

[702] Note that if B is non-empty, then it follows from Lemma 43.1 (0) that $\widetilde{\mathrm{H}}_{-1}(B)=0$. Thus the last possibly non-zero term is $\mathrm{H}_0(X,B)$. On the other hand, if $B=\emptyset$, then we have $\widetilde{\mathrm{H}}_{-1}(B)=\widetilde{\mathrm{H}}_{-1}(\emptyset)=\mathbb{Z}$, and if $X=\emptyset$, then we have $\widetilde{\mathrm{H}}_{-1}(X)=\widetilde{\mathrm{H}}_{-1}(\emptyset)=\mathbb{Z}$. If $B=\emptyset$ the sequence is understood to continue with $\mathrm{H}_0(B,\emptyset)\xrightarrow{\epsilon}\mathrm{H}_{-1}(\emptyset)\to\mathrm{H}_{-1}(X)$.

sequence that $\mathrm{H}_i(X, A) = 0$ for $i \geq 2$. We denote by $\epsilon\colon \mathrm{H}_0(\mathbb{R}^2) \to \mathbb{Z}$ the augmentation map that is an isomorphism by Proposition 41.5. Similarly we denote by $\mu_1\colon \mathrm{H}_0(B_1(-2)) \to \mathbb{Z}$ and $\mu_2\colon \mathrm{H}_0(B_1(2)) \to \mathbb{Z}$ the isomorphisms that are given by the augmentation maps. We obtain the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}_1(X,A) \arrow[r, "\partial"] \arrow[d, "="'] & \mathrm{H}_0(A) \arrow[r, "i_*"] \arrow[d, "\mu_1\oplus\mu_2"', "\cong"] & \mathrm{H}_0(X) \arrow[r] \arrow[d, "\cong"', "\epsilon"] & \mathrm{H}_0(X,A) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathrm{H}_1(X,A) \arrow[r] & \mathbb{Z}^2 \arrow[r] & \mathbb{Z} \arrow[r] & \mathrm{H}_0(X,A) \arrow[r] & 0.
\end{tikzcd}$$

We want to determine the matrix representing the homomorphism $\mathbb{Z}^2 \to \mathbb{Z}$ at the bottom. We do so in the following claim.

Claim. The homomorphism $\epsilon \circ i_* \circ (\mu_1 \oplus \mu_2)^{-1}\colon \mathbb{Z}^2 \to \mathbb{Z}$ is given by the matrix $(1\ 1)$.

We pick a point $P \in B_1(-2)$ and we pick a point $Q \in B_1(2)$. It follows from Proposition 41.5 and Lemma 41.14 that the map

$$\begin{array}{rcl} \Psi\colon \mathbb{Z}^2 & \to & \mathrm{H}_0(A) = \mathrm{H}_0(B_1(-2)) \oplus \mathrm{H}_0(B_1(2)) \\ (m,n) & \mapsto & [m\cdot P + n\cdot Q] \end{array}$$

is the inverse to $\mu_1 \oplus \mu_2$. Now it is easy to verify that $\epsilon \circ i_* \circ (\mu_1 \oplus \mu_2)^{-1} = \epsilon \circ i_* \circ \Psi$ applied to $(1,0)$ and $(0,1)$ gives in both cases the value 1. So the map is described by the matrix $(1\ 1)$. ⊞

Since the top sequence in the above commutative diagram is exact and since the vertical maps are isomorphisms we see that the bottom sequence is also exact. It is now straightforward to see that

$$\mathrm{H}_1(X,A) \;\cong\; \ker\left(\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}\right) \;\cong\; \mathbb{Z} \quad \text{and} \quad \mathrm{H}_0(X,A) \;\cong\; \operatorname{coker}\left(\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}\right) \;\cong\; 0.$$

Now we return to the situation of the example on page 1121. Let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex with $\sigma(0,1) \in B_1(2)$ and $\sigma(1,0) \in B_1(-2)$. We have

$$((\mu_1 \oplus \mu_2)\circ\partial)[\sigma]) \underset{\substack{\uparrow \\ \text{description of } \partial \text{ in Proposition 43.15}}}{=} ((\mu_1\oplus\mu_2)([\partial\sigma]) \;=\; (\mu_1\oplus\mu_2)([\underbrace{\sigma(0,1)}_{\in B_1(2))}] - [\underbrace{\sigma(1,0)}_{\in B_1(-2))}]) \;=\; (1,-1).$$

Since $(1,-1)$ is a generator of the kernel of $\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}$ we see that $[\sigma]$ is in fact a generator of $\mathrm{H}_1(X,A) \cong \mathbb{Z}$. In particular it is non-zero.

Recall that, as defined on page 571, we say that two maps $f, g\colon (X,A) \to (Y,B)$ between pairs of topological spaces are homotopic, if there exists a map $F\colon X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$, and $F(a,t) \in B$ for all $a \in A$ and all $t \in [0,1]$. The following proposition extends Proposition 42.5 to the relative case.

Proposition 43.17. *Let $f, g\colon (X,A) \to (Y,B)$ be two maps between pairs of topological spaces. If f and g are homotopic, then the maps $f_*, g_*\colon \mathrm{C}_*(X,A) \to \mathrm{C}_*(Y,B)$ are chain homotopic, in particular for every $n \in \mathbb{N}_0$ we have*

$$f_* \;=\; g_*\colon\ \mathrm{H}_n(X,A) \;\to\; \mathrm{H}_n(Y,B).$$

Proof. Let $f, g\colon (X, A) \to (Y, B)$ be two homotopic maps between pairs of topological spaces. To prove the proposition we have to look again at the proof of Proposition 42.5. In that proof we explicitly constructed a chain $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(Y)$, $n \in \mathbb{N}_0$ between the maps f_* and g_* from $\mathrm{C}_*(X)$ to $\mathrm{C}_*(Y)$. It follows immediately from the definition of the chain homotopy that it has the property that $\mathrm{C}_n(A)$ gets sent to $\mathrm{C}_{n+1}(B)$. In particular the chain homotopy descends to maps $Q_n\colon \mathrm{C}_n(X, A) \to \mathrm{C}_{n+1}(Y, B)$ which then form a chain homotopy between the maps f_* and g_* from $\mathrm{C}_*(X, A)$ to $\mathrm{C}_*(Y, B)$. It now follows immediately from Lemma 42.2 that $f_* = g_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B)$. ■

We conclude this section with the following result that can be viewed as a generalization of Corollary 42.8 to the relative setting.

Corollary 43.18.
(1) *If $f\colon (X, A) \to (Y, B)$ is a homotopy equivalence of pairs of topological spaces,*[703] *then for every $n \in \mathbb{N}_0$ the induced map $f_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B)$ is an isomorphism.*
(2) *Let X be a topological space.*
 (a) *Let $A \subset B$ be subsets of X. If A is a deformation retract of B, then for every $n \in \mathbb{N}_0$ the inclusion induced map $\mathrm{H}_n(X, A) \to \mathrm{H}_n(X, B)$ is an isomorphism.*
 (b) *If A is a deformation retract of X, then $\mathrm{H}_n(X, A) = 0$ for all $n \in \mathbb{N}_0$.*
 (c) *Let $A \subset Y$ be subsets of X. If Y is a deformation retract of X, then for every $n \in \mathbb{N}_0$ the inclusion induced map $\mathrm{H}_n(Y, A) \to \mathrm{H}_n(X, A)$ is an isomorphism.*

Proof. Let X be a topological space.
(1) We deduce this statement from Proposition 43.17 the same way we deduced Corollary 42.8 from Proposition 42.5.
(2) (a) Let $A \subset B$ be a deformation retract of B. As we mentioned on page 571, the inclusion map $(X, A) \to (X, B)$ is a homotopy equivalence of pairs of topological spaces. Thus the desired statement follows from (1).
 (b) This statement follows from Statement (a) applied to $B = X$ and the trivial observation that $\mathrm{H}_n(X, X) = 0$ for all $n \in \mathbb{N}_0$.
 (c) Let $A \subset Y$ be subsets of X. If Y is a deformation retract of X, then we saw already on page 571 that the inclusion map $(Y, A) \to (X, A)$ is a homotopy equivalence of pairs of topological spaces. Thus the desired statement follows again from (1). ■

The statement of Corollary 43.16 looks already very similar to the statement of Theorem 43.3 that we actually want to prove. Now we "only" have to find the connection between the groups $\mathrm{H}_n(X, A)$ and $\mathrm{H}_n(X/A)$. The notations already look very similar, and they do so for good reasons, since in both cases "one ignores what lies in A". Nonetheless, proving the precise relationship will still require some substantial piece of work.

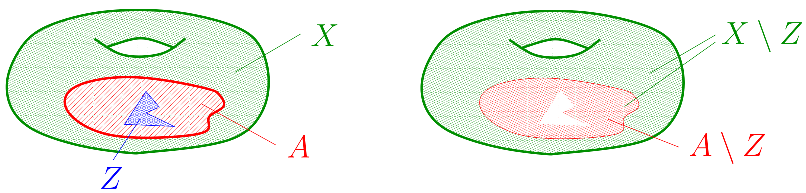
43.6. **The Excision Theorem.** The key step in finding the connection between the groups $\mathrm{H}_n(X, A)$ and $\mathrm{H}_n(X/A)$ is given by the following Excision Theorem 43.19.

[703]We refer to page 571 for the definition.

Theorem 43.19. (Excision Theorem) *Let X be a topological space and let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A. Then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism*

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathrm{H}_n(X, A).$$

The theorem thus says in particular that if we are given a pair of topological spaces (X, A) and if we excise [704] a subset Z from the interior of A, then this does not affect the relative homology groups.



$$\mathrm{H}_n(X, A) \;\cong\; \mathrm{H}_n(X \setminus Z, A \setminus Z)$$

FIGURE 717. Illustration of the Excision Theorem 43.19.

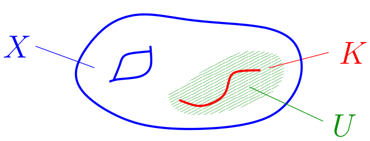
On many occasions we will need the following variation on the Excision Theorem 43.19.

Theorem 43.20. (Excision Theorem) *Let X be a topological space.*

(1) *Let K be a subset of X and let U be a neighborhood of the closure of K. Then the inclusion $(U, U \setminus K) \to (X, X \setminus K)$ induces for each $n \in \mathbb{N}_0$ an isomorphism*

$$\mathrm{H}_n(U, U \setminus K) \xrightarrow{\cong} \mathrm{H}_n(X, X \setminus K).$$

(2) *The hypotheses of (1) are satisfied if X is Hausdorff (e.g. if X is a topological manifold) and if K is a compact subset that is contained in an open set U.*



here we have $\mathrm{H}_n(U, U \setminus K) \xrightarrow{\cong} \mathrm{H}_n(X, X \setminus K)$

FIGURE 718. Illustration of the Excision Theorem 43.20.

PROOF OF THEOREM 43.20 ASSUMING THEOREM 43.19.

(1) We have

$$\mathrm{H}_n(U, U \setminus K) \underset{\substack{\uparrow \\ \text{set } A := X \setminus K \text{ and } Z := X \setminus U}}{=} \mathrm{H}_n(X \setminus Z, A \setminus Z) \underset{\substack{\uparrow \\ \text{since } U \text{ is a neighborhood of the closure of } K \text{ we see} \\ \text{that the closure of } Z \text{ is contained in the interior of } A, \\ \text{thus we can apply the Excision Theorem 43.19}}}{\xrightarrow{\cong}} \mathrm{H}_n(X, A) = \mathrm{H}_n(X, X \setminus K).$$

(2) If X is Hausdorff and if K is a compact subset that is contained in an open set U, then it follows from Lemma 2.17 (2) that K is also a closed subset, hence we have $\overline{K} \subset U$. ∎

[704] The verb "to excise" means to cut out, or in German "ausschneiden".

Before we turn to the proof of the Excision Theorem 43.19 we first want to show that we can now relate the relative homology groups of a pair (X,A) to the homology groups of the quotient space X/A, assuming $A\subset X$ is good. To do so we need the following definition.

Definition. Let X be a topological space and let $A\subset X$ be a[705] subset. We denote by $p\colon (X,A)\to(X/A,A/A)$ the obvious projection map. For each $n\in\mathbb{N}_0$ we define

$$\begin{tikzcd} \mathrm{H}_n(X,A) \arrow[r, "p_*"] \arrow[rr, bend right, "=:\text{ю}"'] & \mathrm{H}_n(X/A,A/A) & \widetilde{\mathrm{H}}_n(X/A) \arrow[l, "\text{ж from Lemma 43.14}"', "\cong"] \end{tikzcd}.$$

The following lemma summarizes two properties of the maps ю.

Lemma 43.21. $(*)$

(1) *Let X be a topological space and let $A\subset X$ be a subset. We denote by $p\colon X\to X/A$ the projection. The following diagram commutes:*[706]

$$\begin{tikzcd} & \mathrm{H}_k(X,A) \arrow[dr, "\text{ю}"] & \\ \widetilde{\mathrm{H}}_k(X) \arrow[ur, "\text{ж}"] \arrow[rr, "p_*"] & & \widetilde{\mathrm{H}}_k(X/A). \end{tikzcd}$$

(2) *We denote by $\mathcal{PairTop}$ the category of pairs of topological spaces that we introduced on page 477. Furthermore as usual we denote by $\mathcal{AbGr}$ the category of abelian groups. The above maps* ю *define a natural transformation from the functor*

$$\begin{array}{rcl} \mathcal{PairTop} & \to & \mathcal{AbGr} \\ (X,A) & \mapsto & \mathrm{H}_n(X,A) \end{array} \quad \text{to the functor} \quad \begin{array}{rcl} \mathcal{PairTop} & \to & \mathcal{AbGr} \\ (X,A) & \mapsto & \widetilde{\mathrm{H}}_n(X/A). \end{array}$$

PROOF $(*)$.

(1) We consider the following diagram

$$\begin{tikzcd} & \mathrm{H}_k(X,A) \arrow[d, "p_*"] \arrow[dr, bend left, "\text{ю}"] & \\ & \mathrm{H}_k(X/A,A/A) & \\ \widetilde{\mathrm{H}}_k(X) \arrow[uur, "\text{ж}"] \arrow[ur, "{[\sigma]\mapsto[\sigma]}"'] \arrow[rr, "p_*"'] & & \widetilde{\mathrm{H}}_k(X/A). \arrow[ul, "\text{Lemma 43.14}"', "\cong"] \end{tikzcd}$$

It follows immediately from the definitions that all three regions of the diagram commute. This observation gives us the desired statement.

(2) This statement follows basically immediately from the definitions. ■

The following proposition is much more interesting than the previous lemma.

[705]Note that the definition also makes sense if A is the empty set.

[706]The left diagonal map is the composition of the natural map $\widetilde{\mathrm{H}}_k(X)\to\mathrm{H}_k(X)$ from Lemma 43.1 with the natural map $\mathrm{H}_k(X)\to\mathrm{H}_k(X,A)$.

Proposition 43.22. *Let X be a topological space and let $A \subset X$ be a subset and let $n \in \mathbb{N}_0$. If A is good, then the natural homomorphism $\mathrm{H}_n(X, A) \xrightarrow{\text{ю}} \widetilde{\mathrm{H}}_n(X/A)$ is an isomorphism.*

Remark. In Corollary 46.17 we will show the analogue of Proposition 43.22 under the slightly different hypothesis that the inclusion map $i\colon A \to X$ is a closed cofibration.

Example. Let X be a topological space, let $A \subset X$ be a subset and let $n \in \mathbb{N}_0$. For better or worse the natural homomorphism $\mathrm{H}_n(X, A) \to \widetilde{\mathrm{H}}_n(X/A)$ is in general not an isomorphism. In fact in general the two groups $\mathrm{H}_n(X, A)$ and $\widetilde{\mathrm{H}}_n(X/A)$ are not even abstractly isomorphic.

For example consider $X = [0, 1]$ and $A = \{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$. In Exercise 25.1 we showed that the quotient space X/A is homeomorphic to the Hawaiian earrings. (See Figure 719 for an illustration.) In Exercise 43.17 we will prove the following two statements:

(1) The group $\mathrm{H}_1(X, A)$ is countable.
(2) The group $\mathrm{H}_1(X/A)$ is uncountable.

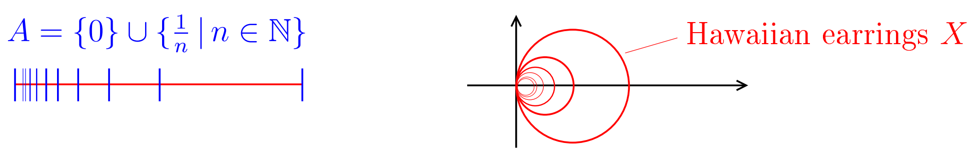


Figure 719

Proof of Proposition 43.22 assuming the Excision Theorem 43.19. Let X be a topological space, let $A \subset X$ be a good subset and let $n \in \mathbb{N}_0$.

First we need to get the annoying case that $A = \varnothing$ out of the way. Note that the map $(X, \varnothing) \to (X/\varnothing, \varnothing/\varnothing)$ is easily seen to be a homeomorphism of pairs of topological spaces. Thus it remains to show that the map ж: $\widetilde{\mathrm{H}}_n(X/\varnothing) \to \mathrm{H}_n(X, \varnothing)$ is an isomorphism. It follows immediately from Lemma 43.1 (4) and Corollary 43.16 that the only dimension one needs to worry about for a second is $n = 0$. In this case we have the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathrm{H}_0(\varnothing/\varnothing) \arrow[d] \arrow[dr, "\cong"] & & \\
0 \arrow[r] & \widetilde{\mathrm{H}}_0(X/\varnothing) \arrow[r, "ж"] \arrow[dr, "ж"'] & \mathrm{H}_0(X/\varnothing) \arrow[r, "\epsilon"] \arrow[d] & \mathbb{Z} \arrow[r] & 0 \\
 & & \mathrm{H}_0(X/\varnothing, \varnothing/\varnothing) \arrow[d] & & \\
 & & 0. & &
\end{tikzcd}$$

It follows from Lemma 43.2 and Corollary 43.16 that the horizontal and the vertical sequences are both exact. The diagonal map on the top right is easily seen to be an isomorphism. Some mild diagram chasing, see Exercise 43.10, now shows that the diagonal map on bottom the is also an isomorphism.

Now we turn to the much more interesting case that $A \neq \emptyset$. We make the following simple preparations:

(1) Since A is good we can pick an open neighborhood U of A in X such that A is a deformation retract of U.
(2) We denote by $p\colon X \to X/A$ the projection map.
(3) We denote by $q\colon X \setminus A \to X/A \setminus A/A$ the restriction of p to $X \setminus A$. For completeness' sake we point out that it follows from Lemma 2.30 that q is continuous.

Now we consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_n(X,A) \arrow[r] \arrow[d, "p_*"] & \mathrm{H}_n(X,U) \arrow[d, "p_*"] & \mathrm{H}_n(X\setminus A, U\setminus A) \arrow[l] \arrow[d, "q_*"] \\
\mathrm{H}_n(X/A, A/A) \arrow[r] & \mathrm{H}_n(X/A, U/A) & \mathrm{H}_n(X/A \setminus A/A, U/A\setminus A/A), \arrow[l]
\end{tikzcd}$$

here all the horizontal maps are induced by inclusions of pairs of topological spaces. It follows easily from the definitions that the diagram commutes. It follows from our hypothesis that A is closed, together with the slightly obscure but rather elementary Lemma 3.29, that the map $q\colon (X \setminus A, U \setminus A) \to (X/A \setminus A/A, U/A \setminus A/A)$ is a homeomorphism of pairs of topological spaces. We obtain from this observation that the vertical map on the right is an isomorphism. Our actual goal is to show that the vertical map on the left is an isomorphism. This follows from the commutativity of the above diagram and from the following claim:

Claim. All horizontal maps are isomorphisms.

We distinguish three different situations:

(1) It follows from the Excision Theorem 43.19 that both the top horizontal and the bottom horizontal map on the right-hand side are isomorphisms.
(2) Since A is deformation retract of U we obtain from Corollary 43.18 that the top left horizontal map is an isomorphism.
(3) Recall that A is a deformation retract of U. It follows from Corollary 18.22 that A/A is a deformation retract of U/A.[707] As in (2) we now see that the bottom-left horizontal map is an isomorphism.

■

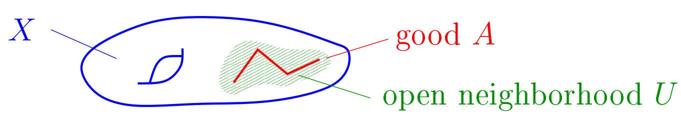


FIGURE 720. Illustration for the proof of Proposition 43.22.

PROOF OF THEOREM 43.3 ASSUMING THE EXCISION THEOREM 43.19. Let X be a topological space and let $A \subset X$ be a good subset. We denote by $i\colon A \to X$ the inclusion and

[707]The diligent reader who checks the statements we refer to will not fail to notice, that Corollary 18.22 rests eventually on the not-so-trivial Theorem 5.16. It is all too often overlooked (see see e.g. earlier version of these notes) that Corollary 18.22 is required and that it is non-trivial.

we denote by $q\colon X \to X/A$ the projection map. We consider the following diagram:

$$\begin{tikzcd} \dots \arrow[r] & \widetilde{\mathrm{H}}_n(A) \arrow[r, "i_*"] & \widetilde{\mathrm{H}}_n(X) \arrow[r, "ж_{X,A*}"] \arrow[dr, "q_*"'] & \mathrm{H}_n(X,A) \arrow[r, "\partial"] \arrow[d, "ю"] & \widetilde{\mathrm{H}}_{n-1}(A) \arrow[r] & \widetilde{\mathrm{H}}_{n-1}(X) \arrow[r, "i_*"] & \dots \\ & & & \widetilde{\mathrm{H}}_n(X/A). & & & \end{tikzcd}$$

We make the following comments and observations:

(1) The horizontal sequence is the long exact sequence coming from Corollary 43.16.
(2) The vertical map is the natural homomorphism that we defined on page 1129. By Lemma 43.21 (1) we know that the triangle commutes.
(3) Since A is good we obtain from Proposition 43.22 that the vertical map ю is an isomorphism.

Now we consider the maps $\widetilde{\partial} := ю^{-1} \circ \partial\colon \widetilde{\mathrm{H}}_n(X/A) \to \widetilde{\mathrm{H}}_{n-1}(A)$. Since ю and ∂ are natural we see that the maps $ю^{-1} \circ \partial$ are also natural. By the above the sequence

$$\dots \longrightarrow \widetilde{\mathrm{H}}_n(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{q_*} \widetilde{\mathrm{H}}_n(X/A) \xrightarrow{\widetilde{\partial}} \widetilde{\mathrm{H}}_{n-1}(A) \longrightarrow \widetilde{\mathrm{H}}_{n-1}(X) \xrightarrow{i_*} \dots$$

is exact. ■

43.7. **The proof of the Excision Theorem 43.19: the idea.** We are still not done with the proof of Theorem 43.3 since we still have to prove the Excision Theorem 43.19. In this section we want to come up with the idea for the proof of the Excision Theorem 43.19. The full details of the proof will then be given in the following section.

Thus let X be a topological space and let $Z \subset A \subset X$ be two subsets such that the closure of Z is contained in the interior of A. Our goal is to show that the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \ \to \ \mathrm{H}_n(X, A).$$

Let us attempt to prove that the map $\mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A)$ is surjective. We take an element in $\mathrm{H}_n(X, A)$ and we choose a representative $a_1\sigma_1 + \dots + a_r\sigma_r$ where $\sigma_i\colon \Delta^n \to X$, $i = 1, \dots, r$ are singular n-simplices in X.

We make the following two basic observations:

(1) If the singular n-simplices $\sigma_1, \dots, \sigma_r$ lie in $X \setminus Z$,[708] then $[a_1\sigma_1 + \dots + a_r\sigma_r]$ evidently lies in the image of the inclusion induced map $\mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A)$.
(2) If a singular n-simplex lies entirely in A, then it represents the zero element in the quotient group $\mathrm{C}_n(X, A) = \mathrm{C}_n(X)/\,\mathrm{C}_n(A)$.

The combination of these two observations proves the following lemma.

Lemma 43.23. *Let X be a topological space and let $Z \subset A$ be two subsets of X. Let $c = a_1\sigma_1 + \dots + a_r\sigma_r$ be a cycle in X such that each σ_i lies entirely in A or it lies entirely in $X \setminus Z$. Then $[c]$ lies in the image of* $\mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A)$.

[708] Given a topological space X and a subset A we say that *a singular simplex $\sigma\colon \Delta^n \to X$ lies in A* if the image of σ lies in A.

Unfortunately not every cycle in $\mathrm{C}_n(X)$ satisfies the hypothesis of the lemma. But we will see in the next section that we can subdivide each singular simplex into smaller singular simplices that lie entirely in A or lie entirely in $X \setminus Z$.

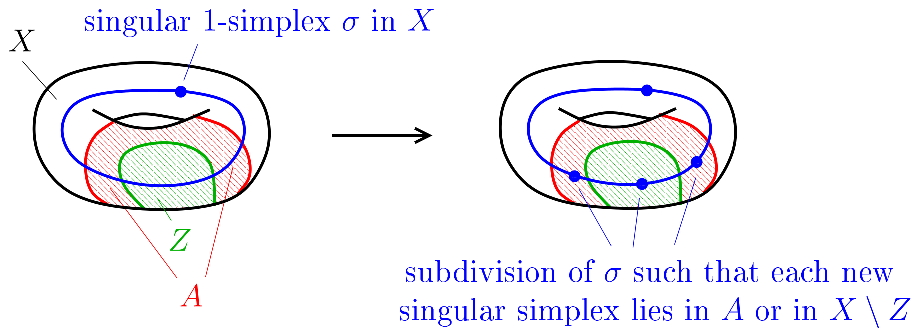


FIGURE 721. Illustration of the key idea of the proof of the Excision Theorem 43.19.

43.8. **The proof of the Excision Theorem 43.19: the full details.** In this chapter we will now turn the vague idea of the last section into a rigorous proof of the Excision Theorem 43.19. This will take a surprisingly long time.

First we introduce a few definitions.

Definition. Let again X be a topological space.

(1) On page 89 we defined a *cover* of X to be a family $\{U_i\}_{i\in I}$ of subsets of X such that $X = \bigcup\limits_{i\in I} U_i$.

(2) We say a cover $\{U_i\}_{i\in I}$ is *comfortable* if $X = \bigcup\limits_{i\in I} \mathring{U}_i$, i.e. if the interiors of the U_i also form a cover of X.

We move straight to the next definition.

Definition. Let again X be a topological space. Given a comfortable cover $\mathcal{U} = \{U_i\}_{i\in I}$ we consider the following subgroup of $\mathrm{C}_n(X)$:

$$\mathrm{C}_n^{\mathcal{U}}(X) := \left\{ \sum_{j=1}^{k} a_j\sigma_j \,\middle|\, \text{for each } j \text{ there exists a } U_i \in \mathcal{U} \text{ such that the image of } \sigma_j \text{ lies in } U_i \right\}.$$

It follows quite easily from the definitions that the restriction of the usual boundary map $\partial_n\colon \mathrm{C}_n(X) \xrightarrow{\partial} \mathrm{C}_{n-1}(X)$ to $\mathrm{C}_n^{\mathcal{U}}(X)$ restricts to a map $\partial_n\colon \mathrm{C}_n^{\mathcal{U}}(X) \xrightarrow{\partial} \mathrm{C}_{n-1}^{\mathcal{U}}(X)$. Since this map is the restriction of the usual boundary map it follows in particular that the composition $\partial_n \circ \partial_{n+1}\colon \mathrm{C}_{n+1}^{\mathcal{U}}(X) \xrightarrow{\partial} \mathrm{C}_{n-1}^{\mathcal{U}}(X)$ is the zero map. Thus we can define

$$\begin{aligned} \mathrm{H}_n^{\mathcal{U}}(X) &:= \mathrm{H}_n\left(C_*^{\mathcal{U}}(X), \partial_*\right) \\ &= \ker\left(\partial_n\colon \mathrm{C}_n^{\mathcal{U}}(X) \to \mathrm{C}_{n-1}^{\mathcal{U}}(X)\right) \big/ \operatorname{im}\left(\partial_{n+1}\colon \mathrm{C}_{n+1}^{\mathcal{U}}(X) \to \mathrm{C}_n^{\mathcal{U}}(X)\right). \end{aligned}$$

The homology groups $\mathrm{H}_n^{\mathcal{U}}(X)$ are thus defined by singular chains which are "small" in the sense that the image of each singular simplex lies in some U_i.

The following proposition says that we can determine the homology of a topological space through such "small" singular chains. As we will see later on many occasions, this proposition is at the heart of many methods for computing homology groups.

Proposition 43.24. *Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a comfortable cover of X. Then the inclusion map*

$$\mathrm{C}_*^{\mathcal{U}}(X) \;\to\; \mathrm{C}_*(X)$$

induces for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n^{\mathcal{U}}(X) \;\xrightarrow{\cong}\; \mathrm{H}_n(X).$$

Remark. In fact a stronger statement holds: the inclusion map $\mathrm{C}_*^{\mathcal{U}}(X) \to \mathrm{C}_*(X)$ is in fact a chain homotopy equivalence. Later, in Section 49.1, we will show that this stronger statement follows in fact from Proposition 43.24 and a purely algebraic argument. Alternatively one can fight one's way through the proof of Proposition 43.24 and upgrade the proof to this stronger statement. This approach is for example pursued in [**Hat02**, Proposition 2.21]. The disadvantage is that it makes the proof even more complicated than it already is.

The idea of the proof of Proposition 43.24 is the one we already described in the previous section. We want to replace systematically each singular n-simplex $\sigma\colon \Delta^n \to X$ by a singular n-chain of the form $\sum\limits_{j=1}^{r} a_j \cdot \sigma_j$ such that the image of each singular simplex σ_j lies in some $U_i \in \mathcal{U}$. As we have seen before in Lemma 41.3 in the 1-dimensional case, an appropriate subdividing procedure does not change the homology class. The systematic approach we are going to use is given by "barycentric subdivision". The key idea is sketched in Figure 722 for the case $n = 2$.

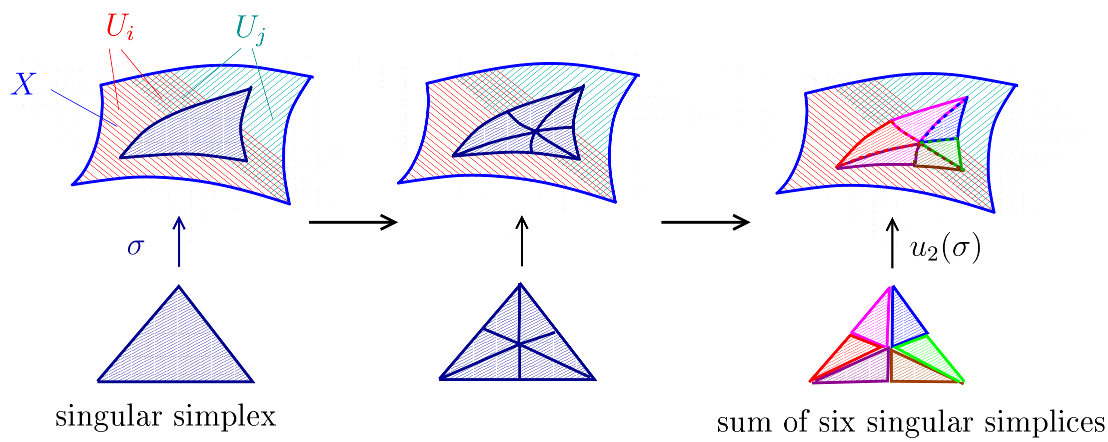


FIGURE 722. Barycentric decomposition of a singular 2-simplex.

Before we can execute this idea we have to introduce a few more definitions.

(1) Given $n \in \mathbb{N}_0$ we denote by $B_n := (\frac{1}{n+1}, \ldots, \frac{1}{n+1}) \in \Delta^n$ the *barycenter*[709] of the standard n-simplex Δ^n.

[709]The word "barycenter" comes from the Greek word "barys" which means heavy. In physics the barycenter of an object is the center of mass, in our case the barycenter of the standard n-simplex is the center of mass if we view Δ^k as a physical object of uniform density.

(2) Given a singular l-simplex $\mu\colon \Delta^l \to \Delta^n$ we consider the singular $(l+1)$-simplex $c_{l+1}(\mu)\colon \Delta^{l+1} \to \Delta^n$ which is defined as the composition of the (inverses) of the following two maps:

$$\begin{array}{rcccl} & & \text{homeomorphism by Proposition 2.43 (3)} & & \text{continuous by Lemma 3.22} \\ & & \downarrow & & \downarrow \\ c_{l+1}(\mu)\colon \Delta^{l+1} & \overset{\cong}{\longleftarrow} & (\Delta^l \times [0,1])/\Delta^l \times \{1\} & \to & \Delta^n \\ ((t_0,\dots,t_l)\cdot(1-s),s) & \leftarrow\!\shortmid & [((t_0,\dots,t_l),s)] & & \\ & & [((t_0,\dots,t_l),s)] & \mapsto & \underbrace{\mu(t_0,\dots,t_l)\cdot(1-s)+B\cdot s}_{\text{contained in } \Delta^n \text{ since } \Delta^n \text{ is convex}}. \end{array}$$

Pictorially speaking the singular $(l+1)$-simplex $c_{l+1}(\mu)$ thus corresponds to the "cone" which is given by the singular l-simplex μ and the barycenter B_n. By Lemma 19.1 this map $\mu \to c_{l+1}(\mu)$ defines a linear map $\mathrm{C}_l(\Delta^n) \to \mathrm{C}_{l+1}(\Delta^n)$ that we also denote by c_{l+1} and which we call the $(l+1)$-*st cone map*.

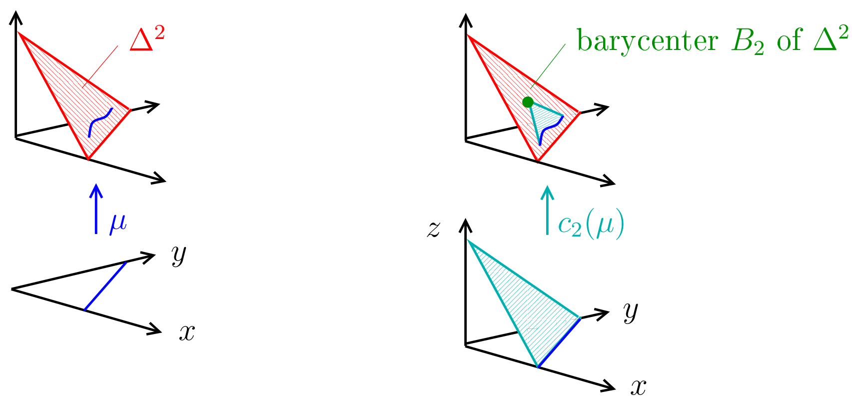


FIGURE 723. Illustration of the cone map.

(3) For $n = 0,1,2,3,\dots$ we now inductively do the following:
 (a) we define an element $\chi_n \in \mathrm{C}_n(\Delta^n)$, and
 (b) for each topological space X we define a homomorphism $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$.

 We do this inductive procedure as follows:
 (i) We set $\chi_0 = \mathrm{id}\colon \Delta^0 \to \Delta^0$ and for each topological space X we define u_0 to be the identity on $\mathrm{C}_0(X)$.
 (ii) Now suppose $\chi_{n-1} \in \mathrm{C}_{n-1}(\Delta^{n-1})$ and that furthermore all the homomorphisms $u_{n-1}\colon \mathrm{C}_{n-1}(X) \to \mathrm{C}_{n-1}(X)$ are already defined. Then we define

$$\chi_n \;:=\; (-1)^n \cdot c_n(u_{n-1}(\underbrace{\partial\,\mathrm{id}_{\Delta^n}}_{\in \mathrm{C}_{n-1}(\Delta^n)})) \;\in\; \mathrm{C}_n(\Delta^n).$$

 Furthermore, for each topological space X we define

$$\begin{array}{rcl} u_n\colon\ \mathrm{C}_n(X) & \to & \mathrm{C}_n(X) \\ \sum\limits_{l=1}^{r} a_l \cdot (\sigma_l\colon \Delta^n \to X) & \mapsto & \sum\limits_{l=1}^{r} a_l \cdot \underset{\substack{\uparrow \\ \text{the map } \sigma_l\colon \Delta^l \to X \text{ induces a map } (\sigma_l)_*\colon\ \mathrm{C}_n(\Delta^l) \to \mathrm{C}_n(X)}}{(\sigma_l)_*}(\underbrace{\chi_n}_{\in \mathrm{C}_n(\Delta^n)}). \end{array}$$

Note that with this notation we have $u_n(\mathrm{id}_{\Delta^n}) = \chi_n$. The map u_n is illustrated for $n = 1$ in Figure 724. For $n = 2$ we already gave a more schematic sketch in Figure 722 without considering signs and orientations.

The definition of χ_1 is illustrated in Figure 724.

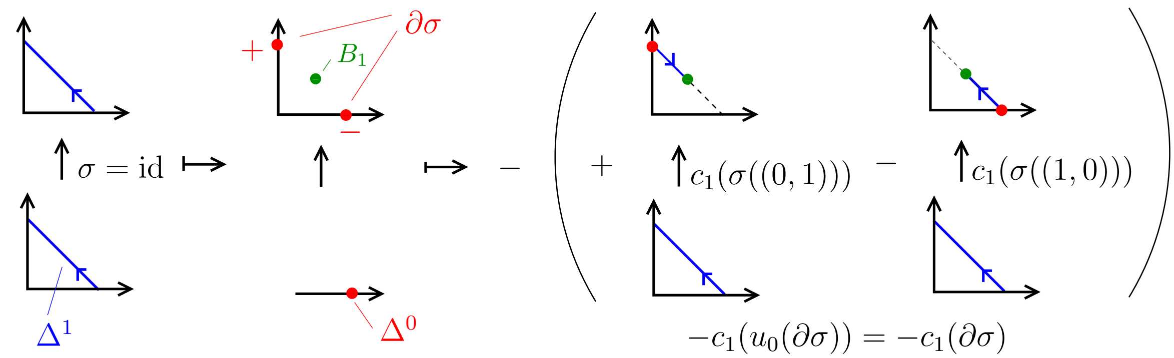


FIGURE 724. Illustration of the definition of first subdivision map.

Before we can continue we need to prove several technical lemmas about the subdivision maps.

Lemma 43.25. *Let X be a topological space. The maps $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$, $n \in \mathbb{N}_0$, form a chain map from the chain complex $(\mathrm{C}_*(X), \partial)$ to itself.*

The statement of Lemma 43.25 is equivalent to the statement that for any singular n-simplex σ in X we have

$$u_{n-1}(\partial\sigma) \;=\; \partial(u_n(\sigma)).$$

Put differently, Lemma 43.25 says that it does not make a difference whether we first take the boundary and then subdivide, or whether we first subdivide and then take the boundary. This statement is sketched in Figure 725. The picture is quite convincing.

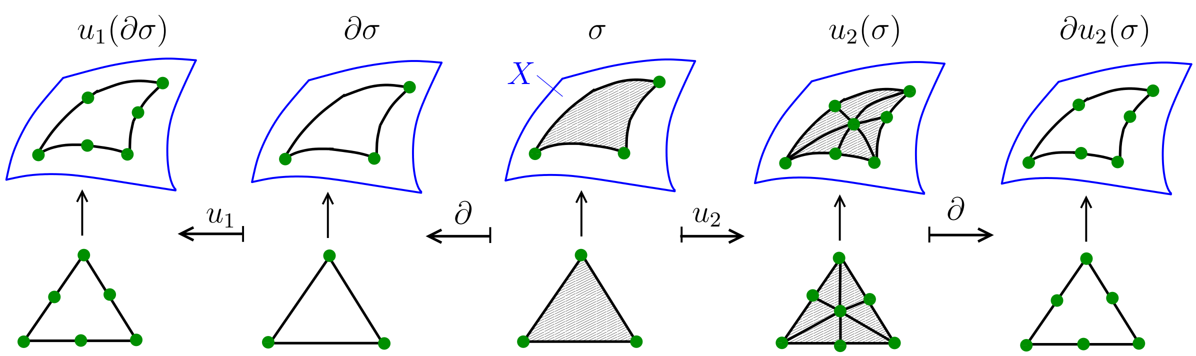


FIGURE 725. Illustration of Lemma 43.25.

PROOF OF LEMMA 43.25 (∗). [710] We adopt the following notation and we make the following remarks:

(1) We denote the barycenter of Δ^n by b.
(2) As on page 1078, given points $x_0,\dots,x_k\in\Delta^n$ we denote by $[x_0,\dots,x_k]$ the singular k-chain $\Delta^k\to\Delta^n$ that is given by[711]

$$\begin{array}{rcl}\Delta^k & \to & \Delta^n\\ (t_0,\dots,t_k) & \mapsto & \sum\limits_{i=0}^k t_ix_i,\end{array}$$

we refer to such a singular k-simplex as a *linear* singular simplex.
(3) We denote by $\mathrm{C}_k^{\mathrm{lin}}(\Delta^n)$ the subgroup of $\mathrm{C}_k(\Delta^n)$ generated by all the linear singular k-simplices. Note that the boundary map $\partial\colon\mathrm{C}_k(\Delta^n)\to\mathrm{C}_{k-1}(\Delta^n)$ restricts to a boundary map $\partial\colon\mathrm{C}_k^{\mathrm{lin}}(\Delta^n)\to\mathrm{C}_{k-1}^{\mathrm{lin}}(\Delta^n)$.
(4) For $x_0,\dots,x_k\in\Delta^n$ we write

$$[x_0,\dots,\widehat{x_i},\dots,x_k] \;:=\; [x_0,\dots,x_{i-1},x_{i+1},\dots,x_k].$$

The proof of the lemma will be broken into three claims.

Claim. For any $\mu\in\mathrm{C}_k^{\mathrm{lin}}(\Delta^n)$ we have

$$\partial(c_{k+1}(\mu)) \;=\; c_k(\partial\mu)+(-1)^{k+1}\cdot\mu \;\in\; \mathrm{C}_k(\Delta^n).$$

We illustrate the claim in Figure 726. To prove the claim we first note that for any

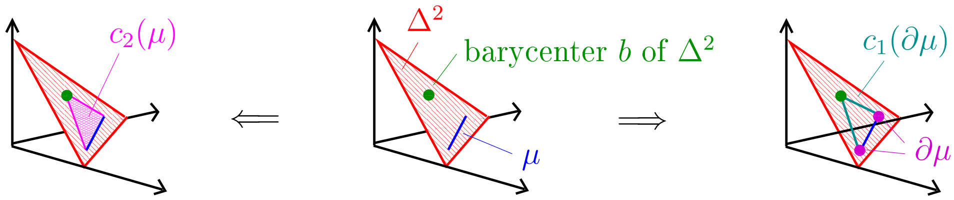


FIGURE 726

choice of points $x_0,\dots,x_{l-1}\in\Delta^n$ we have by definition of c_l that

$$c_l([x_0,\dots,x_{l-1}]) \;=\; [x_0,\dots,x_{l-1},b].$$

[710]We will not do this proof in the lectures. The proof is somewhat lengthy but straightforward. The cleverness hereby does not lie in the proof, but in the cleverly chosen definitions, which turn the verification into a simple calculation.

[711]It is straightforward to show that this map does indeed take values in Δ^n.

It now follows that for any $w_0, \dots, w_k \in \Delta^n$ we have the following equality in $\mathrm{C}_k(\Delta^n)$:

$$\begin{aligned}
\partial(c_{k+1}([w_0,\dots,w_k])) &\overset{\substack{\text{by the above equality} \\ \downarrow}}{=} \partial([w_0,\dots,w_k,b]) \\
&\underset{\substack{\uparrow \\ \text{definition of } \partial \text{ and by the above equality}}}{=} \sum_{j=0}^{k}(-1)^j \cdot [w_0,\dots,\widehat{w_j},\dots,w_k,b] + (-1)^{k+1}\cdot[w_0,\dots,w_k] \\
&\overset{\downarrow}{=} c_k(\partial([w_0,\dots,w_k])) + (-1)^{k+1}\cdot[w_0,\dots,w_k].
\end{aligned}$$

The claim immediately follows from this equality.

Claim. For any $n \in \mathbb{N}_0$ and any $\nu \in \mathrm{C}_n^{\mathrm{lin}}(\Delta^n)$ we have

$$\partial u_n(\nu) \; = \; u_{n-1}(\partial\nu).$$

We will prove this claim by induction on n. The case $n=0$ is trivial. So now suppose we have already shown the case $n-1$. It suffices to show the claim for all linear singular n-simplices. So let $\nu\colon \Delta^n \to \Delta^n$ be a linear singular n-simplex. We then have

$$\begin{aligned}
\partial u_n(\nu) &\overset{\substack{\text{definition of } u_n \text{ and } \chi_n\text{, see page 1135} \\ \downarrow}}{=} \partial(\nu_*(\chi_n)) \overset{\downarrow}{=} (-1)^n\cdot\partial(\nu_*(c_n(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})))) \overset{\substack{\text{since } \partial\circ\nu_*=\nu_*\circ\partial \\ \downarrow}}{=} (-1)^n\cdot\nu_*\big(\partial(c_n(\underbrace{u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})}_{=:\mu}))\big) \\
&\underset{\substack{\uparrow \\ \text{we can apply the previous claim since } \mu := u_{n-1}(\partial\,\mathrm{id}_{\Delta^n}) \in \mathrm{C}^{\mathrm{lin}}_{n-1}(\Delta^n)}}{=} (-1)^n\cdot\nu_*\big(c_{n-1}(\partial(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) + (-1)^n\cdot u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})\big) \\
&\underset{\substack{\uparrow \\ \text{by induction hypothesis we have } \partial u_{n-1}(\rho)=u_{n-2}(\partial\rho) \text{ for } \rho\in\mathrm{C}^{\mathrm{lin}}_{n-1}(\Delta^n)\text{, applied here to } \rho=\partial\,\mathrm{id}_{\Delta^n}}}{=} (-1)^n\cdot\nu_*\big(c_{n-1}(u_{n-2}(\underbrace{\partial\partial\,\mathrm{id}_{\Delta^n}}_{=0}))\big) \; + \; \nu_*(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) \\
&\underset{\substack{\uparrow \\ \text{Lemma 43.26 applied to } f=\nu}}{=} u_{n-1}(\nu_*(\partial\,\mathrm{id}_{\Delta^n})) \underset{\substack{\uparrow \\ \text{by definition}}}{=} u_{n-1}(\partial\nu).
\end{aligned}$$

⊞

The following claim is just a reformulation of the statement of the lemma.

Claim. For any singular n-simplex σ in X we have $u_{n-1}(\partial\sigma) = \partial(u_n(\sigma))$.

So let $\sigma\colon \Delta^n \to X$ be a singular n-simplex in X. Then we have

$$\begin{aligned}
\partial(u_n(\sigma)) &= \partial(u_n(\sigma_*(\mathrm{id}_{\Delta^n}))) \overset{\substack{\text{Lemma 43.26 applied to } f=\sigma \\ \downarrow}}{=} \partial(\sigma_*(u_n(\mathrm{id}_{\Delta^n}))) \overset{\substack{\text{since } \sigma_* \text{ is a chain map} \\ \downarrow}}{=} \sigma_*(\partial(u_n(\mathrm{id}_{\Delta^n}))) \\
&\underset{\substack{\uparrow \\ \text{by the previous claim}}}{=} \sigma_*(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) \underset{\substack{\uparrow \\ \text{Lemma 43.26 applied to } f=\sigma}}{=} u_{n-1}(\sigma_*(\partial\,\mathrm{id}_{\Delta^n})) \underset{\substack{\uparrow \\ \text{since } \sigma_* \text{ is a chain map}}}{=} u_{n-1}(\partial\sigma_*(\mathrm{id}_{\Delta^n})) = u_{n-1}(\partial\sigma).
\end{aligned}$$

■

We continue with a refreshingly straightforward lemma.

Lemma 43.26. *The maps $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$ are natural,*[712] *in particular for any map $f\colon X \to Y$ between topological spaces and any $n \in \mathbb{N}_0$ we have*

$$f_* \circ u_n \; = \; u_n \circ f_*\colon \; \mathrm{C}_n(X) \; \to \; \mathrm{C}_n(Y).$$

As we will see, Lemma 43.26 follows immediately from the definitions.

PROOF (∗). Let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. Then

$$f_*(u_n(\sigma)) \underset{\substack{\uparrow \\ \text{by definition of } u_n(\sigma)}}{=} f_*(\sigma_*(\chi_n)) \underset{\substack{\uparrow \\ \text{since } (f\circ\sigma)_* = f_*\circ\sigma_* \\ \text{as maps } \mathrm{C}_n(\Delta^n)\to\mathrm{C}_n(Y)}}{=} (f\circ\sigma)_*(\chi_n) \underset{\substack{\uparrow \\ \text{by definition of } u_n(f_*(\sigma))}}{=} u_n(f_*(\sigma)).$$

Since the desired equality holds for all generators of $\mathrm{C}_n(X)$ it also holds for all of $\mathrm{C}_n(X)$. ■

The proof of the next lemma is again rather involved.

Lemma 43.27. *Let X be a topological space. The chain maps*

$$u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$$

are chain homotopic to the identity.

PROOF (∗). Showing that the chain maps $(u_n)_{n\in\mathbb{N}}$ are chain homotopic to the identity requires us to find maps $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X), n \in \mathbb{N}_0$ such that for each $n \in \mathbb{N}_0$ the following equality holds:

$$\partial_{n+1}\circ P_n = -P_{n-1}\circ\partial_n + u_n - \mathrm{id}.$$

First we consider the case that X is a convex subset of some $\mathbb{R}^m$. Recall that we defined $\mathrm{C}_{-1}(X) = 0$, accordingly we define $P_{-1}\colon \mathrm{C}_{-1}(X) \to \mathrm{C}_0(X)$ to be the zero map. Let us suppose that we have already found maps $P_{-1},\dots,P_{n-1}$ with the desired properties.

We want to define the next map $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X)$. This map has to have the property that for each singular n-simplex σ we have

$$\partial_{n+1}(P_n(\sigma)) = -(P_{n-1}\circ\partial_n)(\sigma) + (u_n - \mathrm{id})(\sigma).$$

Put differently, to define $P_n(\sigma)$ we have to find a singular $(n+1)$-chain $E_\sigma \in \mathrm{C}_{n+1}(X)$ with

$$\partial E_\sigma = -(P_{n-1}\circ\partial)(\sigma) + (u_n - \mathrm{id})(\sigma),$$

so that we can then define $P_n(\sigma) := E_\sigma$. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex.

(1) For $n = 0$ we have

$$\big(\underbrace{P_{-1}}_{=0}\circ\partial_0 + \underbrace{u_0}_{=\mathrm{id}} - \mathrm{id}\big)(\sigma) = 0 \in \mathrm{C}_0(X),$$

and we put $E_\sigma := 0 \in \mathrm{C}_1(X)$.

---

[712] Let us explain in what sense these maps deserve to be called "natural". Given $n \in \mathbb{N}_0$ we consider the functor C_* from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{ChCplx}$ of chain complexes which is given by

$$\text{topological space } X \mapsto \mathrm{C}_*(X)$$

and

$$(f\colon X\to Y) \mapsto (f_*\colon \mathrm{C}_*(X)\to\mathrm{C}_*(Y)).$$

Given a topological space X we denote by $\Phi_X\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X)$ the map that is given by the n-th subdivision map that we defined on page 1135. Lemmas 43.25 an 43.26 say that these maps define a natural transformation from the functor C_* to itself.

(2) Now suppose that $n > 0$. Since X is convex we know from the discussion on page 1103 that $\mathrm{H}_n(X) = 0$. Therefore to show that an element $c \in \mathrm{C}_n(X)$ is the boundary of some element in $\mathrm{C}_{n+1}(X)$ it suffices to show that $\partial c = 0$. In our specific situation we calculate

$$\begin{array}{rcl} & & \text{by Lemma 43.25 the subdivision maps form a chain map, i.e. } \partial \circ u_n = u_{n-1} \circ \partial \\ & & \downarrow \\ \partial\big(\,(-P_{n-1}\partial_n + u_n - \mathrm{id})\,(\sigma)\big) & = & -\partial P_{n-1}\partial\sigma + u_{n-1}\partial\sigma - \partial\sigma \\ & = & \underbrace{(-\partial P_{n-1} + u_{n-1} - \mathrm{id})}_{=P_{n-2}\partial}(\partial\sigma) \;=\; P_{n-2}(\underbrace{\partial\partial\sigma}_{=0}) \;=\; 0. \end{array}$$

As we had just remarked, this implies that there exists an $E_\sigma \in \mathrm{C}_{n+1}(X)$ with $\partial(E_\sigma) = (P_{n-1}\partial_n - u_{n-1} - \mathrm{id})(\sigma)$.

Now we denote by $P_n\colon\ \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X)$ the map which assigns to each singular n-simplex σ the element $E_\sigma \in \mathrm{C}_{n+1}(X)$. Then it follows from the definition of E_σ that

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; u_n - \mathrm{id}$$

as maps $\mathrm{C}_n(X) \to \mathrm{C}_n(X)$. This concludes the proof of Lemma 43.27 if X is a convex subset of some $\mathbb{R}^m$.

Now we deal with the general case. So let X be a topological space. We consider

$$\begin{array}{rcl} P_n\colon\ \mathrm{C}_n(X) & \to & \mathrm{C}_{n+1}(X) \\ \sum\limits_{l=1}^{r} a_l \cdot (\sigma_l\colon \Delta^n \to X) & \mapsto & \sum\limits_{l=1}^{r} a_l \cdot (\sigma_l)_*(\underbrace{P_n(\mathrm{id}_{\Delta^n})}_{\substack{\uparrow \\ \text{defined since } \Delta^n \text{ is convex}}}). \end{array}$$

It is straightforward to see that this is a chain homotopy from the subdivision map to the identity map. ■

Remark. Later on, in Chapter 80.7, we will generalize the idea behind the proof of Lemma 43.27, to prove the Acyclic Model Theorems.

Definition. Given a topological space X and $m \in \mathbb{N}_0$ we denote by u_n^m the m-fold composition of $u_n\colon\ \mathrm{C}_n(X) \to \mathrm{C}_n(X)$.

Lemma 43.28. *For each topological space X and every $m \in \mathbb{N}$ the maps $\{u_n^m\}_{n\geq 0}$ form a chain map $\mathrm{C}_*(X) \to \mathrm{C}_*(X)$ that is chain homotopic to the identity.*

Proof. Recall that in Lemma 43.25 we already saw that the maps $\{u_n\}_{n\geq 0}$ form a chain map $\mathrm{C}_*(X) \to \mathrm{C}_*(X)$. As we remarked on page 1086, the composition of chain maps is again a chain map. Thus for any $m \in \mathbb{N}_0$ the maps $\{u_n^m\}_{n\geq 0}$ are also chain maps.

By Lemma 43.27 we have $u_* \simeq \mathrm{id}$. By Lemma 42.1 (2) this implies $u_*^2 \simeq u_*$, and by Lemma 42.1 (1) this implies $u_*^2 \simeq \mathrm{id}$. An induction argument now shows that $u_*^m \simeq \mathrm{id}$ for all m. ■

Definition. Given a singular chain c we can write it uniquely as $c = a_1\sigma_1 + \cdots + a_s\sigma_s$ with $a_1, \ldots, a_s \in \mathbb{Z} \setminus \{0\}$ and where the σ_i are pairwise different. We then say $\sigma_1, \ldots, \sigma_s$ are the *pieces* of c.

If $\sigma\colon \Delta^n \to X$ is a singular n-simplex, then the pieces of $u_n(\sigma)$ are evidently "smaller" than the pieces of σ, and by iteratively applying u_n one can hope to make the resulting pieces "arbitrary small". In the following two lemmas we will make it precise what we mean by "arbitrary small". Before we formulate the first lemma we need to recall and introduce several definitions.

Definition.

(1) Given a bounded subset A of $\mathbb{R}^{n+1}$ the *diameter* is defined as

$$\operatorname{diam}(A) \;:=\; \sup\big\{\|a-b\| \,\big|\, a,b \in A\big\} \in \mathbb{R}_{\geq 0}.$$

(2) Given a singular simple $\sigma\colon \Delta^n \to X$ we denote by $|\sigma|$ the image of σ.

Now we can formulate the following lemma.

Lemma 43.29. *For any $x_0,\dots,x_k \in \mathbb{R}^n$ and any piece τ of $u_k([x_0,\dots,x_k])$ we have*[713]

$$\operatorname{diam}(|\tau|) \;\leq\; \tfrac{n}{n+1}\cdot \operatorname{diam}(|[x_0,\dots,x_k]|).$$

FIGURE 727. Illustration of Lemma 43.29.

In the proof of Lemma 43.29 we will need the following elementary[714] geometric lemma.

Lemma 43.30. *Let $y_0,\dots,y_k \in \mathbb{R}^n$.*

(1) *We have the equality*

$$\operatorname{diam}(|[y_0,\dots,y_k]|) \;=\; \max\big\{\|y_i-y_j\| \,\big|\, i,j=1,\dots,k\big\}.$$

(2) *We denote by $b=\frac{1}{k}\sum\limits_{i=0}^{k} y_i$ the barycenter of $y_0,\dots,y_k$. Then for any $i=0,\dots,k$ we have*

$$\|y_i-b\| \;\leq\; \operatorname{diam}(|[y_0,\dots,y_k]|).$$

PROOF OF LEMMA 43.30 $(*)$.

(1) The inequality "$\geq$" follows immediately from the definitions and the observation that each y_i lies in

$$|[y_0,\dots,y_k]| \;=\; \Big\{\sum_{i=0}^{k} t_i\cdot y_i \,\Big|\, t_i\in[0,1] \text{ and } \sum_{i=0}^{k} t_i = 1\Big\}.$$

We turn to the proof of the reverse inequality "$\leq$". So we pick $a,b\in|[y_0,\dots,y_k]|$. By definition we can write $a=\sum\limits_{i=0}^{k} s_iy_i$ and $b=\sum\limits_{i=0}^{k} t_iy_i$ with $s_i,t_i\in[0,1]$ and such that

[713] Recall the definition on page 1078 of the map $[x_0,\dots,x_k]\colon \Delta^k \to \mathbb{R}^n$.

[714] Here "elementary" means that the statement and the proof are basically high school level mathematics which does not mean that the proofs are entirely obvious.

$\sum_{i=0}^{k} s_i = \sum_{i=0}^{k} t_i = 1$. We then have

$$\begin{array}{rcl}\|a-b\| & \overset{\text{since } \sum_{i=0}^{k} t_i = 1}{\downarrow\atop =} & \left\|a - \sum_{i=0}^{k} t_i y_i\right\| \quad = \quad \left\|\sum_{i=0}^{k} t_i(a-t_iy_i)\right\| \quad \leq \quad \sum_{i=0}^{k}\|t_i(a-t_iy_i)\| \\ & \leq & \sum_{i=0}^{k} t_i \cdot \max_{i=0,\dots,k}\{\|a-y_i\|\} \underset{\text{since } \sum_{i=0}^{k} t_i = 1}{\overset{}{=}} \max_{i=0,\dots,k}\{\|a-y_i\|\} \underset{\text{same argument applied to } a=\sum_{j=0}^{k} s_jy_j}{\leq} \max_{i,j=0,\dots,k}\{\|y_j-y_i\|\}\end{array}$$

(2) We denote by b_i the barycenter of $y_0,\dots,\widehat{y_i},\dots,y_k$, i.e.

$$b_i \;=\; \tfrac{1}{k}\sum_{j\neq i} y_j.$$

We then have

$$b \;=\; \tfrac{1}{k+1}y_i + \tfrac{k}{k+1}b_i.$$

It follows that

$$\begin{array}{rcl}\|b-y_i\| & = & \left\|\frac{1}{k+1}y_i + \frac{k}{k+1}b_i - y_i\right\| \;=\; \left\|\frac{k}{k+1}b_i - \frac{k}{k+1}y_i\right\| \\ & & \qquad\qquad\qquad\qquad = \;\frac{k}{k+1}\|b_i-y_i\| \;\leq\; \frac{k}{k+1}\cdot \operatorname{diam}(|[y_0,\dots,y_k]|).\end{array}$$

Thus we have obtained the desired inequality. ■

Now we are ready to provide the proof of Lemma 43.29.

Proof of Lemma 43.29 (∗). We need to show that for any $x_0,\dots,x_k\in\mathbb{R}^n$ and any piece τ of $u_k([x_0,\dots,x_k])$ we have

$$\operatorname{diam}(|\tau|) \;\leq\; \tfrac{n}{n+1}\cdot\operatorname{diam}(|[x_0,\dots,x_k]|).$$

We prove the claim by induction on k. For $k=0$ all the diameters are zero, so the inequality holds trivially. Now suppose that the claim holds for $k-1$. Let $x_0,\dots,x_k\in\Delta^n$ and denote by $b=\frac{1}{k}\sum_{i=0}^{k}x_i$ the barycenter of $x_0,\dots,x_k$. We denote by $\sigma_1,\dots,\sigma_r$ the pieces of $\partial[x_0,\dots,x_k]$.[715]

Now let τ be a piece of $u_k([x_0,\dots,x_k])$. By the definition of the subdivision map given on page 1135 we see that $|\tau| = |[w_0,\dots,w_{k-1},b]|$ where $[w_0,\dots,w_{k-1}]$ is a piece of $u_{k-1}(\sigma_s)$ for some $s\in\{1,\dots,r\}$. By Lemma 43.30 (1) it now suffices to prove the following claim.

Claim.

(1) For any $i,j=0,\dots,k-1$ we have $\|w_i-w_j\|\leq\frac{k}{k+1}\cdot\operatorname{diam}(|[x_0,\dots,x_k]|)$.
(2) For any $i=0,\dots,k-1$ we have $\|w_i-b\|\leq\frac{k}{k+1}\cdot\operatorname{diam}(|[x_0,\dots,x_k]|)$.

[715]Why does it not say that the number of pieces is $k+1$?

Now we provide a proof of these two inequalities. First let $i, j \in \{0, \dots, k-1\}$. Then

$$\|w_i - w_j\| \;\le\; \operatorname{diam}([w_0, \dots, w_{k-1}]) \underset{\substack{\uparrow\\ \text{induction hypothesis}\\ \text{applied to } \sigma_s}}{\le} \tfrac{k-1}{k} \cdot \operatorname{diam}(\sigma_s) \underset{\substack{\uparrow\\ \text{since } |\sigma_s| \subset |[x_0,\dots,x_k]| \text{ and } \frac{k-1}{k} \le \frac{k}{k+1}}}{\le} \tfrac{k}{k+1} \cdot \operatorname{diam}([x_0, \dots, x_k]).$$

Now let $i \in \{0, \dots, k-1\}$. We then have

$$\|w_i - b\| \underset{\substack{\uparrow\\ \text{Lemma 43.30 (2)}}}{\le} \tfrac{k}{k+1} \cdot \operatorname{diam}(|[w_0, \dots, w_{k-1}]|) \underset{\substack{\uparrow\\ \text{since } |[w_0,\dots,w_{k-1}]| \subset |[x_0,\dots,x_k]|}}{\le} \tfrac{k}{k+1} \cdot \operatorname{diam}(|[x_0, \dots, x_k]|)$$

■

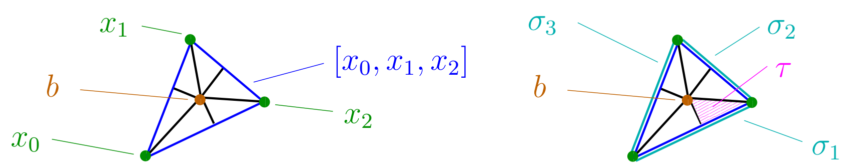


FIGURE 728. Illustration of the proof of Lemma 43.29.

Lemma 43.31. *Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. For every singular n-simplex $\sigma\colon \Delta^n \to X$ there exists an m such that $u_n^m(\sigma)$ lies in $\mathrm{C}_n^{\mathcal{U}}(X)$.*

PROOF. Let X be a topological space let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X and let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. By the Lebesgue Lemma 2.75 there exists a $\delta > 0$ such that for any subset A of Δ^n with diameter $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ such that $A \subset \sigma^{-1}(\mathring{U}_i)$.

We pick an $m \in \mathbb{N}$ such that $\left(\frac{n}{n+1}\right)^m < \delta$. We have

$$\begin{aligned} u_n^m(\sigma) \;=\; u_n^m(\sigma_*(\mathrm{id}_{\Delta^n})) \overset{\substack{\text{Lemma 43.26}\\ \downarrow}}{=} \sigma_*(u_n^m(\mathrm{id}_{\Delta^n})) &\overset{\substack{\text{by Lemma 43.29 and } \left(\frac{n}{n+1}\right)^m < \delta\\ \downarrow}}{=} \sigma_*\begin{pmatrix}\text{simplicial chain such that the image}\\ \text{has diameter less than } \delta\end{pmatrix} \\ &\underset{\substack{\uparrow\\ \text{by definition of } \delta}}{=} \sigma_*\begin{pmatrix}\text{simplicial chain such that the image}\\ \text{of each piece lies in some } \sigma^{-1}(\mathring{U}_i)\end{pmatrix}. \end{aligned}$$

But this means precisely that $u_n^m(\sigma) \in \mathrm{C}_n^{\mathcal{U}}(X)$. ■

Now we can prove Proposition 43.24.

PROOF OF PROPOSITION 43.24. Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. We want to show that the inclusion map $\iota\colon \mathrm{C}_*^{\mathcal{U}}(X) \to \mathrm{C}_*(X)$ induces for each n an isomorphism

$$\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X).$$

We first show that the map $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is surjective. Let $z \in \mathrm{H}_n(X)$ be given. We choose a cycle $c = \sum_{j=1}^{r} a_j \sigma_j \in \mathrm{C}_n(X)$ which represents $z \in \mathrm{H}_n(X)$. It follows from

Lemma 43.31, applied to the singular n-simplices $\sigma_1, \dots, \sigma_r$, that there exists an $m \in \mathbb{N}$ such that $u_n^m(c) \in \mathrm{C}_n^{\mathcal{U}}(X)$. By Lemma 43.28 the maps $\{u_n^m\}_{n \in \mathbb{N}_0}$ form a chain map that is chain homotopic to the identity. By definition this means that there exists a chain homotopy $P_k\colon\ \mathrm{C}_k(X) \to \mathrm{C}_{k+1}(X)$, $k \in \mathbb{N}_0$. Thus we obtain that

$$z \;=\; [c] \;\underset{\substack{\uparrow \\ \text{since } \partial_{n+1}P_n + P_{n-1}\partial_n \;=\; u_n^m - \mathrm{id}}}{=}\; \big[u_n^m(c) - P_{n-1}\overbrace{\partial_n(c)}^{=0} - \partial_{n+1}P_n(c)\big] \;=\; \big[u_n^m(c) - \partial_{n+1}P_n(c)\big] \;\underset{\substack{\uparrow \\ \text{equality in } \mathrm{H}_n(X)}}{=}\; \overbrace{[u_n^m(c)]}^{\in \iota_*(\mathrm{H}_n^{\mathcal{U}}(X))}.$$

Thus we have shown that $\iota_*\colon\ \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is surjective.

The argument that the map $\iota_*\colon\ \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is injective is very similar. Indeed, let $[c] \in \mathrm{H}_n^{\mathcal{U}}(X)$ be an element such that $[c] = 0 \in \mathrm{H}_n(X)$. This means that there exists an $e \in \mathrm{C}_{n+1}(X)$ with $\partial e = c$. From Lemma 43.31 it follows, as above, that there exists an $m \in \mathbb{N}$ with $u_{n+1}^m(e) \in \mathrm{C}_{n+1}^{\mathcal{U}}(X)$. We again choose a chain homotopy $\{P_k\}_{k \in \mathbb{N}_0}$ between the chain maps $\{u_k^m\}_{k \in \mathbb{N}_0}$ and id. Then we have

$$c \;=\; \partial e \;\underset{\substack{\uparrow \\ \text{since } \partial P_{n+1} + P_n\partial = u_{n+1}^m - \mathrm{id}}}{=}\; \partial\big(u_{n+1}^m(e) - \partial P_{n+1}(e) - P_n\partial e\big) \;=\; \partial\underbrace{u_{n+1}^m(e)}_{\in \mathrm{C}_{n+1}^{\mathcal{U}}(X)} \;-\; \partial\underbrace{P_n(c)}_{\in \mathrm{C}_{n+1}^{\mathcal{U}}(X)}.$$

Thus have we have shown that c is indeed the boundary of an $(n+1)$-dimensional singular chain, in $C_{n+1}^{\mathcal{U}}(X)$ i.e. $[c] = 0 \in \mathrm{H}_n^{\mathcal{U}}(X)$. Put differently, we have now proved that $\iota_*\colon\ \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is injective. ■

Using Proposition 43.24 we can now prove the Excision Theorem 43.19.

Proof of the Excision Theorem 43.19. Let X be a topological space. Furthermore let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A. We have to show that the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \;\xrightarrow{\cong}\; \mathrm{H}_n(X, A).$$

We put $B := X \setminus Z$. Note that $A \cap B = A \setminus Z$. It follows from our hypothesis on Z and A that X is the union of the interiors of A and B, put differently, the two sets $\{A, B\}$ is a cover of X.

For each n we consider the following commutative diagram of abelian groups:[716]

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{C}_n(A) & \longrightarrow & \mathrm{C}_n^{\{A,B\}}(X) & \longrightarrow & \mathrm{C}_n^{\{A,B\}}(X)/\,\mathrm{C}_n(A) & \longrightarrow & 0 \\
 & & \downarrow{=} & & \big\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_n(A) & \longrightarrow & \mathrm{C}_n(X) & \longrightarrow & \mathrm{C}_n(X, A) & \longrightarrow & 0.
\end{array}$$

[716]Here we use the definition from page 1133 which in this context says that

$$\mathrm{C}_n^{\{A,B\}}(X) \;:=\; \{\sigma + \tau \,|\, \sigma \in \mathrm{C}_n(A) \text{ and } \tau \in \mathrm{C}_n(B)\}.$$

Note that by definition the horizontal sequences are in fact short exact sequences of chain complexes. We first make the observation that it follows immediately from the definition of $\mathrm{C}_n^{\{A,B\}}(X)$ that the inclusion

$$\mathrm{C}_n(B) \;\to\; \mathrm{C}_n^{\{A,B\}}(X)$$

induces a chain map

$$\mathrm{C}_n(B, A\cap B) \;\to\; \mathrm{C}_n^{\{A,B\}}(X)/\,\mathrm{C}_n(A)$$

that is in fact an isomorphism. Thus in the above diagram we can replace the chain complex $\mathrm{C}_n^{\{A,B\}}(X)/\,\mathrm{C}_n(A)$ by $\mathrm{C}_n(B, A\cap B) = \mathrm{C}_n(X\setminus Z, A\setminus Z)$.

Now we consider the long exact sequence of homology groups arising from the above short exact sequences of chain complexes. By Proposition 43.11 we obtain the following commutative diagram where the horizontal sequences are exact:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_n(A) \arrow[r] \arrow[d, "="] & \mathrm{H}_n^{\{A,B\}}(X) \arrow[r] \arrow[d] & \overbrace{\mathrm{H}_n(B, A\cap B)}^{=\mathrm{H}_n(X\setminus Z, A\setminus Z)} \arrow[r, "\partial"] \arrow[d] & \mathrm{H}_{n-1}(A) \arrow[r] \arrow[d, "="] & \mathrm{H}_{n-1}^{\{A,B\}}(X) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \mathrm{H}_n(A) \arrow[r] & \mathrm{H}_n(X) \arrow[r] & \mathrm{H}_n(X, A) \arrow[r, "\partial"] & \mathrm{H}_{n-1}(A) \arrow[r] & \mathrm{H}_{n-1}(X) \arrow[r] & \dots
\end{tikzcd}$$

From Proposition 43.24 it follows that the second and fifth vertical map are isomorphisms. Since the first and fourth map are evidently isomorphisms it now follows from the Five Lemma 43.12 that the middle map is also an isomorphism. ■

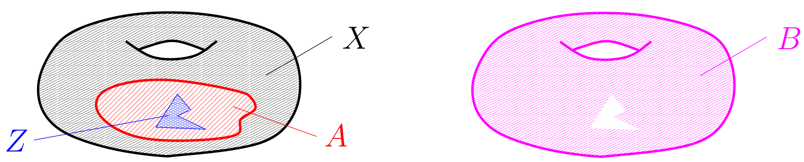


FIGURE 729. Illustration for the proof of the Excision Theorem 43.19.

Exercises for Chapter 43.

Exercise 43.1. Let X be a non-empty topological space. By Lemma 43.1 (4) we know that there exists an isomorphism $\mathrm{H}_0(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_0(X)\oplus\mathbb{Z}$. Show that there is no *natural* isomorphism $\mathrm{H}_0(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_0(X)\oplus\mathbb{Z}$. More precisely, show that it is not possible to assign to each non-empty topological space X an isomorphism $\Phi_X\colon\ \mathrm{H}_0(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_0(X)\oplus\mathbb{Z}$ such that for any map $f\colon X\to Y$ between non-empty topological spaces the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}_0(X) \arrow[r, "\Phi_X"] \arrow[d, "f_*"] & \widetilde{\mathrm{H}}_0(X)\oplus\mathbb{Z} \arrow[d, "f_*\oplus\mathrm{id}_{\mathbb{Z}}"] \\
\mathrm{H}_0(Y) \arrow[r, "\Phi_Y"] & \widetilde{\mathrm{H}}_0(Y)\oplus\mathbb{Z}.
\end{tikzcd}$$

Hint. Let X be the discrete topological space consisting of two points. If such a natural isomorphism Φ_X did exist, what should the map Φ_X do?

Exercise 43.2. Let
$$\mathbb{Z} \xrightarrow{f} B \xrightarrow{g} \mathbb{Z} \oplus \mathbb{Z}_6 \to 0$$
be an exact sequence, i.e. we know that $g \circ f = 0$, we know that $\operatorname{im}(f) = \ker(g)$ and we know that g is an epimorphism. What are the possible isomorphism types of the group B?

Exercise 43.3. We consider the following commutative diagram of homomorphisms between abelian groups:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \arrow[d] \\
0 \arrow[r] & \widetilde{A} \arrow[r] & \widetilde{B} \arrow[r] & \widetilde{C} \arrow[r] & 0.
\end{tikzcd}$$

We suppose that the horizontal sequences are exact and that the maps a and c to the left and right are the zero map. Does it follow that the middle map b is also the zero map?

Exercise 43.4. Let X be a topological space with $\widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}$.

(a) Show that given any $\epsilon \in \{-1, 0, 1\}$ there exists a map $f\colon X \to X$ such that the induced map f_* on $\widetilde{\mathrm{H}}_0(X)$ is given by multiplication by ϵ.

(b) Show that given any map $f\colon X \to X$ the induced map f_* on $\widetilde{\mathrm{H}}_0(X)$ is given by multiplication by some $\epsilon \in \{-1, 0, 1\}$.

Exercise 43.5. Determine the homology groups of $S^2 \vee S^1$.

Exercise 43.6.

(a) Let

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "a"] & B \arrow[r, "j"] \arrow[d, "b"] & C \arrow[r, "k"] \arrow[d, "c"] & D \arrow[r, "l"] \arrow[d, "d"] & E \arrow[d, "e"] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E'
\end{tikzcd}$$

be a commutative diagram such that the horizontal sequences are exact. Show that if a is surjective, if b and d are isomorphisms, and if e is injective, then c is an isomorphism.

(b) Does the statement in (a) also hold, if a injective and if e is surjective ?

Exercise 43.7. Let M be the Möbius band. Determine the relative homology groups of the pair $(M, \partial M)$.

Exercise 43.8. Give examples of two contractible subspaces A and B of $\mathbb{R}^2$ such that $\mathrm{H}_1(A \cup B)$ is infinitely generated.

Exercise 43.9. Let

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "\varphi"] \arrow[d, "a"] & B_* \arrow[r, "\psi"] \arrow[d, "b"] & C_* \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A'_* \arrow[r, "\varphi'"] & B'_* \arrow[r, "\psi'"] & C'_* \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram of homomorphisms between abelian groups. Show that there exists an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \ker(a) \arrow[r, "\varphi"] & \ker(b) \arrow[r, "\psi"] & \ker(c) \arrow[dll, rounded corners] \\
& \operatorname{coker}(a) \arrow[r, "\varphi'"] & \operatorname{coker}(b) \arrow[r, "\psi'"] & \operatorname{coker}(c) \arrow[r] & 0.
\end{tikzcd}$$

This statement is sometimes referred to as the "Snake Lemma".
Hint. Try to prove the statement with as little effort as possible.

Exercise 43.10. We consider the following diagram of homomorphisms between abelian groups:

$$\begin{tikzcd}
& & 0 \arrow[d] & & \\
& & M \arrow[d, "\mu"] \arrow[dr, "\beta\circ\mu"] & & \\
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[dr, "\nu\circ\alpha"'] & B \arrow[r, "\beta"] \arrow[d, "\nu"] & C \arrow[r] & 0 \\
& & N \arrow[d] & & \\
& & 0. & &
\end{tikzcd}$$

We suppose that the horizontal and the vertical sequences are exact.

(a) Show that $\beta \circ \mu$ is a monomorphism if and only if $\nu \circ \alpha$ is a monomorphism.
(b) Show that $\beta \circ \mu$ is an epimorphism if and only if $\nu \circ \alpha$ is an epimorphism.

Exercise 43.11. We consider the following commutative diagram:

$$\begin{tikzcd}
& & \ker(\beta) \arrow[dl, "\varphi"'] \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d, two heads, "\alpha"] & B \arrow[r] \arrow[d, two heads, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0.
\end{tikzcd}$$

Evidently we assume that the horizontal sequences are exact. Furthermore we assume that $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ are epimorphisms. Show that $\gamma\colon C \to C'$ is an isomorphism.

Exercise 43.12. For readers who really really enjoy diagram chases we have the following problem on offer. Suppose we are given the following commutative diagram of

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_* \arrow[r] \arrow[d] & B_* \arrow[r, "h_B"] \arrow[d, "v_B"] & C_* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D_* \arrow[r, "h_D"] \arrow[d, "v_D"] & E_* \arrow[r] \arrow[d] & F_* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_* \arrow[r] \arrow[d] & \mathrm{H}_* \arrow[r] \arrow[d] & J_* \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the vertical and horizontal sequences are exact. We suppose that the chain complex H_* is acyclic, i.e. we suppose that homology groups of H_* vanish. Show that for any $k \in \mathbb{N}_0$ the following diagram commutes up to a minus sign:

$$\begin{tikzcd}
\mathrm{H}_k(D_*) \arrow[r, "v_{D*}"] \arrow[d] & \mathrm{H}_k(G_*) & \mathrm{H}_{k+1}(K_*) \arrow[l, "\partial", "\cong"'] \arrow[d, "\partial"', "\cong"] \\
\mathrm{H}_k(E) \arrow[r, "v_{B*}^{-1}", "\cong"'] & \mathrm{H}_k(B) \arrow[r, "h_{B*}"] & \mathrm{H}_k(C).
\end{tikzcd}$$

Exercise 43.13. Let X be a topological space, let $x_0 \in X$ be a point and let $n \in \mathbb{N}_0$. Show that the map

$$\begin{tikzcd}
\widetilde{\mathrm{H}}_n(X) \arrow[r] \arrow[rrr, bend right, "=:\Phi_{X,x_0}"'] & \mathrm{H}_n(X) \arrow[r, "="] & \mathrm{H}_n(X, \varnothing) \arrow[r] & \mathrm{H}_n(X, \{x_0\})
\end{tikzcd}$$

is an isomorphism.

Exercise 43.14. We say that a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ has *bounded displacement* if there exists a $C \in \mathbb{R}$ such that for any $x \in \mathbb{R}^n$ we have $\|x - f(x)\| \leq C$. Show that any map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of bounded displacement is onto.
Hint. Use the Brouwer Fixed Point Theorem.

Exercise 43.15. Let (X, A) be a pair of topological spaces. Suppose there exists a deformation retraction $r\colon X \times [0,1] \to X$ from X to some subset $Y = r_1(X)$. We write $B = r_1(A)$. Show that the inclusion induced maps $\mathrm{H}_k(Y, B) \to \mathrm{H}_k(X, A)$ are isomorphisms.

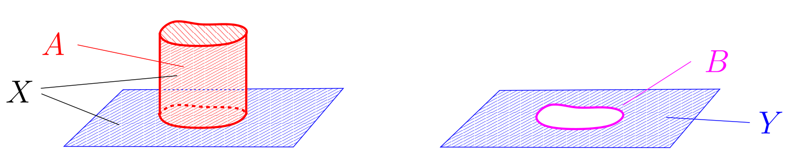


FIGURE 730. Illustration of Exercise 43.15.

Exercise 43.16. Let $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m$ be a nested sequence of topological spaces.

(a) Suppose that for each $k = 0, \ldots, m-1$ we have $\mathrm{H}_n(X_{k+1}, X_k) = 0$ for all $n \in \mathbb{N}_0$. Show that $\mathrm{H}_n(X_m, X_0) = 0$ for all $n \in \mathbb{N}_0$.
(b) Let $n \in \mathbb{N}_0$. Suppose that for each $k = 0, \ldots, m-1$ we have $\mathrm{H}_n(X_{k+1}, X_k) = 0$. Does it also follow that $\mathrm{H}_n(X_m, X_0) = 0$?

Exercise 43.17. We consider $X = [0,1]$ and $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Prove the following two statements:
(a) The group $\mathrm{H}_1(X, A)$ is countable.
(b) The group $\mathrm{H}_1(X/A)$ is uncountable. For psychological reasons it might be convenient to recall that in Exercise 25.1 we showed that the quotient space X/A is homeomorphic to the Hawaiian earrings.

Remark. The homology groups of the Hawaiian earrings are computed in [**EKa00**]. In particular it is shown that the higher homology groups $\mathrm{H}_i(X/A)$ are zero for $i \geq 2$, which is not clear at all, see e.g. [**BaM62**].

Exercise 43.18. Give an example of a topological space X and some non-empty subset A for which there is no exact sequence of the form

$$\ldots \to \widetilde{\mathrm{H}}_{n+1}(X/A) \to \widetilde{\mathrm{H}}_n(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{p_*} \widetilde{\mathrm{H}}_n(X/A) \to \ldots$$

where $i \colon A \to X$ is the inclusion and $p \colon X \to X/A$ is the projection.
Remark. You might want to consult your calculations from Exercise 43.17.

Exercise 43.19. Let X be a non-empty bounded subset of $\mathbb{R}^n$. We pick an $r \in \mathbb{R}_{>0}$ such that X is contained in the bounded n-ball $\overline{B}^n_r$ of radius r. We consider the map

$$\begin{array}{rcl} f \colon S^{n-1} & \to & \mathbb{R}^n \setminus X \\ x & \mapsto & 2r \cdot x. \end{array}$$

Show that the induced map $f_* \colon \mathrm{H}_{n-1}(S^{n-1}) \to \mathrm{H}_{n-1}(\mathbb{R}^n \setminus X)$ is a monomorphism.

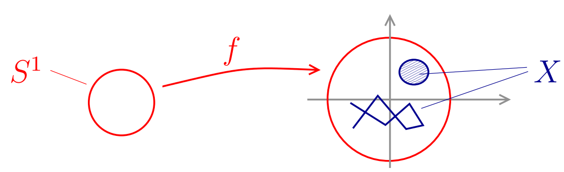


FIGURE 731. Illustration of Exercise 43.19.

Exercise 43.20. Show that for $k \geq 1$ the wedge $S^k \vee S^k$ is not a topological manifold.

Exercise 43.21. Compute the homology groups of the quotient of S^2 obtained by identifying the North Pole N and the South Pole S to a point. We refer to Figure 732 for an illustration.

Exercise 43.22. Let $T = S^1 \times S^1$ be the torus. We consider the subset $A := S^1 \times \{1\}$.
(a) Show that T/A is homeomorphic to the topological space $S^2/(S \cup N)$ considered in Exercise 43.21.
(b) Compute the homology groups of the torus.

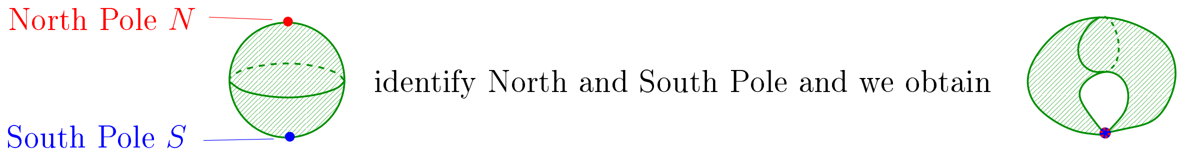


FIGURE 732

(c) Bonus question: How far can you push the above approach if we start out with the Klein bottle instead of the torus?

Remark. We will soon see a more conceptual, less ad hoc, approach to calculating the homology groups of the torus.

44. Applications and variations of the Excision Theorem 43.19 (∗)

As the title of this chapter suggests, in this chapter we will consider some applications of the Excision Theorem 43.19 and we will state and prove several variations thereof.

In particular we will prove several statements on the topology of topological manifolds which extend the results obtain in Section 6.7 from the setting of smooth manifolds to the more general context of topological manifolds. We will also use the opportunity to state a few other results on topological manifolds. At the end of the chapter we will introduce the double of a manifold. This construction makes sense, and is of interest, for smooth and for topological manifolds.

The chapter does not contain any real surprises. The overall message is that the statement of the Excision Theorem 43.19 holds in several convenient alternative settings and that many of the statements which we know and love about the topology of smooth manifolds also hold for topological manifolds. These variations and generalizations of earlier results are non-trivial, but most readers will have a hard time building up enthusiasm for these results. Thus it might be best to have a quick glimpse at the results and then move on to the next section.

44.1. **The local homology groups of a topological space.** We start out with the following definition which at first glance might look somewhat artificial.

Definition. Given a topological space X and a point $x_0 \in X$ the relative homology group $\mathrm{H}_n(X, X \setminus \{x_0\})$ is called the *n-th local homology group* of X at the point x_0.

The main goal of this section is to calculate the local homology groups of topological manifolds and to draw several consequences from this calculation. At this point it is helpful to recall the definition on page 261 of charts of type (i) and (ii) of a topological manifold. For the reader's convenience we illustrate the definitions in Figure 733. We also recall that on page 262 we said that a point x on a topological manifold is a *boundary point* if x does not admit a chart of type (i).

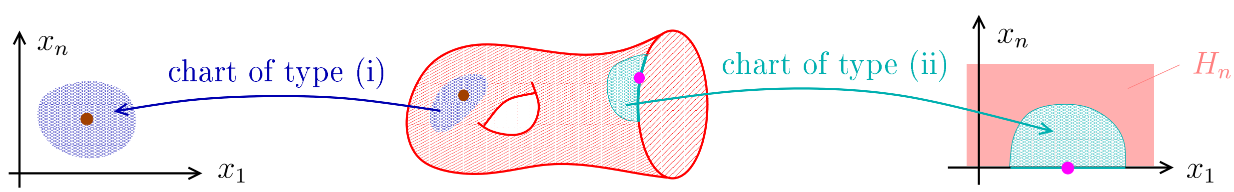


Figure 733

The following lemma calculates the local homology groups of topological manifolds.

Lemma 44.1. *Let M be a k-dimensional topological manifold.*

(1) *Every point on X admits either a chart of type (i) or it admits a chart of type (ii).*
(2) *Given any point $P \in M$, any $l \in \mathbb{N}_0$ and any neighborhood U of P we have*

$$\underbrace{\mathrm{H}_l(U, U\setminus\{P\}) \xrightarrow{\cong} \mathrm{H}_l(M, M\setminus\{P\})}_{\text{induced by the inclusion } (U,U\setminus\{P\}) \to (M,M\setminus\{P\})} \cong \begin{cases} \mathbb{Z}, & \text{if } P \in M \setminus \partial M \text{ and if } l = k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let P be a point on a k-dimensional topological manifold M. We start out with the following claim.

Claim. We have the following isomorphisms:

$$\mathrm{H}_l(U,U\setminus\{P\}) \xrightarrow{\cong} \mathrm{H}_l(M,M\setminus\{P\}) \cong \begin{cases} \mathbb{Z}, & \text{if } P \text{ admits a chart of type (i) and if } l=k, \\ 0, & \text{if } P \text{ admits a chart of type (ii) or if } l \neq k. \end{cases}$$

First we consider the case that P admits a chart of type (i). It follows easily from the definition of an open subset of $\mathbb{R}^k$ that we can pick a chart $\Phi\colon U \to B^k$ for P with $\Phi(P)=0$. We then have the following isomorphisms:

$$\mathrm{H}_l(M,M\setminus\{P\}) \underset{\substack{\uparrow \\ \text{isomorphism by the} \\ \text{Excision Theorem 43.20}}}{\xleftarrow{\cong}} \mathrm{H}_l(U,U\setminus\{P\}) \overset{\substack{\text{isomorphism since } \Phi \text{ is a homeomorphism} \\ \downarrow}}{\xrightarrow{\Phi_*}} \mathrm{H}_l(B^k,B^k\setminus\{0\}) \underset{\substack{\uparrow \\ \text{isomorphism by the long exact} \\ \text{sequence in reduced homology} \\ \text{of the pair } (B^k, B^k\setminus\{0\}) \text{ and the} \\ \text{fact that } \widetilde{\mathrm{H}}_*(B^k)=0 \text{ by page 1106}}}{\xrightarrow{\partial}} \widetilde{\mathrm{H}}_{l-1}(B^k\setminus\{0\}) \underset{\substack{\uparrow \\ \text{isomorphism since} \\ S^{k-1} \text{ is homotopy} \\ \text{equivalent to } B^k\setminus\{0\}}}{\xleftarrow{\cong}} \widetilde{\mathrm{H}}_{l-1}(S^{k-1}).$$

The desired statement now follows from the calculation of the reduced homology groups of spheres given in Proposition 43.4.

Now we consider the case that P admits a chart of type (ii). Similar to the above we can pick a chart $\Phi\colon U \to B^k_{\geq 0} := B^k \cap \mathrm{H}_k$ for P with $\Phi(P)=0$. By arguments that are very similar to the ones used above we see that we have the following isomorphisms:

$$\mathrm{H}_l(M,M\setminus\{P\}) \xleftarrow{\cong} \mathrm{H}_l(U,U\setminus\{P\}) \underset{\cong}{\xrightarrow{\Phi_*}} \mathrm{H}_l(B^k_{\geq 0},B^k_{\geq 0}\setminus\{0\}) \underset{\cong}{\xrightarrow{\partial}} \widetilde{\mathrm{H}}_{l-1}(B^k_{\geq 0}\setminus\{0\}) \underset{\substack{\uparrow \\ \text{one can easily show that } B^k_{\geq 0} \text{ is contractible, hence its} \\ \text{reduced homology vanishes by Lemma 43.1}}}{=} 0.$$

$\boxplus$

Now it is straightforward to prove the statements of the lemma.

(1) It follows from the above claim that $\mathrm{H}_k(M, M\setminus\{P\}) \cong \mathbb{Z}$ if and only if P admits a chart of type (i). This observation immediately implies the promised statement.
(2) The first map is an isomorphism by the Excision Theorem 43.20. The second isomorphism follows immediately from the above claim. ■

44.2. **The topology of topological manifolds.** With the results from Chapter 43 we can now generalize Proposition 6.27 from the case of smooth manifolds to the more general setting of topological manifolds.

Proposition 44.2. *Let X be an n-dimensional topological manifold. The following statements hold:*

(1) *Every point on X admits either a chart of type (i) or it admits a chart of type (ii).*
(2) *If F is equipped with a smooth atlas, i.e. if we view F as a smooth manifold, then the boundary of F as a topological manifold agrees with the boundary of F viewed as a topological manifold.*

(3) *The boundary ∂X is an $(n-1)$-dimensional topological manifold with empty boundary, i.e. we have $\partial(\partial X) = \varnothing$.*
(4) (a) *Any union of components of ∂X is a closed subset of X. In particular ∂X itself is a closed subset of X.*
(b) *If X is compact, then ∂X is also compact.*
(5) *If ∂X is non-empty, then $X \setminus \partial X$ is non-compact.*
(6) *If $N \subset X$ is a proper k-dimensional submanifold, then we have $N \cap \partial X = \partial N$*[717]*, and $N \cap \partial X = \partial N$ is a proper $(k-1)$-dimensional submanifold of ∂X.*
(7) *Let M be an m-dimensional topological manifold with possibly non-empty boundary and let N be an n-dimensional topological manifold with empty boundary. Then $M \times N$ is an $(m+n)$-dimensional topological manifold with $\partial(M \times N) = \partial M \times N$.*

Example. It follows from Proposition 44.2 and the examples of charts for $\overline{B}^n$ given on page 264 that the boundary of the n-dimensional topological manifold is indeed S^{n-1}. We had suspected this for a long time, but only recently did we acquire the tools for providing a proof.

Remark.

(1) For 2-dimensional topological manifolds we had proved some of the statements of Proposition 44.2 in Proposition 18.18.
(2) In Proposition 80.9 we will generalize Proposition 44.2 to the setting that both M and N have non-empty boundary.
(3) Let M and N be two connected n-dimensional topological manifolds with empty boundary. If $M \times [0,1]$ is homeomorphic to $N \times [0,1]$, then evidently the corresponding boundaries are also homeomorphic. It follows from Proposition 44.2 (7) that $\partial(M \times [0,1]) = M \times \{0,1\}$ and $\partial(N \times [0,1]) = N \times \{0,1\}$. Since M and N are connected we obtain easily from the above that M and N are homeomorphic. Even though this sounds quite reasonable, very similar sounding statements can easily be wrong. For example there exists a 3-dimensional topological manifold W (called the *Whitehead manifold*) which is non-compact and has no boundary, which is not homeomorphic to $\mathbb{R}^3$ but such that $W \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$. We refer to [**Kir89**, p. 87] and [**AnR65**] for details.

PROOF. Let X be an n-dimensional topological manifold.

(1) This statement is precisely the statement of Lemma 44.1 (1). The only reason why we wrote down the statement again is to preserve the analogy with Proposition 6.27.
(2) Verbatim the same argument as in the 2-dimensional setting, see the proof of Proposition 18.18 (2) on page 556, shows how to deduce this statement from Statement (1).
(3) First note that it follows from Lemma 2.12 and from Lemma 6.1 (2) that ∂X is Hausdorff and second-countable. Now it remains to show that ∂X admits an $(n-1)$-dimensional atlas. Let $P \in \partial X$. By definition of ∂X there exists a chart $\Phi\colon U \to V$ of type (ii) for X with $P \in U$. Note that Φ is a chart of type (ii) precisely for the

[717] Here ∂N denotes the boundary of N, viewed as a topological manifold in its own right.

points in $\Phi^{-1}(\partial H_n)$. Also note that it follows from (1) and the definition of ∂X that $\Phi^{-1}(\partial H_n) = \partial X \cap U$. In other words, under the obvious identification $\partial H_n = \mathbb{R}^{n-1}$ we see that Φ restricts to an $(n-1)$-dimensional chart of type (i) for the topological space ∂X. This shows that ∂X is an $(n-1)$-dimensional topological manifold with empty boundary.

(4) This statement is precisely the content of Lemma 6.7.
(5) The proof of this statement is verbatim the same as the proof of the corresponding statement for smooth manifolds, see Proposition 6.27 (4).
(6) This statement follows reasonably easily from (1) and the various definitions. We leave it to the reader to fill in the details.
(7) In Proposition 6.5 we already showed that $M \times N$ is an $(m+n)$-dimensional topological manifold. As in the proof of Proposition 6.51 we obtain from (1) that $\partial(M \times N) = \partial M \times N$. ■

Let X be a topological manifold and let Y be a submanifold. It follows almost immediately from the definitions that Y is a topological manifold in its own right. In particular it makes sense to consider its boundary ∂Y and in particular it makes sense to introduce the following definition which can be viewed as a generalization of the definition on page 283.

Definition. Let X be a topological manifold and let Y be a submanifold. We write

$$\partial_0 Y \;:=\; \underset{\uparrow}{\partial Y} \setminus \partial X \qquad \text{and} \qquad \partial_1 Y \;:=\; \underset{\uparrow}{\partial Y} \cap \partial X.$$

here ∂Y denotes the boundary of Y viewed as a topological manifold

The definition is illustrated in Figure 734.

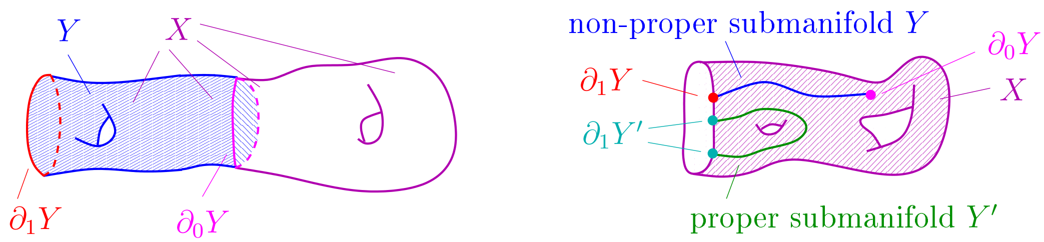


FIGURE 734

The next proposition gives us some results on the boundary of submanifolds. It can be viewed as a generalization of Lemma 6.28 and as an analogue of Proposition 6.30.

Proposition 44.3. *Let X be an m-dimensional topological manifold and let Y be an n-dimensional submanifold. We assume that Y is a closed subset of X. (In most applications Y will be compact).*

(1) *$\partial_0 Y$ and $\partial_1 Y$ are disjoint and each of them is a union of components of ∂Y.*
(2) *If Y is a proper submanifold, then $\partial Y = \partial_1 Y = Y \cap \partial X$.*
(3) *If Y is of codimension zero, then the interior $\mathring{Y}$ of Y as a subset of X agrees with $Y \setminus \partial_0 Y$. In particular $Y \setminus \partial_0 Y$ is an open subset of X.*

(4) *Suppose that Y is of codimension zero and that Y is compact.*
   (a) *The set $Z := X \setminus \mathring{Y}$ is also an n-dimensional submanifold of X and we have $Y \cap Z = \partial_0 Y = \partial_0 Z$.*
   (b) *If X and ∂Y are connected, then $Z = X \setminus \mathring{Y}$ is also connected.*
   (c) *If X is compact, then $Z = X \setminus \mathring{Y}$ is also compact.*

PROOF. The proposition follows easily from making the obvious modifications to the proofs of Lemma 6.28 and Proposition 6.30. The biggest change is that we need to replace Proposition 6.27 by Proposition 44.2. We leave it to the reader to fill in the details. ∎

We conclude this section with the following proposition which is the generalization of Proposition 6.26 from smooth manifolds to topological manifolds.

Proposition 44.4. *Let M be an m-dimensional non-empty topological manifold and let N be an n-dimensional topological manifold. If M is homeomorphic to an open subset of N (e.g. if M is homeomorphic to N), then $m = n$.*

It follows in particular from Proposition 44.4 that the dimension of a non-empty topological manifold is well-defined.

PROOF. By the remark on page 272 it suffices to consider the case that we are given a homeomorphism $f\colon M \to N$ between a non-empty m-dimensional topological manifold M and a topological manifold N. By Proposition 44.2 (5) we know that $M \setminus \partial M$ is non-empty. In particular we can find a point $P \in M \setminus \partial M$. Then

$$\mathbb{Z} \underset{\substack{\uparrow \\ \text{Lemma 44.1}}}{\cong} \mathrm{H}_m(M, M \setminus \{P\}) \underset{\substack{\uparrow \\ \text{since } f \text{ is a homeomorphism}}}{\xrightarrow{\cong}} \mathrm{H}_m(N, N \setminus \{f(P)\}).$$

It now follows from Lemma 44.1 that N is also m-dimensional. ∎

44.3. **The Topological Collar Neighborhood Theorem.** In the next section we will consider some useful variations on the Excision Theorem 43.19 for topological manifolds. To prove these results we will make use of the following following theorem which is an analogue of the Collar Neighborhood Theorem 8.12 for topological manifolds.

Theorem 44.5. (Topological Collar Neighborhood Theorem) *Given any topological manifold M there exists a map $\Phi\colon [0,1] \times \partial M \to M$ with the following four properties:*
(1) *The map Φ is an embedding.*
(2) *The image of $[0,1) \times \partial M$ is an open subset of M.*
(3) *For every $P \in \partial M$ we have $\Phi(0,P) = P$.*
(4) *The image of $[0,1] \times \partial M$ is a closed subset of M.*

Remark.
(1) In the compact case the Topological Collar Neighborhood Theorem 44.5 is also proved in [**Vic94**, Theorem Appendix II.7] and [**Hat02**, p. 253]. The general case, where M is also allowed to be non-compact, is also proved in [**BrownM62**, Theorem 3] and [**Arm70**, Theorem 1]. Note that [**Arm70**, Theorem 1] has a few refinements which could be useful in some applications.

(2) In Exercise 89.1 we will prove a variation on the Topological Collar Neighborhood Theorem 44.5 for the case that M is a codimension-zero submanifold *with corner* of some other topological manifold.

PROOF OF THEOREM 44.5 FOR COMPACT M. We consider M with an "external collar" attached, i.e. we consider $W := M \cup_{\partial M = \{0\} \times \partial M} ([0,1] \times \partial M)$. Let us assume for a second that we have found a homeomorphism $F\colon M \to W$ with $F(\partial M) = \{1\} \times \partial M$. In this fortunate setting one can verify easily, say using Lemma 3.43, that the map

$$\begin{array}{rcl} \Phi\colon [0,1] \times \partial M & \to & M \\ (t,x) & \mapsto & F^{-1}([(1-t,x)]) \end{array}$$

has all the desired properties. Thus it remains to show that such an F actually exists.

We start out our search for such F with the following claim.

Claim. There exist compact subsets $K_1, \dots, K_m \subset \partial M$, embeddings $\Phi_i\colon K_i \times [-1,0] \to M$ and maps $f_i\colon \partial M \to [0,1]$ such that the following are satisfied:

(1) For each $i \in \{1, \dots, m\}$ we have $\Phi_i^{-1}(\partial M) = K_i \times \{0\}$.
(2) For each $i \in \{1, \dots, m\}$ we have $\operatorname{supp}(f_i) \subset K_i$.
(3) We have $f_1 + \cdots + f_m = 1$ on all of ∂M.

It follows from the definition of ∂M, together with Proposition 44.2 (1), that for each $x \in \partial M$ there exists an open neighborhood U_x of x and an embedding $\Phi_x\colon [-1,0] \times U_x \to M$ with $\Phi_x^{-1}(\partial M) = \{0\} \times U_x$. For each x we can and will pick a compact neighborhood K_x with $K_x \subset U_x$. Since M is compact we know by Proposition 44.2 (4b) that ∂M is also compact. Evidently the interiors of the K_x cover all of ∂M. Since ∂M is compact we see that there exist $x_1, \dots, x_m \in \partial M$ such that ∂M is covered by the interiors of the K_{x_i}. Finally note that it follows from Proposition 2.81 together with Lemma 2.78 that there exist maps $f_1, \dots, f_m\colon \partial M \to [0,1]$ such that for each $i \in \{1, \dots, m\}$ we have $\operatorname{supp}(f_i) \subset K_{x_i}$ and such that $f_1 + \cdots + f_m = 1$. Renaming K_{x_i} and Φ_{x_i} as K_i and Φ_i we obtain the desired result. ⊞

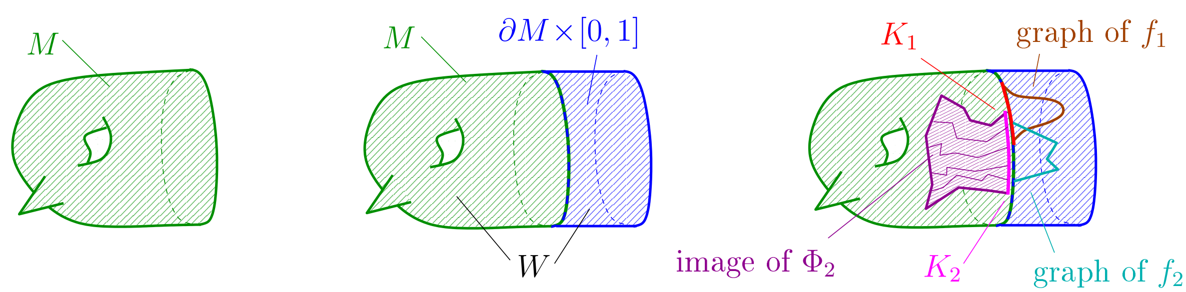


FIGURE 735. First illustration for the proof of Theorem 44.5.

Next, for $i = 0, \dots, m$ we set $g_i := f_1 + \cdots + f_i$ and we set

$$W_i \;:=\; M \cup_{\partial M = \{0\} \times \partial M} \{(t,x) \,|\, x \in \partial M \text{ and } t \in [0, g_i(x)]\}.$$

By Lemma 3.43 (2d) we know that W_0 is homeomorphic to M and evidently $W_m = W$.

Claim. For $i = 1, \dots, m$ there exists a homeomorphism $F_i\colon W_{i-1} \to W_i$.

Let $i \in \{1, \dots, m\}$. We consider the map

$$\begin{array}{rcl} \Theta\colon [-1,1] \times K_i & \to & W \\ (t,x) & \mapsto & \begin{cases} [\Phi_i((t,x))], & \text{if } t \in [-1,0], \\ [(t,x)], & \text{if } t \in [0,1]. \end{cases} \end{array}$$

It follows fairly easily from Lemma 3.43, Lemma 3.45 and Proposition 2.43 (2) that this map is a closed embedding. To simplify the notation we will identify $[-1,1] \times K_i$ with its image in W. The desired homeomorphism $W_{i-1} \to W_i$ is now given by "stretching horizontally" along $[-1,1] \times K_i$. More precisely, we consider the map

$$\begin{array}{rcl} F_i\colon W_{i-1} & \to & W_i \\ x & \mapsto & \begin{cases} x, & \text{if } x \notin [-1,1] \times K_i \\ \left(-1 + \frac{1+g_i(y)}{1+g_{i-1}(y)} \cdot (t+1), y\right), & \text{if } x = (t,y) \in [-1,1] \times K_i. \end{cases} \end{array}$$

It follows easily from the fact that $\operatorname{supp}(f_i) \subset K_i$, Lemma 3.8 (2b) and Lemma 2.35 that this map is continuous and it follows from Proposition 2.43 (2) that this map is in fact a homeomorphism. ⊞

It is now clear that $F := F_m \circ \dots \circ F_1\colon M \to W$ is a homeomorphism which has the feature that $F(\partial M) = \{1\} \times \partial M$. ■

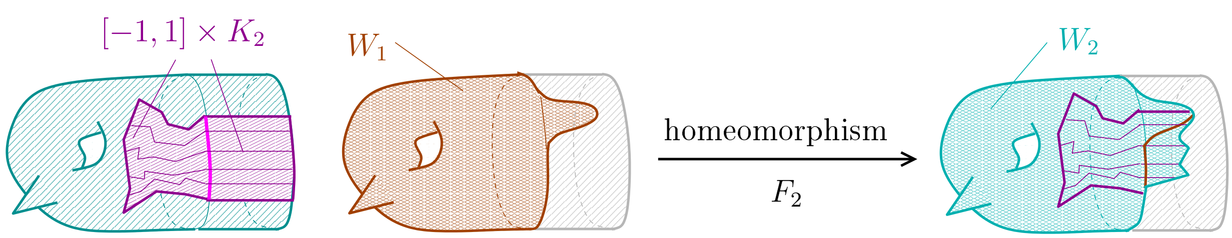


FIGURE 736. Second illustration for the proof of Theorem 44.5.

PROOF OF THE TOPOLOGICAL COLLAR NEIGHBORHOOD THEOREM 44.5. Now let M be any topological manifold. The proof of the Topological Collar Neighborhood Theorem 44.5 in this setting is not that different from the compact case. For space and time reasons we only sketch the argument.

It follows from the fact that ∂M is in particular second-countable together with Theorem 6.57 that there exists an infinite sequence $K_1, K_2, \dots$ of compact subsets, embeddings $\Phi_i\colon K_i \times [-1,0] \to M$ and maps $f_i\colon \partial M \to [0,1]$ such that beyond the aforementioned Statements (1), (2) and (3) the following extra condition is satisfied:

(4) For each $x \in \partial M$ there exists an open neighborhood V such that there exist only finitely many $i \in \mathbb{N}$ with $\operatorname{supp}(f_i) \cap U \neq \emptyset$.

As in the previous argument we now define functions $g_1, g_2, \dots$, subsets $W_1, W_2, \dots$ and homeomorphisms $F_i\colon W_{i-1} \to W_i$. It follows from (4) that the map

$$\begin{array}{rcll} F\colon M & \to & W & \\ x & \mapsto & (F_k \circ \dots \circ F_1)(x) & \quad \text{if } f_i(x) = 0 \text{ for } i \geq k \end{array}$$

is well-defined and continuous. In fact using Proposition 2.45 it is straightforward to see that F is actually a homeomorphism. Furthermore, note that by construction we again have $F(\partial M) = \{1\} \times \partial M$. Thus, as in the proof of the compact case, we are done. ■

The following uniqueness statement for collar neighborhoods is almost verbatim the same as the statement of Proposition 8.13.

Proposition 44.6. (∗) *Let M be a topological manifold and let $f_0, f_1\colon [0,1] \times \partial M \to M$ be two embeddings that are the identity map on $\{0\} \times \partial M = \partial M$. If ∂M is compact, then there exists an isotopy*

$$H\colon M \times [0,1] \;\to\; M$$

rel ∂M from $H_0 = \mathrm{id}_M$ to a homeomorphism $\Phi = H_1\colon M \to M$ with $\Phi \circ f_0 = f_1$. In particular Φ restricts to a homeomorphism

$$\Phi\colon f_0([0,1] \times \partial M) \;\to\; f_1([0,1] \times \partial M).$$

Proof (∗). The proof is sketched in [**KSi69**, Essay I, Theorem A.2]. The proof is also given in [**Arm70**, Theorem 2].[718] ■

Convention. Suppose we are given a topological manifold M with compact boundary ∂M. We pick a map $\Phi\colon [0,1] \times \partial M \to M$ as in the conclusion of the Topological Collar Neighborhood Theorem 44.5. Usually we identify $[0,1] \times \partial M$ with the image in M under the map Φ and we just refer to $[0,1] \times \partial M$ as a "collar neighborhood of ∂M". Sometimes we also find it convenient to use other intervals in our notation, e.g. we might write "let $[-3,0] \times \partial M$ be a collar neighborhood". It is then understood that $\{0\} \times \partial M = \partial M$.

For later we record the following corollary to the Topological Collar Neighborhood Theorem 44.5 which is the complete analogue of Corollary 8.14.

Corollary 44.7. *Let M be an n-dimensional topological manifold. Let $[0,1] \times \partial M$ be a collar neighborhood.*

(1) *The subset $W := M \setminus ([0,1) \times \partial M)$ is an n-dimensional topological manifold with boundary given by $\{1\} \times \partial M$.*
(2) *The subset $W = M \setminus ([0,1) \times \partial M)$ is a deformation retract of M and it is a deformation retract of $M \setminus \partial M$.*
(3) *If M is non-empty, then M is homotopy equivalent to a non-compact n-dimensional topological manifold without boundary.*

Proof. The proof of Corollary 44.7 is basically the same as the proof of Corollary 8.14. We just need to replace the Collar Neighborhood Theorem 8.12 by the Topological Collar Neighborhood Theorem 44.5 and we need to replace Proposition 6.27 by Proposition 44.2. ■

[718] The proof in [**Arm70**] makes me slightly nervous, there is no hypothesis that ∂M is compact, and there is no statement that the image of $[0,1] \times \partial M$ needs to be closed. But that would imply that the "usual" collar neighborhood for $M = \{(x,y) \mid y \geq 0\}$ is equivalent to the one shown in Figure 194, which is certainly *not* the case.

We conclude this discussion of the Topological Collar Neighborhood Theorem 44.5 with the following proposition that looks uncannily similar to Proposition 8.15.

Proposition 44.8. *Let M be an n-dimensional topological manifold (we do not assume that M is connected). Let A and B be disjoint unions of boundary components of M. Furthermore let $f\colon A \to B$ be a homeomorphism. Then the following two statements hold:*

(1) *The topological space*
$$M/a \sim f(a) \quad \text{for } a \in A$$
is an n-dimensional topological manifold.
(2) *If M consists of two components X and Y and if $A \subset X$ and $B \subset Y$, then the images of X and Y in $M/a \sim f(a)$ are codimension-zero submanifolds.*
(3) *If M is compact, then $M/a \sim f(a)$ is also compact.*
(4) *The boundary of $M/a \sim f(a)$ is given by $\partial M \setminus (A \cup B)$.*
(5) *If each component of M contains at least one component of A or B, then $M/a \sim f(a)$ is connected.*

In Chapter 86 we will introduce the notion of an orientation of a topological manifold. To preserve the symmetry with Proposition 8.15 we add the following statement, even though with the present knowledge it does not have any meaning yet.

(6) *Assume that M is oriented. We equip A and B with the boundary orientations, as defined in Proposition 86.19.*
    (a) *If f is orientation-reversing, then $M/a \sim f(a)$ admits an orientation such that the inclusion $M \setminus (A \cup B) \to M/a \sim f(a)$ is orientation-preserving.*
    (b) *If M is connected and if f is not orientation-reversing, then $M/a \sim f(a)$ is non-orientable.*

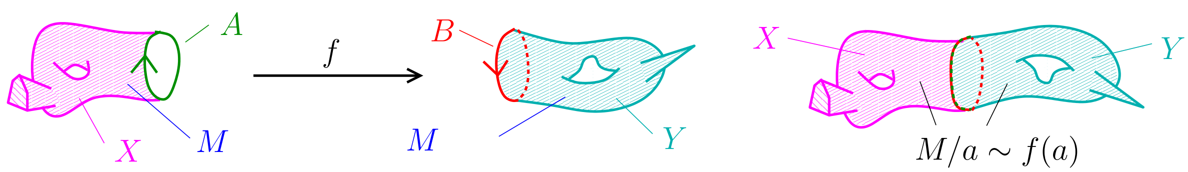


FIGURE 737. Illustration of Proposition 44.8.

PROOF. For the most part the proof of Proposition 44.8 is almost the same as the proof of Proposition 8.15. For statements (1) to (5) we just need to make the following replacements:

(1) We need to replace Proposition 6.27 (3a) by Lemma 6.7.
(2) We need to replace the Collar Neighborhood Theorem 8.12 by the Topological Collar Neighborhood Theorem 44.5.
(3) We need to replace Proposition 6.27 (1) by Proposition 44.2 (1).

The alert reader will notice that in the proof of Proposition 44.8 it is not really necessary to use the Topological Collar Neighborhood Theorem 44.5. For example, as we will see in Exercise 44.6, one can define an atlas for $M/a \sim f(a)$ "by hand". Nonetheless we leave the proof as it is, since the Topological Collar Neighborhood Theorem 44.5 greatly simplifies the discussion.

Finally we will prove Statement (6) in Exercise 86.4. ■

44.4. **Variations on the Excision Theorem 43.19.** In this section we will state and prove two variations on the Excision Theorem 43.19. First we have the following rather technical sounding theorem.

Theorem 44.9. *Let X be a topological space and let $Z \subset A \subset X$ be two subsets. We suppose that there exists a subset $U \subset Z$ such that the closure of U is contained in the interior of A and such that following statements hold:*

(1) *$X \setminus Z$ is a deformation retract of $X \setminus U$*
(2) *$A \setminus Z$ is a deformation retract of $A \setminus U$.*

Then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathrm{H}_n(X, A).$$

The hypotheses of Theorem 44.9 might sound complicated, but as we will see, in practice these conditions are often satisfied.

Example. Let $n \in \mathbb{N}$. As usual we denote by $S^n_{\geq 0}$ and $S^n_{\leq 0}$ the upper and lower hemisphere of S^n and we make the obvious identification $S^{n-1} = S^n_{\geq 0} \cap S^n_{\leq 0}$. It follows from Theorem 44.9 and the choices of X, A, Z and U illustrated in Figure 738 that the inclusion induced map

$$\mathrm{H}_n(S^n_{\geq 0}, S^{n-1}) \to \mathrm{H}_n(S^n, S^n_{\leq 0})$$

is an isomorphism.

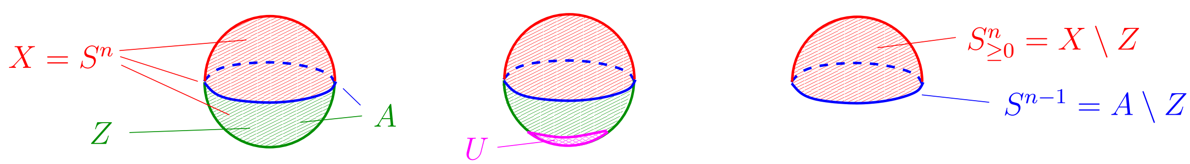


FIGURE 738

We can easily generalize the above example. More precisely, Theorem 44.9 also allows us to prove the following variation on the Excision Theorem 43.19. It says, under mild technical hypotheses, that in the relative homology of an m-dimensional topological manifold we can excise the interior of a codimension-zero submanifold.

Theorem 44.10. (Excision Theorem for Topological Manifolds) *Let X be an m-dimensional topological manifold and let A be a subset of X. Furthermore let B be an m-dimensional submanifold of X that is a closed subset of X. We write $\mathring{B} := B \setminus \partial_0 B$.*[719] *If $\mathring{B} \subset A$, then for every $n \in \mathbb{N}_0$ the inclusion induced map*

$$\mathrm{H}_n\left(X \setminus \mathring{B}, A \setminus \mathring{B}\right) \xrightarrow{\cong} \mathrm{H}_n(X, A)$$

is an isomorphism.

Example. If we take $X = S^n$, $A = S^n_{\leq 0}$ and $B = S^n_{\leq 0}$, then all the hypothesis of the Excision Theorem 44.10 for Topological Manifolds are satisfied and we obtain, as above,

[719]We refer to page 1154 for the definition of $\partial_0 B$.

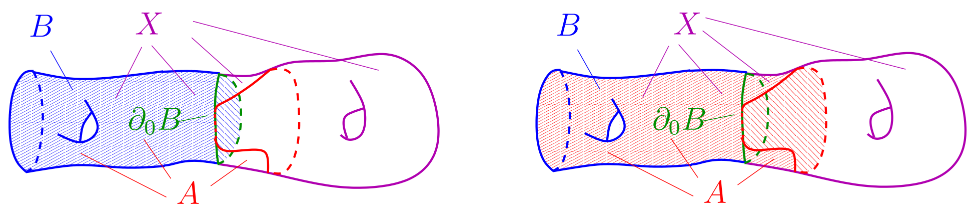


FIGURE 739. Illustration of the Excision Theorem 44.10 for Topological Manifolds.

that the inclusion induced map

$$\mathrm{H}_n(S^n_{\geq 0}, S^{n-1}) \;\to\; \mathrm{H}_n(S^n, S^n_{\leq 0})$$

is an isomorphism.

Remark. In Theorem 89.2 we will generalize the statement of the Excision Theorem 44.10 for Topological Manifolds to the setting that B is a submanifold "with corner".

PROOF OF THEOREM 44.10 ASSUMING THEOREM 44.9. Note that by Proposition 44.3 we know that $\partial_0 B$ is a union of boundary components of B and that $\mathring{B}$ is indeed the interior of B viewed as a subset of X. In particular we can apply the Topological Collar Neighborhood Theorem 44.5 to $\partial_0 B$. We obtain an embedding $\varphi\colon [-1,0]\times\partial_0 B \to B$ that is the identity map on $\{0\}\times\partial_0 B = \partial_0 B$. To simplify the notation we identify $[-1,0]\times\partial_0 B$ with its image in B. We set $Z = \mathring{B} = B\setminus(\{0\}\times\partial_0 B)$ and we set $U = B\setminus((-1,0]\times\partial_0 B)$.

Claim. There exists a deformation retract from $X\setminus U$ to $X\setminus Z$.

First note that we can write $X\setminus U = (X\setminus Z)\cup([-1,0]\times\partial_0 B)$. Furthermore note that $(X\setminus Z)\cap([-1,0]\times\partial_0 B) = \{0\}\times\partial_0 B$. The idea is to use Lemma 20.11 to conclude that the obvious deformation retraction from $[-1,0]\times\partial_0 B$ to $\{0\}\times\partial_0 B$ gives rise to the desired deformation retraction.

We still have to perform the task of verifying the hypotheses of Lemma 20.11, i.e. we need to argue that $X\setminus Z$ and $[-1,0]\times\partial_0 B$ are closed subsets of $X\setminus U$. First note that by Proposition 44.3 we know that $Z = \mathring{B}$ is an open subset of X. It follows that $X\setminus Z$ is a closed subset of $X\setminus U$. Furthermore by Statement (4) of the Topological Collar Neighborhood Theorem 44.5 we know that $[-1,0]\times\partial_0 B$ is a closed subset of X, in particular it is a closed subset of $X\setminus U$. ⊞

Claim. There exists a deformation retract from $A\setminus U$ to $A\setminus Z$.

Since $\mathring{B}\subset A$ we have $A\setminus U = (A\setminus Z)\cup([-1,0]\times\partial_0 B)$. Furthermore note that $(A\setminus Z)\cap([-1,0]\times\partial_0 B) = \{0\}\times\partial_0 B$. Thus basically the same argument as in the proof of the previous claim shows that such a deformation retraction exists. ⊞

The statement of Excision Theorem 44.10 for Topological Manifolds now follows immediately from Theorem 44.9. ■

We conclude this section with the proof of Theorem 44.9.

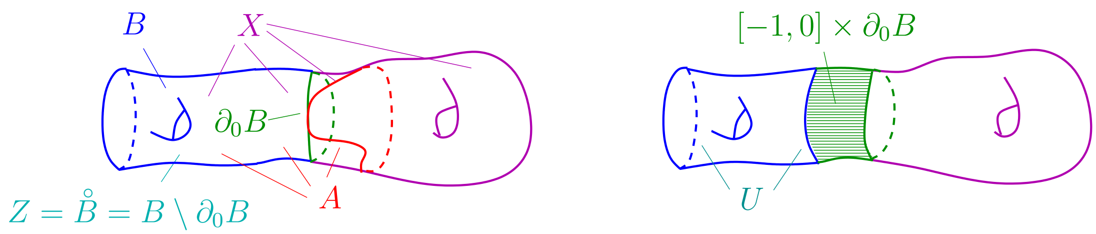


FIGURE 740. Illustration of the proof of the Excision Theorem 44.10.

PROOF OF THEOREM 44.9. We consider the following commutative diagram of inclusion induced maps

$$\begin{tikzcd}
\mathrm{H}_n(X \setminus Z, A \setminus Z) \arrow[r] \arrow[d] & \mathrm{H}_n(X, A) \\
\mathrm{H}_n(X \setminus U, A \setminus U). \arrow[ur, "\cong"'] &
\end{tikzcd}$$

The diagonal map is an isomorphism by the Excision Theorem 43.19. Thus it suffices to show that the vertical map is an isomorphism.

We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_n(A \setminus Z) \arrow[r] \arrow[d, "="] & \mathrm{H}_n(X \setminus Z) \arrow[r] \arrow[d] & \mathrm{H}_n(X \setminus Z, A \setminus Z) \arrow[r] \arrow[d] & \mathrm{H}_{n-1}(A \setminus Z) \arrow[r] \arrow[d, "="] & \mathrm{H}_{n-1}(X \setminus Z) \arrow[d] \\
\mathrm{H}_n(A \setminus Z) \arrow[r] \arrow[d] & \mathrm{H}_n(X \setminus U) \arrow[r] \arrow[d, "="] & \mathrm{H}_n(X \setminus U, A \setminus Z) \arrow[r] \arrow[d] & \mathrm{H}_{n-1}(A \setminus Z) \arrow[r] \arrow[d] & \mathrm{H}_{n-1}(X \setminus U) \arrow[d, "="] \\
\mathrm{H}_n(A \setminus U) \arrow[r] & \mathrm{H}_n(X \setminus U) \arrow[r] & \mathrm{H}_n(X \setminus U, A \setminus U) \arrow[r] & \mathrm{H}_{n-1}(A \setminus U) \arrow[r] & \mathrm{H}_{n-1}(X \setminus U).
\end{tikzcd}$$

Here the horizontal maps are pieces of the long exact sequences of the various pairs of topological spaces provided by Corollary 43.16 (1). Note that by Corollary 43.16 (2) we know that the diagram commutes.

Furthermore the vertical maps are induced by the various inclusion maps. On the upper level the first and the fourth vertical maps are the identity. Furthermore it follows from the hypothesis that $X \setminus Z$ is a deformation retract of $X \setminus U$ and from Corollary 43.18 that the second and the fifth vertical map are also isomorphisms. It follows from the Five Lemma 43.12 that the upper middle vertical map is also an isomorphism.

Using the hypothesis that $A \setminus Z$ is a deformation retract of $A \setminus U$ one can show exactly the same way that the lower middle vertical map is also an isomorphism. Combining the two middle vertical maps we obtain the desired isomorphism. ■

44.5. **The double of topological and smooth manifolds.** In this short section we introduce the double of a topological manifold. This construction associates to a compact manifold with non-empty boundary a closed manifold. This procedure comes in handy on several occasions. In particular sometimes this procedure allows us to reduce the proof of a certain statement for compact manifolds to the closed case. As the reader will notice, this section has only a moderately convincing justification for fitting into this particular chapter, but the material has to go somewhere, and some of the more technical statements in this section make use of the earlier results of this section.

After this apology, let us give the key definition of this section.

Definition. Let M be a topological manifold and let W be the union of some components of ∂M. We refer to

$$\mathrm{D}_W M \;:=\; (M \times \{1\}) \cup_{W\times\{1\}=W\times\{2\}} (M \times \{2\})$$

as the *double of M along W*. We refer to $\mathrm{D}\, M := D_{\partial M} M$ as the *double of M*.

Example. The double of M is given by taking two copies of M and identifying the boundary points "left and right" in the obvious way.

(1) On page 116 we gave an explicit homeomorphism $\overline{B}^n \xrightarrow{\cong} S^n_{\geq 0}$. Using this homeomorphism it is straightforward to see that the double of the n-ball $\overline{B}^n$ is homeomorphic to the n-dimensional sphere S^n. We refer to Figure 741 for an illustration.

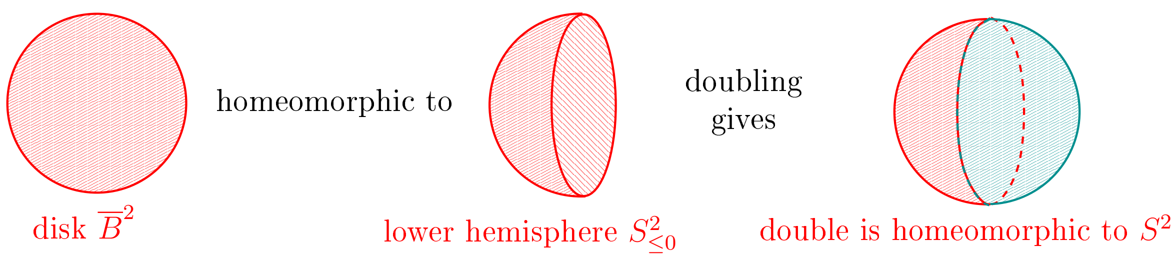


FIGURE 741

(2) Figure 742 makes a convincing case that the double of the torus minus two open disks is homeomorphic to the surface of genus 3. We refer to Figure 742 for an illustration.

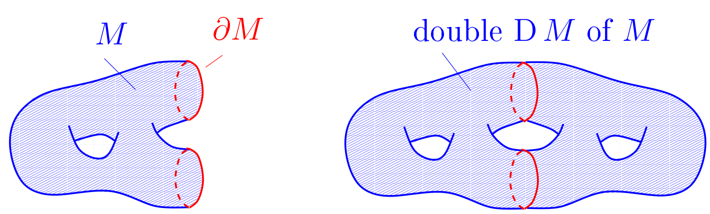


FIGURE 742

(3) In Exercise 44.8 we will see that the double of the Möbius band and the Klein bottle minus an open disk are homeomorphic to well-known topological spaces.

The following definition allows us to relate a topological manifold to its double.

Definition. Let M be a topological manifold and let W be the union of some components of ∂M. We refer to the map

$$i\colon M \xrightarrow{P\mapsto(P,1)} M \times \{1\} \;\to\; (M \times \{1\}) \sqcup (M \times \{2\}) \;\to\; \mathrm{D}_W M$$

as the *natural inclusion*. Furthermore we refer to the map

$$\begin{array}{rcl} r\colon\ \mathrm{D}_W M = ((M\times\{1\})\cup(M\times\{2\}))/\sim & \to & M \\ [(P,i)] & \mapsto & P \end{array}$$

as the *folding map*. Finally we refer to the map

$$\begin{array}{rcl} \rho\colon\ \mathrm{D}_W M & \to & \mathrm{D}_W M \\ [(P,i] & \mapsto & [(P,3-i)] \end{array}$$

as the *natural involution* on $\mathrm{D}_W M$.

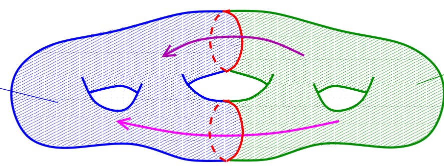


FIGURE 743. Illustration of the folding map $\mathrm{D}\, M \to M$.

The following lemma gives us two useful maps relating a topological manifold M to its doubles $\mathrm{D}_W M$.

Lemma 44.11. *Let M be a topological manifold and let W be the union of some components of ∂M.*

1. *The natural inclusion $i\colon M\to \mathrm{D}_W M$ is an embedding.*
2. *The folding map $\mathrm{D}_W M \to M$ is continuous and it satisfies $r\circ i = \mathrm{id}_M$.*
3. *The natural involution $\mathrm{D}_W M \to \mathrm{D}_W M$ is continuous and it is an involution, i.e. $\rho\circ\rho = \mathrm{id}$.*

PROOF. It follows from Lemmas 3.3 and 3.21 (3) that the natural inclusion i is continuous and that the involution ρ are continuous. It is clear that $\rho\circ\rho = \mathrm{id}$. Furthermore it follows from Lemmas 3.3 and 3.22 that the folding map r is continuous. Clearly we have $r\circ i = \mathrm{id}_M$. This shows in particular that $i\colon M\to i(M)$ is a homeomorphism, i.e. i is an embedding.

Finally it follows from Lemma 3.3, 3.21 (3) and 3.22 that the involution ρ is continuous. It is clear that $\rho\circ\rho = \mathrm{id}$. ■

Convention. Sometimes we will use the fact, established in Lemma 44.11, that the natural inclusion $M\to \mathrm{D}_W M$ is an embedding as an excuse to view M as a subspace of $\mathrm{D}_W M$ and to view $r\colon\ \mathrm{D}_W M\to M$ as a retraction.

The following lemma summarizes a few key properties of the doubling operation.

Lemma 44.12. *Let M be an n-dimensional topological (smooth) manifold and let W be the union of some components of ∂M. The following statements hold:*

1. *The double $\mathrm{D}_W M$ is also an n-dimensional topological (smooth) manifold such that $M\subset \mathrm{D}_W M$ is a (smooth) submanifold and such that M is a closed subset of $\mathrm{D}_W M$.*
2. *The boundary of the double $\mathrm{D}_W M$ is given by two copies of $\partial M\setminus W$. In particular the boundary of the double $\mathrm{D}\, M$ is empty.*
3. *M is compact if and only if the double $\mathrm{D}_W M$ is compact.*
4. *If M is connected and if $W\neq\varnothing$, then the double $\mathrm{D}_W M$ is also connected.*

(5) *If M is an oriented smooth manifold, then $\mathrm{D}_W M$ admits an orientation such that $M \to \mathrm{D}_W M$ is an orientation-preserving embedding. Conversely, if $\mathrm{D}_W M$ is oriented, then M admits an orientation such that the inclusion $M \to \mathrm{D}_W M$ is orientation-preserving.*[720]

(6) *For every $k \in \mathbb{N}_0$ the inclusion induced map*
$$\mathrm{H}_k(M, W) \;\to\; \mathrm{H}_k(\mathrm{D}_W M, M)$$
is an isomorphism.

Proof (∗). Let M be an n-dimensional topological (smooth) manifold. Statements (1), (2) and (4) follow immediately from applying Proposition 8.15 and Proposition 44.8 iteratively to $(W \times \{1\}) \sqcup (W \times \{2\}) \subset (M \times \{1\}) \sqcup (M \times \{2\})$.

We turn to the proof of Statement (3). Note that it follows easily from Lemmas 44.11, 2.40 and 2.16 that M is compact if and only if the double $\mathrm{D}_W M$ is compact.

Next we provide the proof of Statement (5). Thus we suppose that M is an oriented smooth manifold. We equip $M \times \{1\}$ with the obvious orientation. Furthermore we equip $M \times \{2\}$ with the opposite orientation. With these orientations it is straightforward to see that the map $\partial(M \times \{1\}) \xrightarrow{(P,1) \mapsto (P,2)} \partial(M \times \{2\})$ is orientation-reversing. Thus it follows from Proposition 8.15 that the double $\mathrm{D}_W M$ admits an orientation such that $M \to \mathrm{D}_W M$ is an orientation-preserving embedding. Conversely, if $\mathrm{D}_W M$ is oriented, then it follows from Lemma 6.46 (4) that M inherits an orientation such that the inclusion $M \to \mathrm{D}\, M$ is orientation-preserving.

Finally note that Statement (6) is an immediate consequence of the Excision Theorem 44.10 for Topological Manifolds, and the fact, established in (1), that M is a closed subset of $\mathrm{D}\, M$. ■

The following lemma relates some of the algebraic invariants of a manifold M to the invariants of its double $\mathrm{D}\, M$.

Lemma 44.13. *Let M be a topological manifold and let W be the union of some components of ∂M.*

(1) *For every $j \in \mathbb{N}_0$ the inclusion induced map $\mathrm{H}_j(M) \to \mathrm{H}_j(\mathrm{D}_W M)$ is a monomorphism.*
(2) *If ∂M is non-empty and if x_0 is a point in ∂M, then for every $j \in \mathbb{N}_0$ the inclusion induced map $\pi_j(M, x_0) \to \pi_j(\mathrm{D}_W M, x_0)$ is a monomorphism.*

Remark. In Proposition 107.2 we will calculate the homology groups of $\mathrm{D}\, M$ in terms of the homology groups of M.

Proof. Let $i\colon M \to \mathrm{D}_W M$ be the natural inclusion map and let $r\colon \mathrm{D}_W M \to M$ be the folding map. In Lemma 44.11 we saw that both maps are continuous and that $r \circ i = \mathrm{id}_M$. The two statements now follow as usual from the functoriality of homology groups and homotopy groups. More precisely, in both cases we have $r_* \circ i_* = \mathrm{id}_*$, i.e. i_* needs to be a monomorphism. ■

[720] In Lemma 86.18 we will prove an analogous statement for topological manifolds.

In favorable circumstances we can fully control the fundamental group of the double.

Proposition 44.14. *Let M be a compact connected n-dimensional topological manifold, let W be a component of ∂M and let $P \in W$. If the inclusion induced map $\pi_1(W, P) \to \pi_1(M, P)$ is an epimorphism, then the inclusion induced map $\pi_1(M, P) \to \pi_1(\mathrm{D}_W\, M, P)$ is an isomorphism.*

The proof of Proposition 44.14 results on the following group-theoretic lemma, which is precisely the content of Exercise 21.11.

Lemma 44.15. *Let A and G be groups and let $\varphi\colon A \to G$ be an epimorphism. Furthermore let G' be a copy of G. We consider the amalgamated product $G *_A G'$ given by using twice the epimorphism $\varphi\colon A \to G$ and $\varphi\colon A \to G' = G$. In this setting the natural homomorphism $G \to G *_A G'$ is an isomorphism.*

Proof of Lemma 44.15. By Proposition 21.21 there exists a unique homomorphism $\Theta\colon G *_A G' \to G'$ that makes the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, two heads] \arrow[d, two heads] & G \arrow[d] \arrow[ddr, bend left, "\mathrm{id}"] & \\
G' \arrow[r] \arrow[drr, bend right, "\mathrm{id}"'] & G *_A G' \arrow[dr, red, "\Theta"'] & \\
& & G.
\end{tikzcd}$$

In particular we see that the right vertical map is a monomorphism. The left vertical map is by hypothesis an epimorphism. It follows from Lemma 21.22 (3) that the right vertical map is also an epimorphism. Thus we shown that the right vertical map is indeed an isomorphism. ■

Proof of Proposition 44.14. Let $P \in W$. Since W is non-empty and connected we obtain from the Seifert-van Kampen Theorem 22.2 that the inclusions of the two copies of M into the double $\mathrm{D}_W\, M$ induce an isomorphism $\pi_1(\mathrm{D}\, M, P) \cong \pi_1(M, P) *_{\pi_1(W,P)} \pi_1(M, P)$. By our hypothesis the map $\pi_1(W, P) \to \pi_1(M, P)$ is an epimorphism. It follows from the above, together with Lemma 44.15, that the map $\pi_1(M, P) \to \pi_1(\mathrm{D}_W\, M, P)$ is indeed an epimorphism. ■

We conclude this introduction to the double of a topological manifold with the following harmless but at times useful lemma. The lemma follows almost immediately from the definitions, hence we leave it to the conscientious reader to fill in the details.

Lemma 44.16. $(*)$ *Let M be a topological manifold and let W be the union of some components of ∂M. Given $i \in \{1, 2\}$, we denote by $j_i\colon M \to M \times \{i\} \to \mathrm{D}_W\, M$ the obvious map.*

(1) *For every $k \in \mathbb{N}_0$ the map*

$$\begin{array}{rccc} \mathrm{D}\colon & \mathrm{C}_k(M, W) & \to & \mathrm{C}_k(\mathrm{D}_W\, M) \\ & [\underbrace{\sigma}_{\in \mathrm{C}_k(M)}] & \mapsto & j_{1*}(\sigma) - j_{2*}(\sigma) \end{array}$$

is well-defined.

(2) *The maps from* (1) *define a natural chain map, in particular for every* $k \in \mathbb{N}_0$ *these maps induce a map*

$$\mathrm{D}\colon\ \mathrm{H}_k(M,W) \ \to\ \mathrm{H}_k(\mathrm{D}_W\, M)$$

which is called the doubling homomorphism.

(3) *We make the identification* $M = j_1(M\times\{1\})$ *and we write* $M' = j_2(M\times\{2\})$*. The doubling homomorphism has the property that the following diagram commutes*

$$\begin{tikzcd} \mathrm{H}_k(M,W) \arrow[rr, "\mathrm{D}"] \arrow[dr] & & \mathrm{H}_k(\mathrm{D}_W\, M) \arrow[dl] \\ & \mathrm{H}_k(\mathrm{D}_W\, M, M'), & \end{tikzcd}$$

where the diagonal maps are the obvious maps. (As an aside we mention that it follows from the Excision Theorem 44.10 that the left diagonal map is an isomorphism.)

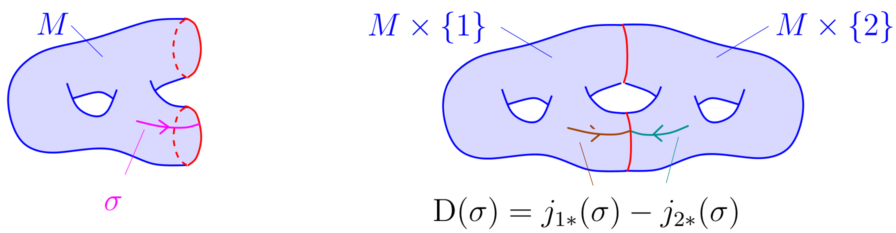


FIGURE 744. Illustration of Lemma 44.16 (1).

Exercises for Chapter 44.

Exercise 44.1. Let M be a topological manifold. Furthermore let $x_1,\dots,x_k$ be distinct points in M. Let $n\in\mathbb{N}_0$. For $i\in\{1,\dots,k\}$ we denote by

$$\nu_i\colon\ \mathrm{H}_n(M, M\setminus\{x_1,\dots,x_k\}) \ \to\ \mathrm{H}_n(M, M\setminus\{x_i\})$$

the map that is induced by the inclusion $M\setminus\{x_1,\dots,x_k\}\to M\setminus\{x_i\}$. Show that the map

$$\bigoplus_{i=1}^{k}\nu_i\colon\ \mathrm{H}_n(M, M\setminus\{x_1,\dots,x_k\}) \ \to\ \bigoplus_{i=1}^{k}\mathrm{H}_n(M, M\setminus\{x_i\})$$

is an isomorphism.
Hint. Use Excision Theorem 43.20.

Exercise 44.2. Let X be a finite topological graph, in other words, let X be a finite 1-dimensional CW-complex. Given any point $x\in X$ compute the local homology groups $\mathrm{H}_*(X, X\setminus\{x\})$.

Exercise 44.3. Let $n_1,\dots,n_k\in\mathbb{N}$. We form the wedge $X := S^{n_1}\vee\cdots\vee S^{n_k}$. We denote by P the wedge point. Compute the local homology groups $\mathrm{H}_*(X, X\setminus\{P\})$.

Exercise 44.4. Let M be a connected topological manifold. Show that $M\setminus\partial M$ is still connected.

Exercise 44.5. Let X be a topological space and let $n \in \mathbb{N}_0$. We consider the following two maps.

(1) Let $p\colon X \times S^1 \to X$ be the projection.
(2) Secondly we consider the homomorphism $\psi\colon \mathrm{H}_n(X \times S^1) \to \mathrm{H}_{n-1}(X)$ that is given by

$$\begin{array}{ccccc} \mathrm{H}_n(X \times S^1) & \to & \mathrm{H}_n(X \times S^1, X \times S^1_{\leq 0}) & \underset{\uparrow}{\overset{\cong}{\longleftarrow}} & \mathrm{H}_n(X \times S^1_{\geq 0}, X \times \{\pm 1\}) \\ & & & \text{isomorphism by the Excision Theorem 44.9} & \\ & & \underset{\substack{\uparrow \\ \text{connecting homomorphism}}}{\to} \mathrm{H}_{n-1}(X \times \{\pm 1\}) & & \underset{\substack{\uparrow \\ \text{projection induced homomorphism}}}{\to} \mathrm{H}_{n-1}(X \times \{1\}) \;=\; \mathrm{H}_{n-1}(X). \end{array}$$

Show that the map

$$p_* \oplus \psi\colon\ \mathrm{H}_n(X \times S^1) \;\to\; \mathrm{H}_n(X) \oplus \mathrm{H}_{n-1}(X)$$

is an isomorphism.
Hint. Consider the long exact sequence of the pair $(X \times S^1, X \times S^1_{\leq 0})$.
Remark. In Lemma 46.22 we will give a different approach to calculating $\mathrm{H}_n(X \times S^1)$.

Exercise 44.6. Let M be an n-dimensional topological manifold and let A and B be disjoint unions of boundary components of M. Furthermore let $f\colon A \to B$ be a homeomorphism. Show, without using the Topological Collar Neighborhood Theorem 44.5, that the topological space

$$M/a \sim f(a) \quad \text{for } a \in A$$

is an n-dimensional topological manifold.

Exercise 44.7. We recall the following definitions:

(1) The suspension of a topological space X is defined to be the topological space

$$\Sigma(X) \;:=\; (X \times [-1,1])/\sim$$

where we identify all points in $X \times \{-1\}$ and we identify all points in $X \times \{1\}$.
(2) A topological space X that has the property that for each $x \in X$ we have

$$\mathrm{H}_k(X, X \setminus \{x\}; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise} \end{cases}$$

is called an *n-dimensional homology manifold.*
(3) We say that an n-dimensional topological manifold X is a *topological homology n-sphere* if for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X; \mathbb{Z}) \cong \mathrm{H}_k(S^n; \mathbb{Z})$.

Let Y be a homology n-sphere.

(a) Show that $Y \times \mathbb{R}$ is an $(n+1)$-dimensional homology manifold.
(b) Show that the suspension $\Sigma(Y)$ of Y is an $(n+1)$-dimensional homology manifold.
(c) Show that if $\pi_1(Y)$ is non-trivial, then the suspension $\Sigma(Y)$ is *not* a topological manifold.

Remark. In Proposition 66.11 we will see that there exists a closed orientable 3-dimensional smooth manifold Y that is a homology 3-sphere, but such that $\pi_1(Y)$ is a non-trivial group. Together with the above this shows that there exist homology manifolds that are not topological manifolds.

Exercise 44.8. What is a familiar topological space that is homeomorphic to the double of the Möbius band? Furthermore, which one is homeomorphic to the double of the Klein bottle minus an open disk?

Exercise 44.9. Let M be a compact connected n-dimensional topological manifold such that ∂M has $k \in \mathbb{N}$ components.

(a) Show that $\operatorname{rank}(\mathrm{H}_1(\mathrm{D}\, M)) \geq k - 1$.
(b) Show that $\pi_1(M)$ admits an epimorphism onto a free group of rank $k - 1$.

45. Homology and self-maps of spheres

By Proposition 43.4 we know that for any $n \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_k(S^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Thus we have determined the homology groups of all finite-dimensional spheres. In this chapter, given $n \in \mathbb{N}_0$, we will determine an explicit singular n-chain which is a generator of $\widetilde{\mathrm{H}}_n(S^n)$. Furthermore, given a map $f\colon S^n \to S^n$ we will get to know several techniques for determining the induced map $f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)$. These results will allow us to prove various interesting results, e.g. the mysterious sounding Hair Ball Theorem. The results of this chapter will also be essential later on in Chapter 48 when we will introduce cellular homology.

45.1. **Explicit generators of relative homology groups.** We start out with the following elementary lemma that we will use explicitly and implicitly on many occasions.

Lemma 45.1. *Let $n \in \mathbb{N}_0$. We have*

$$\widetilde{\mathrm{H}}_k(\overline{B}^n, S^{n-1}) \;\cong\; \begin{cases} \mathbb{Z}, & \textit{if } k = n, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We consider the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$:

$$\dots \to \underbrace{\widetilde{\mathrm{H}}_k(\overline{B}^n)}_{= 0, \text{ see page 1106}} \to \mathrm{H}_k(\overline{B}^n, S^{n-1}) \xrightarrow{\partial} \underbrace{\widetilde{\mathrm{H}}_{k-1}(S^{n-1})}_{\substack{\text{calculated in}\\ \text{Proposition 43.4}}} \to \underbrace{\widetilde{\mathrm{H}}_{k-1}(\overline{B}^n)}_{= 0, \text{ see page 1106}} \to \dots$$

The statement of the lemma is now an immediate consequence of Proposition 43.4. ■

We continue with the following lemma.

Lemma 45.2. *For any $n \in \mathbb{N}_0$ the following two statements hold:*

(1) $\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.
(2) *The identity map* $\mathrm{id}\colon \Delta^n \to \Delta^n$ *represents a generator of* $\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.

Proof. The first statement is an immediate consequence of Lemmas 45.1 and 41.1. Now we turn to the proof of the second statement. We prove the second statement by induction on n. It follows easily from Proposition 41.5 that the statement holds for $n = 0$. Now we suppose that the desired statement holds for $n - 1$ with $n \in \mathbb{N}$.

Our next goal is to find a connection between the two homology groups $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$ and $\mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$. We introduce the following notation:

(1) We consider

$$\Lambda \;:=\; \bigcup_{i=0}^{n-1} \{(t_0, \dots, t_n) \in \Delta^n \,|\, t_i = 0\},$$

i.e. Λ is the union of all $(n-1)$-dimensional faces of Δ^n except for the face given by $t_n = 0$. A generalization of the argument of Exercise 18.24 (c) shows that Λ is a deformation retract of Δ^n. We refer to Figure 745 for an illustration.

(2) We write

$$\mathring{\Lambda} := \bigcup_{i=0}^{n-1} \{(t_0,\dots,t_n)\in\Lambda \,|\, t_i=0 \text{ and } t_n>0\},$$

the set $\mathring{\Lambda}$ is thus the "interior"[721] of Λ.

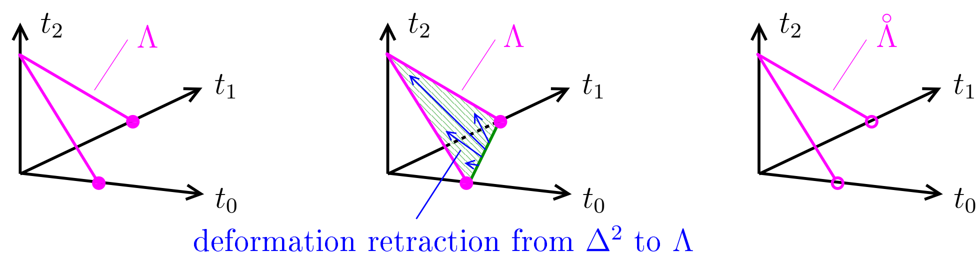


FIGURE 745

In the following we view $\mathbb{R}^n$ as the subset of $\mathbb{R}^{n+1}$ given by setting the last coordinate equal to zero. This point of view allows us to view Δ^{n-1} as a subset of Δ^n. Note that with this convention we have $\partial\Delta^{n-1}\subset\Lambda$.

Claim. We consider the maps

$$\mathrm{H}_n(\Delta^n,\partial\Delta^n) \xrightarrow{\partial} \mathrm{H}_{n-1}(\partial\Delta^n,\Lambda) \xleftarrow{i_*} \mathrm{H}_{n-1}(\Delta^{n-1},\partial\Delta^{n-1}).$$

Here the map on the left is the connecting homomorphism in the long exact sequence of the triple $(\Delta^n,\partial\Delta^n,\Lambda)$ and the right-hand map is induced by the inclusion. Then the following holds:

(1) Both maps are isomorphisms.
(2) $\partial([\mathrm{id}_{\Delta^n}]) = (-1)^n\cdot i_*([\mathrm{id}_{\Delta^{n-1}}])$.

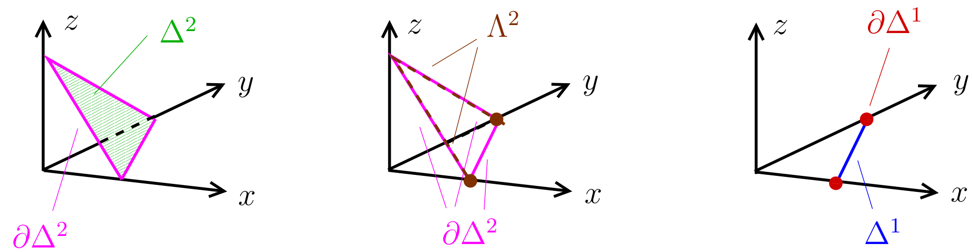


FIGURE 746. Illustration of the pairs $(\Delta^n,\partial\Delta^n)$, $(\partial\Delta^n,\Lambda)$ and $(\Delta^{n-1},\partial\Delta^{n-1})$.

It is clear that a proof of the claim together with the induction hypothesis implies that $[\mathrm{id}_{\Delta^n}]$ is a generator of $\mathrm{H}_n(\Delta^n,\partial\Delta^n)$. In particular a proof of the claim implies Lemma 45.2.

Now we turn to the proof of the claim. Of the two maps in the claim we first consider the map ∂ on the left. It follows from the exact sequence of the triple $(\Delta^n,\partial\Delta^n,\Lambda)$ that

[721] Why is "interior" put in quotes?

the following sequence is exact:

$$\underbrace{\mathrm{H}_n(\Delta^n,\Lambda)}_{\substack{=0 \text{ by Corollary 43.18 since}\\ \Lambda \text{ is a def. retract of } \Delta^n}} \xrightarrow{p} \mathrm{H}_n(\Delta^n,\partial\Delta^n) \xrightarrow{\partial} \mathrm{H}_{n-1}(\partial\Delta^n,\Lambda) \xrightarrow{i_*} \underbrace{\mathrm{H}_{n-1}(\Delta^n,\Lambda).}_{\substack{=0 \text{ by Corollary 43.18 since}\\ \Lambda \text{ is a def. retract of } \Delta^n}}$$

This shows that $\mathrm{H}_n(\Delta^n,\partial\Delta^n) \xrightarrow{\partial} \mathrm{H}_{n-1}(\partial\Delta^n,\Lambda)$ is an isomorphism. Furthermore we have

$$\partial([\mathrm{id}_{\Delta^n}]) \underset{\substack{\uparrow\\ \text{Proposition 43.15}}}{=} [\partial\,\mathrm{id}_{\Delta^n}] = \Big[\underbrace{\sum_{j=0}^{n-1}(-1)^j\,\mathrm{id}\circ i_j^n}_{\text{lies in } \mathrm{C}_{n-1}(\Lambda)} + (-1)^n\cdot \underbrace{\mathrm{id}\circ i_n^n}_{=\mathrm{id}_{\Delta^{n-1}}}\Big] \underset{\substack{\uparrow\\ \text{since we work in } \mathrm{H}_{n-1}(\partial\Delta^n,\Lambda)}}{=} (-1)^n\cdot[\mathrm{id}_{\Delta^{n-1}}].$$

Now we turn our attention to the right-hand map in the statement of the claim. It follows from Theorem 44.9, applied to $X=\partial\Delta^n$, $A=\Lambda$, $Z=\mathring{\Lambda}$ and $U=(0,\dots,0,1)$[722] that the inclusion induced map

$$\mathrm{H}_{n-1}(\Delta^{n-1},\partial\Delta^{n-1}) \;=\; \mathrm{H}_{n-1}(\underbrace{\partial\Delta^n\setminus\mathring{\Lambda}}_{=\Delta^{n-1}},\underbrace{\Lambda\setminus\mathring{\Lambda}}_{=\partial\Delta^{n-1}}) \xrightarrow{i_*} \mathrm{H}_{n-1}(\partial\Delta^n,\Lambda)$$

is an isomorphism. Furthermore we evidently have $i_*([\mathrm{id}_{\Delta^{n-1}}])=[\mathrm{id}_{\Delta^{n-1}}]$.

Thus we have proved that both maps in the claim are isomorphisms and that

$$\partial([\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n\cdot[\mathrm{id}_{\Delta^{n-1}}] \;=\; (-1)^n\cdot i_*([\mathrm{id}_{\Delta^{n-1}}]).$$

This concludes the proof of the claim and thus also of the lemma. ■

In the remainder of this section we explain how to find explicit generators of the local homology groups of a topological manifold. We recall the following lemma from the previous section.

Lemma 44.1. *Let M be a k-dimensional topological manifold. Then for any $P\in M$, any $l\in\mathbb{N}_0$ and any open neighborhood U of P we have*

$$\mathrm{H}_l(M,M\setminus\{P\}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } P\in M\setminus\partial M \text{ and } l=k,\\ 0, & \text{otherwise.}\end{cases}$$

Examples. Our goal now is to give explicit generators for the relative homology groups $\mathrm{H}_k(M,M\setminus\{x\})\cong\mathbb{Z}$. We first do so for open subsets of $\mathbb{R}^k$ and then we "transport" those generators to topological manifolds using charts.

(1) Let $U\subset\mathbb{R}^k$ be an open subset that contains 0. Let $\sigma\colon\Delta^k\to U$ be an injective affine linear map. We suppose that 0 lies in the image of the "interior" $\mathring{\Delta}^k:=\Delta^k\setminus\partial\Delta^k$ of Δ^k. We claim that σ represents a generator of $\mathrm{H}_k(U,U\setminus\{0\})$. To prove this claim we consider the commutative diagram

$$\begin{tikzcd}
\mathrm{H}_k(\Delta^k,\partial\Delta^k) \arrow[rr, "\sigma_*"] \arrow[d, "\sigma_*"] & & \mathrm{H}_k(U,U\setminus\{0\}) \arrow[d, "\cong"] \\
\mathrm{H}_k(\sigma(\Delta^k),\sigma(\partial\Delta^k)) \arrow[r] & \mathrm{H}_k(\mathbb{R}^k,\sigma(\partial\Delta^k)) \arrow[r] & \mathrm{H}_k(\mathbb{R}^k,\mathbb{R}^k\setminus\{0\})
\end{tikzcd}$$

[722]One can easily show that U has the property required in Theorem 44.9.

where the maps that are not induced by σ are induced by the obvious inclusions. The left vertical map is an isomorphism since we know by Proposition 2.43 (3) that $\sigma\colon \Delta^k \to \sigma(\Delta^k)$ is a homeomorphism. The bottom horizontal maps are isomorphisms by Corollary 43.18 since one can easily show that $\sigma(\Delta^k)$ is a deformation retract of $\mathbb{R}^k$ and similarly one can show that $\sigma(\partial\Delta^k)$ is deformation retract of $\mathbb{R}^k \setminus \{0\}$. By Lemma 45.2 we know that id_{Δ^n} represents a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$. It follows from the above that $\sigma \circ \mathrm{id}_{\Delta^n} = \sigma$ represents a generator of $\mathrm{H}_k(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\})$. But the right vertical map is an isomorphism by the Excision Theorem 43.20, hence σ also represents a generator of $\mathrm{H}_k(U, U \setminus \{0\})$.

(2) Let M be a k-dimensional topological manifold and let $P \in M \setminus \partial M$. In the following we want to give an example of an explicit singular k-simplex that represents a generator for $\mathrm{H}_k(M, M \setminus \{P\}) \cong \mathbb{Z}$. We pick a chart $\Phi\colon U \to B^k$ around P with $\Phi(P) = 0$ and we pick an affine linear map $\sigma\colon \Delta^k \to B^k$ as in (1). It follows from (1) together with the proof of Lemma 44.1 that $\Phi^{-1} \circ \sigma\colon \Delta^k \to M$ represents a generator of $\mathrm{H}_k(M, M \setminus \{P\})$. We refer to Figure 747 for an illustration.

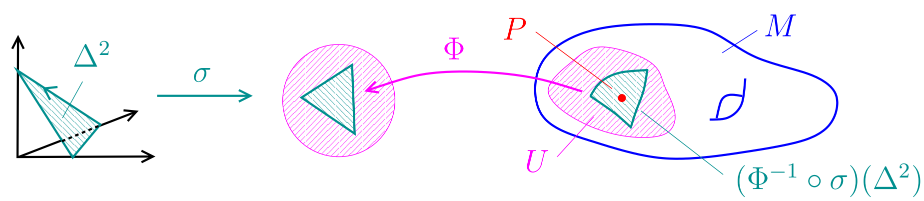


FIGURE 747

45.2. **The standard generator of $\mathrm{H}_n(S^n)$.** In this section we want to find explicit generators of $\mathrm{H}_n(S^n) \cong \mathbb{Z}$ for $n \geq 1$. We consider the map $\alpha\colon \Delta^n \to S^n$ that is given by the concatenation of the following two maps

$$\begin{array}{ccccccc} \Delta^n & \xrightarrow[\text{by Lemma 41.1}]{\text{homeomorphism given}} & \overline{B}^n & \to & S^n & \subset & \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \\ & & x & \mapsto & \left(x, \sqrt{1 - \|x\|^2}\right). & & \end{array}$$

We refer to Figure 748 for an illustration of the definition of the map α. It follows easily from the definitions and Proposition 2.43 (3) that this map α defines a homeomorphism $\Delta^n \to S^n_{\geq 0} = \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} \geq 0\}$ and that it restricts to a homeomorphism $\partial\Delta^n$ to $S^{n-1} = \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} = 0\}$. Furthermore we denote by $\beta\colon \Delta^n \to S^n$

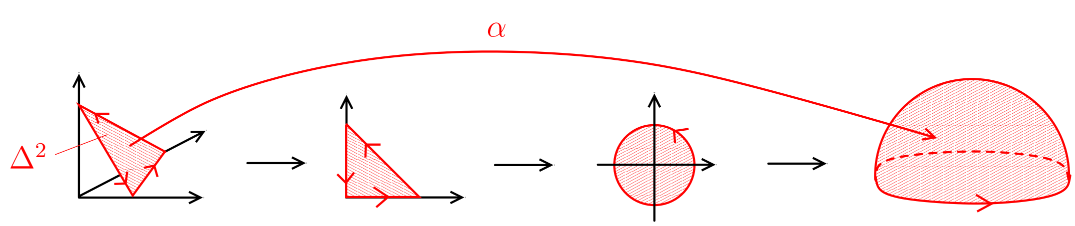


FIGURE 748

the map which is the composition of α with reflection in the hyperplane defined by $x_{n+1} = 0$. The maps $\alpha\colon \Delta^n \to S^n$ and $\beta\colon \Delta^n \to S^n$ both define singular n-chains in S^n. Note that α and β agree on $\partial\Delta^n$. Both maps are sketched in Figure 749.

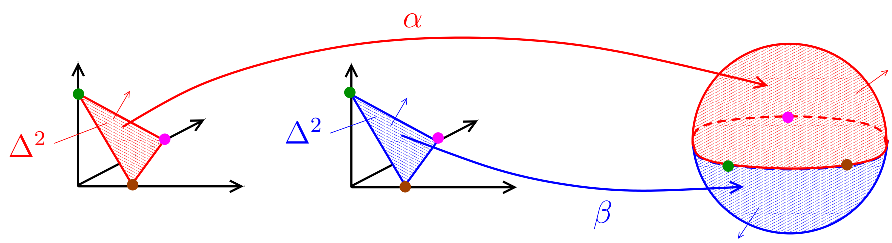


FIGURE 749

Lemma 45.3. *Let $n \geq 1$. The above singular n-chain $\alpha - \beta \in \mathrm{C}_n(S^n)$ is a cycle and it represents a generator of $\widetilde{\mathrm{H}}_n(S^n) = \mathrm{H}_n(S^n)$.*

Definition.

(1) Let $n \geq 1$. We refer to the generator from Lemma 45.3 as the *standard generator of* $\mathrm{H}_n(S^n)$.[723] With great foresight we denote it by $[S^n]$.

(2) We define $[c_1 - c_{-1}] \in \widetilde{\mathrm{H}}_0(S^0)$ to be the *standard generator of* $\widetilde{\mathrm{H}}_0(S^0)$.

PROOF. We start out with the following claim.

Claim. The singular n-chain $\alpha - \beta$ is a cycle.

We denote by ρ the reflection in the hyperplane defined by $x_{n+1} = 0$. We have

$$\partial(\alpha - \beta) \;=\; \partial(\alpha - \rho_*(\alpha)) \underset{\substack{\uparrow \\ \text{since } \rho_* \text{ is a chain map}}}{=} (\mathrm{id} - \rho_*)(\overset{\substack{\text{since } \alpha(\partial\Delta^n) \subset \mathbb{R}^n \times 0 \text{ we have } \partial\alpha \in \mathrm{C}_{n-1}(\mathbb{R}^n \times 0) \\ \downarrow}}{\partial}\alpha) \underset{\substack{\uparrow \\ \text{since } \rho = \mathrm{id} \text{ on } \mathbb{R}^n \times 0 \subset \mathbb{R}^{n+1}}}{=} 0.$$

$\boxplus$

As usual we denote by

$$S^n_{\geq 0} \;:=\; \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} \geq 0\}$$

and

$$S^n_{\leq 0} \;:=\; \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} \leq 0\}$$

the two "hemispheres". By the discussion on page 116 both hemispheres are homeomorphic to the closed ball $\overline{B}^n$, in particular they are contractible which implies by the discussion on page 1106 that their reduced homology groups vanish. We also consider the "equator"

$$S^n_{=0} \;:=\; \{(x_1, \ldots, x_{n+1}) \in S^n \,|\, x_{n+1} = 0\}.$$

[723] It follows easily from Lemma 41.3 and the definitions that the two definitions of standard generator of $\mathrm{H}_1(S^1)$ agree.

Next we consider the following sequence of maps

$$\begin{array}{ccccccc}
 & & \text{isomorphism by the long exact} & & \text{isomorphism by} & & \text{isomorphism since} \\
 & & \text{sequence in reduced homology} & & \text{the example on} & & \alpha \text{ is a homeomorphism} \\
 & & \text{of the pair } (S^n, S^n_{\le 0}) & & \text{on page 1160} & & \\
 & & \downarrow & & \downarrow & & \downarrow \\
\widetilde{\mathrm{H}}_n(S^n) & \to & \mathrm{H}_n(S^n, S^n_{\le 0}) & \xleftarrow{i_*} & \mathrm{H}_n(S^n_{\ge 0}, S^n_{=0}) & \xleftarrow{\alpha_*} & \mathrm{H}_n(\Delta^n, \partial\Delta^n) \\
[\alpha - \beta] & \mapsto & [\alpha - \beta] \underset{\substack{\uparrow \\ \text{since } \beta = 0 \text{ in } \mathrm{C}_n(S^n, S^n_{\le 0})}}{=} [\alpha] & \leftarrow\!\shortmid & [\alpha] & \leftarrow\!\shortmid & [\mathrm{id}_{\Delta^n}].
\end{array}$$

So we have shown that all maps are isomorphisms. Furthermore, by Lemma 45.2 we know that $[\mathrm{id}_{\Delta^n}]$ is a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$. But $[\mathrm{id}_{\Delta^n}]$ and $[\alpha - \beta]$ get sent to the same element in $\mathrm{H}_n(S^n, S^n_{\le 0})$, therefore $[\alpha - \beta]$ must also be generator of $\widetilde{\mathrm{H}}_n(S^n)$. ■

In the following lemma we give an alternative description of the standard generator of $\mathrm{H}_1(S^1)$.

Lemma 45.4. *The singular 1-simplex*

$$\begin{array}{rcl} \sigma\colon \Delta^1 & \to & S^1 \\ (1-t, t) & \mapsto & e^{2\pi \mathrm{i} t} \end{array}$$

is a cycle that represents the standard generator $[S^1]$ *of* $\mathrm{H}_1(S^1) \cong \mathbb{Z}$.

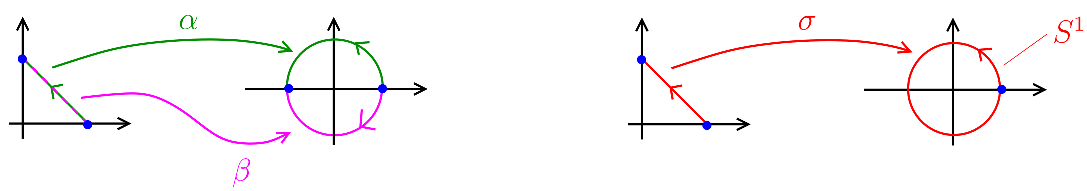


FIGURE 750. Two representatives of the standard generator of $\mathrm{H}_1(S^1)$.

PROOF. We denote by $\alpha, \beta\colon \Delta^1 \to S^1$ the singular 1-simplices in the definition of the standard generator of $\mathrm{H}_1(S^1)$. We have

$$[\alpha - \beta] \underset{\uparrow}{=} [\alpha + \overline{\beta}] \underset{\uparrow}{=} [\sigma] \in \mathrm{H}_1(S^1).$$
$$\text{by Lemma 41.3 (2) and (3)}$$

We refer to Figure 750 for an illustration. ■

Remark. By Lemma 45.4 we know that there exists a singular 1-simplex σ that represents a generator of $\mathrm{H}_1(S^1)$. In Exercise 45.3 we will discuss the question, for which $n \in \mathbb{N}$ does there exist a singular n-simplex $\sigma\colon \Delta^n \to S^n$ that represents a generator of $\mathrm{H}_n(S^n)$.

We conclude this section with a short discussion of several related homology groups and their "standard generators".

Definition. Let $n \in \mathbb{N}$. The *standard generator of* $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ is defined to be the unique element that gets sent to the standard generator of $\widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ under the connecting homomorphism $\mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ of the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$. We denote the standard generator by $[\overline{B}^n]$.

Example. Recall that the standard generator of $\widetilde{\mathrm{H}}_0(S^0)$ is defined as $[c_1 - c_{-1}] \in \widetilde{\mathrm{H}}_0(S^0)$. It follows easily from the explicit description of the connecting homomorphism, see Proposition 43.15, and the explicit description of the boundary of a singular 1-simplex, see page 1079, that the standard generator of $\mathrm{H}_1(\overline{B}^1, S^0)$ is represented by the cycle

$$\begin{array}{rcl} \Delta^1 & \to & \overline{B}^1 \\ (1-t,t) & \mapsto & 2t-1. \end{array}$$

We refer to Figure 751 for an illustration.

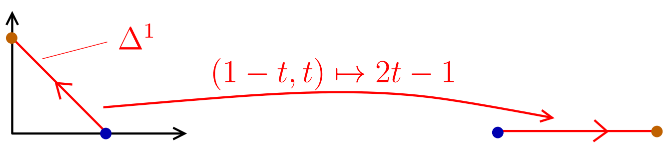


FIGURE 751

We have now introduced several standard generators of homology groups. The following proposition shows the relationship between these standard generators under various maps. The proposition, let alone the proof, are not much fun to read, but it is an absolutely essential proposition since it settles several rather subtle signs.

Proposition 45.5. *Let $n \in \mathbb{N}$. We denote by $f \colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ the homeomorphism from Lemma 41.1 and we denote by $g \colon \overline{B}^n/S^{n-1} \to S^n$ the homeomorphism from page 182. We set $* := g([S^{n-1}])$.*[724] *We consider the following diagram:*

$$\begin{tikzcd}
 & & \widetilde{\mathrm{H}}_n(\overline{B}^n/S^{n-1}) \arrow[r, "g_*"] \arrow[d, "\text{ж}"', "\cong"] & \widetilde{\mathrm{H}}_n(S^n) \arrow[d, "\cong"', "\text{ж}"] & \\
\mathrm{H}_n(\Delta^n, \partial\Delta^n) \arrow[r, "f_*"] \arrow[d, "\partial"] & \mathrm{H}_n(\overline{B}^n, S^{n-1}) \arrow[r] \arrow[ur, "\text{ю}"] \arrow[d, "\partial"] & \mathrm{H}_n(\overline{B}^n/S^{n-1}, [S^{n-1}]) \arrow[r, "g_*"] & \mathrm{H}_n(S^n, *) & \mathrm{H}_n(S^n). \arrow[l] \\
\widetilde{\mathrm{H}}_{n-1}(\partial\Delta^{n-1}) \arrow[r, "f_*"] & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) & & &
\end{tikzcd}$$

The following statements hold:

(1) *The diagram commutes and the top vertical maps are isomorphisms.*
(2) *We have*

$$f_*([\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n \cdot \textit{standard generator } [\overline{B}^n] \textit{ of } \mathrm{H}_n(\overline{B}^n, S^{n-1}).$$

(3) *The standard generator $[\overline{B}^n]$ of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ has the same image in $\mathrm{H}_n(S^n, *)$ as the standard generator $[S^n]$ of $\mathrm{H}_n(S^n)$.*

(4) *The image of the standard generator* $[\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$ *in* $\mathrm{H}_{n-1}(S^{n-1})$ *equals the standard generator* $[S^{n-1}]$ *of* $\mathrm{H}_{n-1}(S^{n-1})$.
(5) *We have*

$$f_*(\partial[\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n \cdot \textit{standard generator } [S^{n-1}] \textit{ of } \mathrm{H}_{n-1}(S^{n-1}).$$

Proof of Proposition 45.5 (∗).

(1) The triangle commutes by definition of ю, see page 1129. Furthermore it follows from Lemma 43.14 and Corollary 43.16 that the two squares commute.
(3),(4) The calculation of these two signs is delicate and tricky. It becomes somewhat easier after we have developed a few more tools. Thus we postpone the proof of these two statements to Section 45.7.
(4) This statement is just a repetition of the definition of the standard generator $[\overline{B}^n]$.
(5) This statement is an immediate consequence of (1), (2) and (4). ■

We conclude this section with the following lemma which gives us another slightly subtle sign calculation.

Lemma 45.6. (∗) *Let* $n \in \mathbb{N}$. *We consider the following sequence of maps*

$$\mathrm{H}_n(S^n) \xrightarrow{\text{я}} \mathrm{H}_n(S^n, S^n_{\geq 0}) \xleftarrow{\cong} \mathrm{H}_n(S^n_{\geq 0}, S^{n-1}) \xrightarrow{\partial} \widetilde{\mathrm{H}}_{n-1}(S^{n-1}).$$

inclusion induced map is an isomorphism by the discussion on page 1160

The image of the standard generator $[S^n] \in \mathrm{H}_n(S^n)$ *under this map is* $(-1)^n \cdot [S^{n-1}]$.

Proof (∗). We recall and introduce the following notation:

(1) Let $f\colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from Lemma 41.1.
(2) Let $p_+\colon S^n_{\geq 0} \to \overline{B}^n$ be the obvious projection and let $q_+\colon \overline{B}^n \to S^n_{\geq 0}$ be the map given by $x \mapsto (x, \sqrt{1-\|x\|^2})$. Evidently p and q are inverses of one another. We define $p_-\colon S^n_{\leq 0} \to \overline{B}^n$ and $q_-\colon \overline{B}^n \to S^n_{\leq 0}$ in an analogous fashion.
(3) Note that $\alpha := q_+ \circ f\colon \Delta^n \to S^n$ and $\beta := q_- \circ f\colon \Delta^n$ are the singular n-simplices that we introduced on page 1173.

Next we consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_n(S^n) \arrow[r, "\text{я}"] & \mathrm{H}_n(S^n, S^n_{\geq 0}) & \mathrm{H}_n(S^n_{\geq 0}, S^{n-1}) \arrow[l, "i"'] \arrow[r, "\partial"] \arrow[d, shift left, "p_+"] & \mathrm{H}_{n-1}(S^{n-1}). \\
& & \mathrm{H}_n(\overline{B}^n, S^{n-1}) \arrow[u, shift left, "q_+"] \arrow[ur, "\partial"'] &
\end{tikzcd}$$

[724]The fact that we use the expression $[S^{n-1}]$ for two very different objects, namely first for the point in $\overline{B}^n/S^{n-1}$ given by the equivalence class S^{n-1} and secondly for the standard generator of $\widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ is unfortunate, but difficult to avoid if one wants to follow the standard conventions.

Note that the triangle to the right commutes by the naturality of the connecting homomorphism, see Corollary 43.16. Now we see that

$$\begin{aligned}
\partial(i_*^{-1}(\text{я}([S^n]))) &\overset{\substack{\text{definition of } [S^n] \\ \downarrow}}{=} \partial(i_*^{-1}(\text{я}([\alpha-\beta]))) \overset{\substack{\text{definition of я} \\ \downarrow}}{=} \partial(i_*^{-1}([\alpha-\beta])) \overset{\substack{\text{since } \beta=0\in \mathrm{C}_n(S^n,S^n_{\le 0}) \\ \downarrow}}{=} \partial(i_*^{-1}([\alpha])) = \partial([\alpha]) \\
&\underset{\substack{\uparrow \\ \text{since } p_+=q_+^{-1}}}{=} \partial(q_{+*}([p_+\circ\alpha])) \underset{\substack{\uparrow \\ \text{since } \alpha=q_+\circ f}}{=} \partial(q_{+*}([f])) \underset{\substack{\uparrow \\ \text{since the triangle} \\ \text{commutes}}}{=} \partial([f]) \underset{\substack{\uparrow \\ \text{Proposition 45.5 (5)}}}{=} (-1)^n\cdot[S^{n-1}].
\end{aligned}$$

■

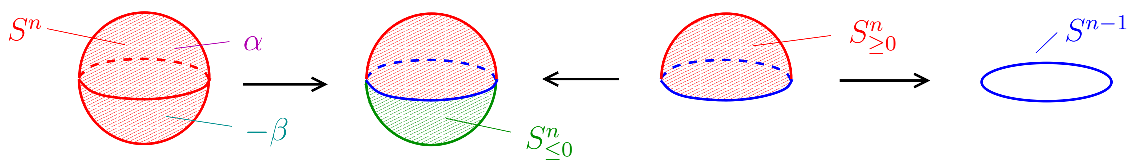


FIGURE 752. Illustration for the proof of Lemma 45.6.

45.3. **Suspensions and homology groups.** In the next section we will study self-maps of spheres. In that section it will be useful to consider the effect of taking suspensions on homology. Since this is going to be interesting for larger classes of topological spaces we dedicate this short section to the study of homology groups of suspensions of topological spaces.

The following definition follows the definition on page 694 and the convention given by Lemma 24.2 (3).

Definition. Let X be a non-empty topological space.

(1) We define the *suspension of* X to be the topological space

$$\Sigma(X) \ := \ (X\times[-1,1])/\sim$$

where we identify all points in $X\times\{-1\}$ to a single point and we identify all points in $X\times\{1\}$ to a single point.

(2) We denote by $p\colon X\times[-1,1]\to\Sigma(X)$ the obvious projection. We write

$$C_-X \ := \ p(X\times[-1,1)) \quad \text{and} \quad C_+X \ := \ p(X\times(-1,1]).$$

By Lemma 24.2 (3) we know that the map $X \xrightarrow{x\mapsto p(x,0)} \Sigma(X)$ is a closed embedding. We use this embedding to view X as a subspace of $\mathrm{C}_\pm X$ and of $\Sigma(X)$.

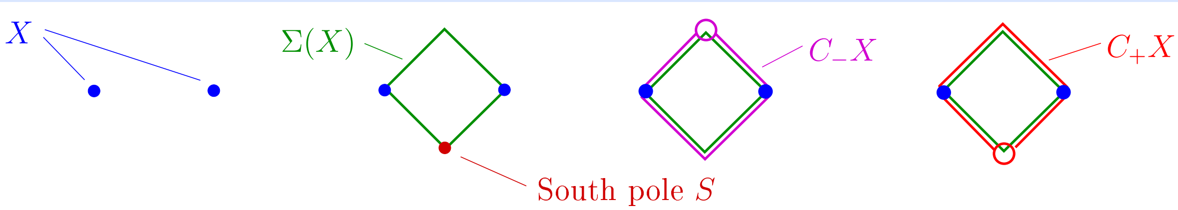


FIGURE 753

In Lemma 24.3 and Exercise 26.5 we calculated the fundamental group of the suspension of a non-empty topological space with finitely many path components. The following proposition, which is the main result of this section, determines the homology groups of a suspension.

Proposition 45.7. *Let X be a non-empty topological space and let $n \in \mathbb{Z}$.*[725]

(1) *The following maps are isomorphisms:*

$$\widetilde{\mathrm{H}}_n(X) \underset{\substack{\uparrow \\ \text{connecting homomorphism of the} \\ \text{pair } (C_+X,X) \text{ from Corollary 43.16}}}{\xleftarrow{\ \partial\ }} \mathrm{H}_{n+1}(C_+X,X) \underset{\substack{\uparrow \\ \text{inclusion induced}}}{\longrightarrow} \mathrm{H}_{n+1}(\Sigma(X),C_-X) \underset{\substack{\uparrow \\ \text{the natural map given} \\ \text{by Lemma 43.1 (3)}}}{\xleftarrow{\ \text{ж}\ }} \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)).$$

(2) *The resulting isomorphism*

$$\Sigma_X \colon \widetilde{\mathrm{H}}_n(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{n+1}(\Sigma(X))$$

is natural. In other words, given any map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{tikzcd}
\widetilde{\mathrm{H}}_n(X) \arrow[r, "\Sigma_X"] \arrow[d, "f_*"'] & \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) \arrow[d, "\Sigma(f)_*"] \\
\widetilde{\mathrm{H}}_n(Y) \arrow[r, "\Sigma_Y"] & \widetilde{\mathrm{H}}_{n+1}(\Sigma(Y)).
\end{tikzcd}$$

Example. Let $n \in \mathbb{N}$ and let X be the disjoint union of n intervals. It follows from Proposition 45.7 together with Lemma 43.1 that $\mathrm{H}_1(\Sigma(X)) \cong \mathbb{Z}^{n-1}$. In Figure 754 we show a generating set for $\mathrm{H}_1(\Sigma(X))$ for $n = 3$. We leave it to the reader to deduce from the proof of Proposition 45.7 that the given cycles do indeed form a generating set for $\mathrm{H}_1(\Sigma(X))$.

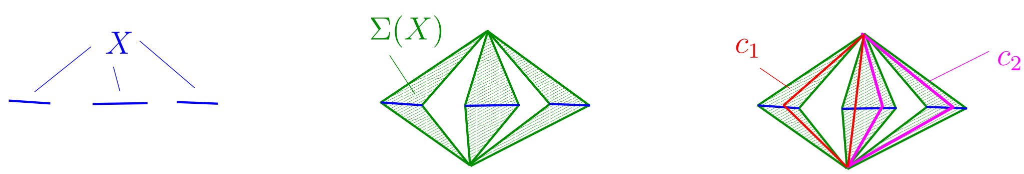


FIGURE 754

PROOF (∗). Let $n \in \mathbb{Z}$.

(1) (a) By the same argument as in the proof of Lemma 24.1 we see that both C_-X and C_+X admit a deformation retraction to a point. It follows from Lemma 18.14 that both are contractible, therefore by Lemma 43.1 (7) the reduced homologies of C_-X and C_+X vanish. Note that it follows in particular that the connecting homomorphism $\partial\colon \mathrm{H}_{n+1}(C_+X,X) \to \widetilde{\mathrm{H}}_n(X)$ is an isomorphism.

(b) We denote by $S := [X \times \{-1\}] \in \Sigma(X)$ the South Pole of the suspension. Note that $\Sigma(X) \setminus S = C_+X$ and $C_-X \setminus S = C_-X \cap C_+X$. Using this observation we obtain from the Excision Theorem 43.19[726] that the second horizontal map is an isomorphism.

[725] Note that the proposition has non-trivial content for $n = -1$.

[726] One can easily verify that the hypotheses for the Excision Theorem 43.19 are satisfied.

(c) Similar to (a) we obtain from Corollary 43.16 that the third horizontal map is an isomorphism.

(2) First note that the connecting homomorphism is natural by Proposition 43.15 (2) and the fact that $\Sigma(f)|_X = f$. It is clear that the second horizontal map is natural. Finally it follows from Lemma 43.1 (3) that the map ж is natural. The combination of these observations shows that Σ_X itself is natural. ■

Example. Let $n \in \mathbb{N}$. Let $\Sigma(S^n) \to S^{n+1}$ be the homeomorphism from page 695. It follows quite easily from the definition of the suspension isomorphism $\Sigma_{S^n}\colon\ \mathrm{H}_n(S^n) \to \mathrm{H}_{n+1}(S^{n+1})$ and Lemma 45.6 that

$$\Sigma_{S^n}([S^n]) \;=\; (-1)^{n+1} \cdot [S^{n+1}] \;\in\; \mathrm{H}_{n+1}(S^{n+1}).$$

We conclude this section with the following corollary.

Corollary 45.8. (∗) *Let X be a topological space that is compact and Hausdorff. Furthermore let $k \in \mathbb{N}_{\geq -1}$. For every $n \in \mathbb{Z}$ there exists a natural isomorphism*

$$\widetilde{\mathrm{H}}_n(X) \;\xrightarrow{\cong}\; \widetilde{\mathrm{H}}_{n+k+1}(S^k \underset{\substack{\uparrow \\ \text{join of } S^k \text{ and } X\text{, see page 207 for the definition}}}{*} X).$$

Here, recall that if X is the empty topological space, then by Lemma 43.1 (0) we have $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$.

Proof (∗). First note that the case $X = \varnothing$ follows from the fact that, by definition, $S^k * \varnothing = S^k$, and the calculation of the reduced homology groups of S^k that we gave in Proposition 43.4.

Now assume that $X \neq \varnothing$. If $k = -1$, then we have, again by definition, the equality $S^{-1} * X = \varnothing * X = X$ which implies the desired statement. Thus it remains to deal with case that $k \geq 0$. To simplify the notation we only consider the case $k = 1$. In this case we have

$$\widetilde{\mathrm{H}}_{n+2}(S^1 * X) \underset{\substack{\uparrow \\ \text{Lemma 3.50}}}{\cong} \widetilde{\mathrm{H}}_{n+2}((S^0 * S^0) * X) \underset{\substack{\uparrow \\ \text{by Proposition 3.52} \\ \text{see page 250 for the definition} \\ \text{of the homeomorphism}}}{\cong} \widetilde{\mathrm{H}}_{n+2}(S^0 * (S^0 * X)) \underset{\substack{\uparrow \\ \text{Exercise 24.3}}}{\cong} \widetilde{\mathrm{H}}_{n+2}(\Sigma(\Sigma(X))) \underset{\substack{\uparrow \\ \text{Proposition 45.7} \\ \text{applied twice}}}{\cong} \widetilde{\mathrm{H}}_n(X).$$

It follows easily from the definitions of the maps that the isomorphism is in fact natural. ■

45.4. **The degree of a self-map of a sphere.** Now that we know the homology groups of spheres, we want to study the maps on homology induced by maps $f\colon S^n \to S^n$. This is interesting in its own right, but it will also play a major role later on in Chapter 48 when we will try to determine the homology groups of CW-complexes.

Throughout the remaining sections of this chapter we will work a lot with the following simple-minded definition.

Definition. Let $(G,+)$ be a group that is isomorphic to $\mathbb{Z}$. Furthermore let $\Phi\colon G \to G$ be a homomorphism. Note that Φ is given by multiplication by a unique integer which we refer to as the *degree* $\deg(\Phi)$.

For completeness' sake we record two basically obvious statements regarding the degree.

Lemma 45.9. *Let $(G,+)$ be a group that is isomorphic to $\mathbb{Z}$.*

(1) *Let $g \in G$ be any non-trivial element. Furthermore let $\Phi\colon G \to G$ be a homomorphism. The degree $\deg(\Phi)$ is uniquely determined by the equality $\Phi(g) = \deg(\Phi)\cdot g$.*
(2) *Let $\varphi\colon G \to H$ be an isomorphism and let $f\colon H \to H$ be a homomorphism. Then $\deg(\varphi^{-1}\circ f\circ\varphi\colon G \to G) = \deg(f\colon H \to H)$.*

Next we relate the above definition to topology.

Definition. Let $n \in \mathbb{N}_0$ and let $f\colon S^n \to S^n$ be map. Since $\widetilde{\mathrm{H}}_n(S^n)$ is isomorphic to $\mathbb{Z}$ we can define the *degree of f* as

$$\deg(f) \;:=\; \deg\big(f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)\big).$$

Example. We consider the map $f\colon S^1 \to S^1$ that is illustrated in Figure 755 on the left.[727] We denote by $\tau\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $\tau(1-t,t) = e^{2\pi \mathrm{i} t}$. By Lemma 45.4 we know that $[\tau] \in \mathrm{H}_1(S^1)$ is a generator.

We pick singular 1-simplices $\sigma_1,\dots,\sigma_5$ as in the middle of Figure 755 and we denote by $\gamma\colon \Delta^1 \to S^1$ the constant singular 1-simplex that sends all points to 1. Then

$$\begin{aligned} f_*([\tau]) \overset{\substack{\text{by Lemma 41.3 (3)}\\ \downarrow}}{=} f_*([\sigma_1+\dots+\sigma_5]) &= \big[\overbrace{\sigma_1\circ f}^{=\gamma} + \overbrace{\sigma_2\circ f}^{=\tau} + \overbrace{\sigma_3\circ f}^{=\gamma} + \overbrace{\sigma_4\circ f}^{=\overline{\tau}} + \overbrace{\sigma_5\circ f}^{=\tau}\big] \\ &= [2\cdot\tau+\overline{\tau}+2\cdot\gamma] \underset{\substack{\uparrow\\ \text{Lemma 41.3 (2)}}}{=} [\tau] = \underbrace{(+1)}_{=:\deg(f)}\cdot[\tau]. \end{aligned}$$

Summarizing we have shown that $\deg(f) = +1$. With almost the same approach we can

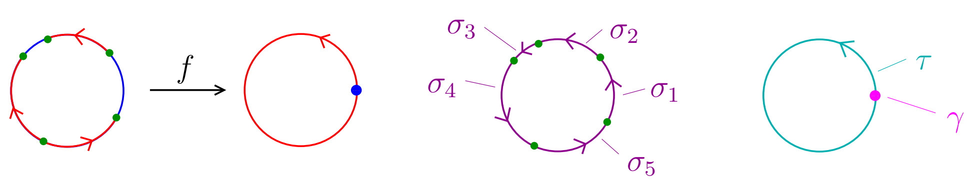
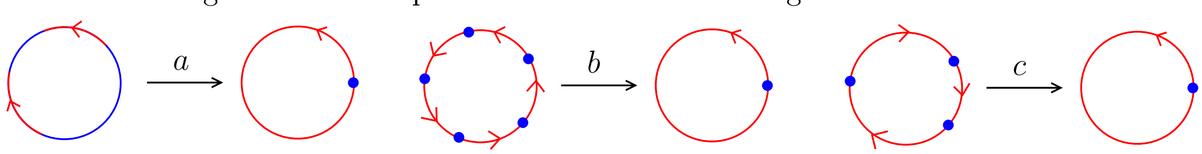


FIGURE 755

calculate the degrees of the maps that are illustrated in Figure 756.



$\deg(a) = 0$ $\deg(b) = +5$ $\deg(c) = -3$

FIGURE 756

[727] More precisely, given an "interval" on S^1 going from $e^{\mathrm{i}\alpha}$ to $e^{\mathrm{i}\beta}$ we mean the singular 1-simplex that is given by $(1-t,t) \mapsto e^{\mathrm{i}(\alpha\cdot(1-t)+\beta\cdot t)}$.

Almost the same arguments that we used in the discussion of the above example gives us the following lemma.

Lemma 45.10. *For any $k \in \mathbb{Z}$ the degree of the map*

$$\begin{array}{rcl} S^1 & \to & S^1 \\ z & \mapsto & z^k \end{array}$$

equals k.

Proof. We denote by $\tau\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $\tau(1-t,t) = e^{2\pi i t}$. By Lemma 45.4 we know that $[\tau] \in H_1(S^1)$ is a generator.

Now we first consider the case $k \geq 1$. For $j = 0, \ldots, k-1$ we denote by $\sigma_j\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $(1-t,t) \mapsto e^{2\pi i(\frac{t+j}{k})}$. Then

$$f_*([\tau]) \underset{\substack{\uparrow \\ \text{by Lemma 41.3 (3)}}}{=} f_*([\sigma_0 + \cdots + \sigma_{k-1}]) = \big[\underbrace{f \circ \sigma_0}_{=\tau} + \cdots + \underbrace{f \circ \sigma_{k-1}}_{=\tau}\big] = k \cdot [\tau].$$

The case $k < 0$ is proved is proved in a similar fashion using Lemma 41.3 (2). Finally the case $k = 0$ follows from the fact that the map $S^1 \to S^1$, $z \mapsto 1$ is constant, hence, as discussed on page 1089, the induced map on first homology is the zero map. ■

Before we move on to the next lemma it is convenient to introduce the following notation.

Notation. As on page 538, given a matrix $A \in O(n+1)$ we denote by $\rho(A)\colon S^n \to S^n$ the map given by $x \mapsto A \cdot x$.

The following lemma summarizes several important properties of the degree of the self-map of a sphere.

Lemma 45.11. *Let $f, g\colon S^n \to S^n$ be two maps. Then the following hold:*

1. $\deg(\mathrm{id}_{S^n}) = 1$.
2. *If f is not surjective, then* $\deg(f) = 0$.
3. *If f is homotopic to g, then* $\deg(f) = \deg(g)$. (*In Proposition 54.8 we will see that the converse to this statement holds.*)
4. $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
5. *If f is a self-homeomorphism of S^n, then* $\deg(f^{-1}) = \deg(f)$.
6. *Given any $A \in O(n)$ we have* $\deg(\rho(A)) = \det(A) \in \{-1, 1\}$.

Examples.

1. If $f\colon S^n \to S^n$ is the reflection in a hyperplane, then it follows from Lemma 45.11 (6) that $\deg(f) = -1$,
2. If $f\colon S^n \to S^n$ is the reflection in the origin i.e. if $f(x) = -x$ for all $x \in S^n$, then it follows from Lemma 45.11 (6) we have $\deg(f) = (-1)^{n+1}$.
3. Let $A \in O(n+1)$. Let $f\colon S^n \to S^n$ be the map that is given by $f(P) = A \cdot P$. If $\det(A) = 1$, then we saw on page 538 that f is actually homotopic to the identity. On the other hand, if $\det(A) = -1$, then it now follows from Lemma 45.11 (1), (3) together with Lemma 45.11 (6) that f is not homotopic to the identity.

Proof.

(1) Since $(\mathrm{id}_{S^n})_* = \mathrm{id}_{\mathrm{H}_n(S^n)}$ we know that $\deg(\mathrm{id}_{S^n}) = 1$.
(2) Suppose that the map $f\colon S^n \to S^n$ misses the point $P \in S^n$. We consider the commutative diagram

$$\begin{tikzcd} & S^n\setminus\{P\} \arrow[dr, hook, "i"] & \\ S^n \arrow[ur, "f"] \arrow[rr, "f"'] & & S^n \end{tikzcd}$$

which gives rise to the commutative diagram

$$\begin{tikzcd} & \mathrm{H}_n(S^n\setminus\{P\}) \arrow[dr, "i_*"] & \\ \mathrm{H}_n(S^n) \arrow[ur, "f_*"] \arrow[rr, "f_*"'] & & \mathrm{H}_n(S^n). \end{tikzcd}$$

By Lemma 3.32 (6) we know $S^n \setminus \{P\}$ is homeomorphic to $\mathbb{R}^n$. Thus we see that $S^n \setminus \{P\}$ is contractible which implies by Corollary 42.8 that $\mathrm{H}_n(S^n \setminus \{P\}) = 0$. But this means that $f_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ factors through the zero group, i.e. f_* is the zero map.
(3) If f is homotopic to g, then it follows from Proposition 42.5 that $\deg(f) = \deg(g)$.
(4) We consider the following diagram

$$\begin{tikzcd} \mathrm{H}_n(S^n) \arrow[r, "f_*", "\cdot\deg(f)"'] \arrow[rr, bend right, "(f\circ g)_*"'] & \mathrm{H}_n(S^n) \arrow[r, "g_*", "\cdot\deg(g)"'] & \mathrm{H}_n(S^n). \end{tikzcd}$$

This diagram commutes since $(f \circ g)_* = f_* \circ g_*$. It follows that the curved map is given by multiplication by $\deg(f) \cdot \deg(g)$, i.e. $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
(5) This statement follows immediately from (1) and (4).
(6) We start out with the following claim.

Claim. Let $\mu\colon S^n \to S^n$ be the reflection in the $(x_{n+1} = 0)$-hyperplane. Then we have $\deg(\mu) = -1$.

To prove the claim we consider again the singular chain

$$\begin{array}{rcccccl} \alpha\colon \Delta^n & \xrightarrow{\cong} & \left\{(x_1,\dots,x_n) \in [0,1]^n \,\middle|\, \sum\limits_{i=1}^n x_i \in [0,1]\right\} & \xrightarrow{\cong} & \overline{B}^n & \to & S^n. \\ (t_0,\dots,t_n) & \mapsto & (t_0,\dots,t_{n-1}) & & x & \mapsto & \big(x, \sqrt{1-\|x\|^2}\big). \end{array}$$

that we had already considered on page 1173. We write $\beta = \mu \circ \alpha$. In Lemma 45.3 we saw that the singular n-chain $\alpha - \beta$ is a generator of $\mathrm{H}_n(S^n)$. Next we calculate

$$\mu_*([\alpha-\beta]) \;=\; [\mu\circ\alpha] - [\mu\circ\beta] \;=\; [\beta] - [\underbrace{\mu\circ\mu}_{=\mathrm{id}}\circ\,\alpha] \;=\; [\beta]-[\alpha] \;=\; -[\alpha-\beta].$$

The map μ_* is therefore given by multiplication with -1. ⊞

Now let $A \in \mathrm{O}(n+1)$. First we consider the case $\det(A) = 1$, i.e. we assume that $A \in \mathrm{SO}(n+1)$. By Lemma 2.65 we know that $\mathrm{SO}(n+1)$ is path-connected, so there exists a map $h\colon [0,1] \to \mathrm{SO}(n+1)$ with $h(0) = \mathrm{id}$ and $h(1) = A$. But then $F(x,t) = h(t) \cdot x$ defines a homotopy from id to $\rho(A)$. It follows from Lemma 45.11 (1) and (3) that $\deg(\rho(A)) = \deg(\mathrm{id}) = 1$.

Finally we consider the case that $\det(A) \neq 1$. Let $R \in \mathrm{O}(n+1)$ be the matrix such that $\rho(R) = \mu$. Now we see that

$$\deg(\rho(A)) = \deg(\rho(A \cdot R \cdot R^{-1})) \underset{\substack{\uparrow \\ \text{by (4)}}}{=} \deg(\rho(A \cdot R)) \cdot \deg(\rho(R)) \underset{\substack{\uparrow \\ \text{since } \det(A \cdot R) = 1 \text{ and by the claim}}}{=} 1 \cdot (-1) = -1. \qquad \blacksquare$$

We have now seen that the circle admits self-maps of arbitrary degrees and that every sphere admits self-maps of degree $-1, 0, 1$. It is natural to ask whether there are also maps on higher-dimensional spheres that have other degrees. The combination of Lemma 45.10 and the following lemma shows that in fact every $k \in \mathbb{Z}$ is realized as the degree of every high-dimensional sphere.

Lemma 45.12. *Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a map. Let $\Sigma(f)\colon \Sigma(S^n) \to \Sigma(S^n)$ be the suspension of f that we defined on page 696. Furthermore let $\Theta\colon \Sigma(S^n) \to S^{n+1}$ be the homeomorphism from page 695. Then*

$$\deg(\Theta \circ \Sigma(f) \circ \Theta^{-1}\colon S^{n+1} \to S^{n+1}) = \deg(f\colon S^n \to S^n).$$

Examples.

(1) Let $k \in \mathbb{Z}$. We view S^2 as a subset of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. We consider the map

$$\begin{aligned} g\colon S^2 &\to S^2 \\ (re^{i\varphi}, w) &\mapsto (re^{i\varphi k}, w). \end{aligned}$$

One can easily verify that the map g equals $\Theta \circ \Sigma(z \mapsto z^k) \circ \Theta^{-1}$. Thus it follows from Lemmas 45.10 and 45.12 that $\deg(g) = \deg(\Sigma(z \mapsto z^k)) = k$.

(2) Let $n \in \mathbb{N}$ and let $k \in \mathbb{Z}$. It follows from Lemmas 45.10 and 45.12 that there exists a self-map $f\colon S^n \to S^n$ of degree k.

Proof. We consider the following diagram

$$\begin{tikzcd} \widetilde{\mathrm{H}}_n(S^n) \arrow[r, "\Sigma_{S^{n+1}}"] \arrow[d, "f_*", "\cdot\deg(f)"'] & \widetilde{\mathrm{H}}_n(\Sigma(S^n)) \arrow[r, "\Theta_*"] \arrow[d, "\Sigma(f)_*"] & \widetilde{\mathrm{H}}_{n+1}(S^{n+1}) \arrow[d, "(\Theta\circ\Sigma(f)\circ\Theta^{-1})_*"] \\ \widetilde{\mathrm{H}}_n(S^n) \arrow[r, "\Sigma_{S^{n+1}}"] & \widetilde{\mathrm{H}}_{n+1}(\Sigma(S^n)) \arrow[r, "\Theta_*"] & \widetilde{\mathrm{H}}_{n+1}(S^{n+1}). \end{tikzcd}$$

First note that it follows from Proposition 45.7 that the square to the left commutes and that the horizontal maps to the left are isomorphisms. It is clear that the square to the right commutes. Furthermore, since Θ is a homeomorphism we see that the horizontal maps to right are isomorphism. Note that it follows immediately from the above discussion together with Lemma 45.9 (2) that $\deg(\Theta \circ \Sigma(f) \circ \Theta^{-1}) = \deg(f)$. $\blacksquare$

The following theorem is an interesting application of our discussion of degrees of self-maps of spheres.

Theorem 45.13. *Let $f\colon S^n \to S^n$ be a map. If $\deg(f) \neq (-1)^{n+1}$, then f has a fixed point.*

Example. Theorem 45.13 says in particular that every map $f\colon S^2 \to S^2$ of degree one has a fixed point. Together with Lemma 45.11 this implies that every map $f\colon S^2 \to S^2$ that is homotopic to the identity has a fixed point. This means that we cannot "deform" the identity to a map that is fixed point free. This is in contrast to the case $n = 1$ where we can "deform" the identity into the map that is given by rotation by a small angle, which is of course a fixed point free map.

PROOF. Let $f\colon S^n \to S^n$ be a map which admits no fixed point. This means that for all $x \in S^n$ we have $f(x) \neq x$. We need to show that $\deg(f) = (-1)^{n+1}$. We will do this by showing that f is homotopic to $-\mathrm{id}$.

Our hypothesis implies that $(1-t)f(x) \neq tx$ for all $t \in [0,1]$ and all $x \in S^n$.[728] Therefore we can consider the map

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (x,t) & \mapsto & \dfrac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}. \end{array}$$

This map is a homotopy between the maps $x \mapsto f(x)$ and $x \mapsto -x$. Thus it follows that

$$\deg(f) \underset{\substack{\uparrow \\ \text{Lemma 45.11 (3)}}}{=} \deg(-\mathrm{id}) \underset{\substack{\uparrow \\ \text{Lemma 45.11 (6)}}}{=} (-1)^{n+1}. \qquad \blacksquare$$

In the following we recall the definition of the "naive" tangent space of a submanifold of $\mathbb{R}^n$.

Definition. Let $M \subset \mathbb{R}^n$ be a submanifold. Given $P \in M \setminus \partial M$ we denote by $\mathrm{V}_P\, M \subset \mathbb{R}^n$ the visual tangent vector space as defined on page 292. A *vector field* on M is a continuous map $v\colon M \to \mathbb{R}^n$ such that for any $P \in M$ we have $v(P) \in \mathrm{V}_P\, M$.

Examples.

(1) Let P be a point on S^n. As discussed on page 293 we have

$$\mathrm{V}_P\, S^n \;=\; \{v \in \mathbb{R}^{n+1} \,|\, \langle P, v\rangle = 0\}.$$

(2) For an odd-dimensional sphere S^{2k-1} a vector field is given by[729]

$$\begin{array}{rcl} v\colon S^{2k-1} & \to & \mathbb{R}^{2k} \\ (x_1, x_2, \dots, x_{2k-1}, x_{2k}) & \mapsto & (-x_2, x_1, \dots, -x_{2k}, x_{2k-1}). \end{array}$$

This vector field has the property that it is nowhere vanishing. In the case of the 1-sphere S^1 this is just the vector field of constant length 1 that points in the "counterclockwise direction".

(3) For an even-dimensional sphere S^{2k} a vector field is similarly given by

$$\begin{array}{rcl} v\colon S^{2k} & \to & \mathbb{R}^{2k+1} \\ (x_1, x_2, \dots, x_{2k-1}, x_{2k}, x_{2k+1}) & \mapsto & (-x_2, x_1, \dots, -x_{2k}, 0). \end{array}$$

[728] This case be seen as follows: if $t \neq \frac{1}{2}$ we have $\|(1-t)f(x)\| = 1 - t \neq t = \|tx\|$. On the other hand, for $t = \frac{1}{2}$ we have $(1-t)f(x) = \frac{1}{2}f(x) \neq \frac{1}{2}x = tx$.

[729] Indeed, for any $P \in S^{2k-1}$ we have $\langle P, v(P)\rangle = 0$, i.e. $v(P)$ does indeed lie in the tangent space $\mathrm{T}_P\, S^{2k-1}$.

This vector field vanishes at the "North Pole" $(0,\dots,0,1)$ and it also vanishes at the "South Pole" $(0,\dots,0,-1)$. Both types of vector fields are sketched in Figure 757.

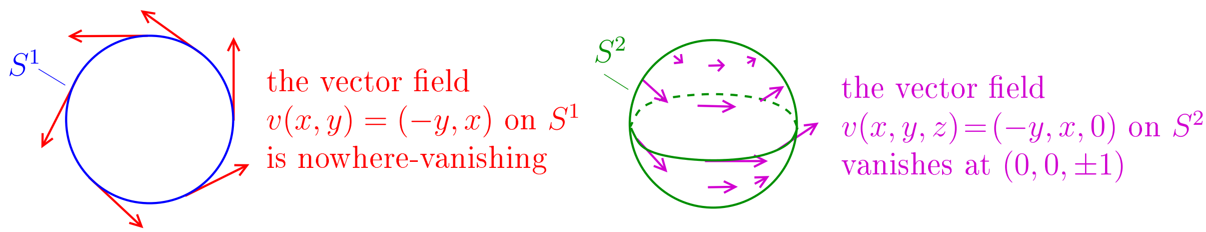


FIGURE 757

One can now ask whether it is possible to find a nowhere vanishing vector field on even-dimensional spheres. The following proposition says that this is not possible.

Theorem 45.14. (Hairy Ball Theorem) *Every vector field on an even-dimensional sphere S^{2k} vanishes on at least one point.*[730]

Remark.

(1) For $n=2$ we get the following "real life application": on the surface of earth there always exists at least one point where the wind does not blow at all. This fact is illustrated at the following website

`https://earth.nullschool.net/`

(2) The "Hairy Ball Theorem" gets its observation that for $n=2$ the theorem says that "you can't comb a hairy ball flat without creating a cowlick".

(3) Contrary to most other languages in German the Hairy Ball Theorem 45.14 is called the "Satz vom (gekämmten) Igel", i.e. it is referred to as the "Combed Hedgehog Theorem".

(4) Evidently the fun never stops. Now that we know that odd-dimensional spheres admit one nowhere vanishing vector field the question arises, how many linearly independent vector fields can we find on a given odd-dimensional sphere. It turns out that there is complete and highly non-trivial answer, see [**Adam62**], [**MillH89**] and [**VogI**] for details.

PROOF. Let v be a vector field on the sphere S^n which is nowhere-vanishing. We have to show that n is odd. We consider the vector field w on S^n that is given by rescaling v to length one, i.e. that is given by

$$\begin{array}{rcl} S^n & \to & \mathbb{R}^{n+1} \\ x & \mapsto & w(x) := \frac{1}{\|v(x)\|}v(x). \end{array}$$

Furthermore we consider the map

$$\begin{array}{rcl} F\colon S^n\times[0,1] & \mapsto & S^n \\ (x,t) & \mapsto & \cos(\pi t)\cdot x+\sin(\pi t)\cdot w(x). \end{array}$$

[730] On S^{2k} we had so far found vector fields which vanish at *two* points. Can we find a vector field on S^{2k} which vanishes on only *one* point? We will discuss this question in Exercise 45.6.

We refer to Figure 758 for an illustration. The map does indeed take values in S^n, since

$$\langle \cos(\pi t)x + \sin(\pi t)w(x), \cos(\pi t)x + \sin(\pi t)w(x)\rangle \underset{\substack{\uparrow \\ \text{since } \langle x, w(x)\rangle = 0}}{=} \cos(\pi t)^2\|x\|^2 + \sin(\pi t)^2\|w(x)\|^2 \underset{\substack{\uparrow \\ \text{since } \|x\| = \|w(x)\| = 1}}{=} 1.$$

The map $F_0 = F(-,0)$ is the identity and the map $F_1 = F(-,1)$ equals F is $-\operatorname{id}$. Put differently, the map F is a homotopy between the two maps id and $-\operatorname{id}$. It follows that

$$1 \underset{\substack{\uparrow \\ \text{Lemma 45.11 (1)}}}{=} \deg(\operatorname{id}) \underset{\substack{\uparrow \\ \text{Lemma 45.11 (3)}}}{=} \deg(-\operatorname{id}) \underset{\substack{\uparrow \\ \text{Lemma 45.11 (6)}}}{=} (-1)^{n+1}.$$

But this means that n is odd. ■

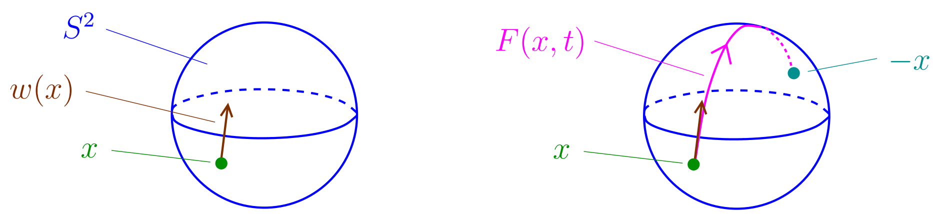


FIGURE 758

45.5. **The local degree I.** As the reader will have noticed, it is not entirely trivial to compute the isomorphism types of the homology groups of a given topological space. In the following two sections we will get to know a few tricks to address this problem.

We start out with the following harmless lemma which is a variation on Lemma 44.1.

Lemma 45.15. *Let U be a subset of $\mathbb{R}^n$. For every point P in the interior of U we have*

$$\mathrm{H}_k(U, U \setminus \{P\}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

PROOF (∗). Since P lies in the interior of U there exists an $r > 0$ such that $B^n_r(P) \subset U$. Now we see that

$$\mathrm{H}_k(U, U\setminus\{P\}) \underset{\substack{\uparrow \\ \text{Excision Theorem 43.20}}}{\leftarrow} \mathrm{H}_l(B_r(P), B_r(P)\setminus\{P\}) \underset{\substack{\uparrow \\ \text{isomorphism by the long exact} \\ \text{sequence in reduced homology} \\ \text{of the pair } (B^n_r(P), B^n_r(P)\setminus\{P\})}}{\xrightarrow{\partial}} \widetilde{\mathrm{H}}_{k-1}(B^n_r(P)\setminus\{P\}) \underset{\substack{\uparrow \\ \text{isomorphism since} \\ S^{n-1} \text{ is homotopy} \\ \text{equivalent to } B^n_r(P)\setminus\{P\}}}{\leftarrow} \widetilde{\mathrm{H}}_{k-1}(S^{n-1}).$$

The lemma now follows from the calculation of $\widetilde{\mathrm{H}}_*(S^{n-1})$ in Proposition 43.4. ■

We move on to the following much more interesting proposition.

Proposition 45.16. *Let W be a subset of $\mathbb{R}^n$ and let A be a subset of W such the origin 0 is contained in the interior of U. Let $f\colon A \to W$ be an injective map with $f(0) = 0$. We suppose that f is a local diffeomorphism at 0. Then the degree of the map*[731]

$$\mathrm{H}_n(W, W\setminus\{0\}) \xleftarrow[\cong]{\iota_*} \mathrm{H}_n(A, A\setminus\{0\}) \xrightarrow{f_*} \mathrm{H}_n(W, W\setminus\{0\})$$

equals $+1$ if f is orientation-preserving at 0 and -1 if f is orientation-reversing at 0.

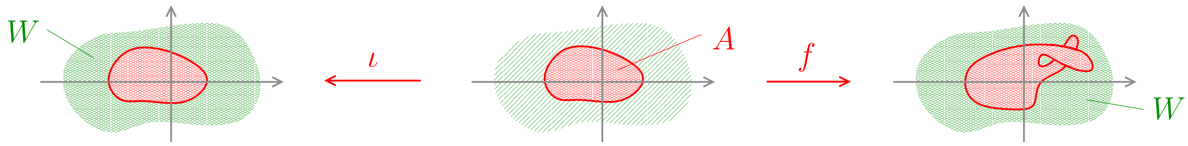


FIGURE 759. Illustration of Proposition 45.16.

In the proof of Proposition 45.16 we will make use of the following two lemmas which we will also use on several other occasions later on.

Lemma 45.17. *Let $f\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. Let $P \in U$. We denote by*

$$\begin{array}{rcl} \lambda\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ x & \mapsto & (\mathrm{D}\, f_P)(x) + f(P) \end{array}$$

the linearization of f at P. There exists an open neighborhood $U' = B^n_r(P)$ of P such that the restrictions of f and λ to maps $(U', U' \setminus \{P\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{f(P)\})$ are homotopic.

Remark. Lemma 45.17 is the key to translating information from smooth methods into topology and later on homology groups.

PROOF. Let $f\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. To simplify the notation we assume that $P = 0$ and that $f(P) = 0$. Furthermore, by restriction to an open ball we might as well assume that U is convex. We write $D := \mathrm{D}\, f_0$. Thus the linearization of f at $P = 0$ is given by the map $\lambda(x) = D \cdot x$. Since f is in particular a local diffeomorphism a $P = 0$ we know by the chain rule, see Proposition 6.15 and 6.17, that D is an invertible matrix.

Since $U \subset \mathbb{R}^n$ is convex we obtain from Proposition 6.16 that there exists a smooth map $A\colon U \to \mathrm{M}(n \times n, \mathbb{R})$ such that $A(0) = \mathrm{D}\, f_0$ and such that $f(x) = A(x) \cdot x$ for all $x \in U$. Since $\det(A(0)) \neq 0$ we see that there exists an $r > 0$ such that $\det(A(x)) \neq 0$ for all $x \in B^n_r(0)$. This implies that the map

$$\begin{array}{rcl} B^n_r(0) \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & A(x \cdot (1-t)) \cdot x \end{array}$$

is a homotopy of the maps f and λ as maps $(B^n_r(0), B^n_r(0) \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ of pairs of topological spaces. ∎

We adopt the following notation:

Notation.

(1) Given a matrix $P \in \mathrm{GL}(n, \mathbb{R})$ we denote by $\rho(P)\colon \mathbb{R}^n \to \mathbb{R}^n$ the map given by $x \mapsto P \cdot x$.

(2) Given a non-zero real number r we denote its sign by $\mathrm{sign}(r) \in \{-1, +1\}$.

[731] By Lemma 45.15 we know that $\mathrm{H}_n(W, W \setminus \{0\}) \cong \mathbb{Z}$, thus it makes sense to talk of the degree of an automorphism of $\mathrm{H}_n(W, W \setminus \{0\})$.

Lemma 45.18. *Let $n \in \mathbb{N}$. Given $A \in \mathrm{GL}(n,\mathbb{R})$ we have*

$$\deg\big(\rho(A)_*\colon \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \to \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\})\big) \;=\; \mathrm{sign}(\det(A))$$

and given $A \in \mathrm{O}(n)$ we have

$$\deg\big(\rho(A)_*\colon \mathrm{H}_n(\overline{B}^n,\overline{B}^n\setminus\{0\}) \to \mathrm{H}_n(\overline{B}^n,\overline{B}^n\setminus\{0\})\big) \;=\; \mathrm{sign}(\det(A)).$$

Proof of Lemma 45.18. For simplicity we only prove the first statement of the lemma. Thus let $A \in \mathrm{GL}(n,\mathbb{R})$. We start out by picking an orthogonal matrix $B \in \mathrm{O}(n)$ such that $\mathrm{sign}(\det(B)) = \mathrm{sign}(\det(A))$. Next we consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \arrow[r, "\mathrm{id}"] \arrow[d, "\rho(A)_*"] & \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \arrow[r, "\partial"] \arrow[d, "\rho(B)_*"] & \mathrm{H}_{n-1}(\mathbb{R}^n\setminus\{0\}) \arrow[d, "\rho(B)_*"] & \mathrm{H}_{n-1}(S^{n-1}) \arrow[l] \arrow[d, "\rho(B)_*"] \\
\mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \arrow[r, "\mathrm{id}"] & \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \arrow[r, "\partial"] & \mathrm{H}_{n-1}(\mathbb{R}^n\setminus\{0\}) & \mathrm{H}_{n-1}(S^{n-1}). \arrow[l]
\end{tikzcd}$$

We make the following observations:

(1) Since $\mathrm{sign}(\det(B)) = \mathrm{sign}(\det(A))$ we obtain from Lemma 2.65 (1) that there exists a path $[0,1] \to \mathrm{GL}(n,\mathbb{R})$ from A to B. Similar to the discussion on page 538 this path gives rise to a homotopy between $\rho(A)$ and $\rho(B)$. We obtain from Proposition 43.17 that $\rho(A)_* = \rho(B)_*$, in other words, the square to the left commutes.
(2) The square in the middle commutes by the naturality of the connecting homomorphism, see Corollary 43.16.
(3) The horizontal maps to the right are induced by the inclusion $S^{n-1} \to \mathbb{R}^n\setminus\{0\}$. It is clear that the square to the right commutes.

Now we see that

$$\deg\big(\mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \xrightarrow{\rho(A)_*} \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\})\big) \overset{\substack{\text{by the above commutative diagram together with Lemma 45.9 (2)}\\ \downarrow}}{=} \deg(\rho(B)\colon S^{n-1} \to S^{n-1})$$
$$\underset{\substack{\uparrow\\ \text{by Lemma 45.11 (6)}}}{=} \det(B) \underset{\substack{\uparrow\\ \text{since } B\in \mathrm{O}(n) \text{ we}\\ \text{have } \det(B)\in\{\pm 1\}}}{=} \mathrm{sign}(\det(B)) \underset{\substack{\uparrow\\ \text{choice of } B}}{=} \mathrm{sign}(\det(A)).$$

■

Proof of Proposition 45.16 (∗). Let W be a subset of $\mathbb{R}^n$ and let A be a subset of W such that the origin 0 is contained in the interior. Let $f\colon A \to W$ be an injective map with $f(0) = 0$. We suppose that f is a local diffeomorphism at 0. Recall that this means that there exists an open neighborhood U of 0 that is contained in A and an open neighborhood V of 0 that is contained in B such that the map $f\colon U \to V$ is a diffeomorphism.

We start out with the following preparations.

(1) We set

$$\epsilon \;:=\; \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } 0,\\ -1, & \text{if } f \text{ is orientation-reversing at } 0.\end{cases}$$

Note that by Lemma 6.44 we have $\epsilon = \mathrm{sign}(\det(\mathrm{D} f_0))$.

(2) By Lemma 45.17 there exists an open neighborhood U' of 0 that is contained in U such that the restrictions of f and $\rho(\mathrm{D}\, f_0)$ to maps $(U', U' \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ are homotopic.

Now we consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_n(W, W\setminus\{0\}) \arrow[ddd, "\cong"'] & \mathrm{H}_n(A, A\setminus\{0\}) \arrow[l, "\cong"'] \arrow[rr, "f_*"] & & \mathrm{H}_n(W, W\setminus\{0\}) \arrow[dl, "\cong"] \arrow[ddd, "\cong"] \\
& \mathrm{H}_n(U', U'\setminus\{0\}) \arrow[u, "\cong"] \arrow[r, "f_*"] \arrow[d, "\mathrm{id}"'] & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \arrow[d, "\mathrm{id}"] & \\
& \mathrm{H}_n(U', U'\setminus\{0\}) \arrow[r, "\rho(\mathrm{D}\, f_0)_*"] \arrow[dl, "\cong"'] & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \arrow[dr, "\cong"] & \\
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \arrow[rrr, "\rho(\mathrm{D}\, f_0)_*"] & & & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}).
\end{tikzcd}$$

We continue with the following clarifications and observations:

(3) The undecorated maps are induced by inclusions. All of them are isomorphisms by the Excision Theorem 43.20.
(4) It follows from (2) and Proposition 43.17 that the central square commutes. All other regions of the diagram commute by the functoriality of homology groups.
(5) By Lemma 45.18 the bottom horizontal map equals multiplication by $\mathrm{sign}(\det(\mathrm{D}\, f_0))$.
(6) It follows from all of the above that the top map is given by multiplication by ϵ. But that is precisely what we wanted to show. ■

45.6. **The local degree II.** As mentioned in the previous section, we want to develop some tools for addressing the following problem: given a map $f\colon X \to Y$ between topological spaces, how can we determine the induced map on homology? For example, let us consider the following question.

Question 45.19. *Let f be a complex polynomial of degree ≥ 1. By Exercise 2.33 we know that the map*

$$\begin{array}{rcl} \Theta(f)\colon S^2 = \mathbb{C} \cup \{\infty\} & \to & \mathbb{C} \cup \{\infty\} = S^2 \\ z & \mapsto & \begin{cases} f(z), & \text{if } z \in \mathbb{C}, \\ \infty, & \text{if } z = \infty \end{cases} \end{array}$$

is continuous. What is the induced map on homology, i.e. what is the degree of $\Theta(f)$?

In this section we will develop a method for determining the degree of a self-map of a sphere which can be applied in many "reasonable" cases. In particular we will be able to answer Question 45.19. The answer will give us an alternative proof of the Fundamental Theorem of Algebra.

We start out with the following observation.

Observation. Let $n \geq 1$. Now let $x \in S^n$ be a point and let U be a neighborhood of x in S^n. We have isomorphisms

$$\begin{array}{ccccc} \mathrm{H}_n(S^n) & \xrightarrow{\cong} & \mathrm{H}_n(S^n, S^n \setminus \{x\}) & \xleftarrow{\cong} & \mathrm{H}_n(U, U \setminus \{x\}). \\ & \uparrow & & \uparrow & \\ & \text{isomorphism by the long exact} & & \text{isomorphism by the} & \\ & \text{of the pair } (S^n, S^n \setminus \{x\}) & & \text{Excision Theorem 43.20} & \end{array}$$

Throughout this section will use frequently that these maps are isomorphisms.

Definition. We say a map $f\colon X \to Y$ between two topological spaces is *nice at a point* $x \in X$ if there exists a neighborhood U of x such that for all $x' \neq x$ in U we have $f(x') \neq f(x)$. We refer to such a neighborhood as a *nice neighborhood of* x.

Examples.

(1) Let $f\colon X \to Y$ be a map between two topological spaces and let $x \in X$. If U is a neighborhood such that $f|_U$ is injective, then U is evidently a nice neighborhood of x.

(2) We consider the functions

$$\begin{array}{rcl} f\colon \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & t^2 \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & \begin{cases} 0, & \text{if } t = 0, \\ t\sin(\frac{1}{t}), & \text{if } t \neq 0. \end{cases} \end{array}$$

For the function f every point on $\mathbb{R}$ is nice, in fact for $x < 0$ we can take $U = (-\infty, 0)$, for $x > 0$ we can take $U = (0, \infty)$ and for $x = 0$ we can take $U = \mathbb{R}$. On the other hand it is straightforward to see that for the function g the point $x = 0$ is not nice.

The following is the key definition of this section.

Definition. Let $n \geq 1$, let $f\colon S^n \to S^n$ be a map and let $x \in S^n$ be a nice point. Let U be a nice neighborhood of x. The *local degree of f at the point x* is defined as the degree of the homomorphism[732]

$$\mathrm{H}_n(S^n) \xrightarrow{\cong} \mathrm{H}_n(S^n, S^n \setminus \{x\}) \xleftarrow{\cong} \mathrm{H}_n(U, U \setminus \{x\}) \xrightarrow{f_*} \mathrm{H}_n(S^n, S^n \setminus \{f(x)\}) \underset{\cong}{\xleftarrow{\cong}} \mathrm{H}_n(S^n).$$

We denote this local degree by $\deg(f, x)$.

Lemma 45.20. *The local degree* $\deg(f, x)$ *of a map* $f\colon S^n \to S^n$ *at a nice point* $x \in S^n$ *is well-defined.*

Proof. Let $n \geq 1$, let $f\colon S^n \to S^n$ be a map and let $x \in S^n$ be a nice point. Let U, U' be nice neighborhoods of x. After replacing U' by $U \cap U'$ we can without loss of generality

[732]Note that f induces a map of pairs $(U, U \setminus \{x\}) \to (S^n, S^n \setminus \{f(x)\})$ since U is nice.

assume that $U' \subset U$. We consider the following commutative diagram

$$\begin{tikzcd}
 & & \mathrm{H}_n(U', U' \backslash \{x\}) \arrow[dl, "\cong"'] \arrow[r, "f_*"] \arrow[dd, "\cong"] & \mathrm{H}_n(S^n, S^n \backslash \{f(x)\}) \arrow[dd, "\mathrm{id}"] & \\
\mathrm{H}_n(S^n) \arrow[r] & \mathrm{H}_n(S^n, S^n \backslash \{x\}) & & & \mathrm{H}_n(S^n). \arrow[ul, "\cong"'] \arrow[dl, "\cong"] \\
 & & \mathrm{H}_n(U, U \backslash \{x\}) \arrow[ul, "\cong"] \arrow[r, "f_*"] & \mathrm{H}_n(S^n, S^n \backslash \{f(x)\}). &
\end{tikzcd}$$

All the non-decorated maps are induced by inclusions. It is clear that the diagram commutes. Also note that the left vertical map is an isomorphism by the Excision Theorem 43.20. It follows immediately from these observations that the local degree defined using U' agrees with the local degree defined using U. ■

Lemma 45.21. *Let $n \geq 1$, let $f \colon S^n \to S^n$ be a map and let $x \in S^n$. If f is a homeomorphism, then for every nice point $x \in S^n$ we have*

$$\deg(f, x) \;=\; \deg(f).$$

Proof. From the hypothesis that f is a homeomorphism we obtain that $U = S^n$ is a nice neighborhood for x. We take $V = S^n$ and we consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_n(S^n) \arrow[r, "f_*"] \arrow[d, "\cong"', "\text{я}"] & \mathrm{H}_n(S^n) \arrow[d, "\text{я}"', "\cong"] \\
\mathrm{H}_n(S^n, S^n \setminus \{x\}) \arrow[r, "f_*"] & \mathrm{H}_n(S^n, S^n \setminus \{f(x)\})
\end{tikzcd}$$

The statement now follows immediately from the definitions. ■

The following proposition can often be used to determine the local degree of a map.

Proposition 45.22. *Let $f \colon S^n \to S^n$ be a map with $n \geq 1$ and let $x \in S^n$.*

(1) *If f is a local homeomorphism around x, then x is nice and*

$$\deg(f, x) \;=\; \pm 1.$$

(2) *If f is a local diffeomorphism around x, then x is nice and*

$$\deg(f, x) \;=\; \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x, \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

Example. We consider the map $f \colon S^1 \to S^1$ given by $(x, y) \mapsto (x, |y|)$. It follows from Proposition 45.22 (2) that for every point on the open upper semicircle the local degree is $+1$ and that for every point on the open lower semicircle the local degree is -1. In Exercise 45.10 we will prove that the local degree at the points ± 1 is in both cases zero.

Now we can provide the proof of Proposition 45.22.

Proof of Proposition 45.22. Let $f \colon S^n \to S^n$ be a map with $n \geq 1$ and let $x \in S^n$.

(1) Suppose that f is a local homeomorphism around the point x. Recall that this means that there exists an open neighborhood U of x and an open neighborhood V of $f(x)$

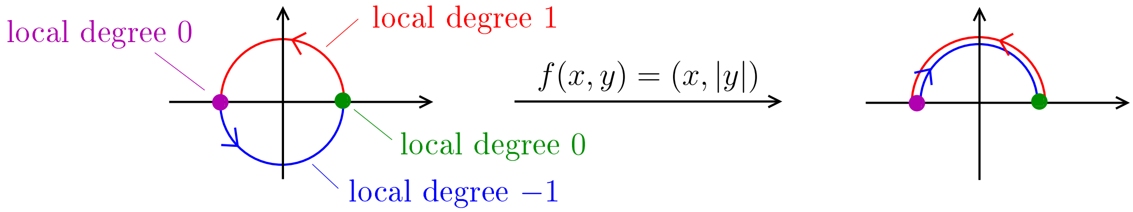


FIGURE 760

such that $f\colon U \to V$ is a homeomorphism. This implies in particular that U is nice. Now we consider the following rather simple diagram:

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "\cong"] & H_n(S^n, S^n\setminus\{x\}) & H_n(U, U\setminus\{x\}) \arrow[l, "\cong"'] \arrow[r] \arrow[d, "f_*"] & H_n(S^n, S^n\setminus\{f(x)\}) & H_n(S^n). \arrow[l, "\cong"'] \\
& & H_n(V, V\setminus\{f(x)\}) \arrow[ur, "\cong"'] & &
\end{tikzcd}$$

It is clear that the diagram commutes. Since the map f restricts to a homeomorphism $(U, U\setminus\{x\}) \to (V, V\setminus\{f(x)\})$ of pairs we see that the vertical map is an isomorphism. Furthermore the diagonal map is an isomorphism by the Excision Theorem 43.20. Thus we see that all horizontal maps are isomorphism. This implies immediately that $\deg(f,x) = \pm 1$.

(2) Let $f\colon S^n \to S^n$ be a map with $n \geq 1$ and let $x \in S^n$. We assume that f is a local diffeomorphism around x. By Lemma 3.32 we know that there exists an $A \in SO(n-1)$ with $A \cdot f(x) = x$. By Exercise 6.28 we know that the map $z \mapsto A \cdot z$ is orientation-preserving and by Lemma 45.11 (6) we know that $\deg(z \mapsto A \cdot z) = 1$. Thus, after replacing f by the $z \mapsto A \cdot f(z)$, we might as well assume that $f(x) = x$.

We make a few further preparations.

(a) We pick a chart $\Phi\colon V \to W$ around x.
(b) Since f is a local diffeomorphism around x there exists an open neighborhood U of x such that $f\colon U \to f(U)$ is a diffeomorphism. Note that after possibly replacing U by $U \cap V \cap f^{-1}(V)$ we can assume that $U \subset V$ and $f(U) \subset V$.
(c) We set $y := \Phi(x)$.
(d) We set

$$\epsilon \ := \ \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x, \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

Next we consider the following diagram

$$\begin{tikzcd}
H_n(S^n) \arrow[rrr, "\cdot \deg(f{,}x)"] \arrow[d, "\text{я}"', "\cong"] & & & H_n(S^n) \arrow[d, "\cong"', "\text{я}"] \\
H_n(S^n, S^n\setminus\{x\}) & H_n(U, U\setminus\{x\}) \arrow[l, "\cong"'] \arrow[rr, "f_*"] \arrow[dr, "f_*"] \arrow[dd, "\Phi_*"] & & H_n(S^n, S^n\setminus\{x\}) \\
& & H_n(f(U), f(U)\setminus\{x\}) \arrow[dr, "\Phi_*"] & \\
H_n(W, W\setminus\{y\}) \arrow[uu, "\Phi_*^{-1}"] \arrow[rrr, bend right, "\cdot\epsilon"'] & H_n(\Phi(U), \Phi(U)\setminus\{y\}) \arrow[l, "\cong"'] \arrow[rr, "(\Phi\circ f\circ\Phi^{-1})_*"] & & H_n(W, W\setminus\{y\}). \arrow[uu, "\Phi_*^{-1}"']
\end{tikzcd}$$

We make the following observations and clarifications.

(a) As usual the undecorated maps are induced by inclusions.

(b) The top rectangle commutes by the definition of $\deg(f, x)$. The middle regions of the diagram commute by the functoriality of relative homology. Finally, note that $\Phi \circ f \circ \Phi^{-1}$ is orientation-preserving at y if and only if f is orientation-preserving at x. Thus we see that the bottom part of the diagram commutes by Proposition 45.16.

(c) It follows from the above together with Lemma 45.9 (2) that $\deg(f, x) = \epsilon$. ■

Using the following proposition we can now determine the degree of a map from local degrees.

Proposition 45.23. *Let $f\colon S^n \to S^n$ be a map where $n > 0$ and let $y \in S^n$ such that $f^{-1}(\{y\})$ consists of finitely many points $x_1, \dots, x_m$.*[733] *Then*[734]

$$\deg(f) \;=\; \sum_{i=1}^{m} \deg(f, x_i).$$

Remark. In Proposition 87.32 we will generalize Proposition 45.23 to a more general setting.

Proof (∗). Since S^n is Hausdorff we can choose disjoint open neighborhoods $U_1, \dots, U_m$ for $x_1, \dots, x_m$ and we choose an open neighborhood V of y such that $f(U_i) \subset V$ for $i = 1, \dots, m$. We consider the following diagram:

[733] Does there always exist such a y?

[734] The hypothesis implies immediately that all the points $x_1, \dots, x_m$ are nice, hence the local degrees are defined.

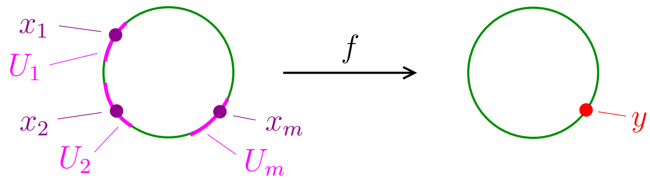


FIGURE 761. Illustration for the proof of Proposition 45.23.

$$
\begin{tikzcd}
\mathrm{H}_n(S^n) \arrow[rrr, "f_*"] \arrow[d, "\oplus\text{я}_i"'] \arrow[drr, "\text{я}"] & & & \mathrm{H}_n(S^n) \arrow[ddd, "\text{я}"', "\cong"] \\
\bigoplus_{i=1}^m \mathrm{H}_n(S^n, S^n\setminus\{x_i\}) & & \mathrm{H}_n(S^n, S^n\setminus f^{-1}(\{y\})) \arrow[ll, "\oplus p_i"', "\cong"] \arrow[ddr, "f_*"] & \\
& & \mathrm{H}_n\left(\bigcup_{i=1}^m U_i, \bigcup_{i=1}^m U_i\setminus\{x_i\}\right) \arrow[u, "\iota"', "\cong"] & \\
\bigoplus_{i=1}^m \mathrm{H}_n(U_i, U_i\setminus\{x_i\}) \arrow[uu, "\oplus r_i", "\cong"'] \arrow[urr, "\cong", "\oplus s_i"'] \arrow[rrr, "\oplus f_{i*}"] & & & \mathrm{H}_n(S^n, S^n\setminus\{y\}).
\end{tikzcd}
$$

We make the following clarifications and observations:

(1) The maps r_i, s_i, p_i and ι are the maps induced by the obvious inclusions of pairs of topological spaces. Furthermore the maps $f_i\colon (U_i, U_i\setminus\{x_i\}) \to (S^n, S^n\setminus\{y\})$ are given by the restriction of f to U_i.
(2) It follows easily from the functoriality of relative homology groups that the diagram commutes.
(3) The argument on page 1092 shows that the diagonal map $\bigoplus_{i=1}^m s_i$ to the bottom left is an isomorphism.
(4) It follows from the Excision Theorem 43.19 that the vertical map ι and that the maps r_i are isomorphisms.
(5) It follows from the fact that the diagram commutes together with (3) and (4) that the horizontal map $\bigoplus_{i=1}^m p_i$ is also an isomorphism.

Since $\text{я}\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n, S^n\setminus\{y\})$ is an isomorphism it suffices to prove the following claim.

Claim. We have

$$\text{я}(f_*([S^n])) \;=\; \sum_{i=1}^m \deg(f, x_i)\cdot \text{я}([S^n]) \;\in\; \mathrm{H}_n(S^n, S^n\setminus\{y\}).$$

We verify the claim by performing the following calculation:

since the diagram commutes and since the maps $\oplus p_i$ and $\oplus r_i$ are isomorphisms

by Lemma 41.12 (1) and (2) and since each r_i is an isomorphism

$$\begin{aligned}
я(f_*([S^n])) &= \Big(\bigoplus_{i=1}^m f_{i*}\Big)\circ\Big(\bigoplus_{i=1}^m r_{i*}\Big)^{-1}\circ\Big(\bigoplus_{i=1}^m я_{i*}\Big)([S^n]) = \Big(\bigoplus_{i=1}^m f_{i*}\Big)\circ\Big(\bigoplus_{i=1}^m r_{i*}^{-1}\circ я_i\Big)([S^n])\\
&= \Big(\bigoplus_{i=1}^m f_{i*}\Big)\Big(\sum_{i=1}^m \underbrace{(r_i^{-1}\circ я_i)([S^n])}_{\in \mathrm{H}_n(U_i,U_i\setminus\{x_i\})}\Big) = \sum_{i=1}^m (f_i\circ r_i^{-1}\circ я_i)([S^n]) = \sum_{i=1}^m \deg(f,x_i)\cdot я([S^n]).
\end{aligned}$$

definition of direct sum of type (II) of homomorphisms, see page 1090

definition of direct sum of type (I) of homomorphisms, see page 1090

definition of local degree, see page 1191 ■

Examples.

(1) The map $f\colon S^1 \to S^1$ given by $f(x,y) = (x,|y|)$ that is illustrated in Figure 760 has degree zero by Lemma 45.11. Taking preimages of points in S^1 one sees that in each case the sum of the local degrees of the preimages is zero.

(2) In Figure 762 we sketch a map $f\colon S^1 \to S^1$ and we choose several points $y \in S^1$. We

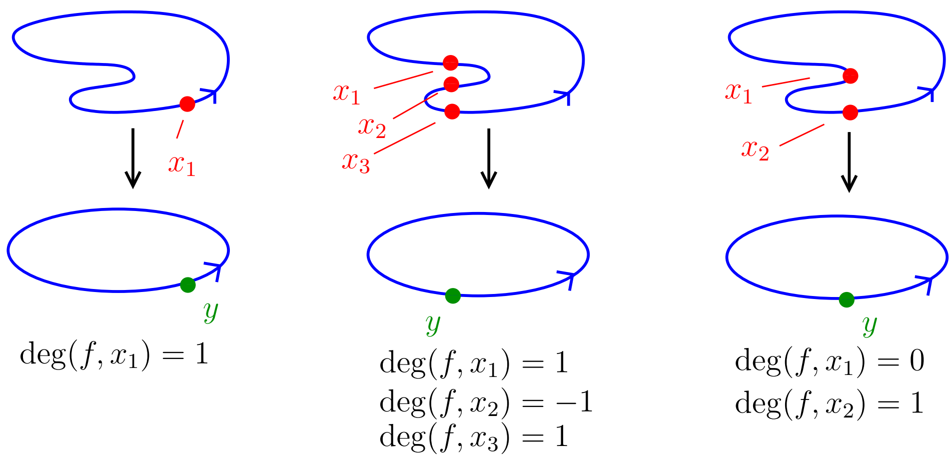


FIGURE 762

use Proposition 45.22 to determine the various degrees at the given points.[735] We see that the number of preimages and the degrees at the points in the preimages vary, but in all three cases the sum of the degrees at the points in the preimage is the same.

(3) We consider the map

$$\begin{aligned} f\colon S^1 &\to S^1\\ z &\mapsto z^m\end{aligned}$$

We had already determined in Lemma 45.10 that $\deg(f) = m$. Now we will show, just for fun, that the degree of f can also easily be determined using Proposition 45.23.

First let $m \neq 0$. Then we choose $y = 1$. The preimages of y are given by $x_j = e^{2\pi \mathrm{i} j/m}$ where $j = 1,\ldots,m$. It follows from Proposition 45.22 (2) that for each

[735] Why is in the third example the degree $\deg(f,x_1)$ equal to zero?

$j \in \{1, \dots, m\}$ we have

$$\deg(f, x_j) = \begin{cases} 1, & \text{if } m > 0, \\ -1, & \text{if } m < 0. \end{cases}$$

Thus it follows from Proposition 45.23 that $\deg(f) = |m| \cdot \operatorname{sign}(m) = m$. If $m = 0$, then $\deg(f) = 0$. The easiest way to see this is to take $y = -1$, since in this case the preimage is the empty set.

As promised, we can now provide an answer to Question 45.19:

Proposition 45.24. *Let f be a complex polynomial of degree ≥ 1. The degree of the map*

$$\begin{aligned} \Theta(f)\colon S^2 = \mathbb{C} \cup \{\infty\} &\to \mathbb{C} \cup \{\infty\} = S^2 \\ z &\mapsto \begin{cases} f(z), & \text{if } z \in \mathbb{C}, \\ \infty, & \text{if } z = \infty \end{cases} \end{aligned}$$

equals the degree of the polynomial f.

SKETCH OF PROOF. Let f be a complex polynomial of degree $n \geq 1$. We also consider the polynomial $g(z) = z^n$. We calculate that

$$\deg(\Theta(f)) \overset{\substack{\text{by Lemma 45.11 (3), since by Exercise 18.5 we know that } \Theta(f) \text{ and } \Theta(g) \text{ are homotopic} \\ \downarrow}}{=} \deg(\Theta(g(z) = z^n)) \underset{\substack{\uparrow \\ \text{follows from Proposition 45.23 and Proposition 45.22, see Exercise 45.7}}}{=} n. \qquad \blacksquare$$

More excitingly, we can provide a new proof for the Fundamental Theorem of Algebra.

Theorem 16.21. (Fundamental Theorem of Algebra) ❀ *Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$.*

PROOF. Let f be a nonconstant polynomial with complex coefficients. In other words, f is a complex polynomial of degree $n \geq 1$. By Proposition 45.24 we know that the degree of the self-map $\Theta(f)$ of $S^2 = \mathbb{C} \cup \{\infty\}$ equals $n \geq 1$. By Lemma 45.11 (2) this implies that $\Theta(f)\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is surjective, in particular 0 lies in the image of $\Theta(f)$. Since $\Theta(f)(\infty) = \infty$ we see that 0 lies in the image of $f\colon \mathbb{C} \to \mathbb{C} \cup \{\infty\}$. In other words, there exists a $z \in \mathbb{C}$ with $\underbrace{\Theta(f)(z)}_{=f(z)} = 0$. $\blacksquare$

45.7. **Proof of Proposition 45.5 (*).** For the tenacious reader we now provide the long-delayed proof of Statements (2) and (3) of Proposition 45.5. For the convenience of the reader we recall the relevant two statements.

Proposition 45.5. *Let $n \in \mathbb{N}$. We denote by $f\colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ the homeomorphism from Lemma 41.1 and we denote by $g\colon \overline{B}^n/S^{n-1} \to S^n$ the homeomorphism from page 182. We set $* := g([S^{n-1}])$. We consider the following sequence of maps:*

$$\mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{f_*} \mathrm{H}_n(\overline{B}^n, S^{n-1}) \longrightarrow \mathrm{H}_n(\overline{B}^n/S^{n-1}, [S^{n-1}]) \xrightarrow{g_*} \mathrm{H}_n(S^n, *) \xleftarrow{\text{я}} \mathrm{H}_n(S^n).$$

The following statements hold:

(2) *We have*

$$f_*([\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n \cdot \underbrace{\textit{standard generator } [\overline{B}^n] \textit{ of } \mathrm{H}_n(\overline{B}^n, S^{n-1})}_{\text{defined by the condition } \partial([\overline{B}^n]) = [S^{n-1}]\text{, see page 1176}}.$$

(3) *The standard generator* $[\overline{B}^n]$ *of* $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ *has the same image in* $\mathrm{H}_n(S^n,*)$ *as the standard generator* $[S^n]$ *of* $\mathrm{H}_n(S^n)$.

In the following we introduce and recall the following notation:

Notation. Let $n \in \mathbb{N}_0$.

(1) Given $P \in S^n$ we denote by $[S^n]_P$ the image of the standard generators $[S^n]$ under the map $\mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n, S^n \setminus \{P\})$.

(2) Let $\Psi\colon \Delta^n \to ◺_n$ be $\Delta^n \to ◺$ and $\Delta^n \to ◺$ and $\Delta^n \to ◺$ the obvious projection. In Lemma 41.1 (1) we saw that Ψ is a homeomorphism. We write $\Phi := \Psi^{-1}\colon ◺_n \to \Delta^n$.

(3) Let $\alpha\colon \Delta^n \to S^n_{\geq 0} \subset S^n$ and $\beta\colon \Delta^n \to S^n_{\leq 0} \subset S^n$ be the two maps defined on page 1173.

(4) We denote by $N = (0,\dots,0,1)$ the North Pole of S^n.

The proof of Proposition 45.5 requires the following lemma.

Lemma 45.25. $(*)$ *Let* $k \in \mathbb{N}_0$ *and let* $f\colon \Delta^k \to S^k$ *be an injective map such that the map* $f \circ \Phi\colon \mathring{◺}_k \to S^k$ *is a smooth embedding.*[736] *Then for every* $Q \in f(\mathring{\Delta}^k)$ *we have the following equality in* $\mathrm{H}_k(S^k, S^k \setminus \{Q\})$*:*

$$[f_*(\mathrm{id}_{\Delta^k})] \;=\; \epsilon \cdot [S^k]_Q \qquad \textit{where } \epsilon \;=\; \begin{cases} +1, & \textit{if } f\circ\Phi\colon \mathring{◺}^k \to S^k \textit{ is orientation-preserving},\\ -1, & \textit{if } f\circ\Phi\colon \mathring{◺}^k \to S^k \textit{ is orientation-reversing}.\end{cases}$$

Proof of Lemma 45.25 $(*)$. First we deal with the special case that $Q = N$ and that $f(\Delta^k) \subset S^k_{\geq 0}$. We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_k(\mathbb{R}^k, \mathbb{R}^k\setminus\{0\}) \arrow[rr, "\cdot\epsilon"] & & \mathrm{H}_k(\mathbb{R}^k, \mathbb{R}^k\setminus\{0\}) \\
\mathrm{H}_k(t_P(◺_k), t_P(◺_k)\setminus\{0\}) \arrow[u, "i_*", "\cong"'] \arrow[rr, "t_X^{-1}\circ\Psi\circ\alpha^{-1}\circ f\circ\Psi^{-1}\circ t_P"] & & \mathrm{H}_k(t_X(◺_k), t_X(◺_k)\setminus\{0\}) \arrow[u, "j_*"', "\cong"] \\
\mathrm{H}_k(◺_k, ◺_k\setminus\{P\}) \arrow[u, "t_P", "\cong"'] \arrow[rr, "\Psi\circ\alpha^{-1}\circ f\circ\Psi^{-1}"] \arrow[dr, "\cong", "f\circ\Psi^{-1}"'] & & \mathrm{H}_k(◺_k, ◺_k\setminus\{X\}) \arrow[u, "t_X"', "\cong"] \arrow[dl, "\cong"', "\alpha\circ\Psi^{-1}"] \\
\mathrm{H}_k(◺_k, \partial◺_k) \arrow[u, "\mu", "\cong"'] & \mathrm{H}_k(S^k, S^k\setminus\{N\}) & \mathrm{H}_k(◺_k, \partial◺_k) \arrow[u, "\mu"', "\cong"] \\
\mathrm{H}_k(\Delta^k, \partial\Delta^k) \arrow[u, "\Psi_*", "\cong"'] \arrow[ur, "f_*"] \arrow[rr, "\cdot\epsilon"] \arrow[uuuu, bend left, "i\circ t_P\circ\Psi"] & & \mathrm{H}_k(\Delta^k, \partial\Delta^k). \arrow[u, "\Psi_*"', "\cong"] \arrow[ul, "\alpha_*"'] \arrow[uuuu, bend right, "j\circ t_X\circ\Psi"']
\end{tikzcd}$$

[736] Note that $\mathring{◺}_k$ is an open subset of $\mathbb{R}^k$, in particular, following the convention established on page 299, we will view it as an oriented k-dimensional smooth manifold.

We make the following clarifications and observations:

(1) We set $P := \Psi(f^{-1}(N)) \in \mathring{\Delta}_k$ and $X := \Psi(\alpha^{-1}(N)) \in \mathring{\Delta}_k$. Evidently we have the equality $(f \circ \Psi^{-1})(P) = N = (\alpha \circ \Psi^{-1})(X)$.
(2) Given $v \in \mathbb{R}^k$ we denote by $t_v \colon \mathbb{R}^k \to \mathbb{R}^k$ the map that is given by $x \mapsto x - v$.
(3) The maps $i \colon t_P(\Delta_k) \to \mathbb{R}^k$ and $j \colon t_X(\Delta_k) \to \mathbb{R}^k$ are the inclusion maps.
(4) The maps $\mu \colon (\Delta_k, \partial\Delta_k) \to (\Delta_k, \Delta_k \setminus \{P\})$ and $\mu \colon (\Delta_k, \partial\Delta_k) \to (\Delta_k, \Delta_k \setminus \{X\})$ are the inclusion maps of pairs of topological spaces.
(5) Since $f(\Delta^k) \subset S^k_{\geq 0}$ we have in particular that $f(\Delta^k) \subset \alpha(\Delta^k)$ which means that the map $\alpha^{-1} \circ f$ is actually defined.
(6) One can easily verify that $\alpha \circ \Psi^{-1}$ is orientation-preserving. As mentioned on page 299, the translation maps t_v are also orientation-preserving. Therefore we see that the map $t_X^{-1} \circ \Psi \circ \alpha^{-1} \circ f \circ \Psi^{-1} \circ t_P$ is orientation-preserving if and only if the map $f \circ \Psi^{-1} = f \circ \Phi$ is orientation-preserving. Thus we obtain from Proposition 45.16 that the top square commutes.
(7) The second square from the top, the top triangle and the two triangles to the left and right commute basically by definition.
(8) It is straightforward to see that the maps $i \circ t_P \circ \Psi$ and $j \circ t_X \circ \Psi$ are homotopic as maps of pairs $(\Delta^k, \partial\Delta^k) \to (\mathbb{R}^k, \mathbb{R}^k \setminus \{0\})$. In particular we obtain from this observation together with Proposition 43.17 that the two sequences of vertical maps on the left and on the right are actually the same map.
(9) Using Corollary 43.18 (2a) and using the Excision Theorem 43.20 one can show fairly easily that all the vertical maps are isomorphisms and we see that the upper diagonal maps are isomorphisms. Since the triangles to the left and right commute we see that all diagonal maps are isomorphisms.
(10) It follows from (6), (7), (8) and (9) that the bottom triangle also commutes.

Now we are basically done with the proof for our special case. Indeed we see that

$$[f_*(\mathrm{id}_{\Delta^k})] \underset{\substack{\uparrow \\ \text{by (10)}}}{=} \epsilon \cdot [\alpha_*(\mathrm{id}_{\Delta^k})] = \epsilon \cdot [\alpha] \underset{\substack{\uparrow \\ \text{since } \beta = 0 \in \mathrm{C}_k(S^n, S^n \setminus \{N\})}}{=} \epsilon \cdot [\alpha - \beta]_Q \underset{\substack{\uparrow \\ \text{definition of } [S^k]}}{=} \epsilon \cdot [S^k]_Q.$$

Now we still need to deal with the general case. In all likelihood few readers will have made it so far. So let us just sketch the ingredients for reducing the general case to the above special case:

(1) By Lemma 3.32 (4) there exists an $A \in \mathrm{SO}(n+1)$ with $A \cdot Q = N$. By the discussion on page 538 the map $\varphi \colon S^n \to S^n$ that is given by $\varphi(x) = A \cdot x$ is homotopic to the identity.
(2) Every map $\varphi \colon \Delta^k \to S^k$ which contains N in its image is homotopic to some map $\psi \colon \Delta^k \to S^k_{\geq 0}$ into the upper hemispheres which has otherwise "the same properties".

We leave it to the reader to fill in the details. ■

Now we can provide the proof of Proposition 45.5 (2) and (3).

PROOF OF PROPOSITION 45.5 (2) AND (3).

(2) We need to show that $f_*([\mathrm{id}_{\Delta^n}]) = (-1)^n \cdot [\overline{B}^n]$. To do so we recall the following maps:

(a) As on page 1077 we denote by $i = i^n_n \colon \Delta^{n-1} \to \Delta^n$ the n-th face map that is given by $i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{n-1}, 0)$.

(b) Below we will denote by ж the natural map from reduced homology to relative homology that we introduced in Lemma 43.14.

We consider the following diagram

$$\begin{tikzcd}[row sep=small]
{[\mathrm{id}_{\Delta^n}]} & \partial[\mathrm{id}_{\Delta^n}] & \partial[\mathrm{id}_{\Delta^n}]=(-1)^n\cdot i_*([\mathrm{id}_{\Delta^{n-1}}]) & (-1)^n\cdot[\mathrm{id}_{\Delta^{n-1}}] \\[-15pt]
\mathrm{H}_n(\Delta^n, \partial\Delta^n) \arrow[r, "\partial"] \arrow[d, "f_*"] & \widetilde{\mathrm{H}}_{n-1}(\partial\Delta^n) \arrow[r, "ж"] \arrow[d, "f_*"] & \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) \arrow[d, "f_*"] & \mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \arrow[l, "i_*"', "\cong"] \arrow[dl, "(f\circ i)_*"] \\
\mathrm{H}_n(\overline{B}^n, S^{n-1}) \arrow[r, "\partial"] & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \arrow[r, "ж"] & \mathrm{H}_{n-1}(S^{n-1}, S^{n-1}\setminus\{N\}) & \mathrm{H}_i(\Delta^{n-1}, \partial\Delta^{n-1}). \arrow[l, "\alpha_*"] \\[-15pt]
{[\overline{B}^n]} & {[S^{n-1}]=[\alpha-\beta]} & {[\alpha-\beta]=[\alpha]} & {[\mathrm{id}_{\Delta^{n-1}}]}
\end{tikzcd}$$

(Here we put on top and above of the various groups the elements which will play a role in the subsequent argument.) Note that the diagram commutes by Corollary 43.16 (2) and Lemma 43.14 (1). Now we see that we have the following equalities in $\mathrm{H}_{n-1}(S^{n-1}, S^{n-1} \setminus \{N\})$:

$$\begin{aligned}
ж(\partial(f_*([\mathrm{id}_{\Delta^n}]))) &\underset{\substack{\uparrow\\ \text{since the diagram commutes}}}{=} f_*(ж(\partial([\mathrm{id}_{\Delta^n}]))) \underset{\substack{\uparrow\\ \text{we have } ж(\partial(\mathrm{id}_{\Delta^n})) = (-1)^n \cdot i_*(\mathrm{id}_{\Delta^{n-1}}) \in \mathrm{C}_{n-1}(\partial\Delta^n, \Lambda) \text{ since}\\ \text{all other summands of } \partial(\mathrm{id}_{\Delta^n}) \text{ are zero in } \mathrm{C}_{n-1}(\partial\Delta^n, \Lambda)}}{=} f_*((-1)^n \cdot i_*([\mathrm{id}_{\Delta^{n-1}}])) = (-1)^n \cdot [\alpha] \\
&= (-1)^n \cdot (f \circ i)_*([\mathrm{id}_{\Delta^{n-1}}]) \underset{\substack{\uparrow\\ \text{we leave it to the reader to verify that } f\circ i\colon \Delta^{n-1}\to S^{n-1} \text{ and } \alpha\colon \Delta^{n-1}\to S^{n-1}\\ \text{are both orientation-preserving, thus the equality follows from Lemma 45.25}}}{=} (-1)^n \cdot \alpha_*([\mathrm{id}_{\Delta^{n-1}}]) \\
&\underset{\substack{\uparrow\\ \text{since } \beta = 0 \in \mathrm{C}_{n-1}(S^n, S^{n-1}\setminus\{N\})}}{=} (-1)^n \cdot ([\alpha - \beta]) \underset{\substack{\uparrow\\ \text{definition of } [S^{n-1}]}}{=} (-1)^n \cdot ж([S^{n-1}]) \underset{\substack{\uparrow\\ \text{definition of } [\overline{B}^n]}}{=} (-1)^n \cdot ж(\partial([\overline{B}^n])).
\end{aligned}$$

As discussed earlier, by the various long exact sequences we know that the maps ∂ and ж are both isomorphisms. Thus we obtain from the above the desired equality $f_*([\mathrm{id}_{\Delta^n}]) = (-1)^n \cdot [\overline{B}^n]$.

(3) We consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{H}_n(\Delta^n, \partial\Delta^n) \arrow[r, "f_*"] \arrow[rrrd] & \mathrm{H}_n(\overline{B}^n, S^{n-1}) \arrow[r] & \mathrm{H}_n(\overline{B}^n/S^{n-1}, [S^{n-1}]) \arrow[r, "g_*"] & \mathrm{H}_n(S^n, *) \arrow[d, "\cong"] & \mathrm{H}_n(S^n). \arrow[l] \arrow[dl] \\
& & & \mathrm{H}_n(S^n, S^n \setminus \{-*\}) &
\end{tikzcd}$$

Let $\Phi \colon ◺_n \to \Delta^n$ be the map that we introduced above. One can easily verify that the map $g \circ f \circ \Phi \colon \mathring{◺}_n \to S^n$ is orientation-preserving if and only if n is even. This observation, together with Lemma 45.25 implies that the image of $[\mathrm{id}_{\Delta^n}]$ in $\mathrm{H}_n(S^n, S^n \setminus \{-*\})$ under the diagonal map agrees with $[S^n]_{-*}$. Since the vertical map is an isomorphism we obtain the desired equality. ∎

Exercises for Chapter 45.

Exercise 45.1. Let $n \in \mathbb{N}_0$ and let $P \in \mathring{\Delta}^n$. Show that $\mathrm{H}_n(\Delta^n, \Delta^n \setminus \{P\}) = \mathbb{Z} \cdot [\mathrm{id}_{\Delta^n}]$.
Hint. Make use of Lemma 45.2.

Exercise 45.2. Let $n \in \mathbb{N}$. Let

$$\begin{array}{rcl} \rho\colon \Delta^n & \to & \Delta^n \\ (t_0, t_1, t_2, \dots, t_n) & \mapsto & (t_1, t_0, t_2, \dots, t_n) \end{array}$$

be the map that swaps the first two coordinates of Δ^n. Show that the induced map $\rho_*\colon \mathrm{H}_n(\Delta^n, \partial\Delta^n) \to \mathrm{H}_n(\Delta^n, \partial\Delta^n)$ is given by multiplication by -1.
Hint. First prove the statement for $n = 1$. Then use an induction argument similar to the proof of Lemma 45.2 (2). Hereby you need to use the naturality of the connecting homomorphism of the long exact sequence in homology of a pair of topological spaces.

Exercise 45.3. For which $n \in \mathbb{N}$ does there exist a singular simplex $\sigma\colon \Delta^n \to S^n$ that represents a generator of $\mathrm{H}_n(S^n)$?

Exercise 45.4. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a map with $f(0) = 0$, with $f(\mathbb{R}^n \setminus \{0\}) \subset \mathbb{R}^n \setminus \{0\}$ and with $f(S^{n-1}) \subset S^{n-1}$. Show that

$$\deg(f_*\colon \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})) = \deg(f_*\colon \mathrm{H}_{n-1}(S^{n-1}) \to \mathrm{H}_n(S^{n-1})).$$

Remark. The analogous statement also holds if we replace $\mathbb{R}^n$ by $\overline{B}^n$.

Exercise 45.5.

(a) Let $n \in \mathbb{N}_{\geq 2}$. Show that given any $k \in \mathbb{N}$ there exists a map $f\colon \overline{B}^n \to \overline{B}^n$ with $f(S^{n-1}) \subset S^{n-1}$ such that

$$\deg\big(f_*\colon \underbrace{\mathrm{H}_n(\overline{B}^n, S^{n-1})}_{\cong\mathbb{Z}} \to \mathrm{H}_n(\overline{B}^n, S^{n-1})\big) = k.$$

(b) Show that the conclusion of (a) does not hold for $n = 1$.

Exercise 45.6. On S^{2k} we had found a vector field which vanishes at *precisely two* points. Does there exist a vector field on S^2 which vanishes on only *one* point?

Exercise 45.7. Let $n \geq 1$. We consider the complex polynomial $g(z) = z^n$. Show that the degree of the map $\Theta(g)\colon S^2 \to S^2$ equals n.
Hint. Use Proposition 45.23 and Proposition 45.22.

Exercise 45.8. Let X be a topological space. Suppose that the suspension $\Sigma(X)$ is a closed n-dimensional topological manifold. Show that X is a topological homology $(n-1)$-sphere, i.e. show that $\mathrm{H}_i(X) \cong \mathrm{H}_i(S^{n-1})$ for all $i \in \mathbb{N}_0$.
Remark. Also have a look at Exercise 45.9.

Exercise 45.9. Let $n \in \mathbb{N}_{\geq 3}$ and let M be a 0-connected topological space with non-trivial fundamental group.

(a) Show that the suspension $\Sigma(M)$ is *not* an n-dimensional topological manifold.
Hint. There are many different approaches one can take. One possibility is to study the fundamental group of $\Sigma(M)$ and to study the fundamental group of $\Sigma(M)$ with some points removed, and to use Exercise 20.3.

(b) Show that the suspension $\Sigma(M)$ is not a topological manifold of any dimension.

Remark. In Exercise 45.8 we just showed that if $\Sigma(M)$ is a closed n-dimensional topological manifold, then M is a topological homology $(n-1)$-sphere, i.e. $\mathrm{H}_i(M) \cong \mathrm{H}_i(S^{n-1})$ for all $i \in \mathbb{N}_0$. Together with (a) and the proof of the Poincaré Conjecture in all dimensions, see Chapter **??**, this shows that if $n \in \mathbb{N}_{\geq 2}$ and if M is a 0-connected n-dimensional topological manifold such that $\Sigma(M)$ is a topological manifold, then M is necessarily homeomorphic to S^n.

Exercise 45.10.

(a) We consider the map $f\colon S^1 \to S^1$ given by $(x,y) \mapsto (x,|y|)$. Show that the local degree at the point -1 is zero.

(b) Let $f\colon S^1 \to S^1$ be a map and let $z \in S^1$ be a nice point. Suppose that $\deg(f,z) = 1$. Does it follow that f is a local homeomorphism at z? It suffices to give a short justification for your answer.

Exercise 45.11. Let $f = f(z)$ be a polynomial with complex coefficients of degree ≥ 1. Let $\Theta(f)$ be the corresponding self-map of $S^2 = \mathbb{C} \cup \{\infty\}$ that we introduced on page 1190. Show that the local degree of $\Theta(f)$ at a root of $f(z)$ equals the multiplicity of the root.

Exercise 45.12. Let $n \in \mathbb{N}$ be even. Show that every map $f\colon \mathbb{RP}^n \to \mathbb{RP}^n$ has a fixed point.
Hint. Apply Lemma 29.11 to get a suitable map $\widetilde{f}\colon S^n \to S^n$. If $\deg(\widetilde{f}) \neq -1$, then we are basically done by Theorem 45.13. What can we do if $\deg(\widetilde{f}) = -1$?

Exercise 45.13. Let $n \in \mathbb{N}_{\geq 2}$. In this exercise we identify S^{n-1} with the equatorial sphere $\{(x,0) \in S^n \,|\, x \in S^{n-1}\}$.

(a) Let $f\colon S^n \to S^n$ be a map with $f(S^n_{\geq 0}) \subset S^n_{\geq 0}$ and with $f(S^n_{\leq 0}) \subset S^n_{\leq 0}$. Show that $\deg(f\colon S^n \to S^n) = \deg(f\colon S^{n-1} \to S^{n-1})$.

(b) Let $f\colon S^n \to S^n$ be a map with $f(S^n_{\geq 0}) \subset S^n_{\leq 0}$ and with $f(S^n_{\leq 0}) \subset S^n_{\geq 0}$. Show that $\deg(f\colon S^n \to S^n) = -\deg(f\colon S^{n-1} \to S^{n-1})$.

Exercise 45.14. Let X be a topological space. Given a homology class $\varphi \in \mathrm{H}_k(X)$ we define

$$\|\varphi\|_{\mathbb{Z}} \;:=\; \inf\Big\{ \sum_{i=1}^{n} |a_i| \,\Big|\, \sigma = \underbrace{\sum_{i=1}^{n} a_i \cdot \sigma_i}_{\text{with } \sigma_i \neq \sigma_j \text{ for } i \neq j} \text{ is a cycle with } [\sigma] = \varphi \in \mathrm{H}_k(X) \Big\}.$$

(a) Let X be a path-connected topological space. Show that for any $\varphi \in \mathrm{H}_1(X)$ we have $\|\varphi\|_{\mathbb{Z}} = 1$.

(b) Let $n \in \mathbb{N}$. Determine $\|[S^n]\|_{\mathbb{Z}}$.

Remark. The notation with the subscript "$\mathbb{Z}$" is used to distinguish this definition from the concept that we will introduce in Exercise 68.16.

Exercise 45.15. Let $f\colon S^n \to S^n$ be an orientation-preserving diffeomorphism. Show that f is homotopic to the identity.
Hint. You can easily arrange that f fixes say the North Pole $N = (0, \ldots, 0, 1)$. Now use Lemma 45.17 and push everything that is "awkward" into the South Pole.

Exercise 45.16. Given $z \in S^1$ we denote by $\Phi_z\colon S^2 \to S^2$ the map that is given by rotating S^2 along the z-axis by the angle z. The map

$$\begin{array}{rcl} G\colon S^1 \times S^2 & \mapsto & S^1 \times S^2 \\ (z, v) & \mapsto & (z, \Phi_z(v)) \end{array}$$

is called the *Gluck twist*. Show that the Gluck twist is homotopic to the identity.
Hint. The exercise builds on Exercise 45.15.
Remark. The Gluck twist is in so far remarkable as it is not diffeotopic to the identity. This non-trivial fact is proved in [**Glu62**, Chapter 5] and [**Wall70**, p. 232].

Exercise 45.17. Let $n \in \mathbb{N}$. Show that every vector field on $\overline{B}^n$ that points outwards on S^{n-1} admits at least one zero in $\overline{B}^n$.

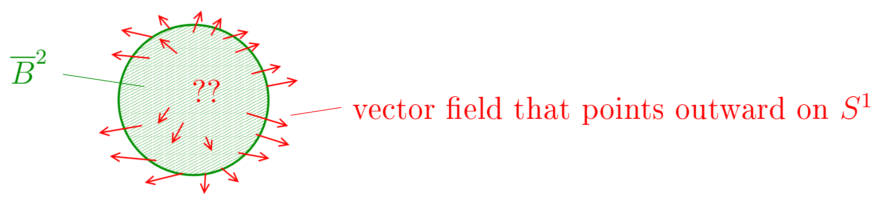


FIGURE 763. Illustration for Exercise 45.17.

Exercise 45.18. Let $k, l \in \mathbb{Z}$. We consider the sphere $S^3 = \{(z, w) \in \mathbb{C}^2 \,|\, |z|^2 + |w^2| = 1\}$ and the self-map

$$\begin{array}{rcl} f\colon S^3 & \to & S^3 \\ (z, w) & \mapsto & (z^k, w^l). \end{array}$$

Determine the degree of f.

Exercise 45.19. Let X be a topological space with subsets A, B, U, V and suppose that we have the following diagram of inclusion maps

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"] & U \\ V & B. \arrow[l, "l"] \arrow[u, "k"'] \end{tikzcd}$$

Suppose that the inclusion induced maps i_*, j_*, k_*, l_* on the first homology groups are isomorphisms. Does it follow that $i_* \circ j_*^{-1} = k_* \circ l_*^{-1}\colon \mathrm{H}_1(V) \to \mathrm{H}_1(U)$?

Exercise 45.20. Let $n \in \mathbb{N}$ and let $f\colon S^n \times S^n \to S^n$ be a map. We pick $x_0, y_0 \in S^n$. We define the *bidegree* of f as

$$\operatorname{bideg}(f) := (p, q) \text{ where } p := \deg\begin{pmatrix} S^n & \xrightarrow{i} & S^n \\ x & \mapsto & f(x, y_0) \end{pmatrix} \text{ and } q := \deg\begin{pmatrix} S^n & \xrightarrow{j} & S^n \\ y & \mapsto & f(x_0, y) \end{pmatrix}.$$

(a) Show that the definition not depend on the choice of x_0 and y_0.

(b) Let $f\colon S^1 \times S^1 \to S^1$ be the map that is given by complex multiplication. Show that the bidegree of f equals $(1,1)$.
Remark. If you already know quaternions, perform the same exercise for the map $f\colon S^3 \times S^3 \to S^3$ that is given by quaternionic multiplication.

(c) Show that the map
$$\mathrm{H}_n(S^n) \oplus \mathrm{H}_n(S^n) \xrightarrow{i_* \oplus j_*} \mathrm{H}_n(S^n \times S^n) \xrightarrow{f_*} \mathrm{H}_n(S^n)$$
is of the form $\varphi \oplus \psi \mapsto p \cdot \varphi + q \cdot \psi$.

Exercise 45.21. Let $n \in \mathbb{N}$ and let G be a group that acts continuously on S^n. We suppose that the action of G is free. Recall that according to the definition on page 185 this means that for every $g \neq e$ and every $x \in S^n$ we have $g \cdot x \neq x$.

(a) Show that if n is even, then the map
$$\begin{array}{rcl} G & \to & \{\pm 1\} \\ g & \mapsto & \deg \left(\begin{array}{rcl} S^n & \to & S^n \\ x & \mapsto & g \cdot x \end{array} \right) \end{array}$$
is a monomorphism.
Remark. This exercise shows in particular that if a group admits a continuous free action on an even-dimensional sphere, then G has order at most 2. This gives in particular a complete answer to Question 16.8. We will give a different proof of this statement in Proposition 55.10.

(b) Show that the conclusion of (a) does not necessarily hold if n is odd.

46. The Mayer–Vietoris sequence and its applications

In this chapter we will state and prove the Mayer–Vietoris sequence for homology groups. It makes often allows us to compute the homology groups of a topological space from the homology groups of appropriate subspaces together with the knowledge of the nature of inclusion induced maps on homology.

The role of the Mayer–Vietoris sequence in homology theory is similar to the role of the Seifert–van Kampen Theorem 22.1 for fundamental groups. The Mayer–Vietoris sequence will give us the tool to compute the homology groups of many topological spaces that we are interested in. In particular we will determine the homology groups of $\mathbb{RP}^2$, the torus and the Klein bottle.

46.1. **Split exact sequences.** We already saw in Theorem 43.3 that homology groups of different topological spaces are sometimes related by a long exact sequence of homology groups. The Mayer–Vietoris sequence that we will introduce later on in this chapter is another instance of such a long exact sequence. Very often one ends up in the situation that one knows the homology groups of two out of three topological spaces and hopefully one knows something about maps betweens these homology groups, and given this data one would like to determine the homology groups of the third topological space from the long exact sequence. Algebraically the question is as follows: suppose we are given an exact sequence

$$\dots \to A_n \to B_n \to \mathrm{C}_n \to A_{n-1} \to \dots$$

of abelian groups where we know the groups A_* and C_*, can we determine the groups B_* from this long exact sequence? The same question of course also arises when the roles of A, B and C are switched.

In this section we will study the simplest case, namely the case of a short exact sequence

$$0 \to A \to B \to C \to 0.$$

If we know A and C, can we read "read off" B from this short exact sequence? In general the answer is no. For example the short exact sequence

$$0 \to \mathbb{Z}_2 \longrightarrow B \longrightarrow \mathbb{Z}_2 \to 0$$

could be of the form

$$0 \to \mathbb{Z}_2 \xrightarrow{x\mapsto(x,0)} \mathbb{Z}_2\oplus\mathbb{Z}_2 \xrightarrow{(x,y)\mapsto y} \mathbb{Z}_2 \to 0$$

but it could also be of the form[737]

$$0 \to \mathbb{Z}_2 \xrightarrow{x\mapsto 2x} \mathbb{Z}_4 \xrightarrow{x\mapsto x} \mathbb{Z}_2 \to 0.$$

In this short section we will give a criterion for when the naive guess $B \cong A \oplus C$ is actually the correct answer.

Definition. We say that a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of (not necessarily abelian) groups *splits*, if there exists a *splitting* $s\colon C \to B$, i.e. a homomorphism $s\colon C \to B$ such that $p \circ s = \mathrm{id}_C$.

[737] Are there any other possibilities for the isomorphism type of B?

For example the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{k\mapsto(k,0)} \mathbb{Z}\oplus\mathbb{Z}_n \xrightarrow{(k,l)\mapsto l} \mathbb{Z}_n \to 0$$

splits, in fact a splitting is given by the map $b \mapsto (0, b)$. On the other hand the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{k\mapsto nk} \mathbb{Z} \to \mathbb{Z}_n \to 0.$$

does not split.[738]

Lemma 46.1. *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups. If C is a free abelian group, then the short exact sequence splits.

Proof. Let $\{c_i\}_{i\in I}$ be a basis of the free abelian group C. For each $i \in I$ we choose a $b_i \in B$ with $p(b_i) = c_i$. By Lemma 19.1 there exists a unique homomorphism $s\colon C \to B$ with $s(c_i) = b_i$ for all i. We have $(p \circ s)(c_i) = c_i$ for each c_i. But then it follows from the uniqueness part of Lemma 19.1 that $p \circ s = \mathrm{id}_C$. The homomorphism s is thus indeed a splitting. ■

The following lemma gives two useful criteria for a short exact sequence to split.

Lemma 46.2. (Splitting Lemma) *Let*

$$\{e\} \to A \xrightarrow{i} B \xrightarrow{p} C \to \{e\}$$

be a short exact sequence of groups. The following two statements are equivalent:

(1) *There exists a homomorphism $t\colon B \to A$ such that $t \circ i = \mathrm{id}_A$.*
(2) *There exists an isomorphism $\Phi\colon B \to A \times C$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & & B \arrow[dr, "p"] \arrow[dd, "\Phi", "\cong"'] & & \\
\{e\} \arrow[r] & A \arrow[ur, "i"] \arrow[dr] & & C \arrow[r] & \{e\} \\
 & & A\times C \arrow[ur] & &
\end{tikzcd}$$

where the lower left diagonal map is the obvious inclusion and the lower right diagonal map is the obvious projection map.

Furthermore, if B is abelian, then the above statements are also equivalent to the following statement:

(3) *The short exact sequence splits.*

Remark. If B is not abelian, then the implication (3) ⇒ (1) of Splitting Lemma 46.2 does in general not hold. For example consider the sequence $0 \to \mathbb{Z}_3 \xrightarrow{\varphi} S_3 \xrightarrow{\sigma} \mathbb{Z}_2 \to 0$ where S_3 is as always the permutation group on the three elements, where $\varphi\colon \mathbb{Z}_3 \to S_3$ is the homomorphism given by $\varphi(1) = (1\,2\,3)$ and where $\sigma\colon S_3 \to \mathbb{Z}_2$ is the unique epimorphism

[738] Why does it not split?

from the permutation group S_3 onto $\mathbb{Z}_2$. A splitting of σ is given by $\varphi(1) = (1\,2)$. Since σ is also the only epimorphism from S_3 onto a non-trivial group we see that the sequence does not split, hence (1) is not satisfied.

Proof.

(1)⇒(2) Let $t\colon B \to A$ be a homomorphism such that $t \circ i = \mathrm{id}_A$. We define

$$\begin{array}{rcl}\Phi\colon B & \to & A \times C \\ b & \mapsto & (t(b), p(b)).\end{array}$$

It follows easily from the exactness of the initial sequence that this map is in fact an isomorphism.[739] It is straightforward to verify that Φ has all the other desired properties.

(2)⇒(1),(3) Now suppose we are given an isomorphism $\Phi\colon B \to A \times C$ such that the given diagram commutes. It follows immediately from the definitions and given properties that the maps

$$B \xrightarrow{\Phi} A \times C \xrightarrow{(a,c)\mapsto a} A \quad \text{and} \quad C \xrightarrow{c\mapsto(0,c)} A \times C \xrightarrow{\Phi^{-1}} B$$

have the required properties.

Finally we assume that B is abelian.

(3)⇒(1) Let $s\colon C \to B$ be a splitting, i.e. let s be a homomorphism such that $p \circ s = \mathrm{id}_C$. We need to find an appropriate homomorphism $t\colon B \to A$. Since the sequence is exact, our task is basically the same as finding an appropriate homomorphism $B \to \ker(p)$. Let $b \in B$. We have

$$p(b - s(p(b))) \;=\; p(b) - (\underbrace{p \circ s}_{=\mathrm{id}})(p(b)) \;=\; p(b) - p(b) \;=0.$$

By the exactness of the short exact sequence there exists a unique $a \in A$ with $i(a) = b - s(p(b))$. We define $t(b) := a$. It follows easily from the fact that p and s are homomorphisms that the map $t\colon B \to A$ is a homomorphism. It remains to show that $t \circ i = \mathrm{id}_A$. Let $a \in A$. We have to show that $t(i(a)) - a = 0$. Since i is injective it suffices to show that $i(t(i(a))) - i(a) = 0$. We have

$$i(t(i(a))) - i(a) \underset{\substack{\uparrow \\ \text{since } t(y) \text{ is the unique element} \\ \text{with } i(t(y)) = y - s(p(y))}}{=} (i(a) - s(p(i(a)))) - i(a) \underset{\substack{\uparrow \\ \text{since } p \circ i = 0}}{=} i(a) - i(a) \;=\; 0.$$

■

In many applications we only need the following corollary that is an immediate consequence of Lemmas 46.1 and 46.2.

Corollary 46.3. *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

[739] In fact the argument is basically the same as in the proof of Lemma 43.1 (4a).

be a short exact sequence of abelian groups. Then the following hold:

(1) *If C is a free abelian group, then $B \cong A \oplus C$.*
(2) *Suppose A and C are free abelian groups. If X is a basis for A and if $Y \subset B$ is a subset such that $p(Y)$ is a basis for C, then $X \sqcup Y$ is a basis for the free abelian group B.*

PROOF. The first statement is an immediate consequence of Lemmas 46.1 and 46.2. The second statement follows easily from Splitting Lemma 46.2 (1) ⇒ (3). We leave the details to the reader. ■

Example. Let B be an abelian group, let $b \in B$ and let $p \colon B \to \mathbb{Z}$ be a homomorphism with $p(b) = 1$. It follows immediately from Corollary 46.3, applied to the short exact sequence $0 \to \ker(p) \to B \to \mathbb{Z} \to 0$, that we can write $B = K \oplus \mathbb{Z} \cdot b$ for some subgroup K that is isomorphic to $\ker(p)$.

The above discussion gives us tools for dealing with short exact sequence. The following lemma allows us to break a long exact sequence into shorter exact sequences.

Lemma 46.4. *Let*

$$\ldots \to A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \ldots$$

be an exact sequence of abelian groups. Then for any n the sequences

$$0 \to \operatorname{coker}\left(A_{n+2} \xrightarrow{f_{n+2}} A_{n+1}\right) \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \ldots$$

and

$$\ldots \to A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} \ker\left(A_{n-1} \xrightarrow{f_{n-1}} A_{n-2}\right) \to 0$$

are also exact. In particular

$$0 \to \operatorname{coker}\left(A_{n+2} \xrightarrow{f_{n+2}} A_{n+1}\right) \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} \ker\left(A_{n-1} \xrightarrow{f_{n-1}} A_{n-2}\right) \to 0$$

is a short exact sequence.

PROOF. The statements follow immediately from the definitions. We leave it to the reader to fill in the details. ■

Example. A typical application is that if we are given an exact sequence

$$\ldots A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n=0} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \xrightarrow{f_{n-2}} A_{n-3} \ldots$$

of abelian groups such that the map f_n is the zero map, then the exact sequence can be split "into the two halves" along that zero map, i.e.

$$0 \to A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \xrightarrow{f_{n-2}} A_{n-3} \ldots$$

and

$$\ldots A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \to 0$$

are also exact sequences. In the remainder of this course we will frequently use this fact without explicitly citing Lemma 46.4.

46.2. **The Mayer–Vietoris sequence.** In the following theorem and throughout the subsequent discussion we will use the following notation: given a topological space D and a subset $C \subset D$, we denote the corresponding inclusion map $C \to D$ by i_C. To avoid pandemonium with subscripts we often drop denote the inclusion induced maps on homology by i_C instead of the more accurate $(i_C)_*$.

Theorem 46.5. (Mayer–Vietoris) [740] *Let X be a topological space and let $A, B \subset X$ be subsets such that $X = \mathring{A} \cup \mathring{B}$. Then the following two statements hold:*

(1) *For each $n \in \mathbb{N}$ there exists a unique homomorphism*
$$\partial_n \colon \mathrm{H}_n(X) \ \to \ \mathrm{H}_{n-1}(A \cap B)$$
which has the following property: for every $a \in \mathrm{C}_n(A)$ and $b \in \mathrm{C}_n(B)$ such that $a + b \in \mathrm{C}_n(X)$ is a cycle we have[741]
$$\partial_n([a+b]) \ = \ [\,\partial a\,] \ = \ -\,[\,\partial b\,] \ \in \ \mathrm{H}_{n-1}(A \cap B).$$
We refer to the maps $\partial_n \colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B)$ as the connecting homomorphisms *corresponding to (X, A, B).*

(2) $(*)$ *The connecting homomorphism of (1) is natural in the following sense. Let X and Y be two topological spaces and let $A, B \subset X$ and $C, D \subset Y$ be subsets with $X = \mathring{A} \cup \mathring{B}$ and $Y = \mathring{C} \cup \mathring{D}$. If $f \colon X \to Y$ is a map with $f(A) \subset C$ and $f(B) \subset D$, then for every $n \in \mathbb{N}$ the following diagram commutes*
$$\begin{tikzcd}
\mathrm{H}_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_*"'] & \mathrm{H}_{n-1}(A \cap B) \arrow[d, "f_*"] \\
\mathrm{H}_n(Y) \arrow[r, "\partial_n"] & \mathrm{H}_{n-1}(C \cap D)
\end{tikzcd}$$
where the horizontal maps are the connecting homomorphisms from (1).

(3) *The following sequence is exact:*
$$\begin{array}{ccccccc}
\ldots \to \mathrm{H}_n(A \cap B) & \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} & \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) & \xrightarrow{i_A + i_B} & \mathrm{H}_n(X) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A \cap B) \to \ldots \\
s & \longmapsto & (i_{A\cap B}(s), -i_{A\cap B}(s)) & & & & \\
 & & s & \longmapsto & i_A(s) + i_B(s) & &
\end{array}$$

Totally analogous statements also hold if we replace the homology groups of $A \cap B, A, B$ and X by the corresponding reduced homology groups.[742]

The long exact sequence in Theorem 46.5 is usually called the *Mayer–Vietoris sequence* of the decomposition $X = A \cup B$.

Example. In Figure 765 we consider the annulus X together with two closed subsets A and B with $X = \mathring{A} \cup \mathring{B}$. Furthermore we show chains $a \in \mathrm{C}_1(A)$ and $b \in \mathrm{C}_1(B)$ such

[740] Walther Mayer (1887-1948) and Leopold Vietoris (1891-2002) were two Austrian mathematicians. Vietoris was the longest living Austrian man ever.

[741] Here in the expression "∂a" we mean by ∂ the boundary map $\partial \colon \mathrm{C}_n(A) \to \mathrm{C}_{n-1}(A)$. In our case we will see that $\partial a \in \mathrm{C}_{n-1}(A \cap B) \subset \mathrm{C}_{n-1}(A)$. A similar statement holds for ∂b.

[742] Note though that for reduced homology groups we might need to use the fact, obtained in Lemma 43.1 (0), that $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$.

Walther Mayer (1887-1948) with some other person

Leopold Vietoris (1891-2002) at 110

FIGURE 764

that $a+b$ form a cycle in $\mathrm{C}_1(X)$. The connecting homomorphism of the Mayer–Vietoris sequence sends $[a+b] \in \mathrm{H}_1(X)$ to $[\partial a] = -[\partial b] \in \mathrm{H}_0(A \cap B)$.

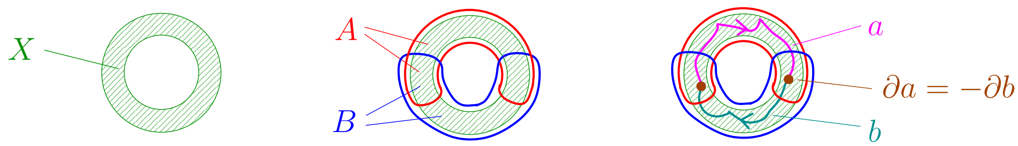


FIGURE 765

Remark.

(1) In many applications some of the inclusions appearing in Mayer–Vietoris Theorem 46.5 are homotopy equivalences, in particular, by Corollary 42.8 many of the inclusion induced maps on homology will be isomorphisms. If that is the case, then it is worth looking up Exercise 41.12 which might greatly simplify the calculations.

(2) Suppose that we are given a topological space X together with an open cover $\{U_i\}_{i\in I}$. If the cover consists of precisely two sets U_1 and U_2, then the Mayer–Vietoris Theorem 46.5 relates the homology groups of X, U_1, U_2 and the intersection $U_1 \cap U_2$. If the open cover consists of more than two sets, then the situation is much more tricky. In this more general setting the homology groups of X are related to the homology groups of the U_i and the homology groups of all finite intersections $U_{i_1} \cap \dots \cap U_{i_m}$ via the "Mayer–Vietoris spectral sequence". We refer to [**LSVJ11, Staf15**], [**Dim04**, Remark 2.3.9], [**TC19**, Chapter 3] and [**BoT82**, Chapter 15] for details. It is also worth having a look at Exercise 46.19.

PROOF. Let X be a topological space and let $A, B \subset X$ be subsets such that $X = \mathring{A} \cup \mathring{B}$. There are two approaches to proving the Mayer–Vietoris Theorem 46.5.

The more elegant approach is as follows: In Exercise 46.1 we prove a purely algebraic statement, namely we construct the algebraic Mayer–Vietoris sequence. In Exercise 46.2 we use this exact sequence together with the Excision Theorem 43.19 and the long exact sequences of pairs of topological spaces to prove the desired statements. This approach allows us to prove Statements (2) and (3) of the Mayer–Vietoris Theorem 46.5 without working again on the "chain level".

In the following we will carry out the second approach. It works on the "chain level" and is thus arguably less conceptual. On the other hand it has the advantage that it is clearer "why it works". After this long preamble, let us remind the reader of the following the notation from page 1133:

$$\mathrm{C}_n^{\{A,B\}}(X) \;:=\; \{a+b \,|\, a \in \mathrm{C}_n(A) \text{ and } b \in \mathrm{C}_n(B)\}.$$

The inclusion maps induce the following sequence of chain maps:

$$0 \to \mathrm{C}_*(A\cap B) \xrightarrow{i_{A\cap B}\oplus -i_{A\cap B}} \mathrm{C}_*(A)\oplus \mathrm{C}_*(B) \xrightarrow{i_A+i_B} \mathrm{C}_*^{\{A,B\}}(X) \to 0.$$

Claim. The above sequence of chain maps is exact.

We prove the claim in the following four steps:

(a) It is clear that each of the maps $\mathrm{C}_*(A\cap B)\to \mathrm{C}_*(A)$ and $\mathrm{C}_*(A\cap B)\to \mathrm{C}_*(B)$ is already injective, in particular the map $i_{A\cap B}\oplus -i_{A\cap B}$ is injective.

(b) It is an immediate consequence of the definition of $\mathrm{C}_*^{\{A,B\}}(X)$ that the map i_A+i_B is surjective.

(c) It follows immediately from the definitions that the composition of both maps is the zero map,[743] i.e. $\mathrm{im}(i_{A\cap B}\oplus -i_{A\cap B})\subset \ker(i_A+i_B)$.

(d) It remains to show that $\ker(i_A+i_B)\subset \mathrm{im}(i_{A\cap B}\oplus -i_{A\cap B})$. Thus let $x\in \mathrm{C}_n(A)$ and $y\in \mathrm{C}_n(B)$ such that $(i_A+i_B)(x+y)=0$. By definition this means that x and y, viewed as elements in $\mathrm{C}_n(X)$, satisfy $x=-y$. This means that x lies in $\mathrm{C}_n(A)$ and also in $\mathrm{C}_n(B)$. Put differently, $x\in \mathrm{C}_n(A)\cap \mathrm{C}_n(B)=\mathrm{C}_n(A\cap B)$. But then we have $(i_{A\cap B}\oplus -i_{A\cap B})(x)=x+y$. ⊞

Next we consider the following maps:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_n(A\cap B) \arrow[r, "i_{A\cap B}\oplus -i_{A\cap B}"] & \mathrm{H}_n(A)\oplus \mathrm{H}_n(B) \arrow[r, "i_A+i_B"] \arrow[dr, "i_A+i_B"'] & \mathrm{H}_n^{\{A,B\}}(X) \arrow[r, "p_n"] \arrow[d, "\Phi_n"] & \mathrm{H}_{n-1}(A\cap B) \arrow[r] & \dots \\
& & & \mathrm{H}_n(X) \arrow[ur, ":=\partial_n"'] & &
\end{tikzcd}$$

We make the following clarifications and observations:

(a) Given $n\in\mathbb{N}$ we denote by $p_n\colon\, \mathrm{H}_n^{\{A,B\}}(X)\to \mathrm{H}_{n-1}(A\cap B)$ the connecting homomorphism of the above short exact sequence of chain complexes, as defined on page 1115.

(b) By Proposition 43.11 the top horizontal sequence is exact.

(c) It follows from Proposition 43.24, together with our hypothesis that $X=\mathring{A}\cup\mathring{B}$, that for every $n\in\mathbb{N}_0$ the inclusion map
$$\mathrm{C}_*^{\{A,B\}}(X) \;\to\; \mathrm{C}_*(X)$$
induces an isomorphism
$$\Phi_n\colon\; \mathrm{H}_n^{\{A,B\}}(X) \;\xrightarrow{\cong}\; \mathrm{H}_n(X).$$

For each $n\in\mathbb{N}_0$ we now write $\partial_n := p_n\circ\Phi_n^{-1}$. It remains to show that these maps have the two desired properties stated in the theorem:

(3) It is clear from the above discussion that the sequence given in (3) is indeed exact.

[743] Note that here we use the minus-sign that appears in the first map.

(1) We leave it to the reader to verify that it follows from the definition of the connecting homomorphism p_n on page 1115 that for every $a \in \mathrm{C}_n(A)$ and $b \in \mathrm{C}_n(B)$ such that $a + b \in \mathrm{C}_n(X)$ is a cycle we have

$$\partial_n([a+b]) \;=\; p_n([a+b]) \;=\; [\partial a] \;=\; -\,[\partial b] \;\in\; \mathrm{H}_{n-1}(A \cap B).$$

Furthermore, the isomorphism $\Phi_n \colon \mathrm{H}_n^{\{A,B\}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ shows that every element in $\mathrm{H}_n(X)$ is of the form $[a+b]$ for some $a \in \mathrm{C}_n(A)$ and $b \in \mathrm{C}_n(B)$, which implies that ∂_n is uniquely determined by the above rule.

(2) It follows basically immediately from the existence and uniqueness of the connecting homomorphisms ∂_n that these maps are also natural.

We leave it to the reader to verify, using Lemma 43.1, that the analogous statements also hold if we replace the homology groups of $A \cap B$, A, B and X by the corresponding reduced homology groups. ■

The following lemma gives an alternative description of the connecting homomorphism of the Mayer–Vietoris sequence.

Lemma 46.6. *Let X be a topological space and let A, B be two open subsets of X with $X = A \cup B$. The connecting homomorphism $\partial_n \colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B)$ equals the map*

$$\mathrm{H}_n(X) \;\xrightarrow{\text{я}}\; \mathrm{H}_n(X,B) \;\underset{\uparrow}{\xleftarrow{\cong}}\; \mathrm{H}_n(A, A \cap B) \;\underset{\uparrow}{\xrightarrow{\partial_n}}\; \mathrm{H}_{n-1}(A \cap B).$$

the inclusion induced map is an isomorphism by the Excision Theorem 43.19 — *connecting homomorphism of the pair $(B, A \cap B)$*

Proof. This equality follows easily from the precise description of the connecting homomorphisms given in the Mayer–Vietoris Theorem 46.5 and in Proposition 43.15. ■

We conclude this section with the following variation on the Mayer–Vietoris Theorem 46.5. This theorem might sound rather unnatural, but it will play an important role in the proof of Theorem 88.19.

Theorem 46.7. (Inverted Mayer–Vietoris Theorem) $(*)$ *Let X be a topological space and let U and V be open subsets. Then for all $n \in \mathbb{N}_0$ there exists a natural homomorphism $\partial_n \colon \mathrm{H}_n(X, U \cup V) \to \mathrm{H}_{n-1}(X, U \cap V)$ such that*

$$\ldots \to \mathrm{H}_n(X, U\cap V) \xrightarrow{i \oplus -i} \mathrm{H}_n(X,U) \oplus \mathrm{H}_n(X,V) \xrightarrow{i+i} \mathrm{H}_n(X, U \cup V) \xrightarrow{\partial} \mathrm{H}_{n-1}(X, U \cap V) \to \ldots$$

is an exact sequence.

Proof $(*)$. Note that as in the proof of Theorem 46.5 we have a short exact sequence

$$0 \;\to\; \mathrm{C}_n(X, U \cap V) \;\to\; \mathrm{C}_n(X,U) \oplus \mathrm{C}_n(X,V) \;\to\; \mathrm{C}_n(X)/\,\mathrm{C}_n^{\{U,V\}}(U \cup V) \;\to\; 0.$$

Thus we get a corresponding long exact sequence of homology groups. It suffices to show that the homology groups of the chain complex on the right are precisely the homology groups of the pair $(X, U \cup V)$. To do so we consider the following commutative pair of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*^{\{U,V\}}(U\cup V) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X)/\,C_*^{\{U,V\}}(U\cup V) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_*(U\cup V) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X, U\cup V) & \longrightarrow & 0. \end{array}$$

We obtain the corresponding long exact sequence of homology groups:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & H_n^{\{U,V\}}(U\cup V) & \longrightarrow & H_n(X) & \longrightarrow & H_n\left(C_*(X)/\,C_*^{\{U,V\}}(U\cup V)\right) & \longrightarrow & H_{n-1}^{\{U,V\}}(U\cup V) & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & H_n(U\cup V) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, U\cup V) & \longrightarrow & H_{n-1}(U\cup V) & \longrightarrow & \ldots \end{array}$$

It follows from Proposition 43.24 that the first and fourth vertical map are isomorphisms. The second map and the fifth map (which for space reasons is not shown) are the identity. Hence it follows from the Five Lemma 43.12 that the third vertical map is also an isomorphism. ■

46.3. **Applications of the Mayer–Vietoris sequence.** As a warm-up we first consider the homology groups of the suspension $\Sigma(X)$ of a topological space X. More precisely, we have the following lemma which basically gives a new proof for Proposition 45.7.

Lemma 46.8. *Let $k \in \mathbb{Z}$. Given any topological space X we have a natural isomorphism*

$$\Sigma_X : \widetilde{H}_k(X) \xrightarrow{\cong} \widetilde{H}_{k+1}(\Sigma(X)).$$

Example. Let $n \in \mathbb{N}_0$. We perform the following calculation:

$$\widetilde{H}_k(S^n) \underset{\substack{\uparrow \\ \text{see page 695}}}{\cong} \widetilde{H}_k(\Sigma(\ldots(\Sigma(S^0)))) \underset{\substack{\uparrow \\ \text{see Lemma 46.8}}}{\cong} \widetilde{H}_{k-n}(S^0) \underset{\substack{\uparrow \\ \text{see page 1092}}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{else.} \end{cases}$$

This recovers the calculation we made in Proposition 43.4.

Remark. Let X be a topological space. In Exercise 46.11 we will see that the two natural isomorphisms $\widetilde{H}_k(\Sigma(X)) \xrightarrow{\cong} \widetilde{H}_{k-1}(X)$ that we constructed in Proposition 45.7 and that we construct in the proof of Lemma 46.8 are actually the same on the nose.

PROOF. Let $k \in \mathbb{Z}$.

(1) Let X be a topological space. We denote by $p\colon X \times [-1,1] \to \Sigma(X)$ the obvious projection. As on page 1178 we consider the subspaces $C_+ := C_+X = p(X \times (-1,1]))$ and $C_- := C_-X = p(X \times [-1,1))$. Note that C_+X and C_-X are both open subsets of $\Sigma(X)$ and that $\Sigma(X) = C_+X \cup C_-X$. We consider the corresponding long exact sequence in reduced homology from the Mayer–Vietoris Theorem 46.5:

$$\begin{array}{ccccccc} \underbrace{\widetilde{H}_{k+1}(C_+)}_{=0} \oplus \underbrace{\widetilde{H}_{k+1}(C_-)}_{=0} & \longrightarrow & \widetilde{H}_{k+1}(\Sigma(X)) & \xrightarrow{\partial_k} & \widetilde{H}_k(C_+ \cap C_-) & \longrightarrow & \underbrace{\widetilde{H}_{k+1}(C_+)}_{=0} \oplus \underbrace{\widetilde{H}_{k+1}(C_-)}_{=0} \\ & & & \nwarrow^{\Sigma_X} & \uparrow i_* & & \\ & & & & \widetilde{H}_k(X). & & \end{array}$$

We make the following clarifications and observations:

(a) Note that by the same argument as in the proof of Lemma 24.1 we see that both $C_+ = C_+X$ and $C_- = C_-X$ admit a deformation retraction to a point. It follows from Lemma 18.14 that both C_+ and C_- are contractible, therefore we know by Lemma 43.1 (7) the reduced homology groups of C_+X and C_-X vanish.

(b) It follows immediately from (1) that the connecting homomorphism

$$\partial\colon \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \;\to\; \widetilde{\mathrm{H}}_k(C_+X \cap C_-X)$$

is in fact an isomorphism.

(c) We consider the following little diagram:

$$\begin{tikzcd}
X \arrow[r, "x\mapsto(x,0)"'] \arrow[rr, bend left, "=:i"] & X\times\{0\} \arrow[r, "p"'] \arrow[d, hook] & C_+X\cap C_-X \\
& X\times(-1,1) \arrow[ur, "p"'] &
\end{tikzcd}$$

The vertical map is evidently a homotopy equivalence. Furthermore one can easily show, either by hand, or using Lemma 3.29, that the diagonal map is a homeomorphism. We obtain that the map $i\colon X \to C_+X \cap C_-X$ is a homotopy equivalence. Thus we obtain from Lemma 43.1 (6) that the induced map $i_*\colon \widetilde{\mathrm{H}}_k(X \times \{0\}) \to \widetilde{\mathrm{H}}_k(C_+X \cap C_-X)$ is an isomorphism.

We define Σ_X to be the map

$$\widetilde{\mathrm{H}}_k(X) \;=\; \widetilde{\mathrm{H}}_k(X \times \{0\}) \;\xrightarrow{i_*}\; \widetilde{\mathrm{H}}_k(C_+X \cap C_-X) \;\xrightarrow{\partial_k^{-1}}\; \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)).$$

(2) It follows easily from the definition of Σ_X and the naturality of the connecting homomorphisms of the Mayer–Vietoris sequence, see Theorem 46.5 (2), that the isomorphism Σ_X is natural. ■

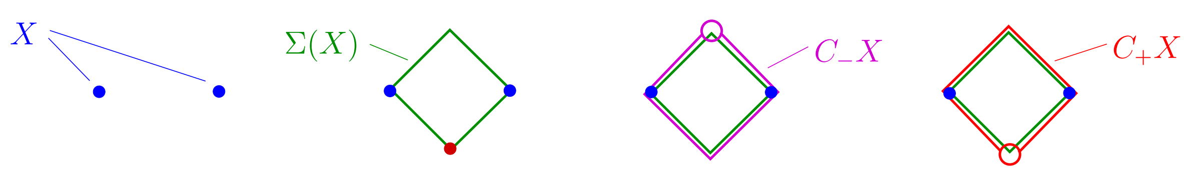


FIGURE 766. Illustration for the proof of Lemma 46.8.

As our next example we want to determine the homology groups of the 2-dimensional projective space $\mathbb{RP}^2$.

Lemma 46.9. *We have*

$$\mathrm{H}_n(\mathbb{RP}^2) \;\cong\; \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}_2, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Remark. In Exercise 46.10 we will determine an explicit cycle that represents the unique non-trivial element of $\mathrm{H}_1(\mathbb{RP}^2)$.

PROOF. In the proof of Lemma 46.9 we use the description from page 196 of $\mathbb{R}\mathrm{P}^2$ as

$$\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\sim \quad \text{where } x \sim y \text{ if } x, y \in \partial\overline{B}^2 \text{ and } x = -y.$$

In the following we denote by p the projection map $\overline{B}^2 \to \overline{B}^2/\sim$. We consider the open subsets

$$A := p\big(\{z \in \overline{B}^2 \,|\, |z| > \tfrac{1}{4}\}\big) \quad \text{and} \quad B := p\big(\{z \in \overline{B}^2 \,|\, |z| < \tfrac{3}{4}\}\big).$$

Evidently we have $\mathring{A} \cup \mathring{B} = A \cup B = \mathbb{R}\mathrm{P}^2$. From the Mayer–Vietoris Theorem 46.5 we obtain the following long exact sequence of reduced homology groups:

$$\dots \to \widetilde{\mathrm{H}}_2(A) \oplus \widetilde{\mathrm{H}}_2(B) \to \widetilde{\mathrm{H}}_2(\mathbb{R}\mathrm{P}^2) \xrightarrow{\partial} \widetilde{\mathrm{H}}_1(A \cap B) \to \widetilde{\mathrm{H}}_1(A) \oplus \widetilde{\mathrm{H}}_1(B) \to \widetilde{\mathrm{H}}_1(\mathbb{R}\mathrm{P}^2) \to \dots$$

Now we have to collect all the available information on the reduced homology groups of

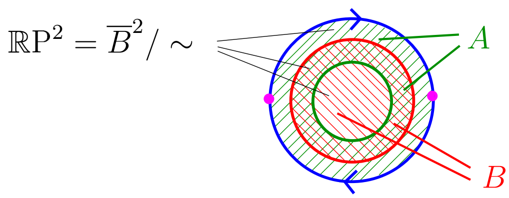


FIGURE 767. Illustration for the proof of Lemma 46.9.

A, B and $A \cap B$. We make the following observations:

(1) We remarked on page 1106 that the reduced homology of a disk vanishes, i.e. we have $\widetilde{\mathrm{H}}_n(B) = 0$ for all $n \in \mathbb{N}_0$.
(2) Note that $p(\frac{1}{2}S^1)$ is a deformation retract of $A \cap B = p(\{z \in \overline{B}^2 \,|\, |z| \in (\frac{1}{4}, \frac{3}{4})\})$. This implies that the map $j\colon S^1 \to A \cap B = p\big(\{z \in \overline{B}^2 \,|\, |z| \in (\frac{1}{4}, \frac{3}{4})\}\big)$ given by $z \mapsto \frac{1}{2}z$ is a homotopy equivalence. It follows from Corollary 42.8 (2) and Proposition 43.4 that $\widetilde{\mathrm{H}}_n(A \cap B) = 0$ for $n \neq 1$ and that $j_*\colon \widetilde{\mathrm{H}}_1(S^1) \to \widetilde{\mathrm{H}}_1(A \cap B)$ is an isomorphism.
(3) We consider the retraction

$$\begin{aligned} r\colon A &\to \{z \in \partial\overline{B}^2\}/z \sim -z \\ p(se^{i\varphi}) &\mapsto e^{i\varphi} \end{aligned}$$

which is easily seen to be a homotopy equivalence and we also consider the homeomorphism

$$\begin{aligned} f\colon \{z \in \partial\overline{B}^2\}/z \sim -z &\to S^1 \\ z &\mapsto z^2. \end{aligned}$$

As in (2) it follows that $\widetilde{\mathrm{H}}_n(A) = 0$ for $n \neq 1$ and that $(f \circ r)_*\colon \widetilde{\mathrm{H}}_1(A) \to \widetilde{\mathrm{H}}_1(S^1)$ is an isomorphism.

The above Mayer–Vietoris sequence is thus in our case of the following form:

$$\begin{tikzcd}[column sep=small]
\dots \arrow[r] & 0 \arrow[r] & \widetilde{\mathrm{H}}_3(\mathbb{R}\mathrm{P}^2) \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \widetilde{\mathrm{H}}_2(\mathbb{R}\mathrm{P}^2) \arrow[r, "\partial"] & \widetilde{\mathrm{H}}_1(A \cap B) \arrow[rr, "i_*"] & & \widetilde{\mathrm{H}}_1(A) \arrow[r, "i_*"] \arrow[d, "\cong"', "(f\circ r)_*"] & \widetilde{\mathrm{H}}_1(\mathbb{R}\mathrm{P}^2) \arrow[r] & 0 \\
& & & & & & \mathrm{H}_1(S^1) \arrow[u, "j_*", "\cong"'] \arrow[rr, "(f\circ r\circ i\circ j)_*", "\cdot \deg(f\circ r\circ i\circ j)"'] & & \mathrm{H}_1(S^1). & &
\end{tikzcd}$$

We obtain immediately that $\widetilde{\mathrm{H}}_n(\mathbb{R}\mathrm{P}^2) = 0$ for $n \geq 3$. Furthermore it follows immediately from the definitions that the map $f \circ r \circ i \circ j \colon S^1 \to S^1$ is given by $z \mapsto z^2$. By Lemma 45.10 this implies that $\deg(f \circ r \circ i \circ j) = 2$. Thus we obtain the following exact sequence

$$0 \;\to\; \widetilde{\mathrm{H}}_2(\mathbb{R}\mathrm{P}^2) \;\to\; \mathbb{Z} \;\xrightarrow{\cdot 2}\; \mathbb{Z} \;\to\; \widetilde{\mathrm{H}}_1(\mathbb{R}\mathrm{P}^2) \;\to\; 0.$$

It follows from the discussion on page 1108 that

$$\widetilde{\mathrm{H}}_2(\mathbb{R}\mathrm{P}^2) \;\cong\; \ker\left(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}\right) \;=\; 0 \quad \text{ and } \quad \widetilde{\mathrm{H}}_1(\mathbb{R}\mathrm{P}^2) \;\cong\; \operatorname{coker}\left(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}\right) \;=\; \mathbb{Z}_2. \qquad \blacksquare$$

Remark. It is a non-trivial, but certainly very amusing exercise to generalize the calculation of Lemma 46.9 to compute the homology groups of any real projective space $\mathbb{R}\mathrm{P}^n$. We cordially invite the reader to try to make an educated guess of what the homology groups of $\mathbb{R}\mathrm{P}^n$ might be. We will compute the homology groups later on in Proposition 48.10 using a somewhat different approach.

46.4. **The Mayer–Vietoris Theorem for manifolds and CW-complexes.** Recall that when we studied fundamental groups we first proved the Seifert–van Kampen Theorem 22.1 for the decomposition of a topological space X into two *open* subsets. Afterwards we showed in Theorem 22.2 and Theorem 37.1 that an analogous statement holds if we decompose a smooth manifold into suitable submanifolds and if we decompose a CW-complex into subcomplexes.

In a similar fashion we now state and prove the "manifold version" and the "CW-complex version" of the Mayer–Vietoris Theorem 46.5.

Theorem 46.10. (Mayer–Vietoris Theorem for Topological Manifolds) *Let X be an m-dimensional topological manifold and let $A, B \subset X$ be two m-dimensional submanifolds such that the following statements hold:*

(a) $X = A \cup B$,
(b) *$A \cap B$ is a union of components of ∂A and it is a union of components of ∂B,*
(c) *A and B are closed subsets of X. (Note that by Lemma 2.17 (2) this condition is satisfied if A and B are compact.)*

The following two statements hold:

(1) *For each $n \in \mathbb{N}_0$ there exists a unique homomorphism $\partial_n \colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B)$ which has the following property: If we are given $a \in \mathrm{C}_n(A)$ and $b \in \mathrm{C}_n(B)$ are such that $a + b \in \mathrm{C}_n(X)$ is a cycle, then*
$$\partial_n([a+b]) \;=\; [\partial a] \;=\; -\,[\partial b] \;\in\; \mathrm{H}_{n-1}(A \cap B).$$
(2) *The following sequence is exact:*
$$\ldots \to \mathrm{H}_n(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i_A + i_B} \mathrm{H}_n(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A \cap B) \to \ldots$$

A totally analogous statement also holds if we replace the homology groups of $A \cap B, A, B$ and X by the corresponding reduced homology groups.

Proof (∗). Let X be an m-dimensional topological manifold and let $A, B \subset X$ be two m-dimensional submanifolds such that$A$ and B are closed subset of M, such that $X = A \cup B$

and such that $A \cap B$ is a union of components of ∂A and also a union of components of ∂B.

We pick maps $f\colon\ [0,1) \times \partial A \to A$ and $g\colon [0,1) \times \partial B \to B$ that are provided by the Topological Collar Neighborhood Theorem 44.5[744] We set

$$U \ := \ A \cup \underbrace{(g([0,1) \times (A \cap B))}_{\subset [0,1) \times B} \quad \text{and} \quad V \ := \ B \cup \underbrace{(f([0,1) \times (A \cap B))}_{\subset [0,1) \times A}.$$

As in the proof of Theorems 20.10 and 22.2 we note that the following statements hold:

(1) U and V are open subsets of X,[745]
(2) A is a deformation retract of U,
(3) B is a deformation retract of V, and
(4) $A \cap B$ is a deformation retract of $U \cap V$.

Now we consider the following diagram

$$\begin{tikzcd}
\dots \arrow[r, dotted] & \mathrm{H}_n(A \cap B) \arrow[r, "i \oplus -i"] \arrow[d] & \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \arrow[r, "i+i"] \arrow[d] & \mathrm{H}_n(X) \arrow[r, dotted, "\partial_n"] \arrow[d, "="] & \mathrm{H}_{n-1}(A \cap B) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \mathrm{H}_n(U \cap V) \arrow[r, "i \oplus -i"] & \mathrm{H}_n(U) \oplus \mathrm{H}_n(V) \arrow[r, "i+i"] & \mathrm{H}_n(X) \arrow[r, "\partial_n"] & \mathrm{H}_{n-1}(U \cap V) \arrow[r] & \dots
\end{tikzcd}$$

where the bottom sequence is given by the Mayer–Vietoris Theorem 46.5. We make the following observations:

(a) The two squares to the left commute since all the relevant maps are given by inclusions.
(b) The vertical maps are isomorphisms by Corollary 42.8.
(c) By (2) we can now define the dotted map $\mathrm{H}_n(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A \cap B)$ such that the third square also commutes.

It now follows from the Mayer–Vietoris Theorem 46.5 that the sequence in (2) is exact. We leave the somewhat technical task of verifying Statement (1) to the reader. ■

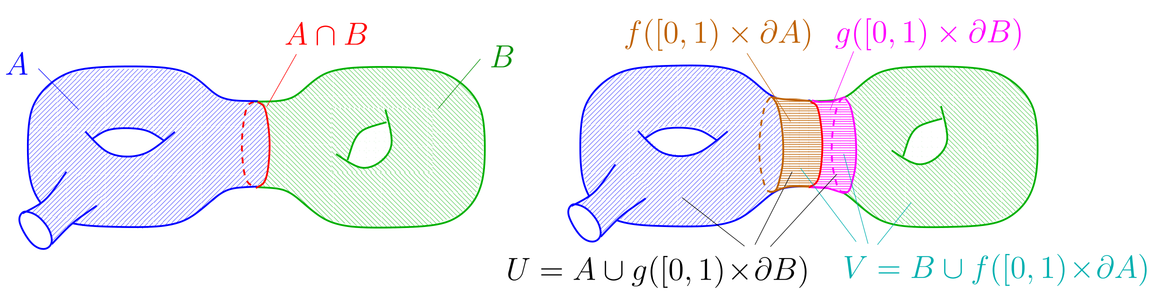


FIGURE 768. Illustration of the proof of Theorem 46.10.

Now we turn to CW-complexes.

[744] Of course, if X is a smooth manifold, then we could also use the Collar Neighborhood Theorem 8.12.
[745] At this point we secretly use our hypothesis that A and B are closed subsets of X.

Theorem 46.11. (Mayer–Vietoris Theorem for CW-complexes) *Let X be a* CW*-complex. If A and B are two subcomplexes of X such that $X = A \cup B$, then exactly the same conclusion as in the Mayer–Vietoris Theorem 46.10 for Manifolds holds.*

PROOF (∗). Let X be a CW-complex and let A and B be two subcomplexes of X with $X = A \cup B$. By Proposition 36.10 (8) there exist open neighborhoods U of A and V of B such that the following statements hold:

(1) A is a deformation retract of U,
(2) B is a deformation retract of V, and
(3) $A \cap B$ is a deformation retract of $U \cap V$.

The proof is now verbatim the same as the proof of the Mayer–Vietoris Theorem 46.10 for Manifolds. ■

Later on in Theorem 74.14 we will generalize the above Mayer–Vietoris theorems to an even more general setting, namely to "excisive triads". But for the time being we will be able to handle most situations with the Mayer–Vietoris theorems that we stated above.

46.5. **The homology groups of the torus and the Klein bottle.** In this section we will determine the homology groups of the torus and the Klein bottle. We do so partly since the results are interesting, but partly also to get to some practice for applying the various Mayer–Vietoris Theorems.

Lemma 46.12. *Let $T = S^1 \times S^1$ be the torus. The following two statements hold.*

(1) *We have the following isomorphisms*

$$\mathrm{H}_n(T) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n = 0, 2, \\ \mathbb{Z}^2, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(2) *The homology groups are generated by the following cycles.*[746]
   (a) *The singular 1-simplices c and d shown in Figure 769 on the left are generators for $\mathrm{H}_1(T) \cong \mathbb{Z}^2$.*
   (b) *The singular 2-chain $\alpha - \beta$ that is given by the singular 2-simplices α and β shown in Figure 769 on the right is a generator for $\mathrm{H}_2(T) \cong \mathbb{Z}$. Note that this singular 2-chain is precisely the one that we had already considered in Figure 704. In particular this answers our question from page 1083, the cycle $\alpha - \beta$ does indeed represent a non-trivial element in $\mathrm{H}_2(T)$.*

Note that Lemma 46.12 (2) shows in particular that the cycles considered in Figure 700 represent non-trivial homology classes.

To increase readability we break the proof of Lemma 46.12 into two parts.

PROOF OF LEMMA 46.12 (1). First we point out that throughout the proof we identify the torus T with $S^1 \times \mathbb{R}/4\mathbb{Z}$ in the obvious way. Next we introduce some notation. We denote by $p\colon S^1 \times \mathbb{R} \to T = S^1 \times \mathbb{R}/4\mathbb{Z}$ the projection map. In the following we write

[746]Here the singular simplices in Figure 769 are understood to be given by the obvious affine maps $\Delta^n \to ([0,1] \times [0,1])/\sim$.

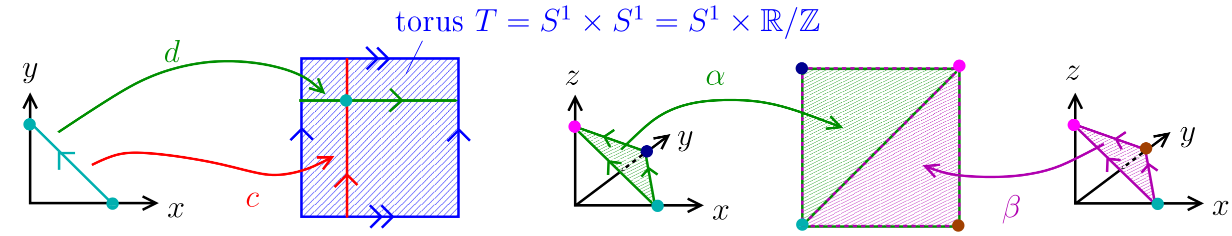


FIGURE 769

$A := p(S^1 \times ([0,1] \cup [3,4]))$ and $B := p(S^1 \times [1,3])$. Evidently we have $T = A \cup B$. Furthermore we write $\mathrm{C}_1 := p(S^1 \times \{1\})$ and $\mathrm{C}_2 := p(S^1 \times \{3\})$. Evidently we have $A \cap B = \mathrm{C}_1 \cup \mathrm{C}_2$.

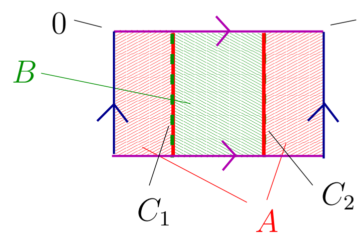


FIGURE 770. Illustration for the proof of Lemma 46.12 (1).

It is fairly easy[747] to verify that the hypotheses for the Mayer–Vietoris Theorem 46.10 for Manifolds are satisfied for $T = A \cup B$. Thus we obtain the following long exact sequence in homology[748]

$$\begin{array}{cccccccc}
 & & \dots & & \to & \mathrm{H}_2(T) & \xrightarrow{\partial} & \\
\to & \begin{array}{c}\mathrm{H}_1(\mathrm{C}_1)\\ \mathrm{H}_1(\mathrm{C}_2)\end{array} & \xrightarrow{\begin{pmatrix} i_{\mathrm{C}_1} & i_{\mathrm{C}_2} \\ -i_{\mathrm{C}_1} & -i_{\mathrm{C}_2}\end{pmatrix}} & \begin{array}{c}\mathrm{H}_1(A)\\ \mathrm{H}_1(B)\end{array} & \xrightarrow{i_A + i_B} & \mathrm{H}_1(T) & \xrightarrow{\partial} & \\
\to & \begin{array}{c}\mathrm{H}_0(\mathrm{C}_1)\\ \mathrm{H}_0(\mathrm{C}_2)\end{array} & \xrightarrow{\begin{pmatrix} i_{\mathrm{C}_1} & i_{\mathrm{C}_2} \\ -i_{\mathrm{C}_1} & -i_{\mathrm{C}_2}\end{pmatrix}} & \begin{array}{c}\mathrm{H}_0(A)\\ H_0(B)\end{array} & \xrightarrow{i_A + i_B} & \mathrm{H}_0(T) & \longrightarrow & 0.
\end{array}$$

Given $x \in [0,4]$ we consider the 0-cycle $Q_x := (1, x)$ and we consider the 1-cycle

$$\begin{array}{rcl} \sigma_x\colon \Delta^1 & \to & S^1 \times \mathbb{R}/4\mathbb{Z} \\ (1-t, t) & \mapsto & e^{2\pi \mathrm{i} t}. \end{array}$$

Let $I \subset \mathbb{R}$ be a compact interval of length less than four. We make the following four observations:

[747]To be completely fair, it might take a few lines to write down a complete argument why A and B are submanifolds of T.

[748]Here we write

$$\begin{array}{c}\mathrm{H}_i(\mathrm{C}_1)\\ \mathrm{H}_i(\mathrm{C}_2)\end{array} \text{ instead of } \mathrm{H}_i(\mathrm{C}_1) \oplus \mathrm{H}_i(\mathrm{C}_2) \quad \text{and} \quad \begin{array}{c}\mathrm{H}_i(A)\\ \mathrm{H}_i(B)\end{array} \text{ instead of } \mathrm{H}_i(A) \oplus \mathrm{H}_i(B).$$

The reason is that we think of direct sums as columns vectors so that we can use the usual matrix notation for describing the maps between the direct sums.

(a) Since the length of I is less than four one obtains almost immediately from Proposition 2.43 (3) that $p\colon S^1 \times I \to p(S^1 \times I)$ is a homeomorphism.
(b) It is a straightforward consequence of Proposition 42.5 that for any two points x, x' in I we have $[Q_x] = [Q_{x'}] \in \mathrm{H}_1(p(S^1 \times I))$ and $[\sigma_x] = [\sigma_{x'}] \in \mathrm{H}_1(p(S^1 \times I))$.
(c) It follows from (1) together with Proposition 41.5 that for each $x \in I$ we have $\mathrm{H}_0(p(S^1 \times I)) = \mathbb{Z} \cdot [\sigma_x]$.
(d) It follows from (1) together with Lemma 45.4 and Proposition 42.5 that for each $x \in I$ we have $\mathrm{H}_1(p(S^1 \times I)) = \mathbb{Z} \cdot [Q_x]$.

It follows from (c) and (d) that we can rewrite the above Mayer–Vietoris sequence as follows:

$$\begin{array}{cccccccc}
 & & & 0 & \to & \mathrm{H}_2(T) & \xrightarrow{\partial} & \\
\to & \begin{matrix} \mathbb{Z} \cdot [\sigma_1] \\ \mathbb{Z} \cdot [\sigma_3] \end{matrix} & \xrightarrow{\begin{pmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2} \end{pmatrix}} & \begin{matrix} \mathbb{Z} \cdot [\sigma_0] \\ \mathbb{Z} \cdot [\sigma_2] \end{matrix} & \xrightarrow{i_A + i_B} & \mathrm{H}_1(T) & \xrightarrow{\partial} & \\
\to & \begin{matrix} \mathbb{Z} \cdot [Q_{x_1}] \\ \mathbb{Z} \cdot [Q_{x_1}] \end{matrix} & \xrightarrow{\begin{pmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2} \end{pmatrix}} & \begin{matrix} \mathbb{Z} \cdot [Q_{x_0}] \\ \mathbb{Z} \cdot [Q_{x_2}] \end{matrix} & \xrightarrow{i_A + i_B} & \mathrm{H}_0(T) & \longrightarrow & 0.
\end{array}$$

To determine the homology groups $\mathrm{H}_2(T)$ and $\mathrm{H}_1(T)$ we now need to understand the maps in the above long exact sequence. For example, it follows immediately from (b) that we have $i_{C_1}([\sigma_1]) = [\sigma_1] = [\sigma_0]$. Basically identical considerations using (b) tell us that we obtain the following long exact sequence:

$$\begin{array}{cccccccc}
 & & & 0 & \to & \mathrm{H}_2(T) & \xrightarrow{\partial} & \\
\to & \begin{matrix} \mathbb{Z} \cdot [\sigma_1] \\ \mathbb{Z} \cdot [\sigma_3] \end{matrix} & \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} & \begin{matrix} \mathbb{Z} \cdot [\sigma_0] \\ \mathbb{Z} \cdot [\sigma_2] \end{matrix} & \xrightarrow{(1\ 1)} & \mathrm{H}_1(T) & \xrightarrow{\partial} & \\
\to & \begin{matrix} \mathbb{Z} \cdot [Q_1] \\ \mathbb{Z} \cdot [Q_3] \end{matrix} & \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} & \begin{matrix} \mathbb{Z} \cdot [Q_0] \\ \mathbb{Z} \cdot [Q_2] \end{matrix} & \xrightarrow{(1\ 1)} & \mathrm{H}_0(T) & \longrightarrow & 0.
\end{array}$$

We will now calculate the homology groups by hand. Alternatively one could now also use Exercise 41.12 to quickly calculate the homology groups. Note that it follows from the discussion on page 1108 that

$$\mathrm{H}_2(T) \;\cong\; \ker\left(\begin{matrix} \mathbb{Z} \\ \mathbb{Z} \end{matrix} \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} \begin{matrix} \mathbb{Z} \\ \mathbb{Z} \end{matrix} \right) \;=\; \mathbb{Z} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \;\cong\; \mathbb{Z}.$$

To determine $\mathrm{H}_1(T)$ we consider the following diagram:

$$\begin{array}{ccccccccc}
0 \to \operatorname{coker}\left(\begin{matrix} \mathbb{Z}\cdot[\sigma_1] \\ \mathbb{Z}\cdot[\sigma_3] \end{matrix}\right. & \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} & \left.\begin{matrix} \mathbb{Z}\cdot[\sigma_0] \\ \mathbb{Z}\cdot[\sigma_2] \end{matrix} \right) & \to & \mathrm{H}_1(T) & \xrightarrow{\partial_1} & \ker\left(\begin{matrix} \mathbb{Z}\cdot[\sigma_0] \\ \mathbb{Z}\cdot[\sigma_2] \end{matrix}\right. & \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} & \left.\begin{matrix} \mathbb{Z}\cdot[Q_1] \\ \mathbb{Z}\cdot[Q_2] \end{matrix} \right) \to 0. \\
 & \cong\uparrow & & & \uparrow\mathrm{id} & & & \mathrm{id}\uparrow & \\
0 \longrightarrow & \begin{matrix} \mathbb{Z}\cdot[\sigma_1] \\ 0 \end{matrix} & & \longrightarrow & \mathrm{H}_1(T) & \xrightarrow{\partial} & & \mathbb{Z}\cdot\begin{pmatrix} [Q_1] \\ -[Q_2] \end{pmatrix} & \longrightarrow 0.
\end{array}$$

It follows from the above discussion and Lemma 46.4 that the top horizontal sequence is exact. Elementary linear algebra shows that the vertical maps are isomorphisms. Finally it is basically clear that the diagram commutes. Since in the bottom the groups to the left and right are both isomorphic to $\mathbb{Z}$ we obtain from Corollary 46.3 that $\mathrm{H}_1(T) \cong \mathbb{Z}^2$. Finally it follows from the above long exact sequence and the discussion on page 1108 that[749]

$$\mathrm{H}_0(T) \;\cong\; \operatorname{coker}\left(\begin{array}{c} \mathbb{Z}\cdot[Q_1] \\ \mathbb{Z}\cdot[Q_3] \end{array} \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} \begin{array}{c} \mathbb{Z}\cdot[Q_0] \\ \mathbb{Z}\cdot[Q_2] \end{array} \right) \;\cong\; \mathbb{Z}.$$

This concludes the proof of statement (1) of the lemma. ■

Now we turn to proof of the second statement of Lemma 46.12.

Proof of Lemma 46.12 (2). We continue with the notation that we introduced in the proof of statement (1) of Lemma 46.12. Next we provide the proof of (2a). Note that $[c] = [\sigma_1] \in \mathrm{H}_1(C)$. Let a and b be the singular simplices in A and B that are indicated in Figure 771. Recall that we have the following short exact sequence:

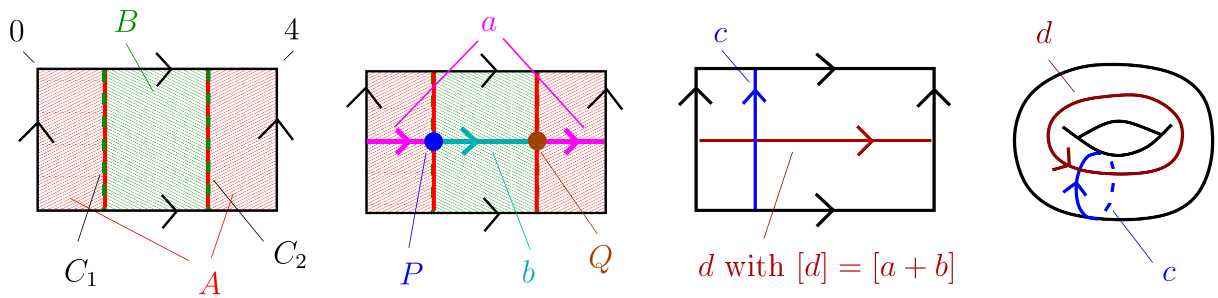


Figure 771

$$0 \longrightarrow \mathbb{Z}\cdot[\sigma_1] \longrightarrow \mathrm{H}_1(T) \xrightarrow{\partial} \mathbb{Z}\cdot\begin{pmatrix} [Q_1] \\ -[Q_2] \end{pmatrix} \longrightarrow 0.$$

Evidently $[c] = [\sigma_1]$ represents a generator of the group on the left-hand side. Furthermore note that $a+b$ is a cycle in T with $a \in \mathrm{C}_1(A)$ and $b \in \mathrm{C}_1(B)$, thus we have

$$\partial_1([a+b]) \underset{\substack{\uparrow \\ \text{see the definition of } \partial_1 \text{ in Theorem 46.10 (1)}}}{=} [\partial a] \;=\; [P-Q] \;=\; \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \begin{array}{l} \mathrm{H}_1(\mathrm{C}_1) = \mathbb{Z} \\ \mathrm{H}_1(\mathrm{C}_2) = \mathbb{Z} \end{array}$$

Thus we see that $\partial_1([a+b])$ is a generator of the group on the right-hand side of the above short exact sequence. It follows from Corollary 46.3 (2) that $[c]$ and $[a+b]$ generate the free abelian group $\mathrm{H}_1(T) \cong \mathbb{Z}^2$. The desired statement now follows from the observation that Lemma 41.3 (3) implies that $[a+b] = [d]$. This concludes the proof of (2a).

Now we turn to the proof of (2b). Let α and β be the two singular 2-simplices that are shown in Figure 769 on the right. It is straightforward to verify that the singular 2-chain $\alpha - \beta$ is indeed a cycle. Note that the proof that $\alpha - \beta$ represents a generator of $\mathrm{H}_2(T)$ is less immediate than the previous argument since the singular simplices α and β do not "fit" into any of the groups of our long exact sequence. In fact, as we will see, the proof that $\alpha - \beta$ represents a generator of $\mathrm{H}_2(T)$ is quite similar to the proof of Lemma 45.3.

[749]Of course we already knew by Proposition 41.5 that $\mathrm{H}_0(T) \cong \mathbb{Z}$.

We denote by A and B the images of $\alpha\colon \Delta^2 \to T$ and $\beta\colon \Delta^2 \to T$. (We refer to Figure 772 for an illustration.) We consider the following maps

$$\begin{tikzcd}
\mathrm{H}_2(T) \arrow[d] \arrow[dr, "{[\alpha-\beta]\mapsto[\alpha-\beta]}"] & & & \mathrm{H}_2(\Delta^2,\partial\Delta^2) \arrow[d, "\cong"'] \arrow[dll, "{[\mathrm{id}_{\Delta^2}]\mapsto[\alpha]}"'] \\
\mathrm{H}_2(T/B) \arrow[r] & \underbrace{\mathrm{H}_2(T/B,B/B)}_{\text{here } [\alpha-\beta]=[\alpha]} & \mathrm{H}_2(A/\partial A,\partial A/\partial A) \arrow[l, "\cong"] & \mathrm{H}_2(\Delta^2/\partial\Delta^2,\partial\Delta^2/\partial\Delta^2). \arrow[l, "\alpha_*"', "\cong"]
\end{tikzcd}$$

We make the following observations:

(i) By Lemma 45.2 we know that $[\mathrm{id}_{\Delta^2}] \in \mathrm{H}_2(\Delta^2,\partial\Delta^2) \cong \mathbb{Z}$ is a generator.
(ii) The two horizontal maps on the right are induced by homeomorphisms of pairs of topological spaces, hence they are isomorphisms.
(iii) The vertical homomorphism on the right is an isomorphism by Proposition 43.22 and Lemma 43.14.
(iv) Note that statements (i), (ii) and (iii) imply that $[\alpha]$ represents a generator of the group $\mathrm{H}_2(T/B,B/B) \cong \mathbb{Z}$.
(v) The singular 2-simplex $\beta\colon \Delta^2 \to B$ is zero in $\mathrm{C}_2(T/B,B/B)$. Therefore we have $[\alpha-\beta] = [\alpha] \in \mathrm{H}_2(T/B,B/B)$.
(vi) Next note that statements (iv) and (v) imply that the image of $[\alpha-\beta] \in \mathrm{H}_2(T) \cong \mathbb{Z}$ under the map $\mathrm{H}_2(T) \to \mathrm{H}_2(T/B,B/B)$ is a generator of $\mathrm{H}_2(T/B,B/B) \cong \mathbb{Z}$. But this implies that $[\alpha-\beta]$ is a generator of $\mathrm{H}_2(T) \cong \mathbb{Z}$. ■

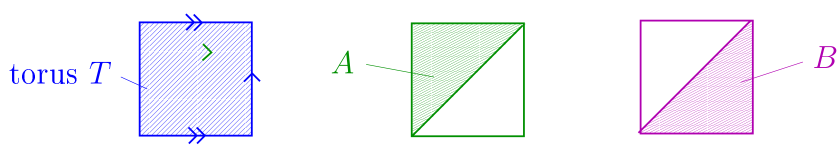


FIGURE 772. Illustration for the proof of Lemma 46.12 (2b).

In Exercise 46.3 we will determine the homology groups of the Klein bottle. More precisely, using the Mayer–Vietoris Theorem 46.10 for Manifolds we will prove the following lemma.

Lemma 46.13. *We denote by K the Klein bottle. Then*

$$\mathrm{H}_n(K) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n=0,\\ \mathbb{Z}\oplus\mathbb{Z}_2, & \text{if } n=1,\\ 0, & \text{otherwise.}\end{cases}$$

Furthermore we can determine explicitly two generators of $\mathrm{H}_1(K)$. More precisely, we consider Figure 773. There the singular 1-simplex d generates the $\mathbb{Z}_2$-summand and the singular 1-simplex c generates a $\mathbb{Z}$-summand of $\mathrm{H}_1(K)$.[750] We leave the verification of this statement to the reader.

[750] It is straightforward to see that c generates an element of infinite order since the "vertical projection" $K \to [0,1]/\sim$ sends c to a generator of $\mathrm{H}_1([0,1]/\sim) = \mathrm{H}_1(S^1)$.

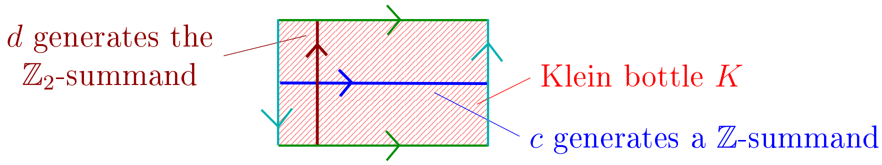


FIGURE 773

Remark. With some effort one could now use the Mayer–Vietoris Theorem 46.10 for Manifolds to compute the homology groups of the surfaces of genus ≥ 2. We will not carry out the argument in this chapter since in Proposition 48.9 we will perform the calculation using an an alternative argument, namely using "cellular homology" which greatly simplifies the calculation. For the impatient reader we state the calculation of the homology groups of surfaces of genus ≥ 2 as Exercise 46.13.

46.6. **The first homology group of a knot complement.** Finally we want to determine the first homology group of the complement of a knot. We recall the relevant definitions from pages 385 and 390.

(1) A *knot* is a 1-dimensional submanifold of S^3 that is diffeomorphic to S^1.
(2) Let K be an oriented knot in S^3. By the Tubular Neighborhood Theorem 8.24 there exists an orientation-preserving smooth embedding $F\colon \overline{B}^2 \times K \to S^3$ such that $\Phi(0, P) = P$ for all $P \in K$. For any $* \in S^1$ we refer to the oriented submanifold $F(S^1 \times \{*\})$ as a *meridian* of the knot K.

We refer to Figure 774 for an illustration of an oriented knot and a meridian. We also introduce one new definition:

(3) Let M be a smooth manifold and let C be a closed oriented curve in M, see page 544 for the definition. Sometimes, by a slight abuse of notation, we denote by C also the element in $\mathrm{H}_1(M)$ that is given by the image of the standard generator $[S^1]$ of $\mathrm{H}_1(S^1)$ under the map $\mathrm{H}_1(S^1) \xrightarrow{\Phi_*} \mathrm{H}_1(C) \to \mathrm{H}_1(M)$.[751]

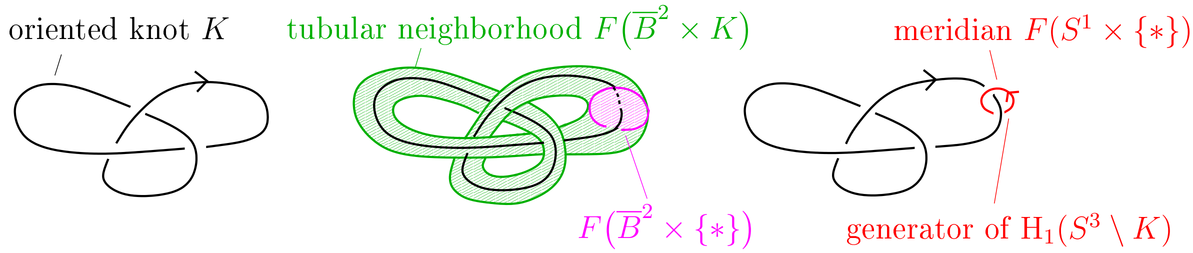


FIGURE 774

Lemma 46.14. *Let $K \subset S^3$ be a knot. Then*[752]
$$\mathrm{H}_0(S^3 \setminus K) \;\cong\; \mathbb{Z} \quad \textit{and} \quad \mathrm{H}_1(S^3 \setminus K) \;=\; \mathbb{Z} \cdot \underset{\substack{\uparrow \\ \text{here } \mu \text{ is any meridian of } K}}{[\mu]}.$$

[751]It follows from Proposition 30.1 and Proposition 42.5 that, up to a sign, the homology class does not depend on the choice of the orientation-preserving diffeomorphism Φ.

Later, in Lemma 68.16, we will compute the remaining homology groups of $S^3 \setminus K$.

Remark. The result of Lemma 46.14 is easy to state but also slightly disappointing since it shows that the first homology groups cannot be used to distinguish different knots. In contrast to this fact we saw in Proposition 27.5 and Proposition 27.6 that the fundamental group can be used to distinguish the trivial knot from the trefoil.

PROOF. Let $K \subset S^3$ be a knot. By the Tubular Neighborhood Theorem 8.24 there exists a smooth embedding $\Phi\colon \overline{B}^2 \times K \to S^3$ with $\Phi(0,P) = P$ for all $P \in K$. To simplify the notation we drop Φ from the notation, in particular we write $\overline{B}^2 \times K$ instead of $\Phi(\overline{B}^2 \times K)$. It is basically clear that the interiors of $S^3 \setminus K$ and $\overline{B}^2 \times K$ cover S^3. Therefore we can apply the Mayer–Vietoris Theorem 46.5 to the decomposition $S^3 = (S^3 \setminus K) \cup (\overline{B}^2 \times K)$ and we obtain the following exact sequence

$$\begin{array}{ccccccccc}
\overbrace{\mathrm{H}_2(S^3)}^{=0} & \longrightarrow & \mathrm{H}_1\big((\overline{B}^2 \setminus \{0\}) \times K\big) & \longrightarrow & \mathrm{H}_1(S^3 \setminus K) \oplus \mathrm{H}_1(\overline{B}^2 \times K) & \longrightarrow & \overbrace{\mathrm{H}_1(S^3)}^{=0} & \longrightarrow & \\
& \longrightarrow & \widetilde{\mathrm{H}}_0((\overline{B}^2 \setminus \{0\}) \times K) & \longrightarrow & \widetilde{\mathrm{H}}_0(S^3 \setminus K) \oplus \widetilde{\mathrm{H}}_0\big(\overline{B}^2 \times K\big) & \longrightarrow & \underbrace{\widetilde{\mathrm{H}}_0(S^3)}_{=0} & \longrightarrow & 0.
\end{array}$$

Now we have to study the homology groups of the various spaces in more detail.[753]

(1) We denote by $[\mu] = S^1 \times \{*\}$ the meridian and we define $[\lambda] = \{*\} \times K$. It follows from Lemma 46.12 (2) that $[\mu]$ and $[\lambda]$ is a basis for $\mathrm{H}_1(S^1 \times K)$. The inclusion $S^1 \times K \to (\overline{B}^2 \setminus \{0\}) \times K$ is a homotopy equivalence. It follows from the above, Proposition 41.5 and Corollary 42.8 that $\mathrm{H}_1\big((\overline{B}^2 \setminus \{0\}) \times K\big) \cong \mathbb{Z} \cdot [\mu] \oplus \mathbb{Z} \cdot [\lambda]$ and that $\widetilde{\mathrm{H}}_0((\overline{B}^2 \setminus \{0\}) \times K) = 0$.

(2) We write $[\gamma] = \{0\} \times K$. The inclusion $\{0\} \times K \to \overline{B}^2 \times K$ is a homotopy equivalence. Similar to (1) we see that $\mathrm{H}_1(\overline{B}^2 \times K) \cong \mathbb{Z} \cdot [\gamma]$ and that $\widetilde{\mathrm{H}}_0(\overline{B}^2 \times K) = 0$.

(3) Under the inclusion induced map

$$\widetilde{\mathrm{H}}_1((\overline{B}^2 \setminus \{0\}) \times K) \;=\; \mathbb{Z} \cdot [\mu] \oplus \mathbb{Z} \cdot [\lambda] \;\to\; \widetilde{\mathrm{H}}_1(\overline{B}^2 \times K) \;=\; \mathbb{Z} \cdot [\gamma]$$

we have $[\mu] \mapsto 0$[754] and we have $[\lambda] \mapsto [\gamma]$.[755].

Thus we obtain an exact sequence of the following form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} \cdot [\mu] \oplus \mathbb{Z} \cdot [\lambda] & \xrightarrow[\cong]{\substack{[\mu] \mapsto ([\mu], 0) \\ [\lambda] \mapsto ([\lambda], [\gamma])}} & \mathrm{H}_1(S^3 \setminus K) \oplus \mathbb{Z} \cdot [\gamma] & \longrightarrow & 0 & \longrightarrow & \\
& \longrightarrow & 0 & \longrightarrow & \widetilde{\mathrm{H}}_0(S^3 \setminus K) \oplus 0 & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

[752]By Proposition 41.5 the first statement just says that $S^3 \setminus K$ is path-connected. This seems obvious from the pictures, but it is nonetheless somewhat non-trivial to prove. For example, as discussed in Corollary 9.14, this statement can be viewed as a consequence of the Transversality Theorem 9.10.

[753]We will do the calculation by hand. Alternatively this calculation is also a good moment to use Exercise 41.12.

[754]This can be seen as follows: the map $\mathrm{H}_1(S^1 \times \{*\}) \to \mathrm{H}_1(\overline{B}^2 \times K)$ factors through $\mathrm{H}_1(\overline{B}^2 \times \{*\}) = 0$.

[755]Why is that?

Now it follows immediately from Corollary 46.3 and Lemma 19.8 (1) that $\mathrm{H}_1(S^3 \setminus K) \cong \mathbb{Z}$ and that $\widetilde{\mathrm{H}}_0(S^3 \setminus K) \cong 0$. Furthermore it follows easily that $[\mu] \in \mathrm{H}_1(S^3 \setminus K) = \mathrm{H}_1(S^3 \setminus K)$ is a generator.[756] ■

Remark.

(1) In the proof of Lemma 46.14 we used the Mayer–Vietoris Theorem 46.5 in a different way than in the previous examples. More precisely, instead of decomposing $S^3 \setminus K$ into smaller pieces, we used that for a decomposition $X = U \cup V$ the Mayer–Vietoris sequence not only relates the homology groups of U, V and $U \cap V$ to the homology groups of X, but also that it relates the homology groups of U to the homology groups of $V, U \cap V$ and X.

(2) The reader might wonder what the approach of Lemma 46.14 can say about the groups $\mathrm{H}_2(S^3 \setminus K)$ and $\mathrm{H}_3(S^3 \setminus K)$. In fact the approach shows that the exists an exact sequence

$$\mathrm{H}_3(S^3 \setminus K) \;\to\; \underbrace{\mathrm{H}_3(S^3)}_{\cong \mathbb{Z}} \;\xrightarrow[=??]{\partial}\; \underbrace{\mathrm{H}_2((\overline{B}^2 \setminus \{0\}) \times K)}_{\cong \mathbb{Z}} \;\to\; \mathrm{H}_2(S^3 \setminus K) \;\to\; 0.$$

Since do not know whether the connecting homomorphism is the zero map, injective or perhaps even an isomorphism we cannot determine $\mathrm{H}_2(S^3 \setminus K)$ and $\mathrm{H}_3(S^3 \setminus K)$. Later, in Chapter 68 will acquire the tools necessary to study this map. This will allow us in Lemma 68.16 to completely determine the isomorphism classes of the homology groups of $S^3 \setminus K$.

The following lemma will be proved in Exercise 46.12.

Lemma 46.15. *Let $X \subset \mathbb{R}^n$ be a non-empty compact subset. We denote by $S^n = \mathbb{R}^n \cup \{\infty\}$ the topological space $\mathbb{R}^n$ with a "point at infinity". Then for any $k \in \mathbb{N}_0$ we have*

$$\mathrm{H}_k(\mathbb{R}^n \setminus X) \;\cong\; \begin{cases} \mathrm{H}_k((\mathbb{R}^n \cup \{\infty\}) \setminus X) \oplus \mathbb{Z}, & \text{if } k = n-1, \\ \mathrm{H}_k((\mathbb{R}^n \cup \{\infty\}) \setminus X), & \text{otherwise.} \end{cases}$$

46.7. **The homology groups of a mapping cone** $(*)$**.** In this section we will discuss the homology groups of a mapping cone. Before we start the discussion we recall the relevant definitions.

(1) On page 692, given a non-empty topological space A we defined the *cone on A* as

$$\mathrm{Cone}(A) \;:=\; (A \times [0,1])/\sim \qquad \text{where } (a,0) \sim (a',0) \text{ for all } a, a' \in A.$$

If $A = \varnothing$, then we defined $\mathrm{Cone}(\varnothing) = \{*\}$ to be the topological space consisting of a single point $*$.

(2) Given a map $f\colon A \to X$ between topological spaces we defined on page 701 the corresponding *mapping cone* as

$$\mathrm{Cone}(f\colon A \to X) \;:=\; (\mathrm{Cone}(A) \sqcup X)/\sim \;\text{ where } (a,1) \sim f(a) \text{ for all } a \in A.$$

[756] Why does this follow?

We refer to $[A\times 0]$ respectively $*$ as the *cone point* of $\mathrm{Cone}(A)$ and of $\mathrm{Cone}(i\colon A\to X)$.

The following lemma gives an elegant calculation of the homology groups of a mapping cone of an inclusion.

Lemma 46.16.

(1) *Let X be a topological space and let $A\subset X$ be a subset. We denote by $i\colon A\to X$ the inclusion map. For any $n\in\mathbb{N}_0$ the two maps*

$$\begin{array}{ccccc} \mathrm{H}_n(X,A) & \underset{\substack{\uparrow \\ \text{induced by the embedding } X\to\mathrm{Cone}(i\colon A\to X)}}{\to} & \mathrm{H}_n(\mathrm{Cone}(i\colon A\to X),\mathrm{Cone}(A)) & \xleftarrow{\text{ж}} & \widetilde{\mathrm{H}}_n(\mathrm{Cone}(i\colon A\to X)) \end{array}$$

are isomorphisms.

(2) *We denote by $\mathcal{PairTop}$ the category of pairs of topological spaces and we denote by $\mathcal{AbGr}$ the category of abelian groups. The composition of the two isomorphisms in (1) define a natural isomorphism from the functor*[757]

$$\begin{array}{rcl} \mathcal{PairTop} & \to & \mathcal{AbGr} \\ (X,A) & \mapsto & \widetilde{\mathrm{H}}_n(\mathrm{Cone}(i\colon A\to X)) \end{array} \quad \text{to the functor} \quad \begin{array}{rcl} \mathcal{PairTop} & \to & \mathcal{AbGr} \\ (X,A) & \mapsto & \mathrm{H}_n(X,A). \end{array}$$

Proof.

(1) Just for kicks we throw in one more map:

$$\begin{tikzcd} & \mathrm{H}_n\big(\mathrm{Cone}(A\xrightarrow{i}X)\setminus\{P\},\mathrm{Cone}(A)\setminus\{P\}\big) \arrow[d, "(b)"] & \\ \mathrm{H}_k(X,A) \arrow[ur, "(a)"] \arrow[r] & \mathrm{H}_k(\mathrm{Cone}(i\colon A\to X),\mathrm{Cone}(A)) & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(i\colon A\to X)). \arrow[l, "\text{ж}", "(c)"'] \end{tikzcd}$$

Since the diagram commutes it suffices to show that the maps (a), (b) and (c) are isomorphisms.

(a) From a slight variation on Lemma 24.8 (1) we obtain a deformation retraction from $\mathrm{Cone}(i\colon A\to X)\setminus\{P\}$ to X which restricts to a deformation retraction from $\mathrm{Cone}(A)$ to A. Thus it follows from Exercise 43.15 that the map (a) is an isomorphism.

(b) It follows from the Excision Theorem 43.19 that the vertical map (b) is an isomorphism.

(c) By Lemma 24.1 we know that the cone $\mathrm{Cone}(A)$ is contractible. It follows from Corollary 43.16 that the map (c) is an isomorphism.

(2) It is clear that the two maps in (1) are natural. Thus it follows that the two maps together define a natural isomorphism. ■

Remark.

(1) It is worth comparing Lemma 46.16 to Lemma 43.14 which says that given a pointed topological space (X,x_0) the reduced homology groups $\widetilde{\mathrm{H}}_i(X)$ are naturally isomorphic to the relative homology groups $\mathrm{H}_i(X,\{x_0\})$.

[757]As in Lemma 24.10 (3) we see that the left-hand side does indeed define a functor.

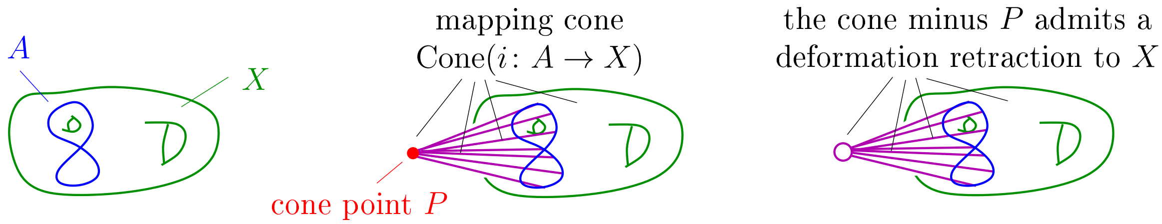


FIGURE 775. Illustration for the proof of Lemma 46.16.

(2) In Proposition 43.22 we showed that, under favorable circumstances, we can replace the relative homology group $\mathrm{H}_n(X, A)$ by the homology group $\widetilde{\mathrm{H}}_n(X/A)$. Lemma 46.16 says that we can replace any relative homology group by the homology group of the mapping cone of the inclusion.
(3) Loosely speaking, Lemma 46.16 says that for all intents and purposes we can usually replace the relative homology groups of a pair (X, A) of topological spaces by the reduced homology of the mapping cone of the inclusion $A \to X$.

PROOF. We had just seen that the map is an isomorphism. It is straightforward to verify that the maps define a natural transformation. ■

We obtain the following convenient corollary which is a very close relative of Proposition 43.22.

Corollary 46.17. *Let X be a topological space and let $A \subset X$ be a subset. We denote by $i\colon A \to X$ the inclusion map. If i is a closed cofibration, then for any $n \in \mathbb{N}_0$ the natural map* ю: $\mathrm{H}_n(X, A) \to \widetilde{\mathrm{H}}_n(X/A)$ *from page 1129 is an isomorphism.*

PROOF. The case that $A = \varnothing$ is basically trivial and is left to the reader. In the following we assume that $A \subset X$ is a non-empty subset. Let $n \in \mathbb{N}_0$. We consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_n(X,A) \arrow[rr, "\text{ю}"] \arrow[dr, "\cong"'] & & \widetilde{\mathrm{H}}_n(X/A) \\
& \widetilde{\mathrm{H}}_n(\mathrm{Cone}(i\colon A \to X)). \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

We make the following clarifications and observations:

(1) The top horizontal map is the natural homomorphism that we introduced on page 1129.
(2) The left diagonal map is the natural *isomorphism* of Lemma 46.16.
(3) The right diagonal map is the map

$$\begin{array}{rcl}\varphi\colon \mathrm{Cone}(i\colon A \to X) & \to & X/A \\ P & \mapsto & \begin{cases} [P], & \text{if } P \in X, \\ [A], & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1] \end{cases}\end{array}$$

that we introduced in Lemma 39.9 (1).
(4) It follows easily from the definitions that the diagram commutes.

(5) Since $i\colon A \to X$ is by hypothesis a closed cofibration we obtain from Lemma 39.9 (1) that φ is a homotopy equivalence. Therefore we obtain from Corollary 42.8 that the right diagonal map is also an isomorphism.

(6) Putting everything together we see that the top map is indeed an isomorphism. ■

Let A be a subset of a topological space X. We denote by $i\colon A \to X$ the corresponding inclusion map. It follows from Corollary 43.16 together with Lemma 46.16 that the inclusion induced maps $i_*\colon \mathrm{H}_k(A) \to \mathrm{H}_k(X)$ fit into the following natural long exact sequence

$$\dots \to \mathrm{H}_k(A) \xrightarrow{i_*} \mathrm{H}_k(X) \to \underbrace{\mathrm{H}_k(X,A)}_{\cong \widetilde{\mathrm{H}}_k(\mathrm{Cone}(i\colon A\to X))} \xrightarrow{\partial} \mathrm{H}_{k-1}(A) \xrightarrow{i_*} \mathrm{H}_{k-1}(X) \to \dots$$

The following lemma generalizes this neat fact to the setting of any map $f\colon A \to X$ between topological spaces.

Lemma 46.18. [758]

(1) *Let $f\colon A \to X$ be a map between topological spaces.*
   (a) *We denote by $j(f)\colon X \to \mathrm{Cone}(f) = (\mathrm{Cone}(A) \sqcup X)/\sim$ the obvious inclusion map.*
   (b) *We denote by U the image of $A \times [0, \frac{3}{4})$ in $\mathrm{Cone}(f)$ and we denote by V the complement of the image of $A \times [0, \frac{1}{4}]$. (See also Figure 776.) We consider the following diagram*[759]

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_k(U)\oplus \mathrm{H}_k(V) \arrow[r] & \mathrm{H}_k(\mathrm{Cone}(f)) \arrow[r, "\partial_k"] & \mathrm{H}_{k-1}(U\cap V) \arrow[r] & \dots \\
 & & \mathrm{H}_k(\mathrm{Cone}(f)) \arrow[u, "\text{ж}"] \arrow[r, "\partial_k", "=:\partial_k(f)"'] & \mathrm{H}_{k-1}(A). \arrow[u, "\cong"'] &
\end{tikzcd}$$

Here the top sequence is the long exact sequence arising from the Mayer–Vietoris Theorem 46.5[760] *and the vertical map is induced by the inclusion $a \mapsto (a, \frac{1}{2})$ which is a homotopy equivalence. From the above diagram we obtain a map*

$$\partial_k(f)\colon \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f)) \to \mathrm{H}_{k-1}(A).$$

With these definitions the following sequence is exact

$$\dots \longrightarrow \mathrm{H}_k(A) \xrightarrow{f_*} \mathrm{H}_k(X) \xrightarrow{j(f)_*} \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f)) \xrightarrow{\partial_k(f)} \mathrm{H}_{k-1}(A) \longrightarrow \dots$$

(2) *The connecting homomorphism in (1) is natural. In particular given any commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\varphi"'] & X \arrow[d, "\psi"] \\
\widetilde{A} \arrow[r, "\widetilde{f}"] & \widetilde{X}
\end{tikzcd}$$

of maps between topological spaces the following diagram commutes:

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}_k(A) \arrow[r, "f_*"] \arrow[d, "\varphi_*"] & \mathrm{H}_k(X) \arrow[r, "j(f)_*"] \arrow[d, "\psi_*"] & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f)) \arrow[r, "\partial_k(f)"] \arrow[d] & \mathrm{H}_{k-1}(A) \arrow[r] \arrow[d, "\varphi_*"] & \dots \\ \dots \arrow[r] & \mathrm{H}_k(\widetilde{A}) \arrow[r, "\widetilde{f}_*"] & \mathrm{H}_k(\widetilde{X}) \arrow[r, "j(\widetilde{f})_*"] & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(\widetilde{f})) \arrow[r, "\partial_k(\widetilde{f})"] & \mathrm{H}_{k-1}(\widetilde{A}) \arrow[r] & \dots . \end{tikzcd}$$

Here the vertical maps between the homology groups of the mapping cones are induced by the maps from Lemma 24.10 (3).

(3) *Let X be a topological space and let $A \subset X$ be a subset. We denote by $i\colon A \to X$ the inclusion map. The following diagram*

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}_k(A) \arrow[r, "i"] \arrow[d, "="] & \mathrm{H}_k(X) \arrow[r] \arrow[d, "="] & \mathrm{H}_k(X,A) \arrow[r, "\partial"] \arrow[d, "\cong"] & \widetilde{\mathrm{H}}_{k-1}(A) \arrow[r] \arrow[d, "="] & \widetilde{\mathrm{H}}_{k-1}(X) \arrow[r] \arrow[d, "="] & \dots \\ \dots \arrow[r] & \mathrm{H}_k(A) \arrow[r, "i"] & \mathrm{H}_k(X) \arrow[r] & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(i)) \arrow[r, "\partial"] & \widetilde{\mathrm{H}}_{k-1}(A) \arrow[r] & \widetilde{\mathrm{H}}_{k-1}(X) \arrow[r] & \dots \end{tikzcd}$$

commutes. Here the middle vertical map is the natural isomorphism from Lemma 46.16, the top sequence is the long exact sequence arising from Corollary 43.16 and the bottom sequence is the exact sequence from (1).

The obvious analogues of the above statements for reduced homology of A and X also hold.

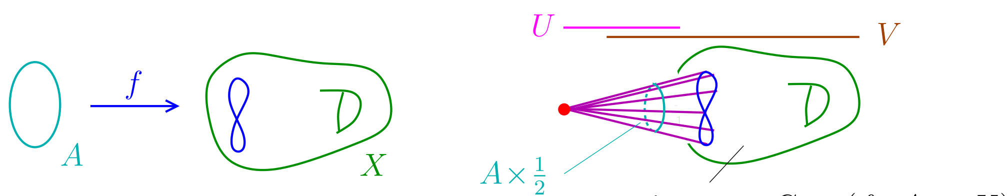


FIGURE 776. Illustration of Lemma 46.18.

PROOF.

[758]In the statement and the proof of the lemma we implicitly use Lemmas 24.2 and 24.9 (2) several times to show that various maps are open and closed embeddings. For example it follows from Lemma 24.2 that the map $j(f)$ given in (1a) is actually a closed embedding.

[759]We leave it to the reader to give a reasonable interpretation of what we say here in the special case that $A = \varnothing$.

[760]Note that U and V are open subsets of $\mathrm{Cone}(f)$.

(1) We consider the following diagram:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_k(U\cap V) \arrow[r] & \mathrm{H}_k(U) \arrow[r, phantom, "\oplus"] & \mathrm{H}_k(V) \arrow[r] & \mathrm{H}_k(\mathrm{Cone}(f)) \arrow[r, "\partial_k"] & \mathrm{H}_{k-1}(U\cap V) \arrow[r] & \dots \\
\dots \arrow[r] & \mathrm{H}_k(A) \arrow[r, "c_*\oplus f_*"] \arrow[u, "\cong"] & \mathrm{H}_k(\{*\}) \arrow[r, phantom, "\oplus"] \arrow[u, "\cong"] & \mathrm{H}_k(X) \arrow[r] \arrow[u, "\cong"'] & \mathrm{H}_k(\mathrm{Cone}(f)) \arrow[r] \arrow[u, "\mathrm{id}"] & \mathrm{H}_{k-1}(A) \arrow[r] \arrow[u, "\cong"'] & \dots \\
\dots \arrow[r] & \mathrm{H}_k(A) \arrow[r, "f_*"] \arrow[u, "\cong"] & \underbrace{\widetilde{\mathrm{H}}_k(\{*\})}_{=0} \arrow[r, phantom, "\oplus"] \arrow[u, "\cong"] & \mathrm{H}_k(X) \arrow[r, "j(f)_*"] \arrow[u, "ж"'] & \widetilde{\mathrm{H}}_k(\mathrm{Cone}(f)) \arrow[r, "\partial_k(f)"] \arrow[u, "ж"] & \mathrm{H}_{k-1}(A) \arrow[r] \arrow[u, "\cong"'] & \dots
\end{tikzcd}$$

We make the following clarifications and observations:

(a) The top sequence is the long exact sequence arising from the Mayer–Vietoris Theorem 46.5 applied to the open cover $\mathrm{Cone}(f) = U \cup V$.

(b) We denote by $*$ the cone point of $\mathrm{Cone}(f)$.

(c) The vertical maps are the almost obvious inclusion maps. The map $c\colon A \to \{*\}$ is the constant map. It follows almost immediately from the definitions together with Proposition 42.5 that the upper part of the diagram commutes.

(d) As in Lemma 24.1 we see that the inclusion $\{*\} \to U$ is a homotopy equivalence. Furthermore, by Lemma 43.1 (7) we know that $\widetilde{\mathrm{H}}_k(\{*\}) = 0$.

(e) It follows now fairly easily from Lemma 43.2 that the lower part of the diagram also commutes and that the lower sequence is exact. But that is what we wanted to show.

(2) This statement follows from the naturality of the connecting homomorphisms of the Mayer–Vietoris sequence, see Theorem 46.5 (2).

(3) We will prove this statement in Exercise 46.18. ■

In Proposition 37.11 we saw how attaching an n-cell to a topological space affects its fundamental group. Basically for free we now get a corresponding statement for homology groups.

Lemma 46.19. *Let X be a topological space, let $n \geq 2$ and let $\varphi\colon S^{n-1} \to X$ be a map. We denote by $i\colon X \to X \cup_\varphi \overline{B}^n$ the obvious map. (Note that by Lemma 3.43 we know that i is a closed embedding.) There exists a natural exact sequence of the form*

$$0 \to \mathrm{H}_n(X) \xrightarrow{i_*} \mathrm{H}_n\big(X \cup_\varphi \overline{B}^n\big) \to \underbrace{\mathrm{H}_{n-1}(S^{n-1})}_{\cong\mathbb{Z}} \xrightarrow{\varphi_*} \mathrm{H}_{n-1}(X) \xrightarrow{i_*} \mathrm{H}_{n-1}\left(X \cup_\varphi \overline{B}^n\right) \to 0.$$

Furthermore, for any $k \neq n-1, n$ the inclusion induced map

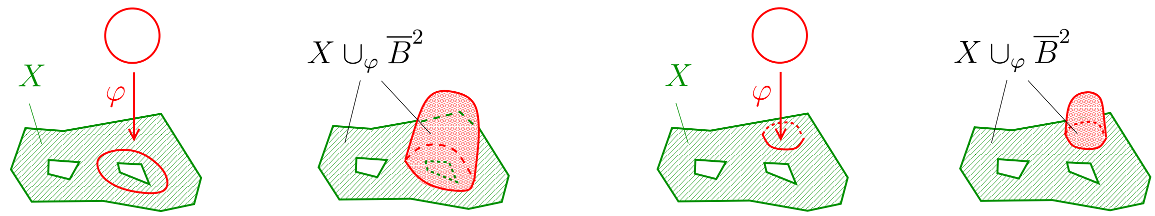
$$i_*\colon \mathrm{H}_k(X) \xrightarrow{\cong} \mathrm{H}_k(X \cup_\varphi \overline{B}^n)$$

is an isomorphism.

Example. Let X be a topological space with finitely generated homology groups. Let $n \geq 2$ and let $\varphi\colon S^{n-1} \to X$ be a map. Loosely speaking Lemma 46.19 says that gluing on a copy of $\overline{B}^n$ along φ has the following effect on homology:

	H_n	H_{n-1}
$\mathrm{H}_{n-1}(S^{n-1}) \xrightarrow{\varphi_*} \mathrm{H}_{n-1}(X)$ has infinite image	unchanged	rank goes down by 1
$\mathrm{H}_{n-1}(S^{n-1}) \xrightarrow{\varphi_*} \mathrm{H}_{n-1}(X)$ has finite image	rank goes up by 1	unchanged.

We illustrate these two different behaviors in Figure 777.



$\varphi_*\colon\ \mathrm{H}_1(S^1) \to \mathrm{H}_1(X)$ has infinite image $\qquad$ $\varphi_*\colon\ \mathrm{H}_1(S^1) \to \mathrm{H}_1(X)$ has trivial image

FIGURE 777

PROOF. On page 991 we made the, almost trivial, observation that there exists a homeomorphism $\mathrm{Cone}(\varphi\colon S^{n-1} \to X) \xrightarrow{\cong} X \cup_\varphi \overline{B}^n$ which is the identity on X. The conclusion of Lemma 46.19 follows immediately from this observation together with Lemma 46.18, the calculation of the homology groups of S^{n-1}, see Proposition 43.4 and the observation that our hypothesis that $n \geq 2$ implies that we do not need to worry about reduced homology groups. ■

The last lemma of this section gives a neat reinterpretation of the connecting homomorphism in the long exact sequence of Lemma 46.18.

Lemma 46.20. (∗) *Let $f\colon A \to X$ be a map between topological spaces. For every $k \in \mathbb{N}_0$ the following diagram commutes:*

$$\begin{tikzcd}
\widetilde{\mathrm{H}}_k(\mathrm{Cone}(f\colon A \to X)) \arrow[r, "\partial"] \arrow[d] & \widetilde{\mathrm{H}}_{k-1}(A) \arrow[d, "\Sigma_A", "\cong"'] \\
\widetilde{\mathrm{H}}_k(\mathrm{Cone}(f\colon A \to X)/X) \arrow[r, "\varphi_*", "\cong"'] & \widetilde{\mathrm{H}}_k(\Sigma(A)).
\end{tikzcd}$$

Here we used the following notation:

(1) *The top horizontal map is the connecting homomorphism of the long exact sequence given in Lemma 46.18.*
(2) *The left vertical map is induced by the obvious projection.*
(3) *The map $\varphi\colon \mathrm{Cone}(f\colon A \to X)/X \to \Sigma(A)$ is the natural homeomorphism from Lemma 24.9 (4).*
(4) *The map $\Sigma_A\colon \widetilde{\mathrm{H}}_{k-1}(A) \to \widetilde{\mathrm{H}}_k(\Sigma(A))$ is the natural isomorphism from Lemma 46.8.*

PROOF (∗). The statement follows immediately from the explicit definition of the maps ∂ and Σ_A via Mayer–Vietoris sequences, from the explicit definition of the map φ, and from the naturality of the connecting homomorphism of the Mayer–Vietoris sequence, see

Theorem 46.5 (2), that the isomorphism Σ_X is natural. We leave it to the reader to fill in the details. ■

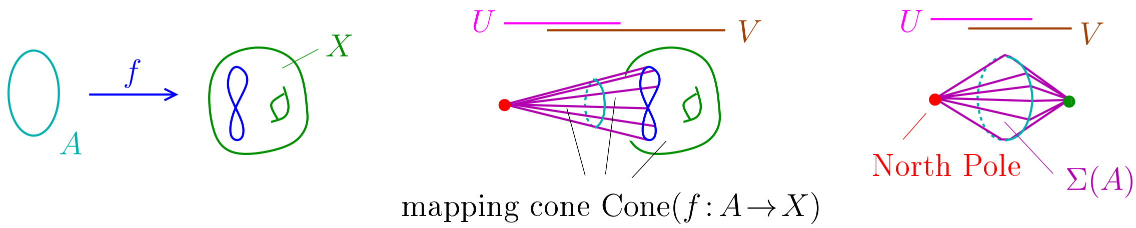


FIGURE 778. Illustration for the proof of Lemma 46.20.

46.8. **The homology groups of a mapping torus** $(*)$. On page 711 we defined the mapping torus of a self-homeomorphism and in Proposition 24.28 we had determined the fundamental group of the mapping torus. In this section we will determine the isomorphism types of the homology groups of a mapping torus.

For convenience we recall from page 711 the definition of the mapping torus. Let X be a topological space and let $f\colon X \to X$ be a map. We refer to

$$\mathrm{Tor}(X,f) \;:=\; (X \times [0,1])/(x,0) \sim (f(x),1)$$

as the *mapping torus of* (X,f). In the following we will use on several occasions Lemma 24.19 which says that for any closed respectively open interval $I \subset \mathbb{R}$ of length less than one, the map

$$\varphi\colon X \times I \xrightarrow{(x,t)\mapsto[(x,t)]} (X \times \mathbb{R})/\sim \xrightarrow{[(x,t)]\mapsto[(x,t)]} \mathrm{Tor}(X,f)$$

is a closed respectively open embedding. Now we can state the main result of this section.

Proposition 46.21. *Let X be a topological space and let $f\colon X \to X$ be a map. For every $n \in \mathbb{N}_0$ we have a natural short exact sequence*[761]

$$0 \to \mathrm{coker}\big(\mathrm{H}_n(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_n(X)\big) \xrightarrow{i} \mathrm{H}_n(\mathrm{Tor}(X,f)) \to \ker\big(\mathrm{H}_{n-1}(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_{n-1}(X)\big) \to 0,$$

where the map i is induced by the embedding $X \xrightarrow{x\,\mapsto\,[(x,0)]} \mathrm{Tor}(X,f)$.

Example. First let $X = S^1$ and let $f\colon S^1 \to S^1$ be the self-homeomorphism that is given by $f(z) = \overline{z} = z^{-1}$. On page 713 we saw that the mapping torus $\mathrm{Tor}(S^1,f)$ is homeomorphic to the Klein bottle. It follows from the discussion on page 1088 the induced map $f_*\colon \mathrm{H}_0(S^1) \to \mathrm{H}_0(S^1)$ is the identity map. Furthermore we showed in Lemma 45.10 that the induced homomorphism $f_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1)$ is given by multiplication by -1. Since $\mathrm{H}_0(S^1) \cong \mathbb{Z}$ and $\mathrm{H}_1(S^1) \cong \mathbb{Z}$ we now see that the short exact sequences of Proposition 46.21 are of the form

$$\begin{array}{lccccccccc}
n=2\colon & & & & 0 & \to & \mathrm{H}_2(\mathrm{Tor}(S^1,f)) & \to & \ker(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}) & \to & 0 \\
n=1\colon & 0 & \to & \mathrm{coker}\,\big(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}\big) & \to & \mathrm{H}_1(\mathrm{Tor}(S^1,f)) & \to & \ker(\mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z}) & \to & 0
\end{array}$$

and

$$\begin{array}{lccccccc}
n=0\colon & 0 & \to & \mathrm{coker}\,\big(\mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z}\big) & \to & \mathrm{H}_0(\mathrm{Tor}(S^1,f)) & \to & 0.
\end{array}$$

[761]Recall that $\mathrm{H}_{-1}(X)$ is defined to be the trivial group.

Using Corollary 46.3 we now deduce that

$$\mathrm{H}_2(\mathrm{Tor}(S^1,f)) \;=\; 0, \quad \mathrm{H}_1(\mathrm{Tor}(S^1,f)) \;\cong\; \mathbb{Z}_2\oplus\mathbb{Z} \quad \text{and} \quad \mathrm{H}_0(\mathrm{Tor}(S^1,f)) \;=\; \mathbb{Z}.$$

This confirms, as it should, the calculation that we had made in Lemma 46.13.

Proof. The proof of the proposition is quite similar to the calculation of the homology groups of the torus.[762] Let X be a topological space and let $f\colon X\to X$ be a homeomorphism. We compute the homology of the following space that is clearly naturally homeomorphic to the mapping torus:

$$M \;:=\; (X\times[0,10])/(x,0)\sim(f(x),10).$$

We denote by $p\colon X\times[0,10]\to M$ the obvious projection map. Furthermore we write $A:=p(X\times([0,4)\cup(6,10]))\subset M$ and $B:=p(X\times(2,8))\subset M$.[763] Finally, given $t\in[0,10]$ we denote by

$$\begin{array}{rcl} i_t\colon X & \to & p(X\times\{t\}) \\ x & \mapsto & p(x,t) \end{array}$$

the obvious inclusion map. We write $i_{3,A}$ for the composition of i_3 with the inclusion map $p(X\times\{3\})\to A$. We denote analogous compositions by the obvious variations in the notation. The proof below crucially relies on the following claim.

Claim. For any $n\in\mathbb{N}_0$ the following two diagrams commute

$$\begin{tikzcd} \mathrm{H}_n(X) \arrow[rr, "\mathrm{id}"] \arrow[dr, "i_{3,B}"'] & & \mathrm{H}_n(X) \arrow[dl, "i_{7,B}"] \\ & \mathrm{H}_n(B) & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{H}_n(X) \arrow[rr, "f_*^{-1}"] \arrow[dr, "i_{3,A}"'] & & \mathrm{H}_n(X). \arrow[dl, "i_{7,A}"] \\ & \mathrm{H}_n(B) & \end{tikzcd}$$

It follows almost immediately from Proposition 42.5 that the two triangles commute. We leave the details to the reader. ⊞

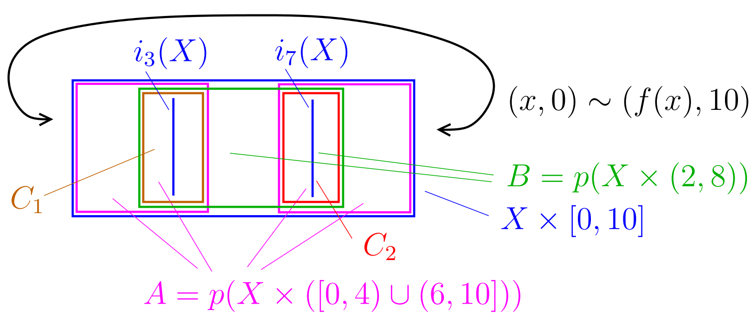


Figure 779. Illustration for the proof of Proposition 46.21.

[762] In fact if one reads the subsequent proof with $X=S^1$ and $f=\mathrm{id}$ in mind, then one obtains almost the same argument as for the torus.

[763] Throughout this proof we will freely use Lemma 24.19 which says that $p\colon X\times[0,1]\to M$, restricted to reasonable open subsets, is in fact an open embedding.

Next we consider the following diagram:[764]

$$
\begin{tikzcd}[column sep=small]
\to & \begin{matrix}\mathrm{H}_n(C_1)\\ \mathrm{H}_n(C_2)\end{matrix} \arrow[r, "{\begin{pmatrix} i_{C_1} & i_{C_2}\\ -i_{C_1} & -i_{C_2}\end{pmatrix}}"] & \begin{matrix}\mathrm{H}_n(A)\\ \mathrm{H}_n(B)\end{matrix} \arrow[r, "{(i_A\ i_B)}"] & \mathrm{H}_n(M) \arrow[r, "\partial"] & \begin{matrix}\mathrm{H}_{n-1}(C_1)\\ \mathrm{H}_{n-1}(C_2)\end{matrix} \arrow[r, "{\begin{pmatrix} i_{C_1} & i_{C_2}\\ -i_{C_1} & -i_{C_2}\end{pmatrix}}"] & \begin{matrix}\mathrm{H}_{n-1}(A)\\ H_{n-1}(B)\end{matrix} & \to \\
\to & \begin{matrix}\mathrm{H}_n(X)\\ \mathrm{H}_n(X)\end{matrix} \arrow[u, "{\begin{pmatrix} i_3 & 0\\ 0 & i_7\end{pmatrix}}"] \arrow[r, "{\begin{pmatrix} \mathrm{id} & f_*\\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}}"] & \begin{matrix}\mathrm{H}_n(X)\\ \mathrm{H}_n(X)\end{matrix} \arrow[u, "{\begin{pmatrix} i_3 & 0\\ 0 & i_7\end{pmatrix}}"'] \arrow[r, "{(i_A\ i_B)}"] & \mathrm{H}_n(M) \arrow[u, "\mathrm{id}"] \arrow[r, "\partial"] & \begin{matrix}\mathrm{H}_{n-1}(X)\\ \mathrm{H}_{n-1}(X)\end{matrix} \arrow[u, "{\begin{pmatrix} i_3 & 0\\ 0 & i_7\end{pmatrix}}"] \arrow[r, "{\begin{pmatrix} \mathrm{id} & f_*\\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}}"] & \begin{matrix}\mathrm{H}_{n-1}(X)\\ H_{n-1}(X)\end{matrix} \arrow[u, "{\begin{pmatrix} i_3 & 0\\ 0 & i_7\end{pmatrix}}"'] & \to \\
\longrightarrow & \mathrm{H}_n(X) \arrow[u, "{\begin{pmatrix} \mathrm{id}\\ -\mathrm{id}\end{pmatrix}}"] \arrow[r, "\mathrm{id}-f_*"] & \mathrm{H}_n(X) \arrow[u, "{\begin{pmatrix} \mathrm{id}\\ 0\end{pmatrix}}"] \arrow[r, "i"] & \mathrm{H}_n(M) \arrow[u, "\mathrm{id}"] \arrow[r, "\partial"] & \mathrm{H}_{n-1}(X) \arrow[u, "{\begin{pmatrix} \mathrm{id}\\ -\mathrm{id}\end{pmatrix}}"] \arrow[r, "\mathrm{id}-f_*"] & \mathrm{H}_{n-1}(X) \arrow[u, "{\begin{pmatrix} \mathrm{id}\\ 0\end{pmatrix}}"] & \longrightarrow
\end{tikzcd}
$$

The above diagram arose as follows:

(1) Note that the interiors of $A = p(X\times([0,4)\cup(6,10])) \subset M$ and $B = p(X\times(2,8)) \subset M$ cover all of M. Furthermore note that we have $A\cap B = C_1 \sqcup C_2$. On the top of the diagram we wrote down the corresponding exact Mayer–Vietoris sequence from Theorem 46.5.

(2) Note that the maps $i_3\colon X \to A$, $i_3\colon X \to C_1$, $i_7\colon X \to C_2$ and $i_7\colon X \to B$ are homotopy equivalences. It follows from Corollary 42.8 that the induced maps on homology are isomorphisms. In particular the vertical maps between the first and the second horizontal sequence are isomorphisms.

(3) Next we want to show that the top part of the diagram commutes. The squares involving $\mathrm{id}\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)$ commute by the functoriality of homology groups and by definition of the connecting homomorphism ∂ in the middle row. It remains to consider the squares which in our excerpt above are to the left and right. The discussion is perhaps less confusing if we explain how the maps on the middle row were found. The task was to find maps $a,b,c,d\colon \mathrm{H}_n(X) \to \mathrm{H}_n(X)$ such that the following equality holds:

$$\underbrace{\begin{pmatrix} i_{C_1,A} & i_{C_2,A}\\ -i_{C_1,B} & -i_{C_2,B}\end{pmatrix}\begin{pmatrix} i_{3,C_1} & 0\\ 0 & i_{7,C_2}\end{pmatrix}}_{=\begin{pmatrix} i_{3,A} & i_{7,A}\\ -i_{3,B} & -i_{7,B}\end{pmatrix}} = \underbrace{\begin{pmatrix} i_{3,A} & 0\\ 0 & i_{7,B}\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}}_{=\begin{pmatrix} i_{3,A}\circ a & i_{3,A}\circ b\\ i_{7,B}\circ c & i_{7,B}\circ d\end{pmatrix}}.$$

It follows from the above claim that a solution is given by $a = \mathrm{id}, b = f_*, c = -\mathrm{id}$ and $d = -\mathrm{id}$.

(4) It is straightforward to see that the bottom squares in the above diagram commute. We leave it to the reader to verify that the bottom sequence is also exact.

As we had just pointed out, the bottom horizontal sequence is exact. The proposition is thus an immediate consequence of Lemma 46.4. ■

[764] Here we use the same notation as on page 1219.

Now we can also easily prove the following lemma.

Lemma 46.22. *Let X be a topological space. We denote by $p\colon X\times S^1\to X$ the natural projection. For every $n\in\mathbb{N}_0$ there exists a natural map $\psi\colon \mathrm{H}_n(X\times S^1)\to \mathrm{H}_{n-1}(X)$ such that the map*

$$\Xi_X := p_*\circ\psi\colon\ \mathrm{H}_n(X\times S^1)\ \xrightarrow{\cong}\ \mathrm{H}_n(X)\oplus \mathrm{H}_{n-1}(X)$$

is a natural isomorphism. Here natural means that for any map $f\colon X\to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}_n(X\times S^1) & \xrightarrow[\cong]{\Xi_X} & \mathrm{H}_n(X)\oplus \mathrm{H}_{n-1}(X)\\
\downarrow{\scriptstyle (f\times\mathrm{id})_*} & & {\scriptstyle f_*}\downarrow \qquad \downarrow{\scriptstyle f_*}\\
\mathrm{H}_n(Y\times S^1) & \xrightarrow[\cong]{\Xi_Y} & \mathrm{H}_n(X)\oplus \mathrm{H}_{n-1}(X).
\end{array}$$

Example. Using an elementary and amusing induction argument one can show quite easily, using Lemma 46.22, that for $k=0,\dots,n$ we have

$$\mathrm{H}_k(n\text{-dimensional torus } (S^1)^n)\ =\ \text{free abelian group of rank } \binom{n}{k}$$

and that all other homology groups vanish.

Remark. We gave a different proof of Lemma 46.22 in Exercise 44.5.

Proof. Let $n\in\mathbb{N}_0$. Note that by Lemma 24.21 the map $\Theta\colon \mathrm{Tor}(X,f:=\mathrm{id})\to X\times S^1$ given by $[(x,t)]\mapsto(x,e^{2\pi i t})$ is a natural homeomorphism. We denote by $i\colon X\to\mathrm{Tor}(f)$ the natural embedding given by $x\mapsto[(x,0)]$ and we denote by $j\colon X\to X\times S^1$ the natural embedding given by $x\mapsto(x,1)$. Next we consider the following diagram:

$$\begin{array}{ccccccccc}
0\to & \mathrm{coker}\big(\mathrm{H}_n(X)\xrightarrow{f_*-\mathrm{id}}\mathrm{H}_n(X)\big) & \xrightarrow{i_*} & \mathrm{H}_n(\mathrm{Tor}(X,f)) & \xrightarrow{\partial} & \ker\big(\mathrm{H}_{n-1}(X)\xrightarrow{f_*-\mathrm{id}}\mathrm{H}_{n-1}(X)\big) & \to 0\\
& \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle\mathrm{id}} & \\
0\longrightarrow & \mathrm{H}_n(X) & \xrightarrow{i} & \mathrm{H}_n(\mathrm{Tor}(X,f)) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(X) & \longrightarrow 0\\
& \downarrow{\scriptstyle\mathrm{id}} & & {\scriptstyle\cong}\downarrow{\scriptstyle\Theta_*} & & \downarrow{\scriptstyle\mathrm{id}} & \\
0\longrightarrow & \mathrm{H}_n(X) & \underset{p_*}{\overset{j_*}{\rightleftarrows}} & \mathrm{H}_n(X\times S^1) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(X) & \longrightarrow 0.
\end{array}$$

We make the following clarifications and observations:

(1) The top sequence is the short exact sequence obtained from Proposition 46.21.
(2) Since $f_*-\mathrm{id}_*=0$ we obtain the two vertical identities on top.
(3) Since Θ is an isomorphism and since $\Theta\circ i=j$ we see that the bottom sequence also commutes.
(4) The map $p\colon X\times S^1\to X$ is the natural map that is given by projection onto the first factor. Note that $p_*\circ i_*=(p\circ i)_*=\mathrm{id}$.

As in the proof of Splitting Lemma 46.2 we now see that the map

$$\begin{array}{rcl}
\Xi_X\colon\ \mathrm{H}_n(X\times S^1) & \to & \mathrm{H}_n(X)\oplus\mathrm{H}_{n-1}(X)\\
\sigma & \mapsto & p_*(\sigma)\oplus\partial(\sigma-(j_*\circ p_*)(\sigma))
\end{array}$$

is an isomorphism. Since all maps involved are natural we see that this isomorphism is in fact natural. ∎

We conclude this long chapter with the following lemma which can be viewed as the analogue of Proposition 24.28.

Lemma 46.23. *Let X be a path-connected topological space and let $f\colon X \to X$ be a map. We denote by $i\colon X \xrightarrow{x \mapsto [(x,0)]} \operatorname{Tor}(X,f)$ the natural embedding and we denote by $q\colon \operatorname{Tor}(X,f) \to S^1$ the natural projection that we introduced on page 711. With this notation the following sequence is natural and exact:*

$$0 \;\to\; \operatorname{coker}\big(\mathrm{H}_1(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_1(X)\big) \;\xrightarrow{i}\; \mathrm{H}_1(\operatorname{Tor}(X,f)) \;\to\; \mathrm{H}_1(S^1) \;\to\; 0.$$

In particular we have an isomorphism

$$\mathrm{H}_1(\operatorname{Tor}(X,f)) \;\cong\; \mathbb{Z} \oplus \operatorname{coker}\big(\mathrm{H}_1(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_1(X)\big).$$

Remark. Later in the Hurewicz Theorem 53.5 we will show that the abelianization of the fundamental group $\pi_1(X,x_0)$ of a pointed path-connected topological space (X,x_0) is naturally isomorphic to the first homology group $\mathrm{H}_1(X)$. Thus we see that Lemma 46.23 is a modest generalization of Exercise 24.18.

Proof. Since X is path-connected we obtain immediately from the discussion on page 1088 that the map $f_* - \mathrm{id}\colon \mathrm{H}_0(X) \to \mathrm{H}_0(X)$ is the zero map. We consider the following maps:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker}\big(\mathrm{H}_1(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_1(X)\big) \arrow[r, "i"] & \mathrm{H}_1(\operatorname{Tor}(X,f)) \arrow[r] \arrow[d, two heads, "p_*"'] & \mathbb{Z} \arrow[r] \arrow[dl, dashed, "\Psi"] & 0. \\
& & \underbrace{\mathrm{H}_1(S^1)}_{\cong\mathbb{Z}}. & &
\end{tikzcd}$$

We make the following observations:

(1) The horizontal map is the short exact sequence from Proposition 46.21.
(2) Since $p \circ i$ is a constant map we see that $p_* \circ i_*$ is the trivial map.
(3) It follows from (1) and (2) that there exists a map $\Psi\colon \mathbb{Z} \to \mathrm{H}_1(S^1)$ that makes the diagram commute.
(4) Furthermore, since X is path-connected one can easily construct a left-inverse to $p\colon \operatorname{Tor}(X,f) \to S^1$. This implies that $p_*\colon \mathrm{H}_1(\operatorname{Tor}(X,f)) \to \mathrm{H}_1(S^1)$ is an epimorphism.
(5) Since $\mathrm{H}_1(S^1) \cong \mathbb{Z}$ and since any epimorphism from $\mathbb{Z} \to \mathbb{Z}$ is an isomorphism we obtain from the above that Ψ is an isomorphism.

This concludes the proof that the sequence in the statement of the lemma is indeed exact.

Finally note that since $\mathbb{Z}$ is free abelian it follows from Corollary 46.3 that we have an isomorphism

$$\mathrm{H}_1(\operatorname{Tor}(X,f)) \;\cong\; \mathbb{Z} \oplus \operatorname{coker}\big(\mathrm{H}_1(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_1(X)\big).$$ ∎

Exercises for Chapter 46.

Exercise 46.1. We consider the following commutative diagram of homomorphisms between abelian groups:

$$\begin{tikzcd}
\ldots \arrow[r] & C''_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f''_{n+1}"] & C'_n \arrow[r, "i_n"] \arrow[d, "f'_n"] & C_n \arrow[r, "p_n"] \arrow[d, "f_n"] & C''_n \arrow[r, "\partial_n"] \arrow[d, "f''_n"] & C'_{n-1} \arrow[r] \arrow[d, "f'_{n-1}"] & \ldots \\
\ldots \arrow[r] & D''_{n+1} \arrow[r, "\partial_{n+1}"] & D'_n \arrow[r, "j_n"] & D_n \arrow[r, "q_n"] & D''_n \arrow[r, "\partial_n"] & D'_{n-1} \arrow[r] & \ldots
\end{tikzcd}$$

We suppose that the horizontal sequences are exact and we suppose that each f''_n is an isomorphism. For each $n \in \mathbb{N}_0$ we set $\Delta_n := \partial_n \circ (f''_n)^{-1} \circ q_n \colon D_n \to C'_{n-1}$. Show that the following sequence is exact

$$\ldots \longrightarrow C'_n \xrightarrow{i_n + f'_n} C_n \oplus D'_n \xrightarrow{f_n + j_n} D_n \xrightarrow{\Delta_n} C'_{n-1} \longrightarrow \ldots$$

Remark. The resulting exact sequence is sometimes called the *algebraic Mayer–Vietoris sequence.* The name will become clear in Exercise 46.2.

Exercise 46.2. Let X be a topological space and let $A, B \subset X$ be subsets such that $X = \mathring{A} \cup \mathring{B}$. Use the algebraic Mayer–Vietoris sequence from Exercise 46.1 together with the Excision Theorem 43.19 and the long exact sequence of the pair of topological spaces to give a new proof of of the Mayer–Vietoris Theorem 46.5 92) and (3). More precisely you need to show that there exists a long exact sequence of the form

$$\ldots \to \mathrm{H}_n(A \cap B) \xrightarrow{i_{A\cap B} \oplus - i_{A \cap B}} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i_A + i_B} \mathrm{H}_n(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A \cap B) \to \ldots$$

where ∂_n is natural.
Hint. Consider the commutative diagram

$$\begin{tikzcd}
A \cap B \arrow[r, "i_{A\cap B}"] \arrow[d, "i_{A\cap B}"'] & A \arrow[d, "i_A"] \\
B \arrow[r, "i_B"] & X.
\end{tikzcd}$$

Exercise 46.3.

(a) We denote by

$$K \;=\; ([0,1] \times [0,1])/\sim \text{ with } (x,0) \sim (x,1) \text{ and } (0,y) \sim (1, 1-y)$$

the Klein bottle. Show that

$$\mathrm{H}_n(K) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z} \oplus \mathbb{Z}_2, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Give an example of an explicit map $f \colon S^1 \to K$ such that the inclusion induced map $f_* \colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(K)$ is a monomorphism. Prove that the map f that you provide does indeed have this property.

Exercise 46.4. Let X be a topological space and let U and V be two open subset of X such that X and $U \cap V$ are path-connected and such that $U \cap V$ is non-empty. Show that U and V are path-connected.
Remark. This is of course just a special case of Exercise 2.55 (a). But it is great fun to prove this statement using the Mayer–Vietoris Theorem 46.5.

Exercise 46.5. Compute the isomorphism types of the homology groups of the real line with two zeros.

Exercise 46.6. Determine the isomorphism types of the homology groups of the topologist's sine curve that we introduced on page 131.

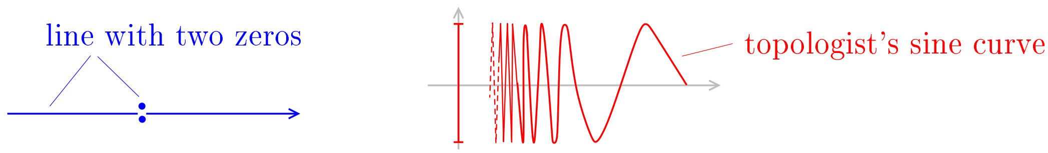


Figure 780. Illustration for Exercises 46.5 and 46.6.

Exercise 46.7. Let $K = (V, E, i, t)$ be a connected abstract graph. Determine the isomorphism types of the homology groups of the topological realization $|K|$.
Hint. Figure 781 shows an example of a convenient covering by open sets.

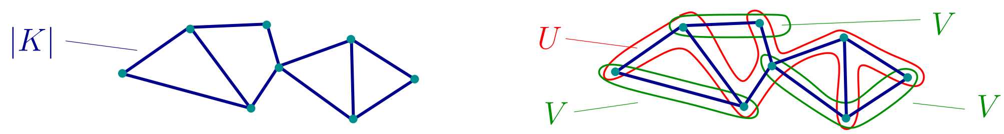


Figure 781. Illustration for Exercise 46.7.

Exercise 46.8. In Lemma 46.12 we computed the homology groups of the torus $T = S^1 \times S^1$ the Mayer–Vietoris Theorem 46.10 for Manifolds. compute the *reduced* homology of the torus using the Mayer–Vietoris sequence with reduced homology groups.
Remark. Depending on how you go about this calculation you can easily end up with a contradictory calculation.

Exercise 46.9. Let $T = S^1 \times S^1$ be the torus. We denote by $p, q\colon S^1 \times S^1 \to S^1$ the two obvious projections. Show that the map

$$\begin{array}{rcl} p_* \oplus q_* \colon \mathrm{H}_1(S^1 \times S^1) & \to & \mathrm{H}_1(S^1) \oplus \mathrm{H}_1(S^1) \\ \varphi & \mapsto & p_*(\varphi) + q_*(\varphi) \end{array}$$

is an isomorphism.
Hint. You could make use of Lemma 46.12 (2).

Exercise 46.10. We consider the real projective plane

$$\mathbb{R}\mathrm{P}^2 = \overline{B}^2 / \sim \quad \text{where } x \sim y \text{ if } x, y \in \partial \overline{B}^2 \text{ and } x = -y.$$

Determine an explicit cycle in $\mathrm{C}_1(\mathbb{R}\mathrm{P}^2)$ that represents the unique non-trivial element of $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2)$.

Exercise 46.11. Let X be a topological space and let $k \in \mathbb{Z}$. Show that the two natural isomorphisms $\widetilde{\mathrm{H}}_k(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{k-1}(\Sigma(X))$ that we constructed in Proposition 45.7 and in the proof of Lemma 46.8 are actually the same.

Exercise 46.12. Let $X \subset \mathbb{R}^n$ be a non-empty compact subset. As in Lemma 2.44 we make the identification $S^n = \mathbb{R}^n \cup \{\infty\}$ of the sphere with the space $\mathbb{R}^n$ together with a "point at infinity".

(a) Show that for every $k \neq n-1$ the inclusion induced map
$$\mathrm{H}_k(\mathbb{R}^n \setminus X) \;\to\; \mathrm{H}_k(\underbrace{(\mathbb{R}^n \cup \{\infty\})}_{=S^n} \setminus X)$$
is an isomorphism.

(b) Show that there exists a short exact sequence
$$0 \;\to\; \mathbb{Z} \;\to\; \mathrm{H}_{n-1}(\mathbb{R}^n \setminus X) \;\to\; \mathrm{H}_{n-1}((\mathbb{R}^n \cup \{\infty\}) \setminus X) \;\to\; 0.$$

(c) Show that the short exact sequence in (b) splits.

Exercise 46.13. Let $g \in \mathbb{N}_{\geq 2}$. Use the Mayer–Vietoris Theorem 46.10 for Manifolds, or some other flavor of the Mayer–Vietoris sequence, to determine the isomorphism types of the surface of genus g.
Hint. In Figure 782 we show two possible approaches to decomposing the surface of genus two.
Remark. We will give the calculation in Proposition 48.9 using "cellular homology".

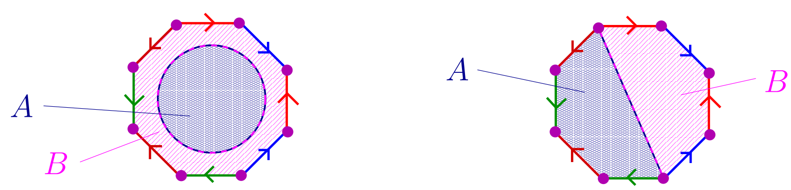


FIGURE 782. Illustration of Exercise 46.13.

Exercise 46.14. Let X be a topological space, let $k \in \mathbb{N}$ and let $n \in \mathbb{N}_0$. What is the isomorphism type of $\mathrm{H}_n(X \times S^k)$ in terms of the homology groups of X?

Exercise 46.15. Let $k \in \mathbb{N}$. We consider the topological space
$$X_k \;:=\; ([-k,k] \,\cup\, S^2 \times \{-k,\dots,k\})/\sim$$
where given $i \in \{-k,\dots,k\}$ we identify $i \in [-k,k]$ with $((0,0,-1),i) \in S^2 \times \{-k,\dots,k\}$. For $i = -k,\dots,k$ we denote by
$$\begin{array}{rcl} f_i \colon S^2 & \to & X_k \\ P & \mapsto & (P,i) \end{array}$$
the various "obvious" inclusion maps. (We refer to Figure 783 for an illustration.)

(a) Show that $\mathrm{H}_0(X_k) \cong \mathbb{Z}$.
(b) Show that $\mathrm{H}_i(X_k) = 0$ for $i \neq 0, 2$.

(c) Let $[S^2] \in \mathrm{H}_2(S^2)$ be the standard generator. Show that the map

$$\begin{array}{rcl} \mathbb{Z}^{2k+1} & \to & 2(X_k) \\ (a_{-k}, \ldots, a_k) & \to & \sum\limits_{i=-k}^{k} a_i \cdot f_{i*}([S^2]) \end{array}$$

is an isomorphism.

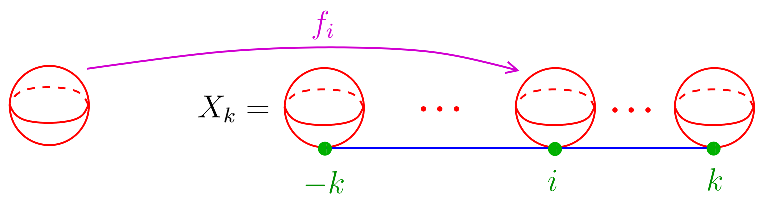


FIGURE 783. Illustration of Exercise 46.15.

Exercise 46.16. Let X be a topological space. We consider the corresponding suspension $\Sigma(X) = (X \times [-1,1])/\sim$ which is given by identifying all points in $X \times \{-1\}$ and by identifying all points in $X \times \{1\}$.

(a) Given a singular k-simplex $\sigma\colon \Delta^k \to X$ we define the two maps

$$\begin{array}{rcl} c_\pm(\sigma)\colon \Delta^{k+1} & \to & \Sigma(X) \\ (t_0, t_1, \ldots, t_{k+1}) & \mapsto & \left\{ \begin{array}{ll} \left(\left[\left(\sigma\left(\frac{1}{1-t_0}(t_1, \ldots, t_{k+1})\right), \pm t_0\right)\right], \right. & \text{if } t_0 < 1, \\ {[X \times \{\pm 1\}]}, & \text{if } t_0 = 1. \end{array} \right. \end{array}$$

Show that the maps $c_-(\sigma)$ and $c_+(\sigma)$ are continuous. (We refer to Figure 784 for an illustration of the definition of $c_\pm(\sigma)$.)
Hint. Consider $\Delta^k \times [0,1]$ and $(\Delta^k \times [0,1])/(\Delta^k \times \{1\})$.

(b) Note that the maps $\sigma \mapsto c_\pm(\sigma)$ define a homomorphisms $c_\pm\colon \mathrm{C}_k(X) \to \mathrm{C}_{k+1}(\Sigma(X))$. We set $s = (-1)^{k+1} \cdot (c_+ - c_-)$. Show that for any $\sigma\colon \Delta^k \to X$ we have

$$\partial s(\sigma) \;=\; \left\{ \begin{array}{ll} s(\partial\sigma), & \text{if } k > 0, \\ {[X \times \{1\}] - [X \times \{-1\}]}, & \text{if } k = 0. \end{array} \right.$$

(c) Let $k \in \mathbb{N}_0$. Show that the map

$$\begin{array}{rcl} \widetilde{\mathrm{H}}_k(X) & \to & \mathrm{H}_{k+1}(\Sigma(X)) \\ \sigma & \mapsto & s(\sigma) \end{array}$$

is well-defined and that it is an isomorphism.

Remark. We refer to c_+, c_- as *cone operators* and we refer to s as the *suspension operator*.
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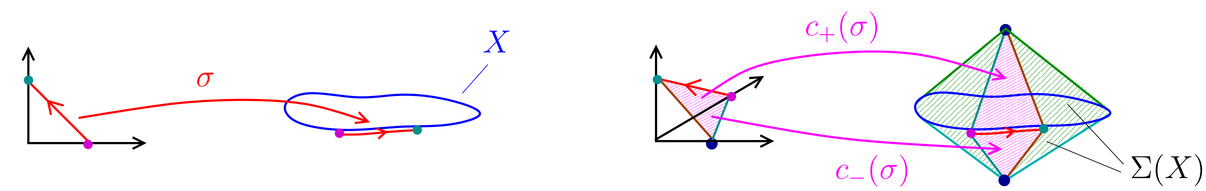

FIGURE 784

Exercise 46.17. Let $k \in \mathbb{N}_{\geq 3}$ and let $m \in \mathbb{N}$. We consider the topological space

$$X_k \;:=\; S^1 \cup \{re^{2\pi \mathrm{i} j/k} \,|\, r \in [0,1] \text{ and } j \in \{0,\dots,k-1\}\}$$

and we consider the homeomorphism

$$\begin{aligned} f_m \colon X_k &\to X_k \\ z &\mapsto e^{2\pi \mathrm{i} m/k} \cdot z. \end{aligned}$$

Determine the homology groups of the mapping torus $\mathrm{Tor}(X_k, f_m)$.

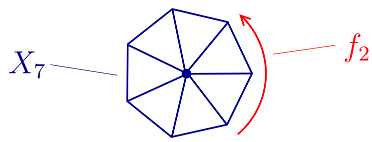

FIGURE 785. Illustration of Exercise 46.17.

Exercise 46.18. Let X be a topological space and let $A \subset X$ be a non-empty subset. We denote by $i \colon A \to X$ the inclusion map. We consider the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & \widetilde{\mathrm{H}}_n(A) \arrow[r, "i"] \arrow[d, "="] & \widetilde{\mathrm{H}}_n(X) \arrow[r] \arrow[d, "="] & \mathrm{H}_n(X,A) \arrow[r, "\partial"] \arrow[d, "\cong"] & \widetilde{\mathrm{H}}_{n-1}(A) \arrow[r] \arrow[d, "="] & \widetilde{\mathrm{H}}_{n-1}(X) \arrow[r] \arrow[d, "="] & \dots \\
\dots \arrow[r] & \widetilde{\mathrm{H}}_n(A) \arrow[r, "i"] & \widetilde{\mathrm{H}}_n(X) \arrow[r] & \widetilde{\mathrm{H}}_n(\mathrm{Cone}(i)) \arrow[r, "\partial"] & \widetilde{\mathrm{H}}_{n-1}(A) \arrow[r] & \widetilde{\mathrm{H}}_{n-1}(X) \arrow[r] & \dots
\end{tikzcd}$$

where the middle vertical map is the natural isomorphism from Lemma 46.16, where the top sequence is the long exact sequence arising from Corollary 43.16. and where the bottom sequence is the exact sequence from Lemma 46.18. Show that the diagram commutes.

Exercise 46.19. Let X be a topological space and let U_1, U_2 and U_3 be open subsets of X with $X = U_1 \cup U_2 \cup U_3$.

(a) We suppose that $U_1 \cap U_2 \cap U_3 = \varnothing$. Show that there exists a long exact sequence of the form

$$\dots \to \begin{matrix} \mathrm{H}_n(U_1 \cap U_2) \\ \mathrm{H}_n(U_2 \cap U_3) \\ \mathrm{H}_n(U_1 \cap U_3) \end{matrix} \xrightarrow{\begin{pmatrix} i_{U_1\cap U_2} & 0 & i_{U_1 \cap U_3} \\ -i_{U_1\cap U_2} & i_{U_2\cap U_3} & 0 \\ 0 & -i_{U_2\cap U_3} & -i_{U_1\cap U_3} \end{pmatrix}} \begin{matrix} \mathrm{H}_n(U_1) \\ \mathrm{H}_n(U_2) \\ \mathrm{H}_n(U_3) \end{matrix} \xrightarrow{i_{U_1}+i_{U_2}+i_{U_3}} \mathrm{H}_n(X) \xrightarrow{\partial_n} \dots$$

(b) Show that in general the conclusion of (a) does not hold if $U_1 \cap U_2 \cap U_3 \neq \varnothing$.

Remark. This exercise is related to the discussion on page 1210.

Exercise 46.20. Let X and Y be topological spaces. We consider the join $X * Y$ as defined on page 207. Show that for each $k \in \mathbb{N}_0$ there exists a short exact sequence of the following form:

$$0 \;\to\; \widetilde{\mathrm{H}}_k(X*Y) \;\to\; \widetilde{\mathrm{H}}_{k-1}(X) \oplus \widetilde{\mathrm{H}}_{k-1}(Y) \;\to\; \widetilde{\mathrm{H}}_{k-1}(X \times Y) \;\to\; 0.$$

Hint. In Exercise 24.11 we gave an explicit homeomorphism

$$X * Y \;\cong\; (\mathrm{Cone}(X) \times Y) \cup_{(X\times\{1\})\times Y = X \times (Y \times\{1\})} (X \times \mathrm{Cone}(Y)).$$

Note though that you need to argue why you can use this decomposition.
Remark. In the Künneth Theorem 58.8 we will learn how to calculate the homology groups of the product $X \times Y$.

Exercise 46.21. Let $X = \mathbb{Z}_n$ be the topological space whose underlying set is $\mathbb{Z}_n$ and whose topology is given by the discrete topology. Let $f\colon X = \mathbb{Z}_n \to X = \mathbb{Z}_n$ be the homeomorphism given by $f(x) = x + 1 \in \mathbb{Z}_n$. Compute the homology groups of the mapping torus $\mathrm{Tor}(X, f)$.

Exercise 46.22. Let $p, q \in \mathbb{N}$ be coprime. We consider the torus knot $T_{p,q}$ as defined on page 777. Compute the homology groups of $S^3 \setminus T_{p,q}$.
Hint. Use the decomposition of $S^3 \setminus T_{p,q}$ given in the proof of Proposition 27.5.

Exercise 46.23. Let K be the Klein bottle minus one open disk. Show that ∂K is not a retract of K.

Exercise 46.24. Let $n \in \mathbb{N}$ and let M be S^2 minus n open disks. Show that the inclusion induced map $\mathrm{H}_1(\partial M) \to \mathrm{H}_1(M)$ is an epimorphism.

Exercise 46.25. We consider the two topological manifolds X and Y that are shown in Figure 786. Use (relative?) homology groups to show that X and Y are not homeomorphic.
Remark. In Exercise 20.6 we attacked the same problem using fundamental groups. It should be much easier to deal with the problem at hand using (relative) homology groups.

FIGURE 786. Illustration of Exercise 46.25.

47. Homology groups and direct limits

In this chapter we will discuss the relationship between homology groups and direct limits. This will allow us to determine the homology groups of suitable nested sequences of topological spaces. In a similar vein we will determine the homology groups of the wedge of arbitrarily many topological spaces. We concludes this chapter with the introduction of the Moore spaces. We will in particular see that all abelian group can appear as homology groups of topological spaces.

For the reader who is in a hurry only few of the results of this chapter are really important. In particular, very soon we will make use of Proposition 47.9. For this proposition we give two proofs, one building on the theme of this chapter. The other proof can be read independently of the rest of the chapter.

47.1. **Direct limits and induced maps.** Given a directed set $(I, \leq)$ and a direct system $(\{K_i\}_{i\in I}, \{\chi_{ij}\}_{i\leq j})$ of abelian groups we write, as on page 733

$$\varinjlim K_i \ := \ \Big(\bigsqcup_{i\in I} K_i\Big)/\sim \quad \text{where } k_i \sim \chi_{ij}(k_i) \text{ for all } i \leq j,\ k_i \in K_i$$

We adopt the following notation, given $k_i \in K_i$ we denote by $[k_i] \in \varinjlim K_i$ the element represented by k_i. We make the following observation that follows immediately from the above definition of the direct limit.

Observation 47.1. *Any element in $\varinjlim K_i$ is of the form $[k_i]$ for some $k_i \in K_i$.*

Definition. Let $(I, \leq)$ be a directed set and let $(\{K_i\}_{i\in I}, \{\chi_{ij}\}_{i\leq j})$ and $(\{L_i\}_{i\in I}, \{\lambda_{ij}\}_{i\leq j})$ be direct systems of abelian groups.

(1) We define a *homomorphism between the above two direct systems* to be a family $\{f_i\colon K_i \to L_i\}_{i\in I}$ of homomorphisms that has the property that for any $i \leq j$ the following diagram commutes:

$$\begin{array}{ccc} K_i & \xrightarrow{f_i} & L_i \\ {\scriptstyle\chi_{ij}}\downarrow & & \downarrow{\scriptstyle\lambda_{ij}} \\ K_j & \xrightarrow{f_j} & L_j. \end{array}$$

(2) Given such a homomorphism we refer to

$$\begin{array}{rcl} \varinjlim K_i & \to & \varinjlim L_i \\ {[k_i]} & \mapsto & [f_i(k_i)] \end{array}$$

as the induced map. Sometimes we denote it by $\varinjlim f_i$.[765][766] It is straightforward to verify that the induced maps are functorial.

[765] It follows immediately from the definitions that this map is well-defined, i.e. the map does not depend on the choice $[k_i]$ of a representative for an element in the direct limit.

[766] More formally one could say that the maps $K_i \to L_i \to \varinjlim L_i$ induce, by the universal property of a direct limit, a map $\varinjlim K_i \to \varinjlim L_i$. It is straightforward to see that this map is exactly the one we wrote down.

Proposition 47.2. *Let $(I, \leq)$ be a directed set. Also let $(\{K_i\}_{i\in I}, \{k_{ij}\}_{i\leq j})$, $(\{L_i\}_{i\in I}, \{l_{ij}\}_{i\leq j})$ and $(\{M_i\}_{i\in I}, \{m_{ij}\}_{i\leq j})$ be direct systems of abelian groups. Furthermore suppose that we are given homomorphisms $f_i\colon K_i \to L_i$ and $g_i\colon L_i \to M_i$ of the direct systems. If for all $i \in I$ the sequence*

$$K_i \xrightarrow{f_i} L_i \xrightarrow{g_i} M_i$$

is exact, then the induced sequence

$$\varinjlim K_i \xrightarrow{\varinjlim f_i} \varinjlim L_i \xrightarrow{\varinjlim g_i} \varinjlim M_i$$

is also exact.

Proof. We write $f = \varinjlim f_i$ and $g = \varinjlim g_i$. We have to prove that $g \circ f = 0$ and we need to show that $\ker(g) \subset \operatorname{im}(f)$. Both statements statements follow our hypothesis together with Observation 47.1 and the fact that in the definition of the induced map between direct limits we can take any representative. We leave the few remaining details to the reader. ■

The following corollary says that direct limits commute with kernels and cokernels.

Corollary 47.3. *Let $(I, \leq)$ be a directed set and let $(\{K_i\}_{i\in I}, \{k_{ij}\}_{i\leq j})$ and $(\{L_i\}_{i\in I}, \{l_{ij}\}_{i\leq j})$ be direct systems of abelian groups. Furthermore suppose that we are given a homomorphism $f_i\colon K_i \to L_i$ of the direct systems. There exist canonical isomorphisms*

$$\varinjlim \ker(f_i\colon K_i \to L_i) \xrightarrow{\cong} \ker\left(\varinjlim K_i \to \varinjlim L_i\right)$$

and

$$\varinjlim \operatorname{coker}(f_i\colon K_i \to L_i) \xrightarrow{\cong} \operatorname{coker}\left(\varinjlim K_i \to \varinjlim L_i\right).$$

Proof. For each $i \in I$ the sequence

$$0 \to \ker(f_i) \to K_i \xrightarrow{f_i} L_i \to \operatorname{coker}(f_i) \to 0.$$

is exact. Evidently these homomorphisms form a homomorphism between the various direct systems. It follows from Proposition 47.2 (applied four times!) that the sequence

$$0 \to \varinjlim \ker(f_i) \to \varinjlim K_i \to \varinjlim L_i \to \varinjlim \operatorname{coker}(f_i) \to 0$$

is also exact. But this observation already implies the desired statements. ■

47.2. **The homology groups of a nested sequence of topological spaces.** In Propositions 25.7 and 40.11 we saw that in favorable situations the homotopy groups of a union of a nested sequence of topological spaces is the direct limit of the corresponding fundamental groups. Now we will see that a similar statement holds for homology groups.

Proposition 47.4. *Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of subsets such that $X = \bigcup_{i\in\mathbb{N}} X_i$ and such that one the following three conditions holds:*

(1) *every compact subset of X is already contained in one of the X_i, or*
(2) *each X_i is open in X, or*

(3) *X is a* CW*-complex and each X_i is a subcomplex.*

Then for any $n \in \mathbb{N}_0$ the inclusion induced maps $\mathrm{H}_n(X_i) \to \mathrm{H}_n(X)$ induce an isomorphism

$$\varinjlim \mathrm{H}_n(X_i) \xrightarrow{\cong} \mathrm{H}_n(X).$$

We remark that it follows easily from the Finiteness Theorem 36.14 that hypothesis (3) implies hypothesis (1). Furthermore we point out that it follows from Lemma 25.8 that hypothesis (2) implies hypothesis (1). Thus it suffices to prove Proposition 47.4 under the hypothesis (1). But in that case the proposition is an immediate consequence of the next two lemmas.

Lemma 47.5. *Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of subsets such that every compact subset of X is already contained in one of the X_i. Then the chain complexes $\mathrm{C}_*(X_i), i \in \mathbb{N}$ together with the inclusion induced chain maps $\mathrm{C}_*(X_i) \to \mathrm{C}_*(X_j)$ for $i \leq j$ form a direct system of chain complexes.*[767] *Furthermore the inclusion induced maps $\mathrm{C}_*(X_i) \to \mathrm{C}_*(X)$ induce an isomorphism*[768]

$$\varinjlim \mathrm{C}_*(X_i) \xrightarrow{\cong} \mathrm{C}_*(X)$$

of chain complexes.

Proof. Given a topological space Y and $n \in \mathbb{N}_0$ we denote by $S_n(Y)$ the set of singular n-simplices in Y. It follows immediately from our hypothesis on the X_i, the fact that Δ^n is compact and Lemma 2.40 that

$$S_n(X) = \bigcup_{i \in \mathbb{N}} S_n(X_i).$$

As in the example on page 730 we now see that

$$\varinjlim \mathrm{C}_n(X_i) = \varinjlim \mathbb{Z}^{(S_n(X_i))} = \mathbb{Z}^{(S_n(X))} = \mathrm{C}_n(X).$$

We leave it to the reader to verify that the inclusion maps also induce an isomorphism of chain complexes. ■

The second lemma required for the proof of Proposition 47.4 is purely algebraic.

Lemma 47.6. *Let $(I, \leq)$ be a directed set and furthermore let $(\{\mathrm{C}_i\}_{i\in I}, \{f_{ij}\}_{i \leq j})$ be a direct system of chain complexes. Then for each $n \in \mathbb{N}_0$ we obtain an induced direct system $(\{\mathrm{H}_n(\mathrm{C}_i)\}_{i\in I}, \{f_{ij*}\}_{i\leq j})$ of abelian groups. Furthermore, given any $n \in \mathbb{N}_0$ the maps $\mathrm{H}_n(\mathrm{C}_i) \to \mathrm{H}_n\left(\varinjlim \mathrm{C}_i\right)$ induce an isomorphism*

$$\varinjlim \mathrm{H}_n(\mathrm{C}_i) \xrightarrow{\cong} \mathrm{H}_n\left(\varinjlim \mathrm{C}_i\right).$$

[767] We refer to page 728 for the definition of a direct system.

[768] The argument of Proposition 25.1 or respectively of page 733 shows that the direct limit of a direct system of chain complexes always exists.

PROOF. We have

$$\begin{aligned}\varinjlim \mathrm{H}_n(\mathrm{C}_i) &= \varinjlim \overbrace{\mathrm{coker}(\mathrm{C}_{i+1} \to \ker(\mathrm{C}_i \to \mathrm{C}_{i-1}))}^{=\mathrm{H}_n(\mathrm{C}_i)} \\ &\underset{\substack{\uparrow \\ \text{Corollary 47.3 applied twice}}}{=} \mathrm{coker}\left(\varinjlim \mathrm{C}_{i+1} \to \ker\left(\varinjlim \mathrm{C}_i \to \varinjlim \mathrm{C}_{i-1}\right)\right) = \mathrm{H}_n\left(\varinjlim \mathrm{C}_i\right).\end{aligned}$$

The fact that this isomorphism is induced by the maps $\mathrm{H}_n(\mathrm{C}_i) \to \mathrm{H}_n(\varinjlim \mathrm{C}_i)$ follows easily from the various definitions. ■

We conclude this short section with a few examples:

Example. For any $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_n(S^\infty) \underset{\substack{\uparrow \\ \text{by Proposition 47.4,} \\ \text{where we view } S^\infty \text{ as a} \\ \text{CW-complex as on page 942}}}{=} \varinjlim_{k\in\mathbb{N}} \mathrm{H}_n(S^k) \underset{\substack{\uparrow \\ \text{Proposition 43.4}}}{=} \varinjlim_{k\in\mathbb{N}} \begin{cases} \mathbb{Z}, & \text{if } n = k, 0, \\ 0, & \text{else} \end{cases} \underset{\substack{\uparrow \\ \text{Lemma 25.2}}}{=} \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{else.} \end{cases}$$

The next lemma is basically also just an example. But since we will need it later on we formulate it as a lemma.

Lemma 47.7. *We consider the real line with infinitely many 2-dimensional spheres attached, i.e. we consider*

$$X := (\mathbb{R} \cup (S^2 \times \mathbb{Z}))/\sim \qquad \text{where } n \sim ((0,0,-1), n) \text{ for } n \in \mathbb{Z}.$$

Then[a]

$$\mathrm{H}_n(X) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}^\infty, & \text{if } n = 2, \\ 0, & \text{else.} \end{cases}$$

In fact we can specify an isomorphism on $\mathrm{H}_2(X)$*. Given* $i \in \mathbb{Z}$ *we denote by* $f_i\colon S^2 \to X$ *the "obvious" i-th inclusion map. Then an isomorphism is given by*

$$\begin{aligned} \mathbb{Z}^\infty \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{Z} &\to \mathrm{H}_2(X) \\ \bigoplus_{i\in\mathbb{Z}} a_i &\mapsto \sum_{i\in\mathbb{Z}} a_i \cdot f_i([S^2]). \end{aligned}$$

[a] Recall that

$$\mathbb{Z}^\infty := \mathbb{Z}^{(\mathbb{N})} = \{(x_1, x_2, \dots) \,|\, x_i \in \mathbb{Z} \text{ but only finitely many } x_i\text{'s are non-zero}\}.$$

PROOF. We view X as a CW-complex in the obvious way. Furthermore, given $k \in \mathbb{N}$ we consider the subcomplex

$$X_k := ([-k,k] \cup S^2 \times \{-k, \dots, k\})/\sim.$$

Note that X_k is the interval $[-k,k]$ with $2k+1$ spheres attached. (We refer to Figure 787 for an illustration.) We have

$$\mathrm{H}_n(X) \underset{\substack{\uparrow \\ \text{Proposition 47.4}}}{=} \varinjlim \mathrm{H}_n(X_k) \underset{\substack{\uparrow \\ \text{see Exercise 46.15}}}{=} \varinjlim \begin{cases} \mathbb{Z}, & \text{if } n=0, \\ \mathbb{Z}^{2k+1}, & \text{if } n=2, \\ 0, & \text{else} \end{cases} \underset{\substack{\uparrow \\ \text{see page 730 and Lemma 25.2}}}{=} \begin{cases} \mathbb{Z}, & \text{if } n=0, \\ \mathbb{Z}^{\infty}, & \text{if } n=2, \\ 0, & \text{else.} \end{cases}$$

The statement about the precise isomorphism of $\mathrm{H}_2(X)$ also follows from the explicit description of the isomorphism Exercise 46.15 and the discussion on page 730. ■

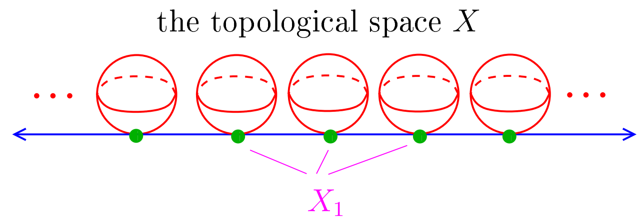

FIGURE 787

47.3. **The homology groups of the wedge of topological spaces.** The following proposition says that under a mild hypothesis "homology groups are additive" under the operation of taking the wedge of two topological spaces. The proposition is very similar in vein to Proposition 20.3.

Proposition 47.8. *Let A and B be two topological spaces and let $a \in A$ and $b \in B$ be good points. We denote by $A \vee B$ the wedge defined using a and b. Then the natural inclusion maps $i\colon A \to A \vee B$ and $j\colon B \to A \vee B$ that we introduced on page 561 induce for every $n \in \mathbb{N}_0$ an isomorphism*

$$i_* \oplus j_*\colon \widetilde{\mathrm{H}}_n(A) \oplus \widetilde{\mathrm{H}}_n(B) \;\to\; \widetilde{\mathrm{H}}_n(A \vee B).$$

Furthermore, if we denote by $p\colon A \vee B \to A$ and $q\colon A \vee B \to A$ the natural projection maps, then an inverse to the above isomorphism is given by

$$p_* + q_*\colon \widetilde{\mathrm{H}}_n(A \vee B) \;\to\; \widetilde{\mathrm{H}}_n(A) \oplus \widetilde{\mathrm{H}}_n(B).$$

PROOF $(*)$. We write $X = A \vee B$ and $x_0 = \{a,b\} \in A \vee B$. We pick an open neighborhood C in A of a such that $\{a\}$ is a deformation retract of C. Furthermore we pick an open neighborhood D of b in B such that b is a deformation retract of D. We consider $U := A \vee D$

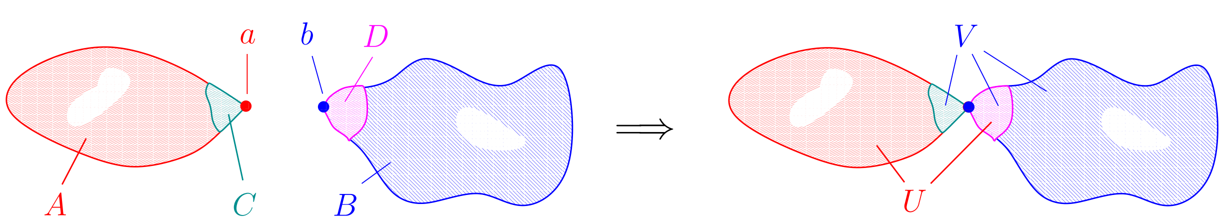

FIGURE 788. Schematic image for the proof of Proposition 47.8.

and $V := B \vee C$. We consider the following diagram of maps

$$\begin{tikzcd}
 & \widetilde{\mathrm{H}}_n(A) \oplus \widetilde{\mathrm{H}}_n(B) \arrow[d, "\cong"] & & & \\
\ldots \arrow[r] & \widetilde{\mathrm{H}}_n(U\cap V) \arrow[r] & \widetilde{\mathrm{H}}_n(U) \oplus \widetilde{\mathrm{H}}_n(V) \arrow[r] & \widetilde{\mathrm{H}}_n(X) \arrow[r] & \widetilde{\mathrm{H}}_{n-1}(U\cap V) \arrow[r] & \ldots
\end{tikzcd}$$

We make the following observations:

(1) The bottom sequence is exact by the Mayer–Vietoris Theorem 46.5.
(2) As in the proof of Proposition 20.3 we see that the intersection $U \cap V$ deformation retracts to x_0, in particular we obtain from Corollary 42.8 that we have the equality $\widetilde{\mathrm{H}}_n(U\cap V) = \widetilde{\mathrm{H}}_{n-1}(U\cap V) = 0$.
(3) By the proof of Proposition 20.3 we know that A is a deformation retract of U and that B is a deformation retract of V. Therefore the vertical maps are isomorphisms by Corollary 42.8.

The combination of these three observations shows that

$$i_* \oplus j_* \colon \widetilde{\mathrm{H}}_n(A) \oplus \widetilde{\mathrm{H}}_n(B) \;\to\; \widetilde{\mathrm{H}}_n(A \vee B)$$

is an isomorphism. It is clear that $(p_* + q_*) \circ (i_* + j_*)$ is the identity on $\widetilde{\mathrm{H}}_n(A) \oplus \widetilde{\mathrm{H}}_n(B)$, so we also get the second isomorphism. ■

Proposition 47.8 has nothing to do with the content of the present chapter, we could easily have proved it in Chapter 46. But odd as it may sound, soon we will be quite interested in the homology groups of the wedge $\bigvee_{i\in I} X_i$ of an arbitrary family of topological spaces. If I is a finite set, i.e. if we take the wedge of finitely many topological spaces, then we can use Proposition 47.8 and an elementary induction argument to determine the homology groups.

It is a priori less clear how one can deal with the case that I is an infinite set. Fortunately the next proposition allows us to handle the general situation. Proposition 47.8 to the wedge of an arbitrary family of topological spaces.

Proposition 47.9. *Let $\{A_k\}_{k\in K}$ be a family of topological spaces. For each $k \in K$ suppose that we are given a good point $a_k \in A_k$.*[769] *We use these points to form the wedge $\bigvee_{k\in K} A_k$. Given $j \in K$ we denote by*

$$i_j \colon A_j \;\to\; \bigvee_{k\in K} A_k \quad \textit{respectively} \quad p_j \colon \bigvee_{k\in K} A_k \;\to\; A_j$$

the natural inclusion map respectively the natural projection map that we introduced on page 561. Then for every $n \in \mathbb{N}_0$ the map

$$\bigoplus_{k\in K} p_{k*} \colon \widetilde{\mathrm{H}}_n\Big(\bigvee_{k\in K} A_k\Big) \;\to\; \bigoplus_{k\in K} \widetilde{\mathrm{H}}_n(A_k)$$

is an isomorphism where the inverse is given by

$$\bigoplus_{k\in K} i_{k*} \colon \bigoplus_{k\in K} \widetilde{\mathrm{H}}_n(A_k) \;\to\; \widetilde{\mathrm{H}}_n\Big(\bigvee_{k\in K} A_k\Big).$$

We like Proposition 47.9 so much that we provide two different proofs. The first proof is quick and slick, but depending on your taste the first proof might have feel slightly unsatisfactory. The second proof is somewhat longer, but it fits in better with the discussion of this chapter. In fact, as an encore we will suggest one more proof of Proposition 47.9 in Exercise 47.2.

First proof of Proposition 47.9. We have the following isomorphisms:

$$\begin{aligned}\widetilde{\mathrm{H}}_n\Big(\bigvee_{k\in K} A_k\Big) &\underset{\substack{\uparrow\\ \text{definition of the}\\ \text{wedge operation}}}{=} \widetilde{\mathrm{H}}_n\Big(\coprod_{k\in K} A_k \Big/ \coprod_{k\in K}\{a_k\}\Big) \underset{\substack{\uparrow\\ \text{since each } a_k\in A_k \text{ is good we see}\\ \text{that } \sqcup_{k\in K}\{a_k\} \text{ is a good subset of } \sqcup_{k\in K} A_k,\\ \text{thus we can apply Proposition 43.22}}}{\xleftarrow{\cong}} \mathrm{H}_n\Big(\coprod_{k\in K} A_k, \coprod_{k\in K}\{a_k\}\Big) \underset{\substack{\uparrow\\ \text{by the obvious generalization}\\ \text{of Lemma 41.14}}}{\xleftarrow{\cong}} \coprod_{k\in K} \mathrm{H}_k(A_k,\{a_k\})\\ &\underset{\substack{\uparrow\\ \text{Lemma 43.14}}}{\xleftarrow{\cong}} \coprod_{k\in K}\widetilde{\mathrm{H}}_k(A_k).\end{aligned}$$

We leave it to the reader to verify that the isomorphism is indeed given by $\bigoplus_{k\in K} p_{k*}$ and that the inverse is the map $\bigoplus_{k\in K} i_{k*}$. ■

The second proof of Proposition 47.9 is almost embarrassingly similar to the proof of Proposition 20.7.

Second proof of Proposition 47.9. Let $\{A_k\}_{k\in K}$ be a family of topological spaces. For each $k\in K$ suppose that we are given a good point $a_k\in A_k$. We use these points to form the wedge $\bigvee_{k\in K} A_k$. Recall that the fact that a_k is good means that

(1) $\{a_k\}$ is a closed subset of A_k and
(2) that there exists an open neighborhood U_k of a_k in A_k such that a_k is a deformation retract of U_k.

We will show that the map

$$\bigoplus_{k\in K} i_k\colon \bigoplus_{k\in K}\widetilde{\mathrm{H}}_n(A_k) \to \widetilde{\mathrm{H}}_n\Big(\bigvee_{k\in K} A_k\Big)$$

is an isomorphism. Once we will have established this fact it is straightforward to show that the inverse is given by $\bigoplus_{k\in K} p_{k*}$.

The idea of the proof is to show that the map is a monomorphism and an epimorphism, in a way which is reminiscent to the approach taken in the proofs of Proposition 47.4.

Claim. The map

$$\bigoplus_{k\in K} i_k\colon \bigoplus_{k\in K}\widetilde{\mathrm{H}}_n(A_k) \to \widetilde{\mathrm{H}}_n\Big(\bigvee_{k\in K} A_k\Big)$$

is an epimorphism.

[769]Recall that by the discussion on page 604 any point on a topological manifold is good and that by Proposition 36.10 (5) any point on a CW-complex is good.

We introduce the following notation:

(1) We write $B := \bigvee_{k\in K} A_k$ and we set $U := \bigcup_{k\in K} U_k \subset B$.

(2) Given a subset $J \subset K$ we write $B_J := \bigvee_{j\in J} A_j$ and we write $\widehat{B}_J := U \cup B_J$.

Given a finite subset $J \subset K$ we have the following commutative diagram

$$\begin{tikzcd}
\bigoplus_{j\in J} \widetilde{\mathrm{H}}_n(A_j) \arrow[r, "\cong"] \arrow[dd, hook] & \widetilde{\mathrm{H}}_n(B_J) \arrow[dr, "\cong"] \arrow[dd] & \\
& & \widetilde{\mathrm{H}}_n(\widehat{B}_J) \arrow[dl] \\
\bigoplus_{k\in K} \widetilde{\mathrm{H}}_n(A_k) \arrow[r] & \widetilde{\mathrm{H}}_n(B). &
\end{tikzcd}$$

We make the following observations and comments:

(1) The maps are either the obvious algebraic inclusion maps or they are the obvious maps induced by inclusions of topological spaces. It follows almost immediately from the definitions that the diagram commutes.
(2) Since J is finite we obtain, as pointed out above, from Proposition 47.8 that the top horizontal map is an isomorphism.
(3) By Exercise 18.32 we know that B_J is a deformation retract of $\widehat{B}_J$. It follows from Corollary 42.8 (4) that the top diagonal map is an isomorphism.
(4) By Exercise 18.33 we know that given any compact subset $L \subset B$ there exists a finite subset $J \subset K$ with $L \subset \widehat{B}_J$.

Now we show that the above map $\bigoplus_{k\in K} i_{k*}$ is an epimorphism. So suppose we are given a homology class $[\sigma] \in \widetilde{\mathrm{H}}_n(B)$. As in the proof of Lemma 47.5 we see that it follows from (4) there exists a finite subset $J \subset K$ such that $\sigma \in \mathrm{C}_n(\widehat{B}_J)$. But considering the above diagram, and using the fact that the top horizontal map is an isomorphism we see that $[\sigma]$ actually lies in the image of $\bigoplus_{k\in K} i_k$. ⊞

Claim. The map

$$\bigoplus_{k\in K} i_k\colon \bigoplus_{k\in K} \widetilde{\mathrm{H}}_n(A_k) \;\to\; \widetilde{\mathrm{H}}_n\Big(\bigvee_{k\in K} A_k\Big)$$

is a monomorphism.

We continue with the notation that we introduced in the proof of the previous claim. Given any two finite subsets $J \subset \widetilde{J} \subset K$ we have the following commutative diagram

$$\begin{tikzcd}
\bigoplus_{j\in J} \widetilde{\mathrm{H}}_n(A_j) \arrow[rr, "\cong"] \arrow[dr, hook] \arrow[dd, hook] & & \widetilde{\mathrm{H}}_n(B_J) \arrow[dr] \arrow[dd] \arrow[drr, bend left] & & \\
& \bigoplus_{j\in \widetilde{J}} \widetilde{\mathrm{H}}_n(A_j) \arrow[rr, "\cong"] \arrow[dl, hook'] & & \widetilde{\mathrm{H}}_n(B_{\widetilde{J}}) \arrow[r, "\cong"] \arrow[dl] & \widetilde{\mathrm{H}}_n(\widehat{B}_{\widetilde{J}}). \arrow[dll, bend left] \\
\bigoplus_{k\in K} \widetilde{\mathrm{H}}_n(A_k) \arrow[rr] & & \widetilde{\mathrm{H}}_n(B) & &
\end{tikzcd}$$

The diagram together with the argument of Exercise 20.7 shows that the map $\bigoplus_{k\in K} i_{k*}$ is also a monomorphism. ■

47.4. **Moore spaces** ($*$)**.** Now that we have played with homology groups the following question arises: which sequences of abelian groups can appear as the sequence of homology groups $\mathrm{H}_*(X)$ of a non-empty topological space? It turns out that anything goes, there are basically no restrictions, except that we know by Corollary 41.15 that $\mathrm{H}_0(X)$ is necessarily a non-trivial free abelian group.

In other words, we have the following proposition.

Proposition 47.10. *If $\{A_n\}_{n\in\mathbb{N}_0}$ is a sequence of abelian groups. If A_0 is a non-trivial free abelian group, then there exists a* CW*-complex Z such that for any $n \in \mathbb{N}_0$ we have $\mathrm{H}_n(Z) \cong A_n$.*

This proposition is hardly the most exciting result of these notes. But the following definition, which we will need in the proof of Proposition 47.10, will in fact turn out to be convenient.

Definition. Let $n \in \mathbb{N}$ and let π be an abelian group. We say that a non-empty topological space X is a *Moore space of type* $\mathrm{M}(\pi, n)$ if the following conditions are satisfied:

(1) X admits a CW-structure,
(2) for any $i \neq n$ we have $\widetilde{\mathrm{H}}_i(X) = 0$,
(3) $\mathrm{H}_n(X) \cong \pi$.
(4) If $n \geq 2$, then we also demand that X is simply connected.

Example. Let $n \in \mathbb{N}$. It follows from Proposition 14.14 and Proposition 43.4 that S^n is a Moore space of type $(\mathbb{Z}, n)$.

Proposition 47.11. *Given any $n \in \mathbb{N}$ and given any abelian group π there exists an $(n+1)$-dimensional* CW*-complex that has no cells in dimensions $1, \dots, n-1$ and that is a Moore space of type* $\mathrm{M}(\pi, n)$*.*

Proof of Proposition 47.10 assuming Proposition 47.11. Let $\{A_n\}_{n\in\mathbb{N}_0}$ be a sequence of abelian groups such that A_0 is a non-trivial free abelian group. In other words, we have $A_0 \cong \mathbb{Z}^{(S)}$ for some non-empty set S.

By Proposition 47.11 we can find for each $n \in \mathbb{N}$ a CW-complex X_n that is a Moore space of type $\mathrm{M}(\pi, n)$. We consider the wedge $Y := \bigvee_{n\in\mathbb{N}} X_n$. By Lemma 36.32 (4) we can equip Y with a CW-structure. It follows from Proposition 47.9 that for every $n \in \mathbb{N}$ we have $\mathrm{H}_n(Y) \cong \mathrm{H}_n(X_n) \cong A_n$ and that $\mathrm{H}_0(Y) \cong \mathbb{Z}$. Since S is non-empty we can pick a point $* \in S$. We define $Z := Y \sqcup (S \setminus \{*\})$ where we equip $S \setminus \{*\}$ with the discrete topology. It follows from Lemma 36.32 that Z is a CW-complex. Finally it follows from Proposition 41.5 and Lemma 41.13 that Z has the promised homology groups. ■

In the proof of Proposition 47.11 we will also need the following little calculation which will be of independent use.

Lemma 47.12. *Let (X,x_0) be a pointed topological space, let $n\in\mathbb{N}$, let $\omega\in \mathrm{H}_n(I^n,\partial I^n)$ and let $f,g\colon (I^n,\partial I^n)\to (X,x_0)$ be two maps. Then*

$$[f_*(\omega)]+[g_*(\omega)] \quad \underset{\substack{\uparrow\\ \text{product of } f \text{ and } g \text{ as defined on page 1055}}}{=} \quad [(f*g)_*(\omega)] \;\in\; \mathrm{H}_n(X,x_0)=\mathrm{H}_n(X).$$

Proof (∗). We write $J=I^n$, $J_1=[0,\frac{1}{2}]\times I^{n-1}$ and $J_2=[\frac{1}{2},1]\times I^{n-1}$. Furthermore we write $K=\partial J\cup(\{\frac{1}{2}\}\times I^{n-1})$. We refer to Figure 789 for an illustration. For $k=1,2$ we denote by $i_k\colon (J_k,\partial J_k)\to (J,K)$ the inclusion map. Finally we denote by $\varphi_k\colon (J_k,\partial J_k)\to (J,\partial J)$ the obvious homeomorphism given by stretching the last coordinate.

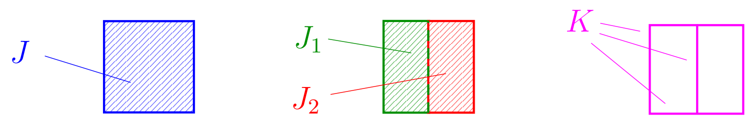

Figure 789. Illustration for the proof of Lemma 47.12.

The map $f*g\colon (J,\partial J)\to (X,x_0)$ factors through a map $(J,K)\to (X,x_0)$ that we also denote by $f*g$. Thus we have the following commutative diagram

$$\begin{tikzcd}
 & (J_1,\partial J_1) \arrow[d, "i_1"] \arrow[dr, "f\circ\varphi_1^{-1}"] & \\
(J,\partial J) \arrow[ur, "\varphi_1"] \arrow[r, "i"] \arrow[dr, "\varphi_2"'] & (J,K) \arrow[r, "f*g"] & (X,x_0) \\
 & (J_2,\partial J_2). \arrow[u, "i_2"'] \arrow[ur, "g\circ\varphi_2^{-1}"'] &
\end{tikzcd}$$

For $k=1,2$ we write $\omega_k=(\varphi_k)_*(\omega)$. Thus it suffices to show that $i_{1*}(\omega_1)+i_{2*}(\omega_2)=i_*(\omega)$. To prove this equality we consider the following commutative diagram[770]

$$\begin{tikzcd}
 & \mathrm{H}_n(J_1,\partial J_1) \arrow[d, "i_1"] \arrow[r, "k_1"] & \mathrm{H}_n(J,\partial J_1\cup J_2) \\
\mathrm{H}_n(J,\partial J) \arrow[ur, "\varphi_1"] \arrow[r, "i"] \arrow[dr, "\varphi_2"'] & \mathrm{H}_n(J,K) \arrow[ur, "j_1"'] \arrow[dr, "j_2"] & \\
 & \mathrm{H}_n(J_2,\partial J_2) \arrow[u, "i_2"'] \arrow[r, "k_2"'] & \mathrm{H}_n(J,J_1\cup\partial J_2),
\end{tikzcd}$$

where all maps, except for φ_1 and φ_2, are induced by the obvious inclusion maps. We have

$$(j_1)(i(\omega)) \;=\; (k_1\circ\varphi_1)(\omega) \;=\; j_1(i_1(\omega_1)) \underset{\substack{\uparrow\\ \text{since } j_1\circ i_2=0}}{=} j_1(i_1(\omega_1))+j_1(i_2(\omega_2))$$

and similarly we show that $(j_2)(i(\omega))=j_2(i_1(\omega_1))+j_2(i_2(\omega_2))$.

[770]To simplify the notation we denote the induced map by the same symbol as the map between pairs of topological spaces, i.e. we drop the subscript $*$ from the notation.

It remains to show that $\ker(j_1)\cap\ker(j_2)=0$. We consider the following diagram

$$\begin{tikzcd}
 & & & \mathrm{H}_n(J,\partial J_1\cup J_2) \\
 & & \mathrm{H}_n(J,K) \arrow[ur, "j_1"] \arrow[dr, "j_2"] & \\
\mathrm{H}_n(J,\partial J_1\cup J_2) \arrow[urr] & & \mathrm{H}_n(J_2,\partial J_2) \arrow[ll, "\cong"'] \arrow[u, "i_2"] \arrow[r, "k_2", "\cong"'] & \mathrm{H}_n(J,J_1\cup\partial J_2).
\end{tikzcd}$$

We make the following clarifications and observations:

(1) The diagonal map to the bottom left of the diagram is induced by the inclusion $(J,\partial J_1\cup J_2)\to(J,K)$.
(2) It follows easily from Corollary 43.18 (2a) and (2c) that the bottom two horizontal maps are isomorphisms.
(3) The diagonal sequence is exact by Proposition 43.15.
(4) It now follows from some mild diagram chasing that $\ker(j_1)\cap\ker(j_2)=0$. ■

Lemma 47.13.

(1) *Let $f\colon S^n\to S^n_1\vee S^n_2$ be the pinching map that we introduced on page 1060. Let $i\in\{1,2\}$. If we denote by $S^n_1\vee S^n_2\to S^n_i=S^n$ the obvious projection, then the induced map $(p_i\circ f)_*\colon \mathrm{H}_n(S^n)\to\mathrm{H}_n(S^n)$ is the identity.*
(2) *Given any $k\in\mathbb{Z}$ there exists a map $g_k\colon S^n\to S^n$ with $g_k(*)=*$ such that $g_{k*}\colon \mathrm{H}_n(S^n)\to\mathrm{H}_n(S^n)$ is given by multiplication by k.*

PROOF.

(1) The first statement follows easily from Lemma 47.12.
(2) The second statement follows almost immediately from Lemma 45.10 together with Proposition 45.7. ■

Now we turn to the proof of Proposition 47.11.

PROOF OF PROPOSITION 47.11. Let $n\in\mathbb{N}$ and let π be an abelian group. Since π is abelian it follows immediately from Lemma 57.16 (1)[771] that there exist sets A and B, a homomorphism $\varphi\colon\mathbb{Z}^{(A)}\to\mathbb{Z}^{(B)}$ and a homomorphism $\rho\colon\mathbb{Z}^{(B)}\to\pi$ such that the following sequence is exact:

$$0\;\to\;\mathbb{Z}^{(A)}\;\xrightarrow{\varphi}\;\mathbb{Z}^{(B)}\;\xrightarrow{\rho}\;\pi\;\to\;0.$$

We let $X:=\bigvee\limits_{a\in A}S^n_a$ and we let $Y:=\bigvee\limits_{b\in B}S^n_b$. By Lemma 36.32 (4) we can equip X with the obvious CW-structure which has precisely one 0-cell and which has precisely one n-cell for each $a\in A$. Similarly we equip Y with the obvious CW-structure. Using Proposition 47.9 and using the standard generator of $\mathrm{H}_n(S^n)$ we can make the obvious identifications $\mathrm{H}_n(X)=\mathbb{Z}^{(A)}$ and $\mathrm{H}_n(Y)=\mathbb{Z}^{(B)}$.

[771]In this case the fact that we refer to a later lemma need not worry us. As the reader can see, the proof of Lemma 57.16 (1) relies only on information that we already know.

Claim. There exists a map $h\colon X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}_n(X) \arrow[r, "h_*"] \arrow[d, "="'] & \mathrm{H}_n(Y) \arrow[d, "="] \\ \mathbb{Z}^{(A)} \arrow[r, "\varphi"] & \mathbb{Z}^{(B)}. \end{tikzcd}$$

This map φ can be constructed explicitly using the following ingredients:

(1) the pinching map $f\colon S^n \to S^n_1 \vee S^n_2$,
(2) the maps $g_k\colon S^n \to S^n$ from Lemma 47.13,
(3) the natural inclusions and projections from page 561,
(4) and the wedge of maps as introduced on page 559.

The notation becomes somewhat cumbersome, thus in a short moment of lassitude we decide to leave it to the reader to fill in the details. A baby example is the content of Exercise 47.3. ⊞

Now let $Z := \mathrm{Cone}(h\colon X \to Y)$ be the mapping cone of the map $h\colon X \to Y$. It follows from the above claim and Lemma 46.18 that we have the following commutative diagram of long exact sequences

$$\begin{tikzcd} \dots \arrow[r] & \overbrace{\mathrm{H}_{n+1}(Y)}^{=0} \arrow[r] \arrow[d] & \mathrm{H}_{n+1}(Z) \arrow[r] \arrow[d, "="] & \mathrm{H}_n(X) \arrow[r, "h_*"] \arrow[d, "="] & \mathrm{H}_n(Y) \arrow[r] \arrow[d, "="] & \mathrm{H}_n(Z) \arrow[r] & \overbrace{\mathrm{H}_{n-1}(X)}^{=0} \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & \mathrm{H}_{n+1}(Z) \arrow[r] & \mathbb{Z}^{(A)} \arrow[r, "\varphi"] & \mathbb{Z}^{(B)} \arrow[r] & \mathrm{H}_n(Z) \arrow[r] & 0 \arrow[r] & \dots. \end{tikzcd}$$

It follows that $\mathrm{H}_n(Z) \cong \mathrm{coker}(\varphi) \cong \pi$. Furthermore, since φ is injective we see that $\mathrm{H}_{n+1}(Z) = 0$. Furthermore we see that all other reduced homology groups of Z are also zero.

Next we equip Z with the natural CW-structure coming from Corollary 36.35. There is one small fly in the ointment. The CW-structure has two 0-cells (namely one corresponding to the tip of the cone and the 0-cell of Y), one 1-cell and otherwise indeed only cells in dimensions n and $n+1$. Let A be the subcomplex that consists of the 1-cell that connects the two 0-cells. By Lemma 36.32 we can view Z/A as a CW-complex which now has only one 0-cell and otherwise only cells in dimension n and $n+1$. By the discussion on page 1108 we know that A is a good subset of Z. It follows from Theorem 43.3 that the reduced homology groups of Z/A coincide with the reduced homology groups of Z. Thus we see that Z/A is the CW-complex of our dreams. ■

Given an abelian group π and $n \in \mathbb{N}$ there are in general infinitely many pairwise non-homeomorphic Moore spaces of type $\mathrm{M}(\pi, n)$. For example the topological spaces $\overline{B}^k$, $k \in \mathbb{N}_0$ are Moore spaces of type $\mathrm{M}(\{e\}, 1)$. It is slightly less absurd to wonder whether Moore spaces are well-defined up to say homotopy equivalence. We formulate this as a question, we will come back to it at a later point.

Question 47.14. *Let π be an abelian group and let $n \in \mathbb{N}$. Are any two Moore spaces of type (π, n) homotopy equivalent?*

Exercises for Chapter 47.

Exercise 47.1. Let X and Y be topological spaces and let $x_0 \in X$ and $y_0 \in Y$ be good points. We write $\widetilde{x}_0 = [(x_0, 0)] \in \Sigma(X)$ and $\widetilde{y}_0 = [(y_0, 0)] \in \Sigma(Y)$.

(a) Show that $\widetilde{x}_0 \in \Sigma(X)$ and $\widetilde{y}_0 \in \Sigma(Y)$ are good points. We use these points to define the wedge $\Sigma(X) \vee \Sigma(Y)$.

(b) Let $i\colon X \to X \vee Y$ and $j\colon Y \to X \vee Y$ be the obvious maps. Show that the map

$$(\Sigma(i) \vee \Sigma(j))_*\colon \mathrm{H}_k(\Sigma(X) \vee \Sigma(Y)) \;\to\; \mathrm{H}_k(\Sigma(X \vee Y))$$

is an isomorphism for each k.
Remark. You could make use of the *natural* isomorphism of Lemma 46.8.

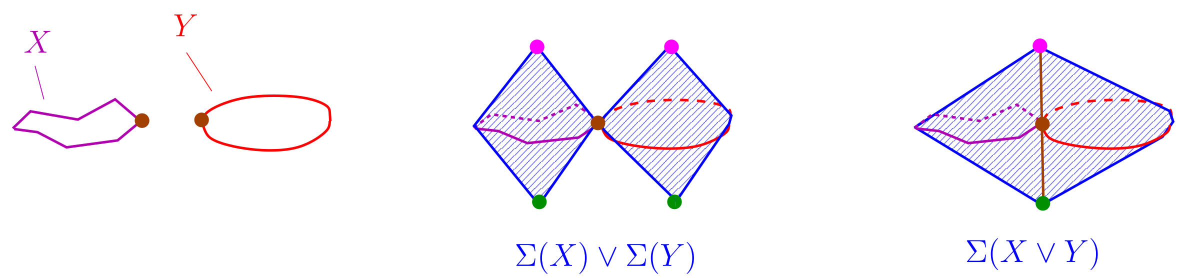

FIGURE 790. Illustration of Exercise 47.1.

Exercise 47.2. Let $\{A_k\}_{k\in K}$ be a family of topological spaces. For each $k \in K$ suppose that we are given a good point $a_k \in A_k$. By definition, see page 604, this means that given any $k \in K$ the following statements hold:

(1) $\{a_k\}$ is a closed subset of A_k and
(2) there exists an open neighborhood U_k of a_k in A_k such that a_k is a deformation retract of U_k.

Provide a third proof of Proposition 47.9 using the following approach. First we write $W := \bigcup\limits_{k\in K}(A_k \setminus \{a_k\})$. Now apply the (natural) reduced Mayer–Vietoris sequences from Theorem 46.5 to the decompositions

$$\bigvee_{k\in K} A_k \;=\; V \cup W \quad \text{and} \quad \bigsqcup_{k\in K} A_k \;=\; \Big(\bigsqcup_{k\in K}(U_k \setminus \{a_k\})\Big) \cup \Big(\bigsqcup_{k\in K}(A_k \setminus \{a_k\})\Big).$$

Remark. Do not forget to make use of the analogue of Exercise 20.7.

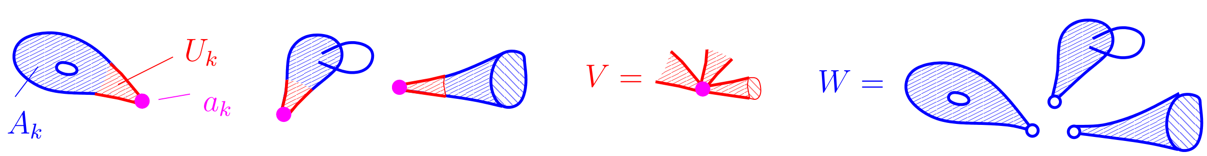

FIGURE 791. Illustration for Exercise 47.2.

Exercise 47.3. Let $n \in \mathbb{N}$ and let $X = S^n \vee S^n$ and let $Y = S^n \vee S^n \vee S^n$. As in the proof of Proposition 47.11 we can make the identification $\mathrm{H}_n(X) = \mathbb{Z}^2$ and $\mathrm{H}_n(Y) = \mathbb{Z}^3$. Show that there exists a map $\varphi\colon X \to Y$ such that the map $\varphi_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is represented by the matrix

$$\begin{pmatrix} 2 & 1 \\ -5 & 1 \\ 0 & 7 \end{pmatrix}.$$

48. Cellular homology

We saw in Theorem 37.9 and Proposition 37.11 that it is quite easy to compute the fundamental group of a finite CW-complex. In this section we will develop a relatively straightforward method for computing the homology groups of a finite CW-complex. We will use this method for finally determining the homology groups of the surfaces of genus ≥ 2 and all the projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$.

48.1. **The definition of cellular homology.** Now we turn, as promised, to the study of homology groups of CW-complexes. We need to recall two definitions:

(1) First recall that in Lemma 41.1 we gave a homeomorphism $f\colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, \partial\overline{B}^n)$. This homeomorphism together with Lemma 45.2 gives us the following isomorphism:
$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\cong} & \mathrm{H}_n(\Delta^n, \partial\Delta^n) & \underset{\cong}{\xrightarrow{f_*}} & \mathrm{H}_n(\overline{B}^n, \partial\overline{B}^n) \\ 1 & \mapsto & [\mathrm{id}_{\Delta^n}] & & \end{array}$$
In the following we will usually identify $\mathrm{H}_n(\overline{B}^n, \partial\overline{B}^n)$ with $\mathbb{Z}$ using the above isomorphism.

(2) Let X be a topological space and let $A \subset X$ be a non-empty subset. As on page 1129 we consider the map
$$\begin{tikzcd} \mathrm{H}_n(X,A) \arrow[r] \arrow[rr, bend right, "\text{ю}"'] & \mathrm{H}_n(X/A, A/A) & \widetilde{\mathrm{H}}_n(X/A). \arrow[l, "\text{Lemma 43.14}"', "\cong"] \end{tikzcd}$$
By Lemma 43.21 we know that the map is a natural homomorphism.

Now we can formulate one of the key lemmas regarding homology groups of CW-complexes.

Lemma 48.1. *For every* CW-*complex X the following statements hold:*

(1) *Given $n \in \mathbb{N}_0$ we denote by I_n the set of n-cells and given $i \in I_n$ we denote by $\Phi_i\colon \overline{B}_i^n \to X^n$ the corresponding characteristic map. Then for every $k \in \mathbb{N}_0$ we have the following commutative diagram where all maps are isomorphisms:*
$$\begin{tikzcd} & \bigoplus\limits_{i\in I_n} \mathrm{H}_k(X^n, X^n \setminus \Phi_i(B_i)) & \\ \bigoplus\limits_{i\in I_n} \mathrm{H}_k(\overline{B}_i^n, \partial\overline{B}_i^n) \arrow[ur, "\bigoplus \Phi_{i*}"] \arrow[rr, "\bigoplus \Phi_{i*}"] \arrow[d, "\bigoplus \text{ю}"] & & \mathrm{H}_k(X^n, X^{n-1}) \arrow[ul] \arrow[d, "\text{ю}"'] \\ \bigoplus\limits_{i\in I_n} \widetilde{\mathrm{H}}_k(\overline{B}_i^n/\partial\overline{B}_i^n) \arrow[rr, "\bigoplus \Phi_{i*}"] \arrow[dr] & & \widetilde{\mathrm{H}}_k(X^n/X^{n-1}). \\ & \widetilde{\mathrm{H}}_k\Big(\bigvee\limits_{i\in I_n} \overline{B}_i^n/\partial\overline{B}_i^n\Big) \arrow[ur, "(\vee\Phi_i)_*"'] & \end{tikzcd}$$
In particular the following holds:

(a) *For $k \neq n$ we have $\mathrm{H}_k(X^n, X^{n-1}) = 0$.*

(b) *The group* $\mathrm{H}_n(X^n, X^{n-1})$ *is a free abelian group where the rank equals the cardinality of the set of* n*-cells in the* CW*-structure.*

(2) *For* $k > n$ *we have* $\mathrm{H}_k(X^n) = 0$.

(3) *For* $k < n$ *the inclusion* $i\colon X^n \to X$ *induces an isomorphism* $i_*\colon \mathrm{H}_k(X^n) \xrightarrow{\cong} \mathrm{H}_k(X)$.

(4) *The inclusion* $i\colon X^n \to X$ *induces an epimorphism* $i_*\colon \mathrm{H}_n(X^n) \twoheadrightarrow \mathrm{H}_n(X)$.

Remark. The statements (3) and (4) of Lemma 48.1 can be viewed as "homology analogues" of Proposition 40.9.

Proof (∗). Let X be a CW-complex.

(1) We consider the following diagram

$$\begin{tikzcd}
\bigoplus\limits_{i\in I_n} \mathrm{H}_k(\overline{B}^n_i, \partial\overline{B}^n_i) \arrow[rr, "\bigoplus \Phi_{i*}", "(e)"'] \arrow[dr, "\bigoplus \Phi_{i*}", "(h)"'] \arrow[ddd, "(a)"', "\bigoplus \text{ю}"] & & \mathrm{H}_k(X^n, X^{n-1}) \arrow[dl, "(i)"] \arrow[ddd, "\text{ю}"', "(d)"] \\
& \bigoplus\limits_{i\in I_n} \mathrm{H}_k(X^n, X^n \setminus \Phi_i(B_i)) \arrow[d, "(g)"', "\bigoplus \text{ю}"] & \\
& \bigoplus\limits_{i\in I_n} \widetilde{\mathrm{H}}_k(X^n/(X^n \setminus \Phi_i(B_i))) & \\
\bigoplus\limits_{i\in I_n} \widetilde{\mathrm{H}}_k\big(\overline{B}^n_i/\partial\overline{B}^n_i\big) \arrow[ur, "\bigoplus \Phi_{i*}", "(f)"'] \arrow[rr, "\bigoplus \Phi_{i*}"] \arrow[dr, "(b)"] & & \widetilde{\mathrm{H}}_k(X^n/X^{n-1}). \\
& \widetilde{\mathrm{H}}_k\Big(\bigvee\limits_{i\in I_n} \overline{B}^n_i/\partial\overline{B}^n_i\Big) \arrow[ur, "(c)", "(\vee\Phi_i)_*"'] &
\end{tikzcd}$$

We make the following observations:

(i) It follows from the definitions that the two triangles at the top and bottom commute.

(ii) It follows almost immediately from the naturality of the maps ю that the square in the middle commutes and that the quadrilateral to the left commutes.

It remains to show that the maps numbered with (a) to (h) are isomorphisms.

(a) This map is an isomorphism by Proposition 43.22.

(b) This map is an isomorphism by Proposition 47.9.

(c) This map is an isomorphism by Lemma 36.33 (1).

(d) It follows from Proposition 43.22 together with Proposition 36.10 (8) that this map is an isomorphism.

(e) It follows from the commutativity of the diagram and the fact that (a) to (d) are isomorphisms that this map is an isomorphism.

(f) This map is an isomorphism by Lemma 36.33 (3).

(g) It follows immediately from Proposition 43.22 together with Proposition 36.10 (8) and Lemma 36.33 (3) that this map is an isomorphism.

(h) It follows from the fact that the quadrilateral commutes and the fact that (a), (f) and (g) are isomorphisms that this map is an isomorphism.

(i) It follows from the fact that the top triangle commutes and the fact that (e) and (h) are isomorphisms that this map is an isomorphism.

The remaining statements of (1) are now an immediate consequence of Lemma 45.1.

(2) We fix $k \in \mathbb{N}$. We want to show that $\mathrm{H}_k(X^n) = 0$ for $n = 0, \dots, k-1$. First of all we have $\mathrm{H}_k(X^0) = 0$ by Proposition 41.5 and Lemma 41.14.[772] Now assume that we already know that $\mathrm{H}_k(X^n) = 0$ for some $n < k-1$. We want to show that $\mathrm{H}_k(X^{n+1}) = 0$. We consider the long exact sequence of the pair (X^{n+1}, X^n):

$$\dots \to \mathrm{H}_{k+1}(X^{n+1}, X^n) \to \underbrace{\mathrm{H}_k(X^n)}_{=0} \to \mathrm{H}_k(X^{n+1}) \to \underbrace{\mathrm{H}_k(X^{n+1}, X^n)}_{\substack{= 0 \text{ by (1) since } n < k\text{-}1 \\ \text{i.e. since } k \neq n+1}} \to \dots$$

Thus we obtain that indeed $\mathrm{H}_k(X^{n+1}) = 0$.

(3) We fix $k \in \mathbb{N}_0$. Let $m > k$. This time we consider the following excerpt from the long exact sequence of the pair (X^{m+1}, X^m):

$$\dots \to \underbrace{\mathrm{H}_{k+1}(X^{m+1}, X^m)}_{= 0 \text{ by (1) since } k < m} \to \mathrm{H}_k(X^m) \to \mathrm{H}_k(X^{m+1}) \to \underbrace{\mathrm{H}_k(X^{m+1}, X^m)}_{= 0 \text{ by (1) since } k < m} \to \dots$$

This implies in particular that for each $k < n \leq l$ we have inclusion induced isomorphisms $\mathrm{H}_k(X^n) \xrightarrow{\cong} \mathrm{H}_k(X^{n+1}) \xrightarrow{\cong} \dots \xrightarrow{\cong} \mathrm{H}_k(X^l)$. In particular, if X is finite-dimensional, then claim (3) follows immediately from this observation. The infinite-dimensional case follows from the finite-dimensional case together with Proposition 47.4.

(4) In (3) we showed that $\mathrm{H}_n(X^{n+1}) \to \mathrm{H}_n(X)$ is an isomorphism. Hence it suffices to show that $\mathrm{H}_n(X^n) \to \mathrm{H}_n(X^{n+1})$ is an epimorphism. To show this we consider the following excerpt from long exact sequence of the pair (X^{n+1}, X^n):

$$\dots \to \mathrm{H}_{n+1}(X^{n+1}, X^n) \to \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^{n+1}) \to \underbrace{\mathrm{H}_n(X^{n+1}, X^n)}_{= 0 \text{ by (1)}} \to \dots$$

We see that the inclusion induced map $\mathrm{H}_n(X^n) \to \mathrm{H}_n(X^{n+1})$ is indeed an epimorphism. ■

Definition. Given a CW-complex X, we denote by $d = d_n$ the map

$$\begin{tikzcd} \mathrm{H}_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[rr, bend right, "=:d=d_n"'] & \mathrm{H}_{n-1}(X^{n-1}) \arrow[r, "j_{n-1}"] & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) \end{tikzcd}$$

where ∂_n is the connecting map in the long exact sequence of the pair (X^n, X^{n-1}) and where j_{n-1} is the map induced by the projection $\mathrm{C}_{n-1}(X^{n-1}) \to \mathrm{C}_{n-1}(X^{n-1}, X^{n-2})$. We refer to d_n as the *cellular boundary map.*[773]

[772] Here we use that X^0 has the discrete topology.

[773] It follows easily from the definitions that d_n is precisely the connecting homomorphism of the long exact sequence of the triple (X^n, X^{n-1}, X^{n-2}).

The name "cellular boundary map" already suggests that the maps d_n have the following important property:

Lemma 48.2. *For all $n \in \mathbb{N}_0$ we have $d_n \circ d_{n+1} = 0$.*

Proof. We consider the following commutative diagram of maps

$$\begin{tikzcd}
& \mathrm{H}_n(X^n) \arrow[dr, "j_n"] & & & \\
\mathrm{H}_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & \mathrm{H}_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) \\
& & & \mathrm{H}_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] &
\end{tikzcd}$$

The map $\partial_n \circ j_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_{n-1}(X^{n-1})$ is the composition of two successive maps in the long exact sequence of the pair (X^n, X^{n-1}), i.e. the map $\partial_n \circ j_n$ is the zero map. Since the diagram commutes we obtain that $d_n \circ d_{n+1} = 0$. ■

Definition. Let X be a CW-complex.

(1) We write $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$ and we refer to $(\mathrm{C}_*^{\mathrm{CW}}(X), d_n)$ as the *cellular chain complex* of the CW-complex X.[774]

(2) We denote the homology groups of the cellular chain complex by $\mathrm{H}_n^{\mathrm{CW}}(X)$ and we refer to these groups as the *cellular homology groups of the* CW-*complex* X.

Remark. Let us summarize the content of Lemma 48.1 (1) in light of the above definition. Given a CW-complex X and given $n \in \mathbb{N}_0$ the n-th cellular chain group $\mathrm{C}_n^{\mathrm{CW}}(X)$ is a free abelian group and the characteristic maps of the n-cells give canonically rise to a basis of this free abelian group.

Example. We consider the n-sphere S^n for $n \geq 2$. On page 935 we showed that we can view S^n as a CW-complex with one 0-cell and one n-cell. The cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X)$ is thus of the form

$$0 \to \underbrace{\mathbb{Z}}_{=\mathrm{C}_n^{\mathrm{CW}}(X)} \to 0 \to \ldots \to 0 \to \underbrace{\mathbb{Z}}_{=\mathrm{C}_0^{\mathrm{CW}}(X)} \to 0.$$

Since $n \geq 2$ both the $\mathbb{Z}$'s are separated by at least one zero group. The cellular boundary maps are therefore necessarily the zero maps and we obtain that

$$\mathrm{H}_i^{\mathrm{CW}}(X) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

These are of course isomorphic to the "usual" homology groups of S^n. In the next section we will see that this is not a coincidence.

The "usual" homology groups define functors from the category of topological spaces to the category of abelian groups. We will now see that in a more restricted setting cellular homology groups also define a functor.

[774] It follows from Lemma 48.2 that $(\mathrm{H}_n(X^n, X^{n-1}), d_n)$ is indeed a chain complex.

Definition.

(1) A map $f\colon X \to Y$ between CW-complexes is called *cellular* if for each $n \in \mathbb{N}_0$ we have $f(X^n) \subset Y^n$.
(2) As before we denote by $\mathcal{CW}$ the category of CW-complexes where the objects are CW-complexes and where the morphisms are cellular maps.

We conclude this section with the following lemma.

Lemma 48.3.

(1) *Every cellular map $f\colon X \to Y$ induces a chain map*
$$f_*\colon \mathrm{C}_*^{\mathrm{CW}}(X) \;\to\; \mathrm{C}_*^{\mathrm{CW}}(Y)$$
and hence also maps
$$f_*\colon \mathrm{H}_*^{\mathrm{CW}}(X) \;\to\; \mathrm{H}_*^{\mathrm{CW}}(Y)$$
on cellular homology.
(2) *For each $n \in \mathbb{N}_0$ the maps*
$$\begin{aligned} X &\mapsto \mathrm{H}_n^{\mathrm{CW}}(X) \\ (f\colon X \to Y) &\mapsto (f_*\colon \mathrm{H}_n^{\mathrm{CW}}(X) \to \mathrm{H}_n^{\mathrm{CW}}(Y)) \end{aligned}$$
define a covariant functor from the category $\mathcal{CW}$ of CW*-complexes to the category $\mathcal{AbGr}$ of abelian groups.*

Proof.

(1) Let $f\colon X \to Y$ be a cellular map. Since the map f is cellular we get for every $n \in \mathbb{N}_0$ an induced map $\mathrm{H}_n(X^n, X^{n-1}) \to \mathrm{H}_n(Y^n, Y^{n-1})$, i.e. we get an induced map $f_*\colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_n^{\mathrm{CW}}(Y)$. It follows almost immediately from the naturality of the connecting homomorphism, see Proposition 43.15 (2), that the above maps $f_*\colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_n^{\mathrm{CW}}(Y)$, $n \in \mathbb{N}_0$ actually form a chain map.
(2) This statement follows easily from the fact that the usual chain complexes and homology groups are covariantly functorial. ■

48.2. **The relationship between cellular and singular homology.** Now we can formulate and prove the following proposition.

Proposition 48.4. *Let $n \in \mathbb{N}_0$.*

(1) *For each* CW*-complex X there exists a uniquely determined isomorphism*
$$\Phi_X\colon \mathrm{H}_n(X) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{CW}}(X)$$
with the property that the following diagram commutes:[775]
$$\begin{tikzcd} & \mathrm{H}_n(X) \arrow[dr, "\Phi_X"] & \\ \mathrm{H}_n(X^n) \arrow[ur] \arrow[dr, "j_n"'] & & \mathrm{H}_n^{\mathrm{CW}}(X). \\ & \ker(d_n) \arrow[ur, two heads] & \end{tikzcd}$$

(2) *Recall that we denote by $\mathcal{CW}$ the category of* CW*-complexes where the objects are* CW*-complexes and where the morphisms are cellular maps. The homomorphisms from* (1) *define a natural isomorphism between the functors $X \mapsto \mathrm{H}_n(X)$ and the functor $X \mapsto \mathrm{H}_n^{\mathrm{CW}}(X)$. In other words, for every cellular map $f\colon X \to Y$ and any $n \in \mathbb{N}_0$ the following diagram commutes*

$$\begin{tikzcd} \mathrm{H}_n(X) \arrow[r, "f_*"] \arrow[d, "\Phi_X"', "\cong"] & \mathrm{H}_n(Y) \arrow[d, "\cong"', "\Phi_Y"] \\ \mathrm{H}_n^{\mathrm{CW}}(X) \arrow[r, "f_*"] & \mathrm{H}_n^{\mathrm{CW}}(Y) \end{tikzcd}.$$

Remark. Later on, in Proposition 49.10, we will give an alternative proof that singular and cellular homology groups are naturally isomorphic. The latter proof has the advantage that it shows a stronger statement, namely it shows in fact that the singular and cellular chain complexes are chain homotopy equivalent.

Example. Even though the cellular boundary maps in the cellular chain complex are still somewhat mysterious we can already use the proposition to do some calculations. As an example, we return to the *n-dimensional complex projective space* $\mathbb{C}\mathrm{P}^n$ which he defined on page 194 as

$$\mathbb{C}\mathrm{P}^n \;:=\; (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\}).$$

In Lemma 36.1 we showed that we can view $\mathbb{C}\mathrm{P}^n$ as a CW-complex which admits exactly one cell in dimensions $0, 2, \dots, 2n$ and which admits no other cells. In particular the cellular chain complex is of the form

$$0 \to \underbrace{\mathrm{C}_{2n}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to \underbrace{\mathrm{C}_{2n-1}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{=0} \to \cdots \to \underbrace{\mathrm{C}_{2}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to \underbrace{\mathrm{C}_{1}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{=0} \to \underbrace{\mathrm{C}_{0}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to 0.$$

Since every other group is trivial it follows that all boundary maps are also zero. In particular the homology groups equal the cellular chain groups. Thus we have shown that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \;\underset{\substack{\uparrow \\ \text{Proposition 48.4}}}{\cong}\; \mathrm{H}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n) \;\cong\; \mathrm{C}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \dots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

It follows immediately from inspecting the cellular chain complex that for any $k \le l$ and any $i \in \{0, \dots, 2k\}$ the inclusion induced map

$$\mathrm{H}_i^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^k; \mathbb{Z}) \;\xrightarrow{\cong}\; \mathrm{H}_i^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^l; \mathbb{Z})$$

is an isomorphism. By the naturality of the isomorphism of Proposition 48.4 the same conclusion also holds for the inclusion induced map

$$\mathrm{H}_i(\mathbb{C}\mathrm{P}^k; \mathbb{Z}) \;\xrightarrow{\cong}\; \mathrm{H}_i(\mathbb{C}\mathrm{P}^l; \mathbb{Z})$$

on singular homology. It follows from Lemma 36.6 (2) that we can view the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$ as a CW-complex that has precisely one cell

[775] Here we again denote by $j_n\colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ the obvious map. In the long exact sequence of the pair (X^n, X^{n-1}) we see that the image of $\mathrm{H}_n(X^n)$ in $\mathrm{H}_n(X^n, X^{n-1})$ lies in the kernel of the connecting homomorphism ∂_n. This implies that the map $j_n\colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ takes values in $\ker(d_n)$.

in each even dimension and no cells in odd dimensions. From the arguments above it now follows that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty) \;\cong\; \mathrm{H}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^\infty) \;\cong\; \mathrm{C}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^\infty) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

Furthermore, as above we see that the inclusion induced map

$$\mathrm{H}_i(\mathbb{C}\mathrm{P}^k;\mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_i(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$$

is an isomorphism for $i \leq k+1$.

Proof of Proposition 48.4. We add a couple of maps to the commutative diagram that we had already encountered in the proof of Lemma 48.2:

$$\begin{tikzcd}
 & & & & & 0 & \\
0 \arrow[drr] & & & \mathrm{H}_n(X^{n+1}) \arrow[urr] & & & \\
 & & \mathrm{H}_n(X^n) \arrow[ur, two heads] \arrow[dr, hook, "j_n"] & & & & \\
\mathrm{H}_{n+1}(X^{n+1},X^n) \arrow[urr, "\partial_{n+1}"] \arrow[rrr, "d_{n+1}"] & & & \mathrm{H}_n(X^n,X^{n-1}) \arrow[rrr, "d_n"] \arrow[dr, "\partial_n"'] & & & \mathrm{H}_{n-1}(X^{n-1},X^{n-2}). \\
 & & & & \mathrm{H}_{n-1}(X^{n-1}) \arrow[urr, hook, "j_{n-1}"'] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

Claim. All the diagonal sequences in the above commutative diagram are exact.

From Lemma 48.1 (1) and (2) we know that

$$\mathrm{H}_n(X^{n+1},X^n) \;=\; 0, \quad \mathrm{H}_n(X^{n-1}) \;=\; 0 \quad \text{and} \quad \mathrm{H}_{n-1}(X^{n-2}) \;=\; 0.$$

It thus follows that the diagonal sequences in the above commutative diagram are just the long exact sequences of the corresponding pairs of topological spaces. In particular all the diagonal sequences are exact. ⊞

Next we prove the following claim.

Claim. The map $j_n\colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n,X^{n-1})$ induces an isomorphism

$$\Phi\colon \mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \;\to\; \ker(d_n)/\operatorname{im}(d_{n+1}) = \mathrm{H}_n^{\mathrm{CW}}(X).$$

We will prove this claim once again by a "diagram chase".[776]

(1) Since $\partial_n \circ j_n = 0$ it follows that $j_n(\mathrm{H}_n(X^n)) \subset \ker(d_n)$.

(2) The map j_{n-1} is injective, i.e. $\ker(d_n) = \ker(\partial_n) = \operatorname{im}(j_n)$, i.e. the map j_n induces an epimorphism

$$\mathrm{H}_n(X^n) \;\to\; \ker(d_n).$$

Since the left triangle commutes it also induces an epimorphism

$$\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \;\to\; \ker(d_n)/\operatorname{im}(d_{n+1}).$$

[776]It is arguably best not to read the remainder of the proof but to go on the chase on one's own.

(3) If an element $j_n(c) \in \mathrm{H}_n(X^n, X^{n-1})$ lies in the image of $d_{n+1} = j_n \circ \partial_{n+1}$, then $j_n(c) = j_n(\partial_{n+1}(d))$ for some $d \in \mathrm{H}_{n+1}(X^{n+1}, X^n)$. Since j_n is injective it follows that $c = \partial_{n+1}(d)$. Thus we have now shown that the map

$$\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \ \to\ \ker(d_n)/\operatorname{im}(d_{n+1})$$

is injective. ⊞

Next we consider the diagram which gives us the definition of the isomorphism Φ:

$$\begin{tikzcd}
\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \arrow[r, "\substack{\text{isomorphism since in the above diagram}\\ \text{the left diagonal sequence is exact}}"] \arrow[d, "\cong", "\substack{\text{by the claim we know that}\\ j_n \text{ induces an isomorphism}}"'] & \mathrm{H}_n(X^{n+1}) \arrow[r, "\substack{\text{isomorphism by}\\ \text{Lemma 48.1 (3)}}"] & \mathrm{H}_n(X). \arrow[d, "=:\Phi"] \\
\ker(d_n)/\operatorname{im}(d_{n+1}) \arrow[rr, "\mathrm{id}"'] & & \mathrm{H}_n^{\mathrm{CW}}(X)
\end{tikzcd}$$

By construction the isomorphism Φ has the property that the following diagram commutes:

$$\begin{tikzcd}
& \mathrm{H}_n(X) \arrow[dr, "\Phi"] & \\
\mathrm{H}_n(X^n) \arrow[ur, two heads] \arrow[dr, "j_n"'] & & \mathrm{H}_n^{\mathrm{CW}}(X). \\
& \ker(d_n) \arrow[ur, two heads] &
\end{tikzcd}$$

Note that the top-left diagonal map is an epimorphism by Lemma 48.1 (4). This implies immediately that Φ is uniquely determined.

Finally we leave it as a straightforward exercise to the reader that the maps Φ define a natural isomorphism. ■

On page 1262 we already saw that Proposition 48.4 can be useful, even if we cannot determine the boundary maps d_n. Another, more theoretical application is, that the cellular chain complex is "much smaller" than the singular chain complex. For example, if X is a CW-complex with finitely many cells, then the chain group $\mathrm{C}_*^{\mathrm{CW}}(X)$ in the cellular chain complex are *finitely generated* free abelian groups. This is in contract to the singular chain groups $\mathrm{C}_*(X)$ which are in almost all cases free abelian groups of uncountable rank.[777]

Using this observation we can now easily prove the following proposition. It can be viewed as an analogue to Proposition 37.13.

Proposition 48.5. *Let X be a CW-complex. The following statements hold:*

(1) *If X contains precisely $d \in \mathbb{N}_0$ cells of dimension n, then $\mathrm{H}_n(X)$ is generated by at most d elements. In particular, if X has no cells of dimension n, then $\mathrm{H}_n(X) = 0$. Similarly, if X contains only countable many cells of dimension n, then $\mathrm{H}_n(X)$ is countable.*

[777]For which topological spaces and which n is the singular chain complex $\mathrm{C}_n(X)$ finitely generated? Can you give a complete answer?

(2) *If X consists only of finitely many cells, then all homology groups of X are finitely generated groups.*
(3) *If X is finite-dimensional, then*
$$\mathrm{H}_k(X) \;=\; 0 \quad \textit{for all } k > \dim(X).$$
(4) *If X is an n-dimensional* CW*-complex, then $\mathrm{H}_n(X)$ is a free abelian group.*

Example. It follows from Proposition 48.5 that for any topological graph G we have $\mathrm{H}_i(G) = 0$ for every $i \geq 2$ and that $\mathrm{H}_1(G)$ is a free abelian group.

Remark.

(1) Later on, in Proposition 49.10, we will give an alternative proof that singular and cellular homology groups are naturally isomorphic. The latter proof has the advantage that it shows a stronger statement, namely it shows in fact that the singular and cellular chain complexes are chain homotopy equivalent.
(2) In Chapter 106, in particular in Proposition 106.8, we will formulate and prove a pretty elementary algebraic statement that allows us in Proposition 106.10 (2) to prove that the "size" of homology groups give a lower bound on the number of cells in a CW-structure. This result can be viewed as a refinement of Proposition 48.5 (1).

Proof. Let X be a CW-complex X.

(1) If X contains precisely $d \in \mathbb{N}_0$ cells of dimension n, then it follows from Lemma 48.1 that $\mathrm{C}_n^{\mathrm{CW}}(X) \cong \mathbb{Z}^d$. By Lemma 19.8 any subgroup U of $\mathbb{Z}^d$ is a free abelian group such that $\mathrm{rank}(U) \leq d$. Thus it follows that
$$\mathrm{H}_n(X) \;\cong\; \underbrace{\ker\big(d_n\colon\ \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_{n-1}^{\mathrm{CW}}(X)\big)}_{\text{free abelian group of rank } \leq d} \big/ \operatorname{im}\big(d_{n+1}\colon\ \mathrm{C}_{n+1}^{\mathrm{CW}}(X) \to \mathrm{C}_n^{\mathrm{CW}}(X)\big)$$
is generated by at most d elements. The argument in the case that X has countably many cells of dimension n is very similar, we just need to use Lemma 1.7 several times.
(2) This statement follows immediately from (1).
(3) This statement is in fact already a consequence of Lemma 48.1 (2).
(4) Now suppose that X is an n-dimensional CW-complex X. We have
$$\mathrm{H}_n(X) \underset{\substack{\uparrow \\ \text{Proposition 48.4}}}{\cong} \mathrm{H}_n^{\mathrm{CW}}(X) \underset{\substack{\uparrow \\ \text{we have } \mathrm{C}_{n+1}^{\mathrm{CW}}(X)=0 \text{ since } X \text{ is } n\text{-dimensional}}}{=} \ker\big(d_n\colon\ \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_{n-1}^{\mathrm{CW}}(X)\big),$$
i.e. $\mathrm{H}_n(X)$ is a subgroup of the free abelian group $\mathrm{C}_n^{\mathrm{CW}}(X)$. It follows from Lemma 19.2 that the group $\mathrm{H}_n(X)$ itself is a free abelian group. ■

By Theorem 64.5, which we already mentioned in a sneak preview on page 936 we know that every compact n-dimensional smooth manifold admits a CW-structure with finitely many cells and such that the maximal dimension of a cell is n. If we combine this result with Proposition 48.5 we obtain immediately the following useful result.

Proposition 64.6. *Let M be a compact n-dimensional smooth manifold. Then the following holds:*

(1) *All homology groups are finitely generated abelian groups.*
(2) *For any $k > n$ we have $\mathrm{H}_k(M) = 0$.*

Remark. We have now shown that for a CW-complex X the singular homology groups $\mathrm{H}_n(X)$ are isomorphic to the cellular homology groups $\mathrm{H}_n^{\mathrm{CW}}(X)$. One might now ask, whether one can generalize this statement, namely whether the singular chain complex $(\mathrm{C}_*(X), \partial_*)$ and the cellular chain complex $(\mathrm{C}_*^{\mathrm{CW}}, d_*)$ are in fact chain homotopic. This is indeed the case. But the proof requires more homological algebra than we have at the moment at our disposal. Therefore we postpone the proof to a later chapter.

48.3. **The cellular boundary maps.** If we want to use Proposition 48.4 to calculate homology groups of CW-complexes we need to develop tools for computing the cellular boundary maps d_n in the cellular chain complex. Let X be a CW-complex and let $n \in \mathbb{N}_0$. Throughout this section we will use the following notation:

(1) We denote by I_n the set of n-cells of X.
(2) Given an n-cell α we denote by $\varphi_\alpha \colon S_\alpha^{n-1} \to X^{n-1}$ the corresponding attaching map and we denote by $\Phi_\alpha \colon \overline{B}_\alpha^n \to X^n$ the corresponding characteristic map.
(3) We denote by

$$\Psi_n \colon \bigvee_{\alpha \in I_n} \overline{B}_\alpha^n / \partial \overline{B}_\alpha^n \quad \to \quad X^n / X^{n-1}$$

the map that is given by

$$x \quad \mapsto \quad \Phi_\alpha(x), \qquad \text{if } x \in \overline{B}_\alpha^n.$$

It is straightforward to verify that Ψ_n is a homeomorphism.
(4) Given $\beta \in I_n$ we denote by

$$i_\beta \colon \overline{B}_\beta^n / \partial \overline{B}_\beta^n \to \bigvee_{\alpha \in I_n} \overline{B}_\alpha^n / \partial \overline{B}_\alpha^n \quad \text{and} \quad p_\beta \colon \bigvee_{\alpha \in I_n} \overline{B}_\alpha^n / \partial \overline{B}_\alpha^n \to \overline{B}_\beta^n / \partial \overline{B}_\beta^n,$$

the obvious inclusion map respectively the obvious projection map. By Proposition 47.9 the maps

$$\bigoplus_{\beta \in I} \mathrm{H}_n\left(\overline{B}_\beta^n / \partial \overline{B}_\beta^n\right) \underset{\bigoplus i_\beta}{\overset{\bigoplus p_\beta}{\rightleftarrows}} \mathrm{H}_n\Big(\bigvee_{\beta \in I_n} \overline{B}_\beta^n / \partial \overline{B}_\beta^n\Big).$$

are inverses of one another.
(5) Given $\alpha \in I_n$ we denote, by a slight abuse of notation, the standard generator of $\mathrm{H}_n(\overline{B}_\alpha^n, \partial \overline{B}_\alpha^n) \cong \mathbb{Z}$, as defined on page 1176, by α. Put differently, we make the identification $\mathrm{H}_n(\overline{B}_\alpha^n, \partial \overline{B}_\alpha^n) = \mathbb{Z} \cdot \alpha$.

(6) By Lemma 48.1 we have the following commutative diagram where all maps are isomorphisms:

$$\begin{tikzcd}
\bigoplus\limits_{i\in I_n} \mathrm{H}_n(\overline{B}_i^n, \partial\overline{B}_i^n) \arrow[rr, "\color{blue}{\Omega_n}:=\bigoplus \Phi_{i*}"] \arrow[d, "\bigoplus \text{ю}"] & & \mathrm{H}_k(X^n, X^{n-1}) \arrow[d, "\text{ю}"'] \\
\bigoplus\limits_{i\in I_n} \widetilde{\mathrm{H}}_n\big(\overline{B}_i^n/\partial\overline{B}_i^n\big) \arrow[r] & \widetilde{\mathrm{H}}_n\Big(\bigvee\limits_{i\in I_n} \overline{B}_i^n/\partial\overline{B}_i^n\Big) \arrow[r, "(\vee\Phi_i)_*"] & \widetilde{\mathrm{H}}_n(X^n/X^{n-1}).
\end{tikzcd}$$

(7) By another abuse of notation we often denote $\Omega_n(\alpha)$ also by α. Put differently, we identify $\mathrm{C}_n^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1})$ with the free abelian group generated by the set I_n of all n-cells.

Our goal now is to determine the matrix of the cellular boundary map

$$d_n\colon\ \underbrace{\overbrace{\mathrm{H}_n(X^n, X^{n-1})}^{=\mathrm{C}_n^{\mathrm{CW}}(X)}}_{\substack{\text{free abelian group}\\ \text{generated by } I_n}} \ \to\ \underbrace{\overbrace{\mathrm{H}_{n-1}(X^{n-1}, X^{n-2})}^{=\mathrm{C}_{n-1}^{\mathrm{CW}}(X)}}_{\substack{\text{free abelian group}\\ \text{generated by } I_{n-1}}}.$$

corresponding to the bases given by the cells. As a warm-up with start out with the case $n = 1$.

Lemma 48.6. *Let X be a CW-complex. We adopt all the notations introduced above.*

(1) *For a 1-cell α with attaching map $\varphi_\alpha\colon \overline{B}^1 \to X^0$ we have*
$$d_1(\alpha) \ = \ \varphi_\alpha(1) - \varphi_\alpha(-1).$$
(2) *If X has precisely one 0-cell, then the cellular boundary map d_1 is the zero map.*

Proof. The second statement of the lemma is evidently a consequence of the first statement. Therefore it suffices to prove the first statement. So let α be a 1-cell with characteristic map $\Phi_\alpha\colon \overline{B}^1 \to X^0$. We have to determine the map

$$\begin{tikzcd}
\mathbb{Z}\cdot\alpha \arrow[r] & \bigoplus\limits_{j\in I_1} \mathrm{H}_1(\overline{B}_j^1, \partial\overline{B}_j^1) \arrow[r, "\Omega_1"] & \mathrm{H}_1(X^1, X^0) \arrow[d, "d_1"] & \\
& & \mathrm{H}_0(X^0, \underbrace{X^{-1}}_{=\varnothing}) \arrow[r, "\Omega_0^{-1}"] & \bigoplus\limits_{i\in I_0} \mathrm{H}_0(\overline{B}_i^0, \underbrace{\partial\overline{B}_i^0}_{=\varnothing}).
\end{tikzcd}$$

Since $X^{-1} = \varnothing$ we see that in the present case we have $d_1 = \partial_1$, i.e. the cellular boundary map d_1 is the connecting homomorphism $\mathrm{H}_1(X^1, X^0) \to \mathrm{H}_0(X^0)$ of the long exact sequence of the pair (X^1, X^0). It follows immediately from the definitions of the connecting homomorphism, as defined in Proposition 43.15 (1), that $d_1(\alpha) = \varphi_\alpha(1) - \varphi_\alpha(-1)$. ■

Example. We consider the topological graph G that is given in Figure 792 with two vertices β_1, β_2 and four edges $\alpha_1, \ldots, \alpha_4$. We view it as a CW-complex in the obvious way. According to Lemma 48.6 the cellular chain complex of X is of the following form graph G

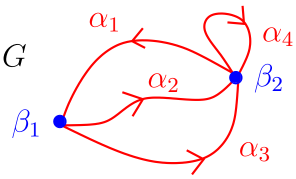

FIGURE 792

$$0 \ \to \ \mathrm{C}_1^{\mathrm{CW}}(G) = \mathbb{Z}^4 \ \xrightarrow{\begin{array}{c|cccc} & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \hline \beta_1 & 1 & -1 & -1 & 0 \\ \beta_2 & -1 & 1 & 1 & 0 \end{array}} \ \mathrm{C}_0^{\mathrm{CW}}(G) = \mathbb{Z}^2 \ \to \ 0.$$

Thus we obtain from Proposition 48.4 that

$$\mathrm{H}_1(G) \cong \ker\left(\mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}} \mathbb{Z}^4\right) \cong \mathbb{Z}^3 \quad \text{and} \quad \mathrm{H}_0(G) \cong \operatorname{coker}\left(\mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}} \mathbb{Z}^4\right) \cong \mathbb{Z}.$$

After this example we return to the discussion of the cellular boundary maps. Now let $n > 1$. We want to determine the cellular boundary map

$$d_n \colon \ \mathrm{H}_n(X^n, X^{n-1}) \ \to \ \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}).$$

Given an n-cell α and an $(n-1)$-cell β we consider the map

$$S_\alpha^{n-1} = \partial \overline{B}_\alpha^n \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \xrightarrow[\cong]{\Psi_{n-1}^{-1}} \bigvee_{j \in I_{n-1}} \overline{B}_j^{n-1}/\partial \overline{B}_j^{n-1} \xrightarrow{p_\beta} \overline{B}_\beta^{n-1}/\partial \overline{B}_\beta^{n-1} \xrightarrow[\cong]{\Theta} S_\beta^{n-1}$$

where $q \colon X^{n-1} \to X^{n-1}/X^{n-2}$ is the quotient map and where $\Theta \colon \overline{B}^{n-1}/\partial \overline{B}^{n-1} \xrightarrow{\cong} S^{n-1}$ denotes the homeomorphism from page 182. Since S_α^{n-1} and S_β^{n-1} are copies of S^{n-1} we can consider the degree of the map as defined on page 1181.

Now we will see that this degree is just the $\beta \times \alpha$–entry of the matrix that represents the cellular boundary map d_n with respect to the bases given by all n-cells and all $(n-1)$-cells. More precisely we have the following proposition:

Proposition 48.7. *Let X be a CW-complex. We use all the notation that we introduced above. Then for each n-cell α we have*

$$d_n(\alpha) \ = \ \sum_{\beta \in I_{n-1}} d_{\alpha,\beta} \cdot \beta$$

where

$$d_{\alpha\beta} \ := \ \text{degree of the map } S_\alpha^{n-1} \xrightarrow{\Theta \circ p_\beta \circ \Psi_{n-1}^{-1} \circ q \circ \varphi_\alpha} S_\beta^{n-1}.$$

PROOF. We write $I := I_n$ and $J := I_{n-1}$. We consider the following diagram[778]:

[778]The strange shape of the diagram is due to space constraints.

$$\begin{tikzcd}
\bigoplus\limits_{i\in I}\mathrm{H}_n(\overline{B}_i^n,\partial\overline{B}_i^n) \arrow[rrrrr, "\Omega_n:=\bigoplus\Phi_{i*}", "\cong"'] \arrow[dr, "\bigoplus\text{ю}", "\cong"'] & & & & & \mathrm{H}_n(X^n,X^{n-1}) \arrow[dl, "\text{ю}"', "\cong"] \arrow[dd, "\partial_n"'] \arrow[dddddd, bend left, "d_n"] \\
\mathrm{H}_n(\overline{B}_\alpha^n,\partial\overline{B}_\alpha^n) \arrow[u] \arrow[d, "\partial_n", "\cong"'] & \bigoplus\limits_{i\in I}\widetilde{\mathrm{H}}_n(\overline{B}_i^n/\partial\overline{B}_i^n) \arrow[rr, "\cong"'] & & \widetilde{\mathrm{H}}_n\Big(\bigvee\limits_{i\in I}\overline{B}_i^n/\partial\overline{B}_i^n\Big) \arrow[r, "\Psi_n", "\cong"'] & \widetilde{\mathrm{H}}_n(X^n/X^{n-1}) & \\
\widetilde{\mathrm{H}}_{n-1}(S_\alpha^{n-1}) \arrow[rrrrr, "\varphi_\alpha"] \arrow[d, "\cdot d_{\alpha\beta}"] & & & & & \widetilde{\mathrm{H}}_{n-1}(X^{n-1}) \arrow[dddl, "q"'] \arrow[dddd, "j_{n-1}"'] \\
\widetilde{\mathrm{H}}_{n-1}(S_\beta^{n-1}) & \widetilde{\mathrm{H}}_{n-1}(\overline{B}_\beta^{n-1}/\partial\overline{B}_\beta^{n-1}) \arrow[l, "\Theta_*"'] & & & & \\
\mathrm{H}_{n-1}(\overline{B}_\beta^{n-1},\partial\overline{B}_\beta^{n-1}) \arrow[u, "\Theta_*\circ\text{ю}"'] & & & \widetilde{\mathrm{H}}_{n-1}\Big(\bigvee\limits_{j\in J}\overline{B}_j^{n-1}/\partial\overline{B}_j^{n-1}\Big) \arrow[ull, "p_\beta"'] & & \\
 & \bigoplus\limits_{j\in J}\widetilde{\mathrm{H}}_{n-1}(\overline{B}_j^{n-1}/\partial\overline{B}_j^{n-1}) \arrow[uu, two heads] \arrow[urr, "\cong"] & & & \widetilde{\mathrm{H}}_{n-1}(X^{n-1}/X^{n-2}) \arrow[ul, "\Psi_{n-1}^{-1}"', "\cong"] & \\
\bigoplus\limits_{j\in J}\mathrm{H}_{n-1}(\overline{B}_j^{n-1},\partial\overline{B}_j^{n-1}) \arrow[uu, two heads] \arrow[ur, "\bigoplus\text{ю}", "\cong"'] & & & & & \mathrm{H}_{n-1}(X^{n-1},X^{n-2}). \arrow[ul, "\text{ю}"', "\cong"] \arrow[lllll, "\cong"', "\Omega_{n-1}^{-1}=\big(\bigoplus\Phi_{j*}\big)^{-1}"]
\end{tikzcd}$$

We make the following claim.

Claim. The diagram commutes.

We consider the various parts of the diagram.

(1) It follows from Lemma 48.1 (1) that the pentagon at the top and the large triangle at the bottom commute.
(2) It follows from the naturality of the connecting homomorphism, see Corollary 43.16 (2), that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}_n(\overline{B}_\alpha^n,\partial\overline{B}_\alpha^n) \arrow[r, "\Phi_\alpha"] \arrow[d, "\partial_n"'] & \mathrm{H}_n(X^n,X^{n-1}) \arrow[d, "\partial_n"] \\
\mathrm{H}_{n-1}(S_\alpha^{n-1}) \arrow[r, "\varphi_\alpha"] & \mathrm{H}_{n-1}(X^{n-1}).
\end{tikzcd}$$

From this observation it follows that the upper rectangle in the diagram commutes.
(3) It follows immediately from the definitions that the pentagon to the left commutes.
(4) It follows again immediately from the definitions that the triangle involving the map p_β commutes.
(5) It follows from Lemma 43.21 (1) that the triangle to the bottom right commutes. ⊞

Finally note that it follows from the definition of the standard generator of $\mathrm{H}_n(\overline{B}^n, \partial \overline{B}^n)$ on page 1176 and from Proposition 45.5 that the two maps

$$\begin{array}{rcll} \partial_n\colon\ \mathrm{H}_n(\overline{B}^n_\alpha, \partial \overline{B}^n_\alpha) & \to & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}_\alpha) & \text{and} \\ \Theta_* \circ \varsigma\colon\ \mathrm{H}_{n-1}(\overline{B}^{n-1}_\beta, \partial \overline{B}^{n-1}_\beta) & \to & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}_\beta) & \end{array}$$

send the standard generator to the standard generator. The reader who is still awake will notice that the proposition follows immediately from the claim and this observation. ■

Example. We consider the surface Σ of genus 2. As usual we view Σ as a CW-complex with one 0-cell P, four 1-cells a, b, c, d and one 2-cell F. We denote by $\varphi\colon S^1 \to \Sigma^1$ the attaching map of the 2-cell F. (We refer to Figure 793 for an illustration.) It follows from the discussion on page 1181 that the degree of the map $p_a \circ \Psi_1^{-1} \circ q \circ \varphi\colon S^1_F \to S^1_a$ is zero. The same conclusion holds for all other 1-cells. It follows from Proposition 48.7 that the

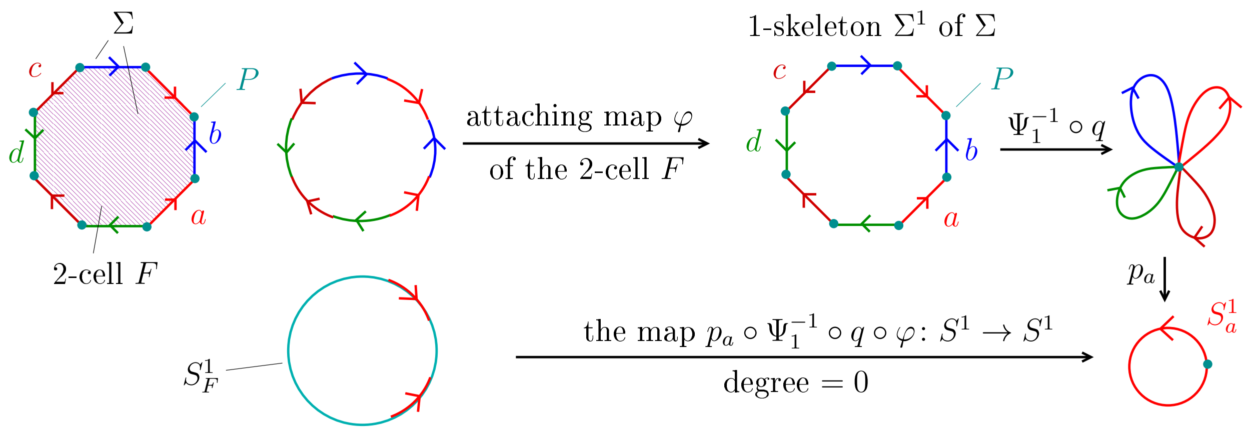

FIGURE 793

cellular boundary map d_2 in the cellular chain complex of Σ is zero. Since Σ has only one 0-cell we know from Lemma 48.6 that the cellular boundary map d_1 is also zero. Thus we see that

$$\mathrm{H}_n(\Sigma) \underset{\substack{\uparrow \\ \text{Proposition 48.4}}}{\cong} \mathrm{H}_n^{\mathrm{CW}}(\Sigma) \underset{\substack{\uparrow \\ \text{since all the cellular} \\ \text{boundary maps are zero}}}{=} \mathrm{C}_n^{\mathrm{CW}}(\Sigma) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

In Figure 794 on the left-hand side we show closed curves a, b, c, d. It follows fairly easily from the above calculation and the explicit isomorphism of Proposition 48.4 that these curves represent a basis for $\mathrm{H}_1(\Sigma) \cong \mathbb{Z}^4$. These curves in turn are isotopic to the curves a', b', c', d' on the right-hand side, which therefore, by Proposition 42.5, also represent a basis for $\mathrm{H}_1(\Sigma) \cong \mathbb{Z}^4$.

In the previous example we determined that $\mathrm{H}_1(\Sigma) \cong \mathbb{Z}^4$ and we gave an explicit basis for $\mathrm{H}_1(\Sigma)$. One might think that we now fully understand the homology groups of Σ. We will try to challenge this idea in Exercise 48.16.

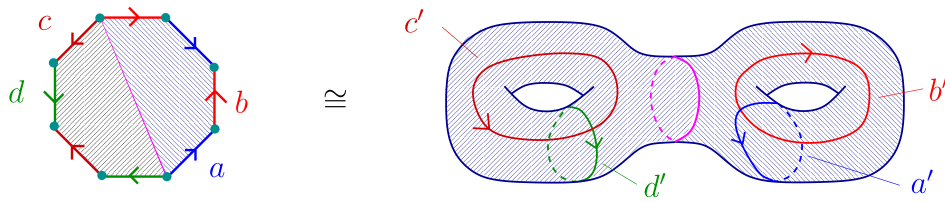

FIGURE 794. A basis for $\mathrm{H}_1(\Sigma)$.

Example. As a reality check we also consider once again the Klein bottle K. We view K as a CW-complex with one 0-cell P, two 1-cells a, b and one 2-cell F. (We refer to Figure 795 for an illustration.) It follows from Proposition 48.7, the discussion on page 1181 and Lemma 48.6 that the cellular chain complex is of the form

$$0 \to \mathbb{Z} \xrightarrow{\begin{pmatrix} -2 \\ 0 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 0 \end{pmatrix}} \mathbb{Z} \to 0.$$

Therefore we see that $\mathrm{H}_2(K) = 0$, $\mathrm{H}_1(K) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ and $\mathrm{H}_0(K) \cong \mathbb{Z}$. This is of course the same result that we had already obtained in Lemma 46.13.

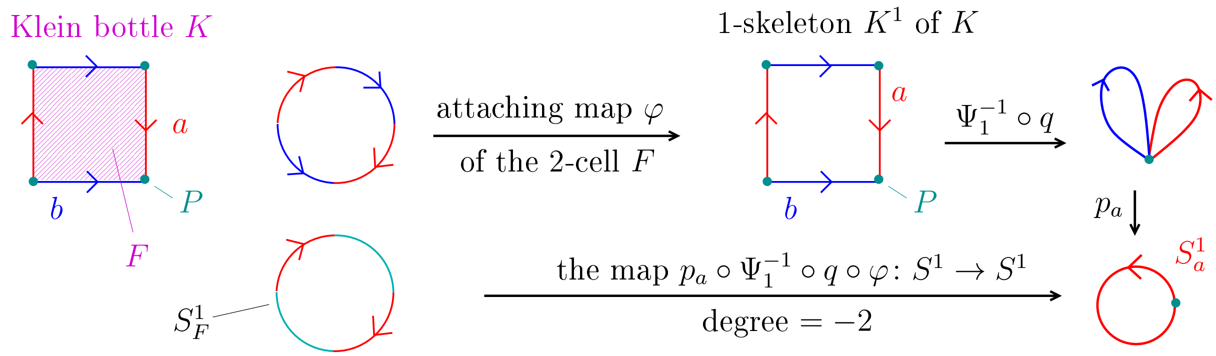

FIGURE 795

48.4. **The homology groups of 2-dimensional smooth manifolds.** Recall that given any $g, m \in \mathbb{N}_0$ we write[779]

$$\Sigma_{g,m} \;:=\; \text{the surface of genus } g \text{ minus } m \text{ open disks}$$

and for $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$ we write

$$N_{k,m} \;:=\; \text{the non-orientable surface of genus } g \text{ minus } m \text{ open disks.}$$

In the Surface Classification Theorem 23.4 we saw that every connected, compact 2-dimensional topological manifold is homeomorphic to either $\Sigma_{g,m}$ for unique $g, m \in \mathbb{N}_0$ or to $N_{k,m}$ for unique $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$.

Our goal in this section will be to determine the homology groups of $\Sigma_{g,n}$ and $N_{k,n}$. Before we consider the general case we will consider the special case $\Sigma_{2,1}$ in great detail.

Example. We consider $\Sigma_{2,1}$ and we equip it with the CW-structure that is illustrated in Figure 796. This CW-structure has two 0-cells P, Q, six 1-cells a, b, c, d, w, z and it has one

[779] Also recall the definition on page 668 what we mean by a "surface minus m open disks".

2-cell F. Given $i = 1, 2$ we denote by A_i the matrix that represents the cellular boundary

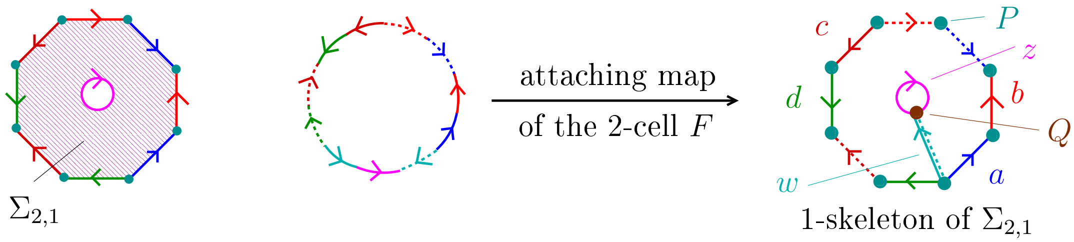

FIGURE 796

map $d_i\colon \mathrm{C}^{\mathrm{CW}}_i(\Sigma_{2,1}) \to \mathrm{C}^{\mathrm{CW}}_{i-1}(\Sigma_{2,1})$ with respect to the given bases.

The argument on page 1270 shows that
$$A_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ w \\ z \end{array} \begin{array}{c} F \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$$
It follows from Lemma 48.6 that we have
$$A_1 = \begin{array}{c} \\ P \\ Q \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & w & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

Some elementary linear algebra shows that

$$\mathrm{H}_n(\Sigma_{2,1}) \underset{\substack{\uparrow \\ \text{Proposition 48.4}}}{\cong} \mathrm{H}^{\mathrm{CW}}_n(\Sigma_{2,1}) = \mathrm{H}_n\Big(0 \to \mathbb{Z} \xrightarrow{A_2} \mathbb{Z}^6 \xrightarrow{A_1} \mathbb{Z}^2 \to 0\Big) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

From this discussion we get in fact a more precise statement. Namely, by the choice of the CW-structure on $\Sigma_{2,1}$ we have that the map inclusion map $\partial\Sigma_{2,1} \to \Sigma_{2,1}$ is in fact a cellular map. Therefore it follows from Proposition 48.4 (2) that the following diagram commutes

$$\begin{tikzcd} \mathrm{H}_1(\partial\Sigma_{2,1}) \arrow[r, "f_*"] \arrow[d, "\cong"'] & \mathrm{H}_1(\Sigma_{2,1}) \arrow[d, "\cong"] \\ \mathbb{Z} = \mathrm{H}^{\mathrm{CW}}_1(\partial\Sigma_{2,1}) \arrow[r, "f_*", "{[z]\mapsto[z]}"'] & \mathrm{H}^{\mathrm{CW}}_1(\Sigma_{2,1}) = \mathbb{Z}^4 \end{tikzcd}$$

where the vertical maps denote the isomorphisms from Proposition 48.4. The 1-cell z is null-homologous in $\mathrm{C}^{\mathrm{CW}}_*(\Sigma_{2,1})$ since we have just seen that it is the boundary of the 2-cell F. Thus we see that the bottom horizontal map is the zero map, but that implies that the top horizontal map is also the zero map.

After the calculations that we just did it we can now easily prove the following lemma.

Lemma 48.8.

(1) *Let $g \in \mathbb{N}_0$. Then the inclusion induced map*
$$\mathrm{H}_1(\partial\Sigma_{g,1}) \to \mathrm{H}_1(\Sigma_{g,1})$$

is the zero map.

(2) *Let M be a smooth manifold and let C be an oriented curve in M. If there exists a $g \in \mathbb{N}_0$ and a map $\psi\colon \Sigma_{g,1} \to M$ such that $\psi(\partial\Sigma_{g,1}) = C$, then $C = 0 \in \mathrm{H}_1(M)$.*[780]

Remark. In Exercise 48.14 we will prove a generalization of Lemma 48.8 (1).

Example. Loosely speaking Lemma 48.8 (2) says that if a curve C is the boundary of a compact orientable surface in M, then C is null-homologous. For example in Figure 797 we see a curve C on a surface M. Since C is the boundary of a compact orientable subsurface contained in M we see that $C = 0 \in \mathrm{H}_1(M)$.

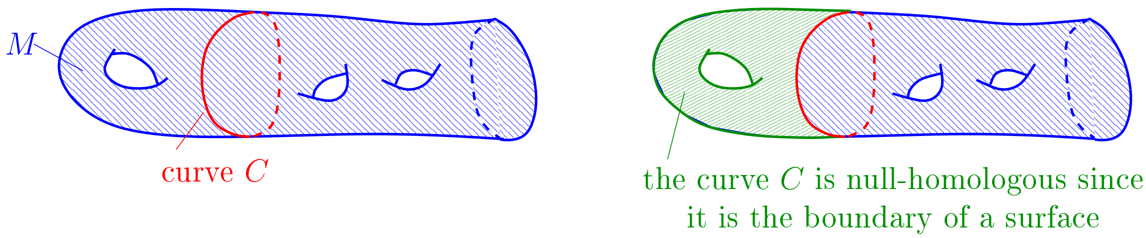

FIGURE 797

PROOF.

(1) For $g = 1$ we proved the statement in the previous example. The general case is proved entirely the same way.
(2) Now let M be a smooth manifold and let C be an oriented curve in M. Suppose there exists a $g \in \mathbb{N}_0$ and a map $\psi\colon \Sigma_{g,1} \to M$ such that $\psi(\partial\Sigma_{g,1}) = C$. We get the commutative diagram

$$\begin{tikzcd} & \mathrm{H}_1(C) \arrow[rrd] & & \\ \mathrm{H}_1(S^1) \arrow[ru] \arrow[rd] & & & \mathrm{H}_1(M). \\ & \mathrm{H}_1(\partial\Sigma_{g,1}) \arrow[r] & \mathrm{H}_1(\Sigma_{g,1}) \arrow[ru, "\psi_*"'] & \end{tikzcd}$$

By the first statement the bottom horizontal map is the zero map. But that shows that the homology class represented by C is also zero. ∎

In the following proposition we give the calculation for the homology groups of all connected, compact 2-dimensional topological manifolds.

[780] Following the convention of page 1223 the curve C defines a homology class in $\mathrm{H}_1(M)$ that we also denote by C.

Proposition 48.9. *For any $g \in \mathbb{N}_0$ we have*

$$\mathrm{H}_n(\Sigma_g) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^{2g}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Furthermore for $m \geq 1$ we have

$$\mathrm{H}_n(\Sigma_{g,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{2g+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

For any $k \in \mathbb{N}$ we have

$$\mathrm{H}_n(N_k) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}^{k-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Furthermore for $m \geq 1$ we have

$$\mathrm{H}_n(N_{k,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{k+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

PROOF. The calculation of the homology groups $\Sigma_{g,m}$ is a slight generalization of the arguments that we provided above and on page 1270. We leave the details to the reader.

We will deal with the homology of $N_{k,m}$ in Exercise 48.1. The calculation is evidently just a variation on the arguments we provided for Σ_2 and $\Sigma_{2,2}$. ∎

Remark. A careful reading of the Proof of Proposition 48.9 gives us some extra information. First of all we endow $\Sigma_{g,m}$ with an orientation. The boundary components of $\Sigma_{g,m}$ then inherit an orientation. By the discussion on page 1223 the boundary components, viewed as oriented curves, then define elements $c_1, \ldots, c_m \in \mathrm{H}_1(\Sigma_{g,m})$. Then there exists an exact sequence

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto \sum c_i} \bigoplus_{i=1}^{m} \mathbb{Z} \cdot c_i \to \mathrm{H}_1(\Sigma_{g,m}) \underset{\substack{\uparrow \\ \text{inclusion induced map}}}{\to} \mathrm{H}_1(\Sigma_g) \to 0.$$

As we saw in Figure 794, a basis for $\mathrm{H}_1(\Sigma_g)$ is given by a pair of curves for each "hole". The above exact sequence shows that a basis for the homology group $\mathrm{H}_1(\Sigma_{g,m}) \cong \mathbb{Z}^{2g+m-1}$ is obtained by a pair of curves for each "hole" together with boundary curves for all except one boundary component. This statement is illustrated in Figure 798.

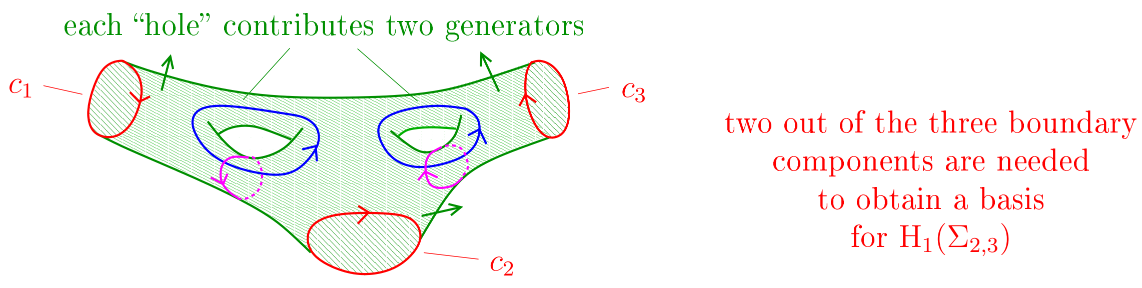

FIGURE 798

We conclude this section with the surface Σ_∞ of infinite genus that we had already encountered on page 740. Using a variation on the proof of Lemma 25.11 one can show, using the above calculations and Proposition 47.4, that

$$H_n(\Sigma_\infty) \;\cong\; \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}^\infty, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

We leave it to the reader to fill in the details.

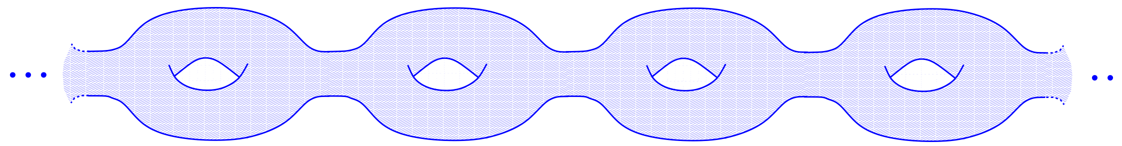

FIGURE 799

48.5. **The homology groups of real projective spaces.** In this section we will compute the homology groups of real projective spaces $\mathbb{RP}^n$. This calculation is significantly trickier than the previous ones, since the calculation of the cellular boundary maps is not as straightforward as in our previous examples. This time we will need the theory of local degrees that we developed in Section 45.6.

Proposition 48.10. *For any $n \in \mathbb{N}_0$ we have*[781]

$$H_k(\mathbb{RP}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd and } k < n, \\ 0, & \text{if } k \text{ is even and } 0 < k \neq 0, \\ 0, & \text{if } k > n, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd.} \end{cases}$$

PROOF (∗). As a reminder we defined

$$\mathbb{RP}^n \;=\; S^n/\sim \qquad \text{where } x \sim -x \text{ for all } x \in S^n.$$

In Lemma 36.1 we already saw that we can view $\mathbb{RP}^n$ as a CW-complex with the following two properties:

(1) the CW-structure has exactly one cell in the dimensions $0, 1, \ldots, n$ and no other cells,
(2) for each $k \leq n$ the k-skeleton of $\mathbb{RP}^n$ is given by $\mathbb{RP}^k$.

Now we recall the definition of this CW-structure. For each $k \in \mathbb{N}_0$ we define

$$\begin{aligned} \varphi_k \colon S^{k-1} = \partial \overline{B}^k \;&\to\; \mathbb{RP}^{k-1} \\ (x_1, \ldots, x_k) \;&\mapsto\; [(x_1, \ldots, x_k)]. \end{aligned}$$

Then we obtain a homeomorphism

$$\begin{aligned} \mathbb{RP}^{k-1} \cup_{\varphi_k} \overline{B}^k \;&\xrightarrow{\cong}\; \mathbb{RP}^k \\ x \;&\mapsto\; \begin{cases} (x, 0), & \text{if } x \in \mathbb{RP}^{k-1} \\ \big[\big(x, \sqrt{1 - \|x\|^2}\big)\big], & \text{if } x \in \overline{B}^k. \end{cases} \end{aligned}$$

[781]For $n = 2$ we thus obtain, as we should, the same result as in Section 46.3.

Thus we see inductively that we can view $\mathbb{RP}^n$ as a CW-complex with a single cell in the dimensions $0,\dots,n$ where the attaching maps are given precisely by the maps $\varphi_1,\dots,\varphi_n$. As usual we denote the characteristic map of the k-cell, which in this case is the obvious map $\overline{B}^k \to \mathbb{RP}^k$, by Φ_k.

Claim. For each $k=1,\dots,n$ the cellular boundary map

$$d_k\colon \mathbb{Z} = \mathrm{C}_k^{\mathrm{CW}}(X) \;\to\; \mathrm{C}_{k-1}^{\mathrm{CW}}(X) = \mathbb{Z}$$

is given by multiplication by $1+(-1)^k$.

Let $k\in\{1,\dots,n\}$. We denote by $\Theta\colon \overline{B}^{k-1}/\partial\overline{B}^{k-1} \to S^{k-1}$ the homeomorphism from page 182 and we denote by $q\colon \mathbb{RP}^{k-1}\to\mathbb{RP}^{k-1}/\mathbb{RP}^{k-2}$ the projection map. By Proposition 48.7 it suffices to show that the degree of the map

$$\begin{tikzcd} S^{k-1} \arrow[r, "\varphi_k"] \arrow[rrrr, bend right=15, "=:\Omega"'] & \mathbb{RP}^{k-1} \arrow[r, "q"] & \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \arrow[r, "\Phi_{k-1}^{-1}"] & \overline{B}^{k-1}/\partial\overline{B}^{k-1} \arrow[r, "\Theta"] & S^{k-1} \end{tikzcd}$$

equals $1+(-1)^k$. (We refer to Figure 800 for an illustration of the map.) We pick the point $y=(0,\dots,0,1)\in S^{k-1}$. The preimages of y under the map Ω are $x_1=(0,\dots,0,1)$ and $x_2=(0,\dots,0,-1)$. We leave it to the reader to verify that[782] $\deg\Omega|_{x_1}=1$ and that $\deg\Omega|_{x_2}=(-1)^n$. The desired statement now follows from Proposition 45.23. ⊞

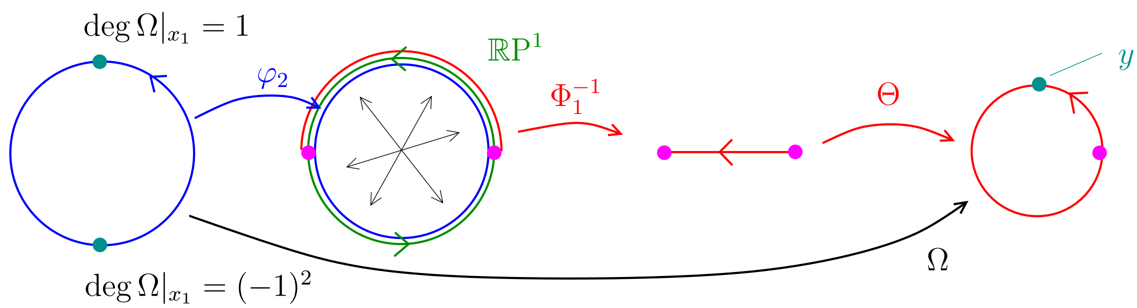

FIGURE 800. Illustration from the proof of Proposition 48.10.

It follows from the claim that the cellular chain complex of X is of the form

$$0 \;\to\; \mathbb{Z} \xrightarrow{1+(-1)^n} \mathbb{Z}\;\dots\;\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \;\to\; 0.$$

It is now straightforward to determine the cellular homology groups which, by Proposition 48.4, are isomorphic to the singular homology groups of $\mathbb{RP}^n$. ■

Remark. Using the approach of the proof of Proposition 48.10, applied to the infinite CW-complex $\mathbb{RP}^\infty$, or alternatively using $\mathbb{RP}^\infty=\bigcup\limits_n\mathbb{RP}^n$ and Proposition 47.4, we see that

[782]It is not entirely easy to struggle through all the orientation conventions and all the maps, but it can indeed be done.

$$\mathrm{H}_k(\mathbb{RP}^\infty) \cong \begin{cases} \mathbb{Z}, & \text{if } k=0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd}, \\ 0, & \text{if } k \text{ is even and } k>0. \end{cases}$$

Remark. We have now seen that the homology groups of CW-complexes can be computed fairly easily using cellular chain complexes. One might ask, why did we introduce singular chain complexes, if it is so much easier to do calculations with cellular chain complexes. There are two related answers:

(1) we also want to study topological spaces that are not CW-complexes,
(2) it is much harder to prove some statements using cellular chain complexes, for example it is very difficult, if not impossible, to show that homeomorphic CW-complexes have isomorphic homology groups, just using the definition of cellular chain complexes.

A general slogan in topology is that for calculations it is usually best to work with "small objects", whereas for proving general properties, it is often best to work with "big objects".

Exercises for Chapter 48.

Exercise 48.1.

(a) Determine the homology groups of $\mathbb{RP}^2\#\mathbb{RP}^2$.
(b) Determine the homology groups of $\mathbb{RP}^2\#\mathbb{RP}^2$ minus an open disk.

Note: In both parts you can make use of the CW-structure for $\mathbb{RP}^2\#\mathbb{RP}^2$ illustrated in Figure 801.

Remark. The homology groups are stated as Proposition 48.9. But it is a very good exercise to try to determine the homology groups without looking at the proposition.

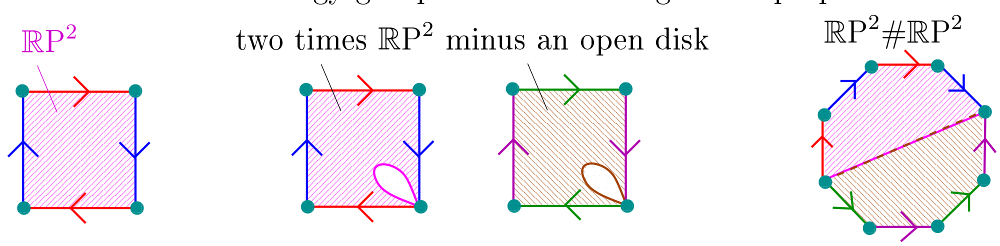

FIGURE 801

Exercise 48.2. Show that given any finitely generated abelian group A there exists a topological space X with $\mathrm{H}_0(X)=\mathbb{Z}$ and $\mathrm{H}_1(X)\cong A$ and with $\mathrm{H}_i(X)=0$ for $i\geq 2$.

Exercise 48.3. Let A be a finitely generated abelian group. Does there exist a topological space X with $\mathrm{H}_0(X)=\mathbb{Z}$, $\mathrm{H}_3(X)\cong A$ and $\mathrm{H}_i(X)=0$ for $i\neq 0,3$?

Exercise 48.4.

(a) Determine $\min\{\#\text{cells of } X \,|\, X \text{ is a CW-complex with } \mathrm{H}_2(X)\cong\mathbb{Z}\oplus\mathbb{Z}_2\}$.
(b) Determine $\min\{\#\text{cells of } X \,|\, X \text{ is a CW-complex with } \mathrm{H}_1(X)\cong\mathbb{Z}\oplus\mathbb{Z}_2\}$.
(c) Determine $\min\{\#\text{cells of } X \,|\, X \text{ is a connected CW-complex with } \pi_1(X)\cong\mathbb{Z}\times\mathbb{Z}_2\}$.

Exercise 48.5.

(a) Let X be the quotient space of S^2 under the identification $x \sim -x$ for all $x \in S^1$. Determine the homology groups of $X = S^2/\sim$.
(b) Let $m, n \in \mathbb{N}$ with $m < n$. As usual we view $\mathbb{RP}^m$ as a subset of $\mathbb{RP}^n$. Compute the homology groups of the quotient space $\mathbb{RP}^n/\mathbb{RP}^m$.

Exercise 48.6. Let C be a closed curve in a smooth manifold M. Suppose that C is the boundary of an embedded compact orientable surface in M. Show that the inclusion induced map $\mathrm{H}_1(C) \to \mathrm{H}_1(M)$ is the zero map.
More concisely, if a closed curve bounds a compact orientable surface, then the closed curve represents the trivial element in homology.

Exercise 48.7.

(a) Let Σ be a compact orientable connected 2-dimensional smooth manifold and let $F \subset \Sigma$ be a 2-dimensional submanifold. We denote by $i\colon F \to \Sigma$ the inclusion map. Show that $i_*(\mathrm{H}_1(F)) \subset \mathrm{H}_1(\Sigma)$ is a summand.
Hint. Consider the long exact sequence of the pair (Σ, F).
(b) Does the conclusion of (a) also hold if we do not assume that Σ is orientable?

Exercise 48.8. Let X be a topological space and let $n \in \mathbb{N}_0$. Suppose X is compact. Does it follow that $\mathrm{H}_n(X)$ is a finitely generated group?

Exercise 48.9. Let $n \in \mathbb{N}_0$. In Lemma 36.1 we showed that $\mathbb{RP}^n$ admits a CW-structure which has exactly one cell in the dimensions $0, 1, \ldots, n$ and no other cells. Show that this is the "smallest" CW-structure on $\mathbb{RP}^n$. More precisely, show that every CW-structure for $\mathbb{RP}^n$ has at least one cell in the dimensions $0, 1, \ldots, n$.
Hint. Consider the (cellular) homology groups of $\mathbb{RP}^n$.

Exercise 48.10. Let $m \leq n$. We denote by $i\colon \mathbb{RP}^m \to \mathbb{RP}^n$ the inclusion map. Determine for each $k \in \mathbb{N}_0$ the induced map $i_*\colon \mathrm{H}_k(\mathbb{RP}^m) \to \mathrm{H}_k(\mathbb{RP}^n)$.

Exercise 48.11. Let n be odd. By Proposition 48.10 we know that $\mathrm{H}_n(\mathbb{RP}^n) \cong \mathbb{Z}$. Show that there exists a singular simplex $\sigma\colon \Delta^n \to \mathbb{RP}^n$ such that $[\sigma]$ is a generator of $\mathrm{H}_n(\mathbb{RP}^n)$.
Hint.

(a) First you need to find an "interesting" cycle σ. You could use the identification $\mathbb{RP}^n = \overline{B}^n/\sim$ from page 196. Or perhaps better, write $\mathbb{RP}^n = \Delta^n/\sim$ for a suitable equivalence relation $\sim$.
(b) To show that the "interesting" cycle σ is actually a generator you could use the map $\overline{B}^n/\sim \to \overline{B}^n/S^{n-1}$ (which is continuous by Lemma 3.28 (2)) and results from Section 45.1.

Exercise 48.12. Let $n \in \mathbb{N}$ be odd. By Proposition 48.10 we know that $\mathrm{H}_n(\mathbb{RP}^n) \cong \mathbb{Z}$. Let $p\colon S^n \to \mathbb{RP}^n = S^n/\{\pm\,\mathrm{id}\}$ be the projection map. Determine the induced map $p_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(\mathbb{RP}^n)$.
Hint. You could use the fact that the isomorphism between cellular and singular homology is natural.

Exercise 48.13. Let $n \in \mathbb{N}$ be odd. By Proposition 48.10 we know that $\mathrm{H}_n(\mathbb{RP}^n) \cong \mathbb{Z}$. Let $A \in \mathrm{GL}(n+1, \mathbb{R})$. We denote by $f\colon \mathbb{RP}^n \to \mathbb{RP}^n$ the map given by $f(x) := A \cdot x$. Determine the degree of the map $f_*\colon \mathrm{H}_n(\mathbb{RP}^n) \to \mathrm{H}_n(\mathbb{RP}^n)$.
Remark. You could make use of the homotopies considered on page 558.

Exercise 48.14. Let $g \in \mathbb{N}_0$ and let $k \in \mathbb{N}_0$. As on page 669 we consider $\Sigma_{g,n}$, i.e. we consider the surface of genus g minus k open disks.

(a) Show that $\mathrm{H}_2(\Sigma_{g,n}, \partial\Sigma_{g,n}) \cong \mathbb{Z}$.
(b) Let $C_1, \dots, C_n$ be the boundary components. We equip $\Sigma_{g,n}$ with an orientation and we equip its boundary $\partial\Sigma_{g,n} = C_1 \sqcup \dots \sqcup C_n$ with the corresponding orientation given by Lemma 6.50. For $i = 1, \dots, n$ we pick an orientation-preserving diffeomorphism $\gamma_i\colon S^1 \to C_i$. We write $[C_i] := \gamma_{i*}([S^1])$ and by a serious abuse of notation we denote by $[C_i]$ also the image of $[C_i]$ under the inclusion induced map $C_i \to \Sigma_{g,n}$. Show that there exists a generator φ of $\mathrm{H}_2(\Sigma_{g,n}, \partial\Sigma_{g,n}) \cong \mathbb{Z}$ such that
$$\underset{\substack{\uparrow \\ \text{connecting homomorphism of the pair } (\Sigma_{g,n}, \partial\Sigma_{g,n})}}{\partial_2}(\varphi) \;=\; [C_1] + \dots + [C_n] \;\in\; \mathrm{H}_1(\partial\Sigma_{g,n}).$$
(c) Show that $\mathrm{H}_1(\Sigma_{g,n}, \partial\Sigma_{g,n}) \cong \mathbb{Z}^{2g}$.
(d) We denote by $i\colon \partial\Sigma_{g,n} \to \Sigma_{g,n}$ and $j\colon \Sigma_{g,n} \to \Sigma_n$ the inclusion maps. Show that the following sequence is exact:
$$0 \;\to\; \mathrm{H}_2(\Sigma_{g,n}) \;\xrightarrow{\partial_2}\; \mathrm{H}_1(\partial\Sigma_{g,n}) \;\xrightarrow{i_*}\; \mathrm{H}_1(\Sigma_{g,n}) \xrightarrow{j_*} \mathrm{H}_1(\Sigma_g) \to 0.$$

Remark. In Corollary 68.11 we will generalize Statement (a) to compact orientable manifolds in a suitable way.

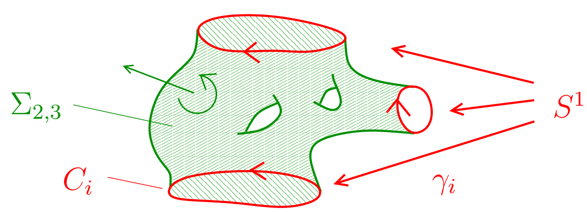

FIGURE 802. Illustration of Exercise 48.14.

Exercise 48.15. Let $k \in \mathbb{N}$ and let $m \in \mathbb{N}_0$. As on page 669 we denote by $N_{k,m}$ the non-orientable surface of genus g minus m open disks.

(a) Determine the relative homology groups $\mathrm{H}_*(N_{k,m}, \partial N_{k,m})$.
(b) Use (a) together with Exercise 48.14 and Proposition 48.9 to show that the topological spaces $\Sigma_{g,n}$ and $N_{k,n}$ are pairwise non-homeomorphic.
Remark. This gives a new, perhaps more natural proof for Theorem 23.6.

Exercise 48.16. In Figure 803 we see to the left the surface Σ of genus two together with four oriented curves a, b, c, d and to the right we see an oriented curve σ. On page 1270 we showed that $\mathrm{H}_1(\Sigma) \cong \mathbb{Z}^4$ and we showed that the curves a, b, c and d represent a basis for $\mathrm{H}_1(\Sigma)$. In particular we can write $[\sigma] \in \mathrm{H}_1(\Sigma)$ uniquely as a linear combination of $[a]$, $[b]$, $[c]$ and $[d]$. Determine the coefficients of this linear combination.

Remark. In Exercise 94.14 we will get to know a charming approach to dealing with problems of this form.

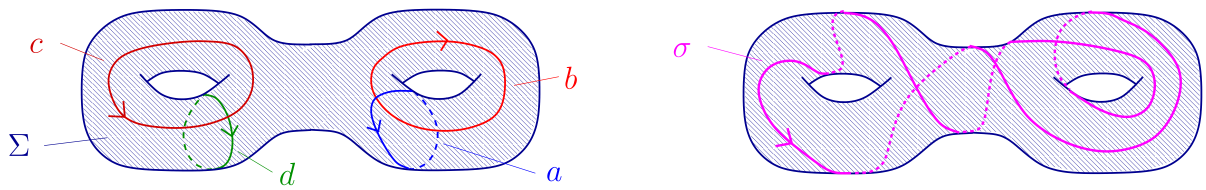

FIGURE 803. Illustration for Exercise 48.16.

49. Chain complexes with isomorphic homology groups

On several occasions we have now shown that certain homology groups are isomorphic, and it is reasonable to ask whether the underlying chain complexes are in fact chain homotopy equivalent. In this chapter we will see that, for purely algebraic reasons, the question can always be answered in the affirmative.

49.1. **Isomorphisms of homology groups are homotopy equivalences.** The goal of this chapter is to prove the following three propositions.

Proposition 49.1. *Suppose* $(\mathrm{C}_*,\partial_*)$ *and* (C'_*,∂'_*) *are two chain complexes of free abelian groups. If* $\gamma_n\colon \mathrm{H}_n(C_*)\to \mathrm{H}_n(C'_*)$, $n\in\mathbb{N}_0$, *is a family of homomorphisms, then there exists a chain map* $f\colon C_*\to C'_*$ *such that for every* $n\in\mathbb{N}_0$ *we have*

$$f_*=\gamma_n\colon\ \mathrm{H}_n(C_*)\ \to\ \mathrm{H}_n(C'_*).$$

Remark. We consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, "n\mapsto(0,n)"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "(m,n)\mapsto m+2n"'] & \mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

We view the top and bottom sequences as chain complexes C_* respectively D_* and we view the vertical maps as a chain map f. It is straightforward to see that f induces the zero map on homology but that it is not chain homotopic to the zero chain map. This shows that the chain map from Proposition 49.1 is in general *not* unique up to chain homotopy.

The second proposition can be viewed as a converse to Lemma 42.2.

Proposition 49.2. *Let* $f\colon C_*\to \mathrm{C}'_*$ *be a chain map between chain complexes of free abelian groups. If* f *induces an isomorphism of all homology groups, then* f *is in fact a chain homotopy equivalence.*

Finally we have the following proposition that is a convenient variation on Proposition 49.2.

Proposition 49.3. *Let* C_* *and* D_* *be two chain complexes of free abelian groups and let* $\iota\colon C_*\to D_*$ *be a chain map with the following two properties:*

(a) *The chain map* $\iota\colon C_*\to D_*$ *induces an isomorphism of homology groups.*
(b) *For each* $k\in\mathbb{N}_0$ *the map* $\iota\colon \mathrm{C}_k\to D_k$ *is a monomorphism and the image is a subsummand of* E_k.

Then there exists a chain map $\pi\colon D_*\to \mathrm{C}_*$ *with the following two properties:*

(1) *For each* $k\in\mathbb{N}_0$ *we have* $\pi_k\circ\iota_k=\mathrm{id}_{\mathrm{C}_k}$.
(2) *The maps* ι *and* π *are chain homotopy homotopy equivalences.*

Before we provide the proofs of the above three proposition let us discuss some consequences. For example we have have the following interesting corollary to Proposition 49.2.

Corollary 49.4. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \longrightarrow & B_* & \longrightarrow & C_* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'_* & \longrightarrow & B'_* & \longrightarrow & C'_* & \longrightarrow & 0 \end{array}$$

be a commutative diagram of short exact sequences of chain complexes. If two of the vertical maps are chain homotopy equivalences, then the third is also a chain homotopy equivalence.

Proof. By Proposition 43.11 we get a commutative diagram of long exact sequences

$$\begin{array}{ccccccccccccc} \dots & \longrightarrow & \mathrm{H}_{n+1}(C_*) & \xrightarrow{\partial} & \mathrm{H}_n(A_*) & \longrightarrow & \mathrm{H}_n(B_*) & \longrightarrow & \mathrm{H}_n(C_*) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(A_*) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & \mathrm{H}_{n+1}(C'_*) & \xrightarrow{\partial} & \mathrm{H}_n(A'_*) & \longrightarrow & \mathrm{H}_n(B'_*) & \longrightarrow & \mathrm{H}_n(C'_*) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(A'_*) & \longrightarrow & \dots \end{array}$$

By our hypothesis, together with Corollary 42.3, we know that always two out of three vertical maps are isomorphisms. But then it follows from the Five Lemma 43.12 that the third vertical map is also an isomorphism. Now Proposition 49.2 implies that the corresponding map of chain complexes is also a chain homotopy equivalence. ■

As we pointed out above, the first time we had mentioned Proposition 49.2 was in the remark after Proposition 43.24. In fact Proposition 49.2 implies immediately that we can strengthen Proposition 43.24 to obtain the following result.

Proposition 49.5. *Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a comfortable cover of X. Then the inclusion map*[783]

$$\mathrm{C}^{\mathcal{U}}_*(X) \;\to\; \mathrm{C}_*(X)$$

is a chain homotopy equivalence.

We conclude this discussion of the consequences of Proposition 49.1 and 49.2 with the following lemma which we will use on several occasions.

Lemma 49.6. *Let (C_*, c_*) be a free chain complex such that for each $k \in \mathbb{N}_0$ the homology group $\mathrm{H}_k(C)$ is finitely generated. Then there exists a chain complex (D_*, ∂_*) with the following two properties:*

(1) *The chain complexes $(\mathrm{C}_*, \partial_*)$ and (D_*, ∂_*) are chain homotopy equivalent.*
(2) *Each chain group D_k is a free abelian group of* finite rank.

Proof. Let $k \in \mathbb{N}_0$. Recall that we assume that $\mathrm{H}_k(C_*)$ is finitely generated. It follows immediately from the classification of finitely generated abelian groups, see Theorem 19.4, that there exists a chain complex D^k_* and an isomorphism $\varphi_k\colon \mathrm{H}_k(D^k_*) \to \mathrm{H}_k(C_*)$ such

[783]Recall that a *comfortable cover of* X is a family $\mathcal{U} = \{U_i\}_{i\in I}$ of subsets of X such that the union of the interiors of the U_i equals X. On page 1133 we defined

$$\mathrm{C}^{\mathcal{U}}_n(X) \;=\; \Big\{ \sum_{j=1}^{k} a_j\sigma_j \;\Big|\; \text{for each } j \text{ there exists a } U_i \in \mathcal{U} \text{ such that the image of } \sigma_j \text{ lies in } U_i \Big\}$$

and we had observed that these groups form a subcomplex of $\mathrm{C}_*(X)$. Note that these chain groups are free abelian groups generated by the union of the sets of singular n-simplices that lie in some U_i.

that only the chain groups D^k_k and D^k_{k+1} are non-zero and such that both chain groups are finitely generated free abelian groups.

We define $D_* = \bigoplus\limits_{k\in\mathbb{N}_0} D^k_*$. It follows immediately from Proposition 49.1 and 49.2 that the chain complex D_* is chain homotopy equivalent to (C_*, c_*). ■

49.2. **Proof of Propositions 49.1 and 49.2.** We start out with the proof of Proposition 49.1.

Proof of Proposition 49.1. Let $(\mathrm{C}_*, \partial_*)$ and (C'_*, ∂'_*) be two chain complexes of free abelian groups. Furthermore let $\gamma_n\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(C'_*)$, $n \in \mathbb{N}_0$ be a family of homomorphisms. We have to show that there exists a chain map $f\colon C_* \to C'_*$ such that for every $n \in \mathbb{N}_0$ we have

$$f_* = \gamma_n\colon\ \mathrm{H}_n(C_*) \ \to\ \mathrm{H}_n(C'_*).$$

We start out by recalling some of the usual notation and basic observations regarding chain complexes. More precisely, for each $n \in \mathbb{N}_0$ we write

$$Z_n \ := \ker(\partial_n\colon\ \mathrm{C}_n \to \mathrm{C}_{n-1}), \quad B_n \ := \ \mathrm{im}(\partial_{n+1}\colon\ \mathrm{C}_{n+1} \to \mathrm{C}_n) \quad \text{ and } \quad H_n \ = \ Z_n/B_n.$$

Recall that these groups form the following two types of short exact sequences

$$\text{(a)} \qquad 0 \ \to\ Z_n \ \hookrightarrow\ \mathrm{C}_n \ \xrightarrow{\partial_n}\ B_{n-1} \ \to\ 0,$$

and

$$\text{(b)} \qquad 0 \ \to\ B_n \ \hookrightarrow\ Z_n \ \to\ H_n \ \to\ 0.$$

Similarly we define Z'_n, B'_n and H'_n for the chain complex C'_*. Evidently these groups also form short exact sequences that we denote by (a′) and (b′).

Since all the chain groups are free abelian it follows from Lemma 19.2 that the subgroups Z_n, B_n, Z'_n and B'_n are also free abelian. In particular the sequences (b) and (b′) are free resolutions of H_n respectively H'_n. It follows from Lemma 57.13 that for each $n \in \mathbb{N}_0$ there exist homomorphisms β_n and ζ_n such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & B_n \arrow[r] \arrow[d, "\beta_n"] & Z_n \arrow[r] \arrow[d, "\zeta_n"] & H_n \arrow[r] \arrow[d, "\gamma_n"] & 0 \\
0 \arrow[r] & B'_n \arrow[r] & Z'_n \arrow[r] & H'_n \arrow[r] & 0.
\end{tikzcd}$$

Our goal is to extend the maps $\zeta_n\colon Z_n \to Z'_n$ to a chain map $f\colon \mathrm{C}_n \to \mathrm{C}'_n$. To do so we consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r, hook] \arrow[dr, hook] \arrow[ddd, "\zeta_n"'] & \mathrm{C}_n \arrow[r, two heads, "\partial"] \arrow[d, dashed, "\psi_n"] & B_{n-1} \arrow[r] \arrow[ddd, "\beta_{n-1}"] & 0 \\
& & Z_n \oplus B_{n-1} \arrow[ur, two heads] \arrow[d, "\zeta_n \oplus \beta_{n-1}"] & & \\
& & Z'_n \oplus B'_{n-1} \arrow[dr, two heads] & & \\
0 \arrow[r] & Z'_n \arrow[r, hook] \arrow[ur, hook] & \mathrm{C}'_n \arrow[r, two heads, "\partial'"'] \arrow[u, dashed, "\psi'_n"'] & B'_{n-1} \arrow[r] & 0.
\end{tikzcd}$$

Here the top sequence is precisely the short exact sequence (a) and the bottom sequence is the short exact sequence (a′). Here the unlabeled maps are the obvious inclusion and projection maps. It follows immediately from Lemmas 46.1 and 46.2 that there exist isomorphisms $\psi_n\colon \mathrm{C}_n \xrightarrow{\cong} Z_n \oplus B_{n-1}$ and $\psi'_n\colon \mathrm{C}'_n \xrightarrow{\cong} Z'_n \oplus B'_{n-1}$ which make the diagram commute. Now we define $f_n\colon \mathrm{C}_n \to \mathrm{C}'_n$ to be vertical homomorphism given by the diagram, i.e. $f_n = (\psi'_n)^{-1} \circ (\zeta_n \oplus \beta_{n-1}) \circ \psi_n$.

Now we need to show that the maps f_n form a chain map. We consider the diagram

$$\begin{tikzcd}
\mathrm{C}_n \arrow[r, "\partial"] \arrow[d, "f_n"] & B_{n-1} \arrow[r, hook] \arrow[d, "\beta_{n-1}"] & Z_{n-1} \arrow[r, hook] \arrow[d, "\zeta_{n-1}"] & \mathrm{C}_{n-1} \arrow[d, "f_{n-1}"] \\
\mathrm{C}'_n \arrow[r, "\partial'"'] & B'_{n-1} \arrow[r, hook] & Z'_{n-1} \arrow[r, hook] & \mathrm{C}'_{n-1}
\end{tikzcd}$$

where all unlabeled maps are the obvious inclusion maps. The middle square commutes by the choice of β_{n-1} and ζ_{n-1}. We had arranged the maps f_n, f_{n-1} in such a way that the squares to the left and right commute. This shows that the rectangle commutes. But that was exactly what we needed to do to show that the maps f_n, $n \in \mathbb{N}_0$ form a chain map.

Finally note that it follows easily from the choice of ζ_n that the chain map f_n induces the given maps γ_n on homology. ■

The remainder of this section will be occupied with the proof of Proposition 49.2. As we will see the proof is significantly trickier than the proof of Proposition 49.1. We start out with the following definition.

Definition. Let (C_n, c_n) and (D_n, d_n) be two chain complexes and let $\varphi\colon C_* \to D_*$ be a chain map. The *algebraic mapping cone of* φ is the chain complex $(\mathrm{M}(\varphi), \partial_*)$ that is given by the chain groups

$$\mathrm{M}(\varphi)_n \;:=\; \begin{matrix} \mathrm{C}_{n-1} \\ D_n \end{matrix} \;:=\; \mathrm{C}_{n-1} \oplus D_n$$

and the boundary maps[784]

$$\partial\colon \overbrace{\begin{matrix} \mathrm{C}_{n-1} \\ D_n \end{matrix}}^{\mathrm{M}(\varphi)_n} \to \overbrace{\begin{matrix} \mathrm{C}_{n-2} \\ D_{n-1} \end{matrix}}^{\mathrm{M}(\varphi)_{n-1}} \qquad \begin{matrix} x \\ y \end{matrix} \mapsto \begin{matrix} -c_{n-1}(x) \\ \varphi_{n-1}(x)+d_n(y) \end{matrix} \qquad \text{which we write as block matrices} \qquad \begin{array}{ccc} \mathrm{C}_{n-1} & D_n & \\ \left(\begin{matrix} -c_{n-1} \\ \varphi_{n-1} \end{matrix}\right. & \left.\begin{matrix} 0 \\ d_n \end{matrix}\right) & \begin{matrix} \mathrm{C}_{n-2} \\ D_{n-1} \end{matrix} \end{array}$$

Remark. Let $f\colon A \to X$ be a map between topological spaces. Recall that on page 701 we defined the corresponding *mapping cone* as

$$\mathrm{Cone}(f\colon A \to X) \ := \ (A \times [0,1] \sqcup X)/\sim$$

where $(a,1) \sim f(a)$ for all $a \in A$ and $(a,0) \sim (b,0)$ for all $a,b \in A$. As an application we showed on page 701 that for an inclusion map $i\colon A \to X$ there exists a natural isomorphism

$$\widetilde{\mathrm{H}}_n(\mathrm{Cone}(i\colon A \to X)) \ \xrightarrow{\cong}\ \mathrm{H}_n(X,A),$$

i.e. we can reinterpret relative homology groups as reduced homology groups of the corresponding mapping cone. Now suppose that A and X are finite CW-complexes and that f is a cellular map. By Corollary 36.35 (3) we can equip the mapping cone $\mathrm{Cone}(f\colon A \to X)$ with a natural CW-structure. In Exercise 49.3 we will show that the corresponding cellular chain complex is isomorphic to the algebraic mapping cone of the chain map $f_*\colon \mathrm{C}_*^{\mathrm{CW}}(A) \to \mathrm{C}_*^{\mathrm{CW}}(X)$.

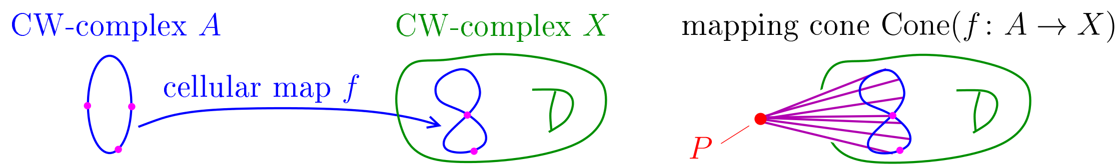

FIGURE 804

Lemma 49.7. *Let $\varphi\colon C_* \to D_*$ be a chain map between two chain complexes (C_n, c_n) and (D_n, d_n). We denote by $i\colon D_n \to \mathrm{M}(\varphi)_n = \mathrm{C}_{n-1} \oplus D_n$ the obvious natural inclusion map and we denote by $p\colon \mathrm{M}(\varphi)_n = \mathrm{C}_{n-1} \oplus D_n \to \mathrm{C}_{n-1}$ the obvious natural projection map. Then the following sequence is exact*

$$\ldots \to \mathrm{H}_n(C_*) \xrightarrow{\varphi_*} \mathrm{H}_n(D_*) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{M}(\varphi)) \xrightarrow{p_*} \mathrm{H}_{n-1}(C_*) \xrightarrow{\varphi_*} \mathrm{H}_{n-1}(D_*) \to \ldots$$

[784]Note that

$$\partial_{n-1} \circ \partial_n \ = \ \begin{pmatrix} -c_{n-2} & 0 \\ \varphi_{n-2} & d_{n-1} \end{pmatrix} \begin{pmatrix} -c_{n-1} & 0 \\ \varphi_{n-1} & d_n \end{pmatrix} \ \underset{\substack{\uparrow \\ \text{since } \varphi \text{ is a chain map and } c_{n-2}\circ c_{n-1}=0 \text{ and } d_{n-1}\circ d_n = 0}}{=} \ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This shows that we have indeed defined a chain complex.

Proof. For each $n \in \mathbb{N}_0$ we consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & D_n \arrow[r, "i_n"] \arrow[d, "d_n"] & \begin{matrix} \mathrm{C}_{n-1} \\ D_n \end{matrix} \arrow[r, "p_n"] \arrow[d, "\partial_n"', "\begin{pmatrix} -c_{n-1} & 0 \\ \varphi_{n-1} & d_n \end{pmatrix}"] & \mathrm{C}_{n-1} \arrow[r] \arrow[d, "c_{n-1}"] & 0 \\
0 \arrow[r] & D_{n-1} \arrow[r, "i_{n-1}"'] & \begin{matrix} \mathrm{C}_{n-2} \\ D_{n-1} \end{matrix} \arrow[r, "p_{n-1}"'] & \mathrm{C}_{n-2} \arrow[r] & 0.
\end{tikzcd}$$

We make the following observations:

(1) the horizontal sequences are exact,
(2) the left-hand square commutes, i.e. the maps i_* define a chain map,
(3) the right-hand square commutes up to a minus sign, i.e. we have

$$c_{n-1} \circ p_n \;=\; -p_{n-1} \circ \partial_n.$$

Fortunately this minus sign in the above equality does not change the fact that the maps p_n induce a map between homology groups.

Despite the minus sign in (3) we can use verbatim the same construction as on page 1115 to obtain connecting homomorphisms $\partial_n\colon \mathrm{H}_n(D_*) \to \mathrm{H}_n(C_*)$, $n \in \mathbb{N}_0$, and verbatim the same proof as in Proposition 43.11 shows that these connecting homomorphisms give rise to a long exact sequence of the form

$$\dots \;\to\; \mathrm{H}_n(C_*) \;\xrightarrow{\partial_n}\; \mathrm{H}_n(D_*) \;\xrightarrow{i_*} \mathrm{H}_n(\mathrm{M}(\varphi)) \;\xrightarrow{p_*} \mathrm{H}_{n-1}(C_*) \;\xrightarrow{\partial_{n-1}}\; \mathrm{H}_{n-1}(D_*) \;\to\; \dots$$

It follows easily from the definition of the connecting homomorphism on page 1115 that for each $n \in \mathbb{N}_0$ we have the equality

$$\partial_n \;=\; \varphi_*\colon\; \mathrm{H}_n(C_*) \;\to\; \mathrm{H}_n(D_*).$$

We have thus now shown that the sequence in the statement of the lemma is exact. ■

The following lemma will be used in the proof of Proposition 49.2.

Lemma 49.8. *Let $(\mathrm{C}_*, \partial_*)$ be a chain complex of free abelian groups such that $\mathrm{H}_n(C_*) = 0$ for all $n \in \mathbb{N}_0$. Then the identity map and the zero map of C_* to itself are chain homotopic.*

Proof. For each $n \in \mathbb{N}_0$ we write as usual

$$Z_n \;=\; \ker(\partial_n\colon\, \mathrm{C}_n \to \mathrm{C}_{n-1}) \quad \text{and} \quad B_n \;=\; \mathrm{im}(\partial_{n+1}\colon\, \mathrm{C}_{n+1} \to \mathrm{C}_n).$$

By our hypothesis on the homology groups of C_* we have $B_n = Z_n$ for all $n \in \mathbb{N}_0$. Thus for each $n \in \mathbb{N}_0$ we have the short exact sequence

$$0 \;\to\; Z_n \;\to\; \mathrm{C}_n \;\xrightarrow{\partial_n}\; Z_{n-1} \;\to\; 0.$$

The group $Z_{n-1} \subset \mathrm{C}_{n-1}$ is a subgroup of a free abelian group, thus it is itself a free abelian group by Lemma 19.2. It follows immediately from Lemmas 46.1 and Splitting Lemma 46.2 that for each $n \in \mathbb{N}_0$ we can find a direct sum decomposition $\mathrm{C}_n = Z_n \oplus A_n$

such that $\partial_n \colon A_n \to B_{n-1} = Z_{n-1}$ is an isomorphism. We denote by $\varphi_n \colon Z_{n-1} \to A_n$ its inverse. Now we define

$$\begin{array}{rcl} P_n \colon \mathrm{C}_n = Z_n \oplus A_n & \to & \mathrm{C}_{n+1} = Z_{n+1} \oplus A_{n+1} \\ z \oplus a & \mapsto & 0 \oplus \varphi_{n+1}(z). \end{array}$$

Using the definitions one can now easily verify that for each $n \in \mathbb{N}_0$ we have

$$P_{n-1} \circ \partial_n + \partial_{n+1} \circ P_n \;=\; \mathrm{id}\colon \underbrace{Z_n \oplus A_n}_{=\mathrm{C}_n} \to \underbrace{Z_n \oplus A_n}_{=\mathrm{C}_n},$$

i.e. the maps $\{P_n\}_{n\geq 0}$ define the desired chain homotopy from the identity to the zero map. For orientation it might be convenient to keep in mind the following diagram:

$$\begin{tikzcd}
C_{n+1} \arrow[r, phantom, "="] & Z_{n+1} & \oplus & A_{n+1} \arrow[dll, "\partial_{n+1}"] \\
C_n \arrow[r, phantom, "="] & Z_n \arrow[urr, "P_n"] & \oplus & A_n \arrow[dll, "\partial_n"] \\
C_{n-1} \arrow[r, phantom, "="] & Z_{n-1} \arrow[urr, "P_{n-1}"] & \oplus & A_{n-1}
\end{tikzcd}$$

This concludes the proof of the lemma. ∎

Now we are finally fully prepared to provide the proof of Proposition 49.2.

Proof of Proposition 49.2. Let $\varphi \colon C_* \to D_*$ be a chain map between chain complexes (C_*, c_*) and (D_*, d_*) of free abelian groups which induces an isomorphism on all homology groups. We need to show that φ is in fact a chain homotopy equivalence.

As above we denote by $(\mathrm{M}(\varphi)_*, \partial_*)$ the mapping cone of φ. We consider the long exact sequence from Lemma 49.7:

$$\dots \;\to\; \mathrm{H}_n(C_*) \;\xrightarrow{\varphi_*}\; \mathrm{H}_n(D_*) \;\xrightarrow{i_*}\; \mathrm{H}_n(\mathrm{M}(\varphi)) \;\xrightarrow{p_*}\; \mathrm{H}_{n-1}(C_*) \;\xrightarrow{\varphi_*}\; \mathrm{H}_{n-1}(D_*) \;\to\; \dots$$

By our hypothesis the maps φ induce isomorphisms of homology groups, thus we see that $\mathrm{H}_n(\mathrm{M}(\varphi)) = 0$ for all $n \geq 0$. By Lemma 49.8 this means that we have a chain homotopy $\{P_n\}_{n\geq 0}$ from the identity map to the zero map. (Note that here we use that by our hypothesis on C_* and D_* each $\mathrm{M}(\varphi)_n = \mathrm{C}_{n-1} \oplus D_n$ is a free abelian group.)

Recall that this means that for each $n \in \mathbb{N}_0$ we have the equality

$$(*) \qquad P_{n-1} \circ \partial_n + \partial_{n+1} \circ P_n \;=\; \mathrm{id}_{\mathrm{C}_{n-1} \oplus D_n}.$$

Using the obvious matrix notation we write

$$P_n \;=\; \begin{pmatrix} P_n^{11} & P_n^{12} \\ P_n^{21} & P_n^{22} \end{pmatrix} \colon \underbrace{\begin{matrix} \mathrm{C}_{n-1} \\ D_n \end{matrix}}_{=\mathrm{M}(\varphi)_n} \;\to\; \underbrace{\begin{matrix} \mathrm{C}_n \\ D_{n+1}. \end{matrix}}_{=\mathrm{M}(\varphi)_{n+1}}$$

Now we can rewrite the above equality $(*)$ and we obtain that

$$\begin{pmatrix} P^{11}_{n-1} & P^{12}_{n-1} \\ P^{21}_{n-1} & P^{22}_{n-1} \end{pmatrix} \circ \begin{pmatrix} -c_{n-1} & 0 \\ \varphi_{n-1} & d_n \end{pmatrix} + \begin{pmatrix} -c_n & 0 \\ \varphi_n & d_{n+1} \end{pmatrix} \circ \begin{pmatrix} P^{11}_{n} & P^{12}_{n} \\ P^{21}_{n} & P^{22}_{n} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{\mathrm{C}_{n-1}} & 0 \\ 0 & \mathrm{id}_{D_n} \end{pmatrix}.$$

Looking at the four entries separately we obtain the following four equalities:

$$\begin{array}{lrcl}
\text{(a)} & -P^{11}_{n-1}\circ c_{n-1} + P^{12}_{n-1}\circ\varphi_{n-1} - c_n\circ P^{11}_n & = & \mathrm{id}_{\mathrm{C}_{n-1}}, \\
\text{(b)} & P^{12}_{n-1}\circ d_n - c_n\circ P^{12}_n & = & 0, \\
\text{(c)} & -P^{21}_{n-1}\circ c_{n-1} + P^{22}_{n-1}\circ\varphi_{n-1} + \varphi_n\circ P^{11}_n + d_{n+1}\circ P^{21}_n & = & 0, \\
\text{(d)} & P^{22}_{n-1}\circ d_n + \varphi_n\circ P^{12}_n + d_{n+1}\circ P^{22}_n & = & \mathrm{id}_{D_n}\,.
\end{array}$$

We make the following observation:

(1) equation (b) implies that the homomorphisms $\{P^{12}_n\colon D_n \to \mathrm{C}_n\}_{n\in\mathbb{N}_0}$ define a chain map $D \to C$.

Now we claim that the chain map $\{P^{12}_n\}_{n\in\mathbb{N}_0}$ is a chain homotopy inverse to the chain map φ. We do so with the following two observations:

(2) equation (a) implies that $P^{12}\circ\varphi \simeq \mathrm{id}_{C_*}$ where the chain homotopy is given by the maps $P^{11}_n\colon\ \mathrm{C}_{n-1} \to C_n$,
(3) whereas equation (d) implies that $\varphi\circ P^{12} \simeq \mathrm{id}_{D_*}$, where the chain homotopy is given by the maps $P^{22}_n\colon D_n \to D_{n+1}$,

The last two statements together imply that φ is indeed a chain homotopy equivalence. ■

We conclude this section with the proof of Proposition 49.3.

Proof of Proposition 49.3. Let C_* and D_* be two chain complexes of free abelian groups and let $\iota\colon C_* \to D_*$ be a chain map with the following two properties:

(a) The chain map $\iota\colon C_* \to D_*$ induces an isomorphism of homology groups.
(b) For each $k \in \mathbb{N}_0$ the map $\iota\colon\ \mathrm{C}_k \to D_k$ is a monomorphism and the image is a subsummand of D_k.

We need to show that there exists a chain map $\pi\colon D_* \to \mathrm{C}_*$ with the following two properties:

(1) For each $k\in\mathbb{N}_0$ we have $\pi_k\circ\iota_k = \mathrm{id}_{\mathrm{C}_k}$.
(2) The maps ι and π are chain homotopy homotopy equivalences.

The key to proving the proposition is the following claim:

Claim. We set $\mathrm{C}_{-1} = D_{-1} = 0$ and we define π_{-1} to be the zero homomorphism. Let $k \in \mathbb{Z}_{\geq -1}$. Suppose that for $j = -1,\dots,k$ we are given maps $\pi_j\colon D_j \to \mathrm{C}_j$ such that for each $j \in \{0,\dots,k\}$ we have the following equalities:

(1) $\pi_j\circ\iota_j = \mathrm{id}_{\mathrm{C}_j}$.
(2) $\partial\circ\pi_{j-1} = \pi_j\circ\partial$.

Then there exists a map $\pi_{k+1}\colon D_{k+1} \to \mathrm{C}_{k+1}$ such that the maps $\pi_0,\dots,\pi_{k+1}$ have the above properties.

First note that by hypothesis we can pick for each $m \in \mathbb{N}$ a submodule $C_m^\dagger$ of D_m such that $D_m = \iota(\mathrm{C}_m) \oplus C_m^\dagger$. For orientation it is perhaps convenient to consider the following maps:

$$\begin{tikzcd} D_{k+1} = \iota(\mathrm{C}_{k+1}) \oplus C_{k+1}^\dagger \arrow[r, "\partial_D"] \arrow[d, shift left, "?\pi_{k+1}?"] & \iota(\mathrm{C}_k) \oplus C_k^\dagger = D_k \arrow[d, shift left, "\pi_k"] \\ \mathrm{C}_{k+1} \arrow[r, "\partial_C"'] \arrow[u, shift left, "\iota_{k+1}"] & \mathrm{C}_k \arrow[u, shift left, "\iota_k"] \end{tikzcd}$$

Since $C_{k+1}^\dagger$ is a subgroup of the free abelian group D_{k+1} we know by Lemma 19.2 that $C_{k+1}^\dagger$ is also a free abelian group. We pick a basis T for the free abelian group $C_{k+1}^\dagger$. Let $t \in T$. Since ι induces an isomorphism of homology groups we know that there exists a $c_{k+1} \in \mathrm{C}_{k+1}$ such that t and $\iota(c_{k+1})$ are homologous. We calculate that

$$\pi_k(\partial_D(t_{k+1})) \underset{\substack{\uparrow \\ \text{since } t_{k+1} \text{ and } \iota(c_{k+1}) \text{ are homologous} \\ \text{we know that } \partial_D(t_{k+1}) = \partial_D(\iota(c_{k+1}))}}{=} \pi_k(\partial_D(\iota_{k+1}(c_{k+1}))) \underset{\substack{\uparrow \\ \text{since } \iota \text{ is a chain map}}}{=} \pi_k(\iota_k(\partial_C(c_{k+1}))) \underset{\substack{\uparrow \\ \text{since } \pi_k \circ \iota_k = \mathrm{id}_{\mathrm{C}_k}}}{=} \partial_C(c_{k+1}).$$

We set $\pi_{k+1}(t) := c_{k+1}$. Since T_{k+1} is a basis for the free abelian group $C_{k+1}^\dagger$ we obtain a homomorphism $\varphi\colon C_{k+1}^\dagger \to \mathrm{C}_{k+1}$. It is now straightforward to verify that the map

$$\begin{array}{rrcl} \pi_{k+1}\colon & D_{k+1} = \iota(\mathrm{C}_{k+1}) \oplus C_{k+1}^\dagger & \to & \mathrm{C}_{k+1} \\ & \iota(c_{k+1}) + d_{k+1} & \mapsto & c_{k+1} + \varphi(d_{k+1}) \end{array}$$

has all the desired properties. ⊞

Since C_* and D_* are chain complexes of free abelian groups and since ι induces an isomorphisms of homology groups we know by Proposition 49.2 that ι is a chain homotopy equivalence. By Property (2) of the claim the maps $\{\pi_n\colon D_n \to \mathrm{C}_n\}_{n\in\mathbb{N}_0}$ define a chain map. By Property (1) of the claim we have for each $k \in \mathbb{N}_0$ that $\pi_* \circ \iota_* = \mathrm{id}_{\mathrm{H}_k(C_*)}$. Thus we see that the chain map π_* also induces isomorphism of the homology groups. So once again we appeal to Proposition 49.2 to conclude that the maps π_* form a chain homotopy equivalence. ■

49.3. **Singular and cellular chain complexes.** In Chapter 48 we introduced the cellular chain complex and the cellular homology groups of a CW-complex. We saw that cellular homology groups are naturally isomorphic to singular homology groups. At that point we did not say anything about the relationship between the cellular chain complex and the singular chain complex. We will rectify this in this section. The extra information that we gain in this section will stand us in good stead in Section 74.2 when we explore the relationship between cellular cohomology and singular cohomology.

First we recall some definitions and some notation. Let X be a CW-complex.

(1) Recall that given $n \in \mathbb{N}_0$ we denote by X^n the n-skeleton of X, i.e. the union of all cells of dimension $\leq n$.
(2) We write $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$ and we denote by $d_n\colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_{n-1}^{\mathrm{CW}}(X)$ the cellular boundary map from page 1259.

(3) We refer to

$$\dots \xrightarrow{d_{n+2}} C^{CW}_{n+1}(X) \xrightarrow{d_{n+1}} C^{CW}_{n}(X) \xrightarrow{d_n} C^{CW}_{n-1}(X) \xrightarrow{d_{n-1}} \dots$$

as the *cellular chain complex.* The corresponding homology groups are denoted by $H^{CW}_n(X)$.

In Proposition 48.4 we had seen that, given a CW-complex X the cellular homology groups $H^{CW}_k(X)$ and the singular homology groups $H_k(X)$ are naturally isomorphic.

In the following we want to relate the cellular chain complex $C^{CW}_*(X)$ to the singular chain complex $C_*(X)$. The aforementioned Proposition 48.4 together with the newly acquired Propositions 49.1 and 49.2 gives us immediately the following corollary.

Corollary 49.9. *Given any* CW*-complex X there exists a chain homotopy equivalence* $C_*(X) \to C^{CW}_*(X)$.

A part of the proof of Corollary 49.9 is pure "algebraic nonsense", in particular we do not get an explicit chain homotopy equivalence. In fact there is no obvious chain homotopy equivalence, in fact there is not even an obvious "interesting" chain map between these two chain complexes. But as we will see in a minute, it turns out that there is a nice "'intermediate" chain complex which maps into both of the above chain complexes:

Definition. Given a CW-complex X and given $n \in \mathbb{N}_0$ we define the[785] *intermediate cellular chain complex*

$$C^{\text{int}}_n(X) \;:=\; \ker\left(C_n(X^n) \xrightarrow{\partial} C_{n-1}(X^n) \to C_{n-1}(X^n, X^{n-1})\right).$$

The following proposition shows that the intermediate cellular chain complex sits comfortably between the singular and the cellular chain complex. The proposition gives us in particular a more conceptual and satisfying proof of Corollary 49.9.

Proposition 49.10. *Let X be a* CW*-complex. The two natural chain maps*

$$i\colon C^{\text{int}}_*(X) \hookrightarrow C_*(X) \quad \text{and} \quad p\colon C^{\text{int}}_*(X) \twoheadrightarrow C^{CW}_*(X),$$

which are given by the obvious inclusion and the obvious projection, are chain homotopy equivalences.

Remark. The proof of the proposition is modelled on the proofs provided in [**SchubH68**, p. 303] or alternatively [**Lü98**, Lemma 4.2] and [**Lü89**, p. 263], see also [**Wall66a**, Lemma 1].

PROOF. By Proposition 49.2 it suffices to show that for each $n \in \mathbb{N}_0$ the maps

$$(1)\ i_*\colon H_n(C^{\text{int}}_*(X)) \to \underbrace{H_n(C_*(X))}_{=H_n(X)} \quad \text{and} \quad (2)\ p_*\colon H_n(C^{\text{int}}_*(X)) \to \underbrace{H_n(C^{CW}_*(X))}_{=H^{CW}_n(X)}$$

are isomorphisms. So let $n \in \mathbb{N}_0$. Before we start with showing that the two maps above are isomorphisms we recall that in Lemma 48.1 we proved the following two convenient facts:

(a) We have $H_{n-1}(X^{n-2}) = 0$ and $H_n(X^{n-1}) = 0$.

[785] It follows almost immediately from the definitions that $C^{\text{int}}_*(X)$ is a subcomplex of $C_*(X)$, in particular it is a chain complex in its own right.

(b) The inclusion $k\colon X^n \to X$ induces an isomorphism $k_*\colon \mathrm{H}_n(X^{n+1}) \xrightarrow{\cong} \mathrm{H}_n(X)$.
(c) The inclusion $k\colon X^n \to X$ induces an epimorphism $k_*\colon \mathrm{H}_n(X^n) \twoheadrightarrow \mathrm{H}_n(X)$.

Claim 1. Let $n \in \mathbb{N}_0$.

(i) The map $i_*\colon \mathrm{H}_n(\mathrm{C}^{\mathrm{int}}_*) \to \mathrm{H}_n(\mathrm{C}_*(X)) = \mathrm{H}_n(X)$ is a monomorphism.
(ii) The map $i_*\colon \mathrm{H}_n(\mathrm{C}^{\mathrm{int}}_*) \to \mathrm{H}_n(\mathrm{C}_*(X)) = \mathrm{H}_n(X)$ is an epimorphism.

We consider the following two commutative diagrams

$$\begin{tikzcd}
\mathrm{C}^{\mathrm{int}}_{n+1}(X) \arrow[r, hook, "j"] \arrow[d, "\partial"] & \mathrm{C}_{n+1}(X^{n+1}) \arrow[r, hook, "k"] \arrow[d, "\partial"] & \mathrm{C}_{n+1}(X) \arrow[d, "\partial"] \\
\mathrm{C}^{\mathrm{int}}_{n}(X) \arrow[r, hook, "j"] & \mathrm{C}_{n}(X^{n+1}) \arrow[r, hook, "k"] & \mathrm{C}_{n}(X)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{C}^{\mathrm{int}}_{n}(X) \arrow[r, hook, "j"] \arrow[d, "\partial"] & \mathrm{C}_{n}(X^{n}) \arrow[r, hook, "k"] \arrow[d, "\partial"] & \mathrm{C}_{n}(X) \arrow[d, "\partial"] \\
\mathrm{C}^{\mathrm{int}}_{n-1}(X) \arrow[r, hook, "j"] & \mathrm{C}_{n-1}(X^{n}) \arrow[r, hook, "k"] & \mathrm{C}_{n-1}(X).
\end{tikzcd}$$

Both statements follow easily from the diagrams together with (b) and (c) above:

(i) Let $\sigma \in \mathrm{C}^{\mathrm{int}}_n(X)$ be a cycle with the property that $i(\sigma)$ is null-homologous. It follows from (b) that $j(\sigma)$ is already null-homologous, i.e. there exists a $\tau \in \mathrm{C}_{n+1}(X^{n+1})$ with $\partial\tau = j(\sigma)$. Note that $j(\sigma) = \sigma \in \mathrm{C}_n(X^n)$. Thus we see that τ actually lies in $\mathrm{C}^{\mathrm{int}}_{n+1}(X)$. But this implies that σ is null-homologous in $\mathrm{C}^{\mathrm{int}}_*(X)$.
(ii) Let $[\sigma] \in \mathrm{H}_n(X)$. By (c) we know that there exists a cycle $\tau \in \mathrm{C}_n(X^n)$ with $[k(\tau)] = [\sigma]$. Since $\partial\tau = 0$ we see that $\tau \in \mathrm{C}^{\mathrm{int}}_n(X)$. Since the bottom-left horizontal map is a monomorphism we see that $\tau \in \mathrm{C}^{\mathrm{int}}_n(X)$ is also a cycle. But this shows that $[\sigma] = [i(\tau)]$. $\boxplus$

Claim 2. Let $n \in \mathbb{N}_0$.

(i) The map $p_*\colon \mathrm{H}_n(\mathrm{C}^{\mathrm{int}}_*) \to \mathrm{H}_n(\mathrm{C}^{\mathrm{CW}}_*(X))$ is a monomorphism.
(ii) The map $p_*\colon \mathrm{H}_n(\mathrm{C}^{\mathrm{int}}_*) \to \mathrm{H}_n(\mathrm{C}^{\mathrm{CW}}_*(X))$ is an epimorphism.

For orientation purposes we consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{C}^{\mathrm{int}}_{n+1}(X) \arrow[r, hook] \arrow[d, "\partial"] & \mathrm{C}_{n+1}(X^{n+1}) \arrow[r, two heads] \arrow[d, "\partial"] & \mathrm{C}_{n+1}(X^{n+1}, X^n) \arrow[d, "d"] \\
\mathrm{C}^{\mathrm{int}}_{n}(X) \arrow[r, two heads, "p"] \arrow[d, "\partial"] & \mathrm{C}_{n}(X^{n}) \arrow[r] \arrow[d, "\partial"] & \mathrm{C}_{n}(X^{n}, X^{n-1}) \arrow[d, "d"] \\
\mathrm{C}^{\mathrm{int}}_{n-1}(X) \arrow[r, two heads, "p"] & \mathrm{C}_{n-1}(X^{n}) \arrow[r] & \mathrm{C}_{n-1}(X^{n-1}, X^{n-2}).
\end{tikzcd}$$

As we will see, the claim follows from (a) and some chasing around in the above diagram:

(i) Let $\sigma \in \mathrm{C}^{\mathrm{int}}_n(X)$ be a cycle. Suppose that $p(\sigma) = 0 \in \mathrm{H}^{\mathrm{CW}}_n(X)$. This means that there exists a $\mu \in \mathrm{C}_{n+1}(X^{n+1}, X^n)$ with $\partial\mu = p(\sigma)$. We pick a $\tau \in \mathrm{C}_{n+1}(X^{n+1})$ that represents σ. Note that $\partial\tau = \sigma + \alpha$ with $\alpha \in \mathrm{C}_n(X^{n-1})$. Since $\partial(\partial\tau) = 0$ and since $\partial\sigma = 0$ we see that $\partial\alpha = 0$. From (a) we obtain that there exists a $\beta \in \mathrm{C}_{n+1}(X^{n-1})$

with $\partial\beta = \alpha$. Note that $\partial(\tau - \beta) = \partial\tau - \partial\beta = \sigma$. Since $\sigma \in \mathrm{C}_n(X^n)$ we conclude that $\tau - \beta \in \mathrm{C}^{\mathrm{int}}_{n+1}(X)$. Thus we have shown that $[\sigma] = 0 \in \mathrm{H}_n(\mathrm{C}^{\mathrm{int}}_*(X))$.

(ii) Given a class in $\mathrm{H}_n(\mathrm{C}^{\mathrm{CW}}_*(X))$ we pick a representative in $\mathrm{C}^{\mathrm{CW}}_n(X)) = \mathrm{C}_n(X^n, X^{n-1})$ for which in turn we pick a lift $\sigma \in \mathrm{C}_n(X^n)$. Note that $\partial\sigma \in \mathrm{C}_{n-1}(X^{n-2})$. But by (a) we know that $\mathrm{H}_{n-1}(X^{n-2}) = 0$. Hence there exists a $\mu \in \mathrm{C}_n(X^{n-2})$ with $\partial\mu = \partial\sigma$. Note that $\partial(\sigma - \mu) = 0$, in particular $\sigma - \mu$ is a cycle in $\mathrm{C}^{\mathrm{int}}_n(X)$. Furthermore note that σ and $\sigma - \mu$ represent the same element in $\mathrm{C}_n(X^n, X^{n-1})$.

■

Exercises for Chapter 49.

Exercise 49.1. We consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, "n\mapsto(0,n)"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "(m,n)\mapsto m+2n"'] & \mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

We view the top and bottom sequences as chain complexes C_* respectively D_* and we view the vertical maps as a chain map f.

(a) Convince yourself that f induces the zero map on homology.
(b) Is the chain map f chain homotopic to the zero chain map?

Exercise 49.2. Suppose $(\mathrm{C}_*, \partial_*)$ and (C'_*, ∂'_*) are two chain complexes. We do *not* assume that the chain groups are necessarily free abelian groups. Suppose that for each $n \in \mathbb{N}_0$ the homology groups $\mathrm{H}_n(C_*)$ and $\mathrm{H}_n(\mathrm{C}'_*)$ are isomorphic. Does it follow that for each $n \in \mathbb{N}_0$ and any abelian group G the cohomology groups $\mathrm{H}^n(\mathrm{C}_*; G)$ and $\mathrm{H}^n(\mathrm{C}'_*; G)$ are also isomorphic?

Exercise 49.3. Let $f\colon A \to X$ be a map between topological spaces. We define the corresponding *mapping cone* as

$$\mathrm{Cone}(f\colon A \to X) \;:=\; ((A \times [0,1]) \sqcup X)/\sim$$

where $(a,1) \sim f(a)$ for all $a \in A$ and $(a,0) \sim (b,0)$ for all $a, b \in A$. Now suppose that A and X are finite CW-complexes and that f is a cellular map.

We equip the mapping cone $\mathrm{Cone}(f\colon A \to X)$ with the natural CW-structure of Corollary 36.35. Show that the corresponding cellular chain complex is isomorphic to the algebraic mapping cone of the chain map $f_*\colon \mathrm{C}^{\mathrm{CW}}_*(A) \to \mathrm{C}^{\mathrm{CW}}_*(X)$.

50. The Jordan Curve Theorem

We start out with the following proposition.

Proposition 50.1.

(1) *For every injective map* $h\colon \overline{B}^k \to S^n$ *we have*

$$\widetilde{\mathrm{H}}_i\big(S^n \setminus h(\overline{B}^k)\big) \;=\; 0 \qquad \textit{for all } i \in \mathbb{N}_0.$$

(2) *For every injective map* $h\colon S^k \to S^n$ *we have*

$$\widetilde{\mathrm{H}}_i\big(S^n \setminus h(S^k)\big) \;\cong\; \begin{cases} \mathbb{Z}, & \textit{if } i = n-k-1, \\ 0, & \textit{otherwise.} \end{cases}$$

Example.

(1) Let $K \subset S^3$ be a knot, i.e. let K be a 1-dimensional submanifold of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is diffeomorphic to S^1. This means in particular that there exists a smooth embedding $h\colon S^1 \to S^3$ with $h(S^1) = K$. In Lemma 46.14 we had already determined that $\mathrm{H}_1(S^3 \setminus K) \cong \mathbb{Z}$. Now we can compute the higher homology groups as well. More precisely, the combination of Proposition 50.1 (2) and Lemma 43.1 says that

$$\mathrm{H}_i(S^3 \setminus K) \;=\; \mathrm{H}_i(S^3 \setminus h(S^1)) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0,1, \\ 0, & \text{otherwise.} \end{cases}$$

(2) In Proposition 50.1 we do not assume that h is in any way smooth. For example there are injective maps $h\colon S^1 \to S^2$ of infinite length, see e.g.

`https://en.wikipedia.org/wiki/Koch_snowflake`,

to which the proposition nonetheless applies. Reduced homology is only defined for non-empty topological spaces. It is implicit in the statement of Proposition 50.1 that $S^n \setminus h(\overline{B}^k)$ is not the empty set. Even this statement is not entirely obvious since, as we have pointed out on several occasions, there are non-injective maps $h\colon \overline{B}^1 \to S^n$ that are in fact surjective.

(3) In Figure 805 we see the image of an injective map $\varphi\colon [0,1] \to S^3 = \mathbb{R}^3 \cup \{\infty\}$ and we see a closed curve $C \subset S^3 \setminus \varphi([0,1])$. It follows from Proposition 50.1 (1) that $\mathrm{H}_1(S^3 \setminus \varphi([0,1])) = 0$, in particular C is null-homologous in $\mathrm{H}_1(S^3 \setminus \varphi([0,1])) = 0$.[786] But on [**Sti93**, p. 152] it is shown that the curve C represents in fact a non-trivial element in $\pi_1(S^3 \setminus \varphi([0,1]))$.

To make the exposition more readable we prove the two parts of Proposition 50.1 separately.

Proof of Proposition 50.1 (1). We write $I = [-1,1]$. By Proposition 2.52 (2) there exists a homeomorphism $I^k \xrightarrow{\cong} \overline{B}^k$. Therefore in the statement of Proposition 50.1 (1) we

[786] In Lemma 48.8 we saw that the boundary curve of an orientable surface with one boundary component is null-homologous. In Exercise 50.1 we will see that the closed curve C is in fact the boundary of an orientable surface that is embedded in $S^3 \setminus \varphi([0,1])$.

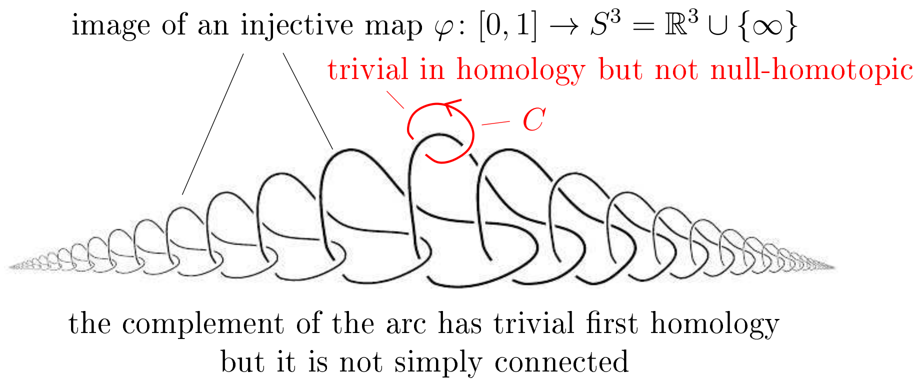

FIGURE 805

can replace $\overline{B}^k$ by I^k. So we now want to show that for any injective map $h\colon I^k \to S^n$ we have

$$\widetilde{\mathrm{H}}_i(S^n \setminus h(I^k)) \;=\; 0 \quad \text{for all } i \in \mathbb{N}_0.$$

We will prove this statement by induction on k. We start out with $k = 0$. So let $h\colon I^0 = \{0\} \to S^n$ be an injective map. It follows from the stereographic projection defined in Lemma 2.44 that $S^n \setminus h(\{0\})$ is homeomorphic to $\mathbb{R}^n$. Hence we obtain from Corollary 42.8 (3) that $\widetilde{\mathrm{H}}_*(S^n \setminus h(\{0\})) = 0$.

Now we do the induction step. We assume we know the desired statement for all injective maps $I^{k-1} \to S^n$. Now let $h\colon I^k \to S^n$ be an injective map. We suppose that the desired conclusion does not hold, i.e. we suppose that $\widetilde{\mathrm{H}}_j(S^n \setminus h(I^k)) \neq 0$ for some j. We pick a non-trivial element $g_0 \in \widetilde{\mathrm{H}}_j(S^n \setminus h(I^k))$.

We write $Q = I^{k-1}$, $J_0 = I$ and we write

$$A_1 \;=\; S^n \setminus h(Q \times [0, \tfrac{1}{2}]) \quad \text{and} \quad A_2 \;=\; S^n \setminus h(Q \times [\tfrac{1}{2}, 1]).$$

Since h is injective we have

$$A_1 \cap A_2 \;=\; S^n \setminus h(I^k) \quad \text{and} \quad A_1 \cup A_2 \;=\; S^n \setminus h(Q \times \{\tfrac{1}{2}\}).$$

From Lemma 2.40 (3) we know that A_1 and A_2 are open subsets of $A_1 \cup A_2$. Therefore we

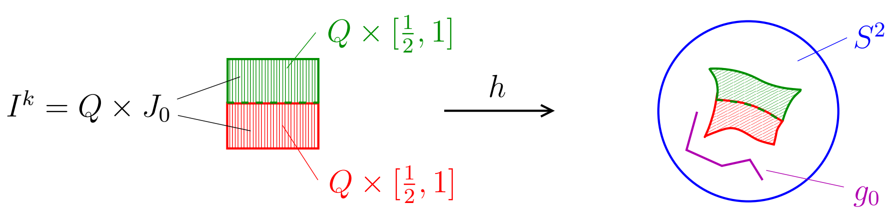

FIGURE 806. Illustration for the proof of Proposition 50.1 (1).

can apply the Mayer–Vietoris Theorem 46.5 and we obtain the long exact sequence

$$\dots \to \underbrace{\widetilde{\mathrm{H}}_{j+1}(A_1 \cup A_2)}_{=0 \text{ by induction}} \to \widetilde{\mathrm{H}}_j(A_1 \cap A_2) \;\to\; \widetilde{\mathrm{H}}_j(A_1) \oplus \widetilde{\mathrm{H}}_j(A_2) \to \underbrace{\widetilde{\mathrm{H}}_i(A_1 \cup A_2)}_{=0 \text{ by induction}} \to \dots$$

So we see that the middle map is an isomorphism. Thus there exists an $i \in \{1,2\}$ such that the image of g_0 is non-trivial in $\widetilde{\mathrm{H}}_j(A_i)$. If $i = 1$, then we set $J_1 := [0, \frac{1}{2}]$, otherwise we set $J_1 := [\frac{1}{2}, 1]$. We denote by

$$\varphi_1 \colon \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_0)) \;\to\; \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_1))$$

the inclusion induced map. By the above exact sequence φ_1 is an epimorphism. We denote by ρ_1 the inverse of the induced isomorphism

$$\widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_0))/\ker(\varphi_1) \;\to\; \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_1)).$$

Now we iterate this process by running the same argument with J_0 replaced by J_1 and so on. For each $m \in \mathbb{N}$ we can find an interval J_m of length $\frac{1}{2^m}$ that is contained in J_{m-1} such that for the inclusion induced homomorphism

$$\varphi_m \colon \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_{m-1})) \;\to\; \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_m))$$

we have that $g_m := \varphi_m(g_{m-1})$ is non-zero. We define ρ_m the same way from φ_m as we had obtained ρ_1 from φ_1.

By the definition of the direct limit we have the following commutative diagram:[787]

$$\begin{tikzcd}
\vdots & \\
\widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_2)) \arrow[u, two heads, "\varphi_3"] \arrow[r] \arrow[ddr, "\rho_2\circ\rho_1"] & \varinjlim \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_m)) \arrow[dd, dashed] \\
\widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_1)) \arrow[u, two heads, "\varphi_2"] \arrow[ur] \arrow[dr, "\rho_1"] & \\
\widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_0)) \arrow[u, two heads, "\varphi_1"] \arrow[uur] \arrow[r, "\text{image of } g_0 \text{ is non-zero}"'] & \dfrac{\widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times I))}{\bigcup\limits_{m\in\mathbb{N}} \ker(\varphi_m \circ \cdots \circ \varphi_1)}
\end{tikzcd}$$

Since each J_m has half the length of J_{m-1} we conclude that the intersection $\bigcap\limits_{m\in\mathbb{N}} J_m$ contains a unique point $t \in [0,1]$.[788] We have

$$\underrightarrow{\lim}\, \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times J_m)) \underset{\substack{\uparrow \\ \text{by Proposition 47.4 since the} \\ \text{subsets } S^n \setminus h(Q\times J_m) \text{ are open}}}{\cong} \widetilde{\mathrm{H}}_j\Big(\bigcap_{m\in\mathbb{N}} S^n \setminus h(Q \times J_m)\Big) \underset{\substack{\uparrow \\ \text{since } h \text{ is injective}}}{=} \widetilde{\mathrm{H}}_j(S^n \setminus h(Q \times \{t\})) \underset{\substack{\uparrow \\ \text{by induction} \\ \text{since } Q\times\{t\} \cong I^{k-1}}}{=} 0.$$

But we have now obtained a contradiction, since the bottom horizontal map of the above commutative diagram has a non-trivial image but it factors through the trivial group. ■

[787]Note that we have

$$\ker(\varphi_1) \subset \ker(\varphi_2 \circ \varphi_1) \subset \ker(\varphi_3 \circ \varphi_2 \circ \varphi_1) \subset \ldots$$

It follows that the union of these subgroups is again a subgroup.

[788]Why does this follow? What results from real analysis do we need to draw that conclusion?

Proof of Proposition 50.1 (2). Recall that we have to show that for every injective map $h\colon S^k \to S^n$ we have

$$\widetilde{\mathrm{H}}_i\big(S^n \setminus h(S^k)\big) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = n-k-1, \\ 0, & \text{otherwise.} \end{cases}$$

We prove this statement by induction on k. For $k=0$ we have

$$\widetilde{\mathrm{H}}_i(S^n \setminus h(S^0)) \underset{\substack{\uparrow \\ \text{since } S^n \setminus h(S^0) \text{ is homotopy} \\ \text{equivalent to } S^{n-1}\text{[789]}}}{=} \widetilde{\mathrm{H}}_i(S^{n-1}) \underset{\substack{\uparrow \\ \text{by Proposition 43.4}}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } i = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Now suppose we know the statement for some $k \in \mathbb{N}$. Suppose we are given an injective map $h\colon S^{k+1} \to S^n$. We denote by $S^{k+1}_{\le 0}$ the lower hemisphere of S^{k+1} and we denote by $S^{k+1}_{\ge 0}$ the upper hemisphere of S^{k+1}. Note that $S^{k+1}_{\le 0} \cap S^{k+1}_{\ge 0} = S^k$. We apply the Mayer–Vietoris Theorem 46.5 to the decomposition of $S^n \setminus h(S^{k+1})$ into the open sets $S^n \setminus h(S^{k+1}_{\le 0})$ and $S^n \setminus h(S^{k+1}_{\ge 0})$. We obtain a long exact sequence[790]

$$\cdots \to \widetilde{\mathrm{H}}_i(S^n \setminus h(S^k)) \to \underbrace{\widetilde{\mathrm{H}}_i((S^n \setminus h(S^{k+1}_{\le 0}))}_{\substack{= 0 \text{ by part (1) and} \\ \text{since } S^{k+1}_{\le 0} \cong \overline{B}^{k+1}}} \oplus \underbrace{\widetilde{\mathrm{H}}_i(S^n \setminus h(S^{k+1}_{\ge 0}))}_{\substack{= 0 \text{ by part (1) and} \\ \text{since } S^{k+1}_{\ge 0} \cong \overline{B}^{k+1}}} \to \widetilde{\mathrm{H}}_i(S^n \setminus h(S^{k+1})) \to \dots.$$

Thus we see that

$$\widetilde{\mathrm{H}}_i(S^n \setminus h(S^{k+1})) \;\cong\; \widetilde{\mathrm{H}}_{i-1}(S^n \setminus h(S^k)) \underset{\substack{\uparrow \\ \text{by induction}}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } i-1 = n-k-1, \\ 0, & \text{otherwise.} \end{cases}$$

■

The following definition is a minute variation on the definition of a Jordan curve given on page 452.

Definition. A *planar Jordan curve* is the image of an injective map $S^1 \to \mathbb{R}^2$.

Now we can formulate the following immediate corollary to Proposition 50.1.

Corollary 50.2. (Jordan Curve Theorem) *For every planar Jordan curve C the complement $\mathbb{R}^2 \setminus C$ consists of two path-connected components.*

Remark.

(1) The Jordan Curve Theorem 50.2 might appear as "self-evident". To disabuse the reader of this idea recall that in Proposition 2.60 we saw that there exist planar Jordan curves with non-zero 2-dimensional Lebesgue measure.
(2) The Jordan Curve Theorem 50.2 was first proved in 1887 by Camille Jordan. Camille Jordan (1838-1922) was a French mathematician who most readers will already have encountered in linear algebra.

[789] Can you give a rigorous proof for the statement that the complement of two points in S^n is homotopy equivalent to S^{n-1}?

[790] Here we use, as in the proof of Proposition 50.1 (1), implicitly the injectivity of h.

PROOF. Let $h\colon S^1 \to \mathbb{R}^2$ be an injective map. We use the inclusion $\mathbb{R}^2 \to \mathbb{R}^2 \cup \{\infty\} = S^2$ to view $\mathbb{R}^2$ as a subset of S^2. Now we see that

$$\widetilde{\mathrm{H}}_0(\mathbb{R}^2 \setminus C) \;=\; \widetilde{\mathrm{H}}_0(\mathbb{R}^2 \setminus h(S^1)) \underset{\substack{\uparrow \\ \text{see Exercise 46.12}}}{\cong} \widetilde{\mathrm{H}}_0(S^2 \setminus h(S^1)) \underset{\substack{\uparrow \\ \text{Proposition 50.1 (2)}}}{\cong} \mathbb{Z}$$

It follows from Lemma 43.1 (5) that $\mathbb{R}^2 \setminus C$ has precisely two path-components. ■

Proposition 50.1, together with Lemma 43.1, says in particular, that if $h\colon S^{n-1} \to S^n$ is an injective map, then

$$\mathrm{H}_i(S^n \setminus h(S^{n-1})) \;\cong\; \begin{cases} \mathbb{Z}^2, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Put differently, homologically the topological space $S^n \setminus h(S^{n-1})$ looks like the disjoint union of two open n-dimensional balls. One might now suspect that $S^n \setminus h(S^{n-1})$ is in fact homeomorphic to the union of two open n-dimensional balls.

This is in fact true in dimension $n = 2$. More precisely Arthur Schönflies[791] proved in 1908 the following theorem. We refer to [**Tn92**, Section 3] for a perhaps more readable proof of the theorem.[792]

Theorem 50.3. (Schönflies Theorem) *Let $h\colon S^1 \to S^2$ be an injective map. Then h extends to a homeomorphism of S^2, i.e. there exists a homeomorphism $H\colon S^2 \to S^2$ such that $H|_{S^1} = h$. In particular both components of $S^2 \setminus h(S^1)$ are homeomorphic to open disks.*

Remark.

(1) The Schönflies Theorem is often also called the Jordan-Schöenflies Theorem.
(2) If $h\colon S^1 \to S^2$ is a smooth embedding, then the conclusion of the Schönflies Theorem 50.3 also follows from the much easier Proposition 23.18.

It is natural to ask whether the conclusion of the Schönflies Theorem also holds in higher dimensions. But this hope breaks down already one dimension higher. More precisely, James Alexander[793] showed in 1924 that there exists an injective map $S^2 \to \mathbb{R}^3$ such that one of the two components of the complement of $h(S^2)$ is not simply connected. The image $h(S^2)$ is called the *Alexander horned sphere*. It is illustrated in Figure 807. A more detailed description of the Alexander horned sphere and the proof that the complement is not simply connected is given in [**Al24**] and also [**Hat02**, p. 170]. In Exercise 50.2 we invite the reader to modify the Example shown in Figure 807 to produce an injective map $S^2 \to \mathbb{R}^3$ such that neither of the two components of the complement of $h(S^2)$ is simply connected.

[791] Arthur Schönflies (1853–1928) was a German mathematician.

[792] More precisely, in [**Tn92**, Section 3] a proof for the following statement is given:
Let $g\colon S^1 \to \mathbb{R}^2$ be an injective map. Then g extends to a homeomorphism of $\mathbb{R}^2$.
We leave it to the reader to deduce Theorem 50.3 from that statement.

[793] James Alexander (1888-1971) was an American mathematician. He also introduced the Alexander polynomial of a knot.

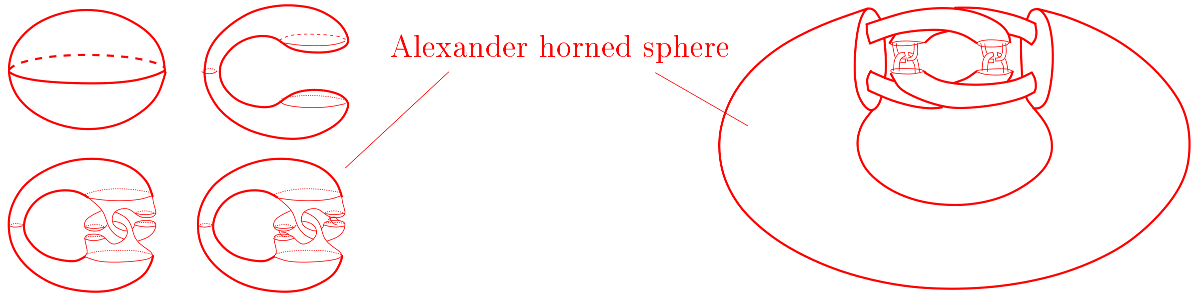

FIGURE 807

We have now seen that injective maps $S^2 \to S^3$ can be pretty "wild". The map that defines the Alexander horned sphere is continuous, but it is certainly not smooth. So what happens if we restrict ourselves to smooth maps? In fact in the smooth setting we have the following theorem.

Theorem 50.4. (Generalized smooth Schönflies Theorem) *Let $h\colon S^{n-1} \to S^n$ be a smooth embedding. If $n \neq 4$, then there exists an orientation-preserving diffeomorphism $H\colon S^n \to S^n$ with $H|_{S^{n-1}} = h$.*

For $n = 2$ the statement follows fairly easily from the classification of smooth 2-dimensional smooth manifolds that we gave in the Surface Classification Theorem 23.4, see e.g. [**Hirs76**, p. 207]. For $n = 3$ this statement was proved by James Alexander [**Al24**]. See also [**Hat3**, Chapter 1] and [**Schul14**, Theorem 3.2.5] and for an exposition of the proof. For $n \geq 5$ this statement was independently proved by Barry Mazur [**Maz59**] in 1959 and by Morton Brown [**BrownM60**] in 1960.

The case $n = 4$ is missing in the Generalized Schönflies Theorem. In fact it is still unknown whether the statement holds for $n = 4$.

Question 50.5. *Is it true that given any smooth embedding $h\colon S^3 \to S^4$ there exists a diffeomorphism $H\colon S^4 \to S^4$ with $H|_{S^3} = h$?*

This question is very much open and there is not even a consensus whether one expects the answer to be yes or no. If you can answer this question you would almost certainly obtain a Fields medal.[794]

We conclude this section with two more consequences of Proposition 50.1. The following theorem was first proved by Luitzen Brouwer [**Brou1911**] in 1911. It can be viewed as a significant strengthening of the Smooth Invariant of Domain Theorem 6.19 (1).

Theorem 50.6. (Topological Invariance of Domain Theorem) *Let U be an open subset of $\mathbb{R}^n$.*

(1) *If $h\colon U \to \mathbb{R}^n$ is an injective map, then $h(U)$ is also an open subset of $\mathbb{R}^n$.*[795]
(2) *If $h\colon U \to \mathbb{R}^m$ be an injective map, then $m \geq n$.*

[794] But hurry up, you can only get a Fields medal if you are under 40.
[795] How can one prove the theorem for $n = 1$ using only the methods from real analysis?

Remark. The Open Mapping Theorem from complex analysis, see [**Lan99**, Theorem II.6.2], says that if $G \subset \mathbb{C}$ is an open subset and if $f\colon G \to \mathbb{C}$ is a non-constant *holomorphic* function, then $f(G)$ is also open. In fact, one can deduce the "Open Mapping Theorem" from the Topological Invariance of Domain Theorem 50.6 since by [**Jä11**, Satz 10] every non-constant holomorphic function on an open subset of $\mathbb{C}$ is locally the power of an injective function.

In the proof of Theorem 50.6 we need the following elementary lemma.

Lemma 50.7. *Let $U \subset \mathbb{R}^n$ be a subset. If U is open in $\mathbb{R}^n$, then every path-component of U is also an open subset of $\mathbb{R}^n$.*

Proof. Let C be a path-component of the subset subset $U \subset \mathbb{R}^n$. Let $P \in C$. Since U is open there exists an $r > 0$ such that $B_r(P) \subset U$. But all points in $B_r(P)$ are path-equivalent to P, hence $B_r(P) \subset C$. ■

Now we can provide the proof of Theorem 50.6.

Proof of Theorem 50.6.

(1) Recall that by the homeomorphism from Lemma 2.44 we can view $\mathbb{R}^n$ as an open subset of $\mathbb{R}^n \cup \{\infty\} = S^n$. So we can view h as a map from U to S^n and we need to show that $h(U)$ is an open subset of S^n.

Let $x \in U$. We pick a closed n-ball B in U that contains x in the interior. It suffices to show that $h(B \setminus \partial B) \subset h(U)$ is open in S^n. Note that $S^n \setminus h(\partial B)$ is open by Lemma 2.40 (3). By Lemma 50.7 it therefore suffices to show that $h(B \setminus \partial B)$ is a path-component of $S^n \setminus h(\partial B)$. We will do so in the following claim.

Claim. The topological space $S^n \setminus h(\partial B)$ has precisely two path-components namely the subsets $h(B \setminus \partial B)$ and $S^n \setminus h(B)$.

By Proposition 50.1 (2) and Lemma 43.1 we know that

$$S^n \setminus h(\partial B) \;=\; h(B \setminus \partial B) \sqcup (S^n \setminus h(B))$$

has two path-components. It suffices to show that the two sets on the right-hand side are path-connected. The first set $h(B \setminus \partial B)$ is path-connected since it is the image of the path-connected topological space $B \setminus \partial B$, while the second set $S^n \setminus h(B)$ is path-connected by Proposition 50.1 (1) and Lemma 43.1. ⊞

(2) Let $h\colon U \to \mathbb{R}^m$ be an injective map with $m \leq n$. We consider the inclusion given by composing h with the inclusion $i\colon \mathbb{R}^m \xrightarrow{x\mapsto(x,0)} \mathbb{R}^n$. By (1) we know that $(i \circ h)(U)$ is an open subset. But a subset of $\mathbb{R}^m$ is open under the inclusion $i\colon \mathbb{R}^m \to \mathbb{R}^n$ only if $m = n$. Thus we have shown that $m \geq n$. ■

Corollary 50.8. *Let M be a non-empty closed n-dimensional topological manifold and let N be a connected n-dimensional topological manifold without boundary. Let $h\colon M \to N$ be a map.*

(1) *If $h\colon M \to N$ is locally injective, then h is also surjective.*
(2) *If h is injective, then h is already a homeomorphism.*

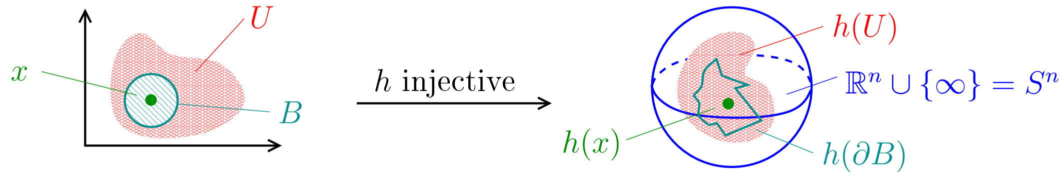

FIGURE 808. Illustration for the proof of Theorem 50.6.

Example. Corollary 50.8 implies in particular that there exists no injective map from the closed smooth manifold S^n into the smooth manifold $\mathbb{R}^n$.[796]

PROOF. The second statement of the corollary is an immediate consequence of the first statement and Proposition 2.43 (3). So it suffices to prove the first statement.

Let $h\colon M \to N$ be a locally injective map from a non-empty closed n-dimensional topological manifold M to a connected n-dimensional topological manifold N without boundary.

The image $h(M)$ is closed in N since M is compact and since N is Hausdorff. Since N is connected and since M is non-empty it suffices to show that $h(M)$ is also open in N.

So let $Q \in h(M)$. We pick $P \in M$ with $h(P) = Q$. We pick a chart $\Phi\colon U \to V$ around P and we pick a chart $\Psi\colon W \to X$ around Q. After possibly replacing U by $U \cap h^{-1}(W)$ we can assume that $h(U) \subset W$. Since h is locally injective we can, after possibly shrinking U even more, assume that $h|_U$ is injective.

From Theorem 50.6 we know that $(\Psi \circ h \circ \Phi^{-1})(V)$ is an open subset of $\mathbb{R}^n$. Since Ψ is a homeomorphism we see that $h(\Phi^{-1}(V)) = h(U)$ is an open neighborhood of Q that is contained in $h(M)$. ■

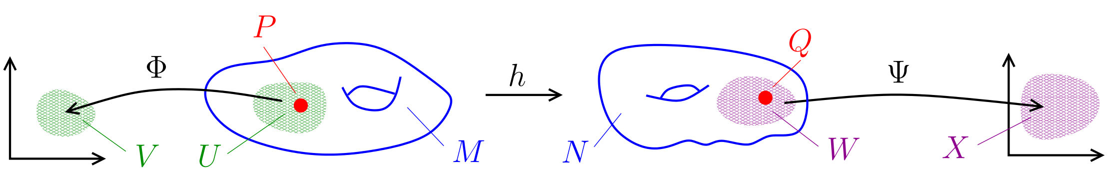

FIGURE 809. Illustration for the proof of Corollary 50.8.

Exercises for Chapter 50.

Exercise 50.1. In Figure 810 we see the image of an injective map $\varphi\colon I \to S^3 = \mathbb{R}^3 \cup \{\infty\}$ and we see a curve $C \subset S^3 \setminus \varphi(I)$. We know from Proposition 50.1 (1) that x is null-homologous. Show that x is the boundary of an orientable surface Σ that is embedded in $S^3 \setminus \varphi(I)$. (In this context this can be seen as the converse to Exercise 48.6.)

Exercise 50.2. Show that there exists an injective map $S^2 \to \mathbb{R}^3$ such that neither of the two components of the complement of $h(S^2)$ is simply connected.
Remark. You could try to modify the example shown in Figure 807.

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[796]How can you show this for $n = 1$, just using methods from real analysis?

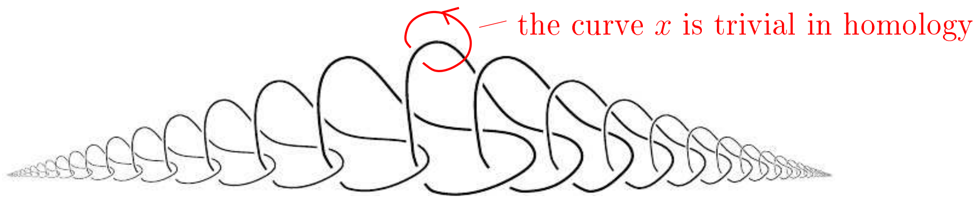

FIGURE 810

51. Topological robotics ❀

In this chapter we will first study the matrix groups SO(3) and SU(2), viewed as topological spaces. In particular we will determine $\pi_1(\mathrm{SO}(3))$ and we will use this result to give an application to topological robotics.

51.1. The matrix groups SO(3) and SU(2).

We recall that Lemma 6.55 implies in particular that we can view SO(3) and SU(2) as closed 3-dimensional smooth manifolds. In this section we will see that the smooth manifolds SO(3) and SU(2) are diffeomorphic to smooth manifolds we are already very familiar with.

We start out our discussion of SO(3) and SU(2) with some definitions. We consider the set of matrices[797]

$$T \;:=\; \big\{B \in \mathrm{SL}(2,\mathbb{C}) \,\big|\, \mathrm{tr}(B) = 0 \text{ and } B = \overline{B}^T\big\}.$$

Note that T is a 3-dimensional real vector space. We equip it with the basis given by the so-called "Pauli matrices"[798]

$$E_1 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad E_3 \;=\; \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

and we equip T with the inner product given by

$$\begin{array}{rcl} T \times T & \mapsto & \mathbb{R} \\ (A,B) & \mapsto & g(A,B) := \frac{1}{2}\,\mathrm{tr}\left(A\overline{B}^T\right). \end{array}$$

We consider the isomorphism

$$\begin{array}{rcl} \Phi\colon \mathbb{R}^3 & \to & T \\ e_i & \mapsto & E_i. \end{array}$$

It follows from a straightforward calculation that Φ defines an isometry from $(\mathbb{R}^3, \langle\,,\,\rangle)$ to (T,g).[799] It follows immediately from this observation that Φ induces an isomorphism $\mathrm{SO}(3) \to \mathrm{SIsom}(T,g)$, where $\mathrm{SIsom}(T,g)$ denotes the group of isometries of (T,g) of determinant $+1$. We use this isomorphism to identify these two groups.

Lemma 51.1. *The map*

$$\begin{array}{rcl} q\colon\ \mathrm{SU}(2) & \to & \mathrm{SIsom}(T,g) = \mathrm{SO}(3) \\ A & \mapsto & (B \mapsto ABA^{-1}) \end{array}$$

is a smooth map between smooth manifolds and it is a well-defined group homomorphism. Furthermore for $A, A' \in \mathrm{SU}(2)$ we have

$$q(A) \;=\; q(A') \quad\iff\quad A \;=\; \pm A'.$$

---

[797] Recall that the trace $\mathrm{tr}(A)$ of a matrix is defined as the sum of all the diagonal entries.

[798] Wolfgang Pauli (1900–1958) was Swiss-American theoretical physicist and one of the pioneers of quantum physics. These matrices were also independently introduced by Paul Dirac. Wikipedia write: "In 1928, building on 2×2-spin matrices which he [Paul Dirac] discovered independently of Wolfgang Pauli's work on non-relativistic spin systems, [. . .] he proposed the Dirac equation as a relativistic equation of motion for the wave function of the electron. This work led Dirac to predict the existence of the positron, the electron's antiparticle [. . .]."

[799] In the calculation we only have to show that $g(E_i,E_j) = \frac{1}{2}\,\mathrm{tr}(E_i\overline{E}_j^T) = \delta_{ij} = \langle e_i, e_j\rangle$ for all choices of $i,j \in \{1,2,3\}$. But this can be verified easily.

Example. Given $t \in \mathbb{R}$ we consider the matrix

$$A = \begin{pmatrix} e^{-\mathrm{i}t} & 0 \\ 0 & e^{\mathrm{i}t} \end{pmatrix} \in \mathrm{SU}(2).$$

We want to determine $q(A) \in \mathrm{SO}(3)$. This means that we have to express the linear map $T \to T$ that is given by $B \mapsto ABA^{-1}$ with respect to the ordered basis $\{E_1, E_2, E_3\}$. A straightforward calculation shows that

$$\begin{aligned} AE_1A^{-1} &= E_1, \\ AE_2A^{-1} &= \begin{pmatrix} 0 & e^{-2\mathrm{i}t} \\ e^{2\mathrm{i}t} & 0 \end{pmatrix} \quad = \quad \cos(2t) \cdot E_2 + \sin(2t) \cdot E_3, \\ & \qquad\qquad\qquad \text{since } e^{\mathrm{i}s} = \cos(s) + \mathrm{i}\sin(s) \\ AE_3A^{-1} &= \begin{pmatrix} 0 & -ie^{-2\mathrm{i}t} \\ \mathrm{i}e^{2\mathrm{i}t} & 0 \end{pmatrix} = -\sin(2t) \cdot E_2 + \cos(2t) \cdot E_3. \end{aligned}$$

But this implies that the homomorphism $T \to T$ given by conjugation with A, with respect to the ordered basis $\{E_1, E_2, E_3\}$ is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2t) & -\sin(2t) \\ 0 & \sin(2t) & \cos(2t) \end{pmatrix}, \text{ put differently, we have } q\begin{pmatrix} e^{-\mathrm{i}t} & 0 \\ 0 & e^{\mathrm{i}t} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2t) & -\sin(2t) \\ 0 & \sin(2t) & \cos(2t) \end{pmatrix}.$$

Proof of Lemma 51.1 (∗). First we have to show that for $A \in \mathrm{SU}(2)$ and $B \in T$ we also have $ABA^{-1} \in T$. This is indeed the case since

$$\operatorname{tr}(ABA^{-1}) = \operatorname{tr}(B) = 0 \quad \text{and} \quad \left(\overline{ABA^{-1}}\right)^T = (\overline{A}^{-1})^T \overline{B}^T \overline{A}^T \underset{\substack{\uparrow \\ \text{since } B = \overline{B}^T \text{ and } A^{-1} = \overline{A}^T}}{=} A\overline{B}^T A^{-1}.$$

It follows easily from the definition that q is a group homomorphism. We also have to show that for $A \in \mathrm{SU}(2)$ we get indeed an isometry of (T, g). So let $B, C \in T$. Then we have indeed

$$\begin{aligned} g(q(A)(B), q(A)(C)) &= \tfrac{1}{2}\operatorname{tr}\left(ABA^{-1} \cdot \overline{ACA^{-1}}^T\right) = \tfrac{1}{2}\operatorname{tr}\left(AB\overbrace{A^{-1} \cdot (\overline{A}^{-1})^T}^{=\mathrm{id}}\overline{C}^T\overline{A}^T\right) \\ &= \tfrac{1}{2}\operatorname{tr}\left(AB\overline{C}^T\overline{A}^T\right) = \tfrac{1}{2}\operatorname{tr}\left(B\overline{C}^T\underbrace{\overline{A}^T A}_{=\mathrm{id}}\right) = \tfrac{1}{2}\operatorname{tr}\left(B \cdot \overline{C}^T\right). \end{aligned}$$

This shows that q defines group homomorphism $\mathrm{SU}(2) \to \mathrm{O}(3)$. By Lemma 2.65 we know that $\mathrm{SU}(2)$ is path-connected. The image $q(\mathrm{SU}(2))$ is a path-connected subset of $\mathrm{O}(3)$ that contains $q(\mathrm{id}) = \mathrm{id}$. It follows again Lemma 2.65 that $q(\mathrm{SU}(2))$ is contained in $\mathrm{SO}(3)$.

Finally let $A, A' \in \mathrm{SU}(2)$. If $A = -A'$ then we clearly have $q(A) = q(A')$. Now suppose that conversely we have $q(A) = q(A')$. We then have in particular $q(A)(E_i) = q(A')(E_i)$ for $i = 1, 2, 3$, i.e. we have $AE_iA^{-1} = A'E_i(A')^{-1}$ for $i = 1, 2, 3$. These equalities give us equations relating the four entries of the matrix A to the four entries of the matrix A'. An elementary calculation then shows that the only solutions are $A = \pm A'$. We leave the details to the reader.

This discussion shows that q is a well-defined map and that it is a group homomorphism. After one has unraveled the definitions it will also have become apparent that the map g is indeed a smooth map between smooth manifolds. ■

The following theorem is the main result of this section.

Theorem 51.2. *We denote by* $q\colon \mathrm{SU}(2) \to \mathrm{SIsom}(T,g) = \mathrm{SO}(3)$ *the map from Lemma 51.1. Then the following two statements hold:*

(1) *The map q is a 2-fold covering map.*
(2) *There exist diffeomorphisms* $f\colon S^3 \to \mathrm{SU}(2)$ *and* $g\colon \mathbb{RP}^3 \to \mathrm{SO}(3)$ *such that the following diagram commutes:*

$$\begin{tikzcd} S^3 \arrow[r, "f", "\cong"'] \arrow[d, "p"'] & \mathrm{SU}(2) \arrow[d, "q"] \\ \mathbb{RP}^3 \arrow[r, "g", "\cong"'] & \mathrm{SO}(3), \end{tikzcd}$$

where we denote by $p\colon S^3 \to \mathbb{RP}^3 = S^3/\pm 1$ *the usual projection map.*

Remark. In Exercise 3.40 we gave a somewhat different homeomorphism between $\mathbb{RP}^3$ and $\mathrm{SO}(3)$. In future discussions we will prefer the homeomorphism from Theorem 51.2.

Convention. We will use the diffeomorphism from Theorem 51.2 to make the identifications $S^3 = \mathrm{SU}(2)$ and $\mathbb{RP}^3 = \mathrm{SO}(3)$.

In the proof of Theorem 51.2 we will need the following technical lemma.

Lemma 51.3. *Let* $q\colon M \to N$ *be a map between two n-dimensional topological manifolds. Suppose the following conditions are satisfied:*

(1) *the map q is surjective,*
(2) *the map q is locally injective, i.e. for each $P \in M$ there exists a neighborhood U such that $q|_U$ is injective,*
(3) *for each point $Q \in N$ the preimage $q^{-1}(Q)$ is a finite set.*

Then q is a covering map.

Proof ($*$). Since q is surjective we only have to show that each $Q \in N$ admits an open neighborhood that is uniformly covered. So let $Q \in N$. We denote its preimages by $P_1, \dots, P_r$. Let $i \in \{1, \dots, m\}$. It follows easily from (2) and the hypothesis that M is a topological manifold that we can pick a closed ball B_i which is a neighborhood around P_i and such that $q\colon B_i \to N$ is injective. Since the topological manifold N is in particular Hausdorff it follows from Proposition 2.43 (3) that $q\colon B_i \to q(B_i)$ is a homeomorphism. It follows from the proof of Corollary 50.8 that $q(\mathring{B}_i)$ is an open neighborhood of Q. It is now straightforward to see that $U := \bigcap_{i=1}^{m} q(\mathring{B}_i)$ is a uniformly covered open neighborhood of Q. ■

Now we can provide the proof of Theorem 51.2.

PROOF OF THEOREM 51.2. Somewhat confusingly the proof of Theorem 51.2 is given in three steps (a), (b) and (c).

(a) One can easily show that the map[800]

$$\begin{array}{rcl} f\colon S^3 & \to & \mathrm{SU}(2) \\ (z,w) & \mapsto & \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{rcl} \widetilde{f}\colon \mathrm{SU}(2) & \to & S^3 \\ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} & \mapsto & (z,w) \end{array}$$

are inverses to one another and that both maps are smooth. This implies that both maps, in particular f, is a diffeomorphism.

(b) Now we want to show that $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a 2-fold covering map. By Lemma 51.1 we know that q is locally injective. But then it follows from Corollary 50.8 and the fact that $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ are closed 3-dimensional smooth manifolds and the fact that $\mathrm{SO}(3)$ is connected (which follows from the Lemma 2.65 (1) together with Lemma 2.72) that q is in fact also a surjection. Since the preimage of each $A \in \mathrm{SO}(3)$ consists of precisely two points in $\mathrm{SU}(2)$ we can apply Lemma 51.3 to deduce that q is indeed a 2-fold covering map.

(c) For any $(z,w) \in S^3$ we have $q(f(-z,-w)) = q(-f(z,w)) = q(f(z,w))$. Thus we see that the map

$$\begin{array}{rcl} g\colon \mathbb{RP}^3 & \to & \mathrm{SO}(3) \\ [(z,w)] & \mapsto & q(f(z,w)) \end{array}$$

is well-defined. It follows immediately from the fact, obtained in (b), that q is surjective, that g is also surjective. Similarly it is straightforward to show, using (b), that the map g is injective. Since q, f and p are local diffeomorphisms and since locally g is given by a composition of q, f, p and their local inverses we see that g is a local diffeomorphism. But by Lemma 6.23 (3) map between smooth manifolds that is a local diffeomorphism and a bijection is in fact a diffeomorphism.

Finally we note that it follows easily from the definitions that $g \circ p = q \circ f$, i.e. the diagram given in the theorem commutes. ■

From Theorem 51.2 and Corollary 16.18 we obtain immediately the following, perhaps initially somewhat surprising, corollary.

Corollary 51.4. *We have*

$$\pi_1(\mathrm{SU}(2)) = 0 \quad \text{and} \quad \pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2.$$

In particular $\mathrm{SU}(2)$ *is the universal covering of* $\mathrm{SO}(3)$.[801]

[800] In Proposition 60.2 we will see that the map f arises naturally if we think of S^3 as the “unit quaternions”.

[801] The fact $\mathrm{SU}(2)$ is the universal covering of $\mathrm{SO}(3)$ plays an important role in theoretical physics and it is essential in understanding the notion of “spin” of an electron. At least that is a slogan that I remember from the days when I studied physics.

Remark. We can actually give an explicit description of the non-trivial element in the fundamental group $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$. More precisely, we consider the loop

$$\begin{array}{rcl} \gamma\colon [0,1] & \to & \mathrm{SO}(3) \\ t & \mapsto & \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi t) & \sin(2\pi t) \\ 0 & -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \end{array}$$

at the base point $\mathrm{id} \in \mathrm{SO}(3)$. The calculation on page 1303 shows that a lift of the path $\gamma\colon [0,1] \to \mathrm{SO}(3)$ to a path $[0,1] \to \mathrm{SU}(2)$ to the starting point $\mathrm{id} \in \mathrm{SU}(2)$ is given by

$$\begin{array}{rcl} \widetilde{\gamma}\colon [0,1] & \to & \mathrm{SO}(3) \\ t & \mapsto & \begin{pmatrix} e^{-\mathrm{i}\pi t} & 0 \\ 0 & e^{\mathrm{i}\pi t} \end{pmatrix}. \end{array}$$

Note that $\widetilde{\gamma}(1) = -\mathrm{id}$, i.e. $\widetilde{\gamma}$ is *not* a loop. Since $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a covering and since $\mathrm{SU}(2)$ is simply connected it follows from Lemma 16.15 that $[\gamma]$ is non-trivial in $\pi_1(\mathrm{SO}(3), \mathrm{id})$.

Corollary 51.4 also raises the following question.

Question 51.5. *Let $n \geq 4$. What is $\pi_1(\mathrm{SO}(n))$?*[802][803]

We will return to this question later on in Proposition 114.11.

51.2. **The belt trick ❀.** In this section we will see that one can visualize the non-trivial element in $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$ using a belt. We proceed as follows.

(1) First of all note that a matrix in $\mathrm{SO}(3)$ is precisely given by an ordered orthonormal basis, i.e. by a choice of an ordered set of three orthogonal vectors v_1, v_2, v_3 of length 1 in $\mathbb{R}^3$.
(2) We equip $\mathrm{SO}(3)$ with the base point given by the identity matrix, i.e. by the standard basis $\{e_1, e_2, e_3\}$.
(3) Now suppose we are given a belt with a buckle, a red edge and a green edge, with a blue top side and a pink lower side.
(4) Any point on a twisted belt defines a triple of orthogonal vectors:
   (a) the first vector lies in the plane defined by the belt and points away from the buckle,
   (b) the second vector also lies in the plane defined by the belt and points towards the red edge,
   (c) the third is orthogonal to the belt and it is attached to the top side of the belt.
(5) In the remainder of the discussion we fix the buckle in the xy-plane such that it points in the x-direction, so that to a point at the buckle we assign the identity matrix.
(6) We say that the end of the belt is in standard position if it lies in a plane parallel to the plane of the buckle and if it points into the x-direction. Put differently, the end of the belt is in standard position precisely if the matrix assigned to it is the identity matrix.

[802] Recall that Lemma 2.65 (1) says that $\mathrm{SO}(n)$ is path-connected.
[803] What is $\pi_1(\mathrm{SO}(2))$?

(7) When we walk on the middle curve of the belt from the buckle towards the other end of the belt, then we obtain a path $[0,1] \to \mathrm{SO}(3)$. In particular, if the end of the belt is in standard position, then the path defined by the belt is in fact a loop in $\mathrm{SO}(3)$ at the base point id.

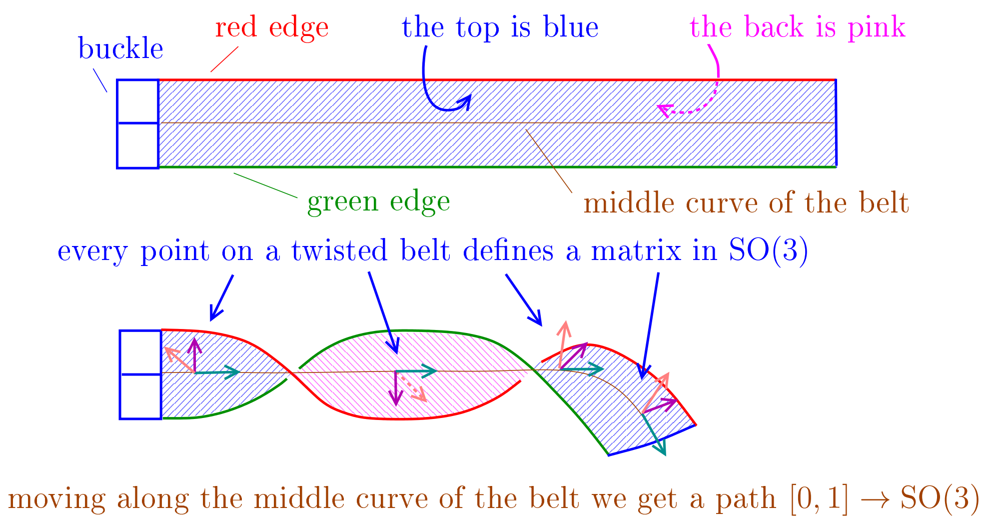

FIGURE 811

(8) When we hold the belt tightly without twisting it, then we assign the identity matrix to every point on the belt, i.e. the belt defines the constant loop $t \mapsto \mathrm{id} \in \mathrm{SO}(3)$.

(9) Now we give the belt, that is still fixed at the buckle, a full twist. As in (7) this defines a loop $\gamma\colon [0,1] \to \mathrm{SO}(3)$, in fact it is straightforward to see that this loop is precisely the loop that we had considered on page 1306. In particular we have visualized the non-trivial element $[\gamma] \in \pi_1(\mathrm{SO}(3), \mathrm{id}) \in \mathbb{Z}_2$.

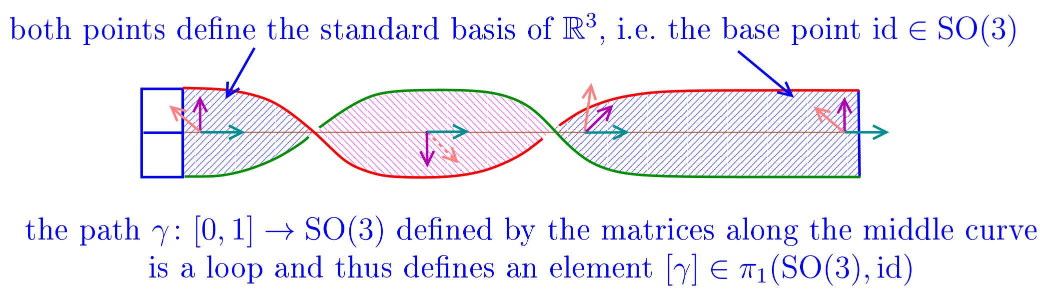

FIGURE 812

(10) The loop $\gamma * \gamma\colon [0,1] \to \mathrm{SO}(3)$ is given by performing another full twist along the belt. It follows from Corollary 51.4 that $[\gamma * \gamma] = [\gamma]^2$ is trivial in $\pi_1(\mathrm{SO}(3), \mathrm{id}) \cong \mathbb{Z}_2$, i.e. $\gamma * \gamma$ is null-homotopic.

(10) One can see that the loop $\gamma * \gamma$ is null-homotopic as follows: one can untwist the belt while keeping the end in standard position, put differently, we can find a path-homotopy from the loop $\gamma * \gamma$ to the trivial loop. This homotopy is illustrated in Figure 814.

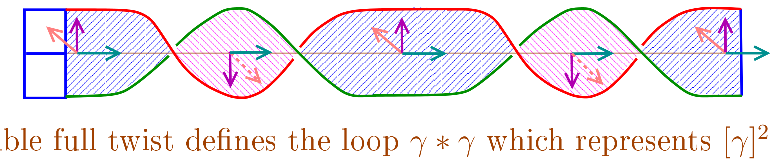

FIGURE 813

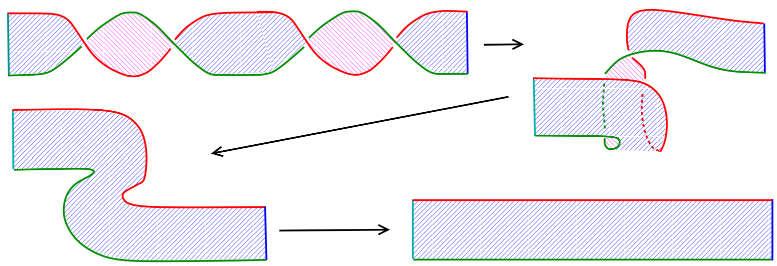

FIGURE 814

The fact that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$ lies at the heart of some other tricks, in particular the "plate trick" and "Dirac's string trick". More information and some videos can be found here:

`https://en.wikipedia.org/wiki/Plate_trick`
`https://www.youtube.com/watch?v=JaIR-cWk_-o`
`https://www.youtube.com/watch?v=JDJKfs3HqRg`
`https://www.youtube.com/watch?v=ICEIgznuHmg`

51.3. **Topological robotics ❀.** In this section we want to discuss a "real-life" application of Corollary 51.4. Let us assume that we are given a solid ball and we are allowed to perform the following operations:

(1) first we can rotate the ball around the z-axis by an angle $\gamma \in S^1$,
(2) then we can rotate the ball around the y-axis by an angle $\beta \in S^1$, finally
(3) we can rotate the ball around the x-axis by an angle $\alpha \in S^1$.

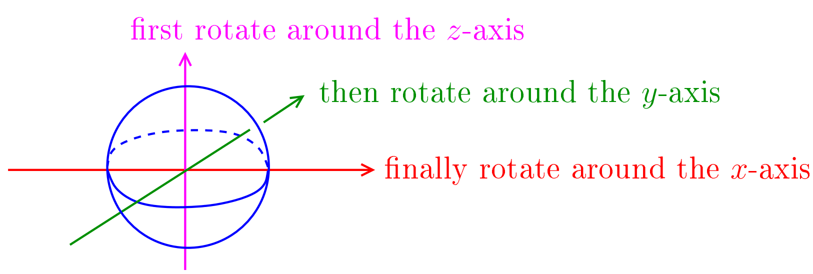

FIGURE 815

Using these three operations, in that order, we can rotate the ball into any position. More precisely, we have the following lemma:

Lemma 51.6. *The map*

$$\begin{array}{rcl}\Phi\colon S^1\times S^1\times S^1 & \to & \mathrm{SO}(3)\\ (\alpha,\beta,\gamma) & \mapsto & \begin{array}{c}\textit{rotation by } \alpha\\ \textit{around the } x\textit{-axis}\end{array} \circ \begin{array}{c}\textit{rotation by } \beta\\ \textit{around the } y\textit{-axis}\end{array} \circ \begin{array}{c}\textit{rotation by } \gamma\\ \textit{around the } z\textit{-axis}\end{array}\end{array}$$

is a surjection.[804]

Remark. One can use Lemma 51.6 and a variation on the argument on page 1303 to give an alternative proof for the statement that $q\colon \mathrm{SU}(2)\to\mathrm{SO}(3)$ is surjective.

Sketch of proof. Let $A\in\mathrm{SO}(3)$ be matrix with columns v_1,v_2,v_3. After a rotation around the z-axis we can assume that v_1 lies in the xz-plane. After a rotation around the y-axis we can then arrange that v_1 lies in fact on the x-axis, i.e. we can arrange that $v_1=e_1$. Since $v_1=e_1,v_2,v_3$ is an orthonormal basis we see that v_2,v_3 lie in the yz-plane. After a rotation around the x-axis we can finally arrange that $v_2=e_2$ and $v_3=e_3$. ∎

This lemma gives rise to the following practical problem: given a matrix $P\in\mathrm{SO}(3)$, can we continuously find parameters $\alpha(P),\beta(P),\gamma(P)\in S^1$ such that these parameters give back the matrix P? More precisely, we have the following question.

Question 51.7. *Does there exist a continuous map $\Psi\colon\mathrm{SO}(3)\to S^1\times S^1\times S^1$ such that $\Phi\circ\Psi=\mathrm{id}_{\mathrm{SO}(3)}$?*

It turns out that the answer is negative. More precisely, we have the following proposition.

Proposition 51.8. *There is no continuous map $\Psi\colon\mathrm{SO}(3)\to S^1\times S^1\times S^1$ such that $\Phi\circ\Psi=\mathrm{id}_{\mathrm{SO}(3)}$.*

This proposition falls into the area of “topological robotics”, more information on this example and on topological robotics can be found in [**Gh14**, p. 76]. Other aspects of topological robotics are discussed in the monograph [**Far08**].

Proof. Let us suppose there exists a continuous map $\Psi\colon\mathrm{SO}(3)\to S^1\times S^1\times S^1$ such that $\Phi\circ\Psi=\mathrm{id}_{\mathrm{SO}(3)}$. Then we get the following commutative diagram

$$\begin{tikzcd} & \overbrace{\pi_1(S^1\times S^1\times S^1)}^{\cong\mathbb{Z}^3} \arrow[dr, "\Phi_*"] & \\ \underbrace{\pi_1(\mathrm{SO}(3))}_{\cong\mathbb{Z}_2} \arrow[ur, "\Psi_*"] \arrow[rr, "\mathrm{id}_*"] & & \underbrace{\pi_1(\mathrm{SO}(3))}_{\cong\mathbb{Z}_2}. \end{tikzcd}$$

By Corollary 51.4 we have that $\pi_1(\mathrm{SO}(3))\cong\mathbb{Z}_2$ and from Corollary 16.18 we know that $\pi_1(S^1\times S^1\times S^1)\cong\mathbb{Z}^3$. Since $\mathbb{Z}_2$ is a finite group and since $\mathbb{Z}^3$ is torsion-free we see that the homomorphism $\Psi_*\colon\pi_1(\mathrm{SO}(3))\to\pi_1(S^1\times S^1\times S^1)$ is the trivial map. But that is not possible since the horizontal map is an isomorphism of a non-trivial group. ∎

[804]The map is also continuous. Why is that?

51.4. **Linkages ❀.** We conclude this chapter with a somewhat unrelated but fun topic that also appears in topological robotics.

(1) We recall that an undirected abstract graph G is a triple (V, E, λ) where V is a non-empty set, E is a set and λ is a map

$$\lambda\colon E \;\to\; \{\text{subsets of } V \text{ with one or two elements}\}.$$

The elements of V are called *vertices of* G and the elements of E are called the *edges of* G. Furthermore, given $e \in E$ the points in $\lambda(e)$ are called the *endpoints of* e.

(2) Given $n \in \mathbb{N}$ we define the *linear graph of length* n to be the undirected abstract graph $G(n)$ which is given by the vertex set $V = \{0, \dots, n\}$ and the edge set $\{e_0, \dots, e_{n-1}\}$ with $\lambda(e_i) = \{i, i+1\}$ for $i = 0, \dots, n-1$. This graph is illustrated in Figure 816.

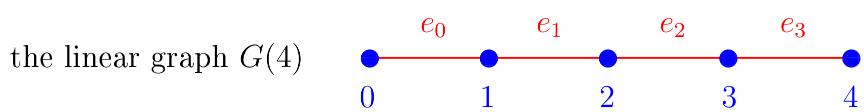

FIGURE 816

Definition.

(1) A *planar linkage* is a triple $(G = (V, E), \lambda, \psi)$ consisting of a finite undirected abstract graph $G = (V, E)$, a function $\lambda\colon E \to \mathbb{R}_{>0}$ and a map $\psi\colon W \to \mathbb{C} = \mathbb{R}^2$ where W is a subset of V. Put differently, λ assigns a length to each edge and ψ represents a subset of the vertices by points in $\mathbb{C}$.

(2) A *configuration* of a planar linkage $(G = (V, E), \lambda, \psi)$ is a map $f\colon V \to \mathbb{R}^2 = \mathbb{C}$ such that $f|_W = \psi$ and such that for every edge e the distance between the endpoints of e is $\lambda(e)$.[805]

(3) The set $C(G = (V, E), \lambda, \psi)$ of all configurations is called the *configuration space* of the linkage. We have an injective map

$$\begin{array}{rcl} C(G = (V, E), \lambda, \psi) & \to & \mathbb{C}^{|V|} := \operatorname{maps}(V \to \mathbb{C}) \\ f & \mapsto & (v \mapsto f(v)) \end{array}$$

thus we can view the configuration space $C(G = (V, E), \lambda, \psi)$ as a topological space.

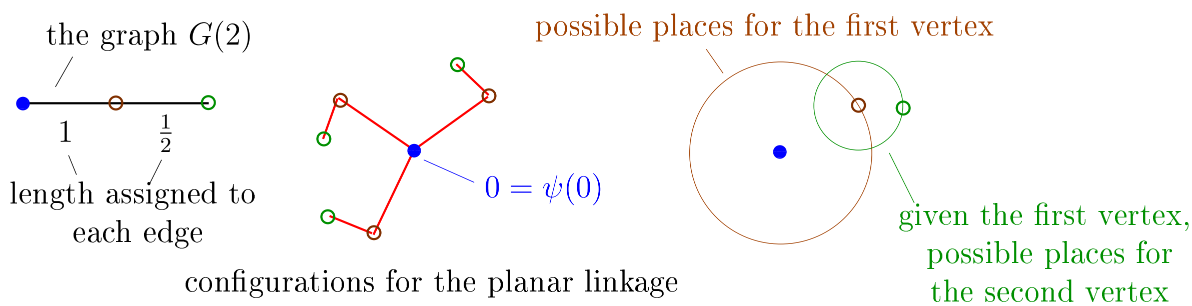

FIGURE 817

In some cases the topological shape of the configuration space of a planar linkage is easy to describe. More precisely, we have the following lemma which is a straightforward

[805] In particular if e has only one endpoint, then we need $\lambda(e) = 0$.

consequence of the definitions and Proposition 2.43 (3). We leave it to the reader to provide the proof.

Lemma 51.9. *Let $G(m)$ be the linear graph with m edges. We consider a planar linkage of the form $(G(m), \lambda, \psi(0) = 0)$. For $i = 1, \dots, m$ we write $r_i = \lambda(e_i)$. Then the map*

$$\begin{array}{rcl} (S^1)^m = S^1 \times \cdots \times S^1 & \to & C(G(m), \lambda, \psi) \\ (z_1, \dots, z_m) & \mapsto & \left(\begin{array}{rcl} \{0, \dots, m\} & \to & \mathbb{C} \\ k & \mapsto & \sum\limits_{i=1}^{k} r_i z_i \end{array} \right) \end{array}$$

is a homeomorphism.

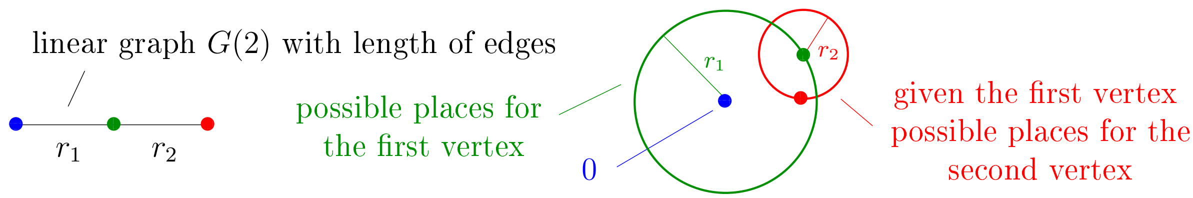

FIGURE 818. Sketch for the proof of Lemma 51.9.

For general linkages the topology of the configuration spaces are much harder to describe than for free planar linkages.

(1) Given $r_1, \dots, r_m > 0$ we denote by $C(r_1, \dots, r_m, P, Q)$ the configuration space corresponding to the planar linkage defined by $G(m)$, $\lambda(e_i) = r_i$ and by the conditions that $\psi(0) = P$ and $\psi(m) = Q$.
 (a) Using elementary arguments one can easily show that

$$C(r_1, r_2, P, Q) \;=\; \begin{cases} \varnothing, & \text{if } r_1 + r_2 < d(P,Q), \\ \text{one point}, & \text{if } r_1 + r_2 = d(P,Q), \\ \text{two points}, & \text{if } r_1 + r_2 > d(P,Q). \end{cases}$$

 (b) Kevin Walker [**Walkb85**, p. 35], see also [**Gh14**, p. 12], showed in 1985 in his bachelor thesis that the surfaces of genus $0, 1, \dots, 4$ are homeomorphic to configuration spaces of the type $(r_1, r_2, r_3, r_4, P, Q)$ for suitable values of $r_1, \dots, r_4$ and points P and Q.

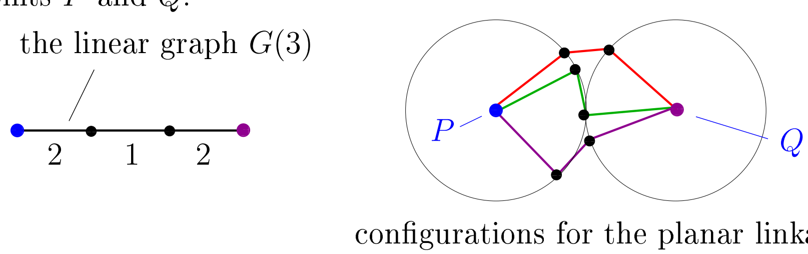

FIGURE 819

(2) Michael Kapovich and John Millson [**KaM02**] showed that given any compact smooth manifold M there exists a planar linkage such that each component of the configuration space is homeomorphic to M. In particular this shows that any compact smooth manifold can be represented by a 2-dimensional physical object.

Exercises for Chapter 51.

Exercise 51.1. Let $q\colon\ \mathrm{SU}(2) \to \mathrm{SO}(3)$ be the homomorphism that we defined in Lemma 51.1. Show that

$$q\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad q\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Remark. This calculation will actually turn out to be useful in Section 66.4.

52. The Hurewicz Theorem in dimension one

In this chapter we want to relate the first homology group to the fundamental group of a path-connected topological space X. Both are defined by mapping 1-dimensional smooth manifolds into X, hence it is reasonable to expect that there is a relationship. But in general these two groups associated to a topological space cannot be the same, since the first homology group is by definition abelian, whereas we saw on many occasions that fundamental groups can be non-abelian.

We start out with the following notation.

Notation. Let X be a topological space. Throughout this chapter we denote by Φ the following bijection:

$$\begin{array}{rcl}\Phi\colon \{\text{paths } [0,1]\to X\} & \to & \{\text{singular 1-simplices in } X\} \\ (\gamma\colon [0,1]\to X) & \mapsto & \left(\begin{array}{rcl}\Delta^1 & \to & X \\ (1-t,t) & \mapsto & \gamma(t)\end{array}\right).\end{array}$$

On many occasions we will use the trivial observation that for any path $\gamma\colon [0,1]\to X$ we have $\Phi(\overline{\gamma}) = \overline{\Phi(\gamma)}$.

We continue with the following lemma.

Lemma 52.1. *Let X be a topological space. If f and g are two paths in X such that the endpoint of f agrees with the starting point of g, then the singular 1-chains $\Phi(f)+\Phi(g)$ and $\Phi(f*g)$ are homologous.*

Proof. The lemma follows quite easily from Lemma 41.3. We refer to Figure 820 for a sketch of the proof. We leave it to the reader to turn the sketch into a proper proof. ■

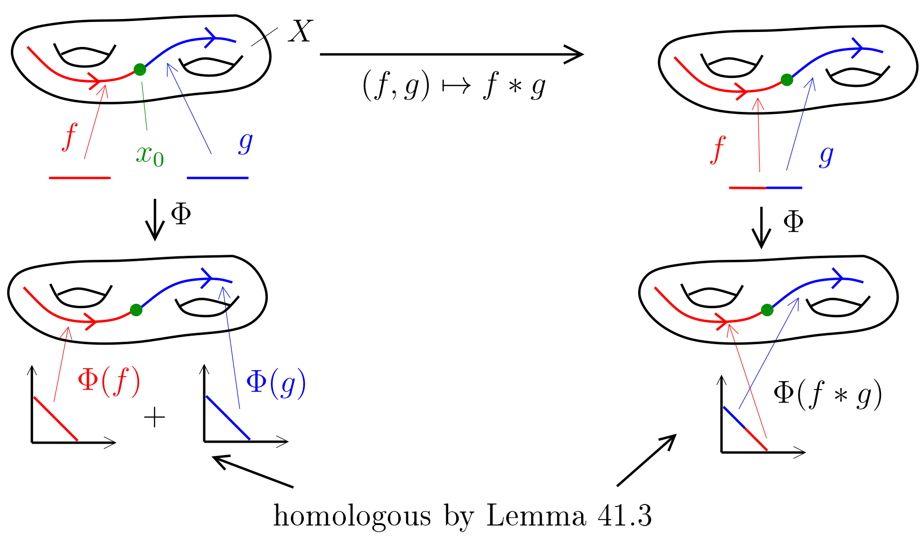

Figure 820. Illustration of the proof of Lemma 52.1.

Now we can easily prove the following proposition.

Proposition 52.2. *Let (X, x_0) be a pointed topological space.*

(1) *If γ is a loop in (X, x_0), then $\Phi(\gamma) \in \mathrm{C}_1(X)$ is a cycle.*
(2) *The map Φ induces a well-defined map*

$$\begin{array}{rcl} \Phi_{(X,x_0)}\colon \pi_1(X, x_0) & \to & \mathrm{H}_1(X) \\ [\gamma\colon [0,1] \to X] & \mapsto & [\Phi(\gamma)]. \end{array}$$

(3) *The map $\Phi_{(X,x_0)}\colon \pi_1(X, x_0) \to \mathrm{H}_1(X)$ is a group homomorphism.*
(4) *Let $\mathcal{PTop}$ be the category of pointed topological spaces and let $\mathcal{Gr}$ be the category of groups. The Hurewicz homomorphisms*

$$\Phi_{(X,x_0)}\colon \pi_1(X, x_0) \ \to \ \mathrm{H}_1(X)$$

define a natural transformation from the functor

$$\begin{array}{rcl} \mathcal{PTop} & \to & \mathcal{Gr} \\ (X, x_0) & \mapsto & \pi_1(X, x_0) \end{array} \quad \text{to the functor} \quad \begin{array}{rcl} \mathcal{PTop} & \to & \mathcal{Gr} \\ (X, x_0) & \mapsto & \mathrm{H}_1(X). \end{array}$$

In plain English this means the following, given a map $f\colon X \to Y$ between topological spaces and given $x_0 \in X$ the following diagram commutes:

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "\Phi_{(X,x_0)}"'] & \pi_1(Y, f(x_0)) \arrow[d, "\Phi_{(Y,f(x_0))}"] \\ \mathrm{H}_1(X) \arrow[r, "f_*"] & \mathrm{H}_1(Y). \end{tikzcd}$$

Proof.

(1) The first statements follows immediately from the definitions and the description of $\partial_1\colon \mathrm{C}_1(X) \to \mathrm{C}_0(X)$ given on page 1079.
(2) We denote by $\sim$ the equivalence relation on $[0, 1]$ that is generated by $0 \sim 1$. We denote by $\alpha\colon \Delta^1 \to [0, 1]/\sim$ the map that is given by $\alpha(1 - t, t) = [t]$. Note that α defines an element in $\mathrm{H}_1([0, 1]/\sim)$. Now suppose that $f, g\colon [0, 1] \to X$ are two path-homotopic loops in x_0. We denote by $\widetilde{f}, \widetilde{g}\colon [0, 1]/\sim \to X$ the induced maps. Then

$$[\Phi(f)] \underset{\substack{\uparrow \\ \text{since } \Phi(f) \text{ and } \widetilde{f}\circ\alpha \text{ are} \\ \text{the same map}}}{=} \big[\widetilde{f} \circ \alpha\big] = \widetilde{f}_*([\alpha]) \underset{\substack{\uparrow \\ \text{by Proposition 42.5, since} \\ \widetilde{f} \text{ and } \widetilde{g} \text{ are homotopic maps } [0,1]/\sim\to X}}{=} \widetilde{g}_*([\alpha]) = \big[\widetilde{g} \circ \alpha\big] = [\Phi(g)].$$

(3) The third statement is an immediate consequence of Lemma 52.1.
(4) The last statement follows easily from the definitions. ∎

Definition. Given a pointed topological space (X, x_0) we refer to the homomorphism

$$\begin{array}{rcl} \Phi_{(X,x_0)}\colon \pi_1(X, x_0) & \to & \mathrm{H}_1(X) \\ [\gamma\colon [0,1] \to X] & \mapsto & [\Phi(\gamma)] \end{array}$$

as the *Hurewicz homomorphism.*[806]

It is clear that the Hurewicz homomorphism is in general neither injective nor surjective:

(1) The Hurewicz homomorphism is in general not injective since $\pi_1(X, x_0)$ is in many cases a non-abelian group whereas $\mathrm{H}_1(X)$ is an abelian group, and it is not possible that a non-abelian group is a subgroup of an abelian group.
(2) The Hurewicz homomorphism is in general not surjective, since $\pi_1(X, x_0)$ depends only on the path-component of x_0, while $\mathrm{H}_1(X)$ "sees" all path-components of X. But as we will see in the next lemma, this turns out to be the only obstacle to Φ being surjective.

Proposition 52.3. *Let X be a topological space and let $x_0 \in X$. If X is path-connected, then the Hurewicz homomorphism*

$$\Phi_{(X,x_0)} \colon \pi_1(X, x_0) \ \to \ \mathrm{H}_1(X)$$

is surjective.

In the proof of Proposition 52.3 we will need the following lemma.

Lemma 52.4. *Let X be a set and let $x_1, \ldots, x_k$ and $y_1, \ldots, y_l$ be elements in X. If*

$$\sum_{i=1}^{k} x_i \ - \ \sum_{i=1}^{l} y_i \ = \ 0 \ \in \ \mathbb{Z}^{(X)},$$

then $k = l$ and there exists a permutation $s \in S_k$ such that $y_i = x_{s(i)}$ for $i = 1, \ldots, k$.

Proof (∗). Let X be a set and let $x_1, \ldots, x_k$ and $y_1, \ldots, y_l$ be elements in X such that

$$\underbrace{\sum_{i=1}^{k} x_i}_{=:a} \ = \ \underbrace{\sum_{i=1}^{l} y_i}_{=:b} \ \in \ \mathbb{Z}^{(X)}.$$

It follows from Lemma 19.1 that there exists a unique homomorphism $\psi \colon \mathbb{Z}^{(X)} \to \mathbb{Z}$ with the property that $\psi(x) = 1$ for all $x \in X$. By definition we have $\psi(a) = k$ and $\psi(b) = l$. So from $a = b$ we deduce that $k = l$.

Note that it follows from applying Lemma 19.1 once again that there exists a unique homomorphism $\psi \colon \mathbb{Z}^{(X)} \to \mathbb{Z}$ with the property that $\psi(x_k) = 1$ and $\psi(x) = 0$ for all $x \in X \setminus \{x_k\}$. By definition we have $\psi(a) > 0$. From $\psi(b) = \psi(a) > 0$ it follows that there exists an $n \in \{1, \ldots, k\}$ such that $y_n = x_k$. We define $s(k) = n$. We obtain an equality

$$\sum_{i=1}^{k-1} x_i \ = \sum_{i \in \{1,\ldots,k\} \setminus \{s(k)\}} y_i \ \in \ \mathbb{Z}^{(X)}$$

and we iterate this procedure to define $s(k-1), \ldots, s(1)$. By construction the map $s \colon \{1, \ldots, k\} \to \{1, \ldots, k\}$ is a bijection ■

Now we can provide the proof of Proposition 52.3.

Proof of Proposition 52.3. We denote by Ψ the inverse of the above bijection

$$\Phi \colon \{\text{paths } [0,1] \to X\} \ \to \ \{\text{singular 1-simplices in } X\}$$

[806] Witold Hurewicz (1904-1956) was a Polish mathematician.

We pick a homology class $z \in \mathrm{H}_1(X)$. Let $c = \sum\limits_{i=1}^{k} a_i\sigma_i$ be a cycle in $\mathrm{C}_1(X)$ that represents z. It follows from Lemma 41.3 that, after possibly replacing a singular chain σ_i by $\overline{\sigma}_i$, we can assume that all the coefficients a_i are positive. Furthermore, since we do not require that the σ_i's are different, we can in fact suppose that $a_1 = \cdots = a_k = 1$. Next note that we have

$$0 \underset{\substack{\uparrow\\ \text{since } c \text{ is a cycle}}}{=} \partial_1(c) = \partial_1\Big(\sum_{i=1}^{k}\sigma_i\Big) \underset{\substack{\uparrow\\ \text{see discussion on page 1079}}}{=} \sum_{i=1}^{k}\sigma_i(0,1) - \sigma_i(1,0) = \sum_{i=1}^{k}\color{blue}{\sigma_i(0,1)} + \sum_{i=1}^{k}\color{red}{-\sigma_i(1,0)}.$$

On the right hand side each "positive summand" needs to cancel with a "negative summand". This means, see also Lemma 52.4, that there exists a permutation $\theta \in S_k$ such that for each i the "endpoint" $\sigma_i(0,1)$ of σ_i is just the "starting point" $\sigma_{s(i)}(1,0)$ of $\sigma_{s(i)}$.

Since X is path-connected we can choose for each $i \in \{1,\dots,k\}$ a path p_i from the base point x_0 to the starting point $\sigma_i(1,0)$ of σ_i.

We obtain the following equality in $\mathrm{H}_1(X)$:

$$\begin{aligned} c = \sum_{i=1}^{k}\sigma_i &\overset{\substack{\text{by Lemma 41.3 the singular chain}\\ \Phi(p_i)+\Phi(\overline{p_i}) = \Phi(p_i)+\overline{\Phi(p_i)}\\ \text{is a null-homologous cycle}\\ \downarrow}}{=} \sum_{i=1}^{k}\sigma_i + \sum_{i=1}^{k}\Phi(p_i) + \sum_{i=1}^{k}\Phi(\overline{p}_i) \overset{\substack{\text{since } \Phi\circ\Psi = \mathrm{id} \text{ and since permutation of the}\\ \text{summands does not change the sum}\\ \downarrow}}{=} \sum_{i=1}^{k}\big(\Phi(p_i) + \Phi(\Psi(\sigma_i)) + \Phi(\overline{p}_{s(i)})\big)\\ &\underset{\substack{\uparrow\\ \text{by Lemma 52.1}}}{=} \sum_{i=1}^{k}\Phi\big(\underbrace{p_i * \Psi(\sigma_i) * \overline{p}_{s(i)}}_{\substack{\text{loop in } (X,x_0)\\ \text{by choice of } s}}\big) \underset{\substack{\uparrow\\ \text{by Lemma 52.1}}}{=} \Phi\Bigg(\underbrace{\prod_{i=1}^{k} p_i * \Psi(\sigma_i) * \overline{p}_{s(i)}}_{\text{loop in } (X,x_0)}\Bigg). \end{aligned}$$

Thus we have shown that $z = [c] \in \Phi(\pi_1(X,x_0))$. This concludes the proof that the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_1(X,x_0) \to \mathrm{H}_1(X)$ is surjective. ■

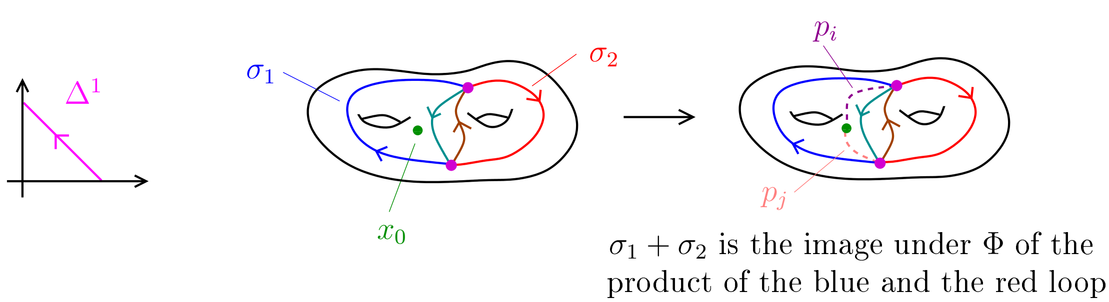

FIGURE 821. Sketch for the proof of Proposition 52.3.

In the following we want to study to what degree the Hurewicz homomorphism fails to be injective. Before we can do so we need to recall some definitions and results from Section 21.5.

Definition. Let π be a group.

(1) For $x, y \in \pi$ we denote by $[x, y] := xyx^{-1}y^{-1}$ the *commutator of x and y.*

(2) We refer to

$$[\pi, \pi] \; := \; \langle\!\langle \{[x, y] \mid x, y \in \pi\} \rangle\!\rangle \; = \; \left\{ \prod_{i=1}^{n} [x_i, y_i] \,\middle|\, x_1, y_1, \ldots, x_n, y_n \in \pi \right\}$$

as the *commutator subgroup* of π.

(3) We refer to $\pi_{\mathrm{ab}} := \pi/[\pi, \pi]$ as the *abelianization* of π.

In Section 21.5 we saw that the abelianization is "the largest abelian quotient" of π. More precisely, if $\alpha\colon \pi \to H$ is a homomorphism from π to an abelian group H, then Proposition 21.20 says that there exists a unique homomorphism $\beta\colon \pi_{\mathrm{ab}} = \pi/[\pi, \pi] \to H$ such that the following diagram commutes:

$$\begin{tikzcd} \pi \arrow[r] \arrow[dr, "\alpha"'] & \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \arrow[d, "\exists!\,\beta"] \\ & H. \end{tikzcd}$$

In particular, given a pointed topological space (X, x_0) the corresponding Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_1(X, x_0) \to \mathrm{H}_1(X)$ induces a homomorphism $\pi_1(X, x_0)_{\mathrm{ab}} \to \mathrm{H}_1(X)$ that we also denote by $\Phi_{(X,x_0)}$ and that we also refer to as the Hurewicz homomorphism. Now we can formulate the following proposition.

Theorem 52.5. (Hurewicz Theorem) *Let (X, x_0) be a pointed topological space. If X is path-connected, then the Hurewicz homomorphism*

$$\Phi_{(X,x_0)}\colon \pi_1(X, x_0)_{\mathrm{ab}} \;\to\; \mathrm{H}_1(X)$$

is a natural[807] *isomorphism.*

Examples.

(1) If we compare our calculations of fundamental groups with our calculations of homology groups, then we see that we had encountered this isomorphism many times.

(a) In Corollary 16.18 we showed that $\pi_1((S^1)^n) \cong \mathbb{Z}^n$, whereas on page 1235 we saw that $\mathrm{H}_1((S^1)^n) \cong \mathbb{Z}^n$.

(b) For $n \geq 2$ we saw in Corollary 16.18 that $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}_2$ and in Lemma 46.9 we had calculated that $\mathrm{H}_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}_2$.

(c) In Proposition 22.3 we showed that

$$\pi_1(\text{Klein bottle}) \;\cong\; \langle x, y \mid yxyx^{-1} \rangle.$$

The abelianization of that group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$ which is isomorphic to the first homology of the Klein bottle that we had determined in Lemma 46.13.

(d) In Proposition 22.3 we showed that for any $g \in \mathbb{N}$ we have

$$\pi_1(\text{surface of genus } g) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \cdot \cdots \cdot [x_g, y_g] \rangle.$$

[807] Whenever we write "natural" one needs to work out what the categories and functors are. In this case we consider the category of path-connected topological spaces, the category of groups and the functors $(X, x_0) \to \pi_1(X, x_0)_{\mathrm{ab}}$ and $(X, x_0) \mapsto \mathrm{H}_1(X)$.

The abelianization of this group is isomorphic to $\mathbb{Z}^{2g}$ which is precisely the first homology group that we had determined in Proposition 48.9.

(e) Let X be a path-connected topological space and let $f\colon X \to X$ be a homeomorphism. By Lemma 46.23 we know that there exists a short exact sequence

$$0 \;\to\; \operatorname{coker}\big(\mathrm{H}_1(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_1(X)\big) \;\xrightarrow{i}\; \mathrm{H}_1(\mathrm{Tor}(X,f)) \;\to\; \mathbb{Z} \;\to\; 0.$$

On the other hand it follows from Proposition 24.28 and the remark on page 720 that there exists an isomorphism $\psi\colon \pi_1(\mathrm{Tor}(X,f)) \cong \pi_1(X) \rtimes \mathbb{Z}$ where the action of $1 \in \mathbb{Z}$ on $\pi_1(X)$ is induced by f. Therefore we can put $\pi_1(\mathrm{Tor}(X,f))$ into a short exact sequence

$$0 \;\to\; \pi_1(X) \;\to\; \pi_1(\mathrm{Tor}(X,f)) \;\to\; \mathbb{Z} \;\to\; 0$$

Now we apply the abelianization functor to these groups and homomorphisms. We obtain the exact sequence[808]

$$\pi_1(X)_{\mathrm{ab}} \;\to\; \pi_1(\mathrm{Tor}(X,f))_{\mathrm{ab}} \;\to\; \mathbb{Z} \;\to\; 0$$

which, using the Hurewicz Theorem 52.5, we can rewrite as

$$\mathrm{H}_1(X) \;\to\; \mathrm{H}_1(\mathrm{Tor}(X,f)) \;\to\; \mathbb{Z} \;\to\; 0.$$

This is, fortunately, consistent with the above short exact sequence.

(2) Note that we now have a new, and depending on one's taste, more satisfying proof of Corollary 27.25: The Hurewicz Theorem 52.5 together with Lemma 46.14 implies that given any knot $K \subset S^3$ the abelianization of $\pi_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$.

Proof of the Hurewicz Theorem 52.5 (∗). Let (X, x_0) be a pointed path-connected topological space. We write $\pi := \pi_1(X, x_0)$. By Proposition 52.2 we know that the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_{\mathrm{ab}} \to \mathrm{H}_1(X)$ is natural. Furthermore it follows from Proposition 52.3 that the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_{\mathrm{ab}} \to \mathrm{H}_1(X)$ is surjective. It remains to show that $\Phi_{(X,x_0)}$ is injective.

In the following we will make the identification $[0,1] = \Delta^1$ that is given by the homeomorphism $t \mapsto (1-t, t)$. In particular we will identify paths $[0,1] \to X$ with singular 1-simplices $\Delta^1 \to X$.

So let $f\colon [0,1] = \Delta^1 \to X$ be a loop in (X, x_0) such that $[f] = 0 \in \mathrm{H}_1(X)$. By definition this means that there exists a 2-chain $\sum\limits_{j=1}^{s} n_j\sigma_j$ in X whose boundary is the singular 1-simplex f. Furthermore, since we do not demand that the σ_j are distinct, we can assume that all n_j are either -1 or 1. Given $j \in \{1,\dots,s\}$ and $k \in \{0,1,2\}$ we write[809]

$$\mu_{(j,k)} \;:=\; \sigma_j \circ i^2_k\colon [0,1] = \Delta^1 \to X \quad \text{and } n_{(j,k)} \;:=\; (-1)^k \cdot n_j.$$

[808] In Exercise 21.6 we saw that if $\varphi\colon A \to B$ is an epimorphism, then the induced map $A_{\mathrm{ab}} \to B_{\mathrm{ab}}$ is also an epimorphism. We had also seen that there exist monomorphisms $\varphi\colon A \to B$ such that the induced map $A_{\mathrm{ab}} \to B_{\mathrm{ab}}$ is not a monomorphism.

[809] Recall that for $j \in \{0,\dots,n\}$ we denote by i^n_j the map

$$\begin{array}{rcl} i^n_j\colon \Delta^{n-1} & \to & \Delta^n \\ (t_0,\dots,t_{n-1}) & \mapsto & (t_0,\dots,t_{j-1},0,t_j,\dots,t_{n-1}). \end{array}$$

We write $A = \{1, \ldots, s\} \times \{0,1,2\}$. We have the following equality in $\mathrm{C}_1(X)$:

$$f \;=\; \partial_1\Big(\sum_{j=1}^{s} n_j\sigma_j\Big) \underset{\substack{\uparrow\\ \text{definition of } \partial_1}}{=} \sum_{j=1}^{s}\sum_{k=0}^{2} n_{(j,k)}\cdot\mu_{j,k} \;=\; \sum_{a\in A} n_a\cdot\mu_a.$$

By comparing the left and the right-hand side of this equality we see that there exists a $b \in A$ with $n_b \cdot \mu_b = 1 \cdot f$. We write $A_\pm = \{a \in A \setminus \{b\} \,|\, n_a = \pm 1\}$. Now we obtain from the above equality that

$$\sum_{a\in A_+} \mu_a \;+\; \sum_{a\in A_-} (-1)\cdot\mu_a \;=\; 0.$$

We make the following choices and introduce the following definitions:

(1) For each point P in X that is a starting or an endpoint of some μ_a we pick a path β_P from x_0 to P. For $P = x_0$ we just pick the constant path.
(2) Given $\mu_{(i,j)} : [0,1] = \Delta^1 \to X$ we write

$$\widehat{\mu}_{(i,j)} \;:=\; \beta_{\text{starting point of } \mu_{(i,j)}} * \mu_{(i,j)} * \overline{\beta}_{\text{endpoint of } \mu_{(i,j)}}.$$

Note that $\widehat{\mu}_{(i,j)}$ is a loop in (X, x_0), in particular it defines an element in the group $\pi = \pi_1(X, x_0)$ which, by a slight abuse of notation, we denote again by $\widehat{\mu}_{(i,j)}$.

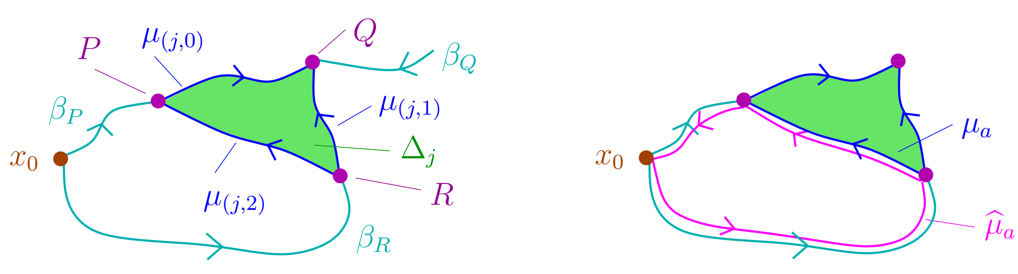

FIGURE 822. Illustration for the proof of the Hurewicz Theorem 52.5.

We make the following elementary observation: for any $(i,j) \in \{1, \ldots, s\} \times \{0,1,2\}$ we have the following equality in $\pi = \pi_1(X, x_0)$:

$$(*) \qquad \widehat{\mu}^{-1}_{(i,j)} \;:=\; \beta_{\text{starting point of } \overline{\mu}_{(i,j)}} * \overline{\mu}_{(i,j)} * \overline{\beta}_{\text{endpoint of } \overline{\mu}_{(i,j)}}.$$

In π_{ab} we have the following equalities:[810]

$$\begin{aligned}
e &\underset{\substack{\uparrow\\ \text{since } \mu_{(j,0)} * \overline{\mu}_{(j,1)} * \mu_{(j,2)} \text{ is null-homotopic}}}{=} \prod_{j=1}^{s} \beta_{\text{starting point of } \mu_{(j,0)}} * (\overbrace{\mu_{(j,0)} * \overline{\mu}_{(j,1)} * \mu_{(j,2)}}^{\text{loop in the starting point of } \mu_{(j,0)}})^{n_j} * \overline{\beta}_{\text{starting point of } \mu_{(j,0)}} \\
&\underset{\substack{\uparrow\\ \text{follows from inserting null-homotopic paths } \overline{\beta}_P * \beta_P\\ \text{together with the above equality } (*)}}{=} \prod_{j=1}^{s} \widehat{\mu}_{(j,0)}^{n_{(j,0)}} \cdot \widehat{\mu}_{(j,1)}^{n_{(j,1)}} \cdot \widehat{\mu}_{(j,2)}^{n_{(j,2)}} \quad = \prod_{a \in \{1,\dots,k\} \times \{0,1,2\}} \widehat{\mu}_a^{n_a} \\
&\underset{\substack{\uparrow\\ \text{since } \{1,\dots,k\} \times \{0,1,2\} = (1,0) \sqcup A_- \sqcup \psi(A_-)\\ \text{since } \mu_{(1,0)} = f \text{ and since } \beta_{x_0} = c_{x_0}}}{=} f \cdot \prod_{a \in A_-} \widehat{\mu}_a^{n_a} \cdot \widehat{\mu}_{\psi(a)}^{n_{\psi(a)}} \quad \underset{\substack{\uparrow\\ \text{for } a \in A_- \text{ we have } \mu_{\psi(a)} = \mu_a \text{ and } n_a = -n_a.}}{=} f \cdot \prod_{a \in A_-} \widehat{\mu}_a^{n_a} \cdot \widehat{\mu}_a^{-n_a} \; = \; f.
\end{aligned}$$

This completes the proof of the proposition.[811] ■

For most topological spaces for which we had computed the fundamental group we had also determined the homology groups. But there is one rather interesting exception, namely the lens spaces. From Corollary 16.18 and the Hurewicz Theorem 52.5 we obtain immediately the following corollary.

Corollary 52.6. *For any coprime $p, q \in \mathbb{N}$ we have*

$$\mathrm{H}_1(\textit{lens space } L(p,q)) \;\cong\; \mathbb{Z}_p.$$

Remark. The Hurewicz Theorem 52.5 is perhaps a mixed blessing. On the one hand it allows us to use the techniques that we had developed in earlier chapters to determine first homology groups. On the other hand it tells us that all the information that is contained in the first homology group is already contained in the fundamental group. So we do not get any new information. In particular, as we saw in Corollary 52.6, the first homology group cannot help us in our long-standing goal of distinguishing lens spaces with the same fundamental group.

Calculating (isomorphism types) of homology groups can get quite involved. But what is often even harder is to determine the maps on homology groups induced by maps between topological spaces. Using Proposition 52.2 (4) we can transfer calculations of induced maps on fundamental groups to obtain the corresponding induced maps on homology groups.

For example we have the following lemma, which later on will be used on many occasions.

[810] In the notation we do not distinguish between loops and elements they represent in π_{ab}.

[811] Where did we actually use that we work with π_{ab} instead of π?

Lemma 52.7. *Let $n \in \mathbb{N}$. We denote by $T = \mathbb{R}^n/\mathbb{Z}^n = (S^1)^n$ the n-dimensional torus.*

(1) *We denote by $\sigma \in \mathrm{H}_1(S^1)$ the standard generator introduced on page 1174. Given $i \in \{1, \dots, n\}$ we denote by*

$$\begin{aligned} p_i \colon S^1 &\to (S^1)^n \\ z &\mapsto (1, \dots, 1, z, 1, \dots, 1) \end{aligned}$$

the obvious i-th inclusion map and we write $z_i = p_{i*}(\sigma)$. Then $e_1, \dots, e_n$ is a basis of $\mathrm{H}_1(T)$.

(2) *Let $A \in \mathrm{GL}(n, \mathbb{Z})$ be a matrix. We denote by $T = \mathbb{R}^n/\mathbb{Z}^n$ the n-dimensional torus. By Lemma 6.35, the map*

$$\begin{aligned} f(A) \colon T = \mathbb{R}^n/\mathbb{Z}^n &\to T = \mathbb{R}^n/\mathbb{Z}^n \\ [v] &\mapsto [A \cdot v] \end{aligned}$$

is a homeomorphism. With respect to the above basis $e_1, \dots, e_n$ of $\mathrm{H}_1(T)$ the induced map $f(A)_* \colon \mathrm{H}_1(T) \to \mathrm{H}_1(T)$ is given by multiplication by the matrix A.

Remark.

(1) In Exercise 84.6 we will study what self-maps of $H_2(S^2 \times S^2)$ can be realized by self-maps of $S^2 \times S^2$.
(2) Exercise 45.20 can be used to give an alternative proof of Lemma 52.7 (2).

Proof.

(1) This statement follows easily from the definitions, Theorem 16.16 and the Hurewicz Theorem 52.5.
(2) Let $\Theta \colon \pi_1(T, 0) \to \mathbb{Z}^n$ be the isomorphism from Theorem 16.16. By Proposition 52.2 (4), Theorem 16.16 and Exercise 16.16 we obtain the following commutative diagram

$$\begin{tikzcd} \mathrm{H}_1(T) \arrow[d, "f(A)"'] & \pi_1(T,0) \arrow[l, "\Phi_{(T,0)}"'] \arrow[r, "\Theta", "\cong"'] \arrow[d, "f(A)_*"] & \mathbb{Z}^n \arrow[d, "v \mapsto A \cdot v"] \\ \mathrm{H}_1(T) & \pi_1(T,0) \arrow[l, "\Phi_{(T,0)}"'] \arrow[r, "\Theta", "\cong"'] & \mathbb{Z}^n. \end{tikzcd}$$

The desired statement follows from the observation that the horizontal isomorphism $\Theta \circ \Phi_{(T,0} \colon \mathrm{H}_1(T) \to \mathbb{Z}^n$ is precisely the one corresponding to the basis $e_1, \dots, e_n$. ■

As another illustration of Proposition 52.2 (4) we also prove the following corollary.

Corollary 52.8. (∗) *Let $p \colon Y \to X$ be a finite-index covering of path-connected topological spaces. Then the cokernel of the map*

$$p_* \colon \mathrm{H}_1(Y) \to \mathrm{H}_1(X)$$

is finite.

Remark. Let $p \colon Y \to X$ be a finite-index covering of path-connected topological spaces. Let $y_0 \in Y$. By Corollary 16.14 the induced map $p_* \colon \pi_1(Y, y_0) \to \pi_1(X, p(y_0))$ is a monomorphism. On the other hand it is in general not true that the induced homomorphism $p_* \colon \mathrm{H}_1(Y) \to \mathrm{H}_1(X)$ is a monomorphism. For example on page 497 we saw that there exists a 2-fold covering map $p\colon T \to K$ from the torus T to the Klein bottle K. By Lemmas 46.12 and 46.13 we know that $\mathrm{H}_1(T) \cong \mathbb{Z}^2$ and that $\mathrm{H}_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. By Lemma 19.8 (2) there is no monomorphism from the group $\mathbb{Z}^2$ to $\mathbb{Z} \oplus \mathbb{Z}_2$.

Proof (∗). We pick a base point y_0 of Y and we write $x_0 = p(y_0)$. We consider the following diagram of maps:

$$\begin{tikzcd}
\pi_1(Y,y_0) \arrow[rr, "p_*"] \arrow[dd, "\Phi_Y"'] \arrow[dr] & & \pi_1(X,x_0) \arrow[dd, "\Phi_X"] \arrow[dr] & \\
& \pi_1(Y,y_0)_{\mathrm{ab}} \arrow[rr, "p_*"] \arrow[dl, "\cong"] & & \pi_1(X,x_0)_{\mathrm{ab}} \arrow[dl, "\cong"] \\
\mathrm{H}_1(Y) \arrow[rr, "p_*"] & & \mathrm{H}_1(X). &
\end{tikzcd}$$

Here it follows from the Hurewicz Theorem 52.5 that the lower diagonal maps are isomorphisms. The "big rectangle in the back" commutes by Proposition 52.2 (4). The upper parallelogram commutes by Proposition 21.20 (7). Since the lower diagonal maps are isomorphisms it follows that the lower parallelogram also commutes.

By Lemma 16.15 and by our hypothesis that p is a finite covering we know that $p_*(\pi_1(Y,y_0))$ is a finite index subgroup of $\pi_1(X,x_0)$, i.e. $p_*\colon \pi_1(Y,y_0) \to \pi_1(X,x_0)$ has a finite cokernel. It follows[812] that $p_*\colon \pi_1(Y,y_0)_{\mathrm{ab}} \to \pi_1(X,x_0)_{\mathrm{ab}}$ also has a finite cokernel. From the commutativity of the lower parallelogram it follows immediately that the map $p_*\colon \mathrm{H}_1(Y) \to \mathrm{H}_1(X)$ has a finite cokernel as well. ■

Exercises for Chapter 52.

Exercise 52.1. Let $T = S^1 \times S^1$ be the torus. Let $\varphi, \psi\colon S^1 \to T$ be two smooth embeddings such that $\varphi_*([S^1]) = \psi_*([S^1]) \in \mathrm{H}_1(T)$. Show that φ and ψ are homotopic maps.

Exercise 52.2. Let $A \in \mathrm{SL}(2,\mathbb{Z})$. Prove the following statements:

(a) If $\operatorname{tr}(A) = 2$, then A is congruent over $\mathbb{Z}$ to a matrix of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.
(b) If $\operatorname{tr}(A) = 2$, then A is congruent over $\mathbb{Z}$ to a matrix of the form $\begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix}$.
(c) If $\operatorname{tr}(A) \in (-2,2)$, then A is a matrix of finite order in the group $\mathrm{SL}(2,\mathbb{Z})$.
(d) If $\operatorname{tr}(A) \notin [-2,2]$, then A has two real eigenvalues λ, λ^{-1} with $|\lambda| > 1$.

Exercise 52.3. Let $A \in \mathrm{SL}(2,\mathbb{Z})$. We consider the map

$$\begin{array}{rcl} f(A)\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [A \cdot v] \end{array}$$

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[812] This can be seen as follows: First we write $G = \pi_1(X,x_0)$ and $H = p_*(\pi_1(Y,y_0))$. Let $g_1, \dots, g_k$ be coset representatives for G/H. One can easily verify that the images of $g_1, \dots, g_k$ in G_{ab} have the property that $G_{\mathrm{ab}} = g_1\,\mathrm{H}_{\mathrm{ab}} \cup \dots \cup g_k\,\mathrm{H}_{\mathrm{ab}}$.

and following the definition on page 711 we consider the corresponding mapping torus

$$\operatorname{Tor}(\mathbb{R}^2/S^2, f(A)) \;=\; ((\mathbb{R}^2/S^2)\times[0,1])/(x,0)\sim(f(A)(v),1).$$

(a) Show that $\operatorname{Tor}(\mathbb{R}^2/S^2, f(A))$ is a closed oriented 3-dimensional smooth manifold.
(b) Determine $\operatorname{rank}(\mathrm{H}_1(\operatorname{Tor}(\mathbb{R}^2/S^2, f(A))$ and $\mathrm{H}_2(\operatorname{Tor}(\mathbb{R}^2/S^2, f(A))$ in terms of the classification of Exercise 52.2.
Remark. You might want to use Proposition 46.21 and Lemma 52.7.

Exercise 52.4. Let $p\colon S^1\times S^1\to S^1$ and $q\colon S^1\times S^1\to S^1$ be the projection onto the first respectively second factor and let $\mu\colon S^1\times S^1\to S^1$ be the multiplication map. Show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}_1(S^1\times S^1) \arrow[rr, "\mu_*"] \arrow[dr, "p_*\oplus q_*"'] & & \mathrm{H}_1(S^1). \\
& \mathrm{H}_1(S^1)\oplus\mathrm{H}_1(S^1) \arrow[ur, "(\varphi,\psi)\mapsto\varphi+\psi"'] &
\end{tikzcd}$$

Exercise 52.5. Let X and Y be path-connected topological spaces. Let $x\in X$ and let $y\in Y$. We denote by $i\colon X\to X\times Y$ the inclusion map that is given by $i(x)=(x,y)$ and similarly we denote by $j\colon Y\to X\times Y$ the inclusion map that is given by $j(y)=(x,y)$. Furthermore we denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the two natural projections Show that the map

$$i_*\oplus j_*\colon\; \mathrm{H}_1(X)\oplus\mathrm{H}_1(Y) \;\to\; \mathrm{H}_1(X\times Y)$$

is an isomorphism and that the inverse is given by $p_*\oplus q_*$.
Hint. Use Proposition 40.8.

53. The Hurewicz Theorem in higher dimensions

The goal of this chapter is to formulate and prove the high-dimensional analogue of the Hurewicz Theorem 52.5 from Chapter 52. Furthermore in this chapter and in Chapter 54 we will discuss several applications of the higher-dimensional Hurewicz Theorem.

53.1. **The Hurewicz homomorphism in higher dimensions.** The Hurewicz Theorem 52.5 evidently raises the question whether there is a connection between higher homotopy groups and higher homology groups. As in the 1-dimensional case we first try to find a promising map from homotopy groups to homology groups.

Throughout the subsequent discussion the following notation is convenient.

Notation. We write $I = [0,1]$.

Next we introduce the following technical definition.

Definition. Let $n \in \mathbb{N}_0$. Let $\omega \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$ be the standard generator as defined on page 1176. Let $\varphi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from page 126. We refer to $\varphi_*(\omega) \in \mathrm{H}_n(I^n, \partial I^n)$ as the *standard generator of* $\mathrm{H}_n(I^n, \partial I^n)$.

Now comes the actual definition we are interested in.

Definition. Let X be a topological space, let $x_0 \in X$ and let $n \geq 1$. We denote by $\omega \in \mathrm{H}_n(I^n, \partial I^n)$ the standard generator as defined above. We refer to the map

$$\begin{array}{rcccl} & & & \text{isomorphism since } n \geq 1 \text{, see page 1125} & \\ & & & \downarrow & \\ \Phi_{(X,x_0)}\colon \pi_n(X,x_0) & \to & \mathrm{H}_n(X,x_0) & \xleftarrow{\cong} & \mathrm{H}_n(X) \\ {[f\colon (I^n, \partial I^n) \to (X,x_0)]} & \mapsto & f_*(\omega) & & \end{array}$$

as the *Hurewicz homomorphism.*[813]

Remark. It follows quite easily from the explicit description of the standard generator of $\mathrm{H}_1(\overline{B}^1, S^0)$ on page 1176 that for $n = 1$ the above definition of the Hurewicz homomorphism agrees with the definition given on page 1314.

In the next lemma we show in particular that the Hurewicz homomorphism is in fact a homomorphism.

Lemma 53.1.

(1) *Let X be a topological space, let $x_0 \in X$ and let $n \geq 1$. The Hurewicz homomorphism*

$$\Phi_{(X,x_0)}\colon \pi_n(X,x_0) \to \mathrm{H}_n(X)$$

is a homomorphism.

[813]It follows from Proposition 42.5 that the Hurewicz homomorphism is indeed well-defined.

(2) *Let X be a topological space and let $n \geq 1$. Furthermore let $\gamma\colon I \to X$ be a path from a point x_0 to some point x_1. The following diagram commutes:*

$$\begin{array}{ccc} \pi_n(X,x_1) & \xrightarrow{\Phi_{(X,x_1)}} & \mathrm{H}_n(X) \\ \gamma_* \downarrow & & \downarrow \mathrm{id} \\ \pi_n(X,x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \mathrm{H}_n(X). \end{array}$$

Here the vertical maps are defined in Propositions 14.11 and 40.5.

(3) *As usual let $\mathcal{PTop}$ be the category of pointed topological spaces and let $\mathcal{Gr}$ be the category of groups. The Hurewicz homomorphisms define a natural transformation from the functor*

$$\begin{array}{rcl} \mathcal{PTop} & \to & \mathcal{Gr} \\ (X,x_0) & \mapsto & \pi_n(X,x_0) \end{array} \quad \text{to the functor} \quad \begin{array}{rcl} \mathcal{PTop} & \to & \mathcal{Gr} \\ (X,x_0) & \mapsto & \mathrm{H}_n(X). \end{array}$$

(4) *Let X be a topological space, let $x_0 \in X$ and let $n \geq 1$. If we take the point of view from page 1059, i.e. if we view elements in $\pi_n(X,x_0)$ as homotopy classes of maps $(S^n,*) \to (X,x_0)$, then the Hurewicz homomorphism is given by the map*

$$\begin{array}{rcl} \pi_n(X,x_0) & \to & \mathrm{H}_n(X) \\ [f\colon (S^n,*) \to (X,x_0)] & \mapsto & f_*(\omega) \end{array}$$

where $\omega \in \mathrm{H}_n(S^n)$ is the standard generator as defined on page 1174.

Proof (∗).

(1) We denote by $\omega \in \mathrm{H}_n(I^n, \partial I^n)$ the standard generator. We need to show that for any two maps $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ we have

$$[f_*(\omega)] + [g_*(\omega)] \;=\; [(f * g)_*(\omega)] \;\in\; \mathrm{H}_n(X,x_0).$$

But we showed precisely this equality in Lemma 47.12.

(2) Let $f\colon (S^n,*) \to (X,x_1)$ be a map. One easily sees that f and f^γ, viewed as maps $S^n \to X$ are homotopic. Thus it follows from Proposition 42.5 that $f_*(\omega) = (f^\gamma)_*(\omega)$. But this means precisely that the given diagram commutes.

(3) This statement follows immediately from the functoriality of homology groups.

(4) This statement is a straightforward consequence of Proposition 45.5 (3). ■

The following lemma gives in particular an affirmative answer to Question 40.17.

Lemma 53.2. *For every $n \in \mathbb{N}$ the Hurewicz homomorphism*

$$\Phi\colon \pi_n(S^n,*) \;\to\; \mathrm{H}_n(S^n) \;\cong\; \mathbb{Z}$$

sends $[\mathrm{id}_{S^n}\colon S^n \to S^n] \in \pi_n(S^n)$ to the standard generator of $\mathrm{H}_n(S^n)$, in particular the Hurewicz homomorphism is an epimorphism.

Proof. We denote by $\omega \in \mathrm{H}_n(S^n)$ the standard generator. By Lemma 53.1 (4) we have

$$\Phi(\underbrace{[\mathrm{id}_{S^n}\colon S^n \to S^n]}_{\in \pi_n(S^n,*)}) \;=\; (\mathrm{id}_{S^n})_*([\omega]) \;=\; [\omega].$$

Thus we see that the standard generator of $\mathrm{H}_n(S^n) \cong \mathbb{Z}$ lies in the image of the Hurewicz homomorphism. But this implies that the Hurewicz homomorphism $\Phi\colon \pi_n(S^n,*) \to \mathrm{H}_n(S^n)$ is an epimorphism. ■

The fairly simple-minded Lemma 53.2 actually already turns out to be quite useful. More precisely, we recall that for $\mathbb{A} = \mathbb{R}$ or $\mathbb{A} = \mathbb{C}$ and $k \leq l$ we can view $\mathbb{A}\mathrm{P}^k$ as a submanifold of $\mathbb{A}\mathrm{P}^l$. Now we can formulate the following application of Lemma 53.2.

Lemma 53.3. *For $k < l$ the submanifold $\mathbb{R}\mathrm{P}^k$ is not a retract of $\mathbb{R}\mathrm{P}^l$.*

Proof. We have

$$\pi_k(\mathbb{R}\mathrm{P}^k) \underset{\substack{\uparrow \\ \text{Proposition 40.13}}}{\cong} \pi_k(S^k) \underset{\substack{\uparrow \\ \text{Lemma 53.2}}}{\neq} 0 \qquad \text{and} \qquad \pi_k(\mathbb{R}\mathrm{P}^l) \underset{\substack{\uparrow \\ \text{Proposition 40.13}}}{\cong} \pi_k(S^l) \underset{\substack{\uparrow \\ \text{Proposition 40.10}}}{=} 0.$$

With this calculation, and the fact that π_n is functorial, the argument of Lemma 15.7 can be easily modified to show that $\mathbb{R}\mathrm{P}^k$ is not a retract of $\mathbb{R}\mathrm{P}^l$. ■

Lemma 53.3 raises the following question.

Question 53.4. *Let $k < l$. Is the complex projective space $\mathbb{C}\mathrm{P}^k$ a retract of $\mathbb{C}\mathrm{P}^l$?*

At this moment though, it is much more interesting to ask whether, given a path-connected topological space X and $n \geq 2$, the Hurewicz homomorphism

$$\Phi_{(X,x_0)}\colon \pi_n(X,x_0) \;\to\; \mathrm{H}_n(X)$$

is necessarily an isomorphism. But browsing through our examples shows that the Hurewicz homomorphism is in general neither a monomorphism nor an isomorphism:

(1) In Lemma 46.12 we showed that $\mathrm{H}_2(S^1 \times S^1) \cong \mathbb{Z}$ is non-zero, whereas on page 1068 we showed that $\pi_2(S^1 \times S^1) = 0$. This shows that the Hurewicz homomorphism $\pi_2(S^1 \times S^1) \to \mathrm{H}_2(S^1 \times S^1)$ is not an epimorphism.

(2) Here are two examples that show that in general the Hurewicz homomorphism is not a monomorphism:
  (a) We consider $X = \mathbb{R}\mathrm{P}^2$. By Proposition 40.13 and Lemma 53.2 we know that $\pi_2(\mathbb{R}\mathrm{P}^2) \cong \pi_2(S^2)$ is non-trivial. On the other hand we saw in Lemma 46.9 that $\mathrm{H}_2(\mathbb{R}\mathrm{P}^2) = 0$.
  (b) We consider $X = S^1 \vee S^2$. It follows immediately from Propositions 43.4 and 47.8 that $\mathrm{H}_2(S^1 \vee S^2) \cong \mathbb{Z}$. On the other hand we know by Proposition 40.13 and the discussion on page 836 that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\pi_2(\widetilde{X})$ where $\widetilde{X}$ is the real line "with infinitely many 2-spheres attached", see e.g. Figure 690 for an illustration. Using the projections onto the various spheres and using Lemma 53.2 one can now show that $\pi_2(\widetilde{X})$ admits an epimorphism onto $\mathbb{Z}^\infty$.[814] It follows from Lemma 19.8 (2) that there is no monomorphism from $\pi_2(X) = \pi_2(\widetilde{X})$ into $\mathrm{H}_2(X) \cong \mathbb{Z}$. In particular the Hurewicz homomorphism $\pi_2(X) \to \mathrm{H}_2(X)$, whatever it is precisely, cannot be a monomorphism.

[814]The proof of this statement is a slight variation on the proof of Exercise 40.3.

The above examples show that we need to put some hypothesis on X for a higher Hurewicz homomorphism to be an isomorphism. This leads us to the following definition.

Definition. Let $k \in \mathbb{N}_0$. We say that a topological space X is *k-connected*, if $\pi_i(X) = 0$ for $i = 0, \ldots, k$.[815][816]

The goal of this chapter is to prove the following theorem which can be viewed as the high-dimensional analogue of the original Hurewicz Theorem 52.5.

Theorem 53.5. (Hurewicz Theorem) *Let (X, x_0) be a pointed topological space. Furthermore let $n \in \mathbb{N}_{\geq 2}$. If X is $(n-1)$-connected*[817]*, then the following two statements hold:*

(1) *we have $\widetilde{\mathrm{H}}_k(X;\mathbb{Z}) = 0$ for $k = 0, \ldots, n-1$, and*
(2) *the Hurewicz homomorphism $\Phi_{(X,x_0)} \colon \pi_n(X, x_0) \to \mathrm{H}_n(X;\mathbb{Z})$ is an isomorphism.*

Remark. On page 1326 we already saw that in general the hypothesis in the Hurewicz Theorem 53.5 that X is $(n-1)$-connected cannot be dropped.

Using the Hurewicz Theorem 53.5 we can now determine some homotopy groups that had been eluding us for a long time.

Corollary 53.6.

(1) *For any $n \in \mathbb{N}$ we have $\pi_n(S^n) \cong \mathbb{Z}$, in fact $\pi_n(S^n, *) = \mathbb{Z} \cdot [\mathrm{id}_{S^n}]$.*
(2) *For any $n \in \mathbb{N} \cup \{\infty\}$ we have $\pi_2(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z}$. In fact, if we make the identification $\mathbb{C}\mathrm{P}^1 = S^2$ from page 197 and if we use the convention from page 194 to consider $\mathbb{C}\mathrm{P}^1 = S^2$ as a subset of $\mathbb{C}\mathrm{P}^n$, then $\pi_2(\mathbb{C}\mathrm{P}^n, *) = \mathbb{Z} \cdot [\mathrm{id}_{\mathbb{C}\mathrm{P}^1}]$.*

PROOF.

(1) Let $n \in \mathbb{N}$. By Proposition 40.10 we know that S^n is $(n-1)$-connected. Thus the statement follows immediately from the Hurewicz Theorems 52.5 and 53.5 and the fact that $\mathrm{H}_n(S^n;\mathbb{Z}) = \mathbb{Z} \cdot [S^n]$.
(2) Let $n \in \mathbb{N} \cup \{\infty\}$. We denote by $i \colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ the obvious inclusion. From Lemma 53.1 (3) we obtain the following commutative diagram

$$\begin{tikzcd}
\pi_2(\mathbb{C}\mathrm{P}^1, *) \arrow[r, "i_*"] \arrow[d, "\Phi_{(\mathbb{C}\mathrm{P}^1,*)}"'] & \pi_2(\mathbb{C}\mathrm{P}^n, *) \arrow[d, "\Phi_{(\mathbb{C}\mathrm{P}^n,*)}"] \\
\mathrm{H}_2(\mathbb{C}\mathrm{P}^1;\mathbb{Z}) \arrow[r, "i_*"] & \mathrm{H}_2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}).
\end{tikzcd}$$

If $n \in \mathbb{N}$, then it follows immediately from Lemma 90.6 that the inclusion induced map $\mathrm{H}_2(\mathbb{C}\mathrm{P}^1;\mathbb{Z}) \to \mathrm{H}_2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is an isomorphism. For $n = \infty$ we need to appeal to the discussion on page 1263 to draw the same conclusion. Finally, since $[\mathbb{C}\mathrm{P}^1] = [S^2]$

[815] For example, X is 0-connected if and only if X is path-connected and *non-empty*. Furthermore, X is 1-connected if and only if it is simply connected.

[816] Since a k-connected topological space is in particularly 0-connected and thus path-connected we see that we can ignore base points.

[817] Recall that by the definition on page 1327 this means that $\pi_i(X) = 0$ for $i = 0, \ldots, n-1$.

is a generator of $\mathrm{H}_2(\mathbb{C}\mathrm{P}^1;\mathbb{Z}) = \mathrm{H}_2(S^2;\mathbb{Z})$ we obtain that $\pi_2(\mathbb{C}\mathrm{P}^n,*) = \mathbb{Z}\cdot[\mathrm{id}_{\mathbb{C}\mathrm{P}^1}]$ from the above commutative diagram. ■

On several occasions we will also need the following straightforward corollary to the Hurewicz Theorem 53.5.

Corollary 53.7. *Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. We suppose that X is simply connected. If given every $k \in \{2,\dots,n\}$ we have $\pi_k(X,x_0) = 0$ or $\mathrm{H}_k(X;\mathbb{Z}) = 0$, then the following two statements hold:*

(1) *for $k = 1,\dots,n$ we have $\pi_k(X,x_0) = \mathrm{H}_k(X;\mathbb{Z}) = 0$,*
(2) *the Hurewicz homomorphism $\pi_{n+1}(X,x_0) \to \mathrm{H}_{n+1}(X;\mathbb{Z})$ is an isomorphism.*

Proof. We prove the corollary by induction on n. First consider the case $n = 1$. Since X is simply connected we obtain from the Hurewicz Theorem 52.5 that $\mathrm{H}_1(X;\mathbb{Z}) = 0$. Furthermore it follows from the Hurewicz Theorem 53.5 that $\Phi_{(X,x_0)}\colon \pi_2(X,x_0) \to \mathrm{H}_2(X;\mathbb{Z})$ is an isomorphism.

Now suppose the statement holds for $n-1$ with $n \geq 2$. Suppose that for every $k \in \{2,\dots,n\}$ we have $\pi_k(X) = 0$ or $\mathrm{H}_k(X;\mathbb{Z}) = 0$. By induction we know that the homotopy and homology groups vanish up to degree $n-1$. Thus we can apply the Hurewicz Theorem 53.5 which says that $\pi_n(X)$ and $\mathrm{H}_n(X;\mathbb{Z})$ are isomorphic. Since one is zero, both are zero. Hence we can apply the Hurewicz Theorem 53.5 once more to see that the Hurewicz homomorphism $\pi_{n+1}(X,x_0) \to \mathrm{H}_{n+1}(X;\mathbb{Z})$ is an isomorphism. ■

As a nice application to Corollary 53.7 we can prove the following lemma.

Lemma 53.8. *Let X be a 1-dimensional* CW*-complex and let $x_0 \in X$ be base point. Then $\pi_n(X,x_0) = 0$ for every $n \geq 2$.*

The content of this lemma is the same as for Lemma 40.15, the only difference is that this time the proof is more satisfying than before.

Proof. Let X be a 1-dimensional CW-complex, let $n \geq 2$ and let $x_0 \in X$ be a base point. If necessary we can replace X by the path-component of x_0. In other words, without loss of generality we can assume that X is path-connected. We denote by $\widetilde{X}$ the universal covering of X. By Proposition 36.37 we know that $\widetilde{X}$ is again a 1-dimensional CW-complex. It follows from Proposition 48.5 (1) that $\mathrm{H}_n(\widetilde{X}) = 0$ for all $n \in \mathbb{N}_{\geq 2}$. Furthermore, by definition of a universal covering we know that $\widetilde{X}$ is simply connected. It follows from Corollary 53.7 that $\pi_n(X,x_0) = 0$ for every $n \geq 2$. ■

We conclude this section with the following proposition.

Proposition 53.9. *Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a homeomorphism with $f(*) = *$. Then the following equality holds in $\pi_n(S^n,*)$:*

$$[f] \;=\; \begin{cases} [\mathrm{id}_{S^n}], & \text{if } f \text{ is orientation-preserving,} \\ -[\mathrm{id}_{S^n}], & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

Remark. Let $n \in \mathbb{N}$. Proposition 53.9 says that every orientation-preserving self-homeomorphism f of S^n with $f(*) = *$ is *homotopic*, rel the base point $*$, to the identity. It is not clear to me whether f is *isotopic*, rel the base point $*$, to the identity. Put differently, can one find a homotopy F from f to id such that each $F_t\colon S^n \to S^n$ is also a homeomorphism?

PROOF. Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a homeomorphism with $f(*) = *$. By the naturality of the Hurewicz homomorphism, see Lemma 53.1 (3), we have the following commutative diagram

$$\begin{tikzcd} \pi_n(S^n,*) \arrow[r, "f_*"] \arrow[d, "\Phi_{(S^n,*)}"'] & \pi_n(S^n,*) \arrow[d, "\Phi_{(S^n,*)}"] \\ \mathrm{H}_n(S^n;\mathbb{Z}) \arrow[r, "f_*"] & \mathrm{H}_n(S^n;\mathbb{Z}). \end{tikzcd}$$

By Proposition 40.10 we know that S^n is $(n-1)$-connected. Therefore the vertical maps are isomorphisms by the Hurewicz Theorems 52.5 and 53.5. The corollary now follows from Proposition 72.2 and the simple observation that $[f] = f_*([\mathrm{id}_{S^n}]) \in \pi_n(S^n,*)$. ■

Now that we are fully convinced of the value of the Hurewicz Theorem 53.5 we turn to the proof thereof. We will discuss more applications in the Chapter 54.

53.2. **The group structure on $\pi_n(X,x_0)$.** Before we turn to the formulation of the high-dimensional Hurewicz Theorem we want to recall some conventions regarding higher homotopy groups and we want to introduce a useful approach to understanding the sum operation in the higher homotopy groups.

First we recall some conventions and notations that apply to any $n \in \mathbb{N}$.

(1) We write $* = (0,\dots,0,1) \in S^n$.
(2) We write $I = [0,1]$.
(3) We denote by $I^n \subset \mathbb{R}^n$ the cube in $\mathbb{R}^n$.
(4) We denote by ∂I^n the boundary of I^n viewed as a subset of $\mathbb{R}^n$.
(5) Let $\varphi\colon (I^n,\partial I^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from page 126.
(6) Let $\psi\colon (\overline{B}^n, S^{n-1}) \to (S^n,*)$ be the map from page 182 that induces the usual identification $\overline{B}^n/S^{n-1} \to S^n$.
(7) As on page 1059 we use φ and ψ to make the identifications

$$\pi_n(X,x_0) \;=\; \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of maps } (I^n,\partial I^n)\to(X,x_0)\\ \| \\ \text{set (of homotopy classes)}\\ \text{of maps } (\overline{B}^n,S^{n-1})\to(X,x_0).\end{array} \;=\; \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of maps } (S^n,*)\to(X,x_0)\end{array}$$

The maps φ and ψ are illustrated in Figure 823.

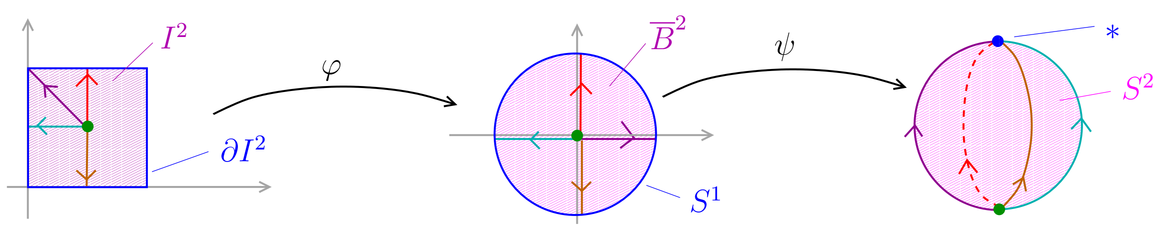


FIGURE 823

Now let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. Recall that on page 1055 we defined the group structure on $\pi_n(X, x_0)$ by considering elements in the homotopy groups as homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$. Under the above identifications it is at times difficult to see what the group structure actually looks like. We start out with considering inverses of homotopy classes.

Lemma 53.10. *Let $n \in \mathbb{N}$ and let (X, x_0) be a pointed topological space.*

(1) *Let $f\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$ be a map. If $\rho\colon \overline{B}^n \to \overline{B}^n$ is the reflection in some hyperplane of $\mathbb{R}^n$, then $-[f] = [f \circ \rho] \in \pi_n(X, x_0)$.*

(2) *Let $f\colon (S^n, *) \to (X, x_0)$ be some map. If $\rho\colon S^n \to S^n$ is the reflection in some hyperplane of $\mathbb{R}^{n+1}$ that contains the line*[818] *$\mathbb{R}\cdot(0, \ldots, 0, 1)$, then we have the equality $-[f] = [f \circ \rho] \in \pi_n(X, x_0)$.*[819]

Proof (∗).

(1) Let $\varphi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from page 126 that we use to identify the two different points of view regarding the homotopy group $\pi_n(X, x_0)$. It follows from Proposition 40.1 (2) and from considering the definition of ρ that if ρ is the reflection in the $(x_2 = 0)$-hyperplane, then $-[f] = [f \circ \rho]$. If ψ is the reflection in some other hyperplane, then it follows from Lemma 18.5 that ψ and ρ are homotopic as maps $(\overline{B}^n, S^{n-1}) \to (\overline{B}^n, S^{n-1})$. But this observation implies $[f \circ \psi] = [f \circ \rho]$ and thus we see that $[f \circ \psi] = -[f]$.

(2) This statement follows easily from statement (1) and from considering the map $\psi\colon (\overline{B}^n, S^{n-1}) \to (S^n, *)$ from page 182 that we use to identify the two different points of view regarding the homotopy group $\pi_n(X, x_0)$. ■

To consider the group structure on the higher homotopy groups we need the following new and fairly harmless definition.

Definition. Let $n \in \mathbb{N}$ and let M be an n-dimensional topological manifold.

(1) We say embeddings[820] $\alpha_1, \ldots, \alpha_m\colon \overline{B}^n \to M$ are (*weakly*) *disjoint* if the images (of the interiors) do not intersect.

(2) Let X be a topological space and let $x_0 \in X$. Given weakly disjoint embeddings $\alpha_1, \ldots, \alpha_m\colon \overline{B}^n \to M$ and maps $f_1, \ldots, f_m\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$ we define[821]

$$\begin{aligned} \Phi(\alpha_1, \ldots, \alpha_m, f_1, \ldots, f_m)\colon M &\to X \\ y &\mapsto \begin{cases} f_i(z), & \text{if } y = \alpha_i(z) \text{ with } z \in \overline{B}^n, \\ x_0, & \text{otherwise.} \end{cases} \end{aligned}$$

The definition of this map is illustrated in Figure 824.

The following proposition gives a versatile tool for dealing with the group structure.

[818]We can only consider such reflections since the point $* = (0, \ldots, 0, 1)$ needs to be fixed.

[819]Note that $\rho(*) = *$, hence $(f \circ \rho)(*) = x_0$.

[820]We refer to page 432 for the definition of an embedding. As discussed on page 433, a smooth embedding of a smooth manifold in the sense of the definition on page 342 is indeed an embedding.

[821]It follows from Lemmas 2.40 and 2.17, 6.28 and Lemma 2.35 (2) that this map is indeed continuous.

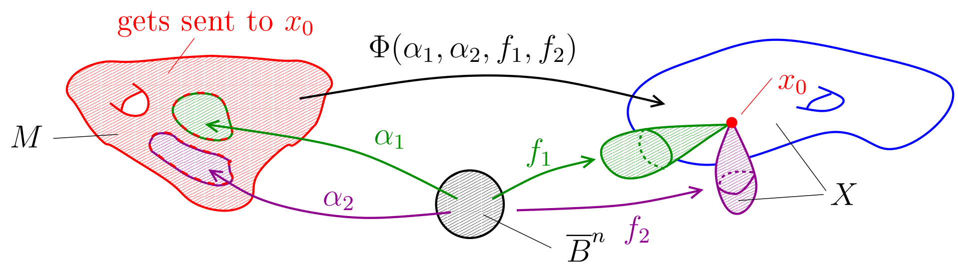


FIGURE 824

Proposition 53.11. *Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_{\geq 2}$. Suppose we are given weakly disjoint smooth embeddings*[822] *$\alpha_1, \ldots, \alpha_m \colon \overline{B}^n \to S^n$ such that the images of the B^n lie in $S^n \setminus \{*\}$. Also suppose we are given maps $f_1, \ldots, f_m \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$. Given $i \in \{1, \ldots, m\}$ we write $\epsilon_i = 1$ if α_i is orientation-preserving otherwise we write $\epsilon_i = -1$. Then we have the following equality in $\pi_n(X, x_0)$:*

$$\big[\underbrace{\Phi(\alpha_1, \ldots, \alpha_m, f_1, \ldots, f_m) \colon (S^n, *) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big] = \sum_{i=1}^{m} \epsilon_i \cdot \big[\underbrace{f_i \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)}_{\in \pi_n(X, x_0)}\big].$$

Example. A slight modification of the map $\overline{B}^n \to S^n$ given on page 182 provides us with an orientation-preserving smooth embedding $\alpha_1 \colon \overline{B}^2 \to S^2$ that sends the disk to the "left hemisphere". Similarly we also obtain an orientation-preserving smooth embedding $\alpha_2 \colon \overline{B}^2 \to S^2$ that sends the disk to the "right hemisphere". It follows from Proposition 53.11 that for any two maps $f_1, f_2 \colon (\overline{B}^2, S^1) \to (X, x_0)$ we have

$$[\Phi(\alpha_1, \alpha_2, f_1, f_2) \colon (S^2, *) \to (X, x_0)] = [f_1 \colon (\overline{B}^2, S^1) \to (X, x_0)] + [f_2 \colon (\overline{B}^2, S^1) \to (X, x_0)].$$

This situation is illustrated in Figure 825. We obtain basically the same picture as we had already obtained on page 1060.

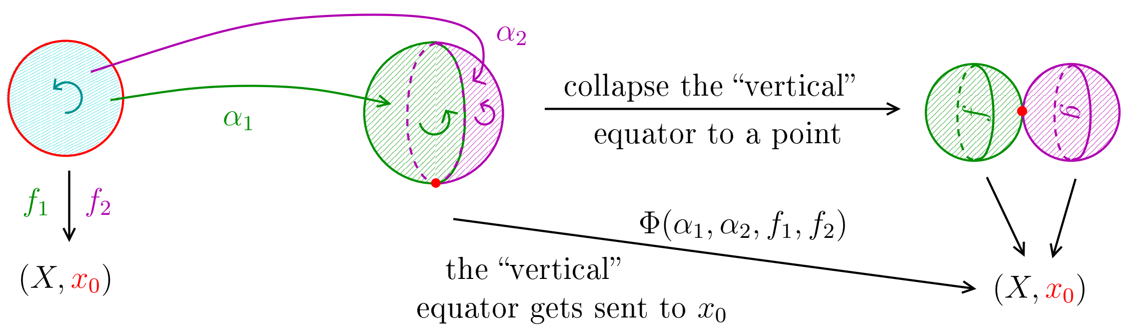


FIGURE 825

Remark. The statement of Proposition 53.11 also holds for embeddings, i.e. we do not necessarily need that the embeddings are smooth. But in that case, in the proof below, one needs to replace the mainstream Theorem 8.36 by the surprisingly difficult Theorem 100.10.

[822] Note that here we work on purpose with "smooth embeddings" instead of "embeddings".

In particular it is not clear to me what goes into the proof of Theorem 100.10 (does it require the Hurewicz Theorem, are we in danger of using circular logic?). Therefore I prefer to stick to smooth embeddings.[823]

In the proof of Proposition 53.11 we need the following lemma which basically says the following: if we isotope the embeddings α_i, then the resulting Φ-maps are homotopic.

Lemma 53.12. (∗) *Let $n \in \mathbb{N}$, let M be an n-dimensional smooth manifold and let Z be a subset of M. Furthermore suppose we are given a topological space X, a base point $x_0 \in X$ and maps $f_1, \dots, f_m \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$. Finally suppose that for $i = 1, \dots, m$ we are given homotopies*

$$\Omega_i \colon \overline{B}^n \times [0,1] \;\to\; M \setminus Z$$

such that for each $t \in [0,1]$ the maps $\Omega_{1,t}, \dots, \Omega_{m,t} \colon \overline{B}^n \to M$ are weakly disjoint embeddings, then the maps

$$\Phi(\underbrace{\Omega_{1,0}, \dots, \Omega_{m,0}}_{\Omega \text{ at } t=0}, f_1, \dots, f_m), \Phi(\underbrace{\Omega_{1,1}, \dots, \Omega_{m,1}}_{\Omega \text{ at } t=1}, f_1, \dots, f_m) \colon (M, Z) \;\to\; (X, x_0)$$

are homotopic as maps of pairs of topological spaces.

Proof (∗). As before it follows from Lemmas 2.40 and 2.17, Lemma 6.28 and Lemma 3.10 that the map

$$\begin{array}{rcl} H \colon M \times [0,1] & \to & X \\ (P, t) & \mapsto & \Phi(\Omega_{1,t}, \dots, \Omega_{m,t}, f_1, \dots, f_m)(P) \end{array}$$

is continuous. This map is then the desired homotopy. ■

Sketch of a proof of Proposition 53.11 (∗). Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_{\geq 2}$. Furthermore let $f_1, \dots, f_m \colon (\overline{B}^n, S^{n-1}) \to (X, x_0)$ be maps. Finally let $\alpha_1, \dots, \alpha_m \colon \overline{B}^n \to S^n$ be weakly disjoint smooth embeddings such that the images of the B^n lie in $S^n \setminus \{*\}$. We start out with the following observations:

(1) Let $\rho \colon \overline{B}^n \to \overline{B}^n$ be the reflection in some hyperplane of $\mathbb{R}^n$. We replace every pair (α_i, f_i) for which α_i is orientation-reversing by the pair $(\alpha_i \circ \rho, f_i \circ \rho)$. It follows immediately from Lemma 53.10 that the desired equality holds for the new data if and only if it holds for the old data.

(2) If the α_i are not disjoint we make them disjoint by shrinking. More precisely, we consider the isotopies

$$\begin{array}{rcl} F_i \colon \overline{B}^n \times [0,1] & \to & M \\ (P, t) & \mapsto & \alpha_i(P \cdot (1 - \tfrac{t}{2})). \end{array}$$

Since the α_i were weakly disjoint we see that the resulting embeddings are now disjoint.

(3) The map $(\overline{B}^n, S^{n-1}) \to (S^n, \{*\})$ that we gave on page 182 restricts to a diffeomorphism $B^n \to S^n \setminus \{*\}$.

[823] Recall that the absence of the word "topological" implies in our setup that smooth embeddings are smooth maps.

Keeping the above three observations in mind we obtain from Lemma 53.12 that it suffices to prove the following statement.

Statement. If $\alpha_1,\dots,\alpha_m\colon \overline{B}^n \to B^n$ are disjoint orientation-preserving smooth embeddings, then the following equality holds in $\pi_n(X,x_0)$:

$$\big[\underbrace{\Phi(\alpha_1,\dots,\alpha_m,f_1,\dots,f_m)\colon (\overline{B}^n,S^{n-1})\to(X,x_0)}_{\in\pi_n(X,x_0)}\big] \;=\; \sum_{i=1}^{m}\big[\underbrace{f_i\colon (\overline{B}^n,S^{n-1})\to(X,x_0)}_{\in\pi_n(X,x_0)}\big].$$

Perhaps the key observation in the proof is that Theorem 8.36 (where we use our hypothesis that $n\geq 2$ and here we use that work with smooth embeddings not just embeddings of balls) together with Lemma 53.12 implies that we only need to prove the statement for a single choice of disjoint orientation-preserving smooth embeddings $\alpha_1,\dots,\alpha_m$. Put differently, it remains to prove the following claim.

Claim. There are disjoint orientation-preserving smooth embeddings $\alpha_1,\dots,\alpha_m\colon \overline{B}^n\to B^n$ such that

$$\big[\underbrace{\Phi(\alpha_1,\dots,\alpha_m,f_1,\dots,f_m)\colon (\overline{B}^n,S^{n-1})\to(X,x_0)}_{\in\pi_n(X,x_0)}\big] \;=\; \sum_{i=1}^{m}\big[\underbrace{f_i\colon (\overline{B}^n,S^{n-1})\to(X,x_0)}_{\in\pi_n(X,x_0)}\big].$$

By induction it suffices prove the claim for $m=2$. Even your famously verbose author does not feel like spelling out all the details. We leave it to the reader to piece together the argument using Figure 826 and the following observations:

(a) by Proposition 53.11 (or alternatively Lemma 40.2) we can "move cubes" in I^n,
(b) by Lemma 18.2 the map $\varphi^{-1}\colon \overline{B}^n\to I^n$ is isotopic to the identity,
(c) the restriction of the map $\varphi\colon I^n\to\overline{B}^n$ to each open cone on one of the sides of the cube I^n is smooth.

Using Lemma 53.12 these moves translate into homotopies of the corresponding maps. ■

53.3. **The proof of the Hurewicz Theorem 53.5.** The proof of the Hurewicz Theorem 53.5 requires some preparations.

Convention. Let $k\in\mathbb{N}_0$. We view Δ^k as a CW-complex in the following way. Given $m\in\{0,\dots,k\}$ the m-cells correspond precisely to all the maps

$$\begin{array}{rcl} \overline{B}^m=\Delta^m & \to & \Delta^k \\ (t_0,\dots,t_m) & \mapsto & \sum\limits_{i=0}^{m} t_i e_{\varphi(i)} \end{array}$$

where φ is a strictly increasing map $\varphi\colon\{0,\dots,m\}\to\{0,\dots,k\}$. Note that the $(k-1)$-skeleton of Δ^k is given by $\partial\Delta^k$.

Definition. Let (X,x_0) be a pointed topological space and let $n\in\mathbb{N}_0$. Given $k\in\mathbb{N}_0$ we define[824]

$$\mathrm{C}_k^{(n)}(X,x_0) \;:=\; \begin{array}{c}\text{the subgroup of the relative chain group } \mathrm{C}_k(X,\{x_0\}) \text{ generated}\\ \text{by all singular simplices } \sigma\colon\Delta^k\to X \text{ with } \sigma(n\text{-skeleton of }\Delta^k)\subset\{x_0\}.\end{array}$$

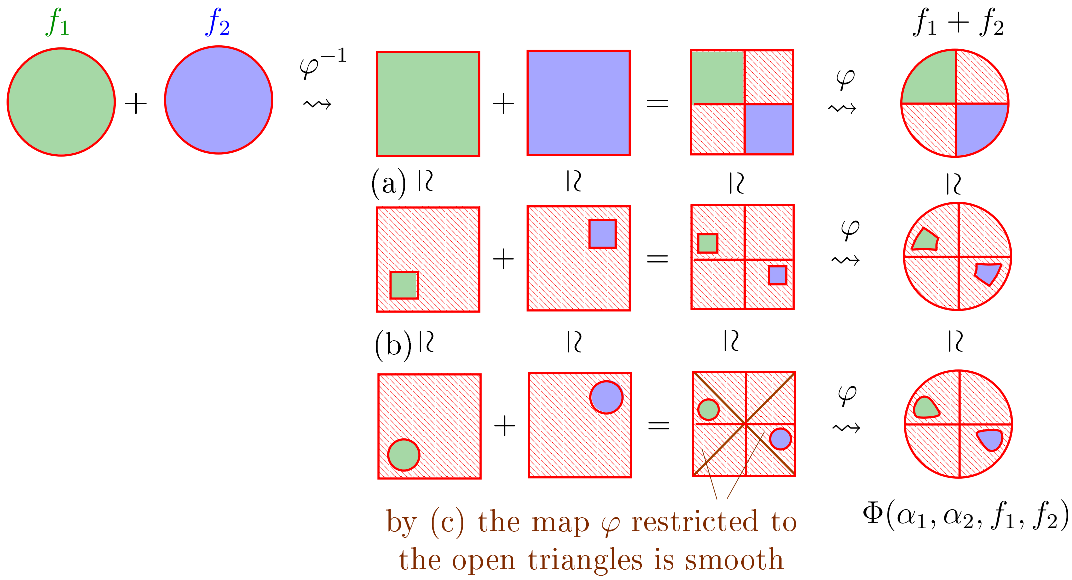


FIGURE 826. Illustration of the proof of Proposition 53.11.

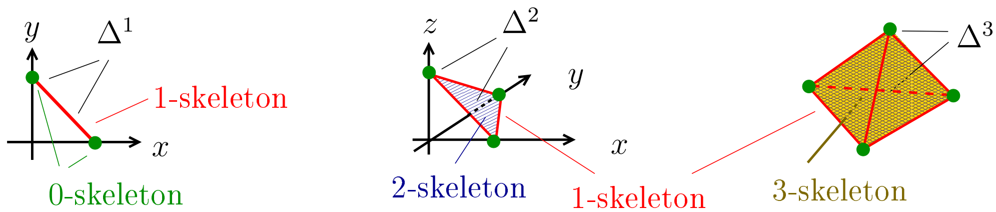


FIGURE 827

These groups define a subcomplex of $C_*(X, \{x_0\})$. In particular we obtain corresponding homology groups $H_k^{(n)}(X, x_0)$ and the inclusion $C_*^{(n)}(X, x_0) \to C_*(X, \{x_0\})$ defines natural maps $H_k^{(n)}(X, x_0) \to H_k(X, \{x_0\})$.

Example. In Figure 828 we show a pointed topological space (X, x_0) and we consider an element $\sigma \in C_1^{(0)}(X, x_0)$ and an element $\tau \in C_2^{(1)}(X, x_0)$. From the pictures it becomes apparent that elements in $C_n^{(n-1)}(X, x_0)$ are "close" to elements in $\pi_n(X, x_0)$. In particular it seems reasonable to expect that $H_n^{(n-1)}(X, x_0)$ is a group that "sits between" $H_n(X; \mathbb{Z})$ and $\pi_n(X, x_0)$.

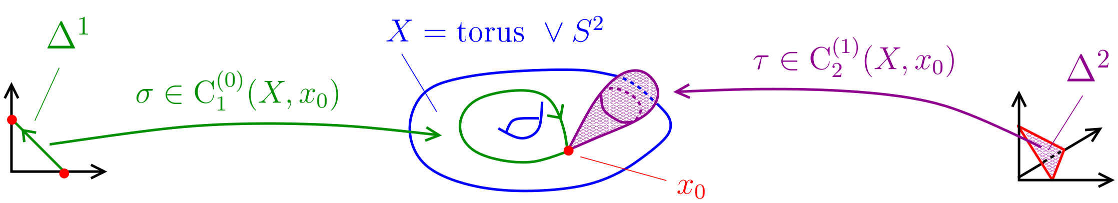


FIGURE 828

[824]Note that the notation $C_k^{(n)}(X, x_0)$ instead of $C_k^{(n)}(X, \{x_0\})$ is on purpose.

In the following lemma we summarize a few basic facts about these new chain groups.

Lemma 53.13. *Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_0$. Furthermore let $k \in \{0, \dots, n\}$. The following statements hold:*

(1) *We have $\mathrm{C}_k^{(n)}(X, x_0) = 0$,*
(2) *we have $\mathrm{H}_k^{(n)}(X, x_0) = 0$,*
(3) *any element in $\mathrm{C}_{k+1}^{(n)}(X, x_0)$ is a cycle.*

Proof. Let $k \in \{0, \dots, n\}$. Put differently, we have k with $n \geq k$. Note that this implies that Δ^k is the n-skeleton of Δ^k. In particular, by definition, $\mathrm{C}_k^{(n)}(X, x_0)$ is the subgroup of $\mathrm{C}_k(X, \{x_0\})$ generated by the constant maps $\Delta^k \to \{x_0\}$, but these are zero in the relative chain group $\mathrm{C}_k(X, \{x_0\})$. This concludes the proof of (1). The remaining two statements are immediate consequences of (1). ■

The following lemma is one of the key ingredients in the proof of the Hurewicz Theorem 53.5

Lemma 53.14. *Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. If X is n-connected, then the inclusion map*

$$\mathrm{C}_*^{(n)}(X, x_0) \quad \to \quad \mathrm{C}_*(X, \{x_0\})$$

is a chain homotopy equivalence. In particular for any $k \in \mathbb{N}_0$ the induced map

$$\mathrm{H}_k^{(n)}(X, x_0) \quad \to \quad \mathrm{H}_k(X, \{x_0\})$$

is an isomorphism.

Proof. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. We suppose that X is n-connected. We recall the following notation from Lemma 42.6: given $t = 0$ and $t = 1$ and $k \in \mathbb{N}_0$ we denote by

$$\begin{array}{rcl} \eta_t \colon \Delta^k & \to & \Delta^k \times [0,1] \\ x & \mapsto & (x, t) \end{array}$$

the obvious inclusion map. We start out with the following claim which basically says that any singular simplices $\Delta^k \to X$ can be homotoped consistently to a singular simplices that lie in $\mathrm{C}_k^{(n)}(X, x_0)$.

Claim. Given any $k \in \mathbb{N}_0$ and given any singular simplex $\sigma \colon \Delta^k \to X$ we can choose a map $P(\sigma) \colon \Delta^k \times [0,1] \to X$ such that the following four statements are satisfied:

(a) we have $P(\sigma) \circ \eta_0 = \sigma \colon \Delta^k \to X$,
(b) the singular k-simplex $P(\sigma) \circ \eta_1 \colon \Delta^k \to X$ lies in $\mathrm{C}_k^{(n)}(X, x_0)$,
(c) if $\sigma \in \mathrm{C}_k^{(n)}(X, x_0)$, then $P(\sigma) = \sigma \circ \pi$ where $\pi \colon \Delta^k \times [0,1] \to \Delta^k$ is the projection map, and
(d) for any $j \in \{0, \dots, k\}$ we have

$$P(\sigma) \circ \underbrace{(i_j^k \times \mathrm{id}_{[0,1]})}_{\Delta^{k-1} \times [0,1] \to \Delta^k \times [0,1]} \;=\; P(\underbrace{\sigma \circ i_j^k}_{\Delta^{k-1} \to X}) \colon \Delta^{k-1} \times [0,1] \to X.$$

We turn to the proof of the claim. Property (d) already suggests that we should prove the claim by induction on k. It is convenient to view $k=-1$ as the (trivial) base case. Now let $k\in\mathbb{N}_0$ and suppose that we have proved the claim for all singular i-simplices with $i<k$. Now let $\sigma\colon\Delta^k\to X$ be a singular simplex. If $\sigma\in \mathrm{C}_k^{(n)}(X,x_0)$, then we set $P(\sigma)=\sigma\circ\pi$. In this case it is straightforward to verify that the conditions (a), (b) and (d) are satisfied.

Now consider the case that $\sigma\notin \mathrm{C}_k^{(n)}(X,x_0)$. We distinguish two cases.

(1) First we consider the case that $k\le n$. By induction the maps $P(\sigma\circ i_j^k)$ have been defined for all j. We set $\partial(\Delta^k\times[0,1]):=(\partial\Delta^k\times[0,1])\cup(\Delta^k\times\{0,1\})$ and we consider the map

$$Q\colon\partial(\Delta^k\times[0,1])\;\to\;X$$

which is defined via σ on $\Delta^k\times\{0\}$, via the maps $P(\sigma\circ i_j^k)$ on $\partial\Delta^k\times[0,1]$[825] and which sends all points in $\Delta^k\times\{1\}$ to x_0.[826] (We refer to Figure 829 for an illustration of this map.) It follows almost immediately from Proposition 2.52 (2) that the pair $(\Delta^k\times[0,1],\partial(\Delta^k\times[0,1]))$ is homeomorphic to $(\overline{B}^{k+1},S^k)$. Since $k\le n$ and since X is n-connected we obtain from Lemma 40.4 that there exists an extension of Q to a map $\Delta^k\times[0,1]\to X$. We define $P(\sigma)$ to be any such extension. We obtain a map $P(\sigma)$ which is easily seen to have all the desired properties.

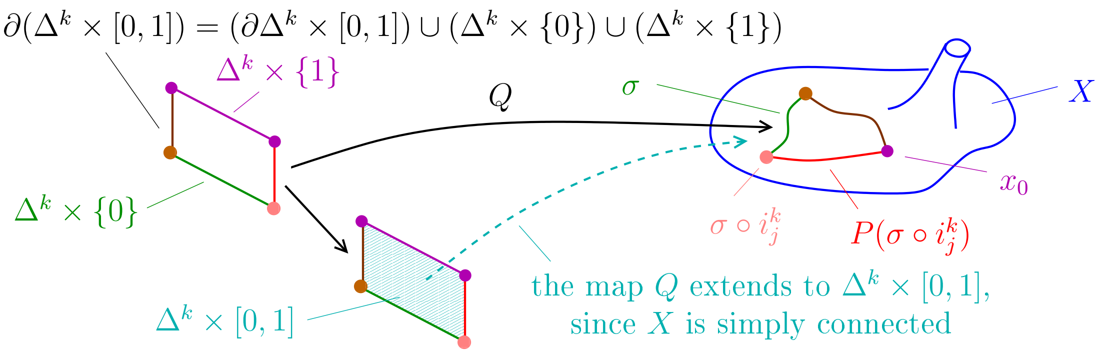


FIGURE 829

(2) Now we consider the case that $k>n$. Note that in this case condition (b) is satisfied "for free" by any map $\Delta^k\times[0,1]\to X$ that satisfies (d). Thus we only have to find a map $P(\sigma)$ that satisfies (a), (c) and (d). This is much easier than in (1). Indeed, similar to (1) we consider the map

$$G\colon(\Delta^k\times\{0\})\cup(\partial\Delta^k\times[0,1])\;\to\;X$$

that is defined via σ on $\Delta^k\times\{0\}$ and via the maps $P(\sigma\circ i_j^k)$ on $\partial\Delta^k\times[0,1]$. (We refer to Figure 830 for an illustration.) Note that by Lemma 38.3 (together with the homeomorphism $\overline{B}^k\cong\Delta^k$) there exists a retraction r from $\Delta^k\times[0,1]$ to the domain of G. Thus combining this retraction r with Q we obtain an extension of Q

[825]Here we use that we assume that the maps $P(\mu)$ are defined for simplices of dimension $<k$.

[826]It follows from Lemma 2.35 (2') that this map is indeed continuous.

to map $\Delta^k \times [0,1] \to X$. We define $P(\sigma)$ to be this extension. It follows basically by construction that all the statements are satisfied. ⊞

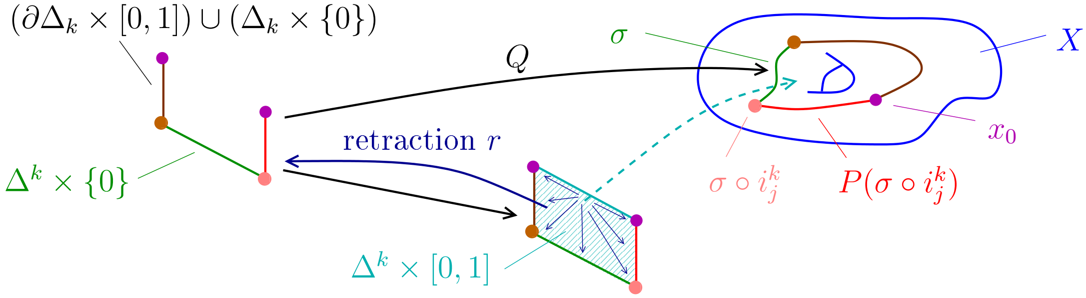


FIGURE 830

Now we turn to the actual proof of the statement that the inclusion map[827]

$$\iota\colon\ \mathrm{C}_*^{(n)}(X, x_0) \ \to\ \mathrm{C}_*(X,\{x_0\})$$

is a chain homotopy equivalence. To do so we consider the map

$$\phi\colon\ \mathrm{C}_*(X,\{x_0\}) \ \to\ \mathrm{C}_*^{(n)}(X, x_0)$$

that is determined by

$$(\sigma\colon \Delta^n \to X) \ \mapsto\ \phi(\sigma) := \underbrace{P(\sigma)\circ\eta_1}_{\Delta^k\to\Delta^k\times[0,1]\to X}.$$

It follows from (b) that ϕ does indeed take values in $\mathrm{C}_*^{(n)}(X, x_0)$ and it follows easily from (d) that ϕ is in fact a chain map. We want to show that ϕ is a chain homotopy inverse to ι. By (c) we have $\phi\circ\iota = \mathrm{id}$. Thus it remains to prove the following claim.

Claim. There exists a chain homotopy from $\iota\circ\phi$ to the identity of $\mathrm{C}_*(X,\{x_0\})$.

Let $k \in \mathbb{N}_0$. We recall that on page 1099 we introduced the singular $(k+1)$-chain

$$\Omega_k \ :=\ \sum_{j=0}^{k} (-1)^j \cdot \underbrace{[v_0,\dots,v_j,w_j,\dots,w_k]}_{\Delta^{k+1}\to\Delta^k\times[0,1]} \ \in\ \mathrm{C}_{k+1}(\Delta^k\times[0,1])$$

where $v_0,\dots,v_k,w_0,\dots,w_k$ denote the vertices of $\Delta^k\times[0,1]$, see page 1099 for details. The key property of these singular chains is that Lemma 42.6 says that for any $k\in\mathbb{N}_0$ we have

$$(*)\qquad \partial_{k+1}(\Omega_k) + \sum_{j=0}^{k}(-1)^j\cdot \underbrace{(i_j^k\times\mathrm{id}_{[0,1]})_*}_{\mathrm{C}_*(\Delta^{k-1}\times[0,1])\to\mathrm{C}_*(\Delta^k\times[0,1])}(\Omega_{k-1}) \ =\ \eta_1-\eta_0 \ \in\ \mathrm{C}_k(\Delta^k\times[0,1]).$$

For any $k\in\mathbb{N}_0$ we consider the map

$$D_k\colon\ \mathrm{C}_k(X,\{x_0\}) \ \to\ \mathrm{C}_{k+1}(X,\{x_0\})$$

that is determined by

$$(\sigma\colon \Delta^k\to X) \ \mapsto\ \underbrace{P(\sigma)_*}_{\mathrm{C}_*(\Delta^k\times[0,1])\to\mathrm{C}_*(X,\{x_0\})}(\underbrace{\Omega_k}_{\in\mathrm{C}_{k+1}(\Delta^k\times[0,1])}).$$

[827] Here and throughout the proof we do not distinguish in the notation between a singular chain in $\mathrm{C}_k(X)$ and the element it represents in $\mathrm{C}_k(X,\{x_0\})$.

We claim that these maps define a chain homotopy from $\iota\circ\phi$ to the identity of $\mathrm{C}_*(X,\{x_0\})$. Thus let $\sigma\colon \Delta^k\to X$ be a singular k-simplex. We need to show that $\partial D\sigma+D\partial\sigma=\phi(\sigma)-\sigma$. Indeed we have

$$\begin{aligned}
\partial_{k+1}(D_k(\sigma))+D_{k-1}(\partial_k(\sigma)) &= \partial_{k+1}(D_k(\sigma))+D_{k-1}\Big(\sum_{j=0}^{k}(-1)^j\cdot\sigma\circ i_j^k\Big)\\
&= \partial_{k+1}(P(\sigma)_*(\Omega_k))+\sum_{j=0}^{k}(-1)^j\cdot P(\sigma\circ i_j^k)_*(\Omega_{k-1})\\
&\underset{\substack{\uparrow\\ \text{in the first summand we used that } P(\sigma)_* \text{ is a chain map and}\\ \text{in the second summand we used that by (d) we have } P(i_j^k\circ\sigma)=P(\sigma)\circ(i_j^k\times\mathrm{id}_{[0,1]})}}{=} P(\sigma)_*\Big(\partial_{k+1}(\Omega_k)+\sum_{j=0}^{k}(-1)^j\cdot(i_j^k\times\mathrm{id}_{[0,1]})_*(\Omega_{k-1})\Big)\\
&\underset{\substack{\uparrow\\ \text{by the above equality } (*)}}{=} P(\sigma)_*(\eta_1-\eta_0) \;=\; P(\sigma)\circ\eta_1-P(\sigma)\circ\eta_0 \underset{\substack{\uparrow\\ \text{by (a) and definition of } \phi}}{=} \phi(\sigma)-\sigma.
\end{aligned}$$

■

Before we can continue with the proof of the Hurewicz Theorem 53.5 we need to introduce another, slightly different point of view, regarding higher homotopy groups.

So let $n\in\mathbb{N}$. We consider the homeomorphism $\eta_n\colon(\Delta^n,\partial\Delta^n)\to(\overline{B}^n,S^{n-1})$ that is given by the composition of the following three maps:[828]

$$\begin{array}{l}
\eta_n\colon \Delta^n \xrightarrow{\cong} \Big\{(x_1,\dots,x_n)\in[0,1]^n\,\Big|\,\sum_{i=1}^n x_i\in[0,1]\Big\} \xrightarrow[\text{with } Q=(\frac{1}{n+1},\dots,\frac{1}{n+1})]{\text{Proposition 2.53 (2)}} \overline{B}^n \xrightarrow[\text{by } A_n]{\text{multiplication}} \overline{B}^n.\\
(t_0,\dots,t_n)\mapsto(t_0,\dots,t_{n-1})
\end{array}$$

where $A_n\in\mathrm{O}(n)$ is a matrix which is chosen in such a way that we have the equality $\eta_n((1,0,\dots,0))=*:=(0,\dots,0,1)$.[829][830] We sketch the definition of η_n in Figure 831.

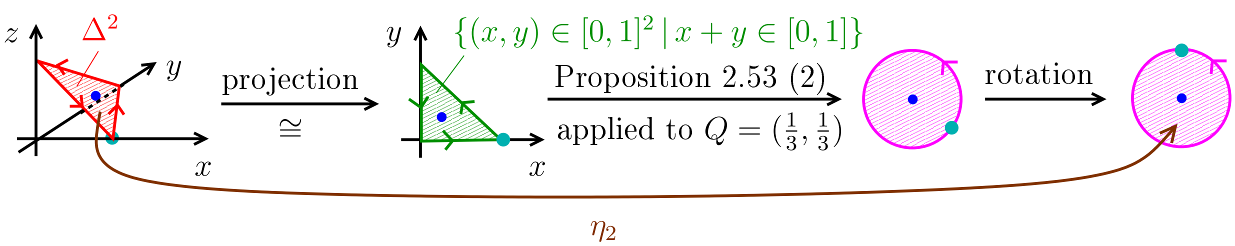


FIGURE 831

We use the homeomorphism $\eta_n\colon(\Delta^n,\partial\Delta^n)\to(\overline{B}^n,S^{n-1})$ to identify these two pairs of topological spaces. In particular we use it to make the identification $\partial\Delta^n=S^{n-1}$. By a slight abuse of notation we write write $*:=(1,0,\dots,0)\in\Delta^n$. In particular for any $n\in\mathbb{N}_0$

[828]Note that the composition of the first two homeomorphisms is precisely the homeomorphism from Lemma 41.1.

[829]It follows from Lemma 3.32 (3) that such a matrix exists.

[830]The map η_n depends on the choice of the matrix A_n, this will not affect us at all.

we can extend the identifications from page 1329 to obtain the following identifications

$$\begin{array}{ccccc}
\pi_n(X,x_0) & = & \text{set (of homotopy classes)} & = & \text{set (of homotopy classes)} \\
 & & \text{of maps } (I^n,\partial I^n)\to(X,x_0) & & \text{of maps } (S^n,*)\to(X,x_0) \\
 & & \| & & \| \\
 & & \text{set (of homotopy classes)} & & \text{set (of homotopy classes)} \\
 & & \text{of maps } (\overline{B}^n,S^{n-1})\to(X,x_0) & & \text{of maps } (\partial\Delta^{n+1},*)\to(X,x_0) \\
 & & \| & & \\
 & & \text{set (of homotopy classes)} & & \\
 & & \text{of maps } (\Delta^n,\partial\Delta^n)\to(X,x_0). & &
\end{array}$$

As above we will use these identifications to pick at each given moment our most convenient point of view.

As an illustration, under the above identifications Lemma 40.4 translates into the following lemma.

Lemma 53.15. *Let (X,x_0) be a pointed topological space and let $n\geq 1$. Furthermore let $f\colon(\partial\Delta^{n+1},*)\to(X,x_0)$ be a map. Then f represents the trivial element in $\pi_n(X,x_0)$ if and only if there exists a map $F\colon\Delta^{n+1}\to X$ so that $F|_{\partial\Delta^{n+1}}=f$.*

The following lemma is the other key technical ingredient in the proof of the Hurewicz Theorem 53.5.

Lemma 53.16. *Let (X,x_0) be a pointed topological space and let $n\in\mathbb{N}_{\geq 2}$. Furthermore let $\sigma\colon\partial\Delta^{n+1}\to X$ be a map that sends the $(n-1)$-skeleton of $\partial\Delta^{n+1}$ to x_0. Then*[831][832]

$$[\sigma\colon(\partial\Delta^{n+1},*)\to(X,x_0)] \;=\; \sum_{j=0}^{n+1}(-1)^j\cdot[\sigma\circ i_j^{n+1}\colon(\Delta^n,\partial\Delta^n)\to(X,x_0)] \;\in\; \pi_n(X,x_0).$$

We refer to Figure 832 for an illustration for $n=1$, even though this case is strictly speaking not allowed.

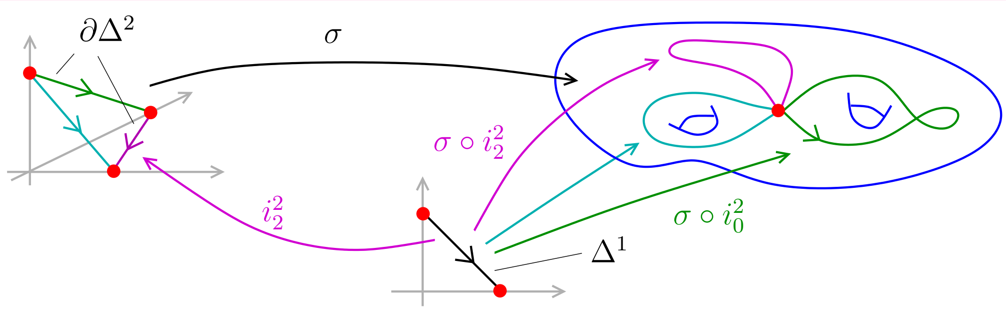


FIGURE 832. Illustration of Lemma 53.16.

PROOF (∗). The reader might already have noticed that the statement of the lemma resembles the statement of Proposition 53.11. Indeed, after some massaging we will be able to deduce the desired statement from Proposition 53.11.

[831]Note that our hypothesis that σ sends the $(n-1)$-skeleton of $\partial\Delta^{n+1}$ to x_0 implies in particular that for all $j\in\{0,\ldots,n+1\}$ we have $(\sigma\circ i_j^{n+1})(\partial\Delta^n)=\{x_0\}$.

[832]Also note that by our base point $*$ is chosen in such a way that it lies on the $(n-1)$-skeleton of $\partial\Delta^{n+1}$.

Given any $k \in \mathbb{N}_0$, let $\eta_k\colon (\Delta^k, \partial\Delta^k) \to (\overline{B}^k, S^{k-1})$ the homeomorphism from page 1338. Using our identifications the desired statement is now equivalent to the following statement.

Statement. Given $j \in \{0, \dots, n+1\}$ we write $\alpha_i = \eta_{n+1} \circ i_j^{n+1} \circ \eta_n^{-1}\colon \overline{B}^n \to S^n = \partial\overline{B}^{n+1}$ and furthermore we write $f_j = \sigma \circ i_j^{n+1} \circ \eta_n^{-1}\colon \overline{B}^n \to X$. Then we have the following equality in $\pi_n(X, x_0)$:

$$\big[\underbrace{\Phi(\alpha_0, \dots, \alpha_n, f_0, \dots, f_n)\colon (S^n, *) \to (X, x_0)}_{\in \pi_n(X,x_0)}\big] \;=\; -\sum_{j=0}^{n+1} (-1)^j \cdot \big[\underbrace{f_j\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)}_{\in \pi_n(X,x_0)}\big].$$

We refer to Figure 833 for an illustration.

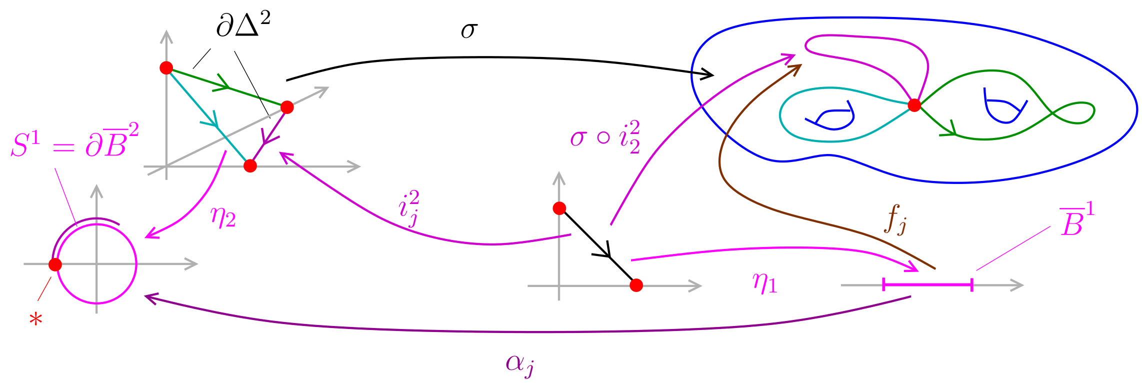


FIGURE 833. Illustration of the proof of Lemma 53.16.

First note that it follows from the discussion on page 2094 that the maps η_k is orientation-preserving if and only if k is even. Furthermore it is fairly straightforward to see that the maps i_j^{n+1} are orientation-preserving if and only if j is even. In summary, the embedding α_j is orientation-preserving if and only if j is odd.[833]

The only reason why we cannot apply Proposition 53.11 immediately is that the maps α_j are embeddings but they are not smooth, so they are not smooth embeddings. Keeping Lemma in mind we see that it suffices to prove the following claim.

Claim. The maps $\alpha_0, \dots, \alpha_{n+1}\colon \overline{B}^n \to S^n$ are isotopic, through weakly disjoint embeddings, to smooth embeddings $\beta_0, \dots, \beta_{n+1}$.

For $j = 0, \dots, n+1$ we write $F_j = i_j^{n+1}(\mathring{\Delta}^n)$ and we write $\overline{F}_j = i_j^{n+1}(\Delta^n)$. Note that F_j is an n-dimensional submanifold of $\mathbb{R}^{n+1}$, we endow it with the usual orientation where the "normal vector sticks outward". Now this claim can be proved using the following observations:

(1) Similar to the discussion on page 1332 we can shrink the weakly disjoint embeddings $\alpha_j\colon \overline{B}^n \to \overline{F}_j$ to disjoint embeddings $\beta_j\colon \overline{B}^n \to F_j$.

[833]Here we used the deep fact that precisely one out of the numbers n, $n+1$ is odd.

(2) Using Lemma 18.2 we can isotope these embeddings $\beta_j\colon \overline{B}^n \to F_j$ to smooth embeddings[834] $\gamma_i\colon \overline{B}^n \to F_j$.

(3) For each $j \in \{0,\dots,n+1\}$ the restriction of $\eta_{n+1}\colon \partial\Delta^{n+1} \to S^n$ to $\eta_{n+1}\colon F_j \to S^n$ is smooth and orientation-preserving.

Together with Figure 834 the reader should have no troubles assembling a full proof of the claim and the lemma using the above observations. ∎

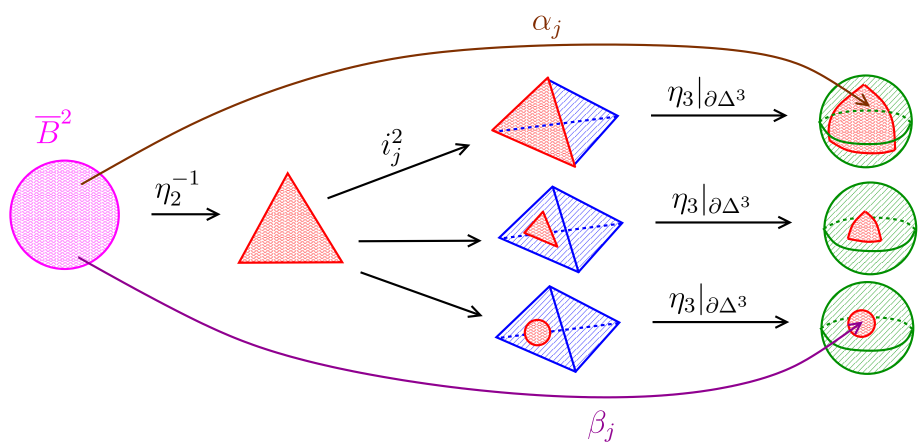


FIGURE 834. Illustration of the proof of Lemma 53.16.

Now we can turn to the proof of the Hurewicz Theorem 53.5.

PROOF OF THE HUREWICZ THEOREM 53.5. Let $n \in \mathbb{N}_{\geq 2}$ and let (X, x_0) be a pointed topological space. We suppose that X is $(n-1)$-connected.

Statement (1) of the theorem can be proved easily. Indeed, let $k \in \{0,\dots,n-1\}$. We have

$$\widetilde{\mathrm{H}}_k(X;\mathbb{Z}) \underset{\substack{\uparrow \\ \text{Lemma 43.14}}}{\cong} \mathrm{H}_k(X,\{x_0\};\mathbb{Z}) \underset{\substack{\uparrow \\ \text{by Lemma 53.14 since} \\ X \text{ is } (n-1)\text{-connected}}}{=} \mathrm{H}_k^{(n-1)}(X,x_0) \underset{\substack{\uparrow \\ \text{by Lemma 53.13 (2)} \\ \text{since } k\leq n-1}}{=} 0.$$

We turn to the much more interesting proof of statement (2). As a reminder, we want to show that the map

$$\pi_n(X,x_0) \xrightarrow{\Phi_{(X,x_0)}} \mathrm{H}_n(X;\mathbb{Z})$$

is an isomorphism. The goal again is to bring the homology groups $\mathrm{H}_n^{(n-1)}(X,x_0)$ into the game. This leads us to the following claim.

Claim.

(1) The map

$$\begin{aligned} \Psi_{(X,x_0)}\colon \pi_n(X,x_0) &\to \mathrm{H}_n^{(n-1)}(X,x_0) \\ [f\colon (\Delta^n,\partial\Delta^n)\to(X,x_0)] &\mapsto \big[\underbrace{f\colon \Delta^n \to X}_{\in \mathrm{C}_n^{(n-1)}(X,x_0)}\big] \end{aligned}$$

[834]Recall that a smooth embedding is by definition smooth.

is well-defined and it is a homomorphism.

(2) The following diagram commutes up to the sign $(-1)^n$:

$$\begin{tikzcd}
\pi_n(X,x_0) \arrow[r, "\Phi_{(X,x_0)}", "{[f]\mapsto f_*([S^n])}"'] \arrow[d, "\Psi_{(X,x_0)}"'] & \mathrm{H}_n(X;\mathbb{Z}) \arrow[d, "\cong"] \\
\mathrm{H}_n^{(n-1)}(X,x_0) \arrow[r, "\cong", "\iota"'] & \mathrm{H}_n(X,\{x_0\};\mathbb{Z})
\end{tikzcd}$$

where ι is induced by the inclusion $\mathrm{C}_*^{(n-1)}(X,x_0) \to \mathrm{C}_n(X,\{x_0\})$.

(3) The right-vertical and the bottom-horizontal maps of the diagram in (2) are isomorphisms.

We prove the claim using a somewhat convoluted logic.[835] We consider the following diagram

$$\begin{tikzcd}
\{\text{set of maps } (\Delta^n,\partial\Delta^n) \to (X,x_0)\} \arrow[rr, "{f\mapsto f_*([\Delta^n])}"] \arrow[dr, two heads] \arrow[d, "{f\mapsto[f]}"'] & & \mathrm{H}_n(X;\mathbb{Z}) \arrow[d, "\cong"] \\
 & \pi_n(X,x_0) \arrow[ur, "\Phi_{(X,x_0)}"'] \arrow[dl, dashed, "\Psi_{(X,x_0)}"'] & \\
\mathrm{H}_n^{(n-1)}(X,x_0) \arrow[rr, "\cong", "\iota"'] & & \mathrm{H}_n(X,\{x_0\};\mathbb{Z})
\end{tikzcd}$$

We make the following observations:

(a) It follows from $n \geq 2$ and from the long exact sequence of homology groups of the pair $(X,\{x_0\})$, see Corollary 43.16, that the right-hand vertical map of both diagrams is an isomorphism.

(b) It follows from Lemma 53.14 and our hypothesis that X is $(n-1)$-connected that the bottom horizontal map ι of both diagrams is an isomorphism.

(c) Let $f\colon (\Delta^n,\partial\Delta^n) \to (X,x_0)$ be a map. The $(n-1)$-skeleton of Δ^n equals $\partial\Delta^n$, thus the map $f\colon \Delta^n \to X$ does indeed define an element in $\mathrm{C}_n^{(n-1)}(X,x_0)$. Furthermore by Lemma 53.13 (3) we know that every element in $\mathrm{C}_n^{(n-1)}(X,x_0)$ is a cycle, thus f defines an element in $\mathrm{H}_n^{(n-1)}(X,x_0)$. This shows that in the above diagram the left vertical map is well-defined.

(d) It is clear that the "outer rectangle" of the above diagram commutes.

(e) It follows from Proposition 45.5 (2) that the top small triangle in the above diagram commutes up to the sign $(-1)^n$.

(f) We want to show that $\Psi_{(X,x_0)}$ is well-defined and that it is a homomorphism. By (b) we know that ι is an isomorphism. Thus it suffices to show these statements for $\iota\circ\Psi_{(X,x_0)}$. But these now follow from (d) and (e) and the fact, see Lemma 53.1 (1), that the Hurewicz homomorphism $\Phi_{(X,x_0)}$ is well-defined and a homomorphism. ⊞

[835] The convoluted logic is forced upon us since proving (1) as a stand-alone statement is actually slightly tricky.

By the claim it suffices to show that the homomorphism

$$\begin{array}{rcl}\Psi_{(X,x_0)}\colon \pi_n(X,x_0) & \to & \mathrm{H}_n^{(n-1)}(X,x_0)\\ [f\colon (\Delta^n,\partial\Delta^n)\to(X,x_0)] & \mapsto & [f\colon \Delta^n\to X]\end{array}$$

is in fact an isomorphism. We will do so by defining an explicit inverse.

Claim.

(1) The map

$$\Xi_{(X,x_0)}\colon\ \mathrm{C}_n^{(n-1)}(X,x_0) \ \to\ \pi_n(X,x_0)$$

that is induced by

$$(\sigma\colon(\Delta^n,\partial\Delta^n)\to(X,x_0)) \ \mapsto\ [\sigma\colon(\Delta^n,\partial\Delta^n)\to(X,x_0)]$$

is well-defined.

(2) The map $\Xi_{(X,x_0)}$ from (1) descends to a well-defined map

$$\mathrm{H}_n^{(n-1)}(X,x_0)\ \to\ \pi_n(X,x_0).$$

(3) The map $\Xi_{(X,x_0)}$ is an inverse to $\Psi_{(X,x_0)}$.

Since the end is near we set out with renewed vigor to prove the claim.

(1) Recall that we know from Proposition 40.3 and our hypothesis $n\geq 2$ that the group on the right is abelian. Therefore it follows from Lemma 19.1 that the map is well-defined.[836]

(2) Let $\sigma\colon\Delta^{n+1}\to X$ be a singular $(n+1)$-simplex that defines an element in the group $\mathrm{C}_{n+1}^{(n-1)}(X,x_0)$. We need to show that $\Xi_{(X,X_0)}(\partial\sigma)=0$. We calculate that we have the following equality in $\pi_n(X,x_0)$:

$$\Xi_{(X,X_0)}(\partial\sigma)=\Xi_{(X,X_0)}\Big(\sum_{j=0}^{n+1}(-1)^j\cdot\sigma\circ i_j^n\Big)\underset{\substack{\uparrow\\ \text{by definition of } \Xi_{(X,x_0)}}}{=}\sum_{j=0}^{n+1}(-1)^j\cdot[\,\overbrace{\sigma\circ i_j^{n+1}}^{(\Delta^n,\partial\Delta^n)\to(X,x_0)}\,]\underset{\substack{\uparrow\\ \text{Lemma 53.16}}}{=}-[\,\overbrace{\sigma|_{\partial\Delta^{n+1}}}^{(\partial\Delta^{n+1},*)\to(X,x_0)}\,]\underset{\substack{\uparrow\\ \text{Lemma 53.15}}}{=}0.$$

(3) It follows immediately from the definitions that $\Xi_{(X,x_0)}\circ\Psi_{(X,x_0)}$ is the identity on $\pi_n(X,x_0)$. Furthermore it follows easily from Lemma 53.13 and the definitions that $\Psi_{(X,x_0)}\circ\Xi_{(X,x_0)}$ is the identity on a generating set for $\mathrm{H}_n^{(n-1)}(X,x_0)$. But by the claim and Lemma 53.1 (1) we know that $\Psi_{(X,x_0)}$ and $\Xi_{(X,x_0)}$ are both homomorphisms. Thus we see that $\Psi_{(X,x_0)}\circ\Xi_{(X,x_0)}$ is actually the identity on all of $\mathrm{H}_n^{(n-1)}(X,x_0)$.

This concludes the proof of the claim and thus we are done with the proof of the Hurewicz Theorem 53.5. ■

Remark. The most awkward aspect of the proof of the Hurewicz Theorem 53.5 is that one needs to go back and forth between the different points of view regarding elements in higher homotopy groups, in particular the fact that we need Proposition 53.11 and Lemma 53.16 is arguably aesthetically somewhat unpleasant.

[836] Put differently, it would be a serious dereliction of duty to use the additive notation for a group structure, if the group was not known to be additive.

In fact there is, in principle, an elegant way around this problem: One could prove the Hurewicz Theorem 53.5 using "cubical singular homology" instead of the usual singular homology. As the name suggests, in this theory the "building blocks" are singular cubes, i.e. maps from $[0,1]^n$ to a topological space X. (This theory is developed in [**Mass91**, Chapter VII.2].) Since cubes are also used in the definition of the product structure on higher homotopy groups I would expect that many of the technical issues we had to deal with vanish magically.

The downside is that one still needs to show that "cubical singular homology" is naturally isomorphic to the usual singular homology. This can be shown using the Acyclic Model Theorem 80.16.

Exercises for Chapter 53. In the following four exercises we need the following definitions.

Definition. Let X be a topological space. We say X is *aspherical* if X is path-connected and if all higher homotopy groups $\pi_n(X)$, $n \geq 2$, vanish.

Exercise 53.1. Let X be a connected 2-dimensional CW-complex with $\pi_2(X) = 0$. Show that X is aspherical.

Exercise 53.2. Let X be a topological space and let U and V be open subsets of X. Let $n \in \mathbb{N}_{\geq 2}$. Suppose that U and V are $(n-1)$-connected and that $U \cap V$ is n-connected. We pick a base point $x_0 \in U \cap V$ and we denote by $i\colon U \to X$ and $j\colon V \to X$ the inclusion maps. Show that the map

$$i_* \oplus j_*\colon \pi_n(U, x_0) \oplus \pi_n(V, x_0) \ \to \ \pi_1(X, x_0)$$

is an isomorphism.

Exercise 53.3. Let X be a CW-complex and let U and V be path-connected subcomplexes of X. Suppose that $U \cap V$ is path-connected and furthermore suppose that the inclusion induced maps $\pi_1(U \cap V) \to \pi_1(U)$ and $\pi_1(U \cap V) \to \pi_1(V)$ are monomorphisms. Finally suppose that U, V and $U \cap V$ are aspherical. Show that X is also aspherical.
Hint. You might want to use Propositions 21.24, 30.6 and Proposition 74.12 (6).

Exercise 53.4. We start out with the following easy-to-digest definition. Let X be a topological space and let $k \in \mathbb{N}_0$. We say X is *k-connected*, if $\pi_i(X) = 0$ for $i = 0, \ldots, k$. Now let M and N be two closed oriented n-dimensional smooth manifolds. Suppose there exists a $k \in \mathbb{N}$ such that M and N are k-connected. Show that the connected sum $M \# N$ is also k-connected.

Exercise 53.5. Let X be an n-dimensional CW-complex and let Y be a subcomplex of X that is homotopy equivalent to S^n. Show that the inclusion induced map $\pi_n(Y) \to \pi_n(X)$ is a monomorphism.

Exercise 53.6. We denote by $*$ suitable base points of the spheres. Now let $f\colon S^n \to S^n$ be a map with $f(*) = *$. We suppose that f is a map of degree $k \in \mathbb{Z}$, i.e. f is a map such that the induced map $f_*\colon \mathrm{H}_n(S^n;\mathbb{Z}) \to \mathrm{H}_n(S^n;\mathbb{Z})$ is given by multiplication by k.

(a) Let $g\colon S^n \to S^m$ be a map. Show that $[g \circ f] = k \cdot [g] \in \pi_n(S^m)$.
(b) Show that if $g\colon S^m \to S^n$ is a map, then it is not necessarily the case that the equality $[f \circ g] = k \cdot [g] \in \pi_m(S^n)$ holds.
Hint. We view S^3 as a subset of $\mathbb{C}^2$ and we make the usual identification $\mathbb{C}\mathrm{P}^1 = S^1$. Now consider the following commutative diagram:

$$\begin{tikzcd} S^3 \arrow[r, "(z,w)\to[(z,w)]"] \arrow[d, "(z,w)\mapsto(\overline{z},\overline{w})"'] & S^2 \arrow[d, "{[(z,w)]\mapsto[(\overline{z},\overline{w})]}"] \\ S^3 \arrow[r, "(z,w)\to[(z,w)]"] & S^2. \end{tikzcd}$$

(c) Let $g\colon S^{m-1} \to S^{n-1}$ be map. As usual we denote by $\Sigma(g)\colon S^m \to S^n$ the suspension of g. Show that $[f \circ \Sigma(g)] = k \cdot [\Sigma(g)] \in \pi_m(S^n)$.

Exercise 53.7. Given $n \in \mathbb{N}$ we denote by $L(n,1) = S^3/\mathbb{Z}_n$ the lens space, as defined on page 502. For which $n \in \mathbb{N}$ is the Hurewicz homomorphism $\pi_3(L(n,1)) \to \mathrm{H}_3(L(n,1);\mathbb{Z})$ an isomorphism?

Exercise 53.8. Let X be a topological homology n-sphere, i.e. X is an n-dimensional topological manifold such that for any $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X) \cong \mathrm{H}_k(S^n)$. We denote by $\Sigma(X)$ the suspension of X as defined on page 694. Show that $\pi_i(\Sigma(X)) = 0$ for $i = 1, \ldots, n$.

Exercise 53.9. Let $n \in \mathbb{N}$. Let $\varphi\colon S^1 \to S^1$ be the map that is given by $z \mapsto z^n$. We consider the topological space $X := \overline{B}^2 \cup_\varphi S^1$. In other words, X is the CW-complex obtained by attaching a 2-cell to S^1 via the attaching map φ.
(a) Compute $\pi_1(X)$.
(b) Compute $\pi_2(X)$.

Exercise 53.10. Let $\varphi\colon S^1 \to S^5$ be a smooth embedding.
(a) Compute $\mathrm{H}_*(S^5 \setminus \varphi(S^1))$.
(b) Show that $\pi_3(S^5 \setminus \varphi(S^1)) \neq 0$.
Remark. For (b) you need to use that φ is a smooth embedding, the statement is in general not true, if we only assume that φ is an injective map. More precisely, in **[Haj19]** it is shown that there exists an injective map $\varphi\colon S^1 \to S^5$ such that $\pi_3(S^5 \setminus \varphi(S^1)) = 0$.

Exercise 53.11. Show that there exists a closed 3-dimensional smooth manifold M such that $\pi_2(M)$ is infinitely generated.
Hint. You could try to imitate the topological space $S^1 \vee S^2$.

Exercise 53.12. Let X be a 0-connected countable CW-complex. Show that $\pi_2(X)$ is countable.
Remark. This answer Question 40.16 in the affirmative for $n = 2$.

54. Applications of the Hurewicz Theorem (∗)

In this chapter we want to spend some more time with the Hurewicz Theorems 52.5 and 53.5. In particular we want to discuss some applications thereof. This chapter is not entirely essential and can also be safely skipped at a first reading.

54.1. **The social choice problem ❀.** In this section we will consider the real-life problem that $n \in \mathbb{N}_{\geq 2}$ people have to agree in a democratic and reasonable way on one of the following:

(1) The temperature setting of a joint office.
(2) A vacation spot in Bavaria.
(3) A vacation spot on Earth.

More generally, the n people are supposed to agree on the following:

(4) A point in some topological space X.

One reasonable approach to solving this problem is the following. Each person gets to pick a point in X. We then need a map f which takes as input the n choices of points in X and which turns these choices into a common choice. In other words, we need a map

$$f\colon X^n \;=\; \underbrace{X \times \cdots \times X}_{n \text{ times}} \;\to\; X.$$

Our interpretation that the common choice should be chosen "democratically and reasonably" is that the map f should have the following three properties:

(a) A small change in choices of the people involved should correspond to a small change in the common choice.
(b) If everybody is of the same opinion, then evidently the common choice should be the point everybody picked.
(c) Everybody should be treated fairly, i.e. the social choice should not depend on the ordering of the n people.

In some cases it is easy to come up with a social choice. For example if X is a convex subset of some $\mathbb{R}^m$, e.g. if $X = \mathbb{R}^m$ itself, then given any $n \in \mathbb{N}$ the map[837][838]

$$\begin{array}{rcl} X^n & \to & X \\ (x_1, \dots, x_n) & \mapsto & \frac{1}{n} \cdot \sum\limits_{i=1}^{n} x_i \end{array}$$

evidently satisfies all of the above conditions. In particular this approach solves the problem for (1). If X is homeomorphic to a convex subset of some $\mathbb{R}^m$, then we can translate our problem into the previous situation and we can find a solution. In particular, Bavaria being to the best of my knowledge homeomorphic to a closed disk, we see that we have a solution to (2).[839]

[837] It follows easily from the convexity of X that the map does indeed take values in X.

[838] For what choices of subsets X of $\mathbb{R}^m$ does the geometric mean $(x_1, \dots, x_n) \mapsto (x_1 \cdot \cdots \cdot x_n)^{\frac{1}{n}}$ define a social choice?

[839] The fact that Bavaria lost the Pfalz in 1946 greatly simplifies finding a solution to (2).

In the following we will now study the third problem in greater detail. At this point it is convenient to introduce the following definition from [**Eckm04**], which is just a formalization of our wish list above.

Definition. Let X be a topological space and let $n \in \mathbb{N}$. A *social choice of type n* is a map

$$f\colon X^n \;=\; \underbrace{X \times \cdots \times X}_{n \text{ times}} \;\to\; X$$

such that the following three axioms are satisfied:

(a) The map f is continuous.
(b) For every $x \in X$ we have $f(x, \ldots, x) = x$.
(c) For every permutation $\sigma \in S_n$ and any $(x_1, \ldots, x_n) \in X^n$ we have

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \;=\; f(x_1, \ldots, x_n).$$

We say that a topological space is *social*, if for every $n \in \mathbb{N}$ it admits a social choice of type n.

Thus the question of whether there exists a "democratic and reasonable" way for n people to settle on a vacation spot on earth can be interpreted as the following question.

Question 54.1. *Does there exist a social choice of type n on S^2?*

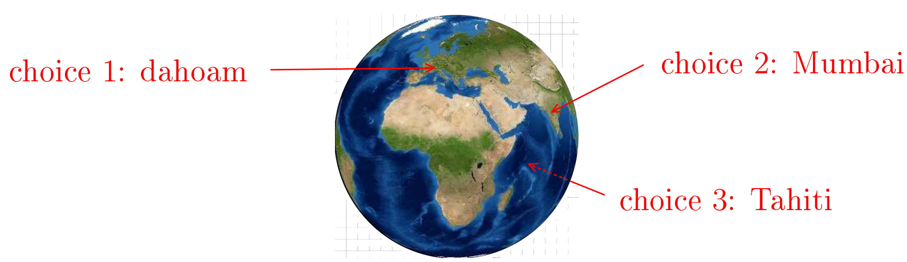


FIGURE 835

As we will see, this question will be short-lived. The key to answer this question is the following proposition which is the main result of this section.

Proposition 54.2.

(1) *Let X be a 0-connected topological space. If X admits a social choice of type n for some $n \in \mathbb{N}_{\geq 2}$, then given any $k \in \mathbb{N}$ the group*[840] *$\pi_k(X)$ is abelian, and it is either zero or infinitely generated.*
(2) *Let X be a non-empty connected* CW*-complex that has the property that for every $k \in \mathbb{N}$ the number of k-cells is finite. If for every $n \in \mathbb{N}_0$ the topological space X admits a social choice of type $n \geq 2$, then for every $k \in \mathbb{N}$ we have $\pi_k(X) = 0$.*

Remark.

(1) It follows from the Whitehead Theorem 119.9 that in Proposition 54.2 (2) we can upgrade the conclusion to the statement that X is actually contractible.

[840]In this section we will only deal with path-connected topological spaces and we will only be interested in the isomorphism type of the higher homotopy groups, thus we will use Proposition 40.5 as an excuse to ignore base points.

(2) In the context of sociology, Proposition 54.2 was first formulated in 1980 by the mathematical economist Graciela Chichilnisky [**Chic80**]. It turns out that the proposition was already proved earlier by Beno Eckmann [**Eckm54**] in 1954. The discussion of this section follows mostly [**Eckm04**]. For more information and some generalizations we refer to [**Eckm04**] and [**Weinb04**].

We are now already in a position to answer Question 54.1 in the negative.

Corollary 54.3. *Let $k \in \mathbb{N}$. The sphere S^k does not admit a social choice for any $n \in \mathbb{N}_{\geq 2}$.*

Proof. By Corollary 53.6 we know that $\pi_k(S^k) \cong \mathbb{Z}$. Since $\mathbb{Z}$ is neither zero nor infinitely generated we obtain from Proposition 54.2 (1) that S^k does not admit a social choice for any $n \in \mathbb{N}_{\geq 2}$. ■

The idea behind the proof of Proposition 54.2 is quite simple. We need to see what effect the existence of a social choice on X has on the higher homotopy groups. Perhaps not surprisingly, the properties of a social choice give rise to very similar properties of the homotopy groups. This leads us to the following definition.

Definition. Let G be a group and let $n \in \mathbb{N}_{\geq 2}$. We say a map $\varphi\colon G^n \to G$ is a *social homomorphism of type n* if the following conditions are satisfied:

(a) The map φ is a homomorphism.
(b) For every $g \in G$ we have $\varphi(g, \ldots, g) = g$.
(c) For every permutation $\sigma \in S_n$ and any $(g_1, \ldots, g_n) \in G^n$ we have

$$\varphi\big(g_{\sigma(1)}, \ldots, g_{\sigma(n)}\big) \;=\; \varphi(g_1, \ldots, g_n).$$

Example. Evidently, if we consider $G = (\mathbb{Q}, +)$ or $G = (\mathbb{R}, +)$, then taking the arithmetic mean defines a social homomorphism.

The following lemma relates social choices on topological spaces to social homomorphisms.

Lemma 54.4. *Let X be a path-connected topological space. If X admits a social choice of some type $n \geq 2$, then for every $k \in \mathbb{N}$ the homotopy group $\pi_k(X)$ admits a social homomorphism of type n.*

Proof. To simplify the notation a little bit we only consider the case $n = 2$. Thus let $f\colon X \times X \to X$ be a social choice of type 2. Let $k \in \mathbb{N}$. We fix a base point $x_0 \in X$. We consider the map

$$\begin{array}{rcl} \Theta\colon \pi_k(X, x_0) \times \pi_k(X, x_0) & \to & \pi_k(X \times X, (x_0, x_0)) \\ ([\alpha\colon (S^k, *) \to (X, x_0)], [\beta\colon (S^k, *) \to (X, x_0)]) & \mapsto & \left[\begin{array}{rcl} (S^k, *) & \to & (X \times X, (x_0, x_0)) \\ z & \mapsto & (\alpha(z), \beta(z)) \end{array}\right]. \end{array}$$

Next we consider the map

$$\varphi\colon \pi_k(X, x_0) \times \pi_k(X, x_0) \xrightarrow{\Theta} \pi_k(X \times X, (x_0, x_0)) \xrightarrow{f_*} \pi_k(X, x_0).$$

It remains to show that $\varphi = f_* \circ \Theta$ is a social homomorphism of type 2. We do so in the following three steps.

(a) By Proposition 40.8 (2) we know that the map Θ is actually a homomorphism. It follows that $\varphi = f_* \circ \Theta$ is a homomorphism.
(b) It follows immediately from the fact that for every $x \in X$ we have $f(x,x) = x$ that for every $g \in \pi_k(X)$ we have $\varphi(g,g) = g$.
(c) Similarly it follows immediately from the fact that for every $x, y \in X$ we have $f(x,y) = f(y,x)$ that for every $g, h \in \pi_k(X, x_0)$ we have $\varphi(g,h) = \varphi(h,g)$. ■

In light of Lemma 54.4 we now need to understand, given $n \in \mathbb{N}_{\geq 2}$, which groups admit social homomorphisms of type n.

Definition. Let G be an abelian group.

(1) Let $n \in \mathbb{N}$. We say G is *strongly divisible by n*, if the map $g \mapsto n \cdot g$ is an isomorphism of the group G.
(2) We say G is *strongly divisible*, if G is strongly divisible by any $n \in \mathbb{N}$.

Remark. Usually an abelian group G is called *divisible* if for every $g \in G$ and $n \in \mathbb{N}$ there exists an $h \in G$ with $n \cdot h = g$. This will also be our definition later on page 1863. Clearly a group which is strongly divisible is also divisible. But the converse does not hold, for example the abelian group $(\mathbb{Q}/\mathbb{Z}, +)$ is divisible but it is not strongly divisible.

The following lemma gives a satisfying characterization of groups that admit a social homomorphism of type n.

Lemma 54.5. *Let G be a group and let $n \in \mathbb{N}_{\geq 2}$. The following two statements are equivalent:*

(1) *The group G admits a social homomorphism of type n.*
(2) *The group G is abelian and it is strongly divisible by n.*

PROOF. In the following let G be a group and let $n \in \mathbb{N}_{\geq 2}$. Since we will shortly see that the groups involved are abelian we will use additive notation throughout the argument.

First we deal with the implication "(2) ⇒ (1)". If G is abelian and if it is strongly divisible by n, then evidently the usual "arithmetic mean formula"

$$(g_1, \ldots, g_n) \;\mapsto\; \tfrac{1}{n}(g_1 + \cdots + g_n) \;:=\; \begin{array}{c}\text{inverse of the isomorphism}\\ G \to G \text{ given by } g \mapsto n \cdot g,\\ \text{applied to } g_1 + \cdots + g_n \in G\end{array}$$

defines a social homomorphism of type n.

Next we prove the much more interesting implication "(1) ⇒ (2)". Let $\varphi\colon G^n \to G$ be a social homomorphism of type $n \geq 2$. Throughout the proof we mean by (a), (b) and (c) the three defining properties of a social homomorphism. First note that it follows immediately from (c) that for any $g \in G$ we have the equalities

$$\varphi(g, 0, \ldots, 0) \;=\; \varphi(0, g, 0, \ldots, 0) \;=\; \varphi(0, \ldots, 0, g) \;\in\; G.$$

We denote this common value by $\psi(g)$. We make the following observations:

(i) It follows immediately from (a) that $\psi\colon G \to G$ is also a homomorphism.

(ii) For any $g, h \in G$ we have

$$\begin{aligned} \psi(g)+\psi(h) &\underset{\text{since } n\geq 2}{=} \varphi(g,0,\dots,0)+\varphi(0,h,0,\dots,0) \underset{\text{by (a)}}{=} \varphi(g,h,0,\dots,0) \\ &\underset{\text{by (a)}}{=} \varphi(0,h,\dots,0)+\varphi(g,0,0,\dots,0) \underset{\text{since } n\geq 2}{=} \psi(h)+\psi(g). \end{aligned}$$

(iii) For any $g \in G$ we have

$$\underbrace{\psi(g)+\dots\psi(g)}_{=n\cdot\psi(g)} \underset{\text{by definition of } \psi}{=} \varphi(g,0,\dots,0)+\dots+\varphi(0,\dots,0,g) \underset{\text{by (a)}}{=} \varphi(g,\dots,g) \underset{\text{by (c)}}{=} g.$$

(iv) Let $g, h \in G$. We have

$$\begin{aligned} g+h &\underset{\text{by (iii)}}{=} \psi(g+h)+\dots+\psi(g+h) \underset{\text{by (i)}}{=} \psi(g)+\psi(h)+\dots+\psi(g)+\psi(h) \\ &\underset{\text{by (ii)}}{=} \psi(h)+\psi(g)+\dots+\psi(h)+\psi(g) \underset{\text{by (i)}}{=} \psi(h+g)+\dots+\psi(h+g) \underset{\text{by (iii)}}{=} h+g. \end{aligned}$$

This shows that G is abelian.

(v) Since G is abelian we know that multiplication by n is a homomorphism of G.

(vi) It follows almost immediately from (i) and (iii) that ψ is an inverse to the map $G \to G$ that is given by $g \mapsto n \cdot g$. Together with (v) this shows that $g \mapsto n \cdot g$ does indeed define an isomorphism of G. ∎

The following lemma basically tells us which finitely generated abelian groups are strongly divisible by some $n \in \mathbb{N}_{\geq 2}$.

Lemma 54.6. *Let $n \in \mathbb{N}_{\geq 2}$.*

(1) *The group $\mathbb{Z}$ is not strongly divisible by n.*
(2) *Let $m \in \mathbb{N}$. The cyclic group $\mathbb{Z}_m$ is strongly divisible by n if and only if m and n are coprime.*
(3) *If a finitely generated group is strongly divisible by n, then it is isomorphic to a direct sum $\bigoplus_{i=1}^{k} \mathbb{Z}_{m_i}$ where each m_i is coprime to n.*

Proof. The first two statements are basically trivial. Finally the last statement follows from the previous two statements together with the classification of finitely generated abelian groups, see Theorem 19.4. ∎

Now we have assembled everything we need for the proof of Proposition 54.2.

Proof of Proposition 54.2.

(1) Let X be a 0-connected topological space that admits a social choice of type n for some $n \in \mathbb{N}_{\geq 2}$. Let $k \in \mathbb{N}$. By Lemma 54.4 we know that the group $\pi_k(X)$ is abelian

and that it is strongly divisible by n. It follows almost immediately from Lemma 54.6 that $\pi_k(X)$ is either zero or infinitely generated.

(2) Let X be a non-empty connected CW-complex that has the property that for every $k \in \mathbb{N}$ the number of k-cells is finite. We assume that X is social, i.e. we assume that given any $n \geq 2$ there exists a social choice of type n on X. By Lemma 54.4 we know that every homotopy group $\pi_k(X)$ is abelian and that it is strongly divisible by n.

Next note that by Proposition 37.13 and our hypothesis that X has finitely many 1-cells we know that $\pi_1(X)$ is finitely generated. It follows from the above together with Lemma 54.6 that $\pi_1(X) = 0$.

Now suppose that we already know that $\pi_i(X) = 0$ for $i = 1, \dots, k$. By the Hurewicz Theorem 53.5 we know that $\pi_{k+1}(X) \cong \mathrm{H}_{k+1}(X)$. Since X has finitely many $(k+1)$-cells we obtain from Proposition 48.5 that $\pi_{k+1}(X) \cong \mathrm{H}_{k+1}(X)$ is finitely generated. Basically the same argument as above shows that $\pi_{k+1}(X) = 0$. Iterating this procedure shows that all homotopy groups of X vanish. ■

We conclude this section with the following question.

Question 54.7. *Does there exist a non-empty* CW*-complex that is social but that is not contractible?*

We will return to this question later on in Section 120.6.

54.2. **The degree of a self-map of a sphere II.** Let $n \in \mathbb{N}_0$ and let $f\colon S^n \to S^n$ be map. On page 1181 we defined the degree of f as follows:

$$\deg(f) \;:=\; \deg\big(f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)\big).$$

In Lemma 45.11 (3) we showed that any two homotopic maps $S^n \to S^n$ have the same degree. Now we have the tools to prove the converse of this statement. In fact we have the following proposition.

Proposition 54.8. *For any $n \in \mathbb{N}$ the map*

$$\begin{array}{rcl} \deg\colon \overbrace{[S^n, S^n]}^{\text{set of homotopy equivalence classes of maps } S^n \to S^n} & \to & \mathbb{Z} \\ [f\colon S^n \to S^n] & \to & \deg(f) \end{array}$$

is an isomorphism. (The set $[S^n, S^n]$ is a monoid through composition of maps.)

Proof. First note that we know by Lemma 45.11 (3) and (4) that the degree map is well-defined and that it is a monoid morphism. It remains to show that the degree map is a bijection. Now let $* \in S^n$ be the standard base point. We consider the following diagram

$$\begin{tikzcd} \langle (S^n,*),(S^n,*)\rangle \arrow[r, "\cong"] \arrow[d, "="] & {[S^n,S^n]} \arrow[r, "\deg(f)"] \arrow[d, "f\mapsto f_*([S^n])"'] & \mathbb{Z} \\ \pi_n(S^n,*) \arrow[r, "\cong", "{[f]\mapsto f_*([S^n])}"'] & \mathrm{H}_n(S^n). \arrow[ur, "\cong", "{k\cdot[S^n]\mapsto k}"'] & \end{tikzcd}$$

We make the following observations and clarifications:

(1) By Corollary 38.8 we know that the map $\langle (S^n,*),(S^n,*)\rangle \to [S^n,S^n]$ is a bijection.
(2) By Theorem 53.5 we know that the bottom horizontal map is an isomorphism.
(3) The diagonal map is an isomorphism by Proposition 43.4.
(4) Finally note that it is clear that the diagram commutes. It follows from this observation and all of the above that the map $\deg\colon [S^n,S^n]\to \mathbb{Z}$ is indeed an isomorphism. ■

54.3. **Paths acting on homotopy groups.** In Propositions 14.11 and 40.5 we already saw that for path-connected topological spaces the *isomorphism types* of the homotopy groups do not depend on the choice of the base point.

In this section we want to discuss this statement in greater detail. We recall the following definition which appeared already in the proof of Proposition 40.5.

Definition. Let X be a topological space and let $n\in\mathbb{N}$. Let $\gamma\colon [0,1]\to X$ be a path from a point x_0 to a point x_1. Given a map $f\colon (I^n,\partial I^n)\to (X,x_1)$ we define the map

$$\begin{array}{rcl} f^\gamma\colon (I^n,\partial I^n) & \to & (X,x_0) \\ (t_1,\dots,t_n) & \mapsto & \begin{cases} f\big(2t_1-\frac{1}{2},\dots,2t_n-\frac{1}{2}\big), & \text{if } (t_1,\dots,t_n)\in\big[\frac{1}{4},\frac{3}{4}\big]^n, \\ \gamma(t), & \text{if } (t_1,\dots,t_n)\in\partial\big(\big[\frac{t}{4},\frac{4-t}{4}\big]^n\big) \text{ for } t\in[0,1]. \end{cases} \end{array}$$

The definition of f^γ is illustrated in Figure 836.

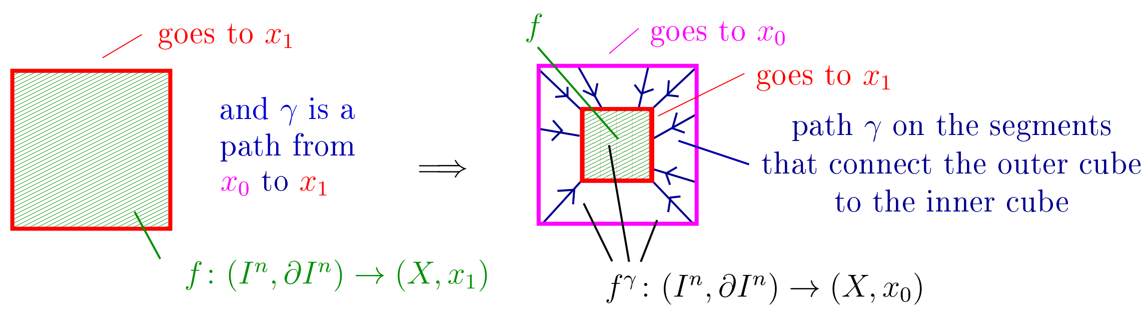


FIGURE 836

Example. We consider the case $n=1$ in the above definition. In this situation the map $f\colon (I,\partial I)\to (X,x_1)$ is a loop. It follows almost immediately from the definitions that the new loop $f^\gamma\colon (I,\partial I)\to (X,x_0)$ equals, up to an orientation-preserving reparametrization, the loop $\gamma * f * \overline{\gamma}$ that is given by the product of the three paths γ, f and $\overline{\gamma}$. We refer to Figure 837 for an illustration.

FIGURE 837

Remark. We recall that on pages 1060 and 1329 we saw that we can interpret elements in a homotopy group $\pi_n(X,x_0)$ as homotopy classes of maps $(S^n,*)\to (X,x_0)$. Now let

$\gamma\colon [0,1] \to X$ be a path from x_0 to x_1 and let $f\colon (S^n,*) \to (X,x_1)$ be a map. We define[841]

$$\begin{array}{rcl} f^\gamma\colon S^n & \to & X \\ \underbrace{\cos(\varphi)\cdot(1,0,\dots,0)+\sin(\varphi)\cdot(0,v)}_{\varphi\in[0,\pi],v\in\mathbb{R}^{n-1}} & \mapsto & \begin{cases} f(\cos(2\varphi)\cdot(1,0,\dots,0)+\sin(2\varphi)\cdot(0,v)), & \text{if } \varphi\in[0,\frac{\pi}{2}] \\ \gamma(2-\frac{2}{\pi}t), & \text{if } \varphi\in[\frac{\pi}{2},\pi]. \end{cases} \end{array}$$

Of course one should only work with the formula if one is forced to at gun point. It is much more illuminating to look at Figure 838 where we try our best to visualize the map $f^\gamma\colon (S^n,*) \to (X,x_0)$. One can easily verify that, under the various identifications, the map f^γ corresponds to the construction on page 1352.

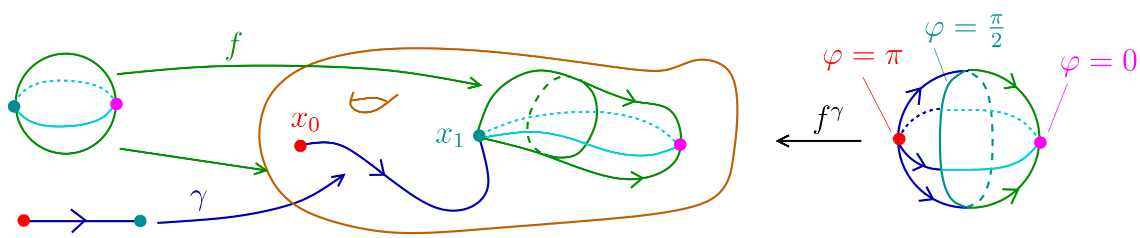


FIGURE 838

Before we continue we introduce the following basically self-explanatory definition.

Definition. Let G be a group and let A be a group (or module or vector space). An *action of G on the group A* (respectively module or vector space) is an action $G \times A \to A$ in the sense of the definition on page 185 such that for each $g \in G$ the map

$$\begin{array}{rcl} A & \to & A \\ a & \mapsto & g\cdot a \end{array}$$

is a group homomorphism (respectively module or vector space homomorphism).[842]

The following proposition can be viewed as a more precise version of Proposition 40.5 and as a generalization of Proposition 15.11.

Proposition 54.9. *Let X be a topological space and let $n \in \mathbb{N}$.*

(1) *If $\gamma\colon [0,1] \to X$ is a path from x_0 to x_1, then the map*
$$\begin{array}{rcl} \gamma_*\colon \pi_n(X,x_1) & \to & \pi_n(X,x_0) \\ [f] & \mapsto & [f^\gamma] \end{array}$$
is a well-defined isomorphism.

(2) *If x_0 and x_1 are two points in X which lie in the same path-component of X, then $\pi_n(X,x_0)$ is isomorphic to $\pi_n(X,x_1)$.*

[841]It follows easily from Lemma 2.35 (2') that this map is continuous.

[842]In fact, since multiplication by g^{-1} is an inverse we see that these maps are actually isomorphisms.

(3) *Let* $\varphi\colon X \to Y$ *be a map between topological spaces.*

(a) *If* $\gamma\colon [0,1] \to X$ *is a path from* x_0 *to* x_1*, then the following diagram commutes*

$$\begin{tikzcd}
\pi_n(X,x_1) \arrow[r, "\gamma_*"] \arrow[d, "\varphi_*"'] & \pi_n(X,x_0) \arrow[d, "\varphi_*"] \\
\pi_n(Y,\varphi(x_1)) \arrow[r, "(\varphi\circ\gamma)_*"] & \pi_n(Y,\varphi(x_0)).
\end{tikzcd}$$

(b) *If* $\varphi_*\colon \pi_n(X,x_0) \to \pi_n(Y,\varphi(x_0))$ *is an isomorphism* (*epimorphism, monomorphism*) *for some base point* $x_0 \in X$*, then for any other base point* x_1 *in the same path component of* X *the map* $\varphi_*\colon \pi_n(X,x_1) \to \pi_n(Y,\varphi(x_1))$ *is also an isomorphism* (*epimorphism, monomorphism*).[843]

(4) *Let* γ *and* δ *be two paths in* X *from* x_0 *to* x_1*. If* γ *and* δ *are path-homotopic in* X*, then the corresponding maps* γ_* *and* δ_* *from* $\pi_n(X,x_1)$ *to* $\pi_n(X,x_0)$ *agree.*

(5) *If* γ *is a path in* X *from* x_0 *to* x_1 *and if* δ *is a path in* X *from* x_1 *to* x_2*, then*[844]

$$(\underbrace{\gamma * \delta}_{\text{path from } x_0 \text{ to } x_2})_* = \gamma_* \circ \delta_*\colon \pi_n(X,x_2) \;\to\; \pi_n(X,x_0).$$

(6) *The map*

$$\begin{array}{rcl} \pi_1(X,x_0) \times \pi_n(X,x_0) & \to & \pi_n(X,x_0) \\ ([\gamma],[f]) & \mapsto & [\gamma]\cdot[f] := [f^\gamma] \end{array}$$

defines an action (*in the sense of the definition on page 185*) *of the group* $\pi_1(X,x_0)$ *on the group* $\pi_n(X,x_0)$*.*

(7) *Let* $f_0, f_1\colon S^n \to X$ *be two maps. We write* $x_0 = f_0(*)$ *and* $x_1 = f_1(*)$*. The following statement holds:*

$$f_0 \text{ and } f_1 \text{ are homotopic} \quad\Longleftrightarrow\quad \begin{array}{c}\text{there exists a path } \gamma \text{ from } x_0 \text{ to } x_1 \\ \text{such that } [f_1^\gamma] = [f_0] \in \pi_n(X,x_0).\end{array}$$

Remark. Let X be a topological space, let x_0, x_1 be two points in X and let γ and δ be two paths from x_0 to x_1. If X is simply connected, then Lemma 14.13 tells us that the two paths γ and δ are in fact homotopic. Therefore it follows from Proposition 54.9 (4) that $\gamma_* = \delta_*\colon \pi_n(X,x_0) \to \pi_n(X,x_1)$. In other words, if X is simply connected, then given any two base points x_0 and x_1 there exists in fact a natural[845] isomorphism $\pi_n(X,x_0) \xrightarrow{\cong} \pi_n(X,x_1)$. In particular in this case it is almost always entirely safe to suppress the base point in the notation. By Proposition 14.14 this observation applies in particular to $X = S^k$ for $k \geq 2$.

Proof.

[843]For $n = 1$ this is basically the content of Proposition 15.11.

[844]Here it is worth remembering that by definition, see page 465, the product path $\gamma * \delta$ is given by first traversing γ and then traversing δ.

[845]As always, when the adjective "natural" gets used it is an amusing exercise to figure out what categories and functors are involved.

(0) Let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1. It is straightforward to see that the map

$$\begin{array}{rcl}\gamma_*\colon \pi_n(X,x_1) & \to & \pi_n(X,x_0)\\ [f] & \mapsto & [f^\gamma]\end{array}$$

is well-defined. It is also elementary, albeit potentially slightly painful, to see that γ_* is a group homomorphism. (See the corresponding discussion in the proof of Proposition 40.5.)

(4),(5) These two statements are basically elementary to verify. Nonetheless it is worth spending a second to convince oneself that in (5) we have the equality $(\gamma*\delta)_* = \gamma_*\circ\delta_*$ instead of $(\gamma*\delta)_* = \delta_*\circ\gamma_*$.

(1) Recall that in Proposition 14.6 (3) we saw that for any path δ the composition $\delta * \overline{\delta}$ is path-homotopic to the constant path. It follows easily from this observation together with (4) and (5) that γ_* is a bijection. Together with (0) we see that for any path $\gamma\colon [0,1]\to X$ from x_0 to x_1 the induced map $\gamma_*\colon \pi_n(X,x_1)\to\pi_n(X,x_0)$ is an isomorphism.

(2) This is an immediate consequence of (1)

(3) Statement (a) follows immediately from the definitions. Statement (b) is a consequence of the commutative diagram of (a) and the fact that the horizontal maps are isomorphisms by (1).

(6) It follows immediately from (4) and (5) that the map defines an action of $\pi_1(X,x_0)$ on the set $\pi_n(X,x_0)$. Furthermore (1) implies that this action is in fact an action on the group $\pi_n(X,x_0)$.

(7) The proof of this statement is a moderately complicated generalization of the proof of Proposition 18.33. We leave it to the reader to fill in the details. ■

54.4. **Group rings.** Before we continue with our study of higher homotopy groups it is convenient to introduce the purely algebraic concept of a group ring.

Definition. Let G be a group and let R be a commutative ring. We denote the free R-module generated by the set G by $R[G]$. Put differently, with the notation from page 580 we have

$$R[G] \;=\; \Big\{ \sum_{i=1}^{m} r_i g_i \,\Big|\, r_1,\dots,r_m\in R \text{ and } g_1,\dots,g_m\in G\Big\}.$$

We equip $R[G]$ with the multiplication that is given by[846]

$$\Big(\sum_{i=1}^{m} r_i g_i\Big)\cdot\Big(\sum_{j=1}^{n} s_j h_j\Big) \;:=\; \sum_{i=1}^{m}\sum_{j=1}^{n} \underbrace{r_i s_j}_{\in R}\cdot\underbrace{(g_i h_j)}_{\in G}.$$

We refer to $R[G]$ as the *group ring of G with R-coefficients.* (For $R=\mathbb{Z}$ we refer to $\mathbb{Z}[G]$ as *the group ring of G*.)

The following lemma summarizes a few basic facts about group rings.

[846] The multiplication is basically given by naive "multiplying out".

Lemma 54.10. *Let G be a group and let R be a commutative ring.*

(1) *The group ring $R[G]$ is an associative ring with a multiplicatively neutral element, namely $1 \cdot e$ where e denotes the trivial element in G.*
(2) *The group ring $R[G]$ is commutative if and only if G is an abelian group.*
(3) *If R is a field, then $R[G]$ is evidently an algebra over the field R.*
(4) *Given a unit $r \in R$ and an element $g \in G$ the product $r \cdot g$ is a unit in $R[G]$.*
(5) *If G has a torsion element, i.e. if there exists a non-trivial $g \in G$ such that $g^k = e$ for some $k \in \mathbb{N}$, then $R[G]$ has zero divisors.*

Proof. All the statements are verified easily. Perhaps the most interesting statement is (5). We will not rob the reader of the pleasure of proving this in Exercise 54.3. ■

Examples. Let R be a commutative ring.

(1) We consider the infinite cyclic group $\langle t \rangle$. In this case the group ring $R[\langle t \rangle]$ is isomorphic to the ring $R[t^{\pm 1}]$ of Laurent polynomials, more precisely the "obvious" map

$$\begin{array}{ccc} R[t^{\pm 1}] & \to & R[\langle t \rangle] \\ \sum\limits_{i=r}^{i=s} a_i t^i & \mapsto & \sum\limits_{i=r}^{i=s} a_i t^i \end{array}$$

is a ring isomorphism.
(2) More generally, if H is a free abelian group of rank m, then any choice of a basis $t_1, \ldots, t_m$ for H gives rise to a natural isomorphism from the multivariable Laurent polynomial ring $R[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ to the group ring $R[H]$.

The group rings of free groups are particularly interesting. Therefore we dedicate a lemma to them.

Lemma 54.11. *Let $\langle T \rangle$ be the free group on a generating set T. Given any ring S and given any map $g\colon T \to S$ there exists a unique ring homomorphism $\varphi\colon \mathbb{Z}[\langle T \rangle] \to S$ with $\psi(t) = g(t)$ for all $t \in T$.*

Remark. Given a commutative ring R the group ring $R[\langle T \rangle]$ is sometimes called the *free ring on the set S over the ring R.*

Proof. It follows immediately from Lemma 19.14 that there exists a unique group homomorphism $\varphi\colon \langle T \rangle \to S^* := \{\text{units of } S\}$ that extends the given map g. It is now straightforward to verify that the map

$$\begin{array}{ccc} \mathbb{Z}[\langle T \rangle] & \to & S \\ \sum\limits_{i=1}^{m} r_i g_i & \mapsto & r_i \cdot \varphi(g_i) \end{array}$$

is a ring homomorphism that extends g and that it is the unique ring homomorphism with this desirable property. ■

As a side remark we mention the following beautiful conjectures. These say in particular that in some sense the statements of Lemma 54.10 (4) and (5) are optimal.

Conjecture 54.12. (Kaplansky Conjectures) *Let G be a torsion-free group and let $\mathbb{F}$ be a field.*

(1) *The elements 0 and 1 are the only idempotents of $\mathbb{F}[G]$, i.e. they are the only elements in $\mathbb{F}[G]$ with $x^2 = x$.*
(2) *The group ring $\mathbb{F}[G]$ is a domain, i.e. it has no non-trivial zero divisors.*
(3) *The only units in $\mathbb{F}[G]$ are the elements of the form $f \cdot g$ with $f \in \mathbb{F} \setminus \{0\}$ and $g \in G$.*
(4) *The group ring $\mathbb{C}[G]$ can be embedded in a skew field.*

Remark.

(1) The conjectures are named after the Canadian mathematician Irving Kaplansky (1917-2006). The first conjecture is often also referred to as the *zero divisor conjecture.* They are formulated in [**Lü02**, Conjecture 10.14], [**Lam91**, p. 95] and [**FrG05**, p. 729]. As is pointed out in [**Lam91**, p. 95], Conjecture (3) implies Conjecture (2). It is elementary to to see that Conjecture (4) implies Conjecture (2) and that Conjecture (2) implies Conjecture (1).
(2) The Kaplansky Conjectures are very attractive since they are easy to state. But it is usually very very hard to prove non-trivial group-theoretic statements that are valid for all (torsion-free) groups. In particular all of the Kaplansky Conjectures are still wide open. We refer to [**Lü09**, Chapter 4.5] for a summary of what is known regarding these conjectures.

We continue with the following elementary lemma.

Lemma 54.13. *Let G be a group and let A be an abelian group (with additive notation for the group structure). Suppose we are given a G-action on the group A. Then the map*[847]

$$\begin{array}{rcl} \mathbb{Z}[G] \times A & \to & A \\ \left(\sum\limits_{i=1}^{m} r_i g_i, a\right) & \mapsto & \sum\limits_{i=1}^{m} r_i \cdot \underbrace{(g_i \cdot a)}_{\in A} \end{array}$$

defines a natural $\mathbb{Z}[G]$-left module structure on A.[848]

Proof. The lemma is indeed elementary and trivial. We leave it to the reader to ponder what "natural" might mean in this context. ■

Before we continue we introduce or recall the following convention:

[847] Here we use a somewhat perilous notation, namely we use the symbol "$\cdot$" with two different meanings. First of all, for $g \in G$ and $a \in A$ we denote by $g \cdot a \in A$ the element defined by the G-action on A. Secondly, for $r \in \mathbb{Z}$ and $b \in A$ we write, as usual, $r \cdot b = \underbrace{b + \cdots + b}_{r\text{-times}}$ for $r \geq 0$ and $r \cdot b = \underbrace{-b - \cdots - b}_{(-r)\text{-times}}$ for $r < 0$.

[848] If R is a non-commutative ring, then one needs to be a little careful when one talks about "R-modules". More precisely, there are now two different concepts one can consider: Given an abelian group M one can demand that for any $r, s \in R$ and $m \in M$ one has $(rs) \cdot m = r \cdot (s \cdot m)$ in which case one says that M is an "R-left module". Alternatively one could demand that for any $r, s \in R$ and $m \in M$ one has $(rs) \cdot m = s \cdot (r \cdot m)$ in which case one says that M is an "R-right module". This discussion is the analogue of the discussion on page 185.

Convention. If a homotopy group is abelian, then sometimes we use additive notation for the group multiplication instead of the multiplicative notation.

This convention, Lemma 54.13, Propositions 40.3 and Proposition 54.9 lead us to the following lemma.

Lemma 54.14. *Let (X, x_0) be a pointed topological space. If $n \geq 2$, then the map*
$$\mathbb{Z}[\pi_1(X, x_0)] \times \pi_n(X, x_0) \quad \to \quad \pi_n(X, x_0)$$
induced by
$$([\gamma], [f]) \quad \mapsto \quad [f^\gamma]$$
defines a natural $\mathbb{Z}[\pi_1(X, x_0)]$-left module structure on $\pi_n(X, x_0)$.

54.5. **Homotopy groups of $S^1 \vee S^2$.** In this section we want to consider the homotopy groups of $S^1 \vee S^2$ since they are quite instructive. On page 1326 we already saw that $\pi_2(S^1 \vee S^2)$ is very big, more precisely, we saw that it admits an epimorphism onto $\mathbb{Z}^\infty$. We will now see that if we take the view of Lemma 54.14, i.e. if we view $\pi_2(S^1 \vee S^2)$ as a module over the group ring $\mathbb{Z}[\pi_1(S^1 \vee S^2)]$, then the apparently messy group $\pi_2(S^1 \vee S^2)$ suddenly gets a simple description.

We explicitly construct $X = S^1 \vee S^2$ by identifying the point $i \in S^1$ with the point $(0, 0, -1) \in S^2$. In the following we evidently use the wedge point as the base point of $X = S^1 \vee S^2$. We refer to it by x_0. Finally we denote by $\gamma\colon [0, 1] \to S^1 \subset X$ the obvious counterclockwise loop that starts at i. Furthermore we denote by $\mathrm{id}_{S^2}\colon S^2 \to S^2 \subset X$ the obvious inclusion map.

Now we can formulate the following proposition which delivers the promised concise description of $\pi_2(S^1 \vee S^2)$.

Proposition 54.15. *To simplify the formulation and the proof of the proposition we use the base point $* = (0, 0, -1)$ for S^2 and we use this base point to define the second homotopy groups. This convention has the advantage that $\mathrm{id}\colon (S^n, *) \to (S^1 \vee S^2, x_0)$ is a map of based topological spaces. Using this convention the homomorphism*
$$\Psi\colon \mathbb{Z}[t^{\pm 1}] \quad \to \quad \pi_2(S^1 \vee S^2, x_0)$$
that is uniquely determined by
$$\sum_{i=1}^{n} a_i \cdot t^i \quad \mapsto \quad \sum_{i=1}^{n} a_i \cdot \left[(\mathrm{id}_{S^2})^{\gamma^i} \right]$$
is an isomorphism of $\mathbb{Z}[t^{\pm 1}] = \mathbb{Z}[\langle t \rangle] = \mathbb{Z}[\pi_1(S^1 \vee S^2, x_0)]$-modules. In particular, as an abelian group, $\pi_2(S^1 \vee S^2)$ is isomorphic to $\mathbb{Z}^\infty$.

Proof. It follows immediately from the definitions that the map Ψ is indeed a homomorphism of $\mathbb{Z}[t^{\pm 1}] = \mathbb{Z}[\langle t \rangle] = \mathbb{Z}[\pi_1(S^1 \vee S^2, x_0)]$-modules. Thus it remains to show that Ψ is an isomorphism of abelian groups.

As on page 836 we see that the universal covering of $X = S^1 \vee S^2$ is given by the map

$$\begin{array}{rcl} p\colon \overbrace{\left(\mathbb{R} \cup (\mathbb{Z} \times S^2)\right) / \, n \sim (n,(0,0,-1))}^{=:\widetilde{X}} & \to & S^1 \vee S^2 \\ P & \mapsto & \begin{cases} e^{2\pi \mathrm{i}(P+\frac{1}{4})}, & \text{if } P \in \mathbb{R} \\ Q, & \text{if } P = (n,Q) \text{ with } n \in \mathbb{Z}, Q \in S^2. \end{cases} \end{array}$$

We pick the base points $x_0 = i \in S^1 \subset X$ and $\widetilde{x}_0 = 0 \in \mathbb{R} \subset \widetilde{X}$. Even though we have seen this covering map on numerous occasions we show it again in Figure 839. We consider the

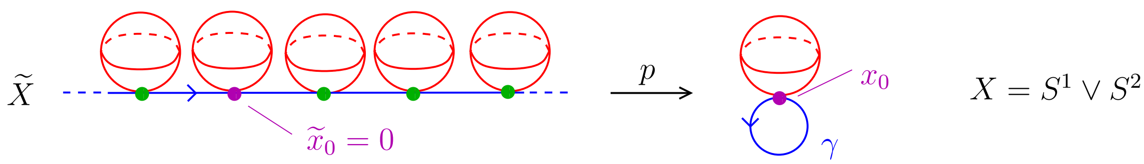


FIGURE 839. First illustration for the proof of Proposition 54.15.

following sequence of maps

$$(*) \qquad \mathbb{Z}[t^{\pm 1}] \xrightarrow{\Psi} \pi_2(X, x_0) \xrightarrow{\Xi} \pi_2(\widetilde{X}, \widetilde{x}_0) \xrightarrow{\Phi_{(\widetilde{X},\widetilde{x}_0)}} \mathrm{H}_2(\widetilde{X}; \mathbb{Z}) \xleftarrow{\Omega} \mathbb{Z}[t^{\pm 1}].$$

Here $\Phi_{(X,x_0)}$ denotes the Hurewicz homomorphism. Since $\widetilde{X}$ is simply connected we obtain from the Hurewicz Theorem 53.5 that it is an isomorphism. The reader will not have failed to notice that we did not define the two maps Ξ and Ω yet. We will do so now.

(1) First we recall that given a map $g\colon (S^2, *) \to (X, x_0)$ there exists by Proposition 29.2 a unique lift $\widetilde{g}\colon (S^2, *) \to (\widetilde{X}, \widetilde{x}_0)$. Now we consider the map

$$\begin{array}{rcl} \Xi\colon \pi_2(X, x_0) & \to & \pi_2(\widetilde{X}, \widetilde{x}_0) \\ {[g\colon (S^2, *) \to (X, x_0)]} & \mapsto & \left[\widetilde{g}\colon (S^2, *) \to (\widetilde{X}, \widetilde{x}_0)\right]. \end{array}$$

By Proposition 40.13 we know that Ξ is well-defined and that it is an isomorphism.

(2) Given $i \in \mathbb{Z}$ we denote by $f_i\colon S^2 \to X$ the "obvious" i-th inclusion map. In Lemma 47.7 we saw that the map

$$\begin{array}{rcl} \Omega\colon \mathbb{Z}[t^{\pm 1}] & \to & \mathrm{H}_2(\widetilde{X}; \mathbb{Z}) \\ \sum\limits_{i=r}^{s} a_i \cdot t^i & \mapsto & \sum\limits_{i=r}^{s} a_i \cdot f_i([S^2]). \end{array}$$

is an isomorphism.

We see in particular that in $(*)$ the three maps to the right are isomorphisms. Therefore it suffices to show that the maps $\Omega^{-1} \circ \Phi_{(X,x_0)} \circ \Xi \circ \Psi$ is the identity. In other words, it suffices to show the following claim.

Claim. For any $n \in \mathbb{Z}$ we have

$$(\Phi_{(X,x_0)} \circ \Xi \circ \Psi)(t^n) \;=\; \Omega(t^n) \;\in\; \mathrm{H}_2(\widetilde{X}; \mathbb{Z}).$$

So let $n \in \mathbb{Z}$. It follows immediately from the definitions of the various maps that the desired statement is equivalent to the statement that

$$\underbrace{\left(\widetilde{(\mathrm{id}_{S^2})^{\gamma^n}}\right)}_{\text{map } S^2 \to \widetilde{X}}{}_*[S^2] \;=\; \underbrace{f_n}_{\text{map } S^2 \to \widetilde{X}}([S^2]) \;\in\; \mathrm{H}_2(\widetilde{X};\mathbb{Z}).$$

Thus by Proposition 42.5 it suffices to show that the two maps $\widetilde{(\mathrm{id}_{S^2})^{\gamma^n}}$ and f_n are homotopic maps from S^2 to $\widetilde{X}$. For those readers who are still awake this should be basically obvious.[849] As a digestion aid we illustrate both maps in Figure 839. ■

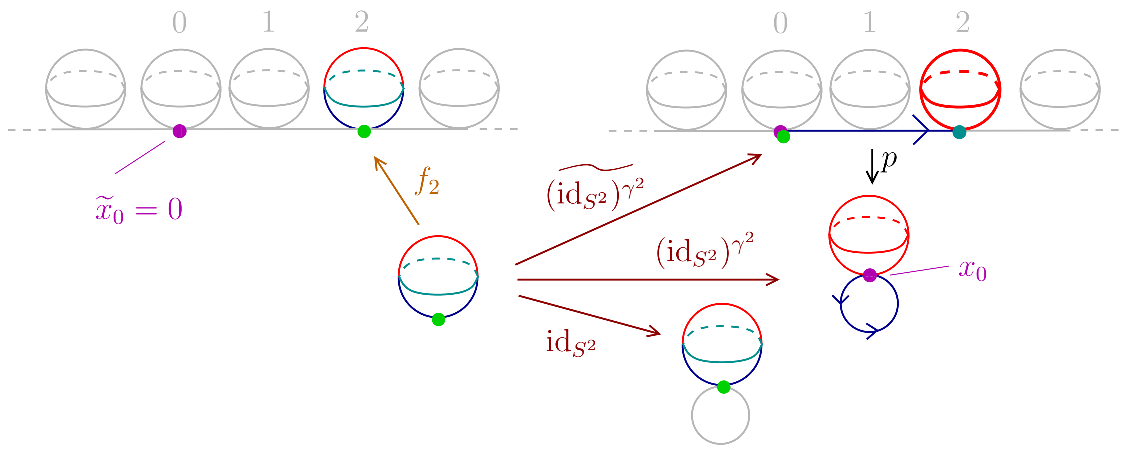


FIGURE 840. Second illustration for the proof of Proposition 54.15.

In Proposition 48.5 we saw that the homology groups of a finite CW-complex are finitely generated abelian groups. Since the discussion on page 1326 we had known that the corresponding statement is not true for homotopy groups. But we now also know that one should really view the homotopy groups of a pointed topological space (X, x_0) as modules over the group ring $\mathbb{Z}[\pi_1(X, x_0)]$. Thus the following question arises.

Question 54.16. *Let X be a finite* CW*-complex. Are the homotopy groups $\pi_n(X, x_0)$, $n \geq 2$, finitely generated $\mathbb{Z}[\pi_1(X, x_0)]$-modules?*

It turns out that the answer is yes if X is simply connected. More precisely, Jean-Pierre Serre **[Ser53]**[850] proved the following theorem.

Theorem 54.17. (Serre) *If X is a* simply-connected *topological space such that all homology groups are finitely generated (for example X could be a compact topological manifold or a finite* CW*-complex), then all homotopy groups are also finitely generated abelian groups.*[851]

[849] In fact this follows from the "⇐"-direction of Proposition 54.9 (7).

[850] Jean-Pierre Serre (*1926) is a French mathematician. He was, and remains, the youngest person ever to be awarded the Fields Medal.

[851] It follows immediately from the Hurewicz Theorem 53.5 that $\pi_2(X)$ is finitely generated. The fact that the higher homotopy groups are also finitely generated is much stronger.

Example. It follows in particular from Serre's Theorem 54.17 that the homotopy groups of spheres are finitely generated. Even this statement is highly non-trivial. In particular it is much better than our only result in this direction, namely Corollary 62.10. which says that homotopy groups of spheres are countable.

Nonetheless, it turns out that the answer to Question 54.16 is negative. In fact, one does not need to wander off particularly far to find such an example:

Proposition 54.18. *The group $\pi_3(S^1 \vee S^2)$ is not finitely generated over the group ring $\mathbb{Z}[\pi_1(S^1 \vee S^2)] = \mathbb{Z}[t^{\pm 1}]$.*

A proof for the proposition is for example sketched in [**Hat02**, Chapter 4.2, Exercise 38].

Remark. The Hilton-Milnor theorem, see [**Hilt55**, Theorem A],[**Miln72**] or [**WhdG78**, Chapter XI], gives a calculation of the homotopy groups of a finite wedge of spheres, more precisely, these homotopy groups are computed in terms of the homotopy groups of the spheres involved. Unfortunately, at least at a first glance, it is not so clear how the module structure over the group ring can be read off from the Hilton-Milnor theorem.

Exercises for Chapter 54.

Exercise 54.1. Let X be a topological space that admits a social choice of some type $n \geq 2$. Is X necessarily connected?

Exercise 54.2. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_{\geq 2}$. We consider the Hurewicz homomorphism $\Phi\colon \pi_n(X, x_0) \to \mathrm{H}_n(X; \mathbb{Z})$. Show that for any $g \in \pi_1(X, x_0)$ and any $\sigma \in \pi_n(X, x_0)$ we have

$$\Phi(g \cdot \sigma) \;=\; \Phi(\sigma) \in \mathrm{H}_n(X; \mathbb{Z}).$$

Exercise 54.3. Let G be a group and let R be a commutative ring. Suppose there exists a non-trivial $g \in G$ such that $g^k = e$ for some $k \in \mathbb{N}$. Show that $R[G]$ has a zero divisor, i.e. show that there exist non-zero elements $p, q \in R[G]$ with $p \cdot q = 0$.
Hint. First consider the case that $G = \langle t \,|\, t^n \rangle$ is a finite cyclic group with $n \geq 2$ elements.

Exercise 54.4. Is the ring $\mathrm{M}(3 \times 3, \mathbb{R})$ of real (3×3)-matrices isomorphic, as a ring, to the real group ring $\mathbb{R}[G]$ of some group G?

Exercise 54.5. Let G be a finite group. We view $\mathbb{Q}[G]$ as a $\mathbb{Q}[G]$-left module via left multiplication. Show that $\mathbb{Q}[G]$ admits a submodule $V \neq 0$ with $V \neq \mathbb{Q}[G]$ that is isomorphic to the trivial $\mathbb{Q}[G]$-module $\mathbb{Q}$ where every $g \in G$ acts via the identity on $\mathbb{Q}$.

Exercise 54.6. Let G be a finite group. Show that there exists a monomorphism from G to some unitary group $\mathrm{U}(n)$.

Exercise 54.7. We consider the ring $\mathrm{M}(2, \mathbb{R})$. Does there exist a finite set T and an epimorphism from the free ring $\mathbb{R}[\langle T \rangle]$ onto $\mathrm{M}(2, \mathbb{R})$?

Exercise 54.8. Let G be a group that acts on an abelian group A. Show that the set

$$\{g \cdot a - a \mid g \in G \text{ and } a \in A\} \ \subset \ A$$

is in fact a subgroup of A.

Exercise 54.9. Let X be a simply connected topological space and let $f \colon X \to X$ be a self-homeomorphism. Recall that on page 711 we introduced the mapping torus

$$M \ := \ (X \times [0,1])/(x,0) \sim (f(x),1).$$

Express the $\mathbb{Z}[\pi_1(M)]$-module $\pi_2(M)$ in terms of the homology groups of X and the action by f on the homology groups of X.

Exercise 54.10. Show that there exists a closed smooth manifold M such that $\pi_2(M)$ is infinitely generated as an abelian group.

55. The Euler characteristic

55.1. **The Euler characteristic and homology groups.** Let X be a finite CW-complex, i.e. let X be a CW-complex with finitely many cells. Recall that on page 983 we defined the Euler characteristic $\chi(X)$ as follows:

$$\chi(X) \;=\; \sum_n (-1)^n \cdot \text{number of } n\text{-cells}.$$

A short glance on the examples in Figure 599 shows that the three different CW-structures for the torus provided in that figure have the same Euler characteristic. The following proposition explains why this is not a coincidence.

Proposition 55.1. *Let X be a finite* CW*-complex, then*[852]

$$\chi(X) \;=\; \sum_n (-1)^n \cdot \operatorname{rank}(\mathrm{H}_n(X)).$$

Examples.

(1) It follows either from the CW-structure of the n-dimensional sphere S^n with one 0-cell and one n-cell given on page 935 or from the calculation of the homology groups in Proposition 43.4 that[853]

$$\chi(S^n) \;=\; 1+(-1)^n \;=\; \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

(2) Let $n \in \mathbb{N}_0$. We want to determine the Euler characteristic of $\overline{B}^n$. One could of course write down a CW-structure for $\overline{B}^n$.[854] But with our knowledge it is quicker to point out that follows immediately from Proposition 55.1 and Corollary 42.8 (3) that $\chi(\overline{B}^n) = 1$.
(3) Given any $k, n \in \mathbb{N}_0$ we deduce from Corollary 42.8, Proposition 43.4 and Proposition 55.1 that $\chi(S^k \times \overline{B}^n) = \chi(S^k) = 1 + (-1)^k$.
(4) It follows either from the obvious generalizations of CW-structure provided in Figure 841 or alternatively from Proposition 48.9 that for the surface Σ_g of genus g and that for the non-orientable surface N_k of genus k we have

$$\chi(\Sigma_g) \;=\; 2-2g \qquad \text{and} \qquad \chi(N_k) \;=\; 2-k.$$

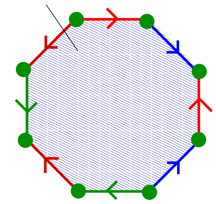
---

[852]As a reminder, if G is a finitely generated abelian group, then it follows from the classification of finitely generated abelian groups, see Theorem 19.4, that G is isomorphic to $\mathbb{Z}^r \oplus T$, where T is finite. We then call r the *rank* of G. Put differently, let $T \subset G$ be the torsion subgroup, then G/T is isomorphic to $\mathbb{Z}^r$ for an r, and this r is just the rank of G.

[853]The statement that $\chi(S^2) = 2$ was proved, in a different language, by Leonhard Euler [**Eul1758**, p. 119]. This equality was already noted before by Francesco Maurolico [**Friedm18**, p. 71] in 1537 and René Descartes [**Rabo10**, p. 447] in 1628.

[854]What is the "simplest" CW-structure you can find for $\overline{B}^n$?

surface Σ_g of genus $g=2$



1	number of 0-cells	1
$2\cdot g$	number of 1-cells	k
1	number of 2-cells	1
$2-2\cdot g$	Euler characteristic	$2-k$

non-orientable surface N_k of genus $k=3$

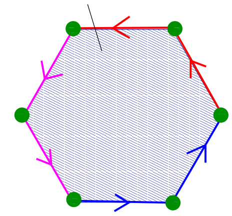


FIGURE 841

(5) It follows immediately from the CW-structures provided in Lemma 36.1 that for any $n \in \mathbb{N}_0$ we have

$$\chi(\mathbb{R}\mathrm{P}^n) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even} \end{cases} \qquad \text{and} \qquad \chi(\mathbb{C}\mathrm{P}^n) = n+1.$$

Before we turn to the proof of Proposition 55.1 we draw the following corollary.

Corollary 55.2. *Let X and Y be finite* CW*-complexes. If X and Y are homotopy equivalent, then $\chi(X) = \chi(Y)$.*

PROOF. The corollary is an immediate consequence of Proposition 55.1 together with Corollary 42.8 (2). ■

Example. On page 672 we introduced two examples of closed orientable 2-dimensional smooth manifolds for which we found it difficult to determine the genus. We show the same surfaces also in Figure 842. Let us first consider the surface on the left: it is given identifying opposite sides of the hexagon E_6. The obvious CW-structures on the hexagon H_6 descends to a CW-structure on $E_6/\sim$. We make the following observations:

(1) the single 2-cell on E_6 defines a 2-cell for $E_6/\sim$,
(2) we identify every 1-cell of E_6 with the opposite 1-cell, thus $E_6/\sim$ has three 1-cells,
(3) we identify all 0-cells that differ by an even number of vertices, thus $E_6/\sim$ has precisely two 0-cells.

Summarizing we see that $E_6/\sim$ has Euler characteristic $2-3+1=0$. We had just seen on page 1363 that the surface of genus g has Euler characteristic $2-2g$. Thus we obtain from the Surface Classification Theorem 23.4 that $E_6/\sim$ is diffeomorphic to the surface of genus zero, i.e. the torus.

The same argument shows that $E_{10}/\sim$ has Euler characteristic $2-5+1=-2$ and that it is diffeomorphic to the surface of genus two. Evidently the same argument shows that if we take a regular $(4g+2)$-gon and if we identify opposite sides, then we obtain the surface of genus g.

It is a great mental exercise to visualize that the surface on the left is a torus and that the surface on the right is a surface of genus two.

The key step in the proof of Proposition 55.1 is the following purely algebraic lemma.

hexagon with opposite sides identified dodecagon with opposite sides identified

in each case there are two equivalence classes of vertices: red and blue

FIGURE 842

Lemma 55.3. *Let*

$$C_* := 0 \to \mathrm{C}_k \xrightarrow{\partial_k} \mathrm{C}_{k-1} \xrightarrow{\partial_{k-1}} \ldots \mathrm{C}_1 \xrightarrow{\partial_1} \mathrm{C}_0 \to 0$$

be a chain complex of finitely generated abelian groups. Then the following equality holds:

$$\sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank} \mathrm{C}_n = \sum_{n=0}^{k} (-1)^n \operatorname{rank} \cdot \mathrm{H}_n(C_*).$$

The same statement holds if the C_i *are vector spaces over a field and if we replace ranks by dimensions.*

PROOF. For the above chain complex (C_n, ∂_n) we write for each $n \in \mathbb{N}_0$[855]

$$Z_n := \ker(\partial_n), \quad B_n := \operatorname{im}(\partial_{n+1}) \quad \text{and} \quad H_n = Z_n/B_n.$$

These groups form the following two types of short exact sequences

$$\text{(a)} \qquad 0 \hookrightarrow Z_n \to \mathrm{C}_n \xrightarrow{\partial_n} B_{n-1} \to 0,$$

and

$$\text{(b)} \qquad 0 \hookrightarrow B_n \to Z_n \to H_n \to 0.$$

[855]It is a common notation to call these groups Z_n and B_n, the names go back to the German words "Zykel" and "Bild".

Now we see that

$$\begin{aligned}
\sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank} C_n &\overset{\substack{\text{by (a) and Lemma 19.6 (2) we know that } \operatorname{rank} C_n=\operatorname{rank} Z_n+\operatorname{rank} B_{n-1}\\ \downarrow}}{=} \sum_{n=0}^{k}(-1)^n\cdot(\operatorname{rank} Z_n+\operatorname{rank} B_{n-1})\\
&\underset{\substack{\uparrow\\ \text{by (b) and Lemma 19.6 (2) we know that } \operatorname{rank} Z_n=\operatorname{rank} B_n+\operatorname{rank} H_n}}{=} \sum_{n=0}^{k}(-1)^n\cdot(\operatorname{rank} B_n+\operatorname{rank} H_n+\operatorname{rank} B_{n-1})\\
&= \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank} H_n + \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank} B_n+\sum_{n=0}^{k}(-1)^n\operatorname{rank} B_{n-1}\\
&\underset{\substack{\uparrow\\ \text{we do the substitution } m=n-1 \text{ and we use that } B_k=0 \text{ and } B_{-1}=0}}{=} \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank} H_n + \sum_{n=0}^{k-1}(-1)^n\cdot\operatorname{rank} B_n-\sum_{m=0}^{k-1}(-1)^m\cdot\operatorname{rank} B_m\\
&= \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank} H_n.
\end{aligned}$$

Basically the same proof also applies if the C_i are vector spaces over a field and if we replace ranks by dimensions. ■

Now we can easily provide the proof of Proposition 55.1.

Proof of Proposition 55.1. Let X be a finite k-dimensional CW-complex. We have

$$\begin{aligned}
\chi(X) &= \sum_{n=0}^{k}(-1)^n\cdot\text{number of } n\text{-cells} &&\overset{\substack{\text{Lemma 48.1}\\ \downarrow}}{=} \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank}(C_n^{\mathrm{CW}}(X))\\
&\underset{\substack{\uparrow\\ \text{Lemma 55.3}}}{=} \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank}(H_n^{\mathrm{CW}}(X)) &&\underset{\substack{\uparrow\\ \text{Proposition 48.4}}}{=} \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank}(H_n(X)).
\end{aligned}$$

■

Using Lemma 55.3 we can also easily prove the following lemma which be will useful on several occasions.

Lemma 55.4. *Let*

$$0 \to A_k \to A_{k-1} \to \dots \to A_1 \to A_0 \to 0$$

be an exact sequence of finitely generated abelian groups. Then

$$\sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank} A_n = 0.$$

Proof. We can view this exact sequence of finitely generated abelian groups as a chain complex whose homology groups vanish. The lemma is therefore an immediate consequence of Lemma 55.3. ■

Definition.

(1) Given a topological space X and $n\in\mathbb{N}_0$ such that $H_n(X)$ is a finitely generated abelian group we refer to

$$b_n(X) := \operatorname{rank}(H_n(X))$$

as the *n-th Betti number of X*.[856]

(2) Let X be a topological space. If all homology groups are finitely generated and if $\sum_n b_n(X)$ is finite, then we define the *Euler characteristic of X* to be

$$\chi(X) \;:=\; \sum_n (-1)^n \cdot b_n(X).$$

It follows from Proposition 55.1 that this new definition of the Euler characteristic of a finite CW-complex agrees with the earlier definition.

Example. Using Proposition 46.21 and using Lemma 55.4 several times we will prove the following statement in Exercise 55.3. Let X be a topological space that is homotopy equivalent to a finite CW-complex. Then for any self-homeomorphism f of X we have

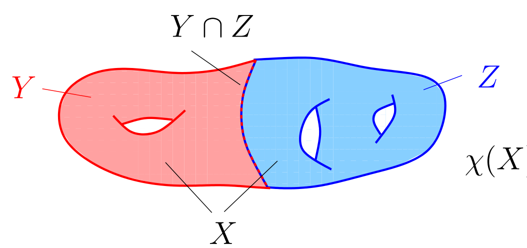
$$\chi(\mathrm{Tor}(X,f)) \;=\; 0.$$

55.2. **Properties of the Euler characteristic.** The Euler characteristic is an invariant which "sees less" than the homology groups. On the other hand we will now see that it behaves better than the homology groups under several natural operations.

The following lemma makes it possible to determine the Euler characteristic of a CW-complex from suitable decompositions.

Lemma 55.5. *Let $X = Y \cup Z$ be a decomposition of a finite* CW*-complex X into two subcomplexes Y and Z. Then the following equality holds*

$$\chi(X) \;=\; \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$



$$\chi(X) = \chi(Y \cup Z) = \chi(Y) + \chi(Z) - \chi(Y \cap Z)$$

FIGURE 843

[856]These numbers are named after the Italian mathematician Enrico Betti (1823-1892)

http://de.wikipedia.org/wiki/Enrico_Betti

who had studied an early version of homology groups.

PROOF. We give two proofs for the lemma:

(1) Given $n \in \mathbb{N}_0$ and given a subset W we denote by $n(W)$ the number of open n-cells that are contained in W. In our case we have
$$n(X) \underset{\substack{\uparrow\\ \text{since every open } n\text{-cell of } X \text{ lies in } Y \text{ or } Z, \text{ the } n\text{-cells that lie in } Y\cap Z \text{ get}\\ \text{counted twice in } n(Y)+n(Z), \text{ so we have to subtract } n(Y\cap Z)}}{=} n(Y)+n(Z)-n(Y\cap Z).$$
By performing the alternating sum over all n we obtain immediately that
$$\chi(X) = \chi(Y)+\chi(Z)-\chi(Y\cap Z).$$
We have thus proved the desired equality.

(2) The lemma also follows from applying Lemma 55.4 and Proposition 55.1 to the long exact sequence of homology groups that we obtain by applying the Mayer–Vietoris Theorem for CW-complexes 46.11 to the decomposition $X = Y \cup Z$. ■

In the following we provide two examples to which we already know the answer by other means. Nonetheless, the approach using Lemma 55.5 gives particularly short and pleasing arguments.

Example. Let $K \subset S^3$ be a knot. By the Tubular Neighborhood Theorem 8.24 there exists a tubular map $\Phi\colon \overline{B}^2 \times K$. We set $X_K := S^3 \setminus \Phi(B^2 \times K)$. By Theorem 64.5 there exists a CW-structure for S^3 such that X_K and $\Phi(\overline{B}^2 \times S^1)$ are subcomplexes. We see that
$$0 \underset{\substack{\uparrow\\ \text{see page 1363}}}{=} \chi(S^3) = \chi\big(X_K \cup \Phi(\overline{B}^2 \times S^1)\big) \underset{\substack{\uparrow\\ \text{Lemma 55.5}}}{=} \chi(X_K) + \chi\big(\overline{B}^2 \times S^1\big) - \chi\big(\partial\overline{B}^2 \times S^1\big) \underset{\substack{\uparrow\\ \text{see page 1363}}}{=} \chi(X_K).$$
Of course this calculation also follows from Lemma 68.16.

Example. As usual, given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ we denote by $\Sigma_{g,n}$ the surface Σ_g of genus g minus n open disks. Now we want to determine the Euler characteristic $\chi(\Sigma_{g,n})$. We could do so by using Propositions 55.1 and 48.9. But here we use an alternative approach. By Theorem 64.5 there exists a CW-structure on Σ_g such that $\Sigma_{g,n}$ and the closed disks that we remove are subcomplexes. It now follows that
$$2-2g \underset{\substack{\uparrow\\ \text{see page 1363}}}{=} \chi(\Sigma_g) = \chi\Big(\Sigma_{g,n} \cup \bigcup_{i=1}^{n} \overline{B}^2\Big) \underset{\substack{\uparrow\\ \text{Lemma 55.5}}}{=} \chi(\Sigma_{g,n}) + \sum_{i=1}^{n} \chi(\overline{B}^2) - \sum_{i=1}^{n} \chi(\partial\overline{B}^2) \underset{\substack{\uparrow\\ \text{see page 1363}}}{=} \chi(\Sigma_{g,n}) + n.$$
Thus we see that $\chi(\Sigma_{g,n}) = 2-2g-n$.

We record the above example in a lemma.

Lemma 55.6.

(1) *Let $g \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$. If we denote by $\Sigma_{g,n}$ the surface Σ_g of genus g minus n open disks, then $\chi(\Sigma_{g,n}) = 2-2g-n$.*

(2) *Let Σ and Σ' be two compact orientable connected 2-dimensional smooth manifolds. If Σ and Σ' have the same number of boundary components and if $\chi(\Sigma) = \chi(\Sigma')$, then Σ and Σ' are diffeomorphic.*

PROOF.

(1) We proved this statement in the example preceding the lemma.
(2) This statement follows immediately from (1) and the classification of compact orientable connected 2-dimensional smooth manifolds, see the Surface Classification Theorem 23.4. ■

We continue with a calculation of the Euler characteristic of a connected sum.

Lemma 55.7. *If M and N are two compact connected n-dimensional smooth manifolds. We have the following equality:*[857]

$$\chi(M\#N) \;=\; \chi(M)+\chi(N)+\begin{cases} -2, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.}\end{cases}$$

PROOF (∗). We perform the following straightforward calculation:

$$\chi(M\setminus B^n) \underset{\substack{\uparrow \\ \text{Lemma 55.5}}}{=} \chi(M)-\chi(\overline{B}^n)+\chi(S^{n-1}) \underset{\substack{\uparrow \\ \text{since } \chi(\overline{B}^n)=1 \text{ and } \chi(S^{n-1})=1+(-1)^{n-1}}}{=} \chi(M)+(-1)^{n-1}.$$

The same way we also see that $\chi(N\setminus B^n)=\chi(N)+(-1)^{n-1}$. It then follows, using the same type of arguments, that

$$\begin{aligned} \chi(M\#N) &= \chi(M\setminus B^n)+\chi(N\setminus B^n)-\chi(S^{n-1}) \\ &= \chi(M)+(-1)^{n-1}+\chi(N)+(-1)^{n-1}-(1+(-1)^{n-1}) \\ &= \chi(M)+\chi(N)-1+(-1)^{n-1}. \end{aligned}$$

For completeness sake we point out that in Proposition 68.13 we will calculate the homology groups of the connected sum of two closed oriented smooth manifolds. Together Proposition 55.1 this gives an alternative proof of the above equality. ■

Examples.

(1) In Figure 220 we saw that the connected sum of two tori is the surface of genus two. By Lemma 55.7 we obtain that[858]

$$\chi(\text{surface of genus two}) \;=\; \underbrace{\chi(\text{torus})}_{=0}+\underbrace{\chi(\text{torus})}_{=0}-2 \;=\; -2.$$

We had obtained the same result on page 1363 using a different method.

(2) Note that from the CW-structure for $\mathbb{R}\mathrm{P}^2$ that we gave in Figure 600 we read off that $\chi(\mathbb{R}\mathrm{P}^2)=1$. It follows from Lemma 55.7 that for any $k\in\mathbb{N}$ we have

$$\chi(k\cdot\mathbb{R}\mathrm{P}^2) \;=\; 1-k.$$

Using the same argument as in Lemma 55.6 one can also show that

$$\chi(k\cdot\mathbb{R}\mathrm{P}^2 \text{ minus } n \text{ open disks}) \;=\; 1-k-n.$$

[857]Strictly speaking, if M and N are both orientable, then we need to assume that M and N are oriented to get a well-defined connected sum.

[858]How many arguments can you find in the lecture notes that show that the Euler characteristic of the torus is zero?

In the following we work our way through more constructions of CW-complexes and the effect on Euler characteristics. First we remind the reader of the following lemma.

Lemma 37.5. *Let X be a finite* CW*-complex and let A be a subcomplex. We equip the quotient X/A with the* CW*-structure given by Lemma 36.32 (3). Then*

$$\chi(X/A) \;=\; \chi(X)-\chi(A)+1.$$

Next let X and Y be two finite CW-complexes. On page 961 we introduced the product CW-structure which by Proposition 36.23 is indeed a CW-structure for the topological space $X\times Y$. The following lemma relates the Euler characteristics of X, Y and $X\times Y$ in an appealing way.[859]

Lemma 55.8. *Let X and Y be two finite* CW*-complexes, then*

$$\chi(X\times Y) \;=\; \chi(X)\cdot\chi(Y).$$

Example. For any finite CW-complex X we have $\chi(S^1\times X)=\chi(S^1)\cdot\chi(X)=0$.

Proof. Let X and Y be two finite CW-complexes. Given $k\in\mathbb{N}_0$ we denote by c_k the number of k-cells of X and we denote by d_k the number of k-cells of Y.

We equip $X\times Y$ with the product CW-structure from page 961. By construction we have

$$\#n\text{-cells of } X\times Y \;=\; \sum_{k+l=n}\#k\text{-cells of } X\cdot\#l\text{-cells of } Y \;=\; \sum_{k+l=n}c_k\cdot d_l.$$

It follows that

$$\chi(X)\cdot\chi(Y) \;=\; \Big(\sum_k(-1)^k\cdot c_k\Big)\cdot\Big(\sum_l(-1)^l\cdot d_l\Big) \;=\; \sum_n(-1)^n\sum_{k+l=n}c_k\cdot d_l \;=\; \chi(X\times Y). \qquad\blacksquare$$

Now let $p\colon\widetilde{X}\to X$ be a k-fold covering of path-connected topological spaces. We saw in Corollary 16.14 that the induced map $p_*\colon\pi_1(\widetilde{X})\to\pi_1(X)$ is a monomorphism and we saw in Lemma 16.15 (3) that $p_*(\pi_1(\widetilde{X}))$ is a subgroup of index k of $\pi_1(X)$. There is no analogous statement for homology groups. For example the projection map $p\colon S^2\to S^2/\sim\,=\mathbb{RP}^2$ is a 2-fold covering, but $\mathrm{H}_2(S^2)=\mathbb{Z}$ and $\mathrm{H}_2(\mathbb{RP}^2)=0$, which shows that the induced map on homology is in general not a monomorphism.

It is therefore perhaps surprising that we already saw in Proposition 37.4 that the Euler characteristic behaves well under finite coverings. For convenience we recall, a slightly simplified version, of the statement of Proposition 37.4.

Proposition 37.4. *Let $p\colon\widetilde{Y}\to Y$ be a finite covering of a finite connected* CW*-complex Y. Then $\widetilde{Y}$ is also a finite* CW*-complex with*

$$\chi(\widetilde{Y}) \;=\; [\widetilde{Y}:Y]\cdot\chi(Y).$$

[859]Note that the lemma also gives an affirmative answer to the question posed in Exercise 37.2.

Example. We consider again the 2-fold covering $p\colon S^2 \to \mathbb{RP}^2$. In this case we have by the calculation on page 1216 or alternatively in Proposition 48.10 that

$$\begin{array}{rcl} \mathrm{H}_0(S^2) & \cong & \mathbb{Z}, \\ \mathrm{H}_1(S^2) & \cong & 0, \\ \mathrm{H}_2(S^2) & \cong & \mathbb{Z}, \\ \mathrm{H}_i(S^2) & \cong & 0, \text{ for } i \geq 3, \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{H}_0(\mathbb{RP}^2) & \cong & \mathbb{Z}, \\ \mathrm{H}_1(\mathbb{RP}^2) & \cong & \mathbb{Z}_2, \\ \mathrm{H}_2(\mathbb{RP}^2) & \cong & 0, \\ \mathrm{H}_i(\mathbb{RP}^2) & \cong & 0, \text{ for } i \geq 3. \end{array}$$

Thus we see that the homology groups are quite different. Nonetheless we observe that $\chi(S^2) = 2$ is indeed twice $\chi(\mathbb{RP}^2) = 1$, as we had determined on page 1364.

In Question 30.7 we asked how the genus of a surface changes if we take finite covers. We had answered this question already in Proposition 31.6 (1). The proof of the following lemma is arguably more conceptual and easier to remember.

Lemma 55.9. *Let Σ be the surface of genus g and let $p\colon \widetilde{\Sigma} \to \Sigma$ be a k-fold connected covering. Then*[860]

$$\mathit{genus}(\widetilde{\Sigma}) \;=\; k \cdot (g-1) + 1.$$

Example. In Figure 549 we illustrate a 3-fold covering of the surface of genus 4 over the surface of genus 2.

Remark. We had given an alternative proof of Lemma 55.9 in Exercise 31.6. The new proof using Euler characteristics is surely much more instructive than our previous proof.

Proof. We denote by $\widetilde{g}$ the genus of $\widetilde{\Sigma}$. The lemma follows immediately from the following two facts:

(1) By the remark on page 1363 we have $\chi(\Sigma) = 2 - 2g$ and $\chi(\widetilde{\Sigma}) = 2 - 2\widetilde{g}$.
(2) By Proposition 37.4 we have $\chi(\widetilde{\Sigma}) = k \cdot \chi(\Sigma)$. ■

Now we can also give a new proof of the following proposition.

Proposition 31.17. *If $g \geq 1$, then the group*

$$\pi_1(\mathit{surface\ of\ genus\ } g) \;\cong\; \langle x_1, y_1, \dots, x_g, y_g \,|\, [x_1, y_1] \cdot \dots \cdot [x_g, y_g] \rangle$$

is not isomorphic to the free group F_{2g} on $2g$ generators.

Proof. Let Σ be a surface of genus $g \geq 1$. We write $\pi = \pi_1(\Sigma)$. Suppose that there exists an isomorphism $\varphi\colon \pi \to F$ where F is the free group on $2g$ generators. Let $\alpha\colon \pi \to \mathbb{Z}_k$ be an epimorphism for some $k \geq 2$.[861] We denote by $p\colon \widetilde{\Sigma} \to \Sigma$ the covering which satisfies $p_*(\pi_1(\widetilde{\Sigma})) = \ker(\alpha)$ which is given by Proposition 29.5. By Lemma 16.15 (3) this is a k-fold

[860] Recall that by the Surface Classification Theorem 23.4 there exists precisely one $\widetilde{g}$ such that the closed, oriented 2-dimensional smooth manifold $\widetilde{\Sigma}$ is diffeomorphic to precisely the surface of genus $\widetilde{g}$.

[861] Why does such an epimorphism exist?

covering. We get the following commutative diagram:

$$\begin{tikzcd}
\pi \arrow[d, "\varphi"', "\cong"] & \ker(\alpha) \arrow[l] \arrow[d, "\varphi"', "\cong"] \arrow[rr, "\cong", "\text{Lemma 55.9}"'] & & \pi_1(\Sigma_{k\cdot(g-1)+1}) \arrow[d, "\cong"] \arrow[rr, "\text{abelianization}"] \arrow[dr] & & \mathbb{Z}^{2k(g-1)+2} \arrow[d, "\cong"] \\
F & \ker(\alpha\circ\varphi^{-1}) \arrow[l] \arrow[rr, "\cong", "\text{Proposition 31.4}"'] & & F_{k(2g-1)+1} \arrow[rr, "\text{abelianization}"] & \mathrm{H}_1(\Sigma_{k\cdot(g-1)+1}) \arrow[ur, "\cong"'] & \mathbb{Z}^{k(2g-1)+1}
\end{tikzcd}$$

It follows from $k \geq 2$ that $2k(g-1)+2 \neq k(2g-1)+1$, hence the two groups on the right cannot by isomorphic. We have thus obtained a contradiction. ■

55.3. **Groups acting on spheres and surfaces.** We recall that on page 185 we said that a group G acts *freely* on a topological space X if for every $g \neq e$ and every $x \in X$ we have $g \cdot x \neq x$. For example for each $k \in \mathbb{N}$ the action

$$\begin{array}{rcl} \mathbb{Z}_k \times S^1 & \to & S^1 \\ (l,z) & \mapsto & e^{2\pi \mathrm{i} l/k} \cdot z \end{array}$$

is free. We can easily generalize this example. More precisely, let S^{2n-1} be an odd-dimensional sphere. We view S^{2n-1} as subset of $\mathbb{R}^{2n} = \mathbb{C}^n$. Then it is straightforward to show that

$$\begin{array}{rcl} \mathbb{Z}_k \times S^{2n-1} & \to & S^{2n-1} \\ (l,(z_1,\dots,z_n)) & \mapsto & (e^{2\pi \mathrm{i} l/k} z_1,\dots,e^{2\pi \mathrm{i} l/k} z_n) \end{array}$$

is a free and continuous action of $\mathbb{Z}_k$ on S^{2n-1}.

It is less clear what finite groups can act freely and continuously on even-dimensional spheres. The group $\mathbb{Z}_2 \cong \{\pm 1\}$ acts of course freely and continuously on any sphere. The following proposition says that $\mathbb{Z}_2$ is in fact the only finite non-trivial group that can act freely and continuously on an even-dimensional sphere. This proposition and the above example give a complete answer to Question 16.8.

Proposition 55.10. *If a finite group G acts freely and continuously on S^{2n}, then G is either trivial or $G \cong \mathbb{Z}_2$.*

Proof. Let G be a finite group that acts freely and continuously on S^{2n}. By Lemma 16.5 the action is also discrete. By Proposition 16.9 the projection map $S^{2n} \to S^{2n}/G$ is a covering of degree $|G|$. We obtain the equalities

$$2 \underset{\substack{\uparrow \\ \text{see page 1110}}}{=} \chi(S^{2n}) \underset{\substack{\uparrow \\ \text{Proposition 37.4}}}{=} |G| \cdot \chi(S^{2n}/G).$$

If follows that $|G|$ divides two, i.e. $|G| = 1$ or $|G| = 2$. In first case the group G is the trivial group and in the second case we have $G \cong \mathbb{Z}_2$.[862] ■

We already gave a different proof of Proposition 55.10 in Exercise 45.21. But whereas the approach of Exercise 45.21 made very much use of the topology of spheres, the approach taken in the above proof generalizes to many other settings:

[862]Why is every group with two elements isomorphic to $\mathbb{Z}_2$?

Proposition 55.11. *Let $g \in \mathbb{N}_0$. If a finite group G acts freely and continuously on the surface Σ of genus g, then the order of G divides $\chi(G) = 2 - 2g$.*

PROOF. The proof of Proposition 55.11 is verbatim the same as the proof of Proposition 55.10. The only change is that we need to use the fact, shown on page 1363, that $\chi(\Sigma) = 2 - 2g$. ■

Example. Let Σ be the surface of genus four. By the remark on page 1363 we know that $\chi(\Sigma) = -6$. By Proposition 55.10 we know that the order of any finite group acting freely and continuously on Σ has to have an order that divides 6. In Figure 844 we sketch a free and continuous action of the group $\mathbb{Z}_3$ on Σ. In Exercise 55.10 we will see that given any n with $n|\chi(\Sigma)$ the cyclic group $\mathbb{Z}_n$ admits a free and continuous action on Σ.

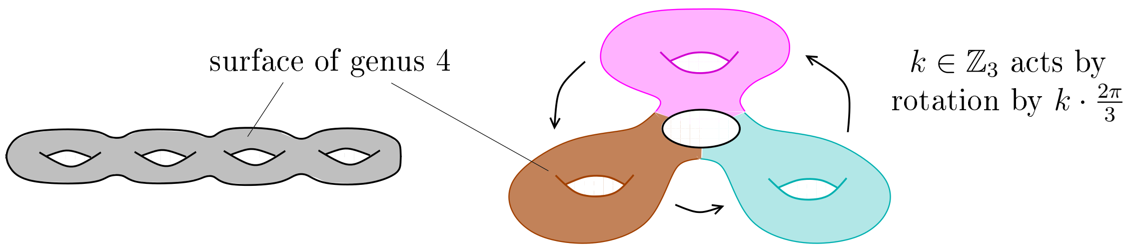


FIGURE 844

55.4. **Topological graphs** (∗). Using the results of this chapter we can now easily prove the following lemma:

Lemma 55.12. *Let G be a connected topological graph with v vertices and e edges. Then*

$$\mathrm{H}_0(G) \cong \mathbb{Z}, \quad \mathrm{H}_1(G) \cong \mathbb{Z}^{e-v+1} \quad \textit{and} \quad \mathrm{H}_i(G) = 0 \textit{ for } i \geq 2.$$

Example. In Figure 845 we see a connected topological graph G with 22 vertices and 24 edges. It follows from Lemma 55.12 that its first homology group is isomorphic to $\mathbb{Z}^3$. In Figure 845 we sketched three cycles which represent a generating set for $\mathrm{H}_1(G) \cong \mathbb{Z}^3$.

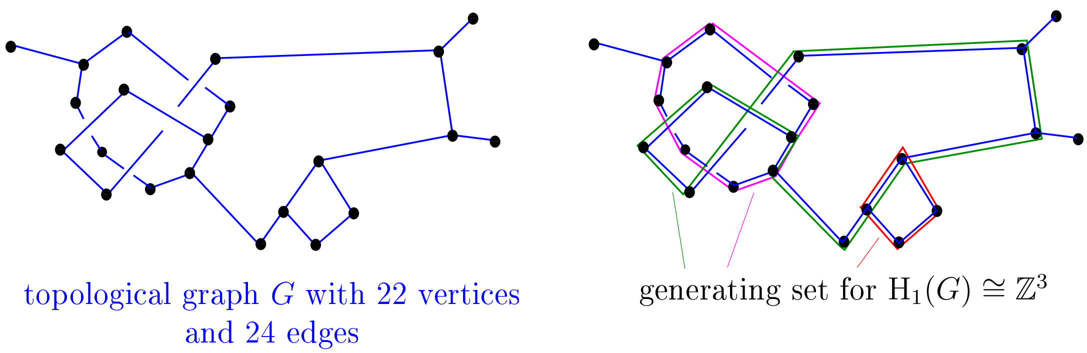


FIGURE 845

PROOF. Let G be a connected topological graph with v vertices and e edges. Recall that by the discussion on page 932 we can view G as a 1-dimensional CW-complex with v 0-cells and e 1-cells.

Now that we have developed so many techniques for computing homology groups it is perhaps not surprising that we can provide three different proofs for the lemma.

(1) First of all, from Proposition 41.5 together with Proposition 36.10 (7) it follows immediately that $\mathrm{H}_0(G) \cong \mathbb{Z}$. Furthermore, from Proposition 48.5 it follows that $\mathrm{H}_i(G) = 0$ for $i \geq 2$ and that $\mathrm{H}_1(G)$ is torsion-free.

   From Proposition 55.1 we obtain that
$$\chi(X) \;=\; \sum_i (-1)^i \cdot \operatorname{rank}(\mathrm{H}_i(X)).$$
   Putting everything together we see that
$$1 - \operatorname{rank}(\mathrm{H}_1(G)) \;=\; \chi(X) \;=\; \#0\text{-cells} - \#1\text{-cells} \;=\; v - e.$$
   Since $\mathrm{H}_1(G)$ is torsion-free we obtain that $\mathrm{H}_1(G) \cong \mathbb{Z}^{e-k+1}$.

(2) We can also modify the argument of (1) as follows: instead of calculating $\mathrm{H}_1(G)$ using the Euler characteristic, we could also have determined that $\mathrm{H}_1(G) \cong \mathbb{Z}^{e-k+1}$ by combining the calculation of the fundamental group in Proposition 20.5 with the isomorphism of the Hurewicz Theorem 52.5.

(3) An alternative proof for Lemma 55.12 is given as follows: we have
$$\underbrace{\widetilde{\mathrm{H}}_j(G) \;\underset{\uparrow}{\cong}\; \widetilde{\mathrm{H}}_j(\text{wedge of } 1-\chi(G) \text{ circles})}_{\text{Proposition 37.8 (1) and Corollary 42.8}} \;\underset{\substack{\uparrow\\ \text{Proposition 47.8}}}{\cong}\; \bigoplus_{i=1}^{1-\chi(G)} \widetilde{\mathrm{H}}_j(S^1) \;\underset{\substack{\uparrow\\ \text{Proposition 43.4}}}{\cong}\; \begin{cases} \mathbb{Z}^{1-\chi(G)}, & \text{if } j=1,\\ 0, & \text{else.}\end{cases}$$

It is a matter of taste which of the three proofs is the nicest. ■

Exercises for Chapter 55.

Exercise 55.1. Show that every CW-structure for the torus with one 0-cell has precisely one 2-cell.

Exercise 55.2.

(a) Let $k \in \mathbb{N}$. By Proposition 17.3 we know that the non-orientable surface N_k admits an essentially unique 2-fold covering $\Sigma \to N_k$ that is orientable. What is the genus of Σ?

(b) Let $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. By Proposition 17.3 we know that the non-orientable surface $N_{k,m}$ admits an essentially unique 2-fold covering $\Sigma \to N_{k,m}$ that is orientable. What is the genus of Σ and what is the number of boundary components of Σ?
   Remark. You could use Exercise 17.2.

Remark. These are precisely the same questions as in Exercise 23.11. But this time you can give a different, shorter proof using Euler characteristics.

Exercise 55.3. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. In Proposition 46.21 we showed that for any $n \in \mathbb{N}_0$ there exists a short exact sequence
$$0 \to \operatorname{coker}\bigl(\mathrm{H}_n(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_n(X)\bigr) \to \mathrm{H}_n(\operatorname{Tor}(X,f)) \to \ker\bigl(\mathrm{H}_{n-1}(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_{n-1}(X)\bigr) \to 0.$$

(a) Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the torus. Given $A \in \mathrm{SL}(2,\mathbb{Z})$ we consider the homeomorphism

$$\begin{array}{rcl} f(A)\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ {[v]} & \mapsto & [Av]. \end{array}$$

Determine the homology groups of $\mathrm{Tor}(T, f(-\mathrm{id}))$.
Remark. You can use the fact that under the usual identification $\mathrm{H}_1(T) = \mathbb{Z}^2$ the induced map $f(A)_*$ is given by multiplication by A.[863]

(b) Does there exist a matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ such that $\mathrm{H}_1(\mathrm{Tor}(T, f(A))) \cong \mathbb{Z}$?

(c) Let X be a topological space that is homotopy equivalent to a finite CW-complex. Show that for any self-homeomorphism f of X we have

$$\chi(\mathrm{Tor}(X, f)) \;=\; 0.$$

Exercise 55.4. For a topological space X and $n \in \mathbb{N}_0$ we denote by $b_n(X) \in \mathbb{N}_0 \cup \{\infty\}$ the rank of $\mathrm{H}_n(X)$. Does there exist a topological space X such that $b_1(X) < \infty$ but such that for a *finite* cover $\widetilde{X}$ of X we have $b_1(\widetilde{X}) = \infty$?

Exercise 55.5. Given $g, n, \in \mathbb{N}_0$ we denote by $\Sigma_{g,n}$ the surface of genus g minus n open disks. Let $g, k, m, n \in \mathbb{N}_0$. Suppose there exists a smooth embedding $\Sigma_{g,m} \to \Sigma_{k,n}$. Show that $g \leq k$.
Hint. Use Proposition 6.30.

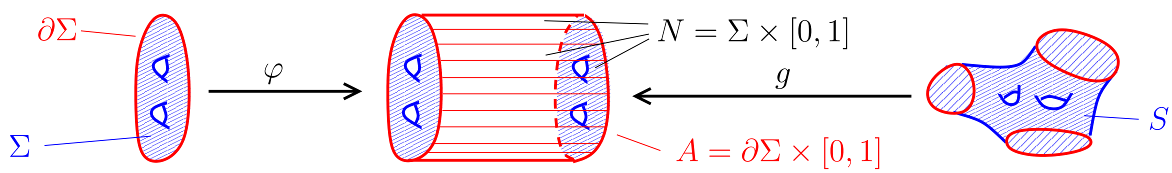


FIGURE 846. Rather schematic illustration of Exercise 66.14.

Exercise 55.6. Let Σ be a 2-dimensional smooth manifold. A *curve system* is a collection of disjoint curves (i.e. closed connected 1-dimensional smooth manifolds) $C_1, \dots, C_n$ on Σ with the following two properties:

(1) None of the C_i is the boundary of a disk on F.
(2) There is no annulus on F such that the boundary is the union of two curves.

In Figure 847 on the right we see a curve system on the surface of genus two consisting of three curves.

(a) Is there a curve system consisting of four curves on the surface of genus two?
(b) Given $g \in \mathbb{N}_0$, what is the maximal number of curves in a curve system on the surface of genus g?

Exercise 55.7. We consider the topological graph G together with the map $f\colon G \to G$ shown in Exercise 55.7. Determine the determinant and the trace of the induced map $f_*\colon\ \mathrm{H}_1(G) \to \mathrm{H}_1(G)$.

[863]Why does this hold?

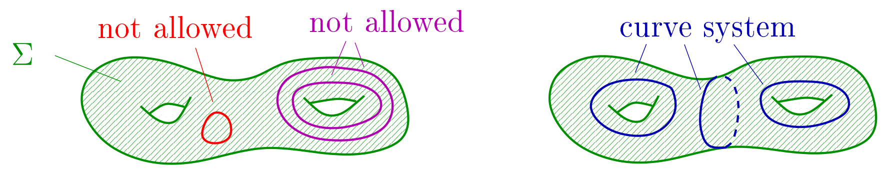


FIGURE 847

Remark. In the proof of Lemma 55.12 we gave three different, more or less explicit, isomorphisms $\varphi\colon \mathrm{H}_1(G) \xrightarrow{\cong} \mathbb{Z}^2$. The challenge now is to find an isomorphism φ that is explicit enough so that one can calculate the map $\varphi \circ f_* \circ \varphi^{-1}\colon \mathbb{Z}^2 \to \mathbb{Z}^2$.

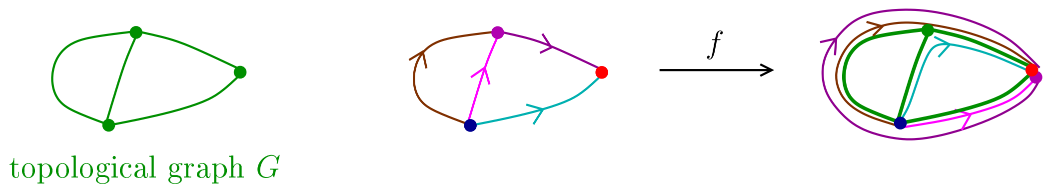


FIGURE 848. Illustration of Exercise 55.7.

Exercise 55.8. Let M be a compact orientable connected 2-dimensional smooth manifold.

(a) Show that there exists a $g \in \mathbb{N}_0$ such that M admits g disjoint curves such that the complement is connected, but such that any $g+1$ disjoint curves disconnect M.
(b) Show that g equals the genus of M, as defined on page 672.

Remark. This exercise gives an intrinsic definition of the genus of a compact orientable connected 2-dimensional smooth manifold.

Exercise 55.9. Let M be a closed non-orientable connected 2-dimensional smooth manifold. We define $k(M) \in \mathbb{N}_0$ to be the maximum $k \in \mathbb{N}_0 \cup \{\infty\}$ such that there exist k smooth embeddings of the Möbius band into M with the following two properties: the images are disjoint and the complement of images is connected.

(a) Let K be the Klein bottle. Show that $k(M) \geq 2$.
(b) Let M be a closed non-orientable connected 2-dimensional smooth manifold.
    (i) Show that $k(M)$ is finite.
    (ii) Show that $k(M)$ equals the genus of M, as defined on page 672.
    Remark. This exercise gives an intrinsic definition of the genus of a compact non-orientable connected 2-dimensional smooth manifold.

Exercise 55.10.

(a) Let $N_3 = \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$ be the non-orientable surface of genus three. Show that given any even $n \in \mathbb{N}$ there exists a connected orientable covering $p\colon \Sigma \to N_3$ of degree n.
(b) Let Σ be the surface of genus g. Show that given any $k \in \mathbb{N}$ that divides $\chi(\Sigma)$ there exists a continuous and free action by the cyclic group $\mathbb{Z}_n$ on Σ.

Remark. This exercise gives in particular a more systematic solution to Exercise 3.33.

Exercise 55.11. Let Σ be the torus minus one open disk. Furthermore let $\alpha, \beta\colon S^1 \to \Sigma$ be two disjoint embeddings with $\alpha_*([S^1]) = \beta_*([S^1]) \in \mathrm{H}_1(\Sigma)$. We denote this homology class by x.

(a) Suppose that $x \neq 0$. Show that α is isotopic to β.
(b) Show that the conclusion of (a) does not necessarily hold, if we drop the hypothesis that $x \neq 0$.

Exercise 55.12. The two player game of "Brussels sprouts" works as follows:

(1) Initially one draws n disjoint crosses on a 2-dimensional smooth manifold.
(2) The two players take turns to do the following move: Join two free ends with a curve, without crossing any existing curve, and then put a short stroke across the new curve to create two new free ends.

Such a game with $n = 2$ is illustrated in Figure 849.

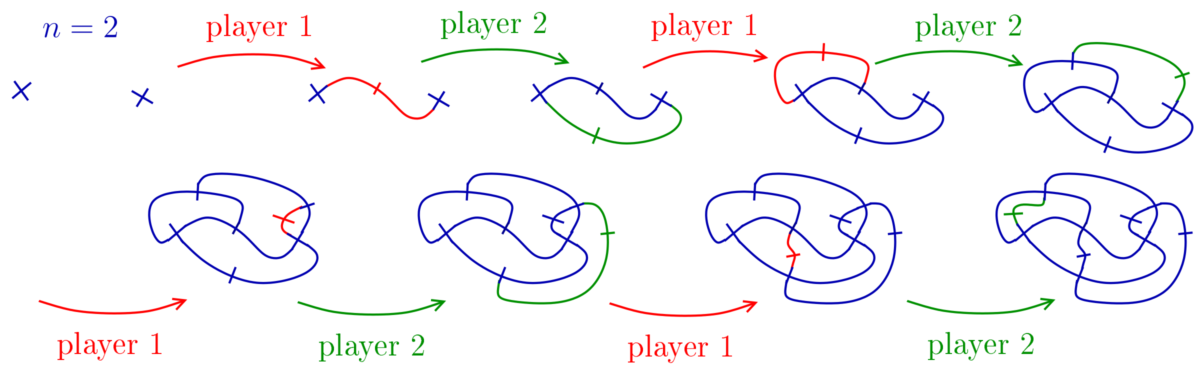


FIGURE 849. Illustration of Exercise 55.12.

(a) Suppose that the game takes place on the plane $\mathbb{R}^2$. Show that if n is odd, then the first player will always win and show that if n is even, then the second player invariably wins.
(b) Show that the conclusion of (a) also holds if the game takes place on a closed orientable 2-dimensional smooth manifold.

Remark. The game of "Brussels sprouts" is a variation on the more interesting game "sprouts" introduced by John Conway and Michal Paterson in the 1967. More information on "Sprouts" and "Brussels sprouts" can be found in [**FiG91**, Chapter 6] and [**Gar89**, Chapter 1].

56. Applications of the Euler characteristic (∗)

In this chapter we show that the fairly simple concept of the Euler characteristic can be used to give answers to several questions about 3-dimensional objects. The reader is warned in that in this chapter at times we will sacrifice complete notational rigor for readability.

56.1. Building a leather football ❀. In this section we will discuss the shapes of footballs and the platonic solids. The results of this section will not be used later on and we will present some of the material in a slightly informal, somewhat less rigorous style.

First we recall that an n-gon on the 2-dimensional sphere S^2 is called *regular* if the isometry group of the sphere acts transitively on the set of vertices and the set of faces of the n-gon. In particular all interior angles are the same. For example the classical leather football, see Figure 850, is made out of regular pentagons and hexagons.

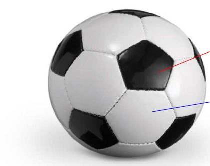


Figure 850

Now we want to determine how many pentagons and hexagons are required to make a football. So suppose a football is made out of p spherical regular pentagons and h spherical regular hexagons where the vertices of the pentagons are precisely the vertices of hexagons. This endows S^2 with a CW-structure. We first note that each 1-cell bounds two 2-cells. Furthermore, elementary spherical geometry, see e.g. [**Bon09**, p. 53] or [**Fen01**, p. 260], says that the interior angle of a spherical pentagon and a spherical hexagon are larger than the corresponding Euclidean angles. Therefore we see that interior angles are larger than[864] $\pi - \frac{2\pi}{5} = \frac{3}{5}\pi$. So at any vertex there are not more than three hexagons and pentagons that meet, which of course implies that there are precisely three hexagons and pentagons that meet. Therefore we see that

$$\begin{aligned} \#\text{2-cells} &= p+h \\ \#\text{1-cells} &= \tfrac{1}{2}(5p+6h) \\ \#\text{0-cells} &= \tfrac{1}{3}(5p+6h). \end{aligned}$$

By Proposition 55.1 and the above calculation of the Euler characteristic of S^2 we now have

$$2 \;=\; \chi(S^2) \;=\; \sum_{n=0}^{2}(-1)^n\cdot\#n\text{-cells} \;=\; \tfrac{1}{3}(5p+6h)-\tfrac{1}{2}(5p+6h)+(p+h) \;=\; \tfrac{1}{6}p.$$

Thus we see that a football has to have precisely 12 pentagons. On the other hand, the number of hexagons is not fixed. For example, the standard football has 20 hexagons, as pictured in Figure 851 on the left. But in principle one could also build a football with zero hexagons, see Figure 852 one the right, and one obtains a spherical dodecahedron. One

[864]Where does that formula come from?

could also build with many hexagons, but they would be much less round since they would have bigger flat parts.

The 1-skeleton of this CW-structure were in fact already "invented by nature", more precisely, there exists a molecule, called the Buckminsterfullerene C_{60}, built out of 60 carbon atoms, which has exactly the same shape as a football. More information can be found at

`https://en.wikipedia.org/wiki/Buckminsterfullerene`

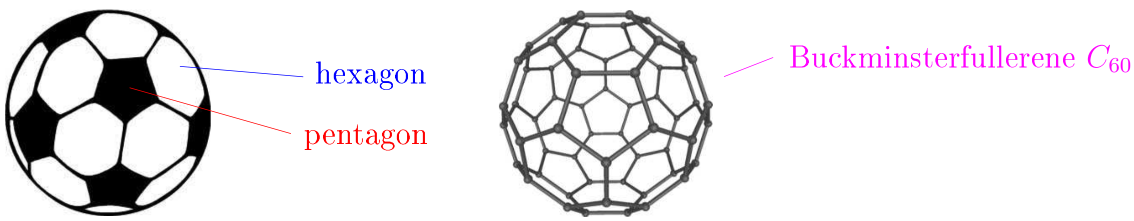


FIGURE 851

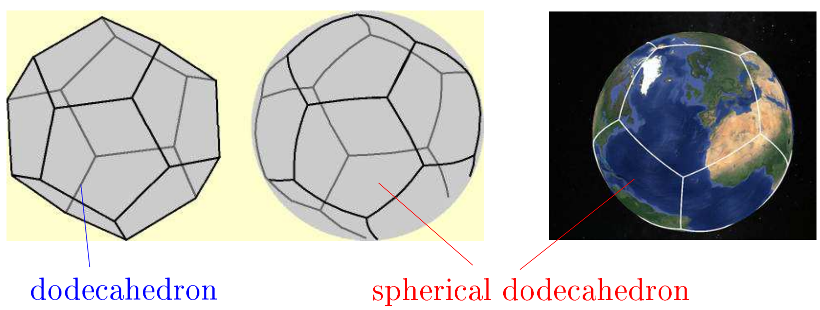


FIGURE 852

56.2. **Platonic solids ❀.** Recall that on page 123 we defined the *convex hull* of a subset $S \subset \mathbb{R}^n$ as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. Since the intersection of convex sets is again convex we see that the convex hull of S is a convex subset of $\mathbb{R}^k$.

Definition.
1. A *convex polyhedron* is the convex hull of finitely many points in $\mathbb{R}^k$.[865][866][867]
2. The *dimension of a convex polyhedron* is defined as the minimal dimension of an affine subspace of $\mathbb{R}^k$ that contains the convex polyhedron.

We leave the task of providing a proof of the following lemma to the meticulous reader.

[865]It is an amusing little exercise to show that this notion of a convex polyhedron agrees with the notion that we had originally defined on page 1016.

[866]In the literature a convex polyhedron is often also called a polytope.

[867]Recall that we saw in Exercise 2.36 (a) that the convex hull of points $P_1, \ldots, P_n \in \mathbb{R}^k$ is given by the set

$$\Big\{ \sum_{i=1}^{n} t_i P_i \,\Big|\, t_1, \ldots, t_i \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^{n} t_n = 1 \Big\}.$$

Lemma 56.1. *If P is a convex polyhedron in $\mathbb{R}^k$, then there exists a unique finite set V such that P is the convex hull of V, but such that for any proper subset W of V, P is not the convex hull of W.*

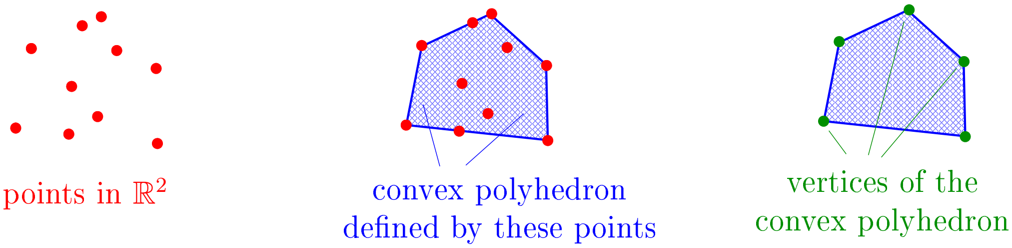


FIGURE 853. Illustration of Lemma 56.1.

Definition. Let P be a convex polyhedron in $\mathbb{R}^k$.

(1) We refer to the finite set of Lemma 56.1 as the *vertices of the convex polyhedron P*.
(2) If P is a k-dimensional convex polyhedron, then an *edge* of the convex polyhedron is the convex hull of two vertices that has the property that it lies on ∂P.[868]
(3) If P is a k-dimensional convex polyhedron, then an *m-dimensional face* of the convex polyhedron is the convex hull of m vertices that has the property that it lies on ∂P. (In particular, by definition a 1-dimensional face is exactly the same as an edge.)
(4) An edge is a 1-dimensional face and sometimes we refer to a 2-dimensional face just as a *face*, without specifying the dimension.
(5) The *valence* of a vertex is defined as the number of edges that contain the vertex.

Example. The notion of a vertex, an edge, a face and the valence of a vertex are illustrated in Figure 854.

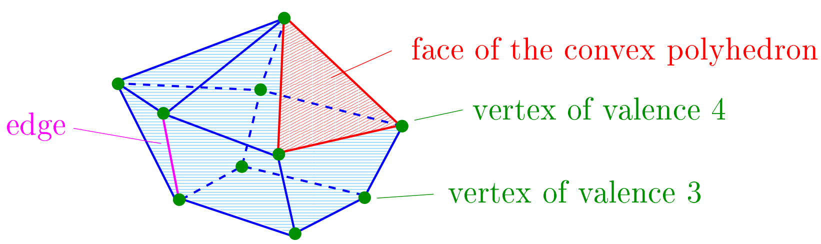


FIGURE 854

Proposition 56.2. (Euler's Formula) *Let P be a k-dimensional convex polyhedron in $\mathbb{R}^k$.*

(1) *The interior $\mathring{P}$ of P is non-empty and all faces of P are of dimension $\leq k-1$.*
(2) *The boundary ∂P admits a CW-structure such that for each $j \in \mathbb{N}_0$ the j-cells are precisely the j-dimensional faces of P.*

[868]Here ∂P is the boundary of P viewed as a subset of $\mathbb{R}^k$.

(3) *We have*

$$\sum_{j=0}^{k-1}(-1)^j\cdot\#j\text{-dimensional faces} \;=\; 1+(-1)^{k-1}.$$

Remark. The 3-dimensional version of Proposition 56.2 gives us that for the boundary of a 3-dimensional convex polyhedron in $\mathbb{R}^3$ we have

$$\text{number of vertices} - \text{number of edges} + \text{number of faces} \;=\; 2.$$

This equality is known as Euler's Formula and was first formulated Leonhard Euler in 1752. Arguably Euler's Formula is historically the first result in topology. A famous book by Imre Lakatos [**Lak76**] is dedicated to the challenge of finding a rigorous proof of Euler's Formula without employing the technical machinery, namely singular homology, that we used.

Proof. We leave the pleasure of proving the first two statements to the reader as Exercise 56.1. Now we calculate that

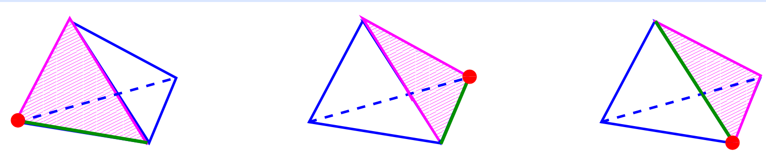
$$\sum_{j=0}^{k-1}(-1)^j\cdot\#j\text{-dimensional faces} \underset{\substack{\uparrow\\ \text{by (1) and (2)}}}{=} \chi(\partial P) \underset{\substack{\uparrow\\ \text{Proposition 2.52 (2)}\\ \text{together with (1)}}}{=} \chi(S^{k-1}) \underset{\substack{\uparrow\\ \text{see page 1363}}}{=} 1+(-1)^{k-1}.$$

■

We move on to the next definitions.

Definition.

(1) Let P be a convex m-dimensional polyhedron in $\mathbb{R}^k$.
   (a) A *rotational symmetry* of P in $\mathbb{R}^k$ is a matrix $A\in \mathrm{SO}(k)$ such that $A\cdot P=P$. The *rotational symmetry group* $\mathrm{Sym}(P)\subset \mathrm{SO}(k)$ is defined as the group of all rotational symmetries of P.
   (b) A *flag* of P is a sequence $F_0\subset F_1\subset\cdots\subset F_{m-1}$ of faces of P such that each F_i is i-dimensional.
   (c) A convex polyhedron P is called *regular* if the rotational symmetry group $\mathrm{Sym}(P)$ acts transitively on the set of flags of P.
(2) Two convex polyhedra P and Q in $\mathbb{R}^k$ are called *equivalent* if there exists an $r>0$, a point $v\in\mathbb{R}^k$, and an orthogonal matrix $A\in \mathrm{SO}(k)$ with $P=A\cdot(r\cdot Q)+v$.



three flags in the regular tetrahedron T

Figure 855

Examples.

(1) Let T be the 3-dimensional polyhedron in $\mathbb{R}^3$ that has four 2-dimensional faces, each of which is an equilateral triangle. We show T together with three flags in Figure 855. Some elementary geometry shows that T is, up to equivalence, uniquely defined and it is regular. We refer to T as the *regular tetrahedron*.

(2) The standard n-simplex Δ^n is an n-dimensional convex polyhedron in $\mathbb{R}^{n+1}$ with vertex set $\{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}$. Each vertex has valence n. It is straightforward to see that Δ^n is regular.

(3) Let $n \in \mathbb{N}_{\geq 3}$ and $r > 0$. The convex hull of the points $r, re^{2\pi\mathrm{i}/n}, \ldots, re^{2\pi\mathrm{i}(n-1)/n}$ in $\mathbb{C} = \mathbb{R}^2$ is a regular convex polyhedron with vertices $r, re^{2\pi\mathrm{i}/n}, \ldots, re^{2\pi\mathrm{i}(n-1)/n}$, where each vertex has valence 2. We leave it to the reader to show that, up to equivalence, these are all regular 2-dimensional convex polyhedra in $\mathbb{R}^2$. We refer to Figure 856 for an illustration.

(4) The convex polyhedron spanned by $\{(\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1)\}$ is a regular 3-dimensional convex polyhedron where each vertex has valence 4. This convex polyhedron is usually called the *regular octahedron*. We refer to Figure 856 for an illustration.

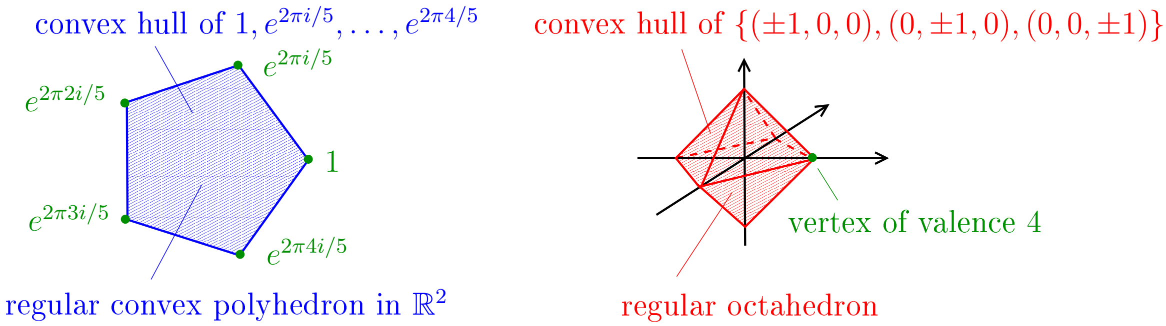


FIGURE 856

(5) We consider the cube $[-1,1]^3 \subset \mathbb{R}^3$. In [**Arm88**, p. 38] it is shown that $\mathrm{Sym}([-1,1]^3)$ is isomorphic to S_4. More precisely, we denote by D the set of the diagonals in the cube, i.e. the segments in $\mathbb{R}^3$ spanned by opposite vertices of the cube. Note that D has precisely four elements. We refer to Figure 857 for an illustration. Then the map

$$\begin{array}{rcl} \mathrm{Sym}([-1,1]^3) & \to & S_4 \cong \text{permutation group of } D \\ A & \mapsto & \left(\begin{array}{ccc} D & \to & D \\ d & \mapsto & A\cdot d \end{array}\right) \end{array}$$

is shown in [**Arm88**, p. 38] to be an isomorphism.[869] It is clear that the cube is a regular convex polyhedron. Since it has six sides it is in our context also referred to as *regular hexahedron*.

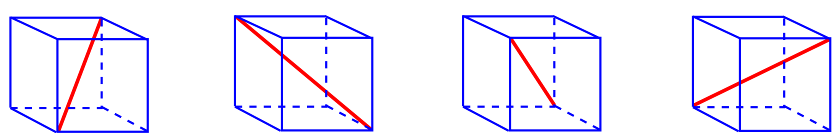


FIGURE 857. The four diagonals in a cube.

(6) Let $z := 1 + \frac{\sqrt{5}}{2}$ be the golden ratio. We denote by D the polyhedron that is spanned by the 20 vertices

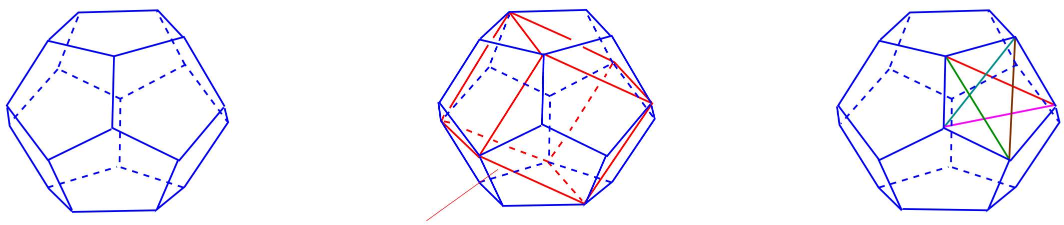
$$(\pm 1, \pm 1, \pm 1), \quad (0, \pm z, \pm \tfrac{1}{z}), \quad (\pm \tfrac{1}{z}, 0, \pm z), \quad \text{and} \quad (\pm z, \pm \tfrac{1}{z}, 0).$$

[869]It is a good exercise to verify this statement.

It is illustrated in Figure 858. By [**Cox48**, p. 53] this is a regular polyhedron. We refer to is as the *regular dodecahedron*. Note that it has 12 faces and 20 vertices. We can inscribe five cubes $C_1,\dots,C_5$ into the regular dodecahedron. In [**Arm88**, p. 40] it is shown that the map

$$\begin{array}{rcl}\mathrm{Sym}(D) & \to & A_5 \cong \text{positive permutations of } \{C_1,\dots,C_5\}\\ A & \mapsto & (C_i \mapsto A\cdot C_i)\end{array}$$

is an isomorphism.



regular dodecahedron

cube inscribed into the regular dodecahedron

the five diagonals of a face correspond precisely to one inscribed cube

FIGURE 858

(7) Let $z := 1 + \frac{\sqrt{5}}{2}$ be the golden ratio. We denote by I the polyhedron that is spanned by the 12 vertices

$$(0,\pm 1,\pm z),\quad (\pm z,0,\pm 1)\quad \text{and}\quad (\pm 1,\pm z,0).$$

By [**Cox48**, p. 52] this is a regular polyhedron. We refer to is as the *regular icosahedron*. Note that the regular dodecahedron has 20 vertices and 12 faces whereas the regular icosahedron has 12 vertices and 20 faces. In fact the regular icosahedron is dual to the regular dodecahedron in the following sense: up to a scaling factor and a rotation, the regular icosahedron is spanned by the centers of the faces of the regular dodecahedron and vice-versa. We refer to Figure 859 for an illustration of this fact. It follows that the symmetry group of the regular icosahedron I is in a suitable sense equal to the symmetry group of the regular dodecahedron D. In particular the group $\mathrm{Sym}(I)$ is isomorphic to A_5.

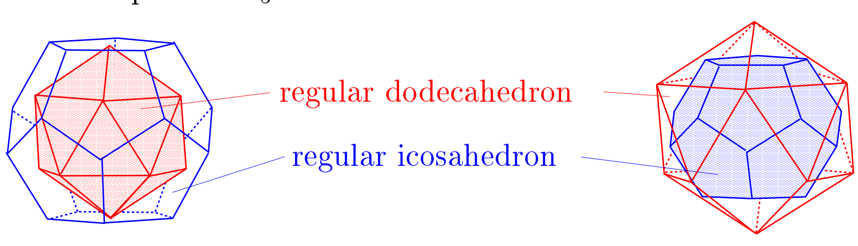


FIGURE 859

The following definition is one of the oldest in the history of mathematics.

Definition. A *Platonic solid* is a regular 3-dimensional convex polyhedron in $\mathbb{R}^3$.

Theorem 56.3. *Up to rotation and stretching the only regular convex 3-dimensional polyhedra in $\mathbb{R}^3$ are the platonic solids illustrated in Figure 860.*[870] *They have the following types of faces and valences and they have the following rotational symmetry groups:*

	tetrahedron	*hexahedron*	*octahedron*	*dodecahedron*	*icosahedron*
number of faces	4	6	8	12	20
shape of face	*3-gon*	*4-gon*	*3-gon*	*5-gon*	*3-gon*
valence of vertex	4	3	4	3	5
rotational symmetry group	A_4	S_4	S_4	A_5	A_5

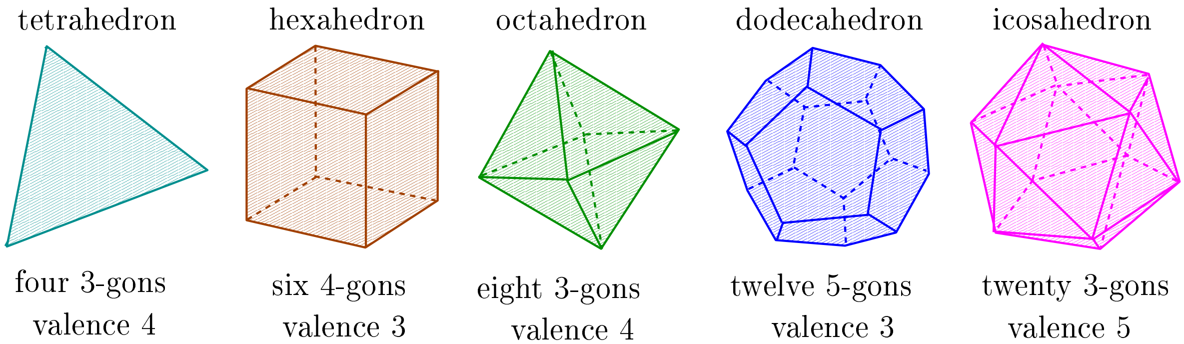


FIGURE 860. The five platonic solids.

SKETCH OF PROOF. We will not provide a full proof for the theorem. We refer to [**Cox48**, Section 6.7] for a full proof that the list of five platonic solids is indeed complete. Furthermore we refer to [**Arm88**, p. 40] or alternatively to [**Aa08**, Chapter 5.5] for a proof of the statement regarding the rotational symmetry groups.[871]

In the following we will show that any platonic solid fits into the table of the theorem. Put differently, we will show that given any platonic solid the number of faces, the shape of faces and the valence of a vertex corresponds to one of the five columns in the above table.

Thus let P be a regular convex 3-dimensional polyhedron in $\mathbb{R}^3$. We make the following observations:

(1) It follows easily from the definition of a regular polyhedron that there exists an m such that each face of P is a regular m-gon.
(2) The number of faces that meet a vertex is the same for every vertex and it is at least three.

The following claim tells us the possible values for m and the number of faces that meet at a vertex.

Claim.

possible values for m	3	4	5
faces that meet at a vertex	$3, 4, 5$	4	3

[871] We had sketched the proofs for the regular hexahedron, the regular dodecahedron and the regular icosahedron.

First note that after a translation we can assume that the sum of all vertices is in the origin. Next we shrink the convex polyhedron so that it fits into the sphere S^2. Then we consider the projection of the vertices and edges of the convex polyhedron onto the sphere, see Figure 861. Recall that the spherical interior angle of a regular m-gon is greater than the Euclidean angle of a regular m-gon. So it follows that in our case the interior angle is greater than the angle of a regular m-gon in $\mathbb{R}^3$, i.e. it is greater than $\pi - \frac{2\pi}{m}$. Since at any vertex at least three faces meet we obtain the restriction that $3(\pi - \frac{2\pi}{m}) < 2\pi$. It is now straightforward to see that we obtain the stated table. ⊞

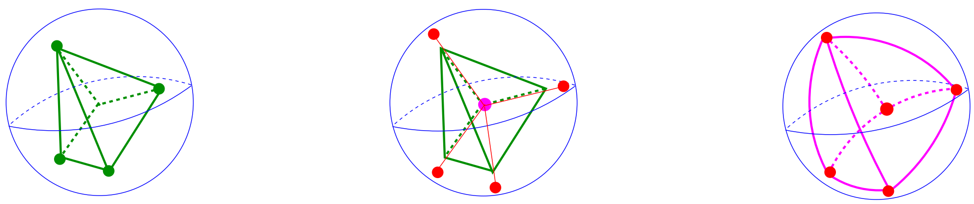


Figure 861

Now we treat the cases $m = 3, 4, 5$ separately. We start out with the case $m = 3$, i.e. we assume that all the faces of the convex polyhedron are equilateral triangles. Let us first consider the case that three triangles meet at a vertex. Then we have

$$\#\text{faces} = k, \quad \#\text{edges} = \tfrac{1}{2}\cdot 3k \quad \text{and} \quad \#\text{vertices} = \tfrac{1}{3}\cdot 3k.$$

Now we see that

$$2 \underset{\substack{\uparrow \\ \text{Euler's Formula 56.2}}}{=} \#\text{vertices} - \#\text{edges} + \#\text{faces} \underset{\substack{\uparrow \\ \text{by the above}}}{=} k - \tfrac{3}{2}k + k = \tfrac{1}{2}k.$$

It follows that $k = 4$. Almost the same calculation shows that if four triangles meet at a vertex, then $k - \frac{3}{2}k + \frac{3}{4}k = 2$, i.e. $k = 8$. Finally by the above claim it remains to consider the case that five triangles meet at a vertex. In this case we see that $k - \frac{3}{2}k + \frac{3}{5}k = 2$, i.e. $k = 20$.

Next we consider the case $m = 4$, i.e. each face is a square. The same argument as above shows that the valence of a vertex can only be three and the same calculation shows that there have to be precisely six faces. Finally if $m = 5$, i.e. if each face is a regular pentagon, then as above we see that the valence of a vertex can only be three and that there have to be precisely twelve faces. ■

On page 1382 we had just seen that there exist infinitely many equivalence classes of regular convex polyhedra in $\mathbb{R}^2$. On the other hand we have just seen in Theorem 56.3 that in $\mathbb{R}^3$ there are only finitely many, namely precisely five equivalence classes of regular convex polyhedra in $\mathbb{R}^3$. This begs the question, what is the number in higher dimensions? It turns out that in $\mathbb{R}^4$ there are six equivalence classes of regular convex polyhedra, whereas in all higher dimensions there are only three equivalence classes of convex polyhedra. A nice exposition of these facts is given on the following website:

`http://math.ucr.edu/home/baez/platonic.html`

56.3. **Planar graphs ❀.** On pages 221 and 226 we defined the notions of an (ordered) abstract graph. By Lemma 4.4 and the discussion on page 480 we know that they have homeomorphic topological realizations. In this section we will mostly deal with undirected abstract graphs. Since this name is rather unwieldy we will refer to them just as graphs. For the convenience of the reader we recall the definition.

Definition. An (*undirected abstract*) *graph* G is a triple (V, E, φ) where V is a non-empty set, E is a set and φ is a map

$$\varphi\colon E \;\to\; \{\text{subsets of } V \text{ with one or two elements}\}.$$

The elements of V are called *vertices of* G and the elements of E are called the *edges of* G.

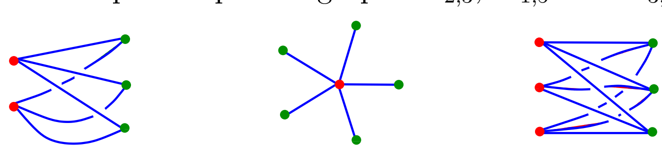
Examples.

(1) For $n \in \mathbb{N}$ we define the *complete graph* K_n as the abstract graph with vertex set $V = \{v_1, \dots, v_n\}$ and edge set $E = \{\{v_i, v_j\} \mid i, j \in \{1, \dots, n\} \text{ with } i \neq j\}$. Furthermore we define $\varphi\colon E \to V$ via $\varphi(e) = e$. Put differently, the complete graph K_n has n vertices and each vertex is connected to any other vertex by precisely one edge. In Figure 863 we show topological realizations of the complete graphs K_1, K_2, K_3, K_4 and K_5.

K_1 K_2 K_3 K_4 K_5

FIGURE 862

(2) For every choice of $m, n \in \mathbb{N}$ we define the *complete bipartite graph* $K_{m,n}$ as the abstract graph which is defined by the vertex set $V = \{x_1, \dots, x_m\} \cup \{y_1, \dots, y_n\}$, the edge set $E = \{\{x_i, y_j\} \mid i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\}\}$ and the map $\varphi\colon E \to V$ via $\varphi(e) = e$. Put differently, the complete bipartite graph $K_{m,n}$ consists of m "red-colored" vertices, n "green-colored" vertices and each "red-colored" vertex is connected to any "green-colored" vertex by precisely one edge. In Figure 863 we show topological realizations of the complete bipartite graphs $K_{2,3}$, $K_{1,5}$ and $K_{3,3}$.



$K_{2,3}$ $K_{1,5}$ $K_{3,3}$

FIGURE 863

Let $G = (V, E, \varphi)$ be a graph. On page 222 we introduced its topological realization $|G|$. It follows from Lemma 4.4 and the discussion on page 932 that $|G|$ is naturally a

1-dimensional CW-complex where the 0-cells correspond to the vertices and the 1-cells correspond to the edges.

Let G be a finite graph. On page 224 we had already mentioned that there exists an embedding of its topological realization $|G|$ into $\mathbb{R}^3$. In Question 4.3 we asked, without really expecting a positive answer, whether one can always find an embedding of $|G|$ into $\mathbb{R}^2$. This leads us to the following dimension.

Definition. We say a finite graph G is *planar* if its topological realization $|G|$ admits an embedding $|G| \to \mathbb{R}^2$.

Examples.

(1) The above graphs K_1, K_2, K_3 and $K_{1,5}$ are planar. The above topological realizations of K_4 and $K_{2,3}$ are not planar. But as we see in Figure 864 both graphs admit planar realizations.

topological realizations of K_4 topological realizations of $K_{2,3}$

FIGURE 864

(2) One can easily verify that given an $m \in \mathbb{N}$ the bipartite graph $K_{2,m}$ is planar.

Now the following question arises.

Question 56.4. *Are the graphs K_5 and $K_{3,3}$ also planar?*

If this question sounds to abstract and bipartite graphs sound too otherworldly let me reformulate the question regarding $K_{3,3}$.

Question 56.5. *Suppose we are in charge of finding efficient bus routes in a given city. Furthermore suppose the city has three main employers A, B and C. After work the employees, naturally, either want to go to the train station α, or the swimming pool β or the shopping mall γ. Is it possible to find bus routes which connect each of the three employers with each of the three destinations such that the bus routes do not intersect?*[872]

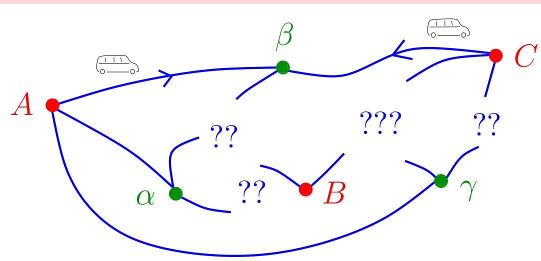


FIGURE 865

To answer that question we need a few more definitions.

[872]It follows easily from Proposition 8.29 that the answer to this question does not depend on the precise location of A, B, C and α, β, γ, as long as they are distinct.

Definition. Let G be a graph.

(1) Given $n \in \mathbb{N}$ we denote by X_n the "circular graph with n vertices and n edges", more precisely, X_n is the graph with

$$V(X_n) = \{1, \ldots, n\} \quad \text{and} \quad E(X_n) = \{\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}, \{n,1\}\}.$$

(2) A *cycle of length n* in G is an injective map $\varphi \colon X_n \to G$ of graphs in the sense of the definition given in Exercise 15.4.

(3) The *girth* of G is the shortest length of a cycle in G. If G has no cycles, then we define its girth as ∞.

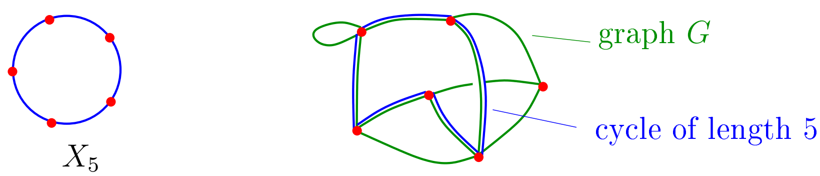


FIGURE 866

Examples.

(1) The girth of the graph shown in Figure 866 to the right is one, since it has an edge where the two endpoints agree.
(2) For any $n \geq 3$ the girth of the complete graphs K_n equals three.
(3) For any $n, m \geq 2$ the girth of the complete bipartite graphs $K_{n,m}$ equals four.

The following gives an obstruction to a graph being planar.

Proposition 56.6. *Let G be a graph with v vertices and e edges. We denote by g the girth of G. If G is planar and if $g < \infty$, then*

$$e \leq \frac{g}{g-2} \cdot (v-2).$$

SKETCH OF PROOF. Let G be a graph with v vertices and e edges. We suppose that the girth g of G is finite. Furthermore we suppose that G is planar, i.e. we suppose that there exists an embedding $\varphi \colon |G| \to \mathbb{R}^2$. We write $X = \varphi(|G|)$. Without loss of generality we can assume that X lies in $\overline{B}^2$.

Now we view X as a subset of the sphere $S^2 = \overline{B}^2/S^1$. We refer to the closures of the components of $S^2 \setminus X$ as *faces* and we denote by f the number of faces. We refer to Figure 867 for an illustration. We make the following observation:

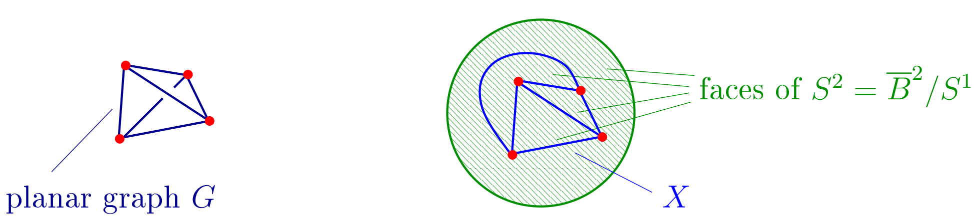


FIGURE 867

(1) The vertices, edges and faces define a CW-structure for $S^2 = \overline{B}^2/S^1$.[873]
(2) We make the following two observations:
   (a) The boundary of each face is a cycle. By the definition of the girth each face is bounded by at least g edges.
   (b) Each edge lies in the boundary of at most two faces.

   Now we deduce that

$$\begin{aligned} g \cdot f &\leq \sum_{\text{faces } f} \text{number of edges in the boundary of } f \\ &= \text{number of pairs } (f,e) \text{ where } f \text{ is a face and } e \text{ an edge in the boundary of } f \\ &= \sum_{\text{edges } e} \text{number of faces that contain } e \text{ in the boundary} \;\leq\; 2e. \end{aligned}$$

   Here the first inequality follows from (a) and the last inequality follows from (b). Summarizing we see that $g \cdot f \leq 2e$.

Now we see that

$$2 \underset{\substack{\uparrow \\ \text{see page 1363}}}{=} \chi(S^2) \;=\; v - e + f \underset{\substack{\uparrow \\ \text{since } g\cdot f \leq 2e}}{\leq} v - e + \tfrac{2}{g}e \;=\; v - \tfrac{g-2}{g}e.$$

Solving for e we get the desired inequality $e \leq \frac{g}{g-2} \cdot (v-2)$. ■

The following proposition gives in particular a negative answer to Question 56.4.

Proposition 56.7. *If a graph G contains the complete graph K_5 or the complete bipartite graph $K_{3,3}$ as a subgraph, then G itself cannot be planar.*

PROOF.

(1) Let $m \in \mathbb{N}_{\geq 3}$. It is straightforward to see that the complete graph K_m has m vertices and $\frac{1}{2}m(m-1)$ edges. Furthermore the girth is three. It is straightforward to verify that the inequality of Proposition 56.6 is only satisfied for $m = 3$ and $m = 4$.
(2) Let $m, n \in \mathbb{N}_{\geq 2}$. Again it easy to verify that the complete bipartite graph $K_{m,n}$ has $m + n$ vertices and $m \cdot n$ edges. Furthermore,as we pointed out above, the girth is four. It is straightforward to verify that the inequality of Proposition 56.6 is only satisfied if $m = 2$ or $n = 2$.
(3) The Proposition now follows from (1) and (2) and the observation that if a graph is planar, then any subgraph is also planar. ■

Remark. Let G be a graph that contains the complete graph K_7. By Proposition 56.7 the graph G cannot be planar. John Conway and Cameron Gordon proved in 1984 the following interesting variation: given any embedding $\varphi\colon |G| \to \mathbb{R}^3$ such that image is a polygon (i.e. it is the union of finitely many line segments) the image $\varphi(|G|)$ contains a non-trivial polygonal knot. More precisely, there exists a closed polygon C in $\varphi(|G|)$ such

[873] Here is the reason why it is the sketch of a proof: we do not prove that each face is in fact homeomorphic to a closed disk. This statement is "clear from the picture" and it can be proved using the Schönflies Theorem 50.3. But we will not attempt to give a rigorous proof.

that $\pi_1(\mathbb{R}^3 \setminus C)$ is not isomorphic to $\mathbb{Z}$. We refer to **[CoG83]** for the precise formulation of the theorem and the proof.

Now we will consider the question, to what degree the converse to Proposition 56.7 holds. First we need to introduce the notion of a subdivision of a graph. From Figure 868 it should become clear what is meant be a subdivision. As so often, the formal definition is a little cumbersome.

Definition. Let $G = (V, E, \varphi\colon E \to \mathcal{P}(V))$ be a graph.

(1) Let $e \in E$ be an edge. We say that a graph $G' = (V', E', \varphi'\colon E' \to \mathcal{P}(V'))$ is a *subdivision of G at e* if $V' = V \sqcup \{w\}$ and $E' = (E \setminus \{e\}) \sqcup \{f_1, f_2\}$ with $w \notin V$ and $f_1, f_2 \notin E$ such that $\varphi'(e') = e$ for $e' \in E \setminus \{e\}$, $\varphi'(f_1) = \{v_1, *\}$ and $\varphi'(f_2) = \{v_2, *\}$ where $\varphi(e) = \{v_1, v_2\}$.
(2) We say a graph G' is a *subdivision of G* if G' is obtained from G by a finite sequence of subdivisions along edges.

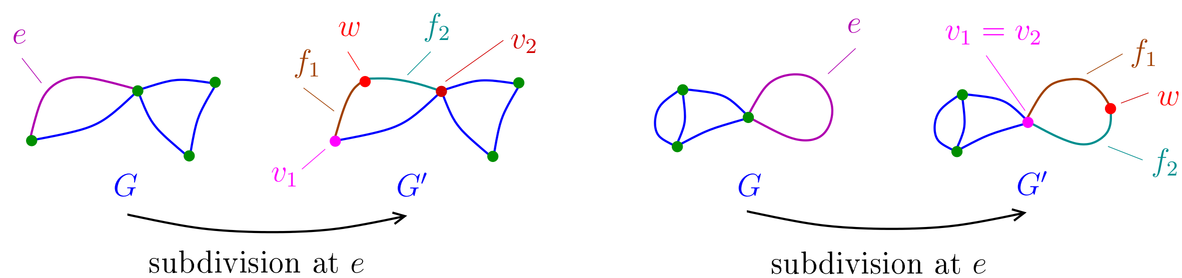


FIGURE 868

The following lemma says in particular that "planarity" is unaffected by subdivisions.

Lemma 56.8. *Let G be a graph. If G' is a subdivision of G, then the topological realizations of G' and G are homeomorphic.*

PROOF. We leave it to the reader to provide the elementary proof of this lemma. ∎

We move on to the final definition of this section.

Definition. We say a graph is *simple* if its girth is at least three.

Put differently, a graph G is simple if the following two conditions are satisfied:

(1) G has no loops, i.e. if there is no edge where the two endpoints agree,
(2) G does not have multiple edges, i.e. there are no two edges with the same endpoints.

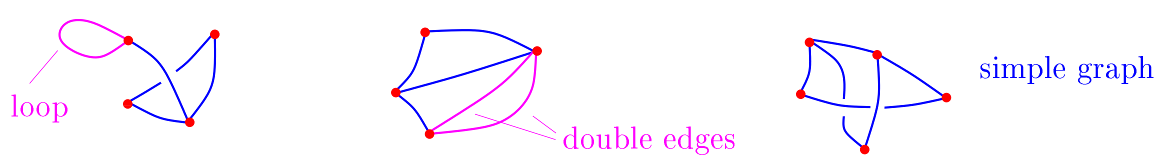
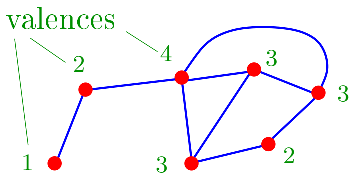


FIGURE 869

In the remainder of this section we restrict ourselves to graphs that are simple. We start out with the following observations.

(1) If a simple graph G admits a vertex v of valence[874] one, then v is the endpoint of a single edge e. If we delete v and e we obtain a new graph G'. An argument as in the proof of Proposition 18.31 shows that the topological realization of G' is homotopy equivalent to the topological realization of G. It is straightforward to see that G is planar if and only if G' is planar.
(2) If a simple graph admits a vertex v of valence two, then v is the endpoint of two edges e_1 and e_2. We can combine e_1 and e_2 into one edge and eliminate v. This way we obtain a new graph with a homeomorphic topological realization.

This shows that for the most part we can restrict ourselves to simple graphs where every vertex has valence at least three.



is homotopy equivalent to

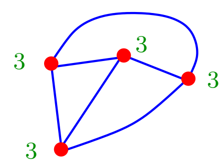


FIGURE 870

If we restrict ourselves to simple graphs such that every vertex has valence at least three, then the graphs K_5 and $K_{3,3}$ turn out to be the only obstruction to being planar. More precisely the following theorem was proved by Kuratowski[875] in 1930. We refer to [**BoM08**, Chapter 10.5] for a proof.

Theorem 56.9. (Kuratowski's Reduction Theorem) *Let G be a finite simple graph such that every vertex has valence at least three. Then the following statements are equivalent:*

(1) *The graph G is planar.*
(2) *The G contains neither a subdivision of K_5 nor a subdivision of $K_{3,3}$ as a subgraph.*

Exercises for Chapter 56.

Exercise 56.1.

(a) Let P be a k-dimensional convex polyhedron in $\mathbb{R}^k$. Show that the interior $\mathring{P}$ of P is non-empty.
(b) Let Q be a k-dimensional convex polyhedron in some $\mathbb{R}^n$. Show that Q is homeomorphic to $\overline{B}^k$.
Hint. Use (a) and Proposition 2.52 (2).
(c) Let P be a k-dimensional convex polyhedron in some $\mathbb{R}^n$. Show that P admits a CW-structure such that the boundary ∂P is a subcomplex and such that for each $j \in \mathbb{N}_0$ the j-cells in ∂P are precisely the j-dimensional faces of P.

[874]Recall that the *valence* of a vertex is defined as the number of edges that contain the vertex.
[875]Kazimierz Kuratowski (1896-1980) was a Polish mathematician.

Exercise 56.2.

(a) Show that there exist injective maps $K_5 \to S^1 \times S^1$ and $K_{3,3} \to S^1 \times S^1$.
(b) Let $n \in \mathbb{N}_{\geq 6}$. What is the smallest g that you can find such that there exists an injective map from K_n to the surface of genus g?

Exercise 56.3. In his last will, a farmer, who owns a rectangular plot, wants to distribute his land among his sons as follows:

(1) every son gets a connected compact field with non-empty area,
(2) every son should be the neighbor of every other son.

What is maximal number of sons for which the farmer's last will can be fulfilled?
Remark. On the margin of the last will the farmer insists that the new distribution should give a CW-structure for the rectangular plot.

FIGURE 871. Illustration of Exercise 56.3.

Exercise 56.4. Find an embedded loop on the edges of the dodecahedron that covers all vertices.
Remark. This is known as Hamilton's "icosian game". This puzzle was actually marketed in the 19th century as a board game.

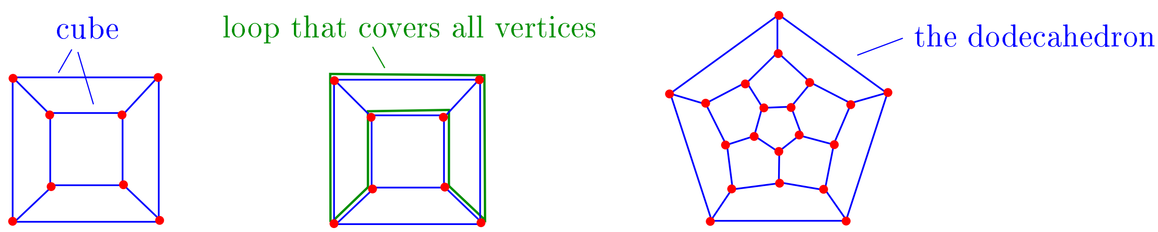


FIGURE 872. Illustration of Exercise 56.4.

57. Homology with coefficients

In this chapter we will introduce the homology groups of a topological space X with coefficients in a ring R. These homology groups will have many applications, for example they will be essential in relating the homology groups of a smooth manifold M to the de Rham cohomology groups of M. They will also be essential in proving theoretic statements, for example we will need to work with $\mathbb{F}_2$-coefficients to prove the higher-dimensional analogue of the Borsuk-Ulam Theorem 16.22. Before we can introduce the homology groups with coefficients we will need to introduce or recall some basic algebraic concepts.

57.1. **The tensor product of abelian groups.** The key to defining homology groups with coefficients is the notion of the tensor product of two abelian groups.

Definition. Let A and B be abelian groups. We define the *tensor product* $A\otimes B$ to be the following abelian group:

$$A\otimes B \;:=\; \left\{ \begin{array}{l} \text{free abelian group which is generated by all} \\ \text{symbols of the form } a\otimes b \text{ with } a\in A,\ b\in B \end{array} \right\} \Big/ N(A,B)$$

where $N(A,B)$ is the subgroup generated by the elements of the form

$$(a+a')\otimes b-(a\otimes b+a'\otimes b) \qquad \text{and} \qquad a\otimes(b+b')-(a\otimes b+a\otimes b')$$

with $a,a'\in A$ and $b,b'\in B$. For $a\in A$ and $b\in B$ we denote, by a slight abuse of notation, the image of $a\otimes b$ in $A\otimes B$ again by $a\otimes b$.

We summarize some elementary properties of the tensor product of two abelian groups in the following lemma.

Lemma 57.1. *Let A and B be two abelian groups.*

(1) *For any $a,a'\in A$ and $b,b'\in B$ we have*

$$\begin{array}{rcl} (a+a')\otimes b &=& a\otimes b+a'\otimes b \;\in\; A\otimes B \\ a\otimes(b+b') &=& a\otimes b+a\otimes b' \;\in\; A\otimes B. \end{array}$$

(2) *For any $a\in A, b\in B$ and $m\in\mathbb{Z}$ we have*

$$(m\cdot a)\otimes b \;=\; a\otimes(m\cdot b) \;=\; m\cdot(a\otimes b).$$

(3) *Any element in $A\otimes B$ is of the form $\sum\limits_{i=1}^{n} a_i\otimes b_i$.*

(4) *Let $n\in\mathbb{N}$. For any $c\in A\otimes\mathbb{Z}_n$ we have $n\cdot c=0\in A\otimes\mathbb{Z}_n$.*

Proof. Let A and B be two abelian groups.

(1) This statement is an immediate consequence of the definition of the tensor product.
(2) Let $a\in A$ and $b\in B$. We first consider the case that $m>0$. We have

$$(m\cdot a)\otimes b \;=\; \Big(\sum_{i=1}^{m} a\Big)\otimes b \;\underset{\text{by (1)}}{\underset{\uparrow}{=}}\; \sum_{i=1}^{m} a\otimes b \;=\; m\cdot(a\otimes b) \;\underset{\text{same argument backwards}}{\underset{\uparrow}{=}}\; \dots \;=\; a\otimes(m\cdot b).$$

It follows easily from (1) that $0 \otimes b = a \otimes 0 = 0$ and that $(-a) \otimes b = a \otimes (-b) = -a \otimes b$. Using these observations we get statement (2) for $m = 0$ and we can easily reduce the case $m < 0$ of (2) to the case $m > 0$ of (2).

(3) It follows immediately from the definition of the tensor product, together with the notational convention of "formal linear combinations" that we introduced on page 580 that any element in $A \otimes B$ is of the form $\sum_{i=1}^{n} r_i \cdot (a_i \otimes b_i)$. But by (2) we know that $r_i \cdot (a_i \otimes b_i) = (r_i \cdot a_i) \otimes b_i$ thus we get the desired result.

(4) Let $n \in \mathbb{N}$ and let $c \in A \otimes \mathbb{Z}_n$. By (3) we can write $c = \sum_{i=1}^{m} a_i \otimes r_i$ with $a_1, \ldots, a_m \in A$ and $r_1, \ldots, r_m \in \mathbb{Z}_n$. Then

$$n \cdot c \;=\; n \cdot \left(\sum_{i=1}^{m} a_i \otimes r_i \right) \;\underset{\substack{\uparrow \\ \text{by statement (2)}}}{=}\; \sum_{i=1}^{m} a_i \otimes \underbrace{n \cdot r_i}_{=0} \;=\; 0.$$

■

Definition. Let A, B and C be abelian groups. We say that a map $f\colon A \times B \to C$ is *bilinear*, if for all $a, a' \in A$, $b, b' \in B$ the following equalities hold:

$$f(a + a', b) \;=\; f(a, b) + f(a', b) \quad \text{and} \quad f(a, b + b') \;=\; f(a, b) + f(a, b').$$

We leave it to the reader to provide the elementary proof of the following lemma.

Lemma 57.2. *Let A and B be abelian groups.*

(1) *The map*

$$\begin{aligned} f\colon A \times B &\to A \otimes B \\ (a, b) &\mapsto a \otimes b \end{aligned}$$

is bilinear.

(2) *Given any abelian group G and any bilinear map $g\colon A \times B \to G$ there exists a unique homomorphism $\varphi\colon A \otimes B \to G$ such that the following diagram commutes:*

$$\begin{tikzcd} A \times B \arrow[r, "f"] \arrow[dr, "g"'] & A \otimes B \arrow[d, "\varphi \text{ unique}"] \\ & G. \end{tikzcd}$$

(3) *Given any abelian group G the map*

$$\begin{aligned} \operatorname{Hom}(A \otimes B, G) &\to \textit{set of bilinear maps } A \times B \to G \\ (\varphi\colon A \otimes B \to G) &\mapsto (A \times B \xrightarrow{f} A \otimes B \xrightarrow{\varphi} G) \end{aligned}$$

is a bijection.

Remark. One can show quite easily that if a homomorphism $g\colon A \times B \to H$ is bilinear and if this homomorphism satisfies the universal property of Lemma 57.2 (2), then there exists an isomorphism $H \xrightarrow{\cong} A \otimes B$. We will not make use of this observation.

Before we state a few more properties of the tensor product it is convenient to introduce the following definition.

Definition. Let $f\colon A \to A'$ and $g\colon B \to B'$ be homomorphisms between abelian groups. We define $f \otimes g$ to be the homomorphism that is given by[876]

$$\begin{array}{rcl} f \otimes g\colon A \otimes B & \to & A' \otimes B' \\ \sum\limits_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum\limits_{i=1}^{n} f(a_i) \otimes g(b_i) \end{array}$$

Now we can continue with the promised lemma on properties of the tensor product. On many occasions we will make use of it without explicitly referring to it.

Lemma 57.3. *Let A, B, C and $A_i, i \in I$ be abelian groups.*

(1) *The maps*

$$\begin{array}{rcl} A \otimes B & \to & B \otimes A \\ \sum\limits_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum\limits_{i=1}^{n} b_i \otimes a_i \end{array} \quad \text{and} \quad \begin{array}{rcl} \Big(\bigoplus\limits_{i\in I} A_i\Big) \otimes B & \to & \bigoplus\limits_{i\in I} A_i \otimes B \\ \Big(\sum\limits_{i=1}^{n} a_i\Big) \otimes b & \mapsto & \sum\limits_{i=1}^{n} a_i \otimes b \end{array}$$

are well-defined and they are natural isomorphisms.

(2) *The map*

$$\begin{array}{rcl} (A \otimes B) \otimes C & \to & A \otimes (B \otimes C) \\ \sum\limits_{j=1}^{n} (a_j \otimes b_j) \otimes c_j & \mapsto & \sum\limits_{j=1}^{n} a_j \otimes (b_j \otimes c_j) \end{array}$$

is well-defined and it is a natural isomorphism.

(3) *Let $k \in \mathbb{N}$. The maps*[877]

$$\begin{array}{rcl} \mathbb{Z}_k \otimes A & \to & A/kA \\ \sum\limits_{i=1}^{m} n_i \otimes a_i & \mapsto & \sum\limits_{i=1}^{m} n_i a_i \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{Z} \otimes A & \to & A \\ \sum\limits_{i=1}^{m} n_i \otimes a_i & \mapsto & \sum\limits_{i=1}^{m} n_i a_i. \end{array}$$

are well-defined and they are natural isomorphisms.

(4) *Suppose $(I, \leq)$ is an ordered set and $(\{A_i\}_{i\in I}, \{\varphi_{ij}\}_{i\leq j})$ is a direct system of abelian groups. Then the map*

$$\varinjlim (A_i \otimes B) \xrightarrow{\cong} \big(\varinjlim A_i\big) \otimes B$$

that is induced by the maps $A_i \to \varinjlim A_i$ is a natural isomorphism.

Proof. We refer to [**Lan93**, Corollary XVI.2.2] for a proof for the distributivity in (1). The other statements in (1) to (3) follow fairly easily from the definitions and Lemma 57.1. We refer to [**Mun84**, Chapter 50] for details. A proof of the last statement is given in [**Bou07**, Chapter II.6.3] or alternatively in [**Mats89**, Theorem A1]. ■

Remark.

(1) As we mentioned in the statement of Lemma 57.3, all the isomorphisms are natural, in the sense that they define natural transformations between appropriate functors. For

[876] It is straightforward to verify that the map $f \otimes g$ is indeed well-defined and that it is a homomorphism.

[877] Note that for an abelian group A and $k \in \mathbb{Z}$ the set $kA := \{k \cdot a \mid a \in A\}$ is a subgroup of A and we can thus form the quotient A/kA.

example if we denote by $\mathcal{AbGr}$ the category of abelian groups, then the isomorphisms of Lemma 57.3 (3) define a natural transformation between the two functors

$$\begin{array}{rcl} \mathcal{AbGr} & \to & \mathcal{AbGr} \\ A & \mapsto & A \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathcal{AbGr} & \to & \mathcal{AbGr} \\ A & \mapsto & \mathbb{Z} \otimes A. \end{array}$$

As an example, for every $k \in \mathbb{N}$ and any abelian group A we get a commutative diagram

$$\begin{tikzcd} \mathbb{Z} \otimes A \arrow[r, "\cong", "n\otimes a \mapsto n\cdot a"'] \arrow[d, "\cdot k \otimes \mathrm{id}"'] & A \arrow[d, "\cdot k"] \\ \mathbb{Z} \otimes A \arrow[r, "\cong", "n\otimes a \mapsto n\cdot a"'] & A. \end{tikzcd}$$

(2) We use these natural isomorphisms from Lemma 57.3 to identify the various groups. Note that Lemma 57.3 (2) allows us to write the tensor product $A \otimes B \otimes C$ unambiguously without having to worry about parentheses.
(3) In Lemma 57.3 we saw that the tensor product commutes with forming direct sums. The analogous statement is not true if we replace direct sums by direct products. More precisely, in Exercise 57.11 we will see that

$$\Big(\prod_{n\in\mathbb{N}} \mathbb{Z}_n\Big) \otimes \mathbb{Q} \neq 0 \qquad \text{but} \qquad \prod_{n\in\mathbb{N}} \underbrace{(\mathbb{Z}_n \otimes \mathbb{Q})}_{=0,\text{ by Lemma 57.3 (3)}} = 0.$$

Examples.

(1) Let $m, n \in \mathbb{N}$. We write $r = \gcd(m,n)$ and $s = m/r$. Then

$$\mathbb{Z}_m \otimes \mathbb{Z}_n \underset{\substack{\uparrow \\ \text{Lemma 57.3 (3)}}}{\cong} \mathbb{Z}_n/m\mathbb{Z}_n = \mathbb{Z}_n/rs\mathbb{Z}_n = \mathbb{Z}_n/r\cdot(s\mathbb{Z}_n) \underset{\substack{\uparrow \\ \text{since } s \text{ is coprime to } n}}{=} \mathbb{Z}_n/r\mathbb{Z}_n \xleftarrow[\cong]{\cdot n/r} Z_r = \mathbb{Z}_{\gcd(m,n)}.$$

(2) Let A be a finitely generated abelian group of rank r. By the classification of finitely generated abelian groups, see Theorem 19.4, we have $A \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}_{a_i}$ where $a_1, \ldots, a_n$ are non-zero natural numbers. For $S = \mathbb{Q}, S = \mathbb{R}$ or $S = \mathbb{C}$ we have

$$A \otimes S \cong \Big(\mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}_{a_i}\Big) \otimes S \underset{\substack{\uparrow \\ \text{Lemma 57.3 (1)}}}{=} (\mathbb{Z}\otimes S)^r \oplus \Big(\bigoplus_{i=1}^n \mathbb{Z}_{a_i} \otimes S\Big) \underset{\substack{\uparrow \\ \text{Lemma 57.3 (3)}}}{\cong} S^r \oplus \bigoplus_{i=1}^n S/a_iS \underset{\substack{\uparrow \\ \text{since } a_i\neq 0 \text{ in } S=\mathbb{Q},\mathbb{R},\mathbb{C}}}{=} S^r.$$

(3) Let A be an abelian group, then

$$\ker\begin{pmatrix} A & \to & A\otimes\mathbb{Q} \\ a & \mapsto & a\otimes 1 \end{pmatrix} = \text{all torsion elements of } A.$$

If A is finitely generated, then this follows easily from the classification of finitely generated abelian groups together with Lemma 57.3. If A is infinitely generated, then we can view it as the direct limit of finitely generated abelian groups and using Lemma 57.3 the statement can be reduced to the finitely generated case. We leave the details to the reader.

(4) Recall that given a set W we denote by $\mathbb{Z}^{(W)}$ the free abelian group generated by W. Let A and B be two sets. We then have

$$\mathbb{Z}^{(A)}\otimes\mathbb{Z}^{(B)} = \Big(\bigoplus_{a\in A}\mathbb{Z}\cdot a\Big)\otimes\Big(\bigoplus_{b\in B}\mathbb{Z}\cdot b\Big) \overset{\substack{\text{Lemma 57.3 (1)}\\ \downarrow}}{=} \bigoplus_{a\in A, b\in B}\mathbb{Z}\cdot a\otimes\mathbb{Z}\cdot b \overset{\substack{\text{Lemma 57.3 (3)}\\ \downarrow}}{=} \bigoplus_{a\in A, b\in B}\mathbb{Z}\cdot(a,b) = \mathbb{Z}^{(A\times B)}.$$

(5) Let A and B be two finitely generated abelian groups. Using examples (1) and (2), the classification of finitely generated abelian groups and Lemma 57.3 (1) and (4) one can now easily show that

$$\operatorname{rank}(A\otimes B) \;=\; \operatorname{rank}(A)\cdot\operatorname{rank}(B).$$

For future reference we record the following lemma:

Lemma 57.4. *Let $f\colon A\to A'$ and $f'\colon A'\to A''$ be homomorphisms and let $g\colon B\to B'$ and $g'\colon B'\to B''$ be homomorphisms between abelian groups. Then we have*

$$(f'\otimes g')\circ(f\otimes g) \;=\; (f'\circ f)\otimes(g'\circ g).$$

Proof. The statement follows immediately from the definitions. ■

The following elementary lemma will play a major role.

Lemma 57.5. *Let R be a commutative ring.*

(1) *Given any abelian group A the map*

$$\begin{array}{rcl} (A\otimes R)\times R & \to & A\otimes R \\ \Big(\sum\limits_{i=1}^{n} a_i\otimes b_i, r\Big) & \mapsto & \sum\limits_{i=1}^{n} a_i\otimes b_i r \end{array}$$

defines an R-module structure on $A\otimes R$.

(2) *If $f\colon A\to B$ is a homomorphism of abelian groups, then the map*

$$\begin{array}{rcl} f\otimes\mathrm{id}\colon A\otimes R & \to & B\otimes R \\ \sum\limits_i a_i\otimes r_i & \mapsto & \sum\limits_i f(a_i)\otimes r_i \end{array}$$

is a homomorphism of R-modules.

(3) *The map*

$$\begin{array}{rcl} A & \mapsto & A\otimes R \\ (f\colon A\to B) & \mapsto & (f\otimes\mathrm{id}\colon A\otimes R\to B\otimes R) \end{array}$$

defines a functor from the category of abelian groups to the category of R-modules.

Proof. Once again the statements follow immediately from the definitions. ■

Lemma 57.5 shows that tensoring with a commutative ring “turns” an abelian group into a module. In particular if we tensor with a field, we obtain a vector space. This is very convenient since sometimes it allows us to reduce problems about abelian groups into problems about vector spaces that we can handle with techniques from linear algebra.

For example we can now give a meaningful definition of the rank of any abelian group.

Definition. The rank of an abelian group A is defined as
$$\operatorname{rank}(A) \ := \ \dim_{\mathbb{Q}}(A \otimes \mathbb{Q}).$$

Remark.

(1) Previously, on page 585 we introduced the rank of finitely generated abelian groups. It follows from the discussion on page 1396 that for finitely generated abelian groups the two definitions of rank agree. Put differently, the above definition is an extension of the earlier notion of rank to all abelian groups.
(2) Let A be an abelian group. We can view A as a $\mathbb{Z}$-module and the above definition of the rank of the abelian group equals the rank of the corresponding $\mathbb{Z}$-module as introduced on page 82.

We conclude this section with the following lemma.

Lemma 57.6. *Let*
$$0 \ \to \ A \ \xrightarrow{i} \ B \ \xrightarrow{p} \ C \ \to \ 0$$
be a short exact sequence and let G be an abelian group. If the short exact sequence splits, then
$$0 \to A \otimes G \xrightarrow{i \otimes \mathrm{id}} B \otimes G \xrightarrow{p \otimes \mathrm{id}} C \otimes G \to 0$$
is also a short exact sequence and it also splits.

Remark. In most applications we will consider a short exact sequence
$$0 \ \to \ A \ \xrightarrow{i} \ B \ \xrightarrow{p} \ C \ \to \ 0$$
where C is a free abelian group. It then follows from Lemma 46.1 that the short exact sequence splits, which then implies by Lemma 57.6 that for any abelian group G the sequence
$$0 \to A \otimes G \xrightarrow{i \otimes \mathrm{id}} B \otimes G \xrightarrow{p \otimes \mathrm{id}} C \otimes G \to 0$$
is also a short exact sequence which furthermore splits.

Proof. According to Splitting Lemma 46.2 we can without loss of generality suppose that the short exact sequence is of the form
$$0 \to A \xrightarrow{a \mapsto (a,0)} A \oplus C \xrightarrow{(a,c) \mapsto c} C \to 0.$$
If we tensor this short exact sequence with G, then using $(A \oplus C) \otimes G = A \otimes G \oplus C \otimes G$ we obtain the sequence
$$0 \to A \otimes G \xrightarrow{\sum\limits_i a_i \otimes g_i \mapsto \left(\sum\limits_i a_i \otimes g_i, 0\right)} A \otimes G \oplus C \otimes G \xrightarrow{\left(\sum\limits_i a_i \otimes g_i, \sum\limits_j c_j \otimes g_j\right) \mapsto \sum\limits_j c_j \otimes g_j} C \otimes G \to 0,$$
which is evidently exact. It is also clear that the short exact sequence splits. ■

57.2. **The homology groups of a topological space with coefficients.** Let

$$\dots \xrightarrow{\partial_{n+1}} \mathrm{C}_n \xrightarrow{\partial_n} \mathrm{C}_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} \mathrm{C}_1 \xrightarrow{\partial_1} \mathrm{C}_0 \to 0$$

be an algebraic chain complex and let G be an abelian group. We tensor the chain complex with G, i.e. we consider the following sequence of homomorphisms:[878]

$$\dots \to \mathrm{C}_n \otimes G \xrightarrow{\partial_n \otimes \mathrm{id}} \mathrm{C}_{n-1} \otimes G \xrightarrow{\partial_{n-1} \otimes \mathrm{id}} \dots \xrightarrow{\partial_2 \otimes \mathrm{id}} \mathrm{C}_1 \otimes G \xrightarrow{\partial_1 \otimes \mathrm{id}} \mathrm{C}_0 \otimes G \to 0.$$

For all i we have

$$(\partial_{i-1} \otimes \mathrm{id}) \circ (\partial_i \otimes \mathrm{id}) \underset{\substack{\uparrow \\ \text{Lemma 57.4}}}{=} (\partial_{i-1} \circ \partial_i) \otimes \mathrm{id} = 0 \otimes \mathrm{id} = 0,$$

i.e. the above sequence of maps is again a chain complex. We denote the corresponding homology groups by $\mathrm{H}_n(C;G) := \mathrm{H}_n(C_* \otimes G, \partial_* \otimes \mathrm{id})$. If we tensor the chain complex C_* with a commutative ring R, then it follows from the discussion in the previous section that we obtain a chain complex of R-modules, i.e. all chain groups are R-modules and all chain maps are R-homomorphisms. In particular also all homology groups are R-modules.

Remark. Most of the results which we have proved for chain complexes of abelian groups hold, with the obvious modifications, also for chain complexes of R-modules. In particular one can easily convince oneself that Lemma 41.7, Lemma 42.2, Corollary 42.3 and Propositions 43.11 and Lemma 43.10 also hold in this more general context. In the remainder of the course we will use these generalizations without further comment.

Now we are ready to define the homology groups of a topological space with coefficients in an abelian group G.

Definition. Let X be a topological space. For an abelian group G we define

$$\mathrm{C}_*(X;G) \;:=\; \mathrm{C}_*(X) \otimes G.$$

As discussed above, we obtain a chain complex and we denote by

$$\mathrm{H}_k(X;G) \;:=\; \mathrm{H}_k(\mathrm{C}_*(X;G), \partial \otimes \mathrm{id})$$

the corresponding homology groups. We call $\mathrm{H}_k(X;G)$ the *k-th homology group of X with G-coefficients*. We define the notions of "cycle", "homologous" and "null-homologous" in exactly the same way as on page 1081.

Example. We consider an electric circuit as a finite undirected abstract graph X where at the vertices we have resistors or capacitors. Kirchhoff's current rule, see e.g.

`https://en.wikipedia.org/wiki/Kirchhoff's_circuit_laws`

states that

> *at any node (vertex) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node,*

or equivalently

> *the algebraic sum of currents in a network meeting at a point is zero.*

[878] Note that if G has also the structure of a commutative ring, then this becomes a chain complex of G-modules.

Put differently, a current defines a 1-cycle in $\mathrm{C}_1(X;\mathbb{R})$.[879]

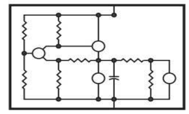
electric circuit 

Figure 873

We start out with the following three observations. Let X be a topological space.

(1) If the coefficients are given by a commutative ring R, e.g. if $R=\mathbb{Z}_n$ or $R=\mathbb{Q},\mathbb{R},\mathbb{C}$, then, as we pointed out above, each $\mathrm{C}_n(X;R)=\mathrm{C}_n(X)\otimes R$ is an R-module, each boundary map is an R-module homomorphism, and each homology group $\mathrm{H}_n(X;R)$ is also an R-module.
(2) From Lemma 57.3 (3) it follows immediately that we have natural isomorphisms $\mathrm{C}_*(X;\mathbb{Z})\xrightarrow{\cong}\mathrm{C}_*(X)$ and $\mathrm{H}_*(X;\mathbb{Z})\xrightarrow{\cong}\mathrm{H}_*(X)$. In the following we will identify these groups and we will go back and forth between these two notations, depending on which one is more convenient.
(3) Let G be an abelian group. Given any singular k-simplex σ, given any $a\in\mathbb{Z}$ and any $g\in G$ we have by definition of the tensor product that $a\sigma\otimes g=\sigma\otimes ng$. From this observation it follows that any element in $\mathrm{C}_k(X;G)$ can be written as

$$\sum_{i=1}^{n}\sigma_i\otimes g_i \quad \text{where } \sigma_1,\dots,\sigma_n \text{ are singular } k\text{-simplices and } g_1,\dots,g_n\in G.$$

Most results for homology groups of topological spaces carry over, with the obvious modifications, to the theory of homology groups with G-coefficients. For example, given an abelian group G we have the following statements:

(1) A map $f\colon X\to Y$ between two topological spaces X and Y induces a chain map
$$f_*\otimes\mathrm{id}\colon\ \mathrm{C}_*(X;G)\to\mathrm{C}_*(Y;G)$$
which in turn, for each $k\in\mathbb{N}_0$, induces a homomorphism
$$(f_*\otimes\mathrm{id})_*\colon\ \mathrm{H}_k(X;G)\to\mathrm{H}_k(Y;G).$$
To simplify the notation we usually denote this induced map on homology groups also by f_*.
(2) The k-th homology group with G-coefficients defines a functor from the category of topological spaces to the category of abelian groups.
(3) If R is a commutative ring, then a map $f\colon X\to Y$ between topological spaces induces a chain map of R-homomorphisms $\mathrm{C}_*(X;R)\to\mathrm{C}_*(Y;R)$ and R-homomorphisms $f_*\colon\ \mathrm{H}_k(X;R)\to\mathrm{H}_k(Y;R)$. Furthermore the k-th homology group with R-coefficients defines a functor from the category $\mathcal{Top}$ of topological spaces to the category $R-\mathcal{Mod}$ of R-modules.

[879] This observation and the figure are due to Ghrist's excellent book on applied topology [**Gh14**].

(4) The augmentation map $\epsilon\colon \mathrm{H}_0(X;G) \to G$ is defined almost the same way as in Lemma 41.4. More precisely, it is given by the map
$$\begin{array}{rcl} \epsilon_X\colon \mathrm{H}_0(X;G) & \to & G \\ \Big[\sum\limits_{i=1}^{k}\sigma_i \otimes g_i\Big] & \mapsto & \sum\limits_{i=1}^{k} g_i. \end{array}$$
If X is path-connected, then the argument of Proposition 41.5 implies immediately that the augmentation map $\epsilon\colon \mathrm{H}_0(X;G) \to G$ is an isomorphism. At times we will use this natural isomorphism to make the identification $\mathrm{H}_0(X;G) = G$.

(5) Precisely as on page 1105 we can use the augmentation map $\epsilon\colon \mathrm{H}_0(X;G) \to G$ to define reduced homology groups $\widetilde{\mathrm{H}}_k(X;G)$ with G-coefficients.

(6) We saw in Proposition 42.5 that if $f,g\colon X \to Y$ are two homotopic maps between topological spaces, then the induced maps $f_*, g_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ are chain homotopic. By tensoring the chain homotopies we see that the corresponding induced maps $\mathrm{C}_*(X) \otimes G \to \mathrm{C}_*(Y) \otimes G$ are also chain homotopic, and we see that for any $n \in \mathbb{N}_0$ we have
$$f_* = g_*\colon \mathrm{H}_n(X;G) \;\to\; \mathrm{H}_n(Y;G).$$
In particular, if X and Y are homotopy equivalent topological spaces, then for any $n \in \mathbb{N}_0$ we have an isomorphism
$$\mathrm{H}_n(X;G) \;\xrightarrow{\cong}\; \mathrm{H}_n(Y;G).$$

(7) Given a pair (X,A) of topological spaces we can tensor the relative chain complex $\mathrm{C}_*(X,A)$ with G. We obtain the *relative chain complex* $\mathrm{C}_*(X,A;G) := \mathrm{C}_*(X,A) \otimes G$. The corresponding homology groups are the *relative homology groups* $\mathrm{H}_k(X,A;G)$.

(8) Let (X,B,A) be a triple of topological spaces. As pointed out on page 1120, the relative chain groups $\mathrm{C}_n(X,B;G)$ are free abelian groups. Thus it follows from the discussion on page 1398 that we obtain a short exact sequence of chain complexes $0 \to \mathrm{C}_*(B,A;G) \to \mathrm{C}_*(X,A;G) \to \mathrm{C}_*(X,B;G) \to 0$. By applying the obvious generalization of Proposition 43.11 to this short exact sequence we obtain the following long exact sequence of homology groups with G-coefficients:
$$\ldots \to \mathrm{H}_n(B,A;G) \xrightarrow{i_*} \mathrm{H}_n(X,A;R) \xrightarrow{p_*} \mathrm{H}_n(X,B;G) \xrightarrow{\partial} \mathrm{H}_{n-1}(B,A;G) \to \ldots$$

(9) We have an obvious version of the Mayer–Vietoris Theorem 46.5 for homology groups with G-coefficients. The proof is basically identical to the proof of Theorem 46.5, we only have to note that when we consider the exact sequence
$$0 \to \mathrm{C}_*(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{C}_*(A) \oplus \mathrm{C}_*(B) \xrightarrow{i_A + i_B} \mathrm{C}_*^{\{A,B\}}(X) \to 0,$$
then it is a consequence of the discussion on page 1398 that the corresponding sequence with G-coefficients is also exact.

(10) We also have an obvious versions of the Excision Theorems 43.19 and 43.20, the proof for G-coefficients is almost verbatim the same as before.

(11) For a CW-complex X we can also consider the corresponding cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X;G) := \mathrm{C}_*^{\mathrm{CW}}(X) \otimes G$ and the corresponding cellular homology groups

$\mathrm{H}_*^{\mathrm{CW}}(X;G)$. Then, as in Proposition 48.4, one can show that there exists a natural isomorphism $\mathrm{H}_*(X;G) \xrightarrow{\cong} \mathrm{H}_*^{\mathrm{CW}}(X;G)$. We can draw the following conclusions:

(a) If X is an n-dimensional CW-complex, then $\mathrm{H}_i(X;G) = 0$ for $i \geq n+1$.

(b) If X has only finitely many n-cells, then given any field $\mathbb{F}$ the vector space $\mathrm{H}_n(X;\mathbb{F})$ is finite-dimensional.

(12) Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subsets such that $X = \bigcup\limits_{i\in\mathbb{N}} X_i$ and such that one the following three conditions holds:

(a) every compact subset of X is already contained in one of the X_i, or

(b) each X_i is open in X, or

(c) X is a CW-complex and each X_i is a subcomplex.

Then a modest modification of the proof of Proposition 47.4 shows that for any $n \in \mathbb{N}_0$ the inclusion induced maps $\mathrm{H}_n(X_i;G) \to \mathrm{H}_n(X;G)$ induce an isomorphism

$$\varinjlim \mathrm{H}_n(X_i;G) \xrightarrow{\cong} \mathrm{H}_n(X;G).$$

(13) If X is a finite CW-complex and if $\mathbb{F}$ is a field, then the argument of Proposition 55.1 shows that

$$\chi(X) = \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(X;\mathbb{F})).$$

Examples.

(1) Let $n \in \mathbb{N}$. As on page 935 we equip S^n with the CW-structure with one 0-cell and one n-cell. For any abelian group G it follows from (10), and similar to the discussion on page 1260, that

$$\mathrm{H}_k(S^n;G) = \mathrm{H}_k^{\mathrm{CW}}(S^n;G) \cong \begin{cases} G, & \text{if } k=0,n, \\ 0, & \text{otherwise.} \end{cases}$$

The same result holds for reduced homology with G-coefficients except that in this case we have $\widetilde{\mathrm{H}}_0(S^n;G) = 0$.

(2) We want to determine homology groups of the real projective plane $\mathbb{RP}^2$ with $\mathbb{F}_2$-coefficients. As usual we equip $\mathbb{RP}^2$ with the CW-structure with one 0-cell, one 1-cell and one 2-cell that we introduced on page 1276. In the proof of Proposition 48.10 we saw that the cellular chain complex is given by

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0,$$

i.e. $\mathrm{H}_2(\mathbb{RP}^2) = 0$, $\mathrm{H}_1(\mathbb{RP}^2) = \mathbb{Z}_2$ and $\mathrm{H}_0(\mathbb{RP}^2) = \mathbb{Z}$. On the other hand, if we tensor the above cellular chain complex with $\mathbb{F}_2$, then using Lemma 57.3 (3) we obtain the chain complex

$$0 \to \mathbb{F}_2 \xrightarrow{\cdot 2 = 0} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \to 0.$$

Using the isomorphism between cellular and singular homology with $\mathbb{F}_2$-coefficients we obtain that $\mathrm{H}_2(\mathbb{RP}^2;\mathbb{F}_2) = \mathbb{F}_2$, $\mathrm{H}_1(\mathbb{RP}^2;\mathbb{F}_2) = \mathbb{F}_2$ and $\mathrm{H}_0(\mathbb{RP}^2;\mathbb{F}_2) = \mathbb{F}_2$.

Note that even though the homology groups of $\mathbb{RP}^2$ with $\mathbb{Z}$-coefficients and $\mathbb{F}_2$-coefficients are quite different, in both cases the alternating sums of the dimensions and ranks give, as promised in Proposition 55.1 and in (10) above, the Euler characteristic. Indeed, we have

$$\operatorname{rank}(\mathrm{H}_0(X)) - \operatorname{rank}(\mathrm{H}_1(X)) + \operatorname{rank}(\mathrm{H}_2(X)) \;=\; 1-0+0 \;=\; 1,$$

and

$$\dim_{\mathbb{F}_2}(\mathrm{H}_0(X;\mathbb{F}_2)) - \dim_{\mathbb{F}_2}(\mathrm{H}_1(X;\mathbb{F}_2)) + \dim_{\mathbb{F}_2}(\mathrm{H}_2(X;\mathbb{F}_2)) \;=\; 1-1+1 \;=\; 1.$$

(3) Basically the same argument as in (3) shows that for any $n \in \mathbb{N}$ we have

$$\mathrm{H}_k(\mathbb{RP}^n;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0,\dots,n, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, it is not difficult to see that for $m < n$ the inclusion induced maps $\mathrm{H}_k(\mathbb{RP}^m;\mathbb{F}_2) \to \mathrm{H}_k(\mathbb{RP}^n;\mathbb{F}_2)$ are isomorphisms for $k = 0,\dots,m$. It follows that for any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\mathbb{RP}^\infty;\mathbb{F}_2) \underset{\substack{\uparrow \\ \text{see page 1402}}}{=} \varinjlim \mathrm{H}_k(\mathbb{RP}^n;\mathbb{F}_2) \underset{\substack{\uparrow \\ \text{by the above}}}{=} \mathbb{F}_2.$$

Remark. Why should we study the homology groups of a topological space with G-coefficients? There are several answers one can give:

(1) At times it is easier to work over a field instead of working over the ring $\mathbb{Z}$. For example, if we only want to determine the Euler characteristic of a CW-complex we can now use a field, even a very simple one like $\mathbb{F}_2$, instead of working with the integers.

(2) If we work over the field $\mathbb{F}_2$ then many calculations become particularly simple, since we do not need to worry about signs. Even in the previous example we already saw that working with $\mathbb{F}_2$-coefficients can simplify calculations.

(3) Recall that given a smooth manifold M and $k \in \mathbb{N}_0$ we denote its de Rham cohomology groups $\mathrm{H}^k_{\mathrm{dR}}(M)$, which is in fact a real vector space. For each k we can now also study the homology groups $\mathrm{H}_k(M;\mathbb{R})$ which are also real vector spaces. The notations $\mathrm{H}^k_{\mathrm{dR}}(M)$ and $\mathrm{H}_k(M;\mathbb{R})$ look similar and the names "de Rham cohomology" and "homology" also sound similar. We will study the relationship between these two objects later on in Chapters 75 and 79.

We conclude this section with the following elementary lemma which follows immediately from the definitions.

Lemma 57.7.

(1) *Let X be a topological space and let R be a commutative ring with neutral element 1_R. The maps*

$$\begin{aligned} \varphi_*\colon\ \mathrm{C}_*(X) &\;\to\; \mathrm{C}_*(X;R) \\ \Big[\sum_{i=1}^{k} a_i\cdot\sigma_i\Big] &\;\mapsto\; \Big[\sum_{i=1}^{k} a_i\cdot\sigma_i \otimes 1_R\Big] \end{aligned}$$

form a chain map, in particular for each $n \in \mathbb{N}_0$ we obtain an induced map

$$\varphi_*\colon\ \mathrm{H}_n(X) \;\to\; \mathrm{H}_n(X;R).$$

(2) *Let X be a topological space and let $\varphi\colon A \to B$ be a homomorphism of abelian groups. Then the maps*

$$\begin{array}{rcl} \varphi_*\colon \; \mathrm{C}_*(X;A) & \to & \mathrm{C}_*(X;B) \\ \Big[\sum\limits_{i=1}^{k} \sigma_i \otimes a_i\Big] & \mapsto & \Big[\sum\limits_{i=1}^{k} \sigma_i \otimes \varphi(a_i)\Big] \end{array}$$

form a chain map, in particular for each $n \in \mathbb{N}_0$ we obtain an induced map

$$\varphi_*\colon \; \mathrm{H}_n(X;A) \;\to\; \mathrm{H}_n(X;B).$$

(3) *Given a topological space X and $n \in \mathbb{N}_0$ the maps $A \mapsto \mathrm{H}_n(X;A)$ from (1) define a covariant functor from the category of abelian groups $\mathcal{A}b\mathcal{G}r$ to itself.*

(4) *The maps from (1) define a natural transformation from "homology with A-coefficients" to "homology with B-coefficients", i.e. given a map $f\colon X \to Y$ of topological spaces we get a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}_n(X;A) & \xrightarrow{f_*} & \mathrm{H}_n(Y;A) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{H}_n(X;B) & \xrightarrow{f_*} & \mathrm{H}_n(Y;B). \end{array}$$

The above statements also generalize in an obvious way to pairs of topological spaces.

57.3. **Exact functors.** The following question naturally arises from the discussion in the previous section:

Question 57.8. *Given a topological space X and an abelian group G, what is the relationship between the usual homology groups of X and the homology groups with G-coefficients?*

The example on page 1402 shows that for better or worse the naive guess that perhaps $\mathrm{H}_k(X;G) \cong \mathrm{H}_k(X) \otimes G$ is in general not correct. More precisely, we saw that

$$\mathbb{F}_2 = \mathrm{H}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \;\neq\; \mathrm{H}_2(\mathbb{R}\mathrm{P}^2) \otimes \mathbb{F}_2 = 0.$$

In the following section we will see that, despite this sobering example, one can always deduce the homology groups $\mathrm{H}_*(X;G)$ of a topological space X from the usual homology groups $\mathrm{H}_*(X)$. But the connection between these homology groups will turn out to be less naive than one might have thought initially.

Before we can hope to answer Question 57.8 we have to introduce many more definitions and algebraic tools. We start out with the following definition from category theory.

Definition. Let $F\colon \mathcal{A}b\mathcal{G}r \to \mathcal{A}b\mathcal{G}r$ be a covariant functor from the category $\mathcal{A}b\mathcal{G}r$ of abelian groups to itself.

(1) We say F is *left-exact*, if for every exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C$$

of abelian groups the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(j)} F(C)$$

is also exact.

(2) We say F is *right-exact*, if for every exact sequence

$$A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

of abelian groups the sequence

$$F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(j)} F(C) \longrightarrow 0$$

is also exact.

(3) We say F is *exact* if it is left-exact and right-exact, put differently, the functor F is exact, if for every exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

of abelian groups the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(j)} F(C) \longrightarrow 0$$

is also exact.

Remark. Let $\varphi\colon A \to B$ be a homomorphism between two abelian groups. Furthermore let $F\colon \mathcal{AbGr} \to \mathcal{AbGr}$ be a covariant functor from the category $\mathcal{AbGr}$ of abelian groups to itself.

(1) If F is a left-exact functor, then "taking F" commutes with "taking kernels", more precisely, if $\varphi\colon A \to B$ is a homomorphism, then by applying F to the exact sequence $0 \to \ker(\varphi) \to A \to B$ we see that $F(\ker(\varphi)) = \ker(\varphi_*\colon F(A) \to F(B))$.

(2) Similarly, if F is right-exact, then "taking F" commutes with "taking cokernels", i.e. if $\varphi\colon A \to B$ is a homomorphism, then by applying F to the exact sequence $A \to B \to \operatorname{coker}(\varphi) \to 0$ we see that $F(\operatorname{coker}(\varphi)) = \operatorname{coker}(\varphi_*\colon F(A) \to F(B))$.

Lemma 57.9. *Let G be an abelian group. If G is free abelian, then tensoring with G is exact.*

Proof. Let

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

be a short exact sequence and let $G \cong \mathbb{Z}^{(S)}$ be a free abelian group. It follows from Lemma 57.3 (1) and (3) that tensoring the above short exact sequence with $G \cong \mathbb{Z}^{(S)}$ results in an exact sequence that is isomorphic to the direct sum of S copies of the original sequence, hence it is again exact. ■

Example. In contrast to the previous lemma we will now see that tensoring with an abelian group G is in general not a *left-exact* functor. For example for any $k \in \mathbb{N}_{\geq 2}$ the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot k} \mathbb{Z} \longrightarrow \mathbb{Z}_k$$

is exact, but after tensoring with $\mathbb{Z}_k$ we obtain, using Lemma 57.3 (3), the sequence

$$0 \longrightarrow \mathbb{Z}_k \xrightarrow[=0]{\cdot k} \mathbb{Z}_k \longrightarrow \mathbb{Z}_k$$

that is not exact.

In contrast to the previous example we will now see that tensoring with an abelian group G is a *right-exact* functor.

Lemma 57.10. *As usual we denote by $\mathcal{AbGr}$ the category of abelian groups. For every abelian group G the functor*

$$\begin{array}{rcl} \mathcal{AbGr} & \to & \mathcal{AbGr} \\ A & \mapsto & A \otimes G \end{array}$$

is right-exact.

Proof. Let

$$A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

be an exact sequence of abelian groups and let G be an abelian group. We have to show that the sequence

$$A \otimes G \xrightarrow{i \otimes \mathrm{id}} B \otimes G \xrightarrow{j \otimes \mathrm{id}} C \otimes G \to 0$$

is also exact.

(a) Since $j\colon B \to C$ is an epimorphism it follows immediately from Lemma 57.1 (3) that the homomorphism

$$\begin{array}{rcl} j \otimes \mathrm{id}\colon B \otimes G & \to & C \otimes G \\ \sum\limits_{i=1}^{n} b_i \otimes g_i & \mapsto & \sum\limits_{i=1}^{n} j(b_i) \otimes g_i \end{array}$$

is also an epimorphism.

(b) By Lemma 57.4 we have $(j \otimes \mathrm{id}) \circ (i \otimes \mathrm{id}) = (j \circ i) \otimes \mathrm{id} = 0$, i.e. we have

$$\mathrm{im}(i \otimes \mathrm{id}\colon A \otimes G \to B \otimes G) \subset \ker(j \otimes \mathrm{id}\colon B \otimes G \to C \otimes G).$$

(c) By (b) it remains to show that

$$\ker(j \otimes \mathrm{id}\colon B \otimes G \to C \otimes G) \subset \mathrm{im}(i \otimes \mathrm{id}\colon A \otimes G \to B \otimes G).$$

Put differently it remains to show that the homomorphism

$$\Psi\colon B \otimes G / \mathrm{im}(i \otimes \mathrm{id}\colon A \otimes G \to B \otimes G) \to C \otimes G$$

that is induced by $j \otimes \mathrm{id}$, i.e. that is given by[880]

$$[b \otimes g] \mapsto j(b) \otimes g,$$

is a monomorphism. We will do so by providing a left-inverse for Ψ, i.e. we will provide a map Φ with $\Phi \circ \Psi = \mathrm{id}$.

Claim. The map

$$\begin{array}{rcl} \varphi\colon C \times G & \to & B \otimes G / \mathrm{im}(i \otimes \mathrm{id}\colon A \otimes G \to B \otimes G) \\ (c, g) & \mapsto & b \otimes g \quad \text{where } b \in B \text{ is an element with } j(b) = c. \end{array}$$

is well-defined.

Let $(c, g) \in C \times G$. Since j is by hypothesis surjective we know that there exists a $b \in B$ with $j(b) = c$. We have to show that $\varphi(c, g)$ is independent of the choice of b. So let $b, b' \in B$ with $j(b) = j(b') = c$, i.e. with $j(b - b') = 0$. From the exactness

[880]Note that it is a consequence of (b) that Ψ is well-defined.

of the original short exact sequence at B it follows that $b-b'=i(a)$ for an $a\in A$. Then we obtain that

$$b\otimes g-b'\otimes g \;=\; (b-b')\otimes g \;=\; i(a)\otimes g\in \operatorname{im}(i\otimes \mathrm{id}\colon A\otimes G\to B\otimes G).$$

Thus we have proved that φ is well-defined. ⊞

The map φ is easily seen to bilinear. It follows from Lemma 57.2 (2) that φ induces a well-defined map

$$\begin{array}{rcl}\Phi\colon C\otimes G & \to & B\otimes G/\operatorname{im}(i\otimes \mathrm{id}\colon A\otimes G\to B\otimes G)\\ \sum\limits_{i=1}^{k} c_i\otimes g_i & \mapsto & \sum\limits_{i=1}^{k}\varphi(c_i,g_i).\end{array}$$

By construction we have $(\Phi\circ\Psi)([b\otimes g])=[b\otimes g]$ for all $b\in B$ and $g\in G$. But the elements $b\otimes g$ generate $B\otimes G$, hence $\Phi\circ\Psi=\mathrm{id}$, i.e. the map Φ is indeed a right-inverse to the map Ψ. ■

57.4. **The G-torsion of an abelian group.** Our final goal is to relate the homology groups $\mathrm{H}_*(C)$ of a chain complex to the homology groups $\mathrm{H}_*(C;G)$ of the same chain complex with G-coefficients. To state the theorem relating these groups we need to introduce the G-torsion of an abelian group.

We start our discussion with the definition of a free abelian group that we already gave on page 580, we also recall some basic facts regarding free abelian groups, and we introduce one new definition.

(1) Let S be a non-empty set. We refer to

$$\mathbb{Z}^{(S)} := \text{all maps from } S \text{ to } \mathbb{Z} \text{ which are non-zero for only finitely many } s\in S$$

as the free abelian group generated by S.

(2) A group G is called *free abelian* if there exists a set S and also an isomorphism $G\xrightarrow{\cong}\mathbb{Z}^{(S)}$.

(3) It follows easily from the definitions that a group G is free abelian if and only if there exists a subset $\mathcal{B}=\{b_i\}_{i\in I}\subset G$ with the following property: For any abelian group A and each choice of elements $a_i\in A, i\in I$ there exists a unique homomorphism $\varphi\colon G\to A$ such that $\varphi(b_i)=a_i$ for all $i\in I$. As on page 582 we call such a subset $\mathcal{B}$ a *basis* of G.

(4) A *free resolution* of an abelian group H is an exact sequence

$$\dots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

where the F_i are free abelian groups. If there exists an $n\in\mathbb{N}_0$ with $F_n\neq 0$ but $F_m=0$ for $m>n$, then we refer to n as the *length* of the resolution. If no such n exists, then we say that the length of the resolution is infinite.

Examples.

(1) Let $m\in\mathbb{N}$, then a free resolution for $H=\mathbb{Z}_m$ is given by

$$0\to\mathbb{Z}\xrightarrow{\cdot m}\mathbb{Z}\to\mathbb{Z}_m\to 0.$$

(2) Let G be a *finitely generated* abelian group. By the classification of finitely generated abelian groups, see Theorem 19.4, there exists an $r \in \mathbb{N}_0$, non-zero natural numbers $a_1, \dots, a_k$ and an isomorphism

$$\Phi\colon G \xrightarrow{\cong} \mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{a_i}.$$

Thus we obtain a free resolution[881]

$$0 \to \mathbb{Z}^k \xrightarrow{0 \oplus \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_k \end{pmatrix}} \mathbb{Z}^r \oplus \mathbb{Z}^k \to \underbrace{\mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{a_i}}_{\cong G} \to 0.$$

In most cases, when we want to work with explicit examples, this type of free resolution will turn out to be the most useful free resolution.

(3) We recall the following basic fact: if M is an abelian group, then by Lemma 19.1 there exists a unique homomorphism $\alpha(M)\colon \mathbb{Z}^{(M)} \to M$ such that for each $m \in M$ we have $\alpha(M)(m) = m$. Evidently this map is an epimorphism. Now let H be an abelian group. We consider

$$\mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0.$$

The group $\mathbb{Z}^{(H)}$ is free abelian and $\alpha(H)$ is an epimorphism. In particular the sequence is exact. We set $\mathrm{H}_1 := \ker\left(\mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H\right)$. Then

$$\mathrm{H}_1 \hookrightarrow \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0$$

is exact. Now we consider

$$\mathbb{Z}^{(\mathrm{H}_1)} \xrightarrow{\alpha(\mathrm{H}_1)} \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0.$$

This sequence is exact and both the groups $\mathbb{Z}^{(\mathrm{H}_1)}$ and $\mathbb{Z}^{(H)}$ are free abelian. Now we set $\mathrm{H}_2 := \ker\left(\mathbb{Z}^{(\mathrm{H}_1)} \xrightarrow{\alpha(H)} H\right)$ and we iterate this procedure. This way we obtain a free resolution of H.[882] Note that this free resolution is unambiguously defined, we did not make any choices along the way. Therefore we refer to this free resolution of H as the *canonical free resolution of H*.

The last example provides the proof for the following lemma.

Lemma 57.11. *Every abelian group (not necessarily finitely generated) admits a canonical free resolution.*

The following definition is one of the key definitions in homological algebra.

Definition. Let G and H be abelian groups. We denote by

$$\dots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

[881]Note that the homomorphism to the left is a monomorphism since $a_1, \dots, a_k$ are non-zero.
[882]Note that this procedure does not stop, i.e. it produces an infinite free resolution.

the canonical free resolution from page 1408. We define the *n-th G-torsion group of H* as

$$\mathrm{Tor}_n(H,G) \ := \ \mathrm{H}_n(F_* \otimes G).$$

The following proposition is the key to calculating torsion-groups.

Proposition 57.12. *Let G and H be abelian groups and let*

$$\dots \ \to \ F_2 \ \xrightarrow{f_2} \ F_1 \ \xrightarrow{f_1} \ F_0 \ \xrightarrow{f_0} \ H \ \to \ 0$$

be a free resolution of H. Then there exists a natural isomorphism

$$\mathrm{Tor}_n(H,G) \ \xrightarrow{\cong} \ \mathrm{H}_n(F_* \otimes G).$$

Example. In many situations we are only interested in determining the isomorphism type of $\mathrm{Tor}_n(H,G)$ where H is a given finitely generated abelian group. In such situations it is usually much more convenient to work with the free resolution given on page 1408. For example, let $m,n \in \mathbb{N}$. We want to determine the isomorphism type of $\mathrm{Tor}_i(\mathbb{Z}_m,\mathbb{Z}_n)$ for $i \in \mathbb{N}_0$. We will do so by doing the following three step process:

(1) We pick the free resolution $0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to \mathbb{Z}_m \to 0$,
(2) we tensor this free resolution of $\mathbb{Z}_m$ with $\mathbb{Z}_n$ and from Lemma 57.3 (3) we obtain the sequence $0 \to \mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_n \to \mathbb{Z}_m \otimes \mathbb{Z}_n \to 0$,
(3) we drop the term to the right and we compute the homology groups.

Now using an argument as in the calculation of the tensor product $\mathbb{Z}_m \otimes \mathbb{Z}_n$ on page 1396 we see that

$$\mathrm{Tor}_1(\mathbb{Z}_m,\mathbb{Z}_n) \ = \ \mathrm{H}_1\left(0 \to \mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_n \to 0\right) \ = \ \ker(\mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_n) \quad \cong \ \mathbb{Z}_{\gcd(m,n)}$$

and

$$\mathrm{Tor}_0(\mathbb{Z}_m,\mathbb{Z}_n) \ = \ \mathrm{H}_0\left(0 \to \mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_n \to 0\right) \ = \ \mathrm{coker}\left(\mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_n\right) \ \cong \ \mathbb{Z}_{\gcd(m,n)}.$$

We will fill in the details in Exercise 57.10.

The proof of Proposition 57.12 will require some effort. We start out with the following definition.

Definition. Let $\alpha\colon H \to H'$ be a homomorphism between abelian groups. Furthermore let F_* be a free resolution of H and let F'_* be a free resolution of H'. An *extension of α to the free resolutions F_* and F'_** is a sequence of homomorphisms $\alpha_i\colon F_i \to F'_i$, $i \in \mathbb{N}_0$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\
\dots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H' & \longrightarrow & 0.
\end{array}$$

Note that the horizontal exact sequences on top and bottom are in particular chain complexes and the maps α_* define a chain map from one complex to the other.

The following lemma shows in particular that extensions of homomorphisms between abelian groups always exist.

Lemma 57.13. *Let $\alpha\colon H \to H'$ be a homomorphism between abelian groups. Furthermore let F_* be a free resolution of H and let F'_* be a free resolution of H'.*

(1) *There exists an extension of α to the free resolutions F_* and F'_*.*
(2) *Any two extensions α_* and α'_*, viewed as chain maps, are chain homotopic.*
(3) *Let $\alpha_n\colon F_n \to F'_n$, $n \in \mathbb{N}_0$ be an extension of $\alpha\colon H \to H$ and let G be an abelian group. The map*

$$\begin{array}{rcl}\alpha_n \otimes \mathrm{id}\colon \mathrm{H}_n(F_* \otimes G) & \to & \mathrm{H}_n(F'_* \otimes G) \\ \left[\sum\limits_{i=1}^{k} c_i \otimes g_i\right] & \mapsto & \left[\sum\limits_{i=1}^{k} \alpha_n(c_i) \otimes g_i\right]\end{array}$$

is well-defined and it is independent of the choice of the extension.

Proof (∗). Let F_* be a free resolution of H and let F'_* be a free resolution of H'. Statements (1) and (2) are both a straightforward consequence of some mild diagram chasing and the flexibility which Lemma 19.1 provides for defining homomorphisms starting from a free abelian group. In the following we fill in all the details.

(1) We have to show that there exists a sequence $\{\alpha_i\}_{i\in\mathbb{N}_0}$ of homomorphisms such that the following diagram commutes:

$$\begin{tikzcd} \dots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\ \dots \arrow[r] & F'_2 \arrow[r, "f'_2"] & F'_1 \arrow[r, "f'_1"] & F'_0 \arrow[r, "f'_0"] & H' \arrow[r] & 0. \end{tikzcd}$$

First we define the homomorphism α_0. We choose a basis $\mathcal{B}_0$ of the free abelian group F_0. For each $x_i \in \mathcal{B}_0$ we choose an $x'_i \in F'_0$ with $f'_0(x'_i) = \alpha(f_0(x_i))$. This is possible, since f'_0 is surjective. By Lemma 19.1 there exists a unique homomorphism $\alpha_0\colon F_0 \to F'_0$ with $\alpha_0(x_i) = x'_i$ for all $i \in I$. It is now evident that the right square commutes.

Now we turn to the definition of the map α_1. We choose a basis $\mathcal{B}_1$ of the free abelian group F_1. Let $x_i \in \mathcal{B}_1$. It follows from the commutativity of the square on the right that

$$f'_0((\alpha_0 \circ f_1)(x_i)) \;=\; \alpha((\underbrace{f_0 \circ f_1}_{=0})(x_i)) \;=\; 0.$$

Since the lower sequence is exact there exists an $x'_i \in F'_1$ with $f'_1(x'_i) = (\alpha_0 \circ f_1)(x_i)$. Again by Lemma 19.1 there exists a unique homomorphism $\alpha_1\colon F_1 \to F'_1$ such that $\alpha_1(x_i) = x'_i$ for all $i \in I$. It is now evident that the "second square from the right" also commutes. It is now clear how inductively we can construct the homomorphisms $\alpha_2, \alpha_3, \dots$.

(2) Now let $\alpha_i, \alpha'_i\colon F_i \to F'_i, i \in \mathbb{N}_0$ be two extensions of α to the free resolutions F_* and F'_*. We have to find a chain homotopy from the chain map α_i to the chain map α'_i. The maps $\beta_i = \alpha_i - \alpha'_i$ are an extension of the zero map $\beta = 0\colon H \to H'$. Therefore it suffices to find a chain homotopy λ from $\beta_i = \alpha_i - \alpha'_i$ to the zero chain map. The construction of the chain homotopy is very similar to the construction of the maps α_i in the first part of the proof.

Recall that we have to find a sequence of homomorphisms λ_i, $i \in \mathbb{N}_0$ such that for any $i \in \mathbb{N}_0$ we have

$$f'_{i+1} \circ \lambda_i + \lambda_{i-1} \circ f_i \;=\; \beta_i.$$

In the following discussion it is convenient to keep an eye on the following diagram that contains all the homomorphisms that we are given and that we need to construct:

$$\begin{tikzcd}
\dots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\beta_2"] \arrow[dl, "\lambda_2"'] & F_1 \arrow[r, "f_1"] \arrow[d, "\beta_1"] \arrow[dl, "\lambda_1"'] & F_0 \arrow[r, "f_0"] \arrow[d, "\beta_0"] \arrow[dl, "\lambda_0"'] & H \arrow[r] \arrow[d, "\beta=0"] \arrow[dl, "\lambda_{-1}"'] & 0 \\
\dots \arrow[r] & F'_2 \arrow[r, "f'_2"'] & F'_1 \arrow[r, "f'_1"'] & F'_0 \arrow[r, "f'_0"'] & H' \arrow[r] & 0.
\end{tikzcd}$$

We put $\lambda_{-1} = 0$. Next we need to find a map $\lambda_0 \colon F_0 \to F'_1$ such that $\beta_0 = f'_1 \circ \lambda_0$. For each $x \in \mathcal{B}_0$ we choose an $x' \in F'_1$ with $f'_1(x') = \beta_0(x)$. The existence of such x' follows from the observation that $f'_0(\beta_0(x)) = \beta(f_0(x)) = 0$ and from the exactness of the lower sequence.

Now we suppose that we have already found maps $\lambda_0, \dots, \lambda_{i-1}$ which have the desired property that

$$(*) \qquad f'_j \circ \lambda_{j-1} + \lambda_{j-2} \circ f_{j-1} \;=\; \beta_{j-1} \quad \text{for } j = 0, \dots, i.$$

We choose again a basis $\mathcal{B}_i$ of the free abelian group F_i. We have to assign to x an element $x' \in F'_{i+1}$ which satisfies the equation

$$f'_{i+1}(x') \;=\; \beta_i(x) - \lambda_{i-1}(f_i(x)).$$

From the exactness of the lower sequence it follows that $\operatorname{im}(f'_{i+1}) = \ker(f'_i)$. Thus it suffices to show that $f'_i \circ (\beta_i - \lambda_{i-1} \circ f_i) = 0$. It turns out that indeed we have

$$\begin{aligned}
f'_i \circ (\beta_i - \lambda_{i-1} \circ f_i) \;&=\; f'_i \circ \beta_i - f'_i \circ \lambda_{i-1} \circ f_i \;\underset{\substack{\uparrow \\ \text{since the } \beta_i\text{'s are a chain map, i.e. } f'_i \circ \beta_i = \beta_{i-1} \circ f_i}}{=}\; \beta_{i-1} \circ f_i - f'_i \circ \lambda_{i-1} \circ f_i \\
&=\; (\beta_{i-1} - f'_i \circ \lambda_{i-1}) \circ f_i \;\underset{\substack{\uparrow \\ \text{by } (*)}}{=}\; \lambda_{i-2} \circ f_{i-1} \circ f_i \;\underset{\substack{\uparrow \\ \text{since } f_{i-1} \circ f_i = 0}}{=}\; 0.
\end{aligned}$$

(3) It is straightforward to verify that the map is well-defined. It follows from (2) and Lemma 42.2 that any two extensions induce the same map on the homology groups. ■

Using Lemma 57.13 we can now prove the following lemma.

Lemma 57.14. *Let*

$$\dots \;\to\; F_2 \;\xrightarrow{f_2}\; F_1 \;\xrightarrow{f_1}\; F_0 \;\xrightarrow{f_0}\; H \;\to\; 0$$

and

$$\dots \;\to\; F'_2 \;\xrightarrow{f'_2}\; F'_1 \;\xrightarrow{f'_1}\; F'_0 \;\xrightarrow{f'_0}\; H \;\to\; 0$$

be two free resolutions of an abelian group H. We pick an extension $\alpha_i, i \in \mathbb{N}$ of the identity map $\mathrm{id}\colon H \to H$ to the free resolutions F_* and F'_*.

(1) *The maps α_* form a chain homotopy equivalence between the two chain complexes F_* and F'_*.*
(2) *Let G be an abelian group. The*

$$\mathrm{H}_n(F_* \otimes G) \xrightarrow{\cong} \mathrm{H}_n(F'_* \otimes G)$$

from Lemma 57.13 (3) is an isomorphism.

Remark. The statement of Lemma 57.14 (1) is sometimes called the *fundamental theorem of homological algebra*, see e.g. [**Lü05**, Satz 6.3].

Proof. We pick an extension $\alpha_i, i \in \mathbb{N}$ of the identity map $\mathrm{id}\colon H \to H$.

(1) We apply Lemma 57.13 to the free resolutions F'_* and F_* (i.e. we swap the roles of the free resolutions) and we obtain maps $\{\beta_i\}_{i\in\mathbb{N}_0}$ that give us the lower half of the following commutative diagram.

$$\begin{tikzcd}
\ldots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\ldots \arrow[r] & F'_2 \arrow[r, "f'_2"] \arrow[d, "\beta_2"] & F'_1 \arrow[r, "f'_1"] \arrow[d, "\beta_1"] & F'_0 \arrow[r, "f'_0"] \arrow[d, "\beta_0"] & H \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\ldots \arrow[r] & F_2 \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] & F_0 \arrow[r, "f_0"] & H \arrow[r] & 0.
\end{tikzcd}$$

The maps $\beta_i \circ \alpha_i$ now provide us with the following commutative diagram:

$$\begin{tikzcd}
\ldots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\beta_2\circ\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\beta_1\circ\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\beta_0\circ\alpha_0"] & H \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\ldots \arrow[r] & F_2 \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] & F_0 \arrow[r, "f_0"] & H \arrow[r] & 0.
\end{tikzcd}$$

This shows that the maps $\beta_i \circ \alpha_i$ define an extension of $\mathrm{id}\colon H \to H$ to the free resolutions F_i and F_i. But the identity maps also define an extension. By Lemma 57.13 the extensions $\{\beta_i \circ \alpha_i\}_{i\in\mathbb{N}_0}$ and $\{\mathrm{id}\}_{i\in\mathbb{N}_0}$ are chain homotopic. The same way one shows that the extensions $\{\alpha_i \circ \beta_i\}_{i\in\mathbb{N}_0}$ and $\{\mathrm{id}\}_{i\in\mathbb{N}_0}$ are chain homotopic.

(2) We had just shown that $\alpha_*\colon F_* \to F'_*$ is a chain homotopy equivalence. One easily verifies that $\alpha_* \otimes \mathrm{id}\colon F_* \otimes G \to F'_* \otimes G$ is also a chain homotopy equivalence. The statement now follows from Corollary 42.3. ■

We are now also in a position to prove Proposition 57.12.

Proof. Let G and H be abelian groups and let (F_*, f_*) be a free resolution of H. We denote by (F'_*, f'_*) the canonical free resolution of H. Then by Lemma 57.14 we have a canonical isomorphism

$$\mathrm{Tor}_n(H, G) \;=\; \mathrm{H}_n(F'_* \otimes G) \xrightarrow{\cong} \mathrm{H}_n(F_* \otimes G). \qquad ■$$

Remark. Using Lemma 57.13 we can now show that the torsion groups $\operatorname{Tor}_n(G, H)$ are "covariantly functorial in both entries". More precisely the following two statements hold:

(1) We fix a group G. Let $\beta\colon H \to H'$ be a homomorphism between two abelian groups. It follows from Lemma 57.13 (3) applied to the canonical free resolutions of H and H' that β induces for each n a canonical homomorphism
$$\beta_*\colon \operatorname{Tor}_n(H, G) \ \to\ \operatorname{Tor}_n(H', G).$$
One can now easily show that the maps given by
$$\begin{array}{rcl} H & \mapsto & \operatorname{Tor}_n(H, G) \\ (\beta\colon H \to H') & \mapsto & (\beta_*\colon \operatorname{Tor}_n(H, G) \to \operatorname{Tor}_n(H', G)) \end{array}$$
define a covariant functor from the category of abelian groups to the category of abelian groups.

(2) We fix a group H. Let $\alpha\colon G \to G'$ be a homomorphism between two abelian groups. Let F_* be the canonical free resolution of H. Then $\mathrm{id} \otimes \alpha$ defines a chain map from $F_* \otimes G$ to $F_* \otimes G'$ and we get an induced map
$$\alpha_*\colon \operatorname{Tor}_n(H, G) \ \to\ \operatorname{Tor}_n(H, G').$$
One can now easily show that the maps given by
$$\begin{array}{rcl} G & \mapsto & \operatorname{Tor}_n(H, G) \\ (\alpha\colon G \to G') & \mapsto & (\alpha_*\colon \operatorname{Tor}_n(H, G) \to \operatorname{Tor}_n(H, G')) \end{array}$$
define a covariant functor from the category of abelian groups to the category of abelian groups.

The following lemma shows that the 0-th torsion groups are objects that we are already familiar with.

Lemma 57.15. *Let G and H be abelian groups, then there exists a natural isomorphism* [883]
$$\operatorname{Tor}_0(H, G) \ \xrightarrow{\cong}\ H \otimes G.$$

In the remainder of this course we will use the isomorphism from Lemma 57.15 to identify the groups $\operatorname{Tor}_0(H, G)$ and $H \otimes G$.

PROOF. Let G and H be abelian groups. Let
$$\cdots \to\ F_2 \ \xrightarrow{f_2}\ F_1 \ \xrightarrow{f_1}\ F_0 \ \xrightarrow{f_0}\ H \ \to\ 0$$
be the canonical free resolution of H. We know from Lemma 57.10 that the sequence
$$(*) \qquad F_1 \otimes G \ \xrightarrow{f_1 \otimes \mathrm{id}}\ F_0 \otimes G \ \xrightarrow{f_0 \otimes \mathrm{id}}\ H \otimes G \ \to\ 0$$

[883]Note that "natural isomorphism" means in this context that given an abelian group G we have natural isomorphisms from the functor $H \mapsto \operatorname{Tor}_0(H, G)$ to the functor $H \mapsto H \otimes G$ and also from the functor $G \mapsto \operatorname{Tor}_0(H, G)$ to the functor $G \mapsto H \otimes G$.

is exact. Now we obtain the following isomorphism

$$\begin{aligned}\mathrm{Tor}_0(H,G) &\overset{\text{by definition}}{=} \mathrm{H}_0\left(\cdots \to F_1\otimes G \xrightarrow{f_1\otimes \mathrm{id}} F_0\otimes G \to 0\right)\\ &= \operatorname{coker}\left(F_1\otimes G \xrightarrow{f_1\otimes \mathrm{id}} F_0\otimes G\right) \underset{\text{isomorphism since } (*) \text{ is exact}}{\xrightarrow{f_0\otimes \mathrm{id}}} H\otimes G.\end{aligned}$$

It is straightforward to verify that these isomorphisms define in fact a natural isomorphism from the functor $H \mapsto \mathrm{Tor}_0(H,G)$ to the functor $H \mapsto H\otimes G$ and also from the functor $G \mapsto \mathrm{Tor}_0(H,G)$ to the functor $G \mapsto H \otimes G$. ■

The following lemma shows that we do not need to worry about higher torsion groups for abelian groups.

Lemma 57.16.

(1) *Every abelian group H admits a free resolution of length* 1.
(2) *For any two abelian groups H and G we have* $\mathrm{Tor}_n(H,G) = 0$ *for* $n \geq 2$.

Proof.

(1) Let H be an abelian group. If H is finitely generated, then we already saw on page 1408 that H admits a free resolution of length 1. Now suppose that H is any abelian group. We have the exact sequence

$$0 \;\to\; \ker\left(\alpha(H)\colon \mathbb{Z}^{(H)} \to H\right) \;\to\; \mathbb{Z}^{(H)} \;\xrightarrow{\alpha(H)}\; H \;\to\; 0$$

where $\alpha(H)$ is defined as in the proof of Lemma 57.11. It follows from Lemma 19.2 that the group $\ker\left(\alpha(H)\colon \mathbb{Z}^{(H)} \to H\right)$, as the subgroup of the free abelian group $\mathbb{Z}^{(H)}$, is itself free abelian. So in this general case we also found a free resolution of length 1.
(2) Let G and H be abelian groups. By (1) there exists a free resolution of length 1, i.e. of the form

$$0 \;\to\; F_1 \;\to\; F_0 \;\to\; H \;\to\; 0.$$

It follows immediately from Proposition 57.12 that $\mathrm{Tor}_n(H,G) \cong \mathrm{H}_n(F_*\otimes G) = 0$ for $n \geq 2$. ■

Definition. Given two abelian groups H and G we now write $\mathrm{Tor}(H,G) := \mathrm{Tor}_1(H,G)$ and we refer to this group as as the *G-torsion of H*.

Remark. For many purposes the long discussion of G-torsion groups can be summarized in the following statement: it follows from Proposition 57.12 that for a free resolution $0 \to F_1 \xrightarrow{i} F_0 \xrightarrow{j} H \to 0$ of an abelian group H we have

$$\mathrm{Tor}(H,G) \;\cong\; \ker\left(F_1\otimes G \xrightarrow{i\otimes \mathrm{id}} F_0\otimes G\right).$$

Now we summarize several properties of the torsion groups $\mathrm{Tor}(H,G)$ in the following lemma.

Lemma 57.17. *Let G, H, $\{\mathrm{H}_i\}_{i\in I}$ and $\{G_j\}_{j\in J}$ be abelian groups. Then the following holds:*

(1) *There exists a natural isomorphism* $\operatorname{Tor}\big(\bigoplus_i \mathrm{H}_i, G\big) \cong \bigoplus_i \operatorname{Tor}(\mathrm{H}_i, G)$.
(2) *There exists a natural isomorphism* $\operatorname{Tor}\big(H, \bigoplus_j G_j\big) \cong \bigoplus_j \operatorname{Tor}(H, G_j)$.
(3) *If H is a torsion-free abelian group, then* $\operatorname{Tor}(H, G) = 0$.
(4) *If G is a torsion-free abelian group, then* $\operatorname{Tor}(H, G) = 0$.
(5) *If T denotes the torsion subgroup*[884] *of H, then* $\operatorname{Tor}(H, G) = \operatorname{Tor}(T, G)$.
(6) *For all $n \in \mathbb{N}$ we have a natural isomorphism* $\operatorname{Tor}(\mathbb{Z}_n, G) \cong \ker\big(G \xrightarrow{\cdot n} G\big)$.
(7) *For all $n, m \in \mathbb{N}$ we have* $\operatorname{Tor}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(n,m)}$.
(8) *Given any two finitely generated abelian groups G and H the corresponding torsion group* $\operatorname{Tor}(H, G)$ *is finite.*[885]

Remark. Using this lemma one can usually determine the G-torsion of an abelian group H without any problems. For example the groups $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$ and $(\mathbb{C}, +)$ are torsion-free abelian groups[886] Thus it follows immediately from Lemma 57.17 (4) that for every abelian group H we have

$$\operatorname{Tor}(H, \mathbb{Q}) \;=\; \operatorname{Tor}(H, \mathbb{R}) \;=\; \operatorname{Tor}(H, \mathbb{C}) \;=\; 0.$$

PROOF.

(1) For each $i \in I$ we choose a free resolution F_*^i of H_i. Then $\bigoplus_i F_*^i$ is a free resolution of $\bigoplus_i \mathrm{H}_i$. The statement follows easily from the natural isomorphisms

$$\Big(\bigoplus_i F_j^i\Big) \otimes G \;\cong\; \bigoplus_i (F_j^i \otimes G)$$

of Lemma 57.3 together with Lemma 41.13 and Proposition 57.12.

(2) We pick a free resolution F_* for H. As in (1) the statement follows easily from the natural isomorphisms

$$F_i \otimes \bigoplus_j G_j \;\cong\; \bigoplus_j F_i \otimes G_j$$

of Lemma 57.3 together with Lemma 41.13 and Proposition 57.12.

(3) If H is a free abelian group, then $0 \to F_0 := H \xrightarrow{\mathrm{id}} H \to 0$ is already a free resolution, and it follows immediately from Proposition 57.12 that $\operatorname{Tor}(H, G) = 0$. Later on we will only use the special case that H is a free abelian group. Therefore we do not provide the proof of the general case of a torsion-free abelian group which is not free. We refer to [**Hat02**, p. 265] for the proof.

(4) By Lemma 57.16 (1) we can pick a free resolution $0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H$ of length 1 for H. First we consider the case that G is a free abelian group. In this case we have

[884] The torsion subgroup of an abelian group is defined as the subgroup of all elements of finite order.
[885] This fact explains the name "torsion group".
[886] Note though that in Lemma 19.9 we saw that the abelian group $(\mathbb{Q}, +)$ is not free abelian.

$G \cong \bigoplus_I \mathbb{Z}$ for an index set I. It follows that

$$\mathrm{Tor}(H,G) = \mathrm{Tor}\left(H, \bigoplus_I \mathbb{Z}\right) \underset{\substack{\uparrow \\ \text{by (1)}}}{=} \bigoplus_I \mathrm{Tor}(H,\mathbb{Z}) \underset{\substack{\uparrow \\ \text{Proposition 57.12}}}{\cong} \bigoplus_I \mathrm{H}_1(0 \to \underbrace{F_1 \otimes \mathbb{Z}}_{=F_1} \to \underbrace{F_0 \otimes \mathbb{Z}}_{=F_0} \to 0) = 0.$$

Now we no longer suppose that G is free abelian, we just suppose that G is an abelian group that is torsion-free. We denote by $\{G_i\}_{i\in I}$ the set of finitely generated subgroups of G. It follows from Theorem 19.4 that each G_i is a free abelian group. We write $i \le j$ if and only if $G_i \subset G_j$. Together with the inclusion maps the groups $\{G_i\}_{i\in I}$ form a direct system. We leave it to the reader to verify that

$$G = \varinjlim G_i.$$

We have

$$\begin{aligned}
\mathrm{Tor}(H,G) &= \mathrm{Tor}(H, \varinjlim G_i) = \mathrm{H}_1\left(0 \to F_1 \otimes \varinjlim G_i \to F_0 \otimes \varinjlim G_i\right) \\
&\underset{\substack{\uparrow \\ \text{by Lemma 57.3 (4)}}}{=} \mathrm{H}_1\left(\varinjlim (0 \to F_1 \otimes G_i \to F_0 \otimes G_i \to 0)\right) \\
&\underset{\substack{\uparrow \\ \text{Lemma 47.6}}}{=} \varinjlim \underbrace{\mathrm{H}_1(0 \to F_1 \otimes G_i \to F_0 \otimes G_i \to 0)}_{\substack{= 0, \text{ by the above, since } G_i \text{ is a} \\ \text{free abelian group}}} = 0.
\end{aligned}$$

(5) If H is a finitely generated abelian group, then it follows from Theorem 19.4 that we can write H as a direct sum $F \oplus T$, where is T the torsion subgroup and where F is a free abelian group. It follows from (1) and (2) that

$$\mathrm{Tor}(H,G) \cong \mathrm{Tor}(F \oplus T, G) \underset{\substack{\uparrow \\ \text{by (1)}}}{\cong} \mathrm{Tor}(F,G) \oplus \mathrm{Tor}(T,G) \underset{\substack{\uparrow \\ \text{by (3)}}}{=} \mathrm{Tor}(T,G).$$

For the proof in the case that H is not finitely generated we once again refer to [**Hat02**, p. 265] for a proof.

(6) We consider the free resolution

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}_n \to 0$$

for $H = \mathbb{Z}_n$. After tensoring with the abelian group G, and using Lemma 57.3 we obtain the sequence

$$0 \to G \xrightarrow{\cdot n} G \to \mathbb{Z}_n \otimes G \to 0.$$

It follows from Proposition 57.12 that $\mathrm{Tor}(\mathbb{Z}_n, G) = \ker\left(G \xrightarrow{\cdot n} G\right)$.[887]

(7) We showed this isomorphism on page 1409. Alternatively one can deduce the statement using (6).

[887] The same argument, together with Lemma 57.15 can be used to give a proof of Lemma 57.3 (3), which back then was perhaps shoved under the carpet a little too quickly.

(8) Let G and H be two finitely generated abelian groups. It is a straightforward consequence of the classification of finitely generated abelian groups, see Theorem 19.4, together with all of the above statements that the corresponding torsion group $\operatorname{Tor}(H, G)$ is finite. ■

57.5. **The Universal Coefficient Theorem.** Before we can state the Algebraic Universal Coefficient Theorem we need to introduce the following definition.

Definition. Let $(\mathrm{C}_*, \partial_*)$ be a chain complex and let G be an abelian group. Throughout this chapter, given $n \in \mathbb{N}_0$, we denote by μ the natural[888] homomorphism[889]

$$\begin{array}{rccc} \mu\colon & \mathrm{H}_n(C_*) \otimes G & \to & \mathrm{H}_n(C_* \otimes G) \\ & \sum\limits_{i=1}^{m} [c_i] \otimes g_i & \mapsto & \Big[\sum\limits_{i=1}^{m} c_i \otimes g_i\Big]. \end{array}$$

Now we can formulate and prove the following theorem which is the key result of the whole chapter.

Theorem 57.18. (The Algebraic Universal Coefficient Theorem) *Let (C_n, ∂_n) be a chain complex of free abelian groups and let G be an abelian group. Then for each $n \in \mathbb{N}_0$ there exists a natural homomorphism $p\colon \mathrm{H}_n(C; G) \to \operatorname{Tor}(\mathrm{H}_{n-1}(C), G)$ such that the following sequence is exact:*

$$0 \to \mathrm{H}_n(C) \otimes G \xrightarrow{\mu} \mathrm{H}_n(C; G) \xrightarrow{p} \operatorname{Tor}(\mathrm{H}_{n-1}(C), G) \to 0.$$

Remark. The fact the homomorphisms μ and p are natural implies that given a chain map $f\colon C \to D$ and a homomorphism $\varphi\colon G \to G'$ between abelian groups the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}_n(C) \otimes G \arrow[r, "\mu"] \arrow[d] & \mathrm{H}_n(C; G) \arrow[r, "p"] \arrow[d] & \operatorname{Tor}(\mathrm{H}_{n-1}(C), G) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{H}_n(D) \otimes G' \arrow[r, "\mu"] & \mathrm{H}_n(D; G') \arrow[r, "p"] & \operatorname{Tor}(\mathrm{H}_{n-1}(D), G') \arrow[r] & 0 \end{tikzcd}$$

commutes. Here the vertical maps are all induced by the chain map $f\colon C \to D$ and the homomorphism $\varphi\colon G \to G'$.

Proof. Let (C_n, ∂_n) be a chain complex of free abelian groups. For each $n \in \mathbb{N}_0$ we write as always $Z_n := \ker(\partial_n)$ and $B_n := \operatorname{im}(\partial_{n+1})$. Note that it follows from Lemma 19.2 that B_n and Z_n are free abelian groups.

[888] Here "natural" means that for a chain map $f_*\colon C_* \to D_*$ and a homomorphism $\varphi\colon G \to G'$ between abelian groups the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}_n(C_*) \otimes G \arrow[r, "\mu"] \arrow[d] & \mathrm{H}_n(C_* \otimes G) \arrow[d] \\ \mathrm{H}_n(D_*) \otimes G' \arrow[r, "\mu"] & \mathrm{H}_n(D_* \otimes G') \end{tikzcd}$$

where the vertical maps are induced by f and φ in the, hopefully, obvious way.

[889] It is straightforward to show that the map is well-defined. We will discuss in Exercise 57.4 whether the "obvious converse" is also well-defined.

We consider the following diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "0"] & \mathrm{C}_n \arrow[r, "\partial_n"] \arrow[d, "\partial_n"] & B_{n-1} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] \arrow[d] & \mathrm{C}_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

One sees easily that this diagram is commutative and that the horizontal sequences are exact. We can and will view the left and the right vertical columns as chain complexes. In other words, the above diagram is a short exact sequence of chain complexes. Recall that the groups B_n are free abelian groups, thus it follows from Lemmas 46.1 that each of the horizontal sequences splits.

Now let G be an abelian group. We tensor the above short exact sequence of chain complexes with G and we obtain the following diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_n \otimes G \arrow[r] \arrow[d, "0"] & \mathrm{C}_n \otimes G \arrow[r, "\partial_n \otimes \mathrm{id}"] \arrow[d, "\partial_n \otimes \mathrm{id}"] & B_{n-1} \otimes G \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \otimes G \arrow[r] \arrow[d] & \mathrm{C}_{n-1} \otimes G \arrow[r, "\partial_{n-1} \otimes \mathrm{id}"] \arrow[d] & B_{n-2} \otimes G \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

This is again a sequence of chain complexes.

Since the original short exact sequences split it follows from Lemmas 57.6 that the horizontal sequences obtained from tensoring with G are still exact. Therefore we can apply Proposition 43.11 to obtain a long exact sequence of homology groups. Since the vertical boundary maps on the left and on the right are the zero maps, we obtain the following long exact sequence

$$\ldots \to B_n \otimes G \xrightarrow{d_n} Z_n \otimes G \to \mathrm{H}_n(C;G) \xrightarrow{\partial_n} B_{n-1} \otimes G \xrightarrow{d_{n-1}} Z_{n-1} \otimes G \to \ldots$$

where the maps d_n are the connecting homomorphisms of the long exact sequence. It follows easily from the definition that the connecting homomorphism $d_n \colon B_n \otimes G \to Z_n \otimes G$ is the map that is given by $i_n \otimes \mathrm{id} \colon B_n \otimes G \to Z_n \otimes G$ where $i_n \colon B_n \to Z_n$ denotes the obvious inclusion map.[890]

[890]Why is this the case?

Using Lemma 46.4 the above discussion implies that we obtain the following short exact sequence

$$0 \to \operatorname{coker}\left(B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}} Z_n \otimes G\right) \to \mathrm{H}_n(C;G) \to \ker\left(B_{n-1} \otimes G \xrightarrow{i_{n-1}\otimes \mathrm{id}} Z_{n-1} \otimes G\right) \to 0.$$

We still have to show that the expressions left and right coincide with the desired expressions. We have natural isomorphisms

$$\underset{\text{by Proposition 57.12 since } B_n \xrightarrow{i_n} Z_n \to \mathrm{H}_n(C) \text{ is a free resolution of } \mathrm{H}_n(C)}{\operatorname{coker}\left(B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}} Z_n \otimes G\right) \overset{\cong}{\underset{\uparrow}{\leftarrow}} \operatorname{Tor}_0(\mathrm{H}_n(C), G)} \ \underset{\text{Lemma 57.15}}{\overset{\cong}{\underset{\uparrow}{\rightarrow}}} \ \mathrm{H}_n(C) \otimes G$$

Similarly we have a natural isomorphism

$$\ker\left(B_{n-1} \otimes G \xrightarrow{i_{n-1}\otimes \mathrm{id}} Z_{n-1} \otimes G\right) \xrightarrow{\cong} \operatorname{Tor}(\mathrm{H}_{n-1}(C), G).$$

We have thus shown the existence of the desired short exact sequences. The statement that the homomorphism p is natural follows easily from the definitions and Lemma 57.13 (2). We leave the verification of the details to the reader. ■

The following theorem is for us the main application of the Universal Coefficient Theorem 57.18.

Theorem 57.19. (Universal Coefficient Theorem) *Let (X, A) be a pair of topological spaces and let G be an abelian group. Then for each $n \in \mathbb{N}_0$ there exists a natural homomorphism $\mathrm{H}_n(X, A; G) \to \operatorname{Tor}(\mathrm{H}_{n-1}(X, A), G)$ such that the following sequence is exact:*

$$0 \to \mathrm{H}_n(X, A) \otimes G \xrightarrow{\mu} \mathrm{H}_n(X, A; G) \to \operatorname{Tor}(\mathrm{H}_{n-1}(X, A), G) \to 0.$$

Remark. Note that the naturality of the maps implies for every map $f\colon (X, A) \to (Y, B)$ between pairs of topological spaces and any homomorphism $\varphi\colon G \to G'$ between abelian groups we obtain the following *commutative* diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}_n(X, A) \otimes G & \xrightarrow{\mu} & \mathrm{H}_n(X, A; G) & \longrightarrow & \operatorname{Tor}(\mathrm{H}_{n-1}(X, A), G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}_n(Y, B) \otimes G' & \xrightarrow{\mu} & \mathrm{H}_n(Y, B; G') & \longrightarrow & \operatorname{Tor}(\mathrm{H}_{n-1}(Y, B), G') & \longrightarrow & 0, \end{array}$$

where the vertical maps are the obvious (?) ones induced by f and φ.

PROOF. Let (X, A) be a pair of topological spaces and let G be an abelian group. As we pointed out on page 1120, the chain groups $\mathrm{C}_n(X, A)$, $n \in \mathbb{N}_0$, are free abelian groups. Thus we can apply the Universal Coefficient Theorem 57.18 and we immediately obtain the desired short exact sequence. ■

Before we consider a few examples let us jot down the following almost immediate corollary.

Corollary 57.20. *Let $f\colon (X,A) \to (Y,B)$ be a map between pairs of topological spaces and let G be an abelian group. Then*

$$\begin{array}{c} \text{the induced map} \\ f_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B) \text{ is} \\ \text{an isomorphism for all } n \in \mathbb{N}_0 \end{array} \quad \Longrightarrow \quad \begin{array}{c} \text{the induced map} \\ f_*\colon \mathrm{H}_n(X,A;G) \to \mathrm{H}_n(Y,B;G) \text{ is} \\ \text{an isomorphism for all } n \in \mathbb{N}_0. \end{array}$$

PROOF. The corollary follows immediately from the naturality of the short exact sequence of the Universal Coefficient 57.19 together with the Five-Lemma 43.12. ∎

Examples.

(1) We consider again the real projective space $\mathbb{RP}^n$. In Proposition 48.10 we saw that

$$\mathrm{H}_k(\mathbb{RP}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k=0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd and } k<n, \\ 0, & \text{if } k \text{ is even and } 0<k\le n, \\ 0, & \text{if } k>n, \\ \mathbb{Z}, & \text{if } k=n \text{ and } n \text{ is odd.} \end{cases}$$

It follows from the Universal Coefficient Theorem 57.19, the discussion on page 1396 and Lemma 57.17 (7) that for any even number $2k \in \{0,\dots,n\}$ we have a short exact sequence

$$0 \;\to\; \underbrace{\mathrm{H}_{2k}(\mathbb{RP}^n)\otimes\mathbb{F}_2}_{\cong 0\otimes\mathbb{F}_2=0} \;\xrightarrow{\mu}\; \mathrm{H}_{2k}(\mathbb{RP}^n;\mathbb{F}_2) \;\to\; \underbrace{\mathrm{Tor}(\mathrm{H}_{2k-1}(\mathbb{RP}^n),\mathbb{F}_2)}_{\cong\mathrm{Tor}(\mathbb{F}_2,\mathbb{F}_2)\cong\mathbb{F}_2} \;\to\; 0$$

whereas for $2k+1 \in \{0,\dots,n\}$ an odd number we have a short exact sequence

$$0 \;\to\; \underbrace{\mathrm{H}_{2k+1}(\mathbb{RP}^n)\otimes\mathbb{F}_2}_{\cong\mathbb{F}_2\otimes\mathbb{F}_2\cong\mathbb{F}_2 \text{ or}\cong\mathbb{Z}\otimes\mathbb{F}_2\cong\mathbb{F}_2} \;\xrightarrow{\mu}\; \mathrm{H}_{2k+1}(\mathbb{RP}^n;\mathbb{F}_2) \;\to\; \underbrace{\mathrm{Tor}(\mathrm{H}_{2k}(\mathbb{RP}^n),\mathbb{F}_2)}_{\cong\mathrm{Tor}(0,\mathbb{F}_2)=0 \text{ or } \cong\mathrm{Tor}(\mathbb{Z},\mathbb{F}_2)=0} \;\to\; 0.$$

Also it follows easily that $\mathrm{H}_k(\mathbb{RP}^n)=0$ for $k>n$. Summarizing we obtain the following result

$$\mathrm{H}_k(\mathbb{RP}^n;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k=0,\dots,n, \\ 0, & \text{otherwise.} \end{cases}$$

Fortunately this calculation agrees with the calculation on page 1403.

(2) As on pages 182 and 196 we make the usual identifications $S^2 = \overline{B}^2/S^1$ and $\mathbb{RP}^2 = \overline{B}^2/z\sim -z$ where $z\in S^1$. We denote by f the obvious projection map[891]

$$f\colon \mathbb{RP}^2 = \overline{B}^2/z\sim -z \;\to\; S^2 = \overline{B}^2/S^1.$$

As in Figure 600 we equip $\mathbb{RP}^2$ with the CW-structure with one 0-cell, one 1-cell and one 2-cell and as on page 935 we equip S^2 with the CW-structure with one 0-cell and one 2-cell. With these CW-structures the above map f is given by collapsing the

[891] Note that this is not a covering map and that this map has nothing to do with the more familiar projection map $S^2 \to \mathbb{RP}^2$.

1-cell to the unique 0-cell of S^2, in particular f is a cellular map. By Proposition 48.4 and the discussion on page 1402 we obtain the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}_2(\mathbb{RP}^2;\mathbb{F}_2) \arrow[r, "f_*"] \arrow[d, "\Phi_{\mathbb{RP}^2}"', "\cong"] & \mathrm{H}_2(S^2;\mathbb{F}_2) \arrow[d, "\cong"', "\Phi_{S^2}"] \\
\mathrm{H}_2^{\mathrm{CW}}(\mathbb{RP}^2;\mathbb{F}_2) \arrow[r, "f_*"] & \mathrm{H}_2^{\mathrm{CW}}(S^2;\mathbb{F}_2).
\end{tikzcd}$$

As we saw on page 1402, the 2-cell is a generator of $\mathrm{H}_2^{\mathrm{CW}}(S^2;\mathbb{F}_2)$ and it gets sent to the 2-cell in S^2 which is, as we saw on page 1260, a generator of $\mathrm{H}_2^{\mathrm{CW}}(\mathbb{RP}^2;\mathbb{F}_2)$. Thus we see that the bottom map is an isomorphism, it follows that the top map is also an isomorphism. Summarizing the short exact sequences of the Universal Coefficient Theorem 57.19 give us the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \overbrace{\mathrm{H}_2(\mathbb{RP}^2)\otimes\mathbb{F}_2}^{=0} \arrow[r, "\mu"] \arrow[d, "=0"', "f_*"] & \overbrace{\mathrm{H}_2(\mathbb{RP}^2;\mathbb{F}_2)}^{\cong\mathbb{F}_2} \arrow[r] \arrow[d, "\cong"', "f_*"] & \overbrace{\mathrm{Tor}(\mathrm{H}_1(\mathbb{RP}^2),\mathbb{F}_2)}^{\cong\mathbb{F}_2} \arrow[r] \arrow[d, "=0"', "f_*"] & 0 \\
0 \arrow[r] & \underbrace{\mathrm{H}_2(S^2)\otimes\mathbb{F}_2}_{\cong\mathbb{F}_2} \arrow[r, "\mu"] & \underbrace{\mathrm{H}_2(S^2;\mathbb{F}_2)}_{\cong\mathbb{F}_2} \arrow[r] & \underbrace{\mathrm{Tor}(\mathrm{H}_1(S^2),\mathbb{F}_2)}_{=0} \arrow[r] & 0.
\end{tikzcd}$$

Note that the two vertical maps to the left and right are the zero map, whereas the middle map is actually an isomorphism. This slightly weird fact will play a role in Proposition 57.26.

In the following corollary we give two instances where the Tor-terms in the Universal Coefficient Theorem 57.19 are zero, i.e. two instances where we can "pull out" the coefficients.

Corollary 57.21.

(1) *Let (X, A) be a pair of topological spaces and let G be a subgroup of $(\mathbb{C}, +)$. Then for each $n \in \mathbb{N}_0$ the map*

$$\mu\colon\ \mathrm{H}_n(X, A;\mathbb{Z})\otimes G \ \to\ \mathrm{H}_n(X, A; G)$$

is a natural isomorphism.

(2) *Let (X, A) be a pair of topological spaces and let $\mathbb{F}$ be a subfield of $\mathbb{C}$. Then for each $n \in \mathbb{N}_0$ we have* $\operatorname{rank}(\mathrm{H}_n(X, A)) = \dim_{\mathbb{F}}(\mathrm{H}_n(X, A;\mathbb{F}))$.

Example. It follows immediately from Proposition 48.10 together with Corollary 57.21 and Lemma 57.3 (3) that

$$\mathrm{H}_k(\mathbb{RP}^{2n+1};\mathbb{R}) \ \cong\ \begin{cases} \mathbb{R}, & \text{if } k = 0, 2n+1, \\ 0, & \text{otherwise.} \end{cases} \qquad \text{and} \qquad \mathrm{H}_k(\mathbb{RP}^{2n};\mathbb{R}) \ \cong\ \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

PROOF.

(1) This statement is an immediate consequence of the Universal Coefficient Theorem 57.19 and the fact, obtained in Lemma 57.17 (4), that the torsion-groups are zero for any subgroup of $(\mathbb{C}, +)$.

(2) To shorten the notation we consider only the case $A = \varnothing$. We have

$$\operatorname{rank}(\mathrm{H}_n(X)) \underset{\substack{\uparrow \\ \text{by the definition on page 1398}}}{=} \dim_{\mathbb{Q}}(\mathrm{H}_n(X) \otimes \mathbb{Q}) \underset{\substack{\uparrow \\ \text{by (1)}}}{=} \dim_{\mathbb{Q}}(\mathrm{H}_n(X)) \otimes \mathbb{Q} \underset{\substack{\uparrow \\ \text{follows fairly easily from } \mathbb{Q} \subset \mathbb{F}}}{=} \dim_{\mathbb{F}}(\mathrm{H}_n(X)) \otimes \mathbb{F}.$$

■

Remark.

(1) Let (X, A) be a pair of topological spaces and let R be a subring of $\mathbb{C}$. Since the map $\mu\colon \mathrm{H}_n(X, A; \mathbb{Z}) \otimes R \to \mathrm{H}_n(X, A; R)$ is natural we can and we will in the following identify the groups $\mathrm{H}_n(X, A; \mathbb{Z}) \otimes R$ and $\mathrm{H}_n(X, A; R)$ using the isomorphism from Corollary 57.21.

(2) Given a pair (X, A) of topological spaces and $n \in \mathbb{N}_0$ we refer to[892]

$$b_n(X, A) \ := \ \operatorname{rank}(\mathrm{H}_n(X, A))$$

as the *n-th Betti number of* (X, A). For $A = \varnothing$ we write of course $b_n(X) := b_n(X, \varnothing)$. Note that it follows from Corollary 57.21 that for any subfield $\mathbb{F}$ of $\mathbb{C}$ we have $b_n(X, A) = \dim_{\mathbb{F}}(\mathrm{H}_n(X, A; \mathbb{F}))$.

(3) Let R be a subring of $\mathbb{C}$. Lemma 57.17 (4) and the Universal Coefficient Theorem 57.18 show that taking the tensor product with R is an exact functor in the sense of the definition on page 1405.

We conclude this section with the following lemma which we will need later on in the generalization of the Borsuk-Ulam Theorem 16.22 to higher dimensions.

Lemma 57.22. *Let $f\colon S^n \to S^n$ be a map and let $k \in \mathbb{N}$. Then the induced map*

$$f_*\colon \ \mathrm{H}_n(S^n; \mathbb{Z}_k) \ \to \ \mathrm{H}_n(S^n; \mathbb{Z}_k)$$

is given by multiplication by $\deg(f)$.

PROOF (∗). Let $f\colon S^n \to S^n$ be a map and let $k \in \mathbb{N}$. Since the short exact sequence of the Universal Coefficient Theorem 57.19 is natural we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}_n(S^n) \otimes \mathbb{Z}_k & \xrightarrow{\mu} & \mathrm{H}_n(S^n; \mathbb{Z}_k) & \longrightarrow & \operatorname{Tor}(\mathrm{H}_{n-1}(S^n), \mathbb{Z}_k) & \longrightarrow & 0 \\
 & & \big\downarrow {\scriptstyle f_* \otimes \mathrm{id}} & & \big\downarrow {\scriptstyle f_*} & & \big\downarrow {\scriptstyle f_*} & & \\
0 & \longrightarrow & \mathrm{H}_n(S^n) \otimes \mathbb{Z}_k & \xrightarrow{\mu} & \mathrm{H}_n(S^n; \mathbb{Z}_k) & \longrightarrow & \operatorname{Tor}(\mathrm{H}_{n-1}(S^n), \mathbb{Z}_k) & \longrightarrow & 0.
\end{array}$$

Note that $\mathrm{H}_{n-1}(S^n)$ is torsion-free, hence $\operatorname{Tor}(\mathrm{H}_{n-1}(S^n), \mathbb{Z}_k) = 0$ by Lemma 57.17. It follows from the exactness of the horizontal sequences that the horizontal maps μ on the left are isomorphisms. By definition the map $f_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is given by multiplication by $\deg(f)$, hence $f_* \otimes \mathrm{id}$ is also given by multiplication by $\deg(f)$. It now follows that the middle vertical map is also given by multiplication by $\deg(f)$. ■

[892] Recall that on page 1398 we defined the rank of an arbitrary abelian group.

57.6. **Splittings of the Universal Coefficient Theorem.** Our goal had been to determine the homology groups $\mathrm{H}_n(C;G)$ only in terms of the original groups $\mathrm{H}_k(C)$ and the abelian group G. The Algebraic Universal Coefficient Theorem 57.18 gave us a short exact sequence

$$0 \to \mathrm{H}_n(C) \otimes G \xrightarrow{\mu} \mathrm{H}_n(C;G) \xrightarrow{p} \mathrm{Tor}(\mathrm{H}_{n-1}(C), G) \to 0.$$

But in general, as we already saw on page 1205, given a short exact sequence, it is not possible to determine the middle group of a short exact sequence from the groups to the left and the right. But we saw in Splitting Lemma 46.2 that if a short exact sequence of abelian groups

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

splits, then the middle group B is the direct sum of the "outer" groups A and C. The question arises, whether the short exact sequence in the Universal Coefficient Theorem 57.18 splits. This is fortunately the case:

Proposition 57.23. *Let (C_n, ∂_n) be a free chain complex*[893] *and let G be an abelian group. Then for any $n \in \mathbb{N}_0$ the short exact sequence*

$$0 \to \mathrm{H}_n(C) \otimes G \xrightarrow{\mu} \mathrm{H}_n(C;G) \to \mathrm{Tor}(\mathrm{H}_{n-1}(C), G) \to 0$$

of the Algebraic Universal Coefficient Theorem 57.18 splits. In particular there exists an isomorphism

$$\mathrm{H}_n(C;G) \cong \mathrm{H}_n(C) \otimes G \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(C), G).$$

PROOF. Let (C_n, ∂_n) be a chain complex of free abelian groups and furthermore let G be an abelian group. According to Splitting Lemma 46.2 it suffices to construct a homomorphism $\eta\colon \mathrm{H}_n(C;G) \to \mathrm{H}_n(C) \otimes G$ such that $\eta \circ \mu$ is the identity on $\mathrm{H}_n(C) \otimes G$.

For each $n \in \mathbb{N}_0$ we write as always $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$. We consider the short exact sequence

$$0 \to Z_n \hookrightarrow \mathrm{C}_n \xrightarrow{\partial_n} B_{n-1} \to 0.$$

The group $B_{n-1} \subset \mathrm{C}_{n-1}$ is a subgroup of a free abelian group, thus it is itself a free abelian group by Lemma 19.2. Therefore we can deduce from Lemma 46.1 that the above short exact sequence splits. In particular, according to Splitting Lemma 46.2, for each $n \in \mathbb{N}_0$ there exists a homomorphism $p_n\colon \mathrm{C}_n \to Z_n$ such that p_n is the identity on $Z_n \subset C_n$.

[893] A chain complex $(\mathrm{C}_*, \partial_*)$ is called *free*, if all the groups C_n are free abelian groups.

Claim. The following diagram commutes:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{C}_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift left, "p_{n+1}"] \arrow[dr, "\partial_{n+1}"'] & \mathrm{C}_n \arrow[r, "\partial_n"] \arrow[d, shift left, "p_n"] \arrow[dr, "\partial_n"'] & \mathrm{C}_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, shift left, "p_{n-1}"] & \dots \\
& Z_{n+1} \arrow[u, hook, shift left] \arrow[d, two heads, "q_{n+1}"] & Z_n \arrow[u, hook, shift left] \arrow[d, two heads, "q_n"] & Z_{n-1} \arrow[u, hook, shift left] \arrow[d, two heads, "q_{n-1}"] & \\
\dots \arrow[r] & \underbrace{\mathrm{H}_{n+1}(C)}_{=Z_{n+1}/B_{n+1}} \arrow[r, "0"] & \underbrace{\mathrm{H}_n(C)}_{=Z_n/B_n} \arrow[r, "0"] & \underbrace{\mathrm{H}_{n-1}(C)}_{=Z_{n-1}/B_{n-1}} \arrow[r, "0"] & \dots
\end{tikzcd}$$

where the maps q_n are just the obvious projection maps.

We start at C_{n+1}. If we go down and then right, then we end up with the zero map to $\mathrm{H}_n(C)$. On the other hand, if we first go right, then we end up in $B_n \subset Z_n$, but then the projection q_n to $\mathrm{H}_n(C) = Z_n/B_n$ is also the zero map. ⊞

The top and the bottom horizontal sequences of the claim are chain complexes. Since the diagram commutes we see that the vertical maps from top to bottom are chain maps.

Now we tensor the top and the bottom horizontal sequences with G and we obtain the following diagram of chain complexes.

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{C}_{n+1} \otimes G \arrow[r, "\partial_{n+1}\otimes \mathrm{id}"] \arrow[d, "(q_{n+1}\circ p_{n+1})\otimes \mathrm{id}"] & \mathrm{C}_n \otimes G \arrow[r, "\partial_n \otimes \mathrm{id}"] \arrow[d, "(q_n\circ p_n)\otimes \mathrm{id}"] & \mathrm{C}_{n-1} \otimes G \arrow[r, "\partial_{n-1}\otimes \mathrm{id}"] \arrow[d, "(q_{n-1}\circ p_{n-1})\otimes \mathrm{id}"] & \dots \\
\dots \arrow[r] & \mathrm{H}_{n+1}(C) \otimes G \arrow[r, "0"] & \mathrm{H}_n(C) \otimes G \arrow[r, "0"] & \mathrm{H}_{n-1}(C) \otimes G \arrow[r, "0"] & \dots
\end{tikzcd}$$

The induced maps on the homology of these two chain complexes give us a homomorphism

$$\eta\colon\ \mathrm{H}_n(C;G) \;=\; \mathrm{H}_n(C_* \otimes G) \;\to\; \mathrm{H}_n(\text{lower sequence}) = \mathrm{H}_n(C) \otimes G.$$

It remains to prove the following claim.

Claim. The composition $\eta \circ \mu$ is the identity on $\mathrm{H}_n(C) \otimes G$.

Let $[\sigma] \otimes g \in \mathrm{H}_n(C) \otimes G$ where $\sigma \in Z_n$. Then

$$(\eta \circ \mu)([\sigma] \otimes g) \;=\; \eta([\sigma \otimes g]) \underset{\substack{\uparrow \\ \text{definition of } \eta}}{=} (q_n(p_n(\sigma))) \otimes g \underset{\substack{\uparrow \\ \text{since } p_n = \mathrm{id} \text{ on } Z_n}}{=} q_n(\sigma) \otimes g \;=\; [\sigma] \otimes g.$$

■

The following immediate corollary to Proposition 57.23 shows in particular that the homology groups with G-coefficients of a topological space X are completely determined by the usual homology groups of X and the abelian group G.

Theorem 57.24. *Let (X, A) be a pair of topological spaces and let G be an abelian group. Then for each $n \in \mathbb{N}_0$ the short exact sequence*

$$0 \;\to\; \mathrm{H}_n(X,A) \otimes G \;\xrightarrow{\mu}\; \mathrm{H}_n(X,A;G) \;\to\; \mathrm{Tor}(\mathrm{H}_{n-1}(X,A),G) \;\to\; 0$$

from the Universal Coefficient Theorem 57.19 splits. In particular there exists an isomorphism

$$\mathrm{H}_n(X, A; G) \;\cong\; \mathrm{H}_n(X, A) \otimes G \oplus \operatorname{Tor}(\mathrm{H}_{n-1}(X, A), G).$$

The following is an immediate corollary to Theorem 57.24.

Corollary 57.25. *Let (X, A) and (Y, B) be topological spaces and let G be an abelian group. Then*

$$\begin{array}{c}\mathrm{H}_n(X,A) \text{ and } \mathrm{H}_n(Y,B) \text{ are}\\ \text{isomorphic for all } n \in \mathbb{N}_0\end{array} \quad\Longrightarrow\quad \begin{array}{c}\mathrm{H}_n(Y,B;G) \text{ and } \mathrm{H}_n(X,A;G) \text{ and}\\ \text{isomorphic for all } n \in \mathbb{N}_0.\end{array}$$

Remark.

(1) Corollary 57.25 is in some sense a disappointment, it shows that homology groups with coefficients are not better at distinguishing topological spaces than ordinary homology groups.
(2) Corollary 57.25 is also an almost immediate consequence of Propositions 49.1 and 49.2.

The question now arises, whether given an abelian group G there exists in fact a *natural* isomorphism

$$\mathrm{H}_n(X; G) \;\cong\; \mathrm{H}_n(X) \otimes G \oplus \operatorname{Tor}(\mathrm{H}_{n-1}(X), G).$$

The discussion on page 490 already shows that this sounds like a dubious idea. Indeed, the following proposition says that such a natural isomorphism does not exist for $G = \mathbb{F}_2$. More precisely, we have the following proposition.

Proposition 57.26. *It is not possible assign to each topology space X an isomorphism*

$$\Phi_X \colon\; \mathrm{H}_n(X; \mathbb{F}_2) \;\to\; \mathrm{H}_n(X) \otimes \mathbb{F}_2 \oplus \operatorname{Tor}(\mathrm{H}_{n-1}(X), \mathbb{F}_2)$$

such that for any map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}_n(X;\mathbb{F}_2) & \xrightarrow[\cong]{\Phi_X} & \mathrm{H}_n(X) \otimes \mathbb{F}_2 & \oplus & \operatorname{Tor}(\mathrm{H}_{n-1}(X), \mathbb{F}_2) \\
\downarrow f_* & & \downarrow f_* & & \downarrow f_* \\
\mathrm{H}_n(Y;\mathbb{F}_2) & \xrightarrow[\cong]{\Phi_Y} & \mathrm{H}_n(Y) \otimes \mathbb{F}_2 & \oplus & \operatorname{Tor}(\mathrm{H}_{n-1}(Y), \mathbb{F}_2).
\end{array}$$

Proof. So suppose instead that such a map Φ_X exists for every topological space X. We consider the map

$$f\colon \mathbb{R}\mathrm{P}^2 = \overline{B}^2/z \sim -z \;\to\; S^2 = \overline{B}^2/S^1$$

from page 1420. We obtain the following diagram:

$$\begin{array}{ccccc}
\overbrace{\mathrm{H}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)}^{\cong\mathbb{F}_2} & \xrightarrow[\cong]{\Phi_{\mathbb{R}\mathrm{P}^2}} & \overbrace{\mathrm{H}_2(\mathbb{R}\mathrm{P}^2) \otimes \mathbb{F}_2}^{=0} & \oplus & \overbrace{\operatorname{Tor}(\mathrm{H}_1(\mathbb{R}\mathrm{P}^2), \mathbb{F}_2)}^{\cong\mathbb{F}_2} \\
f_* \downarrow & & =0 \downarrow f_* & & =0 \downarrow f_* \\
\underbrace{\mathrm{H}_2(S^2;\mathbb{F}_2)}_{\cong\mathbb{F}_2} & \xrightarrow[\cong]{\Phi_{\mathbb{R}\mathrm{P}^2}} & \underbrace{\mathrm{H}_2(S^2) \otimes \mathbb{F}_2}_{\cong\mathbb{F}_2} & \oplus & \underbrace{\operatorname{Tor}(\mathrm{H}_1(S^2), \mathbb{F}_2)}_{=0}.
\end{array}$$

The horizontal maps are by hypothesis isomorphisms. We saw on page 1420 that the vertical map on the left is an isomorphism. On the other hand the two vertical maps on the right are zero since either the domain or the target is zero. Thus this diagram cannot possibly commute. We have thus found a contradiction to our assumption. ■

57.7. **Homological algebra over an arbitrary commutative ring** (∗)**.** A lot of the algebra that we had studied in the previous sections can be generalized from the case of abelian groups, i.e. the case of $\mathbb{Z}$-modules, to modules over a commutative ring. We will not make use of the generalizations that we discuss in this short section.

Throughout this section let R be a commutative ring. Let V and W be two R-modules. We say that an R-homomorphism $f\colon V\times W\to X$ to an R-module X is *R-bilinear*, if for all $v,v'\in V$, $w,w'\in W$ and $r\in R$ the following holds:

$$\begin{array}{rcl} f(v+v',w) &=& f(v,w)+f(v',w), \\ f(rv,w) &=& rf(v,w) \end{array} \quad \text{and} \quad \begin{array}{rcl} f(v,w+w') &=& f(v,w)+f(v,w') \\ f(v,rw) &=& rf(v,w). \end{array}$$

We say an R-bilinear map $f\colon V\times W\to X$ has the *universal R-bilinear property*, if for every R-bilinear map $g\colon V\times W\to Y$ there exists a *unique* R-homomorphism $\varphi\colon X\to Y$ such that the following diagram commutes:

$$\begin{tikzcd} V\times W \arrow[r, "f"] \arrow[dr, "g"'] & X \arrow[d, "\varphi \text{ unique}"] \\ & Y. \end{tikzcd}$$

Using a variation on the definition of the tensor product of abelian groups that we gave on page 1393 one can now fairly easily prove that there exists an R-bilinear map $f\colon V\times W\to X$ which has the universal R-bilinear property.

A variation on the usual "universal property" argument shows that if $f\colon V\times W\to X$ and $f'\colon V\times W\to X'$ are maps that both have the universal R-bilinear property, then there exists a unique isomorphism $\varphi\colon X\to X'$ of R-modules such that $\varphi(f(v,w))=f'(v,w)$ for all $v\in V$ and $w\in W$.

We denote by $V\otimes_R W$ the R-module that satisfies satisfies the above property. (By the universal property we see that $V\otimes_R W$ is uniquely determined up to isomorphism.) Note that $V\otimes_R W$ comes with an R-bilinear map $V\times W\to V\otimes_R W$. Given $v\in V$ and $w\in W$ we denote the image of (v,w) by $v\otimes w$. If $R=\mathbb{Z}$, then it follows from the remark after Lemma 57.2 that $V\otimes_{\mathbb{Z}} W$ is isomorphic to $V\otimes W$.

As before we see that given an R-module W the map $V\mapsto V\otimes_R W$ defines a functor from the category of R-modules to the category of R-modules. The proof of Lemma 57.10 shows that this functor is right-exact.

We define a *free R-module* precisely the same way as we defined a free abelian group. A *free resolution* of an R-module M is an exact sequence

$$\cdots\to\ F_2\ \xrightarrow{f_2}\ F_1\ \xrightarrow{f_1}\ F_0\ \xrightarrow{f_0}\ M\ \to\ 0$$

where each F_i is a free R-module. We can construct the canonical free resolution of an R-module M the same was as we did on page 1408.

Given two R-modules M and N we can define the torsion R-modules $\operatorname{Tor}_n^R(M,N)$ as before. More precisely, we take the canonical free resolution

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

and we define

$$\operatorname{Tor}_n^R(M,N) := \mathrm{H}_n(F_* \otimes_R N).$$

Here the following statements hold:

(1) As in Proposition 57.12 one can show that the isomorphism type of $\operatorname{Tor}_n^R(M,N)$ can be determined using any free resolution for M.
(2) The argument of page 1413 shows that for each $n \in \mathbb{N}_0$ the R-torsion module $\operatorname{Tor}_n^R(M,N)$ is covariantly functorial in both entries.
(3) The proof of Lemma 57.15 can also be generalized to show that given any two R-modules M and N we have a natural isomorphism $\operatorname{Tor}_0^R(M,N) \cong M \otimes_R N$.
(4) The obvious analogues of Lemma 57.17 (1) and (2) also hold in the more general context. Furthermore the obvious analogues of Lemma 57.17 (3) and (4) hold if we replace "torsion-free abelian group" by "free R-module".[894] In general there are no analogues of Lemma 57.17 (5), (6) and (7).
(5) For general commutative rings it is no longer true though that $\operatorname{Tor}_i^R(M,N) = 0$ for $i \geq 2$. For example for $R = \mathbb{Q}[t_1, \ldots, t_k]$ the R-torsion modules can be non-zero for $i = 1, \ldots, k$ and they are only zero for $i > k$. We refer to [**Mac75**, Chapter VII.4] for details.

Now let $(\mathrm{C}_*, \partial_*)$ be a chain complex of R-modules. Furthermore let M be another R-module. We consider the chain complex $C_* \otimes_R M$ of R-modules. It is natural to ask whether one can again determine the homology groups of $C_* \otimes_R M$ from the torsion groups $\operatorname{Tor}_k^R(\mathrm{H}_l(C), M)$ with $k \in \mathbb{N}_0$. If R is a principal ideal domain (usually abbreviated by PID), then almost the same statement as in the Universal Coefficient Theorem 57.18 holds. On the other hand, if R is not a principal ideal, then as we pointed out above, the higher torsion groups are in general non-zero and the problem becomes much more subtle. The torsion-terms and the homology with M-coefficients are then related by a "spectral sequence", see e.g. [**Rot09**, Theorem 10.90] for details.

57.8. **The Eilenberg-Steenrod axioms for homology** $(*)$**.** In this section we give a very short introduction to the concept of (co-) homology theories which was introduced in the seminal book of Eilenberg–Steenrod [**ES52**].

Before we can do so we introduce the following definition.

Definition. Let R be a commutative ring and let S be a set.

(1) An *S-graded R-module* is defined as an R-module M together with a direct sum decomposition $M = \bigoplus_{s \in S} M_s$ into R-modules M_s, $s \in S$. If $x \in M$ lies in some M_s, then we call x *homogeneous* and we refer to $\deg(x) := s$ as the *degree of x*.

[894] Note that in general "R-torsion free module" is not going to be strong enough.

(2) A *morphism* $f\colon M \to N$ *between* S*-graded* R*-modules* M *and* N is an R-module homomorphism such that for each $s \in S$ we have $f(M_s) \subset N_s$. Equivalently, it is a family of R-module homomorphisms $\{f_s\colon M_s \to N_s\}_{s\in S}$.
(3) The *category* S-$\mathcal{GradMod}_R$ *of* S*-graded* R*-modules* is the category where the objects are S-graded R-modules and the morphisms are the ones described in (2).
(4) Since abelian groups are for all intents and purposes the same as $\mathbb{Z}$-modules we define the category S-$\mathcal{GradAb}$ of S-graded abelian groups to be the category of $\mathbb{N}_0$-graded $\mathbb{Z}$-modules.

Example. Given a pair of topological spaces (X, Y) and an abelian group G we write

$$\mathrm{H}_*(X, Y; G) \ := \ \bigoplus_{n\in\mathbb{N}_0} \mathrm{H}_n(X, Y; G)$$

and we view $\mathrm{H}_*(X, Y; G)$ as an $\mathbb{N}_0$-graded $\mathbb{Z}$-module.

The following definition goes back to [**ES52**] and [**Miln62b**].

Definition. A *homology theory* is a covariant functor

$$\mathcal{H}\colon \begin{array}{c}\text{category } \mathcal{PairTop} \text{ of}\\ \text{pairs of topological spaces}\end{array} \ \to \ \begin{array}{c}\text{category } \mathbb{N}_0\text{-}\mathcal{GradAb} \text{ of}\\ \mathbb{N}_0\text{-graded abelian groups}\end{array}$$

that satisfies the following axioms:[895]

(1) (Homotopy invariant) If $f, g\colon (X, A) \to (Y, B)$ are homotopic maps between pairs of topological spaces, then $f_* = g_*$.
(2) (Connecting homomorphism) Given any pair (X, A) of topological spaces and any $n \in \mathbb{N}$ there exists a natural connecting homomorphism $\partial\colon \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A)$[896] such that the sequence

$$\dots \ \to \ \mathcal{H}_{n+1}(X, A) \ \xrightarrow{\partial} \ \mathcal{H}_n(A) \ \to \ \mathcal{H}_n(X) \ \to \ \mathcal{H}_n(X, A) \ \xrightarrow{\partial} \ \mathcal{H}_{n-1}(A) \ \to \ \dots$$

is exact.
(3) (Excision) If U and A are subsets of a topological space X with $\overline{U} \subset \mathring{A}$, then for each $n \in \mathbb{N}_0$ the map $\mathcal{H}_n(X \setminus U, A \setminus U) \to \mathcal{H}_n(X, A)$ is an isomorphism.
(4) (Additivity) Let $\{(X_j, A_j)\}_{j\in J}$ be a family of pairs of topological spaces. We write $X = \bigsqcup_{j\in J} X_j$ and $A = \bigsqcup_{j\in J} A_j$. If given $j \in J$ we denote by $i_j\colon (X_j, A_j) \to (X, A)$ the obvious inclusion map, then for any $k \in \mathbb{N}_0$ the induced map

$$\bigoplus_{j\in J} i_{j*}\colon \bigoplus_{j\in J} \mathcal{H}_k(X_j, A_j) \ \to \ \mathcal{H}_k(X, A)$$

is an isomorphism.
(5) (Dimension axiom) We denote by $\star$ the topological space that consists of a single point. For $n \neq 0$ we have $\mathrm{H}_n(\star) = 0$.

[895] The axioms are usually called Eilenberg-Steenrod axioms.

[896] Here we use the obvious notation that for a topological space Y we write $\mathcal{H}_n(Y) := \mathcal{H}_n(Y, \varnothing)$.

Example. It follows from Proposition 43.17, Corollary 43.16, the Excision Theorem 43.19, the obvious generalization of Lemma 41.14 and Lemma 41.6 that singular homology is indeed a homology theory.

We conclude this chapter with a uniqueness theorem of homology theories. To formulate the theorem we need the following definition.

Definition. We call the category $\mathcal{FullPairCW}$

$$\begin{aligned} \mathrm{Ob}(\mathcal{FullPairCW}) &:= \text{ all pairs of CW-complexes,} \\ \mathrm{Mor}((X,A),(Y,B)) &:= \text{ all continuous maps } f\colon X \to Y \text{ with } f(A) \subset B, \end{aligned}$$

together with the usual composition of maps the *full category of pairs of* CW-*complexes*.[897]

Now we can formulate the following uniqueness statement for homology theories.

Theorem 57.27. *Let $\mathcal{H}$ and $\mathcal{H}'$ be two homology theories and let $\varphi\colon \mathcal{H}_0(\star) \to \mathcal{H}'_0(\star)$ be an isomorphism. Then there exists a natural isomorphism between $\mathcal{H}$ and $\mathcal{H}'$ viewed as functors*

$$\begin{array}{c} \textit{full category } \mathcal{FullPairCW} \textit{ of} \\ \textit{pairs of } \text{CW-}\textit{complexes} \end{array} \to \begin{array}{c} \textit{category } \mathbb{N}_0\text{-}\mathcal{GradAb} \textit{ of} \\ \mathbb{N}_0\textit{-graded abelian groups.} \end{array}$$

Sketch of a proof. Let G be an abelian group. We start our proof with the following observation: Given a pair (X,A) of CW-complexes we saw in Corollary 36.35 (3) that we can equip the mapping cone $\mathrm{Cone}(i\colon A \to X)$ of the inclusion map with a natural CW-structure. Given $n \in \mathbb{N}_0$ we define the n-th cellular homology $\mathrm{H}^{\mathrm{CW}}_n(X,A;G)$ as the reduced cellular homology with G-coefficients of the mapping cone.

Proposition 48.4, together with Lemma 46.16, Proposition 48.4 and the discussion on page 1402 shows that for pairs of CW-complexes there exists a natural isomorphism between singular homology with G-coefficients and cellular homology with G-coefficients. Closer inspection of the proof shows that we can prove this statement only using the above Eilenberg-Steenrod axioms and the fact that $\mathrm{H}_0(\star;G) \cong G$.

Now let $\alpha\colon \mathcal{H}_0(\star) \to G$ be an isomorphism. By the above one can show that there exists a natural isomorphism from $\mathcal{H}$ to $\mathrm{H}^{\mathrm{CW}}_n(-;G)$. The same applies to $\mathcal{H}'$ and we obtain the theorem as long as φ is an isomorphism. This approach to the proof is developed in more detail in [**tD08**, Chapter 12.2]. ■

Remark.

(1) Closely related results are also proved in [**ES52**, p. 100] and [**Hu66**, p. 52], see also [**GH81**, p. 138].
(2) The axioms of a cohomology theory are defined basically the same way with the obvious modifications and the analogue of Theorem 57.27 also holds in this context, see [**DaK01**, Theorem 1.31].
(3) If one drops the dimension axiom, then there are other (co-) homology theories, for example "topological K-theory". We refer to [**Karo08**] for details.

[897] Note that we consider all continuous maps, we do not restrict ourselves to cellular maps. Thus $\mathcal{FullPairCW}$ is a full subcategory of the category $\mathcal{PairTop}$ of pairs of topological spaces.

Exercises for Chapter 57.

Exercise 57.1. Show that the abelianization functor from the category of groups to the category of abelian groups is right-exact and that it is not left-exact.

Exercise 57.2. Let $\mathcal{C}$ be the category of finitely generated abelian groups. Given an abelian group A we denote by $\operatorname{Tor}(A)$ as usual its torsion subgroup and furthermore we denote by $\mathrm{F}A := A/\operatorname{Tor}(A)$ the maximal torsion-free quotient. We consider the following two covariant functors:

$$\begin{array}{rcl} \mathrm{F}\colon \mathcal{C} & \mapsto & \mathcal{C} \\ A & \mapsto & \mathrm{F}A \end{array} \quad \text{and} \quad \begin{array}{rcl} \operatorname{Tor}\colon \mathcal{C} & \to & \mathcal{C} \\ A & \mapsto & \operatorname{Tor}(A). \end{array}$$

(a) Show that the functor F is left-exact.
(b) Show that the functor F is not right-exact.
(c) Is the functor Tor left-exact?
(d) Is the functor Tor is right-exact?

Exercise 57.3. Let A be an $(m \times n)$-matrix over $\mathbb{Z}$ and let R be a commutative ring. Let $\varphi\colon \mathbb{Z} \to R$ be the ring homomorphism given by $\varphi(n) = n \cdot 1_R$. Finally let $\Theta_k\colon \mathbb{Z}^m \otimes R \to R^m$ and $\Theta_n\colon \mathbb{Z}^n \otimes R \to R^n$ be the natural isomorphisms provided by Lemma 57.3. Show that the following diagram commutes:

$$\begin{tikzcd} \mathbb{Z}^n \otimes R \arrow[r, "(v \mapsto Av)\otimes \mathrm{id}_R"] \arrow[d, "\Theta_n"'] & \mathbb{Z}^m \otimes R \arrow[d, "\Theta_m"] \\ R^n \arrow[r, "w \mapsto \varphi(A)w"] & R^m. \end{tikzcd}$$

Exercise 57.4. Let $(\mathrm{C}_*, \partial_*)$ be a chain complex and let G be an abelian group. We consider the following map:

$$\begin{array}{rcl} \mathrm{H}_n(C_* \otimes G) & \to & \mathrm{H}_n(C_*) \otimes G \\ \left[\sum\limits_{i=1}^{n} c_i \otimes g_i\right] & \mapsto & \sum\limits_{i=1}^{n} [c_i] \otimes g_i \end{array}$$

Is this map well-defined?

Exercise 57.5. What are the isomorphism types of the homology groups of $\mathbb{RP}^5$ with $\mathbb{F}_3$-coefficients?

Exercise 57.6. Let $A \to B \to C$ be a short exact sequence of abelian groups and let G be an abelian group. Show that there exists an exact sequence of the form

$$0 \to \operatorname{Tor}(A, G) \to \operatorname{Tor}(B, G) \to \operatorname{Tor}(C, G) \to A \otimes G \to B \otimes G \to C \otimes G \to 0.$$

Hint. Use the Snake Lemma 43.9.
Remark. This exact sequence is sometimes called the *six-term sequence* in homological algebra.

Exercise 57.7. Let $n \in \mathbb{N}$. Let G be an abelian group (which is not necessarily finitely generated). Suppose G admits an element of order g, i.e. $n \cdot g = 0$ and $m \cdot g \neq 0$ for $m = 1, \dots, n-1$. Show that $\operatorname{Tor}(G, \mathbb{Z}_n) \neq 0$.
Remark. You could make use of Exercise 57.6.

Exercise 57.8.

(a) Let G_* be a chain complex. (We do not assume that the chain groups G_k are finitely generated.) Suppose that for every $k \in \mathbb{N}_0$ and every prime $p \in \mathbb{N}$ we have $\mathrm{H}_k(G_*; \mathbb{Q}) = 0$ and $\mathrm{H}_k(G_*; \mathbb{Z}_p) = 0$. Show that for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(G_*) = 0$.
Remark. You might want to use Exercise 57.7.

(b) Let $f\colon C_* \to D_*$ be a chain map between two chain complexes. (We do not assume that the chain groups are finitely generated.) Suppose that for every $k \in \mathbb{N}_0$ and every prime $p \in \mathbb{N}$ the maps $\mathrm{H}_k(C_*; \mathbb{Q}) \to \mathrm{H}_k(D_*; \mathbb{Q})$ and $\mathrm{H}_k(C_*; \mathbb{Z}_p) \to \mathrm{H}_k(D_*; \mathbb{Z}_p)$ are isomorphisms. Show that for every $k \in \mathbb{N}_0$ the map $\mathrm{H}_k(C_*) \to \mathrm{H}_k(D_*)$ is an isomorphism.
Hint. Use the algebraic mapping cone that we introduced on page 1284.

Exercise 57.9. Determine $\operatorname{Tor}(\mathbb{Z}_n, \mathbb{Q}/\mathbb{Z})$.

Exercise 57.10. Let $m, n \in \mathbb{N}$. Show that

$$\operatorname{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \;\cong\; \mathbb{Z}_{\gcd(m,n)}.$$

Exercise 57.11. Show that

$$\Big(\prod_{n \in \mathbb{N}} \mathbb{Z}_n\Big) \otimes \mathbb{Q} \neq 0.$$

Exercise 57.12. Let $k \in \mathbb{Z} \setminus \{0\}$ and let $\varphi_k\colon \Delta^1 \to S^1$ be the map given by $(1-t, t) \mapsto e^{k \cdot 2\pi t \mathrm{i}}$. Furthermore let $i\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1; \mathbb{R})$ be the homomorphism from Lemma 57.7 (1) that is induced by $\sigma \mapsto \sigma \otimes 1$. Show that

$$\big[\tfrac{1}{n} \otimes \varphi\big] \;=\; \tfrac{1}{n} \cdot i_*([S^1]) \;\in\; \mathrm{H}_1(S^1; \mathbb{R}).$$

We refer to Figure 874 for an illustration.

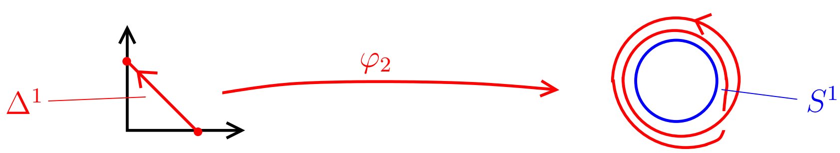


FIGURE 874. Illustration for Exercise 57.12.

Exercise 57.13. Let $T = S^1 \times S^1$ and let K be the Klein bottle.

(a) Does there exist a map $f\colon K \to T$ such that $f_*\colon \mathrm{H}_2(K; \mathbb{F}_2) \to \mathrm{H}_2(T; \mathbb{F}_2)$ is an isomorphism?

(b) Does there exist a map $g\colon T \to K$ such that $g_*\colon \mathrm{H}_2(T; \mathbb{F}_2) \to \mathrm{H}_2(K; \mathbb{F}_2)$ is an isomorphism?

Exercise 57.14. Let $g \in \mathbb{N}$. We denote by N_g the non-orientable surface of genus g as defined on page 206. As a reminder, note that by Lemma 8.33 we know that N_g is diffeomorphic to the connected sum of g copies of $\mathbb{RP}^2$. Show that

$$\mathrm{H}_i(N_g;\mathbb{F}_2) \;\cong\; \begin{cases} 0, & \text{if } i > 2, \\ \mathbb{F}_2, & \text{if } i = 0 \text{ or } i = 2, \\ \mathbb{F}_2^g, & \text{if } i = 1. \end{cases}$$

Remark. We calculated $\mathrm{H}_*(N_g)$ in Proposition 48.9.

58. The Künneth Theorem

In this chapter we want to study the following question:

Question 58.1. *Let X and Y be two topological spaces. What is the connection between the homology groups of X and Y on the one hand and the homology groups of the product topological space $X \times Y$ on the other hand?*

58.1. **The tensor product of chain complexes.** First we want to find some connection between the chain complexes of X, Y and $X \times Y$. To do so we need the notion of the tensor product of chain complexes. Therefore we start out with a short excursion into homological algebra, i.e. into the study of chain complexes.

Lemma 58.2. *Let $\mathcal{C} = (\mathrm{C}_*, \partial_*)$ and $\mathcal{C}' = (\mathrm{C}'_*, \partial'_*)$ be two chain complexes. Given $n \in \mathbb{N}_0$ we define*

$$(\mathcal{C} \otimes \mathcal{C}')_n \;:=\; \bigoplus_{p+q=n} \mathrm{C}_p \otimes \mathrm{C}'_q\,.$$

The consider the maps

$$d\colon (\mathcal{C} \otimes \mathcal{C}')_n \;\to\; (\mathcal{C} \otimes \mathcal{C}')_{n-1}$$

which are uniquely determined by

$$c_p \otimes c'_q \;\mapsto\; \partial c_p \otimes c'_q + (-1)^p \cdot c_p \otimes \partial' c'_q$$

for $c_p \in \mathrm{C}_p$ and $c'_q \in \mathrm{C}'_q$. The maps turn $\mathcal{C} \otimes \mathcal{C}'$ into a chain complex, called the tensor product $\mathcal{C} \otimes \mathcal{C}'$ of the chain complexes $\mathcal{C}$ and $\mathcal{C}'$.

We illustrate the definition of the tensor product of chain complexes in the following diagram:

$$\begin{tikzcd}
\mathrm{C}_0 \otimes \mathrm{C}'_2 \arrow[d, "\mathrm{id} \otimes \partial'_2"] & \mathrm{C}_1 \otimes \mathrm{C}'_2 \arrow[l, "\partial_1 \otimes \mathrm{id}"'] \arrow[d, "-\mathrm{id} \otimes \partial'_2"] & \mathrm{C}_2 \otimes \mathrm{C}'_2 \arrow[l, "\partial_2 \otimes \mathrm{id}"'] \arrow[d, "-\mathrm{id} \otimes \partial'_2"] \\
\mathrm{C}_0 \otimes \mathrm{C}'_1 \arrow[d, "\mathrm{id} \otimes \partial'_1"] & \mathrm{C}_1 \otimes \mathrm{C}'_1 \arrow[l, "\partial_1 \otimes \mathrm{id}"'] \arrow[d, "-\mathrm{id} \otimes \partial'_1"] & \mathrm{C}_2 \otimes \mathrm{C}'_1 \arrow[l, "\partial_2 \otimes \mathrm{id}"'] \arrow[d, "-\mathrm{id} \otimes \partial'_1"] \\
\mathrm{C}_0 \otimes \mathrm{C}'_0 & \mathrm{C}_1 \otimes \mathrm{C}'_0 \arrow[l, "\partial_1 \otimes \mathrm{id}"'] & \mathrm{C}_2 \otimes \mathrm{C}'_0. \arrow[l, "\partial_2 \otimes \mathrm{id}"']
\end{tikzcd}$$

Proof. We still have to show that the maps d are indeed boundary maps of a chain complex, i.e. we have to verify that $d \circ d = 0$. This is the case since for any $c_p \in \mathrm{C}_p$ and $c'_q \in \mathrm{C}'_q$ we have

$$\begin{aligned}
d(d(c_p \otimes c'_q)) &= d(\overbrace{\partial c_p}^{\in \mathrm{C}_{p-1}} \otimes c'_q) + d((-1)^p \cdot c_p \otimes \overbrace{\partial' c'_q}^{\in C'_{q-1}}) \\
&= \underbrace{\partial \partial c_p}_{=0} \otimes c'_q + (-1)^{p-1} \cdot \partial c_p \otimes \partial' c'_q + (-1)^p \cdot \partial c_p \otimes \partial' c'_q + (-1)^p \cdot c_p \otimes \underbrace{\partial' \partial' c'_q}_{=0} \\
&= (-1)^{p-1} \cdot \partial c_p \otimes \partial' c'_q - (-1)^{p-1} \cdot \partial c_p \otimes \partial' c'_q \;=\; 0.
\end{aligned}$$

Thus we have shown that $(\mathcal{C}\otimes\mathcal{C}',d)$ is indeed a chain complex.[898] ■

Example. Let

$$\mathcal{C} \;=\; \dots\to \mathrm{C}_n \xrightarrow{\partial_n} \mathrm{C}_{n-1} \xrightarrow{\partial_{n-1}} \mathrm{C}_{n-2}\to\dots \qquad \text{and} \qquad \mathcal{C}' \;=\; 0 \;\to\; \underset{\substack{\uparrow\\ \text{in degree } k}}{G} \;\to\; 0$$

be two chain complexes where the second consists of a single abelian group. In this case we have

$$(\mathcal{C}_*\otimes\mathcal{C}'_*)_n \;=\; \mathcal{C}_{n-k}\otimes G,$$

i.e. the tensor product is just given by tensoring the chain groups of $\mathcal{C}$ by G and shifting the degrees by $-k$.

We conclude this section with the following lemma.

Lemma 58.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be two chain complexes.*

(1) *Given two chain maps $f\colon\mathcal{C}\to\mathcal{D}$ and $f'\colon\mathcal{C}'\to\mathcal{D}'$ the induced map*

$$\begin{aligned} f\otimes f'\colon \mathcal{C}\otimes\mathcal{C}' &\;\to\; \mathcal{D}\otimes\mathcal{D}' \\ \sum_{i=1}^n c_i\otimes c_i' &\;\mapsto\; \sum_{i=1}^n f(c_i)\otimes f'(c_i') \end{aligned}$$

is also a chain map.

(2) *If $f,g\colon\mathcal{C}\to\mathcal{D}$ and $f',g'\colon\mathcal{C}'\to\mathcal{D}'$ are chain homotopic maps, then the induced maps*

$$f\otimes f',\, g\otimes g'\colon \mathcal{C}\otimes\mathcal{C}' \;\to\; \mathcal{D}\otimes\mathcal{D}'$$

are also chain homotopic.

(3) *If $f\colon\mathcal{C}\to\mathcal{D}$ and $f'\colon\mathcal{C}'\to\mathcal{D}'$ are homotopy equivalences, then the induced map*

$$f\otimes f'\colon \mathcal{C}\otimes\mathcal{C}' \;\to\; \mathcal{D}\otimes\mathcal{D}'$$

is also a homotopy equivalence.

Proof. The proof of the lemma is elementary. For completeness' sake and for a good conscience we provide the proof of the first statement. So let $\mathcal{C}$ and $\mathcal{D}$ be two chain complexes and let $f\colon\mathcal{C}\to\mathcal{D}$ and $f'\colon\mathcal{C}'\to\mathcal{D}'$ be chain maps. Let $c\in C_n$ and $c'\in \mathrm{C}'_l$, then

$$\begin{aligned} \partial(f(c)\otimes f'(c')) &\underset{\substack{\uparrow\\ \text{since } \deg(f(c))=\deg(c)=k}}{=} \partial f(c)\otimes f'(c')+(-1)^k\cdot f(c)\otimes\partial f'(c') \underset{\substack{\uparrow\\ \text{since } f \text{ and } f' \text{ are chain maps}}}{=} f(\partial c)\otimes f'(c')+(-1)^k\cdot f(c)\otimes f'(\partial c') \\ &= (f\otimes f')\big(\partial c\otimes c'+(-1)^k\cdot c\otimes\partial c'\big) \;=\; (f\otimes f')(\partial(c\otimes c')). \end{aligned}$$

■

58.2. **The Eilenberg-Zilber Theorem.** The following theorem, which was first proved in 1953, relates the singular chain complex $\mathrm{C}_*(X\times Y)$ of the product of two topological spaces X and Y to the tensor product $\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)$ of the two individual chain complexes.

---

[898] Note that in the calculation we made use of the extra-term $(-1)^p$ in the definition of the boundary map. If we defined the boundary map "naively" as $d(c_p\otimes c_q')=\partial c_p\otimes c_q'+c_p\otimes\partial' c_q'$, then the result would not have been a chain complex.

Theorem 58.4. (Eilenberg-Zilber) *Let X and Y be topological spaces. There exist natural*[899] *chain maps*

$$\Upsilon\colon\ \mathrm{C}_*(X)\otimes \mathrm{C}_*(Y) \ \to\ \mathrm{C}_*(X\times Y)$$

and

$$\Theta\colon\ \mathrm{C}_*(X\times Y) \ \to\ \mathrm{C}_*(X)\otimes \mathrm{C}_*(Y),$$

with the following three properties:

(1) *The maps are chain homotopy-inverses of one another. In particular, by Corollary 42.3, for each $n\in\mathbb{N}_0$ we have a natural isomorphism*

$$\mathrm{H}_n(X\times Y)\ \xrightarrow[\cong]{\Theta_*}\ \mathrm{H}_n(\mathrm{C}_*(X)\otimes \mathrm{C}_*(Y)).$$

(2) *For any $x\in X$ and any singular k-chain $\sigma\colon \Delta^k\to Y$ the singular chain $\Upsilon(\{x\}\otimes\sigma)$ equals the singular simplex $\Delta^k\to X\times Y$ that is given by $P\mapsto (x,\sigma(P))$.*

(3) *The same statement as in (2) also holds with the roles of X and Y swapped.*

Proof. We postpone this technically intricate proof to Section 80.2. In that section the above statement follows from Theorem 80.2 together with the proof of the Eilenberg-Zilber Theorem provided on page 1962. ■

Since we just postponed the proof of the Eilenberg-Zilber 58.4 to significantly later let us prove a partial result that is much weaker, but that is good enough for many purposes. Namely we will prove an analogous statement for CW-complexes which will be sufficient for the calculations of homology groups that we want to do in this chapter. The proof of this special case is significantly easier to digest than the proof of the general case of the Eilenberg-Zilber Theorem.

Proposition 58.5. *Let X and Y be two CW-complexes with finitely many cells in each dimension. Then there exists a natural*[900] *isomorphism*

$$C_*^{\mathrm{CW}}(X\times Y)\ \xrightarrow{\cong}\ C_*^{\mathrm{CW}}(X)\otimes C_*^{\mathrm{CW}}(Y)$$

of chain complexes.

Remark. Recall that, following the convention of page 1096, given two chain complexes C_* and D_* we write $C_*\simeq D_*$ if they are homotopy equivalent. Suppose that X and Y are

[899]Here "natural" means that for maps $f\colon X\to X'$ and $g\colon Y\to Y'$ of topological spaces we obtain commutative diagrams

$$\begin{tikzcd}
\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y) \arrow[r, "\Upsilon"] \arrow[d, "f_*\otimes f'_*"] & \mathrm{C}_*(X\times Y) \arrow[d, "(f\times g)_*"] \\
\mathrm{C}_*(X')\otimes\mathrm{C}_*(Y') \arrow[r, "\Upsilon"] & \mathrm{C}_*(X'\times Y')
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\mathrm{C}_*(X\times Y) \arrow[r, "\Theta"] \arrow[d, "(f\times g)_*"] & \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y) \arrow[d, "f_*\otimes f'_*"] \\
\mathrm{C}_*(X'\times Y') \arrow[r, "\Theta"] & \mathrm{C}_*(X')\otimes\mathrm{C}_*(Y').
\end{tikzcd}$$

[900]Here the notion "natural" means "natural" in the category of CW-complexes, i.e. the isomorphisms Φ_{XY} are only natural with respect to cellular maps $X\to X'$ and $Y\to Y'$, i.e. the commutative diagram from Footnote 899 only exists for such cellular maps.

CW-complexes which have only finitely many cells in each dimension. Then we have

$$\underset{\text{Corollary 49.9 and Lemma 58.3}}{\mathrm{C}_*(X\times Y) \underset{\uparrow}{\simeq} \mathrm{C}^{\mathrm{CW}}_*(X\times Y)} \underset{\substack{\uparrow\\ \text{Proposition 58.5}}}{\xrightarrow{\cong}} \mathrm{C}^{\mathrm{CW}}_*(X)\otimes \mathrm{C}^{\mathrm{CW}}_*(Y) \underset{\substack{\uparrow\\ \text{Corollary 49.9 and Lemma 58.3}}}{\simeq} \mathrm{C}_*(X)\otimes \mathrm{C}_*(Y).$$

For such X and Y this argument proves the existence of the chain homotopies in the statement of the Eilenberg-Zilber Theorem 58.4. But note that the chain homotopy equivalences of Lemma 58.3 are not natural, hence the above chain equivalences are not natural.

Proof. Let X and Y be two CW-complexes with finitely many cells in each dimension. On page 960 we introduced a CW-structure on $X\times Y$ where the n-cells of $X\times Y$ are precisely of the form $e\times f$ where e is a p-cell of X and f is an $(n-p)$-cell of Y. We quickly recall the definition of the product CW-structure on $X\times Y$:

(1) Using the homeomorphism from Proposition 2.53 (2) we identify the n-dimensional closed ball $\overline{B}^n$ with the n-dimensional cube $I^n=[0,1]^n$.
(2) We put $(X\times Y)^0 = X^0\times Y^0$.
(3) Now we suppose that the $(n-1)$-skeleton $(X\times Y)^{(n-1)}$ has already been defined. We consider a p-cell e in X which is given by a characteristic map $\varphi\colon I^p\to X^p$ and an $(n-p)$-cell f in Y which is given by a characteristic map $\psi\colon I^{n-p}\to Y^{n-p}$. Then we denote by $e\times f$ the n-cell in $X\times Y$ which is given by the attaching map
$$\begin{array}{rcl}\partial(I^n)=(\partial I^p\times I^{n-p})\cup(I^p\times\partial I^{n-p}) & \to & (X\times Y)^{n-1}\\ (x,y) & \mapsto & (\varphi(x),\psi(y)).\end{array}$$
All these n-cells together define the n-skeleton of $X\times Y$.

Since X and Y have only finitely many cells in each dimension the above construction does indeed define a CW-structure on $X\times Y$.

Given $n\in\mathbb{N}_0$ we consider the map
$$C^{\mathrm{CW}}_n(X\times Y) \;\to\; C^{\mathrm{CW}}_p(X)\otimes C^{\mathrm{CW}}_{n-p}(Y)$$
which is induced by the property that for any p-cell e of X and any $(n-p)$-cell f of Y we have[901]
$$[e\times f]\;\mapsto\;[e]\otimes[f].$$
As on page 1397 one can show that this map is an isomorphism. It remains to prove that the above maps form a chain map. This statement in turn is equivalent to the following claim.

Claim. For any p-cell e of X and any q-cell f of Y we have
$$\underbrace{\partial(e\times f)}_{\in C^{\mathrm{CW}}_{p+q-1}(X\times Y)} = \underbrace{\partial e\otimes f}_{\in C^{\mathrm{CW}}_{p-1}(X)\otimes C^{\mathrm{CW}}_q(Y)} + (-1)^p\cdot \underbrace{e\otimes\partial f}_{\in C^{\mathrm{CW}}_p(X)\otimes C^{\mathrm{CW}}_{q-1}(Y)}.$$

It is possible to prove this formula using the description of the cellular boundary map in Proposition 48.7 and by doing a very careful sign-analysis. We illustrate this argument in Figure 875.

[901] We follow the convention of page 1267 where saw that cells naturally give rise to a basis of the cellular chain complex.

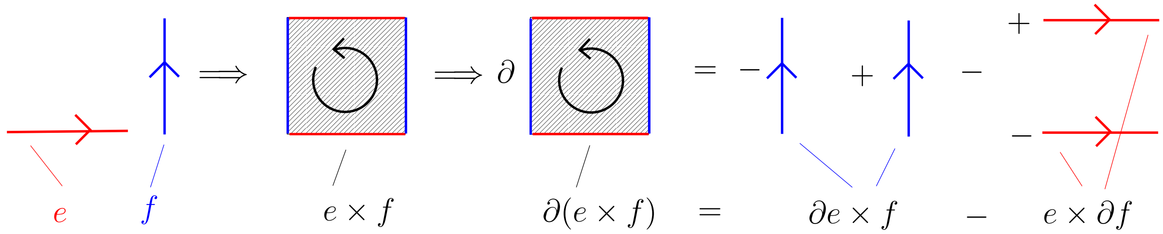


FIGURE 875. The boundary of the product of two 1-cells.

Following [**Bre93**, Theorem 12.2] we will give an alternative proof instead. Given an k-cell σ and a $(k-1)$-cell μ we denote by $[\mu : \sigma]$ the coefficient of μ in $\partial\sigma$. It follows quite easily from Proposition 48.7 that for $p, q \in \mathbb{N}_0$ there exists a unique sign $\epsilon_{p,q} \in \{-1, 1\}$ such that for any CW-complexes K and L, any p-cell e of K, any q-cell f of L and any $(q-1)$-cell g of L we have

$$[e \times g : e \times f] \;=\; \epsilon_{p,q} \cdot [g : f].$$

It follows immediately from the definitions that $\epsilon_{0,q} = 1$ for all q.

Now let e be a p-cell of a CW-complex K and let f be a q-cell of a CW-complex L. We denote by τ the set of $(p-1)$-cells of K and we denote by ϕ the set of $(q-1)$-cells of L. We compute that

$$\begin{aligned}
\partial(e \times f) &= \sum_{\tau}[\tau \times f : e \times f]\cdot(\tau \times f) \;+\; \sum_{\phi}[e \times \phi : e \times f]\cdot(e \times \phi) \\
&= \sum_{\tau}[\tau : e]\cdot(\tau \times f) \;+\; \sum_{\phi}\epsilon_{p,q}\cdot[\phi : f]\cdot(e \times \phi) \;=\; (\partial e) \times f \;+\; \epsilon_{p,q}\cdot e \times (\partial f).
\end{aligned}$$

It remains to show that $\epsilon_{p,q} = (-1)^p$. We will prove this indirectly. More precisely, since we know that $\partial^2 = 0$ in the cellular chain complex we obtain that

$$\begin{aligned}
0 \;=\; \partial\partial(e \times f) \;&=\; \overbrace{(\partial\partial e) \times f}^{=0} + \epsilon_{p-1,q}\cdot(\partial e) \times (\partial f) + \epsilon_{p,q}\cdot(\partial e) \times (\partial f) + \epsilon_{p,q-1}\epsilon_{p,q}\cdot e \times \overbrace{(\partial\partial f)}^{=0} \\
&= \epsilon_{p-1,q}\cdot(\partial e) \times (\partial f) + \epsilon_{p,q}\cdot(\partial e) \times (\partial f) \;=\; (\epsilon_{p-1,q} + \epsilon_{p,q})\cdot(\partial e) \times (\partial f).
\end{aligned}$$

But this equality can hold for all CW-complexes K and L only if $\epsilon_{p,q} = -\epsilon_{p-1,q}$. Since $\epsilon_{0,q} = 1$ we obtain iteratively that $\epsilon_{p,q} = (-1)^p$. ■

58.3. **The Künneth Theorem for chain complexes.** Our goal is to determine the homology groups of the product $X \times Y$ of two topological spaces in terms of the homology groups of X and Y. In light of the Eilenberg-Zilber Theorem 58.4, or alternatively, considering Proposition 58.5, we have to find a connection between the homology groups of two chain complexes $\mathcal{C}$, $\mathcal{C}'$ and the corresponding tensor product $\mathcal{C} \otimes \mathcal{C}'$.

We start out with the following elementary lemma.

Lemma 58.6. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two chain complexes. The map*

$$\begin{array}{rccc} \Omega\colon & \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') & \to & \mathrm{H}_{p+q}(\mathcal{C} \otimes \mathcal{C}') \\ & \sum_{i=1}^{r} [c_i] \otimes [c'_i] & \mapsto & \sum_{i=1}^{r} [c_i \otimes c'_i] \end{array}$$

is well-defined and it is natural with respect to the chain complexes $\mathcal{C}$ and $\mathcal{C}'$.

Proof. Let $\mathcal{C} = (C_n, \partial_n)$ and $\mathcal{C}' = (\mathrm{C}'_n, \partial'_n)$ be two chain complexes. First note that if $c_i \in \mathrm{C}_p$ and $c'_i \in \mathrm{C}_q$ are cycles, then

$$d(c_i \otimes c'_i) \;=\; \underbrace{\partial c_i}_{=0} \otimes c'_i + (-1)^p c_i \otimes \underbrace{\partial c'_i}_{=0} \;=\; 0$$

i.e. $c_i \otimes c'_i \in \mathcal{C} \otimes \mathcal{C}'$ is indeed a cycle. Second, if $c_i + \partial e_i$ is an other representative of $[c_i]$ then

$$[(c_i + \partial e_i) \otimes c'_i] \underset{\substack{\uparrow \\ \text{since } \partial' c' = 0}}{=} [c_i \otimes c'_i + d(e_i \otimes c'_i)] \;=\; [c_i \otimes c'_i].$$

The same way one can show that the value of Ω does not depend on the choice of representative of the class in $\mathrm{H}_q(\mathcal{C}')$. It is straightforward to verify that Ω is in fact a natural map. ∎

Example. In some instances the homomorphism Ω from Lemma 58.6 is an isomorphism. For example let $\mathcal{C} = (C_n, \partial_n)$ be a chain complex of free abelian groups such that all boundary maps $\mathrm{C}_n \to \mathrm{C}_{n-1}$ in $\mathcal{C}$ are the zero map. In this case we have for any chain complex $\mathcal{C}'$ that

$$\bigoplus_{n\in\mathbb{N}_0} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_{n-p}(\mathcal{C}'_*) \underset{\substack{\uparrow \\ \text{since all boundary maps} \\ \text{in } \mathcal{C} \text{ are zero}}}{=} \bigoplus_{n\in\mathbb{N}_0} \mathrm{C}_p \otimes \mathrm{H}_n(\mathcal{C}'_{*-p}) \underset{\substack{\uparrow \\ \text{by Theorem 57.18} \\ \text{and Lemma 57.17 (3)} \\ \text{since all } \mathrm{C}_p \text{ are free abelian}}}{=} \bigoplus_{n\in\mathbb{N}_0} \mathrm{H}_n(\mathrm{C}_p \otimes \mathcal{C}'_{*-p}) \underset{\substack{\uparrow \\ \text{by the example on page 1434} \\ \text{since } \mathcal{C} \text{ is a direct sum} \\ \text{of trivial chain complexes}}}{=} \mathrm{H}_n(\mathcal{C}_* \otimes \mathcal{C}'_*).$$

Now we can formulate the following proposition which is the key algebraic result of this chapter.

Theorem 58.7. (Künneth Theorem for Chain Complexes)[902] *Let $\mathcal{C}$ and $\mathcal{C}'$ be two chain complexes. If $\mathcal{C}$ is free, i.e. if all chain groups of the chain complex $\mathcal{C}$ are free abelian groups, then there exists a natural short exact sequence*

$$0 \;\to\; \bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') \;\xrightarrow{\Omega}\; \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \;\to\; \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(\mathcal{C}), \mathrm{H}_q(\mathcal{C}')) \;\to\; 0.$$

If $\mathcal{C}'$ is also a free chain complex, then this short exact sequence splits.

Remark.

[902]Hermann Künneth (1892-1975) was a German mathematician.

(1) The adjective "natural" means in this context that if $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ are chain maps, then we obtain a commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \bigoplus\limits_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') & \xrightarrow{\Omega} & \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') & \longrightarrow & \bigoplus\limits_{p+q=n} \mathrm{Tor}(\mathrm{H}_p(\mathcal{C}), \mathrm{H}_q(\mathcal{C}')) & \longrightarrow 0 \\ & \big\downarrow {\scriptstyle f_* \otimes f'_*} & & \big\downarrow {\scriptstyle (f \otimes f')_*} & & \big\downarrow {\scriptstyle \mathrm{Tor}(f_*, f'_*)} & \\ 0 \longrightarrow & \bigoplus\limits_{p+q=n} \mathrm{H}_p(\mathcal{D}) \otimes \mathrm{H}_q(\mathcal{D}') & \xrightarrow{\Omega} & \mathrm{H}_n(\mathcal{D} \otimes \mathcal{D}') & \longrightarrow & \bigoplus\limits_{p+q=n} \mathrm{Tor}(\mathrm{H}_p(\mathcal{D}), \mathrm{H}_q(\mathcal{D}')) & \longrightarrow 0. \end{array}$$

Here the middle map is given by Lemma 58.3 (1) and the map on the right is given by the functoriality of the torsion-groups that we had discussed on page 1413.

(2) If $\mathcal{C}'$ is a chain complex which consists of a single abelian group G, i.e. if it is of the form $0 \to G \to 0$ with G in degree 0, then $\mathrm{H}_0(\mathcal{C}') = G$ and all other homology groups are zero. The Künneth Theorem together with the example on page 1434 thus gives us the short exact sequence

$$0 \;\to\; \mathrm{H}_n(\mathcal{C}) \otimes G \;\to \mathrm{H}_n(\underbrace{\mathcal{C} \otimes G}_{=\mathcal{C} \otimes \mathcal{C}'}) \;\to\; \mathrm{Tor}(\mathrm{H}_{n-1}(\mathcal{C}), G) \;\to\; 0,$$

i.e. we recover the Universal Coefficient Theorem 57.18 as a special case of the Künneth Theorem. In fact, as we will see, the proof of the Künneth Theorem is quite similar to the proof of the Universal Coefficient Theorem 57.18.

(3) The fact that the short exact sequence of the Künneth Theorem splits implies that the middle group is isomorphic to the direct sum of the two groups to the left and right. But, as we saw in the discussion of the Universal Coefficient Theorem 57.18, see Proposition 57.26, this isomorphism is not natural.

Proof. Let $\mathcal{C} = (C_n, \partial_n)$ and $\mathcal{C}' = (\mathrm{C}'_n, \partial'_n)$ be two chain complexes such that all C_n are free abelian groups. As we will see, the proof of the theorem is quite similar to the proof of the Universal Coefficient Theorem 57.18.

For $p \in \mathbb{N}_0$ we consider again

$$Z_p \;:=\; \ker(\partial_p\colon\, \mathrm{C}_p \to \mathrm{C}_{p-1}) \quad \text{ and } \quad B_p \;:=\; \mathrm{im}(\partial_{p+1}\colon\, \mathrm{C}_{p+1} \to \mathrm{C}_p).$$

We denote by $i\colon Z_p \to \mathrm{C}_p$ the inclusion map. Furthermore we denote by $\mathcal{Z}$ the chain complex which is given by the groups Z_p where all the boundary maps are the zero map. The same way we obtain from the groups B_p a chain complex $\mathcal{B}$.

Claim. The sequence

$$0 \;\to\; \mathcal{Z} \otimes \mathcal{C}' \;\xrightarrow{i \otimes \mathrm{id}}\; \mathcal{C} \otimes \mathcal{C}' \;\xrightarrow{\partial \otimes \mathrm{id}}\; \mathcal{B} \otimes \mathcal{C}' \;\to\; 0$$

is a sequence of chain maps and it is exact at each degree.

It follows easily from the definitions of the boundary maps in the various chain complexes, see also the diagram on page 1418, that the homomorphisms $i \otimes \mathrm{id}\colon \mathcal{Z} \otimes \mathcal{C}' \to \mathcal{C} \otimes \mathcal{C}'$ and $\partial \otimes \mathrm{id}\colon \mathcal{C} \otimes \mathcal{C}' \to \mathcal{B} \otimes \mathcal{C}$ are chain maps. It remains to show that the sequence is exact at each degree. The argument is exactly the same as on page 1418. For the reader's convenience we recall the argument. First note that by definition the sequences

$$0 \;\to\; Z_p \;\xrightarrow{i}\; \mathrm{C}_p \;\xrightarrow{\partial}\; B_{p-1} \;\to\; 0$$

are exact. Also recall that by hypothesis each C_n is a free abelian group. It follows from Lemma 19.2 that each B_n is also free abelian. By Lemma 46.1 this means that the above short exact sequence splits. But then it follows from Lemma 57.6 that the sequence stays exact even after tensoring it with any C'_q. The claim is an immediate consequence of these observations.

From Proposition 43.11 and the claim we thus obtain the following long exact sequence[903]

$$\cdots \to \mathrm{H}_n(\mathcal{B}\otimes\mathcal{C}') \xrightarrow{d_n} \mathrm{H}_n(\mathcal{Z}\otimes\mathcal{C}') \to \mathrm{H}_n(\mathcal{C}\otimes\mathcal{C}') \to \mathrm{H}_{n-1}(\mathcal{B}\otimes\mathcal{C}') \xrightarrow{d_{n-1}} \mathrm{H}_{n-1}(\mathcal{Z}\otimes\mathcal{C}') \to \dots$$

The connecting homomorphism $d_n\colon \mathrm{H}_n(\mathcal{B}\otimes\mathcal{C}') \to \mathrm{H}_n(\mathcal{Z}\otimes\mathcal{C}')$ from Proposition 43.11 is hereby, as in the proof of the Universal Coefficient Theorem 57.18, induced by the map $i\otimes\mathrm{id}$ where i denotes the inclusion $\mathcal{B}\to\mathcal{Z}$.

Since the boundary maps in the chain complexes $\mathcal{B}$ and $\mathcal{Z}$ are the zero maps it follows from the example on page 1438 that we obtain the following long exact sequence:

$$\dots \xrightarrow{i\otimes\mathrm{id}} \bigoplus_{p+q=n} Z_p\otimes\mathrm{H}_q(\mathcal{C}') \to \mathrm{H}_n(\mathcal{C}\otimes\mathcal{C}') \to \bigoplus_{p+q=n-1} B_p\otimes\mathrm{H}_q(\mathcal{C}') \xrightarrow{i\otimes\mathrm{id}} \bigoplus_{p+q=n-1} Z_p\otimes\mathrm{H}_q(\mathcal{C}') \to \dots$$

From Lemma 46.4 we obtain a short exact sequence

$$\begin{aligned} 0 \;\to\; \bigoplus_{p+q=n} \overbrace{\mathrm{coker}\left(B_p\otimes\mathrm{H}_q(\mathcal{C}') \xrightarrow{i\otimes\mathrm{id}} Z_p\otimes\mathrm{H}_q(\mathcal{C}')\right)}^{\substack{=\mathrm{H}_p(\mathcal{C})\otimes\mathrm{H}_q(\mathcal{C}'),\text{ since}\\ 0\to B_p\to Z_p\to\mathrm{H}_p(\mathcal{C})\to 0\text{ is a free resolution}}} \;\to\; \mathrm{H}_n(\mathcal{C}\otimes\mathcal{C}') \;\to \\ \to \bigoplus_{p+q=n-1} \underbrace{\ker\left(B_p\otimes\mathrm{H}_q(\mathcal{C}') \xrightarrow{i\otimes\mathrm{id}} Z_p\otimes\mathrm{H}_q(\mathcal{C}')\right)}_{\substack{=\mathrm{Tor}(\mathrm{H}_p(\mathcal{C}),\mathrm{H}_q(\mathcal{C}')),\text{ since}\\ 0\to B_p\to Z_p\to\mathrm{H}_p(\mathcal{C})\to 0\text{ is a free resolution}}} \to 0. \end{aligned}$$

This proves the existence of the short exact sequence. Using Lemma 57.13 one can show that this short exact sequence is natural. We leave it to the reader to verify the details.

Finally we suppose that $\mathcal{C}'_*$ is also a free chain complex. We leave it again to the reader to adapt the argument of the proof of Proposition 57.23 to show that the above short exact sequence splits. Alternatively we refer to [**Mun84**, Chapter 58] for details. ■

58.4. **The Künneth Theorem for topological spaces.** Now we return to our initial Question 58.1 of the chapter, namely, what is the relationship between the homology groups of the topological spaces X, Y and $X\times Y$? The following theorem gives a comprehensive answer.

[903] The index goes down by one in the map $\mathrm{H}_n(\mathcal{C}\otimes\mathcal{C}') \to \mathrm{H}_{n-1}(\mathcal{B}\otimes\mathcal{C}')$, since the map is induced by the short exact sequence $0\to Z_p\to\mathrm{C}_p\to B_{p-1}\to 0$.

Theorem 58.8. (Künneth Theorem for Topological Spaces) *Let X and Y be topological spaces. Then there exists a short exact sequence*

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(X) \otimes \mathrm{H}_q(Y) \xrightarrow{\Omega} \mathrm{H}_n(X \times Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X), \mathrm{H}_q(Y)) \to 0$$

with the following four properties:

(1) *Both maps are natural with respect to the topological spaces X and Y.*
(2) *For $p = 0$ the map $\Omega\colon \mathrm{H}_0(X) \otimes \mathrm{H}_q(Y) \to \mathrm{H}_q(X \times Y)$ is the "obvious" map, namely for a point $x \in X$ and a singular cycle σ the image of $[x] \otimes [\sigma] \in \mathrm{H}_0(X) \otimes \mathrm{H}_q(Y)$ under Ω equals the image of $[\sigma]$ under the map $\mathrm{H}_q(Y) \to \mathrm{H}_q(\{x\} \times Y) \to \mathrm{H}_q(X \times Y)$, where the first map is induced by the natural homeomorphism and the second map is induced by the inclusion.*
(3) *The analogue of (2) also holds for $q = 0$.*
(4) *The above short exact sequence splits.*[904]

Remark. Both maps in the short exact sequence of the Künneth Theorem 58.8 for Topological Spaces are difficult to describe explicitly. At least for the former map we will partly get a nice description in Proposition 80.10.

Proof. We consider the following maps:

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \bigoplus\limits_{p+q=n} \mathrm{H}_p(X) \otimes \mathrm{H}_q(Y) \arrow[r, "\Omega"] \arrow[dr, dashed] & \mathrm{H}_n(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)) \arrow[r] \arrow[d, "\cong"', "\Upsilon"] & \bigoplus\limits_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X), \mathrm{H}_q(Y)) \arrow[r] & 0 \\
& & \underbrace{\mathrm{H}_n(\mathrm{C}_*(X \times Y))}_{=H_n(X \times Y)}. \arrow[ur, dashed] & &
\end{tikzcd}$$

Here the sequence on the top is exact and natural by the Künneth Theorem 58.7 for Chain Complexes. The vertical map is a natural isomorphism by the Eilenberg-Zilber Theorem 58.4 together with Corollary 42.3. We obtain the desired short exact sequence from replacing the upper middle term by the lower middle term $\mathrm{H}_n(\mathrm{C}_*(X \times Y)) = \mathrm{H}_n(X \times Y)$. One can easily verify that this short exact sequence has all the desired properties. ■

Examples.

(1) Let X be a topological space. We want to determine the homology groups of $S^1 \times X$. The homology groups of S^1 are of course $\mathbb{Z}$ in the dimensions 0 and 1. For any abelian group A we have by Lemma 57.3 a natural isomorphism $\mathbb{Z} \otimes A = A$ and by Lemma 57.17 we have $\mathrm{Tor}(\mathbb{Z}, A) = 0$. It follows from the Künneth Theorem 58.8 for topological spaces that there exists a natural isomorphism

$$\mathrm{H}_n(S^1 \times X) \xrightarrow{\cong} \mathrm{H}_{n-1}(X) \oplus \mathrm{H}_n(X).$$

This is of course precisely the same result as in Lemma 46.22.

[904] But as in the case of the Universal Coefficient Theorem 57.18, see Proposition 57.26, the splitting is not natural.

It follows fairly easily from Statement (2) that the map $\mathrm{H}_n(X) \to \mathrm{H}_n(S^1 \times X)$ is the map induced by the projection $S^1 \times X \to X$. It is less clear what the map $\mathrm{H}_n(S^1 \times X) \to \mathrm{H}_{n-1}(X)$ should be. Some more effort shows that this map is precisely the map that we had described in detail in Exercise 44.5.

(2) Let $n \in \mathbb{N}$ and let $m \in \mathbb{N}$. We want to determine the homology groups of the generalized torus $(S^m)^n$. In fact using the Künneth Theorem 58.8 for topological spaces one can easily show that for any $k \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}_i((S^m)^n) \cong \begin{cases} \mathbb{Z}^{\binom{n}{k}} = \mathbb{Z}^{\frac{n!}{k!\cdot(n-k)!}} & \text{if } i = km \text{ with } k \in \{0, \dots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

In particular we have

$$\mathrm{H}_i((S^m)^2) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2m, \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } i = m, \\ 0, & \text{otherwise.} \end{cases}$$

We will fill in the details in Exercise 58.1.

(3) Let X and Y be two path-connected topological spaces. Then we have isomorphisms

$$\mathrm{H}_1(X \times Y) \underset{\substack{\uparrow \\ \text{Künneth Theorem 58.8 for topological spaces}}}{\cong} \mathrm{H}_1(X) \otimes \mathrm{H}_0(Y) \oplus \mathrm{H}_0(X) \otimes \mathrm{H}_1(Y) \underset{\substack{\uparrow \\ \text{Proposition 41.5 and Lemma 57.3}}}{=} \mathrm{H}_1(X) \oplus \mathrm{H}_1(Y).$$

This is consistent with the product formula for fundamental groups proved in Proposition 16.20 and with the isomorphism of the Hurewicz Theorem 52.5.

Now we discuss the product $S^2 \times S^4$ in slightly more detail. It follows easily from the Künneth Theorem 58.8 for topological spaces, Proposition 43.4, Proposition 41.5, Lemma 57.3 and Lemma 57.17 that

$$\mathrm{H}_k(S^2 \times S^4) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, 6, \\ 0, & \text{otherwise.} \end{cases}$$

But by the calculation on page 1262 these are precisely the homology groups of the complex projective space $\mathbb{C}\mathrm{P}^3$. Furthermore it follows from Propositions 14.14 and 16.20 that $S^2 \times S^4$ is simply connected and it follows from the discussion on page 997 that $\mathbb{C}\mathrm{P}^3$ is also simply connected. So all the invariants we are aware of, and that we know how to compute[905], agree for $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$. This raises the following question which we will answer in Corollary 90.9.

Question 58.9. *Are the topological spaces $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ homeomorphic?*

---

[905]As of right now we do not know the higher homotopy groups of these two smooth manifolds, but these will not turn out to be very useful.

Exercises for Chapter 58.

Exercise 58.1. Let $n \in \mathbb{N}$ and let $m \in \mathbb{N}$. Show that for every $k \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}_i((S^m)^n) \;\cong\; \begin{cases} \mathbb{Z}^{\binom{n}{k}} = \mathbb{Z}^{\frac{n!}{k!\cdot(n-k)!}} & \text{if } i = km \text{ with } k \in \{0,\dots,n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 58.2. Let $n \in \mathbb{N}_0$. Show that

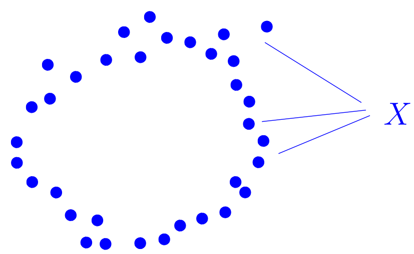
$$\sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \;=\; 0.$$

Remark. There are many ways to prove this equality, one fun way of doing this is by computing the Euler characteristic of the n-dimensional torus in two different ways.

59. Applications of homology groups

59.1. **Persistent homology ❀.** In this section we outline the idea behind "persistent homology". Persistent homology was introduced by Gunnar Carlsson [**Carl09**] about ten years ago. More information and a much more careful treatment of persistent homology can for example be found in [**EdH10**]. Some extra information can also be found in [**Gh14**].

We consider Figure 876. What do we see? If we forget about topology for a second, then we would say that we see a set X of dots that form a loop.



dots that appear to the eye to form a loop

Figure 876

If we view X as a topological space, then it is just a finite discrete subset of $\mathbb{R}^2$. In particular its homology groups are non-zero only in dimension 0. So how can we use the mathematical machine that we have developed over the last two semesters to "see the loop"?

Given $r \in [0,\infty)$ we denote by $X(r)$ the subset of $\mathbb{R}^2$ that is given by replacing each point $x \in X$ by the closed ball $\overline{B}_r(x)$ of radius r around x. The homeomorphism type of $X(r)$ changes with the radius r. For example in Figure 877 we see that at $r = 1$ all but one ball are joined up and we see that $\mathrm{H}_1(X(1)) \cong \mathbb{Z}^3$. We show three cycles c_1, c_2, c_3 in $X(1)$ which form a basis for $\mathrm{H}_1(X(1))$. Two of the generators, namely c_1, c_2 are formed "by accident", namely they arise from balls which are close by. This gets captured by the fact that they become null-homologous in $\mathrm{H}_1(X(2))$ whereas c_3 represents a non-trivial element for "much longer", i.e. for much larger values of r. Put differently, c_3 is a persistent element in homology, hence it captures global information.

Summarizing, given a subset X of $\mathbb{R}^2$ we can study the topological spaces $X(r)$ for any $r \in [0,\infty)$ and furthermore for any $r < s$ we can study the inclusion induced map $\mathrm{H}_n(X(r)) \to \mathrm{H}_n(X(s))$. If there is a non-zero element in $\mathrm{H}_n(X(r))$ which stays non-zero for a "long time", then we say it is persistent and we hope that it captures something interesting.

The same approach can be applied to any finite subset of some $\mathbb{R}^n$. Pretty much any data one has in practice, e.g. blue pixels of a picture, measurements in an experiment, statistical data and so on, can be viewed as a finite subset of some $\mathbb{R}^n$, so in principle one can study its persistent homology. The real question is, whether one captures something significant or not. For example persistent homology was applied in [**SHP16**][906] to study

[906]Here is an easy way to tell that that paper was not written by mathematicians: In a mathematics paper the authors are invariably listed according to the alphabetic order of their last names. In most other sciences this is not the case.

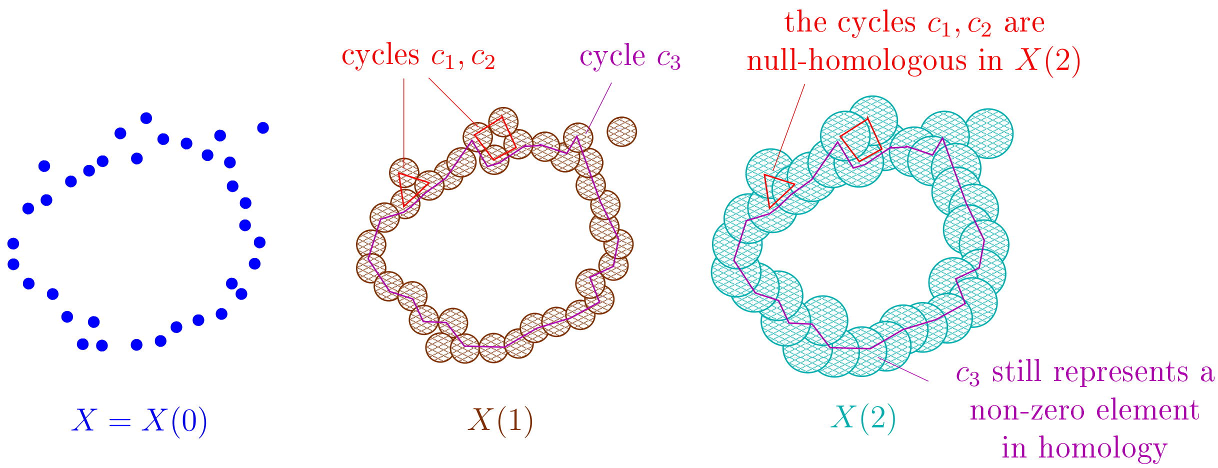


FIGURE 877

the "The topological shape of Brexit". It is perhaps debatable whether one can really obtain anything meaningful about the politics of Brexit using persistent homology.

59.2. **The transfer map.** Before we can give the next application of homology theory, namely the Borsuk-Ulam Theorem in higher dimensions, we need to introduce and study the transfer map in homology corresponding to a finite covering.

Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X. Let $\sigma\colon \Delta^n \to X$ be a singular simplex. Since Δ^n is simply connected[907] it follows from Proposition 29.2 that there exist precisely k different lifts of σ to $\widetilde{X}$, i.e. there exist precisely k different singular simplices $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_k\colon \Delta^n \to \widetilde{X}$ with $p \circ \widetilde{\sigma}_i = \sigma$ for $i = 1, \dots, k$. We write

$$p^*(\sigma) \;:=\; \sum_{i=1}^{k} \widetilde{\sigma}_i.$$

Given any commutative ring R we refer to the map

$$\begin{array}{rccc} p^*\colon & \mathrm{C}_*(X;R) & \to & \mathrm{C}_*(\widetilde{X};R) \\ & \sum\limits_{i=1}^{n} \sigma_i \otimes r_i & \mapsto & \sum\limits_{i=1}^{n} p^*(\sigma_i) \otimes r_i \end{array}$$

as the *transfer map* of the covering. It is straightforward to verify that the transfer map is in fact a chain map. In particular it induces a map

$$p^*\colon\ \mathrm{H}_*(X;R) \;\to\; \mathrm{H}_*(\widetilde{X};R).$$

Finally suppose we are given a subset $A \subset X$. We write $\widetilde{A} := p^{-1}(A)$. One easily verifies that the transfer map $p^*\colon\ \mathrm{C}_n(X;R) \to \mathrm{C}_*(\widetilde{X};R)$ induces a chain map

$$p^*\colon\ \mathrm{H}_*(X,A;R) \;\to\; \mathrm{H}_*(\widetilde{X},\widetilde{A};R).$$

[907]To be precise, we also use that Δ^n is locally path-connected.

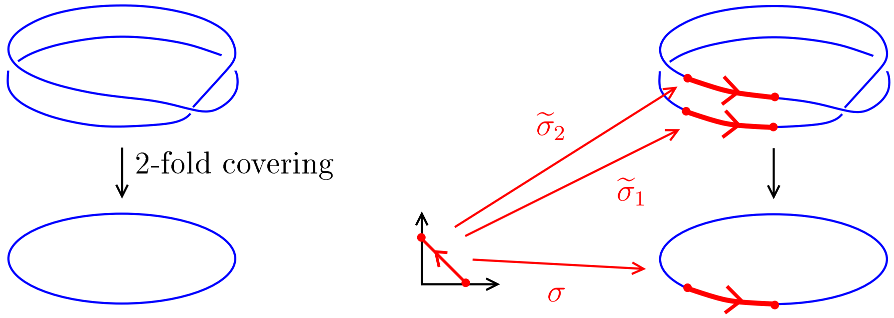


FIGURE 878

Proposition 59.1. *Let $p\colon \widetilde{X} \to X$ be a covering of degree $k \in \mathbb{N}$ of a topological space X and let $n \in \mathbb{N}_0$. Then the following hold:*

(1) *For any commutative ring R the transfer map $p^*\colon \mathrm{C}_n(X;R) \to \mathrm{C}_n(\widetilde{X};R)$ is a monomorphism.*
(2) *For any commutative ring R the map $p_* \circ p^*\colon \mathrm{H}_n(X;R) \to \mathrm{H}_n(X;R)$ is given by multiplication by k.*
(3) *The projection map $p_*\colon \mathrm{H}_n(\widetilde{X};\mathbb{Q}) \to \mathrm{H}_n(X;\mathbb{Q})$ is an epimorphism.*

The analogous statements also hold in the relative setting.

Remark.

(1) If $p\colon \widetilde{X} \to X$ is a finite covering of path-connected topological spaces, then we saw in Corollary 52.8 that the map $p_*\colon \mathrm{H}_1(\widetilde{X}) \to \mathrm{H}_1(X)$ has finite cokernel. This implies in particular[908] that $p_*\colon \mathrm{H}_1(\widetilde{X};\mathbb{Q}) \to \mathrm{H}_1(X;\mathbb{Q})$ is an epimorphism. Therefore we can view Proposition 59.1 (3) as a generalization of Corollary 52.8.
(2) The conclusion of Proposition 59.1 (3) does not hold if we drop the condition that the covering is finite. For example, we have an infinite covering $p\colon \mathbb{R} \to S^1$ but $0 = \mathrm{H}_1(\mathbb{R};\mathbb{Q}) \xrightarrow{p_*} \mathrm{H}_1(S^1;\mathbb{Q}) = \mathbb{Q}$ is not an epimorphism.

PROOF. Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X.

(1) Let $c \in \mathrm{C}_n(X;R)$ be non-trivial. We write $c = \sum_{i=1}^{n} \sigma_i \otimes r_i$ where $\sigma_1, \dots, \sigma_n$ are pairwise different singular simplices in X and where $r_1, \dots, r_n$ are all non-zero. Since each σ_i lifts to k different singular simplices and since for $i \neq j$ the lifts of σ_i and σ_j differ we see immediately from the definition that $p^*(\sigma)$ is also non-zero.
(2) Let R be a commutative ring. We denote by $p^*\colon \mathrm{C}_n(X;R) \to \mathrm{C}_n(\widetilde{X};R)$ the transfer map. Let $\sigma\colon \Delta^m \to X$ be a singular simplex. We denote the corresponding lifts by $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_k$. Then

$$(p_* \circ p^*)(\sigma) \;=\; p_*(\widetilde{\sigma}_1 + \dots + \widetilde{\sigma}_k) \;=\; \underbrace{p \circ \widetilde{\sigma}_1 + \dots + p \circ \widetilde{\sigma}_k}_{k \text{ summands}} \;=\; \underbrace{\sigma + \dots + \sigma}_{k \text{ summands}} \;=\; k \cdot \sigma.$$

[908] Why does this follow?

We thus see that $p_* \circ p^*\colon \mathrm{C}_m(X) \to \mathrm{C}_m(X)$ is multiplication by k for any m. It follows easily that the map

$$\mathrm{H}_n(X;R) \xrightarrow{p^*} \mathrm{H}_n(\widetilde{X};R) \xrightarrow{p_*} \mathrm{H}_n(X;R)$$

is also given by multiplication by k.

(3) Since multiplication by $k \neq 0$ is an isomorphism of any rational vector space we obtain from (1) that $p_* \circ p^*\colon \mathrm{H}_n(X;\mathbb{Q}) \to \mathrm{H}_n(X;\mathbb{Q})$ is an isomorphism. In particular the second map, namely p_*, is an epimorphism.

The proofs in the relative setting are basically identical to the above. ■

Proposition 59.2. *Let $p\colon \widetilde{X} \to X$ be a 2-fold covering. Then there exists a natural long exact sequence of the form*

$$\dots \longrightarrow \mathrm{H}_n(X;\mathbb{F}_2) \xrightarrow{p^*} \mathrm{H}_n(\widetilde{X};\mathbb{F}_2) \xrightarrow{p_*} \mathrm{H}_n(X;\mathbb{F}_2) \xrightarrow{\partial} \mathrm{H}_{n-1}(X;\mathbb{F}_2) \longrightarrow \dots.$$

Here "natural" means that if we are given a commutative diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \widetilde{Y} \arrow[d, "q"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ are 2-fold coverings, then the following diagram commutes:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_n(X;\mathbb{F}_2) \arrow[r, "p^*"] \arrow[d, "f"] & \mathrm{H}_n(\widetilde{X};\mathbb{F}_2) \arrow[r, "p_*"] \arrow[d, "\widetilde{f}"] & \mathrm{H}_n(X;\mathbb{F}_2) \arrow[r, "\partial"] \arrow[d, "\widetilde{f}"] & \mathrm{H}_{n-1}(X;\mathbb{F}_2) \arrow[r] \arrow[d, "f"] & \dots \\
\dots \arrow[r] & \mathrm{H}_n(Y;\mathbb{F}_2) \arrow[r, "q^*"] & \mathrm{H}_n(\widetilde{Y};\mathbb{F}_2) \arrow[r, "q_*"] & \mathrm{H}_n(Y;\mathbb{F}_2) \arrow[r, "\partial"] & \mathrm{H}_{n-1}(Y;\mathbb{F}_2) \arrow[r] & \dots.
\end{tikzcd}$$

Remark. Note that it follows from Lemma 25.16 (3) a 2-fold covering $p\colon \widetilde{Y} \to Y$ together with a map $f\colon X \to Y$ of topological spaces naturally give rise to a commutative diagram as in Proposition 59.2.

Example. Now we will see that we can use Proposition 59.2 to calculate the homology groups $\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ just using the following basic facts:

(1) By the discussion on page 1402 we know that

$$\mathrm{H}_k(S^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k=0 \text{ or } k=n, \\ 0, & \text{otherwise} \end{cases}$$

(2) By the obvious generalization of Proposition 41.5 we know that $\mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) = \mathbb{F}_2$.

(3) Since $\mathbb{R}\mathrm{P}^n$ is an n-dimensional CW-complex we know from the discussion on page 1402 that $\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) = 0$ for $k > n$.

In the following discussion we write $\mathbb{F} = \mathbb{F}_2$ for space reasons. Applying Proposition 59.2 to the 2-fold covering $p\colon S^n \to \mathbb{RP}^n$ we obtain the following long exact sequence:

$$\underbrace{\mathrm{H}_{n+1}(\mathbb{RP}^n;\mathbb{F})}_{=0} \xrightarrow{\partial} \mathrm{H}_n(\mathbb{RP}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_n(S^n;\mathbb{F})}_{=\mathbb{F}} \xrightarrow{p_*} \mathrm{H}_n(\mathbb{RP}^n;\mathbb{F}) \xrightarrow{\partial} \mathrm{H}_{n-1}(\mathbb{RP}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_{n-1}(S^n;\mathbb{F})}_{=0} \to$$

which continues all the way to

$$\ldots \to \mathrm{H}_1(\mathbb{RP}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_1(S^n;\mathbb{F})}_{=0} \xrightarrow{p_*} \mathrm{H}_1(\mathbb{RP}^n;\mathbb{F}) \xrightarrow{\partial} \mathrm{H}_0(\mathbb{RP}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_0(S^n;\mathbb{F})}_{=\mathbb{F}} \xrightarrow{p_*} \underbrace{\mathrm{H}_0(\mathbb{RP}^n;\mathbb{F})}_{=\mathbb{F}} \to 0.$$

Now we can make the following observations:

(1) If we work our way upward through the long exact sequence we see that the connecting homomorphisms give us isomorphisms
$$\mathrm{H}_{n-1}(\mathbb{RP}^2;\mathbb{F}) \underset{\cong}{\xrightarrow{\partial}} \mathrm{H}_{n-2}(\mathbb{RP}^2;\mathbb{F}) \underset{\cong}{\xrightarrow{\partial}} \ldots \underset{\cong}{\xrightarrow{\partial}} \mathrm{H}_1(\mathbb{RP}^2;\mathbb{F}) \underset{\cong}{\xrightarrow{\partial}} \mathrm{H}_0(\mathbb{RP}^2;\mathbb{F}) \cong \mathbb{F}.$$

(2) Now that we know that $\mathrm{H}_{n-1}(\mathbb{RP}^n;\mathbb{F}) \cong \mathbb{F}$ we obtain from the upper end of the long exact sequence that[909]
$$\mathrm{H}_n(\mathbb{RP}^n;\mathbb{F}) \xrightarrow{p^*} \mathrm{H}_n(S^n;\mathbb{F}) \quad \text{and} \quad \mathrm{H}_n(\mathbb{RP}^n;\mathbb{F}) \xrightarrow{\partial} \mathrm{H}_{n-1}(\mathbb{RP}^n;\mathbb{F})$$
are also isomorphisms.

Summarizing we have shown that
$$\mathrm{H}_k(\mathbb{RP}^n;\mathbb{F}) \cong \begin{cases} \mathbb{F}, & \text{if } k = 0,\ldots,n, \\ 0, & \text{otherwise.} \end{cases}$$
This agrees of course with the result that we had obtained on page 1403.

Proof of Proposition 59.2. Let $p\colon \widetilde{X} \to X$ be a 2-fold covering. First we want to show that the promised long exact sequence in homology exists. By Proposition 43.11 it suffices to show that the following sequence of chain maps
$$0 \to C_*(X;\mathbb{F}) \xrightarrow{p^*} C_*(\widetilde{X};\mathbb{F}) \xrightarrow{p_*} C_*(X;\mathbb{F}) \to 0$$
is in fact exact. As in Lemma 16.4 we consider the continuous map
$$\begin{array}{rcl} f\colon \widetilde{X} & \to & \widetilde{X} \\ x & \mapsto & \text{the unique other element of } p^{-1}(p(x)). \end{array}$$

Using Proposition 29.2 we pick for each singular cell σ of X a lift $\widetilde{\sigma}$ to $\widetilde{X}$. We make the following observations which follow easily from the definitions and Proposition 29.2:

(a) given any singular n-simplex σ we have $p_*(\widetilde{\sigma}) = \sigma$,
(b) given any singular n-simplex σ we have $p^*(\sigma) = \widetilde{\sigma} + f_*(\widetilde{\sigma})$,
(c) given any singular simplex μ of $\widetilde{X}$ we have $p_*(f_*(\mu)) = p_*(\mu)$,
(d) any singular singular n-simplex μ of $\widetilde{X}$ is of the form $\mu = \widetilde{\sigma}$ or $\mu = f_*(\widetilde{\sigma})$ for some singular n-simplex σ of X.

[909] Why does this follow, why are there no other possibilities?

Now we turn to the proof of the exactness of the above sequence.

(1) We first show that the map $p_*\colon \mathrm{C}_n(\widetilde{X};\mathbb{F}) \to \mathrm{C}_n(X;\mathbb{F}) = \mathrm{C}_n(X)\otimes\mathbb{F}$ is surjective. So let $c = \sigma_1\otimes b_1 + \cdots + \sigma_k\otimes b_k \in \mathrm{C}_n(X;\mathbb{F})$. It follows from (a) that
$$p_*\big(\widetilde{\sigma}_1\otimes b_1 + \cdots + \widetilde{\sigma}_k\otimes b_k\big) \;=\; \sigma_1\otimes b_1 + \cdots + \sigma_k\otimes b_k \;=\; c.$$
(2) By Proposition 59.1 we know that the transfer map p^* is a monomorphism.

(3) Next we show that $\operatorname{im}(p^*)\subset\ker(p_*)$. For any singular n-simplex σ of $\widetilde{X}$ we have by (c) above that[910] $(p_*\circ p^*)(\sigma) = p_*(\widetilde{\sigma}+f_*(\widetilde{\sigma})) = 2\sigma$, but it follows from Lemma 57.1 (4) that any multiple of two is zero in $\mathrm{C}_*(X;\mathbb{F}) = \mathrm{C}_*(X)\otimes\mathbb{F}$.

(4) Finally we have to show that $\ker(p_*)\subset\operatorname{im}(p^*)$. So let $c\in\ker(p_*)$. By (d) we can write
$$c \;=\; (\widetilde{\sigma}_1\otimes a_1 + f_*(\widetilde{\sigma}_1)\otimes b_1) + \cdots + (\widetilde{\sigma}_k\otimes a_k + f_*(\widetilde{\sigma}_k)\otimes b_k)$$
where $\sigma_1,\dots,\sigma_k$ are pairwise different and where $a_1,\dots,a_k,b_1,\dots,b_k\in\mathbb{F}$. By (c) we have
$$p(c) \;=\; \sigma_1\otimes(a_1+b_1) + \cdots + \sigma_k(a_k+b_k).$$
But this implies that $a_i = -b_i$ for $i=1,\dots,k$. But since we work over $\mathbb{F}$ we actually have $a_i = b_i$ for $i=1,\dots,k$. It follows from (b) that
$$c \;=\; p^*(\sigma_1\otimes a_1 + \cdots + \sigma_k\otimes a_k),$$
as desired.

The fact the long exact sequence is indeed natural follows easily from the definitions and the naturality of the connecting homomorphism, see Lemma 43.10 (2). ■

59.3. **The Borsuk-Ulam Theorem.** Our goal in this section is to prove the following generalization of Theorem 16.22 from the 2-dimensional case to the higher-dimensional situation.

Theorem 59.3. (Borsuk-Ulam) *Let $n\in\mathbb{N}_0$. For every map $f\colon S^n\to\mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $f(x)=f(-x)$.*

Example. If our universe happens to be diffeomorphic to S^3, then there are two antipodal points where the temperature, distance to the Kneitinger and strength of the magnetic field agree.

Remark.

(1) The statement was conjectured by Stanislaw Ulam[911], see [**Bor33**, p. 178]. The theorem was first proved by Karol Borsuk[912] [**Bor33**] in 1933.

[910] That is of course basically the same argument as in the proof of Proposition 59.1.

[911] Stanislaw Ulam (1909-1984) was a Polish mathematician and nuclear scientist. He was one of the fathers of the hydrogen bomb.

[912] Karol Borsuk (1905-1982) was a Polish mathematician. He lost his job at the university of Warsaw after the German invasion in 1939. To survive financially he developed a board game and produced and sold it during the war. Almost all copies of the game were lost during the Warsaw uprising in 1944, but in the 1990s one surviving copy was found and the game was republished. We refer to

https://en.wikipedia.org/wiki/Animal_Husbandry_(game)

for more details.

(2) In Sections 65.6 and 90.3 we will provide two other proofs of the Borsuk-Ulam Theorem 59.3.
(3) The Borsuk-Ulam Theorem has an almost endless string of consequences, many of which lie in combinatorics and graph theory, we refer to [**Mato08, Steinl85, Steinl93**] for an extensive discussion of the Borsuk-Ulam theorem and its corollaries. We will give three applications in Sections 65.6 and 90.3.

The key ingredient in the proof of Theorem 59.3 is the following proposition.

Proposition 59.4. *Let $f\colon S^n \to S^n$ be a map. If $f(-x) = -f(x)$ for all $x \in S^n$, then $\deg(f)$ is odd.*

Proof of Proposition 59.4. Let $f\colon S^n \to S^n$ be a map such that $f(-x) = -f(x)$ for all $x \in S^n$. By Lemma 57.22 it suffices to show that $f_*\colon \mathrm{H}_n(S^n;\mathbb{Z}_2) \to \mathrm{H}_n(S^n;\mathbb{Z}_2)$ is an isomorphism.

We start out with the observation that the map $f\colon S^n \to S^n$ evidently descends to a well-defined map

$$\begin{array}{rcl} \overline{f}\colon \mathbb{R}\mathrm{P}^n & \to & \mathbb{R}\mathrm{P}^n \\ {[x]} & \mapsto & [f(x)]. \end{array}$$

Claim. The diagram

$$\begin{array}{ccccccccccc} \longrightarrow \mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{p^*} & \mathrm{H}_k(S^n;\mathbb{Z}_2) & \xrightarrow{p_*} & \mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & \mathrm{H}_{k-1}(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{p^*} & \mathrm{H}_{k-1}(S^n;\mathbb{Z}_2) \longrightarrow \\ \downarrow \overline{f}_* & & \downarrow f_* & & \downarrow \overline{f}_* & & \downarrow \overline{f}_* & & \downarrow f_* \\ \longrightarrow \mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{p^*} & \mathrm{H}_k(S^n;\mathbb{Z}_2) & \xrightarrow{p_*} & \mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & \mathrm{H}_{k-1}(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{p^*} & \mathrm{H}_{k-1}(S^n;\mathbb{Z}_2) \longrightarrow \end{array}$$

where the horizontal sequences are the exact sequences given by Proposition 59.2, commutes.

By Proposition 43.11 it suffices to show that the following diagram commutes for any $k \in \mathbb{N}_0$:

$$\begin{array}{ccccccc} 0 \longrightarrow \mathrm{C}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{p^*} & \mathrm{C}_k(S^n;\mathbb{Z}_2) & \xrightarrow{p_*} & \mathrm{C}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \longrightarrow 0 \\ \downarrow \overline{f}_* & & \downarrow f_* & & \downarrow \overline{f}_* \\ 0 \longrightarrow \mathrm{C}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{p^*} & \mathrm{C}_k(S^n;\mathbb{Z}_2) & \xrightarrow{p_*} & \mathrm{C}_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \longrightarrow 0. \end{array}$$

Here the right-hand square commutes since by definition we have $p \circ f = \overline{f} \circ p$. Now we argue that the left-hand square commutes. So let $\sigma\colon \Delta^k \to \mathbb{R}\mathrm{P}^n$ be a singular k-simplex. We pick a point $P \in \Delta^k$. Furthermore we pick $Q \in S^n$ with $[Q] = \sigma(1,0,\dots,0) \in \mathbb{R}\mathrm{P}^n$. We denote by $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ the two lifts of σ to S^n. Now we consider the singular simplex $\overline{f} \circ \sigma$. It remains to show that $p^*(\overline{f} \circ \sigma) = f \circ \widetilde{\sigma}_1 + f \circ \widetilde{\sigma}_2$. But this is indeed the case for the following reason:

(1) by construction we have $p \circ f = \overline{f} \circ p$ which implies that $f \circ \widetilde{\sigma}_1$ and $f \circ \widetilde{\sigma}_2$ are both lifts of $\overline{f} \circ \sigma$,
(2) it follows from $f(Q) = -f(Q)$ that $f \circ \widetilde{\sigma}_1$ and $f \circ \widetilde{\sigma}_2$ are two different lifts. ⊞

Now we consider the following diagram:

$$\begin{array}{ccccccccccc}
H_n(S^n;\mathbb{Z}_2) & \xleftarrow{p^*} & H_n(\mathbb{RP}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & H_1(\mathbb{RP}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & H_0(\mathbb{RP}^n;\mathbb{Z}_2) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle \overline{f}_*} & & & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle \overline{f}_*} \\
H_n(S^n;\mathbb{Z}_2) & \xleftarrow{p^*} & H_n(\mathbb{RP}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & H_1(\mathbb{RP}^n;\mathbb{Z}_2) & \xrightarrow{\partial} & H_0(\mathbb{RP}^n;\mathbb{Z}_2).
\end{array}$$

We make the following observations:

(1) the diagram commutes by the previous claim,
(2) all the horizontal maps are isomorphisms by the discussion on page 1448,
(3) by the discussion on page 1088 the right vertical map is an isomorphism.

It follows that the left vertical map is an isomorphism. But that is exactly what we had wanted to prove. ■

Now we turn to the proof of the Borsuk-Ulam Theorem 59.3.

PROOF OF THEOREM 59.3. Let $g\colon S^n \to \mathbb{R}^n$ be a map. Suppose there exists no pair of antipodal points x and $-x$ in S^n with $g(x) = g(-x)$. Put differently, suppose that for any $x \in S^n$ we have $g(x) \neq g(-x)$. We consider the composition of the following two maps

$$\begin{array}{rcccl}
\Phi\colon S^{n-1} \times [0,1] & \to & S^n_{\geq 0} & \to & S^{n-1}. \\
(y,t) & \mapsto & (y\cdot\sqrt{1-t^2}, t) & & \\
 & & x & \mapsto & \underbrace{\frac{g(x)-g(-x)}{\|g(x)-g(-x)\|}}_{\text{defined since } g(x)\neq g(-x)}
\end{array}$$

Now we see that

$$\text{odd number} \overset{\substack{\text{by definition we have } \Phi(-x,0) = -\Phi(x,0)\text{, thus it follows from Proposition 59.4 that } \Lambda(\Phi_0) \text{ is even} \\ \downarrow}}{=} \deg(\Phi_0) \underset{\substack{\uparrow \\ \text{by Lemma 45.11 (3) since } \Phi_0 \text{ and } \Phi_1 \text{ are homotopic}}}{=} \deg(\Phi_1) \underset{\substack{\uparrow \\ \text{by definition of } \Phi_1}}{=} \deg(\text{constant map}) \underset{\substack{\uparrow \\ \text{Lemma 45.11 (2)}}}{=} 0.$$

We have thus obtained a contradiction. ■

59.4. **Applications of the Borsuk-Ulam Theorem I.** In this and in the subsequent section we will state and prove four particularly interesting consequences of the Borsuk-Ulam Theorem 59.3. The first of our applications was first proved by the Soviet mathematicians Lazar Lusternik and Lev Schnirelmann in 1930.

Theorem 59.5. (Lusternik-Schnirelmann Theorem) *Let $A_1, \dots, A_{n+1}$ be closed subsets of S^n with $A_1 \cup \dots \cup A_{n+1} = S^n$. Then there exists an $i \in \{1, \dots, n+1\}$ such that A_i contains antipodal points, i.e. there exists a point x on S^n such that both x and $-x$ lie in A_i.*

Examples.

(1) For $n = 1$ we obtain that if we have a decomposition $S^1 = A \cup B$ with A and B two closed subsets (which do not need to be connected), then the subset A or the subset B contains antipodal points. How can one prove this statement “naively”?

(2) We consider the surface of our earth, we take A_1 to be the closure of the land mass, we take A_2 to be the closure of the Pacific ocean and we take A_3 to be the closure of all seas except for the Pacific. Then at least one of A_1, A_2 or A_3 contains a pair of antipodal points. Which of A_1, A_2 and A_3 is it?

(3) The proposition is optimal in the following sense: it is easy to find three closed subsets A_1, A_2, A_3 that cover all of S^1, but such that none of the A_i contains antipodal points.

PROOF. Let $A_1, \ldots, A_{n+1}$ be closed subsets of S^n with $A_1 \cup \cdots \cup A_{n+1} = S^n$. Without loss of generality we can assume that all A_i are non-empty. (Indeed, at least one of the A_i is evidently non-empty, if one of them is empty we just replace it with the non-empty A_i.) Given a non-empty subset A of S^n and $x \in S^n$ we write

$$d(x, A) \ := \ \inf\{d(x, a) \mid a \in A\},$$

where $d(x, a)$ denotes the distance function on the Riemannian manifold S^n that we introduced on page 880.[913] We make the following two observations:

(1) If A is closed, then $d(x, A) = 0$ if and only if $x \in A$.

(2) It follows easily from Proposition 32.9 that $d(x, A)$ is continuous in x.

Now we consider the function

$$\begin{aligned} f\colon S^n &\to \mathbb{R}^n \\ x &\mapsto (d(x, A_1), \ldots, d(x, A_n)). \end{aligned}$$

By the Borsuk-Ulam Theorem 59.3 there exists an $x \in S^n$ with $f(x) = f(-x)$. We first suppose that there exists an $i \in \{1, \ldots, n\}$ with $d(x, A_i) = 0$. From $f(x) = f(-x)$ it follows that $d(-x, A_i) = 0$ as well. Since A_i is closed it follows that both x and $-x$ lie in A_i.

Now suppose that $d(-x, A_i) = d(x, A_i) \neq 0$ for $i = 1, \ldots, n$. Since $A_1 \cup \cdots \cup A_{n+1} = S^n$ it follows that both x and $-x$ lie in A_{n+1}. ■

Before we can state the next consequence of the Borsuk-Ulam Theorem 59.3 we need to introduce one more definition.

Definition. Let $A \subset \mathbb{R}^n$ be a measurable subset[914] of finite volume. We say a hyperplane $H \subset \mathbb{R}^n$ *bisects* A if for the two half-spaces V, V' determined by H we have the equality $(\nu)(A \cap V) = (\nu)(A \cap V')$.

We continue with the following pretty theorem.

Theorem 59.6. (Ham-Sandwich Theorem) *Let $A_1, \ldots, A_n$ be measurable finite-volume subsets of $\mathbb{R}^n$. Then there exists an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ that bisects $A_1, \ldots, A_n$.*

Remark.

(1) The Ham-Sandwich Theorem gets its name from the following "application". Suppose we are given a sandwich with an upper piece of bread A_1, some ham A_2 and a lower piece of bread A_3. Then using a knife one can cut the sandwich, along a hyperplane,

[913] Put differently, $d(x, a)$ is the infimum of all lengths of piecewise smooth paths on S^n from x to a.

[914] Here "measurable" means Lebesgue-measurable in the sense of say [**Frie16b, Caro00**] or [**RudiW87**, Chapter 2].

into two pieces such that each piece of bread and the ham are bisected. This situation is, more or less convincingly, illustrated in Figure 879.

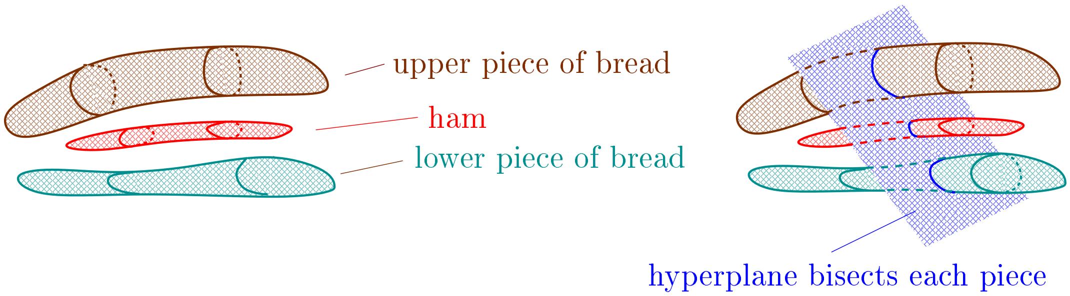


FIGURE 879. Illustration of the "Ham-Sandwich Theorem".

(2) The 2-dimensional version of the Ham-Sandwich Theorem is sometimes referred to as the "Pancake Theorem", since one can bisect the two pancakes along a line into two equal sized halves.

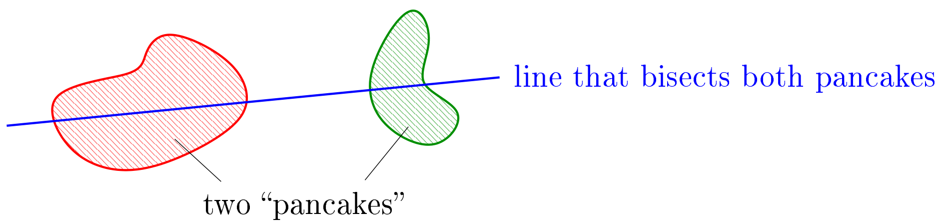


FIGURE 880. Illustration of the "Pancake Theorem".

(3) The Ham-Sandwich Theorem is sometimes also called, more prosaically, the Stone-Tukey theorem after the mathematicians Arthur Stone and John Tukey [**StT42**].

In the proof of the Ham-Sandwich Theorem we will need the following result from measure theory.

Proposition 59.7. *Suppose that for each $t \in \mathbb{R}$ we have a measurable subset X_t of $\mathbb{R}^n$ such that for each $s \leq t \in \mathbb{R}$ we have $X_s \subset X_t$. Then $\bigcup\limits_{t \in \mathbb{R}} X_t$ is also measurable and*

$$\lim_{t \to \infty} (\nu)(X_t) \;=\; (\nu)\Big(\bigcup_{t \in \mathbb{R}} X_t\Big).$$

PROOF OF PROPOSITION 59.7. The proposition can be deduced easily from basic facts of the Lebesgue measure, see [**Caro00**, Chapter 16] or [**Frie16b**, Proposition 6.2 (6)]. ■

Now we can provide the proof of the Ham-Sandwich Theorem.

PROOF OF THE HAM-SANDWICH THEOREM. Let $A_1, \ldots, A_n$ be measurable finite-volume subsets of $\mathbb{R}^n$. Without loss of generality we can assume that the volume of A_n is positive.

Given a point $P \in S^{n-1}$ and $t \in \mathbb{R}$ we write

$$\mathrm{H}_{P,t} \;:=\; \{t \cdot P + w \mid P \cdot w = 0\}$$

and

$$\begin{array}{rcl} V^{+}_{P,t} & := & \{t \cdot P + w \mid P \cdot w \geq 0\} \\ V^{-}_{P,t} & := & \{t \cdot P + w \mid P \cdot w \leq 0\}. \end{array}$$

(We refer to Figure 881 on the left for an illustration.) Put differently, $\mathrm{H}_{P,t}$ is the hyperplane through $t \cdot P$ orthogonal to P, and $V^{\pm}_{P,t}$ are the two half-spaces determined by $\mathrm{H}_{P,t}$. We make the following two observations:

(i) by definition we have $V^{\pm}_{-P,-t} = V^{\mp}_{P,t}$,

(ii) for a subset $A \subset \mathbb{R}^n$ of finite volume we have the equality

$$(\nu)(A \cap V^{\pm}_{P,t}) \underset{\substack{\uparrow \\ \text{since } \mathrm{H}_{P,t} = V^{+}_{P,t} \cap V^{-}_{P,t} \text{ is a set of measure zero}}}{=} (\nu)(A \cap (\mathbb{R}^n \setminus V^{\mp}_{P,t})) \;=\; (\nu)(A) - (\nu)(A \cap V^{\mp}_{P,t}).$$

We continue with the following claim.

Claim. There exists a function

$$t\colon S^{n-1} \;\to\; \mathbb{R}$$

with the following two properties:

(1) for any $P \in S^{n-1}$ we have $(\nu)(A_n \cap V^{+}_{P,t(P)}) = \frac{1}{2}(\nu)(A_n)$, i.e. the hyperplane $\mathrm{H}_{P,t(P)}$ bisects A_n,

(2) for any $P \in S^{n-1}$ we have $t(-P) = -t(P)$,

(3) the function t is continuous.

To prove the claim we first fix $P \in S^{n-1}$. We consider the function

$$\begin{array}{rcl} \mathbb{R} & \to & \mathbb{R}_{\geq 0} \\ t & \mapsto & (\nu)(A_n \cap V^{+}_{P,t}). \end{array}$$

This function is continuous[915] and we have

$$\lim_{t\to\infty} (\nu)(A_n \cap V^{+}_{P,t}) \underset{\substack{\uparrow \\ \text{Proposition 59.7}}}{=} (\nu)\Big(\bigcup_{t\in\mathbb{R}} (A_n \cap V^{+}_{P,t})\Big) \;=\; (\nu)(A_n).$$

and

$$\lim_{t\to-\infty} (\nu)(A_n \cap V^{+}_{P,t}) \underset{\substack{\uparrow \\ \text{by (ii)}}}{=} (\nu)(A_n) - \underbrace{\lim_{t\to-\infty} (\nu)(A_n \cap V^{-}_{P,t})}_{= (\nu)(A_n) \text{ by Proposition 59.7}} \;=\; 0.$$

We deduce from the Intermediate Value Theorem that

$$T := \;\Big\{t \in \mathbb{R} \,\Big|\, (\nu)(A_n \cap V^{+}_{P,t}) \;=\; \tfrac{1}{2}(\nu)(A_n)\Big\}$$

[915] This requires at least a little bit of thought since we do not assume that A is bounded, we only assume it has finite volume. We leave it as a not entirely trivial exercise to prove this statement using basics facts of Lebesgue measures.

is a non-empty set.[916] It is straightforward to see that it is a connected subset of $\mathbb{R}$, i.e. it is an interval. Since $(\nu)(A_n) > 0$ we know that T is a finite interval. We define

$$t(P) \;:=\; \tfrac{1}{2}(\inf(T)+\sup(T)).$$

Note that it follows immediately from the definitions that $t(-P) = -t(P)$. Furthermore with some elementary measure theory one can show that the function $S^{n-1} \to \mathbb{R}$ given by $P \mapsto t(P)$ is continuous. We leave the verification of the details to the reader. ⊞

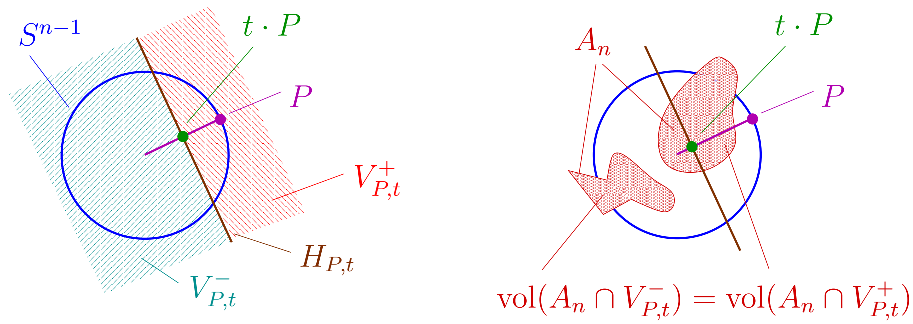


FIGURE 881. Illustration for the proof of the Ham-Sandwich Theorem 59.6

Now we consider the map

$$\begin{aligned} f\colon S^{n-1} &\to \mathbb{R}^{n-1} \\ P &\mapsto \big((\nu)\big(A_1 \cap V^+_{P,t(P)}\big), \dots, (\nu)\big(A_{n-1} \cap V^+_{P,t(P)}\big)\big). \end{aligned}$$

From the continuity of $t\colon S^{n-1} \to \mathbb{R}$ it follows that f itself is continuous. By the Borsuk-Ulam Theorem 59.3, there exists a point $P \in S^{n-1}$ with $f(P) = f(-P)$. We claim that the hyperplane $\mathrm{H}_{P,t(P)}$ has the desired property.

By definition of $t(P)$ the hyperplane $\mathrm{H}_{P,t(P)}$ bisects A_n. Furthermore, for $i = 1, \dots, n-1$ we have

$$(\nu)\big(A_i \cap V^+_{P,t(P)}\big) \underset{\substack{\uparrow \\ \text{since } f(P)=f(-P)}}{=} (\nu)\big(A_i \cap V^+_{-P,t(-P)}\big) \underset{\substack{\uparrow \\ \text{since } t(-P)=-t(P)}}{=} (\nu)\big(A_i \cap V^+_{-P,-t(P)}\big) \underset{\substack{\uparrow \\ \text{since } V^+_{P,s}=V^-_{-P,-s}}}{=} (\nu)\big(A_i \cap V^-_{P,t(P)}\big).$$

This shows that the hyperplane $\mathrm{H}_{P,t(P)}$ also bisects $A_1, \dots, A_{n-1}$. ∎

Remark. The proof for the existence of the hyperplane that bisects the given sets $A_1, \dots, A_n$ is non-constructive, in the sense that even with some flexibility in mathematical rigor one cannot turn it into an algorithm that given $A_1, \dots, A_n$ finds the hyperplane that bisects all of the subsets.

We conclude this section with an application of the Borsuk-Ulam Theorem 59.3 to algebra. First we have the following definition.

Definition. Let R be a commutative ring. The *level* $s(R)$ of R is defined as the smallest number $n \in \mathbb{N}_0$ such that -1 can be written as the sum of n squares in R. If -1 is not the sum of squares in R, then we define $s(R) = \infty$.

[916] One might think that T consists of just a point, but for example if A_n is disconnected, then it can happen that T is in fact an interval that is not just a point.

In 1966 Albrecht Pfister [**Pf65**] showed that the level of a field is either ∞ or a power of 2. Furthermore he showed that any power of 2 appears as the level of a field. In 1977 Manfred Knebusch [**Kneb77**] asked whether every $n \in \mathbb{N}_0$ is the level of some commutative ring. The following theorem, proved by Dai–Lang–Peng [**DLP80**] gives an affirmative answer.

Theorem 59.8. *Given any $n \in \mathbb{N}_0$ there exists a commutative ring R with $s(R) = n$.*

PROOF. We consider the ring $R = \mathbb{R}[x_1^2, \dots, x_n^2]/(1 + x_1^2 + \dots + x_n^2)$. Evidently $s(R) \leq n$. We need to show that $s(R) \geq n$. So suppose that $s(R) < n$. This implies that there exist $f_0(x), f_1(x), \dots, f_{n-1}(x) \in \mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$ such that

$$-1 \;=\; f_1(x)^2 + \dots + f_{n-1}(x)^2 + f_0(x) \cdot (1 + x_1^2 + \dots + x_n^2) \;\in\; \mathbb{R}[x_1, \dots, x_n].$$

Let $j \in \{0, \dots, n-1\}$. Since $f_j(x)$ is a real polynomial we can write $f_j(ix) = p_j(x) + iq_j(x)$ where $p_j(x)$ is an even real polynomial and $q_j(x)$ is an odd real polynomial. Thus if we plug ix into the above equality and if we compare the real parts we obtain the following equality:

$$-1 \;=\; \sum_{j=1}^{n-1} (p_j(x)^2 - q_j(x)^2) + p_0(x)(1 - x_1^2 - \dots - x_n^2).$$

Next we consider the map $Q := (q_1, \dots, q_{n-1})\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$. Since the polynomials $q_1, \dots, q_{n-1}$ are odd we see that for every $a \in S^{n-1}$ we have $Q(-a) = -Q(a)$. Furthermore, by the Borsuk-Ulam Theorem 59.3 there exists an $a = (a_1, \dots, a_n) \in S^{n-1}$ with $Q(-a) = Q(a)$. Note that it follows from the above symmetry of Q that we have $Q(a) = 0$. Now we see that

$$-1 \underset{\substack{\uparrow \\ \text{by the second equality}}}{=} \sum_{j=1}^{n-1} (p_j(a)^2 - q_j(a)^2) + p_0(a)(1 - a_1^2 - \dots - a_n^2) \underset{\substack{\uparrow \\ \text{since } Q(a)=0 \text{ and } a \in S^{n-1}}}{=} \sum_{j=1}^{n-1} p_j(a)^2.$$

Since the right-hand side is non-negative we have thus obtained a contradiction. ■

59.5. **Applications of the Borsuk-Ulam Theorem II: The Necklace Theorem ❀.** We conclude this chapter with another, fun consequence of the Borsuk-Ulam Theorem. As we will see, this application is particularly far-removed from what one would usually consider to be the realm of topology.

The setting is the following: Two thieves steal an open necklace with precious stones of k different types. Conveniently enough, of each type of stone there is an even number. The thieves want to cut the string of the open necklace such that they can distribute the resulting pieces in such a way that both thieves obtain the same number of stones of each type. The question is, what is the minimal number of cuts necessary? In Figure 882 we show two necklaces, one with two types of stones and one with three types of stones. Furthermore we show possible cuts to evenly distribute the stones.

Another open necklace is shown in Figure 883. In this particular example the k different types of stones are grouped together. It is pretty clear that one will need at least k cuts to distribute the stones. It turns out that this is the worst case, more precisely, it turns out that any string of k different types of stones can be distributed evenly using at most k cuts. The following theorem is a formalization of that statement.

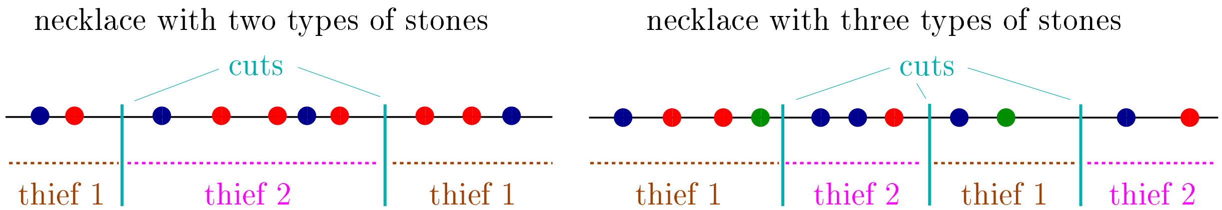


FIGURE 882

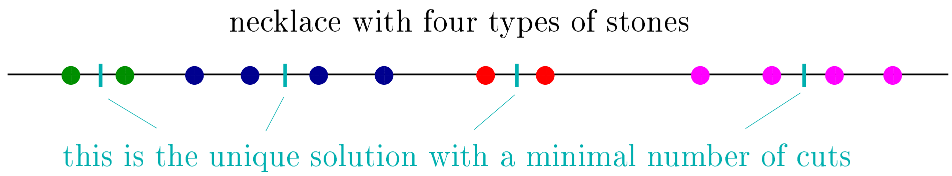


FIGURE 883

Theorem 59.9. (Necklace Theorem) *Let $S_1, \ldots, S_k$ be disjoint finite subsets of the interval $[0,1]$ such that each subset S_i has an even number of elements. There exists a $d \in \{0, \ldots, k\}$ and $0 = z_0 \leq z_1 \leq \cdots \leq z_d \leq z_{d+1} = 1$ such that for each "color" $c \in \{1, \ldots, k\}$ we have*

$$\underbrace{\sum_{i \text{ even}} \#([z_i, z_{i+1}) \cap S_c)}_{\substack{\text{number of } c\text{-colored pieces} \\ \text{in the even intervals}}} = \underbrace{\sum_{i \text{ odd}} \#([z_i, z_{i+1}) \cap S_c)}_{\substack{\text{number of } c\text{-colored pieces} \\ \text{in the odd intervals}}} .$$

Remark.

(1) The Necklace Theorem 59.9 was first formulated and proved by Charles Goldberg and Douglas West [**GW85**] in 1985 using topological methods. The proof we give was found shortly afterwards by Noga Alon and Douglas West [**AW86**].

(2) A topology-free proof of the Necklace Theorem 59.9 was found in 2008 by Frédéric Meunier [**Meu08**].

(3) The Necklace Theorem 59.9 has been generalized by Noga Alon [**Nog87**], again using topological methods, to the case of more than two thieves. It seems like there is no known proof for that more general statement, that does not rely on topology.

(4) The proofs of the Necklace Theorem 59.9 and of the Continuous Necklace Theorem 59.10 are both nicely animated in the following video:

`https://www.youtube.com/watch?v=yuVqxCSsE7c&t=868s`

The Necklace Theorem 59.9 is a priori a discrete problem, far removed from the world of topological spaces. In fact the proof consists of two steps, first we will use the Borsuk-Ulam Theorem 59.3 to prove a "continuous analogue" of the Necklace Theorem 59.9. Afterwards we will show how we can reduce the "discrete Necklace Theorem" from the "continuous Necklace Theorem". Since we will not make use of any of the results of this section we will feel free to use slightly more informal language than usual.

Without further ado, here is the promised continuous analogue of the Necklace Theorem 59.9.

Theorem 59.10. (Continuous Necklace Theorem) *Let $f\colon [0,1] \to \{1,\dots,k\}$ be a function*[917] *such that each preimage $f^{-1}(\{c\})$ is a measurable set (e.g. a union of finitely many intervals). There exists a $d \in \{0,\dots,k\}$ and $0 = y_0 \le y_1 \le \dots < y_d \le y_{d+1} = 1$ such that for each "color" $c \in \{1,\dots,k\}$ we have*

$$\underbrace{\sum_{i \text{ even}} \operatorname{vol}\big([y_i, y_{i+1}) \cap f^{-1}(\{c\})\big)}_{\substack{\text{total length of } c\text{-colored} \\ \text{segments in the even intervals}}} = \underbrace{\sum_{i \text{ odd}} \operatorname{vol}\big([y_i, y_{i+1}) \cap f^{-1}(\{c\})\big)}_{\substack{\text{total length of } c\text{-colored} \\ \text{segments in the odd intervals}}}.$$

The same statement also holds for functions f which are defined on $[0,1)$.

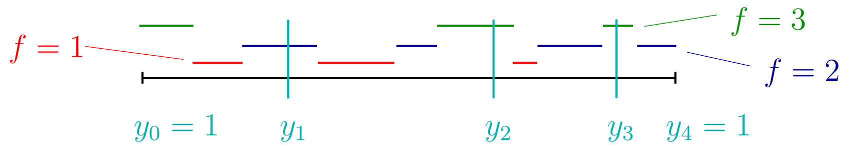


FIGURE 884. Illustration for the Continuous Necklace Theorem 59.10.

PROOF. We adopt the following notation: given $(w_0,\dots,w_k) \in S^k$ we set $x_0 = 0$ and for $i = 1,\dots,k$ we write $x_i := w_0^2 + \dots + w_{i-1}^2$. We make the following observation which is the key to the subsequent proof:

> Let $(w_0,\dots,w_k) \in S^k$. We note that $x_0 = 0 \le x_1 \le \dots \le x_k \le x_{k+1} = 1$. Thus any point in S^k corresponds to a splitting of the interval $[0,1)$ into $k+1$ (possibly empty) intervals. In fact, if we remember the signs of the w_i, then we obtain a way to assign each interval to one of the two thieves, say we award $[x_i, x_{i+1})$ to thief 1 if $w_i < 0$ and to thief 2 if $w_i > 0$. Thus the set of splittings of the interval corresponds to the topological space S^k.

For $c = 1,\dots,k$ we consider the following map:

$$\begin{aligned} g_c\colon S^k &\to \mathbb{R} \\ (w_0,\dots,w_k) &\mapsto \sum_{i \text{ with } w_i > 0} \operatorname{vol}\big(\big[\underbrace{x_i, x_{i+1}}_{\substack{\uparrow \\ \text{here we use the above definition of } x_0,\dots,x_{k+1} \\ \text{as determined by } w_0,\dots,w_k}}\big) \cap f^{-1}(\{c\})\big). \end{aligned}$$

We leave it to the meticulous reader to show, using standard properties of the Lebesgue-measure, that the map g_c is actually continuous.[918]

Next we combine all these maps and we obtain a map $g = (g_1,\dots,g_k)\colon S^k \to \mathbb{R}^k$. By the Borsuk-Ulam Theorem 59.3 there exists a point $(w_0,\dots,w_k) \in S^k$ such that

[917] Evidently for once the convention that all maps are continuous does not apply.

[918] The fact that we take the sum over all i with $w_i > 0$ might make the reader slightly nervous. But note that for $w_i = 0$ we have $x_{i+1} = x_i$, thus $\operatorname{vol}([x_i, x_{i+1}) \cap f^{-1}(\{c\})) = 0$.

$g(-w_0, \ldots, -w_k) = g(w_0, \ldots, w_k)$. This just means that for each $c \in \{1, \ldots, k\}$ we have

$$\underbrace{\sum_{i \text{ with } w_i < 0} \operatorname{vol}\left([x_i, x_{i+1}) \cap f^{-1}(\{c\})\right)}_{=g_c(-w_0,\ldots,-w_k)} = \underbrace{\sum_{i \text{ with } w_i > 0} \operatorname{vol}\left([x_i, x_{i+1}) \cap f^{-1}(\{c\})\right)}_{=g_c(w_0,\ldots,w_k)}.$$

Finally, we consider the numbers $x_0 \leq x_1 \leq \cdots \leq x_{k+1} = 1$. Now we just need to combine adjacent intervals for which the w_i's have the same sign. More precisely, iteratively we remove each x_i for which $w_i \geq 0$ and $w_{i+1} \geq 0$ and we remove each x_i for which $w_i \leq 0$ and $w_{i+1} \leq 0$. It is now fairly elementary to verify that the remaining numbers, let us call them $0 = y_0 < y_1 < \cdots < y_d < y_{d+1} = 1$, have the desired properties. ■

Now we can provide the proof of the original Necklace Theorem 59.9.

Proof of the Necklace Theorem 59.9. Let $S_1, \ldots, S_k$ be disjoint finite subsets of the interval $[0, 1]$ such that each S_i has an even number of elements. Evidently we can assume that $S_1 \cup \cdots \cup S_k \neq \emptyset$. We set $s := \#(S_1 \cup \cdots \cup S_k)$. It is straightforward to see that without loss of equality we can assume that the points break $[0, 1]$ into intervals of the same lengths $\frac{1}{s}$. More precisely, we can assume that $\bigcup_{i=1}^{k} S_i = \{\frac{i}{s} \,|\, i = 0, \ldots, s-1\}$. Now let $f \colon [0, 1) \to \{1, \ldots, k\}$ be the function which is uniquely determined by the property that for each $c \in S_i$ we have $f(x) = i$ for $x \in [c, c + \frac{1}{s})$. By the Continuous Necklace Theorem 59.10 there exists a $d \in \{0, \ldots, k\}$ and $0 = y_0 \leq y_1 \leq \cdots \leq y_d \leq y_{d+1} = 1$ such that for each $c \in \{1, \ldots, k\}$ we have

$$(*) \qquad \sum_{i \text{ even}} \operatorname{vol}\left([y_i, y_{i+1}) \cap f^{-1}(\{c\})\right) = \sum_{i \text{ odd}} \operatorname{vol}\left([y_i, y_{i+1}) \cap f^{-1}(\{c\})\right).$$

If each of the y_i lies in the "lattice" $L := \{\frac{i}{s} \,|\, i = 0, \ldots, s\}$, then we can set $z_i := y_i - \frac{1}{2s}$, $i = 1, \ldots, d$ and we are done. (This setting is shown in Figure 885.) Thus it remains to prove the following claim.

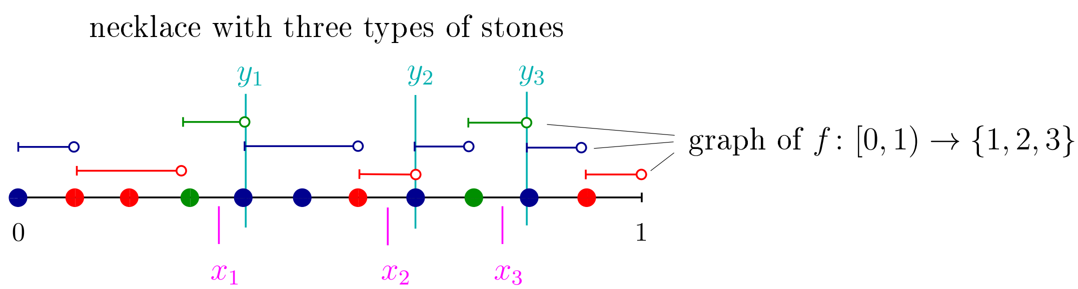


Figure 885. First illustration for the proof of the Necklace Theorem 59.9.

Claim. We can find $0 = y_0 \leq y_1 \leq \cdots \leq y_d \leq y_{d+1} = 1$ which satisfy $(*)$ and such that each y_i lies in $L = \{\frac{i}{s} \,|\, i = 0, \ldots, s\}$.

We prove the claim by induction on the number of y_i that do not lie in the subset $L = \{\frac{i}{s} \,|\, i = 0, \ldots, s\}$. Throughout the proof of the claim we use the following slightly non-standard notation: given $y \in [0, 1]$ we define $\lfloor y \rfloor$ and $\lceil y \rceil$ as the result of rounding y

down and up to the closest point in L. More precisely, we set $\lfloor y \rfloor := \max\{l \in L \,|\, l \le y\}$ and we set $\lceil y \rceil := \min\{l \in L \,|\, y \le l\}$. So suppose there does exist some y_i that does not lie in L. In this case we perform the following three steps:

(1) Among all y_i which do not lie in L we pick a y_i which is closest to a point in L. Without loss of generality we can assume that i is odd.
(2) We define d to be the distance of y_i to some point in L. In other words, if the point on L that is closest to y_i is to the left, then we set $d := y_i - \lfloor y_i \rfloor$. Otherwise we set $d := \lceil y_i \rceil - y_i$.
(3) It follows fairly easily from $(*)$ and the hypothesis that S_c has an even number of elements that there exists a $j \ne i$ such that $f(y_j) = c$ and such that $y_j \notin L$. We distinguish two cases:
   (a) If j is even, then among those we pick the one with a lowest value for $y_j - \lfloor y_j \rfloor$. We replace y_i by $y_i - d \in L$ and we replace y_j by $y_j - d$.
   (b) If j is odd, then among those we pick the one with a lowest value for $\lceil y_j \rceil - y_j$. We replace y_i by $y_i - d \in L$ and we replace y_j by $y_j + d$.

One can easily verify that these new numbers still have all the required properties, except that now y_i also lies in L. By induction we are now done. ⊞ ■

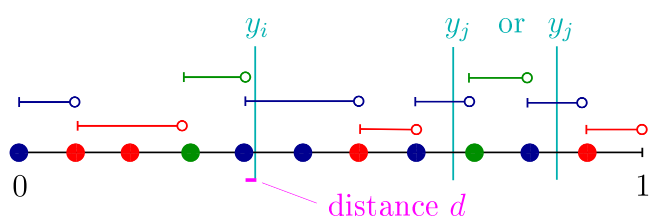


FIGURE 886. Second illustration for the proof of the Necklace Theorem 59.9.

Exercises for Chapter 59.

Exercise 59.1. Let $k \in \mathbb{N}$ and let $l \in \{1, \dots, k-1\}$. Furthermore let $f \colon S^k \to \mathbb{R}^l$ be a map with $f(x) = -f(x)$ for all $x \in S^k$. Show that there exists an $x \in S^k$ with $f(x) = 0$.

Exercise 59.2. We consider the string of jewels shown in Figure 887. Use four cuts to distribute the jewels evenly.

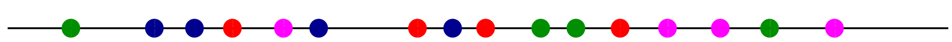


FIGURE 887

Exercise 59.3. We consider the setting of the Continuous Necklace Theorem 59.10. More precisely we are given a map $f \colon [0,1] \to \{1, \dots, k\}$ be a function such that each preimage $f^{-1}(\{c\})$ is a measurable set. For each "color" $c \in \{1, \dots, k\}$ let $y_c \in [0,1]$ be such that $(\nu)\big(f^{-1}(\{c\}) \cap [0, y_c]\big) = (\nu)\big(f^{-1}(\{c\}) \cap [y_c, 1]\big)$. Do these points $y_1, \dots, y_k$, after ordering them, necessarily satisfy the conclusion of the Continuous Necklace Theorem 59.10?

60. Division algebras and generalized projective spaces

60.1. Division algebras. We start out with the following purely algebraic definition.

Definition. An *algebra over a field* $\mathbb{F}$ is an $\mathbb{F}$-vector space A together with an $\mathbb{F}$-bilinear map

$$\begin{array}{rcl} A \times A & \to & A \\ (v,w) & \mapsto & v \cdot w. \end{array}$$

(1) We say that the algebra is *commutative* if for all $a, b \in A$ we have $a \cdot b = b \cdot a$.
(2) We say that the algebra is *associative* if for all $a, b, c \in A$ we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
(3) We say that the algebra *has an identity* if there exists an element $1 \in A$ with $1 \neq 0$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in A$.
(4) If A is an algebra with identity 1, then given $a \in A$ we say that $b \in A$ is *an inverse of* a if $a \cdot b = b \cdot a = 1$.
(5) A *division algebra* is an algebra such that for all $a \neq 0$ and $b \in A$ there exists an $x \in A$ with $a \cdot x = b$ and a $y \in A$ with $y \cdot a = b$.

Examples.

(1) The complex numbers $\mathbb{C} = \mathbb{R} \oplus \mathbb{R} \cdot \mathrm{i}$ with the usual multiplication are an algebra over $\mathbb{R}$ that is commutative and associative, it has an identity, and it is a division algebra.
(2) The set of $(n \times n)$-matrices $\mathrm{M}(n \times n, \mathbb{F})$ over a field $\mathbb{F}$, where the multiplication is given by the usual matrix multiplication, is an associative algebra over $\mathbb{F}$ with an identity. If $n \geq 2$ this algebra is clearly neither commutative nor is it a division algebra.
(3) The real vector space $\mathbb{R}^3$ together with the cross product

$$\begin{array}{rcl} \mathbb{R}^3 \times \mathbb{R}^3 & \to & \mathbb{R}^3 \\ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} & \mapsto & \begin{pmatrix} v_2 w_3 - w_2 v_3 \\ -(v_1 w_3 - w_1 v_3) \\ v_1 w_2 - w_1 v_2 \end{pmatrix} \end{array}$$

is an algebra over $\mathbb{R}$. We recall that the geometric interpretation of the cross product is given as follows: for $v, w \in \mathbb{R}^3$ the cross product $v \times w$ is the unique vector in $\mathbb{R}^3$ that satisfies the following three properties:
(a) the vector is orthogonal to v and w,
(b) the length equals the area of the parallelogram spanned by v and w,
(c) we have $\det(\underbrace{(v\ \ w\ \ v \times w)}_{3 \times 3\text{-matrix}}) \geq 0.$[919]

We make the following observations:
(a) Clearly we have $v \times w = -w \times v$, hence the algebra is non-commutative.
(b) We have

$$(e_1 \times e_1) \times e_2 \;=\; 0 \times e_2 \;=\; 0 \;\neq\; -e_2 \;=\; e_1 \times e_3 \;=\; e_1 \times (e_1 \times e_2),$$

which shows that this algebra is non-associative.

[919] If v and w are non-zero this just means that v, w and $v \times w$ form a positive basis for $\mathbb{R}^3$.

(c) It follows easily from the definition or the geometric interpretation that for $v \neq 0$ there is no vector w with $v \times w = v$, hence the algebra does not have an identity element.

(d) Finally it is straightforward to see that $(\mathbb{R}^3, \times)$ is not a division algebra. More precisely, one can easily show, using the geometric interpretation of the cross product, that given $a, b \in \mathbb{R}^3$ there exists a vector $x \in \mathbb{R}^3$ with $a \times x = b$ if and only if b is orthogonal to a.

We consider the next example in greater detail.

Definition. The *quaternions* $\mathbb{H}$ are defined as the 4-dimensional real vector space $\mathbb{R}^4$ spanned by $1, i, j, k$ and the multiplication which is determined by

(1) $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$,
(2) $1 \cdot \mathrm{i} = \mathrm{i} \cdot 1 = \mathrm{i}, \quad 1 \cdot \mathrm{j} = \mathrm{j} \cdot 1 = \mathrm{j}, \ \ 1 \cdot \mathrm{k} = \mathrm{k} \cdot 1 = \mathrm{k}$, and
(3) $\mathrm{i} \cdot \mathrm{j} = k, \ \mathrm{j} \cdot \mathrm{k} = \mathrm{i}, \ \mathrm{k} \cdot \mathrm{i} = \mathrm{j}$ and $\mathrm{j} \cdot \mathrm{i} = -\mathrm{k}, \ \mathrm{k} \cdot \mathrm{j} = -\mathrm{i}, \ \mathrm{i} \cdot \mathrm{k} = -\mathrm{j}$.[920]

For a quaternion

$$z \ := \ a \cdot 1 + b \cdot \mathrm{i} + c \cdot \mathrm{j} + d \cdot \mathrm{k} \in \mathbb{H}$$

we define the *conjugate*

$$\overline{z} \ := \ a \cdot 1 - b \cdot \mathrm{i} - c \cdot \mathrm{j} - d \cdot \mathrm{k}$$

and the *norm*

$$|z| \ := \ \sqrt{a^2 + b^2 + c^2 + d^2}$$

the same way as we did for complex numbers.

The following lemma summarizes a few key properties of the quaternions. In particular we see that the conjugate and the norm of the quaternions behave in a quite similar fashion as the conjugate and norm of complex numbers.

Lemma 60.1.

(1) *The quaternions are a non-commutative algebra over $\mathbb{R}$ with identity.*
(2) *For any $z \in \mathbb{H}$ we have $\overline{z} + z \in \mathbb{R}$.*
(3) *For all $z, w \in \mathbb{H}$ we have*

$$\overline{\overline{z}} \ = \ z, \qquad \overline{z + w} \ = \ \overline{z} + \overline{w} \qquad \text{and} \qquad \overline{z \cdot w} \ = \ \overline{w} \cdot \overline{z}.$$

(4) *If we make the obvious identification $\mathbb{H} = \mathbb{R}^4$, then the norm $|-|$ on $\mathbb{H}$ agrees with the Euclidean norm $\|-\|$ on $\mathbb{R}^4$.*
(5) *For every $z \in \mathbb{H}$ we have $z \cdot \overline{z} = |z|^2 \in \mathbb{R} = \mathbb{R} \cdot 1$.*
(6) *For every $z \neq 0$ a multiplicative inverse is given by $z^{-1} = \frac{\overline{z}}{|z|^2}$.*
(7) *For all $z, w \in \mathbb{H}$ we have $|z \cdot w| = |z| \cdot |w|$.*
(8) *The quaternions are associative.*
(9) *The quaternions are a division algebra.*

[920]Put differently, we have $\mathrm{i} \cdot \mathrm{j} = \mathrm{k}$ and $\mathrm{j} \cdot \mathrm{i} = -\mathrm{k}$, and both of the equalities hold for *cyclic* permutations of $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$.

FIGURE 888. Brougham Bridge in Dublin

PROOF. Statements (1) to (3) follow from elementary calculations. We provide the details for the remaining statements:

(4) This statement follows immediately from the definition of $|z|$.
(5) This statement follows from a straightforward calculation.
(6) This statement follows immediately from (5).
(7) Let $z_1 = a_1 \cdot 1 + b_1 \cdot \mathrm{i} + c_1 \cdot \mathrm{j} + d_1 \cdot \mathrm{k}$ and $z_2 = a_2 \cdot 1 + b_2 \cdot \mathrm{i} + c_2 \cdot \mathrm{j} + d_2 \cdot \mathrm{k}$ be two quaternions. We see that

$$\begin{aligned} |z_1 \cdot z_2|^2 &= |(a_1 \cdot 1 + b_1 \cdot \mathrm{i} + c_1 \cdot \mathrm{j} + d_1 \cdot \mathrm{k}) \cdot (a_2 \cdot 1 + b_2 \cdot \mathrm{i} + c_2 \cdot \mathrm{j} + d_2 \cdot \mathrm{k})|^2 \\ &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)^2 + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)^2 + \\ &\quad +(a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)^2 + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)^2 \\ &= (a_1^2 + b_1^2 + c_1^2 + d_1^2) \cdot (a_2^2 + b_2^2 + c_2^2 + d_2^2) \;=\; |z_1|^2 \cdot |z_2|^2. \end{aligned}$$

(8) Since the quaternions are an algebra we only have to check associativity for products of three of the generators $\{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\}$ and that can be done easily by hand.
(9) It remains to prove that $\mathbb{H}$ is in fact a division algebra. Thus let $a \neq 0$ and b be two elements in $\mathbb{H}$. We set $y := b \cdot a^{-1}$. Then

$$y \cdot a \;=\; (b \cdot a^{-1}) \cdot a \underset{\substack{\uparrow \\ \text{since by (8) we know that } \mathbb{H} \text{ is associative}}}{=} b \cdot (a^{-1} \cdot a) \;=\; b \cdot 1 \;=\; b.$$

The same way we show that there exists an $x \in \mathbb{H}$ with $a \cdot x = b$. ■

In Theorem 51.2 we saw that $\mathrm{SU}(2)$ is diffeomorphic to S^3. We give a refined version of this statement.

Proposition 60.2.

(1) *The set $S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$ forms a group under multiplication of quaternions.*
(2) (a) *The map $\mathbb{H} \times \mathbb{C} \to \mathbb{H}$ given by $(z, w) \mapsto z \cdot w^{921}$ defines a complex vector space structure on $\mathbb{H}$.*
(b) *A basis for $\mathbb{H}$ viewed as a complex vector space is given by $\{1, \mathrm{j}\}$.*
(c) *Let $z + \mathrm{j} \cdot w \in \mathbb{H}$ with $z, w \in \mathbb{C}$. Left-multiplication by $z + \mathrm{j} \cdot w$ defines a complex linear map $\mathbb{H} \to \mathbb{H}$. With respect to the basis $\{1, \mathrm{j}\}$ this linear map is represented by the matrix $\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$.*
(d) *Left-multiplication by some $h \in \mathbb{H}$ defines a homomorphism of real vector spaces. If $h \neq 0$, then this homomorphism is represented, with respect to the standard basis of $\mathbb{H}$ as a real vector space, by a matrix in $\mathrm{GL}_+(4, \mathbb{R})$.*

(3) *The map*

$$\begin{array}{rcl} f\colon S^3 & \to & \mathrm{SU}(2) \\ (z,w)=z+\mathrm{j}\cdot w & \mapsto & \underbrace{\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}}_{\substack{\text{matrix that represents} \\ \text{left-multiplication by } z+\mathrm{j}\cdot w \text{ with} \\ \textit{respect to the complex basis } \{1,\mathrm{j}\} \textit{ of } \mathbb{H}}} \end{array}$$

is a diffeomorphism and it defines an isomorphism of groups.

Remark. Note that the map $f\colon S^3 \to \mathrm{SU}(2)$ from Proposition 60.2 satisfies in particular

$$f(1)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad f(\mathrm{i})=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\quad f(\mathrm{j})=\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix},\quad \text{and}\quad f(\mathrm{k})=\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}.$$

Recall that in Lemma 51.1 we defined a homomorphism $q\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$. In Exercise 51.1 we calculated $q(f(\mathrm{i}))$, $q(f(\mathrm{j}))$ and $q(f(\mathrm{k}))$.

Proof.

(1) This statement follows immediately from Lemma 60.1 (5), (6) and (7).

(2) (a) This statement follows immediately from the observation that the obvious map $\mathbb{C} \to \mathbb{H}$ is a ring homomorphism.

(b) It follows from $\mathrm{k} = -\mathrm{j}\cdot\mathrm{i}$ that any quaternion $a\cdot 1 + b\cdot\mathrm{i} + c\cdot\mathrm{j} + d\cdot\mathrm{k}$ can be uniquely written as $(a\cdot 1 + b\cdot \mathrm{i}) + \mathrm{j}\cdot(c\cdot 1 - d\cdot\mathrm{i})$ where $a\cdot 1 + b\cdot\mathrm{i}$ and $c\cdot 1 - d\cdot \mathrm{i}$ are complex numbers. This shows that $\{1, j\}$ is a basis for $\mathbb{H}$, viewed as a complex vector space.

(c) First note that it follows from the associativity of quaternions that left-multiplication by a quaternion commutes with right-multiplication by a quaternion, in particular with right-multiplication by a complex number. This shows that left-multiplication by a quaternion is indeed a complex linear map.
Now let us turn to the statement regarding the representative matrix. This statement is elementary, but we think it is nonetheless worth doing. So we consider the basis $\{1, j\}$. Evidently we have $(z+\mathrm{j}\cdot w)\cdot 1 = 1\cdot z + \mathrm{j}\cdot w$. Thus the first column of the matrix is given by $\begin{pmatrix} z \\ w \end{pmatrix}$. More interestingly, we have

$$(z+j\cdot w)\cdot\mathrm{j} \underset{\uparrow}{=} z\cdot\mathrm{j}+\mathrm{j}\cdot w\cdot\mathrm{j} = \mathrm{j}\cdot\overline{z}+\overline{w}\cdot\mathrm{j}\cdot\mathrm{j} \underset{\uparrow}{=} \mathrm{j}\cdot\overline{z}+1\cdot(-\overline{w}).$$

both equalities follow from the fact that $\mathrm{i}\cdot\mathrm{j} = -\mathrm{j}\cdot\mathrm{i}$

Thus the second column of the matrix is given by $\begin{pmatrix} \overline{z} \\ -\overline{w} \end{pmatrix}$.

(d) This statement follows easily from (c).

[921]Here we view $\mathbb{C}$ as a subset of $\mathbb{H}$ in the obvious way. Note that the order in the definition is important since $\mathbb{H}$ is not commutative.

(3) We already saw in Theorem 51.2 that the map $f\colon S^3 \to \mathrm{SU}(2)$ is a diffeomorphism. It follows from the associativity of quaternion multiplication and (2c), or alternatively from a tedious calculation, that the map

$$\begin{array}{rcl} f\colon \mathbb{H} & \to & \mathrm{M}(2\times 2,\mathbb{C}) \\ (z,w) = z + \mathrm{j}\cdot w & \mapsto & \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \end{array}$$

is a ring homomorphism, in particular it preserves the ring structure. This implies that the given map $f\colon S^3 \to \mathrm{SU}(2)$ is a group homomorphism. ■

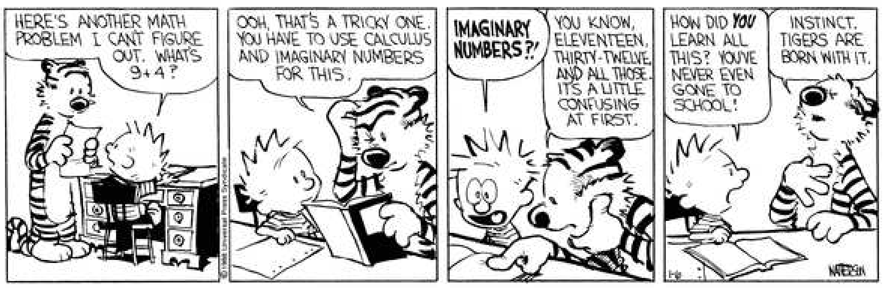


FIGURE 889

Before we move on to the octonions it is helpful to give an alternative description of the quaternions. Namely we can view the Hamiltonians as elements in $\mathbb{C}\oplus\mathbb{C}$ with the multiplication given by

$$(z_1,z_2)\cdot(w_1,w_2) \;:=\; (z_1w_1 - \overline{w}_2z_2,\, z_2\overline{w}_1 + w_2z_1)$$

the conjugate given by

$$\overline{(w,z)} \;:=\; (\overline{w},-z)$$

and the norm given by

$$|(w,z)| \;:=\; \sqrt{(w,z)\cdot\overline{(w,z)}}.$$

One easily verifies that this norm is multiplicative, i.e. given $a,b\in\mathbb{H}=\mathbb{C}\oplus\mathbb{C}$ one has $|a\cdot b| = |a|\cdot|b|$. An elementary calculation shows that the map $(w,z)\mapsto w+z\cdot\mathrm{j}$ defines an isomorphism between the two descriptions of $\mathbb{H}$, furthermore one can verify easily that this isomorphism preserves the conjugate and the norm.

Considering the above description of the quaternions one might hope to be lucky and just try to recycle the definitions:

Definition. The *Cayley octonions* $\mathbb{O}$ (or usually just *octonions*) are defined as $\mathbb{O}=\mathbb{H}\oplus\mathbb{H}$ with the multiplication given by[922]

$$(z_1,z_2)\cdot(w_1,w_2) \;:=\; (z_1w_1 - \overline{w}_2z_2,\, z_2\overline{w}_1 + w_2z_1)$$

the *conjugate* given by

$$\overline{(w,z)} \;:=\; (\overline{w},-z)$$

and the *norm* given by

$$|(w,z)| \;:=\; \sqrt{(w,z)\cdot\overline{(w,z)}}.$$

The following lemma is the analogue of Lemma 60.1.

Lemma 60.3. *The statements of Lemma 60.1* (1) *to* (7) *also hold for octonions. For completeness' sake we spell out the precise statements:*

(1) *The octonions are a non-commutative algebra over $\mathbb{R}$ with identity.*
(2) *For any $z \in \mathbb{O}$ we have $\overline{z}+z \in \mathbb{R}$.*
(3) *For all $z, w \in \mathbb{O}$ we have*

$$\overline{\overline{z}} \;=\; z, \quad \overline{z+w} \;=\; \overline{z}+\overline{w} \quad \text{and} \quad \overline{z\cdot w} \;=\; \overline{w}\cdot\overline{z}.$$

(4) *If we make the obvious identification $\mathbb{O} = \mathbb{H}\oplus\mathbb{H} = \mathbb{R}^4\oplus\mathbb{R}^4 = \mathbb{R}^8$, then the norm $|-|$ on $\mathbb{O}$ agrees with the Euclidean norm $\|-\|$ on $\mathbb{R}^8$.*
(5) *For every $z \in \mathbb{O}$ we have $z\cdot\overline{z} = |z|^2 \in \mathbb{R} = \mathbb{R}\cdot 1$.*
(6) *For every $z \neq 0$ a multiplicative inverse is given by $z^{-1} = \frac{\overline{z}}{|z|^2}$.*
(7) *For all $z, w \in \mathbb{O}$ we have $|z\cdot w| = |z|\cdot|w|$.*

Furthermore, we have the following three statements:

(8) *The octonions are* alternative*, this means that given any $x, y \in \mathbb{O}$ we have the equalities $(x\cdot y)\cdot y = x\cdot(y\cdot y)$ and $(x\cdot x)\cdot y = x\cdot(x\cdot y)$.*
(9) *Given any $x, y \in \mathbb{O}$ the algebra generated by 1, x and y is associative.*
(10) *The octonions are a division algebra.*

Remark.

(1) By the remark on page 82 we know that in an associative ring we can ignore parentheses when it comes to multiplication. Now that we are dealing with the octonions, which are not associative, we have to carefully keep track of parentheses.
(2) Our proof of Lemma 60.3 (9) gets outsourced to a theorem of Artin. An alternative proof for Lemma 60.3 (9), which does not rely on Artin's Theorem, is given in [**CSm03**, p. 76].

Proof. As in the case of Lemma 60.1 the verification of (1) to (3) is totally elementary. The task of filling in the details is once again left to the reader. For later on it is helpful to consider the elementary prof of (4) in the notes:

(4) Given $(z,w) \in \mathbb{O}$ we calculate that

$$|(w,z)|^2 = (w,z)\cdot\overline{(w,z)} = (w,z)\cdot(\overline{w},-z) = (w\overline{w}+\overline{z}z, zw-zw) = \|w\|^2+\|z\|^2.$$

(5) This statement holds by definition of $|-|$.
(6) This statement follows immediately from (5).

Now we turn to the arguably more interesting proofs of the remaining four statements:

(7) It turns out that the proof of this statement is not totally straightforward. Thus let $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbb{O}$. We need the following super-technical claim.

[922] In fact these formulas also work for defining $\mathbb{C}$ as $\mathbb{R}\oplus\mathbb{R}$ where we equip $\mathbb{R}$ with the trivial involution.

Claim. We have

$$\underbrace{z_2\overline{w}_1\overline{z}_1\overline{w}_2 + w_2z_1w_1\overline{z}_2 - z_1w_1\overline{z}_2w_2 - \overline{w}_2z_2\overline{w}_1\overline{z}_1}_{=:S(w_2)} = 0.$$

Evidently, if w_2 is a real number, then we have $S(w_2) = 0$. Now suppose that w_2 is a purely imaginary quaternion, i.e. $w_2 = a \cdot \mathrm{i} + b \cdot \mathrm{j} + c \cdot \mathrm{j}$ for some $a, b, c \in \mathbb{R}$. In this case we calculate that

$$S(w_2) \underset{\text{since } w_2 \text{ is purely imaginary we have } \overline{w}_2 = -w_2}{=} w_2(\underbrace{z_2\overline{w}_1\overline{z}_1 + z_1w_1\overline{z}_2}_{=:c}) - (\underbrace{z_2\overline{w}_1\overline{z}_1 + z_1w_1\overline{z}_2}_{=:c})w_2 = w_2 \cdot c - c \cdot w_2 \underset{\text{we have } \overline{c} = c, \text{ thus } c \text{ is a real number}}{=} 0.$$

Finally note that for any $a, b \in \mathbb{H}$ we have $S(a+b) = S(a)+S(b)$. Since any quaternion is the sum of a real number and a purely imaginary we see that $S(w_2) = 0$ for any quaternion w_2. ⊞

Now we can provide the actual proof that the norm is multiplicative. Indeed, we calculate that

$$\begin{aligned}
|z \cdot w|^2 &= |(z_1, z_2) \cdot (w_1, w_2)|^2 = |(z_1w_1 - \overline{w}_2z_2, z_2\overline{w}_1 + w_2z_1)|^2 \\
&\underset{\text{see (4)}}{=} (z_1w_1 - \overline{w}_2z_2) \cdot \overline{(z_1w_1 - \overline{w}_2z_2)} + (z_2\overline{w}_1 + w_2z_1) \cdot \overline{(z_2\overline{w}_1 + w_2z_1)} \\
&= (z_1w_1 - \overline{w}_2z_2) \cdot (\overline{w}_1\overline{z}_1 - \overline{z}_2w_2) + (z_2\overline{w}_1 + w_2z_1) \cdot (w_1\overline{z}_2 + \overline{z}_1\overline{w}_2) \\
&= (z_1w_1) \cdot (\overline{w}_1\overline{z}_1) - (z_1w_1)(\overline{z}_2w_2) - (\overline{w}_2z_2) \cdot (\overline{w}_1\overline{z}_1) + (\overline{w}_2z_2) \cdot (\overline{z}_2w_2) \\
&\quad + (z_2\overline{w}_1)(w_1\overline{z}_2) + (z_2\overline{w}_1)(\overline{z}_1\overline{w}_2) + (w_2z_1) \cdot (w_1\overline{z}_2) + (w_2z_1) \cdot (\overline{z}_1\overline{w}_2) \\
&\underset{\text{by the claim and the fact that the quaternions are associative}}{=} (z_1w_1) \cdot (\overline{w}_1\overline{z}_1) + (\overline{w}_2z_2) \cdot (\overline{z}_2w_2) + (z_2\overline{w}_1)(w_1\overline{z}_2) + (w_2z_1) \cdot (\overline{z}_1\overline{w}_2) \\
&= (z_1\overline{z}_1 + z_2\overline{z}_2) \cdot (w_1\overline{w}_1 + w_2\overline{w}_2) = |(z_1, z_2)|^2 \cdot |(w_1, w_2)|^2 = |z|^2 \cdot |w|^2.
\end{aligned}$$

(8) Let $x, y \in \mathbb{O}$. We start out with the following claim.

Claim. We have the equalities $(x \cdot y) \cdot \overline{y} = x \cdot (y \cdot \overline{y})$ and $(x \cdot \overline{x}) \cdot y = x \cdot (\overline{x} \cdot y)$.

Let us first prove the first equality. We write $x = (x_1, x_2)$, $y = (y_1, y_2)$ with $x_1, x_2, y_1, y_2 \in \mathbb{H}$. Heroically we calculate that

$$\begin{aligned}
(xy) \cdot \overline{y} &= ((x_1, x_2)(y_1, y_2)) \cdot (\overline{y}_1, -y_2) = (x_1y_1 - \overline{y}_2x_2, y_2x_1 + x_2\overline{y}_1) \cdot (\overline{y}_1, -y_2) \\
&= ((x_1y_1 - \overline{y}_2x_2)\overline{y}_1 + \overline{y}_2(y_2x_1 + x_2\overline{y}_1), (-y_2)(x_1y_1 - \overline{y}_2x_2) + (y_2x_1 + x_2\overline{y}_1)y_1) \\
&\underset{\substack{\text{since } \mathbb{H} \text{ is associative we have } (\overline{y}_2x_2)\overline{y}_1 = \overline{y}_2(y_2x_1) \text{ and } y_2(x_1y_1) = (y_2x_1)y_1 \\ \text{we also use that } \overline{y}_1y_1 = |y_1|^2 \in \mathbb{R}, \text{ hence it commutes with any } z \in \mathbb{H}}}{=} ((|y_1|^2 + |y_2|^2)x_1, (|y_1|^2 + |y_2|^2)x_2) = (|y_1|^2 + |y_2|^2) \cdot (x_1, x_2) \\
&= |y|^2 \cdot x = x \cdot |y|^2 = x \cdot (y\overline{y}).
\end{aligned}$$

The proof of the second equality is basically the same. ⊞

Now, we use (2) to write $\overline{x} = -x + 2a$ and $\overline{y} = -y + 2b$ with $a, b \in \mathbb{R}$. We plug these equalities into the equalities of the claim and, since we are dealing with an algebra over the real numbers, we obtain the desired equalities $(x \cdot y) \cdot y = x \cdot (y \cdot y)$ and $(x \cdot x) \cdot y = x \cdot (x \cdot y)$.

(9) It is a non-trivial theorem of Emil Artin that any alternative algebra $\mathbb{A}$ has the property that given any two $x, y \in \mathbb{A}$ the algebra generated by 1, x and y is associative. A proof is for example provided in [**SchafR66**, Theorem 3.1]. Alternatively see [**Bae01**, p. 155].

(10) It remains to prove that $\mathbb{O}$ is in fact a division algebra. Since the octonions are not associative we can not copy-paste the proof of Lemma 60.1 (9), since the latter proof used the associativity of the quaternions. Fortunately the spartan associativity result given in (8) is enough to give us the desired result.

Let $a, b \in \mathbb{O}$ with $a \neq 0$. We need to show that there exists an $x \in \mathbb{O}$ with $a \cdot x = b$ and a $y \in \mathbb{O}$ with $y \cdot a = b$. We set $x = a^{-1} \cdot b$. We calculate that

$$a \cdot x \;=\; a \cdot (a^{-1} \cdot b) \;=\; a \cdot \left(\tfrac{1}{|a|^2} \cdot \overline{a} \cdot b\right) \;=\; \tfrac{1}{|a|^2} \cdot a \cdot (\overline{a} \cdot b) \underset{\substack{\uparrow \\ \text{by the claim in the proof of (8)}}}{=} \tfrac{1}{|a|^2} \cdot (a \cdot \overline{a}) \cdot b \;=\; b.$$

Basically the same calculation shows that we have $y \cdot a = b$. ■

Remark.

(1) The octonions were discovered[923] by Arthur Cayley in 1845, shortly after Hamilton found the quaternions.

(2) The discussion in the previous pages shows that starting from $\mathbb{R}$ one can construct the complex numbers, which give rise to the quaternions and finally we obtain the octonions. When we move from the complex numbers to the quaternions we loose commutativity and when we go from the quaternions to the octonions we also loose associativity. Evidently the octonions are not commutative, but even worse, one can easily verify that the octonions are not even associative.[924] The octonions are discussed in much greater detail in [**Bae01**], [**CSm03**, Chapter 6] and [**KaS89**, Chapter 6].

(3) The definition of quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ out of the complex numbers and the definition of the octonions $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ out of the quaternions is a general procedure known as the *Cayley-Dickson construction* . This construction is discussed in greater detail in [**Bae01**, Section 2.2]. In particular one can easily see that this process can be extended to define, basically by the same recipe, a 16-dimensional algebra $\mathbb{S} = \mathbb{O} \oplus \mathbb{O}$ with an identity, called the *sedenions*, this algebra is evidently neither commutative nor associative. On the other hand the properties stated in Lemma 60.1 (1)–(6) and Lemma 60.3 (1)–(6) hold the same way, in particular $\mathbb{S}$ has multiplicative inverses, i.e. for each $a \in \mathbb{S} \setminus \{0\}$ there exists a $b \in \mathbb{S}$ with $a \cdot b = b \cdot a = 1$. But the proof of Lemma 60.3 (7) and (8) do not carry over, since in the proof we made use of the fact that the quaternions are actually associative. In fact it turns out that the analogue of

[923] Or were they invented by Arthur Cayley? This question whether mathematics is "discovered" or "invented" is a source for endless philosophical discussions.

[924] To quote from [**Frie12**, p. 5], "here, associativity is obviously a concern".

(7) does not hold for sedenions and that the sedenions are not a division algebra.[925] [926] We refer to [**More97**] for more information on the failure of the sedenions to be a division algebra.

60.2. **Commutative division algebras.** After the discovery of quaternions and the octonions the question arose whether one can also find division algebras on $\mathbb{R}^n$ for some $n \neq 1, 2, 4, 8$. The following theorem, which was first proved by Heinz Hopf [**Hopf40a, Hopf40b, Hopf64**] in 1940, is the best theorem that we can prove in this direction with our present knowledge.

Theorem 60.4. (Hopf Theorem on commutative division algebras) *Any finite-dimensional division algebra over $\mathbb{R}$ which is commutative and which has an identity is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.*

Remark.

(1) Note that in Theorem 60.4 we do not assume that the division algebra is associative. It is a nice exercise in algebra and real analysis to show, using the Fundamental Theorem of Algebra, the weaker statement that if V is a finite-dimensional commutative and *associative* division algebra over $\mathbb{R}$, then V is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.
(2) The statement of Theorem 60.4 does not hold if we drop the "finite dimension" condition. For example the field $\mathbb{R}(t)$ of rational functions over $\mathbb{R}$ is a commutative division algebra, but it is an infinite-dimensional real vector space.
(3) In Theorem 90.23 we will obtain further results on division algebras.

In the proof of Theorem 60.4 we will need the following basic lemma which might already be familiar from earlier algebra courses.

Lemma 60.5. *Let A be a* finite-dimensional *algebra over a field $\mathbb{F}$. The following two statements are equivalent:*

(1) *A is a division algebra,*
(2) *A has no zero-divisors, i.e. for any $a, b \in A$ with $a \cdot b = 0$ we have $a = 0$ or $b = 0$.*

Remark. Let A be a division algebra over a field $\mathbb{F}$. If A is associative, then it is clear that A has no zero-divisors. Indeed, if $a \cdot b = 0$ with $a \neq 0$, then we pick x with $x \cdot a = 1$ and we see that $0 = x \cdot 0 = x \cdot (a \cdot b) = (x \cdot a) \cdot b = 1 \cdot b = b$. The point about Lemma 60.5 is that it applies to non-associative algebras, as long as they are finite-dimensional.

[925] In fact this construction can be iterated and one obtains algebras which are creatively called pathions (32-dimensional), chingons (64-dimensional), routons (128-dimensional), voudons (256-dimensional). We challenge the reader to come up with the explanations for these names. Alternatively the reader can look at [**dMar02**, p. 7].

[926] Perhaps more fruitfully, the Cayley-Dickson construction can be applied to the real numbers to give the complex numbers and it can also be applied to the complex numbers to give the quaternions.

Proof. Let A be a finite-dimensional algebra over a field $\mathbb{F}$. Let $a \in A$ be non-zero. Then the following holds:

$$\begin{aligned}
\text{for any } y \in A \text{ there exists } x \in A \text{ with } ax = y \quad &\Leftrightarrow \text{ the map } A \to A,\ x \mapsto ax \text{ is surjective} \\
&\underset{\substack{\uparrow \\ \text{since } x \mapsto ax \text{ is a homomorphism of finite-dimensional } \mathbb{F}\text{-vector spaces}}}{\Leftrightarrow} \text{ the map } A \to A,\ x \mapsto ax \text{ is injective} \\
&\Leftrightarrow \text{ for any } b \neq 0 \text{ we have } ab \neq 0.
\end{aligned}$$

The same statement holds for left-multiplication. The equivalence of (1) and (2) now follows easily from this observation and the definitions. ■

Now we turn to the proof of Theorem 60.4.

Proof of Theorem 60.4. Let $(A, *)$ be a finite-dimensional division algebra over $\mathbb{R}$ which is commutative and which has an identity. Since A is a finite-dimensional vector space we can, after applying an isomorphism, assume that $A = \mathbb{R}^n$ for some $n \in \mathbb{N}$.

We first consider the case $n = 1$. We define $a := 1 * 1$. Since "$*$" is bilinear we have for all $x, y \in \mathbb{R}$ that $x * y = (x \cdot 1) * (y \cdot 1) = x \cdot (1 * 1) \cdot y = xay$. Since the algebra is a division algebra we obtain from Lemma 60.5 that $a \neq 0$. Then the map $\varphi(x) = ax$ defines an isomorphism from the algebra $(\mathbb{R}, *)$ to the algebra $(\mathbb{R}, \cdot)$.[927]

Now suppose that $n \geq 2$. We first want to show that $n = 2$. To do this we consider the map

$$\begin{array}{rccl} f\colon S^{n-1} & \to & S^{n-1} & \\ x & \mapsto & f(x) & := \dfrac{x * x}{\|x * x\|}. \end{array}$$

This map is indeed well-defined, since for $x \neq 0$ it follows from our hypothesis on $(\mathbb{R}^n, *)$ and from Lemma 60.5 that $x * x \neq 0$. The map f is continuous since the multiplication map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is bilinear, hence continuous.[928]

Since "$*$" is bilinear we have $f(-x) = f(x)$ for all $x \in \mathbb{R}^n$. It follows from Lemma 3.22 that the map f descends to a continuous map

$$\begin{array}{rcl} \overline{f}\colon \mathbb{RP}^{n-1} & \to & S^{n-1} \\ {[x]} & \mapsto & \overline{f}([x]) := f(x). \end{array}$$

Claim. The map $\overline{f}\colon \mathbb{RP}^{n-1} \to S^{n-1}$ is injective.

So let $[x], [y] \in \mathbb{RP}^{n-1} = S^{n-1}/\sim$. We have

$$\begin{aligned}
\underbrace{\overline{f}([x])}_{=f(x)} = \underbrace{\overline{f}([y])}_{=f(y)} \implies f(x) = f(y) &\implies x * x = \alpha^2 \cdot (y * y) \text{ for } \alpha := \sqrt{\|x * x\| \cdot \|y * y\|^{-1}} \\
&\implies 0 = x * x - \alpha^2 \cdot (y * y) \underset{\substack{\uparrow \\ \text{since "}*\text{" is commutative and bilinear}}}{=} (x + \alpha y) * (x - \alpha y) \\
&\underset{\substack{\uparrow \\ \text{Lemma 60.5}}}{\implies} x = \pm\alpha y \underset{\substack{\uparrow \\ \text{since } x, y \in S^{n-1}}}{\implies} x = \pm y \implies [x] = [y].
\end{aligned}$$

⊞

[927] Indeed, for $x, y \in \mathbb{R}$ we have $\varphi(x * y) = \varphi(x \cdot a \cdot y) = a \cdot x \cdot a \cdot y = (a \cdot x) \cdot (a \cdot y) = \varphi(x) \cdot \varphi(y)$.

[928] Indeed, the fact that the map is bilinear implies that there exists a real $n \times n$–matrix P such that for any $v, w \in \mathbb{R}^n$ we have $v \cdot w = v^T P w$, hence the map is continuous.

Since S^{n-1} is closed and connected (here we use that $n \geq 2$) and since $\mathbb{R}\mathrm{P}^{n-1}$ is a closed smooth manifold it follows from Corollary 50.8 that $\overline{f}\colon \mathbb{R}\mathrm{P}^{n-1} \to S^{n-1}$ is in fact a homeomorphism. In particular the fundamental groups of $\mathbb{R}\mathrm{P}^{n-1}$ and S^{n-1} are isomorphic. But it follows from Proposition 14.14 and Corollary 16.18 that this is only possible if $n = 2$.

It remains to show that the algebra $(A, *) = (\mathbb{R}^2, *)$ is isomorphic to the algebra $(\mathbb{C}, \cdot)$. We start out with the following claim.

Claim. There exists a $k \in A \setminus \mathbb{R} \cdot 1$ with $k * k = c \cdot 1$ for some $c \in \mathbb{R}_{<0}$.

We first pick a random element $j \in A \setminus \mathbb{R} \cdot 1$. Then 1 and j are a basis for $A = \mathbb{R}^2$ and therefore we can write $j * j = a \cdot 1 + b \cdot \mathrm{j}$ for some $a, b \in \mathbb{R}$. Then multiplying out and using this equality we see that

$$\underbrace{\left(j - \tfrac{b}{2} \cdot 1\right)}_{=:k} * \underbrace{\left(j - \tfrac{b}{2} \cdot 1\right)}_{=:k} \;=\; \underbrace{\left(a + \tfrac{1}{4} b^2\right)}_{=:c} \cdot 1.$$

We set $k := j - \frac{b}{2} \cdot 1$ and $c := a + \frac{1}{4} b^2$. We still need to show that $c < 0$. Suppose that $c \geq 0$. Then $k * k = c \cdot 1$ implies that $(k + \sqrt{c} \cdot 1) * (k - \sqrt{c} \cdot 1) = k * k - c \cdot 1 = 0$. From Lemma 60.5 we conclude that $k = \pm\sqrt{c} \cdot 1$. But this is not possible since $k = j - \frac{b}{2} \cdot 1$ is linearly independent from 1. ⊞

One can now easily verify that

$$\begin{array}{rcl} A & \to & \mathbb{C} \\ x \cdot 1 + y \cdot \mathrm{k} & \mapsto & x + y \cdot \frac{\mathrm{i}}{\sqrt{-c}} \quad \text{with } x, y \in \mathbb{R} \end{array}$$

is an isomorphism from the algebra $(A, *)$ to the algebra $(\mathbb{C}, \cdot)$. ■

It is natural to ask what other division algebras exist. The following, purely algebraic, theorem classifies *associative* division algebras.

Theorem 60.6. *Any finite-dimensional division algebra over $\mathbb{R}$ which is associative and which has an identity is isomorphic to $\mathbb{R}, \mathbb{C}$ or to $\mathbb{H}$.*

Proof. The statement was first proved by Frobenius in 1878. A short self-contained proof is for example given in [**Pal68**]. ■

Finally let us record the following question.

Question 60.7. *For which $n \in \mathbb{N}$, besides $n = 1, 2, 4, 8$ does $\mathbb{R}^n$ admit the structure of a division algebra?*

We will answer this question in Theorem 90.25.

60.3. **Quaternionic and octonionic projective spaces** (∗)**.** In this section we introduce the quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$ and the we introduce the significantly more subtle octonionic projective space $\mathbb{O}\mathrm{P}^2$. These smooth manifolds get added to our ever expanding list of amusing topological spaces. At a later stage they will play an essential role.

On page 935, given $n \in \mathbb{N}_0$, we introduced the real projective space $\mathbb{R}\mathrm{P}^n$ and the complex projective spaces $\mathbb{C}\mathrm{P}^n$. In the following we will introduce the quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$. Since the quaternions $\mathbb{H}$ are non-commutative we do the constructions carefully.

Lemma 60.8. *Given $v, w \in \mathbb{H}^{n+1} \setminus \{(0, \ldots, 0)\}$ we write $v \sim w$ if there exists a non-zero $h \in \mathbb{H}$ with $h \cdot v = w$. This defines an equivalence relation on $\mathbb{H}^{n+1} \setminus \{(0, \ldots, 0)\}$. The analogous statement also holds for $\mathbb{H}^\infty \setminus \{0\}$.*

Proof.

(1) It is clear that $\sim$ is reflexive.
(2) Since $\mathbb{H}$ is a skew field we see that $\sim$ is symmetric.
(3) Finally suppose that $u \sim v$ and $v \sim w$. Thus there exist $g, h \in \mathbb{H} \setminus \{0\}$ with $gu = v$ and $hv = w$. Then

$$(hg)u \underset{\substack{\uparrow \\ \text{since the quaternions are associative!}}}{=} h(gu) = hv = w.$$

■

Now we can define the quaternionic projective spaces.

Definition. Let $n \in \mathbb{N}_0$.

(1) For $(h_0, \ldots, h_n) \in \mathbb{H}^{n+1} \setminus \{0\}$ we denote by $[h_0 : \ldots : h_n]$ the corresponding equivalence class.
(2) Given $n \in \mathbb{N}$ we define the *quaternionic projective space*

$$\mathbb{HP}^n := (\mathbb{H}^{n+1} \setminus \{(0, \ldots, 0)\}) / \sim .$$

Similarly, we define the *infinite-dimensional quaternionic projective space*

$$\mathbb{HP}^\infty := (\mathbb{H}^\infty \setminus \{0\}) / \sim .$$

The following lemma summarizes some key properties of quaternionic projective spaces. $\mathbb{HP}^n$. We will make use of some of them later on.

Lemma 60.9. *Let $n \in \mathbb{N}_0$.*

(1) *$\mathbb{HP}^n$ is a closed orientable $4n$-dimensional smooth manifold.*
(2) *The map*

$$\begin{array}{rcl} \mathbb{HP}^1 & \to & \mathbb{H} \cup \{\infty\} = \mathbb{R}^4 \cup \{\infty\} \overset{\substack{\text{by Lemma 2.44} \\ \downarrow}}{=} S^4 \\ {[z_0 : z_1]} & \mapsto & \begin{cases} z_0 \cdot z_1^{-1}, & \text{if } z_1 \neq 0, \\ \infty, & \text{if } [z_0 : z_1] = [1 : 0] \end{cases} \end{array}$$

is a homeomorphism.
(3) *$\mathbb{HP}^n$ admits a CW-structure that has exactly one cell in the dimensions $0, 4, 8, \ldots, 4n$.*
(4) *We have $\chi(\mathbb{HP}^n) = n + 1$.*
(5) *$\mathbb{HP}^n$ is 3-connected, i.e. we have $\pi_i(\mathbb{HP}^3) = 0$ for $i = 1, 2, 3$.*
(6) *We have a homeomorphism $\varinjlim \mathbb{HP}^n = \mathbb{HP}^\infty$.*
(7) *We can equip $\mathbb{HP}^\infty$ with a CW-structure that has exactly one cell in the dimensions $0, 4, 8, \ldots$.*

Proof.

(1) This statement is proved in a very similar way as Lemma 12.5.

(2) The proof of this statement is basically the same as the proof of Lemma 3.42 (2).
(3) The proof of this statement is an easy modification of the proof of Lemma 36.1.
(4) This statement follows immediately from (3).
(5) This follows from (3) and Proposition 40.9.
(6) This statement is proved the same way as Lemma 36.5 (1).
(7) This statement is proved basically the same way as Lemma 36.5 (3). ■

Since the octonions are not associative the definition on page 935 cannot be applied to the octonions. But there is some hope in low dimensions. More precisely, as we will see shortly, one can use the octonions to construct smooth manifolds that have all the properties that one would naively expect $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$ to have.

Definition.
(1) We equip[929]
$$T_2 \;:=\; \mathbb{O}^2 \setminus \{(0,0)\}$$
with the equivalence relation[930][931]
$$(x_1,y_1)\sim(x_2,y_2) \quad\Longleftrightarrow\quad \begin{array}{l}\text{there exists a } \lambda\in\mathbb{R}_{>0} \text{ such that}\\ x_1\overline{x_1}=\lambda\cdot x_2\overline{x_2} \text{ and } x_1\overline{y_1}=\lambda\cdot x_2\overline{y_2}.\end{array}$$
We set
$$\mathbb{O}\mathrm{P}^1 \;:=\; T_2/\sim$$
and we refer to $\mathbb{O}\mathrm{P}^1$ as the *octonionic projective line.*
(2) We equip
$$T_3 \;:=\; \big\{(x,y,z)\in\mathbb{O}^3\setminus\{(0,0,0)\}\,\big|\,1,x,y,z \text{ generate an associative algebra}\big\}$$
with the equivalence relation[932]
$$(x_1,y_1,z_1)\sim(x_2,y_2,z_2) \iff \begin{array}{lll}\multicolumn{3}{l}{\text{there exists a } \lambda\in\mathbb{R}_{>0} \text{ such that}}\\ x_1\overline{x_1}=\lambda\cdot x_2\overline{x_2}, & x_1\overline{y_1}=\lambda\cdot x_2\overline{y_2}, & x_1\overline{z_1}=\lambda\cdot x_2\overline{z_2},\\ y_1\overline{y_1}\;=\lambda\cdot y_2\overline{y_2}, & y_1\overline{z_1}=\lambda\cdot y_2\overline{z_2}, & z_1\overline{z_1}=\lambda\cdot z_2\overline{z_2}.\end{array}$$
We set
$$\mathbb{O}\mathrm{P}^2 \;:=\; T_3/\sim$$
and we refer to $\mathbb{O}\mathrm{P}^2$ as the *octonionic projective plane.* In the literature the octonionic projective plane is often also called the *Cayley projective plane.*
(3) We denote the equivalence class of a point $(x,y)\in T_2$ by $[x:y]$ and we denote the equivalence class of a point $(x,y,z)\in T_3$ by $[x:y:z]$.

The following proposition hopefully convinces the reader that the octonion line $\mathbb{O}\mathrm{P}^1$ and the octonion plane $\mathbb{O}\mathrm{P}^2$ deserve their names.

[929]What happens if we use these definitions of $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$, but if we replace the octonions by our more familiar friends $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$?

[930]Note that this is obviously an equivalence relation.

[931]The definition of the equivalence relation is a priori asymmetric in the first and the second variable. As we will see in Exercise 60.6, this sense of asymmetry is an illusion.

[932]As for $\sim$ on T_2, this is basically by definition an equivalence relation.

Proposition 60.10.

(1) *The octonionic projective line* $\mathbb{O}\mathrm{P}^1$ *and the octonionic projective plane* $\mathbb{O}\mathrm{P}^2$ *are both compact.*

(2) *The map*

$$\begin{array}{rcl} \mathbb{O}\mathrm{P}^1 & \to & \mathbb{O} \cup \{\infty\} = \mathbb{R}^8 \cup \{\infty\} \underset{\text{by Lemma 2.44}}{=} S^8 \\ [x:y] & \mapsto & \begin{cases} x \cdot y^{-1}, & \text{if } y \neq 0, \\ \infty, & \text{if } [x:y] = [1:0] \end{cases} \end{array}$$

is well-defined and it is a homeomorphism.

(3) *The octonionic projective plane* $\mathbb{O}\mathrm{P}^2$ *is a closed orientable 16-dimensional smooth manifold.*

(4) *The map*[933]

$$\begin{array}{rcl} \mathbb{O}\mathrm{P}^1 & \to & \mathbb{O}\mathrm{P}^2 \\ [x:y] & \mapsto & [x:y:0] \end{array}$$

is an embedding. We use it to view $\mathbb{O}\mathrm{P}^1$ *as a subspace of* $\mathbb{O}\mathrm{P}^2$.

(5) *The octonionic projective plane* $\mathbb{O}\mathrm{P}^2$ *admits a* CW*-structure with precisely one cell in dimensions 0, 8 and 16, and which has no other cells. Furthermore this* CW*-structure can be chosen such that the 8-skeleton is precisely* $\mathbb{O}\mathrm{P}^1$.

(6) *We have* $\chi(\mathbb{O}\mathrm{P}^2) = 3$.

(7) $\mathbb{O}\mathrm{P}^2$ *is 7-connected, i.e. we have* $\pi_i(\mathbb{O}\mathrm{P}^2) = 0$ *for* $i = 1, 2, \ldots, 7$.

Proof.

(1) It is straightforward to define equivalence relations on S^{15} and on S^{23} such that the octonionic projective line $\mathbb{O}\mathrm{P}^1$ is homeomorphic to $S^{15}/\sim$ and such that the octonionic projective plane $\mathbb{O}\mathrm{P}^2$ is homeomorphic to $S^{23}/\sim$. It follows from Lemma 2.40 that $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$ are both compact.

(2) First we need to show that the given map $\Theta\colon \mathbb{O}\mathrm{P}^1 \to \mathbb{O} \cup \{\infty\}$ is well-defined. Thus suppose that we are given two points (x_1, y_1) and (x_2, y_2) in $\mathbb{O}^2 \setminus \{(0,0)\}$ that are equivalent. One can verify easily that $y_1 \neq 0$ if and only if $y_2 \neq 0$. By definition of Θ we only need to worry about the case that $y_1, y_2 \neq 0$. In this case we do indeed have

$$x_1 y_1^{-1} \underset{\text{Lemma 60.3 (7)}}{=} x_1 \cdot \frac{1}{|y_1|^2} \cdot \overline{y}_1 \underset{\text{Lemma 60.3 (6)}}{=} \frac{1}{y_1 \cdot \overline{y}_1} \cdot (x_1 \cdot \overline{y}_1) \underset{\substack{\text{by definition of “}\sim\text{” we have} \\ y_1\overline{y_1} = \lambda \cdot y_2\overline{y_2} \text{ and } x_1\overline{y_1} = \lambda \cdot x_2\overline{y_2}}}{=} \frac{1}{y_2 \cdot \overline{y}_2} \cdot (x_2 \cdot \overline{y}_2) \underset{\text{same steps backwards}}{=} \ldots = x_2 y_2^{-1}.$$

We have thus shown that Θ is well-defined. It is clear that the map Θ is a surjection. In Exercise 60.5 we will verify that Θ is also an injection. Next note that the argument in the proof of Lemma 3.42 (2) shows that Θ is continuous. We just saw in (1) that $\mathbb{O}\mathrm{P}^1$ is compact. Evidently $\mathbb{O} \cup \{\infty\} = S^8$ is Hausdorff. In summary we obtain from Proposition 2.43 (3) that $\Theta\colon \mathbb{O}\mathrm{P}^1 \to \mathbb{O} \cup \{\infty\}$ is a homeomorphism.

[933] It follows from Lemma 60.3 (9) that this map is well-defined, i.e. given any point $(x_1, y_1) \in T_2$ we have indeed $(x_1, y_1, 0) \in T_3$.

(3) We start out with the following claim:

Claim. The map

$$\begin{array}{rcl}\Phi_z\colon \{[x:y:z]\in \mathbb{O}\mathrm{P}^2\,|\,z\neq 0\} & \to & \mathbb{O}^2\\ {[x:y:z]} & \mapsto & (xz^{-1},yz^{-1})\end{array}$$

is a homeomorphism between an open subset of $\mathbb{O}\mathrm{P}^2$ and $\mathbb{O}^2=\mathbb{R}^{16}$.

We prove the claim in three mostly elementary steps.

(a) Basically the same argument as in (2) shows that the map Φ_z is well-defined.

(b) It follows easily from the definition of the quotient topology that the domain of the map Φ_z is indeed an open subset of $\mathbb{O}\mathrm{P}^2$.

(c) Finally we consider the map

$$\begin{array}{rcl}\Psi_z\colon \mathbb{O}^2 & \to & \{[x:y:z]\in \mathbb{O}\mathrm{P}^2\,|\,z\neq 0\}\\ (x,y) & \mapsto & [x:y:1].\end{array}$$

Note that here we use the non-trivial Lemma 60.3 (9) to conclude that Ψ_z is actually well-defined, i.e. to conclude that $(x,y,1)$ is actually an element in T_3. Furthermore note that it follows from Lemma 3.22 that Ψ_z is continuous. It is clear that $\Phi_z\circ\Psi_z=\mathrm{id}$. Finally note that given $[x:y:z]\in\mathbb{O}\mathrm{P}^2$ with $z\neq 0$ we have

$$\Psi_z(\Phi_z([x:y:z])) = [xz^{-1}:yz^{-1}:1] \underset{\uparrow}{=} [x:y:z]$$

follows with $\lambda=|z|^2$ and from the observation that the hypothesis $(x,y,z)\in T_3$ together with Lemma 60.3 (2) implies that the elements $x,y,z,\overline{x},\overline{y},\overline{z}$ generate an associative algebra

We have thus shown that Ψ_z is an inverse to Φ_z. This implies that Φ_z is indeed a homeomorphism. ⊞

Similarly we define charts Φ_x and Φ_y. It is straightforward to verify that the transition maps between the three charts are smooth and orientation-preserving. In particular these three charts define a smooth atlas for $\mathbb{O}\mathrm{P}^2$. We obtain from Lemma 6.4 that $\mathbb{O}\mathrm{P}^2$ is second-countable. In (1) we showed that $\mathbb{O}\mathrm{P}^2$ is compact. Finally in Exercise 60.7 we will show that $\mathbb{O}\mathrm{P}^2$ is Hausdorff.[934] The combination of all of the above shows that $\mathbb{O}\mathrm{P}^2$ is indeed a closed 16-dimensional smooth manifold. Finally, since the transition maps between the three charts are orientation-preserving we obtain from Lemma 6.45 that $\mathbb{O}\mathrm{P}^2$ is actually orientable.

(4) It follows basically immediately from Exercise 60.6 that the given map $\mathbb{O}\mathrm{P}^1\to\mathbb{O}\mathrm{P}^2$ is an injection. By (1) we know that $\mathbb{O}\mathrm{P}^1$ is compact and by (3) we know that $\mathbb{O}\mathrm{P}^2$ is Hausdorff. Thus it follows from Proposition 2.43 (2) that the given map $\mathbb{O}\mathrm{P}^1\to\mathbb{O}\mathrm{P}^2$ is an embedding.

(5) We view $\mathbb{O}\mathrm{P}^1=S^8$ as a CW-complex with one 0-cell and one 8-cell in the obvious way. Next, similar to the proof of Lemma 36.1 we consider the attaching map

$$\begin{array}{rcl}\varphi\colon S^{15}=\{(x_1,x_2)\in\mathbb{O}^2\,|\,|x_1|^2+|x_2|^2=1\} & \to & \mathbb{O}\mathrm{P}^1\\ (x_1,x_2) & \mapsto & [x_1:x_2].\end{array}$$

[934]An arguably more elegant proof that $\mathbb{O}\mathrm{P}^2$ is Hausdorff is given in [**Lackm12**].

Furthermore we consider the map

$$\begin{array}{rcl}\Theta\colon \mathbb{O}\mathrm{P}^1 \cup_\varphi \overline{B}^{16} & \to & \mathbb{O}\mathrm{P}^2 \\ [P] & \mapsto & \begin{cases} [x_1 : x_2 : 0], & \text{if } P = [x_1 : x_2] \in \mathbb{O}\mathrm{P}^1, \\ [P : 1 - \|P\|], & \text{if } P \in \overline{B}^{16}. \end{cases}\end{array}$$

It follows from Lemma 60.3 (8) that the map actually does take values in $\mathbb{O}\mathrm{P}^2$. We leave the slightly tedious task of verifying that the map is well-defined and a bijection to the reader. The left-hand side is compact and the right-hand side is Hausdorff by (2). Thus it follows from Proposition 2.43 (3) that the map is a homeomorphism. But this implies that $\mathbb{O}\mathrm{P}^2$ admits a CW-structure with precisely three cells, one in dimensions 0, 8 and 16.

(6) Statement (6) is an immediate consequence of (5).

(7) Finally Statement (7) follows from (5) and Proposition 40.9. ■

It is natural to ask whether there exists a smooth manifold that deserves the name $\mathbb{O}\mathrm{P}^n$ for $n \geq 3$. In the proof of Proposition 60.10 we used that $n = 2$ has the property, provided by Lemma 60.3 (9), that for any $x_1, \ldots, x_n \in \mathbb{O}$ the elements $1, x_1, \ldots, x_n$ generate an associative algebra. It is straightforward to see that this statement is not true for $n \geq 3$. Nonetheless, one can wonder whether there exists an alternative way to define a meaningful higher dimensional octonionic projective space. This slightly vague question can be formulated in many different ways. Here is one possible interpretation of this question.

Question 60.11. *Let $n \geq 3$. Does there exist a closed orientable $8n$-dimensional smooth manifold that admits a CW-structure with precisely one cell in dimensions $0, 8, 16, \ldots, 8n$?*

We will answer this question in Exercise 110.2.

Proposition 60.10 says in particular that the octonion plane $\mathbb{O}\mathrm{P}^2$ is a closed orientable 16-dimensional smooth manifold that is 7-connected and that has odd Euler characteristic. The following question is, to the best of my knowledge, open:

Question 60.12. *Does there exist an 8-connected closed orientable smooth manifold with odd Euler characteristic?*

In [**Hoe17**, Theorem 1.2 together with Proposition 4.2] it is shown that the dimension of an 8-connected closed orientable smooth manifold with odd Euler characteristic, if it exists, has to be divisible by 16.

Exercises for Chapter 60.

Exercise 60.1. Let $q_1, q_2 \in S^3 = \{h \in \mathbb{H} \,|\, |h| = 1\}$. We consider the map

$$\begin{array}{rcl}\mathbb{R}^4 = \mathbb{H} & \to & \mathbb{R}^4 = \mathbb{H} \\ z & \mapsto & q_1 z q_2^{-1}.\end{array}$$

Show that the map is linear and that it is represented by a matrix in $\mathrm{SO}(4)$.

Exercise 60.2. Let $n \in \mathbb{Z}$. Show that the degree of the map

$$\begin{array}{rcl} S^3 = \{v \in \mathbb{H} \,|\, \|v\| = 1\} & \to & S^3 \\ z & \mapsto & z^n \end{array}$$

equals n.
Hint. You could use Proposition 45.23.

Exercise 60.3. Let $\mathbb{A} = \mathbb{R} \oplus \mathbb{R} \cdot \mathrm{i} \oplus \mathbb{R} \cdot \mathrm{j} \oplus \mathbb{R} \cdot \mathrm{k}$. We define the multiplication on $\mathbb{A}$ the same way as we did for the quaternions, see page 1462, except that we set $\mathrm{i}^2 = -1 + \epsilon \cdot \mathrm{j}$.

(a) Show that $\mathbb{A}$ is a division algebra.
(b) Show that i has a left-inverse and a right-inverse.
(c) Show that the left-inverse and the right-inverse of i differ for sufficiently small $\epsilon > 0$.

Exercise 60.4. Show that the octonions $\mathbb{O}$ are not associative.

Exercise 60.5. Show that the map

$$\begin{array}{rcl} \mathbb{O}\mathrm{P}^1 & \to & \mathbb{O} \cup \{\infty\} \\ [x:y] & \mapsto & \begin{cases} x \cdot y^{-1}, & \text{if } y \neq 0, \\ \infty, & \text{if } [x:y] = [1:0] \end{cases} \end{array}$$

is an injection.

Exercise 60.6. Let (x_1, y_1) and $(x_2, y_2) \in \mathbb{O}^2 \setminus \{(0,0)\}$. Suppose there exists a $\lambda \in \mathbb{R}_{>0}$ such that $x_1\overline{x_1} = \lambda \cdot x_2\overline{x_2}$ and $x_1\overline{y_1} = \lambda \cdot x_2\overline{y_2}$. Show that there exists a $\mu \in \mathbb{R}_{>0}$ such that $x_1\overline{x_1} = \mu \cdot x_2\overline{x_2}$, $x_1\overline{y_1} = \mu \cdot x_2\overline{y_2}$ and $y_1\overline{y_1} = \mu \cdot y_2\overline{y_2}$.
Hint. Consider norms.

Exercise 60.7. Show that $\mathbb{O}\mathrm{P}^2$ is Hausdorff.
Hint. First show that $\mathbb{O}\mathrm{P}^2$ is in fact homeomorphic to a suitable quotient of the sphere $S^{23} = \{(x,y,z) \in \mathbb{O}^3 \,|\, |x|^2 + |y|^2 + |z|^2 = 1\}$ and then apply Lemma 3.26.

Part VII

Simplicial Complexes

61. Simplicial complexes: Definition, examples and basic properties

In this section we introduce the notion of an abstract simplicial complex. Similar to the notion of an abstract graph these are purely combinatorial objects. The connection to topology comes from the fact that simplicial complexes admit topological realizations which can be used to combinatorially describe many interesting topological spaces. These topological realizations also lead to the notion of a simplicial complex which can be viewed as a special class of CW-complexes with particularly nice properties.

61.1. **Abstract simplicial complexes.** Before we do anything else we recall the following notation that we introduced on page 73.

Notation. Given a set X we denote by $\mathcal{P}(X)$ its *power set*, i.e. $\mathcal{P}(X)$ is the set of all subsets of X.

We continue with the following initially rather abstract definition.

Definition. An *abstract simplicial complex* is a pair (V, S) where V is a set and S is a subset of the power set $\mathcal{P}(V)$ of V such that the following conditions are satisfied:

(1) each $s \in S$ is finite and non-empty,
(2) if $s \in S$, then any non-empty subset of s also lies in S,
(3) given any $v \in V$ the set $\{v\}$ lies in S.

We refer to the elements of V as the *vertices of the abstract simplicial complex* and we refer to the elements of S as the *simplices of the abstract simplicial complex*.

Examples.

(0) The pair $(\varnothing, \varnothing)$ is an abstract simplicial complex, it is called the *empty abstract simplicial complex*. Every other abstract simplicial complex is called *non-empty*.
(1) We consider the set $U = \{A, B, C\}$. It is straightforward to verify that the pair $X := (U, S)$ with $S = \{\{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}\}$ and that the pair $Y := (U, T)$ with $T = \mathcal{P}(U) \setminus \{\varnothing\}$ are abstract simplicial complexes. We illustrate these two abstract simplicial complexes in Figure 890.[935]

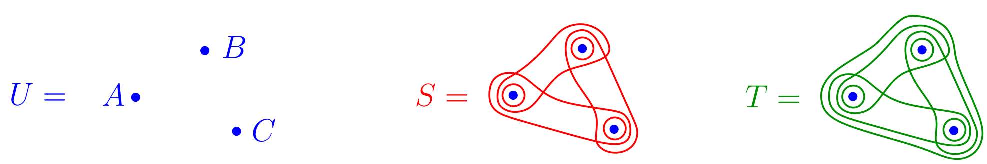


Figure 890

(2) Let $n \in \mathbb{N}$. The pairs
  (a) $D_n := (V, S)$ with $V = \{0, \ldots, n\}$ and $S = \mathcal{P}(V) \setminus \{\varnothing\}$, and
  (b) $S_{n-1} := (V, T)$ with $V = \{0, \ldots, n\}$ and $T = \mathcal{P}(V) \setminus \{\varnothing, V\}$,

  are abstract simplicial complexes.
(3) (a) Given $n \in \mathbb{N}_{\geq 3}$ the pair $R_n = (V, S)$ with
$$V := \mathbb{Z}_n \quad \text{and} \quad S := \{\{k\} \,|\, k \in \mathbb{Z}_n\} \cup \{\{k, k+1\} \,|\, k \in \mathbb{Z}_n\}$$
is an abstract simplicial complex.

[935] To simplify the figures we will not draw the "trivial" simplices that are given by the vertices.

(b) The pair $R_\infty := (V, S)$ with

$$V := \mathbb{Z} \quad \text{and} \quad S := \{\{n\} \,|\, n \in \mathbb{Z}\} \cup \{\{n, n+1\} \,|\, n \in \mathbb{Z}\}$$

is an abstract simplicial complex.

We illustrate these two abstract simplicial complexes in Figure 891.

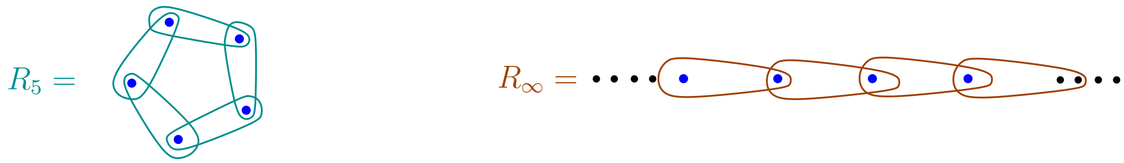


FIGURE 891

Definition. Let $K = (V, S)$ be an abstract simplicial complex.

(1) We say $K = (V, S)$ is *finite* if V is finite.[936] Similarly we say $K = (V, S)$ is *countable* if V is countable.
(2) If $s \in S$ has $k + 1$ elements, then s is called a *k-dimensional simplex*, or shorter, a *k-simplex* of (V, S).
(3) If $K = (V, S)$ is the empty abstract simplicial complex, then we define its dimension as -1. Now suppose that $K = (V, S)$ is non-empty. We define

$$\dim(K) := \max\{\#s - 1 \,|\, s \in S\} \in \mathbb{N}_0 \cup \{\infty\}.$$

In other words, the dimension of $K = (V, S)$ is the maximal dimension of a simplex of $K = (V, S)$.

Examples.

(1) The above abstract simplicial complex (U, S) is 1-dimensional and the above abstract simplicial complex (U, T) is 2-dimensional.
(2) Let $n \in \mathbb{N}$. The abstract simplicial complex D_n is n-dimensional and the abstract simplicial complex S_{n-1} is $(n-1)$-dimensional
(3) The abstract simplicial complexes R_n and R_∞ are one-dimensional.

We continue with a few more harmless definitions and statements.

Definition. Let $K = (V, S)$ be an abstract simplicial complex. Let t be a simplex. A *face of t* is a simplex s with $s \subset t$.

Lemma 61.1. *Let $K = (V, S)$ be an abstract simplicial complex. Furthermore let s and t be two simplices. Then either the intersection $s \cap t$ is empty, or the intersection $s \cap t$ is again a simplex and it is a face of s and t.*

PROOF (∗). Let s and t be two simplices of an abstract simplicial complex $K = (V, S)$. If $s \cap t = \varnothing$ then there is nothing to show. Now suppose that $s \cap t \neq \varnothing$. Evidently $s \cap t$ is a non-empty subset of s and t. By definition of an abstract simplicial complex this means that $s \cap t$ is also a simplex. Furthermore, since $s \cap t \subset s$ and $s \cap t \subset t$ the intersection is a face of s and t. ■

[936] Note that this implies that S is also finite. In fact since every $v \in V$ defines a simplex we see that an abstract simplicial complex is finite if and only if S is finite.

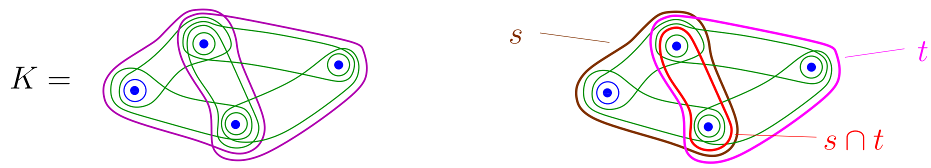


FIGURE 892. Illustration of Lemma 61.1.

Definition. Let $L = (W, T)$ be an abstract simplicial complex. An *abstract subcomplex* of $L = (W, T)$ is a pair $K = (V, S)$ with $V \subset W$, with $S \subset T$ and such that $K = (V, S)$ is an abstract simplicial complex in its own right. On most occasions we just say "subcomplex" instead of "abstract subcomplex".

Examples.

(1) Basically by definition $X = (U, S)$ is a subcomplex of $Y = (U, T)$.
(2) Let $n \in \mathbb{N}$. Again, basically by definition S_{n-1} is a subcomplex of D_n.
(3) Let $K = (V, S)$ be an abstract simplicial complex and let $n \in \mathbb{N}_0$. We refer to V together with all simplices of dimension $\leq n$ as the *n-skeleton of the abstract simplicial complex*. It is basically clear from the definitions that the n-skeleton is a subcomplex. For convenience we also write $K^\infty := K$.
(4) Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$.
   (a) The pair $(s, \mathcal{P}(s) \setminus \{\varnothing\})$ is a subcomplex of K. Often we abbreviate this subcomplex to s.
   (b) The pair $(s, \mathcal{P}(s) \setminus \{\varnothing, s\})$ is a subcomplex of K. Usually we refer to this subcomplex as ∂s.

The following lemma gives us elementary ways to construct subcomplexes.

Lemma 61.2. *Let $\{L_i = (W_i, T_i)\}_{i \in I}$ be a family of abstract simplicial complexes. The following statements hold:*

(1)
$$\bigcap_{i \in I} L_i := \Big(\bigcap_{i \in I} W_i, \bigcap_{i \in I} V_i \Big) \quad \text{and} \quad \bigcup_{i \in I} L_i := \Big(\bigcup_{i \in I} W_i, \bigcup_{i \in I} V_i \Big)$$
are abstract simplicial complexes.
(2) $\bigcap_{i \in I} L_i$ *is a subcomplex of each L_i and each L_i in turn is a subcomplex of* $\bigcup_{i \in I} L_i$.
(3) *If each L_i is a subcomplex of some abstract simplicial complex K, then* $\bigcap_{i \in I} L_i$ *and* $\bigcup_{i \in I} L_i$ *are also subcomplexes of K.*

PROOF. The lemma is close to being a tautology. ■

As the reader will know by now, not only do we want to study objects, we also want to study maps or morphisms between them. The following gives us the type of maps we are now interested in.

Definition. Let $K = (V, S)$ and $L = (W, T)$ be abstract simplicial complexes.

(1) A *simplicial map* $f: K \to L$ is defined as a map $f: V \to W$ such that for every $s \in S$ we have $f(s) \in T$. In particular a simplicial map induces a map $S \to T$.
(2) A *simplicial isomorphism* between $K = (V, S)$ and $L = (W, T)$ is a simplicial map that is a bijection on the set of vertices and on the set of simplices.

Examples.

(1) The identity map of $U = \{A, B, C\}$ defines a simplicial map from $X = (U, S)$ to $Y = (U, T)$.
(2) The map $f: \{0, 1, 2\} \to \{A, B, C\}$ given by $f(0) = A$, $f(1) = B$ and $f(2) = C$ defines a simplicial isomorphism between D_2 and (U, T) and it defines a simplicial isomorphism between S_1 and (U, S). Similarly one sees that the abstract simplicial complex R_3 is simplicially isomorphic to S_1 and (U, S).
(3) The inclusion of a subcomplex into an abstract simplicial complex is, basically by definition, a simplicial map.

It is clear that the composition of simplicial maps is again a simplicial map. Thus it makes sense to introduce the following definition.

Definition. We call the category $\mathcal{AbsSimpCplx}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{AbsSimpCplx}) &:= \text{ all abstract simplicial complexes,} \\ \mathrm{Mor}(K, L) &:= \text{ all simplicial maps from } K \text{ to } L, \end{aligned}$$

together with the usual composition of maps the *category of abstract simplicial complexes.*

For future reference let us record the following lemma.

Lemma 61.3. *Let K and L be two abstract simplicial complexes.*

(1) *If K and L are finite, then there exist only finitely many simplicial maps from K to L.*
(2) *If K is finite and if L is countable, then there exist only countably many simplicial maps from K to L.*

Proof.

(1) This statement is obvious.
(2) Let $K = (V, S)$ and $L = (W, T)$ be two abstract simplicial complexes. We assume that K is finite and that L is countable. It follows from Lemma 1.7 (6) that there exist only countably many maps $V \to W$. Furthermore, since V is finite we see that given any map $V \to W$ there are only finitely many maps $S \to T$ such that the resulting map is a simplicial map. It follows from Lemma 1.7 (3) that there are indeed only countably many simplicial maps. ■

Remark. The reader might have a certain sense of déjà vu: the definition of a 1-dimensional abstract simplicial complex sounds very similar to the definition of an undirected abstract graph which we gave on page 226. In fact some thought shows that the concepts are different since undirected graphs are more flexible. But once we rule out loops and multi-edges (see Figure 893 for a pictorial reminder of the definitions), then the concepts are basically the same. More precisely it follows easily from the definitions that the maps

$$\underbrace{(V,E,\varphi\colon E\to\mathcal{P}(V))}_{\substack{\text{undirected abstract graph}\\ \text{without loops \& multi-edges}}} \longmapsto \underbrace{(V,\varphi(E))}_{\substack{\text{1-dimensional abstract}\\ \text{simplical complex}}} \quad\text{and}\quad \underbrace{(V,S\subset\mathcal{P}(V))}_{\substack{\text{1-dimensional abstract}\\ \text{simplical complex}}} \longmapsto \underbrace{(V,S,\mathrm{id}\colon S\to\mathcal{P}(V))}_{\substack{\text{undirected abstract graph}\\ \text{without loops \& multi-edges}}}$$

define an isomorphism between the "category of undirected abstract graphs without loops and multi-edges" and the "category of 1-dimensional simplicial complexes".

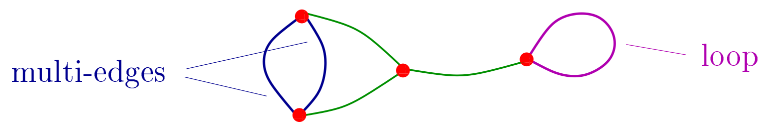


FIGURE 893

61.2. **Ordered abstract simplicial complexes.** We recall the following definition from page 77.

Definition.

(1) A relation that is reflexive, transitive and antisymmetric is called a *partial order*.
(2) A *partially ordered set* is a set together with a partial order.
(3) A partial order that is connex is called a *total order*.
(4) A *totally ordered set* is a set together with a total order.
(5) In a totally ordered set $(P,\leq)$ we write $a<b$ if $a\leq b$ and $a\neq b$.

Examples.

(1) Let X be a set. We consider the power set $\mathcal{P}(X)$. The relation "$\subset$" is a partial order on $\mathcal{P}(X)$.
(2) The usual relation "$\leq$" on $\mathbb{Z}$ turns $\mathbb{Z}$ into a totally ordered set.
(3) Any subset of $\{A,B,\dots,Z\}$ with the usual alphabetical ordering is a totally ordered set.

We continue with the following definition.

Definition.

(1) A *total order* on an abstract simplicial complex (V,S) is a total order on the vertex set S.
(2) An *order* on an abstract simplicial complex (V,S) is a partial order on V which has the property that the restriction to each simplex is actually a total order.
(3) An *ordered abstract simplicial complex* is an abstract simplicial complex together with an order.

Remark.

(1) Every totally ordered abstract simplicial complex is automatically also an ordered simplicial complex.
(2) By the Well-ordering Theorem 1.6 we know that every set admits a total ordering. Thus every abstract simplicial complex can be turned into a totally ordered abstract simplicial complex.

The following definition is the obvious variation on the definition provided on page 1483.

Definition. We call the category $\mathcal{C}$ with

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{C}) & := & \text{all ordered abstract simplicial complexes,} \\ \mathrm{Mor}(K,L) & := & \text{all order-preserving simplicial maps from } K \text{ to } L, \end{array}$$

together with the usual composition of maps the *category* $\mathcal{OrdAbsSimpCplx}$ *of ordered abstract simplicial complexes.*

61.3. **The topological realization of an abstract simplicial complex.** Since we are topologists we want to study abstract simplicial complexes with topological methods. The idea is to associate to an abstract simplicial complex a corresponding topological space. Before we can do so we need to introduce the following notation which is inspired by the definition of the free abelian group $\mathbb{Z}^{(S)}$ that we gave on page 580.

Notation.

(1) Given any set V we write

$$\begin{array}{rcl} \mathbb{R}^V & := & \text{all maps from } V \text{ to } \mathbb{R} \\ \mathbb{R}^{(V)} & := & \text{all maps from } V \text{ to } \mathbb{R} \text{ which are non-zero for only finitely many } v \in V. \end{array}$$

We have the following conventions and observations:

(a) Given $v \in V$ we sometimes denote by v also the map $V \to \mathbb{R}$ which is given by $v \mapsto 1$ and $w \mapsto 0$ for $w \neq v$.

(b) One can easily verify that with this interpretation the set V forms a basis for the real vector space $\mathbb{R}^{(V)}$.

(c) If $V = \{v_1, \dots, v_n\}$ is a finite set, then the map $v_i \mapsto e_i$ defines an isomorphism from $\mathbb{R}^V = \mathbb{R}^{(V)}$ to $\mathbb{R}^n$. In particular, if V is a finite set with a total order, then it follows from Exercise 1.1 that we have a natural isomorphism $\mathbb{R}^V \xrightarrow{\cong} \mathbb{R}^{|V|}$.

(2) If V is a subset of some set W, then we view $\mathbb{R}^{(V)}$ as a subset of $\mathbb{R}^{(W)}$ in the obvious way.

The following definition is the first step in our shift back to topology.

Definition. Let $K = (V,S)$ be some abstract simplicial complex. For a given n-simplex $s = \{v_0, \dots, v_n\}$ we define

$$|s| := \left\{ \alpha \in \mathbb{R}^{(V)} \,\middle|\, \begin{array}{l} (1)\ \alpha(v) \geq 0 \text{ for all } v \in s \\ (2)\ \alpha(v) = 0 \text{ for all } v \notin s \\ (3)\ \text{we have } \sum\limits_{v \in V} \alpha(v) = 1 \end{array} \right\} = \left\{ \underbrace{\sum_{i=0}^{n} t_i \cdot v_i}_{\in \mathbb{R}^{(V)}} \,\middle|\, \begin{array}{l} (1)\ \text{for each } i \text{ we have } t_i \geq 0 \\ (2)\ t_0 + \dots + t_n = 1 \end{array} \right\}$$

Furthermore we set

$$|K| := \bigcup_{s \in S} |s| \subset \mathbb{R}^{(V)}.$$

Sometimes, by a serious abuse of language, we refer to $|s|$ as a *n-simplex in $|K|$.*

Examples.

(1) We consider the abstract simplicial complexes $X = (U, S)$ and $Y = (U, T)$ as above. The corresponding sets $|X|$ and $|Y|$ are the subsets of $\mathbb{R}^U = \mathbb{R}^{\{A,B,C\}} = \mathbb{R}^3$ that are shown in Figure 894.

(2) Let $n \in \mathbb{N}$. Given the abstract simplicial complex R_n the corresponding set $|R_n|$ is a subset of $\mathbb{R}^{\mathbb{Z}_n} = \mathbb{R}^{n-1}$. This means that for $n \geq 5$ we have serious problems with visualizing the set $|R_n|$. The problem is even bigger for the infinite abstract simplicial complex R_∞.

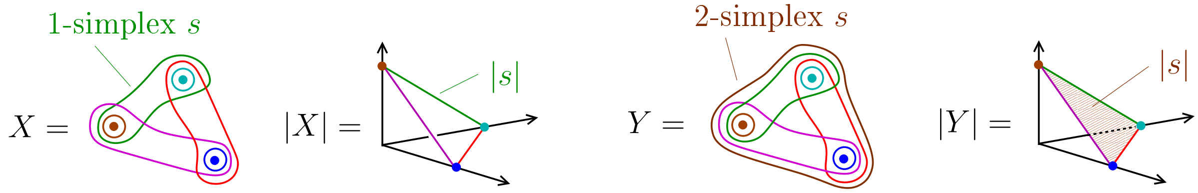


Figure 894

Lemma 61.4. *Given any abstract simplicial complex $K = (V, S)$ we have*

$$|K| \; = \; \left\{ \alpha \in \mathbb{R}^{(V)} \;\middle|\; \begin{array}{ll} (1) & \textit{the set } \{v \in V \,|\, \alpha(v) \neq 0\} \textit{ is a simplex of } K \\ (2) & \textit{for every } v \in V \textit{ we have } \alpha(v) \geq 0 \\ (3) & \textit{we have } \sum\limits_{v \in V} \alpha(v) = 1 \end{array} \right\}.$$

Proof. We leave the elementary task of proving the equality to the reader. Note though that one actually needs to use that $K = (V, S)$ is an abstract simplicial complex. ■

Note that as of right now $|K|$ is just a set. But rest assured, we will equip $|K|$ with a suitable topology shortly.[937] But first let us introduce a convention.

Convention. Let $K = (V, S)$ be an abstract simplicial complex. As discussed above, we can view V as a subset of $\mathbb{R}^{(V)}$. In fact it follows easily from the definitions that $V \subset |K|$. Throughout these notes we use this observation to view V as a subset of $|K|$.

We continue with the following definition.

Definition. Let $K = (V, S)$ be an abstract simplicial complex. Let $k \in \mathbb{N}_0$ and let s be a k-simplex.

(1) If $v_0 \leq \cdots \leq v_k$ is a total order on the set of vertices of s, then we refer to the map

$$\begin{array}{rcl} \Phi_s \colon \Delta^k \; = \; \left\{ (t_0, \ldots, t_k) \in \mathbb{R}^{k+1}_{\geq 0} \,\middle|\, \sum\limits_{i=0}^{k} t_i = 1 \right\} & \to & |K| \\ (t_0, \ldots, t_k) & \mapsto & \sum\limits_{i=0}^{k} t_i \cdot v_i \end{array}$$

as a *characteristic map of the simplex s*.

[937] A first idea might be to equip $\mathbb{R}^{(V)}$ with a suitable topology and then to endow $|K|$ with the subspace topology. But this is not really a good idea since this topology is artificial and does not reflect the structure of K. For example it would be rather unpleasant to have to show that a map out of K is continuous.

(2) If K is actually an ordered abstract simplicial complex, then we use the corresponding order on s and we write $\Phi_s^{\leq}\colon \Delta^k \to |K|$. In this case we can refer to $\Phi_s^{\leq}$ unambiguously as *the* characteristic map of the simplex s.

Remark.

(1) Note that if in the above definition of a characteristic map we use two different orderings of the vertices of s, then we get two maps $\Phi, \Psi\colon \Delta^k \to |K|$ with the same image and such that $\Psi^{-1} \circ \Phi\colon \Delta^k \to \Delta^k$ is a homeomorphism. In other words, any two characteristic maps differ by precomposing by a homeomorphism. In practice this means that any two characteristic maps exhibit the same properties and we do not need to distinguish between different characteristic maps.
(2) Given a k-simplex s we have, by definition, $\Phi_s(\Delta^k) = |s|$.

Definition. Let $K = (V, S)$ be an abstract simplicial complex.

(1) We say that a subset $U \subset |K|$ is open if for every simplex s of K the preimage $\Phi_s^{-1}(U)$ under a characteristic map[938] Φ_s is open. An argument as in Exercise 2.35 shows that this defines a topology on $|K|$.
(2) We refer to the set $|K|$, together with this topology, as the *topological realization* of K.

The following lemma summarizes a few basic properties of the topological realization of an abstract simplicial complex.

Lemma 61.5. *Let K be an abstract simplicial complex.*

(1) *A subset $A \subset |K|$ is closed if and only if for every simplex s of K the preimage $\Phi_s^{-1}(A)$ under a characteristic map Φ_s is closed.*
(2) *The topological realization $|K|$ is Hausdorff.*
(3) *For every k-simplex $s = \{v_0, \dots, v_k\}$ of K any characteristic map $\Phi_s\colon \Delta^k \to |K|$ is a closed embedding. In particular the map $\Phi_s\colon \Delta^k \to |s|$ is a homeomorphism.*

Proof. Let $K = (V, S)$ be an abstract simplicial complex.

(1) This statement follows almost immediately from the definition of the topology on $|K|$ together with the elementary Lemma 1.3 (7).
(2) Let P and Q be two disjoint points on $|K|$. Given $f, g \in \mathbb{R}^{(V)}$ we write

$$d(f,g) \ := \ \sqrt{\sum_{v\in V} (f(v) - g(v))^2}.$$

One can easily verify that this is a metric on $\mathbb{R}^{(V)}$. By Proposition 2.11 we know that there exists an $\epsilon > 0$ such that the open ϵ-balls $B_\epsilon(P)$ and $B_\epsilon(Q)$ are disjoint in $\mathbb{R}^{(V)}$. It remains to show that $B_\epsilon(P) \cap |K|$ and $B_\epsilon(Q) \cap |K|$ are open in $|K|$. By definition of the topology on $|K|$ we need to prove the following claim.

Claim. Let $U := B_\epsilon(R)$ be an ϵ-ball in $\mathbb{R}^{(V)}$. For every simplex s the preimage $\Phi_s^{-1}(U \cap |K|)$ under a characteristic map Φ_s is open.

[938] It follows from the above remark that the statement that $\Phi_s^{-1}(U)$ is open does not depend on the choice of the characteristic map for the simplex s.

Let $s = \{v_0, \dots, v_k\}$ be a k-simplex. We make the following observations:

(a) We have $\Phi_s^{-1}(U \cap |K|) = \Phi_s^{-1}(U)$.

(b) It follows from (a) and the observation that U is an open subset of the topological space $(\mathbb{R}^{(V)}, d)$ that it suffices to show that the map $\Phi_s \colon \Delta^k \to (\mathbb{R}^{(V)}, d)$ is continuous.

(c) We view $\Phi_s \colon \Delta^k \to \mathbb{R}^{(V)}$ as the composition of the maps

$$\Delta^k \xrightarrow{\Phi_s} \mathbb{R}^{k+1} = \mathbb{R}^{(s)} \hookrightarrow \mathbb{R}^{(V)}.$$

We consider the two maps separately.

(i) Note that $\mathbb{R}^{k+1} = \mathbb{R}^{(s)}$ is a finite-dimensional vector space and that the map $\Phi_s \colon \Delta^k \to \mathbb{R}^{(s)}$ is an affine linear map. It follows almost immediately from Lemma 41.1 that this map is continuous.

(ii) It follows easily from the definition of topology on $\mathbb{R}^{k+1}$, via the usual Euclidean metric, and the definition of the metric d on $\mathbb{R}^{(V)}$, together with the simple-minded Exercise 2.14 (a), that the map $\mathbb{R}^{k+1} = \mathbb{R}^{(s)} \to \mathbb{R}^{(V)}$ is continuous.

Thus we see that the map $\Phi_s \colon \Delta^k \to (\mathbb{R}^{(S)}, d)$ itself is continuous. ⊞

(3) Let s be a k-simplex of K. It follows almost immediately from the definitions that the map $\Phi_s \colon \Delta^k \to |K|$ defines a bijection $\Phi_s \colon \Delta^k \to |s|$. It follows immediately from the definition of the topology on $|K|$ that Φ_s is continuous. Evidently Δ^k is compact. By (1) we know that $|K|$ is Hausdorff. So once again Proposition 2.43 (2) rides to our rescue and tells us that Φ_s is indeed an embedding and that the image is a closed subset. ■

The following is the simplicial analogue of the corresponding Lemma 36.7 for CW-complexes.

Lemma 61.6. *Let $K = (V, S)$ be an abstract simplicial complex and let $f \colon |K| \to X$ be a map to some topological space X.*

(1) *If for each k-simplex $s \in S$ with characteristic map $\Phi_s \colon \Delta^k \to |K|$ the composition $f \circ \Phi_s \colon \Delta^k \to X$ is continuous, then the map f is continuous.*

More generally we have the following continuity criterion:

(2) *If for each simplex $s \in S$ there exists a k-simplex t with $s \subset t$ such that for some characteristic map $\Phi_t \colon \Delta^k \to |K|$ the composition $f \circ \Phi_t \colon \Delta^k \to X$ is continuous, then the map f is continuous.*

Proof.

(1) This statement follows immediately from the definition of the topology on $|K|$.

(2) This statement follows immediately from (1) and the observation that characteristic maps are continuous. ■

The following proposition shows that not only do abstract simplicial complexes give rise to topological spaces, but simplicial maps also give rise to continuous maps between the topological realizations.

Proposition 61.7.

(1) *If $f\colon K=(V,S)\to L=(W,T)$ is a simplicial map, then the map*

$$\begin{array}{rcl} |f|\colon |K| & \to & |L| \\ (\alpha\colon V\to\mathbb{R}) & \mapsto & \left(\begin{array}{rcl} W & \to & \mathbb{R} \\ w & \mapsto & \sum\limits_{v\in f^{-1}(w)}\alpha(v)\end{array}\right)\end{array}$$

is well-defined and continuous. We refer to Figure 895 for an illustration.

(2) *Let $\mathcal{AbsSimpCplx}$ be the category of abstract simplicial complexes. The maps*

$$\begin{array}{rcl} K & \mapsto & |K| \\ (f\colon K\to L) & \mapsto & (|f|\colon |K|\to|L|)\end{array}$$

define a covariant functor from the category $\mathcal{AbsSimpCplx}$ of abstract simplicial complexes to the category $\mathcal{Top}$ of topological spaces.

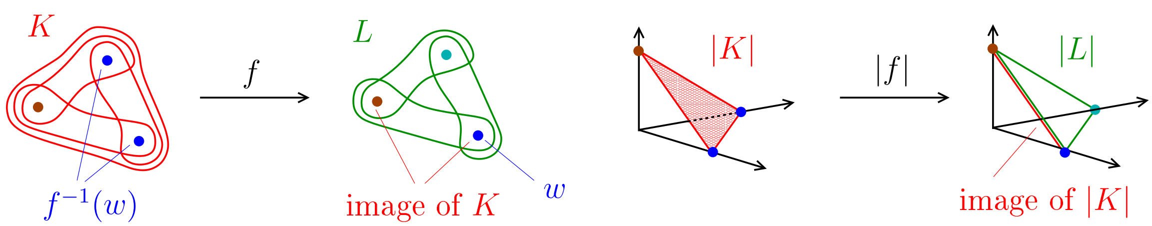


FIGURE 895

Remark.

(1) Let $f\colon K=(V,S)\to L=(W,T)$ be a simplicial map. It is elementary to see that, given a simplex $s=\{v_0,\dots,v_k\}$ of K, the restriction of $|f|$ to $|s|\subset|K|$ is the map

$$\begin{array}{rcl} |f|\colon |s| & \to & |f(s)| \\ \sum\limits_{i=0}^{k} t_i\cdot v_i & \mapsto & \sum\limits_{i=0}^{k} t_i\cdot f(v_i).\end{array}$$

(2) As we will see, the proof of Proposition 61.7 (1) is slightly delicate since the topology of the topological realization of a simplicial complex is designed, see Lemma 61.6, in such a way that it is straightforward to verify that a map *out* of the topological realization is continuous. It is much less clear how one can show that a map *to* a topological realization is continuous. The same health-warning applies also to CW-complexes.

PROOF.

(1) Let $f\colon K=(V,S)\to L=(W,T)$ be a simplicial map. A priori we have written down a map $|f|\colon|K|\to\mathbb{R}^{(W)}$. But it is straightforward to verify, e.g. using Lemma 61.4, that the map $|f|$ does indeed take values in the subset $|L|$. It remains to show that $|f|\colon|K|\to|L|$ is continuous. By Lemma 61.6 (1) it suffices to show that for each m-simplex $s\in S$ with some characteristic map $\Phi_s\colon\Delta^m\to|K|$ the composition $f\circ\Phi_s\colon\Delta^m\to|L|$ is continuous.

Thus suppose we are given such s and Φ_s. We write $t := f(s)$. By definition of a simplicial map we know that t is some n-simplex of T. For clarity we consider the following commutative diagram:

$$\begin{tikzcd}
 & {|K|} \arrow[r, "{|f|}"] & {|L|} \\
\Delta^m \arrow[ur, bend left, "\Phi_s"] \arrow[r, "\Phi_s", "\cong"'] \arrow[drr, bend right, "\Phi_t^{-1}\circ|f|\circ\Phi_s"'] & {|s|} \arrow[u, hook, "i"] \arrow[r, "{|f|}"] & {|t|} \arrow[u, hook, "j"'] \\
 & & \Delta^n. \arrow[u, "\Phi_t", "\cong"']
\end{tikzcd}$$

We make the following clarifications and observations:

(a) We define $|s| \subset |K|$ and $|t| \subset |L|$ as in Lemma 61.5 (3). We equip both with the subspace topology coming from $|K|$ and $|L|$. By the super elementary Lemma 2.30 (2) we know that the inclusion maps $i\colon |s| \to |K|$ and $j\colon |t| \to |L|$ are continuous.

(b) It follows immediately from the definition of $|f|$ that the restriction of $|f|$ to $|s|$ takes values in $|t|$.

(c) We picked a characteristic map $\Phi_t\colon \Delta^n \to |t|$ for t.

(d) By Lemma 61.5 (3) we know that the characteristic map $\Phi_t\colon \Delta^n \to |t|$ is a homeomorphism. In particular the inverse $\Phi_t^{-1}\colon |t| \to \Delta^n$ exists and it is continuous.

(e) It follows almost immediately from the definitions of the maps Φ_s, Φ_t and $|f|$ that the map $\Phi_t^{-1} \circ |f| \circ \Phi_s\colon \Delta^m \to \Delta^n$ is an affine linear map. Hence it follows, say from Lemma 41.1, that the map is continuous.

(f) It follows immediately from the above that the map

$$|f| \circ \Phi_s \;=\; j \circ \Phi_t \circ (\Phi_t^{-1} \circ |f| \circ \Phi_s)\colon \Delta^m \;\to\; Y$$

is indeed continuous.

(2) Let $f\colon K \to L$ and $g\colon L \to M$ be simplicial maps. For any vertex v of M we have $f^{-1}(g^{-1}(v)) = (g \circ f)^{-1}(v)$. It follows immediately from this observation that $|g| \circ |f| = |g \circ f|$. It is now straightforward to see that the given maps define a covariant functor. ■

The following lemma can be viewed as an analogue of Lemma 36.18.

Lemma 61.8. *Let $K = (V, S)$ be a subcomplex of some given abstract simplicial complex $L = (W, T)$.*

(1) *The inclusion $\mathbb{R}^{(V)} \to \mathbb{R}^{(W)}$ restricts to an inclusion $|K| \to |L|$.*

(2) *The inclusion $|K| \to |L|$ is a closed embedding.*

(3) *Let $\{K_i\}_{i\in I}$ be a family of subcomplexes of an abstract simplicial complex K. Then*

$$\Big|\bigcap_{i\in I} K_i\Big| \;:=\; \bigcap_{i\in I} |K_i| \quad \text{and} \quad \Big|\bigcup_{i\in I} K_i\Big| \;:=\; \bigcup_{i\in I} |K_i|.$$

are subcomplexes of K.

Remark. The statement and the proof of Lemma 61.8 is closely related to the content of Lemma 36.18.

Convention. Let $K = (V, S)$ be a subcomplex of some given abstract simplicial complex $L = (W, T)$. We will use Lemma 61.8 to view $|K|$ as a subset of $|L|$. By Lemma 61.8 (2) we know that $|K|$ is in fact a closed subset of $|L|$.

Proof. Let $L = (W, T)$ be a simplicial complex and let $K = (V, S)$ be a subcomplex. We denote by $i\colon K \to L$ the inclusion map.

(1) It follows immediately from the definitions that the inclusion $\mathbb{R}^{(V)} \to \mathbb{R}^{(W)}$ restricts to an inclusion $|K| \to |L|$.
(3) This statement also follows immediately from the definitions. For readers who want to fill in the details, perhaps it is clearest to look at the description given in Lemma 61.4.
(2) Since $i\colon K \to L$ is a simplicial map we obtain from Proposition 61.7 (1) that the induced map $|i|\colon |K| \to |L|$ is continuous. Furthermore it follows easily from the definitions that the map $|i|\colon |K| \to |L|$ is in fact just the inclusion. In the remainder of the proof we will view $|K|$ as a subset of $|L|$.

    By Lemma 2.42 (1) it remains to show that the inclusion $|K| \to |L|$ is a closed map. Thus let $A \subset |K|$ be a closed subset. We need to show that A is also a closed subset of $|L|$. By Lemma 61.5 (1) and (3) it suffices to show that for each $t \in T$ the set $A \cap |t|$ is closed. Note that every simplex, in particular t itself, has only finitely many faces. Let $f_1, \dots, f_m$ be the faces of t that are contained in K. We consider the subsets $|f_1|, \dots, |f_m|$ of $|K|$ and the subset $|t|$ of $|L|$ as defined on page 1485. We make the following observation:

    ($*$) It follows from Lemma 61.5 that the subspace topologies on $|f_i|$ coming from $|f_i| \subset |K|$ and $|f_i| \subset |t| \subset |L|$ agree and that $|f_i|$ is a closed subset of $|K|$ and a closed subset of $|t|$.

    We have

$$
\begin{aligned}
A\cap|t| &\underset{\substack{\uparrow\\ \text{since } A\subset|K|}}{=} A\cap(|K|\cap|t|) \underset{\substack{\uparrow\\ \text{this follows from (3) since } K \text{ is a subcomplex}}}{=} A\cap\bigcup_{i=1}^{m}|f_i| = \bigcup_{i=1}^{m}(A\cap|f_i|)\\
&\underset{\substack{\uparrow\\ \text{since } A \text{ is closed in } |K| \text{ and since}\\ f_1,\dots,f_m\in S \text{ are simplices of } K}}{=} \bigcup_{i=1}^{m}\text{closed subset of } |f_i| \underset{\substack{\uparrow\\ \text{by Lemma 2.38 (2) since by Lemma 61.5 (3) we know}\\ \text{that the } |f_i| \text{ are closed subsets of } |t|}}{=} \bigcup_{i=1}^{m}\text{closed subset of } |t| = \text{closed subset of } |t|.
\end{aligned}
$$

■

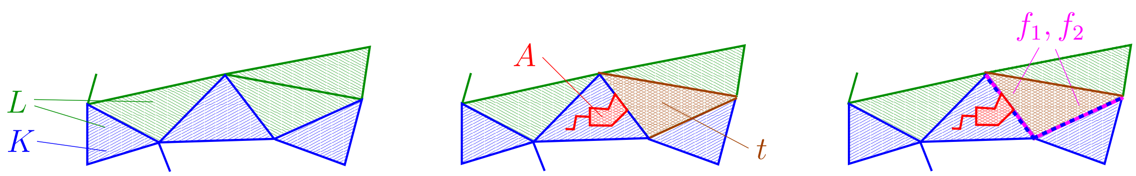


Figure 896. Illustration for the proof of Lemma 61.8.

Example. Let $n \in \mathbb{N}$. We consider again the simplicial complex D_n and its subcomplex S_{n-1}. It follows almost immediately from Lemma 61.5 (3) that the map

$$\begin{array}{rcl} \Psi\colon |D_n| & \to & \Delta^n \\ \sum\limits_{i=0}^{n} t_i \cdot \{i\} & \mapsto & (t_0, \dots, t_n) \end{array}$$

is a homeomorphism. By Proposition 61.7 (2) the map $|S_{n-1}| \to |D_n|$ is an embedding. Thus Ψ restricts to a homeomorphism $\Psi|_{|S_{n-1}|}\colon |S_{n-1}| \to \Psi(|S_{n-1}|)$ but basically by definition we have $\Psi(|S_{n-1}|) = \partial\Delta^n$. This shows that $|S_{n-1}|$ is homeomorphic to $\partial\Delta^n$.

Before we consider some examples we want to consider the topology for finite abstract simplicial complexes in greater detail. We recall that given any finite set X the vector space $\mathbb{R}^{(X)}$ is a finite-dimensional vector space. Furthermore recall that by the discussion on page 116 we know that any finite-dimensional real vector space has a natural topology.

Lemma 61.9. *Let $K = (V, S)$ be an abstract simplicial complex. If K is finite, then the following two conclusions hold:*

(1) *The topological realization $|K|$ is compact.*
(2) *The topology of the topological realization $|K|$, defined on page 1487, agrees with the subspace topology which we obtain from viewing $|K|$ as a subset of the finite-dimensional real vector space $\mathbb{R}^V$.*

Proof. Let $K = (V, S)$ be a finite abstract simplicial complex.

(1) By Lemma 61.5 we know in particular that for each k-simplex s the characteristic map $\Phi_s\colon \Delta^k \to |K|$ is continuous. Since Δ^k is compact we obtain from Lemma 2.40 that $|s| = \Phi_s(\Delta^k)$ is compact. By definition $|K|$ is the union of all the subsets $|s|$. Thus, since K is finite we see that $|K|$ is the union of finitely many compact subsets. By Lemma 2.16 (1) this implies that $|K|$ itself is compact.
(2) We write $V = \{v_1, \dots, v_n\}$ and we identify the vector spaces $\mathbb{R}^V$ and $\mathbb{R}^n$ in the obvious way. We denote by $\mathcal{T}$ the topology on $|K|$ defined above and we denote by $\mathcal{R}$ the subspace topology which comes from $\mathbb{R}^V$. We need to show that the identity map defines a homeomorphism $(|K|, \mathcal{T}) \to (|K|, \mathcal{R})$. We make the following observations:
   (a) For any k-simplex $s \in S$ and any characteristic map $\Phi_s\colon \Delta^k \to |s|$ the corresponding map $\Delta^k \xrightarrow{\Phi_s} |s| \to |K| \to \mathbb{R}^V = \mathbb{R}^n$ is an affine linear map. It follows from Lemma 41.1 that this map is continuous. It follows from this observation together with Lemma 61.6 (1) that the map $(|K|, \mathcal{T}) \to (|K|, \mathcal{R})$ is continuous.
   (b) In (1) we just argued that $(|K|, \mathcal{T})$ is compact.
   (c) The topological space $(|K|, \mathcal{R})$ is evidently Hausdorff.

   It follows from Proposition 2.43 (3) that the identity map is indeed a homeomorphism. ■

As we already noted above, the topological realizations of abstract simplicial complexes have a voracious appetite for dimensions. Fortunately, as well see shortly, in many cases we can reduce the dimensions drastically.

Lemma 61.10. *Let $K = (V, S)$ be a finite abstract simplicial complex and let $\varphi\colon \mathbb{R}^V \to \mathbb{R}^n$ be a linear map such that the restriction of φ to $|K|$ is an injection. Then $\varphi\colon |K| \to \mathbb{R}^n$ is an embedding.*

PROOF. By Lemma 61.9 (1) we know that $|K|$ is compact. Furthermore, note that it follows from Lemma 61.9 (2) that the linear map $\varphi\colon \mathbb{R}^V \to \mathbb{R}^n$ restricts to a continuous map $\varphi\colon |K| \to \mathbb{R}^n$. Since $\mathbb{R}^n$ is Hausdorff it follows from Proposition 2.43 (2) that $\varphi\colon |K| \to \mathbb{R}^n$ is indeed an embedding. ■

Example.

(1) Let $n \in \mathbb{N}_{\geq 3}$. As above we consider the abstract simplicial complex R_n. We consider the linear map $\varphi\colon \mathbb{R}^{\mathbb{Z}_n} \to \mathbb{R}^2$ that is given by $[k] \mapsto e^{2\pi i k/n}$. One can easily verify that the restriction of φ to $|R_n|$ is an injection. Thus we see that $|R_n|$ is homeomorphic to the regular n-gon that is shown in Figure 897.

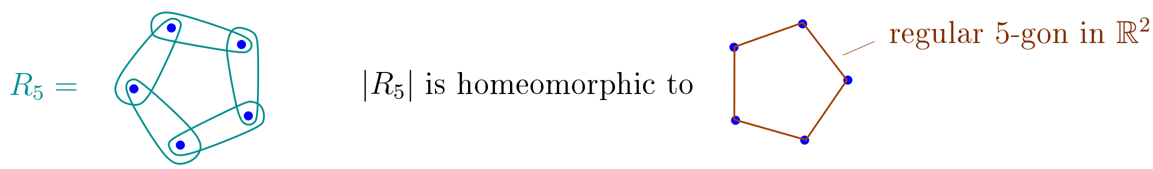


FIGURE 897

(2) Using Proposition 2.45 one can formulate a suitable generalization of Lemma 61.10 to infinite abstract simplicial complexes. We will not work this out in detail, but using this approach one can fairly easily show that the topological realization of the abstract simplicial complex R_∞ is homeomorphic to $\mathbb{R}$. We refer to Figure 898 for a very convincing illustration of this fact.

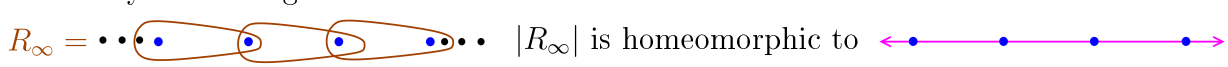


FIGURE 898

Given an n-simplex s of an abstract simplicial complex $K = (V, S)$ we defined on page 1487 the corresponding subset $|s| \subset |K|$, which, somewhat dangerously, we referred to as an n-simplex of $|K|$. The following definition gives us a convenient variation on that definition.

Definition. Let $K = (V, S)$ be an abstract simplicial complex. If we are given an n-simplex $s = \{v_0, \dots, v_n\} \in S$, then we refer to[939]

$$\langle s \rangle := \left\{ \alpha \in \mathbb{R}^{(V)} \;\middle|\; \begin{array}{l} (1)\ \alpha(v) > 0 \text{ for all } v \in s \\ (2)\ \alpha(v) = 0 \text{ for all } v \notin s \\ (3) \text{ we have } \sum\limits_{v \in V} \alpha(v) = 1 \end{array} \right\} = \left\{ \underbrace{\sum_{i=0}^{n} t_i \cdot v_i}_{\in \mathbb{R}^{(V)}} \;\middle|\; \begin{array}{l} (1) \text{ for each } i \text{ we have } t_i > 0 \\ (2)\ t_0 + \dots + t_n = 1 \end{array} \right\}$$

as the corresponding *open n-simplex* of $|K|$.

The following lemma summarizes a few properties of open simplices.

[939]Here we use the convention of page 1486 which allows us to view V as a subset of $\mathbb{R}^{(V)}$.

simplicial complex — the corresponding open simplices

FIGURE 899

Lemma 61.11. *Let $K = (V, S)$ be an abstract simplicial complex.*

(1) *Let $s \in S$ be a k-simplex.*
   (a) *The closure of the open simplex $\langle s \rangle$ is given by $|s|$.*
   (b) *Every characteristic map $\Phi_s \colon \Delta^k \to |s|$ restricts to a homeomorphism $\mathring{\Delta}^k \to \langle s \rangle$.*

(2) *The topological realization $|K|$ is the disjoint union of the open simplices, i.e.*
$$|K| \;=\; \coprod_{s \in S} \langle s \rangle.$$
Put differently, given any $\alpha \in |K|$ there exists a unique simplex s with $\alpha \in \langle s \rangle$.

(3) *For two distinct simplices $s \neq t$ we have $\langle s \rangle \cap \langle t \rangle = \varnothing$.*

(4) *The union of the open simplices is precisely $|K|$.*

(5) *If $J = (U, R)$ is a subcomplex of K, then*
$$|J| \;=\; \coprod_{r \in R} \langle r \rangle \;=\; \bigcup_{r \in R} |r| \qquad \textit{and for any } s \in S \setminus R \textit{ we have } |J| \cap \langle s \rangle = \varnothing.$$
In particular, if s is a simplex of K with $\langle s \rangle \cap |J| \neq \varnothing$, then s is a simplex of J which implies that $|s| \subset |J|$.

(6) *Let s and t be simplices. If $|s| \subset |t|$, then $s \subset t$. In particular, either* $\dim(s) < \dim(t)$ *or $s = t$.*

(7) *Let $s \in S$ be a simplex. If $U \subset |K|$ is an open subset with $U \cap |s| \neq \varnothing$, then $U \cap \langle s \rangle \neq \varnothing$.*

PROOF ($*$).

(1) Let $s \in S$ be a k-simplex and let $\Phi_s \colon \Delta^k \to |s|$ be a characteristic map. By Lemma 61.5 (3) we know that the map $\Phi_s \colon \Delta^k \to |s|$ is a homeomorphism and that $|s|$ is a closed subset of $|K|$. By definition we have $\langle s \rangle \subset |s|$.
   (b) It is clear that Φ_s restricts to a bijection $\mathring{\Delta}^k \to \langle s \rangle$. Thus it follows immediately that Φ_s restricts to a homeomorphism $\mathring{\Delta}^k \to \langle s \rangle$.
   (a) Since $|s|$ is a closed subset of $|K|$ we see that the closure of $\langle s \rangle$ in $|K|$ is contained in $|s|$. It is elementary to see that Δ^k is the closure of $\mathring{\Delta}^k$. Since $\Phi_s \colon \Delta^k \to |s|$ is a homeomorphism it follows almost immediately that $|s| = \Phi_s(\Delta^k)$ is indeed the closure of $\langle s \rangle = \Phi_s(\mathring{\Delta}^k)$.

(2) Let $\alpha \in |K|$. We set $s := \{v \in V \mid \alpha(v) \neq 0\}$. By definition of $|K|$ we know that s is a simplex. Evidently $\alpha \in \langle s \rangle$ and evidently s is unique.

(3) This statement follows immediately from (1).

(4) Conveniently enough this statement also follows immediately from (1).

(5) Let J be a subcomplex of K. If $r \in R$, then evidently we have $\langle r \rangle \subset |r| \subset |J|$. Keeping (1) in mind we see that it remains to show that if we have $s \in S$ with $|J| \cap \langle s \rangle \neq \varnothing$, then $s \in R$. Thus let s be a simplex of K with $|J| \cap \langle s \rangle \neq \varnothing$. It follows from (1), applied to the abstract simplicial complex $J = (U, R)$, that there exists a simplex r of J with $\langle s \rangle \cap \langle r \rangle \neq \varnothing$. But since r is also a simplex of K we obtain from (1), applied to K, that $s = r \in R$.

(6) Let s and t be simplices with $|s| \subset |t|$. We apply (4) to the subcomplex that is defined by s. We immediately obtain that $s \subset t$. If $s \subsetneq t$, then by definition we have $\dim(s) = \#S - 1 < \#T - 1 = \dim(t)$.

(7) Let $s \in S$ be a k-simplex and let $U \subset |K|$ be an open subset with $U \cap |s| \neq \varnothing$. Let $\Phi_s \colon \Delta^k \to |s|$ be a characteristic map. By definition of the topology on $|K|$ and by our hypothesis we know that $\Phi_s^{-1}(U)$ is a non-empty open subset of Δ^k. It is elementary to see that $\Phi_s^{-1}(U) \cap \mathring{\Delta}^k \neq \varnothing$. It now follows from (1b) that $U \cap \langle s \rangle \neq \varnothing$. ■

The following proposition reminds us of Theorem 36.14 which gives an analogous statement for CW-complexes. The proposition can also be viewed as a converse to Lemma 61.9 (1).

Proposition 61.12. *Let L be an abstract simplicial complex.*

(1) *If $X \subset |L|$ is a compact subset, then there exists a finite subcomplex K of L such that $X \subset |K|$.*
(2) *If $|L|$ is compact, then L is a finite abstract simplicial complex.*

Remark. Let K be an abstract simplicial complex. Lemma 61.9 (1) together with Proposition 61.12 shows that $|K|$ is compact if and only if K is finite. Later on in Proposition 62.6 we will prove the related result that $|K|$ is regionally compact if and only if K is "locally finite".

PROOF. Let $L = (W, T)$ be an abstract simplicial complex. Clearly (2) is just a special case of (1). Thus it suffices to prove (1). Now let $X \subset |L|$ be a compact subset. For each simplex t of L with $\langle t \rangle \cap X \neq \varnothing$ we pick a point $y_t \in \langle t \rangle \cap X$. We denote by Y the set given by all the y_t. We equip Y with the subspace topology coming from $|L|$.

Claim.

(1) Every subset of Y, in particular Y itself, is a closed subset of $|L|$.
(2) Y is a discrete subset of $|L|$.

We turn to the proof of the two statements.

(1) Let $A \subset Y$ be a subset. We need to show that A is a closed subset of $|L|$. By Lemma 61.5 (1) and (3) it suffices to show that for every simplex t of $|L|$ the intersection $A \cap |t|$ is closed. By Lemma 61.11 (5) we know that $|t|$ intersects only the interiors of the simplices s with $s \subset t$. Since there are only finitely many simplices[940] contained in t we see that $A \cap |t|$ is finite, in particular it is compact. By Lemma 61.5

[940] This finiteness statement makes the proof of this proposition significantly simpler than the technically slightly tricky proof of Theorem 36.14.

(2) we know that $|t|$ is Hausdorff. Hence we obtain from Lemma 2.17 (2) that $A \cap |t|$ is indeed closed.

(2) We need to show that the subspace topology of Y is the discrete topology. It follows from (1) and Lemma 2.4 that every subset of Y is in fact a closed subset of Y. But this implies that every subset of Y is in fact an open subset of Y. In other words, the subspace topology is indeed the discrete topology. ⊞

It follows from (1) and the definition of the subspace topology that Y is a closed subset of the compact set X. Furthermore by (2) we know that Y is a discrete subset of $|L|$. Since $Y \subset X \subset |L|$ we see that Y is also a discrete subset of X. Since X is compact we obtain from Lemma 2.18 that Y is finite.

We define $S := \{t \in T \,|\, X \cap \langle t \rangle \neq \varnothing\}$. Note that $\#S = \#Y$. Thus, by the above discussion we know that S is finite. Now we set K to be the subcomplex that is given the union of all the simplices $t \in S$.[941] Since t is finite we see that K is a finite subcomplex. Now we have

$$X \overset{\substack{\text{Lemma 61.11 (4)}\\ \downarrow}}{=} X \cap \bigcup_{t \in S} \langle t \rangle = \bigcup_{t \in S} X \cap \langle t \rangle \overset{\substack{\text{by definition of } t\\ \downarrow}}{\subset} \bigcup_{t \in S} \langle t \rangle \subset \bigcup_{t \in S} |s| \overset{\substack{\text{Lemma 61.8 (3)}\\ \downarrow}}{=} |K|. \qquad \blacksquare$$

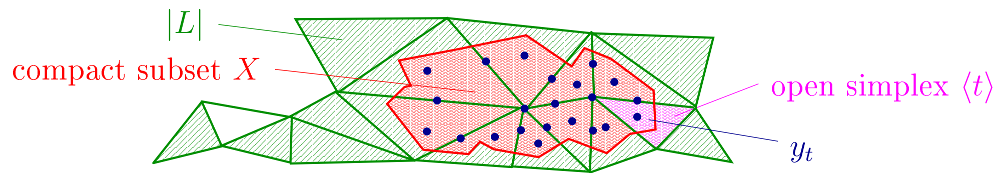


FIGURE 900. Illustration for the proof of Proposition 61.12.

We conclude this section with the following lemma which gives a criterion for a subset of a simplicial complex to be a subcomplex.

Lemma 61.13. (∗) *Let $L = (W,T)$ be an abstract simplicial complex and let $X \subset |L|$ be a closed subset. If for every simplex $t \in T$ we have either $\langle t \rangle \cap X = \varnothing$ or $\langle t \rangle \subset X$, then there exists a subcomplex K of L with $|K| = X$.*

PROOF (∗). We consider

$$V := W \cap X \quad \text{and} \quad S := \{t \in T \,|\, \langle t \rangle \subset X\}.$$

Claim. We claim that (V,S) is a subcomplex of $L = (W,T)$.

Let $s \in S$ be a simplex and let r be a face of s. We need to show that $r \in S$, i.e. we need to show that $\langle r \rangle \subset X$. Basically by definition we have $\langle r \rangle \subset |s|$. Thus it suffices to show that $|s| \subset X$.

Now note note that by Lemma 61.11 we know that $|s|$ is the closure of the subset $\langle s \rangle$. Since X is by hypothesis a closed subset of $|L|$ we obtain from elementary point-set topology, see e.g. Exercise 2.18 (a), that $|s| \subset X$. ⊞

[941] Strictly speaking we view the simplices as subcomplexes, and then take the union as defined in Lemma 61.2.

Now we see that

$$|K| \underset{\substack{\uparrow \\ \text{Lemma 61.11 (5)}}}{=} \bigcup_{t\in S}\langle t\rangle \underset{\substack{\uparrow \\ \text{definition} \\ \text{of } S}}{=} \bigcup_{\substack{t\in T \text{ with} \\ \langle t\rangle\subset X}}\langle t\rangle \underset{\substack{\uparrow \\ \text{by the hypothesis that} \\ \text{either } \langle t\rangle\cap X=\varnothing \text{ or } \langle t\rangle\subset X}}{=} \bigcup_{t\in T}(\langle t\rangle\cap X) \underset{\substack{\uparrow \\ \text{Lemma 61.11 (2)}}}{=} X.$$

■

61.4. **Simplicial complexes.** As on so many other occasions we start out this new section with a definition.

Definition.

(1) Let X be a topological space. A *simplicial structure* for X is a pair $(K, \Theta\colon |K| \to X)$ where the following holds:
   (a) K is an abstract simplicial complex,
   (b) Θ is a homeomorphism between the topological realization $|K|$ of K and the topological space X.

(2) A *simplicial complex* is a pair $(X, (K = (V, S), \Theta\colon |K| \to X))$ given by a topological space X together with a simplicial structure $(K = (V, S), \Theta\colon |K| \to X)$. We refer to X as the *underlying topological space* of the simplicial complex. To keep the notation at a manageable level we will often not distinguish in the notation between a simplicial complex and its underlying topological space.

(3) We say a simplicial complex is *finite* (respectively *countable*) if the corresponding abstract simplicial complex is *finite* (respectively *countable*).

(4) We say a simplicial complex is (*totally*) *ordered* if the corresponding abstract simplicial complex is (*totally*) *ordered.*

(5) Let $(X, (K = (V, S), \Theta\colon |K| \to X))$ be a simplicial complex. We introduce the following notions which are basically self-explanatory:
   (a) Given $v \in V$ we refer to $\Theta(v)$ as a *vertex of the simplicial complex.*
   (b) Let $s \in S$ be a k-simplex.
      (i) We refer to $\Theta(|s|) \subset X$ as a *k-simplex of the simplicial complex.* Furthermore we refer to $\Theta(\langle s\rangle) \subset X$ as an *open k-simplex of the simplicial complex.*
      (ii) If $\Phi_s\colon \Delta^k \to |s|$ is a characteristic map as defined on page 1487, then we refer to $\Theta \circ \Phi_s\colon \Delta^k \to X$ as a *characteristic map of X.*
      (iii) If t is a face of s, then we refer to $\Theta(|t|)$ as a *face of the simplex* $\Theta(|s|)$.
   (c) We define the *dimension of the simplicial complex* to be the dimension of the corresponding abstract simplicial complex.
   (d) Given $n \in \mathbb{N}_0$ we denote by $|K^n| \subset |K|$ the topological realization of the n-skeleton of K. We refer to $X^n := \Theta(|K^n|)$ as the *n-skeleton of the simplicial complex X.*

(6) Let $(Y, (L, \Theta\colon |L| \to Y))$ be a simplicial complex. A *subcomplex* is a simplicial complex $(X, (K, \Omega\colon |K| \to X))$ where X is a subspace of Y, K is a subcomplex of L and where the inclusion $X \to Y$ is precisely the map $\Theta \circ |i| \circ \Omega^{-1}$, where $i\colon K \to L$ is the inclusion map.

Remark.

(1) In many cases the choice of a simplicial structure gets suppressed from the notation.

(2) In the literature a *simplicial structure* is often called a *triangulation*.

Examples.

(1) Given any abstract simplicial complex K its topological realization $|K|$ is a simplicial complex in an obvious way.

(2) On page 1492 we wrote down an explicit homeomorphism $|D_n| \xrightarrow{\cong} \Delta^n$ which restricts to a homeomorphism $|S_{n-1}| \xrightarrow{\cong} \partial\Delta^n$. These two homeomorphisms endow Δ^n and $\partial\Delta^n$ with a simplicial structure, called the *standard simplicial structure*.

(3) We consider the topological space X to the left of Figure 901. To the right we show an abstract simplicial complex K with seven vertices such that X is homeomorphic to $|K|$.[942] We have thus found a simplicial structure for X.

topological space X abstract simplicial complex K

FIGURE 901

Before we continue with more examples of simplicial complexes we want to formulate the following lemma which summarizes a few basic properties of simplices in a simplicial complex.

Lemma 61.14. *Let Y be a simplicial complex.*

(1) *Any k-simplex of Y is homeomorphic to Δ^k.*

(2) *The intersection of two simplices of Y is either empty or it is given by a single face of the two simplices.*

(3) *Let X be a subcomplex of Y and let s be a simplex of Y. If the corresponding open simplex intersects X, then $s \subset X$.*

(4) *If $X \subset Y$ is the union of simplices of X, then Y admits a natural structure of a subcomplex.*

PROOF.

(1) This statement is a reformulation of Lemma 61.5.

(2) This statement is an immediate consequence of Lemma 61.1.

(3) This follows from Lemma 61.11 (5).

(4) This statement follows almost immediately from the definitions. ■

Remark.

(1) In the following a simplicial structure for a topological space is often indicated by a suitable decomposition into simplices, i.e. into subsets which (look like they) are homeomorphic to some Δ^k. Some care though needs to be taken to make sure that one actually does define a simplicial structure. In particular one needs to ensure that

[942] What is the role of the vertices A, F and G? Are they really needed?

the statements of Lemma 61.14 (1) and (2) are satisfied. We refer to Figure 902 for an illustration of examples and non-examples of simplicial complexes.

(2) In the following we often draw a simplicial complex and rely on the good will of the reader to read off the corresponding abstract simplicial complex.

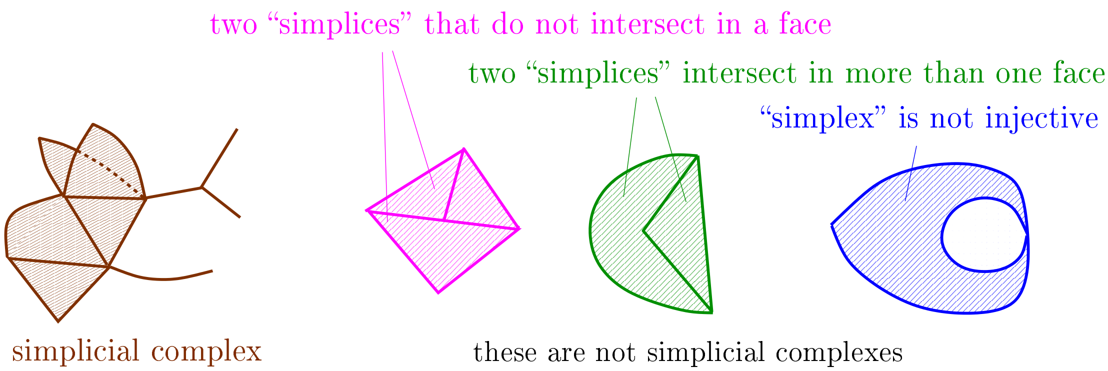


FIGURE 902

We continue with more examples of simplicial complexes.

Examples.

(1) In Figure 903 to the left we show several simplicial structures for the disk $\overline{B}^2$. Furthermore, in Figure 903 to the right we show a simplicial structure for the sphere S^2.

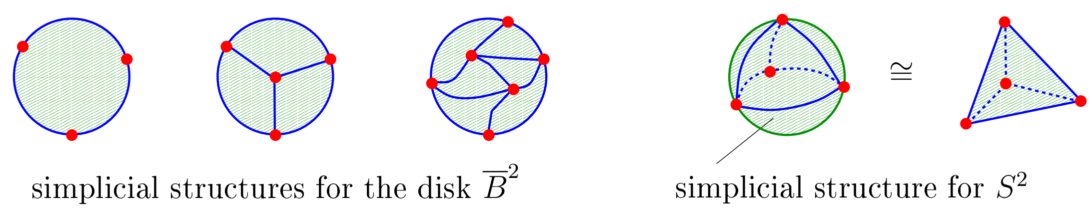


FIGURE 903

(2) In Figure 904 we show the torus $T = ([0,1] \times [0,1])/\sim$ together with three attempts at endowing it with a simplicial structure. Using Lemma 61.14 we see that the two attempts to the left do not define a simplicial structure. Only the fairly complicated attempt to the right actually provides us with a simplicial structure for the torus.

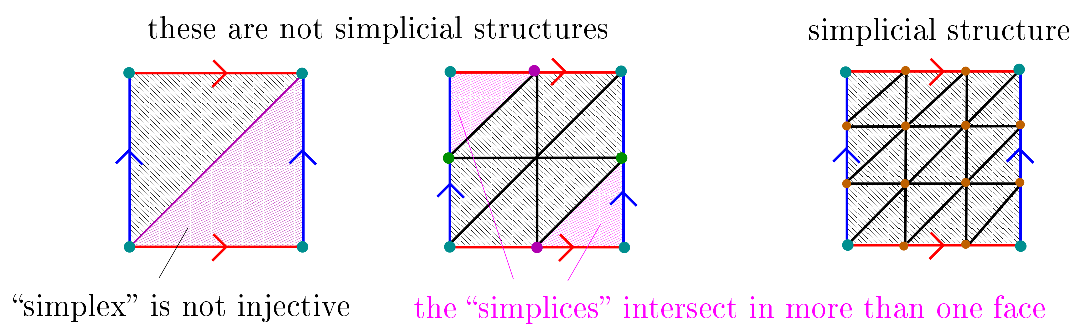


FIGURE 904

(3) Let $n \in \mathbb{N}$. We will now equip the closed ball $\overline{B}^n$ and the sphere S^n with a canonical simplicial structure. To do so we consider the simplicial complex D_n and its subcomplex S_{n-1} that we introduced on page 1480. On page 1492 we saw that there exists a canonical homeomorphism $f\colon |D_n| \to \Delta^n$ which restricts to a homeomorphism $|S_{n-1}| \to \partial\Delta^n$. Furthermore, in Lemma 41.1 we gave an explicit homeomorphism $\Phi\colon \overline{B}^n \to \Delta^n$ that restricts to a homeomorphism $S^{n-1} \to \partial\Delta^n$. We see that $(D_n, \Phi^{-1} \circ f\colon |D_n| \to \overline{B}^n)$ is a simplicial structure for $\overline{B}^n$ and that $(S_{n-1}, \Phi^{-1} \circ f|_{|S_{n-1}|}\colon |S_{n-1}| \to S^{n-1})$ is a simplicial structure for S^{n-1}. We refer to these as the *canonical simplicial structures* of $\overline{B}^n$ and S^{n-1}.

FIGURE 905. Canonical simplicial structures for spheres and balls.

It is by now abundantly clear that most simplicial complexes admit many different simplicial structures. Finally we consider maps between simplicial complexes. What comes next is basically self-explanatory and only written down out of a sense of duty.

Definition. Let $(X, (K, \Theta\colon |K| \to X))$ and $(Y, (L, \Omega\colon |L| \to Y))$ be two simplicial complexes.

(1) A map $f\colon X \to Y$ is called *simplicial* if there exists a simplicial map $g\colon K \to L$ such that $f = \Omega \circ |g| \circ \Theta^{-1}$.
(2) We say the two simplicial complexes are *simplicially isomorphic* if there exist simplicial maps between the two simplicial complexes that are inverses of one another.

Example. If Y is a simplicial complex and X is a subcomplex, then it follows immediately from the definitions that the inclusion $X \to Y$ is a simplicial map.

It follows immediately from the definitions that the composition of two simplicial maps is again simplicial. This leads us to the following definition.

Definition. We call the category $\mathcal{SimpCplx}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{SimpCplx}) &:= \text{ all simplicial complexes,} \\ \mathrm{Mor}(K, L) &:= \text{ all simplicial maps from } K \text{ to } L, \end{aligned}$$

together with the usual composition of maps the *category of simplicial complexes.* Similarly we define the *category of (totally) ordered simplicial complexes.*

We conclude this section with the following basic lemma.

Lemma 61.15. *Let* $(X, (K = (V, S), \Theta\colon |K| \to X))$ *and* $(Y, (L = (W, T), \Omega\colon |L| \to Y))$ *be two simplicial complexes. Given a simplicial map* $\varphi\colon K \to L$ *there exists a unique* *simplicial map $\Phi\colon X \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\Theta"'] & W \arrow[d, "\Omega"] \\ X \arrow[r, "\Phi"] & Y. \end{tikzcd}$$

PROOF. The map Φ is given by $\Omega \circ |\varphi| \circ \Theta^{-1}$. Since Ω and Θ are homeomorphisms we see that Φ is also unique. ∎

61.5. **The product of (abstract) simplicial complexes.** In this section we want to consider the product of (abstract) simplicial complexes. Let us first consider the simplicial complexes $X = Y = [0,1]$ with the "obvious" simplicial structure. As we see in Figure 906, there are two very reasonable looking simplicial structures on the product $[0,1] \times [0,1]$.

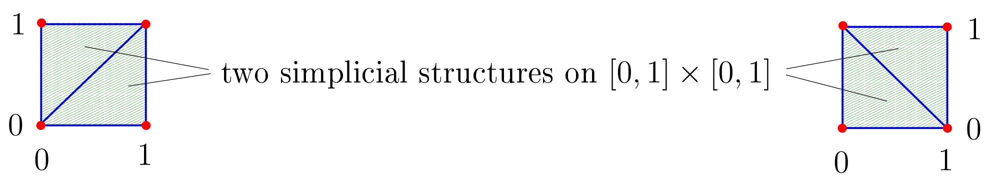


FIGURE 906

It is not clear which of the two simplicial structures shown in Figure 906 is preferable or "more canonical". In fact to break the tie and to end up with a clear winner we need to work with ordered abstract simplicial complexes, as introduced on page 1480.

First we recall the following definition from page 76.

Definition. Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. Given (a, b) and (a', b') in $A \times B$ we define

$$(a, b) < (a', b') \quad \Longleftrightarrow \quad \text{either } a < a' \text{ or } a = a' \text{ and } b < b'$$

and we define $(a, b) \leq (a', b')$ if either $(a, b) = (a', b')$ or $(a, b) < (a', b')$. We refer to "$\leq$" on $A \times B$ as the *lexicographic order on* $A \times B$.

Remark. Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. One can show easily that the lexicographic order is a partial order on $A \times B$ and similarly one can show easily that if $(A, \leq)$ and $(B, \leq)$ totally ordered sets, then the lexicographic order is also a total order.

Definition. Let $K = (V, S)$ and $L = (W, T)$ be two (totally) ordered abstract simplicial complexes. We define the product $K \times L$ of the abstract simplicial complexes to be the abstract simplicial complex[943] that is given by the vertex set $V \times W$ and the simplex set[944]

$$\left\{ \{(v_0, w_0), \dots, (v_k, w_k)\} \;\middle|\; \begin{array}{c} \{v_0, \dots, v_k\} \in S \text{ and } \{w_0, \dots, w_k\} \in T \text{ and} \\ \text{we have } v_0 \leq \dots \leq v_k \text{ and } w_0 \leq \dots \leq w_k \end{array} \right\}.$$

We equip $K \times L$ with the lexicographic order and we view $K \times L$ again as a (totally) ordered abstract simplicial complex.[945]

Before we discuss examples we first state the following lemma which basically reassures us that we are doing something reasonable.

Lemma 61.16. *Let $K = (V, S)$ and $L = (W, T)$ be two ordered abstract simplicial complexes.*

(1) *The two obvious "projection maps" $p\colon K \times L \to K$ and $q\colon K \times L \to L$ are simplicial maps.*
(2) *The map*
$$|p| \times |q|\colon |K \times L| \;\to\; |K| \times |L|$$
is a continuous bijection. If K and L are finite, *then the map is in fact a homeomorphism.*

Thus given two finite simplicial complexes K and L we can view the product also as a simplicial complex.

Remark.

(1) Lemma 36.24 teaches us that it is safer to stick to finite abstract simplicial complexes. In fact that is also all we ever need. For products of arbitrary abstract simplicial complexes we refer to [**FrPi90a**, Proposition 4.3.5] and [**LW69**, Corollary III.5.2]. Note though that one needs to tread carefully to make sure that the definitions and statements are indeed as desired.
(2) As mentioned earlier, the topology of the topological realization of an abstract simplicial complex is designed to make it easy to show that a map out of the topological realization is continuous. Furthermore, as we mentioned in Remark 3.7, the topology of a product of topological spaces makes it easy to find continuous maps to the product. This explains why in Lemma 61.16 we have a continuous map from left to right, but why we struggle to find a continuous map from right to left.

Proof.

(1) It follows immediately from the definitions that the projection maps are indeed simplicial.
(2) It follows from (1) together with Proposition 61.7 and the elementary Lemma 3.6 (1) that the map $|p| \times |q|\colon |K \times L| \to |K| \times |L|$ is continuous. We leave it to the reader to verify that the map is a bijection.

    Finally suppose that K and L are finite. It follows from Proposition 2.43 (3), together with Lemmas 61.5 (1) and 61.9 (1) and Proposition 3.12 that the map is indeed a homeomorphism. ■

Examples.

[943]One can indeed easily verify that this defines an abstract simplicial complex.

[944]To rule out any potential confusion, the $v_0, \ldots, v_k$ and also the $w_0, \ldots, w_k$ do not need to be distinct vertices.

[945]It follows almost immediately from the above remark that with this definition $K \times L$ is in fact a (totally) ordered abstract simplicial complex.

(1) We consider the two totally ordered abstract simplicial complexes K and L that are shown in Figure 907. Here we use the obvious total orders given by $0 < 1 < 2$ and $A < B < C$. We also show the corresponding product $K \times L$.

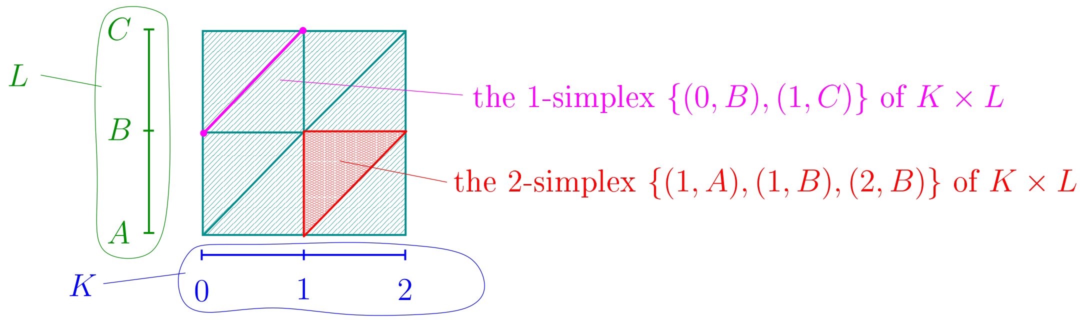


FIGURE 907

(2) We equip S^1 with the canonical simplicial structure defined on page 1500. The product simplicial structure on the torus $S^1 \times S^1$ is simplicially isomorphic to the simplicial structure shown in Figure 904 to the right.

(3) Let I be the abstract simplicial complex given by $(V = \{(0,1)\}, S = \{\{0\},\{1\},\{0,1\})$. As in (1) we use the obvious total order given by $0 < 1$. We consider the product $I \times I$. Given any $n \in \mathbb{N}$ we can now view the cube $[0,1]^n$ as a simplicial complex. For $n = 2$ we obtain the simplicial complex drawn in Figure 906 on the left. We refer to this simplicial structure as the *canonical simplicial structure of* $[0,1]^n$.

In plain English, the following lemma says that the $\mathbb{Z}^n$-translates of the canonical simplicial structure of $[0,1]^n$ give a linear simplicial structure for $\mathbb{R}^n$.

Lemma 61.17. *Let $n \in \mathbb{N}$. We denote by $(K = (V,S), \Theta \colon |K| \to [0,1]^n)$ the canonical simplicial structure of $[0,1]^n$. We define a new simplicial complex $\widetilde{K} = (\widetilde{V}, \widetilde{S})$ where*

$$\widetilde{V} := \mathbb{Z}^n \quad \text{and} \quad \widetilde{S} := \{\{v_0 + w, \ldots, v_k + w\} \mid \{v_0, \ldots, v_k\} \in S \text{ and } w \in \mathbb{Z}^n\}.$$

The (hopefully) obvious map $\widetilde{\Theta} \colon |\widetilde{K}| \to \mathbb{R}^n$ is a homeomorphism, in particular it equips $\mathbb{R}^n$ with the structure of a linear simplicial complex.

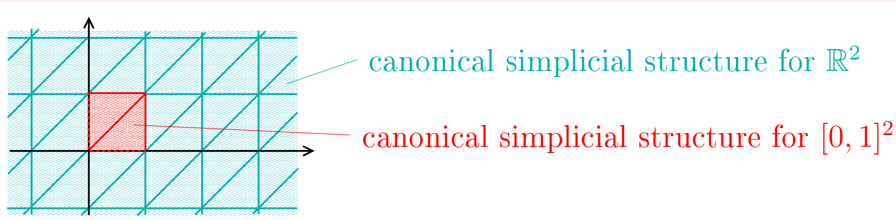


FIGURE 908. Illustration of Lemma 61.17.

PROOF. It is elementary to see that the map $\widetilde{\Theta} \colon |\widetilde{K}| \to \mathbb{R}^n$, once it is written down properly, is continuous and a bijection. It follows from Proposition 2.45 that the map is in fact a homeomorphism. ■

Arguably the most important ways to construct new topological spaces out of given ones are by taking products and by taking quotients. Within reasonable limits we saw in Proposition 36.23 and Lemma 36.32 (3) that both operations can be done with CW-complexes. Furthermore, we just saw in Lemma 61.16 that we can take products of finite simplicial complexes. Quotients on the other hand are a problem. In general, as is illustrated in Figure 909, given a simplicial complex Y and a subcomplex X the simplicial structure of Y does not descend to a simplicial structure on the quotient Y/X.

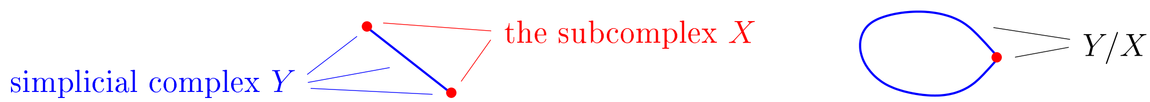


FIGURE 909

Let us jot down the following question.

Question 61.18. *Let Y be a simplicial complex and let X be a subcomplex. Does the quotient Y/X admit a simplicial structure?*

In Corollary 61.26 we will give a partial answer to Question 61.18.

61.6. **The simplicial join of (abstract) simplicial complexes.** In this section we will continue with our task of finding useful constructions of (abstract) simplicial complexes that correspond to constructions that we already know from the world of topological spaces. More precisely, we want to consider the join construction from page 207 that so far we had not seriously used.

Definition. Let X and Y be two non-empty topological spaces. We define the *join* $X*Y$ to be the topological space obtained from $X\times[0,1]\times Y$, equipped with the product topology, by performing the following two types of identifications:

(1) for every $x\in X$ we identify all points in $\{x\}\times\{0\}\times Y$ to a single point and
(2) for every $y\in Y$ we identify all points in $X\times\{1\}\times\{y\}$ to a single point.

Furthermore, if one of X or Y is the empty topological space, then we define $X*Y$ to be the other topological space.

To get into the mood of thinking about joins let us first consider the following two examples.

Examples.

(1) Let X be a non-empty topological space and let $Y=\{1\}$ be the topological space given by a single point 1. We consider the maps

$$\begin{array}{rcl} X*\{1\}=(X\times[0,1]\times\{1\})/\sim & \to & \mathrm{Cone}(X)=(X\times[0,1])/(X\times\{1\}) \\ [(x,t,1)] & \mapsto & [(x,t)] \end{array}$$

One can easily verify that the map is a bijection. Furthermore, using Lemma 3.22 and Lemma 3.21 (3) it is straightforward to see that this map and its inverse are continuous. Thus the map is a homeomorphism.

(2) Let X be a non-empty topological space and let $Y=\{-1,1\}$ be the topological space given by the set with two elements ±1 and the discrete topology. We also consider

the suspension

$$\Sigma(X) \;:=\; (X \times [-1,1])/\sim$$

which is defined by squashing all points in $X \times \{-1\}$ to a single point and by squashing all points in $X \times \{1\}$ to a single point. We consider the map

$$\begin{array}{rcl} X * \{-1,1\} = (X \times [0,1] \times \{-1,1\})/\sim & \to & \Sigma(X) = (X \times [-1,1])/\sim \\ [(x,t,\epsilon)] & \mapsto & [(x, \epsilon \cdot t)] \end{array}$$

With some minor effort one can show that this map is a bijection. Using Lemma 3.22 and Lemma 3.21 (3) it is straightforward to see that this map is continuous. The fact that the inverse is also continuous needs slightly more effort, in fact this argument also requires Lemmas 3.28 and Lemma 3.45. We leave it to the reader to fill in the details. We refer to Figure 910 for $X = S^1$ for an illustration of the homeomorphism.

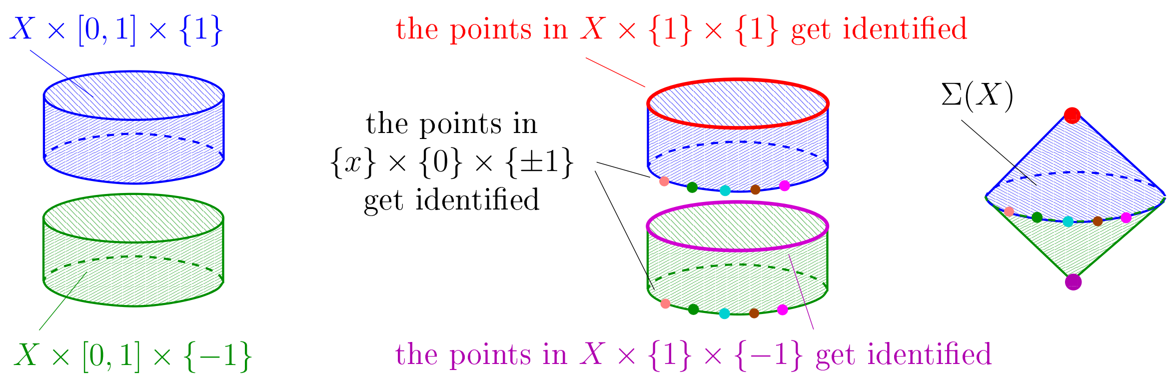


FIGURE 910

For completeness we note that for the empty topological space we have, basically by definition, that $\varnothing * \{1\} = \mathrm{Cone}(\varnothing)$ and $\varnothing * \{\pm 1\} = \Sigma(\varnothing)$

The following lemma gives us a few examples of joins which will play a role later on.

Lemma 61.19.

(1) *Given any $m, n \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl} S^m * S^n & \to & S^{m+n+1} \\ [(x,t,y)] & \mapsto & \underbrace{\left(x \cdot \cos\left(\tfrac{\pi t}{2}\right), y \cdot \sin\left(\tfrac{\pi t}{2}\right)\right)}_{\in \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} = \mathbb{R}^{m+n+2}} \end{array}$$

is a homeomorphism.[946]

(2) *Given any $m, n \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl} \Delta^m * \Delta^n & \to & \Delta^{m+n+1} \\ [(x_0,\dots,x_m,t,y_0,\dots,y_n)] & \mapsto & (t \cdot x_0, \dots, t \cdot x_m, (1-t)\cdot y_0, \dots, (1-t) \cdot y_n) \end{array}$$

is a homeomorphism.

(3) *Given any* $n \in \mathbb{N}_0$ *the map*

$$\begin{array}{rcl} \partial\Delta^n * \{\star\} & \to & \Delta^n \\ [(x,t,\star)] & \mapsto & x\cdot(1-t) + \left(\frac{1}{n+1},\dots,\frac{1}{n+1}\right)\cdot t \end{array}$$

PROOF (*). The proof of (1), (2) and (3) is basically identical. In each case it is straightforward to verify that the map is well-defined and a bijection. It follows from Lemma 3.22 that the given map is continuous. Finally it follows from Lemma 3.48 together with Proposition 2.43 (3) that the given map is in fact a homeomorphism. ■

It turns out that in the simplicial setting we have an analogous construction.

Definition. Let $K_i = (V_i, S_i)$, $i = 1, \dots, m$ be abstract simplicial complexes. We define the *simplicial join of* $K_1, \dots, K_m$ to be the abstract simplicial complex[947][948]

$$K_1 * \cdots * K_m \;:=\; \left(V_1 \sqcup \cdots \sqcup V_m, \left\{ s_1 \sqcup \cdots \sqcup s_m \;\middle|\; \begin{array}{l} \text{for } i = 1,\dots,m \text{ we have } s_i \in \overbrace{S_i \cup \{\varnothing\}}^{\subset \mathcal{P}(V_i)} \\ \text{and there exists at least one } s_i \neq \varnothing \end{array} \right\}\right).$$

This definition is illustrated, with limited success, in Figure 911.

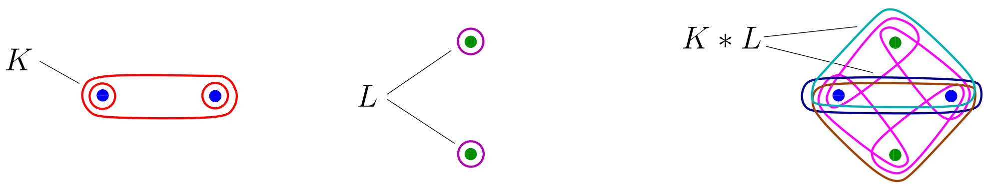


FIGURE 911

Remark. It follows immediately from the definitions that the join construction is associative, i.e. given abstract simplicial complexes K, L and M we have a natural isomorphism $(K * L) * M = K * (L * M)$ of abstract simplicial complexes.

The examples on page 1504 motivate the following definitions.

Definition. Let K be an abstract simplicial complex. Furthermore let $\{1\}$ be the abstract simplicial complex consisting of a single vertex 1 and finally let S_0 be the abstract simplicial complex $S_0 = \{\{0,1\},\{\{0\},\{1\}\}$ that we introduced on page 1480. We refer to

$$\mathrm{Cone}(K) \;:=\; K * \{1\} \qquad \text{respectively} \qquad \Sigma(K) \;:=\; K * S_0$$

as the *simplicial cone on the abstract simplicial complex* K respectively the *simplicial suspension of the abstract simplicial complex* K.

Example. Let $n \in \mathbb{N}_0$. We see that

$$\overbrace{\mathrm{Cone}\left(\{0,\dots,n\}, \mathcal{P}(\{0,\dots,n\})\setminus\{\varnothing\}\right)}^{=\,\mathrm{Cone}(D_n),\text{ see definition of } D_n \text{ on page 1480}} \underset{\substack{\uparrow \\ \text{follows from the definition of the simplicial cone}}}{=} \left(\{0,\dots,n,*\}, \mathcal{P}(\{0,\dots,n,*\}\setminus\{\varnothing\}\right) \underset{\substack{\uparrow \\ \text{induced by } * \mapsto n+1}}{\xrightarrow{\cong}} D_{n+1}.$$

[946]In fact this statement is just the content of Lemma 3.50.

[947]To avoid confusion it is perhaps worth recalling that the empty set is not a simplex.

[948]One can easily verify that this is indeed an abstract simplicial complex.

Lemma 61.20. *Let K and L be two abstract simplicial complexes. If one of K or L is empty, then by definition we have a natural homeomorphism $|K| * |L| \cong |K * L|$. If K and L are non-empty, then the map*

$$\begin{array}{rcl} \Theta\colon \overbrace{(|K| \times [0,1] \times |L|)/\sim}^{=|K|*|L|} & \to & |K * L| \subset \mathbb{R}^{(V \sqcup W)} = \mathbb{R}^{(V)} \times \mathbb{R}^{(W)} \\ [(x, r, y)] & \mapsto & (\underbrace{r \cdot x}_{\in \mathbb{R}^{(V)}}, \underbrace{(1-r) \cdot y}_{\in \mathbb{R}^{(W)}}) \end{array}$$

is a bijection and its inverse Θ^{-1} is continuous.[949] *Furthermore, if K and L are finite, then Θ is in fact a homeomorphism.*

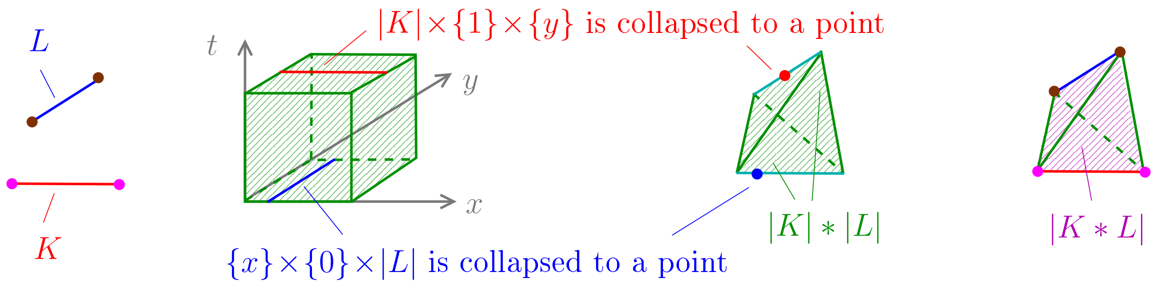


FIGURE 912. Illustration of Lemma 61.20.

PROOF. Let $K = (V, S)$ and $L = (W, T)$ be two abstract simplicial complexes. If one of K or L is empty, then the statement is true basically by definition. Thus let us now assume that K and L are both non-empty. We consider the map

$$\begin{array}{rcl} \Theta\colon \overbrace{(|K| \times [0,1] \times |L|)/\sim}^{=|K|*|L|} & \to & |K * L| \subset \mathbb{R}^{(V \sqcup W)} = \mathbb{R}^{(V)} \times \mathbb{R}^{(W)} \\ [(x, r, y)] & \mapsto & (\underbrace{r \cdot x}_{\in \mathbb{R}^{(V)}}, \underbrace{(1-r) \cdot y}_{\in \mathbb{R}^{(W)}}). \end{array}$$

We leave it to the reader to verify that the map does indeed take values in $|K * L|$ and that it is a bijection. Next we show that the inverse $\Theta^{-1}\colon |K * L| \to |K| * |L|$ is continuous.

Claim. The map $\Theta^{-1}\colon |K * L| \to |K| * |L|$ is continuous.

We use the continuity criterion from Lemma 61.6 (2) to show that Θ^{-1} is continuous. Since K and L are non-empty we see that every simplex of $K * L$ is contained in a simplex of the form $s \sqcup t$ where $s \in S$ and $t \in T$ are simplices. Thus let $s \in S$ be a k-simplex and let $t \in T$ be an l-simplex. Note that by Lemma 61.19 (2) we know that the map

$$\begin{array}{rcl} \Xi\colon \Delta^k * \Delta^l & \to & \Delta^{k+l+1} \\ [(x_0, \dots, x_k, r, y_0, \dots, y_l)] & \mapsto & (r \cdot x_0, \dots, r \cdot x_k, (1-r) \cdot y_0, \dots, (1-r) \cdot y_l) \end{array}$$

[949] This might be viewed as an odd formulation, but it is easier to write down the map Θ than to write down its inverse Θ^{-1}.

is a homeomorphism. Next we pick corresponding characteristic maps $\Phi_s\colon \Delta^k \to |K|$ and $\Phi_t\colon \Delta^l \to |L|$. It follows easily from the definitions that the map

$$\begin{array}{rcccl} \Phi_{s\sqcup t}\colon \Delta^{k+l+1} & \xrightarrow{\Xi^{-1}} & \overbrace{(\Delta^k \times [0,1] \times \Delta^l)/\sim}^{\Delta^k * \Delta^l} & \xrightarrow{\Omega} & |K*L| \subset \mathbb{R}^{(V\sqcup W)} = \mathbb{R}^{(V)} \times \mathbb{R}^{(W)} \\ & & [(x,r,y)] & \mapsto & (\underbrace{r\cdot\Phi_s(x)}_{\in\mathbb{R}^{(V)}}, \underbrace{(1-r)\cdot\Phi_t(y)}_{\in\mathbb{R}^{(W)}}) \end{array}$$

is a characteristic map for the simplex $s \sqcup t$. We obtain the following diagram

$$\begin{tikzcd} \Delta^{k+l+1} \arrow[r, "\Phi_{s\sqcup t}"] \arrow[dr, "\Xi^{-1}"'] & {|K*L|} \arrow[r, "\Theta^{-1}"] & {|K|*|L|} \\ & \Delta^k * \Delta^l \arrow[u, "\Omega"] \arrow[ur, "\Phi_s * \Phi_t"'] & \end{tikzcd}$$

We make the following observations:

(1) It follows immediately from the definitions that the triangle to the left diagram commutes. Furthermore it is equally trivial to see that $\Theta \circ (\Phi_s * \Phi_t) = \Omega$, thus the triangle to the right also commutes.
(2) By Lemma 61.19 (2) we know that the left diagonal map is continuous.
(3) By Lemma 3.47 we know that the right diagonal map is continuous.
(4) It follows from the above that $\Theta^{-1} \circ \Phi_{s\sqcup t}$ is continuous.

It now follows from Lemma 61.6 (2) that $\Theta^{-1}\colon |K*L| \to |K|*|L|$ is continuous. ⊞

In the remainder of the proof we now assume that K and L are finite. We need to show that Θ is a homeomorphism. We already know that Θ is a bijection and that the inverse $\Theta^{-1}\colon |K*L| \to |K|*|L|$ is continuous. Furthermore, by Lemma 61.5 (2) together with Lemma 3.51 we know that $|K|*|L|$ is Hausdorff. Thus it follows, as so often, from Proposition 2.43 (3) that Θ^{-1} is a homeomorphism. Thus Θ is also a homeomorphism. ■

Example. Let $n \in \mathbb{N}_0$. Recall that on page 1500 we gave an explicit simplicial structure $(S_n, \Theta\colon |S_n| \to S^n)$ for S^n. Now let $m,n \in \mathbb{N}$. By Lemma 61.19 (1) we know that S^{m+n+1} is homeomorphic to $S^m * S^n$. This observation, together with Lemma 61.20, gives us lots of different simplicial structures on spheres. We sketch the resulting three simplicial structures for S^2 in Figure 913.

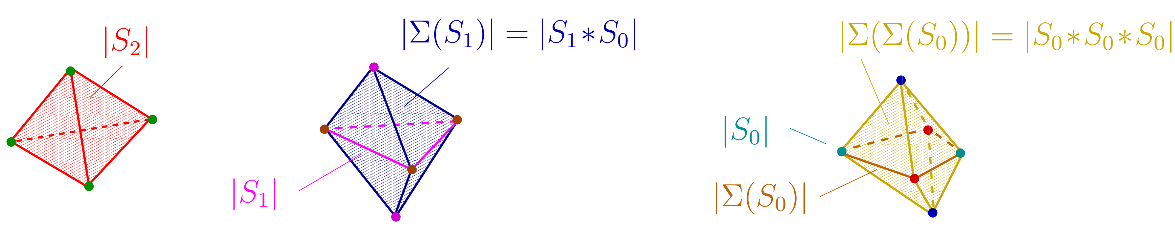


FIGURE 913

Corollary 61.21. *If $X = (K, \Theta\colon |K| \to X)$ is a finite simplicial complex, then the suspension $\Sigma(X)$ and the cone $\mathrm{Cone}(X)$ admit natural simplicial structures where the corresponding abstract simplicial complexes are given by $\Sigma(K)$ and $\mathrm{Cone}(K)$.*

PROOF. The corollary follows immediately from Lemma 61.20 together with the two examples on page 1505. ∎

Remark. In Section 62.4 we will introduce the simplicial analogue of the mapping cylinder and mapping cone.

61.7. **Linear simplicial complexes.** In this short section we discuss an important special case of a simplicial complex.

Definition. A simplicial complex $(X, (K, \Theta\colon |K| \to X))$ is called *linear* if X is a subspace[950] of some $\mathbb{R}^n$ and for each each simplex $s \in K$ the map $\Theta\colon |s| \to X$ is an affine linear map.

Example. Let $n \in \mathbb{N}$. The topological space $\mathbb{R}^n$ together with the canonical simplicial structure defined in Lemma 61.17 is clearly a linear simplicial complex.

Remark.

(1) Different books have different definitions for "simplicial complexes". Our approach mostly follows the approach taken in [**Spa95**, Chapter 3]. In many books what we call a "linear simplicial complex" is basically used as the definition of a simplicial complex, see e.g. [**Matv06**, p. 6] or [**Bre93**, Chapter 21].

(2) Linear simplicial complexes play an important role in computer visualizations. More precisely, in computer visualizations surfaces in $\mathbb{R}^3$ are usually approximated by linear simplicial complexes. These have the nice feature that they can be described by a fairly small amount of data. We refer to Figure 914 for an illustration.

linear simplicial structure for the torus   linear simplicial structure for a bunny

FIGURE 914

Lemma 61.22. *Every finite simplicial complex is simplicially isomorphic to a finite linear simplicial complex.*

Remark. An (abstract) simplicial complex is called *locally finite* if each simplex is the face of only finitely many simplices. With this definition one can formulate a much stronger version of Lemma 61.22: By [**FrPi90a**, Theorem 3.3.15] a simplicial complex is homeomorphic to a linear simplicial complex if and only if it is finite-dimensional, countable and locally finite. We will discuss this statement in Exercise 61.8.

PROOF. By definition a finite simplicial complex is homeomorphic to the topological realization $|K|$ of a finite abstract simplicial complex $K = (V, S)$. In Lemma 61.9 we showed that we can view $|K|$ as a linear simplicial complex in $\mathbb{R}^V = \mathbb{R}^m$. ∎

We adopt the following notation.

[950] In particular we assume that X is endowed with the subspace topology coming from $\mathbb{R}^n$.

Notation. Given[951] $v_0,\dots,v_k\in\mathbb{R}^n$ we write

$$|\{v_0,\dots,v_k\}| \;:=\; \left\{ \underbrace{\sum_{i=0}^{k} t_i\cdot v_i}_{\in\mathbb{R}^n} \;\middle|\; \begin{array}{l}(1)\text{ for each } i \text{ we have } t_i\geq 0\\ (2)\ t_0+\dots+t_k=1\end{array}\right\}.$$

Remark. Let $v_0,\dots,v_k\in\mathbb{R}^n$. Furthermore let $w_0,\dots,w_m\in|\{v_0,\dots,v_k\}|$. An elementary calculation shows that $|\{w_0,\dots,w_m\}|\subset|\{v_0,\dots,v_k\}|$. This inclusion can also be obtained from Exercise 2.36 (a).

Lemma 61.23.

(1) *Let K be a linear simplicial complex in $\mathbb{R}^n$.*
  (a) *Every k-simplex s of K is of the form $s=|\{v_0,\dots,v_k\}|$ for some distinct vertices $v_0,\dots,v_k$.*
  (b) *Every simplex of K is a convex subset of $\mathbb{R}^n$.*
  (c) *Let $v_0,\dots,v_k$ be distinct vertices of K. The following three statements are equivalent:*
    (i) *$|\{v_0,\dots,v_k\}|$ is a simplex of K,*
    (ii) *$\{v_0,\dots,v_k\}$ is the vertex set of a simplex of K,*
    (iii) *$|\{v_0,\dots,v_k\}|\subset K$.*

(2) *Let $f\colon K\to L$ be a simplicial map between two linear simplicial complexes. The restriction of f to any k-simplex $|\{v_0,\dots,v_k\}|$ of K is given by*

$$\begin{array}{rcl}|\{v_0,\dots,v_k\}| & \to & |\{\varphi(v_0),\dots,\varphi(v_k)\}|\\ \sum\limits_{i=0}^{k} t_i\cdot v_i & \mapsto & \sum\limits_{i=0}^{k} t_i\cdot\varphi(v_i).\end{array}$$

Proof.

(1) Almost all of these statements follow easily from the definitions. The only statement which needs a little bit of thought is the fact that (iii) implies (i). Since we will not make use of this statement we feel comfortable with the thought of leaving the details to the reader.
(2) This statement follows immediately from the definition of a simplicial map, see page 1483 and the definition of the topological realization of a simplicial map between abstract simplicial complexes, see Proposition 61.7. ■

61.8. **Simplicial structures and CW-structures I.** It is pretty clear that simplicial complexes and CW-complexes have structural similarities. In particular few readers will be surprised by the following lemma.

Lemma 61.24. *Every (ordered) simplicial complex admits a (natural)* CW*-complex structure where given any $n\in\mathbb{N}$ the n-simplices of the simplicial structure are precisely the n-cells*[952] *of the* CW*-structure. In particular the following statements hold:*

(1) *The simplicial subcomplexes are precisely the cellular subcomplexes,*

[951] We do not assume that $v_0,\dots,v_k$ are distinct.

(2) *For each $n \in \mathbb{N}_0$ the n-skeleton of the simplicial complex equals the n-skeleton of the* CW*-complex.*
(3) *Any simplicial map $f\colon X \to Y$ between simplicial complexes is a cellular map between the corresponding* CW*-complexes.*

Remark. Lemma 61.24 says in particular that there exists a functor

$$F\colon \begin{array}{c}\text{category } \mathcal{OrdAbsSimpCplx} \text{ of}\\ \text{ordered simplicial complexes}\end{array} \to \begin{array}{c}\text{category } \mathcal{CW} \text{ of}\\ \text{CW-complexes}\end{array}$$

which has the property that the functor keeps the underlying topological space fixed. The functor is faithful but not fully faithful. The latter is just a fancy way of saying that there exist cellular maps that are not simplicial. For example, consider the simplicial complex $X = [0, \infty)$ with the vertex set given by $\mathbb{N}_0$. We view X also as a CW-complex with the cellular structure that is induced from the simplicial structure. The map $f\colon X \to X$ given by $x \mapsto 2x$ is cellular but it is not simplicial. We refer to Figure 915 for an illustration.

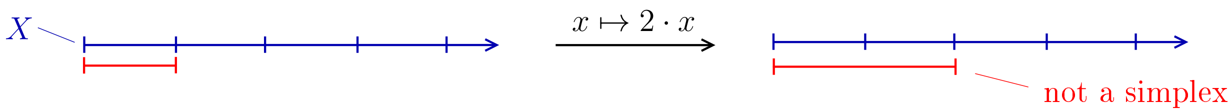


Figure 915

Sketch of proof. Basically by definition of a simplicial complex it suffices to prove the statement for (ordered) abstract simplicial complexes and their topological realizations. The lemma now follows from fleshing out the following steps for a given (ordered) abstract simplicial complex $K = (V, S)$.

(a) Let $n \in \mathbb{N}_0$. Recall that in Lemma 41.1 we had written down a specific homeomorphism $\Psi_n\colon (\overline{B}^n, S^{n-1}) \to (\Delta^n, \partial\Delta^n)$.
(b) Given $n \in \mathbb{N}_0$ we denote by K^n the n-skeleton of K. Recall that K^n is a subcomplex of K.
(c) Given $n \in \mathbb{N}_0$ let I_n be the set of n-simplices of K. For each $i \in I_n$ let $\Theta_i\colon \Delta^n \to |K|$ be a characteristic map as defined on page 1487. Recall that if K is ordered, then we saw on page 1487 that there is a natural choice of a characteristic map.
(d) Given $i \in I_n$ we write $\Phi_i := \Theta_i \circ \Psi_n\colon \overline{B}^n \to |K|$ and $\varphi_i := \Phi_i|_{S^{n-1}}\colon S^{n-1} \to |K^{n-1}|$. Note that φ_i takes values in $|K^{n-1}|$.
(e) Starting from $X^0 := V$ we can build up a CW-complex such that for each $n \in \mathbb{N}_0$ the attaching maps are precisely the maps $\varphi_i\colon S^{n-1} \to |K^{n-1}|$. The maps Ψ_i give rise to a map $\Xi_n\colon |X^n| \to |K^n|$ which is, basically by definition, a bijection.
(f) Eventually we get a CW-complex $X = \varinjlim X_n$ and the bijections Ξ_n can be combined to obtain a bijection $\Xi\colon |X| \to |K|$.
(g) It remains to show that Ξ is continuous. First note that it follows easily from Lemma 61.6 (1) together with Lemma 36.7 (1) that Ξ is continuous. Furthermore, it follows from Lemma 36.7 (4) together with Lemma 61.5 (3) that Ξ^{-1} is also continuous.

[952]Recall that cells and simplices are subsets, so it makes sense to talk of equality.

(h) In summary we have shown that $\Xi\colon |X| \to |K|$ is a homeomorphism.

It follows basically immediately from the construction of the CW-structure on $|K|$ that the properties (1), (2) and (3) are satisfied. ■

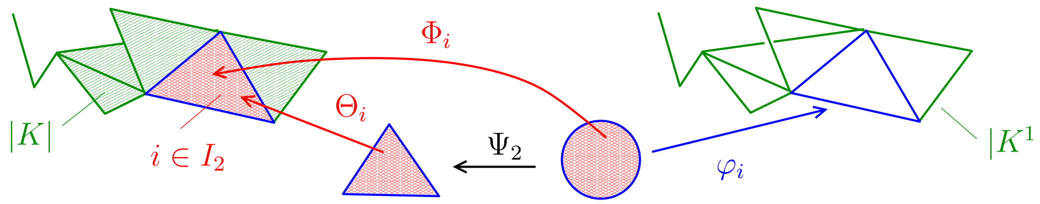


FIGURE 916. Illustration for the proof of Lemma 61.24.

Lemma 61.24 allows us to import many previous results about CW-complexes to the context of simplicial complexes. For example, Proposition 36.10 (2) gives us a new proof that simplicial complexes are Hausdorff. More interesting of course are new results. The following corollary summarizes some of the nicest and most important ones.

Corollary 61.25.

(1) *Simplicial complexes are normal.*
(2) *Simplicial complexes are locally contractible.*[953]
(3) *A simplicial complex is connected if and only if it is path-connected.*
(4) *Let X be a simplicial complex and let A be a subcomplex. The following statements hold:*
  (a) *The inclusion $A \to X$ is a cofibration.*
  (b) *The map*

$$\begin{array}{rcl} \varphi\colon \operatorname{Cone}(i\colon A \to X) & \to & X/A \\ P & \mapsto & \begin{cases} [P], & \text{if } P \in X, \\ [A], & \text{if } P = [(a,t)] \text{ with } a \in A \text{ and } t \in [0,1] \end{cases} \end{array}$$

  is a homotopy equivalence.
  (c) *If A is contractible, then the projection $X \to X/A$ is a homotopy equivalence.*
(5) *If X is a finite simplicial complex, then*

$$\chi(X) \;=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \text{number of } n\text{-simplices of } X.$$

PROOF.

(1),(2) These two statements follow from Lemma 61.24 together with Proposition 36.10 (2) and (6).
(3) This statement follows from (1) combined with Lemma 2.72.
(4) (a) This statement follows from Lemma 61.24 together with Theorem 39.1.
  (b) This statement follows from (a) together with Lemma 39.9 (2).
  (c) This statement follows from (a) together with Lemma 39.9 (3).

[953] In the rather unlikely event that the reader forgot what these properties are good for, recall that by Corollary 29.9 this means in particular that any path-connected simplicial complex admits a universal covering. In Exercise 61.9 we will see that any covering of a simplicial complex is again a simplicial complex.

(5) This statement follows from Lemma 61.24 together with Proposition 55.1. ■

Now we can give a partial answer to Question 61.18.

Corollary 61.26. *Let Y be a simplicial complex and let X be a subcomplex. If Y is finite, then there exists a finite simplicial complex, namely $Y \cup \mathrm{Cone}(X)$, that is homotopy equivalent to Y/X.*

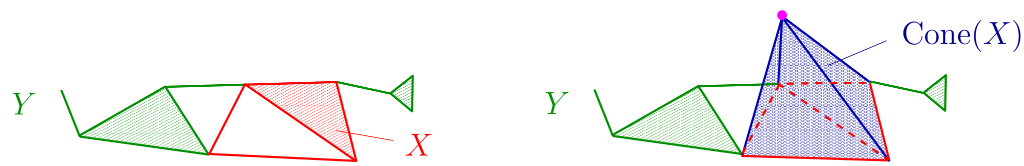


FIGURE 917. Illustration of Corollary 61.26.

Remark. We leave it as an entertainment for long winter nights to figure out whether Lemma 36.8 can be used to generalize Corollary 61.26 to infinite simplicial complexes.

PROOF. Basically by definition we only need to deal with abstract simplicial complexes and their topological realizations. Thus let $L = (W, T)$ be a finite abstract simplicial complex and let $K = (V, S)$ be a subcomplex. We denote by $i\colon K \to L$ the inclusion map. Recall that i is, basically by definition, simplicial. Since K and L are finite we can consider the abstract simplicial complex $\mathrm{Cone}(i\colon K \to L)$ that we will introduce on page 1533. Now we see that

$$|L|/|K| \underset{\substack{\uparrow\\ \text{Corollary 61.25}}}{\xleftarrow{\simeq}} \mathrm{Cone}(|i|\colon |K| \to |L|) = |L| \cup_{|K|} \mathrm{Cone}(|K|)) \underset{\substack{\uparrow\\ \text{Corollary 61.21}}}{\xleftarrow{\cong}} \overbrace{|L| \cup_{|K|} |\,\mathrm{Cone}(K)|}^{=|\,\mathrm{Cone}(i\colon K \to L)|}$$

■

It is pretty clear that there is no naive converse to Lemma 61.24. For example, the CW-complex constructed in the proof of Lemma 61.24 has some special properties:

(1) It follows from Lemma 61.5 (2) that each characteristic map is an embedding.
(2) The CW-structure on the image of a characteristic map of an n-cell is equivalent to the one corresponding to the standard simplicial structure on $\partial\Delta^n$ that we defined on page 1498.

Both types of examples are illustrated in Figure 918. Thus we see that many, arguably most, CW-structures do not come directly from simplicial structures.

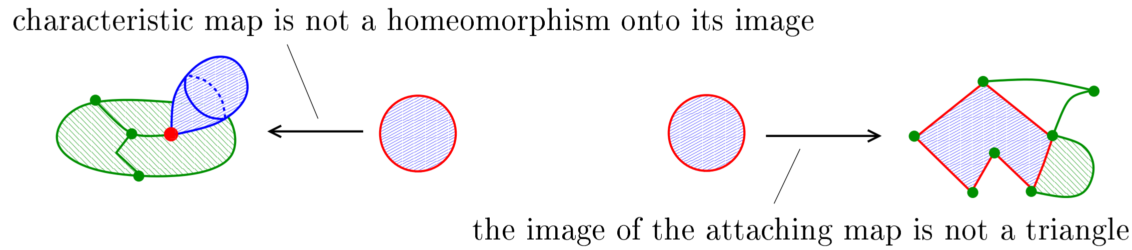


FIGURE 918

It is perhaps more reasonable to ask, when does a CW-complex admit a simplicial structure? For example it is fairly obvious that the two cellular complexes shown in Figure 918 admit a simplicial structure. To formulate the relevant proposition we need the following definition.

Definition. A CW-complex is called *regular* if every characteristic map is an embedding.

Examples.

(1) The CW-complex shown in Figure 918 to the left is non-regular, whereas the CW-complex shown to the right is regular.
(2) Let X be a simplicial complex. We use Lemma 61.24 to view X as CW-complex. It follows from Lemma 61.5 (3) that this CW-complex is regular.

Now we can formulate the promised proposition.

Proposition 61.27. *Every (countable) regular* CW*-complex admits the structure of a (countable) simplicial complex.*

Sketch of proof. We will not make use of this theorem, thus we refer to [**Geo08**, Corollary 5.3.9] or alternatively to [**FrPi90a**, Theorem 3.4.1] for a proof.

Here we only provide a sketch of a sketch of the proof. Thus let X be a regular CW-complex. We turn inductively the skeleta X^n into a simplicial complex. Suppose we already equipped X^{n-1} with a simplicial structure. For each n-cell of X^n we pick a point in the corresponding open cell and we use it to divide said n-cell into a suitable number of n-simplices. If the CW-complex X was countable, then it follows from the construction together with Lemma 1.7 that the simplicial complex is also countable. The reader should have no troubles with turning the above sketch of an argument into a formal proof. ■

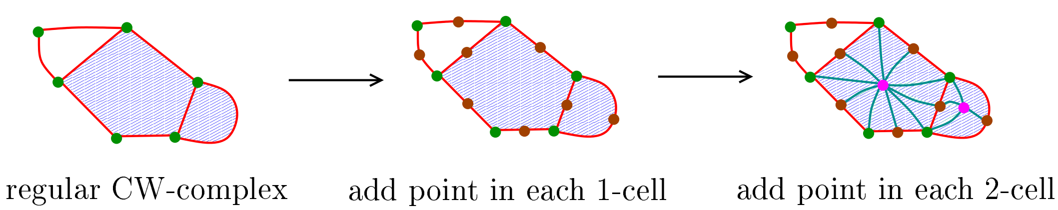


Figure 919. Illustration for the proof of Proposition 61.27.

The relatively strong hypothesis of Proposition 61.27 indicates that it is perhaps too much to hope that every CW-complex admits a simplicial structure. Indeed, this fear turns out to be fully justified:

Proposition 61.28. *There exists a finite* CW*-complex that does not admit a simplicial structure.*

Remark. Proposition 61.28 is rather disappointing. Later on in Proposition 62.12 and Theorem 85.20 we will see that every (finite) CW-complexes is homotopy equivalent to a (finite) simplicial complexes.

PROOF. We consider the function

$$f\colon [0,1] \to \mathbb{R} \qquad t \mapsto \begin{cases} 0, & \text{if } t=0, \\ t\cdot\cos(\frac{2\pi}{t}) & \text{if } t\in(0,1] \end{cases}$$

and we consider the map

$$g\colon [0,1]^2 \to \mathbb{R}^3 \qquad (s,t) \mapsto (s, s\cdot t, f(t)).$$

Note that f assumes its maximum at $t=1$ and that it assumes its minimum at a unique $t_0 \in (0,1)$. We set $X := g([0,1]^2)$. It is not particularly difficult to see that X admits a CW-structure with five 0-cells, five 1-cells and one 2-cell.[954] We refer to Figure 920 for an illustration.

In [**FrPi90a**, p. 128-130] a careful argument is given why X does not admit a simplicial structure. Other examples of finite CW-complexes that do not admit a simplicial structure are given in [**LW69**, p. 81] (with a very sketchy proof) and [**Met67**]. ■

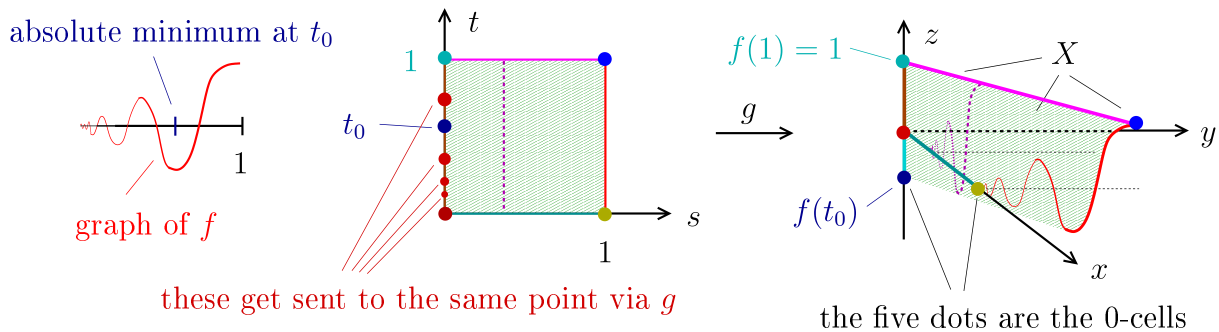


FIGURE 920. Illustration for the proof of Proposition 61.28.

Exercises for Chapter 61.

Exercise 61.1. For each $g \geq 2$ give an explicit simplicial structure for the surface of genus g.

Exercise 61.2. Give an explicit simplicial structure for $\mathbb{R}\mathrm{P}^2$.
Remark. In Exercise 62.5 we will construct an explicit simplicial structure for all $\mathbb{R}\mathrm{P}^n$.

Exercise 61.3. In Figure 904 we gave a simplicial structure for the torus $S^1 \times S^1$ with 18 2-simplices. Does there exist a simplicial structure with less than 18 2-simplices?

Exercise 61.4.
(a) Show that the set of isomorphism classes of finite abstract simplicial complexes is countable.
(b) Show that the set of isomorphism classes of countable abstract simplicial complexes is uncountable.

Exercise 61.5. We consider the abstract simplicial complex $K=(V,S)$ which is given by $V=\mathbb{Z}\cup\{*\}$ and $S=\{\{v\}\,|\,v\in V\}\cup\{\{*,n\}\,|\,n\in\mathbb{Z}\}$. Show that $|K|$ is not homeomorphic to a linear simplicial complex.

[954] The five 0-cells and the five 1-cells are clearly indicated in Figure 920.

Remark. (V, S) is just the simplicial cone on the abstract simplicial complex with vertex set $\mathbb{Z}$ and which has no simplices of dimension ≥ 1.

Exercise 61.6. Let $m, n \in \mathbb{N}_0$. Show that the simplicial join $D_m * D_n$ of the abstract simplicial complexes D_m and D_n is simplicially isomorphic to D_{m+n+1}.

Exercise 61.7. Let K be a finite n-dimensional simplicial complex. Show that K is homeomorphic to a finite linear simplicial complex that is contained in $\mathbb{R}^{2n+1}$.
Remark. In Proposition 56.7 we already showed that there exist finite 1-dimensional simplicial complexes that cannot be embedded into $\mathbb{R}^{2n}$. In fact the van Kampen-Flores Theorem, which can be proved using the Borsuk-Ulam Theorem 59.3, says that given any $n \in \mathbb{N}$ there exists a finite n-dimensional simplicial complex K that does not admit a map $K \to \mathbb{R}^{2n}$ which is a homeomorphism onto its image. We refer to [**Mato08**, Theorem 5.1.1] for details.

Exercise 61.8. Let $K = (V, S)$ be an abstract simplicial complex that is finite-dimensional and countable. Furthermore we assume that it is locally finite, which means that each simplex is the face of only finitely many simplices. Show that $|K|$ is homeomorphic to a linear simplicial complex.
Hint. Use Proposition 2.45.

Exercise 61.9. Let K be a simplicial complex and let $p\colon \widetilde{K} \to K$ be a covering in the sense of the definition on page 493. Show that $\widetilde{K}$ admits a structure of a simplicial complex such that p is simplicial.

Exercise 61.10. Let $K = (V, S)$ be an abstract simplicial complex and let $J = (U, R)$ be a subcomplex such that for any choice of simplices $r \in R$ and $s \in S \setminus R$ we have $r \cap s = \varnothing$. Show that $|J|$ is an open subset of $|K|$.
Remark. Note that by Lemma 61.8 (2) we know that $|J|$ is always a closed subset of $|K|$.

Exercise 61.11. Let $K = (V, S)$ be an abstract simplicial complex. We say $K = (V, S)$ is *connected* if given any $v, v' \in V$ there exist 1-simplices $s_0, \ldots, s_k \in S$ such that $v \in s_0$, such that for every $i \in \{0, \ldots, k-1\}$ we have $s_i \cap s_{i+1} \neq \varnothing$ and such that $v' \in s_k$.[955] Show that the following statements are equivalent:

(a) The abstract simplicial graph K is connected.
(b) Given any two simplices $t, t' \in S$ there exist simplices $s_0, \ldots, s_k \in S$ such that $t \subset s_0$, such that for every $i \in \{0, \ldots, k-1\}$ we have $s_i \cap s_{i+1} \neq \varnothing$ and such that $t' \in s_k$.
(c) The topological realization $|K|$ is connected.

For completeness we mention that Corollary 61.25 (3) says that (c) is equivalent to the following statement:

(d) The topological realization $|K|$ is path-connected.

Remark. The equivalence of (a) and (b) is straightforward.
Remark. For the equivalence of (b) and (c) use Lemma 2.62, Lemma 61.8 (2) and Exercise 61.10.

[955]This definition is modelled on the definition of the connectedness of an abstract graph, see page 221.

Exercise 61.12. Let $K = (V, S)$ and $L = (W, T)$ be abstract simplicial complexes. We denote by X and Y the corresponding CW-complexes as defined in Lemma 61.24. Let $X \otimes Y$ be the topological space that is given by the product CW-structure, see page 963. Show that there exists a homeomorphism $|K \times L| \to X \otimes Y$.
Remark. Note that here we certainly do not assume that K and L are finite.

Exercise 61.13. Let X be a topological space that admits the structure of a 1-dimensional CW-complex. Show that X also admits the structure of a 1-dimensional simplicial complex.

62. The Simplicial Approximation Theorem

In Chapter 61 we covered in great detail the basics of (abstract) simplicial complexes. In this chapter we will formulate and prove the first major result on simplicial complexes, namely the Simplicial Approximation Theorem.

62.1. **The barycentric subdivision.**

In this section we introduce the barycentric subdivision of an abstract simplicial complex. This construction allows us to replace an abstract simplicial complex by a "finer" abstract simplicial complex. This will be a major tool in the proof of several theorems.

Definition. Let $K=(V,S)$ be an abstract simplicial complex.

(1) For each k-simplex $s=\{v_0,\dots,v_k\}\in S$ we define the *barycenter of s* to be the point

$$\underline{s} \;:=\; \sum_{v\in s}\frac{1}{\dim(s)+1}\cdot v \;=\; \frac{1}{k+1}\cdot v_0+\dots+\frac{1}{k+1}\cdot v_k \;\in\; \langle s\rangle \;\subset\; \mathbb{R}^{(V)}.$$

Note that if $v\in V$ is a vertex, then the barycenter of the corresponding 0-simplex $\{v\}$ is v itself.

(2) Given $s,t\in S$ we write $\underline{s}\le\underline{t}$ if $s\subset t$.

The following lemma summarizes a few properties of the barycenters.

Lemma 62.1. $(*)$ *Let $K=(V,S)$ be an abstract simplicial complex.*

(1) *The map $S\mapsto\mathbb{R}^{(V)}$ given by $s\mapsto\underline{s}$ is natural.*
(2) *For simplices $s,t\in S$ with $s\ne t$ we have $\underline{s}\ne\underline{t}$.*
(3) *For each 0-simplex s we have $s=\underline{s}$*[956] *and for each simplex s of dimension ≥ 1 we have $\underline{s}\notin V$.*

Proof $(*)$.

(1) The most difficult part is to figure out what "natural" means in this context. We leave it to the reader to figure this out. Afterwards the statement is basically obvious.
(2) This statement follows from Lemma 61.11 (2).
(3) This statement is trivial. (But it is nonetheless useful.) ■

The following lemma is much more interesting.

Lemma 62.2. *Let $K=(V,S)$ be an abstract simplicial complex.*

(1) *The pair*

$$\operatorname{sd}(K) \;=\; \big(\{\underline{s}\}_{s\in S},\{\{\underline{s_0},\dots,\underline{s_k}\}\,|\,s_0\subsetneq\dots\subsetneq s_k\}\big)$$

is an abstract simplicial complex, called the barycentric subdivision of K.

(2) *The relation "$\le$" on the vertex set $\{\underline{s}\}_{s\in S}$ turns the barycentric subdivision* $\operatorname{sd}(K)$ *into an* ordered *abstract simplicial complex.*

The definition of $\operatorname{sd}(K)$ *is illustrated in Figure 921.*[957]

[956] Here we view, as usual, V as a subset of $\mathbb{R}^{(V)}$.

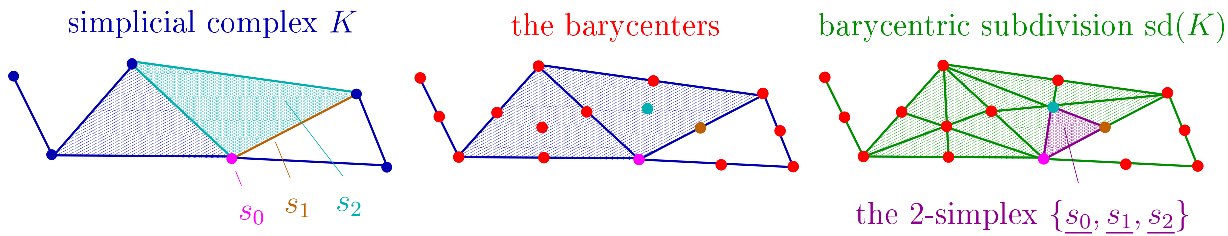


FIGURE 921

Remark.

(1) Perhaps it is helpful to restate the definition of the barycentric subdivision $\mathrm{sd}(K)$ in other words: The vertices of $\mathrm{sd}(K)$ are precisely the barycenters of the original simplices and the simplices of the barycentric subdivision are given by the barycenters of an ascending sequence of simplices in K.
(2) The statement of Lemma 62.2 (2) is quite amazing: ex nihilo we created order where there was chaos.
(3) There is an alternative more formal, but basically equivalent definition of the barycentric subdivision. Namely, given an abstract simplicial complex $K = (V, S)$ one can consider the following abstract simplicial complex:
$$\widetilde{\mathrm{sd}}(K) \;=\; \{S, \{\{s_0, \dots, s_k\} \mid s_0 \subsetneq \cdots \subsetneq\}\}.$$
The map $s \mapsto \underline{s}$ induces a natural isomorphism between the abstract simplicial complex $\widetilde{\mathrm{sd}}(K)$ and the abstract simplicial complex $\mathrm{sd}(K)$. Our first definition is arguably mnemonically superior, the second definition, which is for example used in [**dLon13**, p. 179], is arguably purer.

PROOF. The statements follow immediately from the definitions. ■

Notation. Let $K = (V, S)$ be an abstract simplicial complex. Given $i \in \mathbb{N}$ we refer to
$$\mathrm{sd}^i(K) \;:=\; \underbrace{\mathrm{sd}(\dots\mathrm{sd}(\mathrm{sd}(K))\dots)}_{\mathrm{sd}\text{ applied } i \text{ times}}$$
as the *i-th iterated barycentric subdivision of K.*

The following lemma summarizes the most important properties of the barycentric subdivision of an abstract simplicial complex.

Lemma 62.3. *Let $K = (V, S)$ be an abstract simplicial complex.*
(1) *If K is finite, then $\mathrm{sd}(K)$ is also finite.*
(2) *The dimension of $\mathrm{sd}(K)$ equals the dimension of K.*

[957] Here, as on many other occasions, we blur the difference between an abstract simplicial complex, its topological realization, and a linear simplicial complex that is simplicially isomorphic to the topological realization. In particular the illustrations shown in Figure 921 are dramatically dimensionally reduced.

(3) *The map*

$$\begin{array}{rcl} \mho\colon |\operatorname{sd}(K)| & \to & |K| \\ \alpha & \mapsto & \left(\begin{array}{rcl} V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \sum\limits_{s\in S \text{ with } v\in s} \frac{1}{\dim(s)+1} \cdot \alpha(\underbrace{\underline{s}}_{\uparrow}) \end{array} \right) \end{array}$$

$$\underline{s} \textit{ is a vertex of } \operatorname{sd}(K)\textit{, hence } \alpha(\underline{s}) \textit{ is defined}$$

has the following properties:

(a) *Let σ be a k-simplex of $\operatorname{sd}(K)$. By definition this means that $\sigma = \{\underline{s_0}, \ldots, \underline{s_k}\}$ where $s_0 \subsetneq \cdots \subsetneq s_k$ is an ascending sequence of simplices of K. The restriction of the map $\mho\colon |\operatorname{sd}(K)| \to |K|$ to $|\sigma|$ is given by*[958]

$$\begin{array}{ccc} |\sigma| & \to & |K| \\ \sum\limits_{i=0}^{k} \lambda_i \cdot \underset{\uparrow}{\underline{s_i}} & \mapsto & \sum\limits_{i=0}^{k} \lambda_i \cdot \underset{\uparrow}{\underline{s_i}} \\ \textit{viewed as basis element of } \mathbb{R}^{(\operatorname{sd}(K))} & & \textit{viewed as vector in } \mathbb{R}^{(V)} \end{array}$$

In particular we have $\mho(|\sigma|) \subset |s_k|$ and $\mho(\langle\sigma\rangle) \subset \langle s_k\rangle$.

(b) *The map $\mho$ sends each vertex*[959] *$\underline{s} \in \operatorname{sd}(K)$ to the corresponding point $\underline{s} \in |K|$.*

(c) *Given any k-simplex t of K the map $\mho$ restricts to a homeomorphism*

$$\bigcup_{\substack{s = \{\underline{a_0}, \ldots, \underline{a_{k-1}}, \underline{t}\} \\ \text{a } k\text{-simplex of } \operatorname{sd}(K) \text{ where} \\ a_0 \subsetneq \cdots \subsetneq a_{k\text{-}1} \subsetneq t \\ \text{are simplices of } K}} |s| \xrightarrow{\cong} |t|.$$

(d) *The map $\mho$ is a natural homeomorphism.*

(4) *If $f\colon K = (V,S) \to L = (W,T)$ is a simplicial map between two abstract simplicial complexes, then the map*

$$\{\underline{s}\} \mapsto \{\underline{f(s)}\}$$

defines a simplicial map $f_*\colon \operatorname{sd}(K) \to \operatorname{sd}(L)$. This map has the property that the following diagram commutes:

$$\begin{tikzcd} |\operatorname{sd}(K)| \arrow[r, "\mho_K"] \arrow[d, "|f_*|"] & |K| \arrow[d, "|f|"] \\ |\operatorname{sd}(L)| \arrow[r, "\mho_L"] & |L|. \end{tikzcd}$$

(5) *The maps from (4) turn $K \mapsto \operatorname{sd}(K)$ into a covariant functor from the category of abstract simplicial complexes to the category of (ordered) abstract simplicial complexes.*

[958] Note that this shows that $\mho$ does take values in $|K|$.

[959] Here we use the convention from page 1486 that says that we view the vertex of an abstract simplicial complex as a subset of the topological realization.

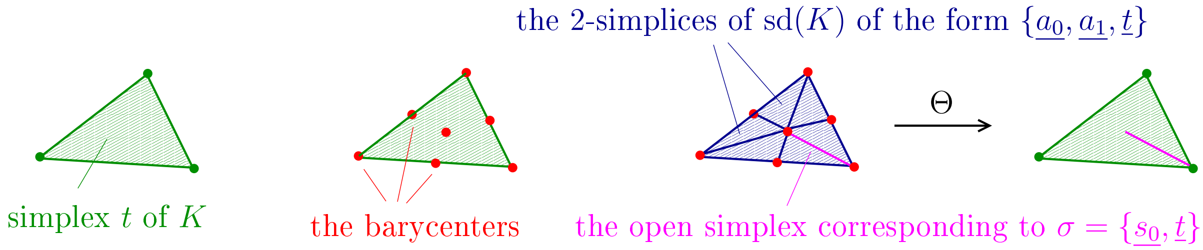


FIGURE 922. Illustration of Lemma 62.3 (3).

Given any abstract simplicial complex K and given any $i \in \mathbb{N}$ we will use the homeomorphism from Lemma 62.3 (3) to make the identification $|\operatorname{sd}^i(K)| = |K|$.

PROOF.

(1) This statement is basically clear.
(2) We will prove this statement in Exercise 62.3.
(3) Now we need to show that $\mho$ has all the promised properties:
   (a) Let $\sigma = \{\underline{s_0}, \dots, \underline{s_k}\}$ be a k-simplex of sd(K). Given a point $P = \sum_{i=0}^{k} \lambda_i \cdot \underline{s_i}$ we calculate that

$$\mho(P) = \mho\Big(\sum_{i=0}^{k} \lambda_i \cdot \underline{s_i}\Big) = \sum_{i=0}^{k} \lambda_i \cdot \mho(\underline{s_i}) = \sum_{i=0}^{k} \lambda_i \cdot \mho\Big(\underbrace{\sum_{v \in s_i} \tfrac{1}{\dim(s_i)+1} \cdot v}_{=\underline{s_i}}\Big) = \sum_{i=0}^{k} \lambda_i \cdot \underline{s_i}.$$

   Note that $\underline{s_0}, \dots, \underline{s_k}$ lie in $|s_k|$. Thus it follows from the calculation, together with the remark on page 1510, that $\mho(|\sigma|) \subset |s_k|$. Furthermore, if $P \in \langle\sigma\rangle$, then $\lambda_k > 0$, which in turn implies almost immediately that $\mho(P) \in \langle s_k \rangle$.[960]
   (b) This statement follows immediately from (a).
   (c) Let t be a k-simplex t of K. We leave it to the reader to verify that $\mho$ restricts to a bijection

$$X := \bigcup_{s = \{\underline{a_0}, \dots, \underline{a_{k-1}}, \underline{t}\}} |s| \xrightarrow{\cong} |t|.$$

   Note that the left-hand side X is actually a finite subcomplex of $|\operatorname{sd}(K)|$. Given a simplex $s = \{\underline{a_0}, \dots, \underline{a_{k-1}}, \underline{t}\}$ we pick a characteristic map $\Phi_s \colon \Delta^k \to |s|$. Furthermore we pick a characteristic map $\Phi_t \colon \Delta^k \to |t|$. It follows almost immediately from the definitions that the map $\Phi_t^{-1} \circ \mho \circ \Phi_s \colon \Delta^k \to \Delta^k$ is an affine linear map, in particular it is continuous. It follows from Lemma 61.6 (1) and Lemma 61.5 that the above map $\mho \colon X \to |t|$ is continuous. It now follows, as always, from Proposition 2.43 (3) together with Proposition 61.12 and Lemma 61.5 that the map $\mho \colon X \to |t|$ is a homeomorphism.

[960]For the proof that $\mho(\langle\sigma\rangle) \subset \langle s_k \rangle$ it is arguably more efficient to work with the initial description of $\mho$.

(d) Using (c) one can easily verify that $\mho$ is a bijection. Furthermore, it follows from Lemma 61.6 (1) together with (c) that $\mho$ is continuous. Finally, it follows from (c) that for each simplex t of K the map $\mho^{-1}\colon |t| \to |\operatorname{sd}(K)|$ is continuous. By Lemma 61.6 (1) this shows that $\mho^{-1}\colon |K| \to |\operatorname{sd}(K)|$ is continuous. We have thus shown that $\mho$ is indeed a homeomorphism. It follows easily from the definitions and Lemma 62.1 (1) that the homeomorphism is in fact natural.

(4) We leave the elementary task of verifying this statement to the reader.

(5) This statement follows almost immediately from the definitions. The only step which requires a short moment of reflection is to show that if $f\colon K \to L$ is a simplicial map, then the induced map $f_*\colon \operatorname{sd}(K) \to \operatorname{sd}(L)$ is actually order-preserving, i.e. it is a morphism in the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes. ■

The following definition is almost self-explanatory.

Definition. Let $(X, (K, \Theta\colon |K| \to X))$ be a simplicial complex and let $i \in \mathbb{N}_0$. We define the *i-th barycentric subdivision* of (X, Θ) to be $(X, (\operatorname{sd}^i(K), \Theta\colon |\operatorname{sd}^i(K)| = |K| \to X))$

For completeness we state the following elementary lemma.

Lemma 62.4. *For every $k \in \mathbb{N}_0$ the barycentric subdivision of a linear simplicial complex in $\mathbb{R}^n$ is again a linear simplicial complex in $\mathbb{R}^n$.*

Proof. This statement follows almost immediately from the definition of the barycentric subdivision. ■

62.2. **Stars and links in simplicial complexes.** Given an (abstract) simplicial complex K and a simplex s we introduce and study the corresponding star $\operatorname{St}(K, s)$ and $\operatorname{Lk}(K, s)$. Both concepts are elementary, but both also play a major role in the study of (abstract) simplicial complexes.

Definition. Let $K = (V, S)$ be an abstract simplicial complex.

(1) Let $s \in S$ be a simplex. We introduce the following subsets of $|K|$:

$$\textit{star}\ \operatorname{St}(K,s) := \bigcup_{t \in S \text{ with } s \subset t} |t| \quad \text{and the} \quad \textit{open star}\ \mathring{\operatorname{St}}(K,s) := \bigcup_{t \in S \text{ with } s \subset t} \langle t \rangle.$$

Furthermore we define the *link* of s to be

$$\operatorname{Lk}(K,s) := \bigcup_{\substack{t \in S \text{ with } s \cup t \in S \\ \text{but with } s \cap t = \varnothing}} |t|.$$

(2) Given a vertex v we use the obvious shorthand notations

$$\operatorname{St}(K,v) := \operatorname{St}(K,\{v\}),\ \ \mathring{\operatorname{St}}(K,v) := \mathring{\operatorname{St}}(K,\{v\}) \ \text{ and } \ \operatorname{Lk}(K,v) := \operatorname{Lk}(K,\{v\}).$$

If K is understood, then we drop it from the notation. Furthermore we extend the definition in the obvious way to simplicial complexes. (We refer to Figures 923 and 924 for illustrations of the definitions.)

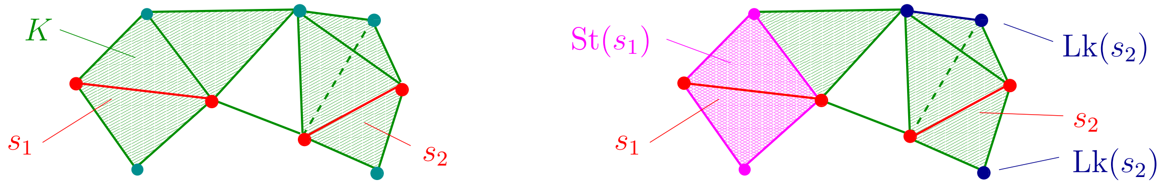


FIGURE 923

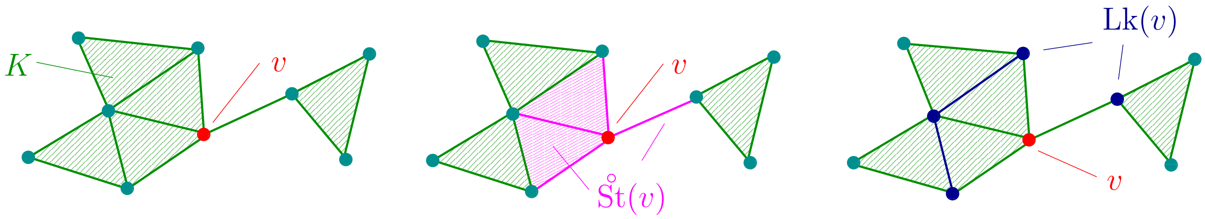


FIGURE 924

Example. Let K be an abstract simplicial complex. We consider the simplicial cone $\operatorname{Cone}(K) = K * \{\star\}$ and the simplicial suspension $\Sigma(K) = K * \{\pm 1\}$ as defined on page 1506. Let $\epsilon \in \{-1, \}$. It follows easily from the definitions that $\operatorname{St}(\Sigma(K), \epsilon)$ is simplicially isomorphic to $\operatorname{Cone}(K)$ and that $\operatorname{Lk}(\Sigma(K), \epsilon) = K$.

Lemma 62.5. *Let $(X, (K = (V, S), \Theta\colon |K| \to X))$ be a simplicial complex.*

(1) *For every simplex $s \in S$ the following statements hold:*
   (a) *We have the equality*
$$\operatorname{Lk}(s) \;=\; \operatorname{St}(s) \setminus \bigcup_{r \in S \text{ with } r \subset s} \mathring{\operatorname{St}}(r).$$
   (b) *The star $\operatorname{St}(s)$ and the link $\operatorname{Lk}(s)$ are subcomplexes of X. The corresponding abstract simplicial complexes are given by*
$$\Big(\bigcup_{\substack{t \in S \text{ with} \\ s \subset t}} t, \{t \in S \,|\, s \cup t \in S\}\Big) \text{ and } \Big(\bigcup_{\substack{t \in S \text{ with} \\ s \subset t}} t \setminus s, \{t \in S \,|\, s \cup t \in S \text{ and } s \cap t = \varnothing\}\Big).$$
   Furthermore $X \setminus \mathring{\operatorname{St}}(s)$ is a subcomplex, the corresponding abstract simplicial complex is given by
$$(V \setminus s, \{t \in S \,|\, t \cap s = \varnothing\}.$$
   (c) *The star $\operatorname{St}(s)$ is naturally simplicially isomorphic to the simplicial join of the link $\operatorname{Lk}(s)$ with the simplex s.*[961962]
   (d) (α) *The open star $\mathring{\operatorname{St}}(s)$ is an open neighborhood of every point $P \in \Theta(\langle s \rangle)$, in particular $\mathring{\operatorname{St}}(s)$ is an open subset of X.*
   (β) *The star $\operatorname{St}(K, s)$ is a closed neighborhood of every point $P \in \Theta(\langle s \rangle)$, in particular $\operatorname{St}(K, s)$ is a closed subset of X.*
   (e) (α) *The star $\operatorname{St}(s)$ admits a deformation retraction to the simplex $\Theta(|s|)$.*

(β) *The open star* $\mathring{\mathrm{St}}(s)$ *admits a deformation retraction to the open simplex* $\Theta(\langle s\rangle)$.

(f) *Both* $\mathrm{St}(s)$ *and* $\mathring{\mathrm{St}}(s)$ *are contractible.*

(g) (α) *The link* $\mathrm{Lk}(s)$ *is a deformation retract of* $\mathrm{St}(s)\setminus\Theta(|s|)$.

(β) *Given any* $P\in\Theta(\langle s\rangle)$ *the subcomplex* $\partial s * \mathrm{Lk}(s)$ *is a deformation retract of* $\mathrm{St}(s)\setminus\{P\}$.

(h) *If* t *is a simplex with* $s\subset t$*, then* $\mathrm{St}(t)\subset\mathrm{St}(s)$.

(2) (a) *Let* $s\in S$ *be a simplex. For* $P\in\Theta(\langle s\rangle)$ *and* $v\in V$ *we have* $P\in\mathring{\mathrm{St}}(v)$ *if and only if* $v\in S$.

(b) *The open stars* $\{\mathring{\mathrm{St}}(v)\}_{v\in V}$ *defined by the vertices are an open cover of* X.

(3) *Given simplices* $s_0,\dots,s_n\in S$ *the following equality holds:*[963]

$$\mathring{\mathrm{St}}(s_0)\cap\dots\cap\mathring{\mathrm{St}}(s_n) \;=\; \begin{cases}\mathring{\mathrm{St}}(s_0\cup\dots\cup s_n), & \textit{if } s_0\cup\dots\cup s_n \textit{ is a simplex,}\\ \varnothing, & \textit{otherwise.}\end{cases}$$

(*Note that in Exercise 62.1 we will see that the analogous statement for the stars does* not *hold.*)

(4) *Given vertices* $v_0,\dots,v_n\in V$ *the following statement holds:*

$$\mathring{\mathrm{St}}(v_0)\cap\dots\cap\mathring{\mathrm{St}}(v_n)\neq\varnothing \quad\Longleftrightarrow\quad \{v_0,\dots,v_n\} \textit{ is a simplex of } K=(V,S).$$

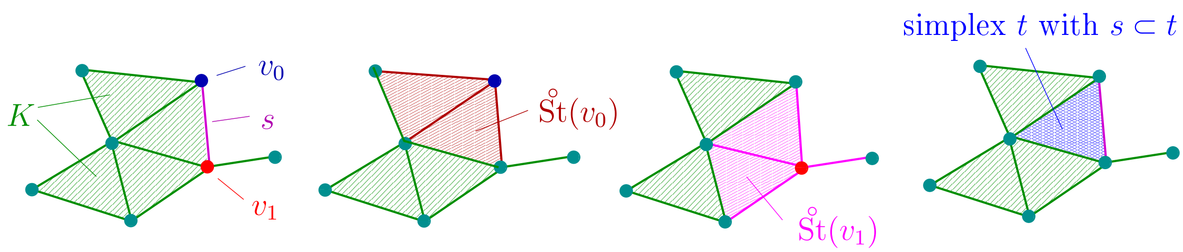


FIGURE 925. Illustration for Lemma 62.5 (3).

[961]Here we view $\mathrm{St}(s)$, $\mathrm{Lk}(s)$ and s as simplicial complexes.

[962] In Exercise 62.2 we will prove the related statement that there exists a natural simplicial isomorphism $\mathrm{Lk}(K,s)*\partial s\cong\mathrm{Lk}(\mathrm{sd}(K),\underline{s})$.

[963]To avoid confusion we point out that here, and in (4), we do not demand that the $s_0,\dots,s_n$ are distinct.

Proof. Basically by definition we can assume that we are dealing with the topological realization $X = |K|$ of an abstract simplicial complex $K = (V, S)$.

(1) Let $s \in S$ be a k-simplex.

(a) We have the following equalities:

$$\begin{aligned}\mathrm{St}(s) \setminus \bigcup_{r\subset s} \mathring{\mathrm{St}}(r) &= \bigcup_{s\subset t} |t| \setminus \bigcup_{r\subset s}\bigcup_{r\subset t} \langle t\rangle \overset{\substack{\text{follows from Lemma 61.11 (5) applied to the subcomplex defined by } t\\ \downarrow}}{=} \bigcup_{s\subset t} \bigsqcup_{u\subset t} \langle u\rangle \setminus \bigcup_{r\subset s}\bigsqcup_{r\subset u} \langle u\rangle \\ &= \bigcup_{\substack{u\in S \text{ with } s\cup u\in S\\ \text{but with } s\cap u = \varnothing}} \langle u\rangle = \bigcup_{\substack{t\in S \text{ with } s\cup t\in S\\ \text{but with } s\cap t = \varnothing}} \bigsqcup_{u\subset t} \langle u\rangle = \underbrace{\bigcup_{\substack{t\in S \text{ with } s\cup t\in S\\ \text{but with } s\cap t = \varnothing}} |t|}_{=\mathrm{Lk}(s)}.\end{aligned}$$

(b) It follows easily from the definitions, (a) and Lemma 61.14 (4) that the star $\mathrm{St}(s)$, that the link $\mathrm{Lk}(s)$ and that $|K| \setminus \mathring{\mathrm{St}}(s)$ are subcomplexes of X and that the corresponding abstract simplicial complexes are the ones we wrote down.

(c) This statement follows immediately from the description of the abstract simplicial complexes given in (b) and the definition of the simplicial join that we gave on page 1506.

(d) First we show that $\mathring{\mathrm{St}}(s)$ is an open subset of $|K|$. By the definition of the topology on $|K|$, see page 1487, we need to show that for every k-simplex u, with some characteristic map $\Phi_u\colon \Delta^k \to |u|$, the preimage $\Phi_u^{-1}(\mathring{\mathrm{St}}(s))$ is an open subset of Δ^k. In fact

$$\Phi_u^{-1}\big(\mathring{\mathrm{St}}(s)\big) \underset{\substack{\uparrow\\ \text{by definition}}}{=} \Phi_u^{-1}\Big(\bigcup_{s\subset t}\langle t\rangle\Big) \underset{\substack{\uparrow\\ \text{Lemma 1.3 (5)}}}{=} \bigcup_{s\subset t}\big(\Phi_u^{-1}(\langle u\rangle)\big) \underset{\substack{\uparrow\\ \text{follows from Lemma 61.11 (2)}}}{=} \begin{cases} \Phi_u^{-1}(\langle t\rangle) = \Delta^k \setminus \partial\Delta^k, & \text{if } s\subset u\\ \varnothing, & \text{else.}\end{cases}$$

Since $\Delta^k \setminus \partial\Delta^k$ and $\varnothing$ are open subsets of Δ^k we are done. A very similar argument also shows that $\mathring{\mathrm{St}}(s)$ is a closed subset of $|K|$.

Now let $P \in \Theta(\langle s\rangle)$. By definition $P \in \mathring{\mathrm{St}}(s) \subset \mathrm{St}(s)$. Since $\mathring{\mathrm{St}}(s)$ is an open subset of $|K|$ we see that $\mathring{\mathrm{St}}(s)$ and $\mathrm{St}(s)$ are both neighborhoods of P.

(e) (α) We write $s = \{v_0, \dots, v_k\}$. Note that by (1) and Lemma 61.8 we can view $\mathrm{St}(s)$ as a simplicial complex in its own right. Now let t be a simplex t with $s \subset t$. We write $t = \{v_0, \dots, v_k, v_{k+1}, \dots, v_l\}$. We consider the map

$$\begin{aligned} F_t\colon |t| \times [0,1] &\to |t| \\ \Big(\sum_{i=0}^{l}\lambda_i \cdot v_i, r\Big) &\mapsto \sum_{i=0}^{k}\lambda_i \cdot v_i + (1-r)\cdot \sum_{i=k+1}^{l}\lambda_i \cdot v_i + r\cdot\Big(\sum_{i=k+1}^{l}\lambda_i\Big)\cdot v_k.\end{aligned}$$

It follows immediately from the definitions that that these maps F_t combine to give a well-defined map $F\colon \mathrm{St}(s) \times [0,1] \to \mathrm{St}(s)$. Furthermore it is fairly straightforward to deduce from the fact that $\mathrm{St}(s)$ is a simplicial complex, Lemma 61.24 and Lemma 36.8 (3) that F is actually continuous. Now that continuity is out of the way it is clear that F is a deformation retraction from $\mathrm{St}(s)$ to $|s|$.

(β) In (α) we just gave an explicit deformation retraction from $\operatorname{St}(s)$ to $|s|$. It follows basically from the definitions that this map restricts to a deformation retraction from $\mathring{\operatorname{St}}(s)$ to the open simplex $\langle s\rangle$.

(f) This statement follows immediately from (e), together with Lemma 61.5 (3) and the observation that Δ^k and $\mathring{\Delta}^k$ are contractible.

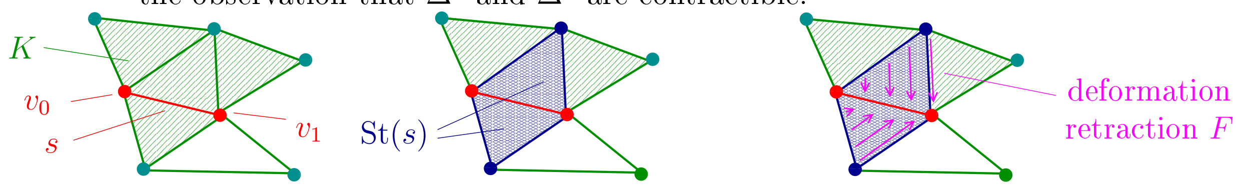


FIGURE 926. Illustration for the proof of Lemma 62.5 (1).

(g) Let $P \in \Theta(\langle s\rangle)$. The proofs that the link $\operatorname{Lk}(s)$ is a deformation retract of $\operatorname{St}(s)\setminus|s|$ and that the subcomplex $\partial s * \operatorname{Lk}(s)$ is a deformation retract of $\operatorname{St}(s)\setminus\{P\}$ is quite similar to the proof of (e). We will fill in the details in Exercise 62.4.

(h) If t is a simplex with $s \subset t$, then it follows immediately from the definitions that $\operatorname{St}(t) \subset \operatorname{St}(s)$.

(2) (a) Let $s \in S$ be a simplex, let $P \in \langle s\rangle$ and let $v \in V$. We have

$$P \in \mathring{\operatorname{St}}(v) = \bigcup_{t\in S \text{ with } v\in t} \langle t\rangle \underset{\uparrow}{\iff} \langle s\rangle \subset \bigcup_{t\in S \text{ with } v\in t} \langle t\rangle \underset{\uparrow}{\iff} v \in s.$$

both statements follow from Lemma 61.11 (2)

(b) Recall that Lemma 61.11 (2) says that given any point $P \in K$ there exists an $s \in S$ with $P \in \langle s\rangle$. By (a) we know that for any vertex v of s we have $P \in \langle s\rangle \subset \mathring{\operatorname{St}}(v)$. This shows that the open stars corresponding to the vertices cover all of K. By (1d) we know that this cover is in fact an open cover.

(3) Let $s_0, \ldots, s_n \in S$. We have the following equality:

$$\bigcap_{i=0}^{n} \mathring{\operatorname{St}}(s_i) \overset{\substack{\text{by definition}\\ \downarrow}}{=} \bigcap_{i=0}^{n} \bigcup_{\substack{t\in S \text{ with}\\ s_i\subset t}} \langle t\rangle \overset{\substack{\text{by Lemma 61.11 (5)}\\ \downarrow}}{=} \bigcup_{\substack{t\in S \text{ with}\\ s_0,\ldots,s_n\subset t}} \langle t\rangle \overset{\text{by definition of an open star}}{=} \begin{cases} \mathring{\operatorname{St}}(s_0\cup\cdots\cup s_n), & \text{if } s_0\cup\ldots\cup s_n\in S,\\ \varnothing, & \text{otherwise.}\end{cases}$$

(4) Given $v_0, \ldots, v_n \in V$ the following holds:

$$\mathring{\operatorname{St}}(v_0)\cap\cdots\cap\mathring{\operatorname{St}}(v_n) \neq \varnothing \underset{\substack{\uparrow\\ \text{by (3)}}}{\Leftrightarrow} \begin{array}{l}\text{there exists a } t\in S\\ \text{with } v_0,\ldots,v_n\in t\end{array} \underset{\substack{\uparrow\\ \text{by definition of an abstract simplicial complex}}}{\Leftrightarrow} \{v_0,\ldots,v_n\} \text{ is a simplex.}$$

■

Definition. An abstract simplicial complex $K = (V, S)$ is called *locally finite* if given any vertex $v \in V$ there exist only finitely many simplices that contain v.

The following proposition can be viewed as an analogue of Proposition 36.11.

Proposition 62.6. *Let $K = (V, S)$ be an abstract simplicial complex. The following three statements are equivalent:*

(1) *The abstract simplicial complex K is locally finite.*
(2) *For every vertex $v \in V$ the star $\mathrm{St}(K, \{v\})$ is a finite simplicial complex.*
(3) *For every vertex $v \in V$ the link $\mathrm{Lk}(K, \{v\})$ is a finite simplicial complex.*
(4) *The topological realization $|K|$ is regionally compact.*

PROOF $(*)$. Let $K = (V, S)$ be an abstract simplicial complex.

We start out with (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). This equivalence of statements follows almost immediately from the definitions and the observation that an abstract simplicial complex is finite if and only if it has finitely many vertices.

We continue with (3) $\Rightarrow$ (4). We need to show that $|K|$ is regionally compact. By Lemma 2.73 (2) together with Lemma 61.5 (2) it suffices to show that every point $P \in |K|$ admits a compact neighborhood. By Lemma 61.11 there exists a simplex $s \in S$ such that P is contained in the open simplex $\langle s \rangle$. By Lemma 62.5 (1d) we know that $\mathrm{St}(K, s)$ is a neighborhood of P. Since $\mathrm{St}(K, s)$ is a finite simplicial complex we obtain from Lemma 61.9 that $\mathrm{St}(K, s)$ is compact.

Finally we prove (4) $\Rightarrow$ (3). Now we assume that $|K|$ is regionally compact. Let $v \in V$ be a vertex. By Lemma 62.5 (1d) we know that the open star $\mathring{\mathrm{St}}(K, \{v\})$ is an open neighborhood of v. Since $|K|$ is regionally compact there exists a compact neighborhood X of v with $X \subset \mathring{\mathrm{St}}(K, \{v\})$. Since X is compact we know by Proposition 61.12 that there exists a finite subcomplex L of K such that X is contained in $|L|$.

Since X is a neighborhood of v we know that there exists an open neighborhood U of v with $v \in U \subset X$. It follows from Lemma 61.11 (7) that for any simplex s with $v \in s$ we have $U \cap \langle s \rangle \neq \varnothing$. But by Lemma 61.11 (5) that implies $|s| \subset |L|$. In other words, we see that $\mathrm{St}(K, \{v\}) \subset |L|$. Since L is finite we see that $\mathrm{St}(K, \{v\})$ is a finite simplicial complex. ∎

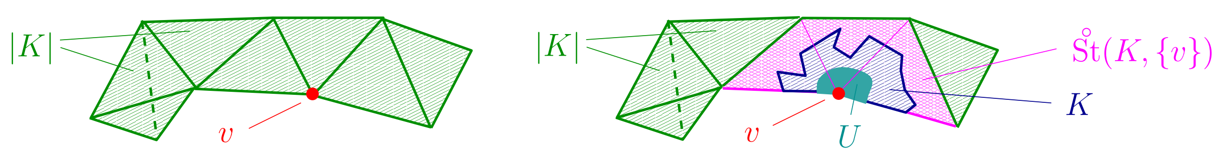


FIGURE 927. Illustration for the proof of Proposition 62.6 (4) $\Rightarrow$ (3).

We recall the following definition from page 141.

Definition. The *diameter* of a non-empty subset A of some $\mathbb{R}^n$ is defined as

$$\mathrm{diam}(A) \;:=\; \sup\{\|a - b\| \mid a, b \in A\} \;\in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

It is pretty clear that the barycentric subdivision of a linear simplicial complex results in more but smaller simplices. The following lemma is one way of formulating precisely that the simplices shrink.

Lemma 62.7. *If X is a finite linear non-empty simplicial complex in $\mathbb{R}^n$, then*

$$\lim_{k\to\infty} \max\{\mathrm{diam}(\underbrace{\mathrm{St}(\mathrm{sd}^k(X),v)}_{\text{the underlying topological space of } X \text{ and } \mathrm{sd}^k(X) \text{ is the same}}) \,|\, v \textit{ is a vertex of } \mathrm{sd}^k(X)\} \;=\; 0.$$

the maximal diameter of a star shrinks

FIGURE 928. Illustration of Lemma 62.7.

PROOF ($*$). Let Y be a finite linear non-empty d-dimensional simplicial complex in $\mathbb{R}^n$. We define the *mesh of* Y as

$$\mathrm{mesh}(Y) \;:=\; \max\{\mathrm{diam}(s) \,|\, s \text{ is a simplex of } Y\}.$$

The lemma follows from Lemma 62.3 (1) and (2), Lemma 62.4 and the following claim.

Claim.

(1) For every vertex v of Y we have $\mathrm{diam}(\mathrm{St}(v)) \leq 2 \cdot \mathrm{mesh}(Y)$.
(2) For every $k \in \mathbb{N}_0$ we have

$$\mathrm{mesh}(\mathrm{sd}^k(Y)) \;\leq\; \left(\tfrac{d}{d+1}\right)^k \cdot \mathrm{mesh}(Y).$$

We turn to the proofs of the two statements:

(1) Let v be a vertex of Y. We need to show that $\mathrm{diam}(\mathrm{St}(v)) \leq 2 \cdot \mathrm{mesh}(Y)$. Thus let $P, Q \in \mathrm{St}(v)$. By definition of the star $\mathrm{St}(v)$ there exist simplices s and t with $v, P \in s$ and with $v, Q \in t$. We see that

$$\|P - Q\| \;\underset{\substack{\uparrow\\ \text{triangle inequality}}}{\leq}\; \|P - v\| + \|Q - v\| \;\underset{\substack{\uparrow\\ \text{since } v,P\in s \text{ and } v,Q\in t}}{\leq}\; \mathrm{diam}(s) + \mathrm{diam}(t) \;\underset{\substack{\uparrow\\ \text{definition of mesh}}}{\leq}\; 2\,\mathrm{mesh}(Y).$$

(2) Once the reader has untangled the definitions it will have become clear that this statement is nothing but a reformulation of Lemma 43.29.

■

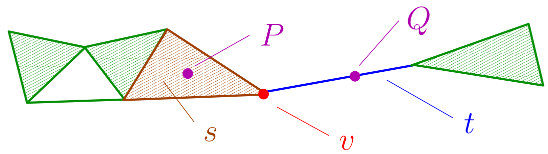


FIGURE 929. Illustration of the proof of Lemma 62.7.

62.3. **The Simplicial Approximation Theorem.** In this section we will state and prove the Simplicial Approximation Theorem which is one of the main tools in the study of simplicial complexes. Before we can state the theorem we need to introduce one last basically self-explanatory notion.

Definition. Let X be an (abstract) simplicial complex and let A be a subcomplex. We refer to (X, A) as a *pair of (abstract) simplicial complexes.*

Now we are ready to formulate the main theorem of this section.

Theorem 62.8. (Simplicial Approximation Theorem) *Let (X, A) and (Y, B) be pairs of simplicial complexes. Let $f\colon (X, A) \to (Y, B)$ be a map of pairs of topological spaces. If X is finite, then there exists an iterated barycentric subdivision $(\widetilde{X}, \widetilde{A})$ of (X, A) and a simplicial map $\widetilde{f}\colon (\widetilde{X}, \widetilde{A}) \to (Y, B)$ with the following properties:*

(1) *The map $\widetilde{f}$ is homotopic to f as a map $(X, A) \to (Y, B)$ of pairs of topological spaces.*
(2) *For every vertex v of $\widetilde{X}$ we have $f(\operatorname{St}(\widetilde{X}, v)) \subset \operatorname{St}(Y, \widetilde{f}(v))$.*

Example. We consider the simplicial complexes X and Y and the map $f\colon X \to Y$ that are illustrated in Figure 930. This map is actually *not* homotopic to a simplicial map.[964] But after a single barycentric subdivision we see that f is homotopic to a simplicial map $\widetilde{f}\colon \operatorname{sd}(X) \to Y$.

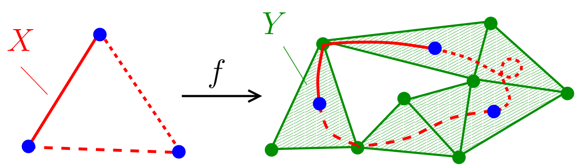


FIGURE 930

Remark.

(1) The first statement of the Simplicial Approximation Theorem 62.8 is of course somewhat similar to the statement of the Cellular Approximation Theorem 38.13. But since simplicial complexes and simplicial maps are more rigid than their cellular siblings there are some important differences:
   (a) On the negative side the Simplicial Approximation Theorem 62.8 only applies to finite domains and we need to allow for several barycentric subdivisions.
   (b) On the plus side the output is much stronger since simplicial maps are much more restrictive than cellular maps. We will make use of this shortly.
(2) The second statement of the Simplicial Approximation Theorem 62.8 essentially says that the new map $\widetilde{f}$ does not stray too far away from the original map f.

Before we give a proof of the Simplicial Approximation Theorem 62.8, let us point out an important consequence. On page 1326 we saw that homotopy groups of fairly simple topological spaces, e.g. $S^1 \vee S^2$, can be uncomfortably large, for example we saw that $\pi_2(S^1 \vee S^2)$ is not necessarily finitely generated. In Question 40.16 we asked whether

[964] Why not?

the homotopy groups of a countable CW-complex are actually countable. The following proposition gives us some evidence for a positive answer.

Proposition 62.9. *If X is a simplicial complex with countably many simplices, then for every $n \in \mathbb{N}$ and every $x_0 \in X$ the group $\pi_n(X, x_0)$ is countable.*

Proof. Let X be a simplicial complex and let $n \in \mathbb{N}$. By the discussion on page 1500 we can pick a finite simplicial structure for S^n. Let $*$ be a vertex of S^n. By Proposition 40.5 it suffices to show that for a single vertex x_0 of X the group $\pi_n(X, x_0)$ is countable.

Next note that by the Simplicial Approximation Theorem 62.8, applied to the maps $(S^n, \{*\}) \to (X, x_0)$ we have a surjection

$$\bigcup_{i \in \mathbb{N}_0} \underbrace{\{\text{set of simplicial maps } (\mathrm{sd}^i(S^n), \{*\}) \to (X, x_0)\}}_{\text{countable set by Lemma 61.3 (2)}} \twoheadrightarrow \underbrace{[(S^n, *), (X, x_0)]}_{\substack{\text{set of homotopy classes} \\ \text{of maps } (S^n, *) \to (X, x_0)}} = \pi_n(X, x_0).$$

By Lemma 61.3 (2) together with Lemma 1.7 (3) the set on the left-hand side is countable. It follows from Lemma 1.7 (3) that the set on the right-hand side, i.e. $\pi_n(X, x_0)$, is also countable. ■

It is not immediately clear how one could possibly generalize the proof of Proposition 62.9 from simplicial complexes to CW-complexes. In particular at the moment we still lack the tools to tackle Question 40.16 completely. We will return to Question 40.16 with Proposition 85.19.

Before we turn to the proof of the Simplicial Approximation Theorem 62.8 let us record the following immediate corollary to Proposition 62.9.

Corollary 62.10. *For every $n, k \in \mathbb{N}$ the group $\pi_n(S^k)$ is countable.*

Now we turn to the proof of the Simplicial Approximation Theorem 62.8.

Proof of the Simplicial Approximation Theorem 62.8. In an attempt to keep the notation at a manageable level we will assume for simplicity that $A = B = \varnothing$. We leave it to the reader to modify the subsequent proof to provide a proof for the general case.

Thus let $(X, (K = (V, S), \Theta\colon |K| \to X))$ and $(Y, (L = (W, T), \Omega\colon |L| \to Y))$ be two simplicial complexes. For convenience we identify V with its image $\Theta(V) \subset X$ and we identify W with its image $\Omega(W) \subset Y$. Furthermore let $f\colon X \to Y$ be a continuous map. We make the following preparations.

(1) Recall that we assume that X is finite. Evidently we can assume that X is non-empty.
(2) It follows from the hypothesis that X is finite and the combination of Lemma 61.9, Lemma 2.40 and Proposition 61.12 that $f(X)$ is contained in a finite subcomplex of Y. Thus we can assume that Y is also a finite simplicial complex.
(3) Since X and Y are finite it follows from Lemma 61.22 that we can and will assume that X and Y are linear simplicial complexes contained in some $\mathbb{R}^m$ respectively some $\mathbb{R}^n$. Recall that this means, by definition, that X and Y are equipped with the subspace topology coming from $\mathbb{R}^m$ and $\mathbb{R}^n$. In particular we can use the usual Euclidean metric on $\mathbb{R}^m$ to view X as a metric space.

We start out with the following claim.

Claim 1. There exists a $k \in \mathbb{N}_0$ such that for every vertex v of $\mathrm{sd}^k(X)$ there exists a $w \in W$ with $f(\mathrm{St}(\mathrm{sd}^k(X), v)) \subset \mathring{\mathrm{St}}(Y, w)$.

Recall that by Lemma 62.5 (2) we know that the open stars $\{\mathring{\mathrm{St}}(w)\}_{w \in W}$, form an open cover of Y. It follows that the sets $U_w := f^{-1}\big(\mathring{\mathrm{St}}(w)\big) \subset X$ form an open cover of X. Since X is compact we obtain from the Lebesgue Lemma 2.75 that there exists an $\epsilon > 0$ such that for every subset A of X with $\mathrm{diam}(A) < \epsilon$ there exists a $w \in W$ with $A \subset U_w$. By Lemma 62.7 there exists a $k \in \mathbb{N}_0$ such that for every vertex v of $\mathrm{sd}^k(X)$ we have $\mathrm{diam}(\mathrm{St}(\mathrm{sd}^k(X), v)) < \epsilon$. In other words, this k has the desired property. ⊞

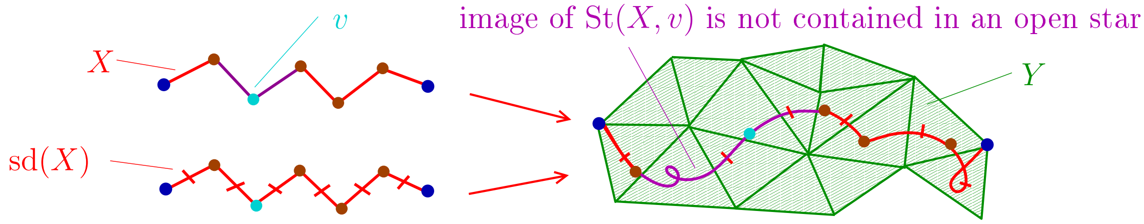


FIGURE 931. Illustration for the proof of the Simplicial Approximation Theorem 62.8

By Lemma 62.3 (1) and Lemma 62.4 we know that $\mathrm{sd}^k(X)$ is again a finite linear simplicial complex in $\mathbb{R}^m$. Thus we can safely replace X by $\mathrm{sd}^k(X)$. We are now in the following setting:

$(*)$ For every $v \in V$ there exists a $w \in W$ with $f(\mathrm{St}(X, v)) \subset \mathring{\mathrm{St}}(Y, w)$. In other words, there exists a map $\varphi\colon V \to W$ with the property that for every vertex $v \in V$ we have $f(\mathrm{St}(X, v)) \subset \mathring{\mathrm{St}}(Y, \varphi(v))$.

We continue the proof with the following claim.

Claim 2. For every simplex $s \in S$ the image $\varphi(s)$ is a simplex in T.

Let $s = \{v_0, \dots, v_k\}$ be a simplex of $K = (V, S)$. Note that

$$\begin{aligned}
&\begin{matrix} v_0, \dots, v_k \text{ form} \\ \text{a simplex in } X \end{matrix} \overset{\text{Lemma 62.5 (4)}}{\Rightarrow} \mathring{\mathrm{St}}(v_0) \cap \ldots \cap \mathring{\mathrm{St}}(v_k) \neq \varnothing \quad \Rightarrow f\big(\mathring{\mathrm{St}}(v_0) \cap \ldots \cap \mathring{\mathrm{St}}(v_k)\big) \neq \varnothing \\
&\underset{\text{Lemma 1.3 (4)}}{\Rightarrow} f(\mathring{\mathrm{St}}(v_0)) \cap \ldots \cap f(\mathring{\mathrm{St}}(v_k)) \neq \varnothing \underset{\text{by choice of } \varphi \text{ we have } f(\mathrm{St}(v_i)) \subset \mathring{\mathrm{St}}(\varphi(v_i))}{\Rightarrow} \mathring{\mathrm{St}}(\varphi(v_0)) \cap \ldots \cap \mathring{\mathrm{St}}(\varphi(v_k)) \neq \varnothing \\
&\underset{\text{Lemma 62.5 (4)}}{\Rightarrow} \{\varphi(v_0), \dots, \varphi(v_k)\} \text{ is a simplex in } T.
\end{aligned}$$

⊞

Claim 2 says that $\varphi\colon V \to W$ defines a simplicial map $K = (V, S) \to L = (W, T)$. It follows from Lemma 61.15 that there exists a simplicial map $g\colon X \to Y$ with $g|_V = \varphi$. We claim that g is the desired map.

First note that it follows immediately from the construction of φ and g that for any vertex v of $\mathrm{sd}(X)$ we have $f(\mathrm{St}(X, v)) \subset \mathrm{St}(Y, g(v))$. Thus it remains to show that g is in fact homotopic to our original map f. To do so we consider the usual interpolation between the maps f and g:[965]

$$\begin{array}{rcl} F\colon X \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & f(x)\cdot(1-t) + g(x)\cdot t. \end{array}$$

It remains to show the map F actually takes values in the subset $Y \subset \mathbb{R}^n$. For each $x \in X$ we know that $f(x) \in Y$ and $g(x) \in Y$. We need to show that for each $t \in [0,1]$ the point $f(x)\cdot(1-t) + g(x)\cdot t$ also lies in Y. In our setting this follows from the observation that simplices are convex, see Lemma 61.23 (1b), together with the following claim.

Claim 3. For every $x \in X$ there exists a $t \in T$ such that both $f(x)$ and $g(x)$ are contained in $\Omega(|t|)$.

Now let $x \in X$. By Lemma 61.11 (2) there exists a unique simplex $s = \{v_0, \ldots, v_k\}$ such that $x \in \Theta(\langle s\rangle)$. We see that

$$\begin{array}{l} x \in \Theta(\langle s\rangle) \overset{\substack{\text{by Lemma 62.5 (3), since } s=\{v_0,\ldots,v_k\}\\ \downarrow}}{\Rightarrow} x \in \mathring{\mathrm{St}}(v_0)\cap\ldots\cap\mathring{\mathrm{St}}(v_k) \quad\Rightarrow\quad f(x) \in f\big(\mathring{\mathrm{St}}(v_0)\cap\ldots\cap\mathring{\mathrm{St}}(v_k)\big) \\ \quad \underset{\substack{\uparrow\\ \text{Lemma 1.3 (4)}}}{\Rightarrow} f(x) \in f\big(\mathring{\mathrm{St}}(v_0)\big)\cap\ldots\cap f\big(\mathring{\mathrm{St}}(v_k)\big) \quad \underset{\substack{\uparrow\\ \text{by choice of } \varphi \text{ we have } f(\mathrm{St}(v_i))\subset\mathring{\mathrm{St}}(\varphi(v_i))}}{\Rightarrow} f(x) \in \mathring{\mathrm{St}}(\varphi(v_0))\cap\ldots\cap\mathring{\mathrm{St}}(\varphi(v_k)) \\ \quad \underset{\substack{\uparrow\\ \text{Lemma 62.5 (3)}}}{\Rightarrow} \text{there exists a } t\in T \text{ which contains } \varphi(s)=\{\varphi(v_0),\ldots,\varphi(v_k)\} \text{ and with } f(x)\in\Omega(\langle t\rangle). \end{array}$$

It remains to show that $g(x) \in \Omega(|t|)$. In fact we see that

$$x \in \Theta(\langle s\rangle) \underset{\substack{\uparrow\\ \text{by definition of } \langle s\rangle \text{ and } |s|}}{\Rightarrow} x \in \Theta(|s|) \underset{\substack{\uparrow\\ \text{follows from the discussion}\\ \text{on page 1489}}}{\Rightarrow} g(x) \in \Omega(|\varphi(s)|) \underset{\substack{\uparrow\\ \text{since } \varphi(s)\subset t}}{\Rightarrow} g(x) \in \Omega(|t|).$$

We have thus found the desired simplex t. ■

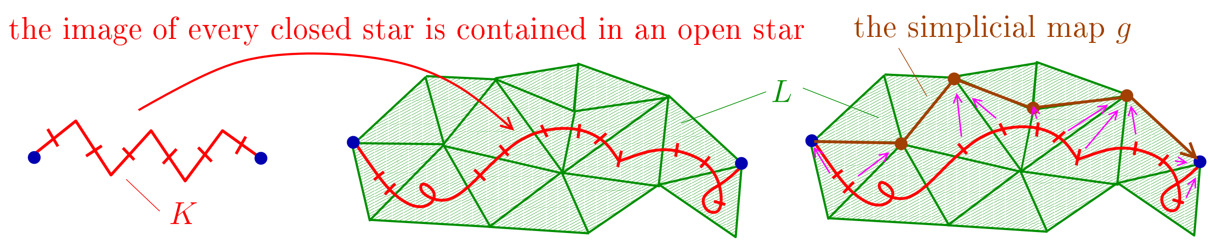


FIGURE 932. Illustration for the proof of Theorem 62.8

62.4. **Mapping cylinders and mapping cones of simplicial maps.** In this section we will see that there exist simplicial analogues of the mapping cylinder and the mapping cone that we introduced on page 697 and 701. The definitions of these simplicial analogues that we give are taken from **[Kas14]**.

[965]Note that we showed in Lemma 18.1 that this map is actually continuous.

Definition. Let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes. The *simplicial mapping cylinder* $\mathrm{Cyl}(f\colon K \to L)$ is the abstract simplicial complex that is given by the following data:

(1) The vertex set is the disjoint union of the vertex set of the barycentric subdivision $\mathrm{sd}(K)$ of K and the vertex set of L.
(2) The set of simplices is given by

$$\left\{ \sigma \sqcup \tau \;\middle|\; \begin{array}{c} \sigma = \varnothing \text{ or } \sigma \text{ a simplex of } \mathrm{sd}(K) \text{ \& } \tau = \varnothing \text{ or } \tau \text{ a simplex of } L \text{ or such that} \\ \text{for every vertex } v = \underset{\substack{\uparrow \\ \text{where } s \text{ is a simplex of } K}}{\underline{s}} \in \sigma \text{ and every vertex } w \in \tau \text{ we have } w \in f(s) \end{array} \right\}.$$

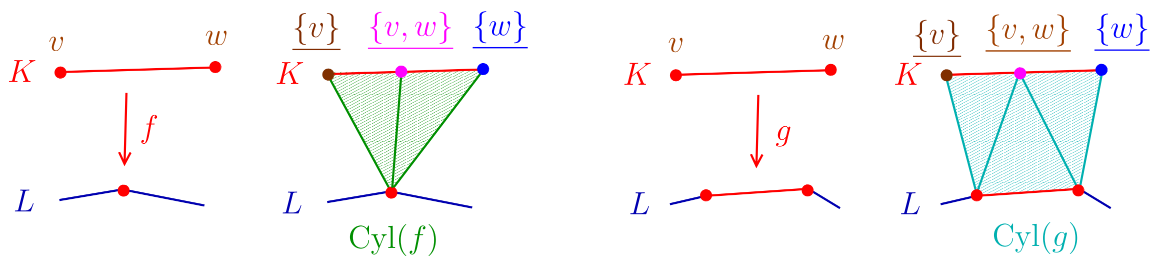


FIGURE 933

Let $f\colon K \to L$ be a simplicial map between two finite abstract simplicial complexes. One might expect that the next lemma states, say under the hypothesis that K is finite, that the topological realization $|\,\mathrm{Cyl}(f\colon K \to L)|$ of the simplicial mapping cylinder is homeomorphic to the mapping cylinder $\mathrm{Cyl}(|f|\colon |K| \to |L|)$ of the map between the topological realizations. We will prove a weaker statement, but which nonetheless says that the simplicial mapping cylinder has all the essential properties we really need.

Lemma 62.11. *Let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes.*

(1) *The simplicial mapping cylinder $\mathrm{Cyl}(f\colon K \to L)$ contains $\mathrm{sd}(K)$ and L as subcomplexes.*

We now assume that K is actually a finite abstract simplicial complex. *We denote by $\iota\colon \mathrm{sd}(K) \to \mathrm{Cyl}(f\colon K \to L)$ the natural inclusion and we denote by $\mho\colon |K| \to |\,\mathrm{sd}(K)|$ the natural homeomorphism from Lemma 62.3.*

(2) *There exists a deformation retraction $R\colon |\,\mathrm{Cyl}(f)| \times [0,1] \to |\,\mathrm{Cyl}(f)|$ from $|\,\mathrm{Cyl}(f)|$ to the subset $|L|$ such that $R_1 \circ \iota \circ \mho = f\colon |K| \to |\,\mathrm{Cyl}(f)|$.*
(3) *The map $|f|\colon |K| \to |\,\mathrm{Cyl}(f)|$ is homotopic to the map $|\iota| \circ \mho\colon |K| \to |\,\mathrm{Cyl}(f)|$.*

Remark. It is not clear to me whether the statement of Lemma 62.11 (2) also holds if K is infinite. For example one might be able to get stronger results using Lemma 38.11. But for our purposes it is good enough to know that the statement holds if K is finite.

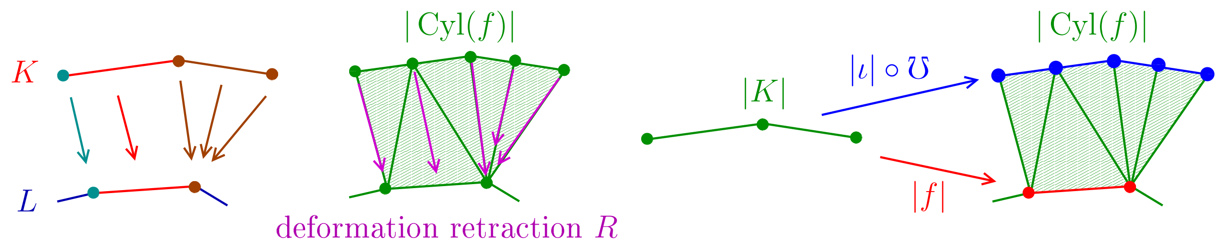


FIGURE 934. Illustration of Lemma 62.11.

SKETCH OF PROOF. Let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes.

(1) By taking $\tau = \varnothing$ respectively $\sigma = \varnothing$ one sees immediately that $\mathrm{sd}(K)$ and L are subcomplexes of $\mathrm{Cyl}(f\colon K \to L)$.

We now assume that K is actually a finite abstract simplicial complex.

(2) Given $r \in \mathbb{N}_0$ we denote by W_r the topological realization of the maximal subcomplex of $\mathrm{Cyl}(f\colon K \to L)$ that contains all vertices of L and that contains all vertices $\mathrm{sd}(K)$ of the form $\underline{s}$ where s is a $\geq r$-simplex of K. We leave it to the reader to verify that for each $r \in \mathbb{N}_0$ there exists a unique homotopy

$$F_r\colon W_r \times [0,1] \;\to\; W_r$$

with the following properties:
(a) The homotopy is stationary on W_{r-1}.
(b) Given any vertex $\underline{s}$ that lies in W_r but that does not lie in W_{r-1} the homotopy is given by $F_r(\underline{s},t) = \underline{s}\cdot(1-t) + \underline{f(s)}\cdot t$.
(c) The restriction of F_r to each simplex takes values in a simplex and it is linear.
Note that F_r is in particular a deformation retraction from W_r to W_{r-1}. The composition of the homotopies $F_{\dim(K)}, \ldots, F_1$ and F_0 has the desired properties.

(3) This statement follows immediately from (2). ■

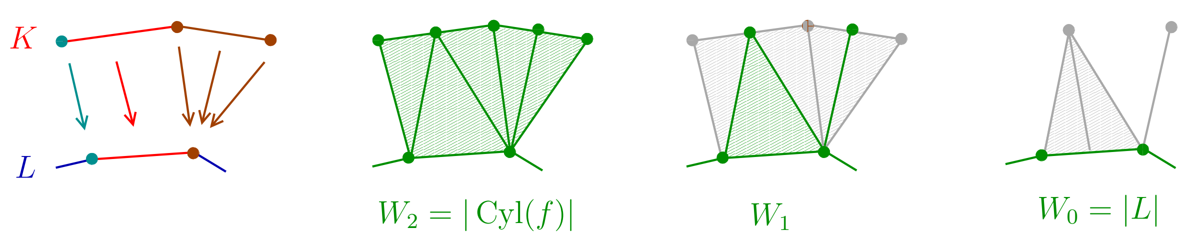


FIGURE 935. Illustration for the proof of Lemma 62.11.

We conclude this section with the following definition.

Definition. Let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes. The *simplicial mapping cone* $\mathrm{Cone}(f\colon K \to L)$ is defined as the union of the simplicial mapping cylinder $\mathrm{Cyl}(f\colon K \to L)$ and the simplicial cone $\mathrm{Cone}(\mathrm{sd}(K))$ that we defined on page 1506.

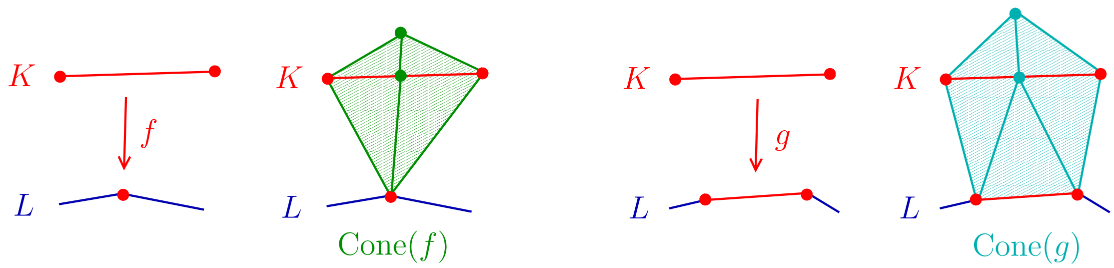


FIGURE 936

In Section 62.5 we will see that the concept of a simplicial mapping cone is indeed useful.

62.5. **Simplicial structures and CW-structures II.** Recall that in Lemma 61.24 we saw that every (ordered) simplicial complex admits a (natural) CW-complex structure where given any $n \in \mathbb{N}$ the n-simplices of the simplicial structure are precisely the n-cells of the CW-structure. The converse of this statement does not hold, in fact in Proposition 61.28 we showed that there exists a CW-complex that does not admit a simplicial structure.

Nonetheless, in this section we will prove a partial converse to Lemma 61.24, namely we will show that every CW-complex is *homotopy equivalent* to a simplicial complex. More precisely, we will prove the following proposition.

Proposition 62.12.

(1) *Every finite* CW*-complex is homotopy equivalent to a finite simplicial complex of the same dimension.*
(2) *Every countable* CW*-complex is homotopy equivalent to a countable simplicial complex of the same dimension.*
(3) *Every* CW*-complex is homotopy equivalent to a simplicial complex of the same dimension.*

Remark. Alternative proofs for Proposition 62.12 are given in [**Hat02**, Theorem 2C.5], [**LW69**, Theorem 6.1], [**WhdJ49b**, Theorem 13] and [**Gra75**, Corollary 16.44].

Most of the work in the proof of Proposition 62.12 is done in the following lemma.

Lemma 62.13. *Let X be a CW-complex that is homotopy equivalent to the topological realization of an abstract simplicial complex K. Furthermore let $\varphi\colon S^{k-1} \to X$ be a map. There exists an $i \in \mathbb{N}_0$ a simplicial map $f\colon \mathrm{sd}^i(S_{k-1}) \to K$ such that $X \cup_\varphi \overline{B}^k$ is homotopy equivalent to the topological realization of the simplicial mapping cone* $\mathrm{Cone}(f\colon \mathrm{sd}^i(S_{k-1}) \to K)|$.

The key ingredient in the proof of Lemma 62.13 is the Homotopy Pushout Theorem which we recall for the reader's convenience.

Theorem 39.18. (Homotopy Pushout Theorem) *Suppose we are given the following commutative diagram of maps between topological spaces:*

$$\begin{tikzcd} Y \arrow[d, "\varphi_Y"'] & A \arrow[l, "f"'] \arrow[r, hook, "i"] \arrow[d, "\varphi_A"] & X \arrow[d, "\varphi_X"] \\ Y' & A' \arrow[l, "f'"'] \arrow[r, hook, "i'"] & X'. \end{tikzcd}$$

If the vertical maps are homotopy equivalences and if the maps i and i' to the right are closed cofibrations, then the induced map

$$Y \cup_A X \;\to\; Y' \cup_{A'} X'$$

between the pushouts is a homotopy equivalence.

Proof of Lemma 62.13. We start out with the following preparations:

(1) By the discussion on page 1492 there exists a homeomorphism $g\colon |D_k| \to \overline{B}^k$ which restricts to a homeomorphism $g\colon |S_{k-1}| \to S^{k-1}$.
(2) Given any abstract simplicial complex L and given any $i \in \mathbb{N}_0$ we always denote by $\mho\colon |L| \to |\operatorname{sd}^i(L)|$ the homeomorphism from Lemma 62.3 (3).
(3) By the Simplicial Approximation Theorem 62.8 there exists an $i \in \mathbb{N}_0$, a simplicial map $f\colon \operatorname{sd}^i(S_{k-1}) \to K$ and a homotopy $F\colon |\operatorname{sd}^i(S_{k-1})| \times [0,1] \to |K|$ such that $F_0 = \theta \circ \varphi \circ g \circ \mho^{-1}$ and such that $F_1 = |f|$.
(4) We denote by $j\colon |K| \to |\operatorname{Cyl}(f\colon \operatorname{sd}^i(S_{k-1}) \to K)|$ the inclusion map. Note that by Lemma 62.11 (2) we know that j is a homotopy equivalence.
(5) Finally note that it follows immediately from Lemma 62.11 (3) that there exists a homotopy $G\colon |\operatorname{sd}^i(S_{k-1})| \times [0,1] \to |\operatorname{Cyl}(f\colon \operatorname{sd}^i(S_{k-1}) \to K))|$ with $G_0 = j \circ |f|$ and such that $G_1 = |\iota| \circ \mho$ where $\iota\colon \operatorname{sd}^{i+1}(S_{k-1}) \to \operatorname{Cyl}(f\colon \operatorname{sd}^i(S_{k-1}) \to K)$ is the natural inclusion.

We consider the following commutative diagram:

$$\begin{tikzcd}
X \arrow[d, "\theta"] & S^{k-1} \arrow[l, "\varphi"'] \arrow[r, hook] \arrow[d, "g^{-1}", "\cong"'] & \overline{B}^k \arrow[d, "g^{-1}", "\cong"'] \\
{|K|} \arrow[d, "\mathrm{id}"] & {|S_{k-1}|} \arrow[l, "\theta\circ\varphi\circ g"'] \arrow[r, hook] \arrow[d, "\mho", "\cong"'] & {|\operatorname{Cone}(S_{k-1}|} \arrow[d, "\mho", "\cong"'] \\
{|K|} \arrow[d, "\mathrm{id}"] & {|\operatorname{sd}^i(S_{k-1})|} \arrow[l, "\theta\circ\varphi\circ g\circ\mho^{-1}"'] \arrow[r, hook] \arrow[d, "x\mapsto(x,0)"] & {|\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))|} \arrow[d, "x\mapsto(x,0)"] \\
{|K|} & {|\operatorname{sd}^i(S_{k-1})| \times [0,1]} \arrow[l, "F"'] \arrow[r, hook] & {|\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))| \times [0,1]} \\
{|K|} \arrow[u, "\mathrm{id}"] \arrow[d, "j"] & {|\operatorname{sd}^i(S_{k-1})|} \arrow[l, "|f|"'] \arrow[r, hook] \arrow[u, "x\mapsto(x,1)"] \arrow[d, "\mathrm{id}"] & {|\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))|} \arrow[u, "x\mapsto(x,1)"] \arrow[d, "\mathrm{id}"] \\
{|\operatorname{Cyl}(f)|} \arrow[d, "\mathrm{id}"] & {|\operatorname{sd}^i(S_{k-1})|} \arrow[l, "j\circ|f|"'] \arrow[r, hook] \arrow[d, "x\mapsto(x,0)"] & {|\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))|} \arrow[d, "x\mapsto(x,0)"] \\
{|\operatorname{Cyl}(f)|} & {|\operatorname{sd}^i(S_{k-1})|} \arrow[l, "G"'] \arrow[r, hook] & {|\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))| \times [0,1]} \\
{|\operatorname{Cyl}(f)|} \arrow[u, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & {|\operatorname{sd}^i(S_{k-1})|} \arrow[l, "|\iota|\circ\mho"'] \arrow[r, hook] \arrow[u, "x\mapsto(x,1)"] \arrow[d, "\mho", "\cong"'] & {|\operatorname{Cone}(\operatorname{sd}^i(S_{k-1}))|} \arrow[u, "x\mapsto(x,1)"] \arrow[d, "\mho", "\cong"'] \\
{|\operatorname{Cyl}(f)|} & {|\operatorname{sd}^{i+1}(S_{k-1})|} \arrow[l, "|\iota|"'] \arrow[r, hook] & {|\operatorname{Cone}(\operatorname{sd}^{i+1}(S_{k-1}))|.}
\end{tikzcd}$$

It follows from (4) and elementary observations that all of the vertical maps are homotopy equivalences. Furthermore note that it follows immediately from Proposition 39.4 and 39.12 that all of the right horizontal maps are closed cofibrations. Thus it follows from the Homotopy Pushout Theorem 39.18, applied altogether eight times, that the induced maps between the various pushouts have homotopy inverses. Combining these eight homotopy equivalences, respectively their homotopy inverses, we obtain the promised homotopy equivalence

$$X \cup_{S^{k-1}} \overline{B}^k \ \to \ \underbrace{|\operatorname{Cyl}(f)| \cup_{|\operatorname{sd}^i(S_{k-1})|} |\operatorname{Cone}(\operatorname{sd}^{i+1}(S_{k-1}))}_{=\operatorname{Cone}(f\colon \operatorname{sd}^i(S_{k-1})\to K)}|. \qquad \blacksquare$$

Proof of Proposition 62.12 (1). Let X be a *finite* CW-complex. An induction argument over the number of cells, using Lemma 62.13, shows that X is homotopy equivalent to a finite simplicial complex of the same dimension. ■

Sketch of a proof of Proposition 62.12 (2) and (3). Now let X be any (countable) CW-complex. Without loss of generality we can and will assume that X is connected and non-empty.

Claim. There exists an ascending sequence $K_{-1} \subset K_0 \subset K_1 \subset K_2 \subset \dots$ of abstract simplicial complexes and a sequence of homotopy equivalences $\theta_n\colon X^n \to |K_n|$, $n \in \mathbb{Z}_{\geq -1}$

such that for each $n \in \mathbb{N}_0$ there exists a homotopy $F_n\colon X^n \times [0,1] \to |K_{n+1}|$ between the maps $X^n \xrightarrow{\theta_n} |K_n| \to |K_{n+1}|$ and $X^n \to X^{n+1} \xrightarrow{\theta_{n+1}} |K_{n+1}|$.

We take K_{-1} to be the empty simplicial complex. Assume that we have already constructed $K_{-1}, \dots, K_n$ and $\theta_{-1}, \dots, \theta_n$. We denote by $\{\varphi_i\colon S^n \to X^n\}_{i \in I}$ the attaching maps of the $(n+1)$-cells of X. By basically the same argument as in the proof of Lemma 62.13 we obtain $d_i \in \mathbb{N}_0$, simplicial maps $f_i\colon \operatorname{sd}^{d_i}(S_n) \to K_n$ and a homotopy equivalence

$$\theta_{n+1}\colon \underbrace{X^n \cup_{\sqcup \varphi_i} \coprod_{i \in I} \overline{B}^{n+1}}_{=X^{n+1}} \;\to\; \underbrace{\left|\operatorname{Cyl}\left(\coprod_{i\in I} f\colon \coprod_{i \in I} \operatorname{sd}^{d_i}(S_n) \to K\right) \cup \coprod_{i \in I} \operatorname{Cone}\left(\operatorname{sd}^{d_i+1}(S_n)\right)\right|}_{=:K_{n+1}}$$

such that the maps $X^n \xrightarrow{\theta_n} |K_n| \to |K_{n+1}|$ and $X^n \to X^{n+1} \xrightarrow{\theta_{n+1}} |K_{n+1}|$ are homotopic. ⊞

We set $K := \bigcup_{n \in \mathbb{N}_0} K_n$. If X is countable, then K is also countable. Thus it remains to show that X is homotopy equivalent to $|K|$.

(1) As on page1010 we equip $[0,\infty)$ with the CW-structure where the 0-skeleton is given by $\mathbb{N}_0$ and we refer to the CW-complex

$$T(X) \;:=\; \bigcup_{n \in \mathbb{N}_0} X^n \times [n, n+1] \quad \subset \quad X \times [0,\infty)$$

as the *telescope of* X.

(2) The homotopies F_n define a map $F\colon T(X) \to |K|$ which is continuous by Lemma 36.7.

(3) Furthermore the projection maps $X^n \times [0,1] \to X^n$ define a map $p\colon T(X) \to X$ which is also continuous by Lemma 36.7.

We pick $P \in X^0$. It follows fairly easily from Proposition 40.11 and the claim that for every $n \in \mathbb{N}$ the maps $F_*\colon \pi_n(T(X),P) \to \pi_n(|K|,P)$ and $p_*\colon \pi_n(T(X),P) \to \pi_n(X,P)$ are isomorphisms. It follows from the Whitehead Theorem 119.9 that $F\colon T(X) \to |K|$ and $p\colon T(X) \to X$ are homotopy equivalences. We have thus shown that X is homotopy equivalent to $|K|$.[966] ■

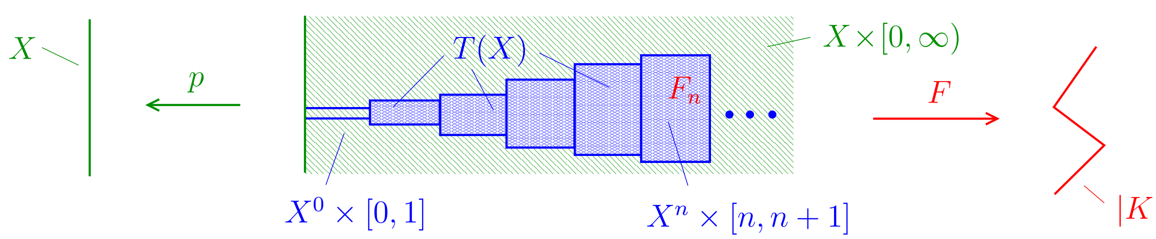


FIGURE 937. Illustration for the proof of Proposition 62.12 (2) and (3).

[966]We leave it to the industrious reader to figure out whether one can avoid the cheap exit via the Whitehead Theorem 119.9 by using say Lemma 38.12 in a clever fashion.

Exercises for Chapter 62.

Exercise 62.1. Let $K = (V, S)$ be an abstract simplicial complex. By Lemma 62.5 (3) we know that for any choice of simplices $s_0, \dots, s_n \in S$ the following equality regarding *open* stars holds:

$$\mathring{\mathrm{St}}(s_0) \cap \cdots \cap \mathring{\mathrm{St}}(s_n) \;=\; \begin{cases} \mathring{\mathrm{St}}(s_0 \cup \cdots \cup s_n), & \text{if } s_0 \cup \cdots \cup s_n \text{ is a simplex,} \\ \varnothing, & \text{otherwise.} \end{cases}$$

Show that in general the analogous statement for stars does *not* hold.

Exercise 62.2. Let K be an abstract simplicial complex and let s be a k-simplex. Show that there exists a natural simplicial isomorphism $\mathrm{Lk}(K, s) * \partial s \cong \mathrm{Lk}(\mathrm{sd}(K), \underline{s})$.

Exercise 62.3. Let $K = (V, S)$ be an abstract simplicial complex. Show that the dimension of the barycentric subdivision $\mathrm{sd}(K)$ equals the dimension of K.

Exercise 62.4. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$.

(a) Show that the link $\mathrm{Lk}(s)$ is a deformation retract of $\mathrm{St}(s) \setminus |s|$.
Hint. Consider the proof of Lemma 62.5 (e).

(b) Let $P \in \langle s \rangle$. Show that $\partial s * \mathrm{Lk}(s)$ is a deformation retract of $\mathrm{St}(s) \setminus \{P\}$.
Hint. Turn the idea of "pushing away from P" into a rigorous argument.

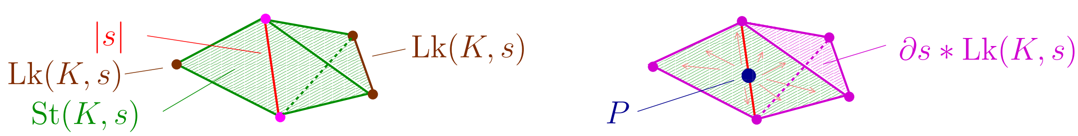


FIGURE 938. Illustration of Exercise 62.4.

Exercise 62.5. Let $n \in \mathbb{N}$. As on page 1480 we consider the abstract simplicial complex $S_n := (V, T)$ with $V = \{0, \dots, n+1\}$ and $T = \mathcal{P}(V) \setminus \{\varnothing, V\}$. Note that given a k-simplex s of S_n the set $V \setminus s$ is an $(n-k)$-simplex. Next we consider the barycentric subdivision $\mathrm{sd}(S_n)$ together with the map

$$\begin{aligned} f\colon \mathrm{sd}(S_n) &\to \mathrm{sd}(S_n) \\ \underline{s} &\mapsto \underline{V \setminus s}. \end{aligned}$$

(a) Show that f is a simplicial map.

(b) Show that $S_n/s \sim f(s)$, interpreted in the obvious way, is an abstract simplicial complex with $2^{n+1} - 1$ vertices.

(c) Sketch the abstract simplicial complex $S_n/s \sim f(s)$ for $n = 1$ and $n = 2$.

(d) Show that the topological realization of $S_n/s \sim f(s)$ is homeomorphic to the real projective space $\mathbb{RP}^n = S^n/x \sim -x$.

Remark. It is not known what is the "smallest" simplicial structure for the real projective spaces $\mathbb{RP}^n$. It is shown in [**ArM91**] that any simplicial structure for $\mathbb{RP}^n$ needs to have at least $\frac{1}{2}(n+1)(n+2)$ vertices. But all known simplicial structures seem to have about 2^n vertices.

Exercise 62.6. Let $n \in \mathbb{N}$ and let $k \in \{1, \ldots, n-1\}$. Use the Simplicial Approximation Theorem 62.8 to show that $\pi_k(S^n) = 0$.
Remark. This statement is of course the content of Proposition 40.10 which we proved using the Cellular Approximation Theorem 38.13.

Exercise 62.7. We say an abstract simplicial complex $K = (V, S)$ is a *flag complex* if the following condition is satisfied:

(∗) If $v_0, \ldots, v_n \in V$ have the property that $\{v_i, v_j\} \in S$ for all $i, j \in \{0, \ldots, n\}$, then $\{v_0, \ldots, v_n\} \in S$.

More loosely speaking, K is a flag complex, if a simplex exists whenever its 1-skeleton exists. We refer to Figure 939 for an illustration of the definition.

(a) Show that every flag complex is determined by its 1-skeleton. More precisely, let $K = (V, S)$ and $L = (W, T)$ be two abstract simplicial complexes. Suppose that $\varphi\colon K^1 \xrightarrow{\cong} L^1$ is a simplicial isomorphism between the 1-skeleta. We suppose that K and L are flag complexes. Show that φ can be extended to a simplicial isomorphism $K \xrightarrow{\cong} L$.

(b) Let K be an abstract simplicial complex. Show that the barycentric subdivision $\mathrm{sd}(K)$ is a flag complex.

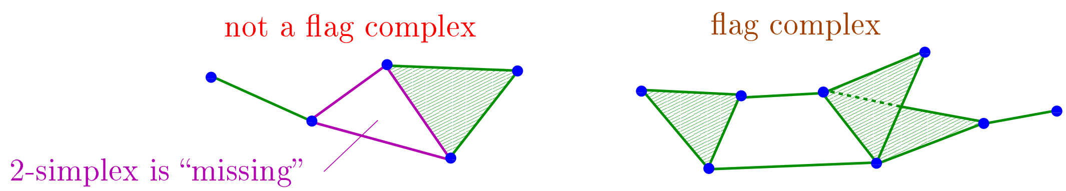


FIGURE 939. Illustration of Exercise 62.7.

63. Simplicial homology

In this chapter we introduce the simplicial homology groups of an (ordered) abstract simplicial complex. As we will see, the definition is refreshingly straightforward. Later in the chapter we discuss the relationship between simplicial and singular homology.

Before we get into details it is convenient to recall the following definition from basic group theory which we will need on numerous occasions throughout this chapter.

Definition. Let $n \in \mathbb{N}$. Given a permutation $\sigma \in S_n$ we refer to[967]

$$\operatorname{sign}(\sigma) \;:=\; \begin{array}{c}\text{determinant of the } (n \times n)\text{-matrix for which the} \\ (i,\sigma(i))\text{-entries are one and all the other entries are zero}\end{array}$$

as the *sign of the permutation*.

The following lemma summarizes the two key properties of the sign of a permutation.

Lemma 63.1. *Let $n \in \mathbb{N}$.*

(1) *The map* $\operatorname{sign}\colon S_n \to \{-1,1\}$ *is a homomorphism.*
(2) *If σ is a transposition, then* $\operatorname{sign}(\sigma) = -1$.

Proof. The proof follows immediately from standard properties of the determinant. ■

Definition. Let X be a finite set and let $\sigma \in \operatorname{Bij}(X)$. We set $n := \#X$. We pick a bijection $\varphi\colon \{1,\dots,n\} \to X$. We define $\operatorname{sign}(\sigma) = \operatorname{sign}(\varphi^{-1} \circ \sigma \circ \varphi)$. It follows easily from Lemma 63.1 that this definition is independent of the choice of the bijection φ.

Throughout this chapter we adopt the following convention.

Convention. Given $n \in \mathbb{N}$ we identify the permutation group $S_n = \operatorname{Bij}\{1,\dots,n\}$ with the group $\operatorname{Bij}\{0,\dots,n-1\}$ in the obvious way.

63.1. **Simplicial homology of an ordered abstract simplicial complex.**

In this section we will define the simplicial chain complex and the simplicial homology groups of an *ordered* abstract simplicial complex. This is going to be surprisingly easy. In the subsequent section we will then deal with abstract simplicial complex that are not equipped with an order.

We start out with the following elementary observation which we will use frequently throughout this chapter.

Observation. Let $(T,\leq)$ be a totally ordered set and let $v_0,\dots,v_n \in T$. By Exercise 1.1 there exists a unique permutation $\sigma \in S_{n+1}$ such that $v_{\sigma(0)} < \cdots < v_{\sigma(n)}$.

After this preliminary remark we can now turn to an interesting definition.

Definition. Let $K = (V,S)$ be an ordered abstract simplicial complex.[968] Given $n \in \mathbb{N}_0$ we define

$$\mathrm{C}_n^{\mathrm{simp},\leq}(K) \;:=\; \text{free abelian group generated by the set of } n\text{-simplices of } K.$$

[967] Surely the reader will have seen the definition of the sign of a permutation. We leave it to the reader to show that the previous definition agrees with our definition.

Furthermore we consider the boundary map

$$\begin{array}{rcl} \partial_n \colon \mathrm{C}_n^{\mathrm{simp},\le}(K) & \to & \mathrm{C}_{n-1}^{\mathrm{simp},\le}(K) \\ \underbrace{\{v_0 < \cdots < v_n\}}_{\substack{\text{any } n\text{-simplex } s \text{ is uniquely} \\ \text{of the form } s=\{v_0,\dots,v_n\} \\ \text{with } v_0<\cdots<v_n}} & \mapsto & \sum\limits_{i=0}^{n} (-1)^i \cdot \{v_0,\dots,v_{i-1},\widehat{v_i},v_{i+1},\dots,v_n\}. \end{array}$$

Lemma 63.2. *Let $K=(V,S)$ be an ordered abstract simplicial complex. Given any $n \in \mathbb{N}_0$ we have $\partial_{n-1} \circ \partial_n = 0$. In other words, $(C_*^{\mathrm{simp},\le}(K), \partial_*)$ is a chain complex.*

Proof. The proof of the lemma is almost verbatim the same as the proof of Proposition 41.2. We leave it to the reader to make the necessary minute modifications. ∎

The reader is by now well aware of the fact that such a lemma leads straight to the following definition.

Definition. Let K be an ordered abstract simplicial complex. We refer to the chain complex $(\mathrm{C}_*^{\mathrm{simp},\le}(K), \partial_*)$ as the *simplicial chain complex.* Given $n \in \mathbb{N}_0$ we define the *n-th simplicial homology group of K* as the n-th homology group of the simplicial chain complex, i.e. we define

$$\mathrm{H}_n^{\mathrm{simp},\le}(K) \;:=\; \frac{\ker\big(\partial_n \colon \mathrm{C}_n^{\mathrm{simp},\le}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp},\le}(K)\big)}{\operatorname{im}\big(\partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathrm{simp},\le}(K) \to \mathrm{C}_n^{\mathrm{simp},\le}(K)\big)}.$$

Example. We consider the 1-dimensional abstract simplicial complex $K=(V,S)$ shown in Figure 940. It has five vertices $\{A,B,C,D,E\}$, it has the corresponding five 0-simplices and it has five 1-simplices. We view $K=(V,S)$ as an ordered abstract simplicial complex in the obvious way, i.e. we have $A<B<C<D<E$. The corresponding simplicial chain complex is given as follows:

$$0 \to \begin{array}{c} \mathbb{Z}\cdot\{A,B\} \\ \mathbb{Z}\cdot\{B,C\} \\ \mathbb{Z}\cdot\{C,D\} \\ \mathbb{Z}\cdot\{D,E\} \\ \mathbb{Z}\cdot\{C,E\} \end{array} \xrightarrow{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}} \begin{array}{c} \mathbb{Z}\cdot\{A\} \\ \mathbb{Z}\cdot\{B\} \\ \mathbb{Z}\cdot\{C\} \\ \mathbb{Z}\cdot\{D\} \\ \mathbb{Z}\cdot\{E\} \end{array} \to 0.$$

We leave it as an elementary exercise in linear algebra to show that

$$\mathrm{H}_n^{\mathrm{simp},\le}(K) \;\cong\; \begin{cases} 0, & \text{if } n>1, \\ \mathbb{Z}, & \text{if } n=0,1. \end{cases}$$

Furthermore one can easily show that

$$\mathrm{H}_1^{\mathrm{simp},\le}(K) \;=\; \ker\big(\partial_1 \colon \mathrm{C}_1^{\mathrm{simp},\le}(K) \to \mathrm{C}_0^{\mathrm{simp},\le}(K)\big) \;=\; \mathbb{Z}\cdot[\{C,D\}+\{D,E\}-\{C,E\}].$$

[968] Recall that according to the definition on page 1484 "ordered" means that the vertex set V is equipped with a partial order such that the restriction to each simplex is a total order.

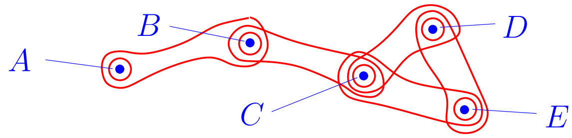


FIGURE 940

As always, not only do we want to associate groups to our given objects, but we really want to construct a functor. This is done in the following lemma.

Lemma 63.3.

(1) *Given an order-preserving simplicial map $f\colon K=(V,S)\to L=(W,T)$ between ordered abstract simplicial complexes we define*

$$\begin{array}{rcl} f_*\colon \mathrm{C}_n^{\mathrm{simp},\le}(K) & \to & \mathrm{C}_n^{\mathrm{simp},\le}(L) \\ \{v_0,\dots,v_n\} & \mapsto & \begin{cases} \{f(v_0),\dots,f(v_n)\}, & \text{if } f(v_0),\dots,f(v_n) \text{ are pairwise different,} \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

These maps form a chain map, thus they induce maps $f_*\colon \mathrm{H}_n^{\mathrm{simp},\le}(K)\to \mathrm{H}_n^{\mathrm{simp},\le}(L)$.

(2) *The maps*

$$\begin{array}{rcl} (K,\le) & \mapsto & \mathrm{C}_*^{\mathrm{simp},\le}(K) \\ (f\colon (K,\le)\to(L,\le)) & \mapsto & (f_*\colon \mathrm{C}_*^{\mathrm{simp},\le}(K)\to \mathrm{C}_*^{\mathrm{simp},\le}(L)) \end{array}$$

define a covariant functor from the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes to the category $\mathcal{ChCplx}$ of chain complexes. In particular, for each $n\in\mathbb{N}_0$ the maps

$$\begin{array}{rcl} (K,\le) & \mapsto & \mathrm{H}_n^{\mathrm{simp},\le}(K) \\ (f\colon (K,\le)\to(L,\le)) & \mapsto & (f_*\colon \mathrm{H}_n^{\mathrm{simp},\le}(K)\to \mathrm{H}_n^{\mathrm{simp},\le}(L)) \end{array}$$

define a covariant functor from the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes to the category $\mathcal{AbGr}$ of abelian groups.

PROOF. The statements follow easily from the definitions. We leave it to the reader to fill in the details. ∎

Naturally the question arises, to what degree do the simplicial homology groups of an ordered abstract simplicial complex depend on the choice of the order. As the following lemma shows, the answer is, not very much.

Lemma 63.4. *Let $K=(V,S)$ be an abstract simplicial complex and let "$\le$" and "$\widetilde{\le}$" be two orders on K. The maps*

$$\begin{array}{rcl} \mathrm{C}_n^{\mathrm{simp},\le}(K) & \to & \mathrm{C}_n^{\mathrm{simp},\widetilde{\le}}(K) \\ \{v_0<\dots<v_n\} & \mapsto & \underbrace{\mathrm{sign}(\sigma)\cdot\{v_{\sigma(0)}\widetilde{<}\dots\widetilde{<}v_{\sigma(n)}\}}_{\substack{\sigma\in S_n \text{ is the unique permutation} \\ \text{such that } v_{\sigma(0)}\widetilde{<}\dots\widetilde{<}v_{\sigma(n)}}} \end{array}$$

define an isomorphism of chain complexes.

Proof. The proof of the lemma is an appealing little calculation that will be done in Exercise 63.1. ■

63.2. **Simplicial homology.** In the last section we introduced simplicial homology for *ordered* abstract simplicial complexes. In this section we will introduce simplicial homology for "unordered" abstract simplicial complexes. Evidently the definition is going to be very similar, but as we will see, the definition is ever so more subtle.

Definition. Let $K=(V,S)$ be an abstract simplicial complex and let $n\in\mathbb{N}_0$.

(1) An *ordered n-simplex of K* is an $(n+1)$-tuple $(v_0,\dots,v_n)\in V^{n+1}$ such that $\{v_0,\dots,v_n\}$ is an n-simplex of $K=(V,S)$.

(2) Given an ordered n-simplex $(v_0,\dots,v_n)$ we denote by $\lfloor(v_0,\dots,v_n)\rfloor:=\{v_0,\dots,v_n\}$. the corresponding n-simplex of K.

(3) Let $n\in\mathbb{N}_0$. We define

$$\widehat{C}_n^{\mathrm{simp}}(K) := \text{free abelian group generated by the set of ordered } n\text{-simplices of } K.$$

Furthermore we define the *n-th simplicial chain group* to be

$$\mathrm{C}_n^{\mathrm{simp}}(K) := \widehat{C}_n^{\mathrm{simp}}(K)/I_n$$

where I_n is the subgroup of $\widehat{C}_n^{\mathrm{simp}}(K)$ generated by all elements of the form

$$(v_0,\dots,v_n)-\mathrm{sign}(\sigma)\cdot(v_{\sigma(0)},\dots,v_{\sigma(n)}) \quad \text{where } \sigma\in S_{n+1}=\mathrm{Bij}\{0,\dots,n\}$$

Given an ordered simplex $(v_0,\dots,v_n)$ we denote its image in $\mathrm{C}_n^{\mathrm{simp}}(K)$ by $[v_0,\dots,v_n]$.

Remark. Let $K=(V,S)$ be an abstract simplicial complex. Since the set of vertices is in a natural bijection with the set of 0-simplices and since the permutation group S_1 is trivial we see that we have a natural identification $\mathrm{C}_0^{\mathrm{simp}}(K)=\mathbb{Z}^{(V)}$.

Lemma 63.5. *Let $K=(V,S)$ be an abstract simplicial complex. If "$\le$" is an order on $K=(V,S)$, then for each $n\in\mathbb{N}_0$ the map*

$$\begin{array}{rcl} \mathrm{C}_n^{\mathrm{simp},\le}(K) & \to & \mathrm{C}_n^{\mathrm{simp}}(K) \\ \{v_0<\dots<v_n\} & \mapsto & [v_0,\dots,v_n] \end{array}$$

is a natural isomorphism.

Proof (∗). Let $n\in\mathbb{N}_0$. First note that it follows immediately from the definitions that the map is natural.

It remains to show that the given map is an isomorphism. We denote by P_n the set of ordered n-simplices, as defined above. Note that by definition we have $\widehat{C}_n^{\mathrm{simp}}(K)=\mathbb{Z}^{(P_n)}$. In the following we denote by Q_n the set of all ordered n-simplices $(v_0,\dots,v_n)$ such that $v_0<\dots<v_n$. By Lemma 19.1 there exist unique homomorphisms as follows:

$$\begin{array}{rcl} \Phi\colon \mathbb{Z}^{(P_n)} & \to & \mathbb{Z}^{(Q_n)} \\ (v_0,\dots,v_n) & \mapsto & \underset{\substack{\uparrow \\ \text{here } \sigma\in S_{n+1} \text{ is the unique permutation} \\ \text{with } v_{\sigma(0)}<\dots<v_{\sigma(n)}}}{\mathrm{sign}(\sigma)}\cdot(v_{\sigma(0)},\dots,v_{\sigma(n)}) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon \mathbb{Z}^{(Q_n)} & \to & \mathbb{Z}^{(P_n)} \\ (v_0,\dots,v_n) & \mapsto & (v_0,\dots,v_n). \end{array}$$

Furthermore we denote by $\pi\colon \mathbb{Z}^{(P_n)} \to \mathbb{Z}^{(P_n)}/I_n = \mathrm{C}_n^{\mathrm{simp}}(K)$ the obvious projection. It follows easily from Lemma 63.1 that $\Phi\colon \mathbb{Z}^{(P_n)} \to \mathbb{Z}^{(Q_n)}$ descends to a homomorphism $\overline{\Phi}\colon \mathbb{Z}^{(P_n)}/I_n \to \mathbb{Z}^{(Q_n)}$. Note that it is an immediate consequence of the definitions that $\overline{\Phi}\circ\pi irc\Psi$ is the identity on the basis Q_n of $\mathbb{Z}^{(Q_n)}$ and that $\pi\circ\Psi\circ\overline{\Phi}$ is the identity on the generating set $\pi(P_n)$ of $\mathbb{Z}^{(P_n)}/I_n$. This shows that $\pi\circ\Psi$ and $\overline{\Phi}$ are isomorphisms. The promised statements follow readily from this observation. ■

Lemma 63.6. *Let K be an abstract simplicial complex.*

(1) *Given any $n\in\mathbb{N}_0$ the group $\mathrm{C}_n^{\mathrm{simp}}(K)$ is a free abelian group where the rank is given by the cardinality of the set S_n of n-simplices of K. In fact, if we choose for each n-simplex $s\in S_n$ an ordered n-simplex $\widetilde{s}$ with $\lfloor\widetilde{s}\rfloor = s$, then $\{[\widetilde{s}]\}_{s\in S_n}$ is a basis for $\mathrm{C}_n^{\mathrm{simp}}(K)$.*

(2) *Let $n\in\mathbb{N}$. There exists a unique homomorphism*
$$\partial_n\colon\ \mathrm{C}_n^{\mathrm{simp}}(K) \ \to\ \mathrm{C}_{n-1}^{\mathrm{simp}}(K)$$
with the property that given any ordered n-simplex $(v_0,\dots,v_n)$ of K we have
$$\partial_n([v_0,\dots,v_n]) \ =\ \sum_{i=0}^{n}(-1)^i\cdot[v_0,\dots,v_{i-1},\widehat{v_i},v_{i+1},\dots,v_n].$$

(3) *For every $n\in\mathbb{N}_0$ we have $\partial_{n-1}\circ\partial_n = 0$.*

Remark. Let $K=(V,S)$ be an abstract simplicial complex and let $n\in\mathbb{N}_0$. Lemma 63.5 and Lemma 63.6 (1) say in particular that $\mathrm{C}_n^{\mathrm{simp}}(K)$ is a free abelian group. If $K=(V,S)$ is an ordered abstract simplicial complex, then this free abelian group has a natural basis. In the absence of an order there is only a natural basis where each element of the basis is well-defined "up to sign".

Example. For an ordered simplex (v_0,v_1,v_2) in an abstract simplicial complex K we see that
$$\partial([v_0,v_1,v_2]) \ =\ [v_1,v_2]-[v_0,v_2]+[v_0,v_1] \ =\ [v_0,v_1]+[v_1,v_2]+[v_2,v_0].$$
In other words, if we consider Figure 941, then we see that the boundary is exactly what we would naively had thought.

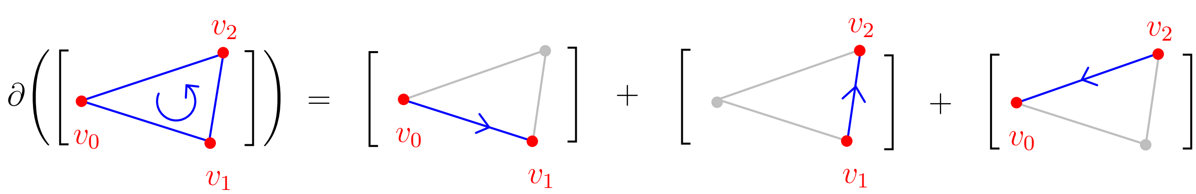


FIGURE 941

PROOF.

(1) It follows from the Well-Ordering Theorem 1.6 that we can equip $K=(V,S)$ with an order. Thus the desired statement is now an almost immediate consequence of Lemma 63.5

(2) Let $n \in \mathbb{N}$. By Lemma 19.1 there exists a unique homomorphism
$$\partial_n\colon \widehat{C}_n^{\text{simp}}(K) \;\to\; \mathrm{C}_{n-1}^{\text{simp}}(K)$$
with the property that given any ordered n-simplex $(v_0,\dots,v_n)$ of K we have
$$\partial_n((v_0,\dots,v_n)) \;=\; \sum_{i=0}^{n}(-1)^i\cdot[v_0,\dots,v_{i-1},\widehat{v_i},v_{i+1},\dots,v_n].$$
It remains to show that ∂_n factors through $\mathrm{C}_n^{\text{simp}}(K)=\widehat{C}_n^{\text{simp}}(K)/I_n$. In other words, we need to show that ∂_n vanishes on I_n. Since the permutation group S_n is generated by transpositions and since transpositions have the sign -1 it remains to prove the following claim.

Claim. For any $0\le k<l\le n$ and any ordered n-simplex $(v_0,\dots,v_n)$ of K we have
$$\partial_n(v_0,\dots,v_l,\dots,v_k,\dots,v_n) \;=\; -\partial_n(v_0,\dots,v_k,\dots,v_l,\dots,v_n).$$
Let $(v_0,\dots,v_n)$ be an ordered n-simplex and let $0\le k<l\le n$. We calculate that
$$\begin{aligned}
&\partial_n(..,v_l,..,v_k,..) \;=\\
&= \sum_{i\ne k,l}(-1)^i\cdot\underbrace{[..,v_l,..,\widehat{v_i},..,v_k,..]}_{=-[..,v_k,..,\widehat{v_i},..,v_l,..]}+(-1)^k\cdot\underbrace{[..,\widehat{v_l},..,v_k,..]}_{\substack{=(-1)^{l-k-1}\cdot[.,v_k,\dots,\widehat{v_l},.]\\ \text{since } l\text{-}k\text{-}1 \text{ transpositions}\\ \text{are used}}}+(-1)^l\cdot\underbrace{[..,v_l,..,\widehat{v_k},..]}_{\substack{=(-1)^{l-k-1}\cdot[.,\widehat{v_k},\dots,v_l,.]\\ \text{since } l\text{-}k\text{-}1 \text{ transpositions}\\ \text{are used}}}\\
&= -\sum_{i\ne k,l}(-1)^i\cdot[..,v_k,..,\widehat{v_i},..,v_l,..]-(-1)^l\cdot[..,v_k,..,\widehat{v_l},..]-(-1)^k\cdot[..,\widehat{v_k},..,v_l,..]\\
&= -\partial_n(..,v_k,..,v_l,..).
\end{aligned}$$

(3) The proof of this lemma is almost verbatim the same as the proof of Proposition 41.2. We leave the pleasant task of copy-pasting the proof to the reader. ■

Lemma 63.6 leads us right away to the following definition.

Definition. Let K be an abstract simplicial complex. We refer to the chain complex $(\mathrm{C}_*^{\text{simp}}(K),\partial_*)$ as the *simplicial chain complex*. Given $n\in\mathbb{N}_0$ we define the *n-th simplicial homology group of K* as the n-th homology group of the simplicial chain complex, i.e. we define
$$\mathrm{H}_n^{\text{simp}}(K) \;:=\; \frac{\ker\big(\partial_n\colon\ \mathrm{C}_n^{\text{simp}}(K)\to\mathrm{C}_{n-1}^{\text{simp}}(K)\big)}{\operatorname{im}\big(\partial_{n+1}\colon\ \mathrm{C}_{n+1}^{\text{simp}}(K)\to\mathrm{C}_n^{\text{simp}}(K)\big)}.$$

Examples.

(1) We consider the 2-dimensional abstract simplicial complex $K=(V,S)$ shown in Figure 942. It has four vertices $\{A,B,C,D\}$, it has the corresponding four 0-simplices, it has four 1-simplices and it has one 2-simplex. The corresponding simplicial chain complex is given as follows:
$$0\;\to\;\mathbb{Z}\cdot[B,C,D]\xrightarrow{\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}}\begin{matrix}\mathbb{Z}\cdot[A,B]\\\mathbb{Z}\cdot[B,C]\\\mathbb{Z}\cdot[B,D]\\\mathbb{Z}\cdot[C,D]\end{matrix}\xrightarrow{\begin{pmatrix}1&0&0&0\\-1&1&1&0\\0&-1&0&1\\0&0&-1&-1\end{pmatrix}}\begin{matrix}\mathbb{Z}\cdot[A]\\\mathbb{Z}\cdot[B]\\\mathbb{Z}\cdot[C]\\\mathbb{Z}\cdot[D]\end{matrix}\;\to\;0.$$

Elementary linear algebra shows that

$$\mathrm{H}_n^{\mathrm{simp}}(K) \cong \begin{cases} 0, & \text{if } n > 0, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

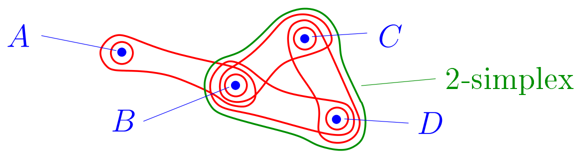


FIGURE 942

(2) If $K = (V, S)$ is an abstract simplicial complex with no simplices of dimension ≥ 1, then it follows immediately from the definitions that for $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_n^{\mathrm{simp}}(K) \cong \begin{cases} \mathrm{C}_0^{\mathrm{simp}}(K) = \mathbb{Z}^{(V)}, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The following is the obvious analogue of Lemma 63.3.

Lemma 63.7.

(1) *Given a simplicial map $f\colon K = (V, S) \to L = (W, T)$ we define*

$$\begin{aligned} f_*\colon \mathrm{C}_n^{\mathrm{simp}}(K) &\to \mathrm{C}_n^{\mathrm{simp}}(L) \\ [v_0, \ldots, v_n)] &\mapsto \begin{cases} [f(v_0), \ldots, f(v_n)], & \text{if } f(v_0), \ldots, f(v_n) \text{ are pairwise different,} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

These maps form a chain map, thus they induce maps $f_*\colon \mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n^{\mathrm{simp}}(L)$.

(2) *The maps*

$$\begin{aligned} K &\mapsto \mathrm{C}_*^{\mathrm{simp}}(K) \\ (f\colon K \to L) &\mapsto (f_*\colon \mathrm{C}_*^{\mathrm{simp}}(K) \to \mathrm{C}_*^{\mathrm{simp}}(L)) \end{aligned}$$

define a covariant functor from the category $\mathcal{AbsSimpCplx}$ of abstract simplicial complexes to the category $\mathcal{ChCplx}$ of chain complexes. For each $n \in \mathbb{N}_0$ the maps

$$\begin{aligned} K &\mapsto \mathrm{H}_n^{\mathrm{simp}}(K) \\ (f\colon K \to L) &\mapsto (f_*\colon \mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n^{\mathrm{simp}}(L)) \end{aligned}$$

define a covariant functor from the category $\mathcal{AbsSimpCplx}$ of abstract simplicial complexes to the category $\mathcal{AbGr}$ of abelian groups.

PROOF. As in the case of the proof of Lemma 63.3 we leave it to the reader to provide the proof. ■

For completeness we state the following elementary lemma.

Lemma 63.8. *Let $K = (V, S)$ be an ordered abstract simplicial complex. The maps*

$$\begin{aligned} \Omega_n\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) &\to \mathrm{C}_n^{\mathrm{simp}}(K) \\ \{v_0 < \cdots < v_n\} &\mapsto [v_0, \ldots, v_n] \end{aligned}$$

define a natural isomorphism of chain complexes. In particular they induce natural isomorphisms

$$\Omega_*\colon\ \mathrm{H}_n^{\mathrm{simp},\leq}(K) \ \to\ \mathrm{H}_n^{\mathrm{simp}}(K).$$

Remark. In the following we will discuss the simplicial homology groups $\mathrm{H}_n^{\mathrm{simp}}(K)$ in greater detail. By Lemma 63.8 the subsequent statements have obvious analogues for the simplicial homology groups $\mathrm{H}_n^{\mathrm{simp},\leq}(K)$.

Proof. A very short moment of reflection shows that the given map is a natural chain map. We deduce from Lemma 63.6 (1) that the chain map is in fact an isomorphism of chain complexes. ■

The next lemma can be viewed as an analogue of Lemma 48.1 (2) and (3). b

Lemma 63.9. *Let K be an abstract simplicial complex and let $m \in \mathbb{N}_0 \cup \{\infty\}$. As usual we denote by K^m the m-skeleton of K. Furthermore let $n \in \mathbb{N}_0$.*

1. *If K has only finitely many n-simplices, then $\mathrm{H}_n^{\mathrm{simp}}(K)$ is finitely generated.*
2. *If $n > m$, then $\mathrm{H}_n^{\mathrm{simp}}(K^m) = 0$.*
3. *If $n < m$, then the inclusion induced map $\mathrm{H}_n^{\mathrm{simp}}(K^m) \to \mathrm{H}_n^{\mathrm{simp}}(K)$ is an isomorphism.*

Analogous statements also hold for ordered abstract simplicial complexes and the homology groups $\mathrm{H}_n^{\mathrm{simp},\leq}(K^m)$ and $\mathrm{H}_n^{\mathrm{simp},\leq}(K)$.

Proof.

1. This statement follows easily from the definitions together with Lemma 19.8 (2b).
2. This statement follows from the observation that $\mathrm{C}_n^{\mathrm{simp}}(K^m) = 0$ for $n > m$.
3. This statement follows almost immediately from the observation that by definition we have $\mathrm{C}_i^{\mathrm{simp}}(K^m) = \mathrm{C}_i^{\mathrm{simp}}(K)$ for $i = 0, \ldots, m$. ■

It is clear that given any finite abstract simplicial complex one can compute the corresponding homology groups by a brute force calculation. But for large examples this can be very inefficient. Furthermore, it is rather unsatisfactory to outsource calculations to a computer. It is much more entertaining to try to do calculations cleverly on one's own. In the following we will get to know a few approaches to calculating simplicial homology groups. We will develop more techniques in Exercises 63.3 and 63.7.

As for singular homology, some calculations, statements and proofs become easier if we consider *reduced* simplicial homology.

Definition. Let K be an abstract simplicial complex.

1. We refer to the map[969]

$$\begin{aligned} \epsilon\colon C_0^{\mathrm{simp}}(K) &\ \to\ \mathbb{Z} \\ \Big[\sum_{i=1}^{m} a_i \cdot [v_i]\Big] &\ \mapsto\ \sum_{i=1}^{m} a_i \end{aligned}$$

as the *augmentation*.

(2) We consider the following generalized[970] chain complex

$$\begin{array}{ccccccccccc} & & & & & & \text{degree } 0 & & \text{degree } -1 & & \\ & & & & & & \downarrow & & \downarrow & & \\ \dots & \xrightarrow{\partial_3} & \mathrm{C}_2^{\mathrm{simp}}(K) & \xrightarrow{\partial_2} & \mathrm{C}_1^{\mathrm{simp}}(K) & \xrightarrow{\partial_1} & \mathrm{C}_0^{\mathrm{simp}}(K) & \xrightarrow{\epsilon} & \mathbb{Z} & \to & 0. \end{array}$$

We introduce the following two definitions.

(a) We refer to this chain complex as the *augmented simplicial chain complex* $\widetilde{\mathrm{C}}_*^{\mathrm{simp}}(K)$ *of* K.

(b) Let $k \in \mathbb{N}_0$. We define the *k-th reduced simplicial homology group* $\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(K)$ *of* K to be the k-th homology group of the augmented chain complex.

Remark. The above definition also makes sense for the empty abstract simplicial complex $\varnothing$. Furthermore, similar to Lemma 43.1 (0) we see that $\mathrm{H}_{-1}^{\mathrm{simp}}(\varnothing) = \mathbb{Z}$ and $\mathrm{H}_i^{\mathrm{simp}}(\varnothing) = 0$ for $i \neq -1$.

The following lemma is the analogue of Lemma 43.1 (4). It basically says that in the following it suffices to calculate reduced simplicial homology groups.

Lemma 63.10. *Let $K = (V, S)$ be an abstract simplicial complex.*

(1) *Given any vertex $v \in V$ the map*

$$\begin{array}{rcl} \widetilde{\mathrm{H}}_0^{\mathrm{simp}}(K) \oplus \mathbb{Z} & \to & \mathrm{H}_0^{\mathrm{simp}}(K) \\ [\sigma] \oplus n & \mapsto & [\sigma] + [n \cdot P] \end{array}$$

is an isomorphism.

(2) *For every $n \geq 1$ we have the equality $\mathrm{H}_n^{\mathrm{simp}}(K) = \widetilde{\mathrm{H}}_n^{\mathrm{simp}}(K)$.*

Proof. The proof is basically identical to the already rather straightforward proof of Lemma 43.1 (4). ■

By Lemma 24.1 we know that the cone on a topological space is contractible, in particular we know by Corollary 42.8 (3) that its reduced homology groups vanish. The following lemma can be viewed as a simplicial analogue of this observation.

Lemma 63.11. *Let K be an abstract simplicial complex. Let* $\mathrm{Cone}(K)$ *be the simplicial cone as defined on page 1506. Given any $k \in \mathbb{N}_0$ we have*

$$\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(\mathrm{Cone}(K)) \;=\; 0.$$

Proof. Let $K = (V, S)$ be an abstract simplicial complex. We set $L := \mathrm{Cone}(K)$. Note that it follows from Lemma 42.4 that it remains to prove that the identity map $\mathrm{id}\colon \mathrm{C}_*^{\mathrm{simp}}(L) \to \mathrm{C}_*^{\mathrm{simp}}(L)$ is homotopic to the zero map. In other words, it suffices to prove the following claim.

[969] Morally the group $\mathbb{Z}$ is the free abelian group on the “empty” simplex.

[970] It requires about one second to verify that this is indeed a generalized chain complex.

Claim. We can find "diagonal" maps P_n, $n \in \mathbb{Z}_{\geq -1}$, to obtain a diagram

$$\begin{tikzcd}
\dots \arrow[r] & C_3^{\text{simp}}(L) \arrow[r, "\partial_3"] \arrow[d, "\text{id}"] \arrow[dl, "P_3"'] & C_2^{\text{simp}}(L) \arrow[r, "\partial_2"] \arrow[d, "\text{id}"] \arrow[dl, "P_2"'] & C_1^{\text{simp}}(L) \arrow[r, "\partial_1"] \arrow[d, "\text{id}"] \arrow[dl, "P_1"'] & C_0^{\text{simp}}(L) \arrow[r, "\partial_0:=\epsilon"] \arrow[d, "\text{id}"] \arrow[dl, "P_0"'] & \mathbb{Z} \arrow[r] \arrow[d, "\text{id}"] \arrow[dl, "P_{-1}"'] & 0 \\
\dots \arrow[r] & C_3^{\text{simp}}(L) \arrow[r, "\partial_3"'] & C_2^{\text{simp}}(L) \arrow[r, "\partial_2"'] & C_1^{\text{simp}}(L) \arrow[r, "\partial_1"'] & C_0^{\text{simp}}(L) \arrow[r, "\partial_0:=\epsilon"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

such that for every $n \in \mathbb{Z}_{\geq -1}$ we have the equality

$$(*) \qquad \partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; \text{id}\,.$$

In the following we denote the cone point of $\text{Cone}(L)$ by "$\star$". To prove the claim we consider the map

$$\begin{array}{rcl} P_{-1}\colon \mathbb{Z} & \to & \text{C}_0^{\text{simp}}(L) \\ n & \mapsto & n\cdot[\star] \end{array} \qquad \begin{array}{c}\text{and for } n\geq 0 \\ \text{we consider} \\ \text{the map}^{971}\end{array} \qquad \begin{array}{rcl} P_n\colon \text{C}_n^{\text{simp}}(L) & \to & \text{C}_{n+1}^{\text{simp}}(L) \\ ][v_0,\dots,v_n] & \mapsto & \begin{cases} [\star, v_0,\dots,v_n], & \text{if all } v_i \neq \star, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

We claim that $(*)$ holds for all $n \in \mathbb{Z}_{\geq -1}$. The case $n = -1$ can be verified easily. Thus it remains to show that $(*)$ holds if $n \in \mathbb{N}_0$. In fact, given $[v_0,\dots,v_n] \in \text{C}_n^{\text{simp}}(L)$ with $v_0,\dots,v_n \in V$ we immediately see that

$$\underbrace{[v_0,\dots,v_n] + \sum_{i=1}^{n+1}(-1)^i\cdot[\star, v_0,\dots,\widehat{v_{i-1}},\dots,v_n]}_{=(\partial_{n+1}\circ P_n)([v_0,\dots,v_n])} + \underbrace{\sum_{i=0}^{n}(-1)^i\cdot[\star, v_0,\dots,\widehat{v_i},\dots,v_n]}_{=(P_{n-1}\circ\partial_n)([v_0,\dots,v_n])} \;=\; [v_0,\dots,v_n].$$

Furthermore, given $[\star, v_0,\dots,v_{n-1}] \in \text{C}_n^{\text{simp}}(L)$ with $v_0,\dots,v_{n-1} \in V$ it is equally straightforward to verify that the equality holds. ■

Next we want to calculate the simplicial homology groups of the abstract simplicial complexes D_n and S_n that we introduced on page 1480. By Lemma 63.10 it suffices to compute the reduced simplicial homology groups.

Lemma 63.12. *Given any $n \in \mathbb{N}$ and given any $k \in \mathbb{N}_0$ we have*

$$\widetilde{\text{H}}_k^{\text{simp}}(D_n) \;=\; 0 \quad \textit{and} \quad \widetilde{\text{H}}_k^{\text{simp}}(S_n) \;\cong\; \begin{cases} \mathbb{Z}, & \textit{if } k = n, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof.

(1) Given any $n \in \mathbb{N}$ and given any $k \in \mathbb{N}_0$ we calculate that

$$\widetilde{\text{H}}_k^{\text{simp}}(D_n) \underset{\substack{\uparrow \\ \text{see page 1506}}}{\cong} \widetilde{\text{H}}_k^{\text{simp}}(\text{Cone}(D_{n-1})) \underset{\substack{\uparrow \\ \text{by Lemma 63.11}}}{=} 0.$$

[971] One can easily verify that the map P_n is actually well-defined.

(2) Let $n \in \mathbb{N}_0$. We consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{C}^{\mathrm{simp}}_n(S_n) \arrow[r, "\partial"] \arrow[d, "="] & \mathrm{C}^{\mathrm{simp}}_{n-1}(S_n) \arrow[r, "\partial"] \arrow[d, "="] & \dots \arrow[r, "\partial"] & \mathrm{C}^{\mathrm{simp}}_0(S_n) \arrow[r, "\epsilon"] \arrow[d, "="] & \mathbb{Z} \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathrm{C}^{\mathrm{simp}}_{n+1}(D_{n+1}) \arrow[r] \arrow[d, "\cong"] & \mathrm{C}^{\mathrm{simp}}_n(D_{n+1}) \arrow[r, "\partial"] \arrow[d] & \mathrm{C}^{\mathrm{simp}}_{n-1}(D_{n+1}) \arrow[r, "\partial"] \arrow[d] & \dots \arrow[r, "\partial"] & \mathrm{C}^{\mathrm{simp}}_0(D_{n+1}) \arrow[r, "\epsilon"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial"] & 0 \arrow[r, "\partial"] & 0 \arrow[r, "\partial"] & \dots \arrow[r, "\partial"] & 0 \arrow[r, "\partial"] & 0 \arrow[r] & 0.
\end{tikzcd}$$

We make the following clarifications and comments:

(a) The top two horizontal sequences are the reduced chain complexes of the abstract simplicial complexes S_n and D_{n+1}.

(b) Recall that by definition the simplices of S_n and D_{n+1} are the same except that D_{n+1} has a single $(n+1)$-simplex whereas S_n has no $(n+1)$-simplices. This observation explains the vertical maps.

(c) The diagram clearly commutes and it is clear that the vertical sequences are exact.

(d) It follows from the above that we can apply Proposition 43.11 and we obtain a long exact sequence of homology groups. Together with (1) we obtain the reduced simplicial homology groups of S_n. ■

In Lemma 46.8 we studied the singular homology groups of a suspension. Statement (2) of the following lemma is the simplicial analogue of Lemma 46.8.

Lemma 63.13. *Let K be a non-empty abstract simplicial complex.*

(1) *Let $n \in \mathbb{N}_0$ and let S_n be the abstract simplicial complex that we introduced on page 1480. For every $k \in \mathbb{N}_0$ there exists a natural isomorphism*

$$\widetilde{\mathrm{H}}^{\mathrm{simp}}_k(K * S_n) \;\cong\; \widetilde{\mathrm{H}}^{\mathrm{simp}}_{k-(n+1)}(K).$$

(2) *We denote by $\Sigma(K)$ the simplicial suspension of K as defined on page 1506. For every $k \in \mathbb{N}_0$ there exists an isomorphism*

$$\widetilde{\mathrm{H}}^{\mathrm{simp}}_k(\Sigma(K)) \;\cong\; \widetilde{\mathrm{H}}^{\mathrm{simp}}_{k-1}(K).$$

Example. Let $n \in \mathbb{N}$. Given any $k \in \mathbb{N}_0$ we see that

$$\widetilde{\mathrm{H}}_k\big(\overbrace{\Sigma(\Sigma(\dots\Sigma(S_0)))}^{n\text{-fold simplicial suspension}}\big) \underset{\substack{\uparrow \\ \text{Lemma 63.13 applied } n \text{ times}}}{\cong} \widetilde{\mathrm{H}}_{k-n}(S_0) \underset{\substack{\uparrow \\ \text{see the discussion on page 1547}}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } k-n=0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since the author has no intention of monopolizing all the fun we will leave the proof of Statement (1) as Exercise 63.2 to the reader. Note that Statement (2) is an immediate consequence of Statement (1) since $\Sigma(K)$ is naturally isomorphic to $K * S_0$. ■

Next we extend the definition of simplicial homology groups in the obvious way to simplicial complexes:

Definition. Let $(X,(K=(V,S),\Theta\colon |K|\to X))$ be a simplicial complex. Given $n\in\mathbb{N}_0$ we define

$$\mathrm{H}_n^{\mathrm{simp}}(X) \ := \ \mathrm{H}_n^{\mathrm{simp}}(K).$$

Similarly, given a simplicial map $f\colon X=(X,\Theta)\to Y=(L,\Omega)$ between two simplicial complexes we define the induced map $f_*\colon \mathrm{H}_n^{\mathrm{simp}}(X)\to \mathrm{H}_n^{\mathrm{simp}}(Y)$ as the induced map $f_*\colon \mathrm{H}_n^{\mathrm{simp}}(K)\to \mathrm{H}_n^{\mathrm{simp}}(L)$.

Finally we introduce simplicial homology with coefficients in the obvious way. More precisely, we have the following definition which is the obvious variation on the definition given in Section 57.2.

Definition. Let G be an abelian group and let $n\in\mathbb{N}_0$. Given an abstract simplicial complex K we set

$$\mathrm{H}_n^{\mathrm{simp}}(K;G) \ := \ \mathrm{H}_n(\mathrm{C}_*^{\mathrm{simp}}(K)\otimes G).$$

If K is actually ordered we set $\mathrm{H}_n^{\mathrm{simp},\le}(K;G):=\mathrm{H}_n(\mathrm{C}_*^{\mathrm{simp},\le}(K)\otimes G)$. Finally given a simplicial complex $(X,(K=(V,S),\Theta\colon |K|\to X))$ we define, following the above convention, $\mathrm{H}_n^{\mathrm{simp}}(X;G):=\mathrm{H}_n^{\mathrm{simp}}(K;G)$.

Surely there is no need to state all the self-evident properties of simplicial homology with coefficients. Let us just mention that it follows from Lemma 57.3 (3) that given any abstract simplicial complex K and given any $n\in\mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{Z}) & \to & \mathrm{H}_n^{\mathrm{simp}}(K) \\ \Big[\sum\limits_{i=0}^{k} s_i\otimes m_i\Big] & \mapsto & \Big[\sum\limits_{i=0}^{k} s_i\cdot m_i\Big] \end{array}$$

is a natural isomorphism. Throughout these notes we will use this isomorphism to make the identification $\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{Z})=\mathrm{H}_n^{\mathrm{simp}}(K)$.

63.3. **The Simplicial Künneth Theorem.** Let $(K,\le)$ and $(L,\le)$ be two ordered abstract simplicial complexes. Recall that on page 1501 we introduced the corresponding product $K\times L$. The following theorem, which can be viewed as the simplicial analogue of the Künneth Theorem 58.8 for Topological Spaces, computes the simplicial homology groups of $K\times L$ in terms of the simplicial homology groups of K and L.

Theorem 63.14. (Simplicial Künneth Theorem) *Let $(K,\le)$ and $(L,\le)$ be two ordered abstract simplicial complexes. There exists a short exact sequence*

$$0\to\bigoplus_{p+q=n}\mathrm{H}_p^{\mathrm{simp},\le}(K)\otimes\mathrm{H}_q^{\mathrm{simp},\le}(L)\xrightarrow{\Omega}\mathrm{H}_n^{\mathrm{simp},\le}(K\times L)\to\bigoplus_{p+q=n-1}\mathrm{Tor}(\mathrm{H}_p^{\mathrm{simp},\le}(K),\mathrm{H}_q^{\mathrm{simp},\le}(L))\to 0$$

with the following four properties:

(1) *Both maps are natural with respect to the ordered abstract simplicial complexes K and L.*
(2) *For $p=0$ the map $\Omega\colon \mathrm{H}_0^{\mathrm{simp},\le}(K)\otimes\mathrm{H}_q^{\mathrm{simp},\le}(L)\to\mathrm{H}_q^{\mathrm{simp},\le}(K\times L)$ is the "obvious" map, namely for any vertex v of K and any simplicial cycle $\sigma\in\mathrm{C}_q^{\mathrm{simp},\le}(L)$ the image of the class $[v]\otimes[\sigma]\in\mathrm{H}_0^{\mathrm{simp},\le}(K)\otimes\mathrm{H}_q^{\mathrm{simp},\le}(L)$ under Ω equals the image of*

$[\sigma]$ under the map $\mathrm{H}_q^{\mathrm{simp},\leq}(L) \to \mathrm{H}_q^{\mathrm{simp},\leq}(\{v\} \times L) \to \mathrm{H}_q^{\mathrm{simp},\leq}(K \times L)$, where the first map is induced by the natural isomorphism and the second map is induced by the inclusion.

(3) *The analogue of (2) also holds for $q = 0$.*
(4) *The above short exact sequence splits.*

We will not really make use of the Simplicial Künneth Theorem 63.14, thus we will only provide a sketch of the proof. A full proof is given in [**FePi11**, Theorem III.6.4].

Sketch of proof. Using the Acyclic Model Theorem 80.16, together with Exercise 63.5, one can prove a simplicial analogue of the Eilenberg-Zilber Theorem 58.4, i.e. one can prove that there exists a natural chain map

$$\Upsilon\colon C_*^{\mathrm{simp},\leq}(K) \otimes C_*^{\mathrm{simp},\leq}(L) \;\to\; C_*^{\mathrm{simp},\leq}(K \times L)$$

with the following three properties:

(1) The chain map is a chain homotopy equivalence.
(2) Given any vertex v of K and given any ordered simplex $(w_0, \ldots, w_k)$ of L we have $\Upsilon([v] \otimes [w_0, \ldots, w_k]) = [(v, w_0), \ldots, (v, w_k)]$.
(3) The analogue of (2) with the roles of K and L swapped also holds.

The precise construction of such a map is given in [**FePi11**, Chapter III.6]. Next we consider the following maps:

$$\begin{tikzcd}
 & & \mathrm{H}_n^{\mathrm{simp},\leq}(\mathrm{C}_*^{\mathrm{simp},\leq}(K)\otimes \mathrm{C}_*^{\mathrm{simp},\leq}(L)) \arrow[dr] \arrow[dd, "\Upsilon", "\cong"'] & & \\
0 \arrow[r] & \bigoplus\limits_{p+q=n} \mathrm{H}_p^{\mathrm{simp},\leq}(K)\otimes \mathrm{H}_q^{\mathrm{simp},\leq}(L) \arrow[ur, "\Omega"'] \arrow[dr, dashed] & & \bigoplus\limits_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p^{\mathrm{simp},\leq}(K), \mathrm{H}_q^{\mathrm{simp},\leq}(L)) \arrow[r] & 0. \\
 & & \underbrace{\mathrm{H}_n^{\mathrm{simp},\leq}(\mathrm{C}_*^{\mathrm{simp},\leq}(K\times L))}_{=H_n(K\times L)} \arrow[ur, dashed] & &
\end{tikzcd}$$

Here the sequence on the top is the natural short exact sequence provided by the Künneth Theorem 58.7 for Chain Complexes. It follows from the above together with Corollary 42.3 that the vertical map is a natural isomorphism. It is now fairly straightforward to see that we obtain the desired short exact sequence from replacing the upper middle term by the lower middle term $\mathrm{H}_n^{\mathrm{simp},\leq}(\mathrm{C}_*^{\mathrm{simp},\leq}(K \times L)) = \mathrm{H}_n^{\mathrm{simp},\leq}(K \times L)$. ■

Example. We consider the ordered abstract simplicial spaces K and L that are shown in Figure 943. Both are simplicially isomorphic to the simplicial circle S_1. Furthermore we consider the "simplicial torus" $K \times L$. It follows from the Simplicial Künneth Theorem 63.14 that

$$\mathrm{H}_i^{\mathrm{simp}}(K \times L) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2, \\ \mathbb{Z}^2, & \text{if } i = 1, \\ 0, & \text{if } i > 2. \end{cases}$$

Furthermore it follows easily from Statement (2) of the Simplicial Künneth Theorem 63.14 that the two singular cycles shown in Figure 943 form a basis for $H_1^{\mathrm{simp},\leq}(K \times L)$.

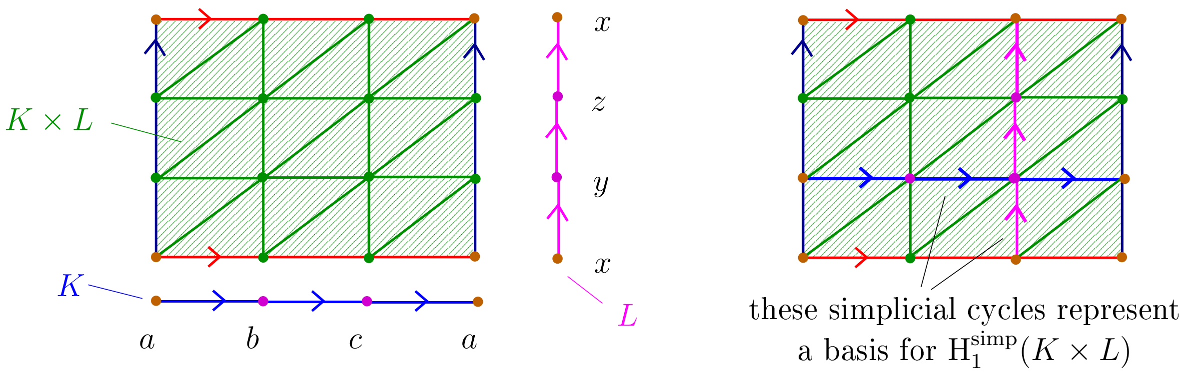


FIGURE 943

63.4. **The topological invariance of simplicial homology I.** The following question arises naturally.

Question 63.15. *Let $(X, (K, \Theta\colon |K| \to X))$ be a simplicial complex and let $n \in \mathbb{N}_0$. Does the isomorphism type of the simplicial homology group $H_n^{\mathrm{simp}}(X)$ only depend on the homeomorphism type of X?*

For example, we know by the discussion on page 1500 and on page 1508 that the topological realizations of S_n and of the n-fold simplicial suspension of S_0 are both homeomorphic to S^n and conveniently enough we know by Lemma 63.12 and the last example of the previous section that the homology groups are indeed isomorphic.

The following "research program" summarizes the traditional approach to tackling Question 63.15.

Program 63.16.

(1) *Show that any two (let us say finite) abstract simplicial complexes with homeomorphic topological realizations are related by a finite sequence of "moves".*
(2) *Show that the isomorphism type of simplicial homology groups is unchanged under these "moves".*

This idea leads straight to the definition of a subdivision of a simplicial complex.

Definition. We say that a simplicial complex $(X, (K = (V, S), \Omega\colon |K| \to X))$ is a *subdivision* of some simplicial complex $(X, (L = (W, T), \Theta\colon |L| \to X)$ if the following two conditions are satisfied:

(1) Every simplex of $|K|$ is contained in some simplex of $|L|$. More precisely, given any simplex $s \in S$ of K there exists a simplex $t \in T$ of L such that $\Omega(|s|) \subset \Theta(|t|)$.
(2) For every k-simplex s of K and every l-simplex t of L with $\Omega(|s|) \subset \Theta(|t|)$ the map $\Phi_t^{-1} \circ \Theta \circ \Omega \circ \Phi_s\colon \Delta^k \to |s| \subset |t| \to \Delta^l$ is affine linear.

Remark.

(1) Note that for linear simplicial complexes condition (3) is automatically satisfied.
(2) The definition of a subdivision we give agrees with the definition in most sources, see e.g. [**Hud69**, p. 5], [**Pon52**, p. 36], [**RS72**, p. 15], [**Gla70**, p. 7], [**Spa95**, p. 121] and [**Ze63b**, p. I.4].

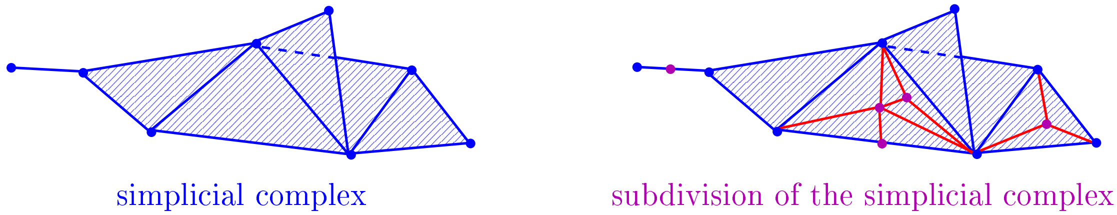


FIGURE 944

(3) In [**Mun84**, p. 83] the definition of a subcomplex has the extra condition that every simplex of X equals the union of finitely many simplices of Y. It follows from [**Hud69**, Lemma 1.2] together with Proposition 61.12 (2) that the above definition agrees with the definition provided in [**Mun84**, p. 83].

(4) The definition of a subdivision is somewhat unsatisfactory. In the spirit that abstract simplicial complexes should be purely combinatorial objects there should be a "topology free" definition of the subdivision of an abstract simplicial complexes, which then mirrors the topological definition given above. We will return to the question of how to work completely combinatorially in Section 64.6.

Examples.

(1) It follows from Lemma 62.3 (3a) and (3c) that the barycentric subdivision $\mathrm{sd}(X)$ of a simplicial complex X is indeed a subdivision in the sense of the above definition.

(2) In Figure 945 we see that a subdivision of the simplicial complex $|S_2|$ is simplicially isomorphic to $\Sigma(S_1)$.

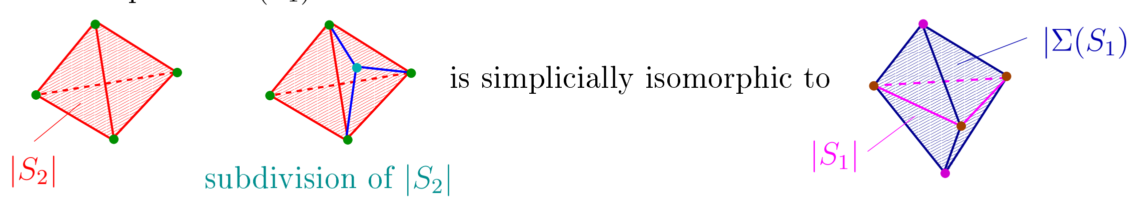


FIGURE 945

(3) Let $n \in \mathbb{N}_0$. By Lemma 61.19 (3) we know that the map

$$\begin{array}{rcl} f\colon \partial\Delta^n * \{\star\} & \to & \Delta^n \\ [(x,t,\star)] & \mapsto & x\cdot(1-t) + \left(\frac{1}{n+1},\dots,\frac{1}{n+1}\right)\cdot t \end{array}$$

is a homeomorphism. It follows easily from this observation that the simplicial complex $(S_{n-1} * \{\star\}, |S_{n-1} * \{\star\}| = \partial\Delta^n * \{\star\} \xrightarrow{f} \Delta^n)$ is a subdivision of the simplicial complex $(D_n, \Omega\colon |D_n| \to \Delta^n)$. We refer to Figure 946 for an illustration.

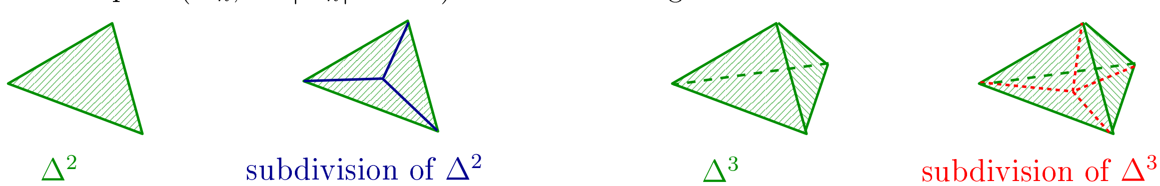


FIGURE 946

For completeness' sake we mention the following result.

Proposition 63.17.

(1) *Let X and Y be two finite linear simplicial complexes. If $|X| = |Y|$, then there exist subdivisions X' of X and Y' of Y which are simplicially isomorphic.*
(2) *If we are given two subdivisions X' and X'' of some finite linear simplicial complex X, then there exists a subdivision Y of X which is a subdivision of X' and X''.*

PROOF.

(1) This statement is proved in [**Gla70**, Corollary I.4] and [**Ze63b**, Corollary I.1].
(2) This statement follows immediately from (1) together with Lemma 61.22. ■

The following theorem gives us part (2) of Program 63.16. We will not provide a proof, instead we refer to [**Mun66a**, Theorem 17.2] for a detailed proof.

Theorem 63.18. *Let X be a simplicial complex and let X' be a subdivision of X. If $\varphi\colon X' \to X$ is a simplicial approximation of the identity map $X' \to X$, then the induced map $\varphi_*\colon\ \mathrm{H}_n^{\mathrm{simp}}(X') \to \mathrm{H}_n^{\mathrm{simp}}(X)$ is an isomorphism for every $n \in \mathbb{N}_0$.*

So an obvious candidate for the "moves" of Program 63.16 is a combination of subdivisions and simplicial isomorphisms. Now that we have dealt with part (2) of Program 63.16, let us try to tackle part (1). In fact this part is the following conjecture, which came to be known as the "Hauptvermutung". It was originally formulated by Heinrich Tietze [**Tie1908**, p. 13f] in 1908. The conjecture was also raised as a question by Ernst Steinitz [**Steini1908**, p. 32] in 1908.

Conjecture 63.19. (Hauptvermutung) *Let X and Y be two finite simplicial complexes. If the underlying topological spaces are homeomorphic, then there exist subdivisions of X and Y which are simplicially isomorphic.*

For better or worse it turned out that the Hauptvermutung was too optimistic. More precisely, the Hauptvermutung was eventually disproved by John Milnor [**Miln61**, Theorems 1 and 2] in 1961.

Theorem 63.20. *There exist two finite linear simplicial complex X and Y for which the underlying topological spaces are homeomorphic, but such that there are no subdivisions of X and Y which are simplicially isomorphic.*

Remark. In fact a few years later Rob Kirby and Larry Siebenmann [**KSi77**] (see also [**Ran96**, Chapter 4]) showed that given any $n \geq 5$ there exist such finite abstract simplicial complexes such that the underlying topological space is a compact n-dimensional topological manifold.

PROOF. The theorem is proved in [**Miln61**, Theorems 1 and 2]. Alternative proofs are given in [**Cohe73**, p. 84], [**Stal65b**, p. 252] and [**Ran96**, Chapter 2]. ■

Even though Part (1) of Program 63.16 did not work out, it turns out that Question 63.15 nonetheless has a positive answer. More precisely we have the following theorem.

Theorem 63.21. *Let X and Y be two simplicial complexes. If the underlying topological spaces X and Y are homeomorphic, then for every $n \in \mathbb{N}_0$ the simplicial homology groups $\mathrm{H}_n^{\mathrm{simp}}(X)$ and $\mathrm{H}_n^{\mathrm{simp}}(Y)$ are isomorphic.*

Using the Simplicial Approximation Theorem 62.8 and several other tools from the theory of simplicial complexes one can give a direct proof of Theorem 63.21. This approach is for example pursued in [**Mun66a**, Corollary 18.2], [**Pra07**, Chapter 2.2] and [**Gib10**, p. 136].

We will instead take an indirect approach to proving Theorem 63.21, namely we will show that given a simplicial complex the *simplicial* and *singular* homology groups are isomorphic. By the functoriality of *singular* homology the isomorphism type of *singular* homology groups only depends on the homeomorphism type of the topological space. With this observation in mind we see that Theorem 63.21 is a consequence of the following theorem.

Theorem 63.22. *Let X be a simplicial complex. For every $n \in \mathbb{N}_0$ there exists a natural*[972] *isomorphism $\Phi_X\colon \mathrm{H}_n^{\mathrm{simp}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$.*

We postpone the proof of Theorem 63.22 to the next section. In fact, in the next section we will formulate and prove a more precise version of Theorem 63.22. But before we delve into the detailed arguments let us first discuss the respective advantages and disadvantages of simplicial homology.

In some textbooks, see e.g. [**?, Mun84, SZ94, HY88**], homology groups are initially introduced via simplicial homology. This approach has several advantages:

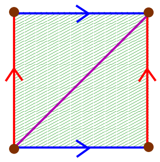
(1) The definition of simplicial homology is much more elementary and intuitive than the definition of singular homology.
(2) One can immediately compute the simplicial homology groups of interesting simplicial complexes, e.g. as we will see in Exercise 63.10, with enough patience and a little bit of cleverness one can compute the simplicial homology groups of abstract simplicial complexes that correspond to the surface of genus g.

Nonetheless there are serious disadvantages to this approach. A first problem, which is already apparent in the case of the torus, is that even for relatively simple examples the simplicial structures tend to have a large number of simplices, thus the corresponding chain complexes tend to be fairly large and brute force computations by hand become unpleasant. It turns out that there are some ways around this, for example one can extend the theory of simplicial homology from simplicial complexes to "Δ-complexes", which tend to have a significantly smaller number of simplices. Loosely speaking a Δ-complex is a topological space that is decomposed into simplices but now we only demand that the interiors of

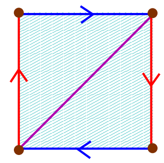
[972]Here "natural" means that if $f\colon X \to Y$ is a simplicial map, then the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}_n^{\mathrm{simp}}(X) \arrow[r, "f_*"] \arrow[d, "\Phi_X"'] & \mathrm{H}_n^{\mathrm{simp}}(Y) \arrow[d, "\Phi_L"] \\ \mathrm{H}_n(X) \arrow[r, "f_*"] & \mathrm{H}_n(Y). \end{tikzcd}$$

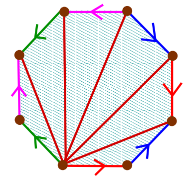
the simplices are embedded. To give a flavor of the theory we show three Δ-complexes in Figure 947. The theory of Δ-complexes is developed in detail in [**Hat02**, Chapter 2.1].



Δ-complex structure for the torus



Δ-complex structure for $\mathbb{R}\mathrm{P}^2$



Δ-complex structure for the surface of genus 2

FIGURE 947

Here are more serious reasons why we, and most modern authors, decided against starting out with simplicial homology and why we do not develop the theory of simplicial homology in greater detail:

(1) For slightly more complicated topological spaces it is rather laborious to compute simplicial homology groups. For example we challenge the reader to compute the simplicial homology groups of the projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ without reverting to cellular homology.
(2) Most topological spaces one wants to study (e.g. the complement of a knot in S^3), do not come with an obvious simplicial structure, it can be quite cumbersome to show that a simplicial structure exists, let alone to give an explicit simplicial structure.
(3) For a given topological space X which admits a simplicial structure we know by Theorem 63.21 that we have a well-defined *isomorphism type* of simplicial homology groups, but there is no single group that we can associate to X.
(4) Simplicial homology is functorial with respect to simplicial maps. Most maps are not simplicial. We do have the Simplicial Approximation Theorem 62.8 as a tool. But to use it in a sensible fashion one needs to show that the simplicial approximations of a given continuous map induce the same map on simplicial homology.

It should have become clear by now that simplicial homology comes with a Pandora's box of issues and problems. Overall, the initial extra effort in developing the theory of singular homology groups eventually pays of handily. The theory of singular homology groups is much more natural and much more flexible than the rather pedestrian theory of simplicial homology groups.

We conclude this section with the following piece of wisdom which was handed down to the author from his advisor:

> *For actual calculations it is best to work with very small objects, for proving general statements it is best to work with very large objects.*

63.5. **The topological invariance of simplicial homology II.** In this section we will first show that given an abstract simplicial complex the simplicial homology groups are naturally isomorphic to singular homology groups. Afterwards we will try to find a natural chain homotopy equivalence.

We start out with the first key result of this section.

Theorem 63.23. *Let $(K, \Theta\colon |K| \to X)$ be a simplicial complex. For each $n_0 \in \mathbb{N}$ there exists a natural isomorphism $\mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n(X)$.*

The proof of Theorem 63.23 builds on a proposition which relates the simplicial chain complex to the cellular chain complex. Before we state the proposition we recall that given a CW-complex we consider $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$ together with the boundary map d_n which is given by the composition

$$\begin{tikzcd} \mathrm{H}_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[rr, bend right, "=:d_n"'] & \mathrm{H}_{n-1}(X^{n-1}) \arrow[r, "j_{n-1}"] & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) \end{tikzcd}$$

where ∂_n is the connecting map in the long exact sequence of the pair (X^n, X^{n-1}) and where j_{n-1} is the natural map from absolute to relative homology.

Now we can formulate the promised proposition.

Proposition 63.24. *Let K be an abstract simplicial complex. Given an ordered n-simplex $s = (v_0, \dots, v_n)$ of K we denote by*

$$\begin{array}{rcl} \Phi_s^{\leq}\colon \Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1}_{\geq 0} \,\middle|\, \sum\limits_{i=0}^{n} t_i = 1 \right\} & \to & |s| \subset |K| = X \\ (t_0, \dots, t_n) & \mapsto & \sum\limits_{i=0}^{n} t_i \cdot v_i \end{array}$$

the corresponding characteristic map. The maps[973]

$$\begin{array}{rcl} \Upsilon_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) & \to & \mathrm{C}_n^{\mathrm{CW}}(|K|) \\ {[s]} & \mapsto & (\Phi_s^{\leq}\colon \Delta^n \to |K|) \end{array}$$

form a natural isomorphism of chain complexes.

Proof of Proposition 63.24. Let $K = (V, S)$ be an abstract simplicial complex. We write $X := |K|$. Given $n \in \mathbb{N}_0$ we denote by S_n the set of ordered n-simplices of K. Next we consider the following diagram

$$\begin{tikzcd} \mathrm{C}_n^{\mathrm{simp}}(K) := \Big(\bigoplus\limits_{s \in S_n} \mathbb{Z} \cdot s \Big) / I_n \arrow[rr, "{[s] \mapsto [\Phi_s^{\leq}]}"] \arrow[dd, "\partial_n"] & & \mathrm{H}_n(X^n, X^{n-1}) = \mathrm{C}_n^{\mathrm{CW}}(X) \arrow[dl, "\partial"] \arrow[dd, "d_n"] \\ & \mathrm{H}_{n-1}(X^{n-1}) \arrow[dr, "j_{n-1}"] & \\ \mathrm{C}_{n-1}^{\mathrm{simp}}(K) := \Big(\bigoplus\limits_{s \in S_{n-1}} \mathbb{Z} \cdot s \Big) / I_{n-1} \arrow[rr, "{[s] \mapsto [\Phi_s^{\leq}]}"] & & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) = \mathrm{C}_{n-1}^{\mathrm{CW}}(X). \end{tikzcd}$$

We make the following clarifications and observations:

(1) If s is an ordered k-simplex, then it follows immediately from the definition of the CW-structure on X that $\Phi_s^{\leq}(\Delta^k) \subset X^k$ and that $\Phi_s^{\leq}(\partial \Delta^k) \subset X^{k-1}$. In particular this shows that $\Phi_s^{\leq}$ represents an element in $\mathrm{H}_k(X^k, X^{k-1})$.

[973]Recall that in Lemma 61.24 we showed that the simplicial complex $|K|$ admits a natural CW-structure.

(2) It follows fairly easily from (1) together with Lemma 19.1 and Proposition 45.22 that the horizontal maps are well-defined.
(3) Basically the same argument as in Lemma 48.1 (1) shows that the horizontal maps are isomorphisms.
(4) As explained above, the top diagonal map $\partial\colon \mathrm{H}_n(X^n, X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1})$ is the connecting homomorphism of the pair (X^n, X^{n-1}) and the bottom diagonal map $j_{n-1}\colon \mathrm{H}_{n-1}(X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$ is the natural map from absolute homology to relative homology. The map d_n is by definition the composition of these two maps. In other words, the triangle commutes by definition.
(5) It follows fairly immediately from the definition of the connecting homomorphism, see Proposition 43.15, and the definition of the maps $\Phi_s^{\leq}$ that the pentagon to the left commutes.
(6) It follows from (4) and (5) that the outer square of the diagram also commutes.
(7) The horizontal maps are basically by definition natural.

The above discussion shows that the horizontal maps define a natural isomorphism of chain complexes. ■

For the reader's convenience we recall the following proposition.

Proposition 48.4. *Given any* CW-*complex X and given any $n \in \mathbb{N}_0$ there exists a uniquely determined isomorphism*

$$\Phi_X\colon \mathrm{H}_n(X) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{CW}}(X)$$

with the property that the following diagram commutes:[a]

$$\begin{tikzcd} & \mathrm{H}_n(X) \arrow[dr, "\Phi_X"] & \\ \mathrm{H}_n(X^n) \arrow[ur] \arrow[dr, "j_n"'] & & \mathrm{H}_n^{\mathrm{CW}}(X). \\ & \ker(d_n) \arrow[ur, two heads] & \end{tikzcd}$$

Furthermore this map Φ_X is natural.

[a]Here we again denote by $j_n\colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ the obvious map. In the long exact sequence of the pair (X^n, X^{n-1}) we see that the image of $\mathrm{H}_n(X^n)$ in $\mathrm{H}_n(X^n, X^{n-1})$ lies in the kernel of the connecting homomorphism ∂_n. This implies that the map $j_n\colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ takes values in $\ker(d_n)$.

Now we can put everything together to obtain a proof of Theorem 63.23.

Proof of Theorem 63.23. It follows immediately from the definitions that we only need to consider the case of the topological realization of an abstract simplicial complex K. We equip $|K|$ with the CW-structure provided by Lemma 61.24. Now given any $n \in \mathbb{N}_0$ we have the following two natural isomorphisms:

$$\underbrace{\mathrm{H}_n^{\mathrm{simp}}(K) \xrightarrow{\cong}}_{\substack{\text{natural isomorphism } \Upsilon_K \\ \text{given by Proposition 63.24}}} \mathrm{H}_n^{\mathrm{CW}}(|K|) \underbrace{\xleftarrow{\cong}}_{\substack{\text{natural isomorphism } \Phi_{|K|} \text{ given} \\ \text{by Proposition 48.4}}} \mathrm{H}_n(|K|).$$

The natural isomorphism $\Phi_X^{-1} \circ \Upsilon_K$ is the natural isomorphism of our desires. ■

One problem with Theorem 63.23 is that it does not give us an explicit isomorphism between the homology groups. This issue gets mostly resolved by the following more sophisticated theorem.

Theorem 63.25. *Let K be an* ordered *abstract simplicial complex. The maps*

$$\begin{array}{rcl} \Theta_n\colon\ \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n(|K|) \\ s & \mapsto & (\Phi_s^{\leq}\colon \Delta^n \to |K|) \end{array}$$

are a natural[974] *chain homotopy equivalence. In particular for each $n \in \mathbb{N}_0$ the induced map*

$$\Theta_*\colon\ \mathrm{H}_n^{\mathrm{simp},\leq}(K) \ \to\ \mathrm{H}_n(|K|)$$

is a natural isomorphism.

Remark. Theorem 63.25 can actually be quite useful in practice. Let us elaborate. In practice, given a topological space X one would like to perform the following three types of tasks:

(1) Determine the isomorphism type of the homology groups $\mathrm{H}_k(X)$.
(2) Determine explicit cycles in $\mathrm{C}_k(X)$ that represent a basis or a generating set for the homology groups $\mathrm{H}_k(X)$.
(3) Calculate induced maps $\mathrm{H}_k(X) \to \mathrm{H}_k(Y)$ with respect to, say, given bases or generating sets for the homology groups $\mathrm{H}_k(X)$ and $\mathrm{H}_k(Y)$.

In fact in many situations, e.g. in applications of the Mayer–Vietoris Theorem 46.5, these problems feed on one another. Even though by now we have a long list of tools, the fact that the singular chain groups $\mathrm{C}_*(X)$ are basically always of infinite, in fact mostly of uncountable, rank makes it hard to attack the above problems directly.

Fortunately, in many situations Theorem 63.25 allows us to translate the above problems into tasks in a simplicial chain complex. For reasonable abstract simplicial complexes, these problems in turn boil down to elementary problems in linear algebra that usually can be solved easily.

Example. We consider the 1-dimensional ordered abstract simplicial complex K shown in Figure 948 which we had already considered on page 1542. By the discussion on page 1542 we know that

$$\mathrm{H}_1^{\mathrm{simp},\leq}(K) \ =\ \ker\big(\partial_1\colon\ \mathrm{C}_1^{\mathrm{simp},\leq}(K) \to \mathrm{C}_0^{\mathrm{simp},\leq}(K)\big) \ =\ \mathbb{Z}\cdot[\{C,D\}+\{D,E\}-\{C,E\}].$$

We write $x = \Theta_1(\{C,D\})$, $y = \Theta_1(\{D,E\})$ and $z = \Theta_1(\{C,E\})$. It follows from Theorem 63.25 and the above discussion that $x+y-z$ represents a generator of $\mathrm{H}_1(|K|)$.[975]

PROOF OF THEOREM 63.25. Let K be an ordered abstract simplicial complex. It follows almost immediately from the definitions that the maps

$$\begin{array}{rcl} \Theta_n\colon\ \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n(|K|) \\ s & \mapsto & (\Phi_s^{\leq}\colon \Delta^n \to |K|) \end{array}$$

[974]Here "natural" means with respect to the category of *ordered* abstract simplicial complexes.
[975]What other methods do you know for proving this statement?

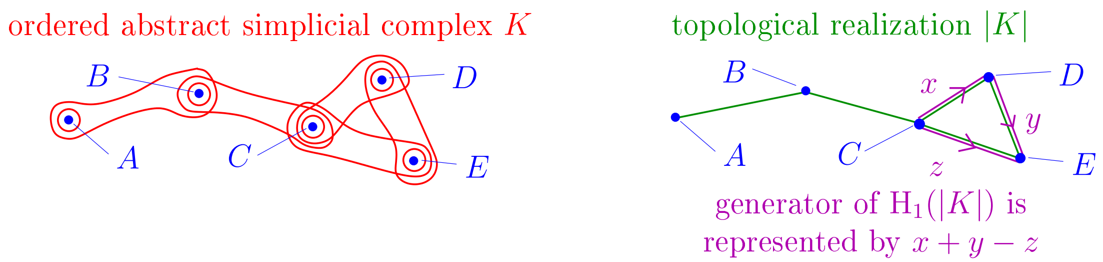


FIGURE 948

define a natural chain map. We leave it to the skeptical reader to verify that this statement is indeed true and basically trivial. It remains to show that the maps Θ_n define a chain homotopy equivalence. By Proposition 49.2 it now suffices to prove the following claim.

Claim. Let K be an ordered abstract simplicial complex. For every $n \in \mathbb{N}_0$ the induced map

$$\Theta_*\colon\ H_n^{\text{simp},\leq}(K) \ \to\ H_n(|K|)$$

is an isomorphism.

We start out with two observations. Let K be an ordered abstract simplicial complex, let $m \in \mathbb{N}_0 \cup \{\infty\}$ and let $n \in \mathbb{N}_0$. We write $X = |K|$. It follows from Lemma 63.9 and Lemma 48.1 that the following statements hold:

(i) The two inclusion induced maps

$$\begin{array}{rcl} H_n^{\text{simp},\leq}(K^m) & \to & H_n^{\text{simp},\leq}(K) \\ H_n(|K^m|) & \to & H_n(|K|) \end{array} \qquad \text{are both isomorphisms whenever } n < m.$$

(ii) If $n > m$, then $\begin{array}{rcl} H_n^{\text{simp},\leq}(K^m) & = & 0 \\ H_n(|K^m|) & = & 0. \end{array}$

It follows immediately from (i) that it suffices to prove the claim for finite dimensional ordered abstract simplicial complexes. As usual, for finite dimensional ordered abstract simplicial complexes we proceed by induction on the dimension. The case of an empty and of a 0-dimensional ordered abstract simplicial complex is trivial. So suppose we have shown the statement for all ordered abstract simplicial complexes of dimension $\leq m-1$. Now let K be an m-dimensional ordered abstract simplicial complex. We write $X := |K|$ and we denote by $i\colon K^{m-1} \to K$ and $i\colon X^{m-1} \to X$ the inclusion maps. It follows from (i) and (ii), the naturality of Θ and our induction hypothesis that we only need to worry about the two maps

$$\Theta_*\colon\ H_m^{\text{simp},\leq}(K) \ \to\ H_m(X) \quad \text{and} \quad \Theta_*\colon\ H_{m-1}^{\text{simp},\leq}(K) \ \to\ H_{m-1}(X).$$

To take care of this map we consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}_m^{\mathrm{simp},\leq}(K) \arrow[r, hook] \arrow[dd, "\Theta_*"] & \mathrm{C}_m^{\mathrm{simp},\leq}(K) \arrow[r, "\partial"] \arrow[d, "\Psi_*", "\cong"'] & \mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K^{m-1}) \arrow[r, "i_*"] \arrow[dd, "\Theta_*", "\cong"'] & \mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K) \arrow[r] \arrow[dd, "\Theta_*"] & 0 \\
& & \mathrm{C}_m^{\mathrm{CW}}(X) \arrow[d, "="] & & & \\
0 \arrow[r] & \mathrm{H}_m(X) \arrow[r] & \mathrm{H}_m(X, X^{m-1}) \arrow[r, "\partial"] & \mathrm{H}_{m-1}(X^{m-1}) \arrow[r, "i_*"] & \mathrm{H}_{m-1}(X) \arrow[r] & 0.
\end{tikzcd}$$

We make the following observations and clarifications:

(1) First we show that the horizontal maps on the top actually make sense. Since $K = K^m$ we see that $\mathrm{H}_m^{\mathrm{simp},\leq}(K)$ is a subgroup of $\mathrm{C}_m^{\mathrm{simp},\leq}(K)$. In fact by definition it equals $\ker(\partial\colon \mathrm{C}_m^{\mathrm{simp},\leq}(K) \to \mathrm{C}_{m-1}^{\mathrm{simp},\leq}(K))$. Furthermore note, that basically by definition, given $s \in \mathrm{C}_m^{\mathrm{simp},\leq}(K)$ we have $\partial s \in \mathrm{C}_{m-1}^{\mathrm{simp},\leq}(K^{m-1})$. Since $\partial \circ \partial = 0$ we see that ∂s defines an element in $\mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K^{m-1})$.

(2) Using (1) and using the explicit definitions of simplicial homology groups it is not hard to see that the top horizontal sequence is exact.

(3) It follows from $X = X^m$ and Lemma 48.1 that $\mathrm{H}_i(X, X^{m-1}) = 0$ for $i \neq m$. Thus we see that the bottom horizontal sequence is a segment of the long exact sequence of the pair (X, X^{m-1}).

(4) It follows easily from the definition of the maps Θ and Ψ and the explicit description of the connecting homomorphism $\partial\colon \mathrm{H}_m(X, X^{m-1}) \to \mathrm{H}_{m-1}(X^{m-1})$ that we gave in Proposition 43.15 that the two squares to the left commute. It follows from the naturality of Θ_* that the square to the right commutes.

(5) It follows from induction that the vertical map $\Theta_*\colon \mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K^{m-1}) \to \mathrm{H}_{m-1}(X^{m-1})$ to the right is an isomorphism.

(6) By Proposition 63.24 we know that Ψ_* is an isomorphism.

(7) It follows from all of the above together with the Five-Lemma 43.12 that the vertical maps $\Theta_*\colon \mathrm{H}_m^{\mathrm{simp},\leq}(K) \to \mathrm{H}_m(X)$ and $\Theta_*\colon \mathrm{H}_{m-1}^{\mathrm{simp},\leq}(K) \to \mathrm{H}_{m-1}(X)$ to the left and right are isomorphisms. ■

Given an abstract simplicial complex K together with an order "$\leq$" on K we denote by $\Omega_{\leq,k}\colon \mathrm{C}_k^{\mathrm{simp},\leq}(K) \to \mathrm{C}_k^{\mathrm{simp}}(K)$ the natural chain isomorphism given by Lemma 63.8. Together with Theorem 63.25 we obtain an isomorphism

$$\mathrm{H}_k^{\mathrm{simp}}(K) \xleftarrow{\Omega_{\leq,k}^{-1}} \mathrm{H}_k^{\mathrm{simp},\leq}(K) \xrightarrow{\Theta_{\leq,k}} \mathrm{H}_k(|K|).$$

A priori this isomorphism depends on the choice of the order "$\leq$". The next lemma shows that fortunately this is not the case.

Lemma 63.26. *Let K be an abstract simplicial complex and let "$\leq$" and "$\subseteq$" be two orders on K. The two chain maps*

$$\begin{array}{rccccc} & \mathrm{C}_k^{\mathrm{simp}}(K) & \xrightarrow{\Omega_{\leq,k}^{-1}} & \mathrm{C}_k^{\mathrm{simp},\leq}(K) & \xrightarrow{\Theta_{\leq,k}} & \mathrm{C}_k(|K|) \\ \textit{and} & \mathrm{C}_k^{\mathrm{simp}}(K) & \xrightarrow{\Omega_{\subseteq,k}^{-1}} & \mathrm{C}_k^{\mathrm{simp},\subseteq}(K) & \xrightarrow{\Theta_{\subseteq,k}} & \mathrm{C}_k(|K|) \end{array}$$

are chain homotopic. In particular they induce the same maps on homology.

Proof ($*$). Let $K = (V, S)$ be an abstract simplicial complex. We start with the following preparations:

(1) For each $i \in \mathbb{N}_0$ we pick once and for all for each i-simplex $s \in S$ an ordered i-simplex $\widetilde{s}$ with $\lfloor \widetilde{s} \rfloor = s$. We denote by $\widetilde{S}_i$ the set given by all these ordered i-simplices.

(2) We introduce the following convention. If for some $i \in \mathbb{N}_0$ we are given for every $\widetilde{s} \in \widetilde{S}_i$ a singular chain $P_i(\widetilde{s}) \in \mathrm{C}_{i+1}(|\widetilde{s}|)$ then we denote by P_i the map

$$\begin{aligned} \mathrm{C}_i^{\mathrm{simp}}(K) &\to \mathrm{C}_{i+1}(|K|) \\ \widetilde{s} &\mapsto P_i(\widetilde{s}) \quad \text{viewed as an element in } \mathrm{C}_{i+1}(|K|) \text{ via the inclusion } |\widetilde{s}| \subset |K| \end{aligned}$$

Now let "$\leq$" and "$\subseteq$" be two orders on $\Delta = (V, S)$.

Claim. Let $l \in \mathbb{N}$. Suppose that for every $i \in \{0, \ldots, l-1\}$ and for every $\widetilde{s} \in \widetilde{S}_i$ we are given a singular chain $P_i(\widetilde{s}) \in \mathrm{C}_{i+1}(|\widetilde{s}|)$ such that for every $i \in \{0, \ldots, l-1\}$ we have the equality

$$(*) \qquad \partial_{i+1} \circ P_i + P_{i-1} \circ \partial_i \;=\; \Theta_{\leq,i} \circ \Omega_{\leq,i}^{-1} - \Theta_{\subseteq,i} \circ \Omega_{\subseteq,i}^{-1} \colon\ \mathrm{C}_i^{\mathrm{simp}}(K) \to \mathrm{C}_i(|K|).$$

Then for every $\widetilde{s} \in \widetilde{S}_l$ we can also find $P_l(s) \in \mathrm{C}_{l+1}(|\widetilde{s}|)$ such that the above equality holds for $i = 0, \ldots, l$.

Let $\widetilde{s} \in \widetilde{S}_l$. Note that we have

$$\begin{aligned} \partial_l\big((-P_{l-1} \circ \partial_l + \Theta_{\leq,l} \circ \Omega_{\leq,l}^{-1} - \Theta_{\subseteq,l} \circ \Omega_{\subseteq,l}^{-1})(\widetilde{s})\big) &\overset{\substack{\text{since } \Theta_{\leq,i} \circ \Omega_{\leq,i}^{-1} \text{ and } \Theta_{\subseteq,i} \circ \Omega_{\subseteq,i}^{-1} \text{ are chain maps} \\ \downarrow}}{=} (-\partial_l \circ P_{l-1} + \Theta_{\leq,l-1} \circ \Omega_{\leq,l-1}^{-1} - \Theta_{\subseteq,l-1} \circ \Omega_{\subseteq,l-1}^{-1})(\partial_l(\widetilde{s})) \\ &\underset{\substack{\uparrow \\ \text{by equality } (*) \text{ for } i = l-1}}{=} (P_{l-2} \circ \partial_{l-1})(\partial_l(\widetilde{s})) = P_{l-2}(\partial_{l-1} \circ \partial_l)(\widetilde{s}) = 0. \end{aligned}$$

Noe that by construction all terms are contained in $\mathrm{C}_l(|\widetilde{s}|) \subset \mathrm{C}_l(|K|)$. For orientation it is helpful to consider the following diagram

$$\begin{tikzcd} & \mathrm{C}_l^{\mathrm{simp}}(\widetilde{s}) \arrow[r, "\partial_l"] \arrow[d] \arrow[dl, "P_l"'] & \mathrm{C}_{l-1}^{\mathrm{simp}}(\widetilde{s}) \arrow[r, "\partial_{l-1}"] \arrow[d] \arrow[dl, "P_{l-1}"'] & \mathrm{C}_{l-1}^{\mathrm{simp}}(\widetilde{s}) \arrow[d] \arrow[dl, "P_{l-1}"'] \\ \mathrm{C}_{l+1}(|\widetilde{s}|) \arrow[r, "\partial_{l+1}"'] & \mathrm{C}_l(|\widetilde{s}|) \arrow[r, "\partial_l"'] & \mathrm{C}_{l-1}(|\widetilde{s}|) \arrow[r, "\partial_{l-1}"'] & \mathrm{C}_{l-2}(|\widetilde{s}|) \end{tikzcd}$$

Note that we are dealing with $l \in \mathbb{N}_0$. This implies that $\mathrm{H}_l(|\widetilde{s}|) = 0$. But this means that every cycle is in fact a boundary. In particular the above cycle is the boundary of some

$P_{l+1}(\widetilde{s}) \in \mathrm{C}_{l+1}(|\widetilde{s}|)$, i.e. we have

$$(\partial_{l+1}(P_{l+1}(\widetilde{s})) \;=\; (-P_{l-1} \circ \partial_l + \Theta_{\leq,l} \circ \Omega_{\leq,l}^{-1} - \Theta_{\subseteq,l} \circ \Omega_{\subseteq,l}^{-1})(\widetilde{s}).$$

By construction all these $P_{l+1}(\widetilde{s})$ have the desired property. ⊞

Now we can easily find the desired chain homotopy. Indeed, for every 0-simplex s we set $P_0(s) = 0 \in \mathrm{C}_1(|s|)$. Since $\Omega_{\subseteq,0} = \Omega_{\subseteq,0}\colon\ \mathrm{C}_0^{\mathrm{simp}}(K) \to \mathrm{C}_0(|K|)$ we see that can start the machine provided by the claim and iteratively we construct maps P_i which are precisely the promised chain homotopy. ■

The following is now an almost immediate corollary to Lemma 63.8, Theorem 63.25 and Lemma 63.26.

Corollary 63.27. *Let K be an abstract simplicial complex. We pick an order "$\leq$" on K. The maps*

$$\Xi_k := \Theta_{\leq,k} \circ \Omega_{\leq,k}^{-1}\colon\ \mathrm{H}_k^{\mathrm{simp}}(K) \;\to\; \mathrm{H}_k(|K|),$$

which by Lemma 63.26 do not depend on the choice of the order "$\leq$", define a natural isomorphism between functors from the category $\mathcal{AbsSimpCplx}$ of abstract simplicial complexes (the key here is that the word "ordered" is missing) to the category $\mathcal{AbGr}$ of abelian groups.

It is worth repeating it:

(1) Given any abstract simplicial complex K and given any $k \in \mathbb{N}_0$ there exists a natural isomorphism $\mathrm{H}_k^{\mathrm{simp}}(K) \to \mathrm{H}_k(|K|)$ of homology groups.
(2) If K is an ordered abstract simplicial, then there is in fact a natural chain homotopy equivalence $\mathrm{C}_*^{\mathrm{simp},\leq}(K) \to \mathrm{C}_*(|K|)$.

The alert reader will have noticed that given an abstract simplicial complex we gave two natural isomorphisms $\mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n(|K|)$. The following proposition shows that the two isomorphisms agree.

Proposition 63.28. *Let K be an abstract simplicial complex and let $n_0 \in \mathbb{N}$. We write $X = |K|$. We denote by*

$$\Upsilon_*\colon\ \mathrm{C}_*^{\mathrm{simp}}(K) \;\to\; \mathrm{C}_*^{\mathrm{CW}}(X) \quad \text{and} \quad \Phi_X\colon\ \mathrm{H}_n(X) \;\to\; \mathrm{H}_n^{\mathrm{CW}}(X)$$

the isomorphisms given by Propositions 63.24 and 48.4. Next we pick an order "$\leq$" on K. We consider the chain homotopy equivalences

$$\Omega\colon\ \mathrm{C}_*^{\mathrm{simp},\leq}(K) \;\to\; \mathrm{C}_*^{\mathrm{simp}}(K) \quad \text{and} \quad \Theta\colon\ \mathrm{C}_*^{\mathrm{simp},\leq}(K) \;\to\; \mathrm{C}_*(|K|)$$

from Lemma 63.8 and Theorem 63.25. With this notation the following equality holds:

$$\Phi_X^{-1} \circ \Upsilon_* \;=\; \Theta_* \circ \Omega_*^{-1}\colon\ \mathrm{H}_n^{\mathrm{simp}}(K) \;\to\; \mathrm{H}_n(X).$$

Notation. Let K be an abstract simplicial complex and let $n_0 \in \mathbb{N}$. By a slight abuse of notation we denote by $\Theta_K\colon\ \mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n(|K|)$ the natural isomorphism given by Theorem 63.23 and Proposition 63.28.

Proof. We write $\Psi_X := \Upsilon_* \circ \Omega_* \circ \Theta_*^{-1}$. We need to show that $\Psi_X = \Phi_X$. To do so we consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_n^{\mathrm{simp},\leq}(K^n) \arrow[rrr, "i_*"] \arrow[dd, "\Theta_*"', "\cong"] & & & \mathrm{H}_n^{\mathrm{simp},\leq}(K) \arrow[dll, "\Theta_*"', "\cong"] \arrow[dd, "\cong"', "\Omega_*"] \\
& \mathrm{H}_n(X) \arrow[dr, "\Psi_X", "\Phi_X"'] & & \\
\mathrm{H}_n(X^n) \arrow[ur, "|i|_*"] \arrow[dr, "j_n"'] & & \mathrm{H}_n^{\mathrm{CW}}(X) & \mathrm{H}_n^{\mathrm{simp}}(K) \arrow[l, "\cong"', "\Upsilon_*"] \\
& \ker(d_n) \arrow[ur, two heads] & &
\end{tikzcd}$$

We make the following clarifications and observations:

(1) The map $i\colon K^n \to K$ is the inclusion map.
(2) The triangle shaped quadrangle to the upper left commutes by the naturality of Θ_*.
(3) By the definition of Ψ_X the quadrangle to the right commutes.
(4) It follows easily from the definitions of the various maps that the "outer diagram" commutes, i.e. the two maps from $\mathrm{H}_n^{\mathrm{simp},\leq}(K^n)$ to $\mathrm{H}_n^{\mathrm{CW}}(X)$ agree.
(5) Since Υ_*, Ω_* and Θ_* are isomorphisms we see that Ψ_X is an isomorphism and we see, in combination with (2), (3) and (4), that Ψ_X makes the parallelogram commute. But by the characterization of Φ_X in Proposition 48.4 this means that $\Psi_X = \Phi_X$. ■

63.6. **Relative simplicial homology** $(*)$**.** In this section, given an (ordered) abstract simplicial complex K together with a subcomplex L we introduce the relative simplicial homology groups $\mathrm{H}_k^{\mathrm{simp}}(K, L)$ and $\mathrm{H}_k^{\mathrm{simp},\leq}(K, L)$. This section contains no surprises and can be skipped easily. We will make little use of relative simplicial homology groups, except that they will come in handy in the proof of the Simplicial Poincaré Duality Theorem 71.4.

Definition. Let K be an abstract simplicial complex and let L be a (possibly empty) subcomplex of K.

(1) Given $k \in \mathbb{N}_0$ we define

$$\mathrm{C}_k^{\mathrm{simp}}(K, L) \;:=\; \mathrm{C}_k^{\mathrm{simp}}(K) / \,\mathrm{C}_k^{\mathrm{simp}}(L).$$

The usual boundary map $\partial_k\colon\, \mathrm{C}_k^{\mathrm{simp}}(K) \to \mathrm{C}_{k-1}^{\mathrm{simp}}(K)$ descends to a boundary map $\partial_k\colon\, \mathrm{C}_k^{\mathrm{simp}}(K, L) \to \mathrm{C}_{k-1}^{\mathrm{simp}}(K, L)$ and we can thus define the corresponding relative simplicial homology groups $\mathrm{H}_k^{\mathrm{simp}}(K, L)$.
(2) If "$\leq$" is an order on K, then we endow L with the corresponding order and we introduce $\mathrm{C}_k^{\mathrm{simp},\leq}(K, L)$ and $\mathrm{H}_k^{\mathrm{simp},\leq}(K, L)$ in the obvious way.

The following lemma is an immediate consequence of Proposition 43.11.

Lemma 63.29. *Let K be an abstract simplicial complex and let L be a subcomplex of K. The sequence*

$$\begin{array}{cccccccccc} \ldots & \to & \mathrm{H}_k^{\mathrm{simp}}(L) & \to & \mathrm{H}_k^{\mathrm{simp}}(K) & \to & \mathrm{H}_k^{\mathrm{simp}}(K,L) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}^{\mathrm{simp}}(L) & \to & \ldots \\ & & & & & & [\sigma] & \mapsto & [\partial_k(\sigma)] & & \end{array}$$

is exact and the connecting homomorphism is natural. The analogous statement also holds if K is equipped with an order.

We conclude this short section with the following lemma.

Lemma 63.30. *Let K be an abstract simplicial complex and let L be a subcomplex of K.*

(1) *Let "$\leq$" be an order on K. The natural maps*

$$\begin{array}{ccc} \mathrm{C}_k^{\mathrm{simp},\leq}(K,L) & \to & \mathrm{C}_k^{\mathrm{simp}}(K,L) \\ [\sigma] & \mapsto & [\Omega(\sigma)] \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{C}_k^{\mathrm{simp},\leq}(K,L) & \to & \mathrm{C}_k(|K|,|L|) \\ [\sigma] & \mapsto & [\Theta(\sigma)] \end{array}$$

are chain homotopy equivalences. Furthermore the natural maps

$$\begin{array}{ccc} \mathrm{H}_k^{\mathrm{simp},\leq}(K,L) & \to & \mathrm{H}_k^{\mathrm{simp}}(K,L) \\ [\sigma] & \mapsto & \Omega(\sigma)] \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{H}_k^{\mathrm{simp},\leq}(K,L) & \to & \mathrm{H}_k(|K|,|L|) \\ [\sigma] & \mapsto & [\Theta(\sigma)] \end{array}$$

are isomorphisms.

(2) *The map $\Xi := \Omega \circ \Theta^{-1} \colon \mathrm{H}_k^{\mathrm{simp}}(K,L) \to \mathrm{H}_k(|K|,|L|)$ does not depend on the choice of the order "$\leq$" on K, and it defines a natural isomorphism.*

(3) *The following diagram commutes:*

$$\begin{array}{ccccccccccc} \ldots \longrightarrow & \mathrm{H}_k^{\mathrm{simp},\leq}(K) & \longrightarrow & \mathrm{H}_k^{\mathrm{simp},\leq}(K,L) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}^{\mathrm{simp},\leq}(L) & \longrightarrow & \mathrm{H}_{k-1}^{\mathrm{simp},\leq}(K) & \longrightarrow \ldots \\ & \Xi\downarrow & & \Xi\downarrow & & \Xi\downarrow & & \Xi\downarrow & \\ \ldots \longrightarrow & \mathrm{H}_k(|K|) & \longrightarrow & \mathrm{H}_k(|K|,|L|) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}(|L|) & \longrightarrow & \mathrm{H}_{k-1}(|K|) & \longrightarrow \ldots \end{array}$$

PROOF. We consider the following diagram:

$$\begin{array}{ccccccccccc} \ldots \longrightarrow & \mathrm{H}_k^{\mathrm{simp}}(L) & \longrightarrow & \mathrm{H}_k^{\mathrm{simp}}(K) & \longrightarrow & \mathrm{H}_k^{\mathrm{simp}}(K,L) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}^{\mathrm{simp}}(L) & \longrightarrow & \mathrm{H}_{k-1}^{\mathrm{simp}}(K) & \longrightarrow \ldots \\ & \Omega\uparrow & & \Omega\uparrow & & \Omega\uparrow & & \Omega\uparrow & & \Omega\uparrow & \\ \ldots \longrightarrow & \mathrm{H}_k^{\mathrm{simp},\leq}(L) & \longrightarrow & \mathrm{H}_k^{\mathrm{simp},\leq}(K) & \longrightarrow & \mathrm{H}_k^{\mathrm{simp},\leq}(K,L) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}^{\mathrm{simp},\leq}(L) & \longrightarrow & \mathrm{H}_{k-1}^{\mathrm{simp},\leq}(K) & \longrightarrow \ldots \\ & \Theta\downarrow & & \Theta\downarrow & & \Theta\downarrow & & \Theta\downarrow & & \Theta\downarrow & \\ \ldots \longrightarrow & \mathrm{H}_k(|L|) & \longrightarrow & \mathrm{H}_k(|K|) & \longrightarrow & \mathrm{H}_k(|K|,|L|) & \xrightarrow{\partial_k} & \mathrm{H}_{k-1}(|L|) & \longrightarrow & \mathrm{H}_{k-1}(|K|) & \longrightarrow \ldots \end{array}$$

It follows easily from the definitions and the naturality of the connecting homomorphisms, see Proposition 43.11, that the diagram commutes. By Lemma 63.29 and by Corollary 43.16 we know that the horizontal sequences are exact. Furthermore by Lemma 63.8 and Theorem 63.25 we know that the outer two vertical maps are isomorphisms. Now it follows from the Five-Lemma 43.12 that the middle vertical maps are also isomorphisms. It follows from Proposition 49.2 that the corresponding chain maps are chain homotopy equivalences. This concludes the proof Statement (1).

The same argument as in the proof of Corollary 63.27 shows that the composition of the middle vertical maps does not depend on the choice of the order "$\leq$".[976]

Finally Statement (3) follows from the fact that the above diagram commutes. ■

63.7. **The Nerve Theorem.** In this section we will state the "Nerve Theorem". In many situations it gives an effective way for showing that a given topological space is homotopy equivalent to a simplicial complex.

Definition. Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a cover of X.

(1) We say the cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X is *good* if for every two choice of $i_1, \ldots, i_m \in I$ the intersection $U_{i_1} \cap \cdots \cap U_{i_m}$ is either empty or contractible.
(2) We say the cover $\mathcal{U}$ of X is *point-finite* if each point $x \in X$ is contained in only finitely many U_i.[977]

Example. In Figure 949 we show the topological space $X = S^1$ with one cover which is not good and three covers which are actually good.

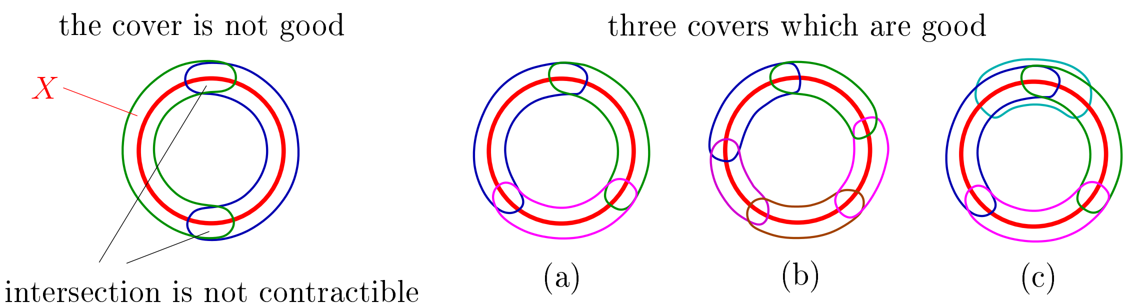


FIGURE 949

As we will see, the following lemma is a straightforward consequence of Lemma 62.5.

Lemma 63.31. *Let $K = (V,S)$ be an abstract simplicial complex. The open stars $\{\mathring{\mathrm{St}}(v)\}_{v\in V}$ are an open good point-finite cover.*

FIGURE 950. Illustration of Lemma 63.31.

PROOF. Let $K = (V,S)$ be an abstract simplicial complex. Let $P \in |K|$. By Lemma 61.11 (2) we know that there exists a unique $s \in S$ with $P \in \langle s\rangle$. Furthermore, by Lemma 62.5 (2) we know that $P \in \mathring{\mathrm{St}}(v)$ if and only if v is a vertex of s. This discussion, together with

[976]Note that Exercise 43.3 shows that one cannot reduce this statement "Five Lemma"-style to the fact that we already know that the composition of the outer two vertical maps is independent of the choice of the order "$\leq$".

[977]Note that this notion is weaker than the notion of a locally finite cover that we introduced on page 89.

Lemma 62.5 (2), shows that the open stars $\overset{\circ}{\mathrm{St}}(v)$ form an open point-finite cover of $|K|$. It remains to show that the cover is good. Thus let $v_0, \ldots, v_n \in V$. We have

$$\overset{\circ}{\mathrm{St}}(v_0) \cap \cdots \cap \overset{\circ}{\mathrm{St}}(v_n) \underset{\substack{\uparrow \\ \text{Lemma 62.5 (3)}}}{=} \begin{cases} \overbrace{\overset{\circ}{\mathrm{St}}(\{v_0, \ldots, v_n\}),}^{\text{contractible by Lemma 62.5 (1f)}} & \text{if } \{v_0, \ldots, v_n\} \text{ is a simplex,} \\ \varnothing, & \text{otherwise.} \end{cases}$$

We have thus shown that the open stars $\{\overset{\circ}{\mathrm{St}}(v)\}_{v \in V}$ are a good point-finite cover. ■

As we will see shortly, Lemma 63.31 has a partial converse, namely the mysteriously named Nerve Theorem. To state this converse we need the following definition.

Definition. Let X be a set and let $\mathcal{U} = \{U_i\}_{i \in I}$ be a family of subsets of X. We refer to the abstract simplicial complex that is given by the vertex set I and the simplex set[978] $\{\{i_1, \ldots, i_r\} \,|\, i_1, \ldots, i_r \in I \text{ with } U_{i_1} \cap \cdots \cap U_{i_m} \neq \varnothing\}$ as the *nerve complex* $N(\mathcal{U})$ of the covering.

Example. In Figure 951 we revisit the four covers of S^1 that we showed in Figure 949. This time we also show the topological realizations of the corresponding nerve complexes. As is clear from the figure, for the cover to the left, which is not good, the topological realization of the nerve complex does not look at all like S^1. On the other hand, for the three covers to the right, which are good, the topological realization of the nerve complex is homeomorphic or at least homotopy equivalent to S^1. As we will see in a minute, that is not a coincidence.

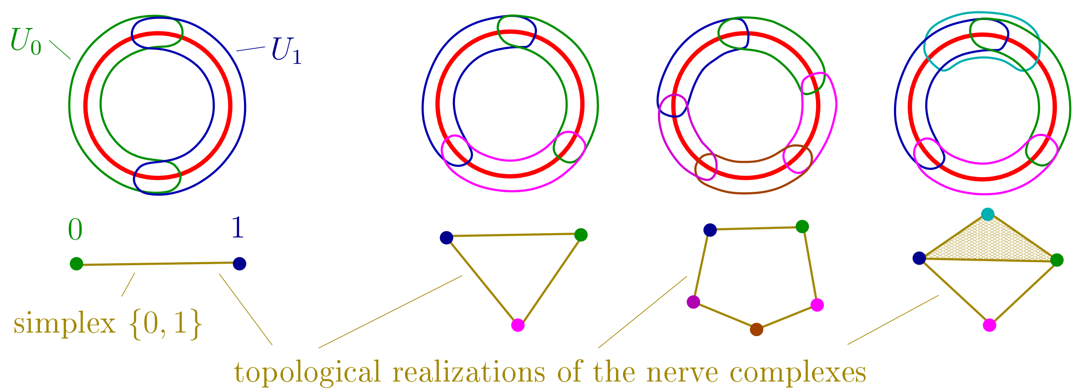


FIGURE 951

To formulate the Nerve Theorem of this section it is convenient to introduce the following definition.

Definition. We say that a map $f\colon X \to Y$ between path-connected topological spaces is a *weak homotopy equivalence* if there exists $x_0 \in X$ such that for every $k \in \mathbb{N}_0$ the induced map $f_*\colon \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$ is an isomorphism.

[978]It follows easily from the definitions that this is indeed an abstract simplicial complex.

By Proposition 40.7 (2) a homotopy equivalence between path-connected topological spaces is a weak homotopy equivalence. The following theorem says, rather amazingly, that for CW-complexes the converse holds.

Theorem 119.9. (Whitehead Theorem) *Let $f\colon X \to Y$ be a map between two connected* CW*-complexes.*[979] *If $f\colon X \to Y$ is a weak homotopy equivalence, then f is a homotopy equivalence.*

As is clear from the numbering, we will prove the Whitehead Theorem 119.9 much later in these notes. But, as we will see, this theorem greatly facilitates the discussion of the Nerve Theorem.

Recall that in Lemma 63.31 we showed that every simplicial complex admits a good cover. We can now formulate and prove the Nerve Theorem which can in particular be viewed as a partial converse to Lemma 63.31.

Theorem 63.32. (Nerve Theorem) *Let X be a path-connected topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open good point-finite cover of X. The following two statements hold:*

(1) *There exists a weak homotopy equivalence $|N(\mathcal{U})| \to X$ from the topological realization of the nerve complex $N(\mathcal{U})$ to X. In fact there exists a weak homotopy equivalence $f\colon |N(\mathcal{U})| \to X$ such that for any simplex $\{i_0,\dots,i_k\}$ of $N(\mathcal{U})$ we have $f(|\{i_0,\dots,i_k\}|) \subset U_{i_0} \cap \dots \cap U_{i_k}$.*
(2) *If X is furthermore a* CW*-complex, then any map $f\colon |N(\mathcal{U})| \to X$ as in (1) is in fact a homotopy equivalence.*

Remark.

(1) The Nerve Theorem has its origins in the work of Karol Borsuk [**Bor48**] from 1948 and Jean Leray [**Ler50**] from 1950. As stated the theorem was basically proved by André Weil [**Weil52**, p. 141] in 1952.
(2) Let X be a CW-complex. It follows from Proposition 36.10 (8) that the conclusion of the Nerve Theorem 63.32 also holds if $\mathcal{U} = \{U_i\}_{i\in I}$ is a family of *subcomplexes* which form a good point-finite cover of X.
(3) There are many variations on the statement and on the proof of the Nerve Theorem 63.32.
   (a) In [**Bj95**, Theorem 10.6] and [**Bj03**, Theorem 6] a Nerve Theorem is formulated for the case that X is a regular CW-complex[980] and the U_i are subcomplexes. In this case the cover does not need to be point-finite.
   (b) Other versions are formulated in [**Nag07**, Theorem 3.3] and [**BoS73**, Theorem 8.2.1].
   (c) An elegant proof of the Nerve Theorem 63.32 for the case of a finite cover is given in [**Koz08**, Theorem 15.21].

Proof. Statement (2) of the Nerve Theorem is an immediate consequence of Statement (1) together with the Whitehead Theorem 119.9. Thus it remains to prove Statement (1).

[979] Note that we do not assume that the map f is cellular.

[980] We refer to page 1514 for the definition of a regular CW-complex, note that simplicial complexes are regular CW-complexes.

In the discussion below we will only sketch an argument. A full proof of Statement (1) is given in [**McCor67**, Theorem 1].

Thus let X be a topological space and let $\mathcal{U} = \{U_v\}_{v\in V}$ be an open good point-finite cover of X. We adopt the following notation:

(1) We set
$$S \;:=\; \{\{v_0,\dots,v_n\}\,|\,v_0,\dots,v_n\in V \text{ with } U_{v_0}\cap\dots\cap U_{v_n}\neq\varnothing\}.$$
Recall that by definition the nerve complex is given by $N(\mathcal{U}) := (V,S)$. For convenience we write $N = N(\mathcal{U})$.
(2) Given $s = \{v_0,\dots,v_n\}\in S$ we set $U_s := U_{v_0}\cap\dots\cap U_{v_n}$.
(3) We consider the barycentric subdivision
$$\mathrm{sd}(N) \;=\; \big(\{\underline{s}\}_{s\in S}, \{\{\underline{s_0},\dots,\underline{s_k}\}\,|\,s_0\subsetneq\dots\subsetneq s_k\}\big).$$
By Lemma 62.3 we have a natural homeomorphism $|\,\mathrm{sd}(N)|\to|N|$ which we use to identify these two topological spaces.

Next we define inductively a map $F\colon |\,\mathrm{sd}(N)|\to X$ which satisfies the following condition:

($\star$) For every simplex $\{\underline{s_0},\underline{s_1},\dots,\underline{s_k}\}$ of $\mathrm{sd}(N)$ we have
$$F\big(\big|\{\underline{s_0},\underline{s_1},\dots,\underline{s_k}\}\big|\big) \;\subset\; U_{s_0}.$$

We construct F inductively as follows:

(1) For each $\underline{s}$ we pick a point $F(\underline{s})\in U_s$.
(2) Suppose that F is defined on the $(n-1)$-skeleton of $|\,\mathrm{sd}(N)|$. We want to extend F to the n-skeleton of $|\,\mathrm{sd}(N)|$. Let $t = \{\underline{s_0},\dots,\underline{s_n}\}$ be an n-simplex of $\mathrm{sd}(N)$ and let $\Phi_t^{\le}\colon \Delta^n\to|\,\mathrm{sd}(N)|$ be the corresponding characteristic map. Note that
$$(F\circ\Phi_t^{\le})(\partial\Delta^n) \;=\; \bigcup_{i=0}^{n}\underbrace{F\big(\big|\{\underline{s_0},\dots,\widehat{\underline{s_i}},\dots,\underline{s_n}\}\big|\big)}_{\subset U_{s_0}\text{ or }\subset U_{s_1}\text{ by }(\star)} \;\underset{\uparrow}{\subset}\; U_{s_0}.$$
by definition we have for $s\subset t$ that $U_t\subset U_s$, in particular we have $U_{s_1}\subset U_{s_0}$

By hypothesis we know that U_{s_0} is contractible. It follows from Proposition 40.7 and Lemma 40.4 together with the fact, shown in Lemma 41.1 (3), that the pair $(\Delta^n,\partial\Delta^n)$ is homeomorphic to $(\overline{B}^n, S^{n-1})$, that $F\circ\Phi_t^{\le}\colon\partial\Delta^n\to U_{s_0}$ extends to a map $G\colon\Delta^n\to U_{s_0}$. Now we define F on $|t| = |\{\underline{s_0},\dots,\underline{s_n}\}|$ to be the map $G\circ\Phi_t^{-1}$. Note that this map still satisfies ($\star$).

It follows almost immediately from Lemma 62.3 and ($\star$) that $F\colon|N| = |\,\mathrm{sd}(N)|\to X$ has the desired property that for every simplex $\{v_0,\dots,v_k\}$ of $N = N(\mathcal{U})$ we have the inclusion $F(|\{v_0,\dots,v_k\}|)\subset U_{v_0}\cap\dots\cap U_{v_k}$.

It remains to show that F is a weak homotopy equivalence. Unfortunately this is where our discussion of the proof peters out. We suggest to use Proposition 119.6 below to show that F is a weak homotopy equivalence. One should be able to employ this proposition using the following two approaches:

(1) One uses the Mayer–Vietoris 22.1 and the HNN-Seifert–van Kampen Theorem 26.3, and suitable analogues for simplicial complexes, to show that F induces an isomorphism of homotopy groups.
(2) One uses cleverly the naturality of the long exact sequence coming from the Mayer–Vietoris Theorem 46.5 (or more elegantly, the naturality of the Mayer–Vietoris spectral sequence which we alluded to on page 1210) to show that the lift $\widetilde{F}$ of F to the universal covers of $|\operatorname{sd}(N)|$ and X induces an isomorphism of all homology groups.

We leave it as a rather challenging exercise to the reader to make the above approach work. Even the case of a finite cover is already rather delicate. ■

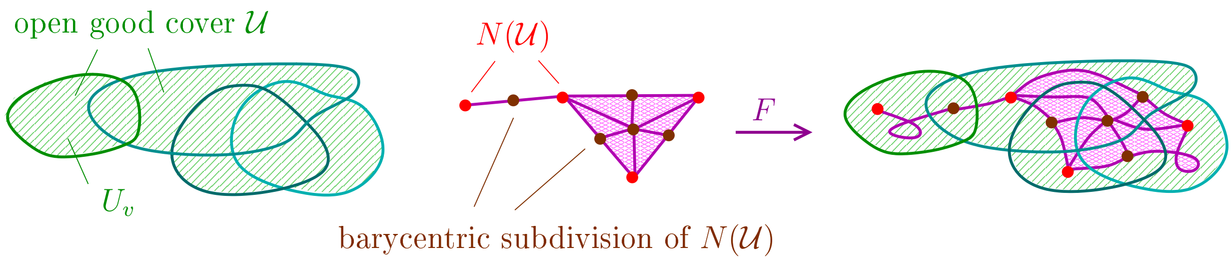


FIGURE 952. Illustration of the proof of Nerve Theorem 63.32.

For completeness we also state the following variation on the Nerve Theorem 63.33.

Theorem 63.33. (Nerve Theorem) *Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open good cover of X. If X is paracompact, then there exists a homotopy equivalence $|N(\mathcal{U})| \to X$.*

PROOF. By definition of a paracompact topological space, see page 144, there exists a partition of unity $\{\varphi_i\}_{i\in I}$ subordinate to the covering. We define

$$\begin{aligned} G\colon X &\to |N(\mathcal{U})| \\ x &\mapsto \left(\begin{array}{ccc} I & \to & \mathbb{R} \\ i & \mapsto & \varphi_i(x) \end{array}\right). \end{aligned}$$

In [**Hat02**, Proposition 4G.2] it is shown that the map G is a homotopy inverse to the map $F\colon |N(\mathcal{U})| \to X$ that we constructed in the proof of the Nerve Theorem 63.32. ■

One of the many fun applications of the Nerve Theorem 63.32 is the following theorem which was first proved by Eduard Helly [**Hel23**] in 1913 using much more elementary techniques. Our proof is arguably technological overkill, but it has the advantage that from our point of view it is very conceptual.

Theorem 63.34. (Helly's Theorem) *Let $n \in \mathbb{N}_0$ and let $U_1, \dots, U_m$ be convex open subsets of $\mathbb{R}^n$. If any choice of $n+1$ of those sets has non-empty intersection, then the intersection of all those sets is non-empty, i.e. we also have*

$$U_1 \cap \cdots \cap U_m \neq \varnothing.$$

Examples.

(1) For $n=1$ and $m=3$ Helly's Theorem 63.34 says that if we are given three open intervals such that any two of them have a non-empty intersection, then the intersection of the three open intervals is non-empty.

(2) For $n=2$ and $m=4$ Helly's Theorem 63.34 says if we are given four convex open subsets of $\mathbb{R}^2$, such that any three of them have non-empty intersection, then the intersection of all four subsets is also non-empty. This situation is illustrated in Figure 953.

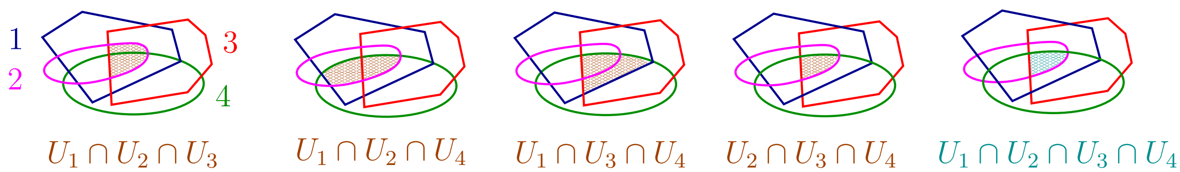


FIGURE 953. Illustration of Helly's Theorem 63.34.

Remark.

(1) In our formulation of Helly's Theorem 63.34 we require that the sets U_i are open. This is in fact just an artifact of our proof. In fact the statement holds without this assumption. We refer to [**Mato02**, Theorem 1.3.2] for an elementary proof of this more general statement.

(2) As we mentioned above, our topological proof of Helly's Theorem 63.34 is strictly speaking more difficult than necessary. But the advantage of the topological approach taken in these notes is that it is easier to generalize this approach to non-convex subsets.

(3) We refer to [**Eckh93**] for a long list of references for proofs of Helly's Theorem and for a discussion of the relevance of Helly's Theorem.

In our proof of Helly's Theorem we will need the following proposition which is interesting in its own right.

Proposition 63.35. *If M is an open subset of $\mathbb{R}^n$, then $\mathrm{H}_k(M)=0$ for $k \geq n$.*

PROOF. The proposition follows from a fun argument using the Mayer–Vietoris Theorem 46.5. We invite the reader to fill in the details in Exercise 63.15. ■

PROOF OF HELLY'S THEOREM 63.34. Let $n \in \mathbb{N}_0$. We prove the theorem by induction on m. For $m \leq n+1$ there is nothing to show. Now suppose that the theorem holds for some $m \geq n+1$. Let $\mathcal{U} := \{U_1, \ldots, U_{m+1}\}$ be a family of convex open subsets of $\mathbb{R}^n$ such that the intersection of any $n+1$ of those subsets has non-empty intersection. By induction hypothesis we know that the following holds:

(i) The intersection of any m of the sets in $\mathcal{U}$ is non-empty.

We need to show that $U_1 \cap \cdots \cap U_{m+1} \neq \varnothing$. We do a proof by contradiction. So we assume that the following holds:

(ii) $U_1 \cap \cdots \cap U_{m+1} = \varnothing$.

Next we consider the corresponding nerve complex $N(\mathcal{U})$.

(1) It follows immediately from (i) and (ii) that the nerve complex $N(\mathcal{U})$ is simplicially isomorphic to the simplicial complex S_{m-1} that we introduced on page 1480.

(2) As we mentioned before, one of the key properties of convex subsets is that any intersection of convex subsets is again convex, in particular any intersection is contractible. Thus $\mathcal{U}$ is an open good finite cover of $U_1 \cup \cdots \cup U_{m+1}$.

Now we see that

$$\mathbb{Z} \underset{\substack{\uparrow\\ \text{Lemma 63.12}}}{\cong} \mathrm{H}^{\mathrm{simp}}_{m-1}(S_{m-1}) \underset{\substack{\uparrow\\ \text{by (1)}}}{\cong} \mathrm{H}^{\mathrm{simp}}_{m-1}(N(\mathcal{U})) \underset{\substack{\uparrow\\ \text{Theorem 63.23}}}{\cong} \mathrm{H}_{m-1}(|N(\mathcal{U})|) \overset{\substack{\text{by the Nerve Theorem 63.32 which we can apply by (2), together with Corollary 42.8}\\ \downarrow}}{\cong} \mathrm{H}_{m-1}\big(\underbrace{U_1 \cup \cdots \cup U_{m+1}}_{\text{open subset of } \mathbb{R}^n}\big) \underset{\substack{\uparrow\\ \text{by Proposition 63.35, since } m-1 \geq n}}{=} 0.$$

We have thus obtained a contradiction to (ii). ■

The Nerve Theorem 63.32 gives a practical tool for showing that a given topological space is homotopy equivalent to a simplicial complex. It is natural to go through familiar classes of topological spaces and to see whether it applies. Fortunately the next proposition shows that it applies to everybody's favorite topological spaces, namely smooth manifolds.

Proposition 63.36.

(1) *Every compact smooth manifold admits an open good finite cover.*

(2) *Every smooth manifold admits an open good point-finite cover.*

Remark. To the best of my knowledge it is not known whether every topological manifold admits an open cover that is good, let alone good and point-finite.

Sketch of proof. The proposition is proved in detail in [**Weil52**, p. 120] and in less detail in [**BoT82**, Theorem 5.1], [**KaL87**, Chapter VI.3] and [**Lee09**, Theorem 10.17]. In the following we sketch the argument in the closed case.

Thus let M be a closed k-dimensional smooth manifold.We point out that by Proposition 9.1 we can view M as a submanifold of some $\mathbb{R}^n$. Given $P \in M$ and $r > 0$ we write $\mathrm{C}_r(P) := B^n_r(P) \cap M$. Furthermore, given $P \in M$ we denote by $\pi_P \colon \mathbb{R}^n \to \mathrm{V}_P\, M$ the orthogonal projection onto the visual tangent space that we introduced on page 292.

We leave it to the reader to show, using the compactness and the definition of a smooth map and using Proposition 6.20, that there exists an $\epsilon > 0$ such that for any $P \in M$ the following statements hold:

(1) the set $\mathrm{C}_\epsilon(P)$ is an open subset of M,

(2) for any $Q \in \mathrm{C}_\epsilon(P)$ the projection $\pi_Q \colon \mathrm{C}_\epsilon(P) \to \mathrm{V}_Q\, M$ is an embedding such that the image is convex.

Since M is compact it follows from (1) that there exist $P_1, \ldots, P_m \in M$ such that the open sets $\mathrm{C}_\epsilon(P_1), \ldots, \mathrm{C}_\epsilon(P_m)$ form an open cover of M. Since the intersection of convex subsets is again convex we obtain from (2) that this open cover is indeed good. ■

We obtain the following corollary from the combination of Proposition 63.36, the Nerve Theorem 63.32 and Lemma 61.24.

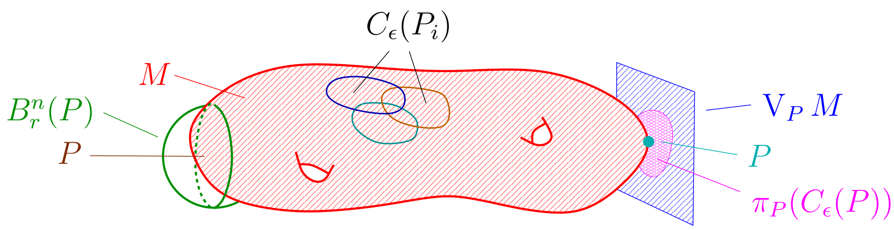


FIGURE 954. Illustration of the proof of Proposition 63.36.

Corollary 63.37. *Every (compact) smooth manifold is weakly homotopy equivalent to a (finite) simplicial complex, in particular it is weakly homotopy equivalent to a (finite) CW-complex.*

Later, in Theorem 64.5, we will see that a much stronger version of Corollary 63.37 holds.

63.8. **The simplicial subdivision map (∗).** Let $(K, \leq)$ be an ordered abstract simplicial complex. By Theorem 63.23 and Lemma 62.3 we know that the simplicial homology groups of K and of its barycentric subdivision $\mathrm{sd}(K)$ are isomorphic.

In this section we will strengthen this result. More precisely, we will give a natural map $\sigma\colon \mathrm{sd}(K) \to K$ and we will give a natural chain map

$$u_*\colon\ \mathrm{C}^{\mathrm{simp},\leq}_*(K) \ \to\ \mathrm{C}^{\mathrm{simp},\leq}_*(\mathrm{sd}(K))$$

such that the chain maps σ_* and u_* are chain homotopy inverses of one another. These two constructions will prove their worth later on when we study continuous self-maps $f\colon |K| \to |K|$. More precisely, by the Simplicial Approximation Theorem 62.8 we know that a given continuous map $f\colon |K| \to |K|$ is homotopic to a simplicial map $g\colon |\,\mathrm{sd}^i(K)| \to |K|$, thus we get an induced chain map $g_*\colon\ \mathrm{C}^{\mathrm{simp},\leq}_*(\mathrm{sd}^i(K)) \to \mathrm{C}^{\mathrm{simp},\leq}_*(K)$. Unfortunately this is a chain map between *different* chain complexes. But if we combine g_* with an i-fold iteration of u_*, then we obtain an *endomorphism* $\mathrm{C}^{\mathrm{simp},\leq}_*(\mathrm{sd}^i(K)) \to \mathrm{C}^{\mathrm{simp},\leq}_*(\mathrm{sd}^i(K))$ of a chain complex.

We start out with the following lemma

Lemma 63.38. *Let $(K, \leq)$ be an ordered abstract simplicial complex.*

(1) *There exists a unique simplicial map $\sigma\colon \mathrm{sd}(K) \to K$ with the following property: given an n-simplex $s = \{v_0 < \cdots < v_n\}$ of K the map σ sends the corresponding vertex $\underline{s}$ of $\mathrm{sd}(K)$ to the vertex v_n of K.*
(2) *The stretching map $\sigma\colon \mathrm{sd}(K) \to K$ is order-preserving. (Here we equip $\mathrm{sd}(K)$ with the natural order introduced in Lemma 62.2 (2).)*
(3) *The stretch map σ is natural. Here "natural" means that given any order-preserving simplicial map $f\colon (K, \leq) \to (L, \leq)$ between two ordered abstract simplicial complexes*

and given any $n \in \mathbb{N}_0$ the following diagram commutes:[981]

$$\begin{tikzcd}
\mathrm{sd}(K) \arrow[r, "f_*"] \arrow[d, "\sigma"'] & \mathrm{sd}(L) \arrow[d, "\sigma"] \\
K \arrow[r, "f_*"] & L.
\end{tikzcd}$$

We refer to this map σ as the stretching map.

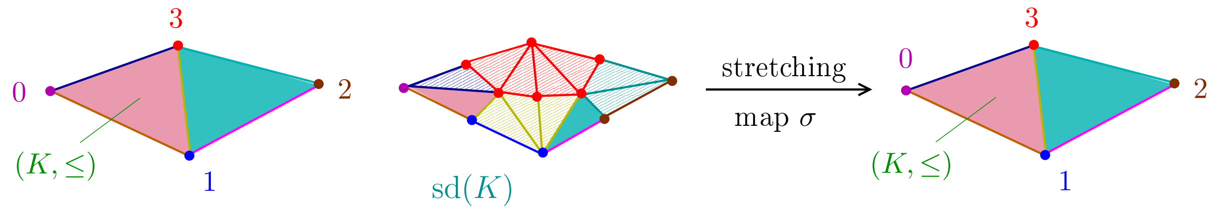


FIGURE 955. Illustration of Lemma 63.38.

PROOF. It follows easily from the definitions that the given map on the set of vertices induces a simplicial map, that this map is order-preserving and that this map is natural. We leave it to the reader to fill in the details. ■

In basically all cases the stretching map $\sigma\colon \mathrm{sd}(K) \to K$ is not a simplicial isomorphism, in particular the induced map on the simplicial chain complexes is not an isomorphism. Our goal now is to show the "best possible result", namely that the induced chain map $\sigma_*\colon \mathrm{C}_*^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \to \mathrm{C}_*^{\mathrm{simp},\leq}(K)$ is a chain homotopy equivalence. We will do so by giving an explicit chain map in the other direction.

Definition. Let $(K, \leq)$ be an ordered abstract simplicial complex.

(1) Given an n-simplex $s = \{v_0 < \cdots < v_n\}$ of K, given[982] $\sigma \in \mathrm{Bij}\{0, \ldots, n\}$ and given $m \in \{0, \ldots, n\}$ we write

$$s_{\sigma,m} \;:=\; \{v_{\sigma(0)}, \ldots, v_{\sigma(m)}\}.$$

Note that $s_{\sigma,n} = s$. Furthermore, we set

$$s_\sigma \;:=\; \{\overbrace{\underline{s_{\sigma,n}}}^{=\underline{s}}, \underline{s_{\sigma,n-1}}, \ldots, \underline{s_{\sigma,0}}\}.$$

Note that s_σ is an n-simplex of $\mathrm{sd}(K)$.

(2) Given an n-simplex s of K we define

$$u_n(s) \;:=\; \sum_{\sigma \in \mathrm{Bij}\{0,\ldots,n\}} \mathrm{sign}(\sigma) \cdot s_\sigma \;\in\; \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)).$$

We refer to the resulting map[983]

$$\begin{array}{rcl} \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \\ s & \mapsto & u_n(s) \end{array}$$

as the *simplicial subdivision map.*

[981]Recall that by Lemma 62.3 (5) we know that the maps $K \mapsto \mathrm{sd}(K)$ define a covariant functor from the category $\mathcal{OrdAbsSimpCplx}$ of ordered abstract simplicial complexes to itself.

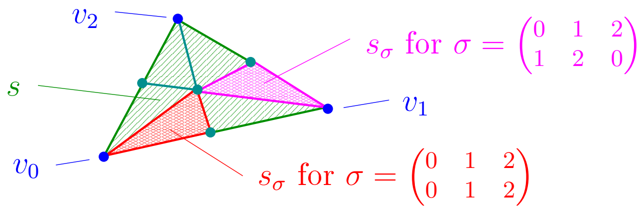


FIGURE 956. Illustration for the proof of Lemma 63.39.

The following lemma can be viewed as an analogue of Lemmas 43.25 and 43.26.

Lemma 63.39. *Let $(K, \leq)$ be an ordered abstract simplicial complex. The simplicial subdivision maps*

$$u_n\colon\ \mathrm{C}_n^{\mathrm{simp},\leq}(K) \ \to\ \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)), \quad n \in \mathbb{N}_0$$

are a natural chain map. Here "natural" means that given any order-preserving simplicial map $f\colon (K, \leq) \to (L, \leq)$ between two ordered abstract simplicial complexes and given any $n \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{C}_n^{\mathrm{simp},\leq}(K) \arrow[r, "f_*"] \arrow[d, "u_n"'] & \mathrm{C}_n^{\mathrm{simp},\leq}(L) \arrow[d, "u_n"] \\ \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \arrow[r, "f_*"] & \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(L)). \end{tikzcd}$$

PROOF (∗). Let $(K, \leq)$ be an ordered abstract simplicial complex. We need to show that the maps

$$u_n\colon\ \mathrm{C}_n^{\mathrm{simp},\leq}(K) \ \to\ \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)), \quad n \in \mathbb{N}_0$$

are a natural chain map. In fact it follows immediately from the definitions and Lemma 62.1 (1) that the maps u_n are natural. Thus it remains to prove that the maps u_n are a chain map. We will do so in the following claim.

Claim. For any $n \in \mathbb{N}_0$ and any n-simplex s of K we have the following equality:

$$\partial u_n(s) \ = \ u_{n-1}(\partial s).$$

We introduce the following notation. Given an n-simplex $t = \{w_0 < \cdots < w_n\}$ and given $i \in \{0, \ldots, n\}$ we write $t(i) := \{w_0 < \cdots < \widehat{w}_i < \cdots < w_n\}$, i.e. $t(i)$ is the $(n-1)$-simplex given by striking out the vertex w_i. Now let $s = \{v_0 < \cdots < v_n\}$ be an n-simplex

[982]Given $n \in \mathbb{N}_0$ we denote by $\mathrm{Bij}\{0, \ldots, n\}$ the group of bijections of the set $\{0, \ldots, n\}$. Note that we have an obvious isomorphism $\mathrm{Bij}\{0, \ldots, n\} \cong S_{n+1}$. Given $\sigma \in \mathrm{Bij}\{0, \ldots, n\}$ we define $\mathrm{sign}(\sigma)$ basically the same way as we did on page 1541.

[983]Recall that $\mathrm{C}_n^{\mathrm{simp},\leq}(K)$ is by definition the free abelian group generated by the set of simplices. Thus to specify a homomorphism $\varphi\colon\ \mathrm{C}_n^{\mathrm{simp},\leq}(K) \to A$ to some abelian group we only need to specify $\varphi(s)$ for each n-simplex s of K.

of K. We perform the following slightly messy calculation:

$$\partial u_n(s) \underset{\text{by definition of } u_n}{=} \partial\Big(\sum_{\sigma\in\mathrm{Bij}\{0,\dots,n\}} \mathrm{sign}(\sigma)\cdot s_\sigma\Big) \underset{\text{definition of } \partial}{=} \sum_{i=0}^{n}(-1)^i\cdot \sum_{\sigma\in\mathrm{Bij}\{0,\dots,n\}} \mathrm{sign}(\sigma)\cdot (s_\sigma)(i) \underset{\text{split the sum into } i=0 \text{ and } i=1,\dots,n}{=}$$

$$= \sum_{\sigma\in\mathrm{Bij}\{0,\dots,n\}} \mathrm{sign}(\sigma)\cdot (s_\sigma)(0) + \sum_{i=1}^{n}(-1)^i\cdot \sum_{\sigma\in\mathrm{Bij}\{0,\dots,n\}} \mathrm{sign}(\sigma)\cdot (s_\sigma)(i)$$

the second summand vanishes since for a fixed $i\in\{1,\dots,n\}$ and for $\mu,\nu\in\mathrm{Bij}\{0,\dots,n\}$ we have $(s_\mu)(i)=(s_\nu)(i)$ if $\mu(i-1)=\nu(i)$, if $\mu(i)=\nu(i-1)$ and if all other values of μ and ν are the same, thus given the fixed $i\in\{1,\dots,n\}$ we can pair up the elements in $\mathrm{Bij}\{0,\dots,n\}$ to give the same contribution $(s_\sigma)(i)$ to the sum, but with opposite sign

$$= \sum_{\sigma\in\mathrm{Bij}\{0,\dots,n\}} \mathrm{sign}(\sigma)\cdot(s_\sigma)(0) = \sum_{\tau\in\mathrm{Bij}\{0,\dots,n-1\}}\sum_{i=0}^{n}(-1)^i\cdot\mathrm{sign}(\tau)\cdot(s(i))_\tau = u_{n-1}(\partial s).$$

follows from considering the bijection

$$\{0,\dots,n\}\times\mathrm{Bij}\{0,\dots,n-1\} \to \mathrm{Bij}\{0,\dots,n\}$$

$$(i,\tau) \mapsto \left(\begin{array}{rcl} \{0,\dots,n\} & \to & \{0,\dots,n\} \\ j & \mapsto & \begin{cases} i, & \text{if } j=0,\\ j-1, & \text{if } j\in\{1,\dots,i\},\\ j, & \text{if } j>i.\end{cases}\end{array}\right)$$

by definition of ∂ and u_{n-1}

■

The following lemma finally relates the stretching maps to the simplicial subdivision maps.

Lemma 63.40. *Let $(K,\le)$ be an ordered abstract simplicial complex. As before we denote by $\sigma\colon \mathrm{sd}(K)\to K$ the stretching map introduced in Lemma 63.38.*

(1) *The induced chain map*
$$\sigma_*\colon\ \mathrm{C}^{\mathrm{simp},\le}_*(\mathrm{sd}(K)) \ \to\ \mathrm{C}^{\mathrm{simp},\le}_*(K)$$
and the simplicial subdivision maps
$$u_n\colon\ \mathrm{C}^{\mathrm{simp},\le}_n(K) \ \to\ \mathrm{C}^{\mathrm{simp},\le}_n(\mathrm{sd}(K)),\qquad n\in\mathbb{N}_0$$
are chain homotopy inverses of one another.

(2) *The maps σ_* and the maps u_* are chain homotopy equivalences.*

(3) *The maps $\sigma_*\colon \mathrm{H}^{\mathrm{simp},\le}_*(\mathrm{sd}(K))\to\mathrm{H}^{\mathrm{simp},\le}_*(K)$ and $u_*\colon \mathrm{H}^{\mathrm{simp},\le}_*(K)\to\mathrm{H}^{\mathrm{simp},\le}_*(\mathrm{sd}(K))$ are natural isomorphisms and they are inverses of one another.*

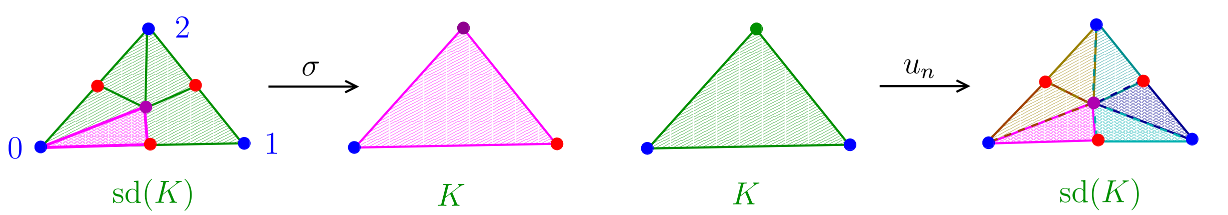


FIGURE 957. Illustration of Lemma 63.40.

Remark. Lemma 63.40 (2) can be viewed as an analogue of Lemma 43.27 where we proved a similar statement for the singular subdivision map.

Proof ($*$). Statement (2) of the lemma is an immediate consequence of Statement (1). Furthermore Statement (3) follows from Statements (1) and (2) together with Corollary 42.3 and the naturality statements of Lemma 63.38 and Lemma 63.39. Thus it remains to prove Statement (1).

Let $(K, \leq)$ be an ordered abstract simplicial complex. It follows easily from the definitions that for each $n \in \mathbb{N}_0$ the map $\sigma_* \circ u_n \colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) \to \mathrm{C}_n^{\mathrm{simp},\leq}(K)$ is the identity. Thus it remains to prove that $u_* \circ \sigma_*$ is chain homotopic to the identity.

The definition of the chain homotopy requires some preparations and extra definitions. Let $n \in \mathbb{N}_0$ and let $t = \{\underline{s_0}, \dots, \underline{s_n}\}$ be an n-simplex of $\mathrm{sd}(K)$ with $\#s_n = n+2$. For convenience we set $s_{-1} := \varnothing$. Since $\#s_n = n+2$ and since for each $i \in \{0, \dots, n\}$ we have $s_{i-1} \subsetneq s_i$ we see that there exists a unique $i \in \{0, \dots, n\}$ with $\#s_i = \#s_{i-1} + 2$. Thus there exist two vertices $v < w$ of K such that $s_i = s_{i-1} \sqcup \{v, w\}$. We say that t can be *expanded upward* if w is larger than every element in s_{i-1}. In this case we set

$$\begin{aligned} \Phi(t) &:= \overbrace{\{\underline{s_0}, \dots, \underline{s_{i-1}}, \underline{s_{i-1} \cup \{w\}}, \underline{s_i}, \dots \underline{s_n}\}}^{(n+1)\text{-simplex of } \mathrm{sd}(K)} \\ \text{and} \qquad \eta(t) &:= \mathrm{sign}(\sigma) \text{ where } \sigma \in \mathrm{Bij}\{0, \dots, n\} \text{ with } \Phi(t) = (s_n)_{\underset{\text{notation from page 1576}}{\underset{\uparrow}{\sigma}}}. \end{aligned}$$

These definitions are illustrated in Figures 958 and 959. For each $n \in \mathbb{N}_0$ we consider the following map

$$\begin{aligned} P_n \colon \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) &\to \mathrm{C}_{n+1}^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \\ t = \{\underline{s_0}, \dots, \underline{s_n}\} &\mapsto \begin{cases} \eta(t) \cdot \Phi(t), & \text{if } \#s_n = n+2 \text{ and if } t \text{ can be expanded upward,} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We leave it to the reader's unrelenting zest for verifying technical details to show that these maps P_n form a chain homotopy between the identity and the chain map $\sigma_* \circ u_*$. ■

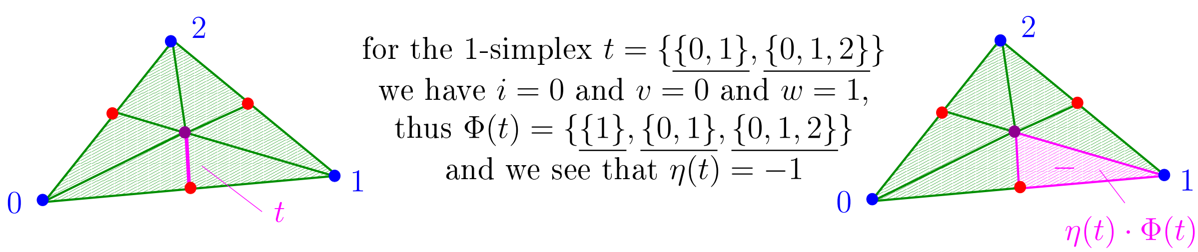


Figure 958. First illustration for the proof of Lemma 63.40.

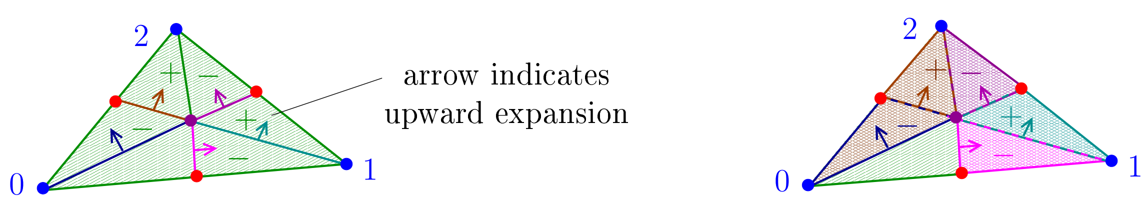


Figure 959. Second illustration for the proof of Lemma 63.40.

We defined the simplicial subdivision map for *ordered* abstract simplicial complexes. As we will see shortly, the choice of an ordering is irrelevant.

Definition. Let $(K, \leq)$ be an abstract simplicial complex.

(1) Let $s = (v_0, \dots, v_n)$ be an ordered n-simplex of K. Given $m \in \{0, \dots, n\}$ we set
$$F_m(s) \;:=\; \{v_0, \dots, v_m\}.$$
Furthermore we define
$$F(s) \;:=\; \big(\underline{F_0(s)}, \underline{F_1(s)}, \dots, \underline{F_n(s)}\big).$$
Note that $F(s)$ is an ordered n-simplex of $\mathrm{sd}(K)$.

(2) Note that any two ordered n-simplices $s = (v_0, \dots, v_n)$ and $t = (w_0, \dots, w_n)$ of K with $\{v_0, \dots, v_n\} = \{w_0, \dots, w_n\}$ there exist a unique $\sigma \in \mathrm{Bij}\{0, \dots, n\}$ such that $v_i = w_{\sigma(i)}$ for all $i = 0, \dots, n$. We set $\mathrm{sign}(s,t) := \mathrm{sign}(\sigma)$.

(3) Given an ordered n-simplex $s = (v_0, \dots, v_n)$ we set
$$u_n(s) \;:=\; \sum_{\substack{t = (w_0, \dots, w_n) \text{ an ordered} \\ n\text{-simplex of } K \text{ with} \\ \{v_0, \dots, v_n\} = \{w_0, \dots, w_n\}}} (-1)^{\mathrm{sign}(s,t)} \cdot F(t)$$

More precisely, we have the following lemma.

Lemma 63.41. *Let K be an abstract simplicial complex.*

(1) *For every $n \in \mathbb{N}_0$ the map*
$$\begin{array}{rcl} \mathrm{C}_n^{\mathrm{simp}}(K) & \to & \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K)) \\ {[s]} & \mapsto & [u_n(s)] \end{array}$$
is well-defined. We refer to this map again as the simplicial subdivision map.

(2) *The maps u_n are natural in the sense that given a simplicial map $f\colon K \to L$ the following diagram commutes*
$$\begin{tikzcd} \mathrm{C}_n^{\mathrm{simp}}(K) \arrow[r, "f_*"] \arrow[d, "u_n"'] & \mathrm{C}_n^{\mathrm{simp}}(L) \arrow[d, "u_n"] \\ \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K)) \arrow[r, "f_*"] & \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(L)). \end{tikzcd}$$

(3) *If K is equipped with an order "$\leq$", then for every $n \in \mathbb{N}_0$ the following diagram commutes:*
$$\begin{tikzcd} \mathrm{C}_n^{\mathrm{simp},\leq}(K) \arrow[r, "u_n"] \arrow[d, "\cong"'] & \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \arrow[d, "\cong"] \\ \mathrm{C}_n^{\mathrm{simp}}(K) \arrow[r, "u_n"] & \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K)). \end{tikzcd}$$
Here the vertical maps are the natural isomorphisms given by Lemma 63.5.

(4) *The maps $u_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K))$ form a chain homotopy equivalence. In fact, if we pick an order on K and we consider the corresponding stretching map*

$\sigma\colon \mathrm{sd}(K) \to K$, then the maps u_n and the maps $\sigma_*\colon \mathrm{C}_*^{\mathrm{simp}}(\mathrm{sd}(K)) \to \mathrm{C}_*^{\mathrm{simp}}(K)$ are chain homotopy inverses.

(5) *The induced maps $u_*\colon \mathrm{H}_n^{\mathrm{simp}}(K) \to \mathrm{H}_n(\mathrm{sd}^{\mathrm{simp}}(K))$ are natural isomorphisms.*

Remark. Note in contrast the stretching maps certainly do depend on the choice of an order.

Proof. The first three statements follow easily from the definitions. The only statement which perhaps needs a little bit of thought is the first statement. We leave it to the reader to fill in the details. Let us turn to the proof of Statement (4). As mentioned before, by the Well-ordering Theorem 1.6 we can pick an order "$\leq$" on K. Let $n \in \mathbb{N}_0$. By (2) we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{C}_n^{\mathrm{simp},\leq}(K) & \xrightarrow{\quad u_n \quad} & \mathrm{C}_n^{\mathrm{simp},\leq}(\mathrm{sd}(K)) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{C}_n^{\mathrm{simp}}(K) & \xrightarrow{\quad u_n \quad} & \mathrm{C}_n^{\mathrm{simp}}(\mathrm{sd}(K)).
\end{array}$$

By Lemma 63.40 we know that the top horizontal maps form a chain homotopy equivalence and that a chain homotopy inverse is given by the stretching maps. It follows that the analogous statements hold for the bottom horizontal maps. Finally Statement (5) follows from (2) and (4) together with Corollary 42.3. ■

In the remainder of this section we want to discuss the relationship of the stretching map and the simplicial division map with various maps on the topological realizations. First we recall that given an abstract simplicial complex K we gave in Lemma 62.3 (3) an explicit natural homeomorphism $\mho\colon |\,\mathrm{sd}(K)| \to |K|$. This allows us to formulate the following lemma.

Lemma 63.42. *Let $(K, \leq)$ be an abstract simplicial complex. We denote by $\sigma\colon \mathrm{sd}(K) \to K$ the stretching map from Lemma 63.38. The map $|\sigma| \circ \mho^{-1}\colon |K| \to |K|$ is homotopic to the identity.*

Proof. Since we will not make use of Lemma 63.42 we outsource the proof thereof to Exercise 63.17. ■

We recall some more maps and statements from earlier on.

(1) Let K be an ordered abstract simplicial complex. By Theorem 63.25 we know that the maps
$$\begin{array}{rcl}\Theta_K\colon \mathrm{C}_n^{\mathrm{simp},\leq}(K) & \to & \mathrm{C}_n(|K|) \\ s & \mapsto & (\Phi_s^{\leq}\colon \Delta^n \to |K|)\end{array}$$
are a natural chain homotopy equivalence.

(2) Given a topological space X and $n \in \mathbb{N}_0$ we denote by $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$ the singular subdivision map as defined in page 1135.

Now we can formulate the following proposition.

Proposition 63.43. *For every ordered abstract simplicial complex $(K,\le)$ and every $n\in\mathbb{N}_0$ the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{C}_n^{\mathrm{simp},\le}(K) \arrow[r, "u_n"] \arrow[dd, "\Theta_K"'] & \mathrm{C}_n^{\mathrm{simp},\le}(\mathrm{sd}(K)) \arrow[d, "\Theta_{\mathrm{sd}(K)}"] \\
& \mathrm{C}_n(|\,\mathrm{sd}(K)|) \arrow[d, "\mho_*"] \\
\mathrm{C}_n(|K|) \arrow[r, "u_n"] & \mathrm{C}_n(|K|).
\end{tikzcd}$$

Furthermore all maps form natural chain homotopy equivalences.

The key step in proving Proposition 63.43 is to give an alternative description of the simplicial subdivision map that resembles the definition of the singular subdivision map which we gave on page 1135. We start out with the following definition.

Definition. Let $(K,\le)$ be an ordered abstract simplicial complex.

(1) Given a k-simplex s of K we define the "coning map"

$$\begin{array}{rcl} c_s\colon\ \mathrm{C}_i^{\mathrm{simp},\le}(\mathrm{sd}(\partial s)) & \to & \mathrm{C}_{i+1}^{\mathrm{simp},\le}(\mathrm{sd}(s)) \\ & \mapsto & \{\underline{s}\}\cup t. \end{array}$$

We refer to Figure 960 for an illustration.

(2) We define[984] $u_0\colon\ \mathrm{C}_0^{\mathrm{simp},\le}(K)\to\mathrm{C}_0^{\mathrm{simp},\le}(\mathrm{sd}(K))$ to be the obvious inclusion. Furthermore, for $n\ge 1$ we define inductively the following maps:

$$\begin{array}{rcl} \widetilde{u}_n\colon\ \mathrm{C}_n^{\mathrm{simp},\le}(K) & \to & \mathrm{C}_n^{\mathrm{simp},\le}(\mathrm{sd}(K)) \\ s & \mapsto & (-1)^n\cdot c_s\big(\underbrace{u_{n-1}(\partial s)}_{\in\mathrm{C}_{n-1}^{\mathrm{simp},\le}(\mathrm{sd}(\partial s))}\big). \end{array}$$

The definition is illustrated in Figure 961.

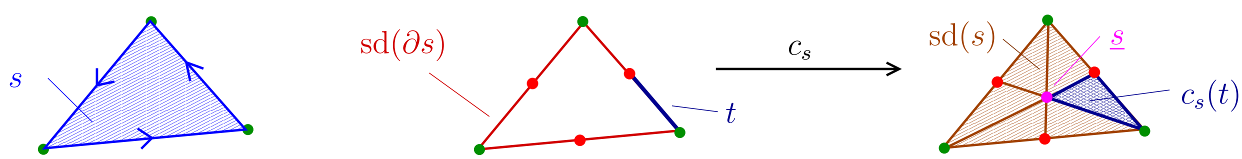


FIGURE 960

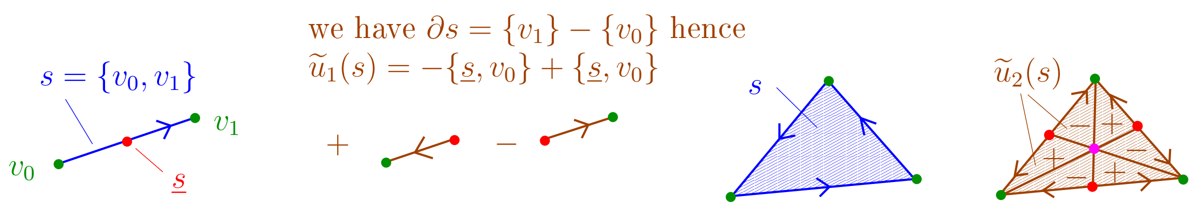


FIGURE 961

The following lemma gives the promised alternative description of the simplicial subdivision maps.

[984] Recall that in Lemma 62.2 (2) we saw that $\mathrm{sd}(K)$ is naturally an ordered abstract simplicial complex.

Lemma 63.44. *Let $(K, \leq)$ be an ordered abstract simplicial complex. Given any n-simplex s of K we have $\widetilde{u}_n(s) = u_n(s)$.*

Proof of Lemma 63.44. Let $(K, \leq)$ be an ordered abstract simplicial complex. We prove the lemma by induction on n. For $n = 0$ the statement is basically obvious. Now suppose that the statement holds for some $n \in \mathbb{N}_0$. Let $s = \{v_0 < \cdots < v_{n+1}\}$ be some $(n+1)$-simplex. Given $i \in \{0, \ldots, n+1\}$ we write $s(i) := \{v_0, \ldots, \widehat{v_i}, \ldots, v_{n+1}\}$. We calculate that

$$
\begin{aligned}
u_{n+1}(s) \;&\overset{\substack{\text{definition of } \widetilde{u}_{n+1}\\ \downarrow}}{=}\; (-1)^{n+1} \cdot c_s(\widetilde{u}_n(\partial s)) \;\overset{\substack{\text{by definition of } \partial s\\ \downarrow}}{=}\; (-1)^{n+1} \cdot c_s\Big(u_n\Big(\sum_{i=0}^{n+1} (-1)^i \cdot s(i)\Big)\Big) \;\overset{\substack{\text{by induction and definition of } c_s\\ \downarrow}}{=} \\
&= (-1)^{n+1} \cdot \sum_{i=0}^{n+1} (-1)^i \cdot \sum_{\sigma \in \mathrm{Bij}\{0,\ldots,n\}} \mathrm{sign}(\sigma) \cdot \{\underline{s}, s(i)_\sigma\} \\
&\underset{\uparrow}{=} \sum_{\tau \in \mathrm{Bij}\{0,\ldots,n+1\}} \mathrm{sign}(\tau) \cdot s_\tau \underset{\substack{\uparrow\\ \text{see definition on page 1576}}}{=} u_n(s).
\end{aligned}
$$

here we use the following bijection:

$$
\begin{aligned}
\Psi\colon \{0, \ldots, n+1\} \times \mathrm{Bij}\{0, \ldots, n\} &\to \mathrm{Bij}\{0, \ldots, n+1\} \\
(i, \sigma) &\mapsto \left(\begin{aligned} \{0, \ldots, n+1\} &\to \{0, \ldots, n+1\} \\ j &\mapsto \begin{cases} j, & \text{if } j \in \{0, \ldots, i-1\}, \\ i, & \text{if } j = n+1, \\ j+1, & \text{if } j \in \{i, \ldots, n\} \end{cases} \end{aligned}\right)
\end{aligned}
$$

and we use that $\mathrm{sign}(\Psi(i, \sigma)) = (-1)^{i+1} \cdot (-1)^{n+2} \cdot \mathrm{sign}(\sigma)$

We have thus proved the induction step. ■

Now we can easily prove Proposition 63.43.

Proof of Proposition 63.43. It follows immediately from Lemma 63.44 together with the definitions of the various maps that the diagram commutes.

By Lemmas 63.40 and 43.27 we know that the horizontal maps form chain homotopy equivalences. Furthermore by Theorem 63.25 the top vertical maps also form chain homotopy equivalences. Finally the map $\mho$ is a homeomorphism. Thus the bottom vertical maps $\mho_*$ also form a chain homotopy equivalence. ■

We conclude with the following "unordered" version of Proposition 63.43.

Proposition 63.45. *For every abstract simplicial complex K and every $n \in \mathbb{N}_0$ the following diagram commutes:*[985]

$$
\begin{tikzcd}
\mathrm{H}_n^{\mathrm{simp}}(K) \arrow[r, "u_n"] \arrow[dd, "\Theta_K"'] & \mathrm{H}_n^{\mathrm{simp}}(\mathrm{sd}(K)) \arrow[d, "\Theta_{\mathrm{sd}(K)}"] \\
& \mathrm{H}_n(|\,\mathrm{sd}(K)|) \arrow[d, "\mho_*"] \\
\mathrm{H}_n(|K|) \arrow[r, "u_n"] & \mathrm{H}_n(|K|).
\end{tikzcd}
$$

Furthermore all maps are natural.

PROOF. The proposition follows easily from the fact that every abstract simplicial complex admits an order together with Lemma 63.8 and Proposition 63.45 and together with Proposition 63.28. ■

Remark. For the record we point out that some of the definitions and results in this section are at least implicit in [**Deo18**, Theorem 4.9.1] and [**Mun66a**, p. 99].

Exercises for Chapter 63.

Exercise 63.1. Let $K = (V, S)$ be an abstract simplicial complex and let "$\leq$" and "$\widetilde{\leq}$" be two orders on K. Show that the maps

$$\begin{array}{rcl} C_n^{\mathrm{simp},\leq}(K) & \to & C_n^{\mathrm{simp},\widetilde{\leq}}(K) \\ \{v_0 < \dots < v_n\} & \mapsto & \underbrace{\mathrm{sign}(\sigma)\cdot\{v_{\sigma(0)}\widetilde{<}\dots\widetilde{<}v_{\sigma(n)}\}}_{\substack{\sigma\in S_n \text{ is the unique permutation} \\ \text{such that } v_{\sigma(0)}\widetilde{<}\dots\widetilde{<}v_{\sigma(n)}}} \end{array}$$

define an isomorphism of chain complexes.

Exercise 63.2. Let K be a non-empty abstract simplicial complex, let $n \in \mathbb{N}_0$ and let S_n be the abstract simplicial complex that we introduced on page 1480. Show that for every $k \in \mathbb{N}_0$ there exists a natural isomorphism

$$\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(K * S_n) \;\cong\; \widetilde{\mathrm{H}}_{k-(n+1)}^{\mathrm{simp}}(K).$$

Exercise 63.3. Let $K = (V, S)$ be an abstract simplicial complex. Recall that on page 1516 we said that $K = (V, S)$ is *connected* if given any $v, v' \in V$ there exist 1-simplices $s_0, \dots, s_k \in S$ such that $v \in s_0$, such that for every $i \in \{0, \dots, k-1\}$ we have $s_i \cap s_{i+1} \neq \varnothing$ and such that $v' \in s_k$. Show that if K is a non-empty connected abstract simplicial complex, then given any abelian group G the map

$$\begin{array}{rcl} \mathrm{H}_0^{\mathrm{simp}}(K;G) & \to & G \\ \left[\sum\limits_{i=0}^{k}\{v_i\}\otimes g_i\right] & \mapsto & \sum\limits_{i=0}^{k} g_i \end{array}$$

is a natural isomorphism

Remark. This is the simplicial analogue of Proposition 41.5.

Exercise 63.4. In this exercise we give a down-to-earth proof of the Simplicial Künneth Theorem 63.14 in the special case that we are dealing with 1-dimensional abstract simplicial complexes.

(a) Let K and L be two ordered 1-dimensional abstract simplicial complexes. Show that the tensor product of the chain complexes $C_*^{\mathrm{simp}}(K)$ and $C_*^{\mathrm{simp}}(L)$, as defined on page 1433, is chain homotopy equivalent to the chain complex $C_*^{\mathrm{simp}}(K \times L)$.
(b) Prove (a) for arbitrary finite dimensional abstract simplicial complexes.

---

[985]Here the vertical maps Θ are the natural isomorphisms introduced on page 1565.

(c) Prove (a) for arbitrary abstract simplicial complexes.

Remark. Together with the Künneth Theorem 58.7 for chain complexes this exercise makes it possible to calculate the simplicial homology groups of $K \times L$ directly in terms of the simplicial homology groups of K and L.

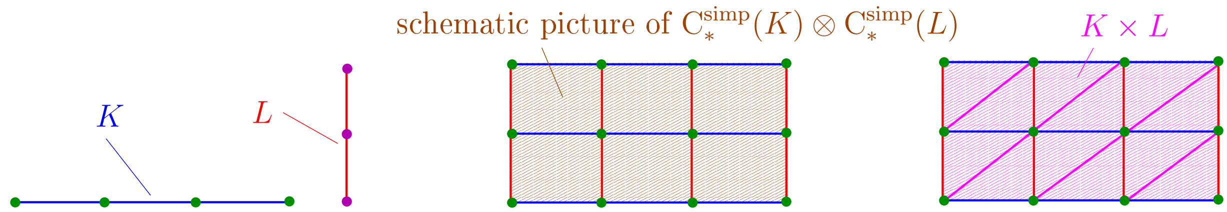


FIGURE 962. Illustration of Exercise 63.4.

Exercise 63.5. Let $p, q \in \mathbb{N}_0$. We consider the product $D_p \times D_q$ of the ordered abstract simplicial complexes defined on page 1480. Show that

$$H_i^{\text{simp}}(D_p \times D_q) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark. This exercise goes into the proof of the Simplicial Künneth Theorem 63.14. In particular you are not allowed to use this theorem.

Exercise 63.6. Show that for any abstract simplicial complex K and any simplex s of K we have for every $i \in \mathbb{N}$ that $H_i^{\text{simp}}(\text{St}(K, s)) = 0$.

Exercise 63.7.

(a) Formulate and prove a Mayer–Vietoris style theorem for simplicial homology, along the lines of the Mayer–Vietoris Theorem 46.5.
(b) Use the exact sequence from (a) to provide a new solution to Exercise 63.2.

Exercise 63.8. Let K be a simplicial complex and let A and B be two subcomplexes with $A \cup B = K$. Show that for each $k \in \mathbb{N}_0$ the natural map $H_k^{\text{simp}}(A, A \cap B) \to H_k^{\text{simp}}(X, B)$ is an isomorphism.
Remark. This statement can be viewed as the Simplicial Excision Theorem. The biggest difference to the singular setting is that in the simplicial setting the proof is almost embarrassingly easy.

Exercise 63.9. Let K be an abstract simplicial complex and let L be a subcomplex of K. Show that for each $k \in \mathbb{N}_0$ there exists a natural isomorphism

$$H_k^{\text{simp}}(K, L) \;\cong\; \widetilde{H}_k^{\text{simp}}(\underbrace{K \cup_L \text{Cone}(L)}_{\text{mapping cone of } L \to K}).$$

Exercise 63.10. Compute the simplicial homology groups of an abstract simplicial complex K such that $|K|$ is homeomorphic to the surface Σ of genus two.
Hint. As we saw in Figure 220, we can view Σ as the connected sum of two tori and you could use Exercise 63.7.

Exercise 63.11. Show that the two simplicial complexes $X = |S_2|$ and $Y = |\Sigma(\Sigma(S_0))|$ which are shown in Figure 963 admit subdivisions X' and Y' which are simplicially isomorphic.

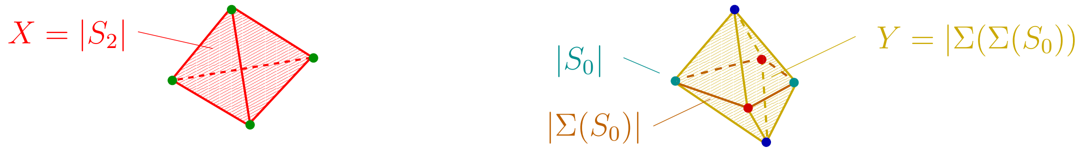


FIGURE 963. Illustration of Exercise 63.11.

Exercise 63.12. Let $K = (V, S)$ be an abstract simplicial complex. Let $\mathcal{U} = \{\mathring{\mathrm{St}}(v)\}_{v \in V}$ be the open cover of the topological realization $|K|$ that is given by the open stars. What is the relationship between the original abstract simplicial complex K and the nerve complex $N(\mathcal{U})$?

Exercise 63.13. Formulate and prove a simplicial analogue of the Mayer–Vietoris Theorem 46.5.
Remark. The proof is significantly easier than the original proof of the Mayer–Vietoris Theorem 46.5.

Exercise 63.14. Let K and L be two ordered abstract simplicial complexes. Show that for each $k \in \mathbb{N}_0$ there exists a short exact sequence of the following form:

$$0 \to \widetilde{\mathrm{H}}_k^{\mathrm{simp}}(K * L) \to \widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(K) \oplus \widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(L) \to \widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(K \times L) \to 0.$$

Remark. This statement is the simplicial analogue of Exercise 46.20. In fact at least if K and L are finite, then we can reduce the simplicial statement to the singular statement. The challenge is to give a purely combinatorial argument.

Exercise 63.15. Let $n \in \mathbb{N}_0$.

(a) Let W be an open subset of $\mathbb{R}^n$ that is the union of finitely many convex open subsets. Show that $\mathrm{H}_k(W) = 0$ for $k \geq n$.
Hint. Use the Mayer–Vietoris Theorem 46.5 and do an induction argument on the number of convex open subsets. As on many other occasions we point out the convenient fact that the intersection of convex subsets of $\mathbb{R}^n$ is again convex.

(b) Let M be an open subset of $\mathbb{R}^n$ and let $\sigma \in \mathrm{C}_n(M)$ be a cycle. Show that there exists an open subset W of $\mathbb{R}^n$ that is the union of finitely many convex open subsets such that $\sigma \in \mathrm{C}_n(W)$ and such that $W \subset M$.

(c) Let M be an open subset of $\mathbb{R}^n$. Show that $\mathrm{H}_k(M) = 0$ for $k \geq n$.

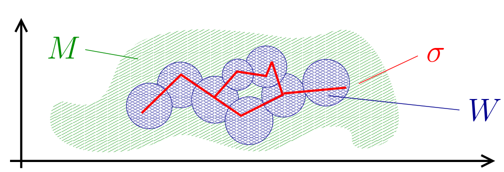


FIGURE 964. Illustration of Exercise 63.15.

Exercise 63.16. Let $P_1, \ldots, P_s$ be points in $\mathbb{R}^2$ with $s \geq 3$. We assume that any three of those points are contained in an open disk of radius r. Show that all the given points are contained in an open disk of radius r.

Exercise 63.17. Let $(K, \leq)$ be an abstract simplicial complex. Let $\sigma\colon \operatorname{sd}(K) \to K$ be the stretching map from Lemma 63.38. Show that the map $|\sigma| \circ \mho^{-1}\colon |K| \to |K|$ is homotopic to the identity.
Remark. Use Corollary 36.9 to overcome any continuity anxieties that might arise.

Exercise 63.18. Let K be an abstract simplicial complex. Show that the maps

$$\begin{array}{rcl} \mathrm{C}_k^{\mathrm{simp}}(K) \otimes \mathbb{Q} & \to & \mathrm{C}_k(|K|) \otimes \mathbb{Q} \\ {[v_0, \ldots, v_k]} \otimes r & \mapsto & \frac{1}{(k+1)!} \cdot \sum\limits_{\sigma \in S_{k+1}} \cdot \left(\begin{array}{rcl} \Delta^k & \mapsto & |K| \\ (t_0, \ldots, t_k) & \mapsto & \sum\limits_{i=0}^{k} t_i \cdot v_{\sigma(i)} \end{array} \right) \otimes r \end{array}$$

are a natural chain homotopy equivalence.

In this chapter we will see that smooth manifolds admit a simplicial structure. In fact we will see that they admit particularly nice simplicial structures. There will be many benefits. For example we can almost immediately draw conclusions on the homology groups and homotopy groups of smooth manifolds.

64.1. **Simplicial structures on smooth manifolds.** We recall the following definition from page 1497.

Definition. A *simplicial structure* for a given topological space X is defined as a pair $(K = (V, S), \Theta \colon |K| \to X)$ where the following holds:

(1) K is an abstract simplicial complex,
(2) Θ is a homeomorphism between the topological realization $|K|$ of K and the topological space X.

Given a simplex $s \in S$ we refer to $\Theta(|s|)$ as a *simplex of* X and we refer to $\Theta(\langle s \rangle)$ as an *open simplex of* X.

In this section we want to consider simplicial structures on smooth manifolds. Since the smooth structure on smooth manifolds is important for many arguments and constructions it is reasonable to consider simplicial structures that reflect the smooth structure. In the following the idea is to consider simplicial structures (K, Θ) such that for each k-simplex the characteristic map $\Theta \circ \Phi_s \colon \Delta^k \to M$ is a smooth embedding. Since Δ^k is not a smooth manifold it takes a little bit of an effort to explain what a "smooth embedding" is supposed to be in this case.

We start out with the following definition. The first part just recalls the definitions introduced on pages 5 and 272.

Definition. Let $n \in \mathbb{N}_0$ and let A be a subset of $\mathbb{R}^n$ for which there exists an open subset U of $\mathbb{R}^n$ with $U \subset A \subset \overline{U}$. Furthermore let M be a smooth manifold.

(1) We say a map $f \colon A \to M$ is *smooth at a point* $P \in A$, if there exists an open neighborhood V of P in $\mathbb{R}^n$ and a smooth map $\widetilde{f} \colon V \to \mathbb{R}^m$ which coincides with f on $A \cap V$. We define $\mathrm{D} f_P := D\widetilde{f}_P \colon \mathbb{R}^n \to \mathrm{T}_{f(P)} M$.[986] We refer to Figure 965 for an illustration.
(2) We say a map $f \colon A \to M$ is an *immersion* if f is smooth at every $P \in A$ and if given any $P \in A$ the differential $\mathrm{D} f_P$ has rank n.[987]
(3) We say a map $f \colon A \to M$ is a *smooth embedding* if it is an immersion and an embedding.

[986] We leave it to the reader to verify, using our restrictive choice of domain A and using Exercise 2.30 (1), that the definition of $\mathrm{D} f_P$ does not depend on the choice of the smooth extension $\widetilde{f}$.

[987] If A is in fact an open subset of $\mathbb{R}^n$, then this definition of an immersion agrees of course with the definition given on page 342.

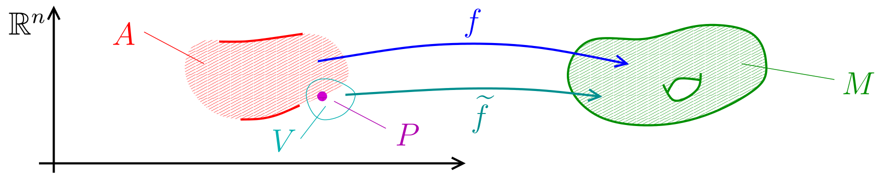


FIGURE 965

Example. Given $k \in \mathbb{N}_0$ we consider the map

$$\begin{array}{rcl} \Xi_k\colon ⊿_k := \left\{ (t_1,\dots,t_k) \in [0,1]^k \,\middle|\, \sum\limits_{j=1}^k t_j \in [0,1] \right\} & \to & \mathbb{R}^{k+1} \\ (t_1,\dots,t_k) & \mapsto & \left(t_1,\dots,t_k, 1-\sum\limits_{j=1}^k t_j\right). \end{array}$$

This map has two interesting properties:

(1) By picking the obvious extension of Ξ_k to a map $\mathbb{R}^k \to \mathbb{R}^{k+1}$ we see that Ξ_k is an immersion.
(2) It follows easily from Proposition 2.43 (3) that the map Ξ_k defines a homeomorphism $\Xi_k\colon ⊿_k \to \Delta^k$.

The combination of (1) and (2) shows that Ξ_k is an embedding.

This leads us to the following definition.

Definition. Let M be a smooth manifold.

(1) Let $k \in \mathbb{N}_0$. We say that a map $\Phi\colon \Delta^k \to M$ is a *smooth embedding* if the composition $\Phi \circ \Xi_k\colon ⊿_k \to M$ is a smooth embedding.
(2) We say that a given simplicial structure $(K, \Theta\colon |K| \to M)$ is *smooth* if for each k-simplex s the corresponding characteristic map $\Theta \circ \Phi_s\colon \Delta^k \to M$ is a smooth embedding.

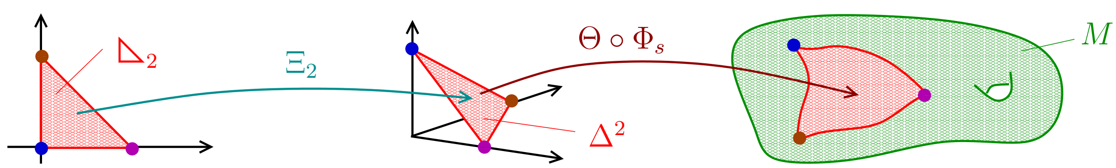


FIGURE 966

Remark. At first glance there are different definitions of "smooth simplicial structure" in the literature. For example our definition of a "smooth simplicial structure" is at first glance different from the one given in [**Mun66a**, p. 81]. But [**Mun66a**, Theorem 8.4] shows that our definition is in fact equivalent to the definition used in Munkres' book. Also, it is not difficult to see that our definition is equivalent to the one used in Whitney's book [**Why57**].

Example. In Figure 967 we show smooth simplicial structures for the closed disk, the annulus, the 2-dimensional sphere and for the torus minus an open disk.

The following lemma contains two basic statements about (smooth) simplicial structures of smooth manifolds.

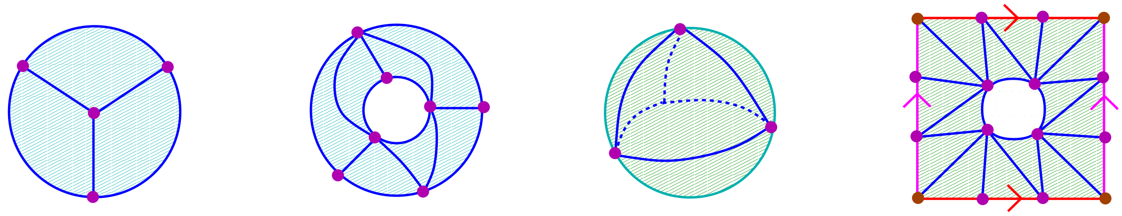


FIGURE 967

Lemma 64.1. *Let M be a smooth manifold which is equipped with a smooth simplicial structure. If M is n-dimensional, then the dimension of each simplex is at most n.*

Remark. In Proposition 66.1 we will prove several other statements regarding simplicial structures on smooth manifolds. In particular we will give an alternative proof of Lemma 64.1. Furthermore we will show that any (smooth) simplicial structure on a manifold M restricts to a (smooth) simplicial structure on the smooth manifold ∂M.

PROOF. Let s be a k-simplex of the given smooth simplicial structure. Combining the characteristic map of the simplex with a chart we obtain a map from a non-empty open subset of $\mathbb{R}^k$ to $\mathbb{R}^n$. If the simplicial structure is smooth this map is in fact an embedding. It follows almost immediately from the invariance of domain, see Theorem 6.19 that $k \leq n$. ■

Now we can formulate the following theorem.

Theorem 64.2.

(1) *Every smooth manifold admits a countable smooth simplicial structure.*
(2) *If M is a smooth manifold with boundary ∂M, then any smooth simplicial structure for ∂M can be* extended *to a smooth simplicial structure for M.*
(3) *If M is a smooth manifold and if N is a proper submanifold of M, then there exists a smooth simplicial structure for M that* restricts *to a smooth simplicial structure on N.*
(4) *If $M = A \cup B$ is a decomposition into two n-dimensional submanifolds such that $A \cap B$ is a union of boundary components of A and a union of boundary components of B, then we can find a smooth simplicial structure on M such that A and B are subcomplexes.*

Remark.

(1) Note that the conclusion of Theorem 64.2 (3) is somewhat weaker than the conclusion of Theorem 64.2 (2), it does not say that we can extend a given smooth simplicial structure on N to a smooth simplicial structure on M.
(2) By [**Bin83**, Theorem XVIII.3.C] any PL-structure on the boundary of a 3-dimensional topological manifold can be extended to a PL-structure on the 3-dimensional topological manifold.
(3) Many more subtle results on finding smooth simplicial structures on smooth manifolds can be found in [**Ve84r**, Chapter 7].

PROOF. We start out with giving precise references for the three statements.

(1) This statement is contained in [**Mun66a**, Theorem 10.6]. For smooth manifolds without boundary the statement is also shown in [**Why57**, Chapter IV.12]. For closed smooth manifolds, with a slightly weaker conclusion, the statement was first proved by Whitehead [**WhdJ40**, Theorem 7] in 1940, building on earlier work of Cairns [**Cai34**]. Other proofs for (some types of) smooth manifolds are given in [**Cai61, Freu39**] and [**Muk15**, Theorem 9.5.1]. A sketch of an argument is also given in [**Thu97**, Theorem 3.10.2].
(2) This statement is also contained in [**Mun66a**, Theorem 10.6], alternatively see the discussion on [**WhdJ40**, p. 822f].
(3) If N is a closed submanifold of M, then this statement can be proved by solving [**Mun66a**, Problem 10.14]. The general case can be deduced from [**Ve84r**, Theorem 7.8], once one has unraveled the statement [**Ve84r**, Theorem 7.8].
(4) First we apply (1) to equip the smooth manifold $A \cap B$ with a simplicial structure. Next we use (2) to extend the smooth simplicial structure to A and B. It is fairly straightforward to verify that putting these two smooth simplicial structures together we obtain a smooth simplicial structure for $M = A \cup B$.

To give the reader a sense of how one can possibly construct smooth simplicial structures on smooth manifolds we provide a sketch for the proof of statement (1). The sketch is partially inspired by the argument given by Whitney [**Why57**, Chapter IV.12].

For simplicity let M be a closed n-dimensional smooth manifold. Recall that by Proposition 9.1 and Proposition 8.1 we can view M as a submanifold of $\mathbb{R}^m$ for some $m \geq n$. By the General Tubular Neighborhood Theorem 10.5 we can pick a tubular neighborhood $p\colon W \to M$. (For the purpose of this sketchy proof we might as well assume that the tubular neighborhood is given by $W = \overline{B}^{m-n} \times M$ and that $p\colon \overline{B}^{m-n} \times M \to M$ is the obvious projection.) We equip $\mathbb{R}^m$ with the canonical simplicial structure defined in Lemma 61.17.

We make the following, ever so slightly vague observation.

Observation. Using a suitable generalization of the Transversality Theorem 9.10 and using the definition of a smooth map and a smooth submanifold ("locally everything is close to being linear") we can shrink the cubes to a suitably small size and we can find an isotopy of M such that the resulting simplicial structure (V, S) for $\mathbb{R}^m$ satisfies the following two conditions:[988]

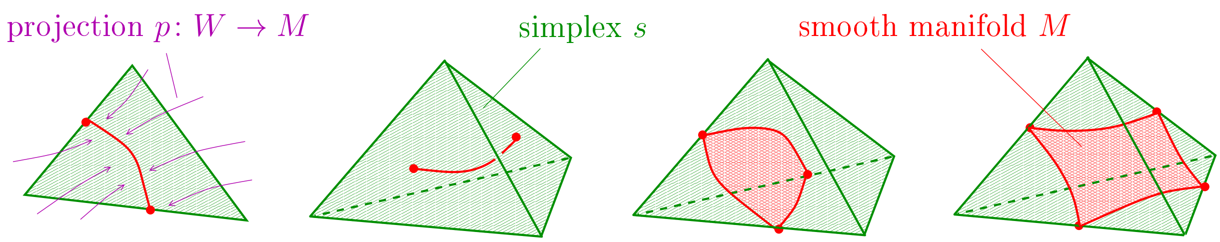
(a) For every simplex $s \in S$ the intersection of s with M is "close" to the transverse[989] intersection of an n-dimensional affine space with s.

[988]In our notation we blur the difference between the abstract simplices of the simplicial structure and the corresponding simplices in $\mathbb{R}^n$.

[989]Here by "transverse" we mean the following: if s is an l-dimensional simplex, then it is contained in an l-dimensional affine subspace. The transverse intersection of an n-dimensional affine space with an l-dimensional affine space in $\mathbb{R}^m$ is of dimension $m + l - n$, where negative dimension is understood to be the empty set.

(b) Every simplex s with $s \cap M \neq \varnothing$ is contained in the tubular neighborhood W and the restriction of the projection $p\colon W \to M$ to s is "close" to being an affine linear map.

We refer to Figure 968 for an illustration.



$n = 1$ and $m = 2$ $\quad$ $n = 1$ and $m = 3$ $\quad$ $n = 2$ and $m = 3$ $\quad$ $n = 2$ and $m = 3$

FIGURE 968

We set $k := m - n$. In the following we spell out two consequences of (a).

(1) No simplex of dimension less than k intersects M.
(2) If a k-simplex s intersects M, then it does so in a unique point $P = P_s$, which we know by (1) lies necessarily in the interior of s.

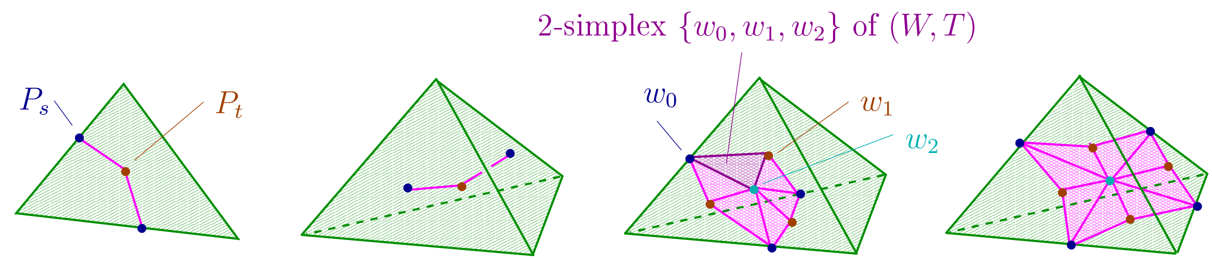
Next we make the following constructions.

(3) For each l-simplex t with $l > k$ which has the property that the corresponding open simplex $\langle t \rangle$ intersects M we pick a point P_t that lies in the intersection of M with the interior of t.
(4) We consider the abstract simplicial complex

$$L := (W, T) \qquad \text{where } \begin{array}{rcl} W & := & \{s \in S \,|\, \langle s \rangle \cap M \neq \varnothing\} \text{ and} \\ T & := & \{(w_0, \ldots, w_l) \,|\, w_0, \ldots, w_l \in S \text{ and } w_0 \subsetneq \ldots \subsetneq w_l\}. \end{array}$$

(5) Let $\Theta\colon |L| \to \mathbb{R}^m$ be the unique map with $\Theta(w) = P_w$ for $w \in W$ and which is affine linear on each simplex $|t|$.

Some of these definitions are illustrated in Figure 969. We continue with the following



$n = 1$ and $m = 2$ $\quad$ $n = 1$ and $m = 3$ $\quad$ $n = 2$ and $m = 3$ $\quad$ $n = 2$ and $m = 3$

FIGURE 969

observations:

(6) It follows from (a) and (b) that the map $p\colon W \to M$ restricts to a homeomorphism $\Theta(|L|) \to M$. Furthermore, for each simplex $t \in T$ the map $p\colon \Theta(|t|) \to M$ is a submersion, i.e. at each point the differential has full rank.
(7) By (6) we now know that $(L = (W, T), p \circ \Theta\colon |L| \to M)$ is a smooth simplicial structure for M.

We refer to Figure 970 for a more "global" illustration of the above argument. Evidently this outline has several serious issues. For example we never made it clear what we mean by "close". Nonetheless we feel that with enough effort one can turn the above sketch of an argument into a proper proof.

We refer to [**Why57**, Chapter IV.12] for a very detailed discussion how a variation on the above approach can be made to work. ∎

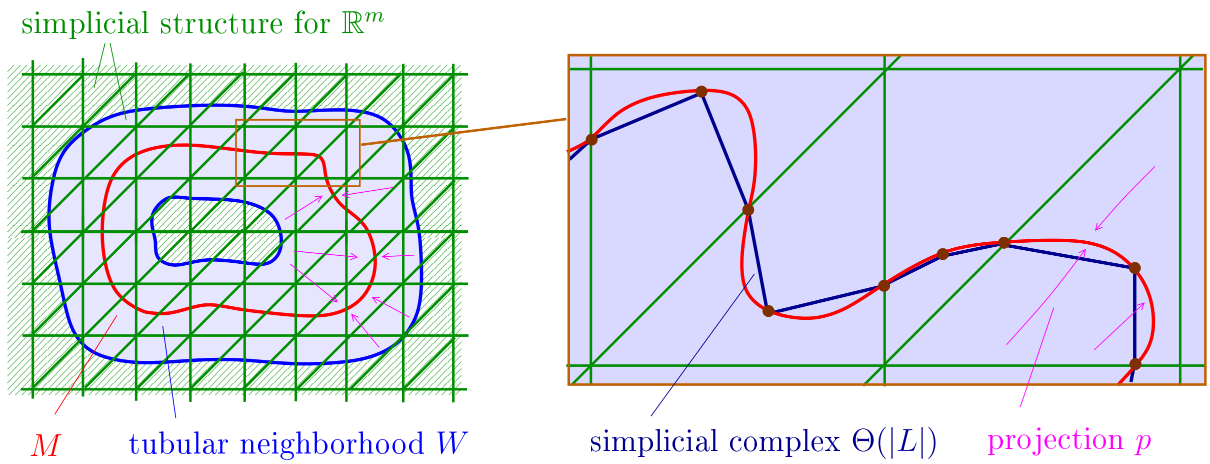


FIGURE 970. Proof of Theorem 64.2 with $n = 1$, $m = 2$ and $k = 1$.

Just for fun we mention that in our sketch of a proof of Theorem 64.2 we basically proved the following proposition which says that any submanifold of $\mathbb{R}^m$ can be "arbitrarily" approximated by a linear simplicial subcomplex of $\mathbb{R}^m$. We will not make use of this proposition, hence we have no qualms about the rather sketchy proof.

Proposition 64.3. *Let M be a closed smooth submanifold of $\mathbb{R}^m$. Given any $\epsilon > 0$ there exists a linear simplicial complex $(L = (W, T), \Theta\colon |L| \to X)$ with $X \subset \mathbb{R}^m$ and an isotopy $F\colon X \times [0, 1] \to \mathbb{R}^m$ with the following properties:*

(1) *We have $F_0 = \mathrm{id}$.*
(2) *The map F_1 is a homeomorphism $F_1\colon X \to M$ and it has the property that $F_1 \circ \Theta$ defines a smooth simplicial structure for M.*
(3) *For every $x \in X$ the map $[0, 1] \to \mathbb{R}^m$ given by $t \mapsto F(x, t)$ is affine linear.*
(4) *For every $x \in X$ we have $\|x - F_1(x)\| < \epsilon$.*

Example. Proposition 64.3 says in particular that we can approximate any knot M in $\mathbb{R}^3$ by a "simplicial knot" X. We refer to Figure 971 for an illustration.

Figure 971

Sketch of proof. We continue with the notation and results in the proof of Theorem 64.2. The $\epsilon > 0$ corresponds to a suitable interpretation of "close" in the formulation of (a) and (b). We set $X := \Theta(|L|)$. The isotopy F is the map which interpolates the identity id_X and the projection $p\colon X \to M$. ∎

In Theorem 63.20 we saw that there exist compact topological spaces which admit two "inequivalent" simplicial structures, in other words, in Theorem 63.20 we saw that the Hauptvermutung 63.19 is false in general. Interestingly, the "smooth" analogue of the Hauptvermutung 63.19 is actually correct. More precisely, the following theorem holds.

Theorem 64.4. *Let M and N be smooth manifolds and suppose we are given two smooth simplicial structures $(K, \Theta\colon |K| \to M)$ and $(L, \Omega\colon |L| \to N)$. If M and N are diffeomorphic, then there are subdivisions of K and L which are simplicially isomorphic.*

Proof. This statement is shown in [**WhdJ40**, Theorem 8] and alternatively in [**Mun66a**, Theorem 10.5] (see also [**Mun75**, Theorem 10.13]). It is perhaps worth pointing out that we do not need to assume that M is compact. ∎

Theorem 64.4 will come in handy at a later stage. But for the foreseeable future it is of no real interest to us.

We conclude this section with picking up some low-hanging fruit that we obtain almost for free from Theorem 64.2. First we have the following theorem.

Theorem 64.5. *Every (compact) n-dimensional smooth manifold M admits a (finite) CW-structure with the following properties:*

(1) *every cell has dimension at most n,*
(2) *the boundary ∂M is a subcomplex.*

Furthermore, if $M = A \cup B$ is a decomposition into two n-dimensional submanifolds such that $A \cap B$ is a union of boundary components of A and a union of boundary components of B, then we can find a CW-structure on M which has all the above properties and such that A and B are subcomplexes.

Remark.

(1) Theorems 64.2 and 64.5 can be viewed as a significant strengthening of Corollary 63.37. Later on, in Theorem 85.12, Proposition 119.12, Proposition 104.12 and Propositions 105.8 we will get to know two more methods for equipping a smooth manifold with CW-structure (possibly up to a homotopy equivalence).
(2) Note that Theorem 64.5 is stated for smooth manifolds and *not* for topological manifolds. The situation for topological manifolds is in fact much more complicated and we will deal with it later on in Section 85.6.

PROOF. The theorem follows immediately from Theorem 64.2 together with Lemma 64.1 and Lemma 61.24. ■

Almost for free we now obtain the following proposition which gives us some control over the "size" of invariants of smooth manifolds.

Proposition 64.6. *Let M be an n-dimensional 0-connected smooth manifold.*

(1) *For every $k > n$ and for every abelian group G we have $\mathrm{H}_k(M;G) = 0$.*
(2) *The higher homotopy groups of M are countable.*

If M is compact then the following further statements hold:

(3) *The fundamental group of M is finitely presented.*
(4) *All homology groups of M are finitely generated abelian groups. Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(M;\mathbb{F})$ are finite-dimensional.*

Remark.

(1) Note that in Exercise 54.10 we saw that there exists a closed smooth manifold M such that $\pi_2(M)$ is infinitely generated as an abelian group. In other words, Proposition 64.6 (2) is in a sense optimal.
(2) On page 2514 we will give an alternative proof for some of the statements of Proposition 85.13 using "handle decompositions".
(3) In Proposition 85.13 we will extend Proposition 64.6 to compact *topological* manifolds.

PROOF.

(1) This statement follows from Theorem 64.5 together with Proposition 48.5 (1) and the discussion on page 1402.
(2) This statement follows from Theorem 64.2 together with Proposition 62.9.
(3) This statement follows from Theorem 64.5 together with Proposition 37.13 (3).
(4) It follows from Theorem 64.5 together with Proposition 48.5 (2) that all homology groups are finitely generated. Furthermore it follows from the discussion on page 1402 that each $\mathrm{H}_k(M;\mathbb{F})$ is finite-dimensional. ■

In fact in Chapter 66 we will see that the existence of smooth simplicial structures on smooth manifolds can be used to extract even more information on homology groups of compact smooth manifolds.

On several occasions we will also study relative homology groups of a compact smooth manifold M, for example we will be interested in the relative homology groups $\mathrm{H}_k(M,\partial M)$. The following lemma, together with Proposition 64.6 (4), gives us some control over these invariants.

Lemma 64.7. (∗) *Let X be a topological space, let $A \subset X$ be a subset and let $k \in \mathbb{N}_0$.*

(1) *If the homology groups $\mathrm{H}_k(X)$ and $\mathrm{H}_{k-1}(A)$ are finitely generated, then so is the relative homology group $\mathrm{H}_k(X,A)$.*
(2) *Let $\mathbb{F}$ be a field. If the homology groups $\mathrm{H}_k(X;\mathbb{F})$ and $\mathrm{H}_{k-1}(A;\mathbb{F})$ are finite dimensional, then so is the relative homology group $\mathrm{H}_k(X,A;\mathbb{F})$.*

Proof (∗). The first statement follows from the long exact sequence of the pair (X, A) together with Lemma 19.6. The second statement is proved in a similar fashion. ■

64.2. **PL-manifolds.** Throughout these notes we have so far only encountered two flavors of manifolds, namely topological manifolds and smooth manifolds. In this section we will introduce a new variety of manifolds, namely we will introduce the concept of PL-manifolds. In a nutshell a PL-manifold is a topological manifold together with a well-behaved simplicial structure.

For the reader's convenience we recall the following definition from page 1554.

Definition. We say that a simplicial complex Y is a *subdivision* of some simplicial complex X if the following three conditions are satisfied:

(1) the underlying topological spaces are the same,
(2) every simplex of Y is contained in some simplex of X,
(3) for every l-simplex t of Y and every k-simplex s of X which contains t the map $\Phi_s^{-1} \circ \Phi_t \colon \Delta^l \to \Delta^k$ is affine linear.

Remark. Let X be a simplicial complex and let Y be a subdivision. In Exercise 64.3 we will see that every k-simplex of X is the union of finitely many simplices of Y of dimension $\leq k$. This shows that our definition of a subdivision agrees with the definition given in [**Mun66a**, p. 83]. It also shows that if we use Lemma 61.24 to view X and Y as CW-complexes, then the identity map $X \to Y$ is cellular.

We continue with the following definition.

Definition. Let X and Y be simplicial complexes. We say a map $f \colon X \to Y$ is *piecewise linear*, if there exist subdivisions X' and Y' such that $f \colon X' \to Y'$ is simplicial.[990]

Remark. The definition of a piecewise linear map is the one given in [**Ze63b**, p. 6] and [**Gla70**, p. 13]. It follows from [**RS72**, Theorem 2.14] and [**RS72**, Example 1.5(4)] that the definition of a piecewise linear map on [**RS72**, p. 5] is equivalent to the above definition. Furthermore, by [**Ze63b**, Lemma I.7] the definition of a piecewise linear map on [**Hud69**, p. 2] is also equivalent to the above definition.

Warning. Many of our favorite reference, e.g. [**Ze63b**], mostly deals with finite simplicial complexes. Thus on several occasions we will also restrict to the case of finite simplicial complex. We leave it to the undaunted reader to figure out to what degree the results of this section also generalize to infinite simplicial complexes.

We start out with some examples of piecewise linear maps.

Examples.

(1) In Figure 972 we show a map $f \colon X \to Y$ between two finite linear simplicial complexes which is not simplicial, but which is piecewise linear.
(2) Let $f \colon X \to Y$ be a map between linear simplicial complexes. If X is a finite linear simplicial complex, then it follows from the Simplicial Approximation Theorem 62.8 that f is homotopic to a piecewise linear map.

[990] Colloquially one usually says "PL" instead of "piecewise linear".

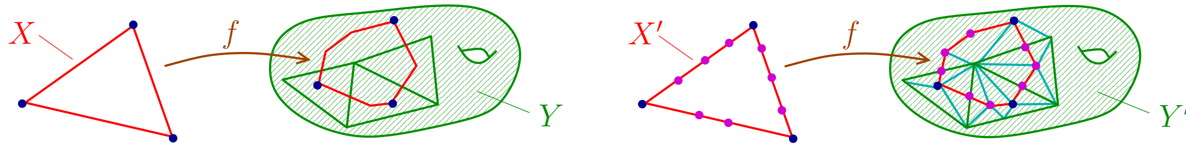


FIGURE 972

From the definitions it is not entirely clear that the composition of two piecewise linear maps is again piecewise linear. Fortunately, with some effort one can show that this is indeed the case.

Proposition 64.8. *The composition of two piecewise linear maps between* finite *simplicial complexes is again piecewise linear.*

PROOF. The proposition is proved in [**Ze63b**, Lemma I.2] and [**Gla70**, Theorem I.6]. ■

Remark. As we hinted at above, both references for the proof of Proposition 64.8 do indeed assume that the simplicial complexes involved are finite. It is not clear to me whether the statement holds for maps between arbitrary simplicial complexes.

We move on to the following basically self-explanatory definition.

Definition. Let X and Y be two simplicial complexes.

(1) A map $f\colon X \to Y$ is called a PL-*homeomorphism* if f is piecewise linear, if f is a bijection and if f^{-1} is also piecewise linear.[991]
(2) We say X and Y are PL-*homeomorphic* if there exists a PL-homeomorphism $X \to Y$.

We continue with two examples of PL-homeomorphisms.

Examples.

(1) If X' is a subdivision of a simplicial complex X, then it follows immediately from the definition that the map $X' \to X$ is a PL-homeomorphism.
(2) If $f\colon X \to Y$ is a map such that there exists subdivision X' of X and Y' of Y such that the map $f\colon X' \to Y'$ is a simplicial isomorphism, then f is evidently a PL-homeomorphism.
(3) (a) In Figure 973 to the left we show 2-dimensional linear simplicial complexes X_1 and Y_1 which are related by adding a vertex to a 2-dimensional simplex. It follows from (1) that X_1 and Y_1 are PL-homeomorphic.
 (b) In Figure 973 to the right we show 2-dimensional simplicial complexes X_2 and Y_2. If we subdivide X_2 and Y_2 by adding the intersection point of the two diagonals, then the resulting two linear simplicial complexes are simplicially isomorphic. By (2) this shows that X_2 and Y_2 are PL-homeomorphic.
(4) Let $n \in \mathbb{N}$. We consider the two linear simplicial complexes $\operatorname{Cone}(\partial\Delta^n)$ and Δ^n. It follows from the discussion on page 1555 that Δ^n admits a subdivision, given by

[991]For the record we state that it follows from Proposition 61.7 that every piecewise linear map is continuous.

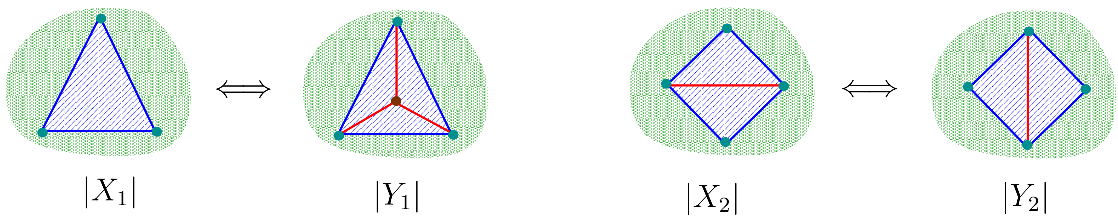


FIGURE 973

adding a single vertex, which is simplicially isomorphic to $\operatorname{Cone}(\partial\Delta^n)$. By (1) this shows that $\operatorname{Cone}(\partial\Delta^n)$ and Δ^n are PL-homeomorphic.

(5) By Proposition 63.17 (1) we know that any two finite linear simplicial complexes X and Y with $|X| = |Y|$ are PL-homeomorphic.

Later on we will need the following elementary lemma.

Lemma 64.9. *If $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ are two* PL-*homeomorphisms between two finite simplicial complexes, then the map*

$$f_1 * f_2\colon X_1 * X_2 \ \to\ Y_1 * Y_2$$

is also a PL-*homeomorphism.*

PROOF. We leave it to the reader to provide the proof. ■

Before we give the definition of a PL-manifold let us introduce the following convention.

Convention. Let $n \in \mathbb{N}_0$. On page 1480 we introduced the abstract simplicial complexes

(1) $D_n := (V, S)$ with $V = \{0, \ldots, n\}$ and $S = \mathcal{P}(V) \setminus \{\varnothing\}$, and
(2) $S_{n-1} := (V, T)$ with $V = \{0, \ldots, n\}$ and $T = \mathcal{P}(V) \setminus \{\varnothing, V\}$.

On page 1492 we saw that there exists a canonical homeomorphism $f\colon |D_n| \to \Delta^n$ which restricts to a homeomorphism $|S_{n-1}| \to \partial\Delta^n$. Unless we say explicitly something else we will always use these homeomorphisms to view Δ^n and $\partial\Delta^n$ as simplicial complexes.

The above convention leads us to our next definition.

Definition. Let $n \in \mathbb{N}$.

(1) A PL *n-ball* is defined as a simplicial complex that is PL-homeomorphic to the simplicial complex Δ^n.
(2) A PL *n-sphere* is defined as a simplicial complex that is PL-homeomorphic to the simplicial complex $\partial\Delta^{n+1}$.

Example. Let $n \in \mathbb{N}_0$. In Exercise 64.2 we will show that $\operatorname{Cone}(\partial\Delta^n)$ is a PL n-ball.

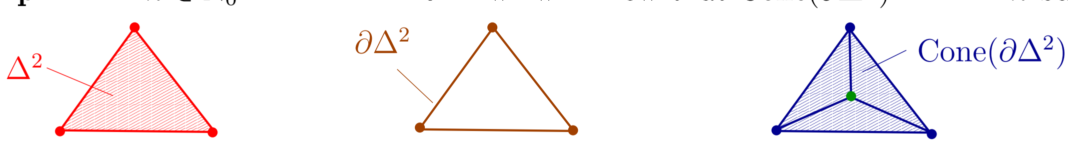


FIGURE 974

Remark. In practice it can be a little tricky to show that a given simplicial complex is in fact a PL n-ball or a PL n-sphere. But in most cases one can verify, with enough patience and dexterity, that if a simplicial complex "looks" like an n-ball or an n-sphere, then it is in fact a PL n-ball respectively a PL n-sphere. For example it is entertaining to show that all the simplicial complexes shown in Figure 975 are in fact PL 2-balls. Shortly we will get to know a more systematic way for finding PL-homeomorphisms.

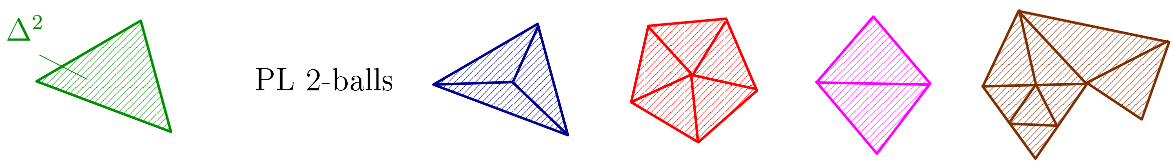


FIGURE 975

Now we can provide the definition of a PL-manifold. The definition is modelled on the definition of a topological manifold that we gave on page 262. The key difference is that we replace the words "neighborhood", "ball" and "homeomorphic" by "simplicial neighborhood", "PL-ball" and "PL-homeomorphic".

Definition.

(1) Let X be a simplicial complex and $P \in X$. A *simplicial neighborhood* is a subcomplex of X that is a neighborhood of P in the usual sense as defined on page 89.
(2) An n-*dimensional* PL-*structure* on a topological space X is defined as a simplicial structure $(K = (V, S), \Theta\colon |K| \to X)$ such that every point $P \in X$ admits a simplicial neighborhood that is a PL n-ball.
(3) An n-*dimensional* PL-*manifold* is a pair (X, Θ) where X is a topological space and Θ is an n-dimensional PL-structure with countably many simplices.

Examples.

(1) In Figure 976 to the left we show a simplicial structure for the annulus $S^1 \times [0, 1]$. The illustrations to the right of Figure 976 should convince the reader that the simplicial structure for the annulus is a PL-structure.

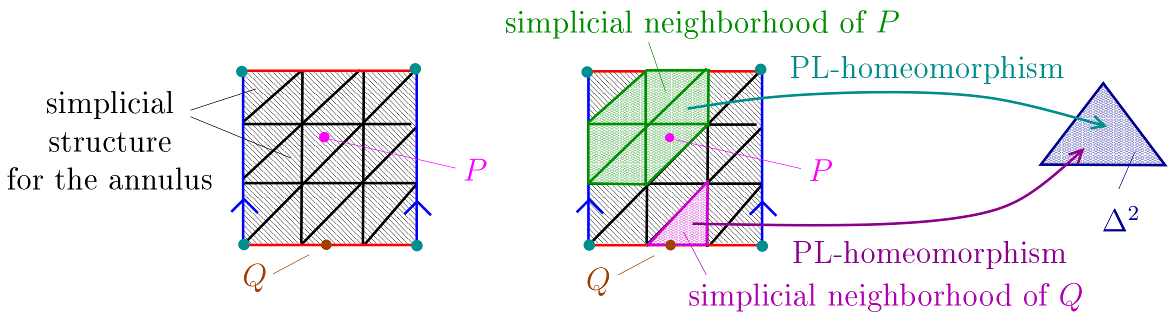


FIGURE 976

(2) Let $n \in \mathbb{N}_0$. The canonical simplicial structure on Δ^n is basically by definition a PL-structure. Recall that in Lemma 41.1 we gave an explicit homeomorphism $\Phi\colon \overline{B}^n \to \Delta^n$. Thus $(D_n, \Phi^{-1}\colon \Delta_n = |D_n| \to \overline{B}^n)$ is a PL-structure for $\overline{B}^n$. Similar to the discussion on page 1500 we will refer to it as the *canonical* PL-*structure* of $\overline{B}^n$.

(3) Let $n \in \mathbb{N}_0$. We consider $\partial\Delta^{n+1}$ with the usual simplicial structure defined on page 1598. Given any vertex v the star $\operatorname{St}(\partial\Delta^n, v)$ is, basically by definition, simplicially isomorphic to $\operatorname{Cone}(\partial\Delta^n)$. It follows from the example on page 1598 that $\operatorname{St}(\partial\Delta^n, v)$ is a PL n-ball. This shows that $\partial\Delta^n$ is an n-dimensional PL-manifold. Recall that in Lemma 41.1 we gave an explicit homeomorphism $\Phi\colon S^n \to \partial\Delta^{n+1}$. This shows that $(S_n, \Phi^{-1}\colon \partial\Delta_{n+1} = |S_n| \to S^n)$ defines a PL-structure on S^n. We refer to it as the *canonical* PL-*structure* on S^n.

Proposition 64.10. *Let X and Y be two simplicial complexes. If X is an n-dimensional* PL*-manifold and if X and Y are* PL*-homeomorphic, then Y is also an n-dimensional* PL*-manifold.*

Example. It follows from Proposition 64.10 that any PL n-ball and any PL n-sphere is indeed an n-dimensional PL-manifold.

PROOF. This proposition is less trivial than it might appear initially. Fortunately a proof is given in [**Gla70**, Theorem II.2]. ■

The following proposition gives an alternative characterization of PL-manifolds.

Proposition 64.11. *Let $(K = (V, S), \Theta\colon |K| \to X)$ be a simplicial complex. The following statements are equivalent:*

(1) *(X, Θ) is an n-dimensional* PL*-manifold.*
(2) *For every vertex $v \in V$ the link* $\operatorname{Lk}(K, v)$ *is a* PL *$(n-1)$-sphere or a* PL *$(n-1)$-ball.*
(3) *For every vertex $v \in V$ the star* $\operatorname{St}(K, v)$ *is a* PL *n-ball.*
(4) *For every k-simplex $s \in S$ the link* $\operatorname{Lk}(K, s)$ *is a* PL *$(n-k-1)$-sphere or a* PL *$(n-k-1)$-ball.*
(5) *For every k-simplex $s \in S$ the star* $\operatorname{St}(K, s)$ *is a* PL *$(n-k)$-ball.*

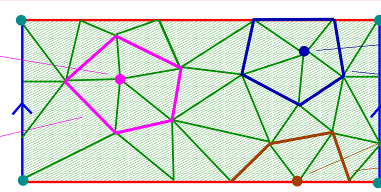


FIGURE 977. Illustration of Proposition 64.11.

PROOF.

(1)⇒(2) This implication is shown in [**Hud69**, p. 26] (alternatively see also [**Ze63b**, p. III.2]).

(2)⇒(3) Let $v \in V$. First we consider the case that the link $\operatorname{Lk}(v)$ is a PL $(n-1)$-sphere. In this case we have the following PL-homeomorphisms:

$$\operatorname{St}(v) \underset{\substack{\uparrow \\ \text{Lemma 62.5 (1c)}}}{\cong} \{v\} * \operatorname{Lk}(v) \underset{\substack{\uparrow \\ \text{since (2) holds}}}{\cong} \{v\} * \text{PL } (n-1)\text{-sphere} \underset{\substack{\uparrow \\ \text{Lemma 64.9}}}{\cong} \{v\} * \partial\Delta^n \underset{\substack{\uparrow \\ \text{see page 1598}}}{\cong} \Delta^n.$$

The case that the link $\operatorname{Lk}(v)$ is a PL $(n-1)$-ball is dealt with the same way, except that this time we use the basically trivial fact, first pointed out on page 1506 that

$\{v\} * \Delta^{n-1}$ is in fact simplicially isomorphic to Δ^n. We have thus shown that (2) implies (3).

(3)⇒(1) This implication follows immediately from Lemma 62.5 (1).

(2)⇒(4) We prove the statement by induction on k. Note that the case $k = 0$ is precisely Statement (2). Now suppose that the statement holds for some $k-1$. Let $K = (V, S)$ be an abstract simplicial complex and let $t = \{v_0, \dots, v_k\}$ be a k-simplex. We consider the $(k-1)$-simplex $s = \{v_0, \dots, v_{k-1}\}$. By induction hypothesis we know that $\mathrm{Lk}(K, s)$ is a PL $(n-k)$-sphere or a PL $(n-k)$-ball. As we pointed out on page 1600, it follows from Proposition 64.10 that $\mathrm{Lk}(K, s)$ is in fact an $(n-k)$-dimensional PL-manifold. Now we see that

$$\mathrm{Lk}(K,t) \underset{\substack{\uparrow \\ \text{follows easily from} \\ \text{the definitions}}}{=} \mathrm{Lk}(\mathrm{Lk}(K,s), v_k) \underset{\substack{\uparrow \\ \text{by induction hypothesis}}}{=} \mathrm{Lk}\left(\begin{matrix}(n-k)\text{-dimensl.} \\ \text{PL-manifold}\end{matrix}, v_k\right) \underset{\substack{\uparrow \\ \text{by (1)⇒(2)}}}{=} \begin{matrix}\text{PL } (n-k-1)\text{-sphere} \\ \text{or a PL } (n-k-1)\text{-ball.}\end{matrix}$$

(4)⇒(5) The proof of this statement is basically identical to the proof of (2)⇒(3).

(5)⇒(3) This implication holds for the trivial reason that every vertex is in particular a 0-simplex. ■

Remark.

(1) The notion of a PL-manifold gets defined differently by different authors. There is the usual question whether or not one wants a manifold to be second-countable or not which in our case translates into the question whether or not one wants the simplicial complex to be countable. But if we ignore this rather technical issue, then we see the following:

(a) The above definition above is basically the one given in [**Hud69**, p. 20] and [**Ze63b**, p. II.10].

(b) It follows from Proposition 64.11 (1)⇔(2) that our definition is equivalent to the definition provided in [**Bry02**, p. 223] and [**Stal67**, Definition 4.4.9].

(c) It follows from Proposition 64.11 (1)⇔(5) that our definition is equivalent to the definition given in [**Geo08**, p. 134].

(d) It is a consequence of [**RS72**, Theorem 2.2] that our definition is equivalent to the one given in [**RS72**, p. 7].

(e) In the literature an n-dimensional PL-manifold is sometimes defined as a pair $(X, \{\Phi_i : U_i \to V_i\})$ where X is an n-dimensional topological manifold and where $\{\Phi_i : U_i \to V_i\}$ is an atlas for X such that given any $i, j \in I$ the corresponding transition map $\Phi_j \circ \Phi_i^{-1} : V_i \cap \Phi_i(U_j) \to V_j \cap \Phi_j(U_j)$ is "piecewise linear". Here, in this context we say that a map $f : U \to V$ between open subsets of $\mathbb{R}^n$ is *piecewise linear* if any point $P \in U$ there exists a linear simplicial complex $X \subset U$ which is a neighborhood of P such that the restriction of f to each simplex of X is linear. This definition is for example used in [**GoS99**, p. 7], [**Rudy16**, p. 16] and [**HM74**, p. 8]. It seems to me that the discussions on [**Hud69**, p. 82] and [**Ded62**, p. 370] show that these two concepts of PL-manifolds basically agree. It would be nice though to have a more direct reference for this statement.

(2) A PL-structure for a topological space is sometimes called a *piecewise linear triangulation*, or shorter, a PL-*triangulation*.

Proposition 64.11, and drawing enough pictures, shows that the following definition captures the right intuition.

Definition. We say a PL-manifold is *closed* if X is compact and for every vertex the link $\operatorname{Lk}(v)$ is a PL-sphere.

We have no intention of going deep into the theory of PL-manifolds. Instead we refer the interested reader to [**Hud69, Ze63b, Gla70, RS72**] for a much more detailed introduction to PL-manifolds. For example these references contain the definition of the boundary a PL-manifold and a discussion of PL-submanifolds and the existence of "regular neighborhoods" (see also the next section) of subcomplexes.

As an aside we mention only one result, namely suppose that M is a PL-manifold. Similar to the statements of Theorem 64.2 (2) and (3), one can extend a given PL-structure on the boundary to a PL-structure on M and one can extend a given PL-structure on a proper submanifold to a PL-structure of M. We refer to [**Arm67**] for the precise statements and proofs.

64.3. **Regular Neighborhoods** (∗)**.** In this short section we introduce the notion of a regular neighborhood of a subcomplex of a simplicial complex. This is an object that gets used commonly in topology and thus we want to give a definition and we want state to some of the main properties of regular neighborhoods. Otherwise we will barely make use of this concept.

First we give a slight generalization of the concept of the barycentric subdivision of an abstract simplicial complex.

Definition. Let $K = (V, S)$ be an abstract simplicial complex and let $J = (U, R)$ be a finite subcomplex. We define the *barycentric subdivision* $\operatorname{sd}_J(K)$ *of* K *away from* J by barycentric subdividing every simplex that is not contained in J. More formally, and much less readably, $\operatorname{sd}_J(K)$ is defined as follows:

$$\begin{aligned} \text{vertices of } \operatorname{sd}_J(K) \;&:=\; U \cup \{\underline{s}\}_{s \in S \setminus R} \\ \text{simplices of } \operatorname{sd}_J(K) \;&:=\; \left\{ r \cup \{\underline{s_0}, \dots, \underline{s_k}\} \;\middle|\; \begin{array}{c} \text{where } r \in R \cup \{\varnothing\} \text{ and } s_0, \dots, s_k \in S \\ \text{such that } r \subsetneq s_0 \subsetneq \dots \subsetneq s_k \end{array} \right\}. \end{aligned}$$

It follows immediately from the definitions that $\operatorname{sd}_J(K)$ is an abstract simplicial complex and that for $J = \varnothing$ we obtain the usual barycentric subdivision $\operatorname{sd}(K)$.

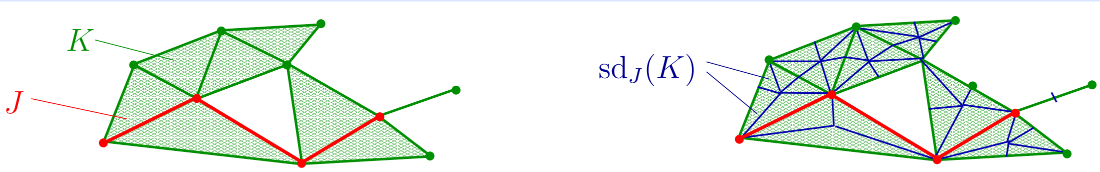


FIGURE 978

The obvious generalization of Lemma 62.3 holds in the above context. But there is certainly no need for formulating the generalizations. Let us move on to the definition of regular neighborhoods.

Definition. Let $(K=(V,S),\Theta\colon |K|\to X)$ be a simplicial complex and let $J=(U,R)$ be a subcomplex. We set

$$S_K(J) \;:=\; \{s\in \mathrm{sd}_J(\mathrm{sd}_J(K))\,|\,\text{there exists an } r\in R \text{ with } r\subset s\}.$$

Now we define

$$N_K(J) \;:=\; \bigcup_{s\in S_K(J)} \Theta(|s|) \quad\text{ and }\quad \mathring{N}_K(J) \;:=\; \bigcup_{s\in S_K(J)} \Theta(\langle s\rangle).$$

We refer to $N_K(J)$ as the *regular neighborhood of J in K*.

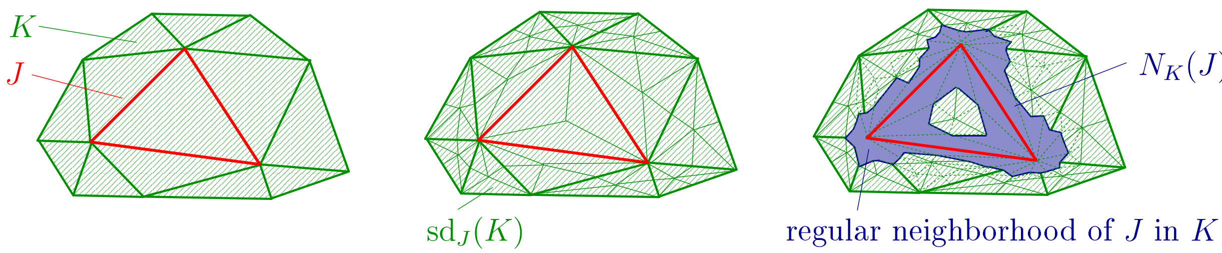


FIGURE 979

Remark. The notion of a regular neighborhood gets defined differently by different authors. Our definition of *the* regular neighborhood of J in K is an example of *a* regular neighborhood as introduced in [**RS72**, p. 33] or alternatively in [**Gla70**, Theorem II.15n].

The following proposition states a few key properties of regular neighborhoods of subcomplexes of PL-manifolds.

Proposition 64.12. *Let $(K=(V,S),\Theta\colon |K|\to M)$ be a compact PL-manifold and let $J=(U,R)$ be a subcomplex with $\Theta(|J|)\subset M\setminus\partial M$.*

(1) *$N_K(J)$ is a compact PL-manifold and it is a submanifold of the topological manifold M.*
(2) (a) *$\mathring{N}_K(J)$ is an open subset of M.*
    (b) *The boundary of $N_K(J)$ as a subset of M equals $N_K(J)\setminus\mathring{N}_K(J)$.*
    (c) *$N_K(J)$ and $\mathring{N}_K(J)$ are both neighborhoods of $\Theta(|J|)$.*
(3) *$\Theta(|J|)$ is a deformation retract of $N_K(J)$.*
(4) *$\partial N_K(J)$ is a deformation retract of $N_K(J)\setminus\Theta(|J|)$.*

SKETCH OF PROOF. Evidently we can just assume that $M=|K|$ and that $\Theta=\mathrm{id}$. As mentioned above, our definition of the regular neighborhood is an example of a regular neighborhood as defined in [**RS72**, p. 33]. Thus we can use the results from [**RS72**].

(1) This statement is explicitly and implicitly proved in [**RS72**, Propostion 3.10].
(2) We leave the fairly routine verification of these statements to the reader.
(3) This statement is proved in [**RS72**, Corollary 3.30 (ii)], see also [**Gla70**, p. 77].
(4) We only provide an outline of a proof. First we define a sequence of abstract simplicial complexes via setting $K_0:=K$ and iteratively via setting $K_{n+1}:=\mathrm{sd}_J(K_n)$ for $n\in\mathbb{N}$. Similarly to Lemma 62.3 (3) we see that these abstract simplicial complexes define again naturally simplicial structures for M. Using a suitable variation

on Lemma 62.7 one can show that $\bigcap_{n\in\mathbb{N}_0} N_{K_n}(J) = |J|$. The key input at this point is [**RS72**, Corollary 3.18] which implies that for every $n \in \mathbb{N}_0$ there exists a homeomorphism $F_n\colon N_{K_n}(J)\times[0,1] \to N_{K_n}(J)\setminus \mathring{N}_{K_{n+1}}(J)$ such that for every $P \in N_{K_n}(J)$ we have $F_n(P,0) = P$ and $F_n(P,1) \in \partial N_{K_{n+1}}(J)$. Next we want to combine all these maps F_n to obtain a homeomorphism $\partial N_K(J)\times[0,1] \to N_K(J)\setminus|J|$. To do so we set $s_0 = 0$, $s_1 = \frac{1}{2}$, $s_2 = \frac{3}{4}$, $s_3 = \frac{7}{8}$ and so on. Furthermore, given $n \in \mathbb{N}_0$ we define $\Xi_n\colon \partial N_K(J) \to \partial N_{K_n}(J)$ via $\Xi_n(P) = F_{n-1}(\dots(F_1(F_0(P,1),1),\dots,1)$. Finally we consider the map

$$\begin{array}{rcl} \partial N_K(J)\times[0,1) & \to & N_K(J)\setminus|J| \\ (P,t) & \mapsto & F_n(\Xi_n(P), 2^{n+1}\cdot(t-s_n)), \quad \text{if } t\in[s_n,s_{n+1}) \text{ for some } n\in\mathbb{N}_0. \end{array}$$

Some thought shows that this map is actually a homeomorphism. The desired statement follows from the observation that $\partial N_K(J) = \partial N_K(J)\times\{0\}$ is a deformation retract of $N_K(J)\times[0,1)$. The more carefully one looks at this outline, the more little exercises one detects which need to be carried out to turn this argument into a proper proof. ■

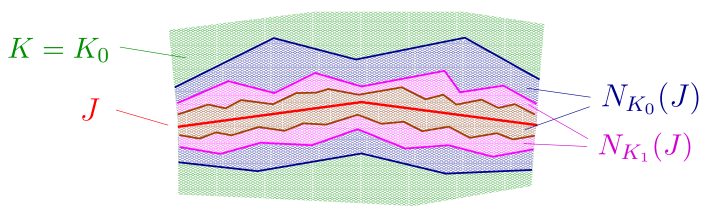


FIGURE 980. Illustration for the proof of Proposition 64.12 (4).

This concludes our discussion of regular neighborhoods. This concept is discussed in much greater detail in [**WhdJ39**], [**RS72**, Chapter 3], [**Gla70**, Chapter III.B], [**Hud69**, Chapter 2] and [**Ze63b**, Chapter 3]. Some properties are also discussed in [**Hem76**, p. 7f]. In particular we refer to [**RS72**, Theorems 3.8 and 3.24] and [**Gla70**, Theorem II.16n] for some uniqueness theorems which explain to what degree the regular neighborhood depends on the choice of simplicial structures on a given PL-manifold. Unfortunately it is not always immediately apparent that the various definitions in the above literature are mutually compatible.

64.4. **The interplay between topological, smooth and PL-manifolds.** Now that the PL-manifolds joined the playground, let us see how they interact with our old friends, the topological and smooth manifolds. Since morally speaking topological manifolds have the least amount of structure it is not surprising that PL-manifolds are indeed topological manifolds:

Proposition 64.13. *Let X be an n-*dimensional PL-manifold. *The following statements hold:*

(1) *X is an n-dimensional topological manifold.*
(2) *A vertex v of the* PL-*structure lies on the boundary of the topological manifold X if and only if the link* $\mathrm{Lk}(v)$ *is a* PL-*ball.*

Remark. Let X be a PL-manifold and let s be a simplex such that all vertices lie on the boundary ∂X of X, viewed as a topological manifold. As we see in Figure 981, this piece of information does not imply that s is actually contained in ∂X.

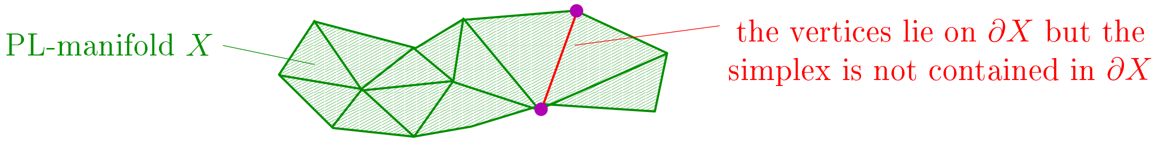


FIGURE 981

PROOF. Let $(K=(V,S),\Theta\colon |K|\to X)$ be an n-*dimensional* PL-*manifold*. Without loss of generality we can assume that $\Theta=\mathrm{id}$, in particular we can assume that $|K|=X$.

(1) In the following we will show that X is an n-dimensional topological manifold.
  (a) By Lemma 61.5 we know that X is Hausdorff.
  (b) By definition of a PL-manifold we know that S is countable, in particular V is countable. Furthermore note that Proposition 64.11 says that for each $v\in V$ the open star $\overset{\circ}{\mathrm{St}}(v)$ is homeomorphic to a subset of $\mathbb{R}^n$. From Lemma 6.1 (1) we obtain that each open star is second-countable. Thus it follows from Lemma 62.5 (2) that X is the countable union of countably many second-countable open subsets. Hence we obtain from Lemma 6.1 (4) that X itself is second-countable.
  (c) By definition of a PL-manifold we know that for each $P\in X$ there exists in particular a neighborhood U of P and a homeomorphism $\Phi\colon U\to \overline{B}^n$. On page 263 we showed that $\overline{B}^n$ itself is an n-dimensional topological manifold. It it now elementary to see that there exists a chart, in the sense of the definition on page 261, for X at P.

  We have thus verified that X is indeed an n-dimensional topological manifold.

(2) Let $v\in V$ be a vertex. We calculate that

$$\begin{aligned}\mathrm{H}_n(X,X\setminus\{v\}) &\underset{\substack{\uparrow\\ \text{by the Excision Theorem 43.20 which}\\ \text{we can apply by Lemma 62.5 (1d)}}}{\cong} \mathrm{H}_n(\mathrm{St}(v),\mathrm{St}(v)\setminus\{v\}) \underset{\substack{\uparrow\\ \text{Lemma 62.5 (1f) and Corollary 43.18 (2a)}}}{\cong} \mathrm{H}_n(\mathrm{St}(v),\mathrm{Lk}(v))\\ &\underset{\substack{\uparrow\\ \text{by the long exact sequence in reduced homology of the pair } (\mathrm{St}(v),\mathrm{Lk}(v))\\ \text{together with Lemma 62.5 (1g)}}}{\cong} \widetilde{\mathrm{H}}_{n-1}(\mathrm{Lk}(v)) \cong \begin{cases}\mathbb{Z}, & \text{if } \mathrm{Lk}(v) \text{ is a PL } (n-1)\text{-sphere},\\ 0, & \text{if } \mathrm{Lk}(v) \text{ is a PL } (n-1)\text{-ball.}\end{cases}\end{aligned}$$

  It follows from Lemma 44.1 that $v\in\partial X$ if and only if $\mathrm{Lk}(v)$ is a PL $(n-1)$-ball. ■

Next we turn to the relationship between smooth manifolds and PL-manifolds. The following theorem says that any smooth manifold admits an essentially natural PL-structure.

Theorem 64.14.

(1) *Every smooth simplicial structure on an n-dimensional smooth manifold is an n-dimensional* PL-*structure.*

(2) *If M is a closed smooth manifold, then for every smooth simplicial structure the corresponding* PL-*manifold is also closed.*
(3) *Every smooth manifold admits the structure of a* PL-*manifold.*
(4) *Any two smooth simplicial structures on a smooth manifold are* PL-*homeomorphic.*

PROOF.

(1) It follows from [**WhdJ40**, Theorem 5] (see also the argument of [**Mun66a**, p. 82]) together with Proposition 64.11 that every smooth simplicial structure on a smooth manifold is a PL-structure in the above sense.
(2) This statement follows immediately from Proposition 64.13, together with fact, proved in Proposition 44.2 (2), that a smooth manifold is closed as a smooth manifold if and only if it is closed as a topological manifold.
(3) This statement follows from (1) together with Theorem 64.2 where we showed that every smooth manifold admits a countable *smooth* simplicial structure.
(4) This statement follows immediately from Theorem 64.4. ■

Theorem 64.14 says in particular that given any $n \in \mathbb{N}_0$ the map

$$\mathfrak{w}_n \colon \left\{ \begin{array}{c} \text{diffeomorphism classes of closed} \\ n\text{-dimensional smooth manifolds} \end{array} \right\} \to \left\{ \begin{array}{c} \text{PL-homeomorphism classes of closed} \\ n\text{-dimensional PL-manifolds.} \end{array} \right\}$$
$$[M] \mapsto [(M, \text{smooth simplicial structure})]$$

is well-defined. It is natural to ask whether $\mathfrak{w}_n$ is a bijection. It turns out that the answer is complicated. More precisely, we have the following theorem.

Theorem 64.15.

(1) *For $n = 1, \ldots, 6$ the map $\mathfrak{w}_n$ is a bijection.*
(2) *For $n = 7$ (and many other dimensions) the map $\mathfrak{w}_n$ is not a monomorphism.*
(3) *For $n = 10$ (and many other dimensions) the map $\mathfrak{w}_n$ is not an epimorphism. In fact there exists a closed 10-dimensional* PL-*manifold that does not admit a single smooth structure.*

PROOF. The theorem is a combination of several major results in topology. Later in Section **??** we will discuss this theorem in much greater detail and we will give all the necessary references. ■

We turn our focus back to the relationship between topological and PL-manifolds. In light of Proposition 64.13 it is perhaps tempting to think that a PL-manifold is the same as a topological manifold together with some simplicial structure. But it turns out that the situation is more delicate.

Theorem 64.16. *There exists a simplicial structure on S^5 that is not a* PL-*structure.*

The key to proving Theorem 64.16 is the following very subtle and very difficult theorem which was proved independently by Jim Cannon [**Cann79**] and Richard Edwards [**Edw80, Edw06**] in the late 1970s. We also refer to [**Dav86**, Corollary VII.2.C] and [**Lat79**] for alternative accounts of the proof.

Theorem 64.17. (Double Suspension Theorem) *Let Y be an n-dimensional* PL-*manifold. If Y is a homology sphere, i.e. if $\mathrm{H}_k(Y;\mathbb{Z}) \cong \mathrm{H}_k(S^n;\mathbb{Z})$ for all k, then the* double *suspension $\Sigma(\Sigma(Y))$ is homeomorphic to S^{n+2}.*

The proof of Theorem 64.16 also requires the following lemma.

Lemma 64.18. *Let $n \in \mathbb{N}_{\geq 2}$ and let M be a closed 0-connected n-dimensional topological manifold. If $\pi_1(M)$ is non-trivial, then the suspension $\Sigma(M)$ is not an $(n+1)$-dimensional topological manifold.*

Proof. We outsourced the proof to the very amusing Exercise 45.9. ■

With these preparations it is now quite easy to provide a proof of Theorem 64.16.

Proof of Theorem 64.16.

(1) By Proposition 66.11 we know that there exists a closed orientable 3-dimensional smooth manifold Y that is a homology 3-sphere, but such that $\pi_1(Y)$ is a non-trivial group.
(2) By Theorem 64.14 we can equip Y with a PL-structure.
(3) It follows from (2) and the Double Suspension Theorem 64.17 that $\Sigma(\Sigma(Y))$ is homeomorphic to S^5.
(4) By (2) we know that Y is equipped with a simplicial structure. We equip the double suspension $\Sigma(\Sigma(Y))$ with the simplicial structure given by "the double suspension" of the simplicial structure. More precisely, we equip $\Sigma(\Sigma(Y))$ with the simplicial structure given by applying Corollary 61.21 twice.
(5) Let N be the "North Pole" of the second suspension. It follows from the discussion on page 1523 that $\mathrm{Lk}(\Sigma(\Sigma(Y)), N) = \Sigma(Y)$.
(6) Since $\pi_1(Y)$ is non-trivial we obtain from Lemma 64.18 that $\Sigma(Y)$ is not even a 4-dimensional topological manifold. Therefore we obtain from Proposition 64.13 that $\mathrm{Lk}(\Sigma(\Sigma(Y)), N) = \Sigma(Y)$ is not a 4-dimensional PL-manifold, let alone a PL 4-sphere.
(7) We obtain from the above, together with Proposition 64.11, that $\Sigma(\Sigma(Y))$ admits a simplicial structure that is not a PL-structure.
(8) In (3) we saw that $\Sigma(\Sigma(Y))$ is homeomorphic to S^5. It follows from (7) that S^5 admits a simplicial structure that is not a PL-structure. ■

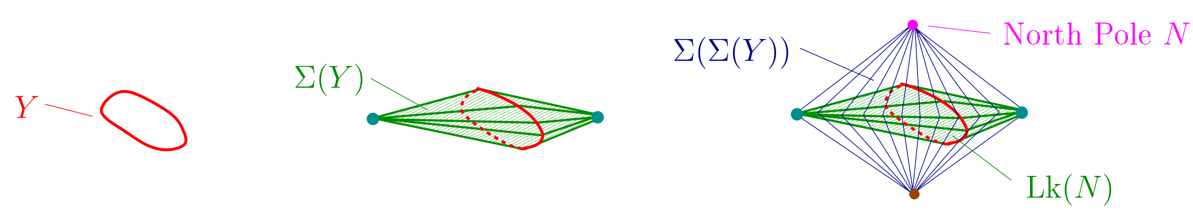


Figure 982. Illustration for the proof of Theorem 64.16.

Towards the end of these notes, in Section **??**, we will study the relationship between topological manifolds, smooth manifolds and PL-manifolds in greater detail.

64.5. **Stellar subdivisions and the Alexander-Newman Theorem.** The notion of a PL-homeomorphism, and thus also the notion of a PL-manifold, relies on the notion of a subdivision which, as discussed on page 1555, is not combinatorial. In the final two sections of this chapter we will discuss how one can turn the theory of PL-homeomorphisms and PL-manifolds into a purely combinatorial theory.

First, given an abstract simplicial complex we introduce, inspired by the definition on page 1522 and by Lemma 62.5 (1b), the combinatorial notion of the star and the link of a simplex.

Definition. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$ be a simplex. We introduce the following three subcomplexes[992] of K:

$$\begin{aligned}
\operatorname{St}(K,s) &:= \Big(\bigcup_{t\in S \text{ with } s\subset t} t,\ \{t\in S \,|\, s\cup t\in S\}\Big) \\
\operatorname{Lk}(K,s) &:= \Big(\bigcup_{t\in S \text{ with } s\subset t} t \setminus s, \{t\in S \,|\, s\cup t\in S \text{ and } s\cap t=\varnothing\}\Big) \\
K\setminus \operatorname{St}(K,\mathring{s}) &:= \begin{cases} (V\setminus\{v\}, \{t\in S \,|\, s\not\subset t\}), & \text{if } s=\{v\} \text{ is a 0-simplex,} \\ (V, \{t\in S \,|\, s\not\subset t\}), & \text{otherwise.}\end{cases}
\end{aligned}$$

We refer to Figure 983 for an illustration of the definition of $K \setminus \operatorname{St}(K, \mathring{s})$.

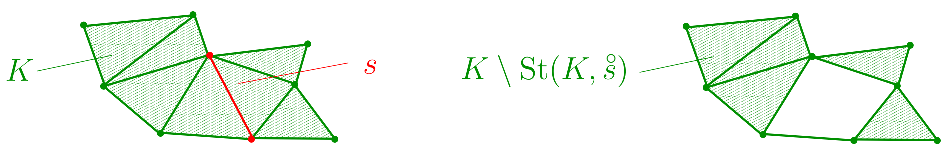


FIGURE 983

Remark. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$.

(1) The vigilant reader will have noticed that we defined $\operatorname{St}(K, s)$ and $\operatorname{Lk}(K, s)$ twice, namely on page 1522 as suitable subsets of $|K|$ and now as abstract subcomplexes. The vigilant reader surely will have no troubles in figuring out which of the two definitions we are working with at any given moment.
(2) The notation "$K \setminus \operatorname{St}(K, \mathring{s})$" is meant to be suggestive, but note that on its own $\mathring{s}$ makes no sense.

Next we recall the following definitions from page 1482 and page 1518.

Definition. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$ be a simplex.
(1) (a) If s is a 0-simplex, then we define $\partial s = (\varnothing, \varnothing)$.
  (b) If s is not a 0-simplex, then we define ∂s to be the abstract simplicial complex $(s, \mathcal{P}(s) \setminus \{\varnothing, s\})$.
(2) We denote by $\underline{s} := \sum\limits_{v\in s} \frac{1}{\dim(s)+1} \cdot v \in |K|$ the barycenter.[993]

The following lemma summarizes several basic relations between the objects we just introduced.

[992]One can easily verify that all three are indeed subcomplexes.

[993]The meticulous reader will notice that in this section, on several occasions, we will use Lemma 62.1.

Lemma 64.19. *Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$. The following statements hold:*

(1) $\mathrm{St}(K, s) = \mathrm{Lk}(K, s) * s$.
(2) *The union of the subcomplexes* $K \setminus \mathrm{St}(K, \mathring{s})$ *and* $\mathrm{St}(K, s) = \mathrm{Lk}(K, s) * s$ *equals* K *and the intersection equals* $\mathrm{Lk}(K, s) * \partial s$.
(3) *The intersection of the abstract simplicial complexes* $K \setminus \mathrm{St}(K, \mathring{s})$ *and* $\mathrm{Lk}(K, s) * \partial s * \underline{s}$ *also equals* $\mathrm{Lk}(K, s) * \partial s$.

Proof. We leave it to the reader to verify that all the statement follow indeed immediately from the definitions. ■

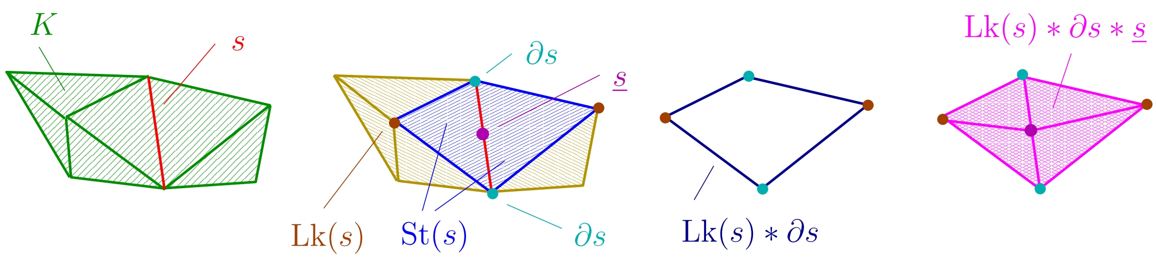


Figure 984. Illustration for Lemma 64.19.

Now we define the notion of a "stellar subdivision at a simplex s". The basic idea is to add a vertex in the interior of s and subdivide all the simplices that contain s accordingly. Slightly more precisely, the "stellar subdivision at a simplex s" is defined as replacing the subcomplex $\mathrm{St}(K, s) = \mathrm{Lk}(K, s) * s$ by the abstract simplicial complex $\mathrm{Lk}(K, s) * \partial s * \underline{s}$. Even more precisely, it is defined as follows.

Definition.

(1) Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$. We define the *stellar subdivision of K at s* to be the abstract simplicial complex[994]

$$\sigma_s(K) \;:=\; (K \setminus \mathrm{St}(K, \mathring{s})) \cup \mathrm{Lk}(K, s) * \partial s * \underline{s}.$$

(2) Let K be an abstract simplicial complex. If K' is a stellar subdivision of K, then we say that K is a *stellar weld* of K'.
(3) We say two abstract simplicial complexes K and L are *stellar equivalent* if there exists a finite sequence $K = K_0, K_1, \ldots, K_m = L$ of abstract simplicial complexes such that each K_{i+1} is simplicially isomorphic to either a stellar subdivision or a stellar weld of K_i.

We illustrate the definition of a stellar subdivision at a 1-simplex in Figure 985 and we illustrate the definition of a stellar subdivision at a 2-simplex in Figure 986.

Remark.

[994] It follows from Lemma 61.2 (1) that $\sigma_s(K)$ is indeed an abstract simplicial complex.

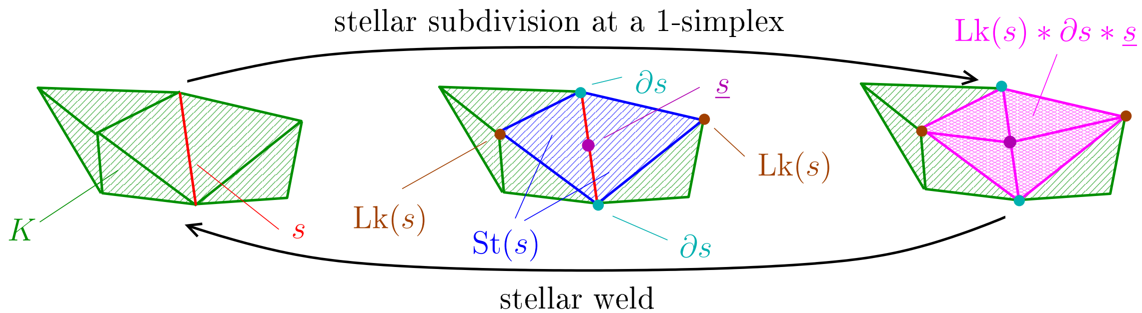


FIGURE 985

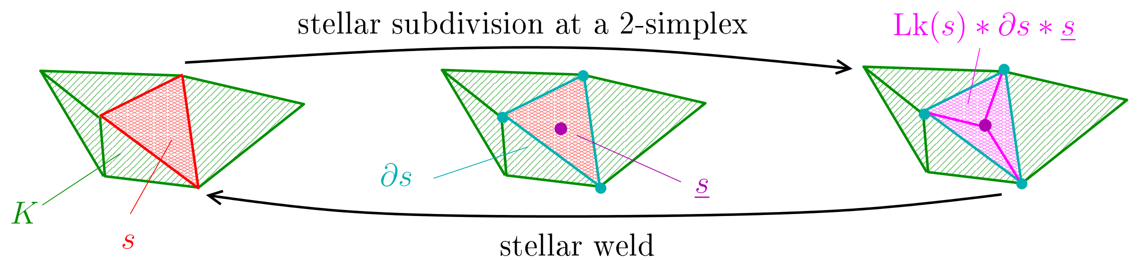


FIGURE 986

(1) We follow the naming conventions used in **[Lic99]** which differ slightly from the ones used in say **[Hud69, Ze63b, RS72]**. More precisely, what we call a *stellar subdivision* is called an *elementary stellar subdivision* in **[Hud69, Ze63b, RS72]**.
(2) It follows easily from the definitions and Lemma 62.1 (3) that the stellar subdivision at a 0-simplex is the identity.

Lemma 64.20. *Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$. If* $\mathrm{Lk}(K, s)$ *is finite, then there exists a natural* PL-*homeomorphism* $|K| \to |\sigma_s(K)|$.

Remark. The proof of Lemma 64.20 shows that $|\sigma_s(K)|$ is essentially a subdivision of $|K|$. This shows that the name "stellar *subdivision*" is justified.

PROOF (∗). Let $K = (V, S)$ be a finite abstract simplicial complex and let $s \in S$. Recall that on page 1598 we saw that there exists a natural PL-homeomorphism $\varphi\colon |s| \to |\partial s * \underline{s}|$ which is the identity on the common subset $|\partial s|$. It follows from Lemmas 61.20 and 64.9, together with our hypothesis that $\mathrm{Lk}(K, s)$ is finite, that the map $\mathrm{id}_{\mathrm{Lk}(K,s)}$ and φ give rise to a natural PL-homeomorphism $f\colon |\,\mathrm{Lk}(K, s) * s| \to |\,\mathrm{Lk}(K, s) * \partial s * \underline{s}|$ which has the following property:

(∗) The map f is the identity on the common subset $|\,\mathrm{Lk}(K, s) * \partial s|$.

Next we consider the two maps

$$\Phi\colon |K| \to |\sigma_s(K)| \qquad\qquad \Psi\colon |\sigma_s(K)| \to |K|$$

$$P \mapsto \begin{cases} P, & \text{if } P \in |K \setminus \mathrm{St}(K, \mathring{s})|, \\ f(P), & \text{if } P \in |\,\mathrm{St}(K, s)| \end{cases} \qquad P \mapsto \begin{cases} P, & \text{if } P \in |K \setminus \mathrm{St}(K, \mathring{s})|, \\ f^{-1}(P), & \text{if } P \in |\,\mathrm{Lk}(K, s) * \partial s * \underline{s}|. \end{cases}$$

We make the following observations:

(1) It follows from Lemma 64.19 (2) and (3) together with ($*$) that both maps are well-defined.
(2) By Lemma 61.8 together with Lemma 2.35 we know that both maps are continuous.
(3) Basically by construction the maps are inverses of one another.

In summary we have shown that both maps are homeomorphisms. It remains to show that $\Phi\colon |K| \to |\sigma_s(K)|$ is in fact a PL-homeomorphism. To do so, note that it follows easily from the definition of the map Φ that the simplicial complex $(\sigma_s(K), \Phi^{-1}\colon |\sigma_s(K)| \to |K|)$ is a subdivision of $|K|$. With respect to this subdivision the map Φ is evidently a simplicial isomorphism. Thus we see that Φ is in fact a PL-homeomorphism. ■

The following rather amazing theorem is one of the key bridges between topology and combinatorics.

Theorem 64.21. (Alexander-Newman Theorem) *Let K and L be two finite abstract simplicial complexes. The following two statements are equivalent:*

(1) *The abstract simplicial complexes K and L are stellar equivalent.*
(2) *The simplicial complexes $|K|$ and $|L|$ are* PL*-homeomorphic.*

Proof. The "(1)⇒(2)"-direction is just the combination of Lemma 64.20 and Proposition 64.8. The much harder "(2)⇒(1)"-direction was first proved by James Alexander [**Al30**, Theorem 15:1] in 1930, building on work of Max Newman [**New26**]. More modern expositions of the proof are given in [**Gla70**, Theorem II.7] and in [**Lic99**, Theorem 4.5]. ■

According to [**Lic99**, p. 311] and [**Mel18**, p. 1] the following question, posed in particular in [**Hud69**, p. 14] and [**Pac91**, Problem 3.3], is still open. An affirmative answer for 2-dimensional abstract simplicial complexes was given by Jürgen Ewald [**Ewa86**] in 1986.

Question 64.22. *Let K and L be two finite abstract simplicial complexes such that the simplicial complexes $|K|$ and $|L|$ are* PL*-homeomorphic. Do there exist iterated stellar subdivisions of K and L which are simplicially isomorphic?*

64.6. **Combinatorial manifolds and Pachner's Theorem.** The Alexander-Newman Theorem 64.21 and Propositions 64.11 and 64.13 (2) motivate the following definitions.

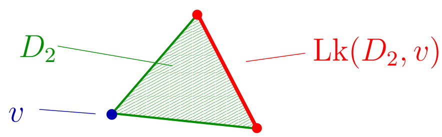
Definition.

(1) We say that two finite abstract simplicial complexes K and L are *combinatorially homeomorphic* if K and L are stellar equivalent.
(2) Let $n \in \mathbb{N}_0$. We say that a finite abstract simplicial complex is a *combinatorial n-ball* if it is combinatorially homeomorphic to D_n and we say it is a *combinatorial n-sphere* if it is combinatorially homeomorphic to S_n.[995]
(3) Let K be a finite abstract simplicial complex. We say K is an *n-dimensional combinatorial manifold* if for every vertex v the link $\operatorname{Lk}(K, v)$ is a combinatorial $(n-1)$-sphere or a combinatorial $(n-1)$-ball.
(4) We say an n-dimensional combinatorial manifold K is *closed* if for every vertex v the link $\operatorname{Lk}(K, v)$ is a combinatorial $(n-1)$-sphere.

.

Examples. Let $n \in \mathbb{N}$. We consider the abstract simplicial complexes D_n and S_n.

(1) For every vertex v of D_n we have, basically by definition, that $\mathrm{Lk}(D_n, v)$ is simplicially isomorphic to D_{n-1}. Thus we see that D_n is an n-dimensional combinatorial manifold.
(2) Similarly, for every vertex v of S_n we have, again basically by definition, that $\mathrm{Lk}(S_n, v)$ is simplicially isomorphic to S_{n-1}. Thus we obtain that S_n is a closed n-dimensional combinatorial manifold.



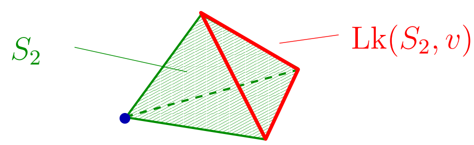


FIGURE 987

Remark.

(1) From what I can see, there is no combinatorial analogue of the notion that a map between two finite abstract simplicial complexes is piecewise linear.
(2) In the literature, see e.g. [**Hud69**, p. 26] and [**Gla70**, p. 18], a *combinatorial manifold* is often defined as a PL-manifold. Even though the concept of a "wrong definition" does not exist, it seems to me that it is more appropriate to define a combinatorial manifold as above, namely to define it immediately as a purely combinatorial object.

The statement of the following proposition, which can be viewed as an analogue of Proposition 64.10, does not come as a surprise.

Proposition 64.23. *Let K and L be two finite abstract simplicial complexes. If K is a (closed) n-dimensional combinatorial manifold and if L is combinatorially homeomorphic to K, then L is a (closed) n-dimensional combinatorial manifold.*

Example. It follows from Proposition 64.23 and the previous examples that every combinatorial n-ball is indeed an n-dimensional combinatorial manifold and that every combinatorial n-sphere is indeed a closed n-dimensional combinatorial manifold.

PROOF. A short moment's thought shows that this statement does indeed need a proof, it does not follow immediately from the definitions. In fact the proposition is a consequence of [**Gla70**, Lemma on p. 19]. ■

By definition the link of any 0-simplex in a closed combinatorial manifold is a combinatorial sphere. The following lemma, which is an analogue of Proposition 64.11 (2)⇒(4), generalizes this statement to higher dimensional simplices.

Lemma 64.24. *Let K be a closed n-dimensional combinatorial manifold. If s is a k-simplex, then the link* $\mathrm{Lk}(K, s)$ *is a combinatorial* $(n-k-1)$*-sphere.*

[995] We refer to page 1480 for the definition of the abstract simplicial complexes S_n and D_n

PROOF. The proof of this lemma is almost the same as the proof of Proposition 64.11 (2)⇒(4). The only slight difference is that now we need to use the fact, mentioned above, that every combinatorial k-sphere is a closed k-dimensional combinatorial manifold. ■

By definition two combinatorially homeomorphic combinatorial manifolds are related by a sequence of stellar subdivisions and stellar welds. In the following we will see that they are also related by the more elegant "bistellar moves".

To introduce the notion of a bistellar move we need the following basic definition.

Definition. Let $K = (V, S)$ be an abstract simplicial complex and let $s \in S$ be a simplex.

(1) Let $m \in \mathbb{N}_0$. We say that $\operatorname{Lk}(K, s) =: (U, R)$ is a *non-trivial m-sphere in K* if the following two conditions are satisfied:
   (a) we have $\#U = m + 2$ and $R = \mathcal{P}(U) \setminus \{\varnothing, U\}$,[996]
   (b) the vertex set U is not a simplex of K.

   In this case we denote by $B(K, s)$ the abstract simplicial complex $(U, \mathcal{P}(U) \setminus \{\varnothing\})$. We illustrate the definition in Figure 988.

(2) We say that $\operatorname{Lk}(K, s)$ is a *non-trivial (-1)-sphere in K* if it is the empty subcomplex. In this case we define $B(K, s)$ to be abstract simplicial complex consisting of the single vertex $\underline{s}$.

As usual, if K is understood then we write $B(s)$ instead of $B(K, s)$.

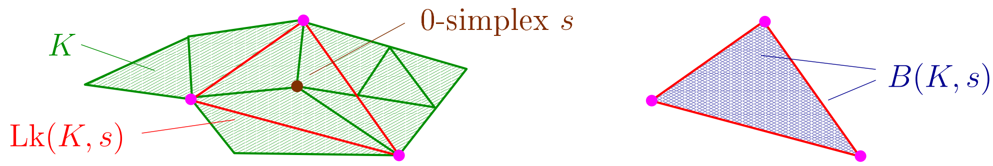


FIGURE 988

We move on to the next lemma which can be viewed as the fourth part of Lemma 64.19.

Lemma 64.25. *Let $K = (V, S)$ be a closed n-dimensional combinatorial manifold and let $s \in S$ be a k-simplex. We suppose that the link $\operatorname{Lk}(K, s)$ is a non-trivial $(n - k - 1)$-sphere in K. Then the intersection of the abstract simplicial complexes $K \setminus \operatorname{St}(K, \mathring{s})$ and $\partial s * B(K, s)$ equals $\operatorname{Lk}(K, s) * \partial s$.*

PROOF. Again we leave the task of verifying the lemma to the reader. ■

Suppose we are in the setting of Lemma 64.25. Loosely speaking the *bistellar move along s* is defined as the result of replacing the subcomplex

$$\operatorname{St}(K, s) \;=\; \underset{\substack{\uparrow \\ \cong D_k}}{s} * \underset{\substack{\uparrow \\ \cong S_{n-k-1}}}{\operatorname{Lk}(K, s)} \quad \text{by the abstract simplicial complex} \quad \underset{\substack{\uparrow \\ \cong S_{k-1}}}{\partial s} * \underset{\substack{\uparrow \\ \cong D_{n-k}}}{B(K, s)}.$$

More precisely, we have the following definition.

[996] This just means that $\operatorname{Lk}(K, s)$ is simplicially isomorphic to the abstract simplicial complex S_m. It is worth recalling that Lemma 64.24 says that the link of a k-simplex in a closed n-dimensional combinatorial manifold is combinatorially homeomorphic to S_{n-k-1}, which is a much weaker condition than demanding that the link is simplicially isomorphic to S_{n-k-1}.

Definition. Let $K = (V, S)$ be a closed n-dimensional combinatorial manifold and let $s \in S$ be a k-simplex such that $\operatorname{Lk}(K, s)$ is a non-trivial $(n-k-1)$-sphere in K. We define

$$\tau_s(K) \ := \ (K \setminus \operatorname{St}(K, \mathring{s})) \cup \partial s * B(K, s).$$

We refer to the unique $(n-k)$-simplex in $B(K, s)$ as the *dual simplex*. We say $\tau_s(K)$ is obtained from K by a *bistellar move along* s.

Examples.

(1) In Figure 989 we show a bistellar move along a 1-simplex s in a 2-dimensional combinatorial manifold.

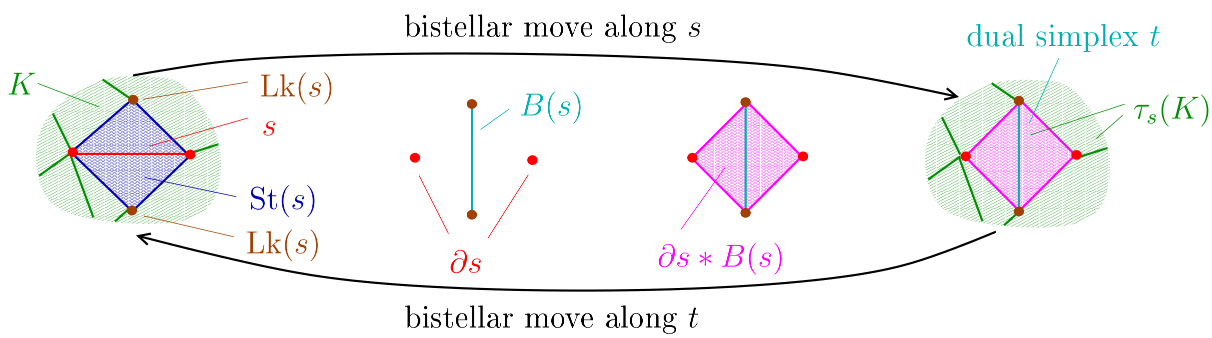


Figure 989

(2) In Figure 990 we show a bistellar move along a 2-simplex s in a 3-dimensional combinatorial manifold.

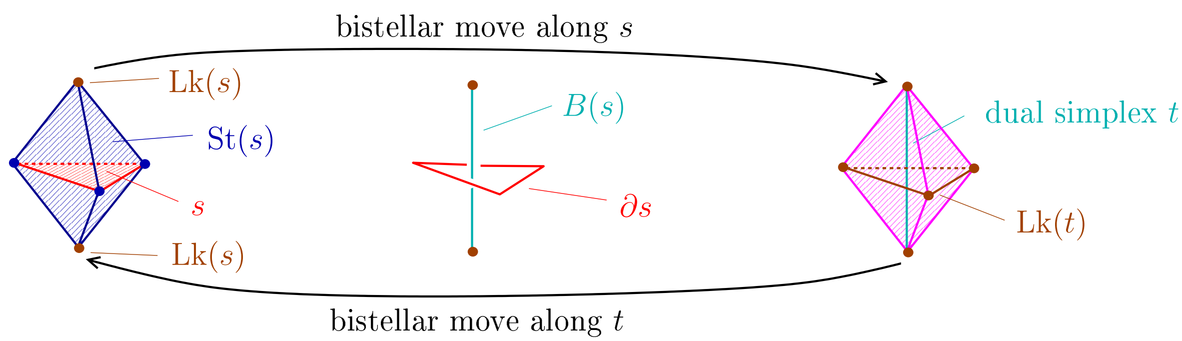


Figure 990

(3) In Figure 991 we show a bistellar move along a 2-simplex s in a 2-dimensional combinatorial manifold. In this case $\operatorname{Lk}(s) = \varnothing$, which by the above convention, is a non-trivial (-1)-sphere.

Note that in each example we see that the bistellar move along the dual simplex actually gets us back to the original abstract simplicial complex. In the next lemma we will see that this is not a coincidence.

The following lemma summarizes a few key properties of bistellar moves.

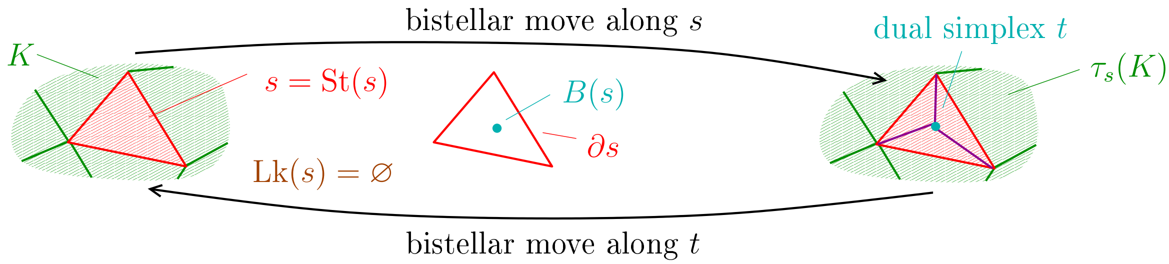


FIGURE 991

Lemma 64.26. *Let $K = (V, S)$ be a closed n-dimensional combinatorial manifold and let $s \in S$ be a k-simplex such that $\mathrm{Lk}(K, s)$ is a non-trivial $(n-k-1)$-sphere in K. We denote by t the dual simplex. The following statements hold:*

(1) *The dual simplex t is an $(n-k)$-simplex in $\tau_s(K)$ with the following properties:*
 (a) $\partial t = \mathrm{Lk}(K, s)$,
 (b) $\mathrm{Lk}(\tau_s(K), t) = \partial s$,
 (c) (i) *if $s = \{v\}$ is a 0-simplex, then $B(\tau_s(K), t) = (\{\underline{t}\}, \{\underline{t}\})$,*
 (ii) *if s is not a 0-simplex, then $B(\tau_s(K), t) = s$,*[997]
 (d) $\tau_s(K) \setminus \mathrm{St}(\tau_s(K), \mathring{t}) = K \setminus \mathrm{St}(K, \mathring{s})$.
 Furthermore ∂s is a non-trivial $(k-1)$-sphere in $\tau_s(K)$.

(2) *$\sigma_s(K)$ is naturally simplicially isomorphic to $\sigma_t(\tau_s(K))$.*

(3) *$\tau_s(K)$ is a closed n-dimensional combinatorial manifold which is naturally combinatorially homeomorphic to K.*

(4) *$\tau_t(\tau_s(K))$*[998] *is naturally simplicially isomorphic to K.*

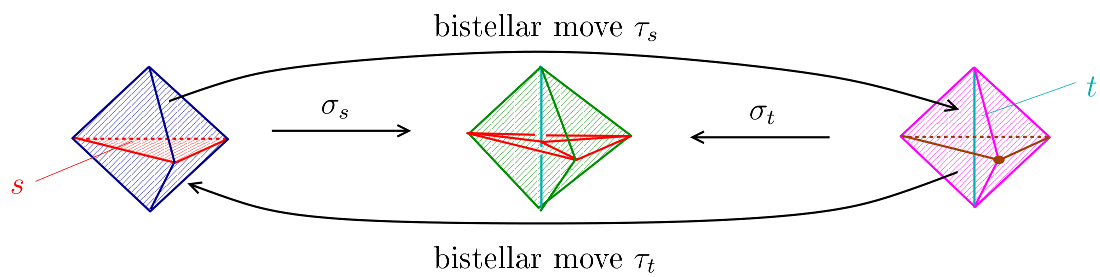


FIGURE 992. Illustration for Lemma 64.26.

Remark. Lemma 64.26 shows that in particular that if two closed n-dimensional combinatorial manifold are bistellar equivalent, then they are also stellar equivalent. Furthermore Lemma 64.26 gives us also the justification for the name “bistellar move”.

PROOF.

(1) We leave the elementary verification of these statements to the reader. Note that the slightly awkward fact that we need to deal with 0-simplices separately is due to

[997] Here we view the simplex s as an abstract simplicial complex in its own right.

[998] It follows from (1) that we can form $\tau_t(\tau_s(K))$.

the fact that already in the definitions on page 1613 and page 1608 we were forced to treat 0-simplices slightly differently.

(2) This statement follows easily from (1). Indeed, we see that

$$\sigma_t(\tau_s(K)) \underset{\substack{\uparrow \\ \text{by definition}}}{=} \overbrace{(\tau_s(K) \setminus \mathrm{St}(\tau_s(K), \mathring{t}))}^{= K \setminus \mathrm{St}(K,\mathring{s}) \text{ by (1)}} \cup \overbrace{\mathrm{Lk}(\tau_s(K), t) * \partial t * \underline{t}}^{=\partial s * \mathrm{Lk}(K,s) * \underline{t} \text{ by (1)}} \underset{\substack{\uparrow \\ \text{by Lemma 64.19 the map } \underline{s} \mapsto \underline{t} \text{ defines a} \\ \text{simplicial isomorphism}}}{\xrightarrow{\cong}} \sigma_s(K).$$

(3) It follows from (2) that $\tau_s(K)$ is naturally stellar equivalent to K. In other words, $\tau_s(K)$ is naturally combinatorially homeomorphic to K. It follows from Proposition 64.23 that $\tau_s(K)$ is also a closed n-dimensional combinatorial manifold.

(4) First assume that s is not a 0-simplex. In this case we see that

$$\tau_t(\tau_s(K)) \underset{\substack{\uparrow \\ \text{by definition}}}{=} \overbrace{(\tau_s(K) \setminus \mathrm{St}(\tau_s(K), \mathring{t}))}^{= K \setminus \mathrm{St}(K,\mathring{s}) \text{ by (1)}} \cup \overbrace{\partial t * B(\tau_s(K), t)}^{=\mathrm{Lk}(K,s) * s \text{ by (1)}} \underset{\substack{\uparrow \\ \text{Lemma 64.19}}}{=} K.$$

Finally we deal with the case that $s = \{v\}$ is a 0-simplex. Note that in this case the vertex set of $\tau_t(\tau_s(K))$ equals $(V \setminus \{v\}) \cup \{\underline{t}\}$. The obvious bijection between the vertex sets of $\tau_t(\tau_s(K))$ and of K, i.e. the map given by $\underline{t} \mapsto v$, induces, by the same argument as above, a simplicial isomorphism $\tau_t(\tau_s(K)) \to K$. ■

Note that Lemma 64.26 (4) implies in particular that "being related by a sequence of bistellar moves" is an equivalence relation. This observation motivates the following definition.

Definition. We say that two closed n-dimensional combinatorial manifold are *bistellar equivalent* if one can be obtained from the other via a finite sequence of bistellar moves and simplicial isomorphisms.

The following theorem can be viewed as a refinement of the Alexander-Newman Theorem 64.21 for closed combinatorial manifolds. The theorem was proved in 1991 by Udo Pachner [**Pac91**].

Theorem 64.27. (Pachner's Theorem) *Let K and L be two closed combinatorial manifolds. The following two statements are equivalent:*

(1) *The abstract simplicial complexes K and L are bistellar equivalent.*
(2) *The abstract simplicial complexes K and L are combinatorially homeomorphic (i.e. they are stellar equivalent).*

Remark.

(1) Pachner's Theorem 64.27 is the reason why bistellar moves are often called *Pachner moves.*
(2) An analogue of Pachner's Theorem 64.27 for combinatorial manifolds that are not closed is given in [**Pac91**, Theorem 6.3] and [**Lic97b**, Theorem 5.10].

(3) The Alexander-Newman Theorem 64.21 and Pachner's Theorem 64.27 can be viewed as an analogue of Theorem 27.21 which says that any two knot diagrams for a given knot are related by a finite sequence of Reidemeister moves.

PROOF. The "(1)⇒(2)"-direction is an immediate consequence of Lemma 64.26. The much more interesting "(2)⇒(1)"-direction is proved in [**Pac91**, Theorem 5.5]. An exposition of the proof is also given in [**Lic99**, Theorem 5.9]. For combinatorial manifolds of dimension ≤ 3 a more elementary proof is given in [**BW96**, Theorem 4.11] and implicitly in [**TV92**]. Furthermore for combinatorial manifolds of dimension ≤ 4 a proof is also given in [**Pac78**]. ■

Let us quickly compare the statements and roles of the Alexander-Newman Theorem 64.21 and Pachner's Theorem 64.27.

(1) The Alexander-Newman Theorem 64.21 is more general in so far as it applies to any two finite simplicial complexes whereas Pachner's Theorem 64.27 only deals with closed combinatorial manifolds.
(2) Now we restrict ourselves to the class of closed n-dimensional combinatorial manifolds.
   (a) A stellar subdivision consists in replacing $\mathrm{Lk}(K,s) * s$ by $\mathrm{Lk}(K,s) * \partial s * \underline{s}$. As should become clear from considering Figure 993, for $n \geq 3$ there are infinitely many simplicial isomorphism types of links of simplices that can appear. Thus we see that even for closed n-dimensional combinatorial manifolds there are infinitely many[999] "combinatorially distinct" stellar subdivisions.
   (b) On the other hand a Pachner move can only be applied if the link $\mathrm{Lk}(K,s)$ is in fact simplicially isomorphic to some S_k, $k = 0,\ldots,n$. Recall that if a Pachner move is possible, then it replaces $\mathrm{Lk}(K,s) * s$ by $\partial s * B(K,s)$. Thus we see that there are only $n+1$ "combinatorially distinct" moves.

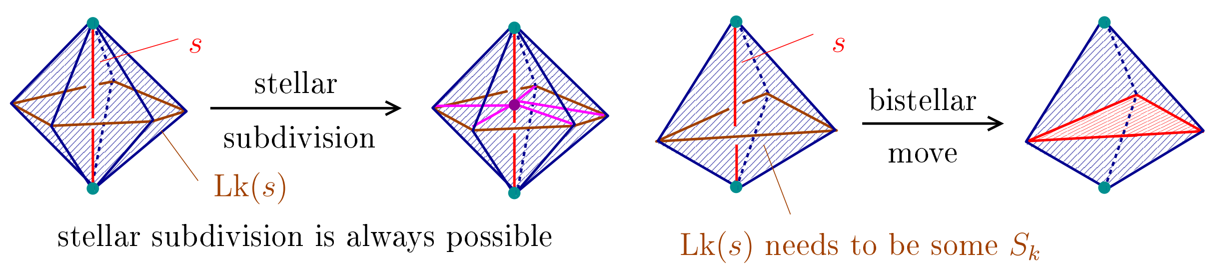


FIGURE 993

[999]We leave it to the reader to fill the expression "combinatorially distinct" with meaning.

Now, at the end the end of this chapter we have obtained the following sequence of maps:

$$
\begin{array}{c}
\{\text{diffeomorphism classes of closed } n\text{-dimensional smooth manifolds}\} \\
\mathfrak{w}_n \Big\downarrow \text{ by Theorem 64.15 a bijection for } n \le 6 \\
\{\text{PL-homeomorphism classes of closed } n\text{-dimensional PL-manifolds}\} \\
\begin{matrix} |K| \\ \updownarrow \\ K \end{matrix} \Big\uparrow \text{ a bijection by the Alexander-Newman Theorem 64.21} \\
\left\{ \begin{matrix} \text{combinatorial homeomorphism classes of} \\ \text{closed } n\text{-dimensional combinatorial manifolds} \end{matrix} \right\} \\
\Big\uparrow \text{ the identity by Pachner's Theorem 64.27} \\
\{\text{bistellar equivalence classes of closed } n\text{-dimensional combinatorial manifolds}\}
\end{array}
$$

This shows in particular that for dimensions ≤ 6 the classification of closed smooth manifolds boils down to (very difficult!) combinatorics.

In particular the sequence of maps gives us an approach to defining invariants of closed n-dimensional PL-manifolds (or equivalently smooth manifolds for $n \le 6$): we "just" need to define an invariant for closed n-dimensional combinatorial manifolds that is invariant under the $n+1$ bistellar moves. This approach to defining invariants has been used by Vladimir Turaev and Oleg Viro [**TV92**] and John Barrett and Bruce Westbury [**BW96**] in the 3-dimensional setting and by Christopher Douglas and David Reutter [**DR18**] in the 4-dimensional setting.

Finally we mention that the significance of Pachner's Theorem 64.27 in mathematics and theoretical physics ("loop quantum gravity theory") is nicely explained in [**Lic97b, Lic99**] and [**Kle18**].

Exercises for Chapter 64.

Exercise 64.1. We equip the smooth manifold $\overline{B}^2$ with the canonical simplicial structure $\Phi\colon \Delta^2 \to \overline{B}^2$ introduced on page 1500. Is this a smooth simplicial structure for the smooth manifold $\overline{B}^2$?

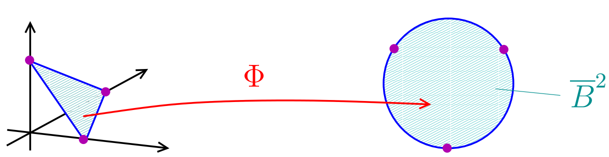


FIGURE 994. Illustration of Exercise 64.1.

Exercise 64.2. Let $n \in \mathbb{N}_0$. Show that $\operatorname{Cone}(\partial\Delta^n)$ is PL-homeomorphic to Δ^n. We refer to Figure 974 for an illustration.

Exercise 64.3. Let X be a simplicial complex and let Y be a subdivision. Show that every k-simplex of X is the union of finitely many simplices of Y of dimension $\le k$.

Exercise 64.4. We consider the triangle Δ_{ABC} shown in Figure 995. Let $P \in \overline{AC}$ and $Q \in \overline{BC}$ be two points such that the segment $\overline{PQ}$ is not parallel to the segment $\overline{AB}$. Let p be the radial projection from C of the segment $\overline{PQ}$ to the base $\overline{AB}$ of the triangle. We equip $\overline{PQ}$ and $\overline{AB}$ with the obvious simplicial structures. Show that the map p is not piecewise linear.
Remark. The fact that the radial projection map p is not piecewise linear is called the "standard mistake" in [**Ze63b**, p. I.6] and [**RS72**, p. 6].

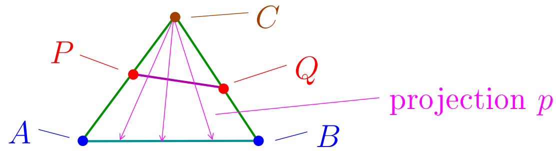


FIGURE 995. Illustration of Exercise 64.4.

Exercise 64.5. We consider the two abstract simplicial complexes K and L shown in Figure 996.

(a) Show that L is not a stellar subdivision of K.
(b) Show "by hand" that K and L are stellar equivalent.

FIGURE 996. Illustration for Exercise 64.5.

Exercise 64.6. Use the Alexander-Newman Theorem 64.21 to prove Pachner's Theorem 64.27 in the 2-dimensional case.

Exercise 64.7. Let K be an abstract simplicial complex. Show that if K is a closed n-dimensional combinatorial manifold, then the maximal dimension of a simplex of K is precisely n.
Hint. You could do an induction on n.

65. The Lefschetz Fixed Point Theorem

In this chapter we will state and prove the Lefschetz Fixed Point Theorem which gives a criterion for a self-map to have a fixed point. This theorem can be viewed as a far reaching generalization of the Brouwer Fixed Point Theorem 43.8.

65.1. **The trace of an endomorphism.** In the formulation of the Lefschetz Fixed Point Theorem we need the notion of the trace of suitable endomorphisms. Even though this notion is surely well-known to the reader we recall the definition and a few properties of the trace of an endomorphism. This short section will facilitate the discussion below. Nonetheless, the author will not hold it against the reader if the reader decides to skip this section.

Definition. Let R be a ring. Given an $(n\times n)$-matrix $A=(a_{ij})$ over R we define its *trace* to be

$$\operatorname{tr}(A) \;:=\; \sum_{i=1}^{n} a_{ii}.$$

Lemma 65.1. *Let R be a ring and let A and B be two $(n\times n)$-matrices over R. If R is commutative, then we have*

$$\operatorname{tr}(A\cdot B) \;=\; \operatorname{tr}(B\cdot A).$$

Proof. We perform the following simple calculation:

$$\operatorname{tr}(A\cdot B) = \operatorname{tr}\Big(\text{matrix with } (i,j)\text{-entry } \sum_{k=1}^{n} a_{ik}\cdot b_{kj}\Big) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}\cdot b_{ki} \underset{\substack{\uparrow\\ \text{since } R \text{ is commutative}}}{=} \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ki}\cdot a_{ik} \overset{\substack{\text{same calculation backwards}\\ \downarrow}}{=} \operatorname{tr}(B\cdot A).$$ ■

Lemma 65.2. *Let R be a commutative domain, let V be a free R-module of rank n and let $\varphi\colon V\to V$ be an endomorphism. We pick a basis for V. Then*

$$\operatorname{tr}(\varphi) \;:=\; \textit{trace of the matrix representing } \varphi \textit{ with the respect to the given basis}$$

does not depend on the choice of the basis. We refer to $\operatorname{tr}(\varphi)$ as the trace of the endomorphism.

Notation. Given a commutative domain R, a free R-module V of finite rank and an endomorphism $\varphi\colon V\to V$, depending on the amount of space we have, we use one of the following three notations:

$$\operatorname{tr}(\varphi) \;=\; \operatorname{tr}(\varphi\colon V\to V) \;=\; \operatorname{tr}(\varphi \circlearrowleft V).$$

Example. Let R be a commutative domain and let V be a free R-module of finite rank. It follows immediately from the definitions that $\operatorname{tr}(\mathrm{id}_V)=\operatorname{rank}(V)$.

Proof. Let R be a commutative domain. Let $\{v_1,\dots,v_m\}$ and $\{w_1,\dots,w_n\}$ be two bases for V. Since R is a commutative domain the rank of V is well-defined, see page 82. Thus we see that $m=n$. Now let $P=(p_{ij})$ be the base change matrix, i.e. $P=(p_{ij})$ is the matrix that is uniquely determined by the condition that given any $i\in\{1,\dots,n\}$ we have

$v_i = \sum\limits_{j=1}^{n} p_{ji} \cdot w_j$. The "opposite" base change matrix is evidently an inverse to P, in particular we see that P is invertible. We denote by A and B the matrices representing φ with respect to the two bases. We see that

$$\operatorname{tr}(B) \underset{\substack{\uparrow \\ \text{an elementary calculation} \\ \text{shows that } B \cdot P = P \cdot A}}{=} \operatorname{tr}(P \cdot A \cdot P^{-1}) \underset{\substack{\uparrow \\ \text{by Lemma 65.1 since } R \text{ is commutative and since all} \\ \text{matrices are square matrices of the same size}}}{=} \operatorname{tr}(P^{-1} \cdot P \cdot A) = \operatorname{tr}(A).$$

■

In many of the applications below we will consider the case that R is a field. But on several occasions the case $R = \mathbb{Z}$ is also of interest.

Lemma 65.3. *Let $\varphi\colon V \to V$ be an endomorphism of a finitely generated free abelian group and let R be a commutative domain. Let $\gamma\colon \mathbb{Z} \to R$ be the natural ring homomorphism with $\gamma(1_{\mathbb{Z}}) = 1_R$ from page 490. Then*

$$\gamma\big(\operatorname{tr}\big(\underbrace{\varphi\colon V \to V}_{\substack{\text{homomorphism of} \\ \text{free } \mathbb{Z}\text{-modules}}}\big)\big) = \operatorname{tr}(\underbrace{\varphi \otimes \mathrm{id}_R\colon V \otimes R \to V \otimes R}_{\substack{\text{homomorphism of free } R\text{-modules} \\ \text{by Lemma 57.5}}}) \in R.$$

Proof. Let $\{v_1, \dots, v_n\}$ be a basis for the free abelian group V and let A be the $(n \times n)$-matrix that represents φ with respect to this basis. It is straightforward to verify that $v_1 \otimes 1, \dots, v_n \otimes 1$ is a basis for the R-module $V \otimes R$ and that $\gamma(A)$ is the matrix that represents the endomorphism $\varphi \otimes \mathrm{id}$. Since the trace commutes with ring homomorphisms we obtain the desired equality $\gamma(\operatorname{tr}(A)) = \operatorname{tr}(\gamma(A))$. ■

Sometimes it is also convenient to consider the trace of an endomorphism of a finitely generated abelian group. This leads us to the following definition.

Definition.

(1) Given a finitely generated abelian group H we refer to
$$\mathrm{F}H := H/\operatorname{Tor}(H) \quad \text{where } \operatorname{Tor}(H) \text{ denotes the torsion subgroup of } H$$
as the *maximal torsion-free quotient* of H.

(2) Given a homomorphism $\alpha\colon A \to B$ between two finitely generated abelian groups we denote the obvious induced map $\mathrm{F}A \to \mathrm{F}B$ by α as well.

Lemma 65.4. *Given a finitely generated abelian group H the maximal torsion-free quotient $\mathrm{F}H = H/\operatorname{Tor}(H)$ of H is a free abelian group.*

Proof. A very elementary exercise shows that $\mathrm{F}H = H/\operatorname{Tor}(H)$ is a torsion-free abelian group. Since H is finitely generated we obtain from Theorem 19.4 that $\mathrm{F}H$ is in fact a free abelian group. ■

Remark.

(1) It follows easily from the definition that the map $H \to \mathrm{F}H$ defines a functor from the category of finitely generated abelian groups to the category of free abelian groups.

(2) It might sound silly to mention Theorem 19.4 explicitly in the proof of Lemma 65.4, but it is worth noting that for commutative rings R which are not PIDs there is usually no meaningful functor from the category of finitely generated R-modules to the category of free R-modules.

(3) The hypothesis that H is finitely generated is of course necessary, for example the maximal torsion-free quotient of the abelian group $(\mathbb{Q},+)$ is again $(\mathbb{Q},+)$, which is certainly not a free abelian group.

Definition. Given an endomorphism $\varphi\colon H\to H$ of a finitely generated abelian group we define

$$\operatorname{tr}(\varphi) \;=\; \operatorname{tr}\left(\begin{array}{ccc} \mathrm{F}H & \to & \mathrm{F}H \\ {[h]} & \mapsto & [\varphi(h)] \end{array}\right).$$

We conclude this elementary section with the following basic lemma which is a variation on Lemma 65.3.

Lemma 65.5. *Let H be a finitely generated abelian group and let $\varphi\colon H\to H$ be an endomorphism. Then the following equality holds:*

$$\operatorname{tr}(\varphi\colon H\to H) \;=\; \operatorname{tr}(\varphi\otimes\mathrm{id}\colon H\otimes\mathbb{Q}\to H\otimes\mathbb{Q}) \;\in\; \mathbb{Q}.$$

Proof. By the classification of finitely generated abelian groups, see Theorem 19.4, we can write $H=F\oplus T$ where F is a free abelian group and T is a torsion group. We denote by ψ the map $F\xrightarrow{f\mapsto f\oplus 0} F\oplus T\xrightarrow{\varphi} F\oplus T\xrightarrow{f\oplus t\mapsto f} F$. Then we see that

$$\operatorname{tr}(\varphi\colon H\to H) \underset{\substack{\uparrow\\ \text{follows easily from the definitions}}}{=} \operatorname{tr}(\psi\colon F\to F) \underset{\substack{\uparrow\\ \text{by Lemma 65.3}}}{=} \operatorname{tr}(\psi\otimes\mathrm{id}\colon F\otimes\mathbb{Q}\to F\otimes\mathbb{Q}) \underset{\substack{\uparrow\\ \text{by Lemma 57.3}}}{=} \operatorname{tr}(\varphi\otimes\mathrm{id}\colon H\otimes\mathbb{Q}\to H\otimes\mathbb{Q}).$$

■

65.2. **The Lefschetz number.** In this section we return to topology.

Definition. Let $\mathbb{F}$ be a field.

(1) Let X be a topological space. We say the homology of X is $\mathbb{F}$-*finite* if all homology groups $\mathrm{H}_i(X;\mathbb{F})$ with $\mathbb{F}$-coefficients are finite-dimensional vector spaces and if all but finitely many homology groups $\mathrm{H}_i(X;\mathbb{F})$ are zero.

(2) Let X be a topological space such that the homology is $\mathbb{F}$-finite. Let $\varphi\colon X\to X$ be a map. We define the $\mathbb{F}$-*Lefschetz number of* φ to be[1000]

$$\Lambda(\varphi,\mathbb{F}) \;:=\; \sum_{n\in\mathbb{N}_0}(-1)^n\cdot\operatorname{tr}(\varphi_*\colon\ \mathrm{H}_n(X;\mathbb{F})\to\mathrm{H}_n(X;\mathbb{F})).$$

Remark. In Exercise 65.1 we will see that the $\mathbb{F}$-Lefschetz number depends only rather mildly on the choice of the field $\mathbb{F}$.

In general it can be tricky to determine (the trace of) an induced map on homology. The following lemma gives us some basic (partial) calculations of Lefschetz numbers.

[1000]Recall that on page 1400 we pointed out that the induced map $\varphi_*\colon\ \mathrm{H}_n(X;\mathbb{F})\to\mathrm{H}_n(X;\mathbb{F})$ is an endomorphism of an $\mathbb{F}$-vector space. Since the vector space $\mathrm{H}_n(X;\mathbb{F})$ is by hypothesis finite-dimensional we see that the trace is in fact defined.

Lemma 65.6.

(1) *If X is a finite* CW*-complex, then for any field $\mathbb{F}$ we have $\Lambda(\mathrm{id}_X, \mathbb{F}) = \chi(X)$.*

(2) *Let X be a 0-connected topological space and let $\varphi\colon X \to X$ be a map. For every field $\mathbb{F}$ we have $\operatorname{tr}(\varphi_*\colon \mathrm{H}_0(X;\mathbb{F}) \to \mathrm{H}_0(X;\mathbb{F})) = 1$.*

(3) *Let $\mathbb{F}$ be a field and let X be a 0-connected topological space with $\mathbb{F}$-finite homology. If $\varphi\colon X \to X$ is a constant map, then $\Lambda(\varphi, \mathbb{F}) = 1$.*

PROOF (∗).

(1) We calculate that

$$\Lambda(\mathrm{id}_X, \mathbb{F}) \underset{\substack{\uparrow \\ \text{by definition}}}{=} \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \operatorname{tr}(\mathrm{id}_* \circlearrowleft \mathrm{H}_n(X;\mathbb{F})) \underset{\substack{\uparrow \\ \text{see example on page 1620}}}{=} \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \dim_{\mathbb{F}}(\mathrm{H}_n(X;\mathbb{F})) \underset{\substack{\uparrow \\ \text{see page 1402}}}{=} \chi(X).$$

(2) This statement follows immediately from the obvious generalizations of Proposition 41.5 and the discussion on page 1088 to homology with $\mathbb{F}$-coefficients.

(3) This statement follows immediately from (2). ■

The following lemma is an immediate and quite useful consequence of Proposition 42.5 and the discussion on page 1401.

Lemma 65.7. *Let $\mathbb{F}$ be a field and let X be a topological space such that the homology is $\mathbb{F}$-finite. Let $\varphi, \psi\colon X \to X$ be two maps. If φ and ψ are homotopic, then the Lefschetz numbers agree, i.e. $\Lambda(\varphi, \mathbb{F}) = \Lambda(\psi, \mathbb{F})$.*

At times it is more efficient to work with the "usual" homology groups, i.e. the homology groups with $\mathbb{Z}$-coefficients. This leads to the following unexciting definition.

Definition.

(1) Let X be a topological space. We say the homology of X is *$\mathbb{Z}$-finite* if all homology groups $\mathrm{H}_i(X)$ are finitely generated abelian groups and if all but finitely many homology groups $\mathrm{H}_i(X)$ are zero.

(2) Let X be a topological space such that the homology is $\mathbb{Z}$-finite. Let $\varphi\colon X \to X$ be a map. We define the *Lefschetz number of* φ to be

$$\Lambda(\varphi) \;:=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \operatorname{tr}(\varphi_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(X)).$$

The following lemma relates the above Lefschetz number $\Lambda(\varphi)$ to the Lefschetz number $\Lambda(\varphi, \mathbb{Q})$.

Lemma 65.8. *Let X be a topological space and let $\varphi\colon X \to X$ be a map.*

(1) *Let $k \in \mathbb{N}_0$. We have $\operatorname{rank}(\mathrm{H}_k(X)) = \dim(\mathrm{H}_k(X;\mathbb{Q}))$. Furthermore, if $\mathrm{H}_k(X)$ is finitely generated, then we have*

$$\operatorname{tr}(\varphi_*\colon \mathrm{H}_k(X) \to \mathrm{H}_k(X)) \;=\; \operatorname{tr}(\varphi_*\colon \mathrm{H}_k(X;\mathbb{Q}) \to \mathrm{H}_k(X;\mathbb{Q})).$$

(2) *If the homology of X is $\mathbb{Z}$-finite, then the homology of X is also $\mathbb{Q}$-finite and we have the equality $\Lambda(\varphi) = \Lambda(\varphi, \mathbb{Q})$.*

Proof. Statement (2) is an immediate consequence of Statement (1). Thus let us prove the first statement. Let $k \in \mathbb{N}_0$. The equality $\operatorname{rank}(\mathrm{H}_k(X)) = \dim(\mathrm{H}_k(X;\mathbb{Q}))$ was shown in Corollary 57.21 (2). Now assume that $\mathrm{H}_k(X)$ is finitely generated. We have the following equalities:

$$\operatorname{tr}\left(\mathrm{H}_k(X) \xrightarrow{\varphi_*} \mathrm{H}_k(X)\right) \underset{\substack{\uparrow \\ \text{by Lemma 65.5}}}{=} \operatorname{tr}\left(\mathrm{H}_k(X)\otimes\mathbb{Q} \xrightarrow{\varphi_*\otimes\mathrm{id}} \mathrm{H}_k(X)\otimes\mathbb{Q}\right) \underset{\substack{\uparrow \\ \text{by the naturality of the short exact sequence of the} \\ \text{Universal Coefficient Theorem 57.19 and the observation that} \\ \text{by Lemma 57.17 (4) the Tor-groups with } \mathbb{Q}\text{-coefficients vanish}}}{=} \operatorname{tr}\left(\mathrm{H}_k(X;\mathbb{Q}) \xrightarrow{\varphi_*} \mathrm{H}_k(X;\mathbb{Q})\right).$$

■

65.3. **The formulation of the Lefschetz Fixed Point Theorems.** Now we can finally formulate the long promised Lefschetz Fixed Point Theorem. This theorem, with somewhat different language, was first proved in 1926 by Solomon Lefschetz [**Lef26**, p. 48].[1001]

Theorem 65.9. (Lefschetz Fixed Point Theorem I) *Let X be a finite simplicial complex and let $\varphi\colon X \to X$ be a map. If there exists a field $\mathbb{F}$ such that the $\mathbb{F}$-Lefschetz number $\Lambda(\varphi,\mathbb{F})$ is non-zero, then φ has a fixed point.*

We will prove the Lefschetz Fixed Point Theorem 65.9 in the next section. First we will see that one of our earlier great successes, namely Theorem 45.13, is just a special case of the Lefschetz Fixed Point Theorem 65.9.

Theorem 45.13. *Let $n \in \mathbb{N}$ and let $\varphi\colon S^n \to S^n$ be a map. If $\deg(\varphi) \neq (-1)^{n+1}$, then φ has a fixed point.*

Proof. We consider the topological space S^n. By the discussion on page 1500 we know that S^n admits the structure of a finite simplicial complex. Let $\varphi\colon S^n \to S^n$ be a map. We calculate that

$$\Lambda(\varphi,\mathbb{Q}) \overset{\substack{\text{by Lemmas 65.8 we have } \Lambda(\varphi,\mathbb{Q})=\Lambda(\varphi) \\ \downarrow}}{=} \underbrace{\operatorname{tr}\left(\mathrm{H}_0(S^n) \xrightarrow{\varphi_*} \mathrm{H}_0(S^n)\right)}_{=1 \text{ by Lemma 65.6 (2)}} + (-1)^n \cdot \underbrace{\operatorname{tr}\left(\mathrm{H}_n(S^n) \xrightarrow{\varphi_*} \mathrm{H}_n(S^n)\right)}_{=\deg(\varphi) \text{ by definition of the degree}} = 1 + (-1)^n \cdot \deg(\varphi).$$

The Lefschetz Fixed Point Theorem 65.9 thus says that if $1 + (-1)^n \cdot \deg(\varphi) \neq 0$, then φ has a fixed point. But that is exactly what we wanted to show. ■

For many applications the following variation on the Lefschetz Fixed Point Theorem 65.9 is particularly useful.

Theorem 65.10. (Lefschetz Fixed Point Theorem II) *Let M be a compact smooth manifold and let $\varphi\colon M \to M$ be a map. If there exists a field $\mathbb{F}$ such that the $\mathbb{F}$-Lefschetz number $\Lambda(\varphi,\mathbb{F})$ is non-zero, then φ has a fixed point.*

Remark.

(1) Later on in Theorem 85.11 we will prove the Lefschetz Fixed Point Theorem also for finite CW-complexes and compact topological manifolds. In fact not only is the statement of Theorem 85.11 more general than the statement of Theorem 65.10, the

[1001] Solomon Lefschetz (1884-1972) was an American mathematician.

proof of Theorem 85.11 will also be self-contained, since we will not make use of Theorem 64.2 for which we provided only a sketch of a proof.
(2) Reading (1) one might be tempted to think that the conclusion of the Lefschetz Fixed Point Theorem holds for every compact topological space. But in Exercise 65.5 we will see that this is certainly *not* the case.
(3) We will prove a generalization of the Lefschetz Fixed Point Theorem 65.10 in the Lefschetz-Hopf Theorem 96.5.

Proof. Let M be a compact smooth manifold. By Theorem 64.2 we know that M admits a finite simplicial structure. Thus we can apply the Lefschetz Fixed Point Theorem 65.9 and we are done. ■

Corollary 65.11. *Let M be a compact connected smooth manifold such that $\mathrm{H}_n(M)$ is torsion for all $n \geq 1$. Then every map $\varphi\colon M \to M$ has a fixed point.*

Proof. Let M be a compact connected smooth manifold such that $\mathrm{H}_n(M)$ is torsion for all $n \geq 1$. We calculate that

$$\Lambda(\varphi, \mathbb{Q}) \overset{\text{by Lemma 65.8}}{=} \underbrace{\operatorname{tr}(\varphi_*\colon \mathrm{H}_0(M) \to \mathrm{H}_0(M))}_{=1 \text{ by Lemma 65.6 (2)}} + \sum_{n\in\mathbb{N}} (-1)^n \cdot \underbrace{\operatorname{tr}(\varphi_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M))}_{=0, \text{ since } \mathrm{H}_n(M) \text{ is torsion}} = 1.$$

It follows from the Lefschetz Fixed Point Theorem 65.10 that φ has a fixed point. ■

Examples.
(1) If we apply Corollary 65.11 to the compact connected smooth manifold $M = \overline{B}^n$, then we recover the Brouwer Fixed Point Theorem 43.8.
(2) In Proposition 48.10 we computed the homology groups of the real projective spaces $\mathbb{R}\mathrm{P}^n$. These calculations show in particular that the hypotheses of Corollary 65.11 are satisfied for *even*-dimensional real projective spaces $\mathbb{R}\mathrm{P}^{2n}$. Thus we see that every self-map $f\colon \mathbb{R}\mathrm{P}^{2n} \to \mathbb{R}\mathrm{P}^{2n}$ has a fixed point. (This gives an alternative solution of Exercise 45.12.) We will consider the case of *odd*-dimensional real projective spaces in Exercise 65.3.

In the next section we will provide a proof of the Lefschetz Fixed Point Theorem 65.9 and afterwards we will consider two more elaborate applications of the Lefschetz Fixed Point Theorem 65.9. There are many other pretty consequences of the Lefschetz Fixed Point Theorem 65.9 which we cannot mention in these modest notes. In fact the book by Brown [**BrownR71**] is completely dedicated to the proof of the Lefschetz Fixed Point Theorem 65.9 and its numerous applications and generalizations.

65.4. **Proof of the Lefschetz Fixed Point Theorem 65.9.** One of the keys to proving the Lefschetz Fixed Point Theorem 65.9 is the following result in linear algebra which is a not-so-distant relative to Lemma 55.3.

Theorem 65.12. (Hopf Trace Formula)

(1) *Let $\mathbb{F}$ be a field. If*

$$C_* := 0 \to \mathrm{C}_k \xrightarrow{\partial_k} \mathrm{C}_{k-1} \xrightarrow{\partial_{k-1}} \dots \xrightarrow{\partial_2} \mathrm{C}_1 \xrightarrow{\partial_1} \mathrm{C}_0 \to 0$$

is a chain complex of finite-dimensional $\mathbb{F}$-vector spaces, then given any chain map $\varphi_*\colon C_* \to \mathrm{C}_*$ we have

$$\sum_{i=0}^{k}(-1)^i \cdot \operatorname{tr}(\varphi_i\colon\ \mathrm{C}_i \to \mathrm{C}_i) \ = \ \sum_{i=0}^{k}(-1)^i \cdot \operatorname{tr}(\varphi_*\colon\ \mathrm{H}_i(C_*) \to \mathrm{H}_i(C_*)).$$

(2) *The analogue of (1) also holds if we start out with a chain complex of finitely generated free abelian groups.*

The key to proving the Hopf Trace Formula 65.12 is the following lemma.

Lemma 65.13. *Let $\mathbb{F}$ be a field. Suppose we are given a commutative diagram of homomorphisms of finite-dimensional $\mathbb{F}$-vector spaces of the following type:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0. \end{array}$$

If the horizontal sequences are exact, then

$$\operatorname{tr}(\beta\colon B \to B) \ = \ \operatorname{tr}(\alpha\colon A \to A) + \operatorname{tr}(\gamma\colon C \to C).$$

Proof. Let $a_1, \dots, a_m$ be a basis for A and let $c_1, \dots, c_n$ be a basis for C. For $i = 1, \dots, n$ we pick a $\overline{c}_i \in B$ with $p(\overline{c}_i) = c_i$. It follows easily from our hypothesis that the horizontal sequences are exact that $i(a_1), \dots, i(a_m), \overline{c}_1, \dots, \overline{c}_n$ form a basis for B. Furthermore it is elementary to show, using the hypothesis that the diagram commutes, that the matrix of β with respect to this basis is of the following form:

$$\begin{pmatrix} \text{matrix that} & * \quad * \\ \text{represents } \alpha & * \quad * \\ 0 \quad 0 & \text{matrix that} \\ 0 \quad 0 & \text{represents } \gamma \end{pmatrix} \qquad \text{where each “}*\text{” indicates some element in } \mathbb{F}.$$

It is now clear that $\operatorname{tr}(\beta) = \operatorname{tr}(\alpha) + \operatorname{tr}(\gamma)$. ■

Now that Lemma 65.13 is in our toolbox the proof of the Hopf Trace Formula 65.12 is almost embarrassingly similar to the proof of Lemma 55.3.

Proof of Hopf Trace Formula 65.12. First we note that Statement (2) follows almost immediately from Statement (1) applied to $\mathbb{F} = \mathbb{Q}$, together with Lemma 65.5 and the Universal Coefficient Theorem 57.18. We leave it to the reader to fill in the details.

Thus it remains to prove Statement (1). For the given chain complex $(\mathrm{C}_*, \partial_*)$ over some field $\mathbb{F}$ we write for each $n \in \mathbb{N}_0$

$$Z_n := \ker(\partial_n), \quad B_n := \operatorname{im}(\partial_{n+1}) \quad \text{and} \quad H_n = Z_n/B_n.$$

We write $\varphi_{B_n} := \varphi_n|_{B_n}$ and $\varphi_{Z_n} := \varphi_n|_{Z_n}$. For symmetry reasons we also write $\varphi_{\mathrm{C}_n} := \varphi_n$ and $\varphi_{H_n} := \varphi_*\colon H_n \to H_n$. These $\mathbb{F}$-vector spaces and homomorphisms form the following two types of commutative diagrams of short exact sequences

(a)
$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "\varphi_{Z_n}"] & \mathrm{C}_n \arrow[r, "\partial_n"] \arrow[d, "\varphi_{\mathrm{C}_n}"] & B_{n-1} \arrow[r] \arrow[d, "\varphi_{B_{n-1}}"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & \mathrm{C}_n \arrow[r, "\partial_n"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

and

(b)
$$\begin{tikzcd}
0 \arrow[r] & B_n \arrow[r] \arrow[d, "\varphi_{B_n}"] & Z_n \arrow[r] \arrow[d, "\varphi_{Z_n}"] & H_n \arrow[r] \arrow[d, "\varphi_{H_n}"] & 0 \\
0 \arrow[r] & B_n \arrow[r] & Z_n \arrow[r, "\partial_n"] & H_n \arrow[r] & 0.
\end{tikzcd}$$

Now we see that

$$\begin{aligned}
\sum_{n=0}^{k}(-1)^n\cdot\operatorname{tr}(\varphi_{\mathrm{C}_n}) &\underset{\text{by (a) and Lemma 65.13 we know that } \operatorname{tr}(\varphi_{\mathrm{C}_n})=\operatorname{tr}(\varphi_{Z_n})+\operatorname{tr}(\varphi_{B_{n-1}})}{=} \sum_{n=0}^{k}(-1)^n\cdot\left(\operatorname{tr}(\varphi_{Z_n})+\operatorname{tr}(\varphi_{B_{n-1}})\right)\\
&\underset{\text{by (b) and Lemma 65.13 we know that } \operatorname{tr}(\varphi_{Z_n})=\operatorname{tr}(\varphi_{B_n})+\operatorname{tr}(\varphi_{H_n})}{=} \sum_{n=0}^{k}(-1)^n\cdot\left(\operatorname{tr}(\varphi_{B_n})+\operatorname{tr}(\varphi_{H_n})+\operatorname{tr}(\varphi_{B_{n-1}})\right)\\
&= \sum_{n=0}^{k}(-1)^n\cdot\operatorname{tr}(\varphi_{H_n}) + \sum_{n=0}^{k}(-1)^n\cdot\operatorname{tr}(\varphi_{B_n}) + \sum_{n=0}^{k}(-1)^n\operatorname{tr}(\varphi_{B_{n-1}})\\
&\underset{\text{we do the substitution } m=n-1 \text{ and we use that } B_k=0 \text{ and } B_{-1}=0}{=} \sum_{n=0}^{k}(-1)^n\cdot\operatorname{tr}(\varphi_{H_n}) + \sum_{n=0}^{k-1}(-1)^n\cdot\operatorname{tr}(\varphi_{B_n}) - \sum_{m=0}^{k-1}(-1)^m\cdot\operatorname{tr}(\varphi_{B_m})\\
&= \sum_{n=0}^{k}(-1)^n\cdot\operatorname{tr}(\varphi_{H_n}).
\end{aligned}$$

■

Now we are fully prepared for tackling the proof of the Lefschetz Fixed Point Theorem 65.

Proof of the Lefschetz Fixed Point Theorem 65.9. Let X be a finite simplicial complex and let $\varphi\colon X\to X$ be a map *without* a fixed point. Furthermore let $\mathbb{F}$ be a field. We need to show that

$$\Lambda(\varphi,\mathbb{F}) \;:=\; \sum_{n\in\mathbb{N}_0}(-1)^n\cdot\operatorname{tr}(\varphi_*\colon\ \mathrm{H}_n(X;\mathbb{F})\to\mathrm{H}_n(X;\mathbb{F})) \;=\; 0.$$

The basic idea for the proof is quite simple:

> We need to upgrade the fact that $\varphi(x)\neq x$ for every $x\in X$ to the statement that for a suitable chain complex C_* that calculates $\mathrm{H}_*(X)$ we also have $\varphi(\sigma)\neq\sigma$ for every element σ of a suitable basis. This implies that the trace on the chain group level is zero, hence we get the desired statement from the Hopf Trace Formula 65.12 (1).

For better or worse turning the idea into a proper proof requires a substantial effort. First of all we note that since X is a finite simplicial complex we can use Lemma 61.9 to view X as a subspace of some $\mathbb{R}^n$. In particular we can and will equip X with the usual Euclidean metric $d(x,y):=\|x-y\|_{\mathbb{R}^n}$.

Claim 1. There exists an iterated barycentric subdivision Y of X such that for every vertex w of Y we have $\varphi(\mathrm{St}(Y,w)) \cap \mathrm{St}(Y,w) = \emptyset$.

The claim is proved in several easy steps:

(1) Since X is a finite simplicial complex we know by Lemma 61.9 that X is compact.
(2) From the compactness of X together with our hypothesis that for every $x \in X$ we have $\varphi(x) \neq x$ we obtain through some elementary arguments, as provided in Exercise 3.11 (b), that there exists an $\epsilon > 0$ such that for every $x \in X$ we have $d(\varphi(x), x) \geq \epsilon$
(3) By Lemma 62.7 there exists a $j \in \mathbb{N}_0$ such that for every vertex v of the iterated barycentric subdivision $\mathrm{sd}^j(X)$ we have $\mathrm{diam}(\mathrm{St}(\mathrm{sd}^j(X), v)) < \frac{\epsilon}{2}$.
(4) It follows from (2) and (3) that $\varphi(\mathrm{St}(\mathrm{sd}^j(X), w)) \cap \mathrm{St}(\mathrm{sd}^j(X), w) = \emptyset$. Indeed, let $x, y \in \mathrm{St}(\mathrm{sd}^j(X), w)$. We calculate that

$$d(\varphi(x), y) \underset{\substack{\uparrow \\ \text{triangle inequality}}}{\geq} d(\varphi(x), x) - d(x,y) \underset{\substack{\uparrow \\ \text{by (2) and (3)}}}{\geq} \epsilon - \tfrac{\epsilon}{2} \geq \tfrac{\epsilon}{2} > 0.$$

This implies that $\varphi(\mathrm{St}(\mathrm{sd}^j(X), w)) \cap \mathrm{St}(\mathrm{sd}^j(X), w) = \emptyset$.

Thus the iterated barycentric subdivision $Y := \mathrm{sd}^j(K)$ has the desired property. ⊞

Next we apply the Simplicial Approximation Theorem 62.8 to the map $\varphi\colon Y \to Y$. We obtain an $i \in \mathbb{N}_{\geq 0}$ and a simplicial map $\psi\colon Z := \mathrm{sd}^i(Y) \to Y$ with the following properties:

(A) The map ψ is homotopic to φ as a map $Y \to Y$.
(B) For every vertex w of $Z = \mathrm{sd}^i(Y)$ we have $\varphi(\mathrm{St}(\underbrace{\mathrm{sd}^i(Y)}_{=Z}, w)) \subset \mathrm{St}(Y, \psi(w))$.

It follows from (A) and Lemma 65.7 that the Lefschetz numbers of φ and ψ agree.

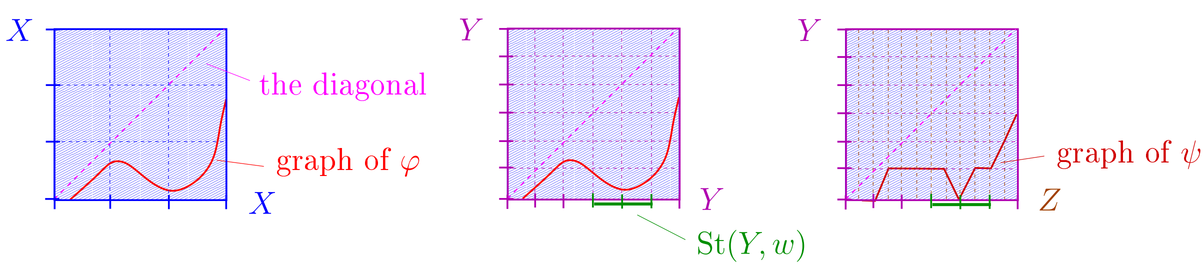


FIGURE 997

The map $\psi\colon Z \to Y$ is a simplicial map between two different simplicial complexes. The following claim gets as close as possible to saying that ψ does not fix a simplex.

Claim 2. For every simplex s of Y and every simplex t of Z with $t \subset s$ we have $\psi(t) \neq s$.

Let s be a simplex of Y and let t be a simplex of Z with $t \subset s$. Let v be a vertex of t. (It might be helpful to consider Figure 998.) We make the following deductions:

v a vertex of the simplex t of Z $\underset{\uparrow}{\Longrightarrow}$ $t \subset \mathrm{St}(Z, v) \implies \varphi(t) \subset \varphi(\mathrm{St}(Z, v))$ $\underset{\uparrow}{\Longrightarrow}$ $\varphi(t) \subset \mathrm{St}(Y, \psi(v))$

by the definition of $\mathrm{St}(Z, v)$, see page 1522 — by (B)

$\underset{\uparrow}{\Longrightarrow}$ $\varphi(t) \cap \varphi(\mathrm{St}(Y, \psi(v))) = \varnothing$ $\underset{\uparrow}{\Longrightarrow}$ $t \cap \mathrm{St}(Y, \psi(v)) = \varnothing$

by Claim 1 — Lemma 1.3 (4)

$\underset{\uparrow}{\Longrightarrow}$ $t \cap \mathrm{St}(Y, \psi(t)) = \varnothing$ $\underset{\uparrow}{\Longrightarrow}$ $\psi(t) \neq s$.

since $\psi(v)$ is a vertex of $\psi(t)$ we have $\mathrm{St}(Y, \psi(t)) \subset \mathrm{St}(Y, \psi(v))$ — we have $t \subset \mathrm{St}(Y, s)$, indeed this follows from $t \subset s$ and from $s \subset \mathrm{St}(Y, s)$, which in turn holds by definition of $\mathrm{St}(Y, s)$

⊞

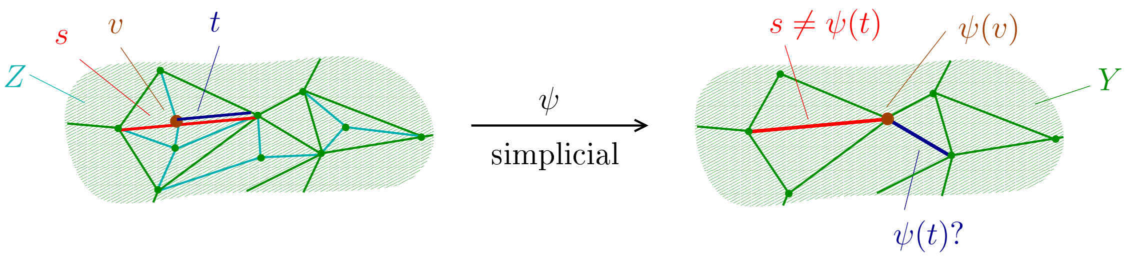


FIGURE 998. Illustration for the proof of Claim 2.

Next, given $n \in \mathbb{N}_0$ let

$$u_n \colon \mathrm{C}_n^{\mathrm{simp}}(Y) \to \mathrm{C}_n^{\mathrm{simp}}(\underbrace{\mathrm{sd}^i(Y)}_{=Z})$$

be the i-fold iterated simplicial subdivision map that we introduced in Lemma 63.41.

Claim 3. For each $n \in \mathbb{N}_0$ the map $u_n \circ \psi_* \colon \mathrm{C}_n^{\mathrm{simp}}(Z) \to \mathrm{C}_n^{\mathrm{simp}}(Z)$ has trace zero.

Let T_n be the set of n-simplices of Z. It follows from Lemma 63.5 that T_n, with some choice of ordering for each $t \in T_n$, is a basis of $\mathrm{C}_n^{\mathrm{simp}}(Z)$. Now let $t \in T_n$. If $\psi(t)$ is not an n-simplex of Y, then by definition we have $\psi_*(t) = 0$. Now suppose that $\psi(t)$ is an n-simplex of Y.

by definition of the simplicial subdivision map

$$\begin{aligned} (u_n \circ \psi_*)(t) &\overset{\downarrow}{=} \text{linear combination of } n\text{-simplices of } Z \text{ contained in } \psi(t) \\ &\underset{\uparrow}{=} \text{linear combination of } n\text{-simplices of } Z \text{ not equal to } t. \end{aligned}$$

here we need to distinguish two cases:[1002]
(1) If t is contained in some n-simplex s of Y, then the conclusion follows from Claim 2.
(2) If t is not contained in any n-simplex of Y, then it is in particular not contained in the n-simplex $\psi(t)$ of Y

$$= \text{linear combination of elements in } T_n \setminus \{t\}.$$

This calculation shows that, with respect to the basis T_n, each diagonal entry of the map $u_n \circ \psi_*$ is zero. In particular the trace of the map $u_n \circ \psi_* \colon \mathrm{C}_n^{\mathrm{simp}}(Z) \to \mathrm{C}_n^{\mathrm{simp}}(Z)$ is zero. $\boxplus$

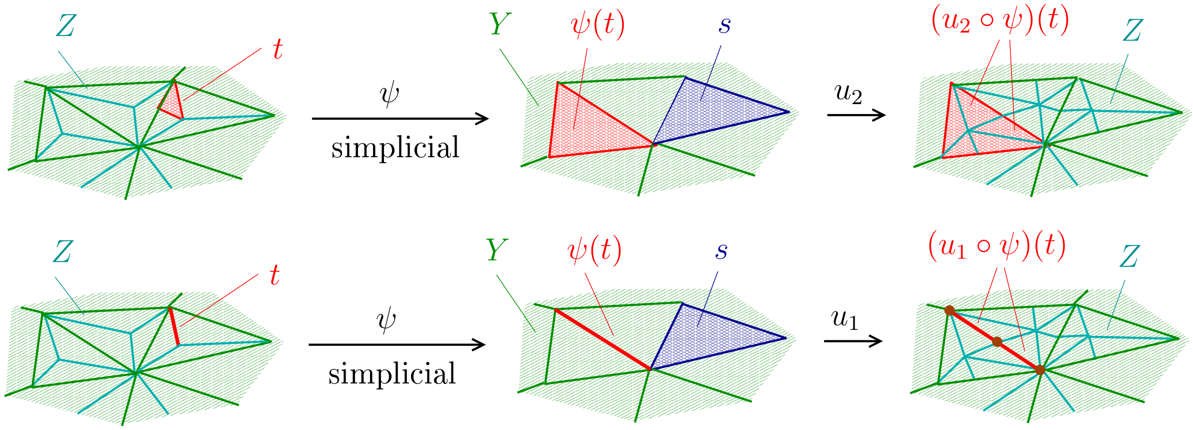


FIGURE 999. Illustration for the proof of Claim 3.

In the following we will use simplicial homology with $\mathbb{F}$-coefficients. The definition should be clear to the reader and it should be clear that some of the statements which we cite below, which were formulated for homology without coefficients, also have obvious generalizations to homology with $\mathbb{F}$-coefficients.

Claim 4. The following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}_n^{\mathrm{simp}}(Z;\mathbb{F}) \arrow[r, "\psi_*"] \arrow[d] & \mathrm{H}_n^{\mathrm{simp}}(Y;\mathbb{F}) \arrow[r, "u_n"] \arrow[d] & \mathrm{H}_n^{\mathrm{simp}}(Z;\mathbb{F}) \arrow[d] \\
\mathrm{H}_n(Z;\mathbb{F}) \arrow[r, "\psi_*"] & \mathrm{H}_n(Z;\mathbb{F}) \arrow[r, "u_n"] & \mathrm{H}_n(Z;\mathbb{F})
\end{tikzcd}$$

here the vertical maps are the natural isomorphism defined on page 1565.

It turns out that there is not much to say. The rectangle to the left commutes by the naturality of the vertical maps and the rectangle to the right commutes by Proposition 63.45. $\boxplus$

[1002]The two cases are illustrated in Figure 999.

Now we can quickly conclude the proof of the Lefschetz Fixed Point Theorem 65.9. Indeed, we see that

$$\begin{aligned}
\Lambda(\varphi,\mathbb{F}) \overset{\text{Lemma 65.7}}{=} \Lambda(\psi,\mathbb{F}) &= \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\operatorname{tr}(\psi_*\circlearrowleft \mathrm{H}_i(Z;\mathbb{F})) \overset{\substack{\text{by Lemma 43.28 the singular subdivision maps } u_* \text{ are chain homotopic}\\ \text{to the identity, thus the induced map on homology is the identity}}}{=} \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\operatorname{tr}(u_*\circ\psi_*\circlearrowleft \mathrm{H}_i(Z;\mathbb{F}))\\
&\underset{\text{by Claim 4}}{=} \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\operatorname{tr}(u_*\circ\psi_*\circlearrowleft \mathrm{H}_i^{\mathrm{simp}}(Z;\mathbb{F})) \underset{\text{by the Hopf Trace Formula 65.12 (1)}}{=} \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\operatorname{tr}(u_*\circ\psi_*\circlearrowleft \mathrm{C}_i^{\mathrm{simp}}(Z;\mathbb{F}))\\
&\underset{\substack{\text{by Lemma 65.3 and since by}\\ \text{definition } \mathrm{C}_i^{\mathrm{simp}}(Z;\mathbb{F})=\mathrm{C}_i^{\mathrm{simp}}(Z)\otimes\mathbb{F}}}{=} \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\operatorname{tr}(u_*\circ\psi_*\circlearrowleft \mathrm{C}_i^{\mathrm{simp}}(Z)) \underset{\text{by Claim 3}}{=} 0.
\end{aligned}$$

We have thus proved that if φ has no fixed points, then $\Lambda(\varphi,\mathbb{F})=0$. ■

65.5. **Vector fields on smooth manifolds.** In this section we will apply the Lefschetz Fixed Point Theorem 65.9 to the study of the (non-)existence of certain vector fields on smooth manifolds.

Let $M\subset\mathbb{R}^n$ be a submanifold without boundary. On page 1185 we defined the notion of a vector field on M. Now we define the notion of a vector field on any smooth manifold.

Definition. Let M be an n-dimensional smooth manifold.

(1) A *vector field on M* is a map v which assigns to each $P\in M$ a vector $v(P)\in \mathrm{T}_P M$ such that the following condition is satisfied: For every chart[1003] $\Phi\colon U\to V$ from an open set U of M to an open set V of $\mathbb{R}^n$ the map

$$\begin{array}{rcl}
V & \to & \mathbb{R}^n\\
P & \mapsto & \mathrm{D}\,\Phi_{\Phi^{-1}(P)}\big(\underbrace{v(\Phi^{-1}(P))}_{\in \mathrm{T}_{\Phi^{-1}(P)} M}\big) \;\subset\; \mathrm{T}_P V \underset{\substack{\uparrow\\ \text{identification from}\\ \text{Proposition 6.36}}}{=} \mathbb{R}^n
\end{array}$$

is continuous. We say that the vector field v is *smooth* if for every chart the above map is smooth.

(2) Given a vector field v on M we say that $x\in M$ is a *zero of v* if $v(x)=0$.

Remark. Let $M\subset\mathbb{R}^n$ be a submanifold without boundary. It follows easily from the definitions that under the usual natural isomorphisms of the tangent spaces $\mathrm{V}_P M \xrightarrow{\cong} \mathrm{T}_P M$ from page 294 and page 491 the two notions of vector fields introduced on page 1185 and above coincide.

On several occasions we will make use of the following basic lemma.

Lemma 65.14. *Let $f\colon M\to N$ be a map between two smooth manifolds and let v be a vector field on N. If f is a local diffeomorphism, then for each $x\in M$ we can define*

[1003] Recall that for a smooth manifold we mean by a "chart" a chart from the given smooth structure.

$(f^*v)(x) = (\mathrm{D}\, f_x)^{-1}(v(f(x)) \in \mathrm{T}_x\, M$. *These vectors define a vector field* f^*v *on* M. *We refer to* f^*v *as the* pullback *of* v *under* f.

PROOF. The lemma follows easily from the definitions. We do not feel like filling in the details. ■

One of the goals of this section is to address the following question.

Question 65.15. *Which closed connected 2-dimensional smooth manifolds admit a nowhere vanishing vector field?*

Example. In Figure 1000 we show vector fields on the projective plane $\mathbb{R}\mathrm{P}^2$, on the Klein bottle and the surface of genus 2. The vector field on the Klein bottle is nowhere-vanishing, whereas the vector field on $\mathbb{R}\mathrm{P}^2$ and the surface of genus 2 both have zeros.

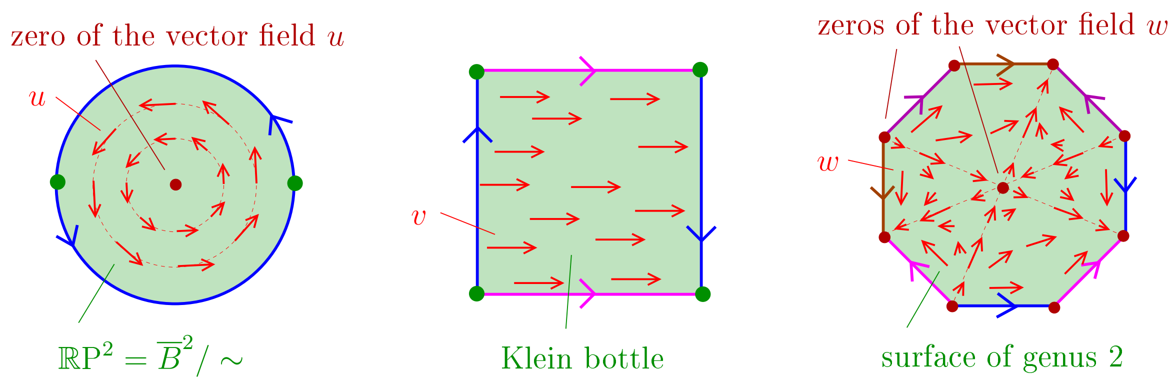


FIGURE 1000

The attentive reader will not have failed to notice that it is the right moment to recall one of the highlights from one of the earlier chapters:

Theorem 45.14. (Hairy Ball Theorem) *Every vector field on an even-dimensional sphere vanishes on at least one point.*

SKETCH OF PROOF. Let v be a nowhere vanishing vector field on a sphere S^n. By "traveling along the great circle defined by v" we can find a homotopy $F\colon S^n \times [0,1] \to S^n$ from the identity to the antipodal map $\rho\colon S^n \to S^n$ given by $\rho(x) = -x$. In particular we see that $(-1)^{n+1} = \deg(\rho) = \deg(\mathrm{id}) = 1$. Thus we see that n is odd. ■

Remark. Regarding Question 65.15 the Hairy Ball Theorem 45.14 says in particular that S^2 does not admit a nowhere vanishing vector field.

To address Question 65.15 it is also helpful to add a few ways to construct new vector fields to our toolbox.

Lemma 65.16.

(1) *Let $f\colon M \to N$ be a map between smooth manifolds which is a local diffeomorphism. Let v be a vector field on N. The map*
$$P \mapsto (\mathrm{D}\, f_P)^{-1}\big(\underbrace{v(f(P))}_{\in \mathrm{T}_{f(P)}(N)}\big)$$
is a vector field on M that we denote by f^*v. A point $P \in M$ is a zero of f^*v if and only if $f(P)$ is a zero of v.
(2) *Let M be a smooth manifold and let v be a vector field on M. Suppose $p\colon \widetilde{M} \to M$ is a covering. Then p^*v is a vector field on $\widetilde{M}$.*
(3) *Let M be a smooth manifold and let v be a vector field on M. Suppose we are given a group G that acts freely, properly and smoothly on M such that for every $g \in G$ we have $(g\cdot\colon M \to M)^*(v) = v$. Then there exists a unique vector field u on M/G such that $v = p^*u$ where $p\colon M \to M/G$ is the projection map.*[1004]

Proof.

(1) The mature reader will have no troubles with providing a proof for this statement.
(2) This statement follows from (1) and Proposition 17.1 (2a).
(3) The proof of this statement is very similar to the proof of (1) and it is also left to the reader. ■

Lemma 65.16 allows us to answer Question 65.15 in some special cases:

Examples.

(1) If v is a vector field on $\mathbb{RP}^2$ with n zeros, then it follows from Lemma 65.16, applied to the 2-fold covering $p\colon S^2 \to \mathbb{RP}^2 = S^2/x \sim -x$ that S^2 admits a vector field with $2n$ zeros. By the Hairy Ball Theorem 45.14 we know that $2n > 0$. In other words, we have just shown that every vector field on $\mathbb{RP}^2$ has at least one zero.
(2) On the smooth manifold $\mathbb{R}^2$ we consider the constant "horizontal" vector field $v(x,y) = (1,0) \in T_{(x,y)}\mathbb{R}^2 = \mathbb{R}^2$. As discussed in Lemma 3.31 and on page 501 we can view the torus and the Klein bottle as quotients of $\mathbb{R}^2$ by a group G that acts freely, properly and smoothly. Furthermore, in both cases one can easily verify that the given G-action preserves the vector field v. This shows that the nowhere vanishing vector field v descends to a nowhere vanishing vector field on the torus and the Klein bottle. For the Klein bottle we obtain precisely the vector field shown in Figure 1000.

The following theorem is the main result of this section.

Theorem 65.17. *Let M be a closed smooth manifold. If M admits a nowhere vanishing vector field, then $\chi(M) = 0$.*

Remark.

(1) On page 1363 we saw that for even n the Euler characteristic $\chi(S^n)$ equals 2. Thus we see that Theorem 65.17 is a generalization of the Hairy Ball Theorem 45.14.

[1004] Recall that by Proposition 6.32 we know that M/G is a smooth manifold and that the projection $p\colon M \to M/G$ is a local diffeomorphism.

(2) For 2-dimensional smooth manifolds a low-tech proof of Theorem 65.17 involving "negative, neutral and positive charges" is given in [**Thu97**, Proposition 1.3.3].
(3) Let M be a closed connected 2-dimensional smooth manifold that is not diffeomorphic to the torus or the Klein bottle. It follows from the calculations on pages 1363 and 1369 that $\chi(M) \neq 0$. Thus we obtain from Theorem 65.17 that M does not admit a nowhere vanishing vector field. Together with the above discussion we now have a complete answer to Question 65.15.
(4) In Proposition 97.4 we will see that the converse to Theorem 65.17 holds. Namely we will see that if M is a closed connected smooth manifold with $\chi(M)$, then M admits a nowhere vanishing vector field.
(5) In Corollary 97.6 we will prove a suitable extension of Theorem 65.17 to the setting of smooth manifolds with non-empty boundary.

The key idea behind the proof of Theorem 65.17 is similar to the proof of the Hairy Ball Theorem 45.14, namely we want to use the nowhere vanishing vector field v on M to construct an interesting self-map of M. This is done in the following proposition.

Proposition 65.18. *Let M be a closed smooth manifold. If there exists a nowhere vanishing vector field, then there exists a map $g\colon M \to M$ that is homotopic to the identity and which has no fixed points.*

Sketch of proof of Proposition 65.18. We sketch two different proofs of the proposition. The first proof is much shorter, but requires more input from analysis and it needs a stronger hypothesis. The second proof is somewhat longer but it is makes fewer demands on analysis.

Let us start with the first proof. To simplify the discussion we assume that the vector field is smooth. As discussed in [**Lee02**, Theorem 17.8], fairly basic results in the theory of differential equations tell us that the vector field v on M defines a flow $\Phi\colon M \times \mathbb{R} \to M$, i.e. a continuous map with the following two properties:

(1) For every $P \in M$ we have $\Phi(P, 0) = P$.
(2) For every $P \in M$ and for every $t \in \mathbb{R}$ we have $(s \mapsto \Phi(P, t+s))' = v(\Phi(P, t))$.

Since $\Phi\colon M \times \mathbb{R}$ is continuous we see that all the maps Φ_t are all homotopic. It follows from the fact that $v(P) \neq 0$ for every $P \in M$ and a compactness argument that there exists an $\epsilon > 0$ such that for every $P \in M$ we have $\Phi(P, \epsilon) \neq P$. In other words, the map $\Phi_\epsilon\colon M \to M$ has no fixed points and it is homotopic to $\Phi_0 = \mathrm{id}$. This concludes the first proof of the proposition.

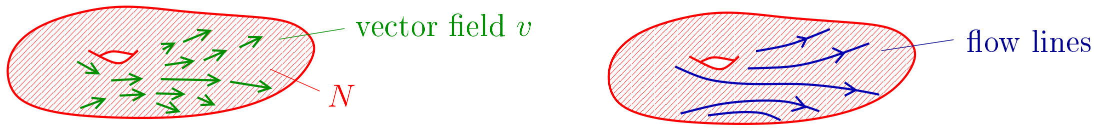


Figure 1001. Second illustration of the proof of Theorem 65.17.

Now we turn to the second proof of the proposition. By Proposition 9.1 (1) together with Proposition 8.1 we can and will view M as a submanifold of some $\mathbb{R}^n$. It follows from Proposition 8.25 that there exists an $\epsilon > 0$ such that the following two conditions are satisfied:

(a) The subset $Z := \{P + w \,|\, P \in M \text{ and } w \in (\mathrm{V}_P\, M)^{\perp} \text{ with } \|w\| \leq \epsilon\}$ of $\mathbb{R}^n$ is a neighborhood of M.
(b) The map

$$\begin{array}{rcll} q\colon Z & \to & M & \\ P+w & \mapsto & P & \text{where } P \in M \text{ and } w \in (\mathrm{V}_P\, M)^{\perp} \text{ with } \|w\| \leq \epsilon \end{array}$$

is well-defined and continuous.

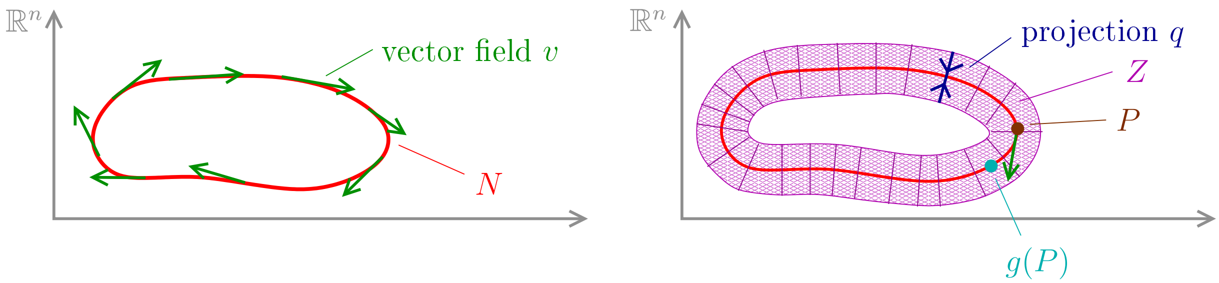


FIGURE 1002. Second illustration of the proof of Theorem 65.17.

Next we consider the map

$$\begin{array}{rcl} F\colon M \times \mathbb{R}_{\geq 0} & \to & \mathbb{R}^n \\ (P,t) & \mapsto & P + t \cdot v(P). \end{array}$$

We make the following observations:

(1) As in Lemma 18.1 we see that F is, unsurprisingly, continuous.
(2) It follows from (a), together with the fact that M is in particular compact and together with some elementary point-set theoretic arguments along the lines of Exercise 3.6, that there exists a $\mu > 0$ such that $F(M \times [0, \mu]) \subset Z$.
(3) We set $g := q \circ F_\mu\colon M \to M$.
(4) By (1) we know that $q \circ F\colon M \times [0, \mu] \to M$ defines a homotopy between the identity and g.
(5) It remains to show that g has no fixed points. Thus let $P \in M$. By (b) we can write $F_\mu(P) = P + \mu \cdot v(P) =: Q + w$ with $Q \in M$ and $w \in (\mathrm{V}_Q\, M)^{\perp}$ with $\|w\| \leq \epsilon$. Since $\mathrm{V}_P\, M \cap \mathrm{V}_P\, M^{\perp} = \{0\}$ and since $v(P)$ is a non-zero vector in $\mathrm{V}_P\, M$ we see that $P \neq Q$. By definition we have $g(P) = Q$. Thus we have shown that $g(P) \neq P$.

This concludes the second proof of the proposition. ■

Now it is straightforward to provide the proof of Theorem 65.17.

PROOF OF THEOREM 65.17. Let M be a closed smooth manifold. Suppose there exists a nowhere vanishing vector field v on M. By Proposition 65.18 there exists a map $g\colon M \to M$

that is homotopic to the identity and which has no fixed points. Now we calculate that

$$\chi(M) \overset{\text{Lemma 65.6 (1)}}{=} \Lambda(\mathrm{id}\colon M \to M) \underset{\text{by Lemma 65.7 since } f \text{ and id are homotopic}}{=} \Lambda(f\colon M \to M) \underset{\text{Lemma 65.8 (2)}}{=} \Lambda(f\colon M \to M, \mathbb{Q}) \overset{\text{by the claim and the Lefschetz Fixed Point Theorem II 65.10}}{=} 0.$$

■

Remark. The approach we used in the proof of Theorem 65.17 is an antediluvian version of the more sophisticated argument using flows of vector fields on smooth manifolds which is studied in detail in [**Lee02**, Chapter 9].

As a teaser we state the following proposition which we will prove later on. This proposition is basically the converse to Theorem 65.17.

Proposition 97.4. *Let M be a closed connected smooth manifold. If $\chi(M) = 0$, then M admits a nowhere vanishing smooth vector field.*

In Chapter 97 we will not only provide a proof of Proposition 97.4 but we will also formulate and prove a significant generalization of Theorem 65.17, namely the Poincaré-Hopf Theorem 97.5 which, given a vector field, relates the type of zeros which occur to the Euler characteristic of the smooth manifold.

65.6. **The Borsuk-Ulam Theorem II ❀.** In this section we will see how to use the Lefschetz Fixed Point Theorem 65.9 to give an alternative proof of the Borsuk-Ulam Theorem 59.3. First we remind the reader of the statement of the Borsuk-Ulam Theorem 59.3.

Theorem 59.3. (Borsuk-Ulam) *Let $n \in \mathbb{N}_0$. For every map $g\colon S^n \to \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $g(x) = g(-x)$.*

Remark. It is worth pointing out that we saw in Sections 59.4 and Section 59.5 that the Borsuk-Ulam Theorem 59.3 has many fascinating consequences.

The key to our second proof of the Borsuk-Ulam Theorem 59.3 is the following proposition.

Proposition 65.19. *Let $n \in \mathbb{N}$ and let $f\colon S^n \to S^n$ be a map. If $f(-x) = -f(x)$ for all $x \in S^n$, then the Lefschetz number $\Lambda(f)$ is even.*

Examples.

(1) Let $A \in \mathrm{O}(n+1)$. We consider the map $f\colon S^n \to S^n$ that is given by $f(x) := A \cdot x$. Evidently we have $f(-x) = -f(x)$ for all $x \in S^n$. Furthermore it follows from Lemma 45.11 (6) together with Lemma 65.6 (2) that $\Lambda(f) = 1 + (-1)^n \cdot \det(A)$.

(2) In Exercise 65.13 we will see that given any $n \in \mathbb{N}$ and given any *even* $m \in \mathbb{Z}$ there exists a map $f\colon S^n \to S^n$ with $\Lambda(f) = m$ and such that $f(-x) = -f(x)$ for all $x \in S^n$.

Remark. The proof of Proposition 65.19 is an expanded write-up of the arguments in [**Deo18**, Theorem 4.9.17] and [**Arm83**, Chapter 9.3].

The following gives us a criterion for showing that a trace is even.

Lemma 65.20. *Let V be a finitely generated free $\mathbb{Z}$-module and let $\rho\colon V \to V$ be an isomorphism with the following two properties:*

(i) *There exists a subset T of V such that $T \cap \rho(T) = \varnothing$ and such that $T \cup \rho(T)$ is a basis for V.*

(ii) *We have $\rho \circ \rho = \mathrm{id}_V$.*

Let $\varphi\colon V \to V$ be a homomorphism. If φ commutes with ρ, then $\mathrm{tr}(\varphi\colon V \to V)$ is even.

Proof. We equip V with the basis $T \sqcup \rho(T)$. It follows immediately from (ii) that

$$\text{matrix representing } \rho \text{ with respect to the basis } T \sqcup \rho(T) \;=\; \begin{array}{cc} & \begin{array}{cc} T & \rho(T) \end{array} \\ \begin{array}{c} T \\ \rho(T) \end{array} & \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \end{array}.$$

Now let $\varphi\colon V \to V$ be a homomorphism that commutes with ρ. We write

$$\text{matrix representing } \varphi \text{ with respect to the basis } T \sqcup \rho(T) \;=:\; \begin{array}{cc} & \begin{array}{cc} T & \rho(T) \end{array} \\ \begin{array}{c} T \\ \rho(T) \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}.$$

Since φ commutes with g we see that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Multiplying out both sides we see in particular that $A = D$. But this implies evidently that the trace of the above matrix representing φ is even. Thus we have shown that $\mathrm{tr}(\varphi)$ is even. ■

Now we can provide the proof of Proposition 65.19.

Proof of Proposition 65.19. Let $f\colon S^n \to S^n$ be a map which has the property that $f(-x) = -f(x)$ for all $x \in S^n$. We denote by $\rho\colon S^n \to S^n$ the map that is given by $\rho(x) = -x$. Note that our hypothesis on f is equivalent to the statement that f commutes with ρ. We need to show that $\Lambda(f)$ is even. Similar to the proof of the Lefschetz Fixed Point Theorem 65.9 on page 1627 we intend to proceed as follows:

> We want to upgrade the hypothesis that $f \circ \rho = \rho \circ f$ to a statement about an induced map on a suitable simplicial chain complex C_* such that we can conclude from Lemma 65.20 and the Hopf Trace Formula 65.12 (2) that $\Lambda(f)$ is even.

To carry out this program we equip S^n with the simplicial structure given by the homeomorphisms

$$S^n \underset{\substack{\uparrow \\ \text{homeomorphism given by Lemma 61.19 (1)}}}{\xleftarrow{\cong}} S^0 * \cdots * S^0 \overset{\substack{\text{here } S_0 \text{ is the abstract simplicial complex with vertex set } \{0,1\} \\ \downarrow}}{=} |S_0| * \cdots * |S_0| \underset{\substack{\uparrow \\ \text{homeomorphism given by Lemma 61.20}}}{\xrightarrow{\cong}} |S_0 * \cdots * S_0|.$$

We introduce the following notation:

(1) We write $K := S_0 * \cdots * S_0$ and we denote by $\Psi\colon |K| \to S^n$ the above homeomorphism.

(2) Given any $k \in \mathbb{N}_0$ we denote by $\mho\colon |\operatorname{sd}^k(K)| \to |K|$ the natural homeomorphism given by Lemma 62.3 (3).
(3) By a slight abuse of notation we denote by $\rho\colon K \to K$ the map that is induced by the reflection on the simplices of all the copies of S_0.
(4) Note that by Lemma 62.3 (5) the simplicial map $\rho\colon K \to K$ induces a simplicial map $\operatorname{sd}^k(K) \to \operatorname{sd}^k(K)$ which we also denote by ρ.

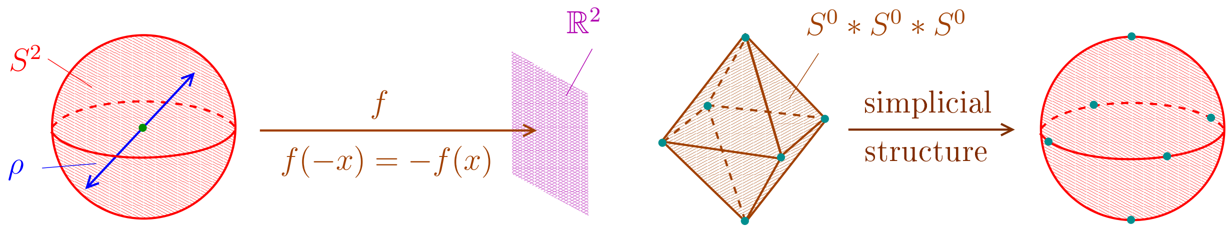


FIGURE 1003. Illustration for the proof of Proposition 65.19.

Claim 1. There exists a $k \in \mathbb{N}_0$ and a *simplicial* map $\varphi\colon \operatorname{sd}^k(K) \to K$ with the following properties:

(1) the map $\Psi \circ |\varphi| \circ \mho^{-1} \circ \Psi^{-1}\colon S^n \to S^n$ is homotopic to f,
(2) the map φ commutes with the maps ρ, more precisely, the following diagram commutes

$$\begin{tikzcd} \operatorname{sd}^k(K) \arrow[r, "\varphi"] \arrow[d, "\rho"'] & K \arrow[d, "\rho"] \\ \operatorname{sd}^k(K) \arrow[r, "\varphi"] & K. \end{tikzcd}$$

As we will see, the proof of Claim 1 follows from a careful reading of the proof of the Simplicial Approximation Theorem 62.8.

(a) First note that Claim 1 of the proof of the Simplicial Approximation Theorem 62.8 shows that there exists a $k \in \mathbb{N}_0$ such that for every vertex v of $\operatorname{sd}^k(K)$ there exists a vertex w of K with $f((\Psi \circ \mho)(\operatorname{St}(|\operatorname{sd}^k(K)|, v))) \subset \Psi(\mathring{\operatorname{St}}(|K|, w))$. We denote by V the vertex set of $\operatorname{sd}^k(K)$ and we denote by W the vertex set of K. By the above there exists a map $\varphi\colon V \to W$ with the following property:
(∗) For every vertex $v \in V$ we have $f((\Psi \circ \mho)(\operatorname{St}(|\operatorname{sd}^k(K)|, v))) \subset \Psi(\mathring{\operatorname{St}}(|K|, \varphi(v)))$.
(b) In the proof of the Simplicial Approximation Theorem 62.8 we have shown that every map $\varphi\colon V \to W$ that satisfies (∗) is a simplicial map and that it has the property that $\Psi \circ |\varphi| \circ \mho^{-1} \circ \Psi^{-1}$ is homotopic to f.
(c) It follows immediately from the definition of Ψ and the naturality of $\mho$ that the following two diagrams commute:

$$\begin{tikzcd} {|K|} \arrow[r, "\rho"] \arrow[d, "\Psi"'] & {|K|} \arrow[d, "\Psi"] \\ S^n \arrow[r, "\rho"] & S^n \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {|\operatorname{sd}^k(K)|} \arrow[r, "\rho"] \arrow[d, "\mho"'] & {|\operatorname{sd}^k(K)|} \arrow[d, "\mho"] \\ {|K|} \arrow[r, "\rho"] & {|K|}. \end{tikzcd}$$

(d) Since all maps called ρ satisfy the condition that $\rho \circ \rho = \mathrm{id}$ we obtain easily from (a) and (c) that we can pick a map $\varphi\colon V \to W$ which satisfies $(*)$ and such that $\varphi \circ \rho = \rho \circ \varphi$.

(e) By (b) we know that φ satisfies (1). Furthermore, by (d) we know that the simplicial map φ also satisfies (2). $\boxplus$

Before we move on to the next claim we introduce and recall some notation:

(1) Now we write $g := \Psi \circ |\varphi| \circ \mho^{-1} \circ \Psi^{-1}$.
(2) We denote by $u_*\colon \mathrm{C}^{\mathrm{simp}}_*(K) \to \mathrm{C}^{\mathrm{simp}}_*(\mathrm{sd}^k(K))$ the k-fold iterated simplicial subdivision map that we introduced in Lemma 63.41.

Claim 2. For every $j \in \mathbb{N}_0$ there exists an isomorphism $\mu\colon \mathrm{H}^{\mathrm{simp}}_j(\mathrm{sd}^k(K)) \to \mathrm{H}_j(S^n)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}^{\mathrm{simp}}_j(\mathrm{sd}^k(K)) \arrow[r, "u_j \circ \varphi_*"] \arrow[d, "\mu"', "\cong"] & \mathrm{H}^{\mathrm{simp}}_j(\mathrm{sd}^k(K)) \arrow[d, "\cong"', "\mu"] \\
\mathrm{H}_j(S^n) \arrow[r, "g_*", "(\Psi\circ|\varphi|\circ\mho^{-1}\circ\Psi^{-1})_*"'] & \mathrm{H}_j(S^n).
\end{tikzcd}$$

For the proof of Claim 2 we consider the following much more elaborate diagram:

$$\begin{tikzcd}
\mathrm{H}^{\mathrm{simp}}_j(\mathrm{sd}^k(K)) \arrow[r, "\varphi_*"] \arrow[dd, "\Theta_{\mathrm{sd}^k(K)}"'] & \mathrm{H}^{\mathrm{simp}}_j(K) \arrow[r, "u_j"] \arrow[dd, "\Theta_K"'] & \mathrm{H}^{\mathrm{simp}}_j(\mathrm{sd}^k(K)) \arrow[d, "\Theta_{\mathrm{sd}^k(K)}"] \\
& & \mathrm{H}_j(|\,\mathrm{sd}^k(K)|) \arrow[d, "\mho_*"] \\
\mathrm{H}_j(|\,\mathrm{sd}^k(K)|) \arrow[r, "|\varphi|_*"] \arrow[d, "(\Psi\circ\mho)_*"'] & \mathrm{H}_j(|K|) \arrow[r, "u_j"] \arrow[dr, "\Psi_*"'] & \mathrm{H}_j(|K|) \arrow[d, "\Psi_*"] \\
\mathrm{H}_j(S^n) \arrow[rr, "g_*", "(\Psi\circ|\varphi|\circ\mho^{-1}\circ\Psi^{-1})_*"'] & & \mathrm{H}_j(S^n).
\end{tikzcd}$$

We make the following observations and clarifications:

(1) The maps Θ are the natural isomorphisms introduced on page 1565.
(2) We denote by $u_*\colon \mathrm{C}_*(|K|) \to \mathrm{C}_*(|K|)$ the k-fold iterated singular subdivision map that we introduced on page 1135.
(3) It follows from Proposition 63.45, applied iteratively k times, that the rectangle to the top right commutes.
(4) The quadrilateral at the bottom commutes by definition.
(5) The triangle at the bottom right commutes by Lemma 43.28.
(6) The square to the top left commutes by the naturality of Θ.

It follows from the above that the isomorphism $\mu := \Psi_* \circ \mho_* \circ \Theta_{\mathrm{sd}^k(K)}$ has the desired property. $\boxplus$

To shorten the notation a little bit we now write $L := \operatorname{sd}^k(K)$. We calculate that

$$\begin{aligned}\Lambda(f) \overset{\substack{\text{by Lemma 65.7 and Claim 1 (1)}\\ \downarrow}}{=} \Lambda(g) &= \sum_{j\in\mathbb{N}_0} (-1)^j \cdot \operatorname{tr}\big(g_* \circlearrowleft \mathrm{H}_i(S^n)\big) \overset{\substack{\text{by Claim 2}\\ \downarrow}}{=} \sum_{j\in\mathbb{N}_0} (-1)^j \cdot \operatorname{tr}\big((u_j\circ\varphi_*) \circlearrowleft \mathrm{H}_i^{\mathrm{simp}}(L)\big)\\ &\underset{\substack{\uparrow\\ \text{by the Hopf Trace Formula 65.12 (2)}}}{=} \sum_{j\in\mathbb{N}_0} (-1)^j \cdot \operatorname{tr}\big((u_j\circ\varphi_*) \circlearrowleft \mathrm{C}_i^{\mathrm{simp}}(L)\big).\end{aligned}$$

At this point we see that it suffices to prove the following claim.

Claim 3. For each $j \in \mathbb{N}_0$ the trace $\operatorname{tr}\big((u_j\circ\varphi_*) \circlearrowleft \mathrm{C}_j^{\mathrm{simp}}(L)\big)$ is even.

Evidently the plan is to apply Lemma 65.20. We make a few more preparations:

(1) We consider the following diagram:

$$\begin{tikzcd} \mathrm{C}_j^{\mathrm{simp}}(L) \arrow[r, "g_*"] \arrow[d, "\rho_*"] & \mathrm{C}_j^{\mathrm{simp}}(K) \arrow[r, "u_j"] \arrow[d, "\rho_*"] & \mathrm{C}_j^{\mathrm{simp}}(L) \arrow[d, "\rho_*"] \\ \mathrm{C}_j^{\mathrm{simp}}(L) \arrow[r, "g_*"] & \mathrm{C}_j^{\mathrm{simp}}(K) \arrow[r, "u_j"] & \mathrm{C}_j^{\mathrm{simp}}(L). \end{tikzcd}$$

It follows from Claim 1 (2) that the square to the left commutes. It follows from the naturality of the simplicial subdivision maps that the square to the right commutes.

(2) Note that it follows immediately from the definition of the map $\rho\colon K \to K$ that $\rho\colon K \to K$ does not fix a single simplex of K and that $\rho\circ\rho = \mathrm{id}_K$. Since the map $K \mapsto L = \operatorname{sd}^k(K) = L$ is functorial we see that $\rho\colon L \to L$ has the same properties, i.e. $\rho\colon L \to L$ does not fix a single simplex of L and it satisfies $\rho\circ\rho = \mathrm{id}_L$.

(3) We denote by S_j the set of j-simplices of L. It follows immediately from (2) that there exists a subset T_j of S_j with the following properties:

$$\text{(i)} \qquad T_j \cap \rho(T_j) \;=\; \varnothing \qquad\qquad \text{and} \qquad\qquad \text{(ii)} \qquad T_j \cup \rho(T_j) \;=\; S_j.$$

We equip each $t \in T_j$ with some order and we use these orders to view T_j as a subset of $\mathrm{C}_j^{\mathrm{simp}}(L)$. It follows from (ii) and Lemma 63.5 that $T_j \cup \rho(T_j)$ is a basis of $\mathrm{C}_j^{\mathrm{simp}}(L)$.

Now we apply Lemma 65.20 to $V = \mathrm{C}_j^{\mathrm{simp}}(L)$, $T = T_j$, $\rho = \rho_*$ and $\varphi = \varphi$. It follows from (1), (2) and (3) that all the hypotheses of Lemma 65.20 are satisfied. Thus we obtain that the trace of $u_j \circ \varphi_*$ is indeed even. ■

The argument for deducing the Borsuk-Ulam Theorem 59.3 from Proposition 65.19 is almost identical to the argument on page 1451 for deducing the Borsuk-Ulam Theorem 59.3 from Proposition 59.4.

Proof of the Borsuk-Ulam Theorem 59.3. Let $g\colon S^n \to \mathbb{R}^n$ be a map. Suppose there exists no pair of antipodal points x and $-x$ in S^n with $g(x) = g(-x)$. Put differently, suppose that for any $x \in S^n$ we have $g(x) \neq g(-x)$. We consider the composition of the following two maps

$$\begin{array}{rcccl} \Phi\colon S^{n-1}\times[0,1] & \to & S^n_{\geq 0} & \to & S^{n-1}. \\ (y,t) & \mapsto & (y\cdot\sqrt{1-t^2},t) & & \\ & & x & \mapsto & \underbrace{\frac{g(x)-g(-x)}{\|g(x)-g(-x)\|}}_{\text{defined since } g(x)\neq g(-x)} \end{array}$$

Now we see that

$$\text{even number} \underset{\substack{\uparrow\\ \text{by definition we have } \Phi(-x,0)=-\Phi(x,0)\text{, thus it follows from Proposition 65.19 that } \Lambda(\Phi_0) \text{ is even}}}{=} \Lambda(\Phi_0) \underset{\substack{\uparrow\\ \text{by Lemma 65.7 since } \Phi_0 \text{ and } \Phi_1 \text{ are homotopic}}}{=} \Lambda(\Phi_1) \underset{\substack{\uparrow\\ \text{by definition of } \Phi_1}}{=} \Lambda(\text{constant map}) \underset{\substack{\uparrow\\ \text{Lemma 65.6 (3)}}}{=} 1.$$

We have thus obtained a contradiction. ■

Exercises for Chapter 65.

Exercise 65.1. Let X be a finite simplicial complex, let $\varphi\colon X\to X$ be a map and let $\mathbb{F}$ be a field. We denote by $\gamma\colon \mathbb{Z}\to\mathbb{F}$ the ring homomorphism that is given by $\gamma(n)=n\cdot 1_{\mathbb{F}}$. Show that

$$\Lambda(X,\mathbb{F}) \;=\; \gamma(\Lambda(X)).$$

Hint. Use Exercise 57.3.

Exercise 65.2. Let $n\in\mathbb{N}$. Show that there exists a compact smooth manifold M and a map $\varphi\colon M\to M$ with Lefschetz number $\Lambda(\varphi)=n$ such that φ has precisely one fixed point.

Exercise 65.3. Let $n\in\mathbb{N}$. Does the real projective space $\mathbb{R}\mathrm{P}^{2n+1}$ admit a self-map without fixed points?

Exercise 65.4. We consider the torus $T:=S^1\times S^1$ embedded in $\mathbb{R}^3$ as shown in Figure 1004. Let $f\colon T\to T$ be the map that is given by reflection in the xy-plane. Is f homotopic to a map without fixed points?

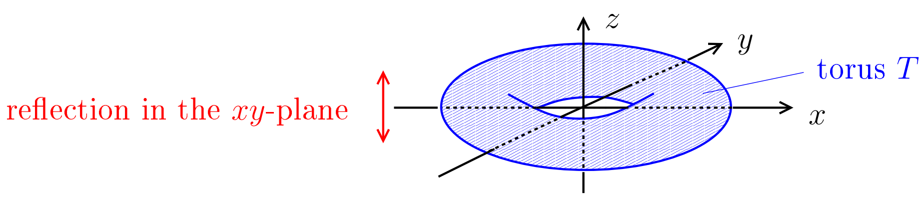


FIGURE 1004. Illustration for Exercise 65.4.

Exercise 65.5. We consider the topological space $X=A\cup B\cup C$ that is shown in Figure 1005. Here A is a circle of radius 3, C is a circle of radius 1 around the origin and B is the image of an embedding $\mathbb{R}\to\mathbb{R}^2$ which spirals towards A and C.

(a) Give a rigorous description of X.
(b) Show that X is compact.

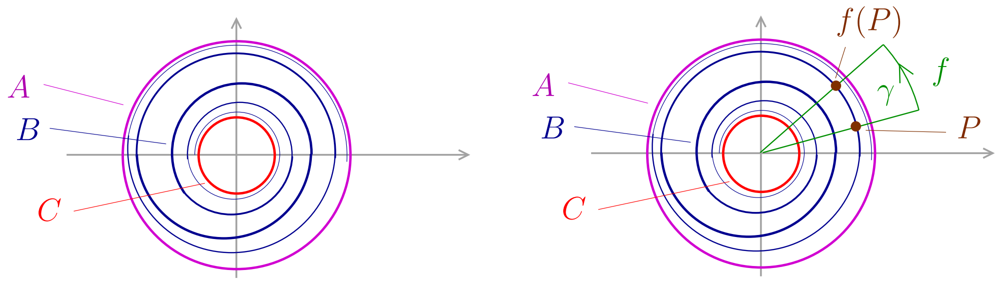


FIGURE 1005. Illustration for Exercise 65.5.

(c) Compute the homology groups of X.

Fix some "angle" $\gamma \in (0, \pi)$. Let $f\colon X \to X$ be the map that is given by rotation by the angle γ on A and C and which is given by sending a point $s \cdot e^{i\varphi}$ on B to the "next" point on B of the form $t \cdot e^{i(\varphi+\gamma)}$ with $s > t$.

(d) Show that f is continuous.
(e) Show that f has no fixed points.
(f) Show that the Lefschetz number $\Lambda(f) \neq 0$.

This long but fun exercise shows that in the formulation of the Lefschetz Fixed Point Theorems it does not suffice to work with compact topological spaces that are say $\mathbb{Q}$-homology finite. We need the topological space to be "sufficiently nice".

Exercise 65.6. We consider the topological space $X = A \cup B$ that is shown in Figure 1006. Let $\varphi\colon X \to X$ be the map that is given by $z \mapsto z^2$ on A and that is given by reflection along the x-axis on B.

(a) Show that the Lefschetz number $\Lambda(\varphi, \mathbb{F}) = 0$ for every field $\mathbb{F}$.
(b) Show that any map ψ that is homotopic to φ actually has a fixed point.

Remark. It is easy to see that X admits a simplicial structure. Thus the exercise shows that the converse to the Lefschetz Fixed Point Theorem 65.9 does not hold.

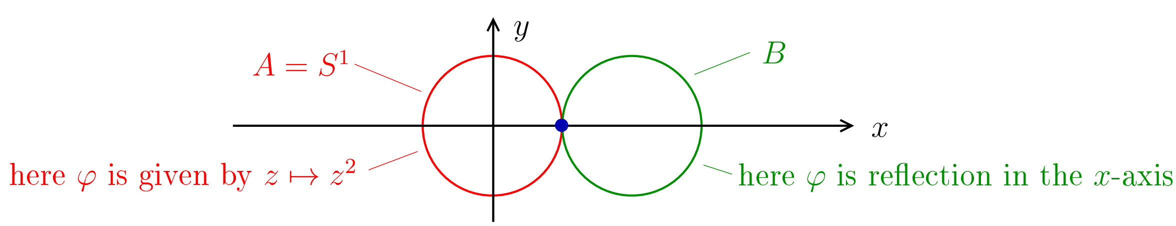


FIGURE 1006. Illustration of Exercise 65.6.

Exercise 65.7.

(1) Let $A \in \mathrm{GL}(n, \mathbb{C})$ be a matrix. We denote by $\varphi_A\colon \mathbb{CP}^{n-1} \to \mathbb{CP}^{n-1}$ the induced map given by $\varphi_A([P]) := [A \cdot P]$.
   (a) Show that the Lefschetz number $\Lambda(\varphi_A) = n$.
   Hint. Use Lemma 2.65 (4). The super conscientious reader might also want to consult the discussion on page 558 to show that a certain map is indeed continuous.

(b) Use (a) to show that A has an eigenvalue.

(2) What does the approach in (1) say about the existence of real eigenvalues of real matrices?

Exercise 65.8. Let $n \in \mathbb{N}_0$ be *even.*

(a) Show that the complex projective space $\mathbb{C}\mathrm{P}^n$ admits a self-map without fixed points.
Remark. In Exercise 90.9 we will prove the much more interesting converse, namely we will show that for every *odd* n every self-map of $\mathbb{C}\mathrm{P}^n$ has a fixed point.

(b) Does there exist a nowhere vanishing vector field on $\mathbb{C}\mathrm{P}^n$?

Exercise 65.9. For which $n \in \mathbb{N}$ does the real projective space $\mathbb{R}\mathrm{P}^n$ admit a nowhere vanishing vector field?

Exercise 65.10. Does the Klein bottle K admit two vector fields v and w such that at every point $P \in K$ the two vectors $v(P), w(P) \in \mathrm{T}_P K$ are linearly independent?

Exercise 65.11. In Figure 1000 we saw that the surface Σ of genus 2 supports a vector field with two zeros. What does such a vector field look like if we consider Σ as a submanifold of $\mathbb{R}^3$ in the usual way?

Exercise 65.12. Does the surface of genus 2 admit a vector field with a single zero?

Exercise 65.13. Let $m \in \mathbb{Z}$ be even.

(a) Show that there exists a map $f\colon S^1 \to S^1$ with Lefschetz number $\Lambda(f) = m$ and such that for every $x \in S^1$ we have $f(-x) = -f(x)$.

(b) Now prove (a) for higher-dimensional spheres.

Exercise 65.14. Let X be a finite CW-complex and let $\varphi\colon X \to X$ be a self-map. Furthermore let $p\colon \widetilde{X} \to X$ be a finite covering. By Lemma 29.11 we know that there exists a map $\widetilde{\varphi}\colon \widetilde{X} \to \widetilde{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X. \end{array}$$

Show that

$$\Lambda(\widetilde{\varphi}\colon \widetilde{X} \to \widetilde{X}) \;=\; [\widetilde{X} : X] \cdot \Lambda(\varphi\colon X \to X).$$

Hint. Use cellular homology and use the Hopf Trace Formula, i.e. use Theorem 65.12.

Exercise 65.15. Let M be a compact topological manifold. Suppose we are given two compact codimension-zero submanifolds A and B such that $A \cap B = \partial A = \partial B$. Furthermore let $f\colon M \to M$ be a map that restricts to self-maps of A and B. Show that

$$\Lambda(f\colon M \to M) \;=\; \Lambda(f\colon A \to A) + \Lambda(f\colon B \to B) - \Lambda(f\colon A \cap B \to A \cap B).$$

Hint. Use the Mayer–Vietoris Theorem 46.10 for Manifolds and use the Hopf Trace Formula, i.e. use Theorem 65.12.
Remark. In Proposition 85.13 we will see that the homology groups of A, B and M are finitely generated and that there exists an n such that they vanish in dimension $> n$.

Exercise 65.16. Let M be a smooth manifold and let v be a vector field on M. Show that the zero set $\{x \in M \mid v(x) = 0\}$ is a closed subset of M.
Hint. Use Lemma 2.6 (2b).

Exercise 65.17. A *Lie group* is a smooth manifold X with empty boundary together with a group structure such that the two maps

$$\begin{array}{rcl} X \times X & \to & X \\ (x, y) & \mapsto & x \cdot y \end{array} \qquad \text{and} \qquad \begin{array}{rcl} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are smooth. For example it is not difficult to show that the matrix groups $\mathrm{SO}(n)$ und $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ with the usual multiplication are Lie groups. Show that the Euler characteristic of any closed Lie group of dimension ≥ 1 is zero.

66. Homology groups of pseudomanifolds and smooth manifolds

Let M be an n-dimensional smooth manifold. By Theorem 64.2 we know that M admits a smooth simplicial structure. In Proposition 64.6 we used this fact to show that for every $k > n$ we have $\mathrm{H}_k(M) = 0$. In this chapter we will squeeze more information out of the existence of a simplicial structure. In particular this will allow us to calculate $\mathrm{H}_n(M;\mathbb{Z})$ and it will allow us to make some statements regarding $\mathrm{H}_{n-1}(M;\mathbb{Z})$.

66.1. **The top-dimensional homology group of a pseudomanifold.** In this chapter we will see that any simplicial structure of an n-dimensional topological manifold has several special properties. This will lead us to the notion of an *n-dimensional pseudomanifold* which can be viewed as a simplicial analogue of a manifold. We will then prove several statements regarding the simplicial homology groups of an n-dimensional pseudomanifold. In the subsequent section we will then use the calculations from this chapter to prove the promised results on homology groups of smooth manifolds.

First we recall and introduce some definitions.

Definition. Let $K = (V, S)$ be an abstract simplicial complex.

(1) Let t be a simplex of K. As on page 1481 we define a *face of t* to be a simplex s with $s \subset t$. In this setting we also say that t *cobounds* s.
(2) Let $s \in S$ be a k-simplex of K. We say s has *order* r, if there exist precisely r simplices of dimension $k+1$ which contain s as a face.

As before we sometimes use the same terms when we talk about topological realizations and simplicial complexes.

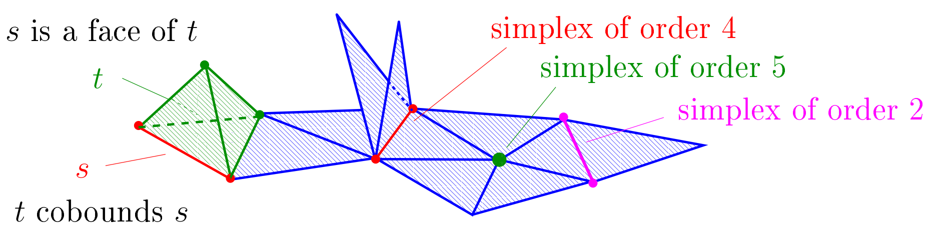


Figure 1007

Proposition 66.1. *Let M be an n-dimensional topological manifold that is equipped with a simplicial structure.*

(1) *There is no simplex of dimension $> n$ and every simplex is the face of an n-simplex.*
(2) *The simplicial structure has finitely many simplices if and only if it has finitely many n-simplices.*
(3) *The simplicial structure on M restricts to a simplicial structure on the $(n-1)$-dimensional topological manifold ∂M. If M is a smooth manifold and if we are given a smooth simplicial structure, then the restriction of the simplicial structure to the smooth manifold ∂M is also a smooth simplicial structure.*

(4) *Let s be an $(n-1)$-simplex of M. Then*

$$\text{order of } s \;=\; \begin{cases} 1, & \text{if } s \text{ lies on the boundary,} \\ 2, & \text{otherwise.} \end{cases}$$

(5) *If M is connected, then given any two n-simplices t and t' there exists a sequence $t = s_0, s_1, \ldots, s_k = t'$ of n-simplices such that any two consecutive simplices have a common $(n-1)$-dimensional face.*

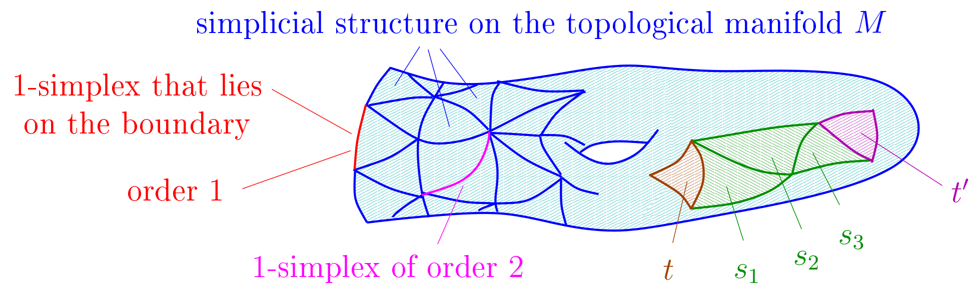


FIGURE 1008. Illustration of Proposition 66.1.

Remark.

(1) In Lemma 64.1 we gave a different proof why the maximal dimension of a simplex of a smooth simplicial structure of an n-dimensional smooth manifold equals n.
(2) A rather different proof of Proposition 66.1 (5) is given in [**Mun84**, Corollary 65.2].

The key idea behind the proof of Proposition 66.1 is to squeeze the maximal amount of information out of the local homology groups. To do so we first recall the following definition from page 1151.

Definition. Given a topological space X and given a point $P \in X$ the relative homology group $\mathrm{H}_j(X, X \setminus \{P\})$ is called the *n-th local homology group* of X at the point P.

We recall the calculation of the local homology groups of a topological manifold.

Lemma 44.1. *Let M be an n-dimensional topological manifold and let $P \in M$. For every $j \in \mathbb{N}_0$ we have*

$$\mathrm{H}_j(M, M \setminus \{P\}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } P \in M \setminus \partial M \text{ and } j = n, \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. Since in a minute we will calculate a different local homology group it is perhaps helpful to remind us of the proof of the lemma. For simplicity we only consider the case that $P \in M \setminus \partial M$. In this case we can pick a chart $\Phi\colon U \to B^n$ for P with $\Phi(P) = 0$. We then have

$$\mathrm{H}_j(M, M\setminus\{P\}) \underset{\uparrow}{\leftarrow} \mathrm{H}_j(U, U\setminus\{P\}) \xrightarrow[\cong]{\Phi_*} \mathrm{H}_j(B^n, B^n\setminus\{0\}) \underset{\uparrow}{\xrightarrow{\partial}} \widetilde{\mathrm{H}}_{j-1}(B^n\setminus\{0\}) \underset{\uparrow}{\leftarrow} \widetilde{\mathrm{H}}_{j-1}(S^{n-1}).$$

Excision Theorem 43.20

isomorphism by the long exact sequence in reduced homology of the pair $(B^n, B^n \setminus \{0\})$ and the fact that $\widetilde{\mathrm{H}}_*(B^n) = 0$ by page 1106

isomorphism since S^{n-1} is homotopy equivalent to $B^n \setminus \{0\}$

The statement now follows from the calculation of the reduced homology groups of spheres given in Proposition 43.4. ■

The next proposition calculates the local homology groups of a simplicial complex. This calculation is also interesting in its own right.

Proposition 66.2. *Let $K = (V, S)$ be an abstract simplicial complex and let $P \in |K|$. By Lemma 61.11 there exists a unique simplex $s \in S$ such that P is a point in the open simplex $\langle s \rangle$. We set $m := \dim(s)$. If $\mathrm{Lk}(K, s)$ is a finite simplicial complex, then for every $j \in \mathbb{N}_0$ we have*[1005][1006]

$$\mathrm{H}_j(|K|, |K| \setminus \{P\}) \;\cong\; \widetilde{\mathrm{H}}_{j-1-m}(\mathrm{Lk}(K, s)).$$

Proof. We perform the following calculation:

by the long exact sequence in reduced homology of the pair $(\mathrm{St}(s), \mathrm{St}(s) \setminus \{P\})$ and the observation, see Lemma 62.5 (1f), that $\mathrm{St}(s)$ is contractible, which implies that its reduced homology vanishes

$$\mathrm{H}_j(|K|, |K| \setminus \{P\}) \;\xleftarrow{\cong}\; \mathrm{H}_j(\mathrm{St}(s), \mathrm{St}(s)\setminus\{P\}) \;\xrightarrow{\cong}\; \widetilde{\mathrm{H}}_{j-1}(\mathrm{St}(s)\setminus\{P\}) \;\cong\; \widetilde{\mathrm{H}}_{j-1}(\partial s * \mathrm{Lk}(s))$$

by Lemma 62.5 (1d) we know that $\mathrm{St}(s)$ is a neighborhood, thus we can apply the Excision Theorem 43.20 (2)

by Lemma 62.5 (1g) together with Corollary 42.8

$$\cong\; \widetilde{\mathrm{H}}_{j-1}(S^{m-1} * \mathrm{Lk}(s)) \;\cong\; \widetilde{\mathrm{H}}_{j-1-m}(\mathrm{Lk}(s)).$$

since s is an m-simplex we know by Lemma 61.5 and Lemma 41.1 (3) that ∂s is homeomorphic to S^{m-1}

by Corollary 45.8 we have $\widetilde{\mathrm{H}}_{l+m}(S^{m-1} * X) \cong \widetilde{\mathrm{H}}_l(X)$ whenever X is compact and Hausdorff, we can apply this result since $\mathrm{Lk}(s)$ is by hypothesis finite ■

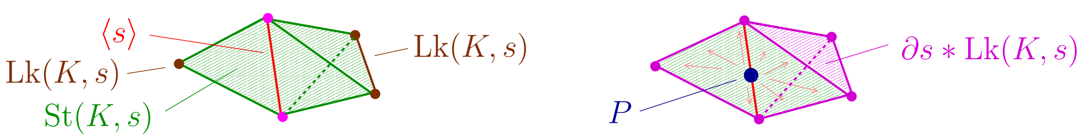


Figure 1009. Illustration for the proof of Proposition 66.2.

Now we can finally provide the proof of Proposition 66.1.

Proof of Proposition 66.1. Basically by definition of a simplicial structure it suffices to consider the case that the n-dimensional topological manifold M is the topological realization $M = |K|$ of some abstract simplicial complex $K = (V, S)$. Note that by Lemma 6.9 we know that the topological manifold $M = |K|$ is regionally compact. Thus it follows from Proposition 62.6 that the abstract simplicial complex K is locally finite and that the link of any vertex is a finite simplicial complex. This implies in particular that we can apply Proposition 66.2 without worries.

(1) It suffices to prove the following claim.

Claim. Given any simplex s, the maximal dimension of a simplex that contains s is precisely given by n.

[1005] It can happen that $\mathrm{Lk}(K, s) = \varnothing$. Recall that by definition we have $\widetilde{\mathrm{H}}_{-1}(\varnothing) \cong \mathbb{Z}$.

[1006] I do not know whether the statement holds in general without the condition that $\mathrm{Lk}(K, s)$ is finite.

Let s be a simplex of K. As we mentioned in the beginning of the proof, we know that K is locally finite. This implies that there exists a simplex t of maximal dimension that contains s. We set $m := \dim(t)$. Now let $P \in \langle t \rangle$ be a point. We calculate that

$$\mathbb{Z} \underset{\substack{\uparrow \\ \text{since there are no simplices of dimension } > m \text{ which contain } t \text{ we see that } \mathrm{Lk}(K,t)=\varnothing}}{\cong} \widetilde{\mathrm{H}}_{-1}(\mathrm{Lk}(K,t)) \overset{\substack{\text{Proposition 66.2 applied to the } m\text{-simplex } t \\ \downarrow}}{\cong} \mathrm{H}_m(|K|,|K|\setminus\{P\}) \overset{\substack{\text{Lemma 44.1 applied to } |K|=M \\ \downarrow}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } P \in M\setminus\partial M \text{ and } m=n, \\ 0, & \text{otherwise.} \end{cases}$$

Comparing the left-hand side with the right-hand side shows that $m = n$. ⊞

(2) This statement is an almost immediate consequence of (1). Indeed, we only have to show that if K has finitely many n-simplices, then it has only finitely many simplices. But every n-simplex has only finitely many faces and by (1) we see that every simplex of K is the face of an n-simplex. Thus we are done.

(3) First note that if we are given a smooth simplicial structure on a smooth manifold that restricts to a simplicial structure on ∂M, then it follows easily from the definitions that this simplicial structure on ∂M is actually a smooth simplicial structure. Thus, to prove (3), it remains to show that the simplicial structure on $|K| = M$ restricts to a simplicial structure on the topological manifold ∂M.

If M is a smooth manifold, then we know by Proposition 6.27 (3a) that ∂M is a closed subset of M. In the general case that M is a topological manifold the same statement follows from Proposition 44.2.

Now that we know that ∂M is a closed subset of M we obtain the desired statement from Lemma 61.13 together with the following claim.

Claim. Let s be an m-simplex of K. Either $\langle s \rangle \cap \partial M = \varnothing$ or $\langle s \rangle \subset \partial M$.

Let s be a simplex. For any point $P \in \langle s \rangle$ we perform almost the same calculation as in (1):

$$\widetilde{\mathrm{H}}_{n-1-m}(\mathrm{Lk}(K,s)) \overset{\substack{\text{Proposition 66.2 applied to the } m\text{-simplex } s \\ \downarrow}}{\cong} \mathrm{H}_n(|K|,|K|\setminus\{P\}) \overset{\substack{\text{Lemma 44.1 applied to } |K|=M \\ \downarrow}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } P \in M\setminus\partial M, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from this calculation that $\langle s \rangle \cap \partial M = \varnothing$ if $\widetilde{\mathrm{H}}_{n-1-m}(\mathrm{Lk}(K,s)) \cong \mathbb{Z}$ and that $\langle s \rangle \subset \partial M$ if $\widetilde{\mathrm{H}}_{n-1-m}(\mathrm{Lk}(K,s)) = 0$. ⊞

(4) Let s be an $(n-1)$-simplex of M. We denote by r the order of s. We pick a point $P \in \langle s \rangle$. Similar to (1) we calculate that

$$\mathbb{Z}^{r-1} \underset{\substack{\uparrow \\ \text{since } s \text{ is an } (n-1)\text{-simplex and since there are no simplices of dimension } > n \text{ we} \\ \text{see that } \mathrm{Lk}(K,s) \text{ consists of } r \text{ points, the isomorphism now follows from Lemma 43.1 (5)}}}{\cong} \widetilde{\mathrm{H}}_0(\mathrm{Lk}(K,s)) \overset{\substack{\text{Proposition 66.2 applied to the } (n-1)\text{-simplex } s \\ \downarrow}}{\cong} \mathrm{H}_n(|K|,|K|\setminus\{P\}) \overset{\substack{\text{Lemma 44.1 applied to } |K|=M \\ \downarrow}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } P \in M\setminus\partial M, \\ 0, & \text{otherwise.} \end{cases}$$

Comparing the left-hand side with the right-hand side gives us the desired statement.

(5) Now we assume that M is in fact connected. We say that two n-simplices t and t' of K are *equivalent* if there exists a sequence $t = s_0, s_1, \dots, s_k = t'$ of n-simplices such that any two consecutive simplices have a common $(n-1)$-dimensional face. Recall that our task is to show that any two n-simplices are equivalent in the above sense.

The key to doing so is the following claim:

Claim. Let t and t' be two n-simplices. If $t \cap t' \neq \varnothing$, then t and t' are actually equivalent in the above sense.

Given $m \in \{0, \dots, n-1\}$ we say that two n-simplices x and x' are *m-equivalent* if there exists a sequence $x = s_0, s_1, \dots, s_k = x'$ of n-simplices such that any two consecutive simplices have a common face of dimension $\geq m$. It suffices to show that if we are given $m \in \{0, \dots, n-2\}$ such that any two n-simplices in K are m-equivalent, then any two n-simplices in K are also $(m+1)$-equivalent. In fact evidently it suffices to show that if we are given two n-simplices x and x' which have an m-simplex s in common, then x and x' are $(m+1)$-equivalent. To do so we first note that since $\dim(x) > \dim(s)$ we can pick a vertex w of $x \cap \mathrm{Lk}(K,s)$ and similarly we can pick a vertex w' of $x' \cap \mathrm{Lk}(K,s)$.

Now we want to show that $\mathrm{Lk}(K,s)$ is connected. To do so we pick a point $P \in \langle s \rangle$ and we perform basically the same calculation as in the previous arguments:

$$\widetilde{\mathrm{H}}_0(\mathrm{Lk}(K,s)) \underset{\substack{\uparrow \\ \text{Proposition 66.2 applied to the } m\text{-simplex } s \text{ and } j=m+1}}{\cong} \widetilde{\mathrm{H}}_{m+1}(|K|, |K| \setminus \{P\}) = \widetilde{\mathrm{H}}_{m+1}(M, M \setminus \{P\}) \underset{\substack{\uparrow \\ \text{by Lemma 44.1 since } m \leq n-2}}{=} 0.$$

It follows from this calculation together with Lemma 43.1 and Exercise 61.11 that the abstract simplicial complex $\mathrm{Lk}(K,s)$ is in fact connected in the sense of the definition on page 1516. This means in our setting that there exist 1-simplices $e_1, \dots, e_k \in S$ of $\mathrm{Lk}(K,s)$ such that the following holds:

$(*)$ We have $w \in e_1$, for every $i \in \{1, \dots, k-1\}$ we have $e_i \cap e_{i+1} \neq \varnothing$ and we have $w' \in e_k$.

Let $i \in \{1, \dots, k\}$. Since e_i is a simplex in $\mathrm{Lk}(K,s)$ there exists a simplex f_i of K with $e_i \cup s \subset f_i$. By (1) we can in fact find an n-simplex $\widetilde{f}_i$ of K with $e_i \cup s \subset f_i \subset \widetilde{f}_i$. We set $\widetilde{f}_0 := x$ and $\widetilde{f}_{k+1} := x'$. Note that it follows from $(*)$ that any two consecutive $\widetilde{f}_i$ and $\widetilde{f}_{i+1}$ have s and a vertex of $\mathrm{Lk}(K,s)$ in common, i.e. they have an $(m+1)$-simplex in common. This shows that x and x' are in fact $(m+1)$-connected. ⊞

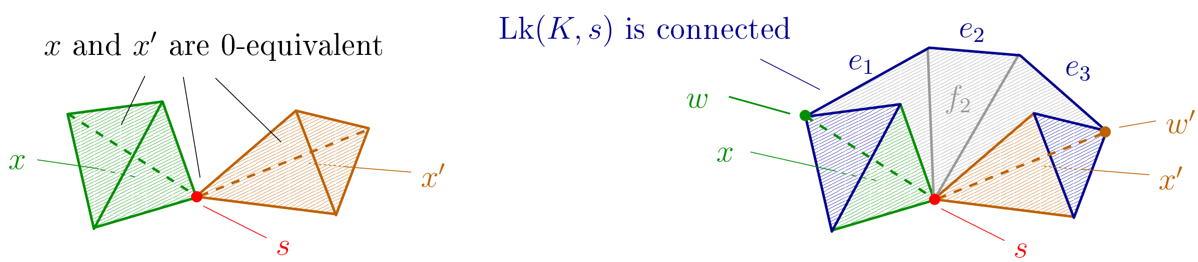


FIGURE 1010. Illustration for the proof of Proposition 66.1 (5).

We pick an equivalence class of n-simplices of K and we denote by J the subcomplex of K given by the union of the n-simplices in said equivalence class. Since J is a subcomplex we know by Lemma 61.8 that $|J|$ is a closed subset of $|K| = M$. Furthermore it follows from the claim together with Exercise 61.10 that $|J|$ is also an open subset of $|K| = M$. By hypothesis $|K| = M$ is connected. Thus we see that $K = J$, which implies that any two n-simplices in K are in fact equivalent. ■

Our main goal is to say something meaningful about homology groups of smooth manifolds. The idea is to use Proposition 66.1 and the fact that we can calculate singular homology groups via simplicial homology groups. Since we will only use the information given by Proposition 66.1 it is convenient to introduce the following definition.

Definition. Let $n \in \mathbb{N}_0$. An *n-dimensional pseudomanifold* is a non-empty abstract simplicial complex K such the following conditions are satisfied:

(1) Every simplex of K is a face of an n-simplex of K.[1007]
(2) Every $(n-1)$-simplex has order one or two.
(3) Given any two n-simplices t and t' there exists a sequence $t = s_0, s_1, \ldots, s_k = t'$ of n-simplices such that any two consecutive simplices have a common $(n-1)$-dimensional face.

Given a pseudomanifold K we define ∂K to be the subcomplex that is given by the union of all $(n-1)$-simplices of order 1. We say a pseudomanifold K is *closed* if $\partial K = \varnothing$ and if K is finite.

Examples.

(1) In Figure 1011 we illustrate the definition of a 2-dimensional pseudomanifold.

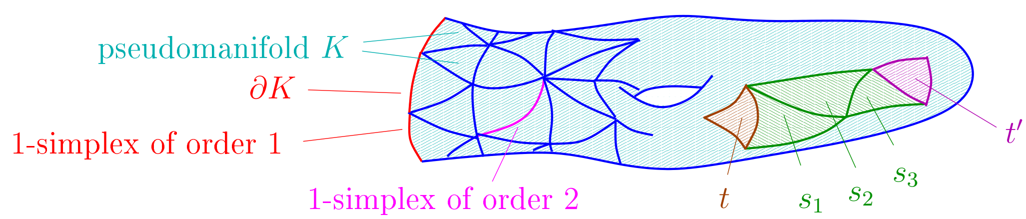


FIGURE 1011

(2) In Figure 1012 we show an abstract simplicial complex that satisfies (1) and (2) but that does not satisfy (3). This example shows that Condition (3) is stronger than requiring ordinary connectedness.

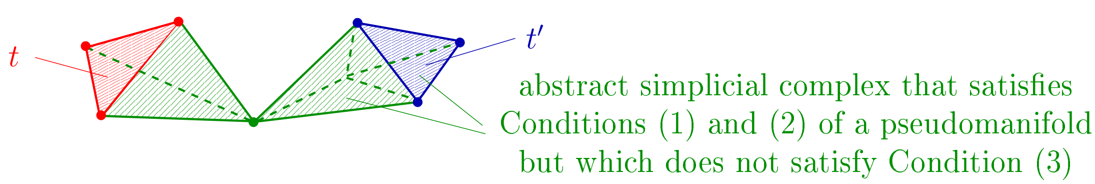


FIGURE 1012

Remark.

[1007] Note that this condition implies that K is necessarily n-dimensional.

(1) The definition of a pseudomanifold we give seems to be the one which is most frequently used, it goes back to work of Luitzen Brouwer [**Brou76**, p. 477] in 1912.
(2) Proposition 66.1 says that if we are given a simplicial structure $(K, \Theta\colon |K| \to M)$ of a connected non-empty n-dimensional topological manifold M, then K is an n-dimensional pseudomanifold.
(3) In Exercise 66.4 we will show that if K is an n-dimensional pseudomanifold with $n \in \{1, 2\}$, then $|K|$ is a closed n-dimensional smooth manifold.
(4) Let K be a finite n-dimensional pseudomanifold. As usual we denote by $\Sigma(K)$ the suspension as defined on page 1506. As we will see in Exercise 66.1, it follows very easily from the definitions that $\Sigma(K)$ is an $(n+1)$-dimensional pseudomanifold with $\partial\Sigma(K) = \Sigma(\partial K)$.

    Now assume that the fundamental group $\pi_1(|K|)$ of the topological realization $|K|$ is non-trivial. It follows from Corollary 61.21 together with Lemma 64.18 that the topological realization $|\Sigma(K)|$ of the suspension $\Sigma(K)$ is not a topological manifold. This shows that for $n \geq 3$ not all n-dimensional pseudomanifolds arise from topological manifolds.
(5) In the literature there exist several, not always entirely equivalent, notions of a pseudomanifold. In general, as we just saw in (3), the topological realizations of pseudomanifolds are not topological manifolds. But many tools and results on topological manifolds can be generalized to the context of (topological realization of) pseudomanifolds, see e.g. [**KiW06**, Chapter 4]. Also note that in the literature an n-dimensional pseudomanifold is sometimes called an *n-circuit*.

Next we recall and expand a definition from page 1544.

Definition. Let $K = (V, S)$ be an abstract simplicial complex.

(1) Let $n \in \mathbb{N}_0$. An *ordered n-simplex of K* is an $(n+1)$-tuple $(v_0, \dots, v_n) \in V^{n+1}$ such that $\{v_0, \dots, v_n\} \in S$. We write $\lfloor (v_0, \dots, v_n) \rfloor := \{v_0, \dots, v_n\}$.
(2) An *oriented n-simplex* is a pair (ϵ, s) where $\epsilon \in \{\pm 1\}$ and s is an ordered n-simplex of K. As in (1) we write $\lfloor (\epsilon, s) \rfloor := \lfloor s \rfloor$ and we refer to $\lfloor (\epsilon, s) \rfloor$ as the *underlying simplex*. Sometimes we also say that (ϵ, s) is an orientation of the n-simplex s.
(3) We say two oriented n-simplices $(\mu, s = (v_0, \dots, v_n))$ and $(\nu, t = (w_0, \dots, w_n))$ are *equivalent* if

$$\mu \;=\; \nu \cdot \underset{\substack{\uparrow \\ \text{sign of } \sigma \text{ as defined on page 1541}}}{\operatorname{sign}(\sigma)} \qquad \text{where } \sigma \in \operatorname{Bij}\{0, \dots, n\} \text{ with } v_i = w_{\sigma(i)} \text{ for } i = 0, \dots, n.$$

    It follows immediately from Lemma 63.1 that this is indeed an equivalence relation on the set of oriented n-simplices.
(4) An *orientation of an n-simplex s* is an equivalence class of oriented n-simplices with underlying simplex s.

Definition. Let $K = (V, S)$ be an abstract simplicial complex and let $(\epsilon, t = (v_0, \dots, v_n))$ be an oriented n-simplex. Let s be a codimension-one face of t. If v_i is the one vertex of t

that is missing in s, then we define the induced orientation of s to be

$$((-1)^i \cdot \epsilon, (v_0, \ldots, \widehat{v_i}, \ldots, v_n)).$$

Example. In Figure 1013 we show the oriented 2-simplex $(+1, (v_0, v_1, v_2))$ together with the orientations induced on all of the three codimension-one faces.

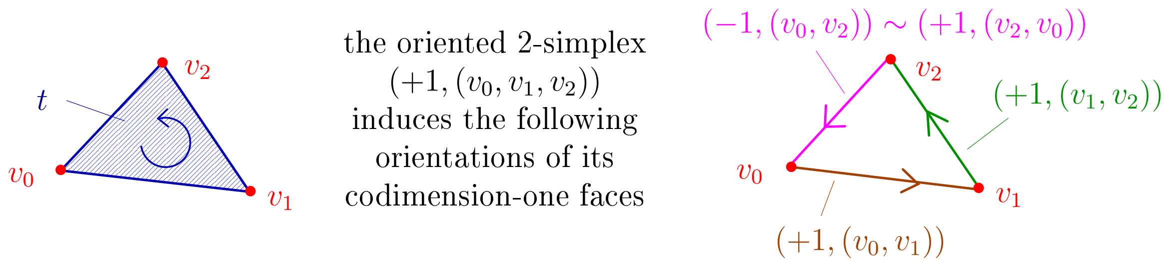


FIGURE 1013

Lemma 66.3. *Let $K = (V, S)$ be an abstract simplicial complex, let t be a simplex of K and let s be a codimension-one face of t. Any two equivalent orientations of t induce equivalent orientations of s.*

PROOF. This statement follows easily from the properties of the sign of a bijection, see Lemma 63.1. We leave it to the reader to fill in the details. ∎

Definition. Let K be an n-dimensional pseudomanifold.

(1) An *orientation* is a choice of an orientation for each n-simplex of K such that for any $(n-1)$-simplex s of order 2 the two cobounding n-simplices induce opposite orientations on s.
(2) We say K is *orientable* if K admits an orientation.

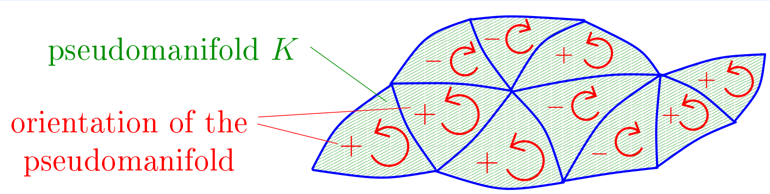


FIGURE 1014

In the following we will study the simplicial homology groups of pseudomanifolds. Before we do so, let us recall the definition of simplicial homology that we gave on page 1542 and in Lemma 63.6.

Definition. Let $K = (V, S)$ be an abstract simplicial complex and let $n \in \mathbb{N}_0$. We set

$$C_n^{\text{simp}}(K) := \left\{ \begin{array}{c} \text{free abelian group generated} \\ \text{by all ordered } n\text{-simplices of } K \end{array} \right\} \Big/ (v_0, \ldots, v_n) \sim \text{sign}(\sigma) \cdot (v_{\sigma(0)}, \ldots, v_{\sigma(n)}).$$

In the discussion below, we will often not distinguish in the notation between an ordered simplex and the element it represents in $\widetilde{\mathrm{C}}_k^{\mathrm{simp}}(K)$. The boundary map is given by

$$\begin{array}{rcl} \partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) & \to & \mathrm{C}_{n-1}^{\mathrm{simp}}(K) \\ (v_0,\dots,v_n) & \mapsto & \sum\limits_{i=0}^{n}(-1)^i\cdot(v_0,\dots,v_{i-1},\widehat{v_i},v_{i+1},\dots,v_n). \end{array}$$

Now we can provide the promised theorem.

Theorem 66.4. *Let K be an n-dimensional pseudomanifold. The following statements hold:*

(1) *We have an isomorphism as follows:*

$$\mathrm{H}_n^{\mathrm{simp}}(K) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } K \text{ is closed and orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if K is closed and orientable, and if $\{(\epsilon_w,Y_w)\}_{w\in W}$ is an orientation of K (here W denotes the set of n-simplices of K), then a generator of $\mathrm{H}_n^{\mathrm{simp}}(K)\cong\mathbb{Z}$ is represented by the simplicial cycle $\sum\limits_{w\in W}\epsilon_w\cdot Y_w\in \mathrm{C}_n^{\mathrm{simp}}(K)$.

(2) *If K is finite, then we also have an isomorphism as follows:*

$$\mathrm{H}_{n-1}^{\mathrm{simp}}(K) \;\cong\; \begin{cases} \text{free abelian group}, & \text{if } K \text{ is orientable or if } \partial K\neq\emptyset, \\ \text{free abelian group}\oplus\mathbb{Z}_2, & \text{if } K \text{ is non-orientable and } \partial K=\emptyset. \end{cases}$$

(3) *If K is finite, then*

$$\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } K \text{ is closed, regardless whether or not } K \text{ is orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if K is in fact closed and if $Y_1,\dots,Y_m$ is a set of ordered n-simplices of K such that for each n-simplex s there exists a unique $i\in\{1,\dots,m\}$ with $s=\lfloor Y_i\rfloor$, then the unique non-zero element of $\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2)\cong\mathbb{F}_2$ is represented by the simplicial cycle $Y_1\otimes 1+\cdots+Y_m\otimes 1\in \mathrm{C}_n^{\mathrm{simp}}(K;\mathbb{F}_2)=\mathrm{C}_n^{\mathrm{simp}}(K)\otimes\mathbb{F}_2$.

Theorem 66.4 leads us to the following definition.

Definition. Let K be a closed n-dimensional pseudomanifold.

(1) If K is equipped with an orientation $\{(\epsilon_w,Y_w)\}_{w\in W}$, then we refer to the simplicial homology class

$$[K] \;:=\; \Big[\sum_{w\in W}\epsilon_w\cdot Y_w\Big] \;\in\; \mathrm{H}_n^{\mathrm{simp}}(K)$$

as the *fundamental class of K*.

(2) We refer to to the unique non-zero element in $\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2)\cong\mathbb{F}_2$ as the $\mathbb{F}_2$-*fundamental class* $[K]_{\mathbb{F}_2}$.

For clarity we break the proof of Theorem 66.4 into three parts. For the proof of the first part it is convenient to introduce the following definitions which we will use on a few other occasions.

Definition. Let $K = (V, S)$ be an n-dimensional abstract simplicial complex such that every $(n-1)$-simplex has order ≤ 2.

(1) We denote by W the set of n-simplices of K.
(2) We denote by E_2 the set of $(n-1)$-simplices of order 2.
(3) Let $\Gamma = (W, E_2, \varphi)$ be the undirected abstract graph with vertex set W, with edge set E_2 and where $\varphi\colon E_2 \to \mathcal{P}(W)$ is the map that associates to each edge $e \in E_2$ the set $\{w \in W \mid e \text{ is a face of } w\}$. Note that for $e \in E_2$ the set $\varphi(e)$ has precisely two elements. This means that Γ is indeed an undirected abstract graph in the sense of the definition on page 226. We call Γ the *dual graph of the abstract simplicial complex* K. We refer to Figure 1015 for an illustration.

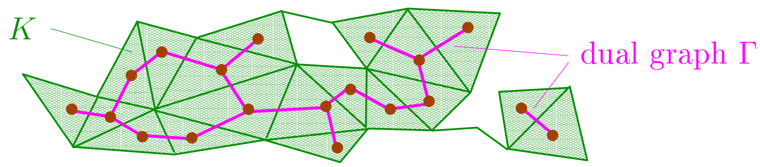


FIGURE 1015

PROOF OF THEOREM 66.4 (1). Let K be an n-dimensional pseudomanifold. Our goal is to calculate $\mathrm{H}_n^{\mathrm{simp}}(K)$. We start out with the following easy but useful observation which we will use throughout this proof without referring to it again explicitly:

$$\mathrm{H}_n^{\mathrm{simp}}(K) \underset{\substack{\uparrow \\ K \text{ has no simplices of dimension } n+1, \text{ hence } \mathrm{C}_{n+1}^{\mathrm{simp}}(K)=0}}{=} \ker\left(\partial_n\colon\ \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right).$$

To determine the right hand side of the above we need to introduce some notation.

(1) We denote by E the set of $(n-1)$-simplices of K. Furthermore we denote by E_1 the set of $(n-1)$-simplices of order 1 and similarly we denote by E_2 the set of $(n-1)$-simplices of order 2. Note that it follows from the hypothesis that K is an n-dimensional pseudomanifold that $E = E_1 \sqcup E_2$.
(2) We denote by W the set of n-simplices of K.
(3) Let $\Gamma = (W, E_2, \varphi)$ be the dual graph of K. Note that part (3) of the definition of a pseudomanifold implies that the undirected abstract graph Γ is connected in the sense of the definition on page 227.
(4) For each $(n-1)$-simplex $e \in E$ we pick, once and for all, an ordered $(n-1)$-simplex X_e with $\lfloor X_e \rfloor = e$. We use Lemma 63.5 to view $\{X_e\}_{e\in E}$ as a basis for $\mathrm{C}_{n-1}^{\mathrm{simp}}(K)$.
(5) We consider the two obvious projections

$$p_1\colon \overbrace{\bigoplus_{e\in E} \mathbb{Z}\cdot X_e}^{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)} \to \bigoplus_{e\in E_1} \mathbb{Z}\cdot X_e \quad \text{and} \quad p_2\colon \overbrace{\bigoplus_{e\in E} \mathbb{Z}\cdot X_e}^{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)} \to \bigoplus_{e\in E_2} \mathbb{Z}\cdot X_e$$

Since $E = E_1 \sqcup E_2$ we see for $s \in \mathrm{C}_{n-1}^{\mathrm{simp}}(K)$ that $s = 0$ if and only if $p_1(s) = 0$ and $p_2(s) = 0$.

Now suppose that for each $w \in W$ we are given an oriented n-simplex Y_w with $\lfloor Y_w \rfloor = w$. We introduce the following conventions.

(6) As above we use Lemma 63.5 to view $\{Y_w\}_{w\in W}$ as a basis for $C_n^{\text{simp}}(K)$.
(7) Given $e \in E$ and $w \in W$ we set[1008]

$$\mu_{ew} \;=\; \begin{cases} 0, & \text{if } e \text{ is not a face of } w, \\ +1, & \text{if } e \text{ is a face of } w \text{ and } [(+1, Y_w)] \text{ induces } [(+1, X_e)], \\ -1, & \text{if } e \text{ is a face of } w \text{ and } [(+1, Y_w)] \text{ induces } [(-1, X_e)]. \end{cases}$$

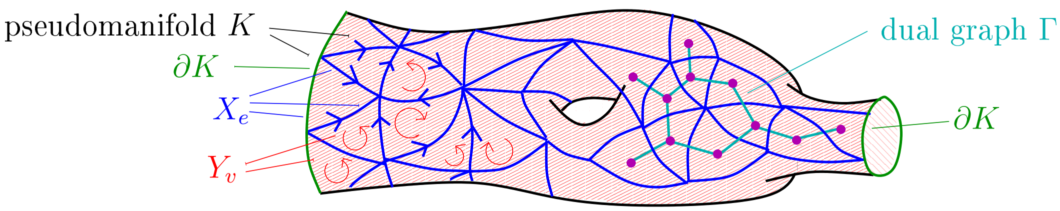


FIGURE 1016. Illustration for the proof of Theorem 66.4.

Claim 1. Given $\sum_{w\in W} a_w \cdot Y_w \in C_n^{\text{simp}}(K)$ we have

(a) $p_1\Big(\partial_n\Big(\sum_{w\in W} a_w \cdot Y_w\Big)\Big) = 0 \;\Leftrightarrow\;$ for all $w \in W$ that cobound an $e \in E_1$ we have $a_w = 0$

(b) $p_2\Big(\partial_n\Big(\sum_{w\in W} a_w \cdot Y_w\Big)\Big) = 0 \;\Leftrightarrow\;$ for all $w \neq w' \in W$ that cobound some $e \in E_2$ we have $\mu_{ew} \cdot a_w = -\mu_{ew'} \cdot a_{w'}$.

We start out our proof of Claim 1 with the following calculation:

$$\partial_n\Big(\sum_{w\in W} a_w \cdot Y_w\Big) = \sum_{w\in W} a_w \cdot \partial_n(Y_w) \underset{\uparrow}{=} \sum_{w\in W} a_w \cdot \sum_{\substack{e\in E \text{ such that} \\ e \text{ is a face of } w}} \mu_{ew} \cdot X_e = \sum_{e\in E} X_e \cdot \sum_{\substack{w\in W \text{ such that} \\ e \text{ is a face of } w}} \mu_{ew} \cdot a_w.$$

follows immediately from the definition of ∂_n, see page 1653, the definition of induced orientations, see page 1652, and the above definition of μ_{ew}

Now we consider the two projections.

(a) Note that an element $s \in C_{n-1}^{\text{simp}}(K)$ lies in the kernel of p_1 if and only if for all $e \in E_1$ the X_e-coefficient of s vanishes. Thus let $e \in E_1$. Let $w \in W$ be the unique n-simplex that cobounds e. Using the above calculation we see that

$$X_e\text{–coefficient of } \partial_n\Big(\sum_{w\in W} a_w \cdot Y_w\Big) \;=\; \mu_{ew} \cdot a_w.$$

The promised result follows from this discussion.

(b) Next let $e \in E_2$. Let $w \neq w'$ be the two n-simplices that cobound e. Similar to the above we see that

$$X_e\text{–coefficient of } \partial_n\Big(\sum_{w\in W} a_w \cdot Y_w\Big) \;=\; \mu_{ew} \cdot a_w + \mu_{ew'} \cdot a_{w'}.$$

[1008]Note that if e is a face of w, then precisely one of the two cases does indeed occur.

As in (a) we see that this calculation gives us the desired result.

Now note that Claim 1 follows almost immediately from the above discussion.

Claim 2. Every element $\sum\limits_{w\in W} a_w \cdot Y_w$ that lies in the kernel of the map

$$\mathrm{C}_n^{\mathrm{simp}}(K) \xrightarrow{\partial_n} \mathrm{C}_{n-1}^{\mathrm{simp}}(K) = \bigoplus_{e\in E} \mathbb{Z}\cdot X_e \xrightarrow{p_2} \bigoplus_{e\in E_2} \mathbb{Z}\cdot X_e$$

has the property that for all $w, w' \in W$ we have $|a_w| = |a_{w'}|$.

We consider the map

$$\begin{array}{rcl} f\colon W & \to & \mathbb{N}_0 \\ w & \mapsto & |a_w|. \end{array}$$

It follows from Claim 1 that for every $e \in E_2$ and $w, w' \in \varphi(e)$ we have $|a_w| = |a_{w'}|$. Since the abstract graph (W, E_2, φ) is connected we obtain from Lemma 4.5 that $f\colon W \to \mathbb{N}_0$ is constant. ⊞

Claim 3. Given any $w_0 \in W$ the map

$$\begin{array}{rcl} \Theta_{w_0}\colon \ker\left(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right) & \to & \mathbb{Z} \\ \sum\limits_{w\in W} a_w \cdot Y_w & \mapsto & a_{w_0} \end{array}$$

is a monomorphism.

It is clear that the given map is a homomorphism. It follows immediately from Claim 2 that the map Θ_{w_0} is a monomorphism. ⊞

Now we can prove Statement (1) of the proposition.

(i) We need to show that if K is infinite, then $\mathrm{H}_n^{\mathrm{simp}}(K) = 0$. So suppose that K is infinite. It follows from the fact that K is finite-dimensional and the fact that every simplex is contained in an n-simplex that W is also infinite. Now let $\sum\limits_{w\in W} a_w \cdot Y_w$ be an element in $\ker\left(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right)$. Since W is infinite and since by definition every element in $\mathrm{C}_n^{\mathrm{simp}}(K)$ is a finite linear combination of ordered simplices we see that there exists a $w_0 \in W$ with $a_{w_0} = 0$. But then it follows from Claim 2 that $a_w = 0$ for all $w \in W$.

(ii) We need to show that if $\partial K \neq \varnothing$, then $\mathrm{H}_n^{\mathrm{simp}}(K) = 0$. Thus suppose that $\partial K \neq \varnothing$. By definition of ∂K this implies that $E_1 \neq \varnothing$. We pick some $e_0 \in E_1$. Let w_0 be the unique n-simplex that cobounds e_0. Now let $\sum\limits_{w\in W} a_w \cdot Y_w$ be an element in $\ker\left(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\right)$. By Claim 1 we have $a_{w_0} = 0$. But then it follows from Claim 2 that $a_w = 0$ for all $w \in W$.

(iii) Now assume that $\mathrm{H}_n^{\mathrm{simp}}(K) \neq 0$. We need to show that K is actually orientable. To do so it suffices to prove the following claim.

Claim. If $\sum_{w\in W} a_w \cdot Y_w$ is a non-zero element in $\ker\big(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K)\to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\big)$, then the orientations[1009] $\{(\mathrm{sign}(a_w),Y_w)\}_{w\in W}$ define an orientation of K.

Let $\sum_{w\in W} a_w \cdot Y_w$ be a non-zero element in $\ker\big(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K)\to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\big)$. Let e be an $(n-1)$-simplex of order 2, i.e. let $e\in E_2$. Let $w\neq w'$ be the two cobounding n-simplices of e. By Claim 1 we know that $\mu_{ew}\cdot a_w = -\mu_{ew'}\cdot a_{w'}$. It follows from Claim 2 that we also have the equality $\mu_{ew}\cdot \mathrm{sign}(a_w) = -\mu_{ew'}\cdot \mathrm{sign}(a_{w'})$. But basically by definition that means that $(\mathrm{sign}(a_w),Y_w)$ and $(\mathrm{sign}(a_{w'}),Y_{w'})$ induce opposite orientations on the $(n-1)$-simplex e. ⊞

(iv) Finally we suppose that K is closed and orientable. Let $\{(\mu_w,Y_w)\}_{w\in W}$ be an orientation of K. We need to show that $\mathrm{H}_n^{\mathrm{simp}}(K)\cong\mathbb{Z}$ and that $\sum_{w\in W}\mu_w\cdot Y_w$ is a generator of $\mathrm{H}_n^{\mathrm{simp}}(K)$.

Since K is by definition non-empty we can pick some $w_0\in W$. We consider the maps

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\Psi} & \ker(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K)\to \mathrm{C}_{n-1}^{\mathrm{simp}}(K)) & \xrightarrow{\Theta_{w_0}} & \mathbb{Z} \\ r & \mapsto & r\cdot \sum_{w\in W}\mu_w\cdot Y_w & & \end{array}$$

It follows almost immediately from Claim 1 that the map Ψ to the left is well-defined, in the sense that it takes values in $\ker(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(K)\to \mathrm{C}_{n-1}^{\mathrm{simp}}(K))$. Furthermore it is clear that the composition $\Theta_{w_0}\circ\Psi$ is given by multiplication by μ_w, in particular the composition $\Theta_{w_0}\circ\Psi$ is an isomorphism. By Claim 3 we know that Θ_{w_0} is a monomorphism. It follows from this discussion that Ψ is an isomorphism. But that is exactly what we needed to show. ■

The proof of Statement (2) of Theorem 66.4 will require the following elementary lemma.

Lemma 66.5. *Let A,B,C and D be abelian groups. We denote by $p_1\colon B\oplus C\to B$ the projection. Now suppose we are given an exact sequence*

$$A \xrightarrow{\alpha} B\oplus C \xrightarrow{\varphi} D$$

of homomorphisms such that the following conditions are satisfied:

(1) *the group $B/p_1(\alpha(A))$ is torsion,*
(2) *the group D is torsion-free,*
(3) *the map $p_1\circ\alpha\colon A\to B$ is a monomorphism.*

Then the map

$$\begin{array}{rcl} \Theta\colon B/p_1(\alpha(A))\oplus\ker(C\xrightarrow{c\mapsto\varphi((0,c))} D) & \to & \dfrac{\ker(\varphi\colon B\oplus C\to D)}{\mathrm{im}(\alpha\colon A\to B\oplus C)} \\ ([b],c) & \mapsto & [(b,c)] \end{array}$$

is an isomorphism.

[1009] Here recall that given a non-zero integer $a\in\mathbb{Z}\setminus\{0\}$ we denote by $\mathrm{sign}(a):=\frac{a}{|a|}\in\{\pm1\}$ its sign. Also note that we know by hypothesis that at least one $a_w\neq 0$, but then it follows from Claim 2 that all $a_w\neq 0$, in particular it makes sense to consider the signs of the a_w.

PROOF OF LEMMA 66.5 (∗). From our hypotheses (1) and (2) and the exactness of the sequence we easily deduce the following statement:

(∗) The map $B \xrightarrow{b \mapsto \varphi((b,0))} D$ is the zero map.

Note that (∗) implies that if $\varphi((b,c)) = 0$, then we also have $\varphi((0,c)) = 0$. From this observation it follows immediately that Θ is an epimorphism. Now suppose we are given $b \in B$ and $c \in \ker(C \to D)$ with $\Theta([b],c) = 0$. This means that there exists an $a \in A$ with $\alpha(a) = (b,c)$. By the exactness of the sequence and by (∗) we know that there exists an $\widetilde{a} \in A$ with $\alpha(\widetilde{a}) = (b,0)$. From (3) we obtain that $a = \widetilde{a}$. Thus we see that $(b,c) = \alpha(a) = \alpha(\widetilde{a}) = (b,0)$. This shows that $[b] = 0 \in B/p_1(\alpha(A))$ and $c = 0$. In other words, we have shown that Θ is a monomorphism. ■

We move on to the proof of Statement (2) of Theorem 66.4.

PROOF OF THEOREM 66.4 (2). Let K be a finite n-dimensional pseudomanifold. Recall that we need to prove the following statements:

(a_1) If K is orientable and closed, then $\mathrm{H}_{n-1}^{\mathrm{simp}}(K)$ is a free abelian group.
(a_2) If $\partial K \neq \emptyset$, then $\mathrm{H}_{n-1}^{\mathrm{simp}}(K)$ is a free abelian group.
(b) If K is non-orientable and if $\partial K = \emptyset$, then $\mathrm{H}_{n-1}^{\mathrm{simp}}(K) \cong F \oplus \mathbb{Z}_2$ where F is a free abelian group.

In the following we will use freely the notation that we introduced in the proof of Statement (1). In particular we consider again the abstract graph $\Gamma = (W, E_2, \varphi)$. Since K is finite we know that Γ is finite, hence by Proposition 4.8 (1) we know that there exists a spanning tree T of Γ. Recall that this means that T is connected, that $\chi(T) = 1$ and, by Proposition 4.8 (2) it means that T contains every vertex. We pick $\widetilde{w} \in W$. By Corollary 4.7 we know that we can write $W = \{\widetilde{w}, w_1, \ldots, w_k\}$ and $E_2 = \{e_1, \ldots, e_k\}$ such that the following statements hold:

(1) For each $i \in \{1, \ldots, k\}$ the vertex w_i is an endpoint of the edge e_i.
(2) If some vertex w_i is an endpoint of some e_j, then $j \leq i$.

We introduce the following notation. We denote by

$$p_1\colon\ \mathrm{C}_{n-1}^{\mathrm{simp}}(K) \ \to\ \bigoplus_{i=1}^{k} \mathbb{Z} \cdot e_i \quad \text{ and } \quad p_2\colon\ \mathrm{C}_{n-1}^{\mathrm{simp}}(K) \ \to \bigoplus_{e \in E \setminus \{e_1,\ldots,e_k\}} \mathbb{Z} \cdot e_i$$

the obvious projection maps.

The key to the proof of Theorem 66.4 (2) is the following observation, which is an immediate consequence of Claim 1 together with the clever choice of $w_1, \ldots, w_k$ and $e_1, \ldots, e_k$.

Observation. The map $\bigoplus_{i=1}^{k} \mathbb{Z} \cdot w_i \xrightarrow{\partial_n} \mathrm{C}_{n-1}^{\mathrm{simp}}(K) \xrightarrow{p_1} \bigoplus_{i=1}^{k} \mathbb{Z} \cdot e_i$ is represented by a matrix of the form

$$D \ := \ \begin{pmatrix} \pm 1 & * & \ldots & * \\ 0 & \pm 1 & * & \vdots \\ 0 & 0 & \pm 1 & * \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}$$

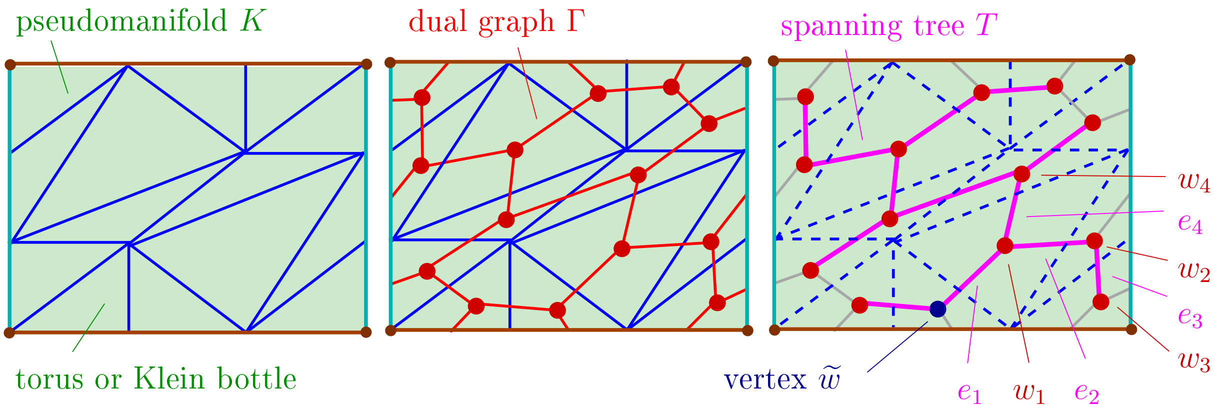


FIGURE 1017. Illustration for the proof of Theorem 66.4 (3).

(a$_1$) First we assume that K is orientable and closed. Let $\{(\epsilon_w, Y_w)\}_{w\in W}$ be an orientation of K. We set $\mu := \sum\limits_{w\in W} \epsilon_w \cdot Y_w$. Since $\epsilon_{\widetilde{w}} \in \{\pm 1\}$ we see that $Y_{w_1}, \dots, Y_{w_k}, \mu$ is a basis for $\mathrm{C}_n^{\mathrm{simp}}(K)$. This time we consider the following maps

$$\begin{tikzcd}
\overbrace{\bigoplus\limits_{i=1}^{k} \mathbb{Z}\cdot Y_{w_i} \oplus \mathbb{Z}\cdot \mu}^{=\mathrm{C}_n^{\mathrm{simp}}(K)} \arrow[rr, "\partial_n", "\mu\mapsto 0"'] \arrow[d, "\text{projection}"] & & \overbrace{\bigoplus\limits_{i=1}^{k} \mathbb{Z}\cdot X_{e_i} \oplus \bigoplus\limits_{e\neq e_1,\dots,e_k} \mathbb{Z}\cdot X_e}^{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)} \arrow[r, "\partial_{n-1}"] \arrow[d, "\text{projection } p_2"] & \mathrm{C}_{n-2}^{\mathrm{simp}}(K) \\
\bigoplus\limits_{i=1}^{k} \mathbb{Z}\cdot Y_{w_i} \arrow[rru, "\partial_n"] \arrow[rr, "\text{matrix } D", "\cong"'] & & \bigoplus\limits_{i=1}^{k} \mathbb{Z}\cdot X_{e_i}. &
\end{tikzcd}$$

As we already discussed in the proof of Statement (1), since K is closed we obtain from Claim 1 that $\partial_n(\mu) = 0$. Thus the image of ∂_n equals the image of the diagonal map. Furthermore, the bottom horizontal map is represented by the matrix D from the observation, in particular we see that the bottom horizontal map is an isomorphism. From Lemma 66.5 we now obtain the following isomorphism:

$$\underbrace{\ker\Big(\bigoplus_{e\neq e_1,\dots,e_k} \mathbb{Z}\cdot X_e \xrightarrow{\partial_{n-1}} \mathrm{C}_{n-2}^{\mathrm{simp}}(K)\Big)}_{\substack{\text{subgroup of a free abelian group thus by}\\ \text{Lemma 19.8 (3) it is a free abelian group}}} \xrightarrow{\cong} \underbrace{\frac{\ker\big(\mathrm{C}_{n-1}^{\mathrm{simp}}(K) \xrightarrow{\partial_{n-1}} \mathrm{C}_{n-2}^{\mathrm{simp}}(K)\big)}{\operatorname{im}\big(\mathrm{C}_{n}^{\mathrm{simp}}(K) \xrightarrow{\partial_{n}} \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\big)}}_{=\mathrm{H}_{n-1}^{\mathrm{simp}}(K)}.$$

(a$_2$) Now we assume that $\partial K \neq \varnothing$, i.e. we assume that $E_1 \neq \varnothing$. We pick an $\widetilde{e} \in E_1$. Let $\widetilde{w} \in W$ be the unique n-simplex that cobounds $\widetilde{e}$. We pick $w_1, \dots, w_k$ and $e_1, \dots, e_k$

accordingly. We consider the following maps

$$\begin{tikzcd}
\overbrace{\bigoplus_{i=1}^{k} \mathbb{Z}\cdot Y_{w_i} \oplus \mathbb{Z}\cdot Y_{\widetilde{w}}}^{=\mathrm{C}_n^{\mathrm{simp}}(K)} \arrow[r, "\partial_n"] \arrow[dr, "\cong", "\begin{pmatrix} D & * \\ 0 & \pm 1 \end{pmatrix}"'] & \overbrace{\bigoplus_{i=1}^{k} \mathbb{Z}\cdot X_{e_i} \oplus \mathbb{Z}\cdot X_{\widetilde{e}} \oplus \bigoplus_{e\neq e_1,\dots,e_k,\widetilde{e}} \mathbb{Z}\cdot X_e}^{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)} \arrow[r, "\partial_{n-1}"] \arrow[d, "\text{projection}"] & \mathrm{C}_{n-2}^{\mathrm{simp}}(K) \\
& \bigoplus_{i=1}^{k} \mathbb{Z}\cdot X_{e_i} \oplus \mathbb{Z}\cdot X_{\widetilde{e}}. &
\end{tikzcd}$$

It follows easily from the Observation and Claim 1 that the diagonal map is an isomorphism. It now follows from Lemma 66.5 that

$$\underbrace{\ker\Big(\bigoplus_{e\neq e_1,\dots,e_k,\widetilde{e}} \mathbb{Z}\cdot X_e \xrightarrow{\partial_{n-1}} \mathrm{C}_{n-2}^{\mathrm{simp}}(K)\Big)}_{\substack{\text{subgroup of a free abelian group thus by} \\ \text{Lemma 19.8 (3) it is a free abelian group}}} \xrightarrow{\cong} \underbrace{\frac{\ker\big(\mathrm{C}_{n-1}^{\mathrm{simp}}(K) \xrightarrow{\partial_{n-1}} \mathrm{C}_{n-2}^{\mathrm{simp}}(K)\big)}{\mathrm{im}\big(\mathrm{C}_{n}^{\mathrm{simp}}(K) \xrightarrow{\partial_{n}} \mathrm{C}_{n-1}^{\mathrm{simp}}(K)\big)}}_{=\mathrm{H}_{n-1}^{\mathrm{simp}}(K)}.$$

But the group on the right hand side is the subgroup of a finitely generated free abelian group, so it follows from Lemma 19.8 (3) that it is itself a finitely generated free abelian group.

(b) Finally we assume that K is closed and non-orientable. First we consider the following maps:

$$\begin{tikzcd}
\overbrace{\bigoplus_{w\in W} \mathbb{Z}\cdot Y_w}^{=\mathrm{C}_n^{\mathrm{simp}}(K)} \arrow[r, hook, "\partial_n"] \arrow[dr, "p_2\circ\partial_n"'] & \overbrace{\bigoplus_{i=1}^{k} \mathbb{Z}\cdot X_{e_i} \oplus \bigoplus_{e\neq e_1,\dots,e_k} \mathbb{Z}\cdot X_e}^{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)} \arrow[d, "\text{projection } p_2"] \\
& \bigoplus_{i=1}^{k} \mathbb{Z}\cdot X_{e_i}.
\end{tikzcd}$$

By our hypothesis that K is closed and non-orientable, together with Statement (1), we know that $\mathrm{H}_n^{\mathrm{simp}}(K) = 0$. This means that the horizontal map is a monomorphism. On the other hand, for rank reasons the diagonal map $p_2 \circ \partial_n$ has a kernel. In other words, there exists some non-zero $\mu = \sum\limits_{w\in W} a_w \cdot Y_w \in \mathrm{C}_n^{\mathrm{simp}}(K)$ with $p_2(\partial_n(\mu)) = 0$. By Claim 2 we know that the value $|a_w|$ is constant. Thus after dividing by this constant (which is non-zero since μ is non-zero) we can arrange that each $|a_w| = 1$.

Since K is by hypothesis closed, i.e. since each edge has order 2, we see, say by the calculation in the proof of Claim 1 or by direct inspection, that for each $e \in E$ the X_e-coefficient of $\partial_n(\mu)$ lies in $\{0, \pm 2\}$. Furthermore since ∂_n is a monomorphism we know that $\partial_n(\mu) \neq 0$. Thus there exists some $f \in E$ such that the f-coefficient of $\partial_n(\mu)$ equals ± 2. From $p_2(\partial_n(\mu)) = 0$ we obtain that $f \neq e_1, \dots, e_k$. Next we consider the following maps:

$$\begin{tikzcd}
\overbrace{\bigoplus_{i=1}^{k}\mathbb{Z}\cdot Y_{w_i}\oplus\mathbb{Z}\cdot\mu}^{=\mathrm{C}_n^{\mathrm{simp}}(K)} \arrow[r, hook, "\partial_n"] \arrow[dr, "\cong", "{\begin{pmatrix} D & * \\ 0 & \pm 2\end{pmatrix}}"'] & \overbrace{\bigoplus_{i=1}^{k}\mathbb{Z}\cdot X_{e_i}\oplus\mathbb{Z}\cdot X_f\oplus\bigoplus_{e\neq e_1,\dots,e_k,f}\mathbb{Z}\cdot X_e}^{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)} \arrow[r, "\partial_{n-1}"] \arrow[d, "\text{projection}"] & \mathrm{C}_{n-2}^{\mathrm{simp}}(K) \\
& \bigoplus_{i=1}^{k}\mathbb{Z}\cdot X_{e_i}\oplus\mathbb{Z}\cdot X_f. &
\end{tikzcd}$$

Finally we note that it follows from the above together with Lemma 66.5 that the following map is an isomorphism:

$$\underbrace{\ker\Big(\bigoplus_{e\neq e_1,\dots,e_k,f}\mathbb{Z}\cdot X_e\to\mathrm{C}_{n-2}^{\mathrm{simp}}(K)\Big)}_{\substack{\text{subgroup of a free abelian group thus by}\\ \text{Lemma 19.8 (3) it is a free abelian group}}}\oplus\underbrace{\mathbb{Z}^{k+1}\Big/\begin{pmatrix} D & * \\ 0 & \pm 2\end{pmatrix}\mathbb{Z}^{k+1}}_{\cong\,\mathbb{Z}_2\text{ by the observation}}\to\underbrace{\frac{\ker\big(\mathrm{C}_{n-1}^{\mathrm{simp}}(K)\to\mathrm{C}_{n-2}^{\mathrm{simp}}(K)\big)}{\operatorname{im}\big(\mathrm{C}_{n}^{\mathrm{simp}}(K)\to\mathrm{C}_{n-1}^{\mathrm{simp}}(K)\big)}}_{=\mathrm{H}_{n-1}^{\mathrm{simp}}(K)}.$$

We have thus finished the proof of Theorem 66.4 (2). ■

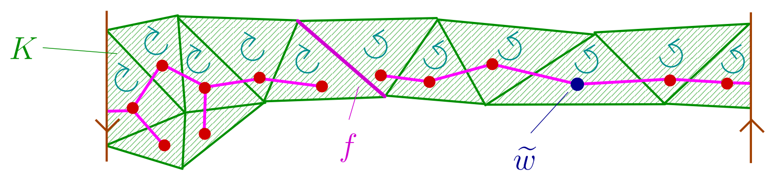


FIGURE 1018. Illustration for the proof of Theorem 66.4 (2).

Finally we get to the proof of the last statement of Theorem 66.4 (3).

PROOF OF THEOREM 66.4 (3). Let K be a finite n-dimensional pseudomanifold. First we point out that it would be a fun exercise to modify the proof of (1) to obtain the promised result. But in these notes we will not pursue this approach. Instead we will do a little trick. First we consider the case that K is closed. In this case we perform the following calculation:

$$\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2)\underset{\substack{\uparrow\\ \text{by the Algebraic Universal Coefficient Theorem 57.18}\\ \text{applied to the chain complex } \mathrm{C}_*^{\mathrm{simp}}(K)}}{\cong}\mathrm{H}_n^{\mathrm{simp}}(K)\otimes\mathbb{F}_2\oplus\mathrm{Tor}(\mathrm{H}_{n-1}^{\mathrm{simp}}(K),\mathbb{F}_2)\underset{\substack{\uparrow\\ \text{by (1) and (3)}}}{\cong}\begin{cases}\overbrace{\mathbb{Z}\otimes\mathbb{F}_2\oplus\mathrm{Tor}(\mathbb{Z}^l,\mathbb{F}_2)}^{\text{if } K \text{ is orientable}}\\ \underbrace{0\otimes\mathbb{F}_2\oplus\mathrm{Tor}(\mathbb{Z}^l\oplus\mathbb{Z}_2,\mathbb{F}_2)}_{\text{if } K \text{ is non-orientable}}\end{cases}\underset{\substack{\uparrow\\ \text{Lemma 57.3 and 57.17}}}{\cong}\mathbb{F}_2.$$

Now suppose that $Y_1,\dots,Y_m$ is a set of ordered n-simplices of K such that for each n-simplex s of K there exists a unique $i\in\{1,\dots,m\}$ with $s=\lfloor Y_i\rfloor$. One easily verifies that the simplicial chain $Y_1\otimes 1+\dots+Y_m\otimes 1\in\mathrm{C}_n^{\mathrm{simp}}(K;\mathbb{F}_2)=\mathrm{C}_n^{\mathrm{simp}}(K)\otimes\mathbb{F}_2$ is a simplicial cycle. Since it is non-zero we see that it equals necessarily the unique non-zero element of $\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2)=\ker(\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2)\to\mathrm{H}_{n-1}^{\mathrm{simp}}(K;\mathbb{F}_2))\cong\mathbb{F}_2$.

Finally if the pseudomanifold K is not closed, then the above approach shows almost immediately that $\mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2)=0$. With a slight sigh of relief we note that we have now finally managed to prove all three statements of Theorem 66.4. ■

66.2. **The top-dimensional homology group of a smooth manifold.** As the reader surely expected, the main application of Theorem 66.4 is the following theorem which gives us lots of interesting extra information on the homology groups of smooth manifolds.

Theorem 66.6. *Let M be a connected non-empty n-dimensional smooth manifold. The following statements hold:*

(1) *We have an isomorphism as follows:*

$$\mathrm{H}_n(M;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } M \text{ is closed and orientable,} \\ 0, & \text{otherwise.}\end{cases}$$

(2) *If M is compact, then we also have an isomorphism as follows:*

$$\mathrm{H}_{n-1}(M;\mathbb{Z}) \;\cong\; \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable or if } \partial M\neq\varnothing, \\ \text{free abelian group} \oplus \mathbb{Z}_2, & \text{if } M \text{ is non-orientable and closed.}\end{cases}$$

In both cases the free abelian group is finitely generated.

(3) *If M is compact, then*

$$\mathrm{H}_n(M;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } M \text{ is closed, regardless whether or not } M \text{ is orientable,} \\ 0, & \text{otherwise.}\end{cases}$$

Examples.

(1) Let $k\in\mathbb{N}$. We consider $k\cdot\mathbb{R}\mathrm{P}^2$, i.e. we consider the connected sum of k copies of $\mathbb{R}\mathrm{P}^2$. By Exercise 8.17 we know that $k\cdot\mathbb{R}\mathrm{P}^2$ is non-orientable. In Proposition 48.9 we showed that

$$\mathrm{H}_1(\underbrace{k\cdot\mathbb{R}\mathrm{P}^2}_{=N_k};\mathbb{Z}) \;\cong\; \mathbb{Z}^{k-1}\oplus\mathbb{Z}_2 \quad\text{ and }\quad \mathrm{H}_2(k\cdot\mathbb{R}\mathrm{P}^2;\mathbb{Z}) \;=\; 0.$$

This is reassuringly consistent with Theorem 66.6.

(2) Recall that the Möbius band M is a compact non-orientable 2-dimensional smooth manifold with non-empty boundary. Since M is homotopy equivalent to S^1 we see that $\mathrm{H}_1(M;\mathbb{Z})\cong\mathrm{H}_1(S^1;\mathbb{Z})\cong\mathbb{Z}$. This example illustrates that in Theorem 66.6 the "$\mathbb{Z}_2$"-term for a non-orientable smooth manifold only appears if M is closed.

(3) In Chapter 86 we will make sense of the notion of an orientable topological manifold. We will prove the generalization of Theorem 66.6 to the setting of topological manifolds in Theorems 87.1, 87.2 and 87.3 and Proposition 87.22. These new results completely subsume Theorem 66.6. Nonetheless it is worth covering the above proof since it is fairly geometric, whereas the latter proofs for topological manifolds are rather unintuitive.

Proof of Theorem 66.6. Let M be a connected non-empty n-dimensional smooth manifold. Note that by Theorem 64.2 we know that M admits a smooth simplicial structure $(K=(V,S),\Theta\colon |K|\to M)$. By Proposition 66.1 we know that K is an n-dimensional pseudomanifold. Evidently now we would like to apply Theorem 66.4 and call it a day.

But to do so we need to show that the hypotheses on the smooth manifolds do indeed match the corresponding hypotheses for pseudomanifolds. Let us start with the good news:

(1) By Lemma 61.9 (1) and Proposition 61.12 we know that M is compact if and only if K is finite.
(2) From Proposition 66.1 (1), (3) and (4) we obtain that $\partial M = \varnothing$ if and only if $\partial K = \varnothing$.

Here is the bad news: We still have to relate the (non-) orientability of the smooth manifold M to the (non-) orientability of the abstract simplicial complex K. This will be done in the subsequent Proposition 66.7. ■

The following proposition gives us the last piece of the proof of Theorem 66.6.

Proposition 66.7. *Let $n \in \mathbb{N}$ and let M be a connected non-empty n-dimensional smooth manifold.*[1010] *Furthermore let $(K = (V, S), \Theta\colon |K| \to M)$ be a smooth simplicial structure. The map*

$$\left\{\begin{array}{c}\textit{orientations of the smooth}\\ \textit{manifold } M \textit{ in the sense of the}\\ \textit{definition on page 297}\end{array}\right\} \xrightarrow{\cong} \left\{\begin{array}{c}\textit{orientations of the}\\ \textit{pseudomanifold } K \textit{ in the sense}\\ \textit{of the definition on page 1652}\end{array}\right\}.$$

$$\textit{orientation of } M \;\mapsto\; \begin{array}{l}\textit{for each } n\textit{-simplex } s \textit{ we pick a corresponding}\\ \textit{ordered simplex } Y_s \textit{ and we consider}\\ \textit{the oriented } n\textit{-simplex } [(\epsilon_s, Y_s)] \textit{ where}\\ \quad \epsilon_s = +1 \textit{ if } \Theta\circ\Phi^{\leq}_{Y_s}\colon \mathring{\Delta}^n \to M \textit{ is orientation-pres.}\\ \quad \epsilon_s = -1 \textit{ if } \Theta\circ\Phi^{\leq}_{Y_s}\colon \mathring{\Delta}^n \to M \textit{ is orientation-rev.}\end{array}$$

is a natural bijection.[1011] *In particular the smooth manifold M is orientable if and only if the pseudomanifold K is orientable.*

Proof of Proposition 66.7 (∗). In the following we will provide a proof of Proposition 66.7. To preserve the author's sanity we will skip a few technical calculations. We leave it to the reader to fill in the details.

Throughout the proof let M be a non-empty n-dimensional smooth manifold and let $(K = (V, S), \Theta\colon |K| \to M)$ be a smooth simplicial structure. First note that the case $n = 0$ is trivial, since every 0-dimensional smooth manifold and every 0-dimensional pseudomanifold is orientable. Thus in the following we assume that $n > 0$. Before we do anything else we recall and introduce a few definitions and we make a few basic comments regarding these definitions.

(a) Let $k \in \mathbb{N}_0$. On page 1075 we introduced
$$\mathring{\Delta}^k \;:=\; \{(t_0, \ldots, t_k) \in \Delta^k \,|\, \text{all coordinates are non-zero}\}.$$
Now we also define
$$\widetilde{\Delta}^k \;:=\; \{(t_0, \ldots, t_k) \in \Delta^k \,|\, \text{at least two coordinates are non-zero}\}.$$

[1010] We do not impose any other conditions, in particular neither do we assume that M is compact nor do we assume that $\partial M = \varnothing$.

[1011] We leave it to the reader to figure out what "natural" means in this context.

(b) Similar to the discussion on page 1075 we view $\mathring{\Delta}^k$ and $\widetilde{\Delta}^k$ in the obvious way[1012] as smooth manifolds. Note that $\mathring{\Delta}^k$ has no boundary and that the boundary of $\widetilde{\Delta}^k$ equals $\widetilde{\Delta}^k \cap \partial\Delta^k$. We equip the smooth manifolds $\mathring{\Delta}^k$ and $\widetilde{\Delta}^k$ with the orientation where for each $P \in \mathring{\Delta}^k$ a basis $v_1, \ldots, v_k$ of $\mathrm{V}_P\,\mathring{\Delta}^k$ is positive if the vectors $\{(1, \ldots, 1), v_1, \ldots, v_k\}$ are a positive basis for $\mathbb{R}^{k+1}$. As always we endow the boundary $\partial\widetilde{\Delta}^k$ of the oriented smooth manifold $\widetilde{\Delta}^k$ with the orientation given by Lemma 6.50.

(c) Note that for any $j \in \{0, \ldots, k\}$ the usual j-th face map $i_j^k\colon \Delta^{k-1} \to \Delta^k$ restricts to an embedding $i_j^k\colon \mathring{\Delta}^{k-1} \to \partial\widetilde{\Delta}^k \subset \widetilde{\Delta}^k$. Using Lemma 6.46 (2) it is straightforward to verify that the map $i_j^k\colon \mathring{\Delta}^{k-1} \to \partial\widetilde{\Delta}^k$ is orientation-preserving if and only if j is even.

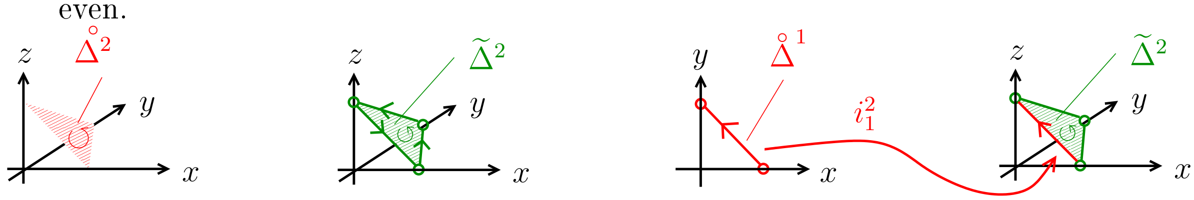


FIGURE 1019

(d) Let $k \in \mathbb{N}_0$ and let $Y = (v_0, \ldots, v_k)$ be an ordered k-simplex of $K = (V, S)$. On page 1487 we introduced the corresponding characteristic map

$$\begin{aligned} \Phi_Y^{\leq}\colon \Delta^k = \Big\{ (t_0, \ldots, t_k) \in \mathbb{R}^{k+1}_{\geq 0} \,\Big|\, \sum_{i=0}^k t_i = 1 \Big\} &\to |K| \\ (t_0, \ldots, t_k) &\mapsto \sum_{i=0}^k t_i \cdot v_i. \end{aligned}$$

(e) It follows almost immediately from the definition of a "smooth" simplicial structure, see page 1589, that for every ordered k-simplex $Y = (v_0, \ldots, v_k)$ the map $\Theta \circ \Phi_Y^{\leq}\colon \widetilde{\Delta}^n \to M$ is an immersion.

Now we turn to the actual proof of the proposition.

First we assume that we are given an orientation of the smooth manifold M. We need to equip the pseudomanifold $K = (V, S)$ with a natural orientation. We say that an ordered n-simplex $Y = (v_0, \ldots, v_n)$ is *positive* if the map $\Theta \circ \Phi_Y^{\leq}\colon \mathring{\Delta}^n \to M$ is orientation-preserving in the sense of the definition on page 299. Since $n \geq 1$ we see easily that, by possibly swapping the order of two vertices, we can find for each n-simplex $s \in S$ a positive ordered n-simplex Y_s with $\lfloor Y_s \rfloor = s$. It remains to prove the following claim.

Claim 1. The oriented n-simplices $[(+1, Y_s)]$ form an orientation of the pseudomanifold K.

Let e be an $(n-1)$-simplex of K of order 2 and let s, t be the two n-simplices that cobound e. We need to show that the orientations $[(+1, Y_s)]$ and $[(+1, Y_t)]$ induce opposite

[1012] Slightly more precisely, a smooth atlas for $\mathring{\Delta}^k$ is given by projection to the first k coordinates, and for $\widetilde{\Delta}^k$ we obtain a smooth atlas by the same projection and an affine linear map that sends points on a component of $\widetilde{\Delta}^k \cap \partial\Delta^k$ into the $(k-1)$-hyperplane.

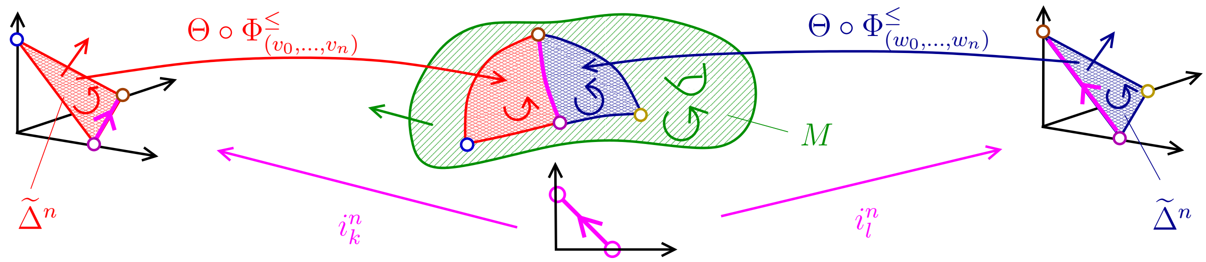


FIGURE 1020. Illustration for the proof of Proposition 66.7.

orientations on e. We write $Y_s = (v_0, \dots, v_n)$ and we write $Y_t = (w_0, \dots, w_n)$. Let v_k be the vertex missing in e and let w_l be the vertex missing in e. We consider the maps

$$\overset{\circ}{\Delta}{}^{n-1} \xrightarrow{i_k^n} \widetilde{\Delta}^n \xrightarrow{\Theta\circ\Phi^{\leq}_{(v_0,\dots,v_n)}} M \xleftarrow{\Theta\circ\Phi^{\leq}_{(w_0,\dots,w_n)}} \widetilde{\Delta}^n \xleftarrow{i_l^n} \overset{\circ}{\Delta}{}^{n-1}.$$

The two maps $\overset{\circ}{\Delta}{}^{n-1} \to M$ are immersions and the images coincide, thus we obtain a self-diffeomorphism of $\overset{\circ}{\Delta}{}^{n-1}$. A little bit of thought, using (c) and using Lemma 6.46 (2), shows that this map is orientation-preserving if and only if $k + l + 1$ is even.[1013] Next we need to associate to each orientation of the pseudomanifold K an orientation of the smooth manifold M. Thus suppose we are given an orientation $\{(\mu_s, Y_s)\}_{s\in S}$ of the pseudomanifold K. Since $n > 0$ we can and will assume that for each n-simplex s we have $\mu_s = 1$.

Claim 2. The smooth manifold $M \setminus \Theta(K^{n-2})$ has a natural orientation.[1014]

Let $P \in M \setminus \Theta(K^{n-2})$. Since every $(n-1)$-simplex is the face of an n-simplex we see that there exists an n-simplex s with $P \in \Theta(|s|)$. Since $P \notin \Theta(K^{n-2})$ we see that there exists a $Q \in \widetilde{\Delta}^n$ such that $P = \Phi^{\leq}_{Y_s}(Q)$. Since $\Phi^{\leq}_{Y_s} \colon \widetilde{\Delta}^n \to M$ is an immersion we know that the induced map $\mathrm{T}_Q\,\widetilde{\Delta}^k \to \mathrm{T}_P\,M$ is an isomorphism. Thus it makes sense to equip $\mathrm{T}_P\,M$ with the image of the positive orientation of $\widetilde{\Delta}^n$ under the above isomorphism. We need to show that the definition of the orientation does not depend on the choice of s. If $P \in M \setminus \Theta(K^{n-1})$, then we know from Lemma 61.11 that s is unique, hence we are done. Otherwise there exists an $(n-1)$-simplex e such that $P \in \Theta(\langle e\rangle)$. If the order of e equals one, then s is again unique and again we are done. Finally if the order of e equals two, then it follows from the hypothesis that $\{(\mu_s, Y_s)\}_{s\in S}$ is an orientation of K. A reasonably elementary calculation shows that the orientation on $\mathrm{T}_P\,M$ does not depend on the choice of the cobounding n-simplex. Finally we note that it is fairly straightforward to show that these orientations of the tangent spaces define an orientation, in the sense of the definition on page 297, of the smooth submanifold $M \setminus \Theta(K^{n-2})$. ⊞

We move on to the next claim.

Claim 3. For every open connected subset U of M the following holds:

[1013] Also note that the fact that $\overset{\circ}{\Delta}{}^n$ is dense in $\widetilde{\Delta}^n$ implies that the map $\Theta \circ \Phi^{\leq}_{Y} \colon \widetilde{\Delta}^n \to M$ is also orientation-preserving.

[1014] Note that by Lemma 61.8 we know that $M \setminus \Theta(K^{n-2})$ is an open subset of M, in particular it is a smooth submanifold of M.

(a) The set $U \cap (M \setminus \Theta(K^{n-2}))$ is connected.
(b) The closure of $U \cap (M \setminus \Theta(K^{n-2}))$ equals U.

For $i = 0, \dots, n-2$ we set $N_i = U \cap (M \setminus \Theta(K^{i-1}))$ and we set $Y_i = \Theta(K^i) \cap N_i$. Note that for each $i \in \{1, \dots, n-2\}$ we have $N_i = N_{i-1} \setminus Y_{i-1}$. It follows easily from the definition of a smooth simplicial structure together with Lemma 61.8 that each N_i is an n-dimensional smooth manifold and that each Y_i is a proper i-dimensional smooth submanifold of N_i. Since $i \leq n-2$ we obtain iteratively from the future Corollary 9.14 and the hypothesis that $N_0 = U$ is path-connected that each N_i is path-connected. Finally note that we know by Exercise 6.21 and the fact that $\dim(Y_i) = i < n = \dim(N_i)$ that the closure of $N_i \setminus Y_i$ is N_i. Note that the statements of Claim 3 follow almost immediately from these considerations.

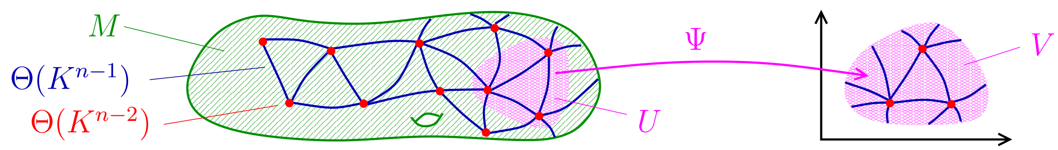


FIGURE 1021. Second illustration for the proof of Proposition 66.7.

Finally let $P \in M$. It follows quite easily from Claim (3a) that we can find a chart $\Psi \colon U \to V$ around P such that V is a connected open subset of $\mathbb{R}^n$ or H_n and such that Ψ is orientation-preserving for some $Q \in M \setminus \Theta(K^{n-2})$. Furthermore it follows from Claim (3a) and Lemma 6.46 (2) that the restriction of Ψ to $U \setminus \Theta(K^{n-2})$ is orientation-preserving at every point. Now we equip all $Q \in U$ with the orientation induced by Ψ and the standard orientation of $\mathbb{R}^n$ respectively H_n. It follows easily from Claim (3b) that these orientations do not depend on the choice of Ψ and that these orientations of the tangent spaces define an orientation of the smooth manifold M.

In the above discussion we constructed natural maps

$$\{\text{orientations of the smooth manifold } M\} \leftrightarrow \{\text{orientations of the pseudomanifold } K\}$$

in both directions. It follows reasonably easily from the definitions that the two maps are inverses of one another, thus both maps are natural bijections. ■

For manifolds with non-empty boundary the following theorem contains lots of extra information.

Theorem 66.8. *Let M be a connected non-empty n-dimensional smooth manifold. The following statements hold:*

(1) *We have an isomorphism as follows:*

$$\mathrm{H}_n(M, \partial M; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } M \text{ is orientable and compact,} \\ 0, & \text{otherwise.} \end{cases}$$

(2) *If M is compact, then we also have an isomorphism as follows:*

$$\mathrm{H}_{n-1}(M, \partial M; \mathbb{Z}) \;\cong\; \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable,} \\ \text{free abelian group} \oplus \mathbb{Z}_2, & \text{if } M \text{ is non-orientable.} \end{cases}$$

In both cases the free abelian group is finitely generated.

(3) *If M is compact, then we have* $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$.

Remark.

(1) Note that if M is a connected non-empty smooth manifold with $\partial M = \varnothing$, then the statements of Theorem 66.8 are contained in Theorem 66.6.

(2) We will prove the generalization of Theorem 66.6 to the setting of topological manifolds in Theorems 87.1 and 87.2 and Proposition 87.22.

Example. Let M be the Möbius band. Recall that M is a compact non-orientable 2-dimensional smooth manifold. In Exercise 43.7 we already saw that

$$\mathrm{H}_k(M, \partial M; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}_2, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

This calculation is of course consistent with Theorem 66.8.

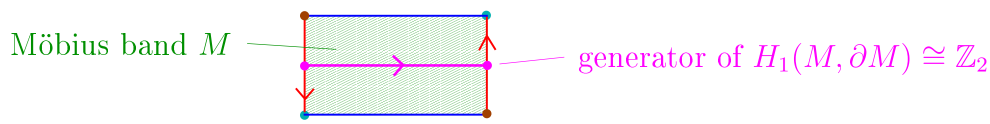


FIGURE 1022

PROOF. Let M be a connected non-empty n-dimensional smooth manifold. We dealt with the case $\partial M = \varnothing$ in Theorem 66.6 so we might as well assume that $\partial M \neq \varnothing$.

In the following we first sketch what we think should be the "correct" proof of Theorem 66.8 (1). Afterwards we will give the full details for an alternative proof.

(a) As before one uses Theorem 64.2 to equip M with some smooth simplicial structure $(K, \Theta\colon |K| \to M)$. It follows from Proposition 66.1 that K is an n-dimensional pseudomanifold such that ∂M corresponds precisely to ∂K.

(b) By Lemma 63.30 we have a natural isomorphism $\mathrm{H}^{\mathrm{simp}}_n(K, \partial K) \xrightarrow{\cong} \mathrm{H}_n(|K|, |\partial K|; \mathbb{Z})$.

(c) It is fairly straightforward to modify the argument of Theorem 66.4 to show that one has an isomorphism

$$\mathrm{H}^{\mathrm{simp}}_n(K, \partial K) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } K \text{ is orientable and finite,} \\ 0, & \text{otherwise.} \end{cases}$$

(d) By Proposition 66.7 we know that M is orientable if and only if K is orientable. The desired statement (1) now follows from the combination of all of the above.

The above approach is surely convincing and it is relatively straightforward to work out the details. Furthermore the same approach can also be employed to settle Statements (2) and (3).

Now we turn to our second approach to proving the proposition. This second approach has the advantage that this time we can provide all the details. Given M we consider its double $\mathrm{D}\,M = M \cup_{\partial M = \partial M'} M'$ where M' is a second copy of M. We refer to page 1163 for details and we refer to Figure 1023 for an illustration. Recall that by Lemma 44.12 we know the following:

(a) The double $\mathrm{D}\,M$ is also an n-dimensional smooth manifold such that $M \subset \mathrm{D}\,M$ is a smooth submanifold.
(b) The boundary of the double $\mathrm{D}\,M$ is empty.
(c) M is orientable if and only if $\mathrm{D}\,M$ is orientable.
(d) M is connected if and only if $\mathrm{D}\,M$ is connected.
(e) M is compact if and only if $\mathrm{D}\,M$ is compact.

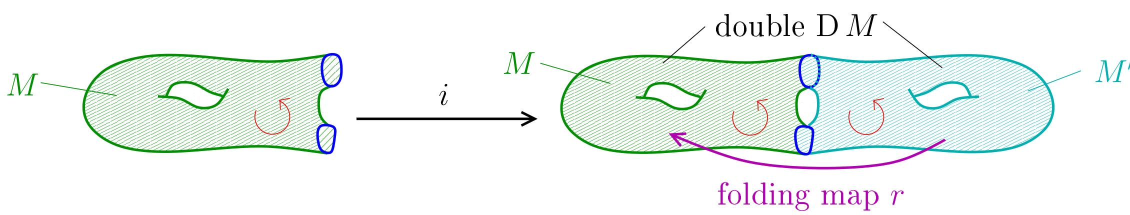


FIGURE 1023

In the following we denote by $i\colon M \to \mathrm{D}\,M$ the natural inclusion and we denote by $r\colon \mathrm{D}\,M \to M$ the folding map that we introduced on page 1164. Given $k \in \mathbb{N}_0$ we consider the following maps:

$$\begin{tikzcd}
 & & & \mathrm{H}_k(M, \partial M) \arrow[d, "\cong"] & & & \\
\ldots \arrow[r] & \mathrm{H}_k(M) \arrow[r, hook, "i_*"] & \mathrm{H}_k(\mathrm{D}\,M) \arrow[l, bend left, "r_*"] \arrow[r] & \mathrm{H}_k(\mathrm{D}\,M, M) \arrow[r, "\partial"] & \mathrm{H}_{k-1}(M) \arrow[r, hook, "i_*"] & \mathrm{H}_{k-1}(\mathrm{D}\,M) \arrow[l, bend left, "r_*"] \arrow[r] & \ldots
\end{tikzcd}$$

We make the following observations:

(f) The horizontal sequence is the long exact sequence in homology corresponding to the pair $(\mathrm{D}\,M, M)$.
(g) Since $r \circ i = \mathrm{id}_M$ we have $r_* \circ i_* = \mathrm{id}_{\mathrm{H}_k(M)}$, which implies that $i_*\colon \mathrm{H}_k(M) \to \mathrm{H}_k(\mathrm{D}\,M)$ is a monomorphism.
(h) By Lemma 44.12 we know that the vertical map is an isomorphism.

Next we recall some of our earlier calculations:

(i) Since $\partial M \neq \varnothing$ we obtain from Theorem 66.6 that $\mathrm{H}_n(M) = 0$ and that $\mathrm{H}_{n-1}(M)$ is a free abelian group.
(j) By Theorem 66.6 we have

$$\mathrm{H}_n(\mathrm{D}\,M) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } \mathrm{D}\,M \text{ is closed and orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

(k) By Theorem 66.6 and since $\partial(\mathrm{D}\,M) = \varnothing$ we have

$$\mathrm{H}_{n-1}(\mathrm{D}\,M) \;\cong\; \begin{cases} \text{free abelian group,} & \text{if } \mathrm{D}\,M \text{ is orientable,} \\ \text{free abelian group} \oplus \mathbb{Z}_2, & \text{if } \mathrm{D}\,M \text{ is non-orientable.} \end{cases}$$

Now we turn to the actual proofs of the three statements of the proposition.

(1) By the above we have an exact sequence

$$0 \longrightarrow \underbrace{\mathrm{H}_n(M)}_{=0} \underset{r_*}{\overset{i_*}{\rightleftarrows}} \mathrm{H}_n(\mathrm{D}\,M) \longrightarrow \underbrace{\mathrm{H}_n(\mathrm{D}\,M, M)}_{\cong \mathrm{H}_n(M,\partial M)} \xrightarrow{\partial} 0.$$

The desired statement now follows from (a)–(h) together with (i) and (j).

(2) By the above we have an exact sequence:

$$0 \longrightarrow \underbrace{\mathrm{H}_{n-1}(M)}_{\text{free abelian}} \underset{r_*}{\overset{i_*}{\rightleftarrows}} \mathrm{H}_{n-1}(\mathrm{D}\,M) \longrightarrow \underbrace{\mathrm{H}_{n-1}(\mathrm{D}\,M, M)}_{\cong \mathrm{H}_{n-1}(M,\partial M)} \xrightarrow{\partial} 0.$$

Since $r_* \circ i_* = \mathrm{id}_{\mathrm{H}_{n-1}(M)}$ we obtain from Splitting Lemma 46.2 and the above discussion that $\mathrm{H}_{n-1}(\mathrm{D}\,M) \cong \mathrm{H}_{n-1}(M, \partial M) \oplus$ free abelian group. The desired statement follows from (a)–(h) together with (k) and Lemma 19.8.

(3) We can deduce this statement from Theorem 66.6 (3) using a slight variation on the above arguments. Alternatively we can deduce (3) from (1) and (2) using the Universal Coefficient Theorem 57.19, the same way that we deduced Theorem 66.6 (3) from Theorem 66.6 (1) and (2). ■

66.3. **Homology groups of lens spaces.** In this section we will apply the results from the previous sections to complete the long overdue calculation of the homology groups of lens spaces.

Lemma 66.9. *Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. For the corresponding lens space $L(p, q)$ we have*

$$\mathrm{H}_n(L(p,q);\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3, \\ 0, & \text{if } n = 2 \text{ or } n > 3, \\ \mathbb{Z}_p, & \text{if } n = 1. \end{cases}$$

Remark. An alternative approach to computing the homology groups of lens spaces will be given in Exercise **??**.

The calculation of this lemma is rather disappointing, it shows that homology groups cannot distinguish between two lens spaces of the form $L(p, q)$ and $L(p, r)$. In particular we have made no progress at all on Question 16.7.

PROOF. We start out with the following observations:

(1) It follows from Proposition 16.9 and the definition of a lens space that $L(p, q)$ is finitely covered by S^3. It follows from $\chi(S^3) = 0$ together with Proposition 37.4 that $\chi(L(p, q)) = 0$.
(2) Note that on page 502 we pointed out that every lens space $L(p, q)$ is a closed orientable connected 3-dimensional smooth manifold.
(3) We already know from Proposition 41.5 that $\mathrm{H}_0(L(p, q)) \cong \mathbb{Z}$.
(4) By Corollary 52.6 we know that $\mathrm{H}_1(L(p, q)) \cong \mathbb{Z}_p$.

(5) By (2) and Proposition 64.6 (or alternatively Theorem 87.3) we know that $\mathrm{H}_n(L(p,q)) = 0$ for $n \geq 4$.
(6) By Theorem 66.6 (1) and (2) (or alternatively by Theorem 87.1) we know that $\mathrm{H}_3(L(p,q)) \cong \mathbb{Z}$.

Thus at the moment we are still missing $\mathrm{H}_2(L(p,q))$. Now note that

$$\operatorname{rank}\mathrm{H}_2(L(p,q)) \overset{\substack{\text{by Proposition 55.1}\\ \downarrow}}{=} \underbrace{\chi(L(p,q))}_{=0} - \underbrace{\operatorname{rank}\mathrm{H}_0(L(p,q))}_{=1} + \underbrace{\operatorname{rank}\mathrm{H}_1(L(p,q))}_{=0} + \underbrace{\operatorname{rank}\mathrm{H}_3(L(p,q))}_{=1} = 0.$$

Furthermore we know by Theorem 66.6 (2) that $\mathrm{H}_2(L(p,q))$ is a finitely generated free abelian group. Therefore we see that $\mathrm{H}_2(L(p,q)) = 0$. ■

66.4. **The Poincaré Homology Sphere.** We start out with the following almost self-explanatory definition of a homology sphere.

Definition.
(1) We say that an n-dimensional topological manifold X is a *topological homology n-sphere* if for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X;\mathbb{Z}) \cong \mathrm{H}_k(S^n;\mathbb{Z})$. Sometimes we also just say *homology n-sphere*.
(2) Sometimes, for disambiguation we also refer to such a topological manifold as an *integral homology n-sphere*.
(3) If X is actually a smooth manifold, then we refer to X as a *smooth homology sphere*.

For the record we state the following lemma which is an immediate consequence of Proposition 41.5 and Theorem 66.6.

Lemma 66.10. *Let M be a smooth manifold. If M is a homology n-sphere, then M is connected and orientable.*

It follows from the classification of 1-dimensional and 2-dimensional topological manifolds that we had stated in Theorem 7.1 and the Surface Classification Theorem 23.4 and the calculation of homology groups in Proposition 48.9 that for $n = 1,2$ every homology n-sphere is homeomorphic to the standard sphere S^n. In 1900 Henri Poincaré [**Poi1900**],[**PGL53**, p. 370] claimed that the same statement also holds for $n = 3$, i.e. he claimed that every homology 3-sphere is in fact homeomorphic to S^3.[1015] Shortly afterwards, in 1904, Henri Poincaré [**Poi1904, PGL53**] himself found a counterexample to this claim. More precisely, he proved the following proposition.

Proposition 66.11. *There exists a 3-dimensional smooth manifold M that is a homology 3-sphere, but such that $\pi_1(M)$ is a non-trivial group.*

Remark. In Exercise 66.13 and in Proposition 107.5 we will prove the higher-dimensional analogue of Proposition 66.11.

[1015] In fact in [**PGL53**, p. 498] he wrote "Pour ne pas trop allonger ce travail, je me bornerai à énoncer le théorème suivant dont la démonstration demanderait quelques développements: Tout polyèdre qui a tous ses nombres de Betti égaux Ã si et tous ses tableaux T_q bilatères est simplement connexe, c'est-à-dire homéomorphe à l'hypersphère.

Now we turn to the proof of Proposition 66.11. By now there are many proofs. In the following we will give Poincaré's example of such a smooth manifold. Later on in Exercise 68.4 we will give a different proof of Proposition 66.11.

To construct Poincaré's example we need to recall a few definitions and results from earlier on.

Definition.

(1) Let $z := 1 + \frac{\sqrt{5}}{2}$ be the golden ratio. As on page 1382 let D be the regular dodecahedron that is the convex hull of the 20 vertices

$$(\pm 1, \pm 1, \pm 1), \quad (0, \pm z, \pm \tfrac{1}{z}), \quad (\pm \tfrac{1}{z}, 0, \pm z), \quad \text{and} \quad (\pm z, \pm \tfrac{1}{z}, 0).$$

Furthermore let $\operatorname{Sym}(D)$ be the subgroup of $\mathrm{SO}(3)$ that preserves the regular dodecahedron, i.e. that preserves the set of these 20 vertices.

(2) Let $T := \{A \in \mathrm{M}(2 \times 2, \mathbb{C}) \,|\, A = \overline{A}^T \text{ and } \operatorname{tr}(A) = 0\}$. This is a 3-dimensional real vector space which we equip with the basis that is given by the matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad E_3 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

Furthermore we equip T with the form $(A, B) \mapsto g(A, B) := \frac{1}{2} \operatorname{tr}\left(A\overline{B}^T\right)$. Next recall that on page 1302 we showed that the map $T \to \mathbb{R}^3$ given by $E_i \to e_i$ defines an isometry $(T, g) \to (\mathbb{R}^3, \langle\ ,\ \rangle)$ which we now use to make the identification $\operatorname{SIsom}(T, g) = \mathrm{SO}(3)$. Furthermore, in Lemma 51.1 we showed that that map

$$\begin{array}{rcl} q\colon \mathrm{SU}(2) & \to & \operatorname{SIsom}(T, g) = \mathrm{SO}(3) \\ A & \mapsto & (B \mapsto ABA^{-1}) \end{array}$$

is a well-defined group homomorphism, that it is an epimorphism and that the kernel is given by $\pm\,\mathrm{id}$.

(3) We make the identification $S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$, i.e. we view S^3 as the set of quaternions of length 1. By Proposition 60.2 (3) we know that the following map is a diffeomorphism and a group isomorphism:

$$\begin{array}{rcl} f\colon S^3 & \to & \mathrm{SU}(2) \\ (z, w) = z + \mathrm{j} \cdot w & \mapsto & \underbrace{\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}}_{\substack{\text{matrix that represents} \\ \text{left-multiplication by } z + \mathrm{j} \cdot w \text{ with} \\ \text{respect to the complex basis } \{1, \mathrm{j}\} \text{ of } \mathbb{H}}} \end{array}$$

(4) We refer to $\Gamma := (q \circ f)^{-1}(\operatorname{Sym}(D)) \subset S^3$ as the *binary icosahedral group.*

Remark. It might be a little confusing that we refer to Γ as the binary *icosahedral* group. As we discussed on page 1383, the symmetry group of a regular icosahedron I equals the symmetry group of a regular dodecahedron D. Therefore in the above definition of the binary icosahedral group we could replace $\operatorname{Sym}(D)$ by $\operatorname{Sym}(I)$, which is more logical and more common. But since it is a little easier to describe $\operatorname{Sym}(D)$ geometrically we prefer to work with the regular dodecahedron.

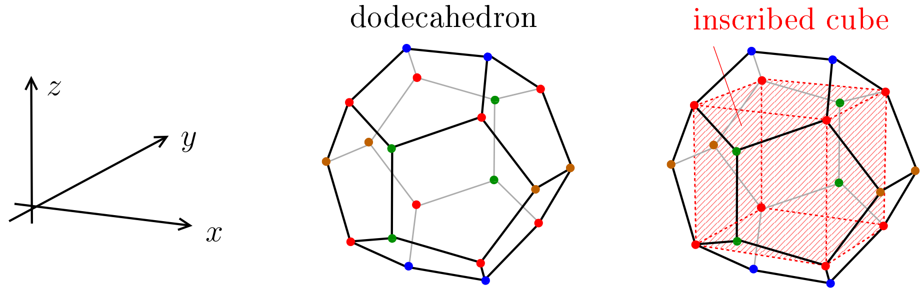


FIGURE 1024

Proposition 66.12.

(1) *The binary icosahedral group* Γ *is a group with 120 elements.*
(2) *The binary icosahedral group* Γ *has trivial abelianization.*
(3) *The quotient* $M := S^3/\Gamma$ *is a smooth homology 3-sphere with non-trivial fundamental group.*

Definition. The smooth manifold S^3/Γ is called the *Poincaré Homology Sphere.*

Remark. The Poincaré Homology Sphere is ubiquitous in topology and can be described in many different ways. The different avatars and also the history of the Poincaré Homology Sphere are discussed in detail in [**SW33**], [**Rolf90**, Chapter 10.E], [**Vol13**] and [**KSc79**]. In fact [**KSc79**] provides altogether *eight* different ways to construct the Poincaré Homology Sphere. In the following we outline a few of these alternative descriptions of the Poincaré Homology Sphere.

(1) In 1904, when Henri Poincaré discovered the eponymous sphere, he gave a different description in terms of gluing two "handlebodies". It is shown in [**SW33**] that this topological space is homeomorphic to the topological space we describe next in (2).
(2) Let D be the "solid" regular dodecahedron. Now identify each pentagon on the boundary with the opposite pentagon via a $\pi/5$ twist in the "clockwise" direction. As is explained in [**KSc79**], the resulting topological space is in fact homeomorphic to the Poincaré Homology Sphere as we defined it above.

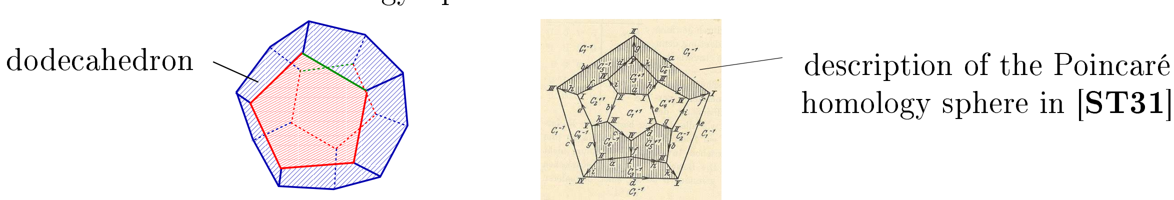


FIGURE 1025

(3) On page **??** we will give two descriptions of the Poincaré Homology Sphere as the result of a "surgery" on a knot respectively a link.

PROOF.

(1) Let $z := 1 + \frac{\sqrt{5}}{2}$ be the golden ratio. Recall that D is the polyhedron that is spanned by the 20 vertices

$$(\pm 1, \pm 1, \pm 1), \quad (0, \pm z, \pm \tfrac{1}{z}), \quad (\pm \tfrac{1}{z}, 0, \pm z), \quad \text{and} \quad (\pm z, \pm \tfrac{1}{z}, 0).$$

In Figures 858 and 1024 we show that one can inscribe five cubes $C_1, \ldots, C_5$ into the regular dodecahedron. As we already mentioned on page 1383, it is shown in [**Arm88**, p. 40] that the map

$$\begin{array}{rcl} \operatorname{Sym}(D) & \to & A_5 \cong \text{positive permutations of } \{C_1, \ldots, C_5\} \\ A & \mapsto & (C_i \mapsto A \cdot C_i) \end{array}$$

is an isomorphism. We consider the maps : $\mathrm{SU}(2) \to \mathrm{SIsom}(T, g) = \mathrm{SO}(3)$ and $f \colon S^3 \to \mathrm{SU}(2)$ that we defined above and we recall that the binary icosahedral group is defined as $\Gamma := (q \circ f)^{-1}(\operatorname{Sym}(D)) \subset S^3$. As we mentioned above, in Lemma 51.1 we showed that $\ker(q \circ f) = \pm 1 \in S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$. Furthermore it follows from Theorem 51.2 (1) that $q \circ f$ is an epimorphism. Since $\operatorname{Sym}(D) \cong A_5$ has $\frac{1}{2} 5! = 60$ elements and since $q \circ f$ has a kernel with two elements we see that Γ is a group with 120 elements.

(2) We need to show that the abelianization of Γ is trivial.[1016] By (1) we know that the group $\Gamma \subset S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$ fits into a short exact sequence

$$1 \;\to\; \{\pm 1\} \;\xrightarrow{\rho}\; \Gamma \;\xrightarrow{q \circ f}\; \operatorname{Sym}(D) \;\to\; 1.$$

By Proposition 21.20 we get an induced exact sequence

$$\{\pm 1\} \;\xrightarrow{\rho}\; \Gamma_{\mathrm{ab}} \;\xrightarrow{q \circ f}\; (\operatorname{Sym}(D))_{\mathrm{ab}} \;\to\; 1.$$

Almost every algebra book worth its money shows that $A_5 \cong \operatorname{Sym}(D)$ is a simple group, see e.g. [**Bog08**, Theorem 11.3]. This fact implies that the abelianization of $\operatorname{Sym}(D)$ is trivial. Thus it suffices to prove the following claim.

Claim. The quaternion $-1 \in S^3 = \{z \in \mathbb{H} = \mathbb{R}^4 \,|\, |z| = 1\}$ represents the trivial element in Γ_{ab}.

First recall that a straightforward explicit calculation, which you and I carried out in Exercise 51.1 and on page 1462, shows that

$$(q \circ f)(\mathrm{i}) = q\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{=:A} \text{ and } (q \circ f)(\mathrm{j}) = q\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{=:B}.$$

It follows immediately from the explicit description of the vertices of the regular dodecahedron D in (1) that the two matrices A and B to the right actually lie in $\operatorname{Sym}(D)$. Next note that it follows from the above calculation, that $\mathrm{i} = (q \circ f)^{-1}(A)$ and $\mathrm{j} = (q \circ f)^{-1}(B)$. In particular we see that that $\mathrm{i}, \mathrm{j} \in (q \circ f)^{-1}(\operatorname{Sym}(D)) = \Gamma$.

[1016] An argument similar to our proof of (2) is also provided in [**Bre93**, p. 354] and in [**Toen17**, p. 133].

By definition of the quaternion multiplication on page 1462 we have

$$[\mathrm{i},\mathrm{j}] \;=\; \underbrace{\mathrm{i}\cdot\mathrm{j}}_{=-\mathrm{j}\cdot\mathrm{i}}\cdot\mathrm{i}^{-1}\cdot\mathrm{j}^{-1} \;=\; -\mathrm{j}\cdot\mathrm{i}\cdot\mathrm{i}^{-1}\cdot\mathrm{j}^{-1} \;=\; -1.$$

We have thus shown that -1 is a commutator of elements in Γ. By definition of Γ_{ab} this implies that -1 represents indeed the trivial element in Γ_{ab}. ⊞

(3) Recall that it follows from Proposition 60.2 (3) that $\Gamma := (q\circ f)^{-1}(\mathrm{Sym}(D))$ is a finite subgroup of the *group* $S^3 = \{z\in\mathbb{H}=\mathbb{R}^4\,|\,|z|=1\}$. Since subgroups act freely on groups we see that Γ acts freely on S^3. Since Γ is a finite subgroup it now follows from Proposition 6.32 that $M := S^3/\Gamma$ is a closed 3-dimensional smooth manifold. It remains to show that $M = S^3/\Gamma$ is a homology sphere. First we perform the following calculation:

$$\mathrm{H}_1(S^3/\Gamma) \underset{\substack{\uparrow\\ \text{by the Hurewicz}\\ \text{Theorem 52.5}}}{\cong} \pi_1(S^3/\Gamma)_{\mathrm{ab}} \underset{\substack{\uparrow\\ \text{by Proposition 14.14}\\ \text{and Theorem 16.16}}}{\cong} \Gamma_{\mathrm{ab}} \underset{\substack{\uparrow\\ \text{by (2)}}}{=} 0.$$

Next note that by Proposition 64.6 we have $\mathrm{H}_i(M) = 0$ for $i\geq 4$. Furthermore note that by construction Γ is contained in $\mathrm{SU}(2)$. Thus we see that the action of Γ on S^3 is orientation-preserving.[1017] Therefore it follows from Proposition 6.47 that $M = S^3/\Gamma$ is actually orientable. This implies in turn by Theorem 66.6 that $\mathrm{H}_3(M)\cong\mathbb{Z}$. Thus it remains to show that $\mathrm{H}_2(M) = 0$. By Theorem 66.6 (2) we know that $\mathrm{H}_2(M)$ is a free abelian group. So it suffices to show that $b_2(M) = 0$. In fact we have

$$b_2(M) \underset{\substack{\uparrow\\ \text{since } b_0(M)=b_3(M)=1\\ \text{and } b_1(M)=0}}{=} \chi(M) \underset{\substack{\uparrow\\ \text{Proposition 37.4}\\ \text{and Theorem 64.5}}}{=} \frac{1}{k}\cdot\chi(S^3) \underset{\substack{\uparrow\\ \text{page 1363}}}{=} 0.$$

■

Remark. Let M be a homology 3-sphere with *finite* fundamental group. As is pointed out in [**Kerv69**, p. 67], it follows from the work of Michio Suzuki [**Su55**] or alternatively John Milnor [**Miln57b**] that $\pi_1(M)$ is isomorphic to the binary icosahedral group. In fact, it follows from the much more fancy Elliptisation Theorem **??** that such M is actually diffeomorphic to the Poincaré Homology Sphere.

We conclude this section with the statement of a conjecture that was formulated by Henri Poincaré [**PGL53**, p. 498] [**Poi1904**].

Conjecture 66.13. (Poincaré Conjecture) *If M is a homology 3-sphere that is simply connected, then M is homeomorphic to S^3.*

We will discuss the Poincaré Conjecture and various variations thereof in Chapter **??**.

[1017] Since the action of Γ is free it follows also from Exercise 3.28 that the action is orientation-preserving.

66.5. **The topology of smooth manifolds.** In this section we provide two results on the topology of smooth manifolds. Let us formulate our first result.

Proposition 66.14. *Let M be a connected non-empty n-dimensional smooth manifold. If M is non-compact or if M has non-empty boundary, then M is homotopy equivalent to an $(n-1)$-dimensional simplicial complex, in particular M is homotopy equivalent to an $(n-1)$-dimensional* CW*-complex.*

Examples.

(1) Given $g \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ we denote as usual by $\Sigma_{g,k}$ the surface of genus g with k open disks removed. In Figure 1026 we sketch a deformation retraction from $\Sigma_{2,2}$ to a CW-complex of Euler characteristic -4.
More generally, almost the same argument shows that $\Sigma_{g,k}$ with $k \geq 1$ admits a deformation retraction to a CW-complex with k 0-cells and $2g+2(k-1)$ 1-cells, i.e. a 1-dimensional CW-complex of Euler characteristic $k-(2g+2(k-1)) = -2g-k+2$. Using Proposition 37.8 (2) we also see that $\Sigma_{g,k}$ is homotopy equivalent to the wedge of $2g+k-1$ circles.

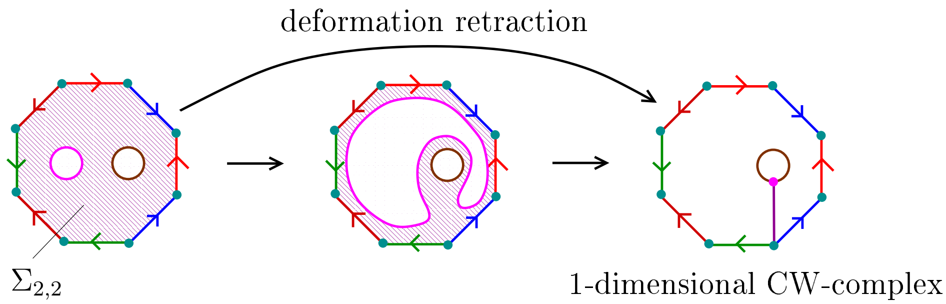


FIGURE 1026

(2) The non-compact smooth manifold that is given by removing a point from the torus is easily seen to be homotopy equivalent to the wedge of two circles.
(3) In Figure 1027 we show the surface M of infinite genus, which is non-compact, and a 1-dimensional CW-complex that is deformation retract of M. It is admittedly not overly obvious why the 1-dimensional CW-complex is indeed a deformation retract of M.

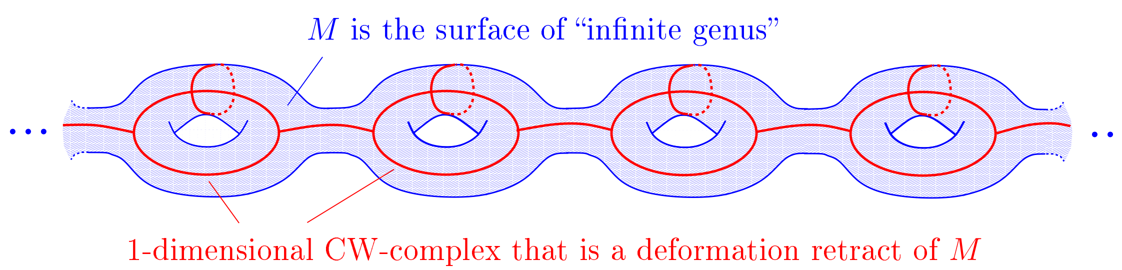


FIGURE 1027

Remark. Let M be a compact connected n-dimensional smooth manifold with non-empty boundary. By Proposition 66.14 we now know that M is homotopy equivalent to a finite

$(n-1)$-dimensional CW-complex which implies by Proposition 48.5 that $H_n(M) = 0$ and that $H_{n-1}(M)$ is a finitely generated free abelian group. This is of course consistent with Theorem 66.6.

The proof of Proposition 66.14 requires some preparations.

Definition. Let $K = (V, S)$ be an n-dimensional abstract simplicial complex such that every $(n-1)$-simplex has order ≤ 2. We denote by W the set of n-simplices and we denote by E the set of $(n-1)$-simplices.

(1) We say an n-simplex is *free* if it admits a codimension one face of order 1. Otherwise we say that the n-simplex is *unfree*.

(2) A *boundary route* is a subgraph $\widetilde{\Gamma} = (W, \widetilde{F}, \varphi)$ of the dual graph[1018] $\Gamma = (W, E_2, \varphi)$ with the following property: every n-simplex of K, i.e. every vertex of the dual graph, is connected in $\widetilde{\Gamma}$ to a free n-simplex.

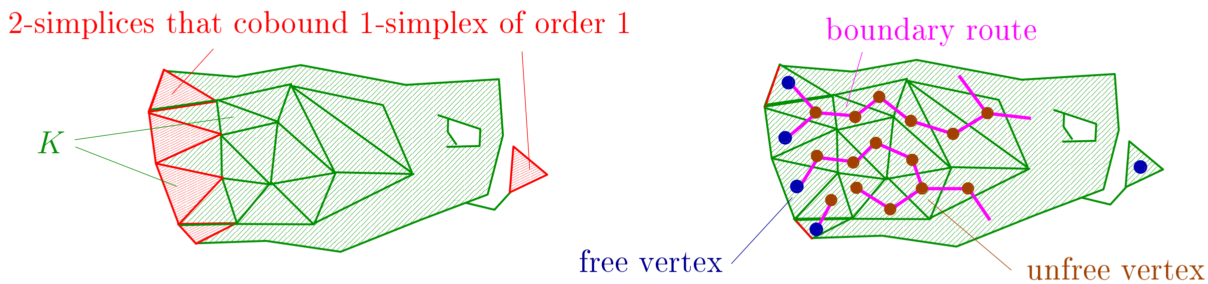


FIGURE 1028

Lemma 66.15. *Let $K = (V, S)$ be an n-dimensional abstract simplicial complex that admits a boundary route. There exists a subcomplex $K' = (V', S')$ of $K = (V, S)$ with the following three properties:*

(1) *$K' = (V', S')$ admits a boundary route.*

(2) *$|K'|$ is a deformation retract of $|K|$.*

(3) *The number of n-simplices of K' is less than the number of n-simplices of K.*

PROOF OF LEMMA 66.15. Let $K = (V, S)$ be an n-dimensional abstract simplicial complex that admits a boundary route $\widetilde{\Gamma} = (W, \widetilde{F}, \varphi)$. Since K is n-dimensional we know that it contains at least one n-simplex and since K admits a boundary route we see that there exists an n-simplex w that is free. By definition this means that w admits a codimension-one face e of order 1. We denote by $f_1, \dots, f_n$ the remaining codimension-one faces of w. We set

$$K' := (V, S \setminus \{w, e\}).$$

Note that K' is indeed a subcomplex of K. A slight generalization of Exercise 18.24 shows that there exists a deformation retraction from $|w|$ to $\bigcup_{i=1}^{n} |f_i|$. It follows from Lemma 61.8 together with Lemma 3.10 that this deformation retraction together with the constant homotopy on $|K'|$ defines a deformation retraction from $|K|$ to $|K'|$.

[1018] We refer to page 1654 for the definition of the dual graph.

Since every $(n-1)$-simplex of K has order ≤ 2 we see see that every n-simplex of K' that contains one of the f_i as a face is a free n-simplex of K'. Using this observation it is now straightforward to see a boundary route for K' is given by $(W\setminus\{w\}, \{\widetilde{f} \in \widetilde{F} \,|\, \widetilde{f} \not\subset w\}, \varphi)$. ■

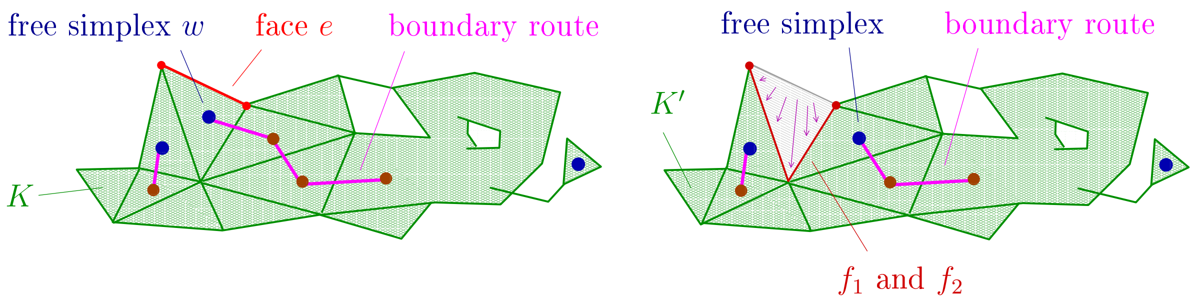


FIGURE 1029. Illustration for the proof of Lemma 66.15.

Now we can provide the proof of Proposition 66.14.

PROOF OF PROPOSITION 66.14 IN THE COMPACT CASE. Let M be a compact connected non-empty n-dimensional smooth manifold with non-empty boundary. Recall that by Theorem 64.2 we know that M admits a simplicial structure $(K = (V, S), \Theta\colon |K| \to M)$ such that ∂M is a subcomplex. We need to show that $|K|$ is homotopy equivalent to a finite $(n-1)$-dimensional simplicial complex.

Let $\Gamma = (W, E_2, \varphi)$ be the dual graph of K. By Proposition 66.1 (5) we know that Γ is connected. Thus it follows from Proposition 4.8 (1) that Γ admits a spanning tree T. By definition T is connected and by Proposition 4.8 we know that a spanning tree contains all vertices. Since $\partial M \neq \varnothing$ we know by Proposition 66.1 that $\partial K \neq \varnothing$. Thus we see that T is in fact a boundary route for K. This means that we can start up the machine given by Lemma 66.15. Iteratively applying Lemma 66.15 we get the desired deformation retraction. ■

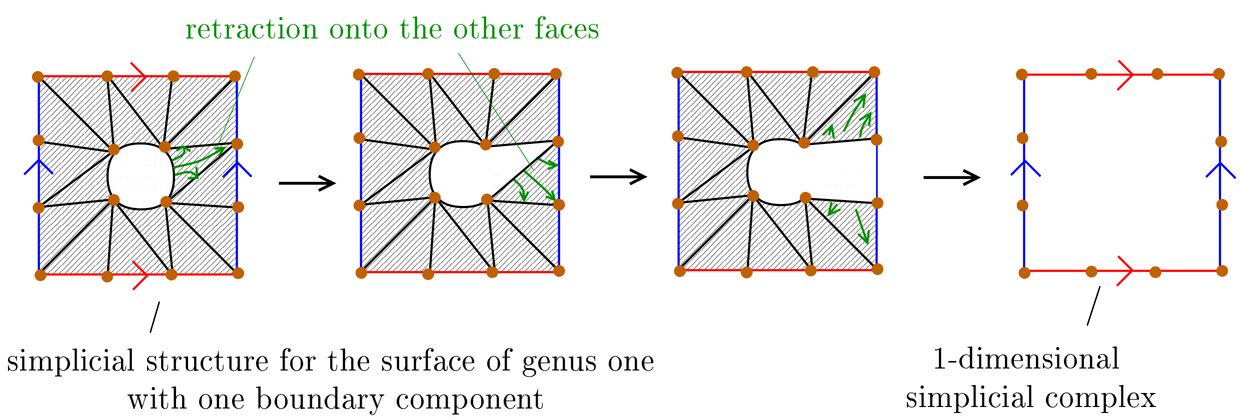


FIGURE 1030. Illustration for the proof of Proposition 66.14.

Proof of Proposition 66.14 in the non-compact case. Let M be a non-compact connected n-dimensional smooth manifold. It follows from [**NR04**, Theorem 2.2], or alternatively [**WhdJ61b**, Lemma 2.1], together with Proposition 62.12 that M is homotopy equivalent to an $(n-1)$-dimensional simplicial complex.

We will not attempt to give a full proof of our own, but we will sketch an outline of a proof. Evidently we again use the fact that by Theorem 64.2 we know that we can equip M with a simplicial structure $(K=(V,S),\Theta\colon |K|\to M)$. It is fairly straightforward to show that there exists a sequence $Y_1,Y_2,\dots$ of non-empty subcomplexes of K with the following properties:

(1) The sequence is nested, i.e. for each $i\in\mathbb{N}$ we have $Y_i\subset Y_{i+1}$.
(2) Each X_i is a finite n-dimensional pseudomanifold.
(3) We have $\bigcup\limits_{i\in\mathbb{N}} X_i = K$.

Now, with a considerable effort one can modify the above arguments used in the compact case to show that there exists a sequence of $(n-1)$-dimensional subcomplexes $X_i\subset Y_i$ with the following properties:

(4) For each $i\in\mathbb{N}$ we have $X_i\subset X_{i+1}$.
(5) Each $|X_i|$ is a deformation retract of $|Y_i|$.

Finally we set $X := \bigcup\limits_{i\in\mathbb{N}} X_i$. Note that X is an $(n-1)$-dimensional subcomplex of Y. It remains to show that X is a deformation retract of Y.

We pick a base point $*\in$. Using the above one can show, reasonably easily, that the inclusion induced map $\pi_n(X,*)\to\pi_n(Y,*)$ is an isomorphism for every $n\in\mathbb{N}$. It follows from Proposition 119.10[1019], which is a variation on the Whitehead Theorem 119.9, that X is indeed a deformation retract of Y. We leave it as a charming challenge to the reader to turn this idea into a rigorous proof. ■

We give a new proof of the following proposition.

Proposition 31.5. *Let M be a connected 2-dimensional smooth manifold. If M is non-compact, then $\pi_1(M)$ is a free group.*

Proof. Let M be a connected 2-dimensional smooth manifold that is non-compact. By Proposition 66.14 we know that M is homotopy equivalent to a 1-dimensional CW-complex. It follows from Proposition 18.16 (2) and Proposition 37.9 (1) that $\pi_1(M)$ is a free group. ■

We conclude this chapter with the following proposition.

Proposition 66.16. *Let M be a compact connected n-dimensional smooth manifold. There exists a* CW*-structure for M with a single n-cell.*

Remark. For fans of small CW-complexes we also recall Proposition 39.11 which says that any 0-connected CW-complex X is homotopy equivalent to a CW-complex Y with a single 0-cell which has the property that for any $n\geq 2$ the number of n-cells of X equals the number of n-cells of Y.

[1019]My apologies for referring to a future result.

In the proof of Proposition 66.16 we will make use of the following lemma.

Lemma 66.17. *Let L be an n-dimensional pseudomanifold that contains two subcomplexes J and K. If K consists of a single n-simplex, if $J \cap K$ consists of a single $(n-1)$-simplex e and if e has order 1 in J, then there exists a homeomorphism $(|L|, |\partial L|) \to (|J|, |\partial J|)$ of pairs of topological spaces.*

SKETCH OF A PROOF OF LEMMA 66.17. Let w be the one n-simplex of J that cobounds e. It is elementary, albeit ever-so-slightly painful, to write down an explicit homeomorphism $|w| \cup |K| \to |w|$ that is the identity on $|\partial w| \setminus (|\partial w| \cap |K|)$. Together with the identity on $|J| \setminus |w|$ this homeomorphism gives us the desired homeomorphism from $|L|$ to $|J|$. It follows almost immediately from the definitions that this homeomorphism restricts to a homeomorphism $|\partial L| \to |\partial J|$. We leave it to the reader to fill in the details. ■

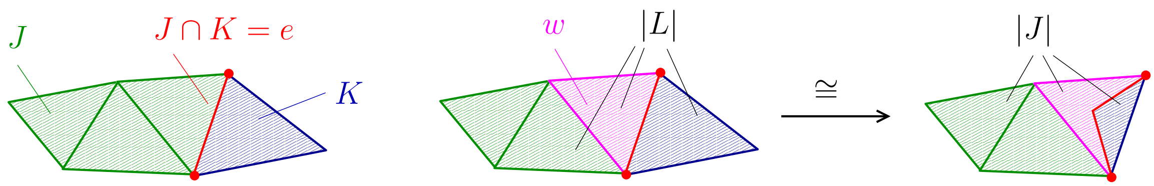


FIGURE 1031. Illustration of Lemma 66.17.

SKETCH OF A PROOF OF PROPOSITION 66.16. We sketch a proof of the proposition. The statement is also proved with fewer details but more authority in [**Mc02**, Theorem 5.1].

Let M be a closed connected non-empty n-dimensional smooth manifold. By Theorem 64.2 we know that M admits a simplicial structure $(K = (V, S), \Theta\colon |K| \to M)$. We need to show that $|K|$ admits a CW-structure with a single n-cell.

Let $\Gamma = (W, E_2, \varphi)$ be the dual graph of K. By Proposition 66.1 (5) we know that Γ is connected. Thus it follows from Proposition 4.8 (1) that Γ admits a spanning tree T. Note that by Proposition 4.8 (2) we know that T contains every vertex of Γ.

Let X be the subcomplex of K that is given by removing all n-simplices and by removing all $(n-1)$ that do not correspond to an edge of the spanning tree T. Note that X is an $(n-1)$-dimensional simplicial complex. Furthermore let L be the abstract simplicial complex that is given by the "disjoint union of the n-simplices of K and gluing them along the $(n-1)$-simplices corresponding to the edges of T". It is not difficult, but also not particularly instructive, to write down a completely rigorous definition of L. Note that L comes with a natural simplicial map $f\colon L \to K$.

Next note that using an induction argument based on Lemma 66.17, and which also makes use of Lemma 41.1, it is rather straightforward to show that there exists a homeomorphism $\Psi\colon (\overline{B}^n, S^{n-1}) \to (\Delta^n, \partial\Delta^n) \to (|L|, |\partial L|)$. Finally, using Proposition 2.43 (3) one can verify that the map

$$\begin{array}{rcl} (\overline{B}^n \sqcup |X|)/\underset{\downarrow}{\overset{\text{where } x \sim (\Psi\circ|f|)(x) \text{ for every } x\in S^{n-1}}{}}\!\!\sim & \to & |K| \\ {[P]} & \mapsto & \begin{cases} (\Psi \circ |f|)(P), & \text{if } P \in \overline{B}^n, \\ P, & \text{if } P \in |X| \subset |K| \end{cases} \end{array}$$

is a homeomorphism. The left-hand side admits an obvious CW-structure with a single n-cell. Thus we have shown that $|K|$ admits a CW-structure with a single n-cell. ■

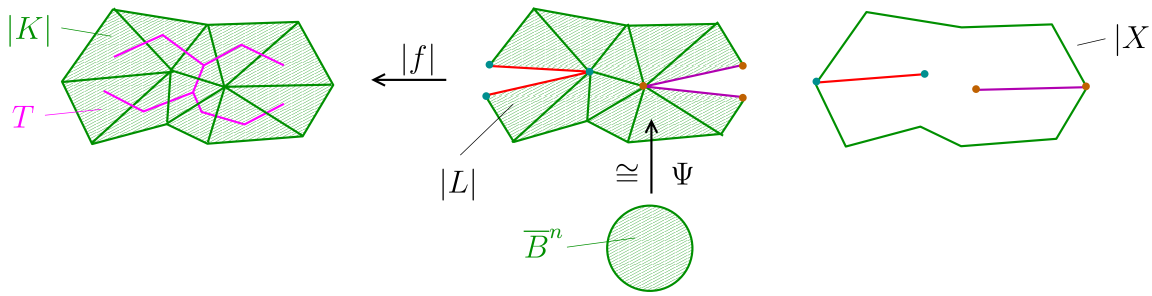


FIGURE 1032. Illustration of the proof of Proposition 66.16.

Exercises for Chapter 66.

Exercise 66.1. Let K be a finite n-dimensional pseudomanifold.

(a) Show that the suspension $\Sigma(K)$, as defined on page 1506, is an $(n+1)$-dimensional pseudomanifold.
(b) Show that if K is closed, then $\Sigma(K)$ is closed.
(c) Show that if K is orientable in the sense of the definition on page 1652, then $\Sigma(K)$ is orientable.

Exercise 66.2. Let K be an n-dimensional pseudomanifold such that ∂K is an $(n-1)$-dimensional pseudomanifold. Let $i\colon \partial K \to K$ be the inclusion map. We consider the cone $K \cup_{\partial K} \operatorname{Cone}(\partial K)$. Show that $\operatorname{Cone}(i\colon \partial K \to K)$ is a closed n-dimensional pseudomanifold.

Exercise 66.3. Let K be an m-dimensional pseudomanifold and let L be an n-dimensional pseudomanifold. We equip K and L with a total order and we define the product $K \times L$ as on page 1501.

(a) Show that $K \times L$ is an $(m+n)$-dimensional pseudomanifold.
(b) Show that $K \times L$ is closed if and only if both K and L are closed.
(c) Show that $K \times L$ is orientable if and only if both K and L are orientable.

Exercise 66.4.

(a) Let K be a 1-dimensional pseudomanifold. Show that $|K|$ is a closed 1-dimensional smooth manifold.
(b) Let K be a 2-dimensional pseudomanifold. Show that $|K|$ is a closed 2-dimensional smooth manifold.

Remark. Part (a) is fairly straightforward. The proof of part (b) is more delicate. One way to prove part (b) is to imitate the proof, given in Propositions 6.8 and 6.21, that the surface Σ_g of genus g is closed 2-dimensional smooth manifold.

Exercise 66.5. Let $K = (V, S)$ be an n-dimensional pseudomanifold.

(a) Show that the barycentric subdivision $\operatorname{sd}(K)$ is an n-dimensional pseudomanifold.

(b) Suppose that K is equipped with an orientation. Let $t = \{\underline{s_0}, \dots, \underline{s_n}\}$ be an n-simplex of $\mathrm{sd}(K)$. For $i = 0, \dots, n$ we denote by v_i the vertex of s_i that is not contained in s_{i-1}. Let $\epsilon \in \{-1, 1\}$ be the sign such that $(\epsilon, (v_0, \dots, v_n))$ matches the given orientation of K. We equip the n-simplex t of $\mathrm{sd}(K)$ with the orientation given by $(\epsilon, (\underline{s_0}, \dots, \underline{s_n}))$. Show that the above orientations of the n-simplices of $\mathrm{sd}(K)$ define an orientation for $\mathrm{sd}(K)$.

We refer to Figure 1033 for an illustration.

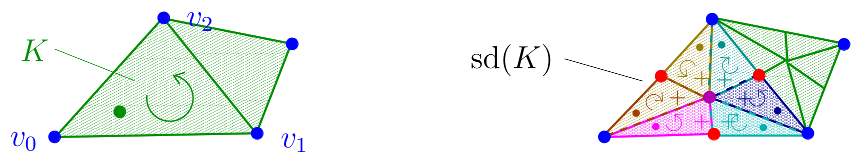


FIGURE 1033. Illustration of Exercise 66.5.

Exercise 66.6. Let M be a compact connected non-empty smooth manifold. Furthermore let $(K = (V, S), \Theta \colon |K| \to M)$ be a simplicial structure for M. Let Γ be the dual graph as defined on page 1654. Is there a relationship between the homology groups $\mathrm{H}_1(\Gamma; \mathbb{Z})$ and the groups $\mathrm{H}_1(M; \mathbb{Z})$ or $\mathrm{H}_1(M, \partial M; \mathbb{Z})$? For example is there a relationship between the ranks or the minimal number of generators?

Exercise 66.7. Let M be a compact orientable n-dimensional smooth manifold. Suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Show that $\mathrm{H}_{n-1}(M, A)$ is a free abelian group.

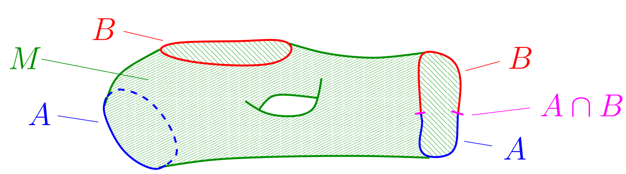


FIGURE 1034. Illustration of Exercise 66.7.

Exercise 66.8. Let M be a compact connected n-dimensional smooth manifold and let $p \neq 2$ be a prime. Determine $\mathrm{H}_n(M, \partial M; \mathbb{F}_p)$ and $\mathrm{H}_n(M; \mathbb{F}_p)$.

Exercise 66.9. Let $k, l \in \mathbb{N}$ and let $N_{k,l}$ be $N_k = k \cdot \mathbb{R}\mathrm{P}^2$ minus l open disks. Determine $\mathrm{H}_1(N_{k,l}, \partial N_{k,l}; \mathbb{Z})$.

Exercise 66.10. Let K be an infinite n-dimensional pseudomanifold such that $\partial K = \emptyset$. Show that $\mathrm{H}_{n-1}^{\mathrm{simp}}(K)$ is a free abelian group.
Remark. This exercise will require a serious effort.

Exercise 66.11. Let $p, q \in \mathbb{N}$ be coprime. Show that every orientation-reversing diffeomorphism of the lens space $L(p, q)$ has at least one fixed point.

Exercise 66.12. Let X be a connected 2-dimensional smooth manifold with infinite fundamental group (We do not assume that X is compact.) Show that X is aspherical, i.e.

show that $\pi_i(X) = 0$ for $i \geq 2$.
Hint. First use the Hurewicz Theorem 53.5 and the results from this section to show that $\pi_2(X) = 0$.
Remark. This exercise gives in particular a new proof of the fact that given a surface Σ_g of genus $g \geq 1$ we have $\pi_i(\Sigma_g) = 0$ for $i \geq 2$. We had initially proved this statement on page 1070 using the rather difficult results from Chapter 33.

Exercise 66.13. Let M be a 3-dimensional smooth homology 3-sphere with non-trivial fundamental group.[1020] Furthermore let $n \geq 4$. We pick a smooth embedding $\varphi\colon \overline{B}^3 \to M$ and we consider $W := (M \setminus \varphi(B^3) \times \overline{B}^{n-3}$. By Proposition 80.9 we can equip W with the structure of an orientable n-dimensional smooth manifold. Show that the double $\mathrm{D}\,W$ is an n-dimensional smooth homology n-sphere with non-trivial fundamental group.

Exercise 66.14. Let Σ be a compact oriented connected 2-dimensional smooth manifold. We consider the product $N := \Sigma \times [0,1]$ and we consider the subset $A := \partial\Sigma \times [0,1] \subset N$. Let $\varphi\colon \Sigma \to \Sigma \times \{0\} \subset N$ be the obvious inclusion map. Now let $g\colon S \to N$ be a map from a compact oriented connected 2-dimensional smooth manifold with $g(\partial S) \subset A$ such that $\varphi_*([\Sigma]) = g_*([S])$. Show that $\chi(S) \leq \chi(\Sigma)$.
Remark. In a slightly simplified version the exercise says that the surface $\Sigma \times \{0\}$ is the "simplest" surface in N representing the homology class $[\Sigma \times \{0\}]$.

[1020]By Proposition 66.11 we know that such M exists.

67. Simplicial cohomology

In this chapter we will introduce the simplicial cohomology groups of an (ordered) abstract simplicial complex. This chapter can be seen as a continuation of our discussions of abstract simplicial complexes in Chapters 61, 62 and 63. The objects in this chapter are for the most part purely combinatorial and for the most part this chapter can be read independently of the chapters on singular cohomology. We will only make use of the purely algebraic statements in Sections 73.1, 73.2 and 73.3.

As for simplicial homology, the study of simplicial cohomology is great fun in its own right. But as we will see, it is also a very convenient tool for calculating singular cohomology groups and for determining the often intractable cup and cap products.

In Chapter 71 we will prove the Simplicial Poincaré Duality Theorem 71.4 for pseudo-manifolds, which gives us in particular a Poincaré Duality theorem for smooth manifolds.

67.1. **Simplicial chain complexes.** In the following section we will introduce simplicial cohomology groups. Before we do so it is convenient to recall several definitions and results on simplicial chain complexes.

We start out with the following definition from page 1542.

Definition. Let $K=(V,S)$ be an *ordered* abstract simplicial complex.[1021] Given $k\in\mathbb{N}_0$ we define

$$\mathrm{C}_k^{\mathrm{simp},\le}(K) := \text{free abelian group generated by the set of } k\text{-simplices of } K.$$

Furthermore we consider the boundary map

$$\begin{array}{rcl} \partial_k\colon \mathrm{C}_k^{\mathrm{simp},\le}(K) & \to & \mathrm{C}_{k-1}^{\mathrm{simp},\le}(K) \\ \underbrace{\{v_0<\dots<v_k\}}_{\substack{\text{any } k\text{-simplex } s \text{ is uniquely} \\ \text{of the form } s=\{v_0,\dots,v_k\} \\ \text{with } v_0<\dots<v_k}} & \mapsto & \sum\limits_{i=0}^{k}(-1)^i\cdot\{v_0,\dots,v_{i-1},\widehat{v_i},v_{i+1},\dots,v_k\}. \end{array}$$

By Lemma 63.2 we know that $(\mathrm{C}_*^{\mathrm{simp},\le}(K),\partial_*)$ is a chain complex, thus we can define the corresponding simplicial homology groups $\mathrm{H}_*^{\mathrm{simp},\le}(K)$.

The above definition is straightforward, but it does have the major disadvantage that it requires the choice of an order on K. In most situations there is no natural choice of an order. In the absence of an order we are led to the following more elaborate definition which initially we gave on page 1544 and in Lemma 63.6.

Definition. Let $K=(V,S)$ be an abstract simplicial complex.

(1) Let $k\in\mathbb{N}_0$. An *ordered k-simplex of* K is a $(k+1)$-tuple $(v_0,\dots,v_k)\in V^{k+1}$ such that $\{v_0,\dots,v_k\}$ is an k-simplex of $K=(V,S)$. In the following we write $\lfloor(v_0,\dots,v_k)\rfloor := \{v_0,\dots,v_k\}$.

[1021] Recall that, according to the definition on page 1484, an *order* on an abstract simplicial complex (V,S) is a partial order on V that has the property that the restriction to each simplex is actually a total order.

(2) Let $k \in \mathbb{N}_0$. We define the *k-th simplicial chain group*

$$\mathrm{C}_k^{\mathrm{simp}}(K) := \begin{matrix}\text{free abelian group on the set}\\ \text{of ordered } k\text{-simplices of } K\end{matrix} \Big/ \begin{matrix}(v_0,\dots,v_k) - \mathrm{sign}(\sigma)\cdot(v_{\sigma(0)},\dots,v_{\sigma(k)})\\ \text{where } \sigma \in S_{k+1} = \mathrm{Bij}\{0,\dots,k\}\end{matrix}$$

(3) Given an ordered simplex $(v_0, \dots, v_k)$ we denote its image in $\mathrm{C}_k^{\mathrm{simp}}(K)$ by $[v_0, \dots, v_k]$.

(4) We consider the map

$$\begin{aligned} \partial_k \colon \mathrm{C}_k^{\mathrm{simp}}(K) &\to \mathrm{C}_{k-1}^{\mathrm{simp}}(K) \\ [v_0, \dots, v_k] &\mapsto \sum_{i=0}^{k} (-1)^i \cdot [v_0, \dots, v_{i-1}, \widehat{v_i}, v_{i+1}, \dots, v_k]. \end{aligned}$$

(5) By Lemma 63.6 we know that $(\mathrm{C}_*^{\mathrm{simp}}(K), \partial_*)$ is a chain complex, thus we can define the corresponding simplicial homology groups $\mathrm{H}_*^{\mathrm{simp}}(K)$.

Notation. In pictures we denote the ordering of the simplices of a 1-simplex in the obvious way by an arrow pointing from the first to the second vertex. Furthermore, given an ordered 2-simplex we indicate the first vertex by a dot and the ordering of the remaining vertices by an oriented 3/4-circle. We refer to Figure 1035 for an illustration.

FIGURE 1035

We recall the following lemma.

Lemma 63.6. *Let K be an abstract simplicial complex. If we choose for each k-simplex $s \in S$ an ordered k-simplex $\widetilde{s}$ with $\lfloor \widetilde{s} \rfloor = s$, then the $[\widetilde{s}]$ form a basis for $\mathrm{C}_k^{\mathrm{simp}}(K)$.*

We conclude this short section with the following proposition that combines several earlier statements on the relationship between the above two simplicial chain complexes and also the singular chain complex of the topological realization.

Proposition 67.1.

(1) *Let $(K = (V, S), \leq)$ be an ordered abstract simplicial complex.*

(a) *We consider the following maps:*

$$\begin{array}{ccccc} \mathrm{C}_k^{\mathrm{simp}}(K) & \xleftarrow{\Omega_{\leq,k}} & \mathrm{C}_k^{\mathrm{simp},\leq}(K) & \xrightarrow{\Theta_{\leq,k}} & \mathrm{C}_k(|K|) \\ [v_0, \dots, v_k] & \longleftarrow & s = \{v_0 < \dots < v_k\} & \longmapsto & (\underbrace{\Phi_s^{\leq}}_{\text{map given by sending } e_i \text{ to } v_i} \colon \Delta^k \to |K|). \end{array}$$

The maps $\Omega_{\leq,k}$ to the left define a natural isomorphism of chain complexes whereas the maps $\Theta_{\leq,k}$ to the right define a natural chain homotopy equivalence.

(b) *There exist maps*

$$\pi_k \colon \mathrm{C}_k(|K|) \to \mathrm{C}_k^{\mathrm{simp},\leq}(K)$$

which have the following two properties:
(i) *The maps π_k form a chain map which form a chain homotopy equivalence.*
(ii) *For each $k \in \mathbb{N}_0$ we have $\pi_k \circ \Theta_k = \mathrm{id}_{\mathrm{C}_k^{\mathrm{simp},\leq}(K)}$.*

(2) *Let K be an abstract simplicial complex and let "$\leq$" and "$\subseteq$" be two orders on K. The two chain maps*

$$\begin{array}{cccccc} & \mathrm{C}_k^{\mathrm{simp}}(K) & \xrightarrow{\Omega_{\leq,k}^{-1}} & \mathrm{C}_k^{\mathrm{simp},\leq}(K) & \xrightarrow{\Theta_{\leq,k}} & \mathrm{C}_k(|K|) \\ \text{and} & \mathrm{C}_k^{\mathrm{simp}}(K) & \xrightarrow{\Omega_{\subseteq,k}^{-1}} & \mathrm{C}_k^{\mathrm{simp},\subseteq}(K) & \xrightarrow{\Theta_{\subseteq,k}} & \mathrm{C}_k(|K|) \end{array}$$

are chain homotopic.

(3) *Let G be an abelian group. The chain homotopy equivalence class of (2) induces a natural isomorphism*

$$\Xi_{k*}\colon\ \mathrm{H}_k^{\mathrm{simp}}(K;G)\ \to\ \mathrm{H}_k(|K|;G).$$

Proof.

(1) (a) This statement is the combination of Lemma 63.8 and Theorem 63.25.
(b) It follows almost immediately from the definitions that for each $k \in \mathbb{N}_0$ the map $\mathrm{C}_k^{\mathrm{simp},\leq}(K) \to \mathrm{C}_k(|K|)$ is a monomorphism and that the image is a summand. The desired statement is now an immediate consequence of (a) and Proposition 49.3.

(2) This statement is the content of Lemma 63.26.

(3) It follows immediately from (2) and a slight generalization of Lemma 42.2 that any choice of order on K induces the same map on homology with G-coefficients, i.e. the map Ξ_{k*} is well-defined. It follows easily from (1) and Corollary 42.3 that the map Ξ_{k*} is a natural isomorphism. ■

67.2. **The definition of the simplicial cohomology groups.** In this section we will introduce the simplicial cohomology groups of an (ordered) abstract simplicial complex. The discussion in the previous section also shows that it will be necessary to consider two slightly different types of simplicial cohomology groups.

We start out with a decidedly dull definition.

Definition. Let K be an (ordered) abstract simplicial complex and let G be an abelian group. For each $k \in \mathbb{N}_0$ we write

$$\mathrm{C}^k_{\mathrm{simp}}(K;G) := \mathrm{Hom}(\mathrm{C}_k^{\mathrm{simp}}(K),G) \quad \text{ and } \quad \mathrm{C}^k_{\mathrm{simp},\leq}(K;G) := \mathrm{Hom}(\mathrm{C}_k^{\mathrm{simp},\leq}(K),G).$$

Elements of $\mathrm{C}^k_{\mathrm{simp}}(K;G)$ and $\mathrm{C}^k_{\mathrm{simp},\leq}(K;G)$ are called *simplicial k-cochains.*

Remark. Let G be an abelian group, let K be an ordered abstract simplicial complex and let $k \in \mathbb{N}_0$.

(1) It follows from Lemma 73.1 that a simplicial k-cochain in $\mathrm{C}^k_{\mathrm{simp},\leq}(K;G)$ is the same data as assigning an element of G to *every* k-simplex. Note that in contrast to the setting of singular cochains one can actually hope to "see" simplicial cochains. For example see Figure 1036 for a simple-minded illustration.

(2) In contrast, it follows easily from Lemma 57.3 that a chain in $C_k^{\text{simp},\leq}(K) \otimes G$ is the same data as assigning an element of G to *finitely many* k-simplices.

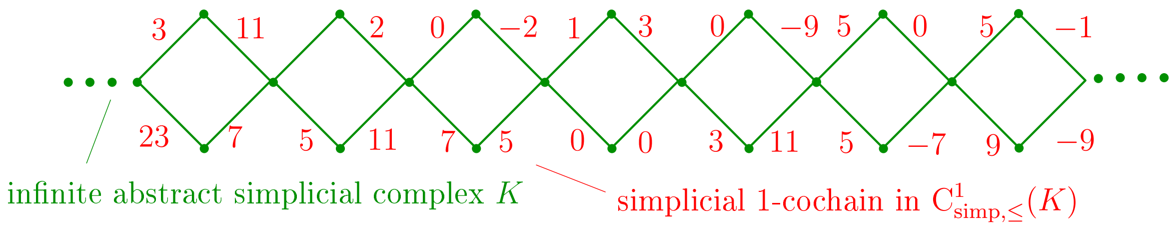


FIGURE 1036

Now we introduce the following notation.

Notation.

(1) Let K be an order abstract simplicial complex and let s be an ordered k-simplex. We denote by $s^*\colon\ C_k^{\text{simp},\leq}(K) \to \mathbb{Z}$ the map that is given by $s^*(s) = 1$ and $s^*(t) = 0$ for any k-simplex $t \neq s$.

(2) Let K be an abstract simplicial complex and let $s = (v_0, \ldots, v_k)$ be an ordered k-simplex. It follows immediately from the definitions that

$$\begin{array}{rcl} s^*\colon\ C_k^{\text{simp}}(K) & \to & \mathbb{Z} \\ {[w_0, \ldots, w_k]} & \mapsto & \begin{cases} 0, & \text{if } \{v_0, \ldots, v_k\} \neq \{w_0, \ldots, w_k\}, \\ \text{sign}(\sigma), & \text{if there exists a } \sigma \in \text{Bij}\{0, \ldots, k\} \\ & \qquad\qquad \text{with } w_i = v_{\sigma(i)} \text{ for} = 0, \ldots, k \end{cases} \end{array}$$

is well-defined, in other words, s^* is a well-defined cochain in $C^k_{\text{simp}}(K;\mathbb{Z})$.

In the following we will mostly study the simplicial cohomology of finite (ordered) abstract simplicial complexes. In this context the following lemma is very helpful.

Lemma 67.2. *Let $k \in \mathbb{N}_0$.*

(1) *Let $K = (V, S)$ be an ordered abstract simplicial complex. If K is* finite, *then the set $\{s^* \mid s \text{ a } k\text{-simplex}\}$ is a basis for $C^k_{\text{simp},\leq}(K;\mathbb{Z})$.*

(2) *Let $K = (V, S)$ be an abstract simplicial complex. We choose for each k-simplex s an ordered k-simplex $\widetilde{s}$ with $\lfloor \widetilde{s} \rceil = s$. If K is finite, then the $\widetilde{s}^*$ form a basis for $C^k_{\text{simp}}(K;\mathbb{Z})$.*

PROOF. This follows immediately from Lemma 73.1 together with the definition of $C_k^{\text{simp},\leq}(K)$ and together with Lemma 63.6. ■

Now that we have gained a little bit of familiarity with cochains let us move on to the definition of the simplicial cochain complex and the simplicial cohomology groups.

Definition. Let K be an abstract simplicial complex and let G be an abelian group. For each $k \in \mathbb{N}_0$ we refer to

$$\delta_k := \partial^*_{k+1} \colon \underbrace{\operatorname{Hom}(\mathrm{C}^{\mathrm{simp}}_k(K), G)}_{=\mathrm{C}^k_{\mathrm{simp}}(K;G)} \to \underbrace{\operatorname{Hom}(\mathrm{C}^{\mathrm{simp}}_{k+1}(K), G)}_{=\mathrm{C}^{k+1}_{\mathrm{simp}}(K;G)}$$

as the *coboundary map.* Any simplicial k-cochain in $\ker(\delta_k \colon \mathrm{C}^k_{\mathrm{simp}}(K;G) \to \mathrm{C}^{k+1}_{\mathrm{simp}}(K;G))$ is called a *simplicial k-cocycle.* We refer to the cochain complex

$$\ldots \leftarrow \mathrm{C}^{k+1}_{\mathrm{simp}}(K;G) \xleftarrow{\delta_k} \mathrm{C}^k_{\mathrm{simp}}(K;G) \xleftarrow{\delta_{k-1}} \mathrm{C}^{k-1}_{\mathrm{simp}}(K;G) \leftarrow \ldots \xleftarrow{\delta_0} \mathrm{C}^0_{\mathrm{simp}}(K;G) \leftarrow 0$$

as the *simplicial cochain complex of K with G-coefficients.* We define the *k-th simplicial cohomology group of K with G-coefficients* as follows:

$$\mathrm{H}^k_{\mathrm{simp}}(K;G) := \mathrm{H}^k(\mathrm{C}^{\mathrm{simp}}_*(K);G) = \frac{\ker(\delta_k \colon \mathrm{C}^k_{\mathrm{simp}}(K;G) \to \mathrm{C}^{k+1}_{\mathrm{simp}}(K;G))}{\operatorname{im}(\delta_{k-1} \colon \mathrm{C}^{k-1}_{\mathrm{simp}}(K;G) \to \mathrm{C}^k_{\mathrm{simp}}(K;G))}.$$

If K is actually an ordered abstract simplicial complex, then we define the cochain complex $\mathrm{C}^*_{\mathrm{simp},\leq}(K;G)$ and the simplicial cohomology groups $\mathrm{H}^k_{\mathrm{simp},\leq}(K;G)$ in the obvious way.

Given a finite abstract simplicial complex it is just an elementary, albeit at times surprisingly intricate, problem in linear algebra to determine the simplicial cohomology groups and to give explicit examples of simplicial cocycles.

Example. We consider the "simplicial circle" C with vertex set $V = \{a,b,c\}$ shown in Figure 1037. We consider the ordered 1-simplices $s := (a,b)$, $t := (b,c)$ and $u := (c,a)$. We see that the simplicial chain complex $\mathrm{C}^{\mathrm{simp}}_*(C;\mathbb{Z})$ is of the following form:[1022]

$$\begin{array}{rcl} 0 \to \mathrm{C}^{\mathrm{simp}}_1(C) & \xrightarrow{\partial_1} & \mathrm{C}^{\mathrm{simp}}_0(C) \to 0 \\ s & \mapsto & b-a \\ t & \mapsto & c-b \\ u & \mapsto & a-c. \end{array} \quad \begin{array}{c} \text{thus the dual} \\ \text{chain complex} \\ \mathrm{C}^*_{\mathrm{simp}}(C;\mathbb{Z}) \\ \text{is given by} \end{array} \quad \begin{array}{rcl} 0 \leftarrow \mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z}) & \xleftarrow{\delta_0} & \mathrm{C}^0_{\mathrm{simp}}(C;\mathbb{Z}) \leftarrow 0 \\ u^*-s^* & \leftarrow\!\shortmid & a^* \\ s^*-t^* & \leftarrow\!\shortmid & b^* \\ t^*-u^* & \leftarrow\!\shortmid & c^*. \end{array}$$

An elementary calculation shows that

$$\mathrm{H}^0_{\mathrm{simp}}(C;\mathbb{Z}) = \ker(\delta_0) = \mathbb{Z} \cdot (a^*+b^*+c^*).$$

We turn to the slightly more tricky calculation of $\mathrm{H}^1_{\mathrm{simp}}(C;\mathbb{Z})$. First note that it is elementary to show that

$$\operatorname{im}\big(\delta_0 \colon \mathrm{C}^0_{\mathrm{simp}}(C;\mathbb{Z}) \to \mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z})\big) = \{k \cdot s^* + l \cdot t^* + m \cdot u^* \,|\, k+l+m = 0\}.$$

[1022]Usually it takes a while to get used to coboundaries. Thus let us show in detail that we have $\delta_0(a^*) = u^* - s^* \in \mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z}) = \operatorname{Hom}(\mathrm{C}^{\mathrm{simp}}_1(C),\mathbb{Z})$. We need to show that $\delta_0(a^*) = u^* - s^*$ are the same homomorphism. By Lemma 63.6 we know that $\{s,t,u\}$ is a basis for $\mathrm{C}^{\mathrm{simp}}_1(C)$. So we need to show that the two homomorphisms take the same values on s,t and u. We calculate that

$$\delta_0(a^*)(s) \underset{\substack{\uparrow \\ \text{definition of } \delta_0}}{=} a^*(\partial_1 s) \underset{\substack{\uparrow \\ \text{definition of } \partial_1}}{=} a^*(b-a) = a^*(b) - a^*(a) = -1 = u^*(s) - s^*(s) = (u^*-s^*)(s).$$

The same way we proceed with t und u.

Now let us consider the cochain $\theta = t^*\colon\ \mathrm{C}_1^{\mathrm{simp}}(C) \to \mathbb{Z}$. (In other words, θ is the cochain that is given by $\theta(s) = 0$, $\theta(t) = 1$ and $\theta(u) = 0$.) An elementary argument, see e.g. Exercise 41.11, shows that the map

$$\begin{array}{rcl} \mathbb{Z} & \xrightarrow{\cong} & \mathrm{H}^1_{\mathrm{simp}}(C;\mathbb{Z}) \;=\; \mathrm{coker}(\delta_0) \\ n & \mapsto & n\cdot[\theta] \end{array}$$

is an isomorphism.[1023] Note that on several occasions throughout this chapter we will make use of the above simplicial cocycle $\theta = t^* \in \mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z})$.

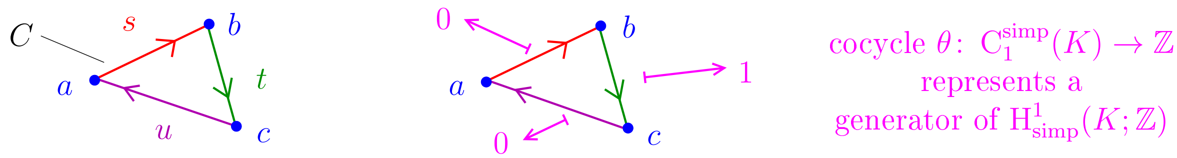


FIGURE 1037

We state the next example as a lemma.

Lemma 67.3. *Let G be an abelian group and let $K = (V,S)$ be an abstract simplicial complex.*

(1) *Given any $g \in G$ the constant cochain $\varphi_g\colon\ \mathrm{C}_0^{\mathrm{simp}}(K) \to G$ given by $\varphi_g(v) = g$ for every $v \in V$ is a cocycle.*
(2) *If K is non-empty and connected in the sense of the definition on page 1516, then $\mathrm{H}^0_{\mathrm{simp}}(K;G) = \{\varphi_g \,|\, g \in G\}$. In particular $\mathrm{H}^0_{\mathrm{simp}}(K;G) \cong G$.*

PROOF.

(1) Let $g \in G$. We need to show that $\delta_0(\varphi_g) = 0 \in \mathrm{C}^1_{\mathrm{simp}}(K) = \mathrm{Hom}(\mathrm{C}_1^{\mathrm{simp}}(K), G)$. In other words, by definition of $\mathrm{C}_1^{\mathrm{simp}}(K)$ we need to show that for every ordered 1-simplex (v_0, v_1) of K we have $(\delta_0\varphi_g)([v_0,v_1]) = 0$. Indeed, we calculate that

$$\delta_0(\varphi_g)([v_0,v_1]) \;\underset{\substack{\uparrow\\ \text{definition of } \delta_0}}{=}\; \varphi_g(\partial_1([v_0,v_1])) \;\underset{\substack{\uparrow\\ \text{definition of } \partial_1}}{=}\; \varphi_g(v_1 - v_0) \;=\; \varphi_g(v_1) - \varphi_g(v_0) \;=\; g - g \;=\; 0.$$

(2) We will prove the second statement in Exercise 67.2. ■

We conclude this section with the following explicit calculation of the coboundary map. This result will come in handy in the following chapter.

Lemma 67.4. *Let $K = (V,S)$ be an abstract simplicial complex and let $(v_0,\dots,v_k)$ be an ordered k-simplex of K. We have*

$$\delta_k([v_0,\dots,v_k]^*) \;=\; (-1)^{k+1}\cdot \sum_{\substack{w \in V \text{ such that}\\ \{v_0,\dots,v_k,w\} \text{ is}\\ \text{a } (k+1)\text{-simplex}}} [v_0,\dots,v_k,w]^* \;\in\; \mathrm{C}^{k+1}_{\mathrm{simp}}(K;\mathbb{Z}).$$

[1023]Of course there are many other choices of a cochain θ that would work. We picked this particular one since later on it works well in the pictures.

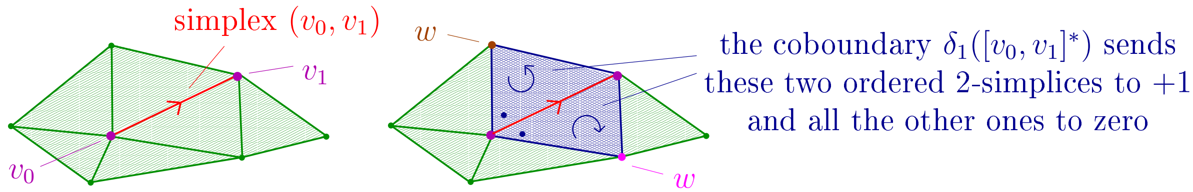


FIGURE 1038. Illustration of Lemma 67.4.

PROOF. We need to show that both sides define the same map $\mathrm{C}^{\mathrm{simp}}_{k+1}(K) \to \mathbb{Z}$. Basically by definition of $\mathrm{C}^{\mathrm{simp}}_{k+1}(K)$ this means that we need to show both sides provide the same value for any given $[y_0, \ldots, y_{k+1}]$. So suppose we are given an ordered $(k+1)$-simplex $(y_0, \ldots, y_{k+1})$ of K. After possibly reordering the y_i we can arrange that there exists an $m \in \{0, \ldots, k\}$ such that $y_i = v_i$ for $i = 0, \ldots, m-1$ and such that $\{y_m, \ldots, y_{k+1}\} \cap \{v_m, \ldots, v_k\} = \varnothing$. We calculate that

$$\begin{aligned}
(\delta_k([v_0, \ldots, v_k]^*)([y_0, \ldots, y_{k+1}]) &\overset{\substack{\text{definition of } \delta_k \\ \downarrow}}{=} [v_0, \ldots, v_k]^*(\partial_{k+1}([y_0, \ldots, y_{k+1}])) \overset{\substack{\text{definition of } \partial_{k+1} \\ \downarrow}}{=} \\
&= \sum_{i=0}^{k+1} (-1)^i \cdot [v_0, \ldots, v_k]^*([y_0, \ldots, \widehat{y}_i, \ldots, y_{k+1}]) \\
&\underset{\substack{\uparrow \\ \text{follows from the choice of } m}}{=} \sum_{i=0}^{k+1} \begin{cases} 0, & \text{if } i < k+1 \text{ or if } m \neq k \\ (-1)^{k+1}, & \text{if } i = k+1 \text{ and if } m = k. \end{cases}
\end{aligned}$$

It follows easily from the definitions that the cochain on the right hand side of the lemma leads to the same result. ■

67.3. **Properties of simplicial cohomology groups.** In the following lemma we will see that simplicial cohomology groups define contravariant functors. Before we can formulate that lemma let us recall a definition from Lemmas 63.3 and 63.7. Namely, given a simplicial map $f\colon K = (V, S) \to L = (W, T)$ between abstract simplicial complexes we consider the following chain map:

$$\begin{aligned}
f_*\colon \mathrm{C}^{\mathrm{simp}}_k(K) &\to \mathrm{C}^{\mathrm{simp}}_k(L) \\
[v_0, \ldots, v_k] &\mapsto \begin{cases} [f(v_0), \ldots, f(v_k)], & \text{if } f(v_0), \ldots, f(v_k) \text{ are pairwise different,} \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

When we are dealing with an order-preserving simplicial map between ordered abstract simplicial complexes, then we define the induced map $\mathrm{C}^{\mathrm{simp},\leq}_k(K) \to \mathrm{C}^{\mathrm{simp},\leq}_k(L)$ in the analogous way.

Lemma 67.5. *Let $k \in \mathbb{N}_0$ and let G be an abelian group.*

(1) *The maps*

$$\begin{array}{rcl} (K,\leq) & \mapsto & \mathrm{H}^k_{\mathrm{simp},\leq}(K;G) \\ (f\colon (K,\leq)\to(L,\leq)) & \mapsto & \left(\begin{array}{rcl} f_*\colon \mathrm{H}^k_{\mathrm{simp},\leq}(K;G) & \to & \mathrm{H}^k_{\mathrm{simp},\leq}(L;G) \\ [\varphi\colon \mathrm{C}^{\mathrm{simp}}_k(K)\to G] & \mapsto & [\varphi\circ f_*] \end{array}\right) \end{array}$$

define a contravariant functor from the category $\mathcal{O}rd\mathcal{A}bs\mathcal{S}imp\mathcal{C}pl\chi$ *of ordered abstract simplicial complexes to the category* $\mathcal{A}b\mathcal{G}r$ *of abelian groups.*

(2) *The maps*

$$\begin{array}{rcl} K & \mapsto & \mathrm{H}^k_{\mathrm{simp}}(K;G) \\ (f\colon K\to L) & \mapsto & \left(\begin{array}{rcl} f_*\colon \mathrm{H}^k_{\mathrm{simp}}(K;G) & \to & \mathrm{H}^k_{\mathrm{simp}}(L;G) \\ [\varphi\colon \mathrm{C}^{\mathrm{simp}}_k(K)\to G] & \mapsto & [\varphi\circ f_*] \end{array}\right) \end{array}$$

define a contravariant functor from the category $\mathcal{A}bs\mathcal{S}imp\mathcal{C}pl\chi$ *of abstract simplicial complexes to the category* $\mathcal{A}b\mathcal{G}r$ *of abelian groups.*

PROOF. The lemma is an immediate consequence of Lemmas 63.3, 63.7 and 73.8. ∎

Example. In Figure 1039 we show the "simplicial annulus" A together with the obvious projection map p to the simplicial circle C. Furthermore we show the simplicial cocycle $\theta\in\mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z})$ that we introduced on page 1688 and we show the corresponding cocycle $p^*(\theta)\in\mathrm{C}^1_{\mathrm{simp}}(A)$.

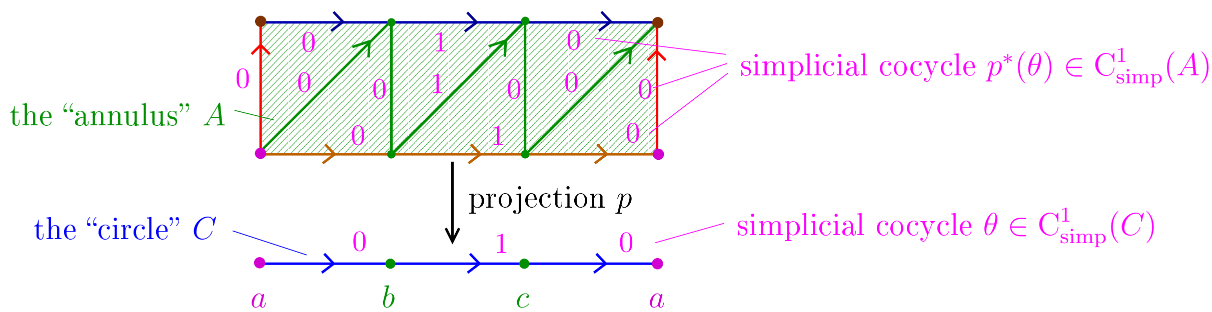


FIGURE 1039

The following proposition relates the two flavors of simplicial cohomology groups to one another and it relates them to the singular cohomology groups of the topological realization.

Proposition 67.6. *Let G be an abelian group.*

(1) *Let $(K,\leq)$ be an ordered abstract simplicial complex.*

(a) *The natural maps*

$$\begin{array}{rcl} \Theta^*_{\leq,k}\colon \mathrm{C}^k(|K|;G) & \to & \mathrm{C}^k_{\mathrm{simp},\leq}(K;G) \\ (\varphi\colon \mathrm{C}_k(|K|)\to G) & \mapsto & \left(\mathrm{C}^{\mathrm{simp}}_{k,\leq}(K)\xrightarrow{\Theta_{\leq,k}}\mathrm{C}_k(|K|)\xrightarrow{\varphi}G\right) \end{array}$$

are a cochain homotopy equivalence. In particular the maps induce natural isomorphisms

$$\Theta^*_{\leq,k}\colon \mathrm{H}^k(|K|;G) \;\to\; \mathrm{H}^k_{\mathrm{simp},\leq}(K;G)$$

(b) *Every simplicial cocycle* $\varphi \in \mathrm{C}^k_{\mathrm{simp},\leq}(K;G)$ *"extends" to a singular cocycle of* $\mathrm{C}^k(|K|;G)$*. More precisely, there exists a cocycle* $\psi \in \mathrm{C}^k(|K|;G)$ *such that* $\Theta^*_{\leq,k}(\psi) = \varphi$*, i.e. such that* $\psi \circ \Theta_{\leq,k} = \varphi\colon\ \mathrm{C}^{\mathrm{simp}}_k(K) \to G$*.*

(2) *Let* K *be an abstract simplicial complex. We pick an order "*$\leq$*" for* K*. The composition of the two maps*

$$\mathrm{H}^k(|K|;G) \xrightarrow{\Theta^*_{\leq}} \mathrm{H}^k_{\mathrm{simp},\leq}(K;G) \xrightarrow{\Omega^{-1}_{\leq,*}} \mathrm{H}^k_{\mathrm{simp}}(K;G)$$

does not depend on the choice of "$\leq$*" and it defines a natural isomorphism*

$$\Xi^*_k\colon\ \mathrm{H}^k(|K|;G) \ \to\ \mathrm{H}^k_{\mathrm{simp}}(K;G).$$

PROOF.

(1) (a) This statement follows immediately from Proposition 67.1 (1) and Lemma 73.8.
(b) Let $\varphi \in \mathrm{C}^k_{\mathrm{simp},\leq}(K;G)$ be a simplicial cocycle. By Proposition 67.1 (2) there exist maps $\pi_k\colon\ \mathrm{C}_k(|K|) \to \mathrm{C}^{\mathrm{simp},\leq}_k(K)$ which have the following two properties:
(i) The maps π_k form a chain map.
(ii) For each $k \in \mathbb{N}_0$ we have $\pi_k \circ \Theta_k = \mathrm{id}_{\mathrm{C}^{\mathrm{simp},\leq}_k(K)}$.
We set $\psi := \pi^*_k(\varphi)$. By (i) this is a cocycle and by (ii) we have $\psi \circ \Theta_{\leq,k} = \varphi$.

(2) This statement follows immediately from Proposition 67.1 (1) and Lemma 73.8. ■

Example. Let us rephrase the statement of Proposition 67.6 (1b). It says that given any simplicial cocycle $\varphi \in \mathrm{C}^k_{\mathrm{simp},\leq}(K;G)$ there exists a singular cocycle $\psi \in \mathrm{C}^k(|K|;G)$ with $[\Theta^*_{\leq,k}(\psi)] = [\varphi] \in \mathrm{H}^k_{\mathrm{simp},\leq}(K;G)$ and which, restricted to the singular simplices provided by the simplices of K, coincides with φ. We try to illustrate this statement in Figure 1040.

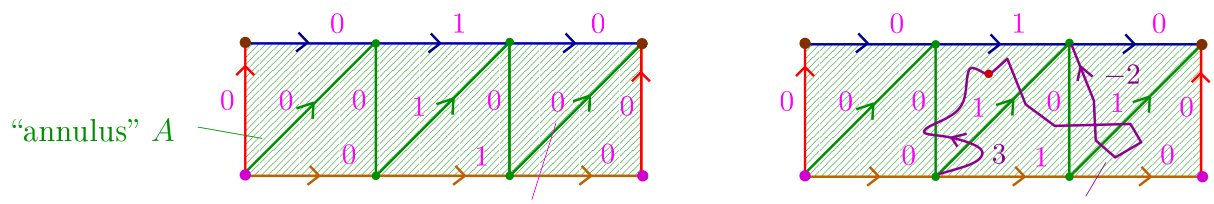


FIGURE 1040

We continue with the following lemma which is just a special case of Lemma 74.6.

Lemma 67.7. *Let* G *be an abelian group and let* $k \in \mathbb{N}_0$*.*

(1) *Given any abstract simplicial complex* K *the map*

$$\begin{array}{rcl}\langle\ ,\ \rangle\colon\ \mathrm{H}^k_{\mathrm{simp}}(K;G) \times \mathrm{H}^{\mathrm{simp}}_k(K) & \to & G \\ ([\varphi],[\sigma]) & \mapsto & \langle [\varphi],[c]\rangle_K := \varphi(\sigma),\end{array}$$

called the Kronecker pairing*, is well-defined and bilinear.*

(2) *Given any simplicial map* $f\colon K \to L$ *between two abstract simplicial complexes, given any* $\sigma \in \mathrm{H}^{\mathrm{simp}}_k(K)$ *and given any* $\varphi \in \mathrm{H}^k_{\mathrm{simp}}(L;G)$ *we have*

$$\langle f^*(\varphi),\sigma\rangle_K \ =\ \langle \varphi, f_*(\sigma)\rangle_L.$$

With the above lemma we can now formulate the following theorem which is an immediate consequence of the purely algebraic Universal Coefficient Theorem 75.12 for Cohomology Groups.

Theorem 67.8. (Universal Coefficient Theorem for Simplicial Cohomology) *Let K be an abstract simplicial complex and let G be an abelian group. For each $k \in \mathbb{N}_0$ there exists a short exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}(\mathrm{H}^{\mathrm{simp}}_{k-1}(K), G) & \longrightarrow & \mathrm{H}^k_{\mathrm{simp}}(K;G) & \xrightarrow{\mathrm{ev}} & \operatorname{Hom}(\mathrm{H}^{\mathrm{simp}}_k(K), G) & \longrightarrow & 0 \\ & & & & \varphi & \mapsto & \left(\begin{array}{rcl} \mathrm{H}^{\mathrm{simp}}_k(K) & \to & G \\ \sigma & \mapsto & \langle \varphi, \sigma \rangle_K \end{array} \right) & & \end{array}$$

which is natural in the abstract simplicial complex K and natural in the abelian group G. This short exact sequence splits and there exists therefore an isomorphism

$$\mathrm{H}^k_{\mathrm{simp}}(K;G) \;\cong\; \operatorname{Ext}(\mathrm{H}^{\mathrm{simp}}_{k-1}(K), G) \oplus \operatorname{Hom}(\mathrm{H}^{\mathrm{simp}}_k(K), G).$$

The analogous statement for ordered abstract simplicial complexes and the corresponding simplicial homology and cohomology groups also holds.

Remark.

(1) The Universal Coefficient Theorem 67.8 for Simplicial Cohomology is a double-edged sword. It allows us to compute the isomorphism types of simplicial cohomology groups in terms of the much more friendly simplicial homology groups. But it also says, somewhat disappointingly, that the isomorphism types of simplicial cohomology groups do not contain any information that is not already contained in the isomorphism types of the simplicial homology groups.

(2) The Universal Coefficient Theorem 67.8 for Simplicial Cohomology is also somewhat less powerful than it might appear. The theorem can be very convenient for determining the isomorphism types of simplicial cohomology groups. But if the Ext-terms are non-zero, then the theorem does not provide an approach to finding explicit cocycles representing cohomology classes.

67.4. **The simplicial cup product.** We had seen that one the remarkable features of singular cohomology is that it comes with a cup product and that it interacts with singular homology via the cap product. Before we attempt to introduce simplicial analogues let us remind us of the definitions from pages 1970, 1981 and 2019.

Definition.

(1) For $i = 0, \ldots, n$ we write $v_i := (0, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$.

(2) For $a_0, \ldots, a_k \in \Delta^n$ we consider the map

$$\begin{array}{rcl} [a_0, \ldots, a_k] \colon \Delta^k & \to & \Delta^n \\ (\lambda_0, \ldots, \lambda_k) & \mapsto & \sum\limits_{j=0}^{k} \lambda_j \cdot a_j. \end{array}$$

(3) Let X be a topological space and let R be a commutative ring.

(a) Let $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ be cochains. We define the *cup product* of φ and ψ to be the singular cochain

$$\begin{array}{rcl} \varphi \cup \psi\colon \mathrm{C}_{p+q}(X) & \to & R \\ (\sigma\colon \Delta^{p+q} \to X) & \mapsto & \underbrace{\varphi(\sigma\circ[v_0,\dots,v_p])}_{\in R} \cdot \underbrace{\psi(\sigma\circ[v_p,\dots,v_{p+q}])}_{\in R}. \end{array}$$

(b) Given a singular cochain $\varphi \in \mathrm{C}^k(X;R)$ and a singular simplex $\sigma\colon \Delta^n \to X$ with $k \le n$ we define the *cap product* of φ and σ to be the singular chain

$$\varphi \,\cap\, \sigma \;:=\; \underbrace{\varphi(\sigma\circ[v_0,\dots,v_k])}_{\in R} \cdot \underbrace{\sigma\circ[v_k,\dots,v_n]}_{\text{map } \Delta^{n\text{-}k} \to X}.$$

For $k > n$ we define $\varphi \cap \sigma = 0$.

In this and in the coming section we will introduce simplicial analogues of the above definitions. Evidently the idea is to imitate the above definitions and to define the cup and cap products on the level of simplicial (co-) chains. A short moment's thought shows that on the (co-) chain level one needs to work with *ordered* abstract simplicial complexes.

First we have the following definition which is clearly inspired by our second definition of the cup product in singular cohomology that we gave on page 1981.

Definition. Let $(K,\le)$ be an ordered abstract simplicial complex and let R be a commutative ring. Given $\varphi \in \mathrm{C}^p_{\mathrm{simp},\le}(K;R)$ and $\psi \in \mathrm{C}^q_{\mathrm{simp},\le}(K;R)$ we consider the *cup product*

$$\begin{array}{rcl} \varphi \cup \psi\colon \mathrm{C}^{\mathrm{simp},\le}_{p+q}(K) & \to & R \\ \{v_0 < \dots < v_p < \dots < v_{p+q}\} & \mapsto & \underbrace{\varphi(\{v_0 < \ldots < v_p\})}_{\in R} \cdot \underbrace{\psi(\{v_p < \ldots < v_{p+q}\})}_{\in R}. \end{array}$$

The following lemma is the analogue of Lemma 81.2.

Lemma 67.9. *Let $(K,\le)$ be an ordered abstract simplicial complex and let R be a commutative ring. Furthermore let $\varphi \in \mathrm{C}^p_{\mathrm{simp},\le}(K;R)$ and $\psi \in \mathrm{C}^q_{\mathrm{simp},\le}(K;R)$. Then*

$$\delta(\varphi\cup\psi) \;=\; \delta\varphi\cup\psi \,+ (-1)^p\cdot\varphi\cup\delta\psi \;\in\; \mathrm{C}^{p+q+1}_{\mathrm{simp},\le}(K;R).$$

In the following we will state several lemmas that are the simplicial counterparts of earlier results on the singular cup product. In most cases the proofs are, with an appropriate change of notation, basically identical. For sake of illustration we provide the proof of Lemma 67.9 and we invite the reader to compare the proof to the proof of Lemma 81.2.

Proof (∗). Let $\varphi \in \mathrm{C}^p_{\mathrm{simp},\le}(K;R)$ and $\psi \in \mathrm{C}^q_{\mathrm{simp},\le}(K;R)$ be simplicial cochains. We have to prove the following equality:

$$\delta(\varphi\cup\psi) \;=\; \delta\varphi\cup\psi \,+ (-1)^p\cdot\varphi\cup\delta\psi \;\in\; \mathrm{C}^{p+q+1}_{\mathrm{simp},\le}(K;R) = \mathrm{Hom}(\mathrm{C}^{\mathrm{simp},\le}_{p+q+1}(K),R).$$

So let $s = \{v_0 < \cdots < v_{p+q+1}\}$ be a simplicial $(p+q+1)$-simplex. We have the following three equalities in R:[1024]

$$\begin{aligned}
\text{(a)}\ (\delta(\varphi\cup\psi))(s) &= (\varphi\cup\psi)\Big(\sum_{j=0}^{p+q+1}(-1)^j\cdot\{v_0<\ldots<\widehat{v_j}<\ldots<v_{p+q+1}\}\Big)\\
&= \sum_{j=0}^{p}(-1)^j\cdot(\varphi\cup\psi)(\{v_0<\ldots<\widehat{v_j}<\ldots<v_{p+q+1}\})\\
&\quad+(-1)^p\cdot\sum_{k=1}^{q+1}(-1)^k\cdot(\varphi\cup\psi)(\{v_0<\ldots<\widehat{v_{k+p}}<\ldots<v_{p+q+1}\})\\
&= \sum_{j=0}^{p}(-1)^j\cdot\varphi(\{v_0<\ldots<\widehat{v_j}<\ldots<v_{p+1}\})\cdot\psi(\{v_p<\ldots<v_{p+q+1}\})\\
&\quad+(-1)^p\cdot\sum_{k=1}^{q+1}(-1)^k\cdot\varphi(\{v_0<\ldots<v_p\})\cdot\psi(\{v_p<\ldots<\widehat{v_k}<\ldots<v_{p+q+1}\}).
\end{aligned}$$

$$\begin{aligned}
\text{(b)}\ (\delta\varphi\cup\psi)(s) &= (\delta\varphi)(\{v_0<\ldots<v_{p+1}\})\cdot\psi(\{v_p<\ldots<v_{p+q+1}\})\\
&=\varphi\Big(\sum_{j=0}^{p+1}(-1)^j\cdot\{v_0<\ldots<\widehat{v_j}<\ldots<v_{p+1}\}\Big)\cdot\psi(\{v_p<\ldots<v_{p+q+1}\})\\
&=\sum_{j=0}^{p+1}(-1)^j\cdot\varphi(\{v_0<\ldots<\widehat{v_j}<\ldots<v_{p+1}\})\cdot\psi(\{v_p<\ldots<v_{p+q+1}\})
\end{aligned}$$

and

$$\begin{aligned}
\text{(c)}\ (-1)^p\cdot(\varphi\cup\delta\psi)(s) &= (-1)^p\cdot\varphi(\{v_0<\ldots<v_p\})\cdot(\delta\psi)(\{v_p<\ldots<v_{p+q+1}\})\\
&=(-1)^p\cdot\varphi(\{v_0<\ldots<v_p\})\cdot\psi\Big(\sum_{k=0}^{q+1}(-1)^k\cdot\{v_p<\ldots<\widehat{v_k}<\ldots<v_{p+q+1}\}\Big)\\
&=(-1)^p\cdot\sum_{k=0}^{q+1}(-1)^k\cdot\varphi(\{v_0<\ldots<v_p\})\cdot\psi(\{v_p<\ldots<\widehat{v_k}<\ldots<v_{p+q+1}\}).
\end{aligned}$$

Now we observe that if we take the sum of (b) and (c), then the $(j = p+1)$-summand in (b) cancels with the $(k = 0)$-summand in (c). But after this observation it is obvious that the desired equality (b)+(c)=(a) holds. ■

In light of Lemma 67.9 it is now straightforward to see that the cup product descends to a map on simplicial cohomology.

Lemma 67.10. *Let R be a commutative ring. For every ordered abstract simplicial complex $(K,\leq)$ the map*

$$\begin{aligned}
\cup\colon\ \mathrm{H}^p_{\mathrm{simp},\leq}(K;R)\times\mathrm{H}^q_{\mathrm{simp},\leq}(K;R) &\to \mathrm{H}^{p+q}_{\mathrm{simp},\leq}(K;R)\\
([\varphi],[\psi])] &\mapsto [\varphi]\cup[\psi]:=[\varphi\cup\psi]
\end{aligned}$$

is well-defined. We refer to this map again as the cup product.

Proof. This lemma follows almost immediately from Lemma 67.9. For more details we refer to the proof of Lemma 81.3. In that proof we just need to replace Lemma 81.2 by Lemma 67.9. ■

[1024] Here in the argument we use on several occasions that for a simplicial n-cochain ϕ and an $(n+1)$-simplex $t = \{w_0<\ldots<w_{n+1}\}$ we have by definition

$$(\delta_{n+1}\phi)(t) \;=\; \phi(\partial_n t) \;=\; \phi\Big(\sum_{l=0}^{n+1}(-1)^l\cdot\{w_0<\ldots<\widehat{w_l}<\ldots<w_{n+1}\}\Big).$$

The simplicial cup product is an interesting invariant in its own right and as we will see shortly, for finite abstract simplicial complexes it can be computed effectively. As for simplicial homology the question arises whether the isomorphism type of the simplicial cup product is an invariant of the topological realization. As we discussed in Sections 63.4 and 63.5, arguably the cleanest way of showing this is by proving that the simplicial cup product corresponds to the singular cup product. This is precisely what we will do in the first part of the following proposition, which can be viewed as an addendum to Proposition 67.6.

Proposition 67.11. *Let K be an abstract simplicial complex and let R be a commutative ring.*

(1) *Let "$\le$" be an order on K. Let $\Theta_{\le,k}\colon \mathrm{C}_k^{\mathrm{simp},\le}(K) \to \mathrm{C}_k(|K|)$ be the natural chain map that we introduced in Proposition 67.1. The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}^p(|K|;R) \times \mathrm{H}^q(|K|;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q}(|K|;R) \\ \Theta_\le^* \downarrow \cong \qquad \cong \downarrow \Theta_\le^* & & \cong \downarrow \Theta_\le^* \\ \mathrm{H}^p_{\mathrm{simp},\le}(K;R) \times \mathrm{H}^q_{\mathrm{simp},\le}(K;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q}_{\mathrm{simp},\le}(K;R). \end{array}$$

(2) *There exists a unique map*

$$\mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}^q_{\mathrm{simp}}(K;R) \xrightarrow{\quad\cup\quad} \mathrm{H}^{p+q}_{\mathrm{simp}}(K;R)$$

such that for every choice of order "$\le$" for K the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}^q_{\mathrm{simp}}(K;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q}_{\mathrm{simp}}(K;R) \\ \Omega_\le^* \downarrow \cong \qquad \cong \downarrow \Omega_\le^* & & \cong \downarrow \Omega_\le^* \\ \mathrm{H}^p_{\mathrm{simp},\le}(K;R) \times \mathrm{H}^q_{\mathrm{simp},\le}(K;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q}_{\mathrm{simp},\le}(K;R). \end{array}$$

In other words, we can define the cup product on $\mathrm{H}^*_{\mathrm{simp}}(K;R)$ in a way that is independent of the choice of an order "$\le$".

(3) *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}^p(|K|;R) \times \mathrm{H}^q(|K|;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q}(|K|;R) \\ \Xi_p^* \downarrow \cong \qquad \cong \downarrow \Xi_q^* & & \cong \downarrow \Xi_{p+q}^* \\ \mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}^q_{\mathrm{simp}}(K;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q}_{\mathrm{simp}}(K;R). \end{array}$$

Remark. Let K be an abstract simplicial complex. In the presence of an order on K we have, by definition, a natural simplicial cup product on the cochain groups $\mathrm{C}^*_{\mathrm{simp},\le}(K;R)$. In the absence of an order we do not have a natural simplicial cup product on the cochain groups $\mathrm{C}^*_{\mathrm{simp}}(K;R)$. But Proposition 67.11 says that we do have a natural simplicial cup product on the cohomology groups $\mathrm{H}^*_{\mathrm{simp}}(K;R)$.

Proof.

(1) It follows follows immediately from the definitions of the simplicial cup product and the definition of the singular cup product that we gave on page 1981 and that we

recalled on page 1693 that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{C}^p(|K|;R) \times \mathrm{C}^q(|K|;R) & \xrightarrow{\quad\cup\quad} & \mathrm{C}^{p+q}(|K|;R) \\
\Theta_\le^* \downarrow \qquad\qquad \downarrow \Theta_\le^* & & \downarrow \Theta_\le^* \\
\mathrm{C}^p_{\mathrm{simp},\le}(K;R) \times \mathrm{C}^q_{\mathrm{simp},\le}(K;R) & \xrightarrow{\quad\cup\quad} & \mathrm{C}^{p+q}_{\mathrm{simp},\le}(K;R).
\end{array}$$

The desired statement follows immediately from this observation.

(2) Surely there is a heroic proof for this statement that does not involve singular cohomology. But to lessen the pain let us give a proof that makes the most use of our previous results. Thus we consider the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}^q_{\mathrm{simp}}(K;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q}_{\mathrm{simp}}(K;R) \\
(\Omega_\le^{-1})_* \uparrow \cong \qquad\qquad \cong \uparrow (\Omega_\le^{-1})_* & & \cong \uparrow (\Omega_\le^{-1})_* \\
\mathrm{H}^p_{\mathrm{simp},\le}(K;R) \times \mathrm{H}^q_{\mathrm{simp},\le}(K;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q}_{\mathrm{simp},\le}(K;R) \\
\Theta_\le^* \uparrow \cong \qquad\qquad \cong \uparrow \Theta_\le^* & & \cong \uparrow \Theta_\le^* \\
\mathrm{H}^p(|K|;R) \times \mathrm{H}^q(|K|;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q}(|K|;R).
\end{array}$$

By Proposition 67.1 together with Lemma 73.5 we see that the composition of the two vertical maps does not depend on the choice of "$\le$". By (1) we know that the lower square commutes. Now we define the cup product on $\mathrm{H}^*_{\mathrm{simp}}(K;R)$ via the singular cup product and we see that it makes the upper square commute for any choice of "$\le$".

(3) This statement follows immediately from the definition of the natural isomorphisms Ξ^*, see Proposition 67.6 (3). ■

Remark. Proposition 67.11 shows in particular that if we are given a simplicial complex $(X, \Theta\colon |K| \to X)$, then we can compute the initially intractable singular cup product on $\mathrm{H}^*(X;R)$ via the simplicial cup product on $\mathrm{H}^*_{\mathrm{simp}}(K;\mathbb{R})$. If K is finite and explicitly given, then at least in principle one can compute the simplicial cup product on $\mathrm{H}^*_{\mathrm{simp}}(K;R)$ by brute force. This approach though is not a cure-all. For example given any $n \in \mathbb{N}$ we will want to determine the cup product on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$. By Theorem 64.2 we know that the complex projective spaces $\mathbb{C}\mathrm{P}^n$ and the real projective spaces $\mathbb{R}\mathrm{P}^n$ admit simplicial structures. But we run into the following issues:

(1) It seems to be difficult to find explicit simplicial structures for the complex projective spaces $\mathbb{C}\mathrm{P}^n$. So we cannot even get started.
(2) Explicit simplicial structures for smooth manifolds tend to have many simplices. For example in Exercise 62.5 we gave an explicit simplicial structure for the real projective space $\mathbb{R}\mathrm{P}^n$ with $2^{n+1}-1$ vertices and it seems difficult to find simplicial structures with a significantly smaller number of vertices.

Later on, in Propositions 90.7 and 90.7 we will compute these cup products using the Simplicial Poincaré Duality Theorem 71.4.

To do calculations of cup products we need to deal with specific cohomology classes and in fact also with explicit cocycles. In the next lemma we will see that closed oriented pseudomanifolds are a rich source of interesting explicit cocycles. Before we can state this lemma it is convenient to recall some definitions from page 1652 and page 1653.

Definition. Let $K = (V, S)$ be an n-dimensional pseudomanifold.

(1) We denote by S_n the set of n-simplices of K.
(2) An orientation of an n-simplex s of K is an equivalence class of a pair $(\epsilon, \widetilde{s})$ where $\epsilon \in \{-1, 1\}$ and $\widetilde{s}$ is an ordered n-simplex with $\lfloor \widetilde{s} \rfloor = s$.
(3) An *orientation of* K is a choice of an orientation for each n-simplex of K such that for any $(n-1)$-simplex s of order 2 the two cobounding n-simplices induce opposite orientations on s.
(4) If K is a closed pseudomanifold that is equipped with an orientation $\{(\epsilon_w, Y_w)\}_{w \in S_n}$, then we refer to the simplicial homology class
$$[K] \;:=\; \Big[\sum_{w \in S_n} \epsilon_w \cdot Y_w\Big] \;\in\; \mathrm{H}_n^{\mathrm{simp}}(K)$$
as the *fundamental class of* K.

We illustrate the definitions for the simplicial torus in Figure 1041.

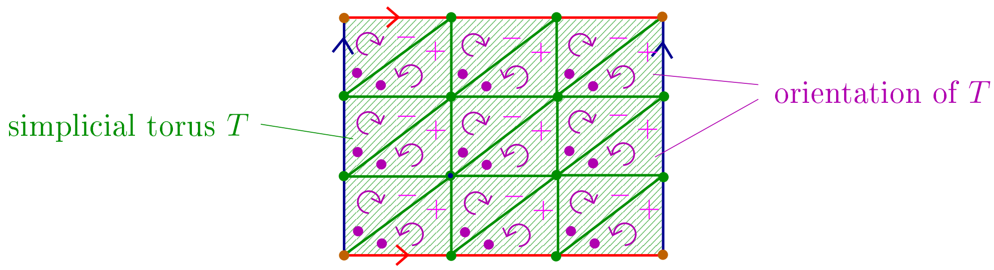


FIGURE 1041

Now we can formulate the following lemma.

Lemma 67.12. *Let $K = (V, S)$ be a closed connected n-dimensional pseudomanifold that is equipped with an orientation $\{(\epsilon_w, Y_w)\}_{w \in S_n}$ in the sense of the definition on page 1652.*

(1) (a) *For any $s, t \in S_n$ we have $[\epsilon_s \cdot Y_s^*] = [\epsilon_t \cdot Y_t^*] \in \mathrm{H}^n_{\mathrm{simp}}(K; \mathbb{Z})$. We refer to this cohomology class as the* dual fundamental class $[K]^* \in \mathrm{H}^n_{\mathrm{simp}}(K; \mathbb{Z})$.
(b) *We have $\mathrm{H}^n_{\mathrm{simp}}(K; \mathbb{Z}) = \mathbb{Z} \cdot [K]^*$.*
(2) *We have $\langle [K]^*, [K] \rangle_K = 1$.*

PROOF.

(1) It follows from Theorem 66.4, and the discussion on page 1659 that the simplicial chain complex of K is of the form
$$0 \;\to\; \underbrace{\mathbb{Z} \cdot \Big[\sum_{w \in S_n} \epsilon_w \cdot Y_w\Big] \oplus \widetilde{C}_n}_{=\mathrm{C}_n^{\mathrm{simp}}(K)} \;\xrightarrow[\partial_n]{\begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}}\; \underbrace{\widetilde{C}_{n-1} \oplus \widehat{C}_{n-1}}_{=\mathrm{C}_{n-1}^{\mathrm{simp}}(K)} \;\to\; \dots$$

such that $D\colon \widetilde{C}_n \to \widetilde{C}_{n-1}$ is an isomorphism. It follows easily that the map

$$\begin{array}{rcl} \mathrm{H}^n_{\mathrm{simp}}(K;\mathbb{Z}) = \operatorname{coker}(\delta_{n-1}) & \to & \operatorname{Hom}\Big(\mathbb{Z}\cdot\Big[\sum\limits_{w\in S_n}\epsilon_w\cdot Y_w\Big],\mathbb{Z}\Big) & \to & \mathbb{Z} \\ \varphi & \mapsto & \text{restriction of } \varphi & & \\ & & \varphi & \mapsto & \varphi\Big(\sum\limits_{w\in S_n}\epsilon_w\cdot Y_w\Big) \end{array}$$

is an isomorphism. The desired statements follow almost immediately from this observation.[1025]

(2) This statement follows immediately from the definitions. ■

Now let us try to determine the simplicial cup product of the simplicial torus.

Example. We consider the simplicial circle C with vertex set $V = \{a,b,c\}$ and the simplicial circle $D = \{x,y,z\}$ with the obvious total order on the vertex sets. Furthermore we consider the simplicial torus $T = C \times D$, where we define the product as on page 1501. We equip T with the orientation shown in Figure 1041. We denote by $p\colon C\times D \to C$ and $q\colon C\times D\to D$ the natural projections. We equip the vertex set of $T = C\times D$ with the lexicographic ordering. In Figure 1042 we show $T = C\times D$ together with the projections.

Let us first consider the simplicial cohomology of T. Let $\theta \in \mathrm{C}^1_{\mathrm{simp}}(C)$ be the simplicial cocycle that we introduced on page 1688. We have the following equalities:

$$\mathrm{H}^i_{\mathrm{simp}}(T;\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z}\cdot[\text{constant cochain } v\mapsto 1], & \text{if } i=0,\\ \mathbb{Z}\cdot[p^*(\theta)]\oplus\mathbb{Z}\cdot[q^*(\theta)], & \text{if } i=1,\\ \mathbb{Z}\cdot[T]^*, & \text{if } i=2,\\ 0, & \text{if } i>2.\end{cases}$$

This can be obtained either through a slightly heroic linear algebra exercise, or, more systematically, through the combination of Lemma 67.12, the Simplicial Künneth Theorem 63.14 and the Universal Coefficient Theorem 67.8 for Simplicial Cohomology. Now we can calculate our first interesting simplicial cup product:

$$[p^*(\theta)]\cup[q^*(\theta)] \;=\; [p^*(\theta)\cup q^*(\theta)] \underset{\substack{\uparrow\\ \text{by inspection}}}{=} [\overbrace{((b,y),(c,y),(c,z))}^{\text{ordered 2-simplex}}{}^*] \underset{\substack{\uparrow\\ \text{by Lemma 67.12}}}{=} [T]^* \;\in\; \mathrm{H}^2_{\mathrm{simp}}(T;\mathbb{Z}).$$

Similar calculations gives us the following table for the cup product:[1026]

$$\begin{pmatrix} [p^*(\theta)]\cup[p^*(\theta)] & [p^*(\theta)]\cup[q^*(\theta)] \\ [q^*(\theta)]\cup[p^*(\theta)] & [q^*(\theta)]\cup[q^*(\theta)]\end{pmatrix} \;=\; \begin{pmatrix} 0 & [T]^* \\ -[T]^* & 0\end{pmatrix}.$$

This is the simplicial analogue of the calculation performed in Lemma 81.14.

In the following we state several properties of the simplicial cup product that are the analogues of the corresponding properties of the singular cup product. First let us start with the analogue of Proposition 81.7.

[1025] The above argument is related to the content of Exercise 73.1.

[1026] In fact the remaining three entries of the matrix can also be obtained easily from Proposition 67.14 and Lemma 67.15.

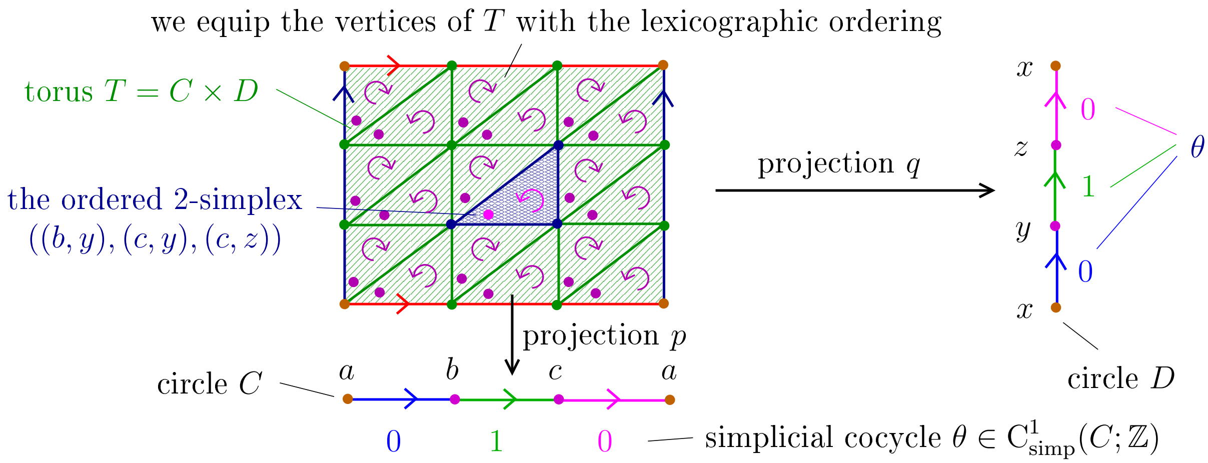


FIGURE 1042

Proposition 67.13. *Let $K = (V, S)$ be an abstract simplicial complex and let R be a commutative ring. The following statements hold:*

(1) *The cup product on $\mathrm{H}^*_{\mathrm{simp}}(K;R)$ is R-bilinear and associative.*

(2) *The abelian group $\mathrm{H}^*_{\mathrm{simp}}(K;R) = \bigoplus\limits_{n\in\mathbb{N}_0} \mathrm{H}^n_{\mathrm{simp}}(K;R)$ together with the map*

$$\cup\colon\ \mathrm{H}^*_{\mathrm{simp}}(K;R) \times \mathrm{H}^*_{\mathrm{simp}}(K;R) \to \mathrm{H}^*_{\mathrm{simp}}(K;R)$$

is a ring where the multiplicatively neutral element is given by

$$1_K \ := \ \big[\text{map } V \to R \text{ given by } v \mapsto 1\big] \ \in \ \mathrm{H}^0_{\mathrm{simp}}(K;R).$$

PROOF. We pick an order "$\le$" for K. As in the proof of Proposition 81.7 we see that the cup product on the cochain level is associative. It now follows easily that all the given statements hold for $(\mathrm{H}^*_{\mathrm{simp},\le}(K;R), \cup)$. By Proposition 67.11 (2) the statements now also hold for $(\mathrm{H}^*_{\mathrm{simp}}(K;R), \cup)$. ■

The following proposition is the analogue of Proposition 81.8.

Proposition 67.14. *Let K be an abstract simplicial complex and let R be a commutative ring. For any $\varphi \in \mathrm{H}^p_{\mathrm{simp}}(K;R)$ and any $\psi \in \mathrm{H}^q_{\mathrm{simp}}(K;R)$ we have*

$$\varphi \cup \psi \ = \ (-1)^{pq} \cdot \psi \cup \varphi \ \in \ \mathrm{H}^{p+q}_{\mathrm{simp}}(K;R).$$

PROOF. We pick an order "$\le$" for K. By Proposition 67.11 (2) it suffices to prove the following claim.

Claim. For any $\varphi \in \mathrm{H}^p_{\mathrm{simp},\le}(K;R)$ and $\psi \in \mathrm{H}^q_{\mathrm{simp},\le}(K;R)$ we have

$$\varphi \cup \psi \ = \ (-1)^{pq} \cdot \psi \cup \varphi \ \in \ \mathrm{H}^{p+q}_{\mathrm{simp},\le}(K;R).$$

The proof of the claim happens at the "cochain level". The argument is basically identical to the proof of Proposition 81.8. The only real difference is that one needs to replace the product topological space $\Delta^n \times [0,1]$ by the product of the ordered abstract simplicial complexes $D_n = (\{0,\ldots,n\},\mathcal{P}(\{0,\ldots,n\}\setminus\{\varnothing\}))$ and $D_1 = (\{0,1\},\{\{0\},\{1\}\})$, as defined in Lemma 61.16. We leave it to the reader to make the necessary modifications to the argument. ■

Finally we have the following lemma which is the simplicial analogue of Lemma 81.10.

Lemma 67.15. *Let R be a commutative ring.*

(1) *Let $f\colon K\to L$ be a simplicial map between two abstract simplicial complexes. For any $\varphi\in \mathrm{H}^p_{\mathrm{simp}}(L;R)$ and $\psi\in\mathrm{H}^q_{\mathrm{simp}}(L;R)$ we have*
$$f^*(\varphi)\cup f^*(\psi) \quad = \quad f^*(\varphi\cup\psi).$$
In particular the map
$$f^*\colon\ \mathrm{H}^*_{\mathrm{simp}}(L;R) \quad\to\quad \mathrm{H}^*_{\mathrm{simp}}(K;R)$$
is a homomorphism of graded rings which satisfies $f^*(1_L)=1_K$.

(2) *The maps*
$$\begin{array}{rcl} K & \mapsto & (\mathrm{H}^*_{\mathrm{simp}}(K;R),\cup) \\ (f\colon K\to L) & \mapsto & (f^*\colon\ \mathrm{H}^*_{\mathrm{simp}}(L;R)\to \mathrm{H}^*_{\mathrm{simp}}(K;R)) \end{array}$$
define a contravariant functor from the category $\mathcal{AbsSimpCplx}$ of abstract simplicial complexes to the category $\mathcal{GrRing}$ of graded rings.

Proof (∗).

(1) Let $f\colon K=(V,S)\to L=(W,T)$ be a simplicial map between two abstract simplicial complexes. By Exercise 1.2 we can pick total orders "$\le_K$" on K and "$\le_L$" on L such that the map $f\colon V\to W$ is order-preserving. It follows basically immediately from the definitions that for $\varphi\in \mathrm{C}^p_{\mathrm{simp},\le_L}(L;R)$ and $\psi\in \mathrm{C}^q_{\mathrm{simp},\le_L}(L;R)$ we have the equality $f^*(\varphi)\cup f^*(\psi)=f^*(\varphi\cup\psi)\in \mathrm{C}^{p+q}_{\mathrm{simp},\le_K}(K)$. The statements now follow from Proposition 67.11 (2).

(2) This statement follows from (1) together with Lemma 67.5 (2). ■

67.5. **The simplicial cap product.** In this section we will introduce the simplicial cap product. In contrast to the simplicial cup product it is perhaps less immediately clear what the simplicial cap product is good for. But it will have its moment of glory in Chapter 68 when we state and prove the Simplicial Poincaré Duality Theorem 71.4.

The following definition of the simplicial cap product is an adaptation of the definition of the singular cap product that we gave on page 2019 and that we recalled on page 1693.

Definition. Let $(K,\le)$ be an ordered abstract simplicial complex and let R be a commutative ring. Given an n-simplex $s=\{v_0<\cdots<v_n\}$, given $k\le n$ and given a simplicial cochain $\varphi\in \mathrm{C}^k_{\mathrm{simp},\le}(K;R)$ we define
$$\varphi\,\cap\, s \;:=\; \underbrace{\varphi(\{v_0<\cdots<v_k\})}_{\in R}\cdot\underbrace{\{v_k<\cdots<v_n\}}_{\in \mathrm{C}^{\mathrm{simp},\le}_{n-k}(K)} \;\in\; \mathrm{C}^{\mathrm{simp},\le}_{n-k}(K;R).$$

We extend this definition to simplicial R-chains and we obtain the *cap product*

$$\begin{array}{rcl} \cap\colon\ \mathrm{C}^k_{\mathrm{simp},\leq}(X;R)\times\mathrm{C}_n^{\mathrm{simp},\leq}(X;R) & \to & \mathrm{C}_{n-k}^{\mathrm{simp},\leq}(X;R) \\ \varphi\cap r\cdot\sigma & \mapsto & r\cdot\underbrace{(\varphi\cap\sigma)}_{\in R}. \end{array}$$

For $k>n$ we define the cap product to be the zero map.

The next lemma says that the cap product defines a map on simplicial (co-) homology.

Lemma 67.16. *Let $(K,\leq)$ be an ordered abstract simplicial complex. Given any commutative ring R and given any $k,n\in\mathbb{N}_0$ the map*

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{simp},\leq}(K;R)\times\mathrm{H}_n^{\mathrm{simp},\leq}(K;R) & \to & \mathrm{H}_{n-k}^{\mathrm{simp},\leq}(K;R) \\ ([\varphi],[\sigma]) & \mapsto & [\varphi\cap\sigma] \end{array}$$

is well-defined.

Proof. The same argument as in the proof of Lemma 83.1 shows that given an n-simplex $s=\{v_0<\cdots<v_n\}$, given $k\leq n$ and given any simplicial cochain $\varphi\in\mathrm{C}^k_{\mathrm{simp},\leq}(K;R)$ we have the following equality:

$$\partial(\varphi\cap\sigma)\ =\ (-1)^k\cdot(-\delta\varphi\cap\sigma\ +\ \varphi\cap\partial\sigma)\ \in\ \mathrm{C}_{n-k-1}(X;R).$$

As in the proof of Lemma 83.2 we see that the lemma follows easily from this equality. ■

The next proposition is an analogue of Proposition 67.11.

Proposition 67.17. *Let K be an abstract simplicial complex and let R be a commutative ring.*

(1) *Let "$\leq$" be an order on K. Let $\Theta_{\leq,k}\colon\ \mathrm{C}_k^{\mathrm{simp},\leq}(K)\to\mathrm{C}_k(|K|)$ be the natural chain map that we introduced in Proposition 67.1. The following diagram commutes:*[1027]

$$\begin{tikzcd} \mathrm{H}^p(|K|;R)\times\mathrm{H}_n(|K|;R) \arrow[r, "\cap"] \arrow[d, "\Theta^*_{\leq}"', "\cong"] & \mathrm{H}_{n-p}(|K|;R) \\ \mathrm{H}^p_{\mathrm{simp},\leq}(K;R)\times\mathrm{H}_n^{\mathrm{simp},\leq}(K;R) \arrow[r, "\cap"] \arrow[u, "\cong", "\Theta_*^{\leq}"'] & \mathrm{H}_{n-p}^{\mathrm{simp},\leq}(K;R). \arrow[u, "\cong"', "\Theta_*^{\leq}"] \end{tikzcd}$$

(2) *There exists unique map*

$$\mathrm{H}^p_{\mathrm{simp}}(K;R)\times\mathrm{H}_n^{\mathrm{simp}}(K;R)\xrightarrow{\ \cap\ }\mathrm{H}_{n-p}^{\mathrm{simp}}(K;R)$$

such that for any choice of order "$\leq$" for K the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}^p_{\mathrm{simp}}(K;R)\times\mathrm{H}_n^{\mathrm{simp}}(K;R) \arrow[r, "\cap"] \arrow[d, "\Omega^*_{\leq}"', "\cong"] & \mathrm{H}_{n-p}^{\mathrm{simp}}(K;R) \\ \mathrm{H}^p_{\mathrm{simp},\leq}(K;R)\times\mathrm{H}_n^{\mathrm{simp},\leq}(K;R) \arrow[r, "\cap"] \arrow[u, "\cong", "\Omega_*^{\leq}"'] & \mathrm{H}_{n-p}^{\mathrm{simp},\leq}(K;R). \arrow[u, "\cong"', "\Omega_*^{\leq}"] \end{tikzcd}$$

In other words, we can define the cap product on $\mathrm{H}^*_{\mathrm{simp}}(K;R)$ and $\mathrm{H}_*^{\mathrm{simp}}(K;R)$ in a way that is independent of the choice of an order "$\leq$".

(3) *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}^p(|K|;R) \times \mathrm{H}_n(|K|;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-p}(|K|;R) \\ \Xi^*\downarrow\cong \qquad \cong\uparrow\Xi_* & & \cong\uparrow\Xi_* \\ \mathrm{H}^p_{\mathrm{simp}}(K;R) \times \mathrm{H}^{\mathrm{simp}}_n(K;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}^{\mathrm{simp}}_{n-p}(K;R). \end{array}$$

Sketch of proof.

(1) It follows almost immediately from the definition that the corresponding diagram on the level of simplicial (co-) chains commutes. This implies that the given diagram commutes.
(2) As in the proof of Proposition 67.11 (2) this statement follows from (1) together with Proposition 67.1, Corollary 42.8 and Lemma 73.5.
(3) This statement follows immediately from the definition of the natural isomorphisms Ξ_* and Ξ^*, see Propositions 67.1 (3) and 67.6 (3). ■

Example. As on page 1698 we consider again the simplicial torus $T = C \times D$. We equip $T = (V, S)$ with the orientation $\{(\epsilon_w, Y_w)\}_{w\in S_2}$ shown in Figures 1041 and 1043 and we denote by $p\colon C \times D \to C$ the natural projection. Furthermore, we again make use of the simplicial cocycle $\theta \in \mathrm{C}^1_{\mathrm{simp}}(C;\mathbb{Z})$ that we first encountered on page 1688. We introduce the following extra notation:

(1) We denote by $j\colon D \to C \times D$ the simplicial map given by $v \mapsto (c, v)$.
(2) We denote by $\sigma := \{x, y\} + \{y, z\} + \{z, x\} \in \mathrm{C}^{\mathrm{simp},\le}_1(D)$ the "obvious" simplicial cycle. Note that it represents the simplicial fundamental class $[D] \in \mathrm{H}^{\mathrm{simp},\le}_1(D)$.

It follows immediately from the definitions that

$$p^*(\theta) \cap \sum_{w\in S_2} \epsilon_w \cdot Y_w \;=\; j_*(\sigma) \;\in\; \mathrm{C}^{\mathrm{simp},\le}_1(C \times D)$$

thus in terms of (co-) homology we have

$$p^*([\theta]) \cap [T] \;=\; j_*([D]) \;\in\; \mathrm{H}^{\mathrm{simp},\le}_1(C \times D).$$

We refer to Figure 1043 for an illustration.

In the following we state three lemmas that summarize the key properties of the simplicial cap product. All of these statements are analogues of singular statements. First, the following lemma is an analogue of Lemma 83.4.

Lemma 67.18. *Let K be a non-empty connected abstract simplicial complex. Given any $\varphi \in \mathrm{H}^p_{\mathrm{simp}}(K;R)$ and $\sigma \in \mathrm{H}^{\mathrm{simp}}_p(K;R)$ the following equality holds:*

$$\varphi \cap \sigma \;=\; \langle \varphi, \sigma\rangle_K \;\in\; H^{\mathrm{simp}}_0(K;R) \underset{\substack{\uparrow \\ \text{as in Exercise 63.3 there is a natural identification}}}{=} R.$$

[1027]Here the expression "the diagram commutes" is understood in the sense of the convention from page 2007.

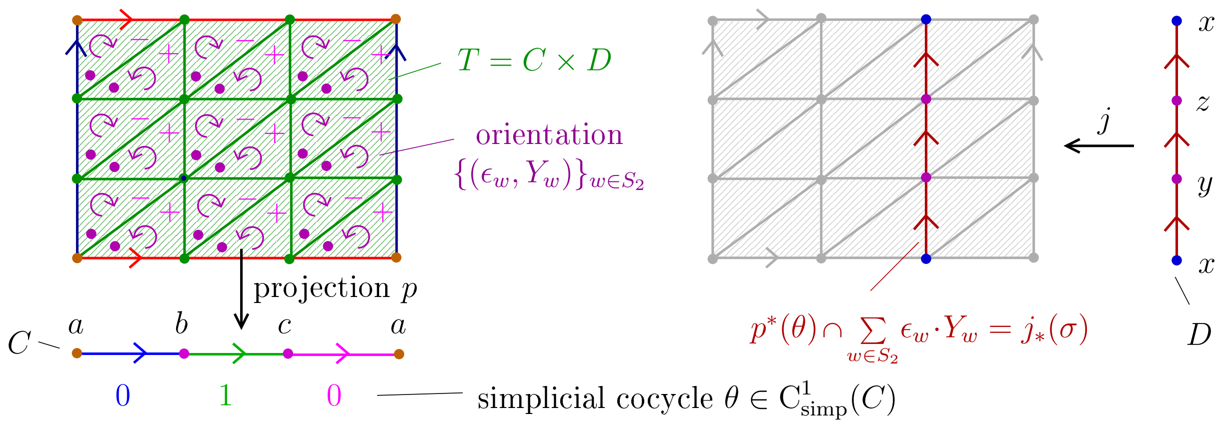


FIGURE 1043

PROOF. First we pick an order "$\leq$" for the abstract simplicial complex K. Next we observe that the statement for $\varphi \in \mathrm{H}^p_{\mathrm{simp},\leq}(K;R)$ and $\sigma \in \mathrm{H}_p^{\mathrm{simp},\leq}(K;R)$ follows immediately from the definitions. Now the statement follows from Proposition 67.17 (2). ■

The following lemma relates the simplicial cup product to the simplicial cap product. It can be viewed as an analogue of Lemma 83.7.

Lemma 67.19. *Let K be an abstract simplicial complex and let R be a commutative ring. For any $\varphi \in \mathrm{H}^p_{\mathrm{simp}}(K;R)$, any $\psi \in \mathrm{H}^q_{\mathrm{simp}}(K;R)$ and any $\sigma \in \mathrm{H}_n^{\mathrm{simp}}(K;R)$ we have*

$$\varphi \cap (\psi \cap \sigma) \;=\; (\psi \cup \varphi) \cap \sigma \;\in\; \mathrm{H}^{\mathrm{simp}}_{n-p-q}(K;R).$$

SKETCH OF PROOF. We pick an order "$\leq$" for K. With basically the same, quite elementary, argument as in the proof of Lemma 83.7 we see that for any $\varphi \in \mathrm{H}^p_{\mathrm{simp},\leq}(K;R)$, any $\psi \in \mathrm{H}^q_{\mathrm{simp},\leq}(K;R)$ and any $\sigma \in \mathrm{H}_n^{\mathrm{simp},\leq}(K;R)$ we have

$$\varphi \cap (\psi \cap \sigma) \;=\; (\psi \cup \varphi) \cap \sigma \;\in\; \mathrm{H}^{\mathrm{simp},\leq}_{n-p-q}(K;R).$$

The desired statement now follows from Proposition 67.17 (2). ■

Finally we have the following analogue of Lemma 83.8.

Lemma 67.20. *Let R be a commutative ring and let $f\colon K \to L$ be a simplicial map between two abstract simplicial complexes. Given any $p, n \in \mathbb{N}_0$ the following diagram commutes*

$$\begin{array}{ccccc}
\mathrm{H}^p_{\mathrm{simp}}(K;R) & \times & \mathrm{H}^{\mathrm{simp}}_n(K;R) & \xrightarrow{\cap} & \mathrm{H}^{\mathrm{simp}}_{n-p}(K;R) \\
{\scriptstyle f^*}\uparrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\mathrm{H}^p_{\mathrm{simp}}(L;R) & \times & \mathrm{H}^{\mathrm{simp}}_n(L;R) & \xrightarrow{\cap} & \mathrm{H}^{\mathrm{simp}}_{n-p}(L;R),
\end{array}$$

i.e. for any $\varphi \in \mathrm{H}^p_{\mathrm{simp}}(L;R)$ and any $\sigma \in \mathrm{H}^{\mathrm{simp}}_n(K;R)$ we have

$$\varphi \cap f_*(\sigma) \;=\; f_*(\underbrace{f^*(\varphi) \cap \sigma}_{\in \mathrm{H}^{\mathrm{simp}}_{n-p}(K;R)}) \;\in\; \mathrm{H}^{\mathrm{simp}}_{n-p}(L;R).$$

Proof. Let $f\colon K=(V,S)\to L=(W,T)$ be a simplicial map between two abstract simplicial complexes. By Exercise 1.2 we can pick total orders "$\leq_K$" on K and "$\leq_L$" on L such that the map $f\colon V\to W$ is order-preserving. It follows basically immediately from the definitions that for any $\varphi\in \mathrm{C}^p_{\mathrm{simp},\leq_L}(L;R)$ and $\sigma\in \mathrm{C}_n^{\mathrm{simp},\leq_K}(K;R)$ we have the equality $f_*(\sigma)=f_*(f^*(\varphi)\cap\sigma)$. The promised statement now follows from Proposition 67.17 (2). ■

We conclude this chapter with a short calculation which in particular shows that at times simplicial cup products can be computed using simplicial cap products.

Example. Let us look one last time at the simplicial torus $T=C\times D$ with the natural projections $p\colon C\times D\to C$ and $q\colon C\times D\to D$. We will now see how we can obtain the simplicial cup product $p^*(\theta)\cup q^*(\theta)$, that we initially computed on page 1698, from the calculation of the simplicial cap product on page 1702. Indeed, under the natural identification $H_0^{\mathrm{simp}}(T;R)=R=H_0^{\mathrm{simp}}(D;R)$ from Exercise 63.3 we have the following equalities:

$$\begin{aligned}\langle p^*([\theta])\cup q^*([\theta]),[T]\rangle_T &\overset{\text{by Lemma 67.18}}{=} (p^*([\theta])\cup q^*([\theta]))\cap[T] \overset{\text{Lemma 67.19}}{=} q^*([\theta])\cap(p^*([\theta])\cap[T]) \overset{\text{see calculation on page 1702}}{=} q^*([\theta])\cap j_*([D])\\ &\underset{\text{by Lemma 67.18}}{=} \langle q^*([\theta]),j_*([D])\rangle_T \underset{\text{by Lemma 67.7 (2)}}{=} \langle[\theta],(q_*\circ j_*)([D])\rangle_D \underset{\text{since } q\circ j=\mathrm{id}_D}{=} \langle[\theta],[D]\rangle_D \underset{\substack{\text{by definition}\\ \text{of } \theta \text{ and } [D]}}{=} 1.\end{aligned}$$

In Chapter 68 we will state the Simplicial Poincaré Duality Theorem 71.4. Given a closed oriented PL-manifold K it allows for the calculation of simplicial cap products of the form $\varphi\cap[K]$. With tricks as above one can use this calculation of a simplicial cap product to determine several interesting simplicial cup products. In particular in Chapter 90 we will use this approach to calculate the (simplicial) cup products of the real and complex projective spaces.

Exercises for Chapter 67.

Exercise 67.1. We consider the abstract simplicial complex K shown in Figure 1044. Determine the isomorphism type of $\mathrm{H}^1_{\mathrm{simp}}(K;\mathbb{Z})$.

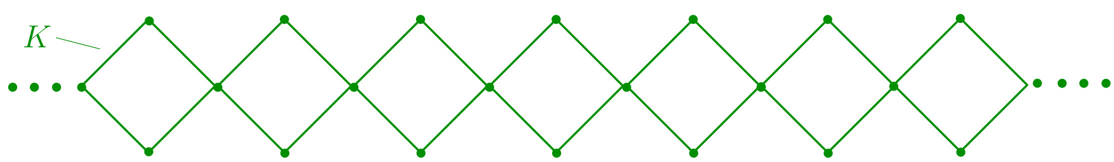


Figure 1044. Illustration for Exercise 67.1.

Exercise 67.2. Let $K=(V,S)$ be a non-empty abstract simplicial complex. We assume that K is connected. Recall that by the definition on page 1516 this means that given any $v,v'\in V$ there exist 1-simplices $s_0,\dots,s_k\in S$ such that $v\in s_0$, such that for every $i\in\{0,\dots,k-1\}$ we have $s_i\cap s_{i+1}\neq\varnothing$ and such that $v'\in s_k$. Now let G be an abelian group. Given any $g \in G$ we consider the simplicial cochain $\varphi_g\colon \mathrm{C}_0^{\mathrm{simp}}(K) \to G$ that is given by $\varphi_g(v) = g$ for every vertex $v \in V$. Show that the map

$$\begin{array}{rcl} G & \to & \mathrm{H}^0_{\mathrm{simp}}(K;G) \\ g & \mapsto & \varphi_g \end{array}$$

is a natural isomorphism.
Remark. This statement can be viewed is the cohomological analogue of Exercise 63.3.

Exercise 67.3. Let K be an abstract simplicial complex and let L be a subcomplex of K. Given an abelian group G and given $k \in \mathbb{N}_0$ we define

$$\mathrm{H}^k_{\mathrm{simp}}(K,L;G) \;:=\; \mathrm{H}^k(\mathrm{C}_*^{\mathrm{simp}}(K,L),G).$$

Show that there exists a natural isomorphism

$$\mathrm{H}^k_{\mathrm{simp}}(K,L;G) \;\cong\; \widetilde{\mathrm{H}}^k_{\mathrm{simp}}(\underbrace{K \cup_L \mathrm{Cone}(L)}_{\text{mapping cone of } L \to K};G).$$

Exercise 67.4. In Figure 1045 we show an abstract simplicial complex K whose topological realization $|K|$ is homeomorphic to the real projective plane $\mathbb{RP}^2$.

(a) Show that for $i = 0,1,2$ we have $\mathrm{H}^i_{\mathrm{simp}}(K;\mathbb{F}_2) \cong \mathbb{F}_2$ and identify in each dimension a cocycle that represents the unique non-zero element.
(b) Compute the simplicial cup product $\mathrm{H}^1_{\mathrm{simp}}(K;\mathbb{F}_2) \times \mathrm{H}^1_{\mathrm{simp}}(K;\mathbb{F}_2) \xrightarrow{\cup} \mathrm{H}^2_{\mathrm{simp}}(K;\mathbb{F}_2)$. In other words, determine whether it is trivial or non-trivial.

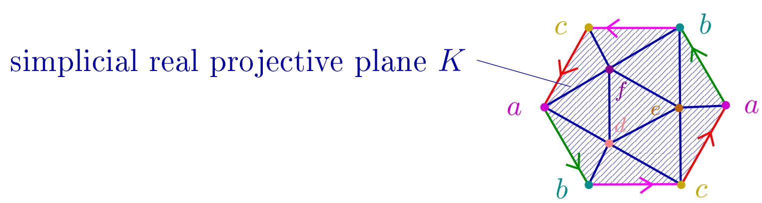


Figure 1045. Illustration for Exercise 67.4.

Exercise 67.5. Let $g \in \mathbb{N}$. First pick a suitable abstract simplicial complex K that has the property that its topological realization is homeomorphic to the surface of genus g. Afterwards compute the cup product

$$\mathrm{H}^1_{\mathrm{simp}}(K;\mathbb{Z}) \times \mathrm{H}^1_{\mathrm{simp}}(K;\mathbb{Z}) \;\to\; \mathrm{H}^2_{\mathrm{simp}}(K;\mathbb{Z}).$$

Exercise 67.6. We consider the topological space

$$X = \overline{B}^2/\sim \qquad \text{where } z \sim z \cdot e^{2\pi \mathrm{i}/3} \text{ for } z \in S^1.$$

(a) Determine the isomorphism types of $\mathrm{H}_i(X;\mathbb{F}_3)$ and $\mathrm{H}^i(X;\mathbb{F}_3)$.
(b) Give an explicit simplicial structure K for X.
(c) Compute the simplicial cup product $\mathrm{H}^1_{\mathrm{simp}}(K;\mathbb{F}_3) \times \mathrm{H}^1_{\mathrm{simp}}(K;\mathbb{F}_3) \xrightarrow{\cup} \mathrm{H}^2_{\mathrm{simp}}(K;\mathbb{F}_3)$.

68. The fundamental class of smooth manifolds

Let M be a compact *orientable* connected non-empty n-dimensional smooth manifold. By Theorem 66.8 we now know that we have $\mathrm{H}_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$. In this chapter we will see that a *choice of an orientation* naturally determines a generator $[M]$ of $\mathrm{H}_n(M, \partial M; \mathbb{Z})$, called the fundamental class of M. As the name already suggests, this homology class is of fundamental importance in the study of smooth manifolds.

68.1. **The definition of the fundamental class.** For our definition of fundamental classes we need to recall the following notation.

Notation. Let $n \in \mathbb{N}_0$. As on page 1075 we consider

$$\begin{array}{rcl} \partial\Delta^n & := & \{(t_0, \dots, t_n) \in \Delta^n \mid \text{at least one coordinate is zero}\} \\ \mathring{\Delta}^n & := & \{(t_0, \dots, t_n) \in \Delta^n \mid \text{all coordinates are non-zero}\}. \end{array}$$

We view $\mathring{\Delta}^n$ as a smooth manifold and we equip it with the orientation defined on page 1075. Loosely speaking it is the orientation given by the "outward pointing vector $(1, \dots, 1)$".

We move on to the following somewhat technical definition.

Definition. Let M be an oriented n-dimensional smooth manifold and let $\Psi \colon \Delta^n \to M$ be a map.

(1) We say that a point $x \in \mathring{\Delta}^n$ is *admissible for* Ψ if $\Psi^{-1}(\Psi(x)) = \{x\}$ and if there exists an open connected neighborhood U of $x \in \mathring{\Delta}^n$ such that the map $\Psi|_U \colon U \to M$ is a smooth embedding.
(2) Given an admissible point x for Ψ we define

$$\operatorname{sign}(\Psi, x) \;=\; \begin{cases} +1, & \text{if } \Psi \text{ is orientation-preserving at } x, \\ -1, & \text{if } \Psi \text{ is orientation-reversing at } x. \end{cases}$$

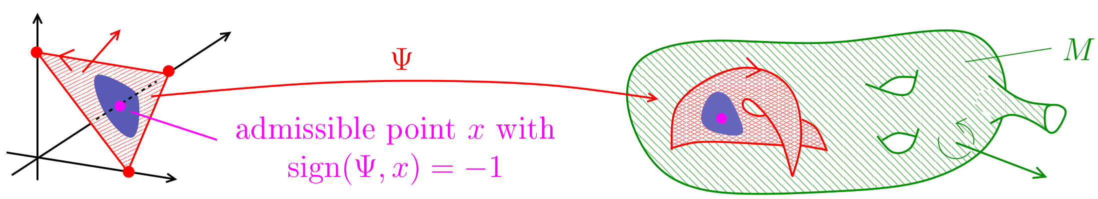


Figure 1046

Example. In many cases we will consider embeddings $\Psi \colon \Delta^n \to M$ such that the restriction of Ψ to $\mathring{\Delta}^n$ is a smooth embedding. In this case every point $x \in \mathring{\Delta}^n$ is admissible for Ψ, and we have

$$\operatorname{sign}(\Psi, x) \;=\; \begin{cases} +1, & \text{if } \Psi \colon \mathring{\Delta}^n \to M \text{ is orientation-preserving}, \\ -1, & \text{if } \Psi \colon \mathring{\Delta}^n \to M \text{ is orientation-reversing}. \end{cases}$$

Also note that it is clear that any non-empty smooth manifold admits such a smooth embedding.

The following theorem is the main technical result of this chapter.

Theorem 68.1. *Let $n \in \mathbb{N}$ and let M be a compact oriented connected n-dimensional smooth manifold.*

(1) *For every $x \in M \setminus \partial M$ the inclusion induced map*

$$\omega_x \colon \underbrace{\mathrm{H}_n(M, \partial M)}_{\cong \mathbb{Z} \text{ by Theorem 66.8}} \to \underbrace{\mathrm{H}_n(M, M \setminus \{x\})}_{\cong \mathbb{Z} \text{ by Lemma 44.1}}$$

is an isomorphism.

(2) *There exists a unique $[M] \in \mathrm{H}_n(M, \partial M)$ with the following property: For every map $\Psi \colon \Delta^n \to M$ and every admissible $x \in \mathring{\Delta}^n$ for Ψ we have the following equality in $\mathrm{H}_n(M, M \setminus \{\Psi(x)\})$:*[1028][1029]

$$\underset{\substack{\uparrow \\ \textit{the map } \mathrm{H}_n(M,\partial M)\to \mathrm{H}_n(M,M\setminus\{\Psi(x)\}) \textit{ as in } (1)}}{\omega_{\Psi(x)}}([M]) = \operatorname{sign}(\Psi, x) \cdot [\, \underbrace{\Psi \colon \Delta^n \to M}_{\in \mathrm{C}_n(M, M \setminus \{\Psi(x)\})} \,].$$

(3) *The uniquely determined element $[M] \in \mathrm{H}_n(M, \partial M)$ from (2) is a generator, in particular we have $\mathrm{H}_n(M, \partial M) = \mathbb{Z} \cdot [M]$.*

Proof of Theorem 68.1 (1) for closed manifolds. Let $n \in \mathbb{N}$ and let M be a closed oriented connected n-dimensional smooth manifold. By Theorem 64.2 we can pick a smooth simplicial structure $(K = (V, S), \lambda \colon |K| \to M)$ for the smooth manifold M. Since V is finite we can easily pick a total order "$\leq$" on the vertex set V.

In the following we will use the results from Section 66.1 to construct an explicit singular cycle in $\mathrm{C}_n(M)$ which represents a generator of $\mathrm{H}_n(M)$. We will use this cycle to prove (1). Afterwards we will show that the homology class $[M]$ that it represents actually has all the properties stated in (2).

In the following we will make heavy use of the objects we introduced in Section 66.1. Even if the reader might no longer be completely familiar with all the technical definitions, the overall flow of the argument should still be easy to follow. Without further ado, let us construct the promised singular cycle.

(a) Given an m-simplex s of K we consider the embedding

$$\begin{array}{rccl} \Phi_s^{\leq} \colon & \Delta^m & \to & |K| \\ & (t_0, \ldots, t_m) & \mapsto & \sum\limits_{i=0}^{m} t_i \cdot \underset{\substack{\uparrow \\ \text{here we use the total order on } V \text{ to order the vertices of } s}}{i\text{-th vertex of } s}. \end{array}$$

(b) We denote by $s_1, \ldots, s_k$ the n-simplices of K.

[1028] Note that by definition of a smooth embedding, see page 342, we know that $\Psi(x) \subset M \setminus \partial M$. This implies that we have an induced map $\mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(M, M \setminus \{\Psi(x)\}$.

[1029] Since x is an admissible point for Ψ we know that $\Psi^{-1}(\Psi(x)) = \{x\}$. Note that this implies in particular that $\Psi(\partial \Delta^n) \subset M \setminus \{\Psi(x)\}$ which in turn implies that Ψ is a cycle in $\mathrm{C}_*(M, M \setminus \{\Psi(x)\})$.

(c) Let $i \in \{1, \ldots, k\}$. It follows from the definition of a smooth simplicial structure, see page 1589, that the map $\lambda \circ \Phi^{\leq}_{s_i} \colon \mathring{\Delta}^n \to M$ is a smooth embedding. Now we set

$$\epsilon_i \ := \ \begin{cases} +1, & \text{if } \lambda \circ \Phi^{\leq}_{s_i} \colon \mathring{\Delta}^n \to M \text{ is orientation-preserving}, \\ -1, & \text{if } \lambda \circ \Phi^{\leq}_{s_i} \colon \mathring{\Delta}^n \to M \text{ is orientation-reversing}. \end{cases}$$

(d) By Proposition 66.7 we know that the oriented n-simplices $[(\epsilon_1, s_1)], \ldots, [(\epsilon_k, s_k)]$ form an orientation, in the sense of the definition on page 1652, for the pseudo-manifold K.

(e) Next we consider the maps

$$\begin{array}{ccccccc} & \overset{\text{isomorphism by Lemma 63.8}}{\downarrow} & & \overset{\text{chain homotopy equivalence by Theorem 63.25}}{\downarrow} & & & \\ \mathrm{C}^{\mathrm{simp}}_m(K) & \leftarrow & \mathrm{C}^{\mathrm{simp},\leq}_m(K) & \rightarrow & \mathrm{C}_m(|K|) & \xrightarrow{\lambda_*} & \mathrm{C}_m(M) \\ \underbrace{[v_0, \ldots, v_m]}_{=[s]} & \leftarrow\!\shortmid & \underbrace{\{v_0 < \ldots < v_m\}}_{=s} & \mapsto & \Phi^{\leq}_s & & \end{array}$$

(f) It follows from (d) together with Theorem 66.4 (1) that $\epsilon_1 \cdot [s_1] + \cdots + \epsilon_k \cdot [s_k]$ is a cycle in the simplicial chain complex $\mathrm{C}^{\mathrm{simp}}_*(K)$ and we know that it represents a generator of $\mathrm{H}^{\mathrm{simp}}_n(K)$.

(g) It now follows from (e) and (f) that $c := \epsilon_1 \cdot \lambda \circ \Phi^{\leq}_{s_1} + \cdots + \epsilon_k \cdot \lambda \circ \Phi^{\leq}_{s_k} \in \mathrm{C}_n(M)$ is a singular cycle in $\mathrm{C}_*(M)$ and that it represents a generator of $\mathrm{H}_n(M)$.

Now we turn to the actual proof of (1). As a reminder, recall that we need to prove the following claim.

Claim. For every $x \in M$ the map

$$\omega_x \colon \ \mathrm{H}_n(M) \ \to \ \mathrm{H}_n(M, M \setminus \{x\})$$

is an isomorphism.

First we take care of the special case of a point $y \in M$ which can be written as $y = (\lambda \circ \Phi^{\leq}_{s_i})(P)$ for some $i \in \{1, \ldots, k\}$ and some $P \in \mathring{\Delta}^n$. We consider the following two maps:

$$\begin{array}{ccccccc} & & & \overset{\text{isomorphism by Lemma 44.1}}{\downarrow} & \overset{=\mathbb{Z}\cdot[\mathrm{id}_{\Delta^n}],\text{ by Exercise 45.1}}{\overbrace{\phantom{\mathrm{H}_n(\Delta^n, \Delta^n \setminus \{P\})}}} & \\ \mathrm{H}_n(M) & \xrightarrow{\omega_y} & \mathrm{H}_n(M, M \setminus \{y\}) & \xleftarrow[\cong]{(\lambda\circ\Phi^{\leq}_{s_i})_*} & \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{P\}) & . \\ {[c]} & \mapsto & [c] \underset{\substack{\uparrow \\ \text{since all the other } \lambda \circ \Phi^{\leq}_{s_j} \text{ are zero in } \mathrm{C}_n(M, M \setminus \{y\})}}{=} [\lambda \circ \Phi^{\leq}_{s_i}] & \longleftarrow\!\shortmid & [\mathrm{id}_{\Delta^n}] & \end{array}$$

The above data shows that the image of $[c]$ in $\mathrm{H}_n(M, M \setminus \{y\})$ equals the image of a generator of $\mathrm{H}_n(\Delta^n, \Delta^n \setminus \{P\})$. Since the map to the right is an isomorphism we see that the map $\omega_y \colon \mathrm{H}_n(M) \to \mathrm{H}_n(M, M \setminus \{y\})$ is an epimorphism. Since M is closed orientable connected and non-empty we know by Theorem 66.8 that the group to the left is also isomorphic to $\mathbb{Z}$. Thus we see that both groups are isomorphic to $\mathbb{Z}$ we see that the map ω_y is actually an isomorphism.

Now let $x \in M$ be any point. Since M is connected we obtain from Proposition 8.29 that there exists a diffeomorphism $f\colon M \to M$ with $f(x) = y$. Now we consider the following diagram:

$$\begin{tikzcd} \mathrm{H}_n(M) \arrow[r, "\omega_x"] \arrow[d, "f_*"', "\cong"] & \mathrm{H}_n(M, M \setminus \{x\}) \arrow[d, "\cong"', "f_*"] \\ \mathrm{H}_n(M) \arrow[r, "\omega_y", "\cong"'] & \mathrm{H}_n(M, M \setminus \{y\}). \end{tikzcd}$$

By the above we know that the bottom map is an isomorphism. Since f is a diffeomorphism the vertical maps are isomorphisms. Since the diagram commutes we see that the top horizontal map is also an isomorphism. ■

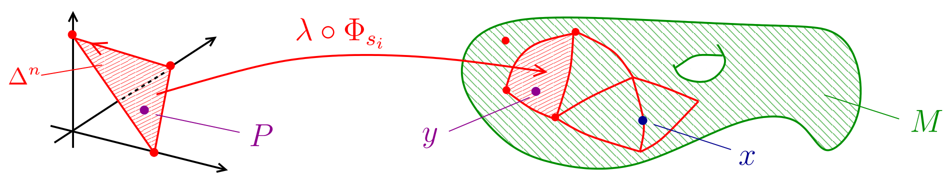


FIGURE 1047. Illustration for the proof of Theorem 68.1 (1).

The proof of Theorem 68.1 (2) is a technically more intricate variation on the proof of Theorem 68.1 (1).

PROOF OF THEOREM 68.1 (2) AND (3) FOR CLOSED MANIFOLDS. We continue with the notation of the proof of Theorem 68.1 (1). Note that the uniqueness part of Statement (2) is basically an immediate consequence of Statement (1). Thus to prove Statement (2) it remains to prove existence. In the following we will now show that

$$[M] := [c] = [\epsilon_1 \cdot \lambda \circ \Phi^{\leq}_{s_1} + \cdots + \epsilon_k \cdot \lambda \circ \Phi^{\leq}_{s_k}] \in \mathrm{H}_n(M)$$

has the desired properties. Before we do so, note though that in (1) we already saw that $[c]$ is a generator of $\mathrm{H}_n(M)$. Thus once we have shown that $[c]$ has the desired properties we have also shown Statement (3).

First let us now prove the following claim.

Claim. For every map $\Psi\colon \Delta^n \to M$ and every admissible $x \in \mathring{\Delta}^n$ we have

$$\omega_{\Psi(x)}([M]) = \operatorname{sign}(\Psi, x) \cdot [\underbrace{\Psi\colon \Delta^n \to M}_{\in \mathrm{C}_n(M, M \setminus \{\Psi(x)\})}].$$

Similar to the proof of the claim in Statement (1) we first deal with a special case. We pick an n-simplex s_i and we consider the corresponding map $\Upsilon := \lambda \circ \Phi^{\leq}_{s_i}\colon \Delta^n \to M$. Note that since we are given a smooth simplicial structure we know that the restriction of $\Upsilon = \lambda \circ \Phi^{\leq}_{s_i}\colon \Delta^n \to M$ to $\mathring{\Delta}^n$ is a smooth embedding. In particular any $x \in \mathring{\Delta}^n$ is admissible. Next note that we have the following equalities in $\mathrm{H}_n(M, M \setminus \{\Upsilon(x)\})$:

$$\omega_{\Upsilon(x)}([M]) \underset{\substack{\uparrow \\ \text{definition of } [M]}}{=} [\epsilon_1 \cdot \lambda \circ \Phi^{\leq}_{s_1} + \cdots + \epsilon_k \cdot \lambda \circ \Phi^{\leq}_{s_k}] \underset{\substack{\uparrow \\ \text{since all other } \Phi^{\leq}_{s_j} \text{ are zero in } \mathrm{C}_n(M, M \setminus \{\Upsilon(x)\})}}{=} [\epsilon_i \cdot \underbrace{\lambda \circ \Phi^{\leq}_{s_i}}_{=\Upsilon}] \underset{\substack{\uparrow \\ \text{since } \epsilon_i = \operatorname{sign}(\Upsilon, x)}}{=} \operatorname{sign}(\Upsilon, x) \cdot [\Upsilon].$$

This concludes the proof of the claim for the special case $\Upsilon = \lambda \circ \Phi^{\leq}_{s_i}$ and any $x \in \mathring{\Delta}^n$.

Now we turn to the general case. Let $\Psi\colon \Delta^n \to M$ be a map and let $x \in \mathring{\Delta}^n$ be an admissible point. By definition we have $\Psi^{-1}(\Psi(x)) = \{x\}$ and by definition there exists an open connected neighborhood U of $x \in \mathring{\Delta}^n$ such that the map $\Psi|_U\colon U \to M$ is a smooth embedding. We pick an orientation-preserving embedding $\theta\colon \overline{B}^n \to U$ with $\theta(0) = x$.

Next we consider the embedding $\widetilde{\Upsilon} := \Upsilon \circ \theta\colon \overline{B}^n \to M$, with $\Upsilon = \lambda \circ \Phi^{\leq}_{s_i}$ as above, and we consider the embedding $\widetilde{\Psi} = \Psi \circ \theta\colon \overline{B}^n \to M$. First we make the following assumption:

($*$) The maps $\widetilde{\Upsilon}$ and $\widetilde{\Psi}$ are both orientation-preserving embeddings or they are both orientation-reversing embeddings.

In this case it follows from Theorem 8.36 that there exists a diffeomorphism f of M with the following two properties:

(i) the map f is homotopic to the identity,
(ii) $f \circ \widetilde{\Upsilon} = \widetilde{\Psi}$.

We consider the following commutative diagram[1030]

$$\begin{tikzcd}
 & & \mathrm{H}_n(M, M\setminus\{\Upsilon(x)\}) \arrow[dd, "f_*", "\cong"'] & \mathrm{H}_n(M) \arrow[l, "\omega_{\Upsilon(x)}"', "\cong"] \arrow[dd, "f_*=\mathrm{id}"'] \\
\mathrm{H}_n(\Delta^n, \Delta^n\setminus\{x\}) \arrow[urr, bend left, "\Upsilon_*"] \arrow[drr, bend right, "\Psi_*"'] & \mathrm{H}_n(\overline{B}^n, \overline{B}^n\setminus\{0\}) \arrow[l, "\cong", "\theta_*"'] \arrow[ur, "\widetilde{\Upsilon}_*"'] \arrow[dr, "\widetilde{\Psi}_*"] & & \\
 & & \mathrm{H}_n(M, M\setminus\{\Psi(x)\}) & \mathrm{H}_n(M). \arrow[l, "\cong"', "\omega_{\Psi(x)}"]
\end{tikzcd}$$

We make the following observations and clarifications:

(a) It follows from (i) and from Proposition 42.5 that the map $f_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)$ is the identity.
(b) Note that by Theorem 68.1 we know that the maps $\omega_{\Upsilon(x)}$ and $\omega_{\Psi(x)}$ are isomorphisms.
(c) Note that it follows immediately from the Excision Theorem 43.20 that the map $\theta_*\colon \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\}) \to \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{x\})$ is an isomorphism.
(d) By (c) we can now set $\sigma := \theta_*^{-1}([\mathrm{id}_{\Delta^n}]) \in \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\})$.

We make the following observations:

$$\begin{array}{lrcll}
 & \omega_{\Upsilon(x)}([M]) & = & \operatorname{sign}(\Upsilon, x) \cdot [\Upsilon\colon \Delta^n \to M] & \text{by the above special case} \\
\Longrightarrow & \omega_{\Upsilon(x)}([M]) & = & \operatorname{sign}(\Upsilon, x) \cdot \widetilde{\Upsilon}_*(\sigma) & \text{by (d), since } \widetilde{\Upsilon}_* = \Upsilon_* \circ \theta_* \\
\Longrightarrow & \omega_{\Psi(x)}([M]) & = & \operatorname{sign}(\Upsilon, x) \cdot \widetilde{\Psi}_*(\sigma) & \text{by (a) and (b)} \\
\Longrightarrow & \omega_{\Psi(x)}([M]) & = & \operatorname{sign}(\Psi, x) \cdot \widetilde{\Psi}_*(\sigma) & \text{by } (*) \\
\Longrightarrow & \omega_{\Psi(x)}([M]) & = & \operatorname{sign}(\Psi, x) \cdot [\Psi\colon \Delta^n \to M] & \text{by (d), since } \widetilde{\Psi}_* = \Psi_* \circ \theta_*.
\end{array}$$

[1030]Since Ψ is admissible we know that $\Psi^{-1}(\Psi(x)) = \{x\}$. This implies that Ψ does indeed induce a map $\Psi_*\colon \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{x\}) \to \mathrm{H}_n(M, M \setminus \{\Psi(x)\})$.

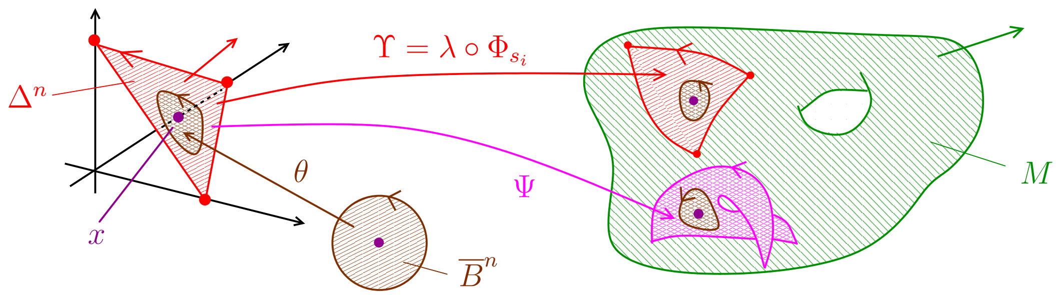


FIGURE 1048. Illustration for the proof of Theorem 68.1 (2).

Finally we suppose that $(*)$ does not hold. Let

$$\begin{array}{rcl} \rho\colon \Delta^n & \to & \Delta^n \\ (t_0, t_1, t_2, \dots, t_n) & \mapsto & (t_1, t_0, t_2, \dots, t_n) \end{array}$$

be the map that swaps the first two coordinates. Since ρ is orientation-preserving we now see that $\widetilde{\Upsilon}$ and $\widetilde{\Psi} \circ \rho$ satisfy $(*)$. Now we see that

$$\begin{array}{rrcll} & \omega_{(\Psi\circ\rho)(\rho(x))}([M]) & = & \operatorname{sign}(\Psi\circ\rho, \rho(x)) \cdot \overbrace{[\Psi\circ\rho\colon \Delta^n \to M]}^{=\Psi_*(\rho_*[\mathrm{id}_{\Delta^n}])} & \text{since } \widetilde{\Upsilon} \text{ and } \widetilde{\Psi}\circ\rho \text{ satisfy } (*) \\ & & & & \text{we can apply the above} \\ \Longrightarrow & \omega_{\Psi(x)}([M]) & = & \operatorname{sign}(\Psi\circ\rho, \rho(x)) \cdot \Psi_*(-[\mathrm{id}_{\Delta^n}]) & \text{by Exercise 45.2} \\ \Longrightarrow & \omega_{\Psi(x)}([M]) & = & (-\operatorname{sign}(\Psi, x)) \cdot \Psi_*(-[\mathrm{id}_{\Delta^n}]) & \text{since } \rho \text{ is orientation-reversing} \\ \Longrightarrow & \omega_{\Psi(x)}([M]) & = & \operatorname{sign}(\Psi, x) \cdot \underbrace{[\Psi\colon \Delta^n \to M]}_{=\Psi_*([\mathrm{id}_{\Delta^n}])}. & \end{array}$$

■

We have now proved Theorem 68.1 for all closed smooth manifolds. It remains to prove Theorem 68.1 for smooth manifold with non-empty boundary.

PROOF OF THEOREM 68.1 FOR MANIFOLDS WITH NON-EMPTY BOUNDARY $(*)$. Let M be a compact oriented connected n-dimensional smooth manifold with non-empty boundary. Let M' be another copy of M. We consider the double $\mathrm{D}\,M = M \cup_{\partial M = \partial M'} M'$ of M, as defined on page 1163. Since $\partial M \neq \varnothing$ we obtain from Lemma 44.12 that $\mathrm{D}\,M$ is a closed oriented n-dimensional smooth manifold such that the map $M \to \mathrm{D}\,M$ is an orientation-preserving embedding.

(1) Let $x \in M \setminus \partial M$. We consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{H}_n(M, \partial M) \arrow[r, "\cong"] \arrow[d] & \mathrm{H}_n(\mathrm{D}\,M, M') \arrow[dr] & \mathrm{H}_n(\mathrm{D}\,M) \arrow[l, "\cong"'] \arrow[d, "\cong"] \\ \mathrm{H}_n(M, M \setminus \{x\}) \arrow[rr, "\cong"] & & \mathrm{H}_n(\mathrm{D}\,M, \mathrm{D}\,M \setminus \{x\}). \end{tikzcd}$$

As we saw in Lemma 44.12 (6), it follows from the Excision Theorem 44.10 that the top left horizontal map is an isomorphism. Basically the same argument shows that the bottom horizontal map is an isomorphism. Furthermore note that in the proof of Theorem 66.8 we saw that the top right horizontal map is an isomorphism. By the above discussion of closed smooth manifolds we know that the right vertical map is an isomorphism. It now follows, as desired, that the left vertical map is also an isomorphism.

(2) By the discussion of the closed case we can consider the class $[\mathrm{D}\,M] \in \mathrm{H}_n(\mathrm{D}\,M)$. We define $[M] \in \mathrm{H}_n(M, \partial M)$ to be the preimage of $[\mathrm{D}\,M]$ under the composition of the above two horizontal isomorphisms. A slight variation on the argument provided in (1) now shows that $[M]$ has all the desired properties. We leave it to the reader to fill in the details. ■

Theorem 68.1 leads us to the following definition.

Definition. Let $n \in \mathbb{N}$ and let M be a compact oriented n-dimensional smooth manifold.

(1) We denote by $N_1, \dots, N_k$ the components of M. Furthermore for $i = 1, \dots, k$ we now denote by $[N_i] \in \mathrm{H}_n(N_i, \partial N_i)$ the homology class from Theorem 68.1. We refer to

$$[M] \;:=\; \sum_{i=1}^{k} [N_i] \;\in\; \mathrm{H}_n(M, \partial M) \underset{\substack{\uparrow \\ \text{isomorphism given by Lemma 41.14}}}{=} \bigoplus_{i=1}^{k} \mathrm{H}_n(N_i, \partial N_i).$$

as the *fundamental class of M*.

(2) We refer to any cycle in $\mathrm{C}_n(M, \partial M)$ that represents the fundamental class $[M]$ as a *fundamental cycle*.

Remark. In Chapter 86 we will introduce the notion of an orientation on a topological manifold and on page 2120 we will introduce fundamental classes for oriented topological manifolds. After a few reinterpretations one sees that Proposition 86.11 together with Theorem 87.1 provide in particular an alternative proof of Theorem 68.1.

Conveniently enough, the proof of Theorem 68.1 actually gives an algorithm for determining explicit fundamental cycles.

Algorithm 68.2. *Let $n \in \mathbb{N}$ and let M be a compact oriented non-empty n-dimensional smooth manifold.*

(1) *By Theorem 64.2 we can pick a smooth simplicial structure $(K = (V, S), |K| \xrightarrow{\lambda} M)$. Since V is finite we can pick a total order "$\leq$" on V.*

(2) *We denote by $s_1, \dots, s_k$ the n-simplices of K.*

(3) *For $i \in \{1, \dots, k\}$ we consider the characteristic map*

$$\begin{array}{rccl} \Phi_{s_i}^{\leq} \colon & \Delta^n & \to & |K| \\ & (t_0, \dots, t_n) & \mapsto & \sum\limits_{j=0}^{n} t_j \cdot \underset{\substack{\uparrow \\ \text{we use the total order on } V \text{ to order the vertices of } s_i}}{j\text{-th vertex of } s_i} \end{array}$$

and we set

$$\epsilon_i \;:=\; \begin{cases} +1, & \text{if } \lambda \circ \Phi_{s_i}^{\leq} \colon \mathring{\Delta}^n \to M \text{ is orientation-preserving,} \\ -1, & \text{if } \lambda \circ \Phi_{s_i}^{\leq} \colon \mathring{\Delta}^n \to M \text{ is orientation-reversing.} \end{cases}$$

(4) *The cycle* $\epsilon_1 \cdot \lambda \circ \Phi^{\leq}_{s_1} + \cdots + \epsilon_k \cdot \lambda \circ \Phi^{\leq}_{s_k} \in \mathrm{C}_n(M)$ *is a fundamental cycle of* M.

PROOF. It is basically clear that it suffices to prove the validity of the algorithm for the *connected* case. Next note that if M is closed, then it follows immediately from the proof of Theorem 68.1 (1) that $\epsilon_1 \cdot \lambda \circ \Phi^{\leq}_{s_1} + \cdots + \epsilon_k \cdot \lambda \circ \Phi^{\leq}_{s_k} \in \mathrm{C}_n(M)$ is a cycle. Furthermore, in the proof of Theorem 68.1 (2) we saw that this cycle represents the fundamental class $[M] \in \mathrm{H}_n(M)$ of M. If M has non-empty boundary, then we equip the double $\mathrm{D}\,M$ with the simplicial structure given by "doubling" the simplicial structure of M. It is now straightforward to verify that the above algorithm also works in that case. We leave it to the reader to fill in the details. ■

Example. In Figure 1049 we consider the annulus $M = ([0,3] \times [0,1])/(0,y) \sim (3,y)$ equipped with an orientation and equipped with a smooth simplicial structure. In Figure 1049 we use two different ways to name the vertices, namely once the vertex set is $V = \{a,\dots,f\}$ and once it is $V = \{A,\dots,F\}$. In both cases we equip these sets with the obvious total order. We now see that the singular chains

$$\begin{array}{rcllllllll} \mu &=& +\Phi^{\leq}_{\{a,d,f\}} & -\Phi^{\leq}_{\{d,e,f\}} & +\Phi^{\leq}_{\{c,e,f\}} & -\Phi^{\leq}_{\{b,c,e\}} & +\Phi^{\leq}_{\{a,b,c\}} & -\Phi^{\leq}_{\{a,b,d\}} & \in & \mathrm{C}_n(M,\partial M) \\ \nu &=& +\Phi^{\leq}_{\{A,B,C\}} & -\Phi^{\leq}_{\{B,C,D\}} & +\Phi^{\leq}_{\{C,D,E\}} & -\Phi^{\leq}_{\{D,E,F\}} & +\Phi^{\leq}_{\{A,E,F\}} & -\Phi^{\leq}_{\{A,B,F\}} & \in & \mathrm{C}_n(M,\partial M) \end{array}$$

are both fundamental cycles of M.

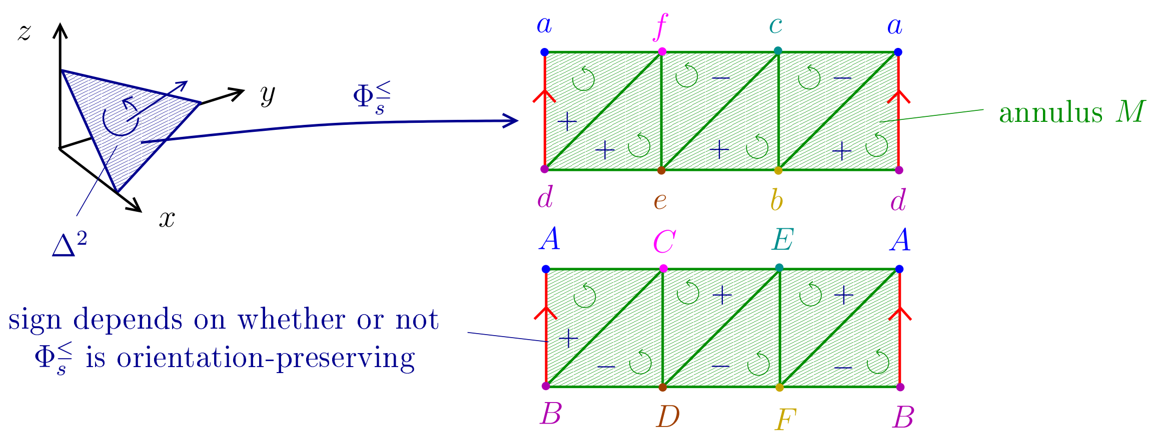


FIGURE 1049

Even though Algorithm 68.2 is very explicit, it also has some drawbacks. For example simplicial structures tend to have many simplices, which in the above algorithm leads to cycles with many summands. In a minute we will state a result that addresses this issue. But first let us prove a lemma which in particular says that, in order to show that we are given the fundamental class of a compact oriented *connected* smooth manifold, we only have to verify the defining property for a single choice of a map $\Psi\colon \Delta^n \to M$ and an admissible point $x \in \mathring{\Delta}^n$.

Lemma 68.3. *Let M be a compact oriented non-empty n-dimensional smooth manifold and let $\varphi \in \mathrm{H}_n(M, \partial M)$. We denote by $N_1, \dots, N_k$ the components of M. The following statements are equivalent:*

(1) *φ is the fundamental class of M.*
(2) *For each $i \in \{1, \dots, k\}$ there exists a map $\Psi \colon \Delta^n \to N_i$ and an admissible point $x \in \mathring{\Delta}^n$ such that the following equality holds in $\mathrm{H}_n(M, M \setminus \{\Psi(x)\})$:*

$$\underbrace{\omega_{\Psi(x)}}_{\text{the map } \mathrm{H}_n(M,\partial M)\to \mathrm{H}_n(M,M\setminus\{\Psi(x)\})}(\varphi) \;=\; \operatorname{sign}(\Psi, x) \cdot [\underbrace{\Psi \colon \Delta^n \to N_i \to M}_{\in \mathrm{C}_n(M, M\setminus\{\Psi(x)\})}].$$

(3) *For each $i \in \{1, \dots, k\}$ there exists a point $y \in N_i \setminus \partial_i N$ such that*

$$\omega_y(\varphi) \;=\; \omega_y([M]).$$

Proof ($*$). A short moment of reflection shows that it suffices to prove the lemma in the case that M is connected. Next we note that the (1)⇒(2)-direction and (1)⇒(3)-direction are of course tautologies. We turn to the proof of the (2)⇒(1)-direction. We have the following two equalities in $\mathrm{H}_n(M, M \setminus \{x\})$:

$$\omega_{\Psi(x)}(\varphi) \underset{\text{by hypothesis}}{=} \operatorname{sign}(\Psi, x) \cdot [\Psi \colon \Delta^n \to M] \underset{\text{by definition of } [M]}{=} \omega_{\Psi(x)}([M]).$$

Since we assume that M is connected we know by Theorem 68.1 that $\omega_{\Psi(x)}$ is an isomorphism. Thus it follows from the above that $\varphi = [M]$.

Finally we consider the (3)⇒(1)-direction. This statement follows again from the fact that ω_y is an isomorphism. ■

The following proposition now gives us a practical criterion for showing that a given cycle represents the fundamental class.

Proposition 68.4. *Let M be a compact oriented connected n-dimensional smooth manifold. Let $r_1 \cdot \sigma_1 + \dots + r_m \cdot \sigma_m \in \mathrm{C}_n(M, \partial M)$ be a cycle. Suppose there exists a $j \in \{1, \dots, m\}$ such that the following three conditions are satisfied:*

(1) *The map $\sigma_j \colon \mathring{\Delta}^n \to M$ is an orientation-preserving smooth embedding.*[1031]
(2) *There exists an $x \in \sigma_j(\Delta^n) \setminus \sigma_j(\partial \Delta^n)$ which is not contained in the image of any other $\sigma_i \colon \Delta^n \to M$.*
(3) *$r_j = 1$.*

Then the cycle $r_1 \cdot \sigma_1 + \dots + r_m \cdot \sigma_m$ represents the fundamental class of the oriented smooth manifold, i.e. we have

$$[r_1 \cdot \sigma_1 + \dots + r_m \cdot \sigma_m] \;=\; [M] \;\in\; \mathrm{H}_n(M, \partial M).$$

Example. It follows easily from Proposition 68.4 and our orientation convention, see page 299, that the singular 1-simplex $\mu \colon \Delta^1 \to S^1$ given by $\mu(1-t, t) = e^{2\pi i t}$ represents the fundamental class of the oriented smooth manifold S^1.

[1031]Note that we do *not* place any conditions on the restriction of σ_j to $\partial \Delta^n$.

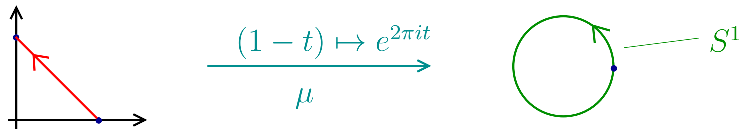


FIGURE 1050

PROOF OF PROPOSITION 68.4 (∗). Let M be a compact oriented connected n-dimensional smooth manifold, let $\sum_{i=1}^{m} r_i \cdot \sigma_i \in \mathrm{C}_n(M, \partial M)$ be a cycle and let $j \in \{1, \dots, m\}$ such that conditions (1)–(3) above are satisfied. We pick $x \in \sigma_j(\mathring{\Delta}^n)$ as in (2). Now we see that we have the following equalities in $\mathrm{H}_n(M, M \setminus \{x\})$:

$$\omega_x([r_1 \cdot \sigma_1 + \dots + r_m \cdot \sigma_m]) = [\overbrace{r_1 \cdot \sigma_1 + \dots + r_m \cdot \sigma_m}^{\in \mathrm{C}_n(M, M\setminus\{x\})}] \underset{\substack{\uparrow \\ \text{since (2) implies that for } i \neq j \text{ we have } \sigma_i(\Delta^n) \subset M \setminus \{x\}}}{=} [r_j \cdot \sigma_j] = [\sigma_j \colon \Delta^n \to M].$$

By hypothesis $\sigma_j \colon \Delta^n \to M$ restricts to an orientation-preserving embedding $\sigma_j \colon \mathring{\Delta}^n \to M$. Since $x \in \sigma_j(\Delta^n) \setminus \sigma_j(\partial\Delta^n)$ we see that x is an admissible point for σ_j. Thus all the hypotheses of Lemma 68.3 hold, and therefore we conclude from Lemma 68.3 that the cycle $r_1 \cdot \sigma_1 + \dots + r_m \cdot \sigma_m \in \mathrm{C}_n(M, \partial M)$ is indeed a fundamental cycle of M. ■

68.2. **Examples of fundamental classes.** In this section we will give explicit cycles representing the fundamental classes of $\overline{B}^n$, of S^n and of surfaces of genus ≥ 1. At this stage it is convenient to recall the following notation.

Notation.

(1) We consider the projection

$$\begin{array}{rcl} p \colon \Delta^n & \to & \text{◺}_n := \left\{ (t_1, \dots, t_n) \in [0,1]^n \,\middle|\, \sum_{j=1}^{n} t_j \in [0,1] \right\} \\ (t_0, \dots, t_n) & \mapsto & (t_0, \dots, t_{n-1}). \end{array}$$

One can easily verify that the restriction $p \colon \mathring{\text{◺}}_n \to \mathring{\Delta}^n$ is orientation-preserving if and only if n is even.

(2) (a) Let $\mu \colon \text{◺}_n \to \overline{B}^n$ be the explicit homeomorphism with $\mu(\frac{1}{n+1}, \dots, \frac{1}{n+1}) = 0$ that is given by Proposition 2.53 (2). Note that the restriction of μ to a map $\mathring{\text{◺}}_n \to B^n$ is not smooth. But it is easily seen to be an orientation-preserving smooth embedding on an open dense subset of $\mathring{\text{◺}}$.

(b) Let $\nu \colon \text{◺}_n \to \overline{B}^n$ be the explicit homeomorphism with $\nu(\frac{1}{n+1}, \dots, \frac{1}{n+1}) = 0$ given in Exercise 41.1. This map has the very neat feature that the restriction $\mathring{\text{◺}}_n \to B^n$ is actually an orientation-preserving diffeomorphism.

The following lemma is an immediate consequence of the above and Lemma 68.3.

Lemma 68.5. *Let $n \in \mathbb{N}$. The cycles $(-1)^n \cdot (\mu \circ p \colon \Delta^n \to \overline{B}^n) \in \mathrm{C}_n(\overline{B}^n, S^{n-1})$ and $(-1)^n \cdot (\nu \circ p \colon \Delta^n \to \overline{B}^n) \in \mathrm{C}_n(\overline{B}^n, S^{n-1})$ are both representatives of the fundamental class $[\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$.*

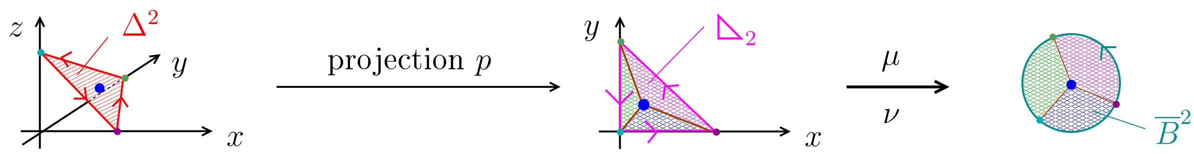


FIGURE 1051

Let $n \in \mathbb{N}$. The eagle-eyed reader will have noticed that by now we have introduced the symbol $[S^n] \in \mathrm{H}_n(S^n)$ twice:

(1) On page 1174 we introduced the "standard generator" $[S^n] = [\alpha - \beta] \in \mathrm{H}_n(S^n)$. Here α is given by

$$\begin{array}{cccccc} \Delta^n & \xrightarrow{\mu\circ p} & \overline{B}^n & \to & S^n & \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \\ & & x & \mapsto & \big(x, \sqrt{1-\|x\|^2}\big) & \end{array}$$

and $\beta\colon \Delta^n \to S^n$ is the composition of α with reflection in the hyperplane defined by $x_{n+1} = 0$.

(2) We endow S^n with the orientation introduced on page 299, namely given $P \in S^n$ we say a basis $v_1, \ldots, v_n$ of $\mathrm{T}_P\, M = \mathrm{V}_P\, M$ is positive if $P, v_1, \ldots, v_n$ is a positive basis for $\mathbb{R}^{n+1}$. With this orientation of S^n we now also have the corresponding fundamental class $[S^n] \in \mathrm{H}_n(S^n)$.

Fortunately the next lemma says that our various sign conventions work out in such a way that the two definitions of $[S^n]$ agree.

Lemma 68.6. *Let $n \in \mathbb{N}$. The standard generator $[S^n] \in \mathrm{H}_n(S^n)$ that we introduced on page 1174 agrees with the fundamental class of the oriented smooth manifold S^n.*

PROOF. It follows from the above discussion, and some elementary arguments that the map $\alpha\colon \Delta^n \to S^n$ is an embedding and that it is an orientation-preserving embedding on an open subset of $\mathring{\Delta}^n$. The desired equality now follows almost immediately from Proposition 68.4 applied to the cycle $\alpha - \beta$. ■

We move on to giving explicit representatives of fundamental classes for surfaces. This will require the introduction of more notation.

Notation. We will give many examples of fundamental classes of 2-dimensional smooth manifolds. Since we want to be very careful what singular simplex we mean by a given picture we adopt the following convention:

A triangle T with a dot in a corner and a directional arrow in the center means the unique affine linear map $\Delta^2 \to T$ which sends $(1,0,0)$ to the vertex decorated with the dot and which respects the orientation of the vertices. We illustrate this notation in Figure 1052. We also take some artistic liberties and allow triangles where the boundary is not given by segments and where the boundary consists of curves. In this case we expect that the boundary curves are parametrized with constant speed. Since the precise definition of the boundary of a singular 2-simplex is often a source of confusion we recall the definition in Figure 1053.

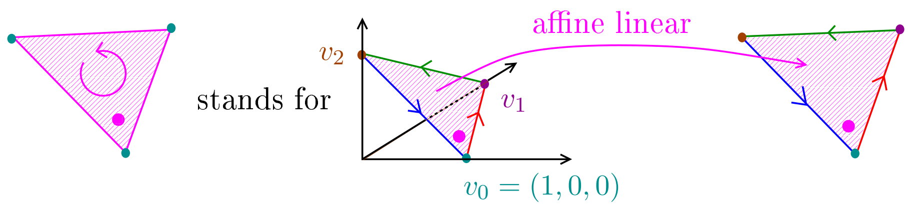


FIGURE 1052

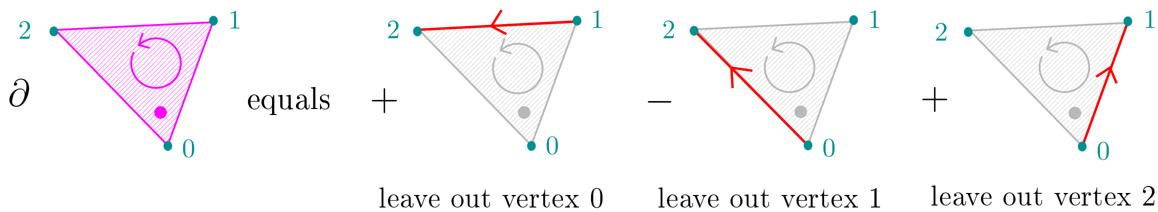


FIGURE 1053

Example. In Figure 1054 we show the torus $T = ([0,1] \times [0,1])/\sim$. We equip T with the orientation coming from the standard orientation on $(0,1) \times (0,1) \subset T$, which we can also view as an open subset of $\mathbb{R}^2$. In Figure 1054 we also show two singular 2-simplices σ_1 and σ_2. It is straightforward to verify that $-\sigma_1 + \sigma_2$ is a cycle. (In the verification it is very helpful to keep Figure 1053 in mind.) The summand σ_2 satisfies the conditions imposed in Proposition 68.4, thus we see that $-\sigma_1 + \sigma_2$ represents a fundamental class of T, in particular it represents a generator of $\mathrm{H}_2(T)$. This is of course the same cycle that we had already initially encountered on page 1083. We already saw in Lemma 46.12 (2) in a somewhat ad hoc argument that it represents a generator of $\mathrm{H}_2(T)$.

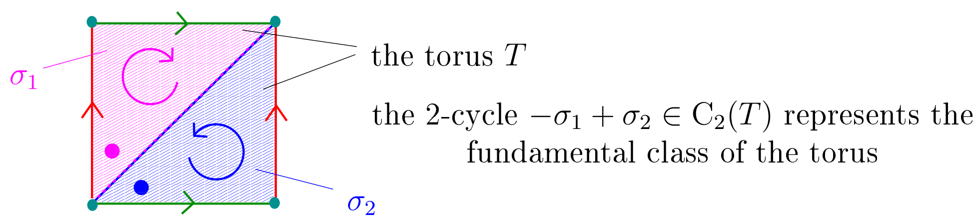


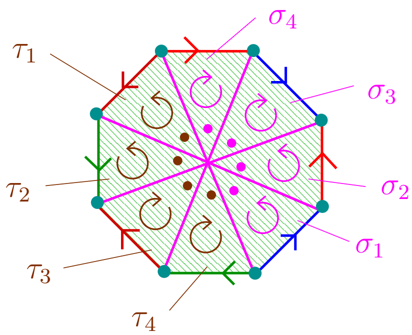
FIGURE 1054

Examples. In Figure 1055 we show the surface of genus 2 with eight singular 2-simplices $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and $\tau_1, \tau_2, \tau_3, \tau_4$. The interior of the octagon is an open subset of $\mathbb{R}^2$ and we equip the surface of genus 2 with the orientation coming from the standard orientation of $\mathbb{R}^2$. It is a fun exercise to verify that the singular 2-chain

$$\sigma_1 + \sigma_2 - \sigma_3 - \sigma_4 \, + \, \tau_1 + \tau_2 - \tau_3 - \tau_4$$

is indeed a cycle. It follows easily from Proposition 68.4 that the singular 2-chain is a fundamental cycle, i.e. it represents a fundamental class of the surface of genus 2.

We will discuss more examples and properties of the fundamental class shortly. But for coherence it is best to first quickly discuss the sightly annoying case of 0-dimensional manifolds.



$$\begin{array}{rl} & \sigma_1+\sigma_2-\sigma_3-\sigma_4 \\ + & \tau_1+\tau_2-\tau_3-\tau_4 \end{array}$$

is a cycle and represents the fundamental class

FIGURE 1055

68.3. **Orientations of 0-dimensional manifolds.** The following definition might initially feel rather artificial.

Definition.

(1) An *orientation for a 0-dimensional smooth manifold* M is a map $\epsilon\colon M \to \{\pm 1\}$.

(2) Let W be an oriented 1-dimensional smooth manifold. We define[1032]

$$\begin{array}{rcl} \epsilon\colon \partial W & \to & \{\pm 1\} \\ P & \mapsto & \begin{cases} -1, & \text{if for every smooth map } f\colon W \to \mathbb{R}_{\geq 0} \text{ with } f(P)=0 \\ & \text{and for every positive vector } v \in \mathrm{T}_P W \text{ we have } v(f) \geq 0, \\ +1, & \text{otherwise.} \end{cases} \end{array}$$

Loosely speaking ∂W is the oriented 0-dimensional smooth manifold given by the set ∂W and the sign $+1$ respectively -1 assigned to the points depending on whether or not the orientation of W sticks “out of” respectively “into” W.[1033]

(3) Let $(M=\{P_1,\dots,P_k\}, \epsilon\colon M \to \{\pm 1\})$ be a compact oriented 0-dimensional smooth manifold. The corresponding *fundamental class* $[M]$ is defined as the homology class

$$[M] \;:=\; \sum_{i=1}^{k} \epsilon(P_i)\cdot [P_i] \;\in\; \mathrm{H}_0(M).$$

(4) Let $k \in \mathbb{N}$, let N be an oriented k-dimensional topological manifold and let $\{P\}$ be the 0-dimensional smooth manifold consisting of a single point P that is equipped with an orientation $\epsilon \in \{-1,1\}$. If $\epsilon = 1$, then we equip the product manifold $\{P\}\times N$ with the unique orientation that turns the obvious map $N \to \{P\}\times N$ into an orientation-preserving diffeomorphism. Furthermore, if $\epsilon = -1$, then we equip the product manifold $\{P\}\times N$ with the unique orientation that turns the map $N \to \{P\}\times N$ into an orientation-reversing diffeomorphism. The same way we define the orientation on $N \times \{P\}$.

Examples.

(1) The definition of the orientation of the boundary of an oriented 1-dimensional smooth manifold is illustrated in Figure 1056

[1032]Recall that on page 291 we defined the tangent space $\mathrm{T}_P W$ as the set of derivations at P.

[1033]We leave it to the reader to verify that the definition and the intuition agree.

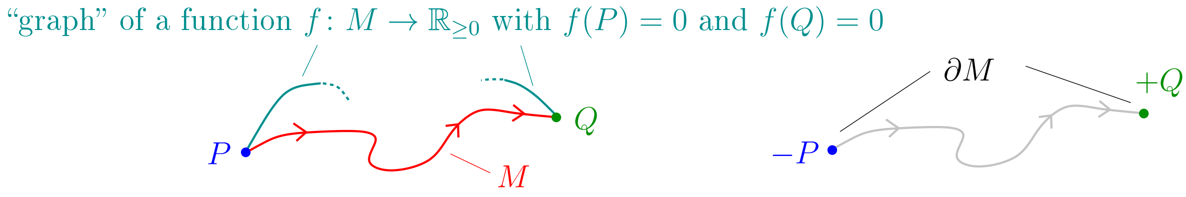


FIGURE 1056

(2) The boundary of the 1-dimensional smooth manifold $[0,1]$, which we equip with the usual orientation coming from $\mathbb{R}$, is easily seen to be given by $(-\{0\}) \cup \{1\}$.

(3) One can easily verify that the statement regarding product orientations from Proposition 6.51 (5a) also holds in this context, i.e. given a 1-dimensional smooth manifold M with boundary and a smooth manifold N without boundary we have the equality $\partial(M \times N) = (\partial M) \times N$ as *oriented* smooth manifolds. In particular we have

$$\partial([0,1] \times N) \;=\; \underbrace{(-\{0\} \times N)}_{=\{0\}\times(-N)} \cup (\{1\} \times N).$$

This now gives a prettier description of the boundary orientations, than the ad hoc description in Proposition 6.51 (5b).

(4) We equip the 0-dimensional sphere $S^0 = \{\pm 1\}$ with the orientation given by $\pm 1 \mapsto \pm 1$. Note that this implies that for any oriented smooth manifold M we have a natural orientation-preserving diffeomorphism $M \times S^0 \xrightarrow{\cong} (-M) \sqcup M$.

Convention. Unless we say anything we will equip any given 0-dimensional manifold M with the orientation given by assigning $+1$ to every point in M.

68.4. **Properties of the fundamental class.** Before we continue with the discussion of the fundamental class, let us consider the one case that we had not dealt with, namely the empty manifold. As we mentioned on page 264, basically by definition the empty set is a topological and smooth manifold of any dimension. Note that for the empty manifold $\varnothing$ we have the *equality* $\mathrm{H}_0(\varnothing) = \mathrm{H}_1(\varnothing) = \mathrm{H}_2(\varnothing) = \cdots = \{0\}$. We define the fundamental class $[\varnothing]$ of the empty manifold to be 0. Note that by the above the fundamental class $[\varnothing]$ is an element in every homology group of $\varnothing$.

Now let us continue with serious mathematics. The following lemma tells us the behavior of the fundamental class under diffeomorphisms.

Lemma 68.7. *Let $n \in \mathbb{N}_0$, let M and N be compact oriented connected n-dimensional smooth manifolds and let $f\colon M \to N$ be a diffeomorphism. We have*[1034]

$$f_*([M]) \;=\; \begin{cases} [N], & \textit{if } f \textit{ is orientation-preserving,} \\ -[N], & \textit{if } f \textit{ is orientation-reversing.} \end{cases}$$

PROOF (∗). If $n = 0$, then the statement follows immediately from the definitions. As we will see, for $n \in \mathbb{N}$ the statement follows basically immediately from the definition property of the fundamental class. Indeed, we set $\epsilon := +1$ if f is orientation-preserving, otherwise

[1034] Note that this statement now also makes sense for the empty manifold.

we set $\epsilon := -1$. Let $\Psi\colon \Delta^n \to M$ be an embedding such that the map $\Psi\colon \mathring{\Delta}^n \to M$ is orientation-preserving. We pick some $x \in \mathring{\Delta}^n$. We consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{H}_n(M,\partial M) \arrow[r, "\omega_{\Psi(x)}"] \arrow[d, "f_*"] & \mathrm{H}_n(M, M\setminus\{\Psi(x)\}) \arrow[d, "f_*"] \\
\mathrm{H}_n(N,\partial N) \arrow[r, "\omega_{(f\circ\Psi)(x)}"] & \mathrm{H}_n(N, N\setminus\{(f\circ\Psi)(x)\}).
\end{tikzcd}$$

Now we see that

$$\begin{array}{rrcll}
& \omega_{\Psi(x)}([M]) & = & \mathrm{sign}(\Psi,x)\cdot[\Psi], & \text{by definition of } [M] \\
\Longrightarrow & \omega_{(f\circ\Psi)(x)}(f_*([M])) & = & \mathrm{sign}(\Psi,x)\cdot[f\circ\Psi], & \text{since the diagram commutes} \\
\Longrightarrow & \omega_{(f\circ\Psi)(x)}(f_*([M])) & = & \epsilon\cdot\mathrm{sign}(f\circ\Psi,x)\cdot[f\circ\Psi], & \text{by definition of } \epsilon \\
\Longrightarrow & f_*([M] & = & \epsilon\cdot[N], & \text{by Lemma 68.3.}
\end{array}$$
■

Given an oriented smooth manifold M we denote, as usual, by $-M$ the same smooth manifold but with the opposite orientation. Now we can formulate the following lemma.

Lemma 68.8. *For every compact oriented n-dimensional smooth manifold M we have*

$$[-M] \;=\; -[M] \;\in\; \mathrm{H}_n(M,\partial M).$$

PROOF. This statement follows basically immediately from Lemma 68.7 since the identity map $M \to -M$ is orientation-reversing. ■

On page 303 we saw that the boundary of an oriented smooth manifold inherits a natural orientation. For compact smooth manifolds the following proposition gives the relationship between the fundamental class of the smooth manifold and of its boundary.

Proposition 68.9. *Let M be a compact oriented n-dimensional smooth manifold with boundary. The connecting homomorphism*

$$\partial\colon\; \mathrm{H}_n(M,\partial M) \;\to\; \mathrm{H}_{n-1}(\partial M)$$

of the long exact sequence in homology of the pair $(M,\partial M)$ has the property that

$$[M] \;\mapsto\; [\partial M].$$

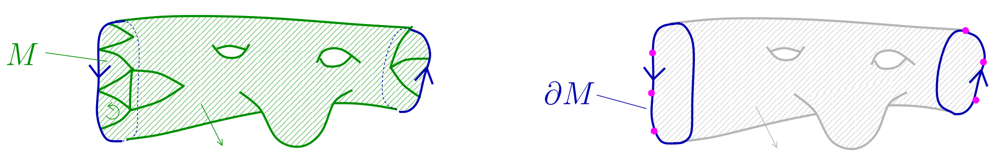


FIGURE 1057. Illustration of Proposition 68.9.

Remark. In Proposition 87.27 we will prove the analogue of Proposition 68.9 for topological manifolds.

PROOF (∗). Let M be a compact oriented n-dimensional smooth manifold with boundary components $N_1,\dots,N_k$. By Theorem 64.2 the smooth manifold M admits a smooth simplicial structure $(K=(V,S),\lambda\colon |K|\to M)$. By Proposition 66.1 (3) we know that there exists a subcomplex $J=(U,R)$ of $=(V,S)$ such that $(J=(U,R),\lambda\colon |J|\to \partial M)$

is a smooth simplicial structure for ∂M. We pick some total order "$\leq$" on V. Next we introduce the following notation for K:

(1) We denote by S_n the set of n-simplices of K.
(2) For each $s \in S_n$ we denote by $\Phi_s^{\leq}\colon \Delta^n \to |K|$ the corresponding characteristic map.
(3) For each $s \in S_n$ we set

$$\epsilon_s := \begin{cases} +1, & \text{if } \lambda \circ \Phi_s^{\leq}\colon \mathring{\Delta}^n \to M \text{ is orientation-preserving,} \\ -1, & \text{if } \lambda \circ \Phi_s^{\leq}\colon \mathring{\Delta}^n \to M \text{ is orientation-reversing.} \end{cases}$$

Next we introduce the analogous notions for the subcomplex J:

(4) We denote by R_{n-1} the set of $(n-1)$-simplices of R.
(5) For each $r \in R_{n-1}$ we denote by $\Psi_r^{\leq}\colon \Delta^{n-1} \to |J|$ the corresponding characteristic map.
(6) For each $r \in R_{n-1}$ we set

$$\mu_r := \begin{cases} +1, & \text{if } \lambda \circ \Psi_r^{\leq}\colon \mathring{\Delta}^{n-1} \to \partial M \text{ is orientation-preserving,} \\ -1, & \text{if } \lambda \circ \Psi_r^{\leq}\colon \mathring{\Delta}^{n-1} \to \partial M \text{ is orientation-reversing.} \end{cases}$$

Now we see that

$$\partial[M] \underset{\text{Algorithm 68.2}}{\overset{\downarrow}{=}} \partial\Big[\sum_{s\in S_n} \epsilon_s \cdot \lambda \circ \Phi_s^{\leq}\Big] \underset{\text{Proposition 43.15}}{\overset{\downarrow}{=}} \Big[\sum_{s\in S_n} \epsilon_s \cdot \lambda \circ \partial_n(\Phi_s^{\leq})\Big] \underset{\text{definition of } \partial_n\colon \mathrm{C}_n(M)\to \mathrm{C}_{n-1}(M)}{\overset{\downarrow}{=}} \Big[\sum_{s\in S_n} \epsilon_s \cdot \lambda \circ \sum_{j=0}^{n} (-1)^j \cdot \Phi_s^{\leq} \circ i_j^n\Big]$$

since we are given a cycle in $\mathrm{C}_n(M,\partial M)$ the contributions of singular simplices for which the image does not lie in ∂M cancel

$$\overset{\downarrow}{=} \Big[\sum_{\substack{s\in S_n \text{ and } j\in\{0,\dots,n\} \\ \text{such that} \\ (\Phi_s^{\leq}\circ i_j^n)(\Delta^{n-1})\subset \partial M}} (-1)^j \cdot \epsilon_s \cdot \lambda \circ \Phi_s^{\leq} \circ i_j^n\Big] \underset{\uparrow}{=} \Big[\sum_{r\in R} \mu_r \cdot \lambda \circ \Psi_r^{\leq}\Big] \underset{\uparrow}{=} [\partial M].$$

by Proposition 66.1 (4) we have a bijection of singular simplices that appear in the sums, furthermore the corresponding signs match by the discussion on page 1664; Algorithm 68.2 ■

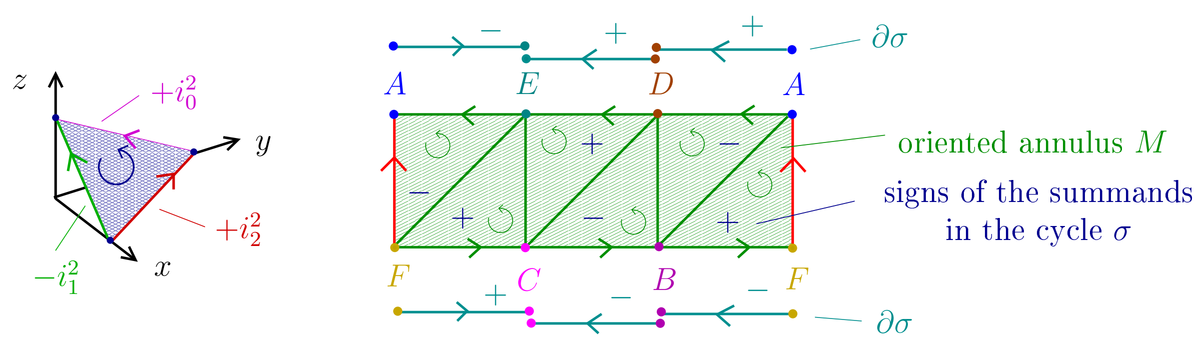


FIGURE 1058. Illustration for the proof of Proposition 68.9.

The following straightforward corollary says that we can "extend" fundamental cycles from the boundary to the total manifold.

Corollary 68.10. (∗) *Let M be a compact oriented n-dimensional smooth manifold. Let A be a union of components of ∂M. We write $B := \partial M \setminus A$. We consider the map*

$$\begin{tikzcd} \mathrm{C}_n(M,\partial M) \arrow[r, "\partial_n"] \arrow[rrrr, bend right, "=:\Theta_A"'] & \mathrm{C}_{n-1}(\partial M) \arrow[r, "\mathrm{id}"] & \mathrm{C}_{n-1}(A \sqcup B) \arrow[r, "\cong"] & \mathrm{C}_{n-1}(A) \oplus \mathrm{C}_{n-1}(B) \arrow[r, two heads] & \mathrm{C}_{n-1}(A). \end{tikzcd}$$

Given any fundamental cycle $\mu \in \mathrm{C}_{n-1}(A)$ there exists a chain $\nu \in \mathrm{C}_n(M)$ such that ν represents a fundamental cycle in $\mathrm{C}_n(M,\partial M)$ and such that $\Theta_A(\nu) = \mu$.

Proof (∗). First note that it is basically clear that it suffices to prove the statement for the case $A = \partial M$. Next, pick a singular chain $\widetilde{\nu} \in \mathrm{C}_n(M)$ that represents a fundamental cycle in $\mathrm{C}_n(M,\partial M)$. By Propositions 68.9 and 43.15 we know that the cycles $\partial_n(\widetilde{\nu})$ and μ are homologous in $\mathrm{C}_{n-1}(\partial M)$. This means that there exists a $\sigma \in \mathrm{C}_n(\partial M)$ with $\partial\sigma = \partial_n(\widetilde{\nu}) - \mu$. Now we set $\nu := \widetilde{\nu} - \sigma \in \mathrm{C}_n(M)$. Since ν and $\widetilde{\nu}$ represent the same element in $\mathrm{C}_n(M,\partial M)$ we see that ν is also a fundamental cycle. Finally we note that by construction we have $\partial_n(\nu) = \partial_n(\widetilde{\nu}) - \partial_n(\sigma) = \mu$. ■

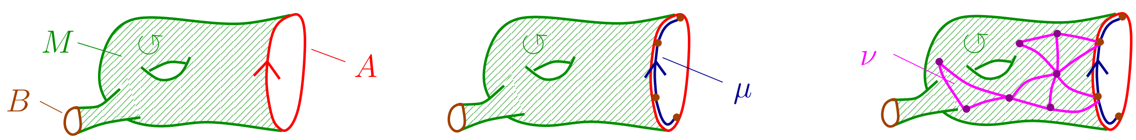


Figure 1059. Illustration of Corollary 68.10.

Next we record the following useful corollary to Proposition 68.9.

Corollary 68.11. *Let M be a compact oriented connected n-dimensional smooth manifold. We denote by $i\colon \partial M \to M$ the inclusion map and we denote by $N_1, \ldots, N_k$ the components of ∂M.*

(1) *We have*

$$\ker\big(i_*\colon \underbrace{\mathrm{H}_{n-1}(\partial M)}_{=\mathbb{Z}\cdot[N_1]\oplus\cdots\oplus\mathbb{Z}\cdot[N_k]} \to \mathrm{H}_{n-1}(M)\big) \;=\; \mathbb{Z}\cdot \underbrace{[\partial M]}_{\substack{\uparrow \\ \text{we have } [\partial M] = [N_1]+\cdots+[N_k]}} \;\subset\; \mathrm{H}_{n-1}(\partial M).$$

(2) *Let $j \in \{1,\ldots,k\}$. We consider the following maps:*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}_{n-1}(\partial M)/\mathbb{Z}\cdot[\partial M] \arrow[r, "i_*"] & \mathrm{H}_{n-1}(M) \arrow[r] & \operatorname{coker}(\mathrm{H}_{n-1}(\partial M) \xrightarrow{i_*} \mathrm{H}_{n-1}(M)) \arrow[r] & 0. \\ \mathbb{Z}\cdot[N_1]\oplus\ldots\oplus\widehat{\mathbb{Z}\cdot[N_j]}\oplus\ldots\oplus\mathbb{Z}\cdot[N_k] \arrow[ur, "\cong"] & & & & \end{tikzcd}$$

The following two statements hold:

(a) *The horizontal sequence is exact and it splits.*

(b) *The vertical map is an isomorphism.*

(3) *If M has precisely one boundary component, then $i_*\colon \mathrm{H}_{n-1}(\partial M) \to \mathrm{H}_{n-1}(M)$ is the zero map.*

(4) *The boundary ∂M is not a retract of M.*

Remark.

(1) The special case that M is a 2-dimensional smooth manifold is foreshadowed by Exercise 48.14.
(2) In Proposition 43.7 we showed that the sphere $S^{n-1} = \partial\overline{B}^{n-1}$ is not a retract of the closed ball $\overline{B}^n$. Corollary 68.11 (3) can thus be viewed as a generalization of Proposition 43.7 from $\overline{B}^n$ to the case of compact oriented connected smooth manifolds with non-empty boundary.

Example. In Figure 1060 we show a compact oriented n-dimensional smooth manifold with three boundary components N_1, N_2 and N_3. Corollary 68.11 says that *any two* boundary components generate a *subsummand* of $\mathrm{H}_{n-1}(M)$. Furthermore in $\mathrm{H}_{n-1}(M)$ we have the equality $[N_1] + [N_2] + [N_3] = 0$.

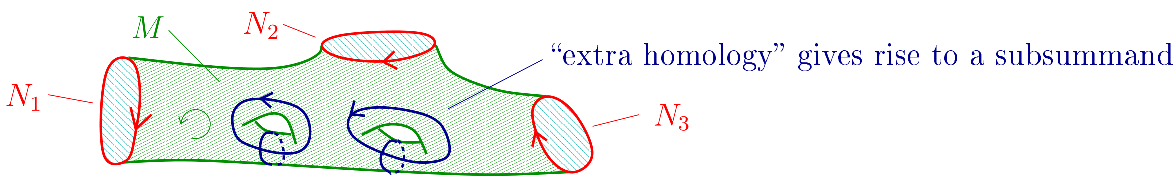


FIGURE 1060

PROOF. Let M be a compact oriented n-dimensional smooth manifold. We denote by $i\colon \partial M \to M$ the inclusion map.

(1) Using Proposition 68.9 and the long exact sequence in homology of the pair $(M, \partial M)$ we obtain that the following sequence is exact:

$$\underbrace{\mathrm{H}_n(M, \partial M)}_{=\mathbb{Z}\cdot[M] \text{ by Theorem 68.1}} \xrightarrow{[M]\mapsto[\partial M]} \mathrm{H}_{n-1}(\partial M) \xrightarrow{i_*} \mathrm{H}_{n-1}(M).$$

Statement (1) is now an immediate consequence.
(2) (a) It follows from (1) that the horizontal sequence is exact. By Theorem 66.6 (2) the group $\mathrm{H}_{n-1}(M)$ is free abelian. It follows from Lemma 19.8 that the group $\operatorname{coker}(i_*\colon \mathrm{H}_{n-1}(\partial M) \to \mathrm{H}_{n-1}(M)) \cong \ker(\mathrm{H}_{n-1}(M) \to \mathrm{H}_{n-1}(M, \partial M))$ is also free abelian. It now follows from Lemma 46.1 that the short exact sequence splits.
(b) It follows immediately from $[\partial M] = [N_1] + \cdots + [N_k]$ that the vertical map is an isomorphism.
(3) Now we assume that ∂M has precisely one component. By Theorem 68.1 we now have $\mathrm{H}_{n-1}(\partial M) = \mathbb{Z} \cdot [\partial M]$. Thus this statement is now an immediate consequence of (1).
(4) Suppose there exists a retraction $r\colon M \to \partial M$, i.e. a map with $r \circ i = \mathrm{id}_{\partial M}$. We consider the following commutative diagram

$$\begin{tikzcd} & \mathrm{H}_{n-1}(M) \arrow[dr, "r_*"] & \\ \mathrm{H}_{n-1}(\partial M) \arrow[ur, "i_*"] \arrow[rr, "(r\circ i)_*=\mathrm{id}"] & & \mathrm{H}_{n-1}(\partial M). \end{tikzcd}$$

We just saw in (1) that i_* is not injective. But this leads immediately to a contradiction. ■

In the next lemma we study the fundamental classes of codimension-zero submanifolds.

Lemma 68.12. *Let M be a compact oriented n-dimensional smooth manifold. Furthermore let $W \subset M$ be a compact n-dimensional submanifold. We write $\mathring{W} = W \setminus \partial W$ and we equip W with the orientation given by Lemma 6.46 (5). The following two statements hold:*

(1) *The inclusion induced map $\mathrm{H}_n(W, \partial W) \to \mathrm{H}_n(M, M \setminus \mathring{W})$ is an isomorphism.*
(2) *The images of the fundamental classes $[M] \in \mathrm{H}_n(M, \partial M)$ and $[W] \in \mathrm{H}_n(W, \partial W)$ under the maps*

$$\mathrm{H}_n(M, \partial M) \underset{\substack{\uparrow \\ \textit{isomorphism by the Excision Theorem 44.10}}}{\longrightarrow} \mathrm{H}_n(M, M \setminus \mathring{W}) \xleftarrow{\cong} \mathrm{H}_n(W, \partial W)$$

agree.

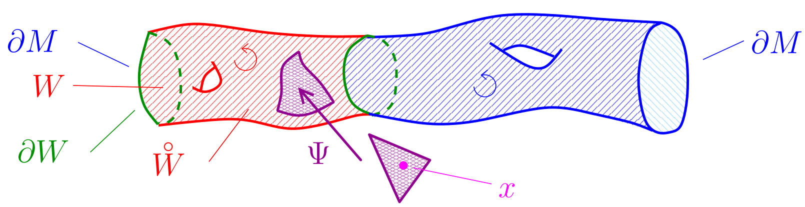


FIGURE 1061. Illustration for the proof of Lemma 68.12.

PROOF (∗).

(1) The first statement is an immediate consequence of the Excision Theorem 44.10.
(2) We denote by $\Phi\colon \mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(W, \partial W)$ the map given in statement (2). We have to show that $\Phi([M])$ is the fundamental class for W. So let $\Psi\colon \Delta^n \to W$ be a map and let $x \in \mathring{\Delta}^n$ be an admissible point. We consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}_n(M, \partial M) \arrow[r] \arrow[rr, bend left, "\Phi"] \arrow[d, "\omega_{\Psi(x)}"'] & \mathrm{H}_n(M, M \setminus \mathring{W}) \arrow[dl] & \mathrm{H}_n(W, \partial W) \arrow[l, "\cong"'] \arrow[d, "\omega_{\Psi(x)}"] \\
\mathrm{H}_n(M, M \setminus \{\Psi(x)\}) & & \mathrm{H}_n(W, W \setminus \{\Psi(x)\}) \arrow[ll, "\cong"', "\text{isomorphism by the Excision Theorem 43.20}"] \\
& \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{x\}). \arrow[ul, "\Psi_*"] \arrow[ur, "\Psi_*"'] &
\end{tikzcd}$$

The desired statement now follows easily from the defining properties of fundamental classes and the above discussions. We leave the few remaining details to the reader. ■

One of the most popular approaches to computing the homology groups of some topological space X is to write X as the union $X = A \cup B$ of "simpler topological spaces A and B" in a suitable way and to use the Mayer–Vietoris Theorem 46.5. The catch of the Mayer–Vietoris Theorem 46.5 is that not only does one need to understand the homology groups of A, B and $A \cap B$, one also needs to understand the inclusion induced maps on homology groups, which can be quite difficult. Fortunately Corollary 68.11 (2) now gives an interesting calculation of an inclusion induced map of homology groups. In the following two sections we will use Corollary 68.11 (2) to study the homology groups of the connected sum of two smooth manifolds and to study the homology groups of knot complements.

68.5. **The homology groups of the connected sum of two smooth manifolds** $(*)$**.** Let M and N be two compact oriented connected non-empty n-dimensional smooth manifolds. We recall the definition of the connected sum $M\#N$ that we gave on page 377. First we pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and we pick an orientation-reversing smooth embedding $\psi\colon \overline{B}^n \to N \setminus \partial N$. Then we defined the connected sum of M and N as follows:

$$M\#N \;:=\; \big(M \setminus \varphi(B^n)\big) \sqcup \big(N \setminus \psi(B^n)\big) \,/\, \big\{\varphi(P) = \psi(P) \,\big|\, P \in S^{n-1}\big\}.$$

We refer to Proposition 8.35 for some key properties of $M\#N$. Recall that in Proposition 20.12 we showed that $\pi_1(M\#N)$ is isomorphic to the free product of $\pi_1(M)$ and $\pi_1(N)$. The following proposition says that a very similar result holds for the homology groups of $M\#N$.

Proposition 68.13. *Let M and N be two closed oriented connected non-empty n-dimensional smooth manifolds. For any $k \in \mathbb{N}_0$ we have*

$$\mathrm{H}_k(M\#N) \;\cong\; \begin{cases} \mathrm{H}_k(M) \oplus \mathrm{H}_k(N), & \text{if } k \neq 0, n, \\ \mathbb{Z}, & \text{if } k = 0, n. \end{cases}$$

Moreover, for $k \neq 0, n$ the inclusion maps induce isomorphisms

$$\mathrm{H}_k(M) \oplus \mathrm{H}_k(N) \;\xleftarrow{\cong}\; \mathrm{H}_k(M \setminus B^n) \oplus \mathrm{H}_k(N \setminus B^n) \;\xrightarrow{\cong}\; \mathrm{H}_k(M\#N)$$

where $B^n \subset M$ and $B^n \subset N$ denote the open balls used in the definition of the connected sum of M and N.

Remark. In Exercise 68.8 we will consider the homology groups of $M\#N$ if we drop the condition that the smooth manifolds are closed.

The key to the proof of Proposition 68.13 is the following lemma.

Lemma 68.14. *Let X and Y be two compact orientable connected n-dimensional smooth manifolds such that $\partial X \cong S^{n-1}$ and $\partial Y \cong S^{n-1}$. We pick a diffeomorphism $\varphi\colon \partial X \to \partial Y$.*

(1) *There exists an isomorphism*

$$\mathrm{H}_n(X \cup_\varphi Y) \;\cong \mathbb{Z}.$$

Furthermore for every $k < n$ the inclusion maps $X \to X \cup_\varphi Y$ and $Y \to X \cup_\varphi Y$ induce an isomorphism

$$\widetilde{\mathrm{H}}_k(X) \oplus \widetilde{\mathrm{H}}_k(Y) \xrightarrow{\cong} \widetilde{\mathrm{H}}_k(X \cup_\varphi Y).$$

(2) *In particular for the case that $Y = \overline{B}^n$ we obtain that for any $k < n$ the inclusion map $X \to X \cup_\varphi \overline{B}^n$ induces an isomorphism*

$$\widetilde{\mathrm{H}}_k(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_k(X \cup_\varphi \overline{B}^n).$$

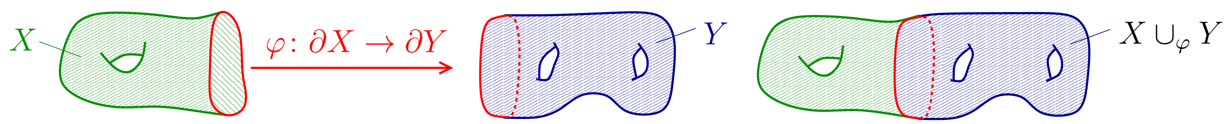


FIGURE 1062. Illustration for Lemma 68.14.

PROOF OF LEMMA 68.14. The second statement is an immediate consequence of the first statement and the fact that for every $k < n$ we have $\widetilde{\mathrm{H}}_k(\overline{B}^n) = 0$. Thus it suffices to prove the first statement. To simplify the notation we write $S^{n-1} = \partial X = \partial Y$. We also write $Z = X \cup_\varphi Y$. It follows from the Mayer-Vietoris Theorem 46.10 for Manifolds[1035] that there exists a long exact sequence

$$\to \widetilde{\mathrm{H}}_k(S^{n-1}) \xrightarrow{i_* \oplus -i_*} \widetilde{\mathrm{H}}_k(X) \oplus \widetilde{\mathrm{H}}_k(Y) \xrightarrow{i_* + i_*} \widetilde{\mathrm{H}}_k(Z) \xrightarrow{\partial_k} \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) \to \dots$$

where i always stands for the corresponding inclusion map. By Proposition 43.4 we have $\widetilde{\mathrm{H}}_n(S^n) \cong \mathbb{Z}$ and $\widetilde{\mathrm{H}}_k(S^n) = 0$ for $k \neq n$. We see immediately that for any $k < n$ the inclusion maps induce an isomorphism $\widetilde{\mathrm{H}}_k(X) \oplus \widetilde{\mathrm{H}}_k(Y) \to \widetilde{\mathrm{H}}_k(Z)$.

Now we turn to the n-th homology of Z. First note that by Theorem 66.8 we have $\widetilde{\mathrm{H}}_n(X) = \widetilde{\mathrm{H}}_n(Y) = 0$. Therefore the above long exact sequence gives rise to an exact sequence

since S^{n-1} is the unique boundary component of X we deduce from Corollary 68.11 that the map $i_* \colon \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \to \widetilde{\mathrm{H}}_{n-1}(X)$ is the zero map, the same way we also see that the map $i_* \colon \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \to \widetilde{\mathrm{H}}_{n-1}(Y)$ is the zero map

$$\underbrace{\widetilde{\mathrm{H}}_n(X) \oplus \widetilde{\mathrm{H}}_n(Y)}_{=0 \text{ by Theorem 66.8}} \to \widetilde{\mathrm{H}}_n(Z) \xrightarrow{\partial_n} \underbrace{\widetilde{\mathrm{H}}_{n-1}(S^{n-1})}_{=\mathbb{Z}\cdot[S^{n-1}]} \xrightarrow{i_* \oplus -i_*} \widetilde{\mathrm{H}}_{n-1}(X) \oplus \widetilde{\mathrm{H}}_{n-1}(Y) \to \widetilde{\mathrm{H}}_{n-1}(Z) \to 0.$$

isomorphism by the above

All the desired statements follow from the above data. ■

Now we can easily provide the proof of Proposition 68.13.

PROOF OF PROPOSITION 68.13. Let M and N be two closed oriented connected non-empty n-dimensional smooth manifolds. As required in the definition of the connected sum of M and N we pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^n \to M$ and we pick an orientation-reversing smooth embedding $\psi\colon \overline{B}^n \to N$. Recall that

$$M \# N \;=\; \big(M \setminus \varphi(B^n)\big) \sqcup \big(N \setminus \psi(B^n)\big) \,/\, \big\{\varphi(P) = \psi(P) \,\big|\, P \in S^{n-1}\big\}.$$

[1035] Note that here secretly we also used Proposition 8.15

We write $X = M \setminus \varphi(B^n)$ and $Y = N \setminus \psi(B^n)$. Note that it follows immediately from Lemma 68.14 (1) that $\mathrm{H}_n(M\#N) = \mathrm{H}_n(X \cup_{S^{n-1}} Y) \cong \mathbb{Z}$. Now let $k < n$. Note that in this case we have

$$\mathrm{H}_k(M) \oplus \mathrm{H}_k(N) \ \underset{\substack{\uparrow \\ \text{Lemma 68.14 (1) applied to} \\ M = X \cup_{S^{n-1}} \overline{B}^n \text{ and } N = Y \cup_{S^{n-1}} \overline{B}^n}}{\xleftarrow{\cong}} \ \mathrm{H}_k(\underbrace{M \setminus \varphi(B^n)}_{=X}) \oplus \mathrm{H}_k(\underbrace{N \setminus \psi(B^n)}_{=Y}) \ \underset{\substack{\uparrow \\ \text{Lemma 68.14 (1) applied to } M\#N = X \cup_{S^{n-1}} Y}}{\xrightarrow{\cong}} \ \mathrm{H}_k(M\#N)$$

■

Examples.

(1) Let $T = S^1 \times S^1$ be the torus. In Figure 220 we saw that the connected sum $T\#T$ is diffeomorphic to the surface Σ_2 of genus 2. More generally, the same argument shows that the connected sum of g copies of the torus T is diffeomorphic to the surface Σ_g of genus g. Together with the calculation of the homology groups of the torus in Lemma 46.12 we now see that

$$\mathrm{H}_k(\Sigma_g) \ \cong \ \mathrm{H}_k(\underbrace{T\# \dots \#T}_{g \text{ copies}}) \ \cong \ \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, \\ \mathbb{Z}^{2g}, & \text{if } k = 1, \\ 0, & \text{if } k > 2. \end{cases}$$

This agrees of course with the result obtained in Proposition 48.9.

(2) We consider the complex manifold $\mathbb{C}\mathrm{P}^2$. On page 1262 we showed that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^2) \ \cong \ \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \\ 0, & \text{otherwise.} \end{cases}$$

As always we equip $\mathbb{C}\mathrm{P}^2$ with the natural orientation coming from Proposition 12.7. We can now form the connected sum $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$. It follows from Proposition 68.13 that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2) \ \cong \ \begin{cases} \mathbb{Z}, & \text{if } k = 0, 4, \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

For the complex projective space $\mathbb{C}\mathrm{P}^2$ we use the common notation that we write $\overline{\mathbb{C}\mathrm{P}}^2$ instead of $-\mathbb{C}\mathrm{P}^2$. Since the homology groups do not depend on the orientation we have the same homology groups for $\overline{\mathbb{C}\mathrm{P}}^2$ and the calculation shows that the homology groups of $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ agree.

Recall that on page 1442 we showed, using the Künneth Theorem 58.8 that

$$\mathrm{H}_k(S^2 \times S^2) \ \cong \ \begin{cases} \mathbb{Z}, & \text{if } k = 0, 4, \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

This fact, and our above calculations shows, that the homology groups of the connected sums $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ and of $S^2 \times S^2$ are all isomorphic. Furthermore, note that by Propositions 14.14, 16.20, 20.12 and the discussion on page 997 we know that all of these topological spaces are simply connected. So this raises the following intriguing question.

Question 68.15. *Are the closed 4-dimensional smooth manifolds $S^2 \times S^2$, $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ homeomorphic?*

We will answer this question in Corollary 101.12 and Proposition 101.13.

68.6. **Homology groups of knot complements** $(*)$**.** We return to the ever-popular study of knots in S^3.

Definition. Let $K \subset S^3$ be a knot. By the Tubular Neighborhood Theorem 8.24 there exists a smooth embedding $\Phi\colon \overline{B}^2 \times K \to S^3$ such that $\Phi(0, P) = P$ for all $P \in K$. We refer to

$$X_K \;:=\; S^3 \setminus \Phi(B^2 \times K)$$

as the *exterior of* K. If K is oriented, then we pick Φ to be orientation-preserving and, following the definition on page 390, we refer to any oriented submanifold $\Phi(S^1 \times \{P\})$ as a *meridian* μ_K *of* K.

The following lemma gives us in particular the long-awaited calculation of the homology groups of knot complements.

Lemma 68.16. *For any knot $K \subset S^3$ the following statements hold:*

(1) *The exterior X_K of the knot K is a compact orientable connected 3-dimensional smooth manifold and the boundary ∂X_K is diffeomorphic to the torus.*
(2) *The diffeomorphism type of X_K as an oriented smooth manifold does not depend on the choice of Φ.*
(3) *The exterior X_K is a deformation retract of the knot complement $S^3 \setminus K$.*
(4) *We pick an orientation for K. For every $n \in \mathbb{N}_0$ we have*[1036]

$$\mathrm{H}_n(S^3 \setminus K) \;\underset{\substack{\uparrow \\ \text{induced by the inclusion } X_K \to S^3 \setminus K}}{\xleftarrow{\;\cong\;}}\; \mathrm{H}_n(X_K) \;=\; \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z} \cdot [\mu_K], & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

where μ_K denotes a meridian of K.

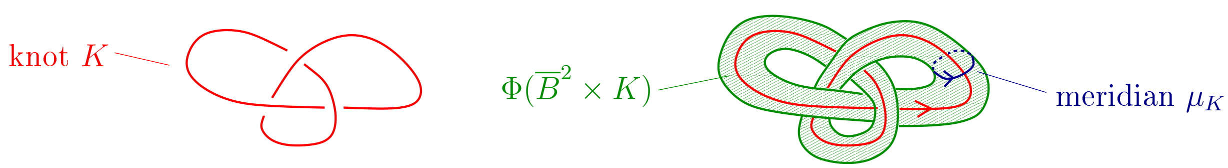


Figure 1063. Illustration for Lemma 68.16.

Remark. In Lemma 46.14 we already showed that $\mathrm{H}_0(X_K) = \mathbb{Z}$ and that $\mathrm{H}_1(X_K) = \mathbb{Z} \cdot [\mu_K]$. In fact in the proof of Lemma 68.16 (4) below we will follow the same approach as in Lemma 46.14, except that now can overcome the hurdle which prevented us back then from computing all homology groups.

[1036] We will give an alternative approach to calculating these homology groups in Exercise 68.5.

PROOF. Let $K \subset S^3$ be an oriented knot and let $P \in K$. We pick a tubular neighborhood $\overline{B}^2 \times K$ for K and we pick $Q \in S^1$. We write $\mu_K = S^1 \times \{P\}$ and $\gamma = \{Q\} \times K$.

(1) This statement is basically a consequence of Proposition 8.2.
(2) This statement is an immediate consequence of Proposition 8.26.
(3) A deformation retraction from $S^3 \setminus K$ to X_K is given by the map

$$\begin{array}{rcl} (S^3 \setminus K) \times [0,1] & \to & X_K = S^3 \setminus (B^2 \times K) \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x \in Y = S^3 \setminus (\overline{B}^2 \times K), \\ (((1-t)r+t)e^{i\alpha}, z), & \text{if } x = (re^{i\alpha}, z) \in \overline{B}^2 \times K. \end{cases} \end{array}$$

It follows easily from Lemma 3.10 and Proposition 8.2 that this map is indeed continuous.

(4) First note that by Corollary 42.8, Lemma 18.14 and by (3) we know that the inclusion $X_K \to S^3 \setminus K$ induces an isomorphism on all homology groups.

Now we proceed with almost the same argument as in the proof of Lemma 46.14. Namely, we consider the decomposition

$$S^3 \;=\; \underbrace{(S^3 \setminus (B^2 \times K))}_{=X_K} \cup_{S^1 \times K} (\overline{B}^2 \times K)$$

and we apply the Mayer-Vietoris Theorem 46.10 with reduced homology groups. We obtain the following long exact sequence:

$$\begin{array}{llllllll} \to & H_3(\partial X_K) & \xrightarrow{i_*-i_*} & H_3(X_K) \oplus H_3(\overline{B}^2 \times K) & \xrightarrow{i_*+i_*} & H_3(S^3) & \xrightarrow{\partial_3} & \\ \to & H_2(\partial X_K) & \xrightarrow{i_*-i_*} & H_2(X_K) \oplus H_2(\overline{B}^2 \times K) & \xrightarrow{i_*+i_*} & H_2(S^3) & \xrightarrow{\partial_2} & \\ \to & H_1(\partial X_K) & \xrightarrow{i_*-i_*} & H_1(X_K) \oplus H_1(\overline{B}^2 \times K) & \xrightarrow{i_*+i_*} & H_1(S^3) & \xrightarrow{\partial_1} & 0. \end{array}$$

Now we throw in everything we know about the above setting:

$$\begin{array}{llllllll} \to & \underbrace{H_3(\partial X_K)}_{=0} & \xrightarrow{i_*-i_*} & H_3(X_K) \oplus \underbrace{H_3(\overline{B}^2 \times K)}_{=0} & \xrightarrow{i_*+i_*} & \underbrace{H_3(S^3)}_{=\mathbb{Z}\cdot[S^3]} & \xrightarrow{\partial_3} & \\ \to & \underbrace{H_2(\partial X_K)}_{=\mathbb{Z}\cdot[\partial X_K]} & \xrightarrow{i_*-i_*} & H_2(X_K) \oplus \underbrace{H_2(\overline{B}^2 \times K)}_{=0} & \xrightarrow{i_*+i_*} & \underbrace{H_2(S^3)}_{=0} & \xrightarrow{\partial_2} & \end{array}$$

by Corollary 68.11 (2) this is the zero map

$$\begin{array}{llllllll} \to & \underbrace{H_1(\partial X_K)}_{=\mathbb{Z}\cdot[\mu_K]\oplus\mathbb{Z}\cdot[\gamma]} & \xrightarrow{i_*-i_*} & H_1(X_K) \oplus \underbrace{H_1(\overline{B}^2 \times K)}_{=\mathbb{Z}\cdot[\gamma]} & \xrightarrow{i_*+i_*} & \underbrace{H_1(S^3)}_{=0} & \xrightarrow{\partial_1} & 0. \end{array}$$

we have $[\mu_K] \mapsto ([\mu_K], 0)$ and $[\gamma] \mapsto (?, [\gamma])$

Note that we obtain from the above discussion that the connecting homomorphism $\partial_3\colon \mathbb{Z} \cdot [S^3] = H_3(S^3) \to H_2(\partial X_K) = \mathbb{Z} \cdot [\partial X_K]$ is an epimorphism. Since every epimorphism $\mathbb{Z} \to \mathbb{Z}$ is necessarily an isomorphism we see that the connecting homomorphism is in fact an isomorphism. Thus we obtain that $H_3(X_K) = H_2(X_K) = 0$. At the lower end we obtain easily, as we explained in the proof of Lemma 46.14, that $H_1(X_K) = \mathbb{Z} \cdot [\mu_K]$. ∎

68.7. **The $\mathbb{F}_2$-fundamental class** (∗)**.** In this final section we introduce the $\mathbb{F}_2$-fundamental class for any compact (not necessarily orientable) smooth manifold. Truth be told, this notion will only play a fairly minor role in the subsequent chapters.

Now let M be a compact connected n-dimensional smooth manifold. Recall that by Theorem 66.8 (3) we know that, regardless of whether or not M is orientable, we have $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$. This leads us to the following definition.

Definition. Let M be a compact n-dimensional smooth manifold. We denote by $N_1, \dots, N_k$ the components of M. We refer to

$$[M]_{\mathbb{F}_2} \;:=\; \sum_{i=1}^{k} \text{the unique non-zero element of } \mathrm{H}_n(N_i, \partial N_i; \mathbb{F}_2) \;\in\; \underbrace{\mathrm{H}_n(M, \partial M; \mathbb{F}_2)}_{= \bigoplus_{i=1}^{k} \mathrm{H}_n(N_i, \partial N_i; \mathbb{F}_2)}$$

as the $\mathbb{F}_2$-*fundamental class of* M.

Remark. In Corollary 87.5 we will generalize this notion to compact *topological* manifolds.

The following proposition is the "$\mathbb{F}_2$"-analogue of some of the results from Section 68.1.

Proposition 68.17. *Let M be a compact n-dimensional smooth manifold.*

(1) *Let $(K = (V, S), \lambda \colon |K| \to M)$ be a smooth simplicial structure for M. We pick a total order "$\leq$" on V. We denote by $s_1, \dots, s_k$ the n-simplices of K. Furthermore for $i \in \{1, \dots, k\}$ we consider the characteristic map $\Phi^{\leq}_{s_i} \colon \Delta^n \to |K|$. With this notation the singular chain*
$$\lambda \circ \Phi^{\leq}_{s_1} \otimes 1 + \dots + \lambda \circ \Phi^{\leq}_{s_k} \otimes 1 \;\in\; \mathrm{C}_n(M) \otimes \mathbb{F}_2 \;=\; \mathrm{C}_n(M; \mathbb{F}_2)$$
is a cycle which represents the $\mathbb{F}_2$-fundamental class of M.

(2) *Let $x \in M \setminus \partial M$.*
   (a) *The image of the $\mathbb{F}_2$-fundamental class $[M]_{\mathbb{F}_2} \in \mathrm{H}_n(M, \partial M; \mathbb{F}_2)$ under the natural map $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \to \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2)$ is the unique non-trivial element.*
   (b) *If M is connected, then the natural map $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \to \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2)$ is an isomorphism.*

Sketch of proof. It is straightforward to see that it suffices to consider the case that M is connected. Furthermore, using a doubling argument one can fairly easily reduce the proof of the statement to the case of a closed n-dimensional smooth manifold M. Next note that a slight variation on the argument of Theorem 66.4 shows that

$$\sigma \;:=\; \lambda \circ \Phi^{\leq}_{s_1} \otimes 1 + \dots + \lambda \circ \Phi^{\leq}_{s_k} \otimes 1 \;\in\; \mathrm{C}_n(M; \mathbb{F}_2)$$

is indeed a cycle in the chain complex $\mathrm{C}_*(M; \mathbb{F}_2) = \mathrm{C}_*(M) \otimes \mathbb{F}_2$. Next note that the argument of the proof of Theorem 68.1 (1) shows that for every $x \in M$ the natural map $\mathrm{H}_n(M; \mathbb{F}_2) \to \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2)$ sends $[\sigma]$ to the unique non-zero element. Furthermore note that by Theorem 66.8 (3) we know that $\mathrm{H}_n(M; \mathbb{F}_2) \cong \mathbb{F}_2$. Finally note that this implies that the natural map $\mathrm{H}_n(M; \mathbb{F}_2) \to \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2)$ is indeed an isomorphism and that $[\sigma]$ is indeed a generator of $\mathrm{H}_n(M; \mathbb{F}_2)$. ■

The following proposition is the $\mathbb{F}_2$-analogue of Proposition 68.4.

Proposition 68.18. *Let M be a compact connected n-dimensional smooth manifold. Let $(\sigma_1 + \cdots + \sigma_m) \otimes 1 \in \mathrm{C}_n(M, \partial M; \mathbb{F}_2) = \mathrm{C}_n(M, \partial M) \otimes \mathbb{F}_2$ be a cycle. Suppose there exists a $j \in \{1, \ldots, m\}$ such that the following two conditions are satisfied:*

1. *The map $\sigma_j \colon \mathring{\Delta}^n \to M$ is a smooth embedding.*
2. *There exists an $x \in \sigma_j(\Delta^n) \setminus \sigma_j(\partial\Delta^n)$ which is not contained in the image of any other $\sigma_i \colon \Delta^n \to M$.*

Then the cycle $(\sigma_1 + \cdots + \sigma_m) \otimes 1$ represents the $\mathbb{F}_2$-fundamental class of M.

Proof. The proof of this proposition is almost the same as the proof of Proposition 68.4. We leave it to the reader to make the necessary modifications. ■

Examples.

(1) Now we will give a representative for the $\mathbb{F}_2$-fundamental class for the real projective plane $\mathbb{R}\mathrm{P}^2$. More precisely, we consider the four singular simplices $\sigma_1, \sigma_2, \tau_1$ and τ_2 of $\mathbb{R}\mathrm{P}^2$ shown in Figure 1064, where we use a hopefully obvious variation on the notation introduced in Figure 1052. It is straightforward to see that the singular 2-chain $(\sigma_1 + \sigma_2 + \tau_1 + \tau_2) \otimes 1 \in \mathrm{C}_2(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$ is a cycle. Using Proposition 68.18 we see that it represents the $\mathbb{F}_2$-fundamental class of the real projective plane.

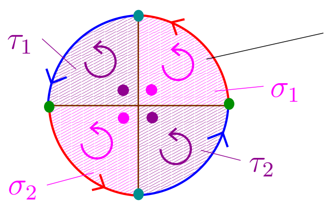


Figure 1064

(2) In Exercise 68.13 we will write down a representative for the $\mathbb{F}_2$-fundamental class of the Klein bottle.

The following proposition is the analogue of Proposition 68.9.

Proposition 68.19. *Let M be a compact n-dimensional smooth manifold. The connecting homomorphism*

$$\partial \colon \mathrm{H}_n(M, \partial M; \mathbb{F}_2) \;\to\; \mathrm{H}_{n-1}(\partial M; \mathbb{F}_2)$$

of the long exact sequence in homology with $\mathbb{F}_2$-coefficients of the pair $(M, \partial M)$ has the property that

$$[M]_{\mathbb{F}_2} \;\mapsto\; [\partial M]_{\mathbb{F}_2}.$$

Proof. The proof is the obvious modification of the previous proof. ■

Finally we have the following analogue of Corollary 68.11.

Corollary 68.20. *Let M be a compact n-dimensional smooth manifold. We denote by $i\colon \partial M \to M$ the inclusion map.*

(1) *We have*
$$\ker\left(i_*\colon \mathrm{H}_{n-1}(\partial M) \to \mathrm{H}_{n-1}(M)\right) = \mathbb{Z}\cdot[\partial M]_{\mathbb{F}_2} \in \mathrm{H}_{n-1}(\partial M;\mathbb{F}_2).$$

(2) *If M has precisely one boundary component, then $i_*\colon \mathrm{H}_{n-1}(\partial M;\mathbb{F}_2) \to \mathrm{H}_{n-1}(M;\mathbb{F}_2)$ is the zero map.*

(3) *The boundary ∂M is not a retract of M.*

Proof. Once again the proof is basically the same as the proof of the analogous statement for ordinary homology groups. ■

Remark. It is perhaps instructive to consider the good old Möbius band M. We denote by $i\colon \partial M \to M$ the inclusion map. Furthermore let $[b] \in \mathrm{H}_1(M)$ and $[c] \in \mathrm{H}_1(\partial M)$ be the generators shown in Figure 1065. We end up with the following situation:

$$\begin{tikzcd}
\mathrm{H}_1(\partial M) \arrow[r, "{[b]\mapsto 2\cdot[c]}"] \arrow[d, "{[\sigma]\mapsto[\sigma\otimes 1]}", "\cong"'] & \mathrm{H}_1(M) \arrow[d, "{[\sigma]\mapsto[\sigma\otimes 1]}", "\cong"'] \\
\mathrm{H}_1(\partial M;\mathbb{F}_2) \arrow[r, "{[b\otimes 1]\mapsto[2\cdot c\otimes 1]=[c\otimes 2]=0}"] & \mathrm{H}_1(M;\mathbb{F}_2).
\end{tikzcd}$$

Note that the vertical maps are isomorphisms by the Universal Coefficient Theorem 57.19. We see that the top horizontal map is an injective map, in fact it is basically given by multiplication by 2. On the other hand we see that the bottom horizontal map is the zero map.

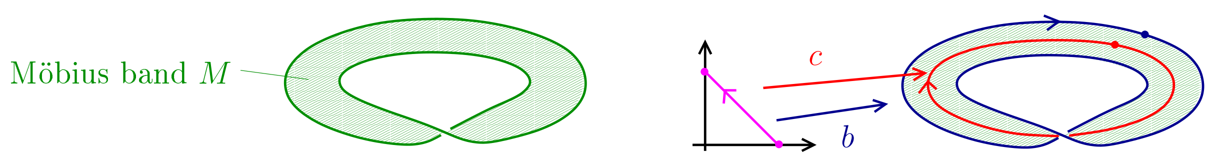


Figure 1065

Exercises for Chapter 68.

Exercise 68.1. We consider the surface $\Sigma = E_8/\sim$ of genus two as shown in Figure 1066. Show that there exists a cycle of the form $\epsilon_1\cdot\sigma_1 + \cdots + \epsilon_6\cdot\sigma_6 \in \mathrm{C}_2(E_8/\sim)$ such that the images of the six singular simplices $\sigma_1,\ldots,\sigma_6$ are precisely the six triangles shown in Figure 1066.

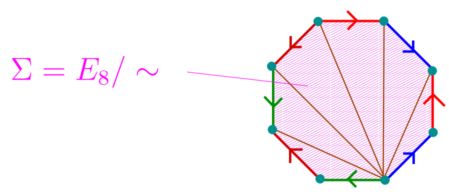


Figure 1066. Illustration of Exercise 68.1.

Exercise 68.2. Let M be a compact oriented connected n-dimensional smooth manifold with *at least two* boundary components. Let C be one of the boundary components. By Corollary 68.11 we know that the inclusion induced map $i_*\colon \mathrm{H}_{n-1}(C) \to \mathrm{H}_{n-1}(M)$ admits a left-inverse, i.e. there exists a homomorphism $\varphi\colon \mathrm{H}_{n-1}(M) \to \mathrm{H}_{n-1}(C)$ such that $\varphi \circ i_* = \mathrm{id}_{\mathrm{H}_{n-1}(C)}$. Also note that in Lemma 23.13 we showed that in the case $n = 2$ a stronger statement holds, namely C is a retract of M. Is it true in general that C is necessarily a retract of M?

Exercise 68.3. Let M be a closed oriented connected n-dimensional smooth manifold and let $C \subset M$ be a closed oriented connected $(n-1)$-dimensional submanifold of M. We assume that C is non-separating. Recall that according to the definition on page 682 this means that $M \setminus C$ is connected. By the Tubular Neighborhood Theorem 8.24 we can pick a tubular neighborhood $[-1,1] \times C$. We write $W := M \setminus ((-1,1) \times C)$.

(a) Show that there exists an exact sequence of the form

$$0 \;\to\; \mathrm{H}_{n-1}(\underbrace{M \setminus ((-1,1) \times C)}_{=W}) \;\to\; \mathrm{H}_{n-1}(M) \;\to\; \mathrm{H}_{n-2}(C) \;\to\; \dots$$

Hint. Consider a Meyer–Vietoris sequence.

(b) Use (a) and Exercise 68.2 to show that the map $i_*\colon \mathrm{H}_{n-1}(C) \to \mathrm{H}_{n-1}(M)$ admits a left-inverse.

Remark. This implies in particular that $i_*([C])$ is a non-trivial element of $\mathrm{H}_{n-1}(M)$.

(c) Does the conclusion of (b) also hold if we replace the hypothesis that M is closed by the hypothesis that M is compact?

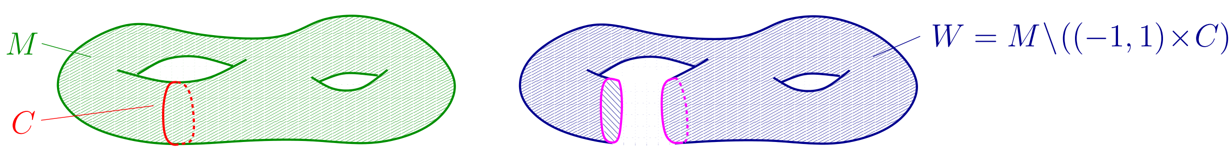


FIGURE 1067

Exercise 68.4. Let $K = \gamma(S^1) \subset S^3$ be a knot and let $\Phi\colon \overline{B}^2 \times K \to S^3$ be a tubular map for K. As on page 1728 we refer to $X_K = S^3 \setminus \Phi(B^2 \times K)$ as the exterior of the knot K and we refer to $\mu_K = \Phi(S^1 \times \{P\})$ as a meridian of K. In Lemma 46.14, or alternatively in Lemma 68.16, we saw that $[\mu_K]$, viewed as an element in $\mathrm{H}_1(X_K)$, is a generator of $\mathrm{H}_1(X_K)$.

(a) Show that there exists an $n \in \mathbb{Z}$ such that the curve

$$\begin{array}{rcl} S^1 & \to & \partial X_K = \Phi(S^1 \times K) \\ z & \mapsto & \Phi(z^n, \gamma(z)) \end{array}$$

represents a generator of $\ker(\mathrm{H}_1(\partial X_K) \to \mathrm{H}_1(X_K))$. We refer to the image of this curve as a *longitude* λ_K of K.

(b) Sketch longitudes for the three knots that are shown in Figure 1068.

(c) Now suppose that we are given two knots K and J. We define X_K, X_J, μ_K, μ_J and λ_K and λ_J as above. We pick a diffeomorphism $\varphi\colon \partial X_K \to \partial X_J$ with $\varphi_*(\mu_K) = \lambda_J$

and $\varphi_*(\lambda_K) = \mu_K$. We denote by $Y = X_K \cup_\varphi X_J$ the result of gluing X_K to X_J along the boundary. Show that Y is a homology sphere, i.e. show that $\mathrm{H}_1(Y) = \mathrm{H}_2(Y) = 0$ and $\mathrm{H}_3(Y) \cong \mathbb{Z}$.

Remark. If K and J are two non-trivial knots, then it follows from Proposition **??** (which in turn is a consequence of the famous Loop Theorem **??**) together with Proposition 21.24, that the fundamental group of Y is always non-trivial. This shows that Y is a homology sphere with non-trivial fundamental group. In other words we have now found an alternative proof of Proposition 66.11.

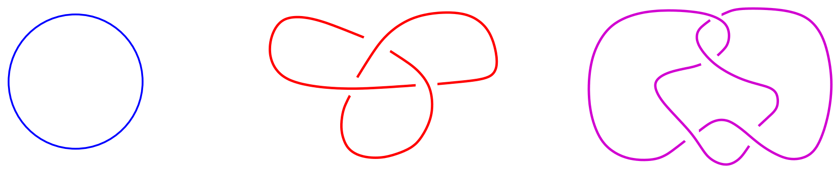


FIGURE 1068

Exercise 68.5. In this exercise we will consider an alternative approach to computing the homology groups of a knot complement. Let $K \subset S^3$ be a knot. We pick a tubular neighborhood $\overline{B}^2 \times K$ and we consider the exterior $X_K = S^3 \setminus (B^2 \times K)$. Now do the following:

(a) Show that the inclusion induced maps
$$\mathrm{H}_*(S^3, K) \xrightarrow{\cong} \mathrm{H}_*(S^3, \overline{B}^2 \times K) \xleftarrow{\cong} \mathrm{H}_*(X_K, \partial X_K)$$
are both isomorphisms.

(b) Compute the homology groups of the pair $(X_K, \partial X_K)$.

(c) Determine, using (b), the homology groups of X_K.

Exercise 68.6. We define an *m-component link* to be a 1-dimensional submanifold of S^3 that is diffeomorphic to the disjoint union of m copies of S^1. We pick a tubular neighborhood $\overline{B}^2 \times L$ and we define the exterior of L as $X_L := S^3 \setminus (B^2 \times L)$. We denote the components of L by $L_1, \ldots, L_m$. For each $i \in \{1, \ldots, m\}$ we pick a point $P_i \in L_i$ and we define $\mu_i := S^1 \times \{P_i\}$.

(a) Show that $\mathrm{H}_1(X_L) = \mathbb{Z} \cdot [\mu_1] \oplus \cdots \oplus \mathbb{Z} \cdot [\mu_m]$.

(b) Determine $\mathrm{H}_2(X_L)$.

Remark. There are basically two ways to go about doing this exercise, you can modify the proof of Lemma 68.16 or you could generalize the approach taken in Exercise 68.5.

Exercise 68.7.

(a) In Figure 1069 we see two 2-component links, namely the unlink and the Hopf link. Are these two links isotopic?

(b) Let $L \subset S^3$ be an m-component link. Show that the weight of the group $\pi_1(S^3 \setminus L)$, as defined in Exercise 27.13, equals m.

FIGURE 1069. Illustration for Exercises 68.6 and 68.7

Exercise 68.8. Let M and N be two oriented, connected n-dimensional smooth manifolds. We suppose that M is closed and that N has *non-empty* boundary. Express the homology groups of $M\#N$ in terms of the homology groups of M and N.

Exercise 68.9. Let $f\colon M \to N$ be a map between two compact orientable connected n-dimensional smooth manifolds with non-empty boundary. We assume that $f\colon \partial M \to \partial N$ is a homeomorphism. Show that f is a surjection.

Exercise 68.10. Let M be a compact oriented n-dimensional smooth manifold and let $p\colon \widetilde{M} \to M$ be a finite covering. By Proposition 17.1 we can equip $\widetilde{M}$ with the structure of a compact oriented smooth manifold such that $p\colon \widetilde{M} \to M$ is an orientation-preserving map. Let $p^*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(\widetilde{M})$ be the transfer map that we introduced on page 1445. Show that $p^*([M]) = \big[\widetilde{M}\big] \in \mathrm{H}_n(\widetilde{M}, \partial\widetilde{M})$.

Exercise 68.11. Let Σ be a closed orientable connected 2-dimensional smooth manifold and let $f\colon \Sigma \to \Sigma$ be a diffeomorphism. We consider the corresponding mapping torus $M := \mathrm{Tor}(\Sigma, f)$. Note that we know by Lemma 24.22 that M is a closed connected 3-dimensional smooth manifold.

(a) Suppose that if f is orientation-preserving, then $\mathrm{rank}(\mathrm{H}_2(M)) = \mathrm{rank}(\mathrm{H}_1(M))$.
(b) Suppose that if f is orientation-reversing, then $\mathrm{rank}(\mathrm{H}_2(M)) = \mathrm{rank}(\mathrm{H}_1(M)) - 1$.

Remark. You might want to use Proposition 46.21.

Exercise 68.12. Give examples of closed oriented connected non-empty smooth manifolds M and N of the same dimension such that $\pi_2(M\#N)$ is not isomorphic to $\pi_2(M) \oplus \pi_2(N)$.

Exercise 68.13. Give an example of a representative for the $\mathbb{F}_2$-fundamental class for the Klein bottle.

Exercise 68.14. Can you find a representative for the $\mathbb{F}_2$-fundamental class of $\mathbb{RP}^2$ with two or three summands?

Exercise 68.15. Let $n \in \mathbb{N}$ and let M be a compact oriented n-dimensional smooth manifold. The image of the fundamental class $[M] \in \mathrm{H}_n(M, \partial M)$ under the natural map $\mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(M, \partial M; \mathbb{R})$ is called the *real fundamental class*, denoted by $[M]_{\mathbb{R}}$. Let $(K = (V, S), \Theta\colon |K| \to M)$ be a smooth simplicial structure. Given an ordered n-simplex $s = (v_0, \dots, v_n)$ of K we denote by $\Phi_s\colon \Delta^n \to |K|$ the corresponding characteristic map given by $(t_0, \dots, t_n) \mapsto t_0 \cdot v_0 + \dots + t_n \cdot v_n$.

(a) Show that

$$\sigma \;:=\; \sum_{\substack{s=(v_0,\dots,v_n) \text{ ordered } n\text{-simplex} \\ \text{such that } \Theta\circ\Phi_s\colon \mathring{\Delta}^n \to M \\ \text{is orientation-preserving}}} (\Theta\circ\Phi_s\colon \Delta^n \to M) \;\in\; \mathrm{C}_n(M)$$

represents a cycle in $\mathrm{C}_n(M,\partial M)$.

(b) Show that $[\sigma]\otimes\frac{2}{n!} \in \mathrm{C}_n(M,\partial M;\mathbb{R}) = \mathrm{C}_n(M,\partial M)\otimes\mathbb{R}$ represents $[M]_{\mathbb{R}}$.

Exercise 68.16. Let X be a topological space. Given a homology class $\varphi \in \mathrm{H}_k(X;\mathbb{R})$ we define

$$\|\varphi\|_{\mathbb{R}} \;:=\; \inf\Big\{ \sum_{i=1}^{n} |r_i| \;\Big|\; \sigma = \underbrace{\sum_{i=1}^{n} \sigma_i\otimes r_i}_{\text{with } \sigma_i\neq\sigma_j \text{ for } i\neq j} \text{ is a cycle with } [\sigma]=\varphi\in \mathrm{H}_k(X;\mathbb{R})\Big\}.$$

In the following let M be a closed oriented connected non-empty n-dimensional smooth manifold M. Let $[M]\in \mathrm{H}_n(M)$ be the fundamental class and let $i\colon \mathrm{H}_1(S^1)\to \mathrm{H}_1(S^1;\mathbb{R})$ be the homomorphism from Lemma 57.7 (1). We define the *simplicial volume of M* as

$$\|M\|_{\mathbb{R}} \;:=\; \|i_*([M])\| \;\in \mathbb{R}_{\geq 0}.$$

(a) Show that $\|S^1\|_{\mathbb{R}} = 0$.
(b) Show that $\|(S^1)^2\|_{\mathbb{R}} = 0$.
(c) Let $n\in\mathbb{N}_{\geq 2}$. What is $\|S^n\|_{\mathbb{R}}$?
(d) Let $f\colon M\to N$ be a map between closed oriented connected non-empty n-dimensional smooth manifold. Show that

$$\|N\|_{\mathbb{R}} \;\leq\; \frac{1}{\deg(f)}\cdot\|M\|_{\mathbb{R}}$$

where $\deg(f)$ denotes the degree of f as defined on page 1737.
(e) Let M and N be two closed oriented connected non-empty n-dimensional smooth manifolds. We suppose that M and N are in fact homotopy equivalent. Show that $\|M\|_{\mathbb{R}} = \|N\|_{\mathbb{R}}$.
(f) Let N be a closed oriented non-empty n-dimensional smooth manifold and let $p\colon M\to N$ be a d-fold covering. Show that $\|M\|_{\mathbb{R}} = d\cdot\|N\|_{\mathbb{R}}$.
(g) Let $g\in\mathbb{N}_0$. As usual we denote by Σ_g the surface of genus $g\geq 1$. Show that $\|\Sigma_g\|_{\mathbb{R}} \leq 2g-2$.
Hint. Make use of (f).
Remark. In [**Frig17**, Corollary 7.5] or alternatively [**Thu78**, Theorem 6.2] it is shown that we also have the reverse inequality, i.e. we have $\|\Sigma_g\|_{\mathbb{R}} = 2g-2$.

Remark. The invariant $\|\varphi\|_{\mathbb{R}}$ is the "real analogue" of the invariant $\|\varphi\|_{\mathbb{Z}}$ introduced in Exercise 45.14.

69. The degree of a map between oriented smooth manifolds

Let $n \in \mathbb{N}$ and let M be a compact oriented connected non-empty n-dimensional smooth manifold. Recall that in Theorem 68.1 and Lemma 68.3 we proved that there exists a unique class $[M] \in \mathrm{H}_n(M, \partial M)$ such that the following holds:

(1) We have $\mathrm{H}_n(M, \partial M) = \mathbb{Z} \cdot [M]$.
(2) For every smooth embedding[1037] $\Psi \colon \Delta^n \to M$ and for every point $x \in \mathring{\Delta}^n$ with $\Psi^{-1}(\Psi(x)) = \{x\}$ we have the following equality in $\mathrm{H}_n(M, M \setminus \{\Psi(x)\})$:

$$\underset{\substack{\uparrow \\ \text{this is the natural map } \mathrm{H}_n(M,\partial M)\to \mathrm{H}_n(M,M\setminus\{\Psi(x)\}), \text{ which} \\ \text{is in fact an isomorphism by Theorem 68.1 (1)}}}{\omega_{\Psi(x)}([M])} \;=\; \operatorname{sign}(\Psi, x) \cdot [\,\underbrace{\Psi \colon \Delta^n \to M}_{\in \mathrm{C}_n(M, M\setminus\{\Psi(x)\})}\,].$$

In this short chapter we will explore the following definition.

Definition. Let $f \colon M \to N$ be a map between two compact oriented connected non-empty n-dimensional smooth manifolds. We assume that $f(\partial M) \subset \partial N$. We denote by $\deg(f) \in \mathbb{Z}$ the unique integer with

$$\underset{\substack{\uparrow \\ \text{since } f(\partial M)\subset \partial N \text{ we have an induced map } f_*\colon \mathrm{H}_n(M,\partial M)\to \mathrm{H}_n(N,\partial N)}}{f_*([M])} \;=\; \deg(f) \cdot [N] \;\in\; \mathrm{H}_n(N, \partial N) = \mathbb{Z} \cdot [N].$$

We refer to $\deg(f)$ as the *degree of f*.

Remark. For a map $f \colon S^n \to S^n$ the above degree is, by definition, exactly the same as the degree that we defined on page 1181.

The following simple lemma generalizes several statements of Lemma 45.11.

Lemma 69.1. *Let $n \in \mathbb{N}$ and let L, M and N be compact oriented connected non-empty n-dimensional smooth manifolds.*

(1) $\deg(\mathrm{id}_M) = 1$.
(2) *For every map $f \colon M \to N$ such that $f(\partial M) \subset \partial N$ we have the following equality:*
$$f_*([\partial M]) \;=\; \deg(f \colon M \to N) \cdot [\partial N] \;\in\; \mathrm{H}_{n-1}(\partial N).$$
(3) *Let $f \colon M \to N$ be a map with $f(\partial M) \subset \partial N$. If f is not surjective, then* $\deg(f) = 0$.
(4) *Let $f, g \colon M \to N$ be two maps with $f(\partial M) \subset \partial N$ and $g(\partial M) \subset \partial N$. If f is homotopic to g, then* $\deg(f) = \deg(g)$.
(5) *For any two maps $f \colon L \to M$ and $g \colon M \to N$ with $f(\partial L) \subset \partial M$ and $g(\partial M) \subset \partial N$ we have* $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

Proof.

(1) This statement is trivial.

[1037] Recall that on page 1589 we defined what it means for a map $\Delta^n \to M$ to be a smooth embedding.

(2) We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_n(M,\partial M) \arrow[r, "\partial_n", "{[M]\mapsto[\partial M],\text{ by Proposition 68.9}}"'] \arrow[d, "f_*"] & \mathrm{H}_{n-1}(\partial M) \arrow[d, "f_*"'] \\
\mathrm{H}_n(N,\partial N) \arrow[r, "\partial_n", "{[N]\mapsto[\partial N],\text{ by Proposition 68.9}}"'] & \mathrm{H}_{n-1}(\partial N)
\end{tikzcd}$$

where the horizontal maps are given by the connecting homomorphisms of the pairs $(M, \partial M)$ and $(N, \partial N)$. By Proposition 43.15 we know that the connecting homomorphism are natural, in other words, the diagram commutes. The statement now follows from Proposition 68.9.

(3) So let $f\colon M \to N$ be a non-surjective map with $f(\partial M) \subset \partial N$. We pick a point P in N that is not in the image of f. Note that $N \setminus \{P\}$ is an open subset of N. Thus, by the discussion on page 281, we can view it as an n-dimensional smooth manifold with $\partial(N \setminus \{P\}) = \partial N \cap (N \setminus \{P\}) = \partial N \setminus \{P\}$. Also note that it follows easily from $n \geq 1$ that $N \setminus \{P\}$ is non-compact.

Now we consider the following very simple commutative diagram:

$$\begin{tikzcd}
\mathrm{H}_n(M,\partial M) \arrow[rr, "f_*"] \arrow[dr, "f_*"'] & & \mathrm{H}_n(N,\partial N) \\
& \underbrace{\mathrm{H}_n(N\setminus\{P\},\partial(N\setminus\{P\}))}_{=0 \text{ by Theorem 66.6, since } N\setminus\{P\} \text{ is non-compact and connected}} \arrow[ur] &
\end{tikzcd}$$

It follows immediately that $f_*\colon \mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(N, \partial N)$ is the zero map.

(4) This statement follows immediately from Proposition 42.5.

(5) Exactly as in the proof of Lemma 45.11 (4) we have

$$(g \circ f)_*([L]) \;=\; g_*(f_*([L])) \;=\; g_*(\deg(f) \cdot [M]) \;=\; \deg(g) \cdot \deg(f) \cdot [N].$$

This shows that $\deg(g \circ f) = \deg(g) \cdot \deg(f)$. ■

We move on to the following definition.

Definition. Let $f\colon M \to N$ be a smooth map between two smooth manifolds of the same dimension. Let $P \in M \setminus \partial M$. We say f is a *local diffeomorphism at P* if there exists an open neighborhood U of P such that $f|_U$ is an embedding.[1038]

The following proposition is much more interesting than the previous lemma. It gives in particular a practical criterion for a map to have degree ± 1.

Proposition 69.2. *Let $n \in \mathbb{N}$ and let $f\colon M \to N$ be a map between two compact oriented connected non-empty n-dimensional smooth manifolds such that $f^{-1}(\partial N) = \partial M$. We suppose that there exists an $x \in M \setminus \partial M$ such that $f^{-1}(f(x)) = \{x\}$ and we suppose that*

[1038] Note that it follows from the Smooth Invariance of Domain, i.e. from Theorem 6.19, that this definition of a "local diffeomorphism at a point" is consistent with the definition given on page 295.

f is a local diffeomorphism at x. Then

$$\deg(f) \;=\; \begin{cases} +1, & \textit{if } f \textit{ is orientation-preserving at } x \\ -1, & \textit{if } f \textit{ is orientation-reversing at } x. \end{cases}$$

PROOF. Let U be an open connected neighborhood of x such that $f\colon U \to N$ is an embedding. We set

$$\epsilon \;:=\; \begin{cases} +1, & \text{if } f\colon U \to N \text{ is orientation-preserving} \\ -1, & \text{if } f\colon U \to N \text{ is orientation-reversing.} \end{cases}$$

Note that we need to show that $\deg(f) = \epsilon$. Since $f^{-1}(\partial N) = \partial M$ we can pick a smooth embedding $\Psi\colon \Delta^n \to U$ such that $x \in \Psi(\mathring{\Delta}^n)$. Note that $\Psi^{-1}(x)$ is an admissible point for Ψ and note that it follows from our hypothesis $f^{-1}(f(x)) = \{x\}$ that $\Psi^{-1}(x)$ is also an admissible point for $f \circ \Psi$. We consider the following commutative diagram[1039]

$$\begin{tikzcd} \mathrm{H}_n(M, \partial M) \arrow[r, "\omega_x", "\cong"'] \arrow[d, "f_*"] & \mathrm{H}_n(M, M \setminus \{x\}) \arrow[d, "f_*"'] \\ \mathrm{H}_n(N, \partial N) \arrow[r, "\omega_{f(x)}", "\cong"'] & \mathrm{H}_n(N, N \setminus \{f(x)\}). \end{tikzcd}$$

After these preparations we can now turn to the actual proof:

since $x \in \Psi(\mathring{\Delta}^n)$ and Ψ is an embedding we see that $\Psi\colon \Delta^n \to M$ defines an element of $\mathrm{H}_n(M, M \setminus \{x\})$

$$\begin{array}{rrcll} & \omega_x([M]) & = & \operatorname{sign}(\Psi, x) \cdot [\Psi\colon \Delta^n \to M] & \text{by definition of } [M] \\ \Rightarrow & f_*(\omega_x([M])) & = & \operatorname{sign}(\Psi, x) \cdot f_*([\Psi]) & \text{applying } f_* \\ \Rightarrow & \omega_{f(x)}(f_*([M])) & = & \operatorname{sign}(\Psi, x) \cdot [f \circ \Psi] & \text{since the diagram commutes} \\ \Rightarrow & \omega_{f(x)}(f_*([M])) & = & \epsilon \cdot \operatorname{sign}(f \circ \Psi, x) \cdot [f \circ \Psi] & \text{since } \operatorname{sign}(f \circ \Psi, x) = \epsilon \cdot \operatorname{sign}(\Psi, x) \\ \Rightarrow & \omega_{f(x)}(\deg(f) \cdot [N]) & = & \epsilon \cdot \omega_{f(x)}([N]) & \text{by definition of } \deg(f) \text{ and } [N] \\ \Rightarrow & \deg(f) \cdot [N] & = & \epsilon \cdot [N] & \text{by Theorem 68.1 (1)} \\ \Rightarrow & \deg(f) & = & \epsilon & \text{since } \mathrm{H}_n(N) = \mathbb{Z} \cdot [N]. \end{array}$$ ■

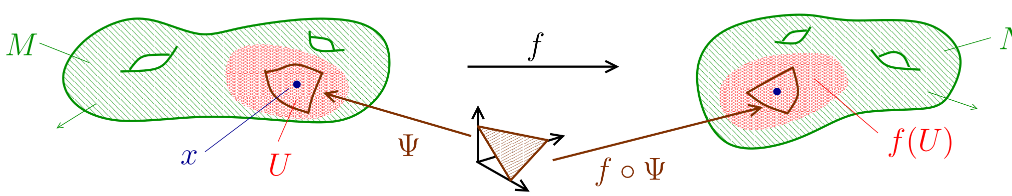


FIGURE 1070. Illustration for the proof of Proposition 69.2.

Proposition 69.3. *Let $n \in \mathbb{N}$ and let M be a closed oriented connected non-empty n-dimensional smooth manifold. For every $k \in \mathbb{Z}$ there exists a map $f\colon M \to S^n$ of degree k.*

PROOF. Let M be a closed oriented connected non-empty n-dimensional smooth manifold and let $k \in \mathbb{Z}$. As we mentioned on page 1184, by Lemmas 45.10 and 45.12 there exists a

[1039] Note that our hypothesis that $f^{-1}(f(x)) = \{x\}$ implies that $f(M \setminus \{x\}) \subset N \setminus \{f(x)\}$. Also note that $x \in M \setminus \partial M$ and $f(x) \in N \setminus \partial N$. In particular this implies that the various inclusion induced maps actually make sense.

map $g\colon S^n \to S^n$ of degree k. It follows from this fact and Lemma 69.1 (4) that it suffices to prove the following claim.

Claim. There exists a map $f\colon M \to S^n$ of degree $+1$.

First we pick an orientation-preserving smooth embedding $\Phi\colon \overline{B}^n \to M$. It follows fairly easily from Lemma 3.28 (2), and Proposition 2.43 (3) together with Lemma 2.40 and Lemma 3.26, that the induced map $\overline{\Phi}\colon \overline{B}^n/S^{n-1} \to M/(M \setminus \Phi(B^n))$ is a homeomorphism. We denote by $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ the explicit homeomorphism from page 182. As we mentioned on page 182, this homeomorphism has the property that the restriction of Ω to B^n is in fact an orientation-preserving embedding. We consider the map $f\colon M \to S^n$ that is given by

$$\begin{tikzcd} M \arrow[r] \arrow[rrr, bend right=10, "=:f"'] & M/(M \setminus \Phi(B^n)) \arrow[r, "\overline{\Phi}^{-1}"] & \overline{B}/S^{n-1} \arrow[r, "\Omega"] & S^n. \end{tikzcd}$$

It follows almost immediately from Proposition 69.2 that f is a map of degree $+1$. ■

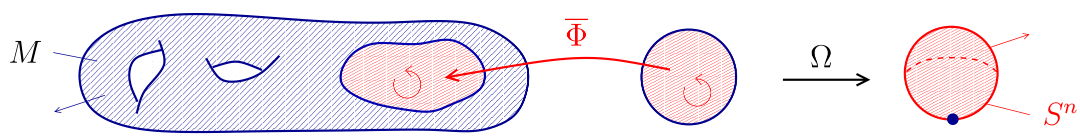


FIGURE 1071. Illustration for the proof of Proposition 69.3.

To put the previous proposition into context we state the following theorem from a later chapter. It says that for maps to S^n the converse to Lemma 69.1 (3) holds.

Theorem ??. (Hopf Theorem) *Let M be a closed oriented connected non-empty n-dimensional smooth manifold and let $f, g\colon M \to S^n$ be two maps. If* $\deg(f) = \deg(g)$*, then the maps f and g are actually homotopic.*

For the record we also state the following open question, see [**Kir97**, Problem 5.26], which was initially posed by Heinz Hopf.

Question 69.4. (Hopf) *Let M be a closed orientable smooth manifold M. Is every map $f\colon M \to M$ of degree ± 1 a homotopy equivalence?*

In the following discussion, given $g \in \mathbb{N}_0$, we denote as usual by Σ_g the surface of genus g. We move on to our next list of interesting examples of maps of no-zero degree.

Lemma 69.5. *Let $k \in \mathbb{N}_0$. For every $g \geq k$ there exists a degree-one map from Σ_g to Σ_k.*

PROOF. We prove the lemma for $g = 2$ and $k = 1$. It should be clear how to modify the discussion for other values of $g \geq k$. Let us consider the map from the surface Σ_2 of genus two to the torus Σ_1 that is shown in Figure 1072. In fact in Figure 1072 we show the map twice, on top we show a slightly informal description of the map, at the bottom we give a more formal description. Note that the hypotheses of Proposition 69.2 are satisfied. Thus we see that $\deg(f) = 1$. ■

The following question arises naturally.

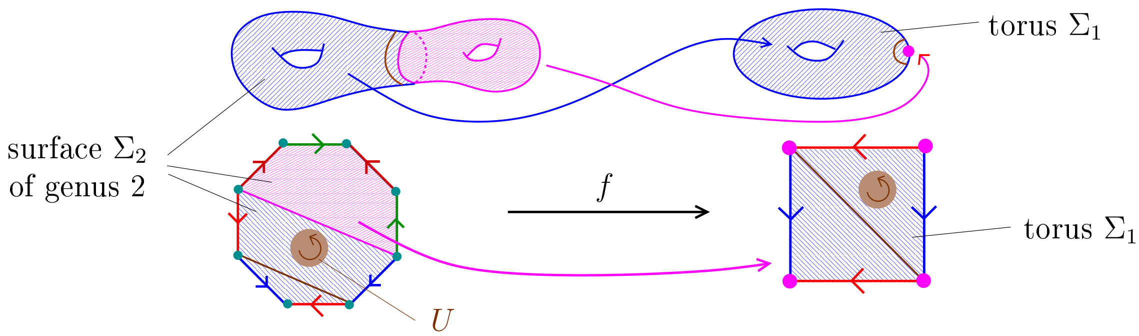


FIGURE 1072

Question 69.6. *Suppose that $g < k$. Does there exist a degree-one map from Σ_g to Σ_k?*

We will be able to give an answer to this question later on in Corollary 69.10.

But first let us move on to the next proposition, which is a generalization of Proposition 69.2.

Proposition 69.7. *Let $f\colon M \to N$ be a map between two compact oriented connected non-empty n-dimensional smooth manifolds such that $f^{-1}(\partial N) = \partial N$.*

(1) *Let $y \in N \setminus \partial N$ such that the following condition is satisfied:*
   (∗) *The preimage $f^{-1}(\{y\})$ consists of finitely many points $x_1, \dots, x_m$ and f is a local diffeomorphism around each x_i.*

   Then
$$\deg(f) \;=\; \sum_{i=1}^{m} \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x_i \\ -1, & \text{if } f \text{ is orientation-reversing at } x_i. \end{cases}$$

(2) *If f is smooth, then every regular value in $N \setminus \partial N$ satisfies (∗).*

(3) *If f is smooth, then the set of regular values in $N \setminus \partial N$ has full measure in N.*[1040]

Remark. In Proposition 87.32 we state a result that generalizes Proposition 69.7 (1) in the sense that we no longer demand that f is a local diffeomorphism around each x_i.

Example. Let M be a compact oriented n-dimensional smooth manifold and let W be the union of some components of ∂M. As on page 1163 we consider the double

$$\mathrm{D}_W\, M \;=\; (M \times \{1\}) \cup_{W\times\{1\}=W\times\{2\}} (M \times \{2\})$$

of M along W. By Lemma 44.12 we know that $\mathrm{D}_W\, M$ is naturally a compact oriented n-dimensional smooth manifold. We consider the folding map

$$\begin{aligned} f\colon \mathrm{D}_W\, M \;&\to\; M \\ [(P,i)] \;&\mapsto\; P. \end{aligned}$$

If we take any $x \in M \setminus \partial M$, then we see that $f^{-1}(P)$ consists of precisely two points, namely $x_1 = [(x,1)]$ and $x_2 = [(x,2)]$. The folding map is an orientation-preserving local

[1040] We refer to page 316 for the definition of a subset of full measure in N and, more usefully, we refer to Proposition 6.62 for the main properties of subsets of full measure.

diffeomorphism around x_1 and it is an orientation-reversing local diffeomorphism around x_2. Thus we obtain from Proposition 69.7 that the degree of the folding map is zero.

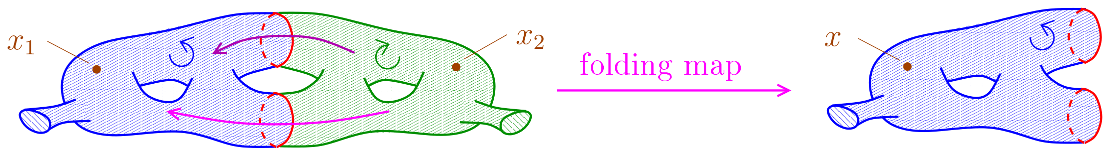


FIGURE 1073

Since the two parts of Proposition 69.7 are of very different nature we break the proof of Proposition 69.7 into two parts.

PROOF OF PROPOSITION 69.7 (1) (∗). Let $f\colon M \to N$ be a map between two compact oriented connected non-empty n-dimensional smooth manifolds such that $f^{-1}(\partial N) = \partial M$. Let $y \in N \setminus \partial N$ such that $f^{-1}(\{y\})$ consists of finitely many points $x_1, \dots, x_m$ and such that f is a local diffeomorphism around each x_i.

For $i = 1, \dots, m$ we pick an open connected neighborhood U_i of x_i such that $f\colon U_i \to N$ is an embedding. Note that we can arrange that for any $i \neq j$ we have $x_j \notin U_i$. For $i = 1, \dots, m$ we set

$$\epsilon_i \;:=\; \begin{cases} +1, & \text{if } f\colon U_i \to N \text{ is orientation-preserving} \\ -1, & \text{if } f\colon U_i \to N \text{ is orientation-reversing.} \end{cases}$$

Note that we need to show that $\deg(f) = \epsilon_1 + \dots + \epsilon_m$.

Let $i \in \{1, \dots, m\}$. Since $y \in N \setminus \partial N$ and since by hypothesis we have $f^{-1}(\partial N) = \partial M$ we know that $x_i \in M \setminus \partial M$. Thus we can pick a smooth embedding $\Psi_i\colon \Delta^n \to U_i$ such that $x_i \in \Psi_i(\mathring{\Delta}^n)$. We consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}_n(M,\partial M) \arrow[rr, "=:\mu"] \arrow[dr, "\oplus\omega_{x_i}"'] \arrow[dd, "f_*"] & & \mathrm{H}_n(M, M\setminus\{x_1,\dots,x_m\}) \arrow[dl, "\cong", "\oplus\nu_i"'] \arrow[dd, "f_*"] \\
& \bigoplus_{i=1}^m \mathrm{H}_n(M, M\setminus\{x_i\}) & \\
\mathrm{H}_n(N,\partial N) \arrow[rr, "\omega_{f(x)}", "\cong"'] & & \mathrm{H}_n(N, N\setminus\{y\}).
\end{tikzcd}$$

We make the following clarifications and observations:

(1) The map $\mu\colon \mathrm{H}_n(M,\partial M) \to \mathrm{H}_n(M, M\setminus\{x_1,\dots,x_m\})$ and furthermore all the maps $\nu_i\colon \mathrm{H}_n(M, M\setminus\{x_1,\dots,x_m\}) \to \mathrm{H}_n(M, M\setminus\{x_i\})$ are induced by the natural maps on the pairs of topological spaces. Note that here we use that $x_1,\dots,x_m \in M\setminus\partial M$, that $y \in N\setminus\partial N$ and that $f(\partial M) \subset \partial N$.

(2) It follows easily from the Excision Theorem 43.20, see also Exercise 44.1, that the diagonal map $\bigoplus_{i=1}^m \nu_i\colon \mathrm{H}_n(M, M\setminus\{x_1,\dots,x_m\}) \to \bigoplus_{i=1}^m \mathrm{H}_n(M, M\setminus\{x_i\})$ is an isomorphism.

(3) Let $i \in \{1,\dots,m\}$. Note that we arranged that $\Psi_i(\partial\Delta^n) \subset M\setminus\{x_1,\dots,x_m\}$. In particular $\Psi_i\colon \Delta^n \to M$ defines an element in $\mathrm{H}_n(M, M\setminus\{x_1,\dots,x_m\})$.

Now we see that

$$
\begin{array}{rrcll}
 & \bigoplus_{i=1}^{m}\omega_{x_i}([M]) & = & \bigoplus_{i=1}^{m}\operatorname{sign}(\Psi_i,x_i)\cdot[\Psi_i] & \text{by definition of } [M] \\
\Rightarrow & \Big(\bigoplus_{i=1}^{m}\omega_{x_i}\Big)([M]) & = & \Big(\bigoplus_{i=1}^{m}\nu_i\Big)\Big(\sum_{j=1}^{m}\operatorname{sign}(\Psi_j,x_j)\cdot[\Psi_j]\Big) & \text{since } \nu_i([\Psi_j])=\delta_{ij}\cdot[\Psi_i] \\
\Rightarrow & \mu([M]) & = & \sum_{j=1}^{m}\operatorname{sign}(\Psi_j,x_j)\cdot[\Psi_j] & \text{since } \oplus\nu_i \text{ is an isomorphism} \\
 & & & & \text{and since the triangle commutes} \\
\Rightarrow & f_*(\mu([M])) & = & \sum_{j=1}^{m}\operatorname{sign}(\Psi_j,x_j)\cdot f_*([\Psi_j]) & \text{applying } f_* \\
\Rightarrow & \omega_{f(x)}(f_*([M])) & = & \sum_{j=1}^{m}\operatorname{sign}(\Psi_j,x_j)\cdot[f\circ\Psi_j] & \text{since the diagram commutes} \\
\Rightarrow & \omega_{f(x)}(f_*([M])) & = & \sum_{j=1}^{m}\epsilon_j\cdot\operatorname{sign}(f\circ\Psi_j,x_j)\cdot[f\circ\Psi_j] & \text{since } \operatorname{sign}(f\circ\Psi_j,x_j)=\epsilon_j\cdot\operatorname{sign}(\Psi_j,x_j) \\
\Rightarrow & \omega_{f(x)}(\deg(f)\cdot[N]) & = & \sum_{j=1}^{m}\epsilon_j\cdot\omega_{f(x)}([N]) & \text{by definition of } \deg(f) \text{ and } [N] \\
\Rightarrow & \deg(f)\cdot[N] & = & \sum_{j=1}^{m}\epsilon_j\cdot[N] & \text{by Theorem 68.1 (1)} \\
\Rightarrow & \deg(f) & = & \sum_{j=1}^{m}\epsilon_j & \text{since } \mathrm{H}_n(N,\partial N)=\mathbb{Z}\cdot[N]. \quad\blacksquare
\end{array}
$$

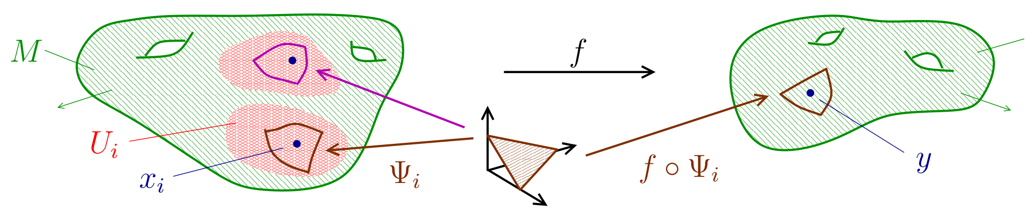


Figure 1074. Illustration for the proof of Proposition 69.7 (1).

Proof of Proposition 69.7 (2) and (3)(∗). Let $f\colon M\to N$ be a smooth map between two compact oriented connected non-empty n-dimensional smooth manifolds M and N such that $f^{-1}(\partial N)=\partial M$.

(2) Let $y\in N\setminus\partial N$ be a regular value. It follows from Lemma 6.54 (2) that $f^{-1}(\{y\})$ consists of finitely many points $x_1,\dots,x_m$. Let $i\in\{1,\dots,m\}$. Note that it follows from $f^{-1}(\partial N)=\partial M$ that $x_i\in M\setminus\partial M$. Since x_i is a regular point we obtain from Lemma 6.54 (1) that f is a local diffeomorphism around x_i.

(3) By Sard's Theorem 6.63 together with Proposition 6.62 we know that the set of regular values of f that are contained in $N\setminus\partial N$ has full measure in N. ■

The following proposition gives us many examples of degrees of maps between smooth manifolds.

Proposition 69.8. *Let $p\colon\widetilde{M}\to M$ be a finite covering of compact oriented connected non-empty smooth manifolds. If p is orientation-preserving, then*

$$\deg(p) \;=\; [\widetilde{M}:M].$$

Proof. It turns out that there are two quick proofs for the proposition. In fact the desired statement follows immediately from Proposition 69.7. Alternatively, if we denote by $p^*\colon \mathrm{H}_n(M,\partial M) \to \mathrm{H}_n(\widetilde{M},\partial\widetilde{M})$ the transfer map that we introduced on page 1445, then we also see that

$$p_*([\widetilde{M}]) \underset{\substack{\uparrow \\ \text{Exercise 68.10}}}{=} p_*(p^*([M])) \underset{\substack{\uparrow \\ \text{Proposition 59.1}}}{=} [\widetilde{M}:M]\cdot[M].$$

■

The following proposition is the key to answering Question 69.6.

Proposition 69.9. *Let $f\colon M \to N$ be a map between compact oriented connected non-empty n-dimensional smooth manifolds such that $f(\partial M) \subset \partial N$. If $\deg(f) = \pm 1$, then $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism.*

The statement of Proposition 69.9 is somewhat surprising, in so far as it takes as input fairly simple information, namely the induced map on n-th homology, which is an abelian group, and gives as output information about fundamental groups, which are usually much harder to deal with. Also it is a priori not clear why the homology in the maximal dimension should have any impact on fundamental groups.

Proof. Let $f\colon M \to N$ be a map between compact oriented connected non-empty n-dimensional smooth manifolds such that $f(\partial M) \subset \partial N$. We fix a base point $P \in M$ and we write $Q = f(P)$. Furthermore we write $\pi = \pi_1(N,Q)$ and $\Gamma = f_*(\pi_1(M,P))$.

By Proposition 29.5[1041] there exists a path-connected covering $p\colon (\widetilde{N},\widetilde{Q}) \to (N,Q)$ such that $p_*(\pi_1(\widetilde{N},\widetilde{Q})) = \Gamma$ and such that $[\widetilde{N}:N] = [\pi:\Gamma]$. By Proposition 17.1 we can equip the covering space $\widetilde{N}$ with the structure of a compact oriented smooth manifold such that the projection $p\colon \widetilde{N} \to N$ is an orientation-preserving map and such that we have $\partial\widetilde{N} = p^{-1}(\partial N)$. Since by design we know that $p_*(\pi_1(\widetilde{N},\widetilde{Q})) = \Gamma = f_*(\pi_1(M,P))$ we obtain from Proposition 29.2 that the map $f\colon (M,P) \to (N,Q)$ lifts to a map $\widetilde{f}\colon (M,P) \to (\widetilde{N},\widetilde{Q})$. We can summarize the situation in the following simple commutative diagram

$$\begin{tikzcd} & \widetilde{N} \arrow[d, "p"', "\text{covering of degree } [\pi:\Gamma]"] \\ M \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & N. \end{tikzcd}$$

First suppose that Γ is a subgroup of π of infinite index. It follows immediately from $[\widetilde{N}:N] = [\pi:\Gamma]$ and Lemma 16.3 (5) that $\widetilde{N}$ is non-compact. Therefore we obtain from Theorem 66.6 that $\mathrm{H}_n(\widetilde{N}) = 0$. But this shows that $f_* = p_* \circ \widetilde{f}_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(N)$ factors through the trivial group. Hence $\deg(f) = 0$. But this is a contradiction to our hypothesis that $\deg(f) = \pm 1$.

So now we know that Γ is necessarily a finite-index subgroup of π. It follows from $[\widetilde{N}:N] = [\pi:\Gamma]$ and Lemma 16.3 (4) that $\widetilde{N}$ is compact. Therefore, by the above we now know that $\widetilde{N}$ is a compact oriented connected n-dimensional smooth manifold. We deduce

[1041] Here we use that according to the discussion on pages 816 any smooth manifold is locally simply connected.

that

$$\pm 1 \;=\; \deg(f) \underset{\substack{\uparrow \\ \text{Lemma 69.1}}}{=} \deg(\widetilde{f})\cdot\deg(p) \underset{\substack{\uparrow \\ \text{Proposition 69.8}}}{=} \deg(\widetilde{f})\cdot[\widetilde{N}:N] \;=\; \deg(\widetilde{f})\cdot[\Gamma\colon\pi].$$

This shows that $[\Gamma\colon \pi]=1$, i.e. $\Gamma=\pi$. ■

Now we can give a negative answer to Question 69.6.

Corollary 69.10. *Let $g,k\in\mathbb{N}_0$. If there exists a degree-one map from Σ_g to Σ_k, then $g\geq k$.*

Proof. Suppose that there exists a degree-one map from Σ_g to Σ_k. By Proposition 69.9 we obtain an epimorphism $\pi_1(\Sigma_g)\to\pi_1(\Sigma_k)$. In particular we obtain an epimorphism from the abelianization $\pi_1(\Sigma_g)$ onto the abelianization of $\pi_1(\Sigma_k)$. By Proposition 22.3 the former is isomorphic to $\mathbb{Z}^{2g}$ and the latter is isomorphic to $\mathbb{Z}^{2k}$. It follows from Lemma 19.6 that $g\geq k$. ■

We conclude this chapter with several questions. Recall that it follows from Lemma 69.5 that there exists a degree-one map $S^2\times S^2\to S^4$. It is natural to ask whether the converse holds. This leads us to the following question which we will answer in Lemma 84.5.

Question 69.11. *Does there exist a degree-one map from the 4-dimensional sphere S^4 to $S^2\times S^2$?*

Later, in Corollary 88.15, we will also address the following question.

Question 69.12. *Does there exist a degree-one map from $S^1\times S^2$ to $\mathbb{RP}^3$?*

It is also interesting to study degrees of self-maps of smooth manifolds. For example note that it follows from Lemmas 45.10 and 45.12 that given any $k\in\mathbb{Z}$ and given any $n\in\mathbb{N}$ there exists a map $f\colon S^n\to S^n$ of degree k. Also note that it follows easily from Proposition 69.8 that the torus $T=S^1\times S^1$ admits a self-map of any degree. This naturally leads us to the question whether similar phenomena hold for surfaces of higher genus.

Question 69.13.

(1) *Let $g\geq 2$. Does there exist a self-map $f\colon\Sigma_g\to\Sigma_g$ of degree -1?*
(2) *Let $g\geq 2$ and let $k\geq 2$. Does there exist a self-map $f\colon\Sigma_g\to\Sigma_g$ of degree k?*
(3) *Does there exist a self-map $f\colon\mathbb{CP}^2\to\mathbb{CP}^2$ of degree -1?*

We will address the first question in Exercise 69.3 and we will address the last two questions later on in Proposition 81.17 and Corollary 90.10.

Finally we conclude this chapter with two conjectures that were formulated by Heinz Hopf in 1931. We refer to **[Hau87, Neum53]** for some more information and partial results.

Conjecture 69.14. (Hopf Conjecture) *Let M be a closed oriented connected smooth manifold and let $f\colon M\to M$ be a map.*

(1) *If $\deg(f)=\pm1$, then $f_*\colon\pi_1(M)\to\pi_1(M)$ is an isomorphism.*
(2) *If $\deg(f)=\pm1$, then f is a homotopy equivalence.*

Remark. Let M be a closed oriented connected smooth manifold and let $f\colon M \to M$ be a map with $\deg(f) = \pm 1$.

(1) By Proposition 69.9 we know that f_* is an epimorphism. Thus the Hopf Conjecture 69.14 (2) leads to the group-theoretic question, under what circumstances is a surjective automorphism of a group also injective. In particular this leads to the definition of a Hopfian group that we gave on page 862.
(2) In Proposition **??** we will see that in general f is not homotopic to a homeomorphism. Thus in general "homotopy equivalence" is the best we can hope for. Note though that if M is in fact aspherical, see page 1344 for the definition, then the Borel Conjecture **??** predicts that f is actually homotopic to a homeomorphism.

Exercises for Chapter 69.

Exercise 69.1. Let M be a compact oriented n-dimensional smooth manifold with non-empty boundary. Show that there exists a map $f\colon M \to \overline{B}^n$ with $f(\partial M) \subset S^{n-1}$ of degree $+1$.
Remark. For $n \geq 2$ one can combine this result with Exercise 45.5 to get maps of arbitrary degree.

Exercise 69.2. Given an example of a map $f\colon M \to N$ between closed oriented connected non-empty n-dimensional smooth manifolds such that $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism, but such that the degree of f is zero.

Exercise 69.3. Given $g \in \mathbb{N}$ we denote by Σ_g the surface of genus g.

(a) Let $g \geq 2$. Does there exist a self-map $f\colon \Sigma_g \to \Sigma_g$ of degree -1?
(b) Let $g \geq 1$ and let $k \geq 2$. Does there exist a self-map $f\colon \Sigma_g \to \Sigma_g$ of degree k?

Exercise 69.4. Given $g \in \mathbb{N}$ we denote by Σ_g the surface of genus g. Show that if there exists a map $f\colon \Sigma_g \to \Sigma_k$ of non-zero degree, then $g \geq k$.

Exercise 69.5. Let $f\colon M \to N$ be a map between two closed connected non-empty n-dimensional smooth manifolds. Furthermore let $p\colon \widetilde{N} \to N$ be a finite covering. As in Lemma 25.16 we consider the pullback

$$\widetilde{M} \;:=\; f^*\widetilde{N} \;=\; \{(x,y) \in M \times \widetilde{N} \,|\, f(x) = p(y)\}$$

together with the following commutative diagram:

$$\begin{tikzcd} \widetilde{M} \arrow[r, "(x,y)\mapsto y"', "=:F"] \arrow[d, "=:q"', "(x,y)\mapsto x"] & \widetilde{N} \arrow[d, "p"] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

Note that by Lemma 25.16 (1) we know that q is again a covering with $\deg(q) = \deg(p)$.

(a) We suppose that M and N are oriented. We equip the smooth manifolds $\widetilde{M}$ and $\widetilde{N}$ with the corresponding orientations given by Proposition 17.1. Show that $\deg(F\colon \widetilde{M} \to \widetilde{N}) = \deg(f\colon M \to N)$.

(b) Suppose that the induced map $f_*\colon\ \mathrm{H}_n(M;\mathbb{F}_2) \to \mathrm{H}_n(N;\mathbb{F}_2)$ is an isomorphism. Show that $F_*\colon\ \mathrm{H}_n(\widetilde{M};\mathbb{F}_2) \to \mathrm{H}_n(\widetilde{N};\mathbb{F}_2)$ is also an isomorphism.

Exercise 69.6. Let M be the torus minus one open disk and let N be S^2 minus three open disks.

(a) Show that there is no degree one map $f\colon M \to N$.
(b) Show that there is no degree one map $g\colon N \to M$.

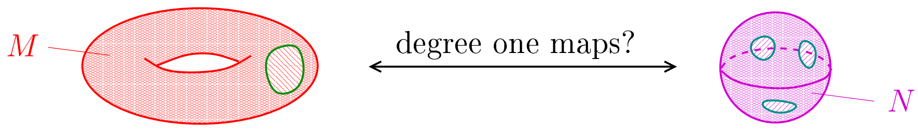


FIGURE 1075. Illustration for Exercise 69.6.

Exercise 69.7. Given $m \in \mathbb{N}$ we denote by N_m the non-orientable surface of genus m, as defined on page 206. Let $k \in \mathbb{N}$.

(a) Let $g \geq k$. Show that there exists a map $f\colon N_g \to N_k$ such that the induced map $\mathrm{H}_2(N_g;\mathbb{F}_2) \to \mathrm{H}_2(N_k;\mathbb{F}_2)$ is an isomorphism.
(b) Let $g < k$. Show that there does not exist a map $f\colon N_g \to N_k$ such that the induced map $\mathrm{H}_2(N_g;\mathbb{F}_2) \to \mathrm{H}_2(N_k;\mathbb{F}_2)$ is an isomorphism.
Remark. Recall that in Exercise 57.14 we saw that $\mathrm{H}_1(N_m;\mathbb{F}_2) \cong \mathbb{F}_2^m$.

70. Homology classes and smooth (sub-)manifolds

In Chapter 68 we saw that every compact oriented n-dimensional smooth manifold N comes naturally with a fundamental class $[N] \in \mathrm{H}_n(N, \partial N)$. In Chapters 68 and 69 we already saw several interesting applications of the fundamental class and we now feel like we have a good understanding of fundamental classes.

In this chapter we want to study to which degree homology classes can be understood in terms of fundamental classes. More precisely, we want to study the following two questions:

(1) Given a topological space X and a homology class $\varphi \in \mathrm{H}_n(X)$, can we always find a closed oriented n-dimensional smooth manifold N and a map $f\colon N \to X$ such that $f_*([N]) = \varphi$?
(2) If X is in fact a smooth manifold, can we find an $f\colon N \to X$ as in (1) that is actually a smooth embedding?

70.1. **Realizing homology classes by smooth manifolds.**

Let us start out with the following friendly definition.

Definition. Let X be a topological space, let $A \subset X$ be a subset and let $\varphi \in \mathrm{H}_n(X, A)$ be a homology class. We say φ *can be realized by a smooth manifold* if there exists a compact oriented n-dimensional smooth manifold N and a map $f\colon (N, \partial N) \to (X, A)$ such that $f_*([N]) = \varphi$.

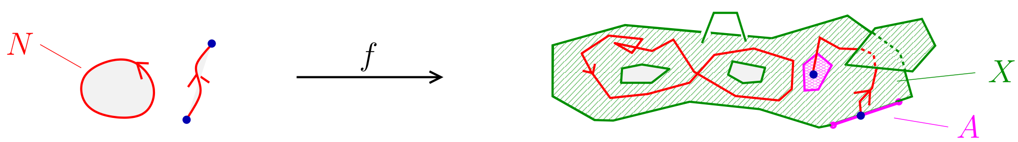


Figure 1076

Examples. Let (X, x_0) be a path-connected pointed topological space and let $n \in \mathbb{N}$. Recall that the Hurewicz homomorphism is defined as the map

$$\begin{array}{rcl} \Phi_{(X,x_0)}\colon \pi_n(X, x_0) & \to & \mathrm{H}_n(X) \\ [f\colon S^n \to X] & \mapsto & f_*([S^n]). \end{array}$$

Thus we see that every homology class that lies in the image of the Hurewicz homomorphism is realized by a smooth manifold. But as we remarked on page 1326, in general the Hurewicz homomorphism is certainly not an epimorphism.

In this and the subsequent section we want to study the following question.

Question 70.1. *Let X be a topological space and let $A \subset X$ be a subset. For which $n \in \mathbb{N}_0$ can homology classes in $\mathrm{H}_n(X, A)$ be realized by smooth manifolds?*

The case $n = 0$ is evidently trivial. So let us continue with the case $n = 1$.

Proposition 70.2. *Let X be a topological space. Every class in $\mathrm{H}_1(X)$ is realized by a smooth manifold.*

PROOF. The proposition is basically an immediate consequence of Proposition 52.3. Nonetheless we prefer to do prove the proposition here "by hand", since this proof will also give us some ideas for how to approach the case of higher-dimensional homology classes.

Thus suppose we are given a homology class $\varphi \in \mathrm{H}_1(X)$. We pick a singular chain $c = a_1 \cdot \sigma_1 + \cdots + a_k \cdot \sigma_k \in \mathrm{C}_k(X)$ that represents φ. Note that we do not necessarily demand that the σ_j are different. In fact note that by "taking multiple copies of the singular simplices σ_j" we can and will arrange that $a_j \in \{-1, 1\}$, $j = 1, \ldots, k$. Next we will see that we can arrange that all $a_j = 1$. To do so, given a singular 1-simplex $\sigma\colon \Delta^1 \to X$ we denote by $\overline{\sigma}\colon \Delta^1 \to X$ the singular 1-simplex that is given by $\overline{\sigma}(t, 1-t) := \sigma(1-t, t)$. In Lemma 41.3 we saw that $-\sigma$ and $\overline{\sigma}$ are homologous. Thus, after replacing each summand $-\sigma_j$ by $\overline{\sigma}_j$ we can assume that all $a_j = 1$.

Next note that we have

$$0 \underset{\substack{\uparrow\\ \text{since } c \text{ is a cycle}}}{=} \partial_1(c) = \partial_1\Big(\sum_{j=1}^{k} \sigma_j\Big) \underset{\substack{\uparrow\\ \text{by definition of } \partial_1\text{, see also page 1079}}}{=} \sum_{j=1}^{k} \sigma_j(0,1) - \sigma_j(1,0) = \sum_{j=1}^{k} \sigma_j(0,1) + \sum_{j=1}^{k} -\sigma_j(1,0).$$

On the right hand side each "positive summand" needs to cancel with a "negative summand". As in the proof of Proposition 52.3 this means, see also Lemma 52.4, that there exists a permutation $\theta \in S_k$ such that for each $j \in \{1, \ldots, k\}$ the "endpoint" $\sigma_j(0,1)$ of σ_j is just the "starting point" $\sigma_{\theta(j)}(1,0)$ of $\sigma_{\theta(j)}$. We introduce the following notation:

(1) We consider the topological space
$$N := (\{1, \ldots, k\} \times \Delta^1)/\sim \quad \text{where for } j \in \{1, \ldots, k\} \text{ we have } \sigma_j(0,1) \sim \sigma_{\theta(j)}(1,0).$$
(2) For $j = 1, \ldots, k$ we denote by $\Psi_j\colon \Delta^1 \xrightarrow{x \mapsto (j,x)} \{1, \ldots, k\} \times \Delta^1$ the j-th inclusion map.
(3) We denote by $p\colon \{1, \ldots, k\} \times \Delta^1 \to N$ the projection map.
(4) It follows immediately from the definitions and Lemma 3.22 that the map
$$\begin{aligned} f\colon N = (\{1, \ldots, k\} \times \Delta^1)/\sim \;&\to\; X \\ [(j, z)] \;&\mapsto\; \sigma_j(z) \end{aligned}$$
is well-defined and continuous.

It is now straightforward to verify that N can be equipped with the structure of a closed 1-dimensional smooth manifold and that we can equip N with an orientation such that for each $j \in \{1, \ldots, k\}$ the map $p \circ \Psi_j\colon \mathring{\Delta}^1 \to N$ is an orientation-preserving embedding. Finally we see that

$$f_*([N]) \underset{\substack{\uparrow\\ \text{follows from Proposition 68.4}\\ \text{applied to any of the maps } p\circ\Psi_j}}{=} f_*([p \circ \Psi_1 + \cdots + p \circ \Psi_k]) \underset{\substack{\uparrow\\ \text{by definition of } f \text{ and } p\circ\Psi_j}}{=} [\sigma_1 + \cdots + \sigma_k] = c.$$

■

Evidently the idea now is to work our way up to higher dimensions. In fact we have the following proposition.

Proposition 70.3. *Let X be a topological space. Every class in $\mathrm{H}_2(X)$ is realized by a smooth manifold.*

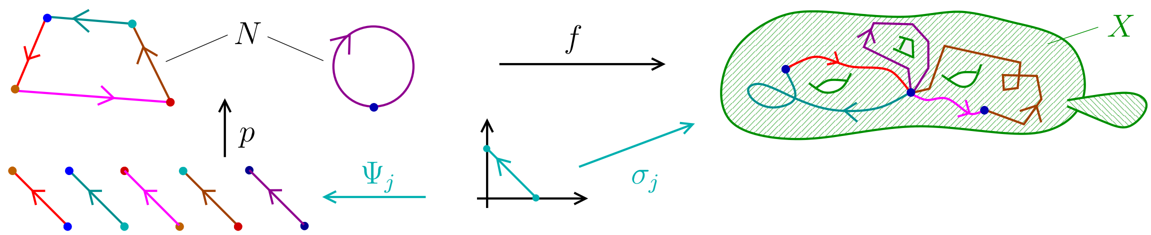


FIGURE 1077. Illustration of the proof of Proposition 70.2.

PROOF. The basic idea of the proof of Proposition 70.3 is similar to the proof of Proposition 70.2. Nonetheless, ironing out the details is much more messy, hence we postpone the proof of the case $n = 2$ to Section 70.2. Alternatively see also [**Cal09**, p. 3] for a quick sketch of the proof. ∎

Remark. In Lemma 70.9 we will see that any naive attempt of generalizing the ideas behind the proofs of Propositions 70.2 and 70.3 to higher dimensions runs into serious problems.

We move on to the following proposition that summarizes a few generalities on realizing homology classes by smooth manifolds.

Proposition 70.4. *Let $k \in \mathbb{N}$.*

(1) *Let X be a topological space and let $A \subset X$ be a subset. We suppose that X is path-connected. If $A = \varnothing$ or if $k \geq 2$, then every homology class in $\mathrm{H}_k(X, A)$ that can be realized by a smooth manifold can also be realized by a* connected *smooth manifold.*
(2) *Suppose that for every topological space X every homology class in $\mathrm{H}_k(X)$ can be realized by a smooth manifold. Then also for every pair (X, A) of topological spaces every relative homology class in $\mathrm{H}_k(X, A)$ can be realized by a smooth manifold.*

PROOF. In order not to disrupt the flow of the conversation too much we postpone the somewhat technical proof of Proposition 70.4 to Section 70.3. ∎

Note that Propositions 70.2 and 70.3 together with Proposition 70.4 give us a complete, and affirmative, answer to Question 70.1 for the dimensions $k = 1$ and $k = 2$.

We move on to the following much more fancy theorem which was first proved by René Thom [**Tho54b, Tho07**] in 1954.[1042] The proof of the theorem uses methods that go well beyond our present abilities. We will not make use of the theorem.

Theorem 70.5. (Thom's Theorem I)

(1) *If X is a topological space, then every homology class of dimension ≤ 6 can be realized by a smooth manifold.*
(2) *Given any $n \geq 7$ there exists a topological space X and a homology class in $\mathrm{H}_n(X)$ that is not realized by a smooth manifold.*
(3) *Given any $n \in \mathbb{N}_0$ there exists a $k \in \mathbb{N}$ such that for any topological space X and any homology class $\sigma \in \mathrm{H}_n(X)$ the class $k \cdot \sigma$ can be realized by a smooth manifold.*

[1042] René Thom (1923-2002) was a French mathematician who received the Fields Medal in 1958.

Proof. As mentioned before, the theorem is proved in [**Tho54b**, Theorem III.3, III.4 and III.9]. A more modern account of the proof is given in Theorems IV.7.35, IV.7.36, IV.7.37 and IV.7.38 together with Remark IV.7.40 of [**Rudy98**]. In fact the topological space X in (2) is given by the $(n-7)$-fold suspension of the Eilenberg-Maclane space $K(\mathbb{Z}_3, 2)$, which we will introduce later in Chapter 120. ■

For completeness we also mention $\mathbb{F}_2$-coefficients.

Theorem 70.6. (Thom's Theorem II) *Let X be a path-connected topological space and let $A \subset X$ be a subset. Given any $\varphi \in \mathrm{H}_n(X, A; \mathbb{F}_2)$ there exists a compact connected n-dimensional smooth manifold N and a map $f\colon N \to X$ with $f(\partial N) \subset A$ such that $f_*([N]_{\mathbb{F}_2}) = \varphi$.*

Proof. First let us consider the case $A = \varnothing$ and for the time being let us allow possibly disconnected smooth manifolds. In this setting, for $n = 1$ and $n = 2$ the statement follows from straightforward modifications of the proofs of Propositions 70.2 and 70.3. In fact, much more interestingly, the statement now holds in all dimensions by the above work of René Thom [**Tho54b**, Theorem III.2], see also [**Rudy98**, Theorem IV.7.33]. The statements about relative homology and connectedness follow from the obvious variations on Proposition 70.4. ■

We conclude this section with the following remark.

Remark. In Chapter 86 we will introduce the notion of an orientation of a topological manifold. Furthermore in Chapter 87 we will introduce the notion of the fundamental class $[N] \in \mathrm{H}_n(N, \partial N)$ of a compact oriented n-dimensional topological manifold N. It now makes sense to ask whether a homology class can be realized by a topological manifold. By [**Rudy98**, Theorem IV.7.35 and Remark IV.3.740] there exist homology classes in any dimension ≥ 7 that cannot by realized by topological manifolds. But by [**Rudy98**, Theorem IV.7.38] there do exist homology classes that can be realized by topological manifolds but which cannot be realized by smooth manifolds.

70.2. **Proof of Proposition 70.3.** In this section we prove the following proposition from the last section.

Proposition 70.3. *Let X be a topological space. Every class in $\mathrm{H}_2(X)$ is realized by a smooth manifold.*

The proof of Proposition 70.3 is modelled on the proof of Proposition 70.2. But, as we will see, the proof is technically more difficult.

We start out our proof with the more general setting of some n-dimensional class. Later on we will specialize to the case that $n = 2$. It is even worthwhile keeping the case $n = 1$ in mind, since even for $n = 1$ the argument below differs slightly from the argument provided in the proof of Proposition 70.2.

Thus let X be a topological space, let $n \in \mathbb{N}$ and let $c = a_1 \cdot \sigma_1 + \cdots + a_k \cdot \sigma_k$ be an n-dimensional singular chain in X. We do not necessarily demand that the σ_j are different.

In fact note that by "taking multiple copies of the singular simplices σ_j" we can and will arrange that $a_j \in \{-1, 1\}$, $j = 1, \dots, k$.[1043] We set $T := \{1, \dots, k\} \times \{0, \dots, n\}$.

Now assume that c is a cycle. We see that we have the following equality in the free abelian group $C_{n-1}(X)$:

$$0 \underset{\substack{\uparrow \\ \text{since } c \text{ is a cycle}}}{=} \partial_1(c) = \partial_1\Big(\sum_{j=1}^{k} a_j \cdot \sigma_j\Big) = \sum_{j=1}^{k}\sum_{r=0}^{n} (-1)^r \cdot a_j \cdot \sigma_j \circ \underset{\substack{\uparrow \\ \text{here } i_r^n\colon \Delta^{n-1} \to \Delta^n \text{ is the } r\text{-th face map}}}{i_r^n} = \sum_{(j,r)\in T} (-1)^r \cdot a_j \cdot \sigma_j \circ i_r^n.$$

It follows from Lemma 52.4 that there exists a decomposition $T = T_1 \sqcup T_2$ into two disjoint subsets together with a bijection

$$\begin{array}{rcl} \Theta = (\theta_1, \theta_2)\colon T_1 & \to & T_2 \\ (j, r) & \mapsto & \Theta(j, r) = (\theta_1(j, r), \theta_2(j, r)) \end{array}$$

such that the following condition is satisfied: For every $(j, r) \in T_1$ we have

$$\begin{array}{rrcl} & \sigma_j \circ i_r^n & = & \sigma_{\theta_1(j,r)} \circ i^n_{\theta_2(j,r)} \\ \text{and} & (-1)^r \cdot a_j & = & -(-1)^{\theta_2(j,r)} \cdot a_{\theta_1(j,r)}. \end{array}$$

Next we introduce some notation:

(1) We consider the topological space

$$Y(\Theta) := (\{1, \dots, k\} \times \Delta^n)/\sim \quad \begin{array}{l} \text{where for each } (j, r) \in T_1 \text{ and each } P \in \Delta^{n-1} \\ \text{we have } (j, i_r^n(P)) \sim (\theta_1(j, r), i^n_{\theta_2(j,r)}(P)). \end{array}$$

Informally speaking $Y(\Theta)$ is given by taking one n-simplex for each singular simplex σ_j and gluing these simplices along $(n-1)$-dimensional faces.[1044]

(2) For $j = 1, \dots, k$ we denote by $\Psi_j\colon \Delta^n \xrightarrow{x \mapsto (j,x)} \{1, \dots, k\} \times \Delta^n$ the j-th inclusion map.

(3) We denote by $p\colon \{1, \dots, k\} \times \Delta^n \to Y(\Theta)$ the projection map.

(4) It follows immediately from the definitions and Lemma 3.22 that the map

$$\begin{array}{rcl} f\colon Y(\Theta) = (\{1, \dots, k\} \times \Delta^n)/\sim & \to & X \\ {[(j, P)]} & \mapsto & \sigma_j(P) \end{array}$$

is well-defined and continuous.

(5) We set

$$\mu := \sum_{j=1}^{k} a_j \cdot p \circ \Psi_j \in C_n(Y(\Theta)).$$

We can now formulate the following lemma.

Lemma 70.7.

(1) *$\mu \in C_n(Y(\Theta))$ is a cycle.*

[1043] Note that if n is even, then it follows from Exercise 41.10 that we cannot arrange that all a_j are of the same sign. This is one, but not the main, reason why the proof of Proposition 70.3 is more difficult than the proof of Proposition 70.2.

[1044] It might seem that $Y(\Theta)$ has an "obvious simplicial structure", but that is not necessarily the case, for example after gluing there might be two n-dimensional simplices that intersect in more than just one $(n-1)$-dimensional face.

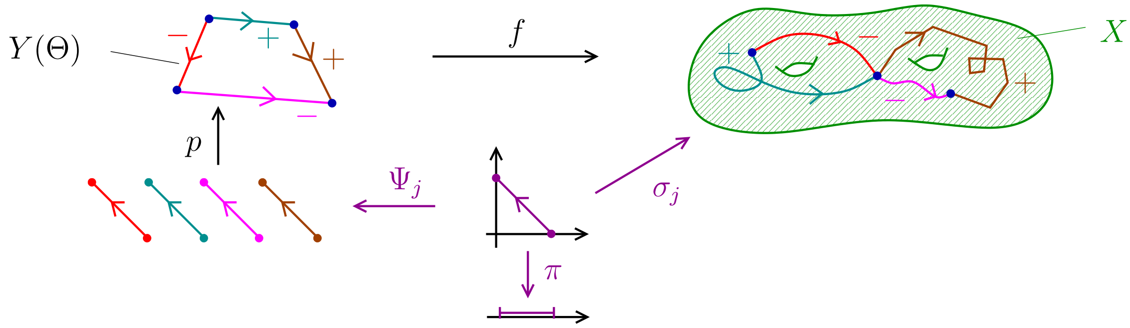


FIGURE 1078

(2) $f_*([\mu]) = [c] \in \mathrm{H}_n(X)$.

PROOF. First we show that μ is a cycle. This follows from the following straightforward calculation:

$$\begin{aligned}
\partial\mu \;&=\; \partial\Big(\sum_{j=1}^{k} a_j \cdot p\circ\Psi_j\Big) \;=\; \sum_{j=1}^{k}\sum_{r=0}^{n} (-1)^r \cdot a_j \cdot p\circ\Psi_j\circ i_r^n \;=\; \sum_{(j,r)\in T} (-1)^r \cdot a_j \cdot p\circ\Psi_j\circ i_r^n \\
&\underset{\substack{\uparrow \\ \text{since } T = T_1 \sqcup T_2 \text{ and since } \Theta=(\theta_1,\theta_2)\colon T_1\to T_2 \text{ is a bijection}}}{=} \sum_{(j,r)\in T_1} (-1)^r \cdot a_j \cdot p\circ\Psi_j\circ i_r^n + \sum_{(j,r)\in T_1} (-1)^{\theta_2(j,r)} \cdot a_{\theta_1(j,r)} \cdot p\circ\Psi_{\theta_1(j,r)}\circ i^n_{\theta_2(j,r)} \\
&= \sum_{(j,r)\in T_1} \underbrace{(-1)^r \cdot a_j \cdot p\circ\Psi_j\circ i_r^n + (-1)^{\theta_2(j,r)} \cdot a_{\theta_1(j,r)} \cdot p\circ\Psi_{\theta_1(j,r)}\circ i^n_{\theta_2(j,r)}}_{=0 \text{ since } (-1)^r\cdot a_j = -(-1)^{\theta_2(j,r)}\cdot a_{\theta_1(j,r)} \text{ and } p\circ\Psi_j\circ i_r^n = p\circ\Psi_{\theta_1(j,r)}\circ i^n_{\theta_2(j,r)}\colon \Delta^{n-1}\to Y(\Theta)} \;=\; 0.
\end{aligned}$$

It follows immediately from the definitions that $f_*(\mu) = c \in \mathrm{C}_n(X)$. In particular we have $f_*([\mu]) = [c] \in \mathrm{H}_n(X)$. ■

We move on to the next lemma, which together with Lemma 70.7 provides the long-desired proof of Proposition 70.3. Note that, as a bonus, we also obtain a new proof of Proposition 70.2.

Lemma 70.8. *We continue with the above notation. If $n = 1$ or if $n = 2$, then the topological space $Y(\Theta)$ admits the structure of a closed oriented 2-dimensional smooth manifold such that $[\mu] \in \mathrm{H}_2(Y(\Theta))$ is the corresponding fundamental class.*

Remark. In Lemma 70.9 below we will see that for $n = 3$ the topological space $Y(\Theta)$ is in general not a topological manifold.

SKETCH OF A PROOF OF LEMMA 70.8. The proof of Lemma 70.8 for the case $n = 1$ is very elementary and thus left to the reader.

In the following we sketch a proof of Lemma 70.8 for the case $n = 2$. As we will see, the proof is quite similar to the proof, given in Propositions 6.8 and 6.21, that the surface Σ_g of genus g is indeed a closed orientable 2-dimensional smooth manifold.

First note that it follows easily from Lemma 3.21 (4) that $Y(\Theta)$ is compact. We leave it to the reader to verify, say using Lemma 3.26, that $Y(\Theta)$ is Hausdorff.

Next we construct a finite atlas for $Y(\Theta)$. To do so we introduce the following notation:

- Let $\pi\colon \Delta^2 \to ◺_2$ be the projection map. Note that in Lemma 41.1 we saw that π is a homeomorphism.
- We denote by $v_0 = (1,0,0)$, $v_1 = (0,1,0)$ and $v_2 = (0,0,1)$ the three vertices of Δ^2.
- Let $j \in \{0,1,2\}$. In the following will define two open subsets X_j and Y_j of Δ^2. In the subsequent definition we denote by w, w' the two vertices that are not equal to v_j. Now we introduce the following notation:
  - ▷ We denote by X_j the union of the open interval from w to w' and the open triangle in Δ^2 spanned by w, w' and the barycenter $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ of Δ^2.
  - ▷ We denote by Y_j the intersection of $\Delta^2 \subset \mathbb{R}^3$ with the open ball in $\mathbb{R}^3$ of radius $\frac{1}{2}$ around the j-th vertex v_j.

  We refer to Figures 1079 and 1080 for an illustration of the above definitions.

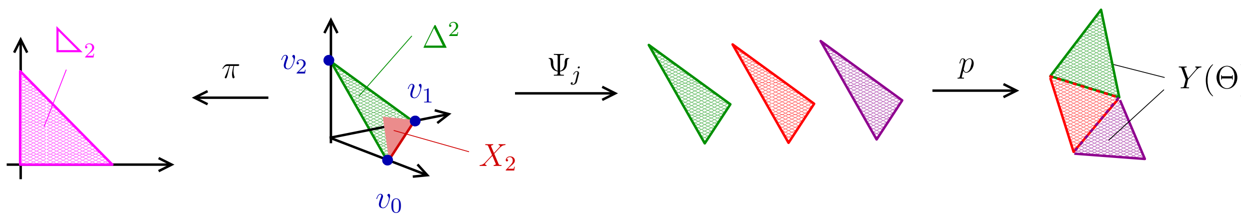


Figure 1079

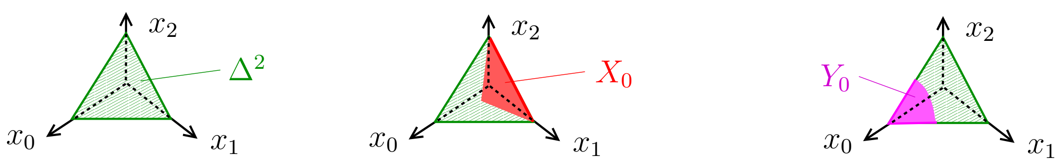


Figure 1080

Next we sketch the construction of three types of charts for the topological space $Y(\Theta)$. We leave it to the reader to fill in the details in the discussions below.

(1) For each $j \in \{1,\dots,k\}$ we let $\alpha_j := \pi \circ \Psi_j^{-1} \circ p^{-1}\colon U_j := p(\mathring{\Delta}_j) \to V_j := \mathring{◺}_2 \subset \mathbb{R}^2$ be the "obvious" chart.

(2) Let $(j,r) \in T_1$. We write $\Theta(j,r) =: (j', r')$. Similar to the proof of Proposition 6.8 one can easily write down an explicit chart

$$\beta_{j,r}\colon U_{j,r} := \underbrace{p(\Psi_i(X_r)) \cup p(\Psi_{j'}(X_{r'}))}_{\subset Y(\Theta)} \;\to\; V_{j,r} \subset \mathbb{R}^2$$

such that the maps $\beta_{j,r} \circ p \circ \Psi_j^{-1} \circ \pi^{-1}\colon ◺_2 \to \mathbb{R}^2$ and $\beta_{j,r} \circ p \circ \Psi_{j'}^{-1} \circ \pi^{-1}\colon ◺_2 \to \mathbb{R}^2$ are both affine linear and such that they are both orientation-preserving if and only if $(-1)^r \cdot a_j = 1$ and such that they are both orientation-reversing if and only if $(-1)^r \cdot a_j = -1$.

(3) Let $j \in \{1,\dots,k\}$ and let $r \in \{0,1,2\}$. An open neighborhood of $p(\Psi_j(v_r))$ is given by

$$\widetilde{U}_{j,r} \;:=\; \bigcup_{\substack{(j',r') \in T \text{ such that} \\ p(\Psi_{j'}(v_{r'})) = p(\Psi_j(v_r))}} p(\Psi_{j'}(Y_{r'})).$$

Again similar to the proof of Propositions 6.8 one can now write down an explicit chart $\gamma_{j,r}\colon \widetilde{U}_{j,r} \to B^2_{\frac{1}{2}}$ such that on each $p(\Psi_r(Y,s))$ the map is a concatenation of π, $(p\circ\Psi_r)^{-1}$, translation, rotation and a map of the form $e^{\mathrm{i}t} \mapsto e^{\mathrm{i}wt}$ for some $w \in \mathbb{R}_{\geq 0}$. In fact we can take $w = \frac{2\pi}{n}$ where $n \in \mathbb{N}$ is the number of terms (j',r') that appear in the above definition of $\widetilde{U}_{j,r}$.

We leave it to the reader to verify that all of these maps are homeomorphisms and that they form an atlas for $Y(\Theta)$. Note that by Lemma 6.4 (2) we now also know that $Y(\Theta)$ is second-countable. We have thus verified that $Y(\Theta)$ is a closed 2-dimensional topological manifold.

Next note that, as in the proof of Proposition 6.21, one can now verify easily that the transition maps between the given charts are indeed smooth and that they are orientation-preserving. Not surprisingly, once again we leave the details to the reader. Thus in summary we obtain from the above discussion, together with Lemmas 6.4 and 6.45, that $Y(\Theta)$ is a smooth manifold that is endowed with an orientation such that all charts are orientation-preserving.

Recall that in Lemma 70.7 we showed that $\mu = a_1\cdot p\circ\Psi_1 + \dots + a_k\cdot p\circ\Psi_k \in C_2(Y(\Theta))$ is a cycle. It follows almost immediately from Proposition 68.4 that μ represents the fundamental class of the closed oriented smooth manifold $Y(\Theta)$. ■

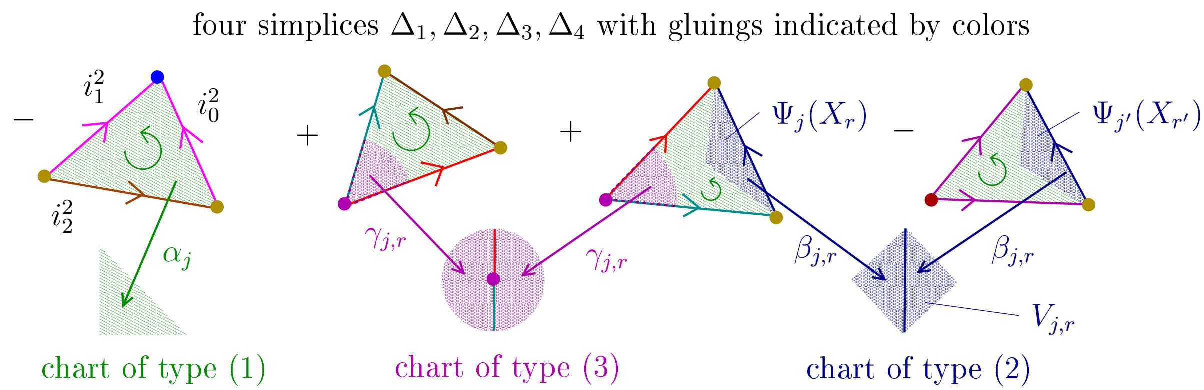


FIGURE 1081. Illustration for the proof of Lemma 70.8.

We conclude this section with the following lemma which shows that the above approach will not work in higher dimensions.

Lemma 70.9. *Given any $n \in \mathbb{N}_{\geq 3}$ there exists a topological space X together with a cycle $c = a_1\cdot\sigma_1 + \dots + a_k\cdot\sigma_k \in C_n(X)$ such that for any choice of Θ the topological space $Y(\Theta)$ is not a closed topological manifold, let alone a closed smooth manifold.*

Sketch of proof. Let $n \in \mathbb{N}_{\geq 3}$ and let M be the $(n-1)$-dimensional torus. Furthermore let $(K, \Xi\colon |K| \to M)$ be a smooth simplicial structure for M, which we know exists by Theorem 64.2, or by some more down-to-earth argument.

By Propositions 66.1 and 66.7 we know that K is a closed orientable $(n-1)$-dimensional pseudomanifold. Next we consider the suspension $\Sigma(K)$, as defined on page 1506. Note that by Corollary 61.21 we know that $X := |\Sigma(K)|$ is homeomorphic to $\Sigma(M)$.

Next recall that by the fairly straightforward Exercise 66.1 we know that $\Sigma(K)$ is a closed n-dimensional pseudomanifold that admits an orientation $\{(\epsilon_1, Y_1), \dots, (\epsilon_k, Y_k)\}$. Now let $\varphi = \epsilon_1 \cdot Y_1 + \dots + \epsilon_k \cdot Y_k \in \mathrm{C}_n^{\mathrm{simp}}(\Sigma(K))$ be the cycle from Theorem 66.4. Next note that after picking some total order on the vertex set of $\Sigma(K)$ we obtain from Lemmas 63.8 and Theorem 63.25 a corresponding singular cycle $c = a_1 \cdot \sigma_1 + \dots + a_k \cdot \sigma_k \in \mathrm{C}_n(X) = \mathrm{C}_n(|\Sigma(K)|)$. We set $T := \{1, \dots, k\} \times \{0, \dots, n\}$. We leave it to the dedicated reader to verify that there is essentially only one choice of a decomposition $T = T_1 \sqcup T_2$ and a bijection $\Theta = (\theta_1, \theta_2)\colon T_1 \to T_2$ such that the condition from page 1752 holds, i.e. such that for every $(j, r) \in T_1$ we have

$$\begin{array}{rrcl} & \sigma_j \circ i_r^n & = & \sigma_{\theta_1(j,r)} \circ i_{\theta_2(j,r)}^n \\ \text{and} & (-1)^r \cdot a_j & = & -(-1)^{\theta_2(j,r)} \cdot a_{\theta_1(j,r)}. \end{array}$$

Furthermore we leave it to the reader to verify that $Y(\Theta)$ is actually homeomorphic to $|\Sigma(K)| \cong \Sigma(M)$.

Finally we point out that it follows from Exercise 45.8, together with the fact that $\mathrm{H}_1(M) = \mathrm{H}_1((S^1)^n) \neq 0$ and together with our hypothesis $n \geq 3$ that $X = |\Sigma(K)| \cong \Sigma(M)$ is not a closed topological manifold. ■

70.3. **Proof of Proposition 70.4.** The sole goal of this section is to prove Proposition 70.4. For convenience we prove the two parts of the proposition separately.

Proposition 70.4.

(1) *Let $k \in \mathbb{N}$, let X be a topological space and let $A \subset X$ be a subset. We suppose that X is path-connected. If $A = \varnothing$ or if $k \geq 2$, then every homology class in $\mathrm{H}_k(X, A)$ that can be realized by a smooth manifold can also be realized by a* connected *smooth manifold.*

Proof of Proposition 70.4 (1). To simplify the discussion we only consider the case that $A = \varnothing$. The case $A \neq \varnothing$ is proved in a completely analogous fashion, the only difference is that the notation becomes even more messy. We will point out though explicitly where we need that $k \geq 2$ for $A \neq \varnothing$.

Let X be a path-connected topological space and let $\theta \in \mathrm{H}_k(X)$ be a homology class. Suppose there exists a closed oriented k-dimensional smooth manifold W and a map $f\colon W \to X$ with $f_*([W]) = \theta$. If W is connected, then there is nothing to show. In the following we will deal with the case that W has precisely two components M and N. The general case is dealt with the same way.

We make the following preparations.

(1) As usual we denote by $\overline{B}_3^k \subset \mathbb{R}^k$ the closed ball of radius 3 around the origin. We pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}_3^k \to M$ and we pick an orientation-reversing smooth embedding $\psi\colon \overline{B}_3^k \to N$. We write $P := \varphi(0)$ and $Q := \psi(0)$.

(2) We use the restrictions of φ and ψ to $\overline{B}^k$ to form, following the definition on page 377, the connected sum $M\#N$, i.e. we consider

$$M\#N \;=\; ((M \setminus \varphi(B^k)) \sqcup (N \setminus \psi(B^k)))/\sim \qquad \text{where } \varphi(z) \sim \psi(z) \text{ for } z \in S^{k-1}.$$

(3) We recall a few statements from Proposition 8.35:
   (a) The connected sum $M\#N$ is a closed k-dimensional smooth manifold which can be oriented in such a way that the two natural inclusions $M \setminus \varphi(B^k) \to M\#N$ and $N \setminus \psi(B^k) \to M\#N$ are both orientation-preserving smooth embeddings.
   (b) If $k \geq 2$ or if at least one of M or N is closed, then $M\#N$ is connected.
   Note though that for $A \neq \varnothing$ we have to deal with the connected sum of compact manifolds M and N that are not necessarily closed. So here the argument breaks down for $A \neq \varnothing$ and $k = 1$, since the connected sum of two copies of $\overline{B}^1$ is actually not connected.

(4) We consider the topological space $Z := (M \sqcup [-1,1] \sqcup N)/\sim$ where we make the identifications $P \sim -1$ and $1 \sim Q$.

(5) We denote by $i\colon M \to Z$ and $j\colon N \to Z$ the obvious maps. It is easy to see that they are in fact embeddings. Next we denote by $p\colon Z \to M$ the map that is given by the identity on M and which sends all other points to P. Similarly we define $q\colon Z \to N$.

(6) We consider the following maps:

$$\begin{tikzcd}[column sep=small]
 & & \mathrm{H}_k(M) \oplus \mathrm{H}_k(N) \arrow[dll, "\text{isomorphism since inclusions are homotopy equivalences}"'] \arrow[d, "i_*\oplus j_*"] \\
\mathrm{H}_k((M\sqcup[-1,1))/\sim)\oplus\mathrm{H}_k((N\sqcup(-1,1])/\sim) \arrow[rr, "\text{isomorphism by the}", "\text{Mayer–Vietoris Theorem 46.5}"'] & & \mathrm{H}_k(Z).
\end{tikzcd}$$

It follows that the right diagonal map $i_* \oplus j_*\colon \mathrm{H}_k(M) \oplus \mathrm{H}_k(N) \to \mathrm{H}_k(Z)$ is an isomorphism. It is now straightforward to see that the inverse of $i_* \oplus j_*$ is given by $p_* \oplus q_*\colon \mathrm{H}_k(Z) \to \mathrm{H}_k(M) \oplus \mathrm{H}_k(N)$.

(7) We consider the map

$$\begin{array}{rcl}
\Theta\colon M\#N & \to & Z \\
x & \mapsto & \begin{cases} [x], & \text{if } x \in M \setminus \varphi(B_3^k), \\ [\varphi(3(r-2)\cdot y)], & \text{if } x = \varphi(r\cdot y) \text{ with } r \in [2,3] \text{ and } y \in S^{k-1}, \\ [-r+1], & \text{if } x = \varphi(r\cdot y) \text{ with } r \in [1,2] \text{ and } y \in S^{k-1}, \\ [r-1], & \text{if } x = \psi(r\cdot y) \text{ with } r \in [1,2] \text{ and } y \in S^{k-1}, \\ [\psi(3(r-2)\cdot y)], & \text{if } x = \psi(r\cdot y) \text{ with } r \in [2,3] \text{ and } y \in S^{k-1}, \\ [x], & \text{if } x \in N \setminus \psi(B_3^k). \end{cases}
\end{array}$$

It follows easily from Lemma 2.35 (2) that $\Theta\colon M\#N \to Z$ is actually continuous.

(8) It follows almost immediately from Proposition 69.2 together with (3a) that the maps $p \circ \Theta \colon M\#N \to M$ and $q \circ \Theta \colon M\#N \to N$ both have degree one. In other words, we have the equality

$$(p_* \oplus q_*)(\Theta_*([M\#N])) \;=\; [M] + [N] \;\in\; \mathrm{H}_k(M) \oplus \mathrm{H}_k(N) = \mathrm{H}_k(M \sqcup N) = \mathrm{H}_k(W).$$

(9) By hypothesis the topological space X is path-connected. This implies that there exists a path $\gamma \colon [-1,1] \to X$ with $\gamma(-1) = P$ and $\gamma(1) = Q$. Using γ we can extend the map $f \colon W = M \sqcup N \to X$ to a map $g \colon Z \to X$ in the obvious way.

By (3) the connected sum $M\#N$ is a closed oriented *connected* smooth manifold. Thus it remains to prove the following claim.

Claim. We have $(g \circ \Theta)_*([M\#N]) = \theta \in \mathrm{H}_k(X)$.

It follows from the above that we have the following diagram:

$$\begin{tikzcd}
 & \mathrm{H}_k(M) \oplus \mathrm{H}_k(N) \arrow[r, "="] \arrow[d, shift left, "i_* \circ j_*", "\cong"'] \arrow[dr, "f_*"] & \mathrm{H}_k(M \sqcup N) = \mathrm{H}_k(W) \arrow[d, "f_*"] \\
\mathrm{H}_k(M\#N) \arrow[r, "\Theta_*"] & \mathrm{H}_k(Z) \arrow[r, "g_*"] \arrow[u, shift left, "p_* \oplus q_*", "\cong"'] & \mathrm{H}_k(X).
\end{tikzcd}$$

It follows from the definition of g together with (6) that the diagram commutes. Now we see that we have the following equalities in $\mathrm{H}_k(X)$:

$$(g \circ \Theta)_*([M\#N]) \;\underset{\substack{\uparrow \\ \text{by (8) and (6)}}}{=}\; g_*(i_*([M]) + j_*([N])) \;\underset{\substack{\uparrow \\ \text{since the diagram commutes}}}{=}\; f_*([M]) + f_*([N]) \;=\; f_*(\underbrace{[M \sqcup N]}_{=[W]}) = \theta.$$

■

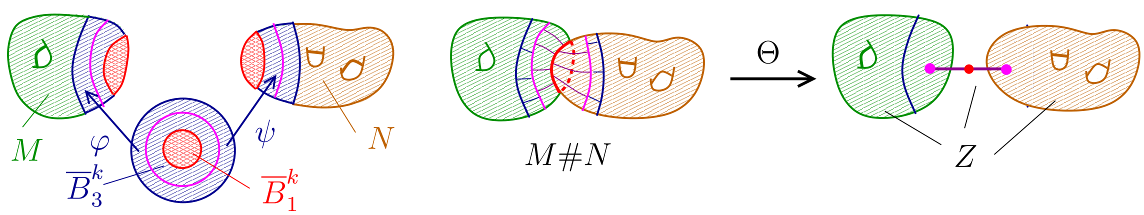


FIGURE 1082. Illustration for the proof of Proposition 70.4 (1).

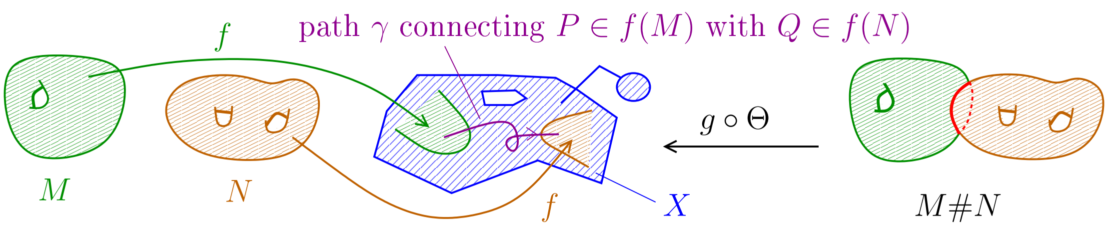


FIGURE 1083. Second Illustration for the proof of Proposition 70.4 (1).

We move on to the second statement of Proposition 70.4.

Proposition 70.4.

(2) *Let $k \in \mathbb{N}$. Suppose that for every topological space X every homology class in $\mathrm{H}_k(X)$ can be realized by a smooth manifold. Then also for every pair (X, A) of*

topological spaces every relative homology class in $\mathrm{H}_k(X,A)$ can be realized by a smooth manifold.

PROOF. Let $k \in \mathbb{N}$. We make the following hypothesis:

($*$) For every topological space Y every homology class in $\mathrm{H}_k(Y)$ can be realized by a smooth manifold.

Now let (X,A) be a pair of topological spaces and let $\varphi \in \mathrm{H}_k(X,A)$ be a relative homology class. We denote by $i\colon A \to X$ the inclusion map. We consider the mapping cone

$$\mathrm{Cone}(i\colon A \to X) \;=\; (\underbrace{\mathrm{Cone}(A)}_{=(A\times[0,1])/A\times\{0\}} \sqcup X)/\sim \;\text{ where } [(a,1)] \sim i(a) = a \text{ for } a \in A.$$

We denote by $*[A \times \{0\}] \in \mathrm{Cone}(i\colon A \to X)$ the cone point. We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_k(M,\partial M) \arrow[r, "\cong"] \arrow[dr, "g_*"] \arrow[ddr, "g_*"'] \arrow[dd, "(r\circ g)_*"] & \mathrm{H}_k(W,N) \arrow[d, "g_*"] & \mathrm{H}_k(W) \arrow[l, "\text{я}"'] \arrow[dd, "g_*"] \\
& \color{red}{\mathrm{H}_k(\mathrm{Cone}(i\colon A \to X), \mathrm{Cone}(A))} & \\
\mathrm{H}_k(X,A) \arrow[r, "\cong"] & \mathrm{H}_k(\mathrm{Cone}(i\colon A \to X)\setminus\{*\}, \mathrm{Cone}(A)\setminus\{*\}) \arrow[u, "\cong"] \arrow[l, bend left, "\cong"', "r_*"] & \mathrm{H}_k(\mathrm{Cone}(i\colon A \to X)). \arrow[ul, "\text{я}", "\cong"']
\end{tikzcd}$$

We make the following explanations, clarifications and observations:

(1) By Lemma 24.9 (2) and (3) we can and will view $\mathrm{Cone}(A)$ and X as subsets of $\mathrm{Cone}(i\colon A \to X)$.
(2) All maps that are not specified explicitly are inclusion induced maps.
(3) The maps я are the natural maps from absolute to relative homology.
(4) Note that by Lemma 24.9 (6) there exists a natural deformation retraction from $\mathrm{Cone}(i\colon A \to X) \setminus \{*\}$ to X. We denote by $r\colon \mathrm{Cone}(i\colon A \to X) \setminus \{*\} \to X$ the corresponding retraction. Note that it restricts to a retraction from $\mathrm{Cone}(A) \setminus \{*\}$ to A.
(5) By the proof of Lemma 46.16 we know that the four maps at the bottom, shown in blue, are isomorphisms.
(6) By (5) together with our hypothesis ($*$) we know that there exists a closed oriented k-dimensional smooth manifold W and a map $g\colon W \to \mathrm{Cone}(i\colon A \to X)$ such that я$(g_*([W]))$ equals the image of our given class φ in the central red group $\mathrm{H}_k(\mathrm{Cone}(i\colon A \to X), \mathrm{Cone}(A))$.
(7) Since $\mathrm{Cone}(i\colon A \to X) \setminus \{*\})$ and $\mathrm{Cone}(A) \setminus A$ are open subsets that cover $\mathrm{Cone}(i)$ we obtain from Exercise 6.36 that there exist two compact codimension-zero submanifolds M and N of W such that $M \cup N = W$, $M \cap N = \partial M = \partial N$, such that $M \subset g^{-1}(\mathrm{Cone}(i\colon A \to X) \setminus \{*\})$ and such that $N \subset g^{-1}(\mathrm{Cone}(A) \setminus A)$.

(8) By (7) we have $g(N) \subset \mathrm{Cone}(A)$ and $g(\partial M) \subset \mathrm{Cone}(A)$. This implies that the two maps in relative homology denoted by g_*, namely the diagonal map to the left and the vertical map in the middle, actually do exist.

(9) By Lemma 68.12 (2) we know that the images of $[M]$ and $[W]$ in $\mathrm{H}_k(W, N)$ agree.

It is clear that the diagram commutes. It is now straightforward to verify that the compact oriented n-dimensional smooth manifold M together with the map $r \circ g\colon (M, \partial M) \to (X, A)$ has all the desired properties. ■

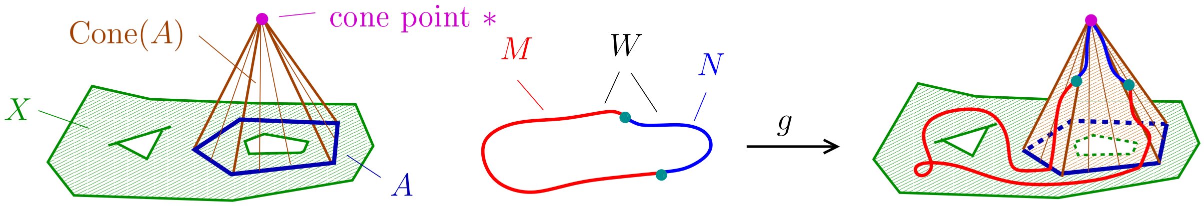


FIGURE 1084. Illustration for the proof of Proposition 70.4 (2).

70.4. **Representing homology classes by submanifolds I.** In the previous sections we studied the following definition.

Definition. Let X be a topological space, let $A \subset X$ be a subset and let $\varphi \in \mathrm{H}_k(X, A)$ be a homology class. We say φ *can be realized by a smooth manifold* if there exists a compact oriented k-dimensional smooth manifold N and a map $f\colon (N, \partial N) \to (X, A)$ such that $f_*([N]) = \varphi$.

If the topological space X we are interested in is in fact a smooth manifold it is natural to ask whether one can actually find maps f that are smooth embeddings. This leads us to the following definition.

Definition. Let M be a smooth manifold and let A be a union of components of ∂M.

(1) Let N be a compact oriented proper k-dimensional submanifold of M with $\partial N \subset A$. By a slight abuse of notation, we denote by $[N]$ the image of the fundamental class $[N]$ under the inclusion induced map $\mathrm{H}_k(N, \partial N) \to \mathrm{H}_k(M, A)$.

(2) We say that a homology class $\varphi \in \mathrm{H}_k(M, A)$ *is represented by a submanifold* if there exists a compact oriented proper k-dimensional submanifold N with $\partial N \subset A$ such that $[N] = \varphi$.

Examples.

(1) We first consider the 2-dimensional torus $T = S^1 \times S^1$. Under the usual isomorphism $\mathrm{H}_1(S^1 \times S^1) \cong \mathbb{Z}^2$ the basis element $(1, 0)$ is represented by the oriented submanifold $S^1 \times \{1\}$ and the basis element $(0, 1)$ is represented by the oriented submanifold $\{1\} \times S^1$. This situation is illustrated in Figure 1086 on the left.

(2) We consider the surface Σ_2 of genus 2. As in Figure 220 we view Σ_2 as the connected sum $T \# T$ of two tori. If we use the calculation on page 1727 to determine an isomorphism $\mathrm{H}_1(\Sigma_2) \cong \mathbb{Z}^4$, then it follows from (1) that the basis elements are

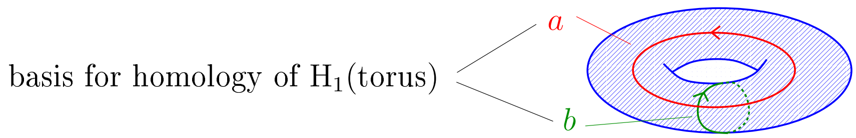
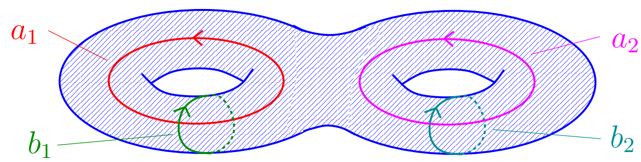


FIGURE 1085

represented by the submanifolds a_1, b_1, a_2, b_2 that are illustrated in Figure 1086.[1045] [1046]



basis for homology of $\mathrm{H}_1(\text{surface of genus 2})$

FIGURE 1086

(3) Let $L = L_1 \cup \cdots \cup L_m$ be an m-component link in S^3. Those readers who solved Exercise 68.6 will see that the m meridians represent a basis for $\mathrm{H}_1(S^3 \setminus L) \cong \mathbb{Z}^m$ and that any choice of $m-1$ tori $S^1 \times L_i$ represent a basis for $\mathrm{H}_2(S^3 \setminus L) \cong \mathbb{Z}^{m-1}$.

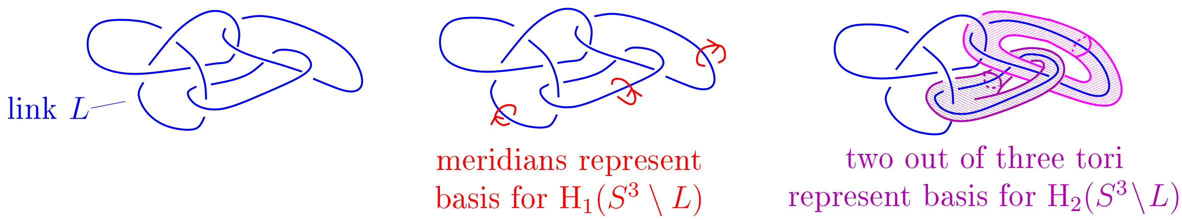


FIGURE 1087

(4) Any zero class $0 \in \mathrm{H}_k(M)$ is represented by the empty submanifold.

(5) We consider the complex projective space $\mathbb{C}\mathrm{P}^n$. We saw on page 935 that for $k \leq n$ the map

$$\begin{array}{rcl} \mathbb{C}\mathrm{P}^k & \to & \mathbb{C}\mathrm{P}^n \\ [v_0 : \ldots : v_k] & \mapsto & [v_0 : \ldots : v_k : 0 : \ldots : 0] \end{array}$$

is a smooth embedding. We use this map to view $\mathbb{C}\mathrm{P}^k$ as a submanifold of $\mathbb{C}\mathrm{P}^n$. It follows from the discussion on page 1262 and the natural isomorphism of Proposition 48.4 that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \;\cong\; \begin{cases} \mathbb{Z} \cdot [\mathbb{C}\mathrm{P}^i], & \text{if } k = 2i \text{ for some } i \in \{0, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

[1045] Another point of view is that the submanifolds a_1, b_1, a_2, b_2 are given by deforming the curves x_1, y_1, x_2, y_2 that are shown in Figure 409.

[1046] What do these submanifolds look like in the formal picture of the surface of genus 2 as the quotient $E_8/\sim$ of the Euclidean octagon?

Put differently, the generators of the non-trivial homology groups of the complex projective space $\mathbb{CP}^n$ are represented by the complex projective spaces $\mathbb{CP}^i$ with $i = 0, \dots, n$.

(6) Let M be a compact oriented connected k-dimensional smooth manifold. By Theorem 66.8 we have $\mathrm{H}_n(M, \partial M) = \mathbb{Z} \cdot [M]$. It follows from Lemma 68.8 that the classes $\pm[M]$ are represented by M, viewed as its own submanifold, equipped with the two possible orientations. Since M is connected we know by Exercise 6.24 that M admits no other non-empty n-dimensional proper submanifold. Thus we see that the classes $k \cdot [M]$ with $k \notin \{\pm 1\}$ are not represented by submanifolds.

For completeness' sake we state the following lemma.

Lemma 70.10. *Let M be a smooth manifold, let A be a union of components of ∂M and let $\varphi \in \mathrm{H}_k(M, A)$. If N is a proper submanifold that represents φ, then ∂N represents the class $\partial_k(\varphi) \in \mathrm{H}_{k-1}(A)$ where $\partial_k\colon \mathrm{H}_k(M, A) \to \mathrm{H}_{k-1}(A)$ is the connecting homomorphism of the long exact sequence of the pair (M, A).*

Proof (∗). We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_k(M, A) \arrow[r, "\partial_k", "\varphi \mapsto \partial_k(\varphi)"'] & \mathrm{H}_{k-1}(A) \\
\mathrm{H}_k(N, \partial N) \arrow[u, "i_*"] \arrow[r, "\partial_k", "{[N] \mapsto [\partial N]}"'] & \mathrm{H}_{k-1}(\partial N) \arrow[u, "i_*"']
\end{tikzcd}$$

where the horizontal maps are given by the connecting homomorphisms and where the vertical maps are induced by the inclusion maps. By Proposition 43.15 we know that the connecting homomorphism are natural, in other words, the diagram commutes. The statement now follows from Proposition 68.9. ■

We are naturally lead to the following question which can be viewed as a refinement of Question 70.1 in the context of smooth manifolds.

Question 70.11. *Let M be a smooth manifold and let A be a union of components of ∂M. For which $k \in \mathbb{N}_0$ can homology classes in $\mathrm{H}_k(M, A)$ be represented by submanifolds?*

The following proposition, which can be viewed as analogue of Proposition 70.4 (2) says that the relative problem is equivalent to the absolute problem.

Proposition 70.12. *Let $k \in \mathbb{N}$. Suppose that for every (orientable) n-dimensional smooth manifold every homology class in $\mathrm{H}_k(M)$ can be represented by a submanifold. Then also for every (orientable) n-dimensional smooth manifold M and every union A of components of ∂M every relative homology class in $\mathrm{H}_k(M, A)$ can be represented by a submanifold.*

Proof (∗). Let A be the union of some components of ∂M and let $\sigma \in \mathrm{H}_k(M, A)$. Let M' be a second copy of M. As on page 1163 we consider the double of M along A, i.e. we consider $\mathrm{D}_A M = M \cup_{A=A'} M'$. Note that by Lemma 44.12 we know that $\mathrm{D}_A M$ is an (orientable) n-dimensional smooth manifold which contains A as a submanifold. Next let $D\colon \mathrm{H}_k(M, A) \to \mathrm{H}_k(\mathrm{D}_A M)$ be the doubling homomorphism given by Lemma 44.16.

By our hypothesis there exists a closed oriented submanifold W of $\mathrm{D}_A M$ such that $[W] = D(\sigma)$. It follows from the Transversality Theorem 9.10 and Proposition 42.5 that we can arrange that W is transverse to A. Now we set $N := W \cap M$. It follows from Lemma 6.52 (3) that N is a proper submanifold of W. It remains to prove the following claim.

Claim. We have $[N] = \sigma \in \mathrm{H}_k(M, A)$.

We denote by $i\colon W \to \mathrm{D}_A M$ the inclusion map. Now we consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_k(N, \partial N) \arrow[r, "\alpha", "\cong"'] \arrow[d, "i_*"] & \mathrm{H}_n(W, W \setminus \mathring{N}) \arrow[d, "i_*"] & \mathrm{H}_k(W) \arrow[l, "\beta"'] \arrow[d, "i_*"] \\
\mathrm{H}_k(M, A) \arrow[r, "\gamma", "\cong"'] \arrow[rr, bend right, "D"'] & \mathrm{H}_k(\mathrm{D}_A M, M') & \mathrm{H}_k(\mathrm{D}_A M). \arrow[l, "\delta"']
\end{tikzcd}$$

Here α, β, γ and δ are the obvious maps. Now we make the following observations:

$$\begin{array}{rrcll}
 & i_*([W]) & = & D(\sigma) & \text{by choice of } W \\
\Longrightarrow & i_*(\beta([W])) & = & \delta(D(\sigma)) & \text{since the diagram commutes} \\
\Longrightarrow & i_*(\alpha([N]) & = & \gamma(\sigma) & \text{by Lemmas 68.12 and 44.16 (3)} \\
\Longrightarrow & \gamma(i_*([N])) & = & \gamma(\sigma) & \text{since the diagram commutes} \\
\Longrightarrow & i_*([N]) & = & \sigma & \text{since } \gamma \text{ is an isomorphism by the Excision Theorem 44.10.} \blacksquare
\end{array}$$

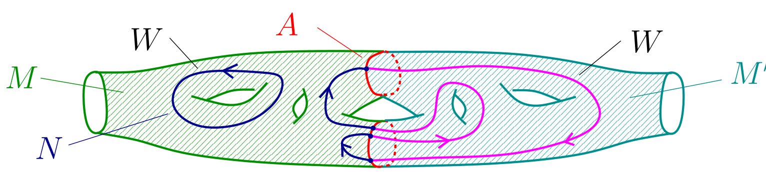


FIGURE 1088. Illustration for the proof of Proposition 70.12.

One obvious approach to Question 70.11 is as follows: in Section 70.1 we already saw that in many cases, given a class $\varphi \in \mathrm{H}_k(M, A)$, there exists a compact oriented k-dimensional smooth manifold N together with a map $f\colon (N, \partial N) \to (X, A)$ such that $f_*([N]) = \varphi$. The question now is the following: can we replace f by some proper embedding? The following proposition, which we will prove in Section 70.6, addresses this issue:

Proposition 70.13. *Let $n \in \mathbb{N}$, let M be an n-dimensional smooth manifold and let A be a union of components of ∂M. Furthermore let N be a compact oriented k-dimensional smooth manifold and let $\varphi\colon (N, \partial N) \to (M, A)$ be a map.*

(1) *If $n > 2k$, then $\varphi\colon (N, \partial N) \to (M, A)$ is homotopic, as a map of pairs of topological spaces, to a proper embedding $\psi\colon (N, \partial N) \to (M, A)$. In particular by Proposition 8.1 (2) $\psi(N)$ is a proper submanifold of M and by Proposition 43.17 we have $\varphi_*([N]) = [\psi(N)] \in \mathrm{H}_k(M, A)$.*
(2) *If $n = 2k$, then there exists a compact oriented proper k-dimensional submanifold W of N with $\partial W \subset A$ such that $\varphi_*([N]) = [W] \in \mathrm{H}_k(M, A)$.*[1047]

It is worth stating the following immediate corollary to Propositions 70.2 and 70.13.

Corollary 70.14. *Let M be an m-dimensional smooth manifold with $m \geq 2$ and let A be the union of some components of ∂M. Given any homology class in $\sigma \in \mathrm{H}_1(M, A)$ there exists a compact oriented proper 1-dimensional submanifold P with $\partial P \subset A$ such that $[P] = \sigma$.*

Example. Let M be a 2-dimensional smooth manifold and let $\sigma \in \mathrm{H}_1(M)$ be a homology class. For fun we illustrate in Figure 1089 the three main steps in the proof of Proposition 70.13 in this special case.

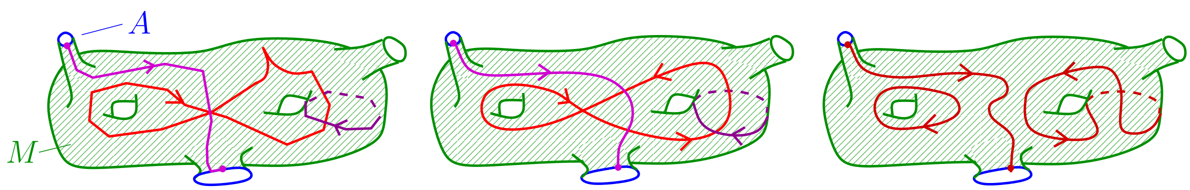


Figure 1089

Next we want to consider the question whether we can represent a homology class by a *connected* submanifold. For tori we have the following ambiguous answer.

Lemma 70.15. *Let $T = S^1 \times S^1$ be the torus. We denote by $x = [S^1 \times \{1\}]$ and $y = [\{1\} \times S^1]$ the obvious basis for $\mathrm{H}_1(T)$. A non-zero class $r \cdot x + s \cdot y$ can be represented by a connected submanifold if and only if $\gcd(r, s) = 1$.*

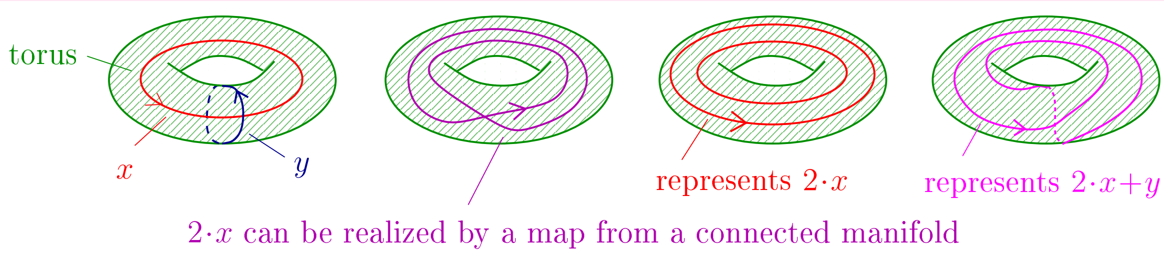


Figure 1090. Illustration for Lemma 70.15.

Proof. We, i.e. you, will provide the proof in Exercise 70.4. For fun we will deal with the analogous problem for surfaces of higher genus in Exercise 70.5. ■

The following proposition, which can be viewed as an analogue of Proposition 70.4 (1), shows that in most settings we will not have troubles finding connected submanifolds. As for the previous proposition we postpone the proof to Section 70.7.

Proposition 70.16. *Let $n \in \mathbb{N}$, let M be a connected n-dimensional smooth manifold and let A be a union of components of ∂M. Furthermore let N be a compact oriented*

[1047]Note that we do not claim that W is diffeomorphic to N.

k-dimensional submanifold with $\partial N \subset N$. We suppose that $n-k \geq 2$. If $n \geq 2$ or if $A = \emptyset$, then there exists a compact oriented connected *submanifold W of M with $\partial W \subset A$ such that $[W] = [N] \in \mathrm{H}_k(M, A)$.*

Remark. As we just saw in Lemma 70.15, in general the conclusion of Proposition 70.16 fails to hold if $k = n-1$. Note thought that in this setting some control over connectedness can be obtained using [**Tur02**, Lemma 1.2].

The combination of Propositions 70.3, 70.13 and 70.16 gives us the following pretty corollary.

Corollary 70.17. *Let M be an n-dimensional smooth manifold with $n \geq 4$ and let $A \subset \partial M$ be a union of boundary components. Every class in $\mathrm{H}_2(M, A)$ is represented by a connected submanifold.*

Remark. Let M be a compact 4-dimensional smooth manifold and let $\sigma \in \mathrm{H}_2(M)$. By Corollary 70.17 it makes sense to the define the *genus of σ*:

$$\mathrm{genus}(\sigma) \;:=\; \begin{array}{c}\text{minimal genus of a closed oriented connected} \\ \text{smooth submanifold } \Sigma \text{ of } M \text{ with } [\Sigma] = \sigma.\end{array}$$

As an example let us consider the 4-dimensional smooth manifold $\mathbb{C}\mathrm{P}^2$. By the discussion on page 1761 we know that $\mathrm{H}_2(\mathbb{C}\mathrm{P}^2) = \mathbb{Z} \cdot [\mathbb{C}\mathrm{P}^1]$. Let $n \in \mathbb{Z}$. We consider

$$W_n \;:=\; \{[x : y : z] \in \mathbb{C}\mathrm{P}^2 \,|\, x^n + y^n + z^n = 0\}.$$

In Exercise 12.3 we showed that W_n is a closed 1-dimensional complex submanifold of the 2-dimensional complex manifold $\mathbb{C}\mathrm{P}^2$. In particular this shows that W_n is a closed 2-dimensional submanifold of $\mathbb{C}\mathrm{P}^2$. In Exercise 94.6 we will show that $[W_n] = n \cdot [\mathbb{C}\mathrm{P}^1]$. As is pointed out in [**GoS99**, p. 38], it is a consequence of the "adjunction formula", that W_n is diffeomorphic to the surface of genus $\frac{1}{2}(n-1)(n-2)$. In particular we have the inequality $\mathrm{genus}(n \cdot [\mathbb{C}\mathrm{P}^1]) \leq \frac{1}{2}(n-1)(n-2)$. The Thom Conjecture predicted that equality holds. This conjecture was proved in 1994 by Peter Kronheimer and Tom Mrowka [**KrM94**] using "Seiberg-Witten Theory". In general though it is still a very difficult problem to determine the genus of the homology class of a given smooth manifold.

Corollary 70.17 suffers from the rather annoying defect that it does not apply to the case of 3-dimensional smooth manifolds. Perhaps rather surprisingly this gets taken care of by coming "from the other direction", namely this case gets taken care of by the following proposition.

Proposition 98.4. *Let M be a compact orientable n-dimensional smooth manifold and let $A \subset \partial M$ be a union of boundary components. Every class in $\mathrm{H}_{n-1}(M, A)$ can be represented by a submanifold.*

Sketch of proof. As the number already suggests, we will not prove this statement in this chapter. In the following we will just provide the flavor of the argument which is meant to serve as an amuse-bouche. For simplicity we assume that M is closed. Now suppose we are given a homology class $\sigma \in \mathrm{H}_{n-1}(M) = \mathrm{H}_{n-1}(M; \mathbb{Z})$. By the Poincaré Duality Theorem 88.1 we have an isomorphism $\mathrm{PD}_M \colon \mathrm{H}_{n-1}(M; \mathbb{Z}) \to \mathrm{H}^1(M; \mathbb{Z})$. Thus we obtain the Poincaré dual $\mathrm{PD}_M(\sigma) \in \mathrm{H}^1(M;\mathbb{Z})$. We will see that cohomology classes in $\mathrm{H}^1(M;\mathbb{Z})$ give rise to smooth maps $f\colon M \to K(\mathbb{Z},1) = S^1$. (Here $K(G,n)$ stands for an Eilenberg-Maclane space of type $K(G,n)$, the precise definition is irrelevant right now, except that in the discussion below it indicates some similarities between initially quite different looking arguments.) The desired submanifold is then given by the preimage of a regular value of f. ■

The analogue for codimension-two homology classes also holds:

Proposition ??. *Let M be a compact orientable n-dimensional smooth manifold and let $A \subset \partial M$ be a union of boundary components. Every class in $\mathrm{H}_{n-2}(M,A)$ can be represented by a submanifold.*

Sketch of proof. The logic of the proof is somewhat similar to the idea behind the proof of Proposition 98.4. This time the Poincaré Duality Theorem 88.1 gives us an isomorphism $\mathrm{PD}_M\colon \mathrm{H}_{n-2}(M;\mathbb{Z}) \to \mathrm{H}^2(M;\mathbb{Z})$. Thus the given homology class in $\mathrm{H}_{n-2}(M;\mathbb{Z})$ corresponds to a cohomology class in $\mathrm{H}^2(M;\mathbb{Z})$. We will see that such cohomology classes give rise to maps $f\colon M \to K(\mathbb{Z},2) = \mathbb{C}\mathrm{P}^\infty$. Since M is compact we obtain a map $f\colon M \to \mathbb{C}\mathrm{P}^m$ for sufficiently large m. In this occasion the desired submanifold of M is given by the preimage of $\mathbb{C}\mathrm{P}^{m-1} \subset \mathbb{C}\mathrm{P}^m$. ■

Optimistically one might hope that the pattern of the previous two propositions extends to codimension-three homology classes. But Statement (2) of the following theorem shows that this is not the case.

Theorem 70.18. (Thom's Theorem III)

(1) *Let M be a compact orientable n-dimensional smooth manifold and let A be a union of boundary components. Given $k \le 6$ with $k < n$ every class in $\mathrm{H}_k(M,A)$ can be represented by a submanifold.*
(2) *There exists a 10-dimensional closed smooth manifold*[1048] *and a class in $\mathrm{H}_7(M)$ that cannot be represented by a submanifold.*

Proof.

(1) First let us deal with the case that $A = \varnothing$. In this case, for $k \le 5$ the statement can be found in [**Tho54a**, Théorème 6] and also in [**Tho54b**, Théorème II.27]. The statement for $k = 6$ can be found in the footnote on page 173 of [**Tho07**]. The relative case now follows from the above together with Proposition 70.12.
(2) This statement was proved by Christian Bohr, Bernhard Hanke and Dieter Kotschick, see [**BHK02**, Theorem 1]. Their work builds on earlier work of René Thom [**Tho54b**, p. 62f]. ■

For completeness' sake we now also discuss homology with $\mathbb{F}_2$-coefficients. Given a compact (possibly non-orientable) proper k-dimensional submanifold N of a manifold M

[1048] It follows from Statement (1) together with Propositions 98.4 and **??** that for a compact orientable n-dimensional smooth manifold of dimension $n \le 9$ any homology class of dimension $< n$ can be represented by a submanifold. Thus the example in Statement (2) is "dimensionally minimal".

we denote by $[N]_{\mathbb{F}_2}$ the image of the $\mathbb{F}_2$-fundamental class N under the inclusion induced map $\mathrm{H}_k(N,\partial N;\mathbb{F}_2)\to \mathrm{H}_k(M,\partial M;\mathbb{F}_2)$. We say N *represents the homology class* $[N]_{\mathbb{F}_2}$.

Surely it does not come as a surprise that we have the following analogue of Corollaries 70.14 and 70.17 for $\mathbb{F}_2$-coefficients. We will skip the proof since it is essentially the same.

Proposition 70.19. *Let M be an n-dimensional smooth manifold and let $A\subset \partial M$ be a union of boundary components.*

(1) *If $n\geq 2$, then every class in $\mathrm{H}_1(M,A;\mathbb{F}_2)$ can be represented by a submanifold.*
(2) *If $n\geq 4$, then every class in $\mathrm{H}_2(M,A;\mathbb{F}_2)$ can be represented by a connected submanifold.*

The following proposition is an analogue of Proposition 98.4.

Proposition ??. *Let M be a compact n-dimensional smooth manifold and let A be a union of boundary components. Every class in $\mathrm{H}_{n-1}(M,A;\mathbb{F}_2)$ can be represented by a submanifold.*

Remark. This result will have its moment of glory when we will employ it in our proof, provided in Section 117.5, that $\pi_{n+2}(S^n)\cong \mathbb{Z}_2$ for $n\geq 3$.

Sketch of proof. The proof of this proposition is quite similar to the proof of Proposition **??**. The Poincaré Duality Theorem 88.6 with $\mathbb{F}_2$-coefficients gives us an isomorphism $\mathrm{PD}_M\colon \mathrm{H}_{n-1}(M;\mathbb{F}_2)\to \mathrm{H}^1(M;\mathbb{F}_2)$. Thus given the homology class in $\mathrm{H}_{n-1}(M;\mathbb{F}_2)$ corresponds to a cohomology class in $\mathrm{H}^1(M;\mathbb{F}_2)$. We will see that such cohomology classes give rise to maps $f\colon M\to K(\mathbb{F}_2,1)=\mathbb{R}\mathrm{P}^\infty$. Since M is compact we obtain a map $M\to\mathbb{R}\mathrm{P}^m$ for sufficiently large m. Now the desired submanifold is given by the preimage of $\mathbb{R}\mathrm{P}^{m-1}\subset\mathbb{R}\mathrm{P}^m$. ■

Remark. Note that this author is not aware of an analogue of Proposition **??**.

Finally let us see once again what René Thom has to say:

Theorem 70.20. (Thom's Theorem IV) *Let M be a compact n-dimensional smooth manifold and let A be a union of boundary components.*

(1) *If $n\leq 5$, then any class in $\mathrm{H}_{n-2}(X,A;\mathbb{F}_2)$ can be represented by a submanifold.*
(2) *If $n\leq 7$, then any class in $\mathrm{H}_{n-3}(X,A;\mathbb{F}_2)$ can be represented by a submanifold.*
(3) *For $k\leq\lfloor\frac{n}{2}\rfloor$ any class in $\mathrm{H}_k(X,A;\mathbb{F}_2)$ can be represented by a submanifold.*

Proof. In the case $A=\varnothing$ the three statements are precisely the content of [**Tho07**, Théorème II.26]. The relative case follows from the $\mathbb{F}_2$-analogue of Proposition 70.12. Finally note that Statement (3) also follows from Theorem 70.5 together with the $\mathbb{F}_2$-analogue of Proposition 70.13 (2). ■

We conclude this section with the following remark.

Remark. As we already mentioned on page 1751, later in Chapter 86 we will introduce the notion of an orientation of a topological manifold and in Chapter 87 we will introduce the notion of the fundamental class $[N]\in\mathrm{H}_n(N,\partial N)$ of a compact oriented n-dimensional topological manifold N. Thus it makes sense to ask which homology classes of a given topological manifold can be represented by submanifolds. This question is discussed in [**FNOP19**].

In the remainder of this chapter we will provide the proofs of Proposition 70.13 and 70.16.

70.5. **Rearranging maps from smooth manifolds** $(*)$**.** Let (X, A) be a pair of topological spaces. In this section we will state and prove a proposition which allows us to replace certain compact oriented smooth manifolds M and maps $f\colon (M, \partial M) \to (X, A)$ by a different compact oriented smooth manifold N together with a map $g\colon (N, \partial N) \to (X, A)$ such that $f_*([M]) = g_*([N])$. This procedure will come in handy in the proofs of Proposition 70.13 and 70.16.

The following definition is a special case of a more general concept introduced on page 284.

Definition. Let M be a compact n-dimensional smooth manifold. We say M is *decomposed into submanifolds A and B* if the following conditions hold:

(1) $M = A \cup B$.
(2) A and B are both compact n-dimensional submanifolds of M.
(3) We have $A \cap B = \partial_0 A = \partial_0 B$.

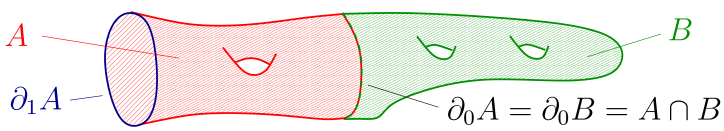


FIGURE 1091

Now we can state the one result of this section.

Proposition 70.21. *Let M and $\widetilde{M}$ be two compact oriented n-dimensional smooth manifolds. Suppose that M is decomposed into two submanifolds A and B and that $\widetilde{M}$ is decomposed into two submanifolds $\widetilde{A}$ and $\widetilde{B}$. We equip $A \cap B$ and $\widetilde{A} \cap \widetilde{B}$ with the orientation coming from A respectively $\widetilde{A}$. Suppose that we are given an orientation-preserving diffeomorphism $\Theta\colon A \cap B \to \widetilde{A} \cap \widetilde{B}$. Next we set*

$$N := (A \sqcup \widetilde{B})/x \sim \Theta(x) \text{ for } x \in A \cap B \quad \text{ and } \widetilde{N} := (\widetilde{A} \sqcup B)/\Theta(y) \sim y \text{ for } y \in A \cap B.$$

Note that it follows from Proposition 8.15 that N and $\widetilde{N}$ are both compact oriented n-dimensional smooth manifolds such that the obvious maps $A \to N$ and $\widetilde{A} \to \widetilde{N}$ are orientation-preserving embeddings. Next let (X, S) be a pair of topological space and furthermore let $f\colon (M, \partial M) \to (X, S)$ and $\widetilde{f}\colon (\widetilde{M}, \partial\widetilde{M}) \to (X, S)$ be two maps such that we have $\widetilde{f} \circ \Theta = f\colon A \cap B \to X$. We define[1049]

$$g\colon N = (A \sqcup \widetilde{B})/\sim \;\to\; X \qquad [P] \mapsto \begin{cases} f(P), & \text{if } P \in A, \\ \widetilde{f}(P), & \text{if } P \in \widetilde{B} \end{cases} \qquad \text{and} \qquad \widetilde{g}\colon \widetilde{N} = (\widetilde{A} \sqcup B)/\sim \;\to\; X \qquad [P] \mapsto \begin{cases} \widetilde{f}(P), & \text{if } P \in \widetilde{A}, \\ f(P), & \text{if } P \in B. \end{cases}$$

With this notation we have

$$f_*([M]) + \widetilde{f}_*([\widetilde{M}]) \;=\; g_*([N]) + \widetilde{g}_*([\widetilde{N}]) \;\in\; H_n(X, S).$$

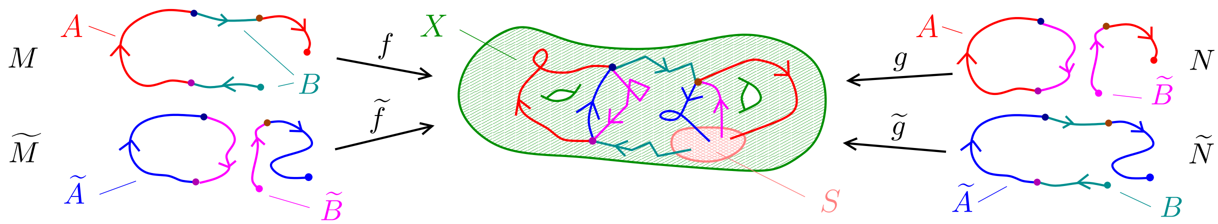


FIGURE 1092. Illustration for Proposition 70.21.

PROOF. We pick a fundamental cycle $\mu \in \mathrm{C}_{n-1}(A \cap B)$ for the closed oriented $(n-1)$-dimensional smooth manifold $A \cap B$. We consider the map

$$\begin{tikzcd} \mathrm{C}_n(A,\partial A) \arrow[r, "\partial_n"] \arrow[rrr, bend right=10, "=:\omega_A"'] & \mathrm{C}_{n-1}(\overbrace{\partial_0 A \sqcup \partial_1 A}^{=\partial A}) \arrow[r, "\cong"] & \mathrm{C}_{n-1}(\overbrace{A \cap B}^{=\partial_0 A}) \oplus \mathrm{C}_{n-1}(\partial_1 A) \arrow[r, two heads] & \mathrm{C}_{n-1}(A \cap B). \end{tikzcd}$$

By Corollary 68.10 we know that there exists a singular chain $\nu_A \in \mathrm{C}_n(A)$ such that ν_A represents a fundamental cycle in $\mathrm{C}_n(A,\partial A)$ and such that $\omega_A(\nu_A) = \mu$. With the same notation we also obtain some $\nu_B \in \mathrm{C}_n(B)$ with $\omega_B(\nu_B) = \mu$.

Next we consider $\widetilde{\mu} := \Theta_*(\mu) \in \mathrm{C}_{n-1}(\widetilde{A} \cap \widetilde{B})$. As above we pick corresponding singular chains $\widetilde{\nu}_{\widetilde{A}} \in \mathrm{C}_n(\widetilde{A})$ and $\widetilde{\nu}_{\widetilde{B}} \in \mathrm{C}_n(\widetilde{B})$. Now we see that we have the following equalities in $\mathrm{H}_n(X,S)$:

it follows from $\omega_A(\nu_A) = \mu = \omega_B(\nu_B)$ that $\nu_A - \nu_B$ is a cycle in $\mathrm{C}_n(M,\partial M)$
using Lemma 68.3 (3)⇒(1) one now can easily verify that $\nu_A - \nu_B$ is a fundamental cycle of M,
the same argument shows that $\widetilde{\nu}_{\widetilde{A}} - \widetilde{\nu}_{\widetilde{B}}$ is a fundamental cycle of $\widetilde{M}$

$$\begin{aligned} f_*([M]) + \widetilde{f}_*([\widetilde{M}]) &\overset{\downarrow}{=} f_*([\nu_A - \nu_B]) + \widetilde{f}_*([\widetilde{\nu}_{\widetilde{A}} - \widetilde{\nu}_{\widetilde{B}}]) \qquad = [f_*(\nu_A) - f_*(\nu_B) + \widetilde{f}_*(\widetilde{\nu}_{\widetilde{A}}) - \widetilde{f}_*(\widetilde{\nu}_{\widetilde{B}})] \\ &= [f_*(\nu_A) - \widetilde{f}_*(\widetilde{\nu}_{\widetilde{B}}) + \widetilde{f}_*(\widetilde{\nu}_{\widetilde{A}}) - f_*(\nu_B)] \underset{\uparrow}{=} [f_*(\nu_A) - \widetilde{f}_*(\widetilde{\nu}_{\widetilde{B}})] + [\widetilde{f}_*(\widetilde{\nu}_{\widetilde{A}}) - f_*(\nu_B)] \end{aligned}$$

it follows fairly easily from $\widetilde{f} \circ \Theta = f$ and $\widetilde{\mu} = \Theta_*(\mu)$ that
$f_*(\nu_A) - \widetilde{f}_*(\widetilde{\nu}_{\widetilde{B}})$ and $\widetilde{f}_*(\widetilde{\nu}_{\widetilde{A}}) - f_*(\nu_B)$ are both cycles in $\mathrm{C}_n(X,S)$

$$\underset{\uparrow}{=} [g_*(\nu_A) - g_*(\widetilde{\nu}_{\widetilde{B}})] + [\widetilde{g}_*(\widetilde{\nu}_{\widetilde{A}}) - \widetilde{g}_*(\nu_B)] \;=\; g_*(\underbrace{[\nu_A - \widetilde{\nu}_B]}_{=[N]}) + \widetilde{g}_*(\underbrace{[\widetilde{\nu}_{\widetilde{A}} - \widetilde{\nu}_B]}_{=[\widetilde{N}]}).$$

by definition of g and $\widetilde{g}$

■

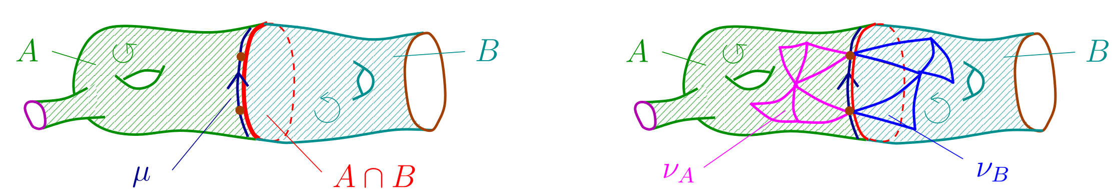


FIGURE 1093. Illustration for the proof of Proposition 70.21.

[1049] It follows from $\widetilde{f} \circ \Theta = f$ that both maps are continuous.

70.6. **Proofs of Proposition 70.13.** For the reader's convenience we recall the statement of Proposition 70.13.

Proposition 70.13. *Let $n \in \mathbb{N}$, let M be an n-dimensional smooth manifold and let A be a union of components of ∂M. Furthermore let N be a compact oriented k-dimensional smooth manifold and let $\varphi\colon (N, \partial N) \to (M, A)$ be a map.*

(1) *If $n > 2k$, then $\varphi\colon (N, \partial N) \to (M, A)$ is homotopic, as a map of pairs of topological spaces, to a proper embedding $\psi\colon (N, \partial N) \to (M, A)$. In particular $\psi(N)$ is a proper submanifold of M and we have $\varphi_*([N]) = [\psi(N)] \in \mathrm{H}_k(M, A)$.*

(2) *If $n = 2k$, then there exists a compact oriented proper k-dimensional submanifold W of N with $\partial W \subset A$ such that $\varphi_*([N]) = [W] \in \mathrm{H}_k(M, A)$.*

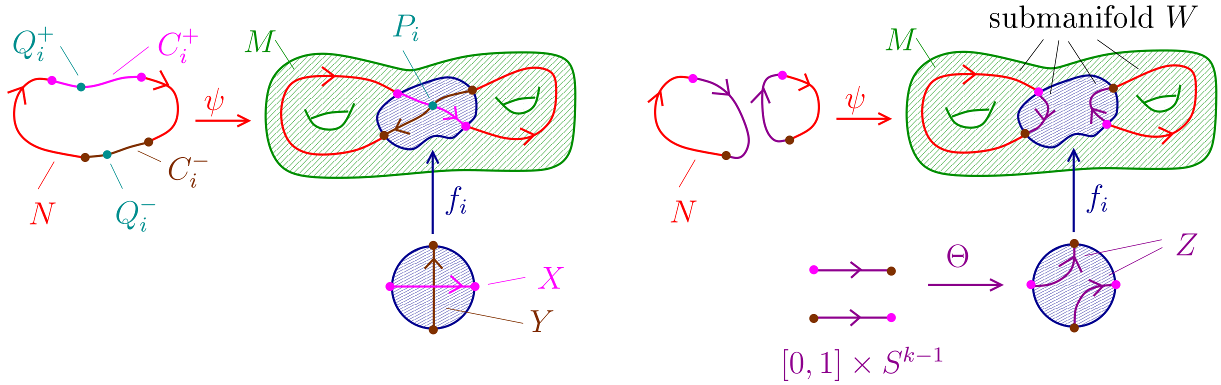


FIGURE 1094. Illustration for the proof of Proposition 70.13.

PROOF. Let $n \in \mathbb{N}$, let M be an n-dimensional smooth manifold and let A be a union of components of ∂M. Furthermore let N be a compact oriented k-dimensional smooth manifold and let $\varphi\colon (N, \partial N) \to (M, A)$ be a map.

First let us consider the case $n > 2k$. In this case it follows from Theorem 9.15 (3) that $\varphi\colon (N, \partial N) \to (M, A)$ is homotopic, as a map of pairs of topological spaces, to a proper embedding $\psi\colon (N, \partial N) \to (M, A)$. It follows from Proposition 8.1 (2) that $\psi(N)$ is a proper submanifold of M and that $\psi\colon N \to \psi(N)$ is a diffeomorphism. Thus we see that

$$\varphi_*([N]) \underset{\substack{\uparrow \\ \text{Proposition 43.17}}}{=} \psi_*([N]) \underset{\substack{\uparrow \\ \text{since } \psi \text{ is a diffeomorphism}}}{=} [\psi(N)] \in \mathrm{H}_k(M, A).$$

Now let us deal with the much more subtle case $n = 2k$. From Theorem 9.15 (2) we now obtain that the map φ is homotopic, as a map of pairs of topological spaces, to a proper immersion $\psi\colon N \to M$ which has furthermore the property that there exist points $P_1, \dots, P_m$ in $M \setminus \partial M$ with the following properties:

(1) For each $i \in \{1, \dots, m\}$ the preimage $\psi^{-1}(P_i)$ consists of precisely two points $Q_i^\pm$ which furthermore satisfy that $\psi_*(\mathrm{T}_{Q_i^+} N) + \psi_*(\mathrm{T}_{Q_i^-} N) = \mathrm{T}_{P_i}(M)$.

(2) The restriction of ψ to $N \setminus \{Q_1^\pm, \dots, Q_m^\pm\}$ is injective.

In the following we write

$$X := \left\{ (x, 0) \in \overline{B}^{2k} \,\middle|\, x \in \overline{B}^k \right\} \quad \text{and} \quad Y := \left\{ (0, y) \in \overline{B}^{2k} \,\middle|\, y \in \overline{B}^k \right\}.$$

We view X and Y as submanifolds of $\overline{B}^{2k}$. We equip X and Y with the obvious orientation. It follows easily from Proposition 8.1 (2) together with Lemma 6.52 (1) that we can find the following:

(1) Disjoint k-dimensional submanifolds $\mathrm{C}_1^{\pm}, \dots, \mathrm{C}_m^{\pm}$ of N such that for each $i \in \{1, \dots, m\}$ we have $Q_i^{\pm} \in \mathring{C}_i^{\pm} = \mathrm{C}_i^{\pm}$. Note that the restriction of ψ to any $\mathring{C}_i^{\pm}$ is in fact a smooth embedding.

(2) Disjoint embeddings $f_1, \dots, f_m \colon \overline{B}^{2k} \to M \setminus \partial M$ such that for each $i \in \{1, \dots, m\}$ we have

$$\psi(\mathrm{C}_i^+) \;=\; f_i(X), \quad \psi(\mathrm{C}_i^-) \;=\; f_i(Y), \quad \text{ and } f_i(\overline{B}^{2k}) \cap \psi(N) \;=\; f_i(X) \cup f_i(Y).$$

Furthermore, after possibly precomposing the maps f_i with reflections in hyperplanes we can assume that the maps $X \to f_i(X) \xleftarrow{\psi} \mathrm{C}_i^+$ and $Y \to f_i(Y) \xleftarrow{\psi} \mathrm{C}_i^-$ are orientation-preserving.

Claim. There exists a proper smooth embedding $\Theta \colon [0,1] \times S^{k-1} \to \overline{B}^{2k}$ with the following three properties:

(1) For $t \in [0, \frac{1}{4}]$ and $z \in S^{k-1}$ we have $\Theta(t,z) = ((1-t)z, 0) \in \mathbb{R}^k \times \mathbb{R}^k$. Furthermore the map $\Theta_0 \colon \{0\} \times S^{k-1} \to S^{k-1} \times \{0\}$ is orientation-preserving.

(2) For $t \in [\frac{3}{4}, 1]$ and $z \in S^{k-1}$ we have $\Theta(t,z) = (0, t \cdot z) \in \mathbb{R}^k \times \mathbb{R}^k$. Furthermore the map $\Theta_0 \colon \{1\} \times S^{k-1} \to \{0\} \times S^{k-1}$ is orientation-reversing.

To prove the claim first note that, using Lemma 6.13 one can easily find a smooth function $\mu \colon [0,1] \to [0,1]$ such that $\mu(t) = t$ for $t \in [0, \frac{1}{4}]$, and such that $\mu(t) = 1$ for $t \in [\frac{3}{4}, 1]$ and such that $\mu'(t) > 0$ for any $t \in (0, \frac{3}{4})$. We leave it to the reader to verify that the following map has all the desired properties:

$$\begin{array}{rcl} \Theta \colon [0,1] \times S^{k-1} & \to & \overline{B}^{2k} \subset \mathbb{R}^k \times \mathbb{R}^k \\ (t,z) & \mapsto & (\underbrace{(1-\mu(t)) \cdot z}_{\in \mathbb{R}^k}, \underbrace{(1-\mu(1-t)) \cdot z}_{\in \mathbb{R}^k}). \end{array}$$

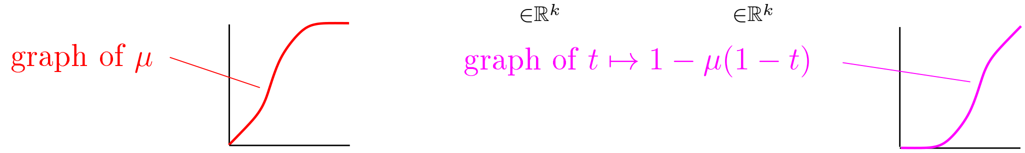


FIGURE 1095

Next we set $Z := \Theta([0,1] \times S^{k-1})$ and we set

$$W \;:=\; \psi\Big(\underbrace{N \setminus \bigcup_{i=1}^{m} (\mathring{C}_i^+ \cup \mathring{C}_i^-)}_{=:\widehat{N}}\Big) \cup \bigcup_{i=1}^{m} f_i(Z).$$

We leave it to the reader to verify that W is in fact a compact proper k-dimensional submanifold of M with $\partial W \subset A$ which can be oriented in such a way that the embedding $\psi \colon \widehat{N} \to W$ is orientation-preserving. It remains to prove the following claim.

Claim. We have $\varphi_*([N]) = [W] \in \mathrm{H}_k(X, A)$.

As usual we denote by $-W$ the same smooth manifold but with the opposite orientation. By Lemma 68.8 we have $[-W]=-[W]$. Thus it suffices to show that $\varphi_*([N])+[-W]=0$. Now we perform the following calculation:

$$\varphi_*([N])+[-W] \overset{\substack{\text{by Proposition 43.17 since } \psi \text{ is homotopic to } \varphi\\ \downarrow}}{=} \psi_*([N])+[-W] = \psi_*\Big(\Big[\underbrace{\widehat{N}\cup\bigcup_{i=1}^{m}(C_i^+\cup C_i^-)}_{N}\Big]\Big)+\Big[\underbrace{-\psi(\widehat{N})\cup-\bigcup_{i=1}^{m}f_i(Z)}_{=W}\Big]$$

$$\overset{\substack{\text{Proposition 70.21}\\ \downarrow}}{=} \underbrace{(\psi\cup\psi)_*([\overbrace{\widehat{N}\cup_{\partial_0\widehat{N}=\partial_0\widehat{N}}-\widehat{N}}^{\text{the double } D_{\partial_0\widehat{N}}\widehat{N}}])}_{\substack{\text{the map } \psi\cup\psi \text{ factors through the}\\ \text{folding map } D_{\partial_0\widehat{N}}\widehat{N}\to\widehat{N}\\ \text{which has degree zero by}\\ \text{the discussion on page 1742}}} + \sum_{i=1}^{m}\underbrace{(\psi\cup f_i)_*((Q_i^+\cup Q_i^-)\cup Z)}_{\substack{=0,\text{ since } \psi\cup f_i \text{ takes values in } f_i(\overline{B}^{2k})\\ \text{and since } H_k(f_i(\overline{B}^{2k}))=0}} = 0.$$

■

70.7. **The internal connected sum and the proof of Proposition 70.16.** We start with the slightly technical construction of an internal connected sum of two oriented submanifolds of a given smooth manifold. This construction is interesting in its own right, and it will be the key ingredient for the proof of Proposition 70.16.

Construction 70.22. *Let M be a connected n-dimensional smooth manifold and let N_0 and N_1 be two compact oriented proper k-dimensional submanifolds of M. We suppose that $n-k\geq 2$. We pick points $P_0\in N_0\setminus\partial N_0$ and $P_1\in N_1\setminus\partial N_1$. We proceed with the following steps:*

(1) *Suppose we are given a smooth embedding $\gamma\colon[0,1]\to M\setminus\partial M$ with the following properties:*
   (a) *$\gamma(0)=P_0$ and $\gamma(1)=P_1$.*
   (b) *If we set $C:=\gamma([0,1])$, then $C\cap N_0=\{P_0\}$ and $C\cap N_1=\{P_1\}$.*
   (c) *C hits N_0 and N_1 transversally, in the sense that $T_{P_0}C\cap T_{P_0}N_0=\{0\}$ and $T_{P_1}C\cap T_{P_1}N_1=\{0\}$.*

   The courageous reader is challenged to show, by a variation on the proof of Corollary 9.14, that it follows from our hypothesis $n-k\geq 2$ that such γ always exists.
(2) *Next we write $D:=\{(x,0)\in\mathbb{R}^k\times\mathbb{R}^{n-1-k}\,|\,x\in\overline{B}^k\}\subset\overline{B}^{n-1}$. Once again we challenge the courageous reader to show, say using a suitable variation on the Tubular Neighborhood Theorem 8.24, that there exists an embedding $\Phi\colon\overline{B}^{n-1}\times C\to M$ such that $\Phi(\overline{B}^{n-1}\times C)\cap N_0=\Phi(D\times\{P_0\})$ and such that $\Phi(\overline{B}^{n-1}\times C)\cap N_1=\Phi(D\times\{P_1\})$.*
(3) *Using Exercise 6.9 one can easily find a smooth embedding $\Theta\colon S^{k-1}\times[0,1]\to\overline{B}^{k-1}\times C$ with the following two properties:*
   (a) *For $z\in S^{k-1}$ and $t\in[0,\frac14]$ we have $\Theta(z,t)=((1-t)\cdot z,P_0)$.*
   (b) *For $z\in S^{k-1}$ and $t\in[\frac34,1]$ we have $\Theta(z,t)=(t\cdot z,P_1)$.*

   We set $Z:=\Theta(S^{k-1}\times[0,1])$.

There are now two cases to consider.

Case (1) *Among the two maps $\Phi\colon D\times\{P_0\}\to N_0$ and $\Phi\colon D\times\{P_1\}\to N_1$ one is orientation-preserving and one is orientation-reversing. In this case we set*

$$N_0\#_\gamma N_1 \;:=\; (N_0\setminus\Phi(D\times\{P_0\}))\cup(N_1\setminus\Phi(D\times\{P_1\})))\cup\Phi(D\times C).$$

Case (2) *If both maps $\Phi\colon D\times\{P_0\}\to N_0$ and $\Phi\colon D\times\{P_1\}\to N_1$ are orientation-preserving or if both are orientation-reversing then we add a "half-twist" to get us back into the game. More precisely, we consider the map* [1050]

$$\begin{array}{rcl}\Omega\colon D\times C & \to & \overline{B}^{n-1}\times C\\ ((\underbrace{x_1,x_2,\dots,x_k}_{\in\mathbb{R}^k}),\underbrace{(0,0)}_{\in\mathbb{R}\times\mathbb{R}^{n-k-2}}),\gamma(t)) & \mapsto & ((\underbrace{(\cos(t)\cdot x_1,x_2,\dots,x_k)}_{\in\mathbb{R}^k},\underbrace{(\sin(t)\cdot x_1,0)}_{\in\mathbb{R}\times\mathbb{R}^{n-k-2}}),\gamma(t)).\end{array}$$

Note that Ω is the identity on $D\times\{P_0\}$ and that Ω restricts to an orientation-reversing self-diffeomorphism of $D\times\{P_1\}$. Now we set

$$N_0\#_\gamma N_1 \;:=\; (N_0\setminus\Phi(D\times\{P_0\}))\cup(N_1\setminus\Phi(D\times\{P_1\}))\cup(\Phi\circ\Omega)(D\times C).$$

It is straightforward to verify that in both cases $N_0\#_\gamma N_1$ is a k-dimensional submanifold of M. Furthermore one sees easily that in both cases we can equip the smooth manifold $N_0\#_\gamma N_1$ with an orientation in such a way that the inclusions $N_0\setminus\Phi(D\times\{P_0\})\to N_0\#_\gamma N_1$ and $N_1\setminus\Phi(D\times\{P_1\})\to N_0\#_\gamma N_1$ are both orientation-preserving. We refer to $N_0\#_\gamma N_1$ as an internal connected sum of N_0 and N_1 along γ *or as a* tubing of N_0 and N_1 along γ.[1051]

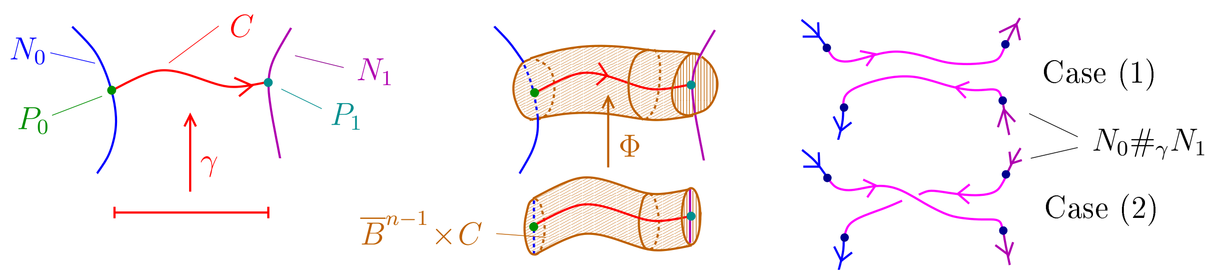


FIGURE 1096. Illustration of Construction 70.22.

The following lemma justifies the name "internal connected sum".

Lemma 70.23. *We continue with the above notation. The internal connected sum $N_01\#_\gamma N_1$, viewed as a smooth manifold in its own right, is diffeomorphic to the connected sum $N_0\# N_1$.*

PROOF. The lemma can be proved fairly easily using the Collar Neighborhood Theorem 8.12 applied to $N_0\setminus\Phi(D\times\{P_0\})$ and $N_1\setminus\Phi(D\times\{P_1\})$. We leave it to the reader to fill in the details. ■

Example. In Figure 1097 we show two disjoint oriented knots J and J. By definition both are submanifolds of S^3. We also show a path γ that connects K to J and we show a corresponding internal connected sum $K\#_\gamma J$. On page 784 we introduced the connected

[1050] Note that here we use again, in a most subtle way, our hypothesis $n-k\geq 2$.

[1051] The construction also depends on the choice of Φ and Θ. We do not claim that the isotopy type of $N_0\#_\gamma N_1$ is well-defined.

sum $K\#J$. The resulting image, see e.g. Figure 507 looks quite similar to what we see in Figure 1097. With some effort one can show that given any two oriented knots the corresponding connected sum $K\#J$ is smoothly isotopic to any internal connected sum $K\#_\gamma J$. In fact sometimes this approach is used as the definition of the connected sum of two oriented knots.

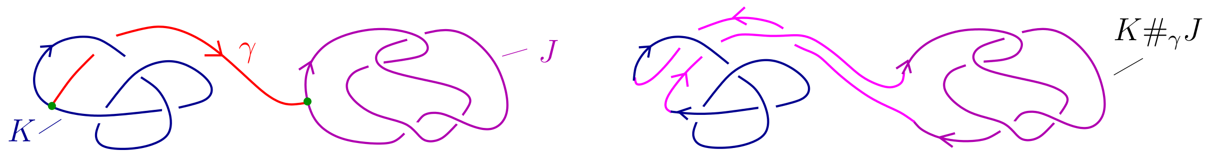


FIGURE 1097

We recall the statement of proposition that we are actually supposed to proof.

Proposition 70.16. *Let $n \in \mathbb{N}$, let M be a connected n-dimensional smooth manifold and let A be a union of components of ∂M. Furthermore let N be a compact oriented k-dimensional submanifold with $\partial N \subset N$. We suppose that $n-k \geq 2$. If $n \geq 2$ or if $A = \varnothing$, then there exists a compact oriented* connected *submanifold W of M with $\partial W \subset A$ such that $[W] = [N] \in \mathrm{H}_k(M, A)$.*

SKETCH OF PROOF. To simplify the notation we assume that N has precisely two components N_0 and N_1. As explained in Construction 70.22, since $n-k \geq 2$ we can perform an internal connected sum $N_0\#_\gamma N_1$. As in the proof of Proposition 70.13 one can deduce, using Proposition 70.21, that $[N_0\#_\gamma N_1] = [N_0] + [N_1] \in \mathrm{H}_k(M, A)$.

Next, as we pointed out in Lemma 70.23, the internal connected sum $N_0\#_\gamma N_1$ is diffeomorphic to the connected sum $N_0\#N_1$. If $n \geq 2$, then it follows immediately from Proposition 8.35 that the connected sum $N_0\#N_1$ is connected. Finally, if $A = \varnothing$, then we know that both N_0 and N_1 are closed. Thus it follows again from Proposition 8.35 that $N_0\#N_1$ is connected. ■

Exercises for Chapter 70.

Exercise 70.1. Let M be a closed 3-dimensional smooth manifold and let $\sigma \in \mathrm{H}_2(M)$ be a homology class. By Proposition 70.3 there exists a closed oriented 2-dimensional smooth manifold F and a map $\varphi\colon F \to M$ such that $\varphi_*([F]) = \varphi$. Can you modify the proof of Proposition 70.13 to show that there exists a closed oriented 2-dimensional smooth manifold G and an *embedding* $\psi\colon G \to M$ such that $\psi_*([G]) = \sigma$? Identify the issues you run into and try to resolve them.

Exercise 70.2. Let $n \in \mathbb{N}$. As on page 194 we consider the complex projective space $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$. By the discussion on page 1262 we know that

$$\mathrm{H}_k(\mathbb{CP}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k \in \{0, 2, \dots, 2n\}, \\ 0, & \text{otherwise.} \end{cases}$$

We consider the map $f\colon \mathbb{CP}^n \to \mathbb{CP}^n$ given by $f([z_0 : \cdots : z_n]) = [\overline{z}_0 : \cdots : \overline{z}_n]$. For each $k \in \{0, 2, \dots, 2n\}$ determine the degree of the induced map $f_*\colon \mathrm{H}_k(\mathbb{CP}^n) \to \mathrm{H}_k(\mathbb{CP}^n)$.

Exercise 70.3. Let $g \in \mathbb{N}_0$. As usual we denote by Σ_g the surface of genus g. Let M be a smooth manifold and let $P \in M$. Now let $k \in \mathbb{N}_0$. Show that if there exists a map $f\colon \Sigma_k \to M \times \Sigma_g$ with $f_*([\Sigma_k]) = [\{P\} \times \Sigma_g] \in \mathrm{H}_2(M \times \Sigma_g)$, then $k \geq g$.

Exercise 70.4. Let $T = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ be the torus. We denote by $x = [S^1 \times \{1\}]$ and $y = [\{1\} \times S^1]$ the obvious basis for $\mathrm{H}_1(T)$. Let $(r, s) \neq (0, 0) \in \mathbb{Z}^2$. Show that the following two statements are equivalent:

(a) The homology class $r \cdot x + s \cdot y$ can be represented by a connected submanifold of T.
(b) $\gcd(r, s) = 1$.

Hint. For the "(a)⇒(b)"-direction you might want to use the classification of 2-dimensional smooth manifolds.

Exercise 70.5. Let $g \in \mathbb{N}$. We consider the surface Σ of genus g. Let $\varphi \neq 0 \in \mathrm{H}_1(\Sigma)$. Show that the following two statements are equivalent:

(a) φ can be represented by a connected submanifold of Σ.
(b) φ is a primitive class in $\mathrm{H}_1(\Sigma)$.

Remark. The case $g = 1$ is evidently just the content of Exercise 70.4.

71. The Simplicial Poincaré Duality Theorem

In this chapter we will prove one of the most beautiful and powerful theorems in the theory of abstract simplicial complexes, namely the Simplicial Poincaré Duality Theorem 71.4. This theorem has many applications, arguably the most interesting ones concern the singular (co-) homology groups of smooth manifolds.

71.1. Simplicial Homology Manifolds.

To state the Simplicial Poincaré Duality Theorem we will need the following notion.

Definition.

(1) We say that an abstract simplicial complex L is an *simplicial homology m-sphere* if for every $i \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(L) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = m, \\ 0, & \text{else.} \end{cases}$$

Furthermore we say that L is a *simplicial homology ball* if $\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(L) = 0$ for all $i \in \mathbb{N}_0$.

(2) We say that an abstract simplicial complex K is an n-dimensional *simplicial homology manifold* if for every $k \in \mathbb{N}_0$ and every k-simplex s of K the link $\mathrm{Lk}(K, s)$ is a simplicial homology $(n-1-k)$-sphere or if it is a simplicial homology ball.

(3) We say a simplicial homology manifold K is *closed*, if K is finite and if for every k-simplex s of K the link $\mathrm{Lk}(K, s)$ is a simplicial homology $(n-1-k)$-sphere.

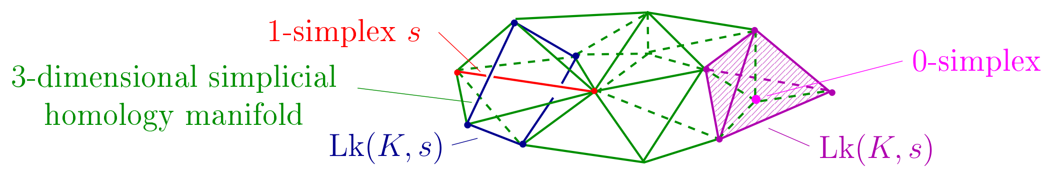


Figure 1098

Remark. Note that by the discussion on page 1549 the empty abstract simplicial complex is a simplicial homology (-1)-sphere. It follows in particular that every simplex in a closed n-dimensional simplicial homology manifold has dimension at most n.

The following lemma gives a reformulation of the definition of a simplicial homology manifold.

Lemma 71.1. *Let K be an abstract simplicial complex. As always we denote by $\mathrm{sd}(K)$ its barycentric subdivision. The following two statements are equivalent:*

(1) *K is a closed n-dimensional simplicial homology manifold.*
(2) *For every vertex v[1052] of the barycentric subdivision $\mathrm{sd}(K)$ the link $\mathrm{Lk}(\mathrm{sd}(K), v)$ is a simplicial homology $(n-1)$-sphere.*

Proof. The lemma follows immediately from the second part of the following claim.

Claim. Let K be an abstract simplicial complex and let s be a k-simplex of K.

[1052] It is worth remembering that by definition the vertices of $\mathrm{sd}(K)$ are precisely the barycenters of the simplices of K.

(a) There is a simplicial isomorphism $\operatorname{Lk}(K,s) * \partial s \cong \operatorname{Lk}(\operatorname{sd}(K),\underline{s})$.
(b) For each $i \in \mathbb{N}_0$ we have $\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(\operatorname{Lk}(K,s)) \cong \widetilde{\mathrm{H}}_{i+k}^{\mathrm{simp}}(\operatorname{Lk}(\operatorname{sd}(K),\underline{s}))$.

Let us prove the two statements.

(a) This statement is actually the content of Exercise 62.2. For the reader's convenience we sketch the proof. So let us consider the map
$$\begin{array}{rcl} \operatorname{Lk}(K,s) * \partial s & \to & \operatorname{Lk}(\operatorname{sd}(K),\underline{s}) \\ v & \mapsto & \begin{cases} \underline{s \setminus \{v\}}, & \text{if } v \in \partial s, \\ \underline{s \sqcup \{v\}}, & \text{if } v \in \operatorname{Lk}(K,s). \end{cases} \end{array}$$
It is elementary to verify that this map on the level of vertices induces an isomorphism of abstract simplicial complexes.

(b) For any $i \in \mathbb{N}_0$ we have
$$\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(\operatorname{Lk}(K,s)) \underset{\substack{\uparrow \\ \text{Lemma 63.13 (1)}}}{\cong} \widetilde{\mathrm{H}}_{i+k}^{\mathrm{simp}}(\operatorname{Lk}(K,s) * \partial s) \underset{\substack{\uparrow \\ \text{by (a)}}}{\cong} \widetilde{\mathrm{H}}_{i+k}^{\mathrm{simp}}(\operatorname{Lk}(\operatorname{sd}(K),\underline{s})).$$
■

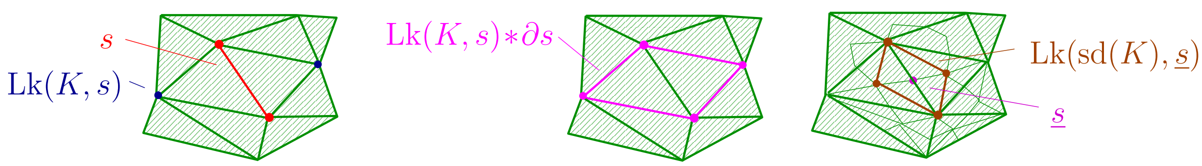


FIGURE 1099. Illustration of the proof of Lemma 71.1.

Slightly annoyingly the notion of a "simplicial homology manifold" is now the third definition of a "manifold-like" structure on an abstract simplicial complex. Namely on page 1650 we introduced the notion of a pseudomanifold and on page 1599 we introduced the notion of a PL-manifold. For the reader's and also the author's convenience we now recall these two earlier notions. Thus let K be a finite non-empty abstract simplicial complex.

(1) K is a closed n-dimensional pseudomanifold if the following conditions are satisfied:
 (a) Every simplex of K is a face of an n-simplex of K.
 (b) Every $(n-1)$-simplex has order two, i.e. it is the face of precisely two n-dimensional faces.
 (c) Given any two n-simplices t and t' there exists a sequence $t = s_0, s_1, \ldots, s_k = t'$ of n-simplices such that any two consecutive simplices have a common $(n-1)$-dimensional face.
(2) By Proposition 64.11 we know that K is a closed n-dimensional PL-manifold if and only if for every k-simplex $s \in S$ the link $\operatorname{Lk}(K,s)$ is a PL $(n-k-1)$-sphere, i.e. $|\operatorname{Lk}(K,s)|$ is PL-homeomorphic to the simplicial complex $\partial\Delta^{n-k}$.[1053]

[1053] By the Alexander-Newman Theorem 64.21 the notion of a PL-manifold is basically equivalent to the notion of a combinatorial manifold that we introduced on page 1611. Since the notion of a PL-manifold is more commonly used we stick in the discussion to PL-manifold even though the notion of a combinatorial manifold would be slightly more appropriate.

The following proposition summarizes everything anybody ever wanted to know about the relationship between this new concept and our two earlier concepts.

Proposition 71.2.

(1) *A closed connected non-empty n-dimensional simplicial homology manifold K is a closed n-dimensional pseudomanifold with $\partial K = \varnothing$.*
(2) *There exists a closed pseudomanifold that is not a simplicial homology manifold.*
(3) *Every closed non-empty n-dimensional* PL*-manifold is a closed n-dimensional simplicial homology manifold.*
(4) *There exists a closed simplicial homology manifold that is not a* PL*-manifold.*

Remark. Slightly more informally Proposition 71.2 can be summarized as follows:

$$\text{PL-manifold} \quad \underset{\not\Longleftarrow}{\Longrightarrow} \quad \text{simplicial homology manifold} \quad \underset{\not\Longleftarrow}{\Longrightarrow} \quad \text{pseudomanifold.}$$

Proof.

(1) Let K be a closed connected non-empty n-dimensional simplicial homology manifold. We want to show that K is a closed n-dimensional pseudomanifold. The well-versed reader will easily spot that arguments below are at times almost identical to the arguments provided in the proof of Proposition 66.1.

Let us prove that the three conditions of a pseudomanifold, which we recalled above, are satisfied.

(a) Let s be a k-simplex of K. By hypothesis $\widetilde{\mathrm{H}}^{\mathrm{simp}}_{n-k-1}(\mathrm{Lk}(K,s)) \neq 0$. This implies immediately that s is the face of an n-simplex.

(b) Let s be an $(n-1)$-simplex of K. By hypothesis $\widetilde{\mathrm{H}}^{\mathrm{simp}}_{0}(\mathrm{Lk}(K,s)) \cong \mathbb{Z}$, which implies by Lemma 63.10 that $\mathrm{H}^{\mathrm{simp}}_{0}(\mathrm{Lk}(K,s)) \cong \mathbb{Z}^2$. Recall that on page 1776 we pointed out that the maximal dimension of a simplex of K is n. Since s is $(n-1)$-dimensional we see that $\mathrm{Lk}(K,s)$ is a 0-dimensional simplicial complex. Now we see that it consists of precisely two vertices, which implies that there are precisely two n-simplices which contain s as a face.

(c) We say that two n-simplices t and t' of K are *equivalent* if there exists a sequence $t = s_0, s_1, \ldots, s_k = t'$ of n-simplices such that any two consecutive simplices have a common $(n-1)$-dimensional face. Recall that our task is to show that any two n-simplices are equivalent in the above sense.
The key to doing so is the following claim:

Claim. Let t and t' be two n-simplices. If $t \cap t' \neq \varnothing$, then t and t' are actually equivalent in the above sense.

Given $m \in \{0, \ldots, n-1\}$ we say that two n-simplices x and x' are *m-equivalent* if there exists a sequence $x = s_0, s_1, \ldots, s_k = x'$ of n-simplices such that any two consecutive simplices have a common face of dimension $\geq m$. It suffices to show that if we are given $m \in \{0, \ldots, n-2\}$ such that any two n-simplices in K are m-equivalent, then any two n-simplices in K are also $(m+1)$-equivalent. In fact evidently it suffices to show that if we are given two n-simplices x and x' which have an m-simplex s in common, then x and x' are $(m+1)$-equivalent. To do so we first note that since $\dim(x) > \dim(s)$ we can pick a vertex w of $x \cap \mathrm{Lk}(K,s)$ and similarly we can pick a vertex w' of $x' \cap \mathrm{Lk}(K,s)$.

By hypothesis we know that $\widetilde{\mathrm{H}}_0^{\mathrm{simp}}(\mathrm{Lk}(K,s)) = 0$. It follows from this fact together with Lemma 43.1 and Exercise 61.11 that the abstract simplicial complex $\mathrm{Lk}(K,s)$ is in fact connected. This means in our setting that there exist 1-simplices $e_1,\dots,e_k \in S$ of $\mathrm{Lk}(K,s)$ such that the following holds:

(∗) We have $w \in e_1$, for every $i \in \{1,\dots,k-1\}$ we have $e_i \cap e_{i+1} \neq \varnothing$ and we have $w' \in e_k$.

Let $i \in \{1,\dots,k\}$. Since e_i is a simplex in $\mathrm{Lk}(K,s)$ there exists a simplex f_i of K with $e_i \cup s \subset f_i$. By (a) we can in fact find an n-simplex $\widetilde{f}_i$ of K with $e_i \cup s \subset f_i \subset \widetilde{f}_i$. We set $\widetilde{f}_0 := x$ and $\widetilde{f}_{k+1} := x'$. Note that it follows from (∗) that any two consecutive $\widetilde{f}_i$ and $\widetilde{f}_{i+1}$ have s and a vertex of $\mathrm{Lk}(K,s)$ in common, i.e. they have an $(m+1)$-simplex in common. This shows that x and x' are in fact $(m+1)$-connected. ⊞

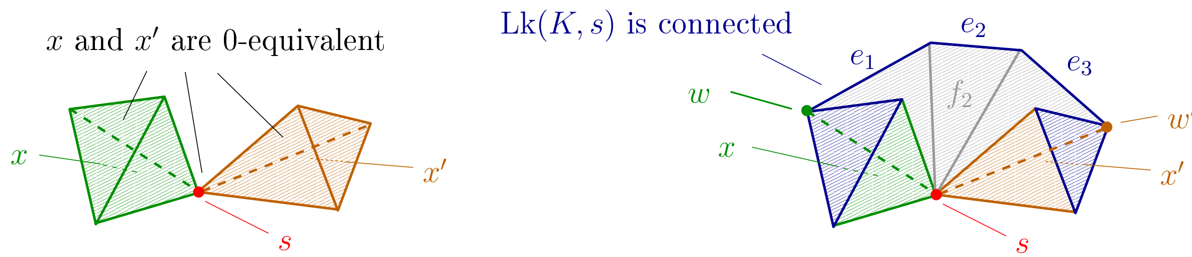


FIGURE 1100. Illustration for the proof of Proposition 71.2 (1).

Now let t and t' be any two n-simplices of K. By hypothesis K is connected. Together with (a) this implies easily that there exists a sequence of n-simplices $t = u_0, u_1, \dots, u_k = t'$ such that any two consecutive n-simplices share at least one vertex. By the claim any two consecutive n-simplices are equivalent. Hence t and t' are equivalent.

(2) Let K be a closed n-dimensional pseudomanifold. It follows from the discussion on page 1651 that the suspension $\Sigma(K) = K * \{\pm 1\}$ is a closed $(n+1)$-dimensional pseudomanifold. By the discussion on page 1523 we know that $\mathrm{Lk}(\Sigma(K),\{-1\}) = K$. Thus if $\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(K) \neq 0$ for two different dimensions, e.g. if $K = S_1 \times S_1$ is the simplicial torus, then $\Sigma(K)$ is not a simplicial homology manifold.

(3) Let K be a closed n-dimensional PL-manifold and let s be a k-simplex. We see that

$$\widetilde{\mathrm{H}}_i^{\mathrm{simp}}(\mathrm{Lk}(K,s)) \underset{\substack{\uparrow \\ \text{by Theorem 63.23}}}{\cong} \widetilde{\mathrm{H}}_i(|\mathrm{Lk}(K,s)|) \underset{\substack{\uparrow \\ \text{since } \mathrm{Lk}(K,s) \text{ is a PL } (n-k-1)\text{-sphere we know} \\ \text{that } |\mathrm{Lk}(K,s)| \text{ is homeomorphic to } \partial\Delta^{n-k} \text{ in particular} \\ |\mathrm{Lk}(K,s)| \text{ is homeomorphic to } S^{n-k-1}}}{\cong} \widetilde{\mathrm{H}}_i(S^{n-k-1}) \cong \begin{cases} \mathbb{Z}, & \text{if } i = n-k-1, \\ 0, & \text{else.} \end{cases}$$

(4) By Proposition 66.11 we know that there exists a closed orientable 3-dimensional smooth manifold Y that is a homology 3-sphere, but such that $\pi_1(Y)$ is a non-trivial

group. By Theorem 64.2 and Theorem 64.14 we can equip Y with a PL-structure $(K, \Theta\colon |K| \to Y)$. We consider the suspension $\Sigma(K) = K * \{\pm 1\}$. We want to show that $\Sigma(K)$ is a closed 4-dimensional simplicial homology manifold but that it is not a PL-manifold. It turns out that the simplices of $\Sigma(K) = K * \{\pm 1\}$ come in the following three flavors:

(a) Any k-simplex s of K is also a k-simplex of $\Sigma(K)$. We have, almost by definition, that $\mathrm{Lk}(\Sigma(K), s) = \Sigma(\mathrm{Lk}(K, s))$. It follows from this observation, together with (1) and Lemma 63.13 (2) that $\mathrm{Lk}(\Sigma(K), s)$ is a simplicial homology $(3-k)$-sphere.

(b) Given a k-simplex s of K we obtain the $(k+1)$-simplices $s \cup \{-1\}$ and $s \cup \{1\}$ of $\Sigma(K)$. Note that it follows almost immediately from the various definitions that $\mathrm{Lk}(\Sigma(K), s \cup \{-1\}) = \mathrm{Lk}(K, s) = \mathrm{Lk}(\Sigma(K), s \cup \{1\})$. Thus we obtain from (1) that $\mathrm{Lk}(\Sigma(K), s \cup \{-1\})$ and $\mathrm{Lk}(\Sigma(K), s \cup \{1\})$ are both simplicial homology $(3-(k+1))$-spheres.

(c) Finally we need to consider the 0-simplices ± 1 of $\Sigma(K) = K * \{\pm 1\}$. As we discussed on page 1523, we have $\mathrm{Lk}(\Sigma(K), \{\pm 1\}) = K$. Since K is a homology 3-sphere we see that $\mathrm{Lk}(\Sigma(K), \{\pm 1\}) = K$ are simplicial homology 3-spheres. But since $\pi_1(|K|) = \pi_1(Y)$ is by construction a non-trivial group we see that $|\mathrm{Lk}(\Sigma(K), \{\pm 1\})| = |K|$ are not homeomorphic to the 3-sphere, let alone are $\mathrm{Lk}(\Sigma(K), \{\pm 1\})$ PL 3-spheres or PL-balls.

Thus we see that $\Sigma(K)$ is a 4-dimensional simplicial homology manifold, but $\Sigma(K)$ is not a PL-manifold. ■

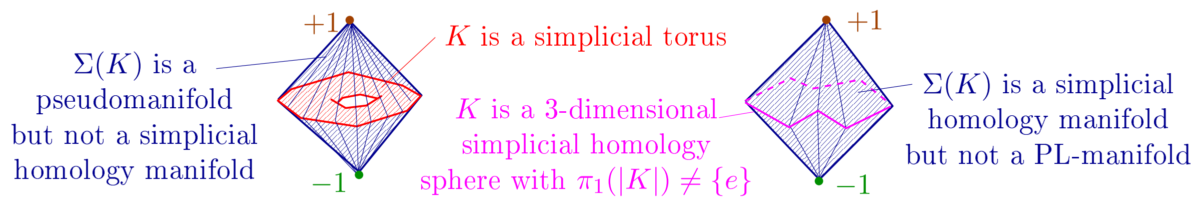


FIGURE 1101. Illustration for the proof of Proposition 71.2 (2) and (4).

We conclude this section with the following lemma.

Lemma 71.3. *Let K be an abstract simplicial complex. If $|K|$ is an n-dimensional topological manifold with $\partial(|K|) = \varnothing$, then K is an n-dimensional simplicial homology manifold.*

PROOF. Let s be a k-simplex of K. We need to show that $\mathrm{Lk}(K, s)$ is a simplicial homology $(n-k-1)$-sphere. We pick a point P in the corresponding open simplex $\langle s \rangle$. We perform the following calculation:

$$\widetilde{\mathrm{H}}^{\mathrm{simp}}_i(\mathrm{Lk}(K,s)) \underset{\substack{\uparrow \\ \text{by Proposition 66.2}}}{\cong} \widetilde{\mathrm{H}}_{i+k+1}(|K|, |K| \setminus \{P\}) \underset{\substack{\uparrow \\ \text{by Lemma 44.1, since } |K| \text{ is an } n\text{-dimensional} \\ \text{topological manifold with } \partial(|K|) = \varnothing}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } i+k+1 = n, \\ 0, & \text{else.} \end{cases}$$

As a parenthetical remark we point out that alternatively we could have proved the lemma using Lemma 71.1 instead of using Proposition 66.2. ■

71.2. **The statement of the Simplicial Poincaré Duality Theorem.** Let K be a closed connected n-dimensional simplicial homology manifold. By Proposition 71.2 (1) we know that K is in particular a closed n-dimensional pseudomanifold. In particular it makes sense to talk of orientations of K. We recall the notion of a fundamental class.

(1) If K is oriented, then by the definition on page 1653 we have a fundamental class $[K] \in \mathrm{H}_n^{\mathrm{simp}}(K)$. In the following, given a commutative ring R we denote by $[K]$ also the image of $[K]$ under the natural map $\mathrm{H}_n^{\mathrm{simp}}(K) \xrightarrow{[\sigma] \mapsto [\sigma \otimes 1_R]} \mathrm{H}_n^{\mathrm{simp}}(K;R)$.
(2) Regardless of whether or not K is orientable we defined on page 1653 the $\mathbb{F}_2$-fundamental class $[K]_{\mathbb{F}_2} \in \mathrm{H}_n^{\mathrm{simp}}(K;\mathbb{F}_2)$.

The following theorem is the main theorem on simplicial (co-) homology groups of simplicial homology manifolds.

Theorem 71.4. (Simplicial Poincaré Duality Theorem) *Let K be a closed n-dimensional simplicial homology manifold and let $k \in \mathbb{N}_0$. If K is orientable, then for every choice of orientation and every commutative ring R the map*

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{simp}}(K;R) & \to & \mathrm{H}_{n-k}^{\mathrm{simp}}(K;R) \\ \varphi & \mapsto & \varphi \cap [K] \end{array}$$

is an isomorphism. Regardless of whether or not K is orientable the map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{F}_2) & \to & \mathrm{H}_{n-k}^{\mathrm{simp}}(K;\mathbb{F}_2) \\ \varphi & \mapsto & \varphi \cap [K]_{\mathbb{F}_2} \end{array}$$

is an isomorphism.

Remark. One might think or hope that the Simplicial Poincaré Duality Theorem 71.4 also holds for closed (orientable) pseudomanifolds, but as we will see in Exercise 71.1 this is not the case. Nonetheless, it turns out that on pseudomanifolds one can introduce more refined (co-) homology groups, namely intersection homology, that do satisfy an analogue of the Poincaré Duality Theorem. We refer to [**Ban07**] for details.

The proof of the Simplicial Poincaré Duality Theorem 71.4 is evidently non-trivial and somewhat lengthy. Thus we will first draw several interesting conclusions before we head to the proof.

Example. Let K be a closed orientable n-dimensional simplicial homology manifold. Let us see what the Simplicial Poincaré Duality Theorem 71.4 tells us for $k = 1$. We have the following isomorphisms:

$$\begin{array}{rcl} \mathrm{H}_{n-1}^{\mathrm{simp}}(K) & \underset{\substack{\uparrow \\ \text{Poincaré Duality Theorem 71.4}}}{\cong} & \mathrm{H}^1_{\mathrm{simp}}(K;\mathbb{Z}) \underset{\substack{\uparrow \\ \text{Universal Coefficient Theorem 67.8}}}{\cong} \mathrm{Hom}(\mathrm{H}_1^{\mathrm{simp}}(K),\mathbb{Z}) \oplus \mathrm{Ext}(\overbrace{\mathrm{H}_0^{\mathrm{simp}}(K)}^{\text{free abelian group by Exercise 63.3}},\mathbb{Z}) \\ & \underset{\substack{\uparrow \\ \text{Lemma 75.5 (3)}}}{\cong} & \mathrm{Hom}(\mathrm{H}_1^{\mathrm{simp}}(K),\mathbb{Z}) \underset{\substack{\uparrow \\ \text{by Lemma 63.9 (1) we know that the group the group } \mathrm{H}_1^{\mathrm{simp}}(K) \text{ is finitely} \\ \text{generated, hence it is isomorphic to } \mathbb{Z}^k \oplus T \text{ for some } k \in \mathbb{N} \text{ and some finite group } T}}{\cong} \mathrm{Hom}(\mathbb{Z}^k \oplus T,\mathbb{Z}) \cong \mathbb{Z}^k. \end{array}$$

We have thus shown that $\mathrm{H}_{n-1}^{\mathrm{simp}}(K)$ is a free abelian group. In the more general setting of closed orientable pseudomanifolds we had obtained the same result in Theorem 66.4, but using a rather different argument.

Our richest source of simplicial homology manifolds are smooth manifolds. We obtain the following corollary to the Simplicial Poincaré Duality Theorem 71.4.

Theorem 71.5. (Poincaré Duality Theorem) *Let M be a closed n-dimensional smooth manifold and let $k \in \mathbb{N}_0$. If M is orientable, then for every choice of orientation and every commutative ring R the map*

$$\begin{array}{rcl} \mathrm{H}^k(M;R) & \to & \mathrm{H}_{n-k}(M;R) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism. Regardless of whether or not M is orientable the map

$$\begin{array}{rcl} \mathrm{H}^k(M;\mathbb{F}_2) & \to & \mathrm{H}_{n-k}(M;\mathbb{F}_2) \\ \varphi & \mapsto & \varphi \cap [M]_{\mathbb{F}_2} \end{array}$$

is an isomorphism.

Proof. It follows easily from Lemma 83.9 that we can assume that M is actually connected. Recall that by Theorem 64.2 we know that M admits a smooth simplicial structure $(K = (V,S), f\colon |K| \to M)$. Without loss of generality we can assume that $|K| = M$. We denote by $\Xi_*\colon \mathrm{H}_k^{\mathrm{simp}}(K;R) \to \mathrm{H}_k(M;R)$ and $\Xi^*\colon \mathrm{H}^k(M;R) \to \mathrm{H}_k^{\mathrm{simp}}(K;R)$ the natural isomorphisms provided by Proposition 67.1 (1) and 67.6 (3). By Lemma 71.3 we know that K is a closed n-dimensional simplicial homology manifold.

Let us first deal with the case that M is equipped with a natural orientation. By Proposition 66.7 we know that the simplicial homology manifold K inherits an orientation. Next we consider the following diagram:

$$\begin{tikzcd} \mathrm{H}^k_{\mathrm{simp}}(K;R) \arrow[rr, "\cap[K]", "\cong"'] & & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;R) \arrow[d, "\Xi_*", "\cong"'] \\ \mathrm{H}^k(M;R) \arrow[u, "\Xi^*", "\cong"'] \arrow[rr, "\cap[M]"] & & \mathrm{H}^{\mathrm{simp}}_{n-k}(M;R). \end{tikzcd}$$

We make the following observations:

(1) By Algorithm 68.2 we know that $\Xi_*([K]) = [M]$.
(2) It follows from (1) and Proposition 67.17 (3) that the diagram commutes.
(3) The top horizontal map is an isomorphism by the Simplicial Poincaré Duality Theorem 71.4.

Since the vertical maps are isomorphisms we obtain from the above that the bottom horizontal map is, as promised, an isomorphism.

The proof for the statement regarding $\mathbb{F}_2$-coefficients is almost identical. We leave it to the reader to make the necessary straightforward modifications to the above argument. ■

There are many applications of the Poincaré Duality Theorem 71.5. In the following we will state two results. Since we have a predilection for smooth manifolds we state them in the context of smooth manifolds, but the statements also hold in the admittedly more general setting of simplicial homology manifolds.

Theorem 88.6. *Let M be a closed n-dimensional smooth manifold and let $k \in \mathbb{N}_0$.*

1. *We have* $\dim_{\mathbb{F}_2}(\mathrm{H}_k(M;\mathbb{F}_2)) = \dim_{\mathbb{F}_2}(\mathrm{H}_{n-k}(M;\mathbb{F}_2))$.
2. *If M is orientable, then for any field $\mathbb{F}$ we have* $\dim_{\mathbb{F}}(\mathrm{H}_k(M;\mathbb{F})) = \dim_{\mathbb{F}}(\mathrm{H}_{n-k}(M;\mathbb{F}))$.

Proposition 88.7. *The Euler characteristic of any closed odd-dimensional smooth manifold is zero.*

As the reader will have noticed, the numbers of these results correspond to Chapter 88. In fact, in Chapter 88 we will prove a more general Poincaré Duality Theorem 88.1 that applies also to topological manifolds, possibly with boundary. In Section 88.2 we discuss many consequences of the Poincaré Duality Theorem 88.1. For closed smooth manifolds all statements in Section 88.2 can also be deduced from the above Poincaré Duality Theorem 71.5. In particular this way we obtain the above results. To avoid repetition, and for the sake of brevity, we do not spell out the proofs twice.

There are many other applications of the Poincaré Duality Theorem 71.5. Arguably one of the most interesting applications is that the Poincaré Duality Theorem 71.5 allows us to compute the cup product of the real and complex projective spaces. This will be done in Chapter 90.

Remark. The Poincaré Duality Theorem 71.5 has two significant limitations and one aesthetic issue.

1. The theorem only deals with closed smooth manifolds. In fact the approach taken in the proof of Poincaré Duality Theorem 71.4 can be generalized, with some effort, to deal with compact (oriented) smooth manifolds with boundary. We refer to [**SZ94**, Satz 14.8.8] for details.
2. The Poincaré Duality Theorem 71.5 applies only to smooth manifolds. There is though some room for improvement. More precisely, the Simplicial Poincaré Duality Theorem 71.4 together with Lemma 71.3 shows that we have Poincaré Duality for any closed oriented topological manifold that admits a simplicial structure. But by Manolescu's Theorem 85.33 we know that for every $n \geq 5$ there exists a closed orientable n-dimensional topological manifold that does not admit a simplicial structure. Thus the Poincaré Duality Theorem 71.4 does not imply that Poincaré Duality holds for all closed topological manifolds.
3. Even for smooth manifolds the proof of the Poincaré Duality Theorem 71.5 is aesthetically problematic since it makes use of the somewhat artificial choice of a simplicial structure.

All three issues will be addressed by the Poincaré Duality Theorem 88.1 that we will state and prove in Chapter 88. Nonetheless, the (proof) of the Poincaré Duality Theorems 71.4 and 71.5 also have two advantages over the Poincaré Duality Theorem 88.1:

1. The proofs in this chapter are much more visual, whereas the proof of the Poincaré Duality Theorem 88.1 is rather formal and it is difficult to get an intuition for "why" and "how" it works.

(2) More importantly, much later in Chapter **??** we will introduce a new invariant, namely Reidemeister torsion. The intermediate results in our proof of Poincaré Duality Theorem 71.4 will show that Reidemeister torsion of closed smooth manifolds is symmetric.

71.3. **The barycentric subdivision.** The proof of the Simplicial Poincaré Duality Theorem 71.4 relies on a clever use of the barycentric subdivision of the simplicial homology manifold. In this section we will recall the main definitions and statements on barycentric subdivisions which we will need later on in this chapter.

We recall the following definitions from pages 1518 and 1580.

Definition. Let $K = (V, S)$ be an abstract simplicial complex.

(1) For each k-simplex $s = \{v_0, \dots, v_k\} \in S$ we define the *barycenter of* s to be the point

$$\underline{s} \;:=\; \sum_{v \in s} \tfrac{1}{\dim(s)+1} \cdot v \;=\; \tfrac{1}{k+1} \cdot v_0 + \dots + \tfrac{1}{k+1} \cdot v_k \;\in\; \mathbb{R}^{(V)}.$$

(2) The *barycentric subdivision of* K is defined to be the following abstract simplicial complex:

$$\mathrm{sd}(K) \;=\; \big(\{\underline{s}\}_{s \in S}, \{\{\underline{s_0}, \dots, \underline{s_k}\} \,|\, s_0 \subsetneq \dots \subsetneq s_k\}\big).$$

(3) Given an ordered k-simplex $(v_0, \dots, v_k)$ of K we consider the following ordered k-simplex of $\mathrm{sd}(K)$:

$$F(v_0, \dots, v_k) \;:=\; \big(\underline{\{v_0\}}, \underline{\{v_0, v_1\}}, \dots, \underline{\{v_0, \dots, v_k\}}\big).$$

(4) Given an ordered k-simplex $s = (v_0, \dots, v_k)$ of K we set

$$u_k(s) \;:=\; \sum_{\sigma \in \mathrm{Bij}\{0,\dots,k\}} (-1)^{\mathrm{sign}(\sigma)} \cdot [F(v_{\sigma(0)}, \dots, v_{\sigma(k)})] \;\in\; \mathrm{C}_k^{\mathrm{simp}}(\mathrm{sd}(K)).$$

We refer to the resulting map

$$\begin{array}{rcl} \mathrm{C}_k^{\mathrm{simp}}(K) & \to & \mathrm{C}_k^{\mathrm{simp}}(\mathrm{sd}(K)) \\ {[s]} & \mapsto & u_k(s) \end{array}$$

as the *simplicial subdivision map*. By Lemma 63.41 (1) we know that these subdivision maps are actually well-defined and that they form a chain map.

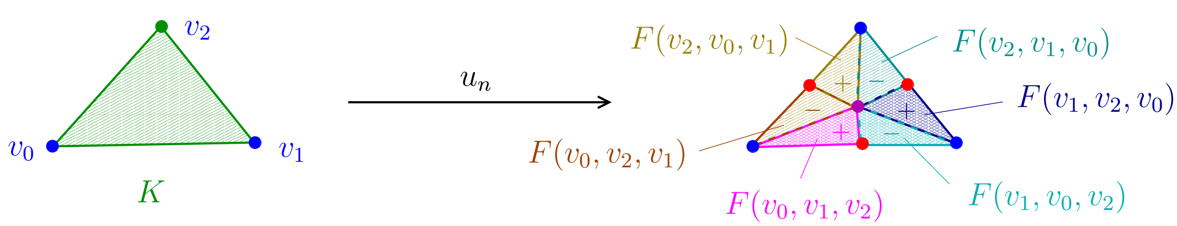


FIGURE 1102

Remark. Let $K = (V, S)$ be an abstract simplicial complex. Note that it follows easily from the definitions that given any n-simplex t of $\mathrm{sd}(K)$ there exists a unique ordered n-simplex $(v_0, \dots, v_n)$ of K with $\lfloor F(v_0, \dots, v_n) \rfloor = t$.

The following lemma shows that subdivision maps induce isomorphisms on simplicial (co-) homology groups.

Lemma 63.38. *Let $(K, \leq)$ be an ordered abstract simplicial complex.*

(1) *There exists a unique simplicial map $\sigma\colon \operatorname{sd}(K) \to K$, called the* stretching map, *with the following property: given an n-simplex $s = \{v_0 < \cdots < v_n\}$ of K the map σ sends the corresponding vertex $\underline{s}$ of $\operatorname{sd}(K)$ to the vertex v_n of K.*

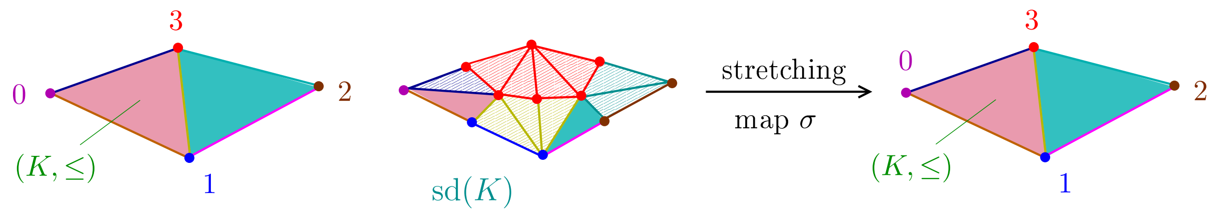


FIGURE 1103. Illustration of Lemma 63.38.

The following lemma is an excerpt of Lemma 63.41 together with a little dose of Lemma 73.8 (4).

Lemma 63.41. *Let K be an abstract simplicial complex. We pick an order on K and we consider the corresponding stretching map $\sigma\colon \operatorname{sd}(K) \to K$.*

(5) *For every choice of some $k \in \mathbb{N}_0$ and of some abelian group G the two induced maps $u_*\colon \mathrm{H}_k^{\mathrm{simp}}(K;G) \to \mathrm{H}_k^{\mathrm{simp}}(\operatorname{sd}(K);G)$ and $u^*\colon \mathrm{H}^k_{\mathrm{simp}}(\operatorname{sd}(K);G) \to \mathrm{H}^k_{\mathrm{simp}}(K;G)$ are isomorphisms and the inverses are given by σ_* and σ^*.*

Next we turn to barycentric subdivisions of pseudomanifolds. At this point it is convenient to introduce the following definition.

Definition. Let $K = (V, S)$ be a closed oriented n-dimensional pseudomanifold. Given an ordered n-simplex t of K we write

$$\operatorname{sign}(t) := \begin{cases} +1, & \text{if the oriented } n\text{-simplex } [(+1,t)] \text{ corresponds to the orientation,} \\ -1, & \text{otherwise.} \end{cases}$$

The following lemma contains everything anybody ever wanted to know about barycentric subdivisions of closed pseudomanifolds.

Lemma 71.6. *Let $K = (V, S)$ be a closed n-dimensional pseudomanifold.*

(1) *The barycentric subdivision $\operatorname{sd}(K)$ is a closed n-dimensional pseudomanifold.*
(2) *Suppose that K is equipped with an orientation. Let t be an n-simplex of $\operatorname{sd}(K)$. As we pointed out on page 1784, there exists a unique ordered n-simplex $(v_0, \ldots, v_n)$ of K with $\lfloor F(v_0, \ldots, v_n) \rfloor = t$. We equip the n-simplex t of $\operatorname{sd}(K)$ with the orientation given by $(\operatorname{sign}(t), F(v_0, \ldots, v_n))$.*
   (a) *The above orientations of the n-simplices define an orientation for $\operatorname{sd}(K)$.*
   (b) *We have $u_*([K]) = [\operatorname{sd}(K)] \in \mathrm{H}_n^{\mathrm{simp}}(\operatorname{sd}(K))$.*

(3) *If K is oriented, then given any $k \in \mathbb{N}_0$ and given any commutative ring R the following diagram commutes:*

$$\begin{tikzcd} \mathrm{H}^k_{\mathrm{simp}}(K;R) \arrow[r, "\cap[K]"] & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;R) \arrow[d, "\cong"', "u_*"] \\ \mathrm{H}^k_{\mathrm{simp}}(\mathrm{sd}(K);R) \arrow[u, "u^*", "\cong"'] \arrow[r, "\cap[\mathrm{sd}(K)]"] & \mathrm{H}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K);R). \end{tikzcd}$$

The analogues of (2) and (3) hold for arbitrary closed n-dimensional pseudomanifolds if we work with $\mathbb{F}_2$-coefficients and the simplicial $\mathbb{F}_2$-fundamental classes $[K]_{\mathbb{F}_2}$ and $[\mathrm{sd}(K)]_{\mathbb{F}_2}$.

Sketch of proof. Statements (1) and (2a) follow almost immediately from the definitions, the proof is outsourced to Exercise 66.5. Furthermore Statement (2b) can be deduced easily from the definition of the fundamental class on page 1653 and the definition of the subdivision map $u_n\colon \mathrm{C}^{\mathrm{simp}}_n(K) \to \mathrm{C}^{\mathrm{simp}}_n(\mathrm{sd}(K))$. We leave it to the reader to fill in the details.

Next we turn to the proof of (3). We pick an order "$\le$" on K and we denote by $\sigma\colon \mathrm{sd}(K) \to K$ the corresponding stretching map. Note that by the above Lemma 63.41 (5) we know that $u_* = \sigma_*^{-1}$ and $u^* = (\sigma^*)^{-1}$. It follows that it suffices to show that the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}^k_{\mathrm{simp}}(K;R) \arrow[r, "\cap[K]"] \arrow[d, "\sigma^*"', "\cong"] & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;R) \\ \mathrm{H}^k_{\mathrm{simp}}(\mathrm{sd}(K);R) \arrow[r, "\cap[\mathrm{sd}(K)]"] & \mathrm{H}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K);R). \arrow[u, "\cong", "\sigma_*"'] \end{tikzcd}$$

Now let $\varphi \in \mathrm{H}^k_{\mathrm{simp}}(K;R)$. We see that we have the following equalities:

$$\sigma_*(\sigma^*(\varphi) \cap [\mathrm{sd}(K)]) \underset{\substack{\uparrow \\ \text{by Lemma 67.20, here we use} \\ \text{that } \sigma\colon \mathrm{sd}(K)\to K \text{ is a simplicial map}}}{=} \varphi \cap \sigma_*([\mathrm{sd}(K)]) \underset{\substack{\uparrow \\ \text{by (2b)}}}{=} \varphi \cap [K].$$

Finally we leave it to the reader to verify that the $\mathbb{F}_2$-analogues also hold. ■

71.4. **The dual chain complex I.** In this section, given a pseudomanifold K, we will introduce the dual chain complex $\mathrm{D}^{\mathrm{simp}}_*(K)$. This is the main new concept needed for the proof of the Simplicial Poincaré Duality Theorem 71.4.

Definition. Let $K = (V,S)$, let $s \in S$ be a k-simplex and let $v_1,\dots,v_l \in V$ such that $s \cup \{v_1,\dots,v_l\}$ is a $(k+l)$-simplex of K. We set

$$G(s \mid v_1,\dots,v_l) \;=\; \overbrace{(\underline{s}, \underline{s\cup\{v_1\}},\dots,\underline{s\cup\{v_1,\dots,v_l\}})}^{\text{ordered } l\text{-simplex of } \mathrm{sd}(K)}.$$

Remark. Note that for an ordered k-simplex $s = (v_0,\dots,v_k)$ of K we have by definition $G(\{v_0\} \mid v_1,\dots,v_k) = F(v_0,\dots,v_k)$.

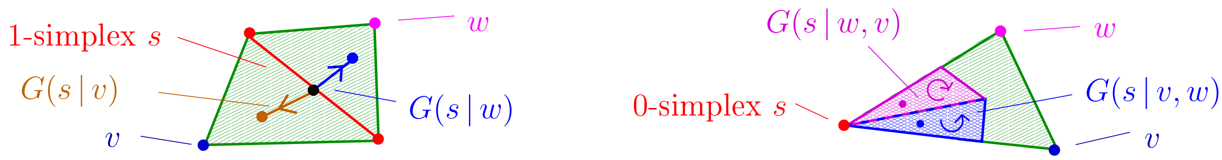


FIGURE 1104

Definition. Let $K = (V, S)$ be a closed oriented n-dimensional pseudomanifold.

(1) Let $s = (v_0, \dots, v_k)$ be an ordered k-simplex of K. We define the *dual chain* of s to be the following simplicial chain in $C^{\text{simp}}_{n-k}(\text{sd}(K))$:

$$s^\dagger := \sum_{\substack{(v_{k+1},\dots,v_n)\in V^{n-k} \text{ such that} \\ \{v_0,\dots,v_k,v_{k+1},\dots,v_n\}\in S_n}} \text{sign}((v_0,\dots,v_k,v_{k+1},\dots,v_n)) \cdot [\, \underbrace{G(s\,|\,v_{k+1},\dots,v_n)}_{\text{ordered } (n\text{-}k)\text{-simplex of sd}(K)} \,].$$

(2) We define

$$D^{\text{simp}}_{n-k}(K) \;=\; \begin{array}{c}\text{subgroup of } C^{\text{simp}}_{n-k}(\text{sd}(K)) \text{ generated by the } s^\dagger, \\ \text{where } s \text{ runs over all } k\text{-simplices of } K.\end{array}$$

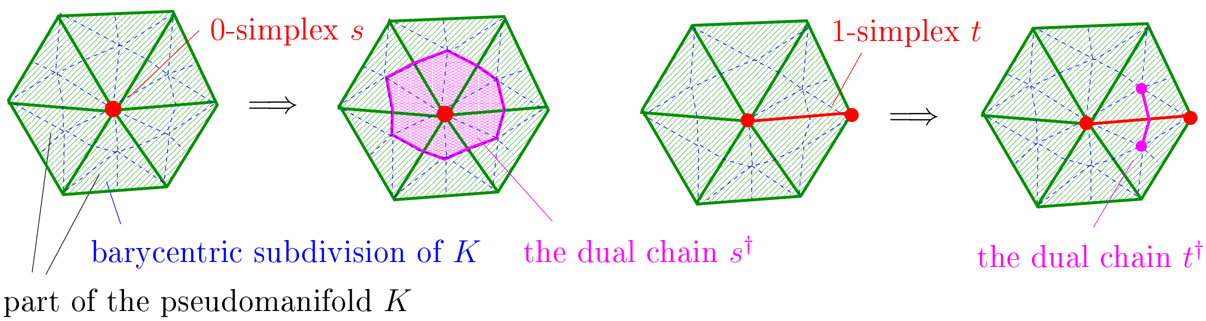


FIGURE 1105. The dual chains of a 0-simplex and a 1-simplex.

Lemma 71.7. *Let $K = (V, S)$ be a closed oriented n-dimensional pseudomanifold and let $k \in \mathbb{N}_0$.*

(1) *Given two ordered k-simplices s and $\widetilde{s}$ with $[s] = [\widetilde{s}] \in C^{\text{simp}}_k(K)$ we have $s^\dagger = \widetilde{s}^\dagger$.*

(2) *Let "$\le$" be an order on $K = (V, S)$. The dual chains*

$$\{s^\dagger \,|\, s = (v_0, \dots, v_k) \text{ an ordered } k\text{-simplex of } K \text{ with } v_0 < \dots < v_k\}$$

form a basis for $D^{\text{simp}}_{n-k}(K)$.

PROOF $(*)$. The first statement of the lemma follows easily from the definitions. Thus let us turn to the proof of the second statement. We pick an order "$\le$" on $K = (V, S)$. We set

$$Y \;:=\; \{(v_0, \dots, v_k) \in V^{k+1} \,|\, \{v_0, \dots, v_k\} \in S_k \text{ and } v_0 < \dots < v_k\}.$$

Recall that by Proposition 66.1 we know that in a closed n-dimensional pseudomanifold any simplex is the face of an n-simplex. It follows that given any $y = (v_0, \dots, v_k) \in Y$ we can pick an ordered $(n-k)$-simplex $\widetilde{y} = (w_0, \dots, w_{n-k})$ of K such that $v_k = w_0$, such that

$\{v_0, \ldots, v_k = w_0, \ldots, w_{n-k}\} \in S_n$ and such that $w_1 < \cdots < w_{n-k}$. We make the following observation:

$(*)$ It follows easily from the definition of an order that for $y_1 \neq y_2 \in Y$ we have $\widetilde{y}_1 \neq \widetilde{y}_2$.

Now we consider the following maps:

$$\begin{tikzcd}
\bigoplus\limits_{y\in Y} \mathbb{Z}\cdot y \arrow[r, "y\mapsto y^\dagger"] \arrow[d, "\substack{y\\ \downarrow \\ \widetilde{y}}"] & \mathrm{D}^{\mathrm{simp}}_{n-k}(K) \arrow[r, hook, "\iota"] & \mathrm{C}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K)) \arrow[d, bend right, "\Phi^{-1}"'] \\
\bigoplus\limits_{\{\widetilde{y}\,|\,y\in Y\}} \mathbb{Z}\cdot \widetilde{y} & & \bigoplus\limits_{\substack{(w_0,\ldots,w_{n-k})\\ \text{ordered}\\ (n\text{-}k)\text{-simplex of } K}} \mathbb{Z}\cdot(w_0,\ldots,w_{n-k}) \arrow[ll, two heads, "\text{projection}"', "p"] \arrow[u, "\Phi"', "\substack{F(w_0,\ldots,w_{n-k})\\ \uparrow \\ (w_0,\ldots,w_{n-k})}"]
\end{tikzcd}$$

We make the following observations:

(a) The right vertical map is an isomorphism. This follows from the observation, which we basically already mentioned on page 1784, that given any ordered $(n-k)$-simplex $(\underline{s}_0, \ldots, \underline{s}_{n-k})$ of $\mathrm{sd}(K)$ with $s_0 \subsetneq s_1 \subsetneq \cdots \subsetneq s_{n-k}$ there exists a unique ordered n-simplex $(w_0, \ldots, w_{n-k})$ of K with $F(w_0, \ldots, w_{n-k}) = (\underline{s}_0, \ldots, \underline{s}_{n-k})$.

(b) It follows immediately from the definitions of the various maps that the diagram commutes.

Now we turn to the actual proof of the second statement of the lemma. First note that it follows from (1) that $\{y^\dagger\}_{y\in Y}$ form a generating set for $\mathrm{D}^{\mathrm{simp}}_{n-k}(K)$. Secondly note that it follows from (a) and (b) that the map $p \circ \Phi^{-1} \circ \iota\colon \mathrm{D}^{\mathrm{simp}}_{n-k}(K) \to \bigoplus\limits_{\{\widetilde{y}\,|\,y\in Y\}} \mathbb{Z}\cdot\widetilde{y}$ sends $y^\dagger$ to $\widetilde{y}$. It follows from $(*)$ that the $\{y^\dagger\}_{y\in Y}$ are linearly independent. Thus we have shown that the given dual chains $\{y^\dagger\}_{y\in Y}$ are a basis for $\mathrm{D}^{\mathrm{simp}}_{n-k}(K)$. ■

To fully appreciate our next lemma it is worth recalling the following lemma.

Lemma 67.4. *Let $K = (V, S)$ be an abstract simplicial complex. Given any ordered k-simplex $s = (v_0, \ldots, v_k)$ we have the following equality:*

$$\delta_k([v_0, \ldots, v_k]^*) \;=\; (-1)^{k+1} \cdot \sum_{\substack{w\in V \text{ such that}\\ \{v_0,\ldots,v_k,w\} \text{ is}\\ \text{a } (k+1)\text{-simplex}}} [v_0, \ldots, v_k, w]^* \;\in\; \mathrm{C}^{k+1}_{\mathrm{simp}}(K;\mathbb{Z}).$$

Now we can state the promised lemma. Its statement perfectly matches the statement of Lemma 67.4.

Lemma 71.8. *Let $K = (V, S)$ be an abstract simplicial complex that is an orientable n-dimensional pseudomanifold.*

(1) *Given any ordered k-simplex $s = (v_0, \dots, v_k)$ we have the following equality:*

$$\partial_{n-k}(\underbrace{[v_0, \dots, v_k]^\dagger}_{\in C^{\text{simp}}_{n-k}(\text{sd}(K))}) = \sum_{\substack{w \in V \text{ such that} \\ \{v_0, \dots, v_k, w\} \in S_{k+1}}} [v_0, \dots, v_k, w]^\dagger \in C^{\text{simp}}_{n-k-1}(\text{sd}(K)).$$

(2) *The groups $D^{\text{simp}}_*(K)$ form a subcomplex of $C^{\text{simp}}_*(\text{sd}(K))$.*

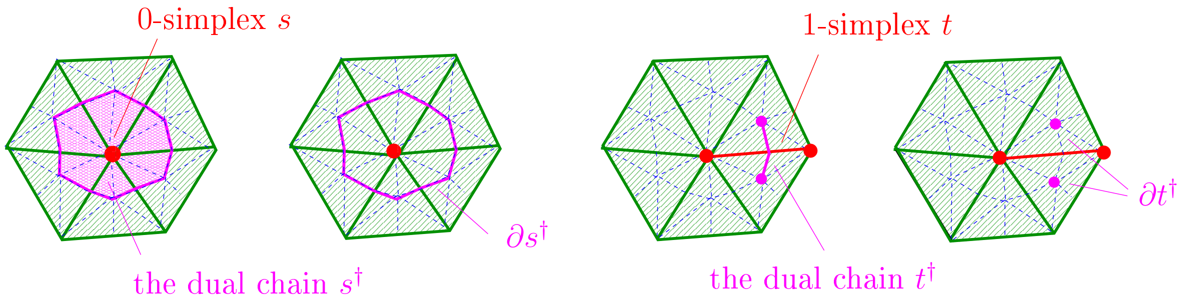


FIGURE 1106. Illustration of Lemma 71.8.

PROOF. The second statement is an immediate consequence of the first statement. Thus we just need to prove the first statement. We start out with the following elementary observation.

Observation. Let $s = (v_0, \dots, v_k)$ be some ordered k-simplex of K and furthermore let $(v_{k+1}, \dots, v_n) \in V^{n-k}$ such that $s \cup \{v_{k+1}, \dots, v_n\} = \{v_0, \dots, v_k, v_{k+1}, \dots, v_n\} \in S_n$. Given any $i \in \{1, \dots, n-k-1\}$ we have by definition

$$\begin{array}{c} G(s \,|\, v_{k+1}, .., v_n) \text{ with} \\ i\text{-th vertex removed} \end{array} = (\underline{s}, .., \underline{s \cup \{v_{k+1}, .., v_{k+i-1}\}}, \underline{s \cup \{v_{k+1}, .., v_{k+i+1}\}}, .., \underline{s \cup \{v_{k+1}, .., v_n\}}).$$

Furthermore it follows immediately from the definitions that we have

$$\begin{aligned} G(s \,|\, v_{k+1}, .., v_n) \text{ with 0-th vertex removed} &= G(s \cup \{v_{k+1}\} \,|\, v_{k+2}, \dots, v_n) \\ G(s \,|\, v_{k+1}, .., v_n) \text{ with } (n-k)\text{-th vertex removed} &= G(s \,|\, v_{k+1}, v_{k+2}, \dots, v_{n-1}). \end{aligned}$$

Now we turn to the actual proof of the first statement. Thus let $s = (v_0, \ldots, v_k)$ be an ordered k-simplex of K. We see that

definition of $s^\dagger$ and definition of the simplicial boundary map

$$\partial s^\dagger \overset{\downarrow}{=} \sum_{\substack{(v_{k+1},\ldots,v_n)\in V^{n\text{-}k} \text{ such that} \\ \{v_0,\ldots,v_k,v_{k+1},\ldots,v_n\}\in S_n}} \sum_{i=0}^{n-k} (-1)^i \cdot \operatorname{sign}(v_0,\ldots,v_n)\cdot \begin{bmatrix} G(s\,|\,v_{k+1},\ldots,v_n) \text{ with} \\ i\text{-th vertex removed} \end{bmatrix}$$

$$= \sum_{\substack{(v_{k+1},\ldots,v_n)\in V^{n\text{-}k} \text{ such that} \\ \{v_0,\ldots,v_k,v_{k+1},\ldots,v_n\}\in S_n}} \sum_{i=0 \text{ or } i=n\text{-}k} (-1)^i \cdot \operatorname{sign}(v_0,\ldots,v_n)\cdot \begin{bmatrix} G(s\,|\,v_{k+1},\ldots,v_n) \text{ with} \\ i\text{-th vertex removed} \end{bmatrix}$$

given a fixed $(v_{k+1},\ldots,v_n) \in V^{n-k}$ with $\{v_0,\ldots,v_k,v_{k+1},\ldots,v_n\} \in S_n$ and given $i \in \{1,\ldots,n-k-1\}$ it follows almost immediately from the above observation that the contributions by $((v_{k+1},\ldots,v_{k+i},v_{k+i+1},\ldots,v_n),i)$ and $((v_{k+1},\ldots,v_{k+i+1},v_{k+i},\ldots,v_n),i+1)$ cancel, in other words, for $i = 1,\ldots,n-k-1$ all terms cancel

$$= \sum_{\substack{(v_{k+1},\ldots,v_n)\in V^{n\text{-}k} \text{ such that} \\ \{v_0,\ldots,v_k,v_{k+1},\ldots,v_n\}\in S_n}} \operatorname{sign}(v_0,\ldots,v_n)\cdot \begin{bmatrix} G(s\,|\,v_{k+1},\ldots,v_n) \text{ with} \\ 0\text{-th vertex removed} \end{bmatrix}$$

by definition of a closed pseudomanifold every $(n-1)$-simplex has order 2, since the pseudomanifold is oriented we obtain from the above observation that all contributions for $i = n-k$ cancel

$$= \sum_{\substack{v_{k+1}\in V \text{ such that} \\ \{v_0,..,v_k,v_{k+1}\}\in S_{k+1}}} \sum_{\substack{(v_{k+2},..,v_n)\in V^{n\text{-}k\text{-}1} \text{ such that} \\ \{v_0,..,v_k,v_{k+1},v_{k+2},..,v_n\}\in S_n}} \operatorname{sign}(v_0,..,v_n)\cdot[G(s\cup\{v_{k+1}\}\,|\,v_{k+2},..,v_n)]$$

follows from fact that any subset of a simplex is again a simplex, we also use the above observation to rewrite the summand

$$= \sum_{\substack{w\in V \text{ such that} \\ \{v_0,\ldots,v_k,w\}\in S_{k+1}}} [v_0,\ldots,v_k,w]^\dagger.$$

■

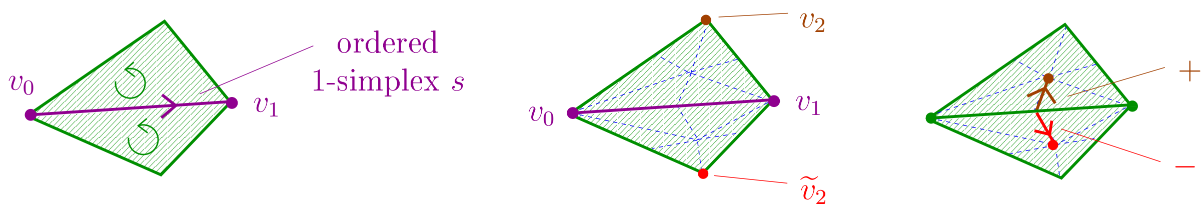


FIGURE 1107. Illustration for the proof of Lemma 71.8.

Using 71.8 we can now easily prove the following proposition which is one of the key steps in the proof of the Simplicial Poincaré Duality Theorem 71.4.

Proposition 71.9. *Let $K = (V,S)$ be a closed oriented n-dimensional pseudomanifold.*

(1) *For each $k \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl} \Lambda\colon\ \operatorname{Hom}(\mathrm{C}_k^{\mathrm{simp}}(K),\mathbb{Z}) & \to & \mathrm{D}_{n-k}^{\mathrm{simp}}(K) \\ \sum\limits_{i=1}^m a_i\cdot \underset{\uparrow}{s_i^*} & \mapsto & \sum\limits_{i=1}^m a_i\cdot s_i^\dagger \end{array}$$

here s_i is a k-simplex of K and we define $s_i^*\colon \mathrm{C}^k_{\mathrm{simp}}(K)\to\mathbb{Z}$ as on page 1686 *is a natural isomorphism of abelian groups.*

(2) *For each $k \in \mathbb{N}_0$ the following diagram commutes:*

$$\begin{tikzcd}
\operatorname{Hom}(C^{\text{simp}}_k(K), \mathbb{Z}) \arrow[r, "s^* \mapsto s^\dagger", "\Lambda"'] \arrow[d, "(-1)^k \cdot \delta_k = (-1)^k \cdot \partial^*_{k+1}"] & D^{\text{simp}}_{n-k}(K) \arrow[d, "\partial_{n-k}"] \\
\operatorname{Hom}(C^{k+1}_{\text{simp}}(K), \mathbb{Z}) \arrow[r, "s^* \mapsto s^\dagger", "\Lambda"'] & D^{\text{simp}}_{n-k-1}(K).
\end{tikzcd}$$

(3) *The maps*

$$\begin{array}{rcl} \Lambda_* \colon \mathrm{H}^k_{\text{simp}}(K; \mathbb{Z}) & \to & \mathrm{H}_{n-k}(D^{\text{simp}}_*(K)) \\ \left[\sum\limits_{i=1}^m a_i \cdot s_i^*\right] & \mapsto & \left[\sum\limits_{i=1}^m a_i \cdot s_i^\dagger\right] \end{array}$$

are well-defined and they are natural isomorphisms.

PROOF.

(1) First note that K is by definition a finite abstract simplicial complex. Thus it follows immediately from Lemma 67.2 and Lemma 71.7 (2) that the maps Λ are isomorphisms. Basically by definition the maps Λ are natural.

(2) Let $k \in \mathbb{N}_0$ and let $s = (v_0, \ldots, v_k)$ be an ordered k-simplex of K. We calculate that

$$\partial_{n-k}(\Lambda(s)) = \partial_{n-k}([v_0, .., v_k]^\dagger) \overset{\substack{\text{Lemma 71.8} \\ \downarrow}}{=} \sum_{\substack{w \in V \text{ such that} \\ \{v_0, \ldots, v_k, w\} \in S_{k+1}}} [v_0, \ldots, v_k, w]^\dagger \overset{\substack{\text{Lemma 67.4} \\ \downarrow}}{=} (-1)^k \cdot \Lambda(\delta_k([v_0, .., v_k]^*)).$$

(3) In (2) we showed that the maps Λ are cochain maps "up to sign", i.e. they are generalized cochain maps in the sense of the discussion on page 1819. As we pointed out on page 1819, this sign issue is completely irrelevant and we still get a well-defined map on cohomology groups. Furthermore, by (1) we know that the maps on each cochain level are isomorphisms, thus we get in fact an isomorphism of cohomology groups. ■

It is a good moment to keep stock of where we are in the proof of the Simplicial Poincaré Duality Theorem 71.4. Let K be a closed oriented n-dimensional simplicial homology manifold and let $k \in \mathbb{N}_0$. Let us consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}^k_{\text{simp}}(K; \mathbb{Z}) \arrow[rrr, "\cap [K]"] \arrow[ddd, "{[\Sigma a_i s_i^*] \mapsto [\Sigma a_i s_i^\dagger]}"', "\cong"] & & & \mathrm{H}^{\text{simp}}_{n-k}(K; \mathbb{Z}) \arrow[dl, "u_*", "\cong"'] \arrow[ddd, "\cong"', "u_*"] \\
& \mathrm{H}^k_{\text{simp}}(\mathrm{sd}(K); \mathbb{Z}) \arrow[ul, "u^*"', "\cong"] \arrow[r, "\cap [\mathrm{sd}(K)]"] & \mathrm{H}^{\text{simp}}_{n-k}(\mathrm{sd}(K); \mathbb{Z}) \arrow[ddr, "\mathrm{id}"] & \\
& & & \\
\mathrm{H}_{n-k}(D^{\text{simp}}_*(K)) \arrow[rrr, "\text{induced by the inclusions } D^{\text{simp}}_i(K) \to C^{\text{simp}}_i(\mathrm{sd}(K))"] & & & \mathrm{H}^{\text{simp}}_{n-k}(\mathrm{sd}(K); \mathbb{Z}).
\end{tikzcd}$$

Our main goal is to show that the top horizontal map is an isomorphism. By now we already know the following:

(1) By Proposition 71.9 we now know that the left vertical map is an isomorphism.
(2) By Lemma 63.41 we know that the subdivision maps u_* and u^* are isomorphisms.
(3) By Lemma 71.6 (3) we know that the upper quadrilateral commutes.

Thus to prove the core of the Simplicial Poincaré Duality Theorem 71.4 it remains to carry out the following steps:

(4) We need to show that the lower left part of the diagram commutes.
(5) We need to show that the bottom horizontal map is an isomorphism.

As we will see in a second, Step (4) follows from a fairly straightforward calculation. Step (5) is much more difficult and we postpone Step (5) to the next section. Note though that Step (5) is particularly interesting since so far we have only used that K is a closed n-dimensional pseudomanifold. We did not yet use the hypothesis that we are dealing with a simplicial homology manifold.

As promised, here is the lemma that takes care of Step (4).

Lemma 71.10. *Given any closed oriented n-dimensional pseudomanifold K the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{Z}) \arrow[d, "\cong", "{\substack{[\Sigma a_i s_i^*] \\ \mapsto \\ [\Sigma a_i s_i^\dagger]}}"'] & & \mathrm{H}^k_{\mathrm{simp}}(\mathrm{sd}(K);\mathbb{Z}) \arrow[ll, "u^*"', "\cong"] \arrow[d, "\cap[\mathrm{sd}(K)]"] \\
\mathrm{H}_{n-k}(\mathrm{D}^{\mathrm{simp}}_*(K)) \arrow[rr] & & \mathrm{H}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K);\mathbb{Z}).
\end{tikzcd}$$

Proof. Let $K=(V,S)$ be a closed oriented n-dimensional pseudomanifold. Given two k-simplices s and t of K we write $\underline{s}\leq\underline{t}$ if $s\subset t$. As we pointed out in Lemma 62.2, this order on the vertex set of $\mathrm{sd}(K)$ turns $\mathrm{sd}(K)$ naturally into an ordered abstract simplicial complex. In the following we use the natural isomorphisms $\mathrm{C}^{\mathrm{simp},\leq}_k(\mathrm{sd}(K))\to\mathrm{C}^{\mathrm{simp}}_k(\mathrm{sd}(K))$ and $\mathrm{C}^k_{\mathrm{simp}}(\mathrm{sd}(K);\mathbb{Z})\to\mathrm{C}^k_{\mathrm{simp},\leq}(\mathrm{sd}(K);\mathbb{Z})$ to identify these two (co-) chain complexes and throughout the proof we will work with $\mathrm{C}^{\mathrm{simp}}_k(\mathrm{sd}(K))$ and $\mathrm{C}^k_{\mathrm{simp}}(\mathrm{sd}(K))$.

We set

$$\mu\;:=\sum_{\substack{(u_0,\dots,u_n)\text{ ordered}\\ n\text{-simplex of }K}}\mathrm{sign}(u_0,\dots,u_n)\cdot[\underline{\{u_0\}},\underline{\{u_0,u_1\}},\dots,\underline{\{u_0,\dots,u_n\}}]\;\in\;\mathrm{C}^{\mathrm{simp}}_n(\mathrm{sd}(K)).$$

It follows immediately from Lemma 71.6 and Theorem 66.4 that μ is a simplicial cycle that represents the simplicial fundamental class $[\mathrm{sd}(K)]\in\mathrm{H}^{\mathrm{simp}}_n(\mathrm{sd}(K))$.

Next we pick an order "$\leq$" on K and we denote by $\sigma\colon\,\mathrm{sd}(K)\to K$ the corresponding stretching map, as defined on page 1785. Recall that by Lemma 63.41 we know that

$\sigma^*\colon \mathrm{H}^k_{\text{simp}}(K;\mathbb{Z}) \to \mathrm{H}^k_{\text{simp}}(\mathrm{sd}(K);\mathbb{Z})$ and $u^*\colon \mathrm{H}^k_{\text{simp}}(\mathrm{sd}(K);\mathbb{Z}) \to \mathrm{H}^k_{\text{simp}}(K;\mathbb{Z})$ are inverses of one another. Thus it remains to prove the following claim.[1054]

Claim. The following diagram commutes:[1055]

$$\begin{tikzcd}
\mathrm{C}^k_{\text{simp}}(K;\mathbb{Z}) \arrow[r, "\sigma^*"] \arrow[d, "\cong", "\substack{\Sigma a_i s_i^* \\ \downarrow \\ \Sigma a_i s_i^\dagger}"'] & \mathrm{C}^k_{\text{simp}}(\mathrm{sd}(K);\mathbb{Z}) \arrow[d, "\cap\mu"] \\
\mathrm{D}^{\text{simp}}_{n-k}(K) \arrow[r, hook] & \mathrm{C}^{\text{simp}}_{n-k}(\mathrm{sd}(K)).
\end{tikzcd}$$

The verification of the claim consists of a careful, but at the end of the day elementary, calculation. Note that by Lemma 67.2 (2) it suffices to show that for every ordered k-simplex $s = (v_0, \dots, v_k)$ of K with $v_0 < \dots < v_k$ we have the following equality:

$$\sigma^*(s^*) \cap \mu \;=\; s^\dagger \;\in\; \mathrm{C}^{\text{simp}}_{n-k}(\mathrm{sd}(K)).$$

To verify this equality it is helpful to first do one specific calculation separately. Namely let $(z_0, \dots, z_k)$ be some other ordered k-simplex of S. We perform the following little calculation:

$$\begin{array}{rcl}
\sigma^*(s^*)([\underline{\{z_0\}}, \dots, \underline{\{z_0, \dots, z_k\}}]) & \overset{\substack{\text{definition of } \sigma^* \\ \downarrow}}{=} & s^*(\sigma_*([\underline{\{z_0\}}, \dots, \underline{\{z_0, \dots, z_k\}}])) \\
\underset{\uparrow}{=} \begin{cases} (v_0, \dots, v_k)^*(z_0, \dots, z_k), & \text{if } z_0 < \dots < z_k, \\ 0, & \text{otherwise} \end{cases} & = & \begin{cases} 1, & \text{if } (v_0, \dots, v_k) = (z_0, \dots, z_k), \\ 0, & \text{otherwise.} \end{cases}
\end{array}$$

by the definition of the stretching map we see that $\{\sigma(\underline{\{z_0\}}), \dots, \sigma(\underline{\{z_0, \dots, z_k\}})\}$ is a k-simplex only if $z_0 < \dots < z_k$, furthermore if $z_0 < \dots < z_k$, then $\sigma(\underline{\{z_0\}}, \dots, \underline{\{z_0, \dots, z_k\}})) = (z_0, \dots, z_k)$

Now we can verify that $\sigma^*(s^*) \cap \mu = s^\dagger$. Indeed:

$$\sigma^*(s^*) \cap \mu \underset{\substack{\uparrow \\ \text{definition of } \mu}}{=} \sigma^*(s^*) \cap \Big(\sum_{\substack{(z_0, .., z_n) \text{ ordered} \\ n\text{-simplex of } K}} \operatorname{sign}(z_0, .., z_n) \cdot [\underline{\{z_0\}}, \underline{\{z_0, z_1\}}, .., \underline{\{z_0, .., z_n\}}] \Big)$$

by definition of the cap product, here we use that $\underline{\{z_0\}} < \underline{\{z_0, z_1\}} < \dots < \underline{\{z_0, .., z_n\}}$

$$\begin{array}{l}
\underset{\downarrow}{=} \displaystyle\sum_{\substack{(z_0, .., z_n) \text{ ordered} \\ n\text{-simplex of } K}} \operatorname{sign}(z_0, .., z_n) \cdot \sigma^*(s^*)([\underline{\{z_0\}}, .., \underline{\{z_0, .., z_k\}}]) \cdot \underbrace{[\underline{\{z_0, .., z_{k+1}\}}, .., \underline{\{z_0, .., z_n\}}]}_{=G(\{z_0, .., z_k\} \,|\, z_{k+1}, .., z_n)} \\
\underset{\uparrow}{=} \displaystyle\sum_{\substack{(z_0, .., z_n) \text{ ordered} \\ n\text{-simplex of } K \text{ with} \\ (z_0, .., z_k) = (v_0, .., v_k)}} \operatorname{sign}(z_0, .., z_n) \cdot G(\underbrace{\{v_0, .., v_k\}}_{=s} \,|\, z_{k+1}, .., z_n) \underset{\uparrow}{=} s^\dagger.
\end{array}$$

by the above calculation

by definition of $s^\dagger$, see page 1787

[1054]Here, as on many other occasions we will use the remark on page 1552 to make the identification $\mathrm{C}^{\text{simp}}_*(\mathrm{sd}(K)) = \mathrm{C}^{\text{simp}}_*(\mathrm{sd}(K);\mathbb{Z})$.

[1055]It follows from Proposition 71.9 and the formula in the proof of Lemma 67.16 that the two vertical maps are (co-) chain maps “up to the sign $(-1)^k$”.

■

71.5. **The dual chain complex II.** As mentioned in the last section, the following proposition is the last major puzzle-piece in the proof of the Simplicial Poincaré Duality Theorem 71.4.

Proposition 71.11. *Let K be a closed n-dimensional simplicial homology manifold.*

(1) *The inclusion map $\mathrm{D}_*^{\mathrm{simp}}(K) \to \mathrm{C}_*^{\mathrm{simp}}(\mathrm{sd}(K))$ is a chain homotopy equivalence.*
(2) *For every abelian group G the induced maps $\mathrm{H}_k(\mathrm{D}_*^{\mathrm{simp}}(K) \otimes G) \to \mathrm{H}_k^{\mathrm{simp}}(\mathrm{sd}(K); G)$ are isomorphisms.*

Remark.

(1) Note that in Proposition 71.11 we finally work with simplicial homology manifolds and not just with pseudomanifolds.
(2) Note that in Proposition 71.11 we do not demand that K is orientable.
(3) In Section 71.7 we will see that the dual chain complex $\mathrm{D}_*^{\mathrm{simp}}(K)$ can be viewed as the cellular chain complex of a "dual CW-structure" on $|K|$. The proof of Proposition 71.11 is in fact similar in spirit to the proof, given in Proposition 48.4 and Corollary 49.9, that the cellular chain complex of a CW-complex X is chain homotopy equivalent to the singular chain complex of X.

Definition. Let $K = (V, S)$ be an n-dimensional abstract simplicial complex.

(1) Given $k \in \mathbb{N}_0$ we define

$$X^k \;:=\; \begin{array}{c}\text{all simplices of } \mathrm{sd}(K) \text{ of the form } G(s \,|\, v_1, \ldots, v_l)\\ \text{where } s \text{ is a simplex of } S \text{ of dimension at least } n-k\end{array}$$

It follows immediately from the definitions that X^k can be viewed as a k-dimensional subcomplex of $\mathrm{sd}(K)$ and that $X^n = \mathrm{sd}(K)$.

(2) Let $k \in \mathbb{N}_0$. Similar to the discussion on page 1259 we consider the map

$$\begin{array}{rccc} d_k\colon & \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) & \to & \mathrm{H}_k^{\mathrm{simp}}(X^{k-1}, X^{k-2}) \\ & \underset{\substack{\uparrow \\ \text{where } \sigma \in \mathrm{C}_k^{\mathrm{simp}}(X^k)}}{[\sigma]} & \mapsto & [\partial_k(\sigma)]. \end{array}$$

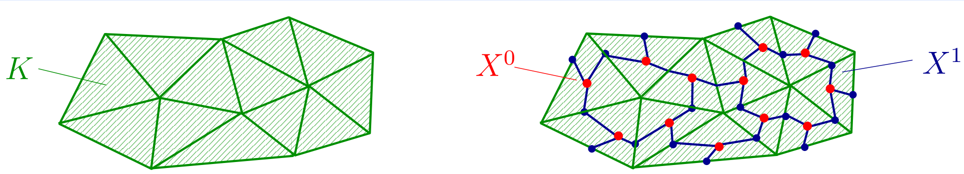


FIGURE 1108

The following lemma can be viewed as a partial analogue of Lemma 48.1.

Lemma 71.12. *Let $K = (V, S)$ be a closed n-dimensional simplicial homology manifold and let $k \in \mathbb{N}_0$.*

(1) *For $i \neq k$ we have $\mathrm{H}_i^{\mathrm{simp}}(X^k, X^{k-1}) = 0$.*

(2) *The map*

$$\begin{array}{rcl}\Theta\colon\ \mathrm{D}_k^{\mathrm{simp}}(K) & \to & \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) \\ \sigma & \mapsto & [\sigma]\end{array}$$

is well-defined, i.e. every chain $\sigma \in \mathrm{D}_k^{\mathrm{simp}}(K)$ *does indeed represent a cycle in* $\mathrm{C}_k^{\mathrm{simp}}(X^k, X^{k-1})$. *Furthermore the map* Θ *is an isomorphism.*

(3) *The maps* Θ *introduced in* (2) *define an isomorphism of chain complexes*

$$(\mathrm{D}_k^{\mathrm{simp}}(K), \partial_k) \ \to\ (\mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}), d_k).$$

PROOF. Let $K = (V, S)$ be a closed n-dimensional simplicial homology manifold. Given $l \in \mathbb{N}_0$ we denote by S_l the set of l-simplices of K. Given an $(n-k)$-simplex $s \in S_{n-k}$ we define the following:[1056]

$$\begin{array}{rcl} Z(s) & = & \{\text{ all simplices of } \mathrm{sd}(K) \text{ of the form } G(t\,|\,v_1, \dots, v_l) \text{ with } s \subset t\} \ \subset\ X^k \\ \partial Z(s) & = & \{\text{ all simplices of } \mathrm{sd}(K) \text{ of the form } G(t\,|\,v_1, \dots, v_l) \text{ with } s \subsetneq t\} \ \subset\ X^{k-1}. \end{array}$$

FIGURE 1109

Claim 1. Let s be an $(n-k)$-simplex of K. Given any $l \in \mathbb{N}$ we have

$$\widetilde{\mathrm{H}}_l^{\mathrm{simp}}(Z(s)) \ =\ 0 \qquad \text{and} \qquad \widetilde{\mathrm{H}}_l^{\mathrm{simp}}(\partial Z(s)) \ \cong\ \begin{cases} \mathbb{Z}, & \text{if } l = k-1, \\ 0, & \text{otherwise.} \end{cases}$$

In particular $\partial Z(s)$ is a simplicial homology $(k-1)$-sphere.

Let s be an $(n-k)$-simplex of K. We start out with the following little observation:

($*$) Note that $Z(s)$ and $\mathrm{sd}(\partial s)$ are subcomplexes of $\mathrm{St}(\mathrm{sd}(K), \underline{s})$ and that $\partial Z(s)$ and $\mathrm{sd}(\partial s)$ are subcomplexes of $\mathrm{Lk}(\mathrm{sd}(K), \underline{s})$. It is now elementary to verify that the natural maps on the vertex sets induce simplicial isomorphisms

$$Z(s) * \mathrm{sd}(\partial s) \xrightarrow{\cong} \mathrm{St}(\mathrm{sd}(K), \underline{s}) \quad \text{and} \quad \partial Z(s) * \mathrm{sd}(\partial s) \xrightarrow{\cong} \mathrm{Lk}(\mathrm{sd}(K), \underline{s}).$$

We leave it to the reader to go over the details. It might be helpful to have a look at Figure 1109 to do a reality check.

[1056] Note that it follows immediately from the definitions that $Z(s)$ are precisely the simplices that appear as summands in $s^\dagger \in \mathrm{C}_k^{\mathrm{simp}}(K)$ and that $\partial Z(s)$ are precisely the simplices that appear as summands in $\partial s^\dagger \in \mathrm{C}_{k-1}^{\mathrm{simp}}(K)$.

Now let us turn to the actual calculation of the simplicial homology groups of $Z(s)$ and $\partial Z(s)$. Thus let $l \in \mathbb{N}_0$. We perform the following two calculations:

$$\widetilde{\mathrm{H}}^{\mathrm{simp}}_{l}(Z(s)) \underset{\uparrow}{=} \widetilde{\mathrm{H}}^{\mathrm{simp}}_{l+n-k}(Z(s) * \mathrm{sd}(\partial s)) \underset{\uparrow}{=} \widetilde{\mathrm{H}}^{\mathrm{simp}}_{l+n-k}(\overbrace{\mathrm{St}(\mathrm{sd}(K),\underline{s})}^{=\mathrm{Cone}(\mathrm{Lk}(\mathrm{sd}(K),\underline{s}))}) \underset{\uparrow}{=} 0.$$

since s is an $(n-k)$-simplex we obtain from Lemma 63.41 that $\widetilde{\mathrm{H}}^{\mathrm{simp}}_{n-k-1}(\mathrm{sd}(\partial s)) \cong \mathbb{Z}$ and that $\widetilde{\mathrm{H}}^{\mathrm{simp}}_{i}(\mathrm{sd}(\partial s)) = 0$ for $i \neq n-k-1$, the statement now follows from Exercise 63.14 — by $(*)$ — by Lemma 62.5 together with Lemma 63.11

by $(*)$

$$\widetilde{\mathrm{H}}^{\mathrm{simp}}_{l}(\partial Z(s)) \overset{\downarrow}{=} \widetilde{\mathrm{H}}^{\mathrm{simp}}_{l+n-k}(\partial Z(s) * \mathrm{sd}(\partial s)) \overset{\downarrow}{=} \widetilde{\mathrm{H}}^{\mathrm{simp}}_{l+n-k}(\mathrm{Lk}(\mathrm{sd}(K),\underline{s})) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } l+n-k=n-1, \\ 0, & \text{otherwise.} \end{cases}$$

by our hypothesis that K is an n-dimensional simplicial homology manifold together with Lemma 71.1 we know that $\mathrm{Lk}(\mathrm{sd}(K),\underline{s})$ is an $(n-1)$-dimensional simplicial homology sphere ⊞

Claim 2. Let $k \in \mathbb{N}_0$. The map

$$\bigoplus_{s \in S_{n-k}} j_s \colon \bigoplus_{s \in S_{n-k}} \mathrm{C}^{\mathrm{simp}}_{*}(Z(s), \partial Z(s)) \;\to\; \mathrm{C}^{\mathrm{simp}}_{*}(X^k, X^{k-1}),$$

induced by the inclusions, is an isomorphism of chain complexes.

Since each of the inclusion induced maps $\mathrm{C}^{\mathrm{simp}}_{*}(Z(s), \partial Z(s)) \to \mathrm{C}^{\mathrm{simp}}_{*}(X^k, X^{k-1})$ is a chain map we see that the given map is indeed a chain map. Thus it suffices to show that the map is an isomorphism in each degree. Now let $l \in \mathbb{N}_0$. We consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus\limits_{s \in S_{n-k}} \mathrm{C}^{\mathrm{simp}}_{l}(\partial Z(s)) \arrow[r, "\alpha"] \arrow[d, two heads, "=:\varphi"] & \bigoplus\limits_{s \in S_{n-k}} \mathrm{C}^{\mathrm{simp}}_{l}(Z(s)) \arrow[r] \arrow[d, two heads, "=:\psi"] & \bigoplus\limits_{s \in S_{n-k}} \mathrm{C}^{\mathrm{simp}}_{l}(Z(s), \partial Z(s)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{C}^{\mathrm{simp}}_{l}(X^{k-1}) \arrow[r] & \mathrm{C}^{\mathrm{simp}}_{l}(X^{k}) \arrow[r] \arrow[d, "p"] & \mathrm{C}^{\mathrm{simp}}_{l}(X^{k}, X^{k-1}) \arrow[r] & 0. \\
 & & \bigoplus\limits_{s \in S_{n-k}} \bigoplus\limits_{(v_0,\ldots,v_l)} \mathbb{Z}\cdot G(s|v_0,\ldots,v_l) & &
\end{tikzcd}$$

We make the following clarifications and observations:

(a) The horizontal sequences are evidently exact.
(b) The vertical maps between the first and second row are induced by the inclusions $\partial Z(s) \to X^{k-1}$ and $Z(s) \to X^k$.
(c) It is basically clear that the middle vertical map ψ is an epimorphism.
(d) Let $G(t \,|\, v_0, \ldots, v_l)$ be an l-simplex in X^{k-1}. By definition of X^{k-1} this means that t is a simplex of K of dimension $\geq n-(k-1)$. For any $(n-k)$-simplex s with $s \subset t$ we have by definition that $G(t \,|\, v_0, \ldots, v_l)$ is a simplex of $\partial Z(s)$. This shows that the left vertical map φ is also an epimorphism.
(e) The vertical map p is the natural projection given by the fact that the bottom group is generated by a subset of the set of l-simplices of X^k.

(f) It follows easily from the definitions that $\ker(p \circ \psi) \subset \operatorname{im}(\alpha)$.
(g) It follows from (f) that $\ker(\psi) \subset \operatorname{im}(\alpha)$.

It now follows from (a), (b), (c) and (g) together with an elementary diagram chase, see Exercise 43.11, that the right vertical map is indeed an isomorphism. ⊞

Claim 3. For every ordered $(n-k)$-simplex s of K we have $\mathbb{Z} \cdot [s^\dagger] = \mathrm{H}_k^{\mathrm{simp}}(Z(s), \partial Z(s))$.

Let $s = (v_0, \dots, v_{n-k})$ be an ordered $(n-k)$-simplex of K. By Proposition 71.2 we know that K is a closed n-dimensional pseudomanifold. By definition this implies that s is the face of an n-simplex. We pick vertices $w_1, \dots, w_k$ of K such that $(v_0, \dots, v_{n-k}, w_1, \dots, w_k)$ is an n-simplex of K. Next we consider the following diagram:

$$\begin{tikzcd}
\overbrace{\widetilde{\mathrm{H}}_k^{\mathrm{simp}}(Z(s))}^{=0 \text{ by Claim 1}} \arrow[r] & \mathrm{H}_k^{\mathrm{simp}}(Z(s), \partial Z(s)) \arrow[r, "\partial_k", "\cong"'] & \overbrace{\widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(\partial Z(s))}^{\cong\mathbb{Z} \text{ by Claim 1}} \arrow[r] \arrow[d, hook, "\iota"] & \overbrace{\widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(Z(s))}^{=0 \text{ by Claim 1}} \\
& \mathbb{Z} \arrow[u, "n\mapsto n\cdot[s^\dagger]", "=:\varphi"'] & \widetilde{\mathrm{C}}_{k-1}^{\mathrm{simp}}(\partial Z(s)). \arrow[l, two heads, "\text{projection } p"] &
\end{tikzcd}$$

We make the following clarifications and observations:

(a) It follows from Claim 1 that the connecting homomorphism ∂_k on the top is an isomorphism.
(b) Note that $\partial Z(s)$ is a $(k-1)$-dimensional simplicial complex. Therefore we have the *inclusion* $\iota\colon \widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(\partial Z(s)) \to \widetilde{\mathrm{C}}_{k-1}^{\mathrm{simp}}(\partial Z(s))$.
(c) It follows immediately from the definitions that $\partial s^\dagger$ lies in $\mathrm{C}_{k-1}^{\mathrm{simp}}(\partial Z(s))$. In particular $s^\dagger$ defines an element $\mathrm{H}_k^{\mathrm{simp}}(Z(s), \partial Z(s))$. In other words, the vertical map φ is defined.
(d) We denote by $p\colon \mathrm{C}_{k-1}^{\mathrm{simp}}(\partial Z(s)) \to \mathbb{Z}$ the projection from the free abelian group $\mathrm{C}_{k-1}^{\mathrm{simp}}(\partial Z(s))$ onto the $G(s \cup \{w_{k+1}\} \mid w_{k+2}, \dots, w_n)$-coefficient.
(e) It follows easily from Lemma 71.8 (1) together with the various definitions that the map $p \circ \iota \circ \partial_k \circ \varphi\colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by ± 1.
(f) By Claim 1 we have $\widetilde{\mathrm{H}}_{k-1}^{\mathrm{simp}}(\partial Z(s)) \cong \mathbb{Z}$.
(g) It follows easily from (d) and (e) that $\partial_k \circ \varphi$ is an isomorphism. Together with (1) we see that φ is an isomorphism. ⊞

As the reader will have noticed, the above claims basically take care of the three statements of the proposition:

(1) It follows from Claim 1 and 2 that for $i \neq k$ we have $\mathrm{H}_i^{\mathrm{simp}}(X^k, X^{k-1}) = 0$.
(2),(3) We consider the following diagram:

$$\begin{tikzcd}
\mathrm{D}_k^{\mathrm{simp}}(K) \arrow[r, "s^\dagger\mapsto[s^\dagger]", "\cong"'] \arrow[d, "\partial_k"] & \bigoplus\limits_{s \in S_{n-k}} \mathrm{H}_k^{\mathrm{simp}}(Z(s), \partial Z(s)) \arrow[r, "\cong"'] & \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) \arrow[d, "d_k"] \\
\mathrm{D}_{k-1}^{\mathrm{simp}}(K) \arrow[r, "s^\dagger\mapsto[s^\dagger]", "\cong"'] & \bigoplus\limits_{s \in S_{n-k+1}} \mathrm{H}_{k-1}^{\mathrm{simp}}(Z(s), \partial Z(s)) \arrow[r, "\cong"'] & \mathrm{H}_{k-1}^{\mathrm{simp}}(X^{k-1}, X^{k-2}).
\end{tikzcd}$$

We make the following clarifications and observations:

(a) By Lemma 71.7 we know that the $\{s^\dagger\}_{s\in S_{n-l}}$ form a basis for each $\mathrm{D}_l^{\mathrm{simp}}(K)$.
(b) The composition of the horizontal maps equals the map Θ that is induced by the maps $\mathrm{D}_k^{\mathrm{simp}}(K) \to \mathrm{C}_k^{\mathrm{simp}}(X^k) \to \mathrm{C}_k^{\mathrm{simp}}(X^k, X^{k-1})$.
(c) It follows immediately from the definitions that the diagram commutes.
(d) By Claim 3 we know that the left horizontal maps are isomorphisms and by Claim 2 we know the right horizontal maps are isomorphisms.

This discussion shows that the maps

$$\begin{array}{rcl} \Theta\colon\ \mathrm{D}_k^{\mathrm{simp}}(K) & \to & \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) \\ \sigma & \mapsto & [\sigma] \end{array}$$

define an isomorphism $(\mathrm{D}_*^{\mathrm{simp}}(K), \partial_*) \to (\mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}), d_k)_{k\in\mathbb{N}_0}$ of chain complexes. ■

The following lemma is also a partial analogue of Lemma 48.1.

Lemma 71.13. *Let $K = (V, S)$ be a closed n-dimensional simplicial homology manifold and let $k \in \mathbb{N}_0$.*

(1) *For every $i > k$ we have $\mathrm{H}_i^{\mathrm{simp}}(X^k) = 0$.*
(2) *For every $i < k$ the inclusion induced map $\mathrm{H}_i^{\mathrm{simp}}(X^k) \to \mathrm{H}_i^{\mathrm{simp}}(\mathrm{sd}(K))$ is an isomorphism.*

Proof.

(1) Note that X^k is an abstract simplicial complex of dimension k. It follows immediately from the definition of simplicial homology that $\mathrm{H}_i^{\mathrm{simp}}(K) = 0$ for $i > k$.
(2) We fix $i \in \mathbb{N}_0$. Let $m > i$. We consider the following excerpt from the long exact sequence in simplicial homology of the pair (X^{m+1}, X^m):

$$\dots \to \underbrace{\mathrm{H}_{i+1}^{\mathrm{simp}}(X^{m+1}, X^m)}_{=0 \text{ by Lemma 71.12 (1) since } i<m} \to \mathrm{H}_i^{\mathrm{simp}}(X^m) \to \mathrm{H}_i^{\mathrm{simp}}(X^{m+1}) \to \underbrace{\mathrm{H}_i^{\mathrm{simp}}(X^{m+1}, X^m)}_{=0 \text{ by Lemma 71.12 (1) since } i<m} \to \dots$$

Now let $k > i$. The above discussion immediately implies that the following inclusion induced maps are isomorphisms: $\mathrm{H}_i^{\mathrm{simp}}(X^k) \xrightarrow{\cong} \mathrm{H}_i^{\mathrm{simp}}(X^{k+1}) \xrightarrow{\cong} \dots \xrightarrow{\cong} \mathrm{H}_i^{\mathrm{simp}}(X^n)$. The desired statement follows from the fact that $X^n = \mathrm{sd}(K)$. ■

Now we are finally in a position to provide the long overdue proof of Proposition 71.11.

Proof of Proposition 71.11. Let K be a closed n-dimensional simplicial homology manifold and let G be an abelian group. Recall that we need to prove the following two statements:

(1) The inclusion map $\mathrm{D}_*^{\mathrm{simp}}(K) \to \mathrm{C}_*^{\mathrm{simp}}(\mathrm{sd}(K))$ is a chain homotopy equivalence.
(2) The induced maps $\mathrm{H}_k(\mathrm{D}_*^{\mathrm{simp}}(K) \otimes G) \to \mathrm{H}_k^{\mathrm{simp}}(\mathrm{sd}(K); G)$ are isomorphisms.

Similarly to the proof of Proposition 48.4 we consider the following diagram:

$$\begin{tikzcd}
 & & & & 0 & & \\
 & 0 \arrow[dr] & & \mathrm{H}_k^{\mathrm{simp}}(X^{k+1}) \arrow[ur] & & & \\
 & & \mathrm{H}_k^{\mathrm{simp}}(X^k) \arrow[ur, two heads, "j_*"] \arrow[dr, hook] & & & & \\
\mathrm{H}_{k+1}^{\mathrm{simp}}(X^{k+1}, X^k) \arrow[urr, "\partial_{k+1}"] \arrow[rrr, "d_{k+1}"] & & & \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) \arrow[rrr, "d_k"] \arrow[drr, "\partial_k"'] & & & \mathrm{H}_{k-1}^{\mathrm{simp}}(X^{k-1}, X^{k-2}). \\
 & & & & & \mathrm{H}_{k-1}^{\mathrm{simp}}(X^{k-1}) \arrow[ur] &
\end{tikzcd}$$

We make the following observations:

(1) It follows from Lemma 63.29, i.e. from the long exact sequence in simplicial homology, together with Lemma 71.12 (1) and Lemma 71.13 (1) that all diagonal sequences are exact.

(2) It follows from the definition of d_k that the map $\mathrm{H}_k^{\mathrm{simp}}(X^k) \to \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1})$ takes values in $\ker(d_k\colon\ \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) \to \mathrm{H}_{k-1}^{\mathrm{simp}}(X^{k-1}, X^{k-2}))$.

We move on to the main diagram of this proof:

$$\begin{tikzcd}
\mathrm{D}_{k+1}^{\mathrm{simp}}(K) \arrow[rrrr, "\partial_{k+1}"] \arrow[dddd, "\Theta", "\cong"'] & & & & \ker\big(\mathrm{D}_k^{\mathrm{simp}}(K) \xrightarrow{\partial_k} \mathrm{D}_{k-1}^{\mathrm{simp}}(K)\big) \arrow[dddd, "\Theta", "\cong"'] \arrow[dll, "j_*"'] \arrow[dddlll, bend right, "j_*"'] \\
 & & & 0 & \\
 & & \mathrm{H}_k^{\mathrm{simp}}(\mathrm{sd}(K)) \arrow[ur] & & \\
 & \mathrm{H}_k^{\mathrm{simp}}(X^k) \arrow[ur, two heads, "j_*"] \arrow[drrr, hook] & & & \\
\mathrm{H}_{k+1}^{\mathrm{simp}}(X^{k+1}, X^k) \arrow[ur, "\partial_{k+1}"] \arrow[rrrr, "d_{k+1}"'] & & & & \ker\big(\mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) \xrightarrow{d_k} \mathrm{H}_{k-1}^{\mathrm{simp}}(X^{k-1}, X^{k-2})\big).
\end{tikzcd}$$

We make the following clarification and observations:

(3) All the maps called j_* are induced by inclusions on the level of chain groups. In fact note that every map in the diagram is either induced by taking boundaries or it is induced by an inclusion. It follows almost immediately from this observation that the diagram commutes.

(4) The vertical maps are isomorphisms by Lemma 71.12.

(5) By (1) the map $\mathrm{H}_k^{\mathrm{simp}}(X^k) \to \ker(d_k\colon\ \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) \to \mathrm{H}_{k-1}^{\mathrm{simp}}(X^{k-1}, X^{k-2}))$ is a monomorphism.

(6) It follows from (1), together with Lemma 71.13, that the diagonal blue sequence

$$\mathrm{H}^{\mathrm{simp}}_{k+1}(X^{k+1}, X^k) \xrightarrow{\partial_{k+1}} \mathrm{H}^{\mathrm{simp}}_{k}(X^k) \xrightarrow{j_*} \mathrm{H}^{\mathrm{simp}}_{k}(\mathrm{sd}(K)) \to 0$$

is exact.

An amusing little diagram chase now shows that the sequence

$$\mathrm{D}^{\mathrm{simp}}_{k+1}(K) \xrightarrow{\partial_{k+1}} \ker\left(\partial_k \colon \mathrm{D}^{\mathrm{simp}}_{k}(K) \to \mathrm{D}^{\mathrm{simp}}_{k-1}(K)\right) \xrightarrow{j_*} \mathrm{H}^{\mathrm{simp}}_{k}(\mathrm{sd}(K)) \to 0$$

is also exact. In other words we have proved the following statement:

(0) The inclusion map $\mathrm{D}^{\mathrm{simp}}_{*}(K) \to \mathrm{C}^{\mathrm{simp}}_{*}(\mathrm{sd}(K))$ induces for each $k \in \mathbb{N}_0$ an isomorphism $\mathrm{H}_k(\mathrm{D}^{\mathrm{simp}}_{*}(K)) \to \mathrm{H}^{\mathrm{simp}}_{k}(\mathrm{sd}(K))$.

Now let us turn to the two statements we actually need to prove.

(1) It follows from (0) and Proposition 49.2 that the chain map $\mathrm{D}^{\mathrm{simp}}_{*}(K) \to \mathrm{C}^{\mathrm{simp}}_{*}(\mathrm{sd}(K))$ is in fact a chain homotopy equivalence.

(2) Finally note that it follows either from (1), or from (0) together with the Algebraic Universal Coefficient Theorem 57.18 and the Five Lemma 43.12, that the maps $\mathrm{H}_k(\mathrm{D}^{\mathrm{simp}}_{*}(K); G) \to \mathrm{H}^{\mathrm{simp}}_{*}(\mathrm{sd}(K); G)$ are also isomorphisms. Alternatively we could have carried through the above arguments with G-coefficients without any hiccup. ■

71.6. **The proof of the Simplicial Poincaré Duality Theorem 71.4.** In this section we combine the results from the previous sections to finally provide the proof of the Simplicial Poincaré Duality Theorem 71.4. For the most part we are now just saying that the program, set out on page 1791 has been completed successfully.

Let K be a closed n-dimensional pseudomanifold. First we assume that K is orientable and that we are given an orientation for K. In the following we will only work with the commutative ring $R = \mathbb{Z}$. The proof for the general case is basically the same, except that one has to carry slightly more notation around. As on page 1791 we consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{Z}) \arrow[rrr, "\cap[K]"] \arrow[dd, "\cong", "{[\Sigma a_i s_i^*] \mapsto [\Sigma a_i s_i^\dagger]}"'] & & & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;\mathbb{Z}) \arrow[dl, "u_*"', "\cong"] \arrow[dd, "u_*", "\cong"'] \\
& \mathrm{H}^k_{\mathrm{simp}}(\mathrm{sd}(K);\mathbb{Z}) \arrow[ul, "u^*"', "\cong"] \arrow[r, "\cap[\mathrm{sd}(K)]"] & \mathrm{H}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K);\mathbb{Z}) \arrow[dr, "\mathrm{id}"] & \\
\mathrm{H}_{n-k}(\mathrm{D}^{\mathrm{simp}}_*(K)) \arrow[rrr, "\text{induced by the inclusions } \mathrm{D}^{\mathrm{simp}}_i(K) \to \mathrm{C}^{\mathrm{simp}}_i(\mathrm{sd}(K))", "\cong"'] & & & \mathrm{H}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K);\mathbb{Z}).
\end{tikzcd}$$

In the last sections we have shown the following:

(1) By Proposition 71.9 we now know that the left vertical map is an isomorphism.

(2) By Lemma 63.41 we know that the subdivision maps u_* and u^* are isomorphisms.

(3) By Lemma 71.6 (3) we know that the upper quadrilateral commutes.

(4) By Lemma 71.10 we know that the lower left part of the diagram commutes.

(5) Finally we now know by Proposition 71.11 that the bottom horizontal map is an isomorphism.

The combination of the above five statements implies that the map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{Z}) & \to & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;\mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [K] \end{array}$$

is, as promised and desired, an isomorphism.

Finally suppose that we are dealing with any closed n-dimensional pseudomanifold, in other words we drop the hypothesis that K is orientable. We need to show that the map

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{simp}}(K;\mathbb{F}_2) & \to & \mathrm{H}^{\mathrm{simp}}_{n-k}(K;\mathbb{F}_2) \\ \varphi & \mapsto & \varphi \cap [K]_{\mathbb{F}_2} \end{array}$$

is an isomorphism. The argument is almost identical to the previous argument. In fact the only place where we ever really used the orientation was in the definition of $s^\dagger$ on page 1787 and the resulting map $\mathrm{C}^k_{\mathrm{simp}}(K;\mathbb{Z}) \to \mathrm{D}^{\mathrm{simp}}_k(K)$ that is given by $s^* \mapsto s^\dagger$. A little bit of thought shows that the same recipe defines a map $\mathrm{C}^k_{\mathrm{simp}}(K;\mathbb{F}_2) \to \mathrm{D}^{\mathrm{simp}}_k(K) \otimes \mathbb{F}_2$ and the argument of Proposition 71.9 shows that this map is actually an isomorphism of cochain complexes. In the remainder of the argument we did not really use the orientation, so these parts stay virtually untouched. For example Proposition 71.11 says that the maps $\mathrm{H}_{n-k}(\mathrm{D}^{\mathrm{simp}}_*(K) \otimes \mathbb{F}_2) \to \mathrm{H}^{\mathrm{simp}}_{n-k}(\mathrm{sd}(K);\mathbb{F}_2)$ are isomorphisms. We leave it to the reader to turn this sketch of a proof into a proper argument.

71.7. **The dual CW-structure of a smooth manifold** $(*)$**.** Let M be a closed smooth manifold that is equipped with a smooth simplicial structure $(K = (V,S), \Theta\colon |K| \to M)$. Some of the figures above, e.g. Figures 1105 and 1109, suggest that the dual chains correspond to an alternative decomposition of M into cells. As we will see shortly, that is indeed the case.

First we recall some of the notation that we introduced earlier.

Notation. Let $K = (V,S)$ be an n-dimensional pseudomanifold.

(1) Given $m \in \mathbb{N}_0$ we denote by S_m the set of m-simplices of K.
(2) Let $s \in S_m$ be an m-simplex and let $v_1,\dots,v_l \in V$ such that $s \cup \{v_1,\dots,v_l\} \in S_{m+l}$. As on page 1786 we consider the following ordered l-simplex of $\mathrm{sd}(K)$:
$$G(s\,|\,v_1,\dots,v_l) \;:=\; \big(\underline{s}, \underline{s\cup\{v_1\}}, \dots, \underline{s\cup\{v_1,\dots,v_l\}}\big).$$
(3) Let $k \in \mathbb{N}_0$ and let $s \in S_{n-k}$ be an $(n-k)$-simplex. As on page 1795 we consider
$$Z(s) \;:=\; \{\text{ all simplices of } \mathrm{sd}(K) \text{ of the form } G(t\,|\,v_1,\dots,v_l) \text{ with } s \subset t\}.$$
Furthermore, as on page 1794 we consider the following:
$$X^k \;:=\; \text{the subcomplex of } \mathrm{sd}(K) \text{ given by all } Z(s) \text{ with } s \in S_{n-k}.$$

Now we can formulate the main result of this section.

Proposition 71.14. *Let M be a closed n-dimensional smooth manifold. Furthermore let $(K=(V,S),\Theta\colon |K| \to M)$ be a smooth simplicial structure.*

(1) *For each $(n-k)$-simplex s we have a homeomorphism $|Z(s)| \cong \overline{B}^k$.*

(2) *There exists a* CW*-structure on* M *such that for each* $k \in \mathbb{N}_0$ *the following holds:*
   (a) *The* k*-cells are given by the* $|Z(s)|$ *with* $s \in S_{n-k}$.
   (b) *The* k*-skeleton is given by* $|X^k|$.

(3) *If we equip* M *with the* CW*-structure from* (2)*, then there exists a natural isomorphism* $\mathrm{D}_k^{\mathrm{simp}}(K) \to \mathrm{C}_k^{\mathrm{CW}}(M)$ *of chain complexes.*

We refer to the CW-structure described in Proposition 71.14 as the *dual* CW*-structure* of the smooth manifold. We refer to Figure 1110 for an illustration.

FIGURE 1110. Illustration of Proposition 71.14.

SKETCH OF PROOF. Let M be a closed n-dimensional smooth manifold. Furthermore let $(K = (V,S), \Theta\colon |K| \to M)$ be a smooth simplicial structure. Note that we know by Theorem 64.14 that $(K = (V,S), \Theta\colon |K| \to M)$ is a closed n-dimensional PL-manifold. This implies that we might as well prove the statement under the hypothesis that we are dealing with a closed n-dimensional PL-manifold. Furthermore note that it follows immediately from the definition of a PL-manifold, see page 1599, together with Lemma 62.3 that the barycentric subdivision $\operatorname{sd}(K)$ is also a closed n-dimensional PL-manifold.

Now let us turn to the proof of the three statement.

(1) One could prove this statement in one go, but it seems didactically opportune to first consider the two simplest cases.
   (a) Let $s = \{v_0\}$ be a 0-simplex of K. We have the following homeomorphisms:

$$|Z(\{v_0\})| \underset{\substack{\uparrow \\ \text{follows immediately} \\ \text{from the definitions}}}{=} |\operatorname{St}(\operatorname{sd}(K), \underline{\{v_0\}})| \underset{\substack{\uparrow \\ \text{by Proposition 64.11 since } \operatorname{sd}(K) \text{ is} \\ \text{an } n\text{-dimensional PL-manifold}}}{=} |\text{PL } n\text{-ball}| \underset{\substack{\uparrow \\ \text{since PL-homeomorphic} \\ \text{implies homeomorphic}}}{\cong} \overline{B}^n.$$

   (b) Next let $s = \{v_0, v_1\}$ be a 1-simplex of K. Note that

$$\begin{aligned} |Z(\{v_0,v_1\})| &\overset{\substack{\text{follows easily from the definitions} \\ \downarrow}}{=} |\operatorname{St}(\operatorname{Lk}(\operatorname{sd}(K), \underline{\{v_0\}}), \underline{\{v_0,v_1\}})| \overset{\substack{\text{by Proposition 64.11 since } \operatorname{sd}(K) \text{ is a} \\ \text{closed } n\text{-dimensional PL-manifold} \\ \downarrow}}{=} |\operatorname{St}(\text{PL } (n-1)\text{-sphere}, \underline{\{v_0,v_1\}})| \\ &\underset{\substack{\uparrow \\ \text{as we pointed out on page 1600, it follows from Proposition 64.10 that a} \\ \text{PL } (n-1)\text{-sphere is a closed } (n-1)\text{-dimensional PL-manifold}}}{=} |\operatorname{St}(\text{closed } (n-1)\text{-dimensional PL-manifold}, \underline{\{v_0,v_1\}})| \\ &\underset{\substack{\uparrow \\ \text{Proposition 64.11}}}{\cong} |\text{PL } (n-1)\text{-ball}| \underset{\substack{\uparrow \\ \text{since PL-homeomorphic implies homeomorphic}}}{\cong} \overline{B}^{n-1}. \end{aligned}$$

(c) Finally let $s = \{v_0, \dots, v_k\}$ be any k-simplex of K. Similar to (b) we see that $Z(s)$ is the star of an iterated sequence of k links. Applying Proposition 64.11 multiple times we see that $Z(s)$ is homeomorphic to $\overline{B}^{n-k}$. We leave it to the reader to write down this notationally messy argument.

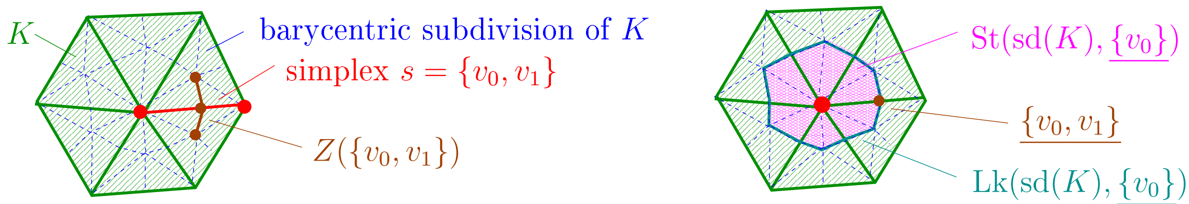


FIGURE 1111. Illustration for the proof of Proposition 71.14 (1b).

(2) This statement follows fairly immediately from (1). We leave it to the reader to fill in the details.
(3) Given each $k \in \mathbb{N}_0$ we consider the map

$$\begin{array}{ccccc} \mathrm{D}_k^{\mathrm{simp}}(K) & \to & \mathrm{H}_k^{\mathrm{simp}}(X^k, X^{k-1}) & \underset{\cong}{\xrightarrow{\text{natural isomorphism given by Lemma 63.30}}} & \mathrm{H}_k(|X^k|, |X^{k-1}|) = \mathrm{C}_k^{\mathrm{CW}}(M). \\ \sigma & \mapsto & [\sigma] & & \end{array}$$

Note that by Lemma 71.12 we know that the left maps are well-defined and that they form an isomorphism of chain complexes. We obtain immediately from Lemma 63.30 (3) that the right maps also form an isomorphism of chain complexes. The composition of the two chain isomorphisms thus gives us the promised natural isomorphism of chain complexes. ■

Exercises for Chapter 71.

Exercise 71.1. Let T be the simplicial 2-dimensional torus. By Exercise 66.1 we know that the suspension $\Sigma(T)$ is a closed orientable 3-dimensional pseudomanifold. Show that the conclusion of the Simplicial Poincaré Duality Theorem 71.4 does not hold for $\Sigma(T)$.

Exercise 71.2. Let K be a simplicial complex. Show that K is an n-dimensional simplicial homology manifold if and only if the topological realization $|K|$ is an n-dimensional homology manifold. Recall by the definition on page 1168 a topological space X is called an n-dimensional homology manifold if for each $x \in X$ we have

$$\mathrm{H}_k(X, X \setminus \{x\}; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 71.3.

(a) Let M be a connected closed 3-dimensional smooth manifold. Show that if M is non-orientable, then $\mathrm{H}_1(M; \mathbb{Z})$ is infinite.
(b) Is the hypothesis in (a) that M is closed necessary?

Exercise 71.4. Let M and N be two closed orientable connected non-empty 3-dimensional smooth manifolds such that $\pi_1(M)$ is isomorphic to $\pi_1(N)$. Show that for all $i \in \mathbb{N}_0$ there exists an isomorphism $\mathrm{H}_i(M;\mathbb{Z}) \cong \mathrm{H}_i(N;\mathbb{Z})$.

Exercise 71.5. Let W be a closed orientable connected 4-dimensional smooth manifold with Euler characteristic $\chi(W) = -3$.

(a) Show that W admits a finite connected covering $\widetilde{W}$ with $\chi(\widetilde{W}) < -27$.
(b) Is the hypothesis in (a) that W is orientable necessary?

Exercise 71.6. Let M be a closed orientable connected odd-dimensional smooth manifold and let $f\colon M \to M$ be a map such that the induced map $f_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)$ is the identity. Show that the Lefschetz number

$$\Lambda(\varphi) \;=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \mathrm{tr}(\varphi_*\colon\, \mathrm{H}_n(M) \to \mathrm{H}_n(M)),$$

equals zero.

Exercise 71.7. Let $n \in \mathbb{N}$ and let M be a closed oriented connected non-empty n-dimensional smooth manifold. Suppose that there exists a map $f\colon S^n \to M$ of degree ± 1, i.e. such that $f_*([S^n]) = [M] \in \mathrm{H}_n(M)$. Show that $\mathrm{H}_i(M) = 0$ for $i = 1, \dots, n-1$.
Remark. In Proposition 69.9 we showed under the same hypotheses that we also have $\pi_1(M) = 0$.

Exercise 71.8.

(1) Give a suitable definition of an n-dimensional simplicial homology manifold *with boundary.*
(2) Let K be a connected orientable n-dimensional simplicial homology manifold with non-empty boundary ∂K. We consider $M := K \cup_{\partial K} \mathrm{Cone}(\partial K)$, i.e. we consider the simplicial mapping cone of the inclusion $L \to K$.
  (a) Show that $M = K \cup_{\partial K} \mathrm{Cone}(\partial K)$ is a *closed* orientable n-dimensional pseudo-manifold.
  (b) Show that

$$\mathrm{H}_i(\mathrm{D}_k(M)) \;\cong\; \begin{cases} 0, & \text{if } k = n,\\ \mathrm{H}_i^{\mathrm{simp}}(\mathrm{sd}(K)), & \text{otherwise.}\end{cases}$$

  Remark. This is a non-trivial variation on Proposition 71.11.
  (c) Use the above together with Proposition 71.9 and Exercise 67.3 to show that for every $i \in \mathbb{N}_0$ there exists a natural isomorphism $\mathrm{H}^i_{\mathrm{simp}}(K, \partial K;\mathbb{Z}) \to \mathrm{H}^{\mathrm{simp}}_{n-i}(K;\mathbb{Z})$.

Part VIII

Singular Cohomology

72. Chirality

In this section we will take a short break from overly abstract concepts and we study chirality of objects in $\mathbb{R}^3$, knots and topological manifolds. We will run into several questions which will motivate us to develop new tools in the study of topological spaces.

72.1. **Chiral objects.** In physics or chemistry an object X in $\mathbb{R}^3$ is called *amphichiral* (or often short *achiral*) if it agrees with its mirror image, otherwise it is called *chiral*. To make this definition more precise we have to explain what we mean by "mirror image" and what we mean by "agree".

First of all, given a hyperplane $H \subset \mathbb{R}^3$ through the origin we denote by $\rho_H : \mathbb{R}^3 \to \mathbb{R}^3$ the reflection in the hyperplane H.[1057] Sometimes in this chapter we will extend ρ_H to a map from $S^3 = \mathbb{R}^3 \cup \{\infty\}$ to itself by setting $\rho_H(\infty) = \infty$.

The following lemma says that any two reflections in a hyperplane through the origin differ by multiplication by a matrix in $\mathrm{SO}(3)$.

Lemma 72.1. *If H and G are two hyperplanes in $\mathbb{R}^3$ through the origin, then there exists a matrix $A \in \mathrm{SO}(3)$ such that for all $v \in \mathbb{R}^3$ we have*

$$\rho_G(v) \;=\; A \cdot \rho_H(v).$$

Sketch of proof. With respect to the standard basis of $\mathbb{R}^3$ the maps ρ_G and ρ_H are represented by orthogonal matrices P and Q with determinant -1. The matrix $A = PQ^{-1}$ then lies in $\mathrm{SO}(3)$ and for any $v \in \mathbb{R}^3$ we have $\rho_G(v) = Pv = A \cdot Qv = A \cdot \rho_H(v)$. ■

The above discussion leads us to the following definitions.

Definition. Let X be a subset of $\mathbb{R}^3$.

(1) We say another subset Y of $\mathbb{R}^3$ *agrees with* X if there exists a matrix $A \in \mathrm{SO}(3)$ and a point $P \in \mathbb{R}^3$ such that $X = A \cdot Y + P = \{A \cdot y + P \,|\, y \in Y\}$.
(2) For every hyperplane $H \subset \mathbb{R}^3$ we refer to $\rho_H(X)$ as the *mirror image* of X.[1058]
(3) We say X is *amphichiral* if its mirror image agrees with X, otherwise we say X is *chiral*.

Examples.

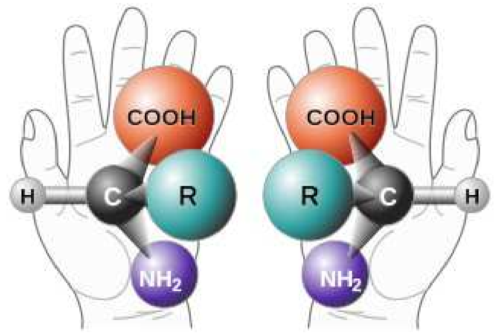
(1) Human hands are chiral, indeed, the left-hand does not agree with the right-hand. In fact the term chirality is derived from the Greek word for hand, $\chi\epsilon\iota\rho$.[1059]
(2) Many molecules are chiral, see e.g. Figure 1112 for an example of a chiral molecule.[1060]

[1057] How is the reflection in a hyperplane defined?

[1058] To be precise, the definition of the mirror image depends on the choice of the hyperplane. By Lemma 72.1 reflections in two different hyperplanes differ by multiplication by a matrix in SO(3), i.e. the mirror images with respect to different hyperplanes agree in the above sense.

[1059] The same word is also the etymological root for the German word "Chirurg".

[1060] In fact, how can you show that the molecule is chiral?



source: `https://en.wikipedia.org/wiki/Chirality`

FIGURE 1112

72.2. **Chiral knots.** Now we turn our focus to knots. We recall some of the relevant definitions from pages 385 and 780.

Definition.

(1) A *knot* is a 1-dimensional submanifold of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is diffeomorphic to S^1.
(2) Given a knot $K \subset S^3$ the reflection of K in any hyperplane of $\mathbb{R}^4$ is called *mirror* of K and denoted by K^{mir}. Note that in Exercise 27.1 we showed that, up to a smooth isotopy, the definition of K^{mir} does not depend on the choice of the hyperplane.

Example. In Figure 1113 we show the trefoil and its mirror and we also show the figure-8 knot and its mirror. In both cases the question arises whether the original knot and its mirror are smoothly isotopic.

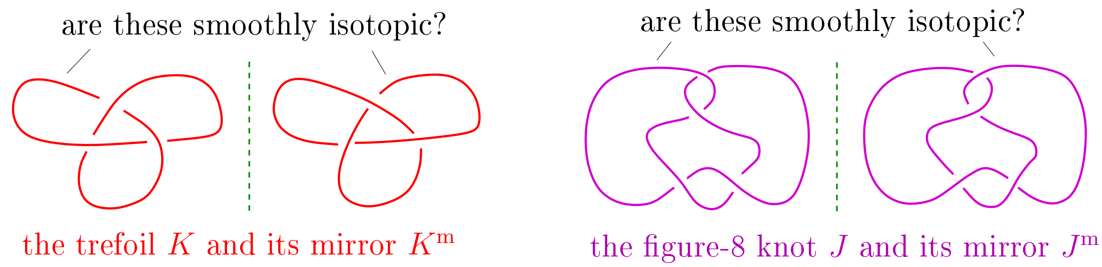


FIGURE 1113

Next we recall the following definition from page 781.

Definition. We say that a knot K in $\mathbb{R}^3$ is *amphichiral* if it is smoothly isotopic to its mirror image, otherwise we call the knot K *chiral*.

Example. In Figure 504 we showed that the figure-8 knot J is smoothly isotopic to its mirror image J^*, i.e. the figure-8 knot is amphichiral.

We recall the following question on which so far we have made zero progress.

Question 27.9. *Is the trefoil chiral?*

The question that naturally arises is, what does this discussion have to do with homology groups? How can we address this question using the tools of algebraic topology? In the following section we will turn Question 27.9 into a question about homology groups.

72.3. **Chiral topological manifolds.**

Definition. Let M be a compact oriented topological manifold. We say that M is *amphichiral* if M admits an orientation-reversing self-homeomorphism, otherwise we say that M is *chiral.*

In the next proposition, which is a variation on Lemma 68.7, we will give several criteria for a self-homeomorphism to be orientation-preserving.

Proposition 72.2. *Let M be a compact oriented connected n-dimensional topological manifold and let $f\colon M \to M$ be a homeomorphism. Then the following three statements are equivalent:*

(1) *The map f is orientation-preserving in the sense of the definition on page 2105.*
(2) *There exists a point $x \in M \setminus \partial M$ and an open neighborhood U of x, e.g. $U = M$, such that*[1061]

$$f_*(\underbrace{\text{orientation at } x}_{\in \mathrm{H}_n(U, U\setminus\{x\};\mathbb{Z})}) = \underbrace{\text{orientation at } f(x)}_{\in \mathrm{H}_n(f(U), f(U)\setminus\{f(x)\};\mathbb{Z})}.$$

(3) *We have $f_*([M]) = [M]$.*

Suppose M is in fact a smooth manifold and that f is a diffeomorphism. We equip M with the orientation in the sense of Section 6.11 coming from Proposition 86.11. Then the above are equivalent to the following statement:

(4) *The map f is orientation-preserving in the sense of Section 6.11.*

Analogously the above statements also hold if in (1) and (4) we replace "orientation-preserving" by "orientation-reversing" and if in (2) and (3) we add a minus sign.

For the most part the proof of Proposition 72.2 boils down to the following lemma which is also useful in its own right.

Lemma 72.3. *Let $f\colon M \to N$ be a map between compact oriented connected n-dimensional topological manifolds such that $f(\partial M) \subset \partial N$. Let $x \in M \setminus \partial M$ and let U be an open neighborhood of x with the property that f restricts to a homeomorphism $f\colon U \to f(U)$ and such that $f^{-1}(f(x)) = \{x\}$. Then*

$$\begin{aligned} \deg(f\colon M \to N) &= \deg(f_*\colon \mathrm{H}_n(M, M\setminus\{x\};\mathbb{Z}) \to \mathrm{H}_n(N, N\setminus\{f(x)\};\mathbb{Z}) \\ &= \deg(f_*\colon \mathrm{H}_n(U, U\setminus\{x\};\mathbb{Z}) \to \mathrm{H}_n(f(U), f(U)\setminus\{f(x)\};\mathbb{Z}). \end{aligned}$$

Proof. Let M and N be compact connected n-dimensional topological manifolds that are equipped with orientations $\{\mu_x\}_{x\in M\setminus\partial M}$ and $\{\nu_x\}_{x\in N\setminus\partial N}$. Furthermore let f, $x \in M \setminus \partial M$

[1061]Recall that by the construction of Lemma 86.17 the orientation on M induces an orientation on any open subset.

and U as above. We consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}_n(M,\partial M;\mathbb{Z}) \arrow[r, "f_*"] \arrow[d] & \mathrm{H}_n(N,\partial N;\mathbb{Z}) \arrow[d] \\
\mathrm{H}_n(M,M\setminus\{x\};\mathbb{Z}) \arrow[r, "f_*"] & \mathrm{H}_n(N,N\setminus\{f(x)\};\mathbb{Z}) \\
\mathrm{H}_n(U,U\setminus\{x\};\mathbb{Z}) \arrow[r, "f_*"] \arrow[u] & \mathrm{H}_n(f(U),f(U)\setminus\{f(x)\};\mathbb{Z}). \arrow[u]
\end{tikzcd}$$

We make the following observations:

(1) the top vertical maps are isomorphisms by Theorem 87.1,
(2) the bottom vertical maps are isomorphisms by the Excision Theorem 43.20,
(3) it follows from Corollary 87.4 that the vertical maps send the fundamental classes $[M]$ and $[N]$ to the orientations μ_x respectively $\nu_{f(x)}$.

The desired equalities follow easily from these observations.[1062] ■

Now we can also provide a proof for Proposition 72.2.

Sketch of the proof (∗). The equivalence of (1), (2) and (3) follows quite easily from Lemma 72.3. Now suppose M that is in fact a smooth manifold and that f is a diffeomorphism. The equivalence of (2) and (4) is a straightforward consequence of Lemma 86.12 and the Excision Theorem 43.20. Once again we leave it to the reader to fill in the details. ■

Examples. Let $n \in \mathbb{N}_0$.

(1) It follows from Lemma 45.11 (6) that the map

$$\begin{array}{rcl} \rho\colon S^n & \to & S^n \\ (x_0,\dots,x_{n-1},x_n) & \mapsto & (x_0,\dots,x_{n-1},-x_n), \end{array}$$

i.e. the map that is given by reflection in the $x_n = 0$ hyperplane, is orientation-reversing. This shows that the sphere S^n is amphichiral.

(2) Let M be a closed, oriented topological manifold. It follows from (1) and the naturality of the Künneth Theorem 58.8 for topological spaces that the map

$$\begin{array}{rcl} S^n\times M & \to & S^n\times M \\ (x,P) & \mapsto & (\rho(x),P), \end{array}$$

is orientation-reversing. This shows that the product of the sphere S^n with any closed, oriented topological manifold is amphichiral.

Now we want to relate the notion of amphichiral knots to the notion of amphichiral smooth manifolds. Before we can do so we recall the following definition.

Definition. Let $K\subset\mathbb{R}^3$ be a knot. We pick a tubular neighborhood $\overline{B}^2\times K$ as provided by the Tubular Neighborhood Theorem 8.24. We refer to $X_K := S^3\setminus(B^2\times K)$ as the *exterior of K*.[1063]

[1062]Where on earth did we use the hypothesis that $f^{-1}(f(x)) = \{x\}$?

FIGURE 1114

The following proposition together with Proposition 72.2 now turns Question 27.9 into a question about topological spaces.

Proposition 72.4. *Let $K \subset \mathbb{R}^3$ be a knot. If K is amphichiral, then the knot exterior X_K is also amphichiral.*

Remark. We had just seen in Proposition 72.4 that the exterior of an amphichiral knot is amphichiral. The converse to that statement is also true, i.e. if the exterior of a knot is amphichiral, then the knot is also amphichiral. But this direction is much harder to prove. To the best of my knowledge it follows from a very difficult theorem by Cameron Gordon and John Luecke [**GL89**, p. 371] from 1989.

PROOF. Let $K \subset \mathbb{R}^3$ be an amphichiral knot. By the Tubular Neighborhood Theorem 8.24 we can pick a tubular neighborhood $\nu K = \overline{B}^2 \times K$. As on page 1728 we consider the knot exterior $X_K := S^3 \setminus (B^2 \times K)$. By Proposition 72.2 it suffices to show that there exists a homeomorphism $f\colon X_K \to X_K$ with $f_*([X_K]) = -[X_K]$.

We denote by ρ the reflection in a hyperplane of $\mathbb{R}^3$. As usual we extend ρ to a self-homeomorphism of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ by setting $\rho(\infty) = \infty$. We write $K^{\mathrm{mir}} = \rho(K)$ and $\nu K^{\mathrm{mir}} = \rho(\nu K)$. By hypothesis there exists a smooth isotopy from K^{mir} to K. By the Isotopy Extension Theorem 8.27 this smooth isotopy extends to a smooth diffeotopy of S^3. This means that we have a smooth map

$$F\colon S^3 \times [0,1] \;\to\; S^3$$

such that $F_0 = \mathrm{id}$, such that each F_t is a diffeomorphism and such that $F_1(K^{\mathrm{mir}}) = K$. We write $\Phi := F_1\colon S^3 \to S^3$. Then $\Phi(\nu K^{\mathrm{mir}})$ is also a tubular neighborhood for K and by Proposition 8.26 there exists an orientation-preserving diffeomorphism $\Psi\colon S^3 \to S^3$ with $\Psi(\Phi(\nu K^{\mathrm{mir}})) = \nu K$. We write $f := \Psi \circ \Phi \circ \rho$.

[1063]As we pointed out in Lemma 68.16, the exterior X_K is a compact orientable smooth manifold and the diffeomorphism type of X_K as an oriented smooth manifold is well-defined, i.e. it is independent of the choice of the tubular neighborhood.

Now we consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}_3(S^3) \arrow[r, "\rho_*"] \arrow[d, "\cong"] & \mathrm{H}_3(S^3) \arrow[r, "\Phi_*"] \arrow[d, "\cong"] & \mathrm{H}_3(S^3) \arrow[r, "\Psi_*"] \arrow[d, "\cong"] & \mathrm{H}_3(S^3) \arrow[d, "\cong"] \\
\mathrm{H}_3(S^3, \nu K) \arrow[r, "\rho_*"] & \mathrm{H}_3(S^3, \nu K^{\mathrm{mir}}) \arrow[r, "\Phi_*"] & \mathrm{H}_3(S^3, \Phi(\nu K^{\mathrm{mir}})) \arrow[r, "\Psi_*"] & \mathrm{H}_3(S^3, \nu K) \\
\mathrm{H}_3(X_K, \partial X_K) \arrow[r, "\rho_*"] \arrow[u, "\cong"] \arrow[rrr, bend right, "f_* = (\Psi\circ\Phi\circ\rho)_*"'] & \mathrm{H}_3(X_{K^{\mathrm{mir}}}, \partial X_{K^{\mathrm{mir}}}) \arrow[r, "\Phi_*"] \arrow[u, "\cong"] & \mathrm{H}_3(\Phi(X_{K^{\mathrm{mir}}}), \Phi(\partial X_{K^{\mathrm{mir}}})) \arrow[r, "\Psi_*"] \arrow[u, "\cong"] & \mathrm{H}_3(X_K, \partial X_K). \arrow[u, "\cong"]
\end{tikzcd}$$

Here the upper left vertical map is an isomorphism by the long exact sequence of the pair $(S^3, \nu K)$ and the fact that $\mathrm{H}_i(\nu K) = 0$ for $i = 2, 3$ since $\nu K = K \times B^2$ is homotopy equivalent to S^1. Exactly the same argument shows that in fact all upper vertical maps are isomorphisms. The lower vertical maps are induced by the inclusion of pairs of topological spaces. All of them are isomorphisms by the Excision Theorem for Smooth Manifolds 44.10.

By Proposition 72.2 we are done once we have shown that the composition of three bottom horizontal maps is given by multiplication by -1. By the commutative diagram it suffices to prove the analogous statement for the composition of the top three horizontal maps. It follows from Lemma 45.11 (6) that the top-left horizontal map is given by multiplication by -1. It is a consequence of Proposition 42.5 that the top-middle horizontal map is the identity, since Φ is homotopic to the identity. Finally using Proposition 72.2 we see that the top-right horizontal map is the identity since Ψ is orientation-preserving. ■

Now we continue with our discussion of examples of amphichiral smooth manifolds.

Examples.

(1) We had just seen on page 1809 that topological manifolds of the type $S^n \times M$ are amphichiral. It follows that the torus $S^1 \times S^1$ is amphichiral.

(2) The surface of genus 2 is amphichiral. Pictorially an orientation-reversing self-homeomorphism is given by "reflection in the yz-plane", see Figure 1115 on the left. In the "octagon-picture" an orientation-reversing self-homeomorphism is given by the reflection f in the line illustrated in Figure 1115 on the left. It follows easily from Proposition 72.2 applied to the point $x = 0$ and U an open disk around x together with Lemma 86.2 that the map f is indeed orientation-reversing.

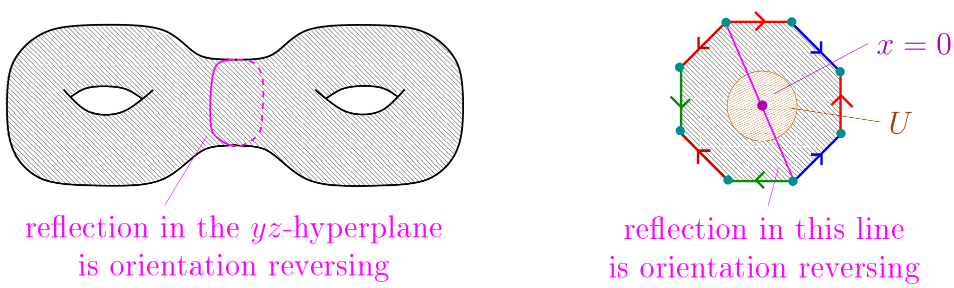


FIGURE 1115

(3) All other compact oriented 2-dimensional smooth manifolds are also amphichiral. We show some of the orientation-reversing maps in Figure 1116. In Exercise 72.2 we will discuss how these maps can be described rigorously.

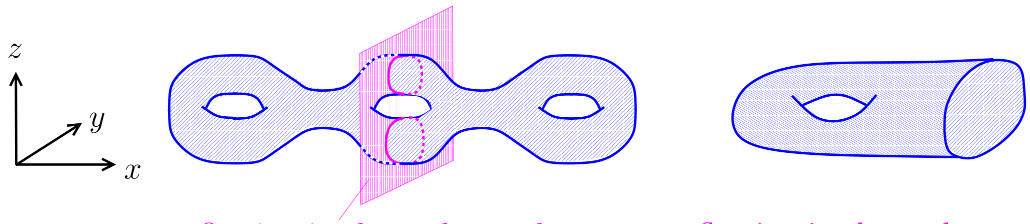


FIGURE 1116

(4) Example (2) can also be generalized as follows: if M is a compact oriented topological manifold, then its double $\mathrm{D}\,M = (M \sqcup M')/\sim$ is amphichiral. Indeed, the map $f\colon \mathrm{D}\,M \to \mathrm{D}\,M$ that swaps the two copies of M is orientation-reversing. This follows immediately from the definition of the orientation of $\mathrm{D}\,M$ that we gave in Lemma 86.18.

The long list of examples of amphichiral smooth manifolds raises the following question.

Question 72.5. *Are there compact oriented topological manifolds that are chiral?*

We end this chapter with a short discussion of lens spaces. Let $p \in \mathbb{N}$ and let $q \in \mathbb{N}$ be coprime to p. On page 502 we defined the lens space

$$L(p,q) \;=\; S^3/\mathbb{Z}_p \;=\; \big\{(z_1,z_2) \in \mathbb{C}^2 \,\big|\, |z_1|^2 + |z_2|^2 = 1\big\}/(z_1,z_2) \sim (e^{2\pi \mathrm{i}/p} z_1, e^{2\pi \mathrm{i} q/p} z_2).$$

Furthermore, on page 502 we had equipped $L(p,q)$ with the orientation which turns the obvious projection map $S^3 \to L(p,q)$ into an orientation-preserving map.

Now let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. In Lemma 16.6 we showed that if $q \equiv \pm r^{\pm 1} \bmod p$, then $L(p,q)$ and $L(p,r)$ are diffeomorphic.[1064] The following lemma is now a slight refinement of Lemma 16.6.

Lemma 72.6. *Let $p \in \mathbb{N}$ and let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$.*

(1) *In the case that $q \equiv r^{\pm 1} \bmod p$ there exists an orientation-preserving diffeomorphism $L(p,q) \to L(p,r)$.*
(2) *On the other hand, if $q \equiv -r^{\pm 1} \bmod p$, then there exists an orientation-reversing diffeomorphism $L(p,q) \to L(p,r)$.*

[1064] We still don't know whether the converse holds.

PROOF. Let $p \in \mathbb{N}$ and let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. We suppose that $q \equiv \pm r^{\pm 1} \bmod p$. As in the proof of Lemma 16.6 we define a diffeomorphism

$$\Phi\colon S^3 \to S^3$$

as follows:

$$\begin{array}{llll} \text{Case 1: } q \equiv r \phantom{^{-1}} \bmod p. & (z_1,z_2) & \mapsto & (z_1,z_2) \\ \text{Case 2: } q \equiv -r \phantom{^{-1}} \bmod p. & (z_1,z_2) & \mapsto & (z_1,\overline{z_2}) \\ \text{Case 3: } q \equiv r^{-1} \bmod p. & (z_1,z_2) & \mapsto & (z_2,z_1) \\ \text{Case 4: } q \equiv -r^{-1} \bmod p. & (z_1,z_2) & \mapsto & (z_2,\overline{z_1}) \end{array}$$

In all four cases it is straightforward to verify that the map descends to a diffeomorphism $L(p,q) \to L(p,r)$. It follows fairly easily from Lemma 45.11 and Proposition 72.2 that the diffeomorphism Φ is orientation-preserving in cases 1 and 3 and that it is orientation-reversing in cases 2 and 4.[1065] The lemma now follows from the fact that the projection maps $S^3 \to L(p,q)$ and $S^3 \to L(p,r)$ are orientation-preserving. ■

Example. Note that $-2^{-1} \equiv -3 \equiv 2 \bmod 5$. Thus we obtain from Lemma 72.6 that there exists an orientation-reversing diffeomorphism $L(5,2) \to L(5,2)$, i.e. the lens space $L(5,2)$ is amphichiral.

The argument of the previous example does not work for the lens space $L(3,1)$. So the following question arises:

Question 72.7. *Is the lens space $L(3,1)$ chiral?*

With our present knowledge we are not able to answer Questions 27.9, 72.5 and 72.7. The key to answering these three questions will be the cohomology groups that we introduce in Chapter 73.

Exercises for Chapter 72.

Exercise 72.1. Why is the molecule shown in Figure 1117 chiral? How can you show rigorously that the two molecules left and right cannot be turned into one another using a translation and a matrix in SO(3)?

Exercise 72.2.

(a) Give a rigorous proof for the statement that the surface of genus 3 is amphichiral. More precisely, give a precise description of the surface of genus 3 (e.g. as a dodecagon with appropriate identifications) and indicate carefully an orientation-reversing self-homeomorphism.
(b) Do the same exercise for the genus 2 surface with two open disks removed.

[1065] For example in case 3 the map is given by $(x_1,x_2,x_3,x_4) \mapsto (x_3,x_4,x_1,x_2)$ which is the composition of two reflections in hyperplanes, namely in the $x_1 = x_3$ and the $x_2 = x_4$ hyperplane.

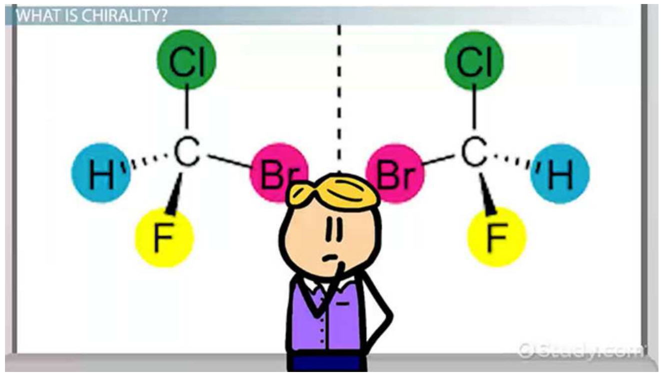


Figure 1117

73. The cohomology groups of a topological space

In this chapter we will introduce the cohomology groups $\mathrm{H}^k(X;G)$ of a topological space together with a choice of an abelian group G. These study of the cohomology will keep us busy for a very long time. For many mathematicians it takes a long time till they feel fully comfortable with cohomology group since elements of cohomology groups are much less visual than elements of homology groups. Perhaps, before we embark on our new adventure it is good quote John von Neumann who said

"In mathematics you don't understand things. You just get used to them."

73.1. **The Hom-functor.**

We start out with the following obvious notation.

Notation. Given a group A and an abelian group G we write

$$\operatorname{Hom}(A,G) \;=\; \{\text{all group homomorphisms } A \to G\}.$$

The set of homomorphisms $\operatorname{Hom}(A,G)$ has the structure of an abelian group with the sum of two homomorphisms $\varphi,\psi\colon A \to G$ given by the group homomorphism that is defined by $(\varphi+\psi)(a) := \varphi(a)+\psi(a)$.

Example. We leave it to the reader to provide a proof for the statement that for any $n \in \mathbb{N}$ and any abelian group G we have a natural isomorphism $\operatorname{Hom}(\mathbb{Z}_n,G) \cong \ker(G \xrightarrow{\cdot n} G)$. Note that this implies in particular that for any $m \in \mathbb{N}$ we have $\operatorname{Hom}(\mathbb{Z}_n,\mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(n,m)}$.

On several occasions we will implicitly and explicitly use the following lemma which is an immediate consequence of Lemma 19.1.

Lemma 73.1. $(*)$ *Let G be an abelian group. If S is a non-empty set, then the homomorphism*[1066][1067][1068]

$$\begin{array}{rcl} \operatorname{Hom}(\mathbb{Z}^{(S)},G) & \to & \{\textit{all maps from the set } S \textit{ to } G\} \\ (\varphi\colon \mathbb{Z}^{(S)} \to G) & \mapsto & \begin{pmatrix} S & \to & G \\ s & \mapsto & \varphi(s) \end{pmatrix} \end{array}$$

is an isomorphism. In particular the map

$$\begin{array}{rcl} \operatorname{Hom}(\mathbb{Z},G) & \to & G \\ \varphi & \mapsto & \varphi(1) \end{array}$$

is an isomorphism.

[1066]Recall that on page 580, given a non-empty set S we defined

$$\mathbb{Z}^S \;:=\; \text{all maps from } S \text{ to } \mathbb{Z}.$$

and we defined the free abelian group generated by S to be the subgroup

$$\mathbb{Z}^{(S)} \;:=\; \text{all maps from } S \text{ to } \mathbb{Z} \text{ which are non-zero for only finitely many } s \in S.$$

Following the notation introduced on page 580 we view each $s \in S$ as an element in $\mathbb{Z}^{(S)}$.

[1067]Why do we write $\mathbb{Z}^{(S)}$ in the lemma? Should it not be $\mathbb{Z}^S$?

[1068]Note that the group structure on G turns both sides of the lemma into a group, so the given map clearly respects this group structure, i.e. the given map is a homomorphism.

Let G be an abelian group and let $f\colon A \to B$ be a group homomorphism. Then f induces a homomorphism

$$\begin{array}{rcl} f^*\colon \operatorname{Hom}(B,G) & \to & \operatorname{Hom}(A,G) \\ (\varphi\colon B \to G) & \mapsto & \left(\begin{array}{rcl} A & \to & G \\ a & \mapsto & \varphi(f(a)) \end{array}\right). \end{array}$$

Example. Let G be an abelian group.

(1) If A is a subgroup of a group B and if $i\colon A \to B$ is the inclusion map, then for each homomorphism $\varphi\colon B \to G$ the homomorphism $i^*\varphi\colon A \to G$ is just the restriction of φ to $A \subset B$.

(2) If S is a subset of a set T, then we have a natural inclusion $i\colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ and it follows from Lemma 73.1 that the map $i^*\colon \operatorname{Hom}(\mathbb{Z}^{(T)},G) \to \operatorname{Hom}(\mathbb{Z}^{(S)},G)$ is an epimorphism. Indeed, we just extend the map $S \to G$ to a map $T \to G$ by assigning the trivial element in G to all elements in $T \setminus S$.

It follows immediately from the definitions that

$$(\mathrm{id}_A)_* = \mathrm{id}_{\operatorname{Hom}(A,G)}, \quad \text{for all groups } A$$

and

$$(g \circ f)^* = f^* \circ g^* \quad \text{for all homomorphisms } f\colon A \to B \text{ and } g\colon B \to C.$$

Put differently, given an abelian group G the maps

$$\begin{array}{rcl} A & \mapsto & \operatorname{Hom}(A,G) \\ (f\colon A \to B) & \mapsto & (f^*\colon \operatorname{Hom}(B,G) \to \operatorname{Hom}(A,G)) \end{array}$$

define a *contravariant* functor from the category of groups to the category of abelian groups.

Remark. If R is a commutative ring, for example if $R = \mathbb{Z}_n, R = \mathbb{Z}, R = \mathbb{Q}$ or $R = \mathbb{R}$, then for any group C the set of homomorphisms $\operatorname{Hom}(C,R)$ forms an R-module in an obvious way. More precisely, given a homomorphism $\varphi\colon C \to R$ and $r \in R$ we define $r\varphi$ to be the homomorphism given by $c \mapsto r \cdot \varphi(c)$. Now we see easily that the above maps $A \mapsto \operatorname{Hom}(A,R)$ and $f \mapsto f^*$ define a contravariant functor from the category of groups to the category of R-modules.

The following lemma might be familiar from linear algebra. We refer to [**Rot09**, Theorem 2.32(i)] for more details.

Lemma 73.2. *Let $\{A_i\}_{i\in I}$ be a family of groups and let G be an abelian group. Then the map*

$$\begin{array}{rcl} \operatorname{Hom}\left(\bigoplus\limits_{i\in I} A_i, G\right) & \to & \prod\limits_{i\in I} \operatorname{Hom}(A_i,G) \\ f & \mapsto & \prod\limits_{i\in I} f|_{A_i} \end{array}$$

is an isomorphism.

The following rather technical lemma will be used later on.

Lemma 73.3. *Let* $\{\{A_i\}_{i\in\mathbb{N}}, \{\varphi_{ij}\colon A_i \to A_j\}_{i\le j}\}$ *be a direct system of groups and let* G *be an abelian group. Then*

$$\{\{\mathrm{Hom}(A_i, G)\}_{i\in\mathbb{N}}, \{\varphi_{ij}^*\colon \mathrm{Hom}(A_j, G) \to \mathrm{Hom}(A_i, G)\}_{i\le j}\}$$

is an inverse system of abelian groups (or modules if G *is a commutative ring), and we have a natural isomorphism*

$$\mathrm{Hom}\left(\varinjlim A_i, G\right) \xrightarrow{\cong} \varprojlim \mathrm{Hom}(A_i, G)$$

of abelian groups (or modules if G *is a commutative ring).*

Proof. As so often for arguments regarding limits, the proof follows mostly from unraveling the definitions. We refer to [**Rot09**, Proposition 5.26] for more details. ■

We continue with the following lemma which we will use on many occasions.

Lemma 73.4. *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups and let G *be an abelian group. If* C *is free abelian, then the dual sequence*

$$0 \to \mathrm{Hom}(C, G) \xrightarrow{p^*} \mathrm{Hom}(B, G) \xrightarrow{i^*} \mathrm{Hom}(A, G) \to 0$$

is also exact.

Proof. It follows from the hypothesis that C is a free abelian and from Lemma 46.1 that the initial short exact sequence splits. But this means that, according to Splitting Lemma 46.2, that there exists an isomorphism $\Phi\colon B \to A \oplus C$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "="] & B \arrow[r, "p"] \arrow[d, "\Phi", "\cong"'] & C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & A \arrow[r, "a\mapsto(a,0)"] & A \oplus C \arrow[r, "(a,c)\mapsto c"] & C \arrow[r] & 0.
\end{tikzcd}$$

By applying the contravariant $\mathrm{Hom}(-, G)$ functor we obtain the upper half of the following commutative diagram:

$$\begin{tikzcd}
0 & \mathrm{Hom}(A,G) \arrow[l] & \mathrm{Hom}(B,G) \arrow[l, "i^*"'] & \mathrm{Hom}(C,G) \arrow[l, "p^*"'] & 0 \arrow[l] \\
0 & \mathrm{Hom}(A,G) \arrow[l] \arrow[u, "="'] & \mathrm{Hom}(A\oplus C,G) \arrow[l, "(a\mapsto(a,0))^*"'] \arrow[u, "\Phi^*"', "\cong"] & \mathrm{Hom}(C,G) \arrow[l, "((a,c)\mapsto c)^*"'] \arrow[u, "="'] & 0 \arrow[l] \\
0 & \mathrm{Hom}(A,G) \arrow[l] \arrow[u, "="'] & \mathrm{Hom}(A,G)\oplus\mathrm{Hom}(C,G) \arrow[l, "f\oplus g\mapsto f"'] \arrow[u, "(f\oplus)\mapsto(a\oplus c\mapsto f(a)+g(c))"'] & \mathrm{Hom}(C,G) \arrow[l, "g\mapsto 0\oplus g"'] \arrow[u, "="'] & 0. \arrow[l]
\end{tikzcd}$$

One can easily verify that the lower half of the diagram commutes. Furthermore it is evident that the bottom sequence is exact. Finally it is straightforward to see that the vertical maps are isomorphisms we obtain that the upper sequence is also exact. But that is exactly what we wanted to prove. ■

73.2. **Cochain complexes.** Now we introduce the notion of a cochain complex and of its cohomology groups. A cochain complex is exactly the same as a chain complex except that the boundary map now increases the degree by one instead of lowering it by one.

Definition.

(1) A *cochain complex* is a sequence

$$\leftarrow D^n \xleftarrow{\delta_{n-1}} D^{n-1} \xleftarrow{\delta_{n-2}} \dots \xleftarrow{\delta_1} D^1 \xleftarrow{\delta_0} D^0 \leftarrow 0$$

of maps between abelian groups such that for all $i \in \mathbb{N}_0$ we have $\delta_{i+1} \circ \delta_i = 0$. We refer to the maps δ_* as *coboundary maps.*

(2) We define the *n-th cohomology group* of a cochain complex (D^*, δ_*) to be

$$\mathrm{H}^n(D) := \frac{\ker(\delta_n \colon D^n \to D^{n+1})}{\mathrm{im}(\delta_{n-1} \colon D^{n-1} \to D^n)}.$$

Example. It follows from [**Lee02**, Theorem 14.24] that the groups

$$\Omega_n(M) := \{\text{all smooth } n\text{-forms on } M\}, \quad n \in \mathbb{N}_0$$

together with the differentials $d_n \colon \Omega_n(M) \to \Omega_{n+1}(M)$ define a cochain complex. The corresponding cohomology groups are called the de Rham cohomology groups of M. We will return to this example in more detail in Chapter 78.

Now we will see that most of the definitions and algebraic results for chain complexes can be generalized almost verbatim to cochain complexes.

Definition.

(1) A *cochain map* $f \colon C^* \to D^*$ between two cochain complexes consists of a family $\{f_n \colon \mathrm{C}^n \to D^n\}_{n \in \mathbb{N}_0}$ of maps that commute with the coboundary maps, i.e. such that for all $n \in \mathbb{N}_0$ we have

$$\underbrace{\delta_n}_{\delta_n \colon D^n \to D^{n+1}} \circ f_n = f_{n+1} \circ \underbrace{\delta_n}_{\delta_n \colon \mathrm{C}^n \to C^{n+1}} \colon \mathrm{C}^n \to D^{n+1}.$$

(2) Let $f, g \colon C^* \to D^*$ be two cochain maps between two cochain complexes C^* and D^*. A *cochain homotopy* $P = \{P_n\}_{n \in \mathbb{N}_0}$ between the cochain maps f and g consists of a family $\{P_n \colon \mathrm{C}^n \to D^{n-1}\}_{n \in \mathbb{N}_0}$ of maps such that for each $n \in \mathbb{N}_0$ we have

$$\delta_{n-1} \circ P_n + P_{n+1} \circ \delta_n = f_n - g_n.$$

If there exists a cochain homotopy between cochain maps f and g, then we say that f and g are *cochain homotopic.*

Remark. For orientation we summarize in the following diagram all objects arising in the definition of a cochain homotopy:

$$\begin{tikzcd}
{} & C^{n+1} \arrow[l, "\delta_{n+1}"'] \arrow[d, "f_{n+1},g_{n+1}"] \arrow[dr, "P_{n+1}"] & C^n \arrow[l, "\delta_n"'] \arrow[d, "f_n,g_n"] \arrow[dr, "P_n"] & C^{n-1} \arrow[l, "\delta_{n-1}"'] \arrow[d, "f_{n-1},g_{n-1}"] & {} \arrow[l, "\delta_{n-2}"'] \\
{} & D^{n+1} \arrow[l, "\delta_{n+1}"] & D^n \arrow[l, "\delta_n"] & D^{n-1} \arrow[l, "\delta_{n-1}"] & {} \arrow[l, "\delta_{n-2}"]
\end{tikzcd}$$

The statements of the next lemma surely do not come as much of a surprise.

Lemma 73.5.

(1) *If $f\colon C^* \to D^*$ is a cochain map between cochain complexes C^* and D^*, then the map*
$$\begin{array}{rcl} f_*\colon\ \mathrm{H}^n(C) & \to & \mathrm{H}^n(D) \\ {[c]} & \mapsto & [f_n(c)] \end{array}$$
is well-defined.

(2) *Two cochain maps that are cochain homotopic induce the same map on cohomology groups.*

(3) *Cochain complexes together with the cochain maps form a category, called the* category of cochain complexes.

Proof. The first statement is proved the same way as Lemma 41.7 and the second statement is proved the same way as Lemma 42.1. The last statement follows immediately from the definitions. ■

Remark. Let C^* and D^* be cochain complexes. On a few rare occasions we will need to deal with the following slight generalization of a cochain map. A *generalized cochain maps* is a family $\{f_n\colon \mathrm{C}^n \to D^n\}_{n\in\mathbb{N}_0}$ of maps that commute with the coboundary maps up to sign, i.e. such that for every $n \in \mathbb{N}_0$ there exists an $\epsilon \in \{-1,1\}$ such that $\delta_n \circ f_n = f_{n+1} \circ \delta_n$. One can easily verify that the conclusion of Lemma 73.5 (1) also holds for generalized cochain maps.

One of the most important algebraic constructions in Algebraic Topology II was that we could associate to a short exact sequence of chain complexes a long exact sequence of homology groups. We have an almost identical construction for short exact sequences of cochain complexes and their cohomology groups. We will discuss this construction in detail. So let
$$0 \to C^* \xrightarrow{f} B^* \xrightarrow{g} A^* \to 0$$
be a short exact sequence of cochain complexes. In particular for each $n \in \mathbb{N}_0$ we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C^{n+1} \arrow[r, "f_{n+1}"] & B^{n+1} \arrow[r, "g_{n+1}"] & A^{n+1} \arrow[r] & 0 \\
0 \arrow[r] & C^n \arrow[r, "f_n"] \arrow[u, "\delta_n"] & B^n \arrow[r, "g_n"] \arrow[u, "\delta_n"] & A^n \arrow[r] \arrow[u, "\delta_n"] & 0
\end{tikzcd}$$

where the horizontal sequences are exact. Now we will define a *connecting homomorphism*

$$\delta\colon\ \mathrm{H}^n(A)\ \to\ \mathrm{H}^{n+1}(C).$$

The definition is almost verbatim the same as the definition on page 1115 of the connecting homomorphism in homology. More precisely, we perform the following steps:

(1) Let $z \in \mathrm{H}^n(A)$.
(2) We pick $\alpha \in A^n$ with $z = [\alpha]$.
(3) By the surjectivity of g_n we have $\alpha = g_n(\beta)$ for some $\beta \in B^n$.
(4) From the commutativity of the diagram and the exactness of the horizontal sequences we see that $\delta_n(\beta) = f_{n+1}(\gamma)$ for some uniquely determined $\gamma \in C^{n+1}$.
(5) We put $\delta_n(z) := [\gamma] \in \mathrm{H}^{n+1}(C)$.

The following proposition is now the obvious analogue of Lemma 43.10 and Proposition 43.11. We leave it to the reader to make the necessary changes to the proof.

Proposition 73.6.

(1) *The connecting homomorphisms are well-defined and they are indeed homomorphisms.*[1069]
(2) *The connecting homomorphisms are natural in the sense that commuting diagrams of short exact sequences of cochain complexes give rise to commuting diagrams of cohomology groups involving the connecting homomorphisms.*
(3) *If*

$$0\ \to\ C^*\ \xrightarrow{f}\ B^*\ \xrightarrow{g}\ A^*\ \to\ 0$$

is a short exact sequence of cochain complexes, then

$$\dots \to \mathrm{H}^n(C) \xrightarrow{f_*} \mathrm{H}^n(B) \xrightarrow{g_*} \mathrm{H}^n(A) \xrightarrow{\delta} \mathrm{H}^{n+1}(C) \xrightarrow{f_*} \mathrm{H}_{n+1}(B) \xrightarrow{g_*} \dots$$

forms an exact sequence.

Finally the following lemma says that cohomology commutes with direct sums and direct products. The proof of the lemma is identical to the proof of Lemma 41.13 that we did not give.

Lemma 73.7. *If $\{C_a\}_{a\in A}$ is a family of cochain complexes, then the family of inclusion maps $C_a \to \bigoplus\limits_{a\in A} C_a$, $a \in A$, and $C_a \to \prod\limits_{a\in A} C_a$, $a \in A$, induce isomorphisms*

$$\bigoplus_{a\in A} \mathrm{H}^n(C_a)\ \xrightarrow{\cong}\ \mathrm{H}^n\Big(\bigoplus_{a\in A} C_a\Big) \quad \text{and} \quad \prod_{a\in A} \mathrm{H}^n(C_a)\ \xrightarrow{\cong}\ \mathrm{H}^n\Big(\prod_{a\in A} C_a\Big).$$

73.3. **Dual cochain complexes.** Now we will present the arguably most important source of cochain complex.

Definition. Let

$$(C_*, \partial_*)\colon\ \ \to\ C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0\ \to\ 0$$

[1069] Furthermore, if the cochain complexes are cochain complexes over a commutative ring R, then the connecting homomorphism is in fact an R-module homomorphism.

be a chain complex and let G be an abelian group. For each $n \in \mathbb{N}_0$ we write

$$\delta_n \;:=\; \partial_{n+1}^*\colon\, \operatorname{Hom}(C_n, G) \;\to\; \operatorname{Hom}(C_{n+1}, G).$$

We obtain the following cochain complex[1070]

$$\leftarrow\, \operatorname{Hom}(C_{n+1}, G) \xleftarrow{\delta_n} \operatorname{Hom}(C_n, G) \xleftarrow{\delta_{n-1}} \dots \xleftarrow{\delta_1} \operatorname{Hom}(C_1, G) \xleftarrow{\delta_0} \operatorname{Hom}(C_0, G) \leftarrow 0.$$

We refer to it as *cochain complex dual to* (C_*, ∂_*) and we denote it by $\operatorname{Hom}(C_*, G)$. We define the *$n$-th cohomology group of the chain complex* (C_*, ∂_*) *with G-coefficients* as the cohomology of the cochain complex $(\operatorname{Hom}(C_*, G), \delta_*)$, i.e. we define

$$\mathrm{H}^n(C; G) \;:=\; \frac{\ker(\delta_n\colon \operatorname{Hom}(C_n, G) \to \operatorname{Hom}(C_{n+1}, G))}{\operatorname{im}(\delta_{n-1}\colon \operatorname{Hom}(C_{n-1}, G) \to \operatorname{Hom}(C_n, G))}.$$

Remark.

(1) We had just defined $\delta_n := \partial_{n+1}^*$, i.e. the coboundary map in the cochain complex is the dual of the boundary map in the chain complex. This is the same definition that is used in most books on algebraic topology, see e.g. [**Mun84**, p. 251] or [**Hat02**, p. 189]. In the book by Bredon [**Bre93**, p. 321] the coboundary is defined as $\delta_n := (-1)^{n+1} \cdot \partial_{n+1}^*$.

The different conventions do not affect the definition of the cohomology groups, but they will affect the sign conventions in some formulas later on. In particular if one cites results from other books one needs to be careful about the sign conventions used.

(2) If R is a commutative ring and if (C_*, ∂_*) is a chain complex, then it follows immediately from the remark on page 1816 that the corresponding cohomology groups $\mathrm{H}^n(C; R)$ come naturally with the structure of R-modules.

Many of the statements about chain complexes and their homology groups hold in a slightly modified version also for cohomology groups. For example we have the following purely algebraic lemma.

Lemma 73.8. *Let G be an abelian group.*

(1) *Let $f = \{f_n\colon C_n \to D_n\}_{n \in \mathbb{N}_0}$ be a chain map between chain complexes C_* and D_*, then for each $n \in \mathbb{N}_0$ the map*

$$\begin{aligned} f^*\colon\, \operatorname{Hom}(D_n, G) \;&\to\; \operatorname{Hom}(C_n, G) \\ (\varphi\colon D_n \to G) \;&\mapsto\; \big(\underbrace{C_n \xrightarrow{f_n} D_n \xrightarrow{\varphi} G}_{=\varphi \circ f_n = f_n^* \varphi}\big) \end{aligned}$$

is a cochain map, in particular it induces a well-defined map

$$f^*\colon\, \mathrm{H}^n(D; G) \;\to\; \mathrm{H}^n(C; G).$$

(2) *For each $n \in \mathbb{N}_0$ the maps*

$$C_* \;\mapsto\; \mathrm{H}^n(C; G) \quad \textit{and} \quad f \;\mapsto\; f^*$$

[1070] It follows the functoriality property of $\operatorname{Hom}(-, G)$ that $\delta_n \circ \delta_{n-1} = \partial_{n+1}^* \circ \partial_n^* = (\partial_n \circ \partial_{n+1})^* = 0$, i.e. this is indeed a cochain complex.

define a contravariant *functor from the category* $\mathcal{ChCplx}$ *of chain complexes to the category* $\mathcal{AbGr}$ *of abelian groups. (If G is a commutative ring, then this is in fact a functor to the category of G-modules.)*

(3) *Let $f, g\colon C_* \to D_*$ be two chain maps. If they are chain homotopic, then the induced cochain maps $f^*, g^*\colon \operatorname{Hom}(D_*, G) \to \operatorname{Hom}(C_*, G)$ are cochain homotopic and therefore the induced maps $f^*, g^*\colon \mathrm{H}^n(D;G) \to \mathrm{H}^n(C,G)$ are identical.*

(4) *If $f_*\colon C_* \to D_*$ is a chain homotopy equivalence, then for every $n \in \mathbb{N}_0$ the induced maps $f^*\colon \mathrm{H}^n(D;G) \to \mathrm{H}^n(C;G)$ are isomorphisms.*

PROOF.

(1) Let $f = \{f_n\colon C_n \to D_n\}_{n\in\mathbb{N}_0}$ be a chain map. This means that for each $n \in \mathbb{N}_0$ we have

$$f_{n-1} \circ \partial_n \;=\; \partial_n \circ f_n$$

which implies by the fact that $A \mapsto \operatorname{Hom}(A, G)$ is a *contravariant* functor that

$$\underbrace{\partial_n^*}_{=\delta_{n-1}} \circ f_{n-1}^* \;=\; f_n^* \circ \underbrace{\partial_n^*}_{=\delta_{n-1}}.$$

We have thus shown that f^* is a cochain map. The second statement follows immediately from Lemma 73.5.

(2) This statement follows easily from the definitions.

(3) Let $\{P_n\colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ be a chain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between f and g. Recall that this means that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; f_n - g_n.$$

We set $Q_n := P_{n-1}^*\colon \operatorname{Hom}(D_n, G) \to \operatorname{Hom}(C_{n-1}, G)$. It follows again from the fact that $A \mapsto \operatorname{Hom}(A, G)$ is a *contravariant* functor that

$$\underbrace{P_n^*}_{=Q_{n+1}} \circ \underbrace{\partial_{n+1}^*}_{=\delta_n} + \underbrace{\partial_n^*}_{=\delta_{n-1}} \circ \underbrace{P_{n-1}^*}_{=Q_n} \;=\; f_n^* - g_n^*.$$

This shows that the maps $Q_n, n \in \mathbb{N}_0$ form a cochain homotopy. It follows from Lemma 73.5 that the induced maps on cohomology groups agree.

(4) The last statement follows immediately from (2) and (3). ■

In Lemma 57.7 (2) we basically saw that for any chain complex C_* and any homomorphism $f\colon A \to B$ of abelian groups the maps

$$\begin{aligned} f_*\colon C_* \otimes A &\;\to\; C_* \otimes B \\ \Big[\sum_{i=1}^k \sigma_i \otimes a_i\Big] &\;\mapsto\; \Big[\sum_{i=1}^k \sigma_i \otimes f(a_i)\Big] \end{aligned}$$

form a chain map, in particular for each $n \in \mathbb{N}_0$ we obtain an induced map

$$f_*\colon \mathrm{H}_n(C_* \otimes A) \;\to\; \mathrm{H}_n(C_* \otimes B).$$

Similarly, it is straightforward to check that the maps

$$\begin{array}{rcl} f_*\colon \operatorname{Hom}(C_n, A) & \to & \operatorname{Hom}(C_n, B) \\ \varphi & \mapsto & f \circ \varphi \end{array}$$

define a cochain map, in particular we get for each $n \in \mathbb{N}_0$ an induced map

$$f_*\colon \operatorname{H}^n(C_*; A) \to \operatorname{H}^n(C_*; B)$$

The following lemma basically says that these maps have all the properties one could possibly hope for. The lemma can be viewed as a cohomological analogue of Lemma 57.7.

Lemma 73.9. (∗)

(1) *Given a chain complex C_* and $n \in \mathbb{N}_0$ the maps*

$$\begin{array}{rcl} A & \mapsto & \operatorname{H}^n(C_*; A) \\ (f\colon A \to B) & \mapsto & (f_*\colon \operatorname{H}^n(C_*; A) \to \operatorname{H}^n(C_*; B)) \end{array}$$

form a covariant functor from the category of abelian groups to the category of abelian groups. In other words, "cohomology is covariant in the coefficients".

(2) *Given a group homomorphism $f\colon A \to B$ the induced maps f_* define a natural transformation from the contravariant functor $C_* \mapsto \operatorname{H}^n(C_*; A)$ to the contravariant functor $C_* \mapsto \operatorname{H}^n(C_*; B)$.*

Proof. The two statements follow basically immediately from the definitions. ■

73.4. **The singular cohomology groups of topological spaces.** Now we apply the somewhat dry algebra from the last section to our favorite chain complex, namely the singular chain complex of a pair of topological spaces.

Definition. Let (X, A) be a pair of topological spaces and let G be an abelian group.

(1) For each $n \in \mathbb{N}_0$ we write

$$\operatorname{C}^n(X, A; G) := \operatorname{Hom}(\operatorname{C}_n(X, A), G) \text{ and } \delta_n := \partial_{n+1}^*\colon \operatorname{C}^n(X, A; G) \to \operatorname{C}^{n+1}(X, A; G).$$

We use the following notation:

(a) We refer to the elements of $\operatorname{C}^n(X, A; G)$ as *singular n-cochains*,

(b) we refer to the elements in the kernel of $\delta_n\colon \operatorname{C}^n(X, A; G) \to \operatorname{C}^{n+1}(X, A; G)$ as *singular n-cocycles* and

(c) we refer to the elements in the image of $\delta_{n-1}\colon \operatorname{C}^{n-1}(X, A; G) \to \operatorname{C}^n(X, A; G)$ as *singular n-coboundaries.*

When $n \in \mathbb{N}_0$ is understood then sometimes we just write δ instead of δ_n.

(2) We refer to the cochain complex

$$\ldots \leftarrow \operatorname{C}^{n+1}(X, A; G) \xleftarrow{\delta_n} \operatorname{C}^n(X, A; G) \xleftarrow{\delta_{n-1}} \operatorname{C}^{n-1}(X, A; G) \leftarrow \ldots \xleftarrow{\delta_0} C^0(X, A; G) \leftarrow 0$$

as the *singular cochain complex of (X, A) with G-coefficients.* We define the *n-th singular cohomology group of (X, A) with G-coefficients*[1071] as follows

$$\operatorname{H}^n(X, A; G) := \operatorname{H}^n(\operatorname{C}_*(X, A); G) = \frac{\ker(\delta_n\colon \operatorname{C}^n(X, A; G) \to \operatorname{C}^{n+1}(X, A; G))}{\operatorname{im}(\delta_{n-1}\colon \operatorname{C}^{n-1}(X, A; G) \to \operatorname{C}^n(X, A; G))}.$$

(3) If $A = \varnothing$ then we just write $\mathrm{C}^n(X;G) = \mathrm{C}^n(X,\varnothing;G)$ and similar for cohomology groups.
(4) If $G = \mathbb{Z}$ then often we drop the coefficients from the notation, e.g. often we write $\mathrm{H}^n(X,A) = \mathrm{H}^n(X,A;\mathbb{Z})$.

Remark. As we had already pointed out on page 1821, if R is a commutative ring, e.g. if $R = \mathbb{Z}_n, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $R = \mathbb{C}$, then the cohomology groups $\mathrm{H}^n(X,A;R)$ can in fact be viewed as R-modules and the coboundary maps are R-module homomorphisms.

In the following we first discuss the definition and examples of the absolute cohomology groups $\mathrm{H}^*(X;G)$. A little later we will discuss the arguably even more confusing relative cohomology groups $\mathrm{H}^*(X,A;G)$. We start this discussion of absolute cohomology groups with the following remark.

Remark. Let G be an abelian group.

(1) Let X be a topological space and let $n \in \mathbb{N}_0$. We denote by $S_n(X)$ the set of singular n-simplices. We then have natural identifications

$$\mathrm{C}^n(X;G) \;=\; \mathrm{Hom}(\mathrm{C}_n(X),G) \underset{\substack{\uparrow \\ \text{by definition } \mathrm{C}_n(X)=\mathbb{Z}^{(S_n(X))}}}{=} \mathrm{Hom}\left(\mathbb{Z}^{(S_n(X))},G\right) \underset{\substack{\uparrow \\ \text{identification given by Lemma 73.1}}}{=} \text{maps from } S_n(X) \text{ to } G.$$

Put differently, a singular n-cochain is precisely the same as assigning to each singular n-simplex an element in G. We will use this bijection throughout the course without explicitly referring to it.

(2) Let X be a topological space. We recall that, following the discussion on page 1078, we can identify the singular 0-simplices in X with the points in X. According to the discussion in (1) we can now make the identification

$$\mathrm{C}^0(X;G) \;=\; \{\text{all maps from } X \text{ to } G\}.$$

Before we develop the theory of cohomology groups in detail we first want to consider some explicit examples. As a starter we consider cochains and cocycles on the topological spaces $\mathbb{R}$ and S^1.

Examples.

(1) We start out with the topological space $X = \mathbb{R}$. We consider the two functions

$$\begin{array}{rcl} \alpha_{\mathbb{R}}\colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} \alpha_{\mathbb{Z}}\colon \mathbb{R} & \to & \mathbb{Z} \\ x & \mapsto & \lfloor x \rfloor := \max\{n \in \mathbb{Z} \,|\, n \le x\}. \end{array}$$

As pointed out above, these two functions define singular 0-cochains. We denote these 0-cochains again by $\alpha_{\mathbb{R}}$ and $\alpha_{\mathbb{Z}}$. For any singular 1-simplex $\mu\colon \Delta^1 \to \mathbb{R}$ we have

$$(\delta\alpha_{\mathbb{R}})(\mu) \underset{\substack{\uparrow \\ \text{definition of } \delta}}{=} \alpha_{\mathbb{R}}(\partial\mu) \underset{\substack{\uparrow \\ \text{see page 1079}}}{=} \alpha_{\mathbb{R}}(\mu(0,1)-\mu(1,0)) \underset{\substack{\uparrow \\ \text{definition of } \alpha_{\mathbb{R}}}}{=} \mu(0,1)-\mu(1,0)$$

[1071] In most cases we drop the adjective "singular" and we just talk of the n-th cohomology group of (X,A) with coefficients in G. We only refer to it as singular cohomology when we want to distinguish it from de Rham cohomology.

and, slightly more interesting, it follows immediately from the above discussion and the definition of $\alpha_{\mathbb{Z}}$ that

$$(\delta\alpha_{\mathbb{Z}})(\mu) \;=\; \alpha_{\mathbb{Z}}(\partial\mu) \;=\; \begin{cases} \#\{n\in\mathbb{Z}\,|\,\mu(1,0)<n\leq\mu(0,1)\}, & \text{if } \mu(0,1)\geq\mu(1,0)\\ -\#\{n\in\mathbb{Z}\,|\,\mu(0,1)<n\leq\mu(1,0)\}, & \text{if } \mu(1,0)\geq\mu(0,1).\end{cases}$$

In a somewhat informal way we can say that the cochain $\delta\alpha_{\mathbb{Z}}\colon \mathrm{C}_1(\mathbb{R})\to\mathbb{Z}$ applied to a singular 1-simplex μ measures how many integers the simplex μ "crosses from left to right". We refer to Figure 1118 for an illustration of $\delta\alpha_{\mathbb{Z}}$.

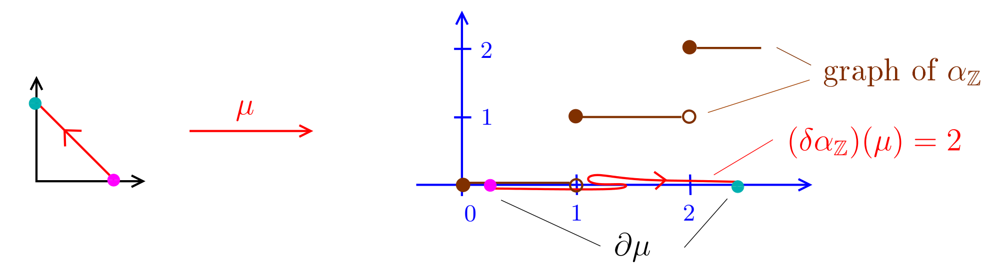


Figure 1118

(2) Now we consider the topological space $X=S^1$. As usual we denote by $p\colon\mathbb{R}\to S^1$, $t\mapsto e^{2\pi i t}$ the universal covering of S^1. Let $\sigma\colon\Delta^1\to S^1$ be a singular 1-simplex. Since Δ^1 is simply connected it follows from Proposition 29.5 that σ lifts to a map $\widetilde{\sigma}\colon\Delta^1\to\mathbb{R}$. The lift is not unique, but it is straightforward to see that any two lifts differ by addition by an integer. For a singular 1-simplex $\sigma\colon\Delta^1\to S^1$ we pick a lift $\widetilde{\sigma}\colon\Delta^1\to\mathbb{R}$ of σ and we define

$$\theta_{\mathbb{R}}(\sigma) \;:=\; (\delta\alpha_{\mathbb{R}})(\underset{\substack{\uparrow\\ \in\,\mathrm{C}_1(\mathbb{R})}}{\widetilde{\sigma}}) \;=\; \widetilde{\sigma}(0,1)-\widetilde{\sigma}(1,0)$$

and we also define

$$\theta_{\mathbb{Z}}(\sigma) \;:=\; (\delta\alpha_{\mathbb{Z}})(\widetilde{\sigma}) \;=\; \begin{cases} \#\{n\in\mathbb{Z}\,|\,\widetilde{\sigma}(1,0)<n\leq\widetilde{\sigma}(0,1)\}, & \text{if } \widetilde{\sigma}(0,1)\geq\widetilde{\sigma}(1,0)\\ -\#\{n\in\mathbb{Z}\,|\,\widetilde{\sigma}(0,1)<n\leq\widetilde{\sigma}(1,0)\}, & \text{if } \widetilde{\sigma}(1,0)\geq\widetilde{\sigma}(0,1).\end{cases}$$

Since any two lifts of σ differ by addition by an integer we see that $\theta_{\mathbb{R}}(\sigma)$ and $\theta_{\mathbb{Z}}(\sigma)$ are both well-defined, i.e. independent of the choice of the lift $\widetilde{\sigma}$. The definitions of $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are illustrated in Figure 1119.[1072]

As we had explained on page 1824, these two maps $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ actually define singular 1-cochains in $\mathrm{C}^1,\mathbb{R})$ and $\mathrm{C}^1;\mathbb{Z})$.

[1072] Here is an alternative description of the singular 1-cochain θ: We denote by $\Phi\colon[0,1]\to\Delta^1$ the homeomorphism that is given by $\Phi(t)=(1-t,t)$. Then for any singular 1-simplex $\sigma\colon\Delta^1\to S^1$ we have

$$\theta_{\mathbb{R}}(\sigma) \;=\; \frac{1}{2\pi\mathrm{i}}\int\limits_{\sigma\circ\Phi}\frac{1}{z}\,dz,$$

where the right-hand side denotes the path integral of the holomorphic function $\frac{1}{z}$ over the continuous path $\sigma\circ\Phi\colon[0,1]\to S^1$. We leave it as an amusing exercise to verify this equality using complex analysis.

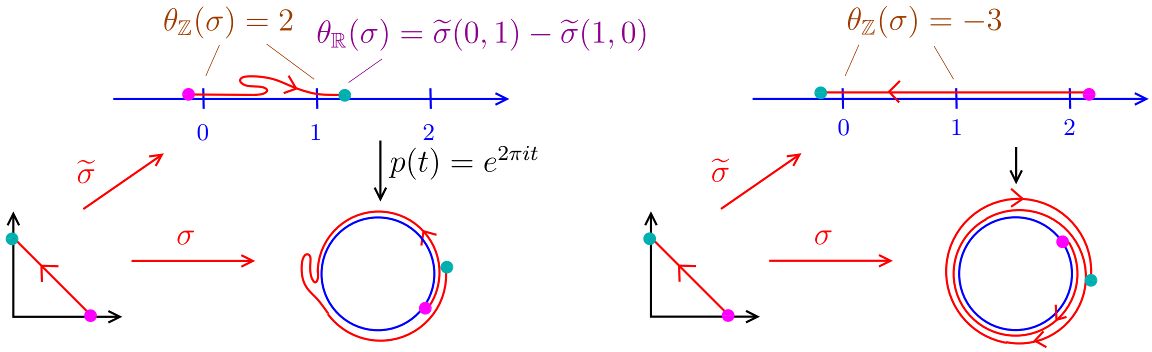


FIGURE 1119

As an example we consider the singular 1-simplex

$$\mu\colon \Delta^1 \to S^1 \quad \text{that is defined by} \quad \mu(1-t,t) = e^{2\pi i t}.$$

A lift of μ to the above universal covering $\mathbb{R} \to S^1$ is given by the singular 1-simplex

$$\widetilde{\mu}\colon \Delta^1 \to \mathbb{R} \quad \text{that is defined by} \quad \widetilde{\mu}(1-t,t) = t.$$

It follows immediately from the definitions that $\theta_{\mathbb{R}}(\mu) = \theta_{\mathbb{Z}}(\mu) = 1$.

The following lemma shows that $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are actually quite interesting singular 1-cochains.

Lemma 73.10. *We continue with the notation from the previous example.*

(1) *The singular 1-cochains $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are cocycles, in particular $\theta_{\mathbb{R}}$ defines an element in $\mathrm{H}^1(S^1;\mathbb{R})$ and $\theta_{\mathbb{Z}}$ defines an element in $\mathrm{H}^1(S^1;\mathbb{Z})$.*
(2) *The singular 1-cochain cochains $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are not the coboundary of a singular 0-cochain, i.e. the cohomology classes $[\theta_{\mathbb{R}}] \in \mathrm{H}^1(S^1;\mathbb{R})$ and $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$ are non-trivial.*

PROOF.

(1) We will first prove that the singular 1-cochain $\theta_{\mathbb{R}}$ is a cocycle. Thus we have to show that $\delta_1\theta_{\mathbb{R}} = 0 \in \mathrm{C}^2(S^1,\mathbb{R}) = \mathrm{Hom}(\mathrm{C}_2(S^1),\mathbb{R})$. So let $\sigma\colon \Delta^2 \to S^1$ be a singular 2-simplex. We have to show that $(\delta_1\theta_{\mathbb{R}})(\sigma) = \theta_{\mathbb{R}}(\partial\sigma) = 0$.

As before we denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$ the universal covering of S^1. Since Δ^2 is simply connected we can apply Proposition 29.2 to obtain a lift of $\sigma\colon \Delta^2 \to S^1$ to a map $\widetilde{\sigma}\colon \Delta^2 \to \mathbb{R}$. We refer to Figure 1120 for an illustration. Then

$$\theta_{\mathbb{R}}(\partial\sigma) \underset{\substack{\uparrow \\ \text{definition of } \partial}}{=} \theta_{\mathbb{R}}\Big(\sum_{j=0}^{2}(-1)^j\cdot\sigma\circ i_j^2\Big) \underset{\substack{\uparrow \\ \text{by definition of } \theta_{\mathbb{R}} \text{ and} \\ \text{since } \widetilde{\sigma}\circ i_j^2 \text{ is a lift of } \sigma\circ i_j^2}}{=} (\delta\alpha_{\mathbb{R}})\Big(\overbrace{\sum_{j=0}^{2}(-1)^j\cdot\widetilde{\sigma}\circ i_j^2}^{=\partial\widetilde{\sigma}}\Big) \underset{\substack{\uparrow \\ \text{by definition of } \delta\alpha_{\mathbb{R}} \text{ and } \partial\widetilde{\sigma}}}{=} \alpha_{\mathbb{R}}(\partial(\partial\widetilde{\sigma})) \underset{\substack{\uparrow \\ \text{since } \partial\circ\partial = 0}}{=} 0.$$

Exactly the same argument applies to $\theta_{\mathbb{Z}}$.

(2) We still need to show that the singular 1-cochain cochains $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are not the coboundary of a singular 0-cochain. So let β be a singular 0-cochain. As usual we

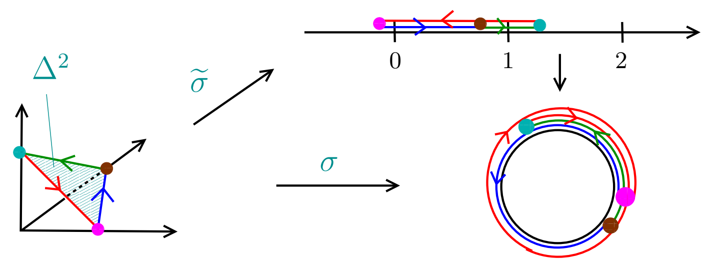


FIGURE 1120

view β as a function $S^1 \to \mathbb{R}$. We consider again the singular 1-simplex $\mu\colon \Delta^1 \to S^1$ that is defined by $\mu(1-t,t) = e^{2\pi i t}$. Then

$$(\delta_0\beta)(\mu) \;=\; \beta(\,\partial(\mu)) \;=\; \beta\big(\underbrace{\mu(0,1)}_{=1\in S^1} - \underbrace{\mu(1,0)}_{=1\in S^1}\big) \;=\; 0.$$

But in the previous example we saw that $\theta_{\mathbb{R}}(\mu) = \theta_{\mathbb{Z}}(\mu) = 1$. ■

Remark. Note that in Lemma 73.10 we have just given a mysteriously simple proof that $\mathrm{H}^1(S^1;\mathbb{R}) \neq 0$ and also that $\mathrm{H}^1(S^1;\mathbb{Z}) \neq 0$.

We continue with another example of cochains.

Example. We return to the topological space $X = \mathbb{R}$. We fix a point $y \in \mathbb{R}$. We consider the cochain

$$\varphi_y\colon\ \mathrm{C}_0(\mathbb{R}) \;\to\; \mathbb{Z}$$

that is determined by the function

$$\begin{array}{rcl} \mathbb{R} & \to & \mathbb{Z} \\ x & \mapsto & \begin{cases} 0, & \text{if } x \le y, \\ 1, & \text{else.} \end{cases} \end{array}$$

For any singular 1-simplex $d\colon \Delta^1 \to X = \mathbb{R}$ we have

$$(\delta_0\varphi_y)(d) \;=\; \varphi_y(\partial d) \underset{\substack{\uparrow \\ \text{since } \partial d = 1\cdot d(0,1) + (-1)\cdot d(1,0) \\ \text{and since } \varphi_y\colon \mathrm{C}_0(\mathbb{R}) \to \mathbb{Z} \text{ is linear}}}{=} \varphi_y(d(0,1)) - \varphi_y(d(1,0)) \;=\; \begin{cases} 0, & \text{if } d(0,1), d(1,0) \le y, \\ 0, & \text{if } d(0,1), d(1,0) \ge y, \\ 1, & \text{if } d(1,0) \le y < d(0,1) \\ -1, & \text{if } d(0,1) \le y < d(1,0). \end{cases}$$

In a slightly informal way we can say that the homomorphism $\delta_0\varphi_y\colon \mathrm{C}_1(\mathbb{R}) \to \mathbb{Z}$ applied to a singular 1-simplex d equals $+1$ if the simplex "crosses y from right to left" and it equals -1 if the simplex "crosses y from left to right". We will make use of this singular 1-cochain on several occasions, hence we give it its own notation, namely we write

$$\gamma_y \;:=\; \delta_0\varphi_y \;\in\; \mathrm{C}^1(\mathbb{R};\mathbb{Z}).$$

We illustrate the singular 1-cochain $\gamma_y = \delta_0\varphi_y \in \mathrm{C}^1(\mathbb{R};\mathbb{Z})$ in Figure 1121.

After this long discussion of explicit cochains and cohomology classes we want to turn to more general results. The only two instances where we could compute homology groups

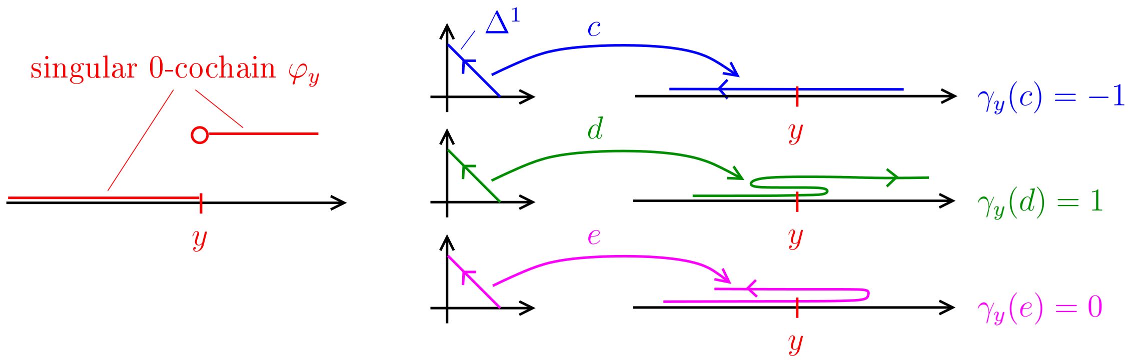


Figure 1121

without any problems was the 0-th homology group of a path-connected space, see Proposition 41.5, and the homology groups of a point, see Lemma 41.6. Not surprisingly we can also deal with the same situations with cohomology groups.

First of all we have the following analogue of Proposition 41.5.

Proposition 73.11. *Let X be a non-empty path-connected topological space and let G be an abelian group. Then the map*

$$\begin{array}{rcl} G & \to & \mathrm{H}^0(X;G) \\ g & \mapsto & [\text{0-cochain that assigns to each } x \in X \text{ the value } g] \end{array}$$

is a well-defined map that is in fact an isomorphism. For any point $P \in X$ an inverse is given by the map

$$\begin{array}{rcl} \mathrm{H}^0(X;G) & \to & G \\ [f] & \mapsto & f(P) \end{array}$$

Remark. At times we will use the isomorphism from Proposition 73.11 to identify the 0-th cohomology $\mathrm{H}^0(X;G)$ of a path-connected topological space X with G.

Proof. We consider the "right-hand end" of the chain complex of $\mathrm{C}_*(X)$ and of the cochain complex $\mathrm{Hom}(\mathrm{C}_*(X), G)$:

$$\mathrm{C}_1(X) \xrightarrow{\partial_1} \mathrm{C}_0(X) \to 0 \quad \text{and} \quad \mathrm{Hom}(\mathrm{C}_1(X), G) \xleftarrow{\delta_0=\partial_1^*} \mathrm{Hom}(\mathrm{C}_0(X), G) \leftarrow 0.$$

Now we calculate that

$$\begin{aligned}
\mathrm{H}^0(X;G) &= \ker\left\{\begin{array}{ccc} \mathrm{Hom}(\mathrm{C}_0(X),G) & \to & \mathrm{Hom}(\mathrm{C}_1(X),G) \\ (f\colon \mathrm{C}_0(X)\to G) & \mapsto & (f\circ\partial_1\colon \mathrm{C}_1(X)\to G)\end{array}\right\} \\
&\overset{\substack{\text{since the singular 1-simplices form a basis of } \mathrm{C}_1(X)\\ \downarrow}}{\underset{\substack{\uparrow \\ \text{by Lemma 73.1 we have the} \\ \text{natural identification} \\ \mathrm{Hom}(\mathrm{C}_0(X),G) = \text{maps from } X \text{ to } G}}{=}} \{\text{maps } f\colon X\to G \mid \underbrace{f(\partial_1(\sigma))}_{=f(\sigma(0,1))-f(\sigma(1,0))} = 0 \text{ for all singular 1-simplices } \sigma\colon \Delta^1\to X\} \\
&\underset{\substack{\uparrow \\ \text{since } X \text{ is path-connected} \\ \text{and since } \Delta^1\cong[0,1]}}{=} \text{all constant maps } X\to G \underset{\substack{\uparrow \\ \text{since } X \text{ is non-empty}}}{=} G.
\end{aligned}$$

The remaining statements of the proposition can be verified easily. ■

In Exercise 73.3 we will prove the following lemma which can be viewed as an analogue of Lemma 41.6.

Lemma 73.12. *Let $X=\{x\}$ be a topological space that consists of a single point x. For any abelian group G we have*

$$\mathrm{H}^n(X;G) = \begin{cases} 0, & \text{if } n\geq 1, \\ G, & \text{if } n=0.\end{cases}$$

We want to conclude this section with a short discussion of relative cohomology classes. Before we do so we make the following observations:

Observation.

(1) Given an abelian group C, a subgroup D and an abelian group G we have a natural isomorphism

$$\begin{array}{ccc} \mathrm{Hom}(C/D,G) & \xrightarrow{\cong} & \text{all homomorphisms } C\to G \text{ that vanish on } D \\ (\varphi\colon C/D\to G) & \mapsto & (C\to C/D\xrightarrow{\varphi} G).\end{array}$$

We will use this isomorphism to identify the groups left and right.

(2) If S is a set and T is a subset, then for any abelian group G the identification

$$\mathrm{Hom}(\mathbb{Z}^{(S)},G) = \text{all maps from } S \text{ to } G$$

from Lemma 73.1 restricts to an isomorphism

$$\big\{f\in\mathrm{Hom}(\mathbb{Z}^{(S)},G)\,\big|\, f \text{ vanishes on } \mathbb{Z}^{(T)}\big\} = \text{all maps } S\to G \text{ that are trivial on } T.$$

(3) It follows immediately from Lemma 73.1 and from (1) and (2) that given a pair of topological spaces (X,A) and an abelian group G we have for every $n\in\mathbb{N}_0$ a natural isomorphism

$$\underbrace{\mathrm{Hom}(\mathrm{C}_n(X,A),G)}_{=\mathrm{C}^n(X,A;G)=\mathrm{Hom}(\mathrm{C}_n(X)/\mathrm{C}_n(A),G)} = \text{all maps } S_n(X)\to G \text{ that are trivial on } S_n(A).$$

We will often use this natural isomorphism to identify the left-hand side with the right-hand side.

Examples.

(1) In Exercise 73.4 we will modify the argument of Proposition 73.11 to show that for any path-connected topological space with non-empty subset A we have

$$\mathrm{H}^0(X, A; G) \; = \; 0 \quad \text{for any abelian group } G.$$

(2) We write $I = [0,1]$. In this example we want to construct a non-trivial element in $\mathrm{H}^1(I, \partial I)$. We consider the function

$$\begin{array}{rcl} f\colon I & \to & \mathbb{Z} \\ x & \mapsto & \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}) \\ 1, & \text{otherwise} \end{cases} \end{array}$$

which we use to define the singular 1-cochain

$$\begin{array}{rcl} \rho\colon \mathrm{C}_1(I) & \to & \mathbb{Z} \\ (\sigma\colon \Delta^1 \to I) & \mapsto & f(\sigma(0,1)) - f(\sigma(1,0)). \end{array}$$

Loosely speaking, similar to the singular 1-cochain φ_y from page 1827 the singular 1-cochain $\rho(\sigma)$ measures whether σ crosses $\frac{1}{2}$, and if yes, in what direction. We refer to Figure 1122 for an illustration. Since Δ^1 is connected and since $\{0,1\}$ is discrete it follows from Lemma 2.61 that any singular 1-simplex $\sigma\colon \Delta^1 \to \partial I = \{0,1\}$ is constant. In particular in this case we have $\rho(\sigma) = 0$. Thus we see, using the above observation, that ρ defines a map

$$\rho\colon \mathrm{C}_1(I, \partial I) \; \to \; \mathbb{Z}.$$

In Exercise 73.8 we will see that ρ is a singular 1-cocycle and that it defines a non-trivial element in $\mathrm{H}^1(I, \partial I; \mathbb{Z})$.

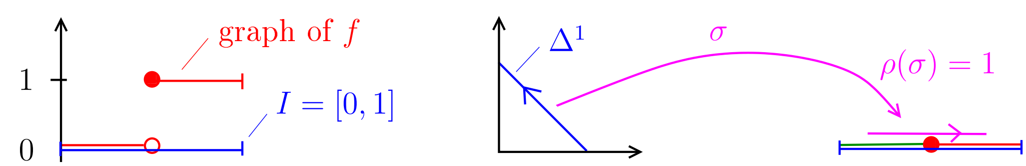


Figure 1122

73.5. **Basic properties of cohomology groups.** Now we will see that maps between pairs of topological spaces induce maps on cohomology groups.

Definition. Let G be an abelian group and let $f\colon (X,A) \to (Y,B)$ be a map between pairs of topological spaces. Recall that f induces a chain map $f_*\colon \mathrm{C}_*(X,A) \to \mathrm{C}_*(Y,B)$. For each $n \in \mathbb{N}_0$ we denote by

$$f^*\colon \mathrm{H}^n(Y, B; G) \; \to \; \mathrm{H}^n(X, A; G)$$

the induced map of cohomology groups that we obtain from Lemma 73.8 (1).

Example. We will see in Exercise 73.6 that any map $f\colon X \to Y$ between two path-connected non-empty topological spaces induces an isomorphism $f^*\colon \mathrm{H}^0(Y;G) \to \mathrm{H}^0(X;G)$. A very similar argument shows that if Y is path-connected and X is non-empty, then $f^*\colon \mathrm{H}^0(Y;G) \to \mathrm{H}^0(X;G)$ is a monomorphism.

Notation. Let X be a topological space and let A be a subset. Furthermore let G be an abelian group. We denote by $i\colon A \to X$ the inclusion map. Sometimes we refer to $i^*\colon \mathrm{H}^n(X;G) \to \mathrm{H}^n(A;G)$ as the *restriction map.*

We summarize some basic properties of the induced maps in the following lemma.

Lemma 73.13. *Let G be an abelian group and let $n \in \mathbb{N}_0$.*

(1) *The maps*

$$\begin{array}{rcl} (X,A) & \mapsto & \mathrm{H}^n(X,A;G) \\ (f\colon (X,A) \to (Y,B)) & \mapsto & (f^*\colon \mathrm{H}^n(Y,B;G) \to \mathrm{H}^n(X,A;G)) \end{array}$$

define a contravariant functor from the category of pairs of topological spaces to the category of abelian groups. (If G is a commutative ring, then this is in fact a functor to the category of G-modules.)

(2) *Let $f,g\colon (X,A) \to (Y,B)$ be maps between pairs of topological spaces. If f and g are homotopic, then they induce the same maps*

$$f^* = g^*\colon \mathrm{H}^n(Y,B;G) \to \mathrm{H}^n(X,A;G).$$

(3) *If $f\colon X \to Y$ is a homotopy equivalence between two topological spaces, then the induced map*

$$f^*\colon \mathrm{H}^n(Y;G) \xrightarrow{\cong} \mathrm{H}^n(X;G)$$

is an isomorphism. In particular the cohomology groups of homotopy equivalent topological spaces are isomorphic.

(4) *Let X be a topological space, let $A \subset B$ be subsets of X and let $n \in \mathbb{N}_0$. If A is a deformation retract of B, then the inclusion induced map*

$$\mathrm{H}^n(X,B;G) \to \mathrm{H}^n(X,A;G)$$

is an isomorphism. Similarly, if B is a deformation retract of X, then the inclusion induced map

$$\mathrm{H}^n(X,A;G) \to \mathrm{H}^n(B,A;G)$$

is an isomorphism.

Example. Lemma 73.13 (4), together with Lemma 73.12 and the discussion on page 548 implies that for any $n \geq 1$, any abelian group G and any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}^n(\mathbb{R}^k;G) = \mathrm{H}^n\left(\overline{B}^k;G\right) = \mathrm{H}^n\left(B^k;G\right) = \mathrm{H}^n\left([0,1]^k;G\right) = \mathrm{H}^n\left(\Delta^k;G\right) = 0.$$

Proof.

(1) This statement follows immediately from Lemma 73.8 (2) together with the fact that the map $(X,A) \mapsto \mathrm{C}_*(X,A)$ defines a covariant functor from the category *PairTop* of pairs of topological spaces to the category *ChCplx* of chain complexes.
(2) This statement is an immediate consequence of Proposition 43.17 together with Lemma 73.8 (3).
(3) This statement follows from (1) and (2).
(4) This statement is proved almost verbatim the same way as Corollary 43.18 (2a) and (2b). ■

Many of the statements about homology groups can be modified to give corresponding statements about cohomology groups. For example we have the following lemma.

Lemma 73.14. *Let X be a topological space and let G be an abelian group.*

(1) *We denote by $\{X_j\}_{j\in J}$ the path-components of X. The inclusion maps $\iota_j\colon X_j \to X$, $j \in J$, induce an isomorphism*
$$\prod_{j\in J}\iota_j^*\colon\ \mathrm{H}^n(X;G) \ \xrightarrow{\cong}\ \prod_{j\in J}\mathrm{H}^n(X_j;G).$$
(2) *Suppose that X has finitely many path-components $X_1,\dots,X_k$. If for $j=1,\dots,k$ we pick a point $P_j \in X_j$, then the map*
$$\begin{array}{rcl} \mathrm{H}^0(X;G) & \to & G^k = \overbrace{G\times\cdots\times G}^{k \text{ times}} \\ {[f]} & \mapsto & (f(P_1),\dots,f(P_k)) \end{array}$$
is an isomorphism.

Proof. Let X be a topological space with path-components $\{X_j\}_{j\in J}$ and let G be an abelian group. We have the following isomorphisms of cochain complexes:
$$\mathrm{Hom}(\mathrm{C}_*(X),G) \ \underset{\substack{\uparrow\\ \text{Lemma 41.14}}}{\xrightarrow{\left(\bigoplus_{j\in J}\iota_{j*}\right)^*}}\ \mathrm{Hom}\Big(\bigoplus_{j\in J}\mathrm{C}_*(X_j),G\Big) \ \underset{\substack{\uparrow\\ \text{Lemma 73.2}}}{\xrightarrow{\cong}}\ \prod_{j\in J}\mathrm{Hom}(\mathrm{C}_*(X_j),G).$$
The first statement now follows easily from this observation together with the fact that cohomology commutes with direct products, see Lemma 73.7.

The second statement follows from the first statement together with the calculation of 0-th cohomology groups of path-connected topological spaces in Proposition 73.11. ■

The following lemma is the analogue of Proposition 43.15.

Lemma 73.15. *Given a triple (X,B,A) of topological spaces and an abelian group G we obtain a long exact sequence*
$$\dots \xrightarrow{\delta_{n-1}} \mathrm{H}^n(X,B;G) \xrightarrow{p^*} \mathrm{H}^n(X,A;G) \xrightarrow{i^*} \mathrm{H}^n(B,A;G) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(X,B;G) \xrightarrow{p^*} \dots$$
hereby the maps are given as follows:

(1) *p^* is the map induced by the map $p\colon (X,A)\to(X,B)$,*
(2) *i^* is the map induced by the map $i\colon (B,A)\to(X,A)$,*
(3) *the connecting homomorphism $\delta_n\colon \mathrm{H}^n(B,A;G)\to\mathrm{H}^{n+1}(X,B;G)$ is natural and it is given by the map*
$$\begin{array}{rcl} \mathrm{H}^n(B,A;G) & \to & \mathrm{H}^{n+1}(X,B;G) \\ {[f\colon\ \mathrm{C}_n(B,A)\to G]} & \mapsto & \left[\begin{array}{rcl} \mathrm{C}_{n+1}(X,B) & \to & G \\ \underset{\substack{\uparrow\\ \textit{with } \sigma\in \mathrm{C}_{n+1}(X)}}{\sigma} & \mapsto & f(q(\partial_n(\sigma))) \end{array}\right] \end{array}$$

here $q\colon \mathrm{C}_n(X) \to \mathrm{C}_n(B)$ *is the map given by sending all singular simplices that do not lie in* B *to zero.*

Remark.

(1) An equivalent approach to expressing the connecting homomorphism is given as follows:
   (a) take a cocycle $f \in \mathrm{C}^n(B, A; G) = \mathrm{Hom}(\mathrm{C}_n(B, A), G)$,
   (b) denote by $\pi\colon \mathrm{C}_n(B) \to \mathrm{C}_n(B, A)$ the projection map,
   (c) denote by $\widetilde{f}$ the result of extending $f \circ \pi\colon \mathrm{C}_n(B) \to G$ to a map $\mathrm{C}_n(X) \to G$ by sending all singular simplices that do not lie in B to zero (sometimes we call $\widetilde{f}$ the *trivial extension* of $f \circ \pi$),
   (d) then consider the coboundary $\delta\widetilde{f} \in \mathrm{C}^{n+1}(X; G)$, since f is a cocycle it follows that $\delta\widetilde{f}$ actually defines an element in $\mathrm{C}^{n+1}(X, B; G)$ that we also denote by $\delta\widetilde{f}$, and which turns out to be a cocycle,
   (e) then $\delta[f] = [\delta\widetilde{f}] \in \mathrm{H}^{n+1}(X, B; G)$.

(2) One of the difficulties with the long exact sequence of cohomology groups is that all maps seem to point the wrong way. You should use the Lucky Luke approach to cohomology: if Rantanplan (i.e. your intuition) points one way, you should take the opposite direction.

FIGURE 1123. The Rantanplan rule.

(3) The most frequent application of Lemma 73.15 is that given a pair (X, B) of topological spaces, i.e. given the triple $(X, B, \varnothing)$ and given an abelian group G we obtain a natural long exact sequence

$$\ldots \xrightarrow{\delta_{n-1}} \mathrm{H}^n(X, B; G) \xrightarrow{p^*} \mathrm{H}^n(X; G) \xrightarrow{i^*} \mathrm{H}^n(B; G) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(X, B; G) \xrightarrow{p^*} \ldots$$

PROOF. Let (X, B, A) be a triple of topological spaces. As in the proof of Proposition 43.15 we note that

$$0 \to \mathrm{C}_*(B, A) \xrightarrow{i_*} \mathrm{C}_*(X, A) \xrightarrow{p_*} \mathrm{C}_*(X, B) \to 0$$

is a short exact sequence of chain complexes. Now let G be an abelian group. Recall that we pointed out on page 1120 that the chain groups $\mathrm{C}_*(X, B)$ are free abelian groups. Thus we deduce from Lemma 73.4 that the following induced sequence of cochain complexes is also exact:

$$0 \to \operatorname{Hom}(C_*(X,B),G) \xrightarrow{p^*} \operatorname{Hom}(C_*(X,A),G) \xrightarrow{i^*} \operatorname{Hom}(C_*(B,A),G) \to 0.$$

The lemma now follows easily from Proposition 73.6 and the construction of the connecting homomorphism. We leave the details to the reader. ∎

Examples.

(1) Let X be a path-connected topological space and let A be any *non-empty* subset A of X. We denote by $i\colon A \to X$ the inclusion map. By the above remark we have a long exact sequence

$$0 \to H^0(X,A;G) \to H^0(X;G) \underset{\substack{\uparrow \\ \text{monomorphism, see page 1830}}}{\xrightarrow{i^*}} H^0(A;G) \to \dots$$

Thus it follows that $H^0(X,A;G) = 0$. In Exercise 73.4 we will provide an alternative proof for this fact using only the definitions of cohomology groups.

(2) Let $b \in \mathbb{R}_{>0}$. We consider the long exact sequence in cohomology with $\mathbb{Z}$-coefficients of the pair $(\mathbb{R}, \mathbb{R}\setminus[-b,b])$. Together with (1) and Lemma 73.14 we obtain the following commutative diagram where the top horizontal sequence is exact:

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathbb{R};\mathbb{Z}) \arrow[r, "i^*"] \arrow[d, "{[f]\mapsto f(-b-1)}"] & H^0(\mathbb{R}\setminus[-b,b];\mathbb{Z}) \arrow[r, "\delta"] \arrow[d, "{[f]\mapsto (f(-b-1),f(b+1))}"] & H^1(\mathbb{R},\mathbb{R}\setminus[-b,b];\mathbb{Z}) \arrow[r] & H^1(\mathbb{R};\mathbb{Z}) \arrow[d, "="] \\ & \mathbb{Z} \arrow[r, "{m\mapsto(m,m)}"'] & \mathbb{Z}^2 & & 0. \end{tikzcd}$$

The cokernel of the bottom horizontal map is easily seen to be isomorphic to $\mathbb{Z}$ and it is generated by the element $(0,1) \in \mathbb{Z}^2$. Put differently, the cokernel of the map $i^*\colon H^0(\mathbb{R};\mathbb{Z}) \to H^0(\mathbb{R}\setminus[-b,b];\mathbb{Z})$ is represented by the singular 0-cochain given by

$$\begin{aligned} \psi\colon \mathbb{R}\setminus[-b,b] &\to \mathbb{Z} \\ x &\mapsto \begin{cases} 0, & \text{if } x \le -b, \\ 1, & \text{if } x \ge b. \end{cases} \end{aligned}$$

Now we want to determine $\delta\psi \in H^1(\mathbb{R},\mathbb{R}\setminus[-b,b];\mathbb{Z})$. It is perhaps helpful to consider the following commutative diagram of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}(C_1(\mathbb{R},\mathbb{R}\setminus[-b,b]),\mathbb{Z}) \arrow[r, "p^*"] & \operatorname{Hom}(C_1(\mathbb{R}),\mathbb{Z}) \arrow[r, "i^*"] & \operatorname{Hom}(C_1(\mathbb{R}\setminus[-b,b]),\mathbb{Z}) \arrow[r] & 0 \\ 0 \arrow[r] & \operatorname{Hom}(C_0(\mathbb{R},\mathbb{R}\setminus[-b,b],\mathbb{Z}) \arrow[r, "p^*"] \arrow[u, "\delta_0"] & \operatorname{Hom}(C_0(\mathbb{R}),\mathbb{Z}) \arrow[r, "i^*"] \arrow[u, "\delta_0"] & \underbrace{\operatorname{Hom}(C_0(\mathbb{R}\setminus[-b,b],\mathbb{Z})}_{\text{contains }\psi} \arrow[r] \arrow[u, "\delta_0"] & 0. \end{tikzcd}$$

We start with the map $\psi\colon \mathbb{R}\setminus[-b,b] \to \mathbb{Z}$. We need to pick an extension to a map $\mathbb{R} \to \mathbb{Z}$. We might as well take the map $\varphi_0\colon \mathbb{R} \to \mathbb{Z}$ from page 1827. Then it follows from the discussion of the connecting homomorphism that the coboundary $\gamma_0 := \delta\varphi_0$

represents an element in $\mathrm{H}^1(\mathbb{R}, \mathbb{R} \setminus [-b, b]; \mathbb{Z})$, and that $\delta\psi = [\gamma_0]$ is a generator of $\mathrm{H}^1(\mathbb{R}, \mathbb{R} \setminus [-b, b]; \mathbb{Z}) \cong \mathbb{Z}$.

73.6. **Reduced cohomology groups.** We saw on several occasions that the reduced homology groups $\widetilde{\mathrm{H}}_k(X)$ of a non-empty topological space can be convenient for expressing calculations and results. In this last section we quickly introduce the reduced cohomology groups.

Definition. Let X be a topological space. By Lemma 41.4 (1) we know that the following is a chain complex

$$\begin{array}{ccccccccccc} \dots & \xrightarrow{\partial_3} & \mathrm{C}_2(X) & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \xrightarrow{\epsilon} & \mathbb{Z} & \to & 0. \\ & & & & & & \sum\limits_{i=1}^{k} n_i\sigma_i & \mapsto & \sum\limits_{i=1}^{k} n_i. & & \end{array}$$

Let G be an abelian group. In a fashion similar to reduced homology we now define the *k-th reduced cohomology group $\widetilde{\mathrm{H}}^k(X;G)$ of X* to be the k-th cohomology group of the above chain complex.

The following lemma is the analogue of Lemma 43.1, and Lemma 43.14. For brevity's sake we leave the proof to the reader.

Lemma 73.16. *Let G be an abelian group.*

(1) *Let $f\colon X \to Y$ be a map between two topological spaces. For every $n \in \mathbb{N}_0$ the map f induces a map $f^*\colon \widetilde{\mathrm{H}}^n(Y;G) \to \widetilde{\mathrm{H}}^n(X;G)$ in the obvious way. For every $n \in \mathbb{N}_0$ the maps*
$$\begin{array}{rcl} X & \mapsto & \widetilde{\mathrm{H}}^n(X;G) \\ (f\colon X \to Y) & \mapsto & (f^*\colon \widetilde{\mathrm{H}}^n(Y;G) \to \widetilde{\mathrm{H}}^n(X;G)) \end{array}$$
define a functor from the category of topological spaces to the category of abelian groups.

(2) *For each $n \in \mathbb{N}_0$ the maps*
$$\begin{array}{rcl} \mathrm{H}^n(X;G) & \to & \widetilde{\mathrm{H}}^n(X;G) \\ [\varphi] & \mapsto & [\varphi] \end{array}$$
define a natural transformation from the functor $X \mapsto \mathrm{H}^n(X;G)$ to the functor that is given by $X \mapsto \widetilde{\mathrm{H}}^n(X;G)$.

(3) *For every non-empty topological space X we have*
$$\mathrm{H}^n(X;G) \cong \begin{cases} \widetilde{\mathrm{H}}^0(X;G) \oplus G, & \text{if } n = 0, \\ \widetilde{\mathrm{H}}^n(X;G), & \text{if } n \neq 0. \end{cases}$$
The isomorphism is natural for $n \geq 1$.

(4) *If X is a topological space with $n \in \mathbb{N}$ path-components, then $\widetilde{\mathrm{H}}^0(X;G) \cong G^{n-1}$.*

(5) *Given a topological space X, a point $x_0 \in X$ and $n \in \mathbb{N}_0$ we consider the map*

$$\mathrm{H}^n(X,\{x_0\};G) \longrightarrow \mathrm{H}^n(X,\varnothing;G) \xrightarrow{=} \mathrm{H}^n(X;G) \longrightarrow \widetilde{\mathrm{H}}^n(X;G).$$

For any $n \in \mathbb{N}_0$ this map forms a natural isomorphism from the contravariant functor $(X,x_0) \mapsto \widetilde{\mathrm{H}}_n(X;G)$ to the contravariant functor $(X,x_0) \mapsto \mathrm{H}^n(X,x_0;G)$.

(6) *Any homotopy equivalence between two topological spaces induces isomorphisms of reduced cohomology.*

Finally the following lemma plays the role of Corollary 43.16.

Lemma 73.17. *Let X be a topological space and let $B \subset X$ be a non-empty subset. Then given any abelian group G there exists a natural long exact sequence*

$$\dots \xrightarrow{\delta_{n-1}} \mathrm{H}^n(X,B;G) \xrightarrow{p^*} \widetilde{\mathrm{H}}^n(X;G) \xrightarrow{i^*} \widetilde{\mathrm{H}}^n(B;G) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(X,B;G) \xrightarrow{p^*} \dots$$

Proof. The lemma follows immediately from Lemma 73.15 applied to the triple $(X,B,\{P\})$ where P is a point in B, together with Lemma 73.16 (5). ■

73.7. **Isomorphic homology groups implies isomorphic cohomology groups.** In this short section we will see that the isomorphism type of cohomology groups cannot be used to extract more information from chain complexes than we already extracted from homology groups.

Proposition 73.18. *Suppose $(\mathrm{C}_*,\partial_*)$ and (C'_*,∂'_*) are two chain complexes of free abelian groups. Suppose that for each $n \in \mathbb{N}_0$ the homology groups $\mathrm{H}_n(C_*)$ and $\mathrm{H}_n(C'_*)$ are isomorphic. Then for each $n \in \mathbb{N}_0$ and any abelian group G the cohomology groups $\mathrm{H}^n(C;G)$ and $\mathrm{H}^n(C';G)$ are also isomorphic.*

Proof. Suppose $(\mathrm{C}_*,\partial_*)$ and (C'_*,∂'_*) are two chain complexes of free abelian groups. By our hypothesis there exist isomorphisms $\gamma_n\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(C'_*)$, $n \in \mathbb{N}_0$. By Proposition 49.1 these isomorphisms can be realized by a chain map $f\colon C_* \to C'_*$. But by Proposition 49.2 this chain map is a chain homotopy equivalence. The statement of the proposition follows immediately from Lemma 73.8. ■

This proposition gives us a rather sobering corollary. It says that if our only goal is to distinguish (pairs of) topological spaces, then cohomology groups do not contain more information than homology groups.

Corollary 73.19. *Let (X,A) and (Y,B) be topological spaces and let G be an abelian group. Then*

$$\begin{array}{c}\mathrm{H}_n(X,A) \text{ and } \mathrm{H}_n(Y,B) \text{ are}\\ \text{isomorphic for all } n \in \mathbb{N}_0\end{array} \Longrightarrow \begin{array}{c}\mathrm{H}^n(Y,B;G) \text{ and } \mathrm{H}^n(X,A;G) \text{ are}\\ \text{isomorphic for all } n \in \mathbb{N}_0.\end{array}$$

Remark. We will see later on in Exercise 75.8 that the analogous statement with the roles of homology and cohomology swapped does not hold. In particular there exist topological spaces X and Y such that $\mathrm{H}^n(X;\mathbb{Z}) \cong \mathrm{H}^n(Y;\mathbb{Z})$ for all $n \in \mathbb{N}_0$ but such that $\mathrm{H}_1(X;\mathbb{Z})$ is not isomorphic to $\mathrm{H}_1(Y;\mathbb{Z})$.

PROOF. This corollary follows immediately from Proposition 73.18 and the observation from page 1120 that relative singular chain groups are free abelian groups. ■

The previous corollary does not specify how the isomorphism of the cohomology groups is given. If we already start out with a map $f\colon (X,A) \to (Y,B)$ that induces an isomorphism of homology groups, then we can be more precise, namely we have the following corollary which is similar in spirit to Corollary 57.20.

Corollary 73.20. *Let $f\colon (X,A) \to (Y,B)$ be a map between pairs of topological spaces and let G be an abelian group. Then*

$$\begin{array}{ccc} \text{the induced map} & & \text{the induced map} \\ f_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B) \text{ is} & \Longrightarrow & f^*\colon \mathrm{H}^n(Y,B;G) \to \mathrm{H}^n(X,A;G) \text{ is} \\ \text{an isomorphism for all } n \in \mathbb{N}_0 & & \text{an isomorphism for all } n \in \mathbb{N}_0. \end{array}$$

Remark. In Proposition 75.14 we will see that the converse to Corollary 73.20 also holds.

PROOF. Let $f\colon (X,A) \to (Y,B)$ be a map between pairs of topological spaces and let G be an abelian group. If $f_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B)$ is an isomorphism for all $n \in \mathbb{N}_0$, then it follows from Proposition 49.2 that $f_*\colon \mathrm{C}_*(X,A) \to \mathrm{C}_*(Y,B)$ is already a chain homotopy equivalence. (Here, as in the previous proof we use that relative singular chain groups are free abelian groups.) Now we get the desired conclusion from Lemma 73.8. ■

The upshot of this short section is that we will need to work to justify spending so many pages on cohomology.

Exercises for Chapter 73.

Exercise 73.1. Let $f\colon B \to C$ be a homomorphism between abelian groups such that $\mathrm{im}(f)$ is a summand of C. Let $\{k_1,\dots,k_m\}$ be a generating set for $\ker(f)$. Finally suppose that we are given homomorphisms $\varphi_1,\dots,\varphi_m\colon B \to \mathbb{Z}$ such that for all $i,j \in \{1,\dots,m\}$ we have $\varphi_i(k_j) = \delta_{ij}$. Show that the map

$$\begin{array}{rcl} \Phi\colon \mathbb{Z}^m & \to & \mathrm{coker}(f\colon \mathrm{Hom}(C,\mathbb{Z}) \to \mathrm{Hom}(B,\mathbb{Z})) \\ e_i & \mapsto & [\varphi_i] \end{array}$$

is an isomorphism.

Exercise 73.2. Let $\varphi\colon C_1 \to C_0$ be a homomorphism between two finitely generated free abelian groups.

(a) Suppose that A is an $(n \times m)$-matrix that represents φ. Show that the transpose A^T represents the map $\varphi^*\colon \mathrm{Hom}(C_0,\mathbb{Z}) \to \mathrm{Hom}(C_1,\mathbb{Z})$.
(b) We consider the chain complex $C_* := 0 \to C_1 \xrightarrow{\varphi} C_0 \to 0$. Show that

$$\mathrm{Tor}(\mathrm{H}^1(C_*;\mathbb{Z})) \;\cong\; \mathrm{Tor}(\mathrm{H}_1(C)) \quad \text{and} \quad \mathrm{FH}^1(C_*;\mathbb{Z}) \;\cong\; \mathrm{FH}_1(C).$$

Remark. Statement (b) can be seen as a special case of the Universal Coefficient Theorem 75.12 for Cohomology Groups that we will prove later on.

Exercise 73.3. Let $X = \{x\}$ be a topological space that consists of a single point x. Show that for any abelian group G we have

$$\mathrm{H}^n(X;G) \;\cong\; \begin{cases} 0, & \text{if } n \geq 1, \\ G, & \text{if } n = 0. \end{cases}$$

Exercise 73.4. Let X be a path-connected topological space and let A be a non-empty subset of X. Show that

$$\mathrm{H}^0(X,A;G) \;=\; 0,$$

using only the definitions of cohomology groups.

Exercise 73.5. Let X be a path-connected topological space and let A and B be two disjoint non-empty subsets of X. Give an explicit example of a map $f\colon X \to \mathbb{Z}$ such that the map

$$\begin{array}{rcl} \mathrm{C}_1(X, A \cup B) & \to & \mathbb{Z} \\ [\sigma\colon \Delta^1 \to X] & \mapsto & f(\sigma((0,1)) - f(\sigma((1,0)) \end{array}$$

is cocycle that represents a non-zero element in $\mathrm{H}^1(X,A;\mathbb{Z})$.

Exercise 73.6. Show that any map $f\colon X \to Y$ between two path-connected non-empty topological spaces induces an isomorphism $f^*\colon \mathrm{H}^0(Y;G) \to \mathrm{H}^0(X;G)$.

Exercise 73.7. Let X be a topological space. Is the 0-th homology group $\mathrm{H}_0(X;\mathbb{Z})$ necessarily isomorphic to the 0-th cohomology group $\mathrm{H}^0(X;\mathbb{Z})$?

Exercise 73.8. We write $I = [0,1]$. As on page 1830 we consider the function

$$\begin{array}{rcl} f\colon I & \to & \mathbb{Z} \\ x & \mapsto & \begin{cases} 0, & \text{if } x \in [0,\frac{1}{2}) \\ 1, & \text{otherwise.} \end{cases} \end{array}$$

This map gives rise to the following singular 1-cochain:

$$\begin{array}{rcl} \rho\colon \mathrm{C}_1(I) & \to & \mathbb{Z} \\ (\sigma\colon \Delta^1 \to I) & \mapsto & f(\sigma(0,1)) - f(\sigma(1,0)). \end{array}$$

As we pointed out on page 1830, the map ρ actually defines a cochain $\rho\colon \mathrm{C}_1(I,\partial I) \to \mathbb{Z}$.

(a) Show that $\rho \in \mathrm{C}_1(I,\partial I)$ is a singular 1-cocycle.
(b) Show that ρ defines a non-trivial element in $\mathrm{H}^1(I,\partial I;\mathbb{Z})$.

Exercise 73.9. Let $X = \mathbb{R}^2$. We consider the singular 2-cochain $\varphi\colon \mathrm{C}_2(\mathbb{R}^2;\mathbb{F}_2)$ that is given by

$$\varphi(\sigma) \;:=\; \begin{cases} 1, & \text{if } (0,0) \text{ lies in the interior of the convex hull} \\ & \text{of the three points } \sigma(1,0,0), \sigma(0,1,0), \sigma(0,0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Is φ a cocycle?

Exercise 73.10. Let $f\colon S^2 \to \mathbb{F}_2$ be a function such that $f(-x) = 1 - f(x)$ for all $x \in S^2$. Given a singular simplex $\tau\colon \Delta^1 \to S^2$ we write $\mu(\tau) = 0$ if f is constant on $\tau(\Delta^1)$ and otherwise we write $\mu(\tau) = 1$. We denote by $p\colon S^2 \to \mathbb{RP}^2 = S^2/x \sim -x$ the projection map. We consider the following cochain

$$\begin{array}{rcl} \varphi\colon \mathrm{C}_2(\mathbb{RP}^2) & \to & \mathbb{F}_2 \\ (\sigma\colon \Delta^2 \to \mathbb{RP}^2) & \mapsto & \begin{array}{c}\text{pick a lift } \widetilde{\sigma}\colon \Delta^2 \to S^2 \text{ of } \sigma \text{ and calculate} \\ \mu(\widetilde{\sigma} \circ i_0^2) + \mu(\widetilde{\sigma} \circ i_1^2) + \mu(\widetilde{\sigma} \circ i_2^2).\end{array} \end{array}$$

(a) Show that φ is well-defined, i.e. show that $\varphi(\sigma)$ does not depend on the choice of the lift $\widetilde{\sigma}$.
Remark. Note that if $\widetilde{\sigma}\colon \Delta^2 \to S^2$ is a lift of σ, then the only other one is given by $\nu \circ \widetilde{\sigma}$ where $\nu\colon S^2 \to S^2$ is the map defined by $\nu(x) = -x$ for all $x \in S^2$.
(b) Show that φ is a cocycle in $\mathrm{C}^2(\mathbb{RP}^2; \mathbb{F}_2)$.
(c) Does φ represent a non-zero cohomology class?

74. Properties of cohomology groups

After the disappointment of Corollary 73.19 one might be tempted to drop cohomology groups and move on to a different subject. But later on we will see that cohomology groups can be very interesting after all. The key, as we will see, is that cohomology is a contravariant functor whereas homology is a covariant functor.

Therefore we now persevere with the study of cohomology groups. In this chapter we want to extend the following deep theorems and theories from the homological setup to the cohomological setting:

(1) The Excision Theorem 43.19,
(2) the isomorphism between singular homology and cellular homology from Proposition 48.4,
(3) the Mayer-Vietoris Theorem 46.5.

These theorems will allow us in particular to give many more calculations of cohomology groups.

74.1. **Excision theorems for cohomology groups.** We start out with the analogue of the Excision Theorem 43.19.

Theorem 74.1. (Excision Theorem for Cohomology Groups) *Let X be a topological space and let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A. Then the inclusion*

$$i\colon (X \setminus Z, A \setminus Z) \;\to\; (X, A)$$

induces for each $n \in \mathbb{N}_0$ and every abelian group G an isomorphism

$$i^*\colon \mathrm{H}^n(X, A; G) \;\xrightarrow{\cong}\; \mathrm{H}^n(X \setminus Z, A \setminus Z; G).$$

Proof. In the original Excision Theorem 43.19 we showed that the inclusion induced map

$$i_*\colon \mathrm{H}_n(X \setminus Z, A \setminus Z) \;\to\; \mathrm{H}_n(X, A)$$

is an isomorphism for every $n \in \mathbb{N}_0$. But then it follows immediately from Corollary 73.20 that the inclusion induced maps

$$i^*\colon \mathrm{H}^n(X, A; G) \;\xrightarrow{\cong}\; \mathrm{H}^n(X \setminus Z, A \setminus Z; G).$$

are also isomorphisms. ■

Remark. Using Corollary 73.20 we also obtain the “cohomological” versions of the various other excision theorems, e.g. Theorem 43.20 and Theorem 44.10. For example, let X be an m-dimensional topological manifold and let B be a compact m-dimensional submanifold of X. We write $\mathring{B} := B \setminus \partial_0 B$.[1073] For every $n \in \mathbb{N}_0$ and any abelian group G the inclusion induced map

$$\mathrm{H}^n(X, B; G) \;\xrightarrow{\cong}\; \mathrm{H}^n\left(X \setminus \mathring{B}, \partial_0 B; G\right)$$

is an isomorphism.

[1073] We refer to page 1154 for the definition of $\partial_0 B$.

Psychologically it is often helpful to replace relative homology groups by the homology groups of a quotient space. In Proposition 43.22 we had seen that we can perform this switch for reasonable subsets. The following proposition does the same for cohomology groups.

Proposition 74.2. *Let G be an abelian group. Let X be a topological space and let $A \subset X$ be a non-empty subset. We denote by $p\colon (X, A) \to (X/A, A/A)$ the obvious projection map. We consider the following map:*

$$\begin{tikzcd} \widetilde{\mathrm{H}}^n(X/A;G) \arrow[r, "\text{Lemma 73.16 (5)}"', "\cong"] \arrow[rr, bend right] & \mathrm{H}^n(X/A, A/A; G) \arrow[r, "p^*"] & \mathrm{H}^n(X, A; G) \end{tikzcd}$$

The following two statements hold:

1. *The composition of the above maps from left to right define a natural transformation from the contravariant functor $(X, A) \mapsto \widetilde{\mathrm{H}}^n(X, A)$ to the contravariant functor $(X, A) \mapsto \mathrm{H}^n(X, A; G)$.*
2. *If A is a non-empty good subset[1074], then the above map $\widetilde{\mathrm{H}}^n(X/A; G) \to \mathrm{H}^n(X, A; G)$ is an isomorphism.*

Proof. At this stage we have all the tools to rewrite the proof of Proposition 43.22. In particular we can use the above Excision Theorem 74.1 for Cohomology Groups to replace the Excision Theorem 43.19. We leave it to the reader to verify that nothing goes wrong. ■

Example. Let G be an abelian group. From the calculation of the cohomology groups of finitely many points, see Lemmas 73.12 and 73.14, we obtain that

$$\mathrm{H}^k(S^0; G) \;\cong\; \begin{cases} G^2, & \text{if } k = 0, \text{ and} \\ 0, & \text{if } k \neq 0. \end{cases}$$

Similar to the proof of Proposition 43.4 we can use Proposition 74.2 to calculate, via an induction argument, that for $n \geq 1$ we have

$$\mathrm{H}^k(S^n; G) \;\cong\; \begin{cases} G, & \text{if } k = 0 \text{ or } k = n, \text{ and} \\ 0, & \text{if } k \neq 0, n. \end{cases}$$

We will give an alternative calculation of the cohomology groups of spheres in the next section using cellular cohomology.

74.2. **Cellular cohomology.** In Chapter 48 we introduced the cellular homology of a CW-complex. We saw that it is naturally isomorphic to singular homology and we saw that in many examples it is straightforward to compute cellular homology. Now we will repeat the trick and introduce cellular cohomology.

First we recall some definitions and some notation. Let X be a CW-complex.

1. Recall that given $n \in \mathbb{N}_0$ we denote by X^n the n-skeleton of X.
2. We write $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$ and we denote by $d_n\colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_{n-1}^{\mathrm{CW}}(X)$ the cellular boundary map from page 1259.

[1074] Recall according to the definition on page 1107 this means that A is closed and that there exists an open neighborhood U of A in X such that A is a deformation retract of U.

(3) We refer to

$$\dots \xrightarrow{d_{n+1}} \mathrm{C}_{n+1}^{\mathrm{CW}}(X) \xrightarrow{d_{n+1}} \mathrm{C}_{n}^{\mathrm{CW}}(X) \xrightarrow{d_{n}} \mathrm{C}_{n-1}^{\mathrm{CW}}(X) \xrightarrow{d_{n-1}} \dots$$

as the *cellular chain complex*.

(4) Let G be an abelian group. We remark that for a finite CW-complex we have

$$\operatorname{Hom}\big(\mathrm{C}_n^{\mathrm{CW}}(X), G\big) \underset{\substack{\uparrow \\ \text{Lemmas 48.1 and 73.1}}}{\cong} \text{maps from the set of } n\text{-cells to } G \;=\; G^{\#\, n\text{-cells}}.$$

We refer to elements in $\operatorname{Hom}\big(\mathrm{C}_n^{\mathrm{CW}}(X), G\big)$ as *cellular n-cochains*.

Example. Given a CW-complex X an integral cellular 1-cochain in $\operatorname{Hom}\big(\mathrm{C}_1^{\mathrm{CW}}(X), \mathbb{Z}\big)$ assigns to *every* 1-cell an integer. We refer to Figure 1124 for an illustration. In contrast an element in $\mathrm{C}_1^{\mathrm{CW}}(X)$ can be viewed as a formal linear combination of 1-cells which is the same as saying that an element in $\mathrm{C}_1^{\mathrm{CW}}(X)$ assigns a non-zero integer to only *finitely many* 1-cells.

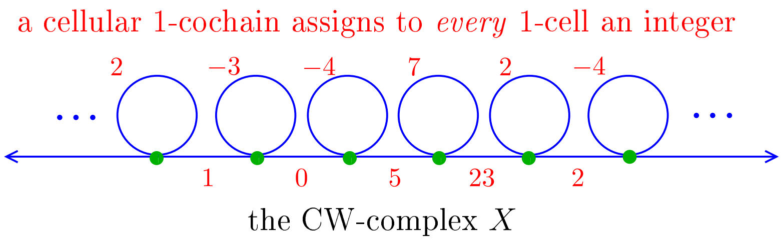


Figure 1124

Definition. Let G be an abelian group and let $n \in \mathbb{N}_0$.

(1) Let X be a CW-complex. We write $f_n := d_{n+1}^*$. We refer to

$$\dots \leftarrow \operatorname{Hom}\big(\mathrm{C}_{n+1}^{\mathrm{CW}}(X), G\big) \xleftarrow{f_n} \operatorname{Hom}\big(\mathrm{C}_{n}^{\mathrm{CW}}(X), G\big) \leftarrow \operatorname{Hom}\big(\mathrm{C}_{n-1}^{\mathrm{CW}}(X), G\big) \xleftarrow{f_{n-1}} \dots$$

as the *cellular cochain complex*. We define the *n-th cellular cohomology group of X with G-coefficients* as

$$\mathrm{H}^n_{\mathrm{CW}}(X;G) := \mathrm{H}^n\big(\mathrm{C}_*^{\mathrm{CW}}(X), G\big) = \frac{\ker\big(f_n\colon \operatorname{Hom}(\mathrm{C}_n^{\mathrm{CW}}(X), G) \to \operatorname{Hom}(\mathrm{C}_{n+1}^{\mathrm{CW}}(X), G)\big)}{\operatorname{im}\big(f_{n-1}\colon \operatorname{Hom}(\mathrm{C}_{n-1}^{\mathrm{CW}}(X), G) \to \operatorname{Hom}(\mathrm{C}_{n}^{\mathrm{CW}}(X), G)\big)}.$$

(2) If $f\colon X \to Y$ is a cellular map, then by Lemma 48.3 we get an induced chain map $f_*\colon \mathrm{C}_*^{\mathrm{CW}}(X) \to \mathrm{C}_*^{\mathrm{CW}}(Y)$ and thus we obtain from Lemma 73.8 (1) an induced map $f^*\colon \mathrm{H}^n_{\mathrm{CW}}(Y;G) \to \mathrm{H}^n_{\mathrm{CW}}(X;G)$.

For completeness' sake we jot down the following fairly self-evident lemma.

Lemma 74.3. $(*)$ *Let G be an abelian group and let $n \in \mathbb{N}_0$. The maps*

$$X \mapsto \mathrm{H}^n_{\mathrm{CW}}(X) \quad \text{and} \quad (f\colon X \to Y) \mapsto (f^*\colon \mathrm{H}^n_{\mathrm{CW}}(Y;G) \to \mathrm{H}^n_{\mathrm{CW}}(X;G))$$

define a contravariant functor from the category of CW*-complexes to the category of abelian groups.*

Proof. This lemma follows immediately from Lemmas 48.3 and 73.8 (2). ∎

Perhaps not surprisingly we have the following theorem which plays a similar role as Proposition 48.4.

Proposition 74.4. *For every* CW*-complex X, any abelian group G and any $n \in \mathbb{N}_0$ there exists a natural isomorphism*[1075]

$$\mathrm{H}^n(X;G) \xrightarrow{\cong} \mathrm{H}^n_{\mathrm{CW}}(X;G).$$

Remark. An alternative proof of this theorem is given in [**Hat02**, p. 203].

First proof of Proposition 74.4. Let X be a CW-complex. Recall that on page 1290 we introduced the intermediate cellular chain complex that is given by

$$\mathrm{C}^{\mathrm{int}}_n(X) \;:=\; \ker\left(\mathrm{C}_n(X^n) \xrightarrow{\partial} \mathrm{C}_{n-1}(X^n) \to \mathrm{C}_{n-1}(X^n, X^{n-1})\right) \quad \text{for } n \in \mathbb{N}_0.$$

We consider the two *natural* chain maps

$$i\colon \mathrm{C}^{\mathrm{int}}_*(X) \hookrightarrow \mathrm{C}_*(X) \quad \text{and} \quad p\colon \mathrm{C}^{\mathrm{int}}_*(X) \twoheadrightarrow \mathrm{C}^{\mathrm{CW}}_*(X),$$

which are given by the obvious inclusion and the obvious projection. In Proposition 49.10 we had seen that these two chain maps are chain homotopy equivalences. Thus it follows from Lemma 73.8 (4) that we get isomorphisms

$$\mathrm{H}^n(X;G) = \mathrm{H}^n(\mathrm{C}_*(X);G) \underset{\cong}{\xrightarrow{i^*}} \mathrm{H}^n(\mathrm{C}^{\mathrm{int}}_*(X);G) \underset{\cong}{\xleftarrow{p^*}} \mathrm{H}^n(\mathrm{C}^{\mathrm{CW}}_*(X);G) = \mathrm{H}^n_{\mathrm{CW}}(X;G).$$

As pointed out above, the two chain maps i and p are natural. Therefore we see that the above isomorphism $\mathrm{H}^n(X;G) \to \mathrm{H}^n_{\mathrm{CW}}(X;G)$ is in fact natural. ■

Alternative proof of Proposition 74.4 without "natural". Let X be a CW-complex. We denote by $\mathrm{C}_* = \mathrm{C}_*(X)$ the singular chain complex and we denote by $D_* = \mathrm{C}^{\mathrm{CW}}_*(X)$ the cellular chain complex. In Proposition 48.4 we showed that all the homology groups of the chain complexes C_* and D_* are isomorphic. It follows from Proposition 73.18 that for any abelian group G and any $n \in \mathbb{N}_0$ the cohomology groups $\mathrm{H}^n(X;G) = \mathrm{H}^n(\mathrm{C}_*;G)$ and $\mathrm{H}^n_{\mathrm{CW}}(X;G) = \mathrm{H}^n(D_*;G)$ are isomorphic. This approach to proving Proposition 74.4 circumvents the use of Proposition 49.10. But it is not clear how this approach can be used to show that there exists a *natural* isomorphism. Of course for many applications this is not an issue. ■

Examples. Let G be an abelian group.

(1) We use cellular cohomology to once again compute the cohomology groups of spheres. As we saw on page 935 we can view S^n, $n \geq 1$, as a CW-complex with one 0-cell and one other cell of dimension n. Furthermore the cellular boundary maps are the zero maps.[1076] Thus we see that the cellular chain complex is of the form

$$0 \to \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \to 0.$$

[1075]Here "natural" means that there exists a natural transformation between the contravariant functors $X \mapsto \mathrm{H}^n(X;G)$ and $X \mapsto \mathrm{H}^n_{\mathrm{CW}}(X;G)$ from the category of CW-complexes to the category of abelian groups.

[1076]For $n \geq 2$ this follows by definition, for $n = 1$ this is a consequence of Lemma 48.6.

The dual cochain complex is then given by

$$0 \leftarrow \underbrace{\operatorname{Hom}(\mathbb{Z},G)}_{=G} \xleftarrow{0} \cdots \xleftarrow{0} \underbrace{\operatorname{Hom}(\mathbb{Z},G)}_{=G} \leftarrow 0.$$

Summarizing, precisely as on page 1841 we obtain that

$$\mathrm{H}^k(S^n;G) \cong \begin{cases} G, & \text{if } k=0,n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let Γ be a connected finite topological graph. It follows from Proposition 18.31 (2) that Γ is homotopy equivalent to a CW-complex X with one 0-cell and $\chi(\Gamma)+1$ 1-cells, see Figure 1125. The unique cellular boundary map in the cellular chain complex of X is zero by Lemma 48.6. Summarizing we see that for any $n \in \mathbb{N}_0$ we have

$$\mathrm{H}^n(\Gamma;G) \underset{\substack{\uparrow \\ \text{Lemma 73.13}}}{\cong} \mathrm{H}^n(X;G) \underset{\substack{\uparrow \\ \text{Proposition 74.4}}}{\cong} \mathrm{H}^n_{\mathrm{CW}}(X;G) \underset{\substack{\uparrow \\ \text{since the coboundary map is zero}}}{\cong} \operatorname{Hom}(\mathrm{C}^{\mathrm{CW}}_n(X);G) \cong \begin{cases} G, & \text{if } n=0, \\ G^{\chi(\Gamma)+1}, & \text{if } n=1, \\ 0, & \text{if } n\geq 2. \end{cases}$$

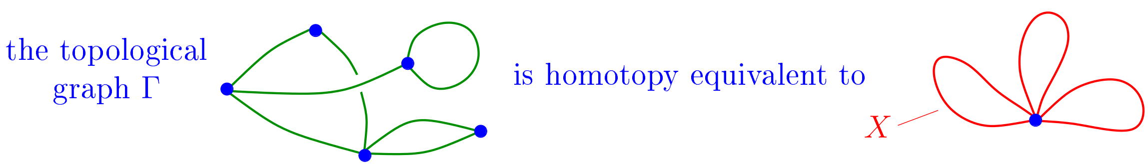


FIGURE 1125

(3) Given $g \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$ we denote by $\Sigma_{g,m}$ the surface of genus g minus m open disks. On page 1675 we had argued that for $m \geq 1$ the topological space $\Sigma_{g,m}$ is homotopy equivalent to the wedge of $2g+m-1$ circles. The previous example, together with Lemma 73.13, thus gives us the calculation of the cohomology groups of $\Sigma_{g,m}$ for $m \geq 1$.

(4) Let Σ_g be the surface of genus g. In Figure 600 we showed that we can view Σ_g as a CW-complex with one 0-cell, $2g$ 1-cells and one 2-cell. On page 1270 we saw that the cellular boundary maps are all zero. Hence for any abelian group G and any $n \in \mathbb{N}_0$ we have

$$\mathrm{H}^n(\Sigma_g;G) \underset{\substack{\uparrow \\ \text{Proposition 74.4}}}{\cong} \mathrm{H}^n_{\mathrm{CW}}(\Sigma_g;G) = \operatorname{Hom}\left(\mathrm{C}^{\mathrm{CW}}_n(\Sigma_g);G\right) \cong \begin{cases} G, & \text{if } n=0 \text{ or } n=2, \\ G^{2g}, & \text{if } n=1, \text{ and} \\ 0, & \text{if } n \neq 0,1,2. \end{cases}$$

In all the examples so far the cohomology groups are isomorphic to the corresponding homology groups that we had calculated earlier. Now we will see an example where the homology and cohomology groups differ.

Examples.

(1) We consider the n-dimensional projective space $\mathbb{RP}^n$. In the proof of Proposition 48.10 we saw that there exists a CW-structure on $\mathbb{RP}^n$ such that the corresponding cellular

chain complex is of the following form:

$$\begin{array}{c} 0 \;\to\; \mathbb{Z} \xrightarrow{\cdot(1+(-1)^n)} \mathbb{Z} \;\to\; \dots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \;\to\; 0 \\ \uparrow \\ \text{degree } n \end{array}$$

which allowed us to compute the homology groups of $\mathbb{R}\mathrm{P}^n$. Now let G be an abelian group. We dualize this chain complex, i.e. we apply the functor $\mathrm{Hom}(-,G)$. Using the isomorphism $\mathrm{Hom}(\mathbb{Z},G) \to G$ from Lemma 73.1 we see that the cochain complex is isomorphic to the following cochain complex:

$$0 \;\leftarrow\; G \xleftarrow{\cdot(1+(-1)^n)} G \;\leftarrow\; \dots \xleftarrow{0} G \xleftarrow{\cdot 2} G \xleftarrow{0} G \;\leftarrow\; 0.$$

For $G = \mathbb{Z}$ we thus obtain from Proposition 74.4 that for any $k \in \{0,\dots,n\}$ we have

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is even and } k \neq 0, \\ 0, & \text{if } k \text{ is odd and } k \neq n, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd} \end{cases} \quad \text{but} \quad \mathrm{H}_k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd and } k \neq n, \\ 0, & \text{if } k \text{ is even and } k \neq 0, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd} \end{cases}$$

where on the right-hand side we recall the calculation of the homology groups using the above cellular chain complex, see also Proposition 48.10. Therefore we see that for $n \geq 2$ the isomorphism types of the homology groups and the cohomology groups of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{Z}$-coefficients do *not* coincide.

(2) To add to the confusion we also compute the cohomology of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients. The corresponding dual cellular cochain complex is then given by

$$0 \;\to\; \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \;\leftarrow\; \dots \;\leftarrow\; \mathbb{F}_2 \xleftarrow{\cdot 2 = 0} \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \;\to\; 0$$

and we see that

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0,\dots,n, \\ 0, & \text{otherwise.} \end{cases}$$

If we compare this result with the calculation on page 1403 of the homology of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients we see that we get the same result. More precisely, the homology groups and the cohomology groups of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients are isomorphic.

One can easily modify the above argument and apply it to $\mathbb{R}\mathrm{P}^\infty$, which we can view as a CW-complex with precisely one cell in each dimension, see 942. We then obtain easily that for each $k \in \mathbb{N}_0$ we have

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2) \;\cong\; \mathbb{F}_2$$

(3) We recall that in Lemma 36.1 we showed that we can view $\mathbb{C}\mathrm{P}^n$ as a CW-complex which admits exactly one cell in dimensions $0, 2, \dots, 2n$ and which admits no odd-dimensional cells. It follows easily from Proposition 74.4 that

$$\mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \dots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 36.6 (2) we can view the infinite-dimensional complex projective space $\mathbb{CP}^\infty$ as a CW-complex that has precisely one cell in each even dimension and no cells in odd dimensions. Using Proposition 74.4 we obtain that

$$\mathrm{H}^k(\mathbb{CP}^\infty;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k \text{ even,} \\ 0, & \text{if } k \text{ odd.} \end{cases}$$

Given $k < l \in \mathbb{N} \cup \{\infty\}$ we have an inclusion $\mathbb{CP}^k \to \mathbb{CP}^l$. Since the isomorphism of Proposition 74.4 is natural and since the cellular chain complexes of $\mathbb{CP}^n$ and $\mathbb{CP}^\infty$ agree up to the dimension $2n+1$ we obtain almost immediately that the inclusion induced maps $\mathrm{H}^k(\mathbb{CP}^\infty;\mathbb{Z}) \to \mathrm{H}^k(\mathbb{CP}^n;\mathbb{Z})$ are actually isomorphisms up to dimension $2n$.

We conclude this section with the following fairly reasonably immediate corollary to Proposition 74.4. This corollary can be viewed as an analogue of Proposition 48.5 and Proposition 64.6.

Corollary 74.5. *Let X be a topological space. Suppose we are in one of the following two situations:*

(1) *X is homotopy equivalent to a retract of a finite* CW*-complex,*
(2) *X is a compact topological manifold.*

Then for every $n \in \mathbb{N}_0$ the cohomology group $\mathrm{H}^n(X;\mathbb{Z})$ is a finitely generated group.

Proof. Statement (2) is an immediate consequence of Statement (1) and Theorem 85.12. We will prove Statement (1) in Exercise 74.1. ■

74.3. **Explicit generators of cohomology groups.** In the previous section we had used cellular cohomology to compute the isomorphism types of the cohomology groups of many different topological spaces. But this calculation of the isomorphism types does not give us any explicit generators of the cohomology groups. For example on page 1825 we introduced a singular 1-cochain $\theta_{\mathbb{Z}}\colon\ \mathrm{C}_1(S^1) \to \mathbb{Z}$ and we showed in Lemma 73.10 that $\theta_{\mathbb{Z}}$ defines a non-zero cohomology class $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$. By the discussion on page 1844 we now also know that $\mathrm{H}^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$, but we do not know whether $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$ is a generator.

On the other hand in earlier chapters we had obtained a pretty good understanding of generators of homology groups. The idea now is to translate information on homology groups to information on cohomology groups. The key tool relating homology and cohomology is given by the Kronecker pairing that we now introduce.

Lemma 74.6. *Let C_* be a chain complex.*

(1) *Let G be an abelian group. The map*

$$\begin{array}{rcl} \langle\ ,\ \rangle\colon\ \mathrm{H}^n(C;G) \times \mathrm{H}_n(C) & \to & G \\ ([\varphi],[c]) & \mapsto & \langle [\varphi],[c]\rangle := \underset{\substack{\uparrow \\ \textit{here } \varphi \in \mathrm{Hom}(C_n,G) \textit{ and } c \in C_n, \textit{ so } \varphi(c) \in G \textit{ is defined}}}{\varphi(c)} \end{array}$$

is well-defined and bilinear.[1077] *In particular, if (X,A) is a pair of topological spaces, then the map*

$$\begin{array}{rcl} \langle\ ,\ \rangle_X\colon\ \mathrm{H}^n(X,A;G)\times \mathrm{H}_n(X,A) & \to & G \\ ([\varphi],[c]) & \mapsto & \langle[\varphi],[c]\rangle_X := \varphi(c) \end{array}$$

is well-defined and bilinear. We refer to both pairings as the Kronecker pairing.[1078][1079]

(2) (∗) *If $\varphi\colon G_1\to G_2$ is a group homomorphism then the following diagram commutes:*

$$\begin{tikzcd} \mathrm{H}^n(C;G_1)\times \mathrm{H}_n(C) \arrow[r, "\langle -,-\rangle"] \arrow[d, "\varphi_*\times \mathrm{id}"] & G_1 \arrow[d, "\varphi"] \\ \mathrm{H}^n(C;G_2)\times \mathrm{H}_n(C) \arrow[r, "\langle -,-\rangle"] & G_2. \end{tikzcd}$$

(3) *Let $f\colon C_*\to D_*$ be a chain map between two chain complexes and let G be an abelian group. Then for any $c\in \mathrm{H}_n(C)$ and $\varphi\in \mathrm{H}^n(D;G)$ we have*

$$\langle f^*(\varphi),c\rangle \;=\; \langle \varphi, f_*(c)\rangle.$$

In particular, suppose that $f\colon (X,A)\to(Y,B)$ is a map between two pairs of topological spaces. Let $c\in \mathrm{H}_n(X,A)$ and let $\varphi\in \mathrm{H}^n(Y,B;G)$. Then

$$\langle f^*(\varphi),c\rangle_X \;=\; \langle \varphi, f_*(c)\rangle_Y.$$

Proof. Let C_* be a chain complex.

(1) Let G be an abelian group. Let $\varphi\in \mathrm{Hom}(C_n,G)$ be a cocycle and let $c\in C_n$ be a cycle. Furthermore let $\alpha\in\mathrm{Hom}(C_{n-1},G)$ and let $d\in \mathrm{C}_{n+1}$. Then the following equalities hold in G:

$$(\varphi+\delta\alpha)(c) \underset{\substack{\uparrow\\ \text{definition of }\delta}}{=} \varphi(c)+\alpha(\partial c) \underset{\substack{\uparrow\\ \text{since } c \text{ is a cycle}}}{=} \varphi(c) \quad\text{and}\quad \varphi(c+\partial d) \underset{\substack{\uparrow\\ \text{definition of }\delta}}{=} \varphi(c)+(\delta\varphi)(d) \underset{\substack{\uparrow\\ \text{since } \varphi \text{ is a cocycle}}}{=} \varphi(c).$$

This calculation shows that the definition of the Kronecker pairing does not depend on the choices of the representatives of the cohomology and the homology classes. Finally note that it is clear that the Kronecker pairing is bilinear.

(2) This statement follows immediately from the definitions.

(3) The statement also follows immediately from the definitions. Indeed, if $[c]\in\mathrm{H}_n(C)$ and $[\varphi]\in \mathrm{H}^n(D;G)$, then

$$\langle f^*([\varphi]),[c]\rangle = \langle [f^*\varphi],[c]\rangle = (f^*\varphi)(c) = \varphi(f_*(c)) = \langle[\varphi],[f_*(c)]\rangle = \langle[\varphi],f_*([c])\rangle.$$

[1077] In general, given a commutative ring R a map $\Theta\colon U\times V\to W$ between R-modules is called *bilinear*, if for all $u,u'\in U$, $v,v'\in V$ and $r\in R$ we have

$$\begin{array}{rclcrcll} \Theta(u+u',v) &=& \Theta(u,v)+\Theta(u',v) & \text{and} & \Theta(ru,v) &=& r\Theta(u,v), & \text{and} \\ \Theta(u,v+v') &=& \Theta(u,v)+\Theta(u,v') & \text{and} & \Theta(u,rv) &=& r\Theta(u,v). & \end{array}$$

In our special case we consider the case of $\mathbb{Z}$-modules.

[1078] Often we drop the subscript X in the notation.

[1079] Leopold Kronecker (1823–1891) was a German mathematician who worked on number theory. I don't know how his name got attached to this pairing.

The statement regarding maps between pairs of topological spaces is an immediate consequence of the purely algebraic statement. ■

Now we state the lemma which will allow us in many situations to show that given cocycles form a basis for cohomology groups.

Lemma 74.7. *Let X be a topological space and let $n \in \mathbb{N}_0$. Suppose that $\mathrm{H}^n(X;\mathbb{Z})$ and $\mathrm{H}_n(X;\mathbb{Z})$ are free abelian groups of rank k and suppose we are given $\varphi_1,\dots,\varphi_k \in \mathrm{H}^n(X;\mathbb{Z})$ and $c_1,\dots,c_k \in \mathrm{H}_n(X;\mathbb{Z})$. If*

$$\det\big(\underbrace{\langle \varphi_i, c_j\rangle_{i,j=1,\dots,k}}_{=\,k\times k\text{–matrix}}\big) \;=\; \pm 1,$$

then $\varphi_1,\dots,\varphi_k \in \mathrm{H}^n(X;\mathbb{Z})$ and $c_1,\dots,c_k \in \mathrm{H}_n(X;\mathbb{Z})$ are both bases.

Example. Let $X = S^1$. We denote by $\theta_{\mathbb{Z}}\colon \mathrm{C}_1(S^1) \to \mathbb{Z}$ the singular 1-cocycle that we introduced on page 1825. We consider the singular cycle $\mu\colon \Delta^1 \to S^1$ that is given by $(1-t,t) \mapsto e^{2\pi i t}$. As on page 1826 we see that

$$\langle [\theta_{\mathbb{Z}}],[\mu]\rangle \;=\; \theta_{\mathbb{Z}}(\mu) \;=\; \alpha_{\mathbb{Z}}\left(\text{boundary of lift } \begin{array}{rcl}\widetilde{\mu}\colon \Delta^1 & \to & \mathbb{R}\\ (1-t,t) & \mapsto & t\end{array} \text{ of } \mu\right) \;=\; 1.$$

From Proposition 43.4 we know that $\mathrm{H}_1(S^1;\mathbb{Z}) \cong \mathbb{Z}$ and by the discussion on page 1841 we know that $\mathrm{H}^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$. Therefore it follows from Lemma 74.7 that $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$ is in fact a generator.[1080] In particular we have an equality $\mathrm{H}^1(S^1;\mathbb{Z}) = \mathbb{Z}\cdot[\theta_{\mathbb{Z}}]$.

Remark. Lemma 74.7 is really quite generous, it gives us as output that both sets are bases. In particular to apply the lemma we do not even have to verify that the c_i form a basis.

Lemma 74.7 is an immediate consequence of the following purely algebraic lemma.

Lemma 74.8. *Let A and B be two free abelian groups of rank k, let $a_1,\dots,a_k \in A$ and let $b_1,\dots,b_k \in B$. Furthermore let $\langle\ ,\ \rangle\colon A\times B \to \mathbb{Z}$ be a bilinear pairing. If*

$$\det(\langle a_i, b_j\rangle_{i,j=1,\dots,k}) \;=\; \pm 1,$$

then both $a_1,\dots,a_k$ and $b_1,\dots,b_k$ are bases.

Proof. We denote by S the free abelian group generated by the set $\{a_1,\dots,a_k\}$ and we denote by T the free abelian group generated by the set $\{b_1,\dots,b_k\}$. We denote by $i\colon S \to A$ and by $j\colon T\to B$ the obvious maps given by $a_i \mapsto a_i$ and $b_i \mapsto b_i$. We denote by $\Phi\colon A \to \mathrm{Hom}(B,\mathbb{Z})$ the homomorphism given by $a \mapsto (b \mapsto \langle a,b\rangle)$.[1081] The claim follows from the following three observations:

(1) Let $\gamma\colon G \to H$ be a homomorphism between two finitely generated free abelian groups of the same rank. Lemma 19.8 (5) says that if γ is an epimorphism, then γ it is already an isomorphism.

[1080] It also follows from Lemma 74.7 that the singular 1-simplex $\mu\colon \Delta^1 \to S^1$ represents a generator for $\mathrm{H}_1(S^1) \cong \mathbb{Z}$. We had obtained that result in Lemma 45.4.

[1081] The fact that $\langle\,,\,\rangle$ is bilinear implies that $b \mapsto \langle a,b\rangle$ is a homomorphism and that $\Phi\colon A \to \mathrm{Hom}(B,\mathbb{Z})$ is a homomorphism.

(2) We consider the following sequence of homomorphisms
$$S \xrightarrow{i} A \xrightarrow{\Phi} \mathrm{Hom}(B,\mathbb{Z}) \xrightarrow{j^*} \mathrm{Hom}(T,\mathbb{Z}).$$
With respect to the basis $a_1,\dots,a_k$ of S and the dual basis $b_1^*,\dots,b_k^*$ of $\mathrm{Hom}(T,\mathbb{Z})$ this linear map is represented by the matrix $\langle a_i, b_j\rangle_{i,j=1,\dots,k}$. Thus by our hypotheses this map is an isomorphism. Furthermore by our hypotheses each of the individual groups is a free abelian group of rank k.

(3) Since the map in (2) is an isomorphism it is in particular an epimorphism. Starting from the right and using (1) three times in a row we see that all maps are in fact isomorphisms. In particular we have $S = A$. The same argument turned around shows that $T = B$. ■

Now we will use Lemma 74.7 to determine explicit bases for the first cohomology groups of surfaces. We start out with the torus.

Lemma 74.9. *We consider the two projection maps*
$$\begin{array}{rcl} p\colon S^1\times S^1 & \to & S^1 \\ (v,w) & \mapsto & v \end{array} \quad \textit{and} \quad \begin{array}{rcl} q\colon S^1\times S^1 & \to & S^1 \\ (v,w) & \mapsto & w. \end{array}$$
Then $p^*([\theta_{\mathbb{Z}}])$ and $q^*([\theta_{\mathbb{Z}}])$ form a basis for $\mathrm{H}^1(S^1\times S^1;\mathbb{Z}) \cong \mathbb{Z}^2$.

Proof. We denote as usual by $\mu\colon \Delta^1 \to S^1$ our favorite singular 1-cycle of S^1 that is given by $\mu(1-t,t) = e^{2\pi i t}$. For simplicity we write $\theta = \theta_{\mathbb{Z}}$. We denote by
$$\begin{array}{rcl} i\colon S^1 & \to & S^1\times S^1 \\ z & \mapsto & (z,1) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon S^1 & \to & S^1\times S^1 \\ z & \mapsto & (1,z) \end{array}$$
the two obvious inclusion maps. We calculate that

since $\langle [p^*\theta],[i_*(\mu)]\rangle = (p^*\theta)(i_*(\mu)) = \theta(p_*(i_*(\mu)))$ and similarly for the other terms
$$\begin{pmatrix} \langle [p^*\theta],[i_*(\mu)]\rangle & \langle [p^*\theta],[j_*(\mu)]\rangle \\ \langle [q^*\theta],[i_*(\mu)]\rangle & \langle [q^*\theta],[j_*(\mu)]\rangle \end{pmatrix} \overset{\downarrow}{=} \begin{pmatrix} \theta(p_*(i_*(\mu))) & \theta(p_*(j_*(\mu))) \\ \theta(q_*(i_*(\mu))) & \theta(q_*(j_*(\mu))) \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} \theta(\mu) & \theta(1) \\ \theta(1) & \theta(\mu) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
since $p\circ i = \mathrm{id}$ and $q\circ j = \mathrm{id}$ and since $p\circ j \equiv 1 \in S^1$ and $q\circ i \equiv 1 \in S^1$
here we denote by 1 also the constant singular 1-simplex given by $(1-t,t)\mapsto 1$

We know from Lemma 46.12 that $\mathrm{H}_1(S^1\times S^1;\mathbb{Z}) \cong \mathbb{Z}^2$. Furthermore we saw on page 1844 that $\mathrm{H}^1(S^1\times S^1;\mathbb{Z}) \cong \mathbb{Z}^2$. The above calculation, together with Lemma 74.7 shows that $p^*([\theta_{\mathbb{Z}}])$ and $q^*([\theta_{\mathbb{Z}}])$ form a basis for $\mathrm{H}^1(S^1\times S^1;\mathbb{Z})$. ■

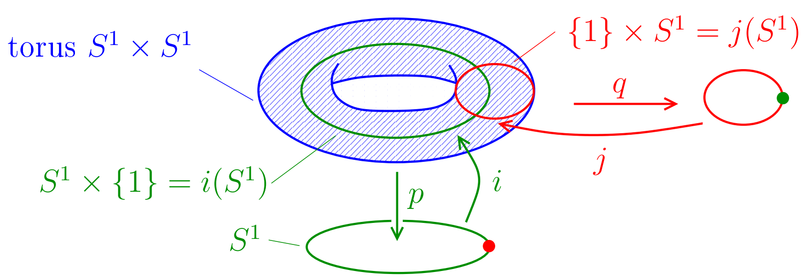


Figure 1126

Together with Proposition 73.11 we have now found explicit generators for the cohomology groups of the torus in dimensions 0 and 1. The following question arises:

Question 74.10. *Can we find an explicit generator of* $\mathrm{H}^2(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}$*?*

Now we turn to surfaces of higher genus. As usual, given $g \in \mathbb{N}$ we denote by Σ_g the surface of genus g. In Figure 1127 we show projection maps $p_i \colon \Sigma_g \to S^1 \times S^1$, $i = 1, \ldots, g$.[1082] The following lemma is a generalization of Lemma 74.9.

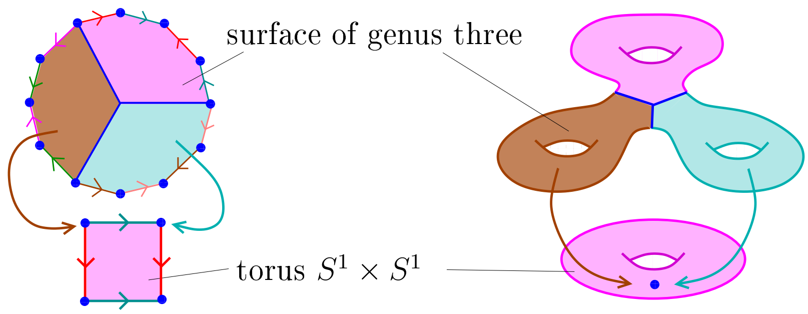


FIGURE 1127

Lemma 74.11. *We denote by* $\alpha, \beta \in \mathrm{H}^1(S^1 \times S^1; \mathbb{Z})$ *the basis given by Lemma 74.9. A basis for* $\mathrm{H}^1(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}^{2g}$ *is given by* $p_i^*(\alpha), p_i^*(\beta)$, $i = 1, \ldots, g$.

PROOF. As we will see, the proof of the lemma is very similar to the proof of Lemma 74.9. For $i = 1, \ldots, g$ we write $\varphi_i = p_i^*(\alpha)$ and $\psi_i := p_i^*(\beta)$. We consider the singular 1-cycles $x_1, \ldots, x_g$ and $y_1, \ldots, y_g$ that are indicated in Figure 1128. As in the proof of Lemma 74.9 it follows easily from the definitions that (for $g = 2$) we have

$$\begin{pmatrix} \langle \varphi_1, x_1 \rangle & \langle \varphi_1, y_1 \rangle & \langle \varphi_1, x_2 \rangle & \langle \varphi_1, y_2 \rangle \\ \langle \psi_1, x_1 \rangle & \langle \psi_1, y_1 \rangle & \langle \psi_1, x_2 \rangle & \langle \psi_1, y_2 \rangle \\ \langle \varphi_2, x_1 \rangle & \langle \varphi_2, y_1 \rangle & \langle \varphi_2, x_2 \rangle & \langle \varphi_2, y_2 \rangle \\ \langle \psi_2, x_1 \rangle & \langle \psi_2, y_1 \rangle & \langle \psi_2, x_2 \rangle & \langle \psi_2, y_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The lemma now follows immediately from Lemma 74.7. ■

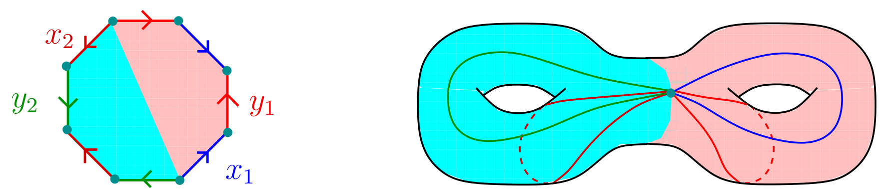


FIGURE 1128. Illustration for the proof of Lemma 74.11.

74.4. **Excisive triads and the Mayer–Vietoris Theorem for cohomology groups.** Before we state and prove the Mayer–Vietoris Theorem for cohomology groups it is helpful to introduce the following definition.

[1082] In fact for $g = 2$ we already saw this map in Figure 307.

Definition.

(1) A *triad of topological spaces* is a triple (X, A_1, A_2) where X is a topological space and where both A_1 and A_2 are subsets of X.[1083]
(2) A map $f\colon (X, A_1, A_2) \to (Y, B_1, B_2)$ of triads is a map $f\colon X \to Y$ with $f(A_1) \subset B_1$ and $f(A_2) \subset B_2$.
(3) Given a triad (X, A_1, A_2) of topological spaces we consider the following subgroup of $\mathrm{C}_n(A_1 \cup A_2)$:[1084]

$$\mathrm{C}_n^{\{A_1,A_2\}}(A_1 \cup A_2) := \left\{ \sum_{j=1}^{k} a_j\sigma_j \in \mathrm{C}_n(A_1 \cup A_2) \;\middle|\; \begin{array}{l} \text{for each } j \text{ there exists an } i \in \{1,2\} \\ \text{such that the image of } \sigma_j \text{ lies in } A_i \end{array} \right\}.$$

The usual boundary map on the groups $\mathrm{C}_*(A_1 \cup A_2)$ restricts to a boundary map on $\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2)$, so we can view $\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2)$ as a chain complex. We denote the corresponding homology groups by $\mathrm{H}_*^{\{A_1,A_2\}}(A_1 \cup A_2)$.
(4) We say a triad (X, A_1, A_2) is *excisive* if the inclusion map

$$\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2) \;\to\; \mathrm{C}_*(A_1 \cup A_2)$$

is a chain homotopy equivalence of chain complexes.

Example. In Exercise 74.2 we will see that the triad $(X, A_1, A_2) := (\mathbb{R}, (-\infty, 0], (0, \infty))$ is *not* excisive.

After this negative example we want to collect some positive examples of excisive triads.

Proposition 74.12. *Let (X, A_1, A_2) be a triad of topological spaces. The triad is excisive if one of the following conditions is satisfied:*

(1) *$A_1 \subset A_2$ or $A_2 \subset A_1$,*
(2) *$A_1 = \varnothing$ or $A_2 = \varnothing$ or $A_1 = A_2$,*
(3) *$A_1 \cup A_2 = \mathring{A}_1 \cup \mathring{A}_2$ where we denote by $\mathring{A}_1$ and $\mathring{A}_2$ the interiors of A_1 and A_2 in $A_1 \cup A_2$,*
(4) *A_1 and A_2 are both open subsets of $A_1 \cup A_2$,*
(5) *$A_1 \cup A_2$ is a topological manifold and A_1 and A_2 are submanifolds such that each component of $A_1 \cap A_2$ is a boundary component of A_1 and a boundary component of A_2, and such that A_1 and A_2 are closed subsets,*
(6) *X is a CW-complex and A_1 and A_2 are both subcomplexes.*

In the proof of Proposition 74.12 we will need the following lemma which gives us several criteria for a triad to be excisive.

[1083] This definition should not be confused with the definition of a *triple* of topological spaces, which was defined as a triple (X, B, A) where X is a topological space and $A \subset B \subset X$ are subsets. In a triad we do not demand that the third set is contained in the second set.

[1084] The definition is basically the same as on page 1133 except that this time we do not demand that the interiors of A_1 and A_2 cover all of X.

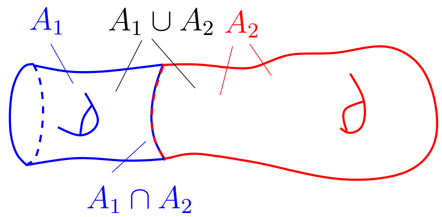


FIGURE 1129. Illustration of Proposition 74.12 (5).

Lemma 74.13. *Let (X, A_1, A_2) be a triad of topological spaces. The following statements are equivalent:*

(1) *the triad is excisive,*
(2) *for every $n \in \mathbb{N}_0$ the inclusion induced map*
$$\mathrm{H}_n^{\{A_1,A_2\}}(A_1 \cup A_2) \;\to\; \mathrm{H}_n(A_1 \cup A_2)$$
is an isomorphism,
(3) *for every $n \in \mathbb{N}_0$ the inclusion induced map*
$$\mathrm{H}_n(A_1, A_1 \cap A_2) \;\to\; \mathrm{H}_n(A_1 \cup A_2, A_2)$$
is an isomorphism.

Now suppose that the triad is excisive.

(a) *Given any abelian group G the statements (2) and (3) also hold for G-coefficients.*
(b) *Given any abelian group G the analogue of (2) also holds for cohomology groups, i.e. given any $n \in \mathbb{N}_0$ and given any abelian group G the inclusion induced map $\mathrm{H}^n(A_1 \cup A_2, A_2; G) \to \mathrm{H}^n(A_1, A_1 \cap A_2; G)$ is an isomorphism.*

PROOF. Let (X, A_1, A_2) be a triad of topological spaces. The equivalence of (1) and (2) is an immediate consequence of Corollary 42.3 and Proposition 49.2.

Now we turn to the proof that (2) and (3) are equivalent. We write $Y = A_1 \cup A_2$. We consider the following sequences of chain complexes:

$$\begin{array}{ll} \text{(I)} & \qquad 0 \;\to\; \mathrm{C}_*^{\{A_1,A_2\}}(Y) \;\to\; \mathrm{C}_*(Y) \;\to\; \mathrm{C}_*(Y)/\,\mathrm{C}_*^{\{A_1,A_2\}}(Y) \;\to\; 0 \\ \text{(II)} & 0 \;\to\; \underbrace{\mathrm{C}_*(A_1)/\,\mathrm{C}_*(A_1 \cap A_2)}_{=\mathrm{C}_*(A_1, A_1 \cap A_2)} \;\to\; \underbrace{\mathrm{C}_*(Y)/\,\mathrm{C}_*(A_2)}_{=\mathrm{C}_*(Y, A_2)} \;\to\; \mathrm{C}_*(Y)/\,\mathrm{C}_*^{\{A_1,A_2\}}(Y) \;\to\; 0. \end{array}$$

The top sequence is exact by definition. It is easy to see that the bottom sequence is also exact. We can thus apply Proposition 43.11 to get the corresponding long exact sequences of homology groups:

$$\text{(I)} \;\to \mathrm{H}_n\left(\mathrm{C}_*^{\{A_1,A_2\}}(Y)\right) \to \mathrm{H}_n(Y) \to \mathrm{H}_n\left(\mathrm{C}_*(Y)/\,\mathrm{C}_*^{\{A_1,A_2\}}(Y)\right) \xrightarrow{\partial} \mathrm{H}_{n-1}\left(\mathrm{C}_*^{\{A_1,A_2\}}(Y)\right) \to$$
$$\text{(II)} \;\to \mathrm{H}_n(A_1, A_1 \cap A_2) \to \mathrm{H}_n(Y, A_2) \to \mathrm{H}_n\left(\mathrm{C}_*(Y)/\,\mathrm{C}_*^{\{A_1,A_2\}}(Y)\right) \xrightarrow{\partial} \mathrm{H}_{n-1}(A_1, A_1 \cap A_2) \to$$

But using these long exact sequences we see that

$$\text{(2) holds} \underset{\substack{\uparrow \\ \text{long exact sequence (I)}}}{\iff} \mathrm{H}_n\left(\mathrm{C}_*(Y)/\,\mathrm{C}_*^{\{A_1,A_2\}}(Y)\right) = 0 \text{ for all } n \in \mathbb{N}_0 \underset{\substack{\uparrow \\ \text{long exact sequence (II)}}}{\iff} \text{(3) holds.}$$

Finally we turn to the proofs of the addenda (a) and (b). Thus suppose that (X, A_1, A_2) is excisive and that G is an abelian group.

(a) The fact that the map in (2) is also an isomorphism with G-coefficients follows from basically the same argument as above. Furthermore, if (2) holds for G-coefficients, then again the same argument as above shows that also (3) holds with G-coefficients.
(b) This statement follows from the above "(1) ⇒ (3)" implication together with Corollary 73.20. ■

Now we turn to the proof of Proposition 74.12.

Proof of Proposition 74.12. Let (X, A_1, A_2) be a triad of topological spaces.

(1) If $A_1 \subset A_2$, then $\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2) = \mathrm{C}_*(A_2) = \mathrm{C}_*(A_1 \cup A_2)$, so the triad is excisive. The same argument applies if $A_2 \subset A_1$.
(2) This case is just a special case of (1).
(3) If $A_1 \cup A_2 = \mathring{A}_1 \cup \mathring{A}_2$, then the desired statement follows immediately from Proposition 49.5.
(4) The case that A_1 and A_2 are both open subsets of $A_1 \cup A_2$ is a special case of (3).
(5) Suppose that $A_1 \cup A_2$ is a compact topological manifold and that A_1 and A_2 are submanifolds such that each component of $A_1 \cap A_2$ is a boundary component of A_1 and a boundary component of A_2. The Excision Theorem 44.10 for topological manifolds implies that for every $n \in \mathbb{N}_0$ the inclusion induced map
$$\mathrm{H}_n(A_1, A_1 \cap A_2) \;\to\; \mathrm{H}_n(A_1 \cup A_2, A_2)$$
is an isomorphism. By Lemma 74.13 (3) ⇒ (1) this implies that the triad (X, A_1, A_2) is excisive.
(6) Suppose that X is a CW-complex and that A_1 and A_2 are both subcomplexes. The proof of the Excision Theorem 44.10 for Topological Manifolds can easily be modified to give the corresponding statement for CW-complexes. More precisely, in the proof of the Excision Theorem 44.10 for Topological Manifolds we need to replace the Collar Neighborhood Theorem 8.12 by Proposition 36.10 (8). Furthermore we need to replace Proposition 6.27 (3) by Lemma 36.18 (2). We then proceed as in (5). We leave it to the reader to fill in the details. ■

Now we can formulate the following theorem, which in view of Proposition 74.12 is a generalization of the various Mayer–Vietoris Theorems 46.5, 46.10, and 46.11 that we had formulated earlier.

Theorem 74.14. (Mayer–Vietoris Theorem) *Given any excisive triad (X, A, B) with $X = A \cup B$ there exists a natural*[1085] *long exact sequence*
$$\cdots \to \mathrm{H}_n(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i_A + i_B} \mathrm{H}_n(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A \cap B) \to \dots$$
where the various maps denoted by i are the obvious inclusion induced maps. The same long exact sequence also exists for homology with coefficients in any abelian group G.

Proof. Let (X, A, B) be a triad with $X = A \cup B$. In the proof of the original Mayer–Vietoris Theorem 46.5 we showed that the sequence

$$0 \to C_*(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} C_*(A) \oplus C_*(B) \xrightarrow{i_A + i_B} C_*^{\{A,B\}}(X) \to 0$$

of chain maps is exact. We deduce from Proposition 43.11 that we get a long exact sequence of the form

$$\cdots \to H_n(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} H_n(A) \oplus H_n(B) \xrightarrow{i_A + i_B} H_n^{\{A,B\}}(X) \xrightarrow{p_n} H_{n-1}(A \cap B) \to \dots$$

Recall that the connecting homomorphism is natural by Proposition 43.11. Now we suppose that (X, A, B) is in fact excisive. By definition, and since $X = A \cup B$, this means that the inclusion map

$$C_*^{\{A,B\}}(X) \to C_*(X)$$

is a chain homotopy equivalence, hence for every $n \in \mathbb{N}_0$ it induces a natural isomorphism

$$\Phi_n \colon H_n^{\{A,B\}}(X) \xrightarrow{\cong} H_n(X).$$

For each $n \in \mathbb{N}_0$ we now replace $H_n^{\{A,B\}}(X)$ by $H_n(X)$ and we write $\partial_n := p_n \circ \Phi_n^{-1}$ and we finally get the desired natural long exact sequence.

The argument for homology with G-coefficients is very similar, see also page 1401. We leave the details to the dedicated reader. ■

We also have the following analogue for cohomology groups.

Theorem 74.15. (Mayer–Vietoris Theorem for Cohomology Groups) *Let (X, A, B) be an excisive triad with $X = A \cup B$ and let G be an abelian group. Then there exists a natural[1086] long exact sequence*

$$\cdots \to H^n(X; G) \xrightarrow{i^* \oplus i^*} H^n(A; G) \oplus H^n(B; G) \xrightarrow{i^* \oplus -i^*} H^n(A \cap B; G) \to H^{n+1}(X; G) \to \dots$$

where the maps i^* are the restrictions to the various subspaces.

Proof. Let (X, A, B) be an excisive triad with $X = A \cup B$ and let G be an abelian group. We start out with the following claim.

Claim. The inclusion map

$$C_*^{\{A,B\}}(X) \to C_*(X)$$

induces for each $n \in \mathbb{N}_0$ a natural isomorphism

$$H^n(X; G) \to H^n\left(C_*^{\{A,B\}}(X); G\right).$$

[1085] Here "natural" means that given a map $f \colon (X, A, B) \to (Y, C, D)$ of excisive triads with $X = A \cup B$ and $Y = C \cup D$ the following diagram commutes:

$$\begin{array}{ccccccccccc} \dots \longrightarrow & H_n(A \cap B) & \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} & H_n(A) \oplus H_n(B) & \xrightarrow{i_A + i_B} & H_n(X) & \xrightarrow{\partial_n} & H_{n-1}(A \cap B) & \longrightarrow \dots \\ & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\ \dots \longrightarrow & H_n(C \cap D) & \xrightarrow{i_{C\cap D} \oplus -i_{C\cap D}} & H_n(C) \oplus H_n(D) & \xrightarrow{i_C + i_D} & H_n(Y) & \xrightarrow{\partial_n} & H_{n-1}(C \cap D) & \longrightarrow \dots \end{array}$$

[1086] The meaning of "natural" is evidently totally analogous to the interpretation given in Footnote 1085.

The claim follows immediately from the definition of an excisive triad together with Lemma 73.8 (3).

Now we turn to the actual proof of the Mayer–Vietoris Theorem for Cohomology Groups. Exactly as in the proof of the Mayer–Vietoris Theorem 74.14 we start out with the fact that the following sequence is exact:

$$0 \to \mathrm{C}_*(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{C}_*(A) \oplus \mathrm{C}_*(B) \xrightarrow{i_A + i_B} \mathrm{C}_*^{\{A,B\}}(X) \to 0.$$

The groups $\mathrm{C}_*^{\{A,B\}}(X)$ are clearly free abelian groups. Thus we deduce from Lemma 73.4 and Proposition 73.6 that there exists a natural long exact sequence

$$\to \underbrace{\mathrm{H}^n\left(\mathrm{C}_*^{\{A,B\}}(X); G\right)}_{=\mathrm{H}^n(X;G) \text{ by the claim}} \to \mathrm{H}^n(A;G) \oplus \mathrm{H}^n(B;G) \to \mathrm{H}^n(A \cap B; G) \to \underbrace{\mathrm{H}^{n+1}\left(\mathrm{C}_*^{\{A,B\}}(X); G\right)}_{=\mathrm{H}^{n+1}(X;G) \text{ by the claim}} \to \qquad \blacksquare$$

As an application of the Mayer–Vietoris Theorem 74.15 for Cohomology Groups we will determine the cohomology groups of the wedge of two topological spaces. We will not make use of this example.

Lemma 74.16. $(*)$ *Let X and Y be topological spaces and let $x \in X$ and $y \in Y$ be good points.*[1087] *We use these points to define the wedge $X \vee Y$. Then $(X \vee Y, X, Y)$ is an excisive triad.*

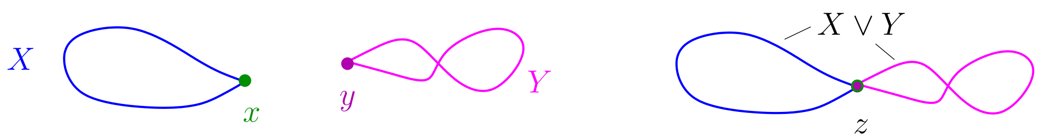


FIGURE 1130

PROOF $(*)$. We denote by z the point in $X \vee Y$ that is given by identifying x with y. For each $k \in \mathbb{N}_0$ we consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}_k(Y,\{y\};\mathbb{Z}) & \longrightarrow & \mathrm{H}_k(X,\{x\};\mathbb{Z}) \oplus \mathrm{H}_k(Y,\{y\};\mathbb{Z}) & \longrightarrow & \mathrm{H}_k(X,\{x\};\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}_k(Y,\{y\};\mathbb{Z}) & \longrightarrow & \mathrm{H}_k(X \vee Y,\{z\};\mathbb{Z}) & \longrightarrow & \mathrm{H}_k(X \vee Y, Y;\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

where all the maps are the obvious maps. The top horizontal sequence is evidently exact. The bottom horizontal sequence is an excerpt from the long exact sequence of the triple $(\{y\} = \{z\}, Y, X \vee Y)$, see Proposition 43.15. The excerpt shown is also exact since $(Y, \{y\})$ is a retract of $(X \vee Y, \{z\})$, which implies that for any $l \in \mathbb{N}_0$ the inclusion induced map $\mathrm{H}_l(Y,\{y\};\mathbb{Z}) \to \mathrm{H}_l(X \vee Y, \{z\};\mathbb{Z})$ is a monomorphism.

The middle vertical map in the above diagram is an isomorphism by Proposition 47.8 and the left vertical map is the identity. It follows from the Five Lemma 43.12 that the right vertical hand map is an isomorphism. We obtain from Lemma 74.13 (3) $\Rightarrow$ (1) that $(X \vee Y, X, Y)$ is an excisive triad. $\blacksquare$

[1087] Recall that according to the definition on page 604 we say that a point x in a topological space X is *good*, if $\{x\}$ is a closed subset and if there exists an open neighborhood U of x such that $\{x\}$ is a deformation retract of U.

We prove the following proposition that is the analogue of Proposition 47.9.

Proposition 74.17. $(*)$ *Let $\{A_k\}_{k\in K}$ be a family of topological spaces. For each $k \in K$ suppose that we are given a good point $a_k \in A_k$. Let G be an abelian group. Given $j \in K$ we denote by*

$$i_j\colon A_j \to \bigvee_{k\in K} A_k \quad \text{respectively} \quad p_j\colon \bigvee_{k\in K} A_k \to A_j$$

the natural inclusion map respectively the natural projection map that we introduced on page 561. Then for every $n \in \mathbb{N}_0$ the map

$$\prod_{k\in K} i_k^*\colon \widetilde{\mathrm{H}}^n\Big(\bigvee_{k\in K} A_k; G\Big) \ \to \ \prod_{k\in K} \widetilde{\mathrm{H}}^n(A_k; G)$$

is an isomorphism where the inverse is given by

$$\prod_{k\in K} p_k^*\colon \prod_{k\in K} \widetilde{\mathrm{H}}^n(A_k; G) \ \to \ \widetilde{\mathrm{H}}^n\Big(\bigvee_{k\in K} A_k; G\Big)$$

Proof $(*)$.

(1) First we consider the case we are given just two topological spaces. By Lemma 74.16 we can use the Mayer–Vietoris Theorem 74.15 for Cohomology Groups to prove, completely analogously to Proposition 47.8, the desired statement.
(2) The case that we are dealing with finitely many topological spaces follows from (1) by induction.
(3) The case that K is infinite is more tricky. There are two possible approaches:
   (a) In the proof of Proposition 47.9, that deals with the corresponding statement for homology groups, we gave two different proofs. The first proof generalizes to the present setting. It is not clear to me how the second proof can be generalized. But fortunately providing one proof is enough.[1088]
   (b) Alternatively we can consider the map

$$\bigoplus_{k\in K} i_{k*}\colon \bigoplus_{k\in K} \widetilde{C}_n(A_k) \ \to \ \widetilde{C}_n\Big(\bigvee_{k\in K} A_k\Big).$$

   By Proposition 47.9 this chain map induces an isomorphism of chain complexes. It follows from Proposition 49.2 that this map is in fact a chain homotopy equivalence. It follows from Lemma 73.8 that the above map induces an isomorphism of dual cochain complexes. The desired statement now follows from throwing Lemmas 73.2 and 73.7 into the bargain. The same logic applies to the projections. ■

Recall that in Lemma 46.8 we showed that for any $k \in \mathbb{Z}$ and any topological space X we have a natural isomorphism

$$\Sigma_X\colon \widetilde{\mathrm{H}}_k(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)).$$

Not surprisingly we have a totally analogous statement in cohomology:

[1088] In fact the proof of Proposition 47.9 that we sketched in Exercise 47.2 can also be generalized to the present context.

Lemma 74.18. (∗) *Let $k \in \mathbb{Z}$ and let G be an abelian group. Given any topological space X we have a natural isomorphism*

$$\Sigma_X \colon \widetilde{\mathrm{H}}^k(\Sigma(X); G) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{k+1}(X; G)$$

which has the property that for any $\varphi \in \mathrm{H}^k(X;G)$ and any $\sigma \in \mathrm{H}_k(X)$ we have

$$\langle \Sigma_X(\varphi), \Sigma_X(\sigma)\rangle_{\Sigma(X)} \;=\; \langle \varphi, \sigma\rangle_X \;\in\; G.$$

Proof (∗). The construction of the isomorphism Σ_X is basically the same as the construction of the corresponding isomorphism in cohomology. In the Lemma 46.8 we just need to replace the Mayer–Vietoris Theorem 46.5 by the Mayer–Vietoris Theorem 74.15 for Cohomology Groups. We leave it to the reader to fill in the details. ■

Example. Let $n \in \mathbb{N}$. Let $\Sigma(S^n) \to S^{n+1}$ be the homeomorphism from page 695. On page 1180 we showed that

$$\Sigma_{S^n}([S^n]) \;=\; (-1)^{n+1} \cdot [S^{n+1}] \;\in\; \mathrm{H}_{n+1}(S^{n+1}; \mathbb{Z}).$$

From the equality in Lemma 74.18 we obtain the corresponding equality

$$\Sigma_{S^n}([S^n]^*) \;=\; (-1)^{n+1} \cdot [S^{n+1}]^* \;\in\; \mathrm{H}^{n+1}(S^{n+1}; \mathbb{Z}).$$

Exercises for Chapter 74.

Exercise 74.1. Let X be a topological space which is homotopy equivalent to a retract of a finite CW-complex. Show that for every $n \in \mathbb{N}_0$ the cohomology group $\mathrm{H}^n(X;\mathbb{Z})$ is a finitely generated group.

Exercise 74.2. Show that the triad $(X, A_1, A_2) := (\mathbb{R}, (-\infty, 0], (0, \infty))$ is *not* excisive.

Exercise 74.3. Use cellular cohomology to determine the isomorphism types of the cohomology groups of the Klein bottle with $\mathbb{F}_2$-coefficients, with $\mathbb{F}_3$-coefficients and with $\mathbb{Z}$-coefficients.

Exercise 74.4. Let $\{A_k\}_{k\in K}$ be a family of CW-complexes. Suppose that for each $k \in K$ suppose that we are given a point in the 0-skeleton of A_k. Provide a proof for Proposition 74.17 using cellular cohomology. More precisely, provide a proof along the following lines:

(a) Equip $X := \bigvee\limits_{k\in K} A_k$ with the CW-structure provided by Lemma 36.32.
(b) Determine the relationship between the reduced cellular chain complexes of X and of the A_i.
(c) Make use of Proposition 49.10.
(d) Make use of Lemmas 73.2 and 73.7.

75. The Universal Coefficient Theorems for Cohomology Groups

75.1. **The Ext group.** One of the questions that arises from the last chapter is, how can we explain the erratic relationship between homology groups and cohomology groups that we saw on page 1845. Furthermore it would be interesting to know whether given a chain complex one can determine the cohomology groups with G-coefficients from the usual homology group.

These questions are purely in the realm of algebra. More precisely, let $(\mathrm{C}_*, \partial_*)$ be a chain complex of free abelian groups and let G an abelian group. We want to find a connection between the cohomology groups $\mathrm{H}^*(C; G)$ and the homology groups $\mathrm{H}_*(C)$. This problem, and later on its solution, is very similar to the problem we faced in Chapter 57 where we showed that we can determine homology groups of a chain complex with G-coefficients from the usual homology group of the chain complex.

As usual we denote by $\mathcal{AbGr}$ the category of abelian groups. On page 1405 we defined what it means for a *covariant* functor $F\colon \mathcal{AbGr} \to \mathcal{AbGr}$ to be exact, left-exact and right-exact. Basically the same way we can introduce these notions for *contravariant* functors $F\colon \mathcal{AbGr} \to \mathcal{AbGr}$.

Definition. A contravariant functor $F\colon \mathcal{AbGr} \to \mathcal{AbGr}$ is called *left-exact*[1089] if for all exact sequences

$$A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

of abelian groups the sequence

$$0 \longrightarrow F(C) \xrightarrow{F(j)} F(B) \xrightarrow{F(i)} F(A)$$

is also exact. Furthermore a contravariant functor $F\colon \mathcal{AbGr} \to \mathcal{AbGr}$ is called *right-exact*, if for all exact sequences

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C$$

the sequence

$$F(C) \xrightarrow{F(j)} F(B) \xrightarrow{F(i)} F(A) \longrightarrow 0$$

is also exact.

In Lemma 57.10 we saw that the covariant functor $- \otimes G$ is right-exact but that in general it is not left-exact. With the contravariant functor $\mathrm{Hom}(-, G)$ the situation is precisely the converse. More precisely, in the next lemma we will show that this functor is left-exact and immediately afterwards we will give an example that shows that the contravariant functor $\mathrm{Hom}(-, G)$ is in general not right-exact.

Lemma 75.1. *For any abelian group G the contravariant functor $\mathrm{Hom}(-, G)$ is left-exact.*

[1089] Admittedly one could also refer to this definition as "right"-exact, since the direction changes. But the definition we give is the one which is commonly used in the literature.

PROOF (*). Let G be an abelian group and let

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be an exact sequence of abelian groups. We need to show that the sequence

$$0 \longrightarrow \operatorname{Hom}(C,G) \xrightarrow{\beta^*} \operatorname{Hom}(B,G) \xrightarrow{\alpha^*} \operatorname{Hom}(A,G)$$

is also exact. It is clear that β^* is injective and that $\alpha^* \circ \beta^* = (\beta \circ \alpha)^* = 0$, i.e. we have $\operatorname{im}(\beta^*) \subset \ker(\alpha^*)$. It remains to show that $\ker(\alpha^*) \subset \operatorname{im}(\beta^*)$. So let $f\colon B \to G$ be a homomorphism such that $\alpha^*(f) = 0$. We consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] \arrow[d, "f"] \arrow[dr, "\pi"] & C \arrow[r] & 0. \\
& G & B/\alpha(A) \arrow[l, "\overline{f}", "f(b) \leftarrow\!\shortmid [b]"'] \arrow[u, "\overline{\beta}"', "{[b] \mapsto \beta(b)}"]
\end{tikzcd}$$

We make the following observations:

(1) Since $\alpha^*(f)$ we know that f vanishes on $\alpha(A)$. We denote by $\overline{f}\colon B/\alpha(A) \to G$ the resulting homomorphism given by $[b] \mapsto f(b)$.
(2) We denote $\pi\colon B \to B/\alpha(A)$ the projection.
(3) By the exactness of the original sequence the homomorphism β induces an isomorphism $\overline{\beta}\colon B/\alpha(A) \to C$.
(4) By construction the above diagram commutes.
(5) Now we see that $f = \overline{f} \circ \overline{\beta}^{-1} \circ \beta$, i.e. we have $f = \beta^*(\overline{f} \circ \overline{\beta}^{-1})$. ■

Now we give the example, promised above, that shows that the contravariant functor $\operatorname{Hom}(-,G)$ is in general not right-exact.

Example. We consider the chain complex

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{p} \mathbb{Z}_2$$

where $p\colon \mathbb{Z} \to \mathbb{Z}_2$ denotes the obvious projection map. We apply the contravariant functor $\operatorname{Hom}(-,\mathbb{F}_2)$. The resulting cochain complex is of the form[1090]

$$\mathbb{F}_2 \xrightarrow[\cong]{p^*} \mathbb{F}_2 \xrightarrow[=0]{(\cdot 2)^*} \mathbb{F}_2 \to 0.$$

Since this cochain complex is not exact it follows that the functor $\operatorname{Hom}(-,\mathbb{F}_2)$ is not right-exact.

In Chapter 57.4 we introduced the torsion groups $\operatorname{Tor}(H,G)$ which "measure" to what degree the functor $-\otimes G$ fails to be a left-exact functor. The same way we now introduce the so called Ext-groups $\operatorname{Ext}(H,G)$ which "measure" to what degree the functor $\operatorname{Hom}(-,G)$ fails to be right-exact.

[1090]Here we use the identification $\operatorname{Hom}(\mathbb{Z},G) = G$ from Lemma 73.1 and we use that multiplication by 2 is the zero endomorphism of $\mathbb{F}_2$.

Definition. Let G and H be abelian groups. We denote by

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

the canonical free resolution from page 1408. We define the *n-th G-Ext-group of H* as

$$\mathrm{Ext}_n(H,G) := \mathrm{H}^n(F_*;G).$$

Remark. Let $n \in \mathbb{N}_0$. On page 1413 we saw that $\mathrm{Tor}_n(H,G)$ is covariantly functorial in both G and H. Now we will see that the groups $\mathrm{Ext}_n(H,G)$ are contravariantly functorial in H and covariantly functorial in G:

(1) Let $\alpha\colon H \to H'$ be a homomorphism between abelian groups and let G be an abelian group. Then it follows from Lemmas 57.13 and 73.8 together with the definition of the G-Ext groups that α induces a homomorphism
$$\alpha^*\colon \mathrm{Ext}_n(H',G) \to \mathrm{Ext}_n(H,G).$$
It is now straightforward to show that
$$\begin{array}{rcl} H & \mapsto & \mathrm{Ext}_n(H,G) \\ (\alpha\colon H \to H') & \mapsto & (\alpha^*\colon \mathrm{Ext}_n(H',G) \to \mathrm{Ext}_n(H,G)) \end{array}$$
is a *contravariant* functor from the category of abelian groups to the category of abelian groups.

(2) On the other hand, if $\beta\colon G \to G'$ is a homomorphism between abelian groups then one sees easily that β induces a homomorphism
$$\beta_*\colon \mathrm{Ext}_n(H,G) \to \mathrm{Ext}_n(H,G')$$
Furthermore one can easily prove that
$$\begin{array}{rcl} G & \mapsto & \mathrm{Ext}_n(H,G) \\ (\beta\colon G \to G') & \mapsto & (\beta_*\colon \mathrm{Ext}_n(H,G) \to \mathrm{Ext}_n(H,G')) \end{array}$$
is a *covariant* functor from the category of abelian groups to the category of abelian groups.

The following proposition is the analogue of Proposition 57.12. The proof is almost the same as the proof of Proposition 57.12. We leave it to the reader to make the necessary changes.

Proposition 75.2. *Let G and H be abelian groups and let*

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

be a free resolution of H. Then for each $n \in \mathbb{N}_0$ there exists a canonical isomorphism

$$\mathrm{Ext}_n(H,G) \xrightarrow{\cong} \mathrm{H}^n(F_*,G).$$

In particular if

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

is a free resolution of H of length one, then there exist canonical isomorphisms

$$\operatorname{Ext}_1(H,G) \xrightarrow{\cong} \operatorname{coker}\left(\operatorname{Hom}(F_0,G) \xrightarrow{f_1^*} \operatorname{Hom}(F_1,G)\right)$$

and

$$\operatorname{Ext}_0(H,G) \xrightarrow{\cong} \ker\left(\operatorname{Hom}(F_0,G) \xrightarrow{f_1^*} \operatorname{Hom}(F_1,G)\right)$$

In Lemma 57.16 we saw that the higher torsion groups of abelian groups vanish. The same statement, with basically the same proof, also holds for Ext groups. More precisely, we have the following corollary.

Corollary 75.3. *For every choice of abelian groups G and H we have $\operatorname{Ext}_n(H,G)=0$ for $n\geq 2$.*

Proof. As we pointed out in the proof of Lemma 57.16 any abelian group H admits a free resolution of length 2:

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0.$$

The corollary now follows immediately from Proposition 75.2. ■

In Lemma 57.15 we saw that the 0-th Torsion functor is an old acquaintance. More precisely we saw that for all abelian groups G and H there exists a natural isomorphism $\operatorname{Tor}_0(H,G) \xrightarrow{\cong} H\otimes G$. The following lemma gives us the analogous result for the 0-th Ext functor.

Lemma 75.4. *Let G and H be abelian groups, then there exists a natural*[1091] *isomorphism*

$$\operatorname{Ext}_0(H,G) \xrightarrow{\cong} \operatorname{Hom}(H,G).$$

Proof (∗). The proof of Lemma 75.4 is, not surprisingly, rather similar to the proof of Lemma 57.15. Let G and H be abelian groups. Let

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

be the canonical free resolution of H. We know from Lemma 75.1 that the sequence

$$(*) \qquad 0 \to \operatorname{Hom}(H,G) \xrightarrow{f_0^*} \operatorname{Hom}(F_0,G) \xrightarrow{f_1^*} \operatorname{Hom}(F_1,G)$$

is exact. Now we obtain the following isomorphism

$$\begin{aligned}\operatorname{Ext}_0(H,G) &\overset{\substack{\text{by definition}\\ \downarrow}}{=} \mathrm{H}^0\left(\cdots \leftarrow \operatorname{Hom}(F_1,G) \xleftarrow{f_1^*} \operatorname{Hom}(F_0,G) \leftarrow 0\right)\\ &= \ker\left(\operatorname{Hom}(F_1,G) \xleftarrow{f_1^*} \operatorname{Hom}(F_0,G)\right) \underset{\substack{\uparrow\\ \text{isomorphism since } (*) \text{ is exact}}}{\xrightarrow{f_0^*}} \operatorname{Hom}(H,G).\end{aligned}$$

It is straightforward to verify that these isomorphisms are in fact natural. ■

[1091]Here by "natural" we mean that these isomorphisms define a natural isomorphism from the contravariant functor $H\mapsto \operatorname{Ext}_0(H,G)$ to the contravariant functor $H\mapsto \operatorname{Hom}(H,G)$ and that they also define a natural transformation from the covariant functor $G\mapsto \operatorname{Ext}_0(H,G)$ to the covariant functor $G\mapsto \operatorname{Hom}(H,G)$.

Definition. Given abelian groups G and H we refer to

$$\mathrm{Ext}(H,G) \;:=\; \mathrm{Ext}_1(H,G)$$

as *the* G-Ext-group of H.

Remark. Let G and H be two abelian groups. An *extension of* H *by* G is a short exact sequence of the form

$$0 \longrightarrow G \xrightarrow{\alpha} K \xrightarrow{\beta} H \longrightarrow 0.$$

We say that two such extensions given by K and K' are *equivalent* if there exists a homomorphism $\varphi\colon K \to K'$ that makes the following diagram commute

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r, "\alpha"] \arrow[d, "="] & K \arrow[r, "\beta"] \arrow[d, "\varphi"] & H \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & G \arrow[r, "\alpha'"] & K' \arrow[r, "\beta'"] & H \arrow[r] & 0.
\end{tikzcd}$$

In [**HS97**, Theorem II.3.4] it is shown that there is a natural bijection from the set of equivalence classes of extensions of H by G with the set $\mathrm{Ext}(H,G)$. This fact is the origin of the name "Ext-group".

In light of Corollary 75.3 and Lemma 75.4 the remaining task is to compute the Ext-group $\mathrm{Ext}(H,G) = \mathrm{Ext}_1(H,G)$. The following lemma allows us in particular to determine the Ext-group for all finitely generated abelian groups.

Lemma 75.5. *Let G, H and $G_1, \dots, G_k, \mathrm{H}_1, \dots, \mathrm{H}_k$ be abelian groups. The following statements hold:*

(1) *There exists a natural isomorphism*

$$\mathrm{Ext}\left(\bigoplus_{i=1}^{k} \mathrm{H}_i, G\right) \;\cong\; \bigoplus_{i=1}^{k} \mathrm{Ext}(\mathrm{H}_i, G).$$

(2) *There exists a natural isomorphism*

$$\mathrm{Ext}\left(H, \bigoplus_{i=1}^{k} G_i\right) \;\cong\; \bigoplus_{i=1}^{k} \mathrm{Ext}(H, G_i).$$

(3) *If H is a free abelian group, then* $\mathrm{Ext}(H,G) = 0$.
(4) *For all $n \in \mathbb{N}$ we have a natural isomorphism* $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/nG$.
(5) *If H is a finitely generated abelian group, then we have a natural isomorphism*

$$\mathrm{Ext}(H,\mathbb{Z}) \cong \textit{torsion subgroup of } H := \{h \in H \mid \textit{there exists an } n \in \mathbb{N} \textit{ with } nh = 0\}.$$

In particular $\mathrm{Ext}(H,\mathbb{Z})$ *is a finite group.*
(6) *For $n, m \in \mathbb{N}$ we have* $\mathrm{Ext}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(n,m)}$.

Remark.

(a) The first two statements of the lemma also generalize to the case of infinitely many groups, but some care needs to be taken. Indeed, some of the finite direct sums are secretly finite direct products and hence they generalize to infinite direct products. More precisely, the following holds:

(1') If $\{\mathrm{H}_i\}_{i\in I}$ is a family of abelian groups and if G is an abelian group, then
$$\operatorname{Ext}\Big(\bigoplus_{i\in I}\mathrm{H}_i, G\Big) \;\cong\; \prod_{i\in I}\operatorname{Ext}(\mathrm{H}_i, G).$$

(2') If H is an abelian group and if $\{G_i\}_{i\in I}$ is a family of abelian groups, then
$$\operatorname{Ext}\Big(H, \prod_{i\in I}G_i\Big) \;\cong\; \prod_{i\in I}\operatorname{Ext}(H, G_i).$$

We refer to [**Rot09**, Propositions 7.21 and 7.22] for a proof. We will not make use of these more general statements.

(b) The statements of Lemma 75.5 look very similar to the statements of Lemma 57.17. But there are some differences:
   (i) In Lemma 57.17 we saw that for every abelian group we have $\operatorname{Tor}(H,\mathbb{Z}) = 0$ whereas we have now shown in Lemma 75.5 (5) that for every finite abelian group we have $\operatorname{Ext}(H,\mathbb{Z}) \cong H$.
   (ii) In Lemma 57.17 we saw that $\operatorname{Tor}(H,G) = 0$ not only whenever H is a free abelian group but also for all torsion-free abelian groups H. Here the situation is different, the statement of Lemma 75.5 (3) does not generalize to torsion-free abelian groups. For example in [**Rot09**, p. 428] it is shown that there exists an abelian group G such that $\operatorname{Ext}(\mathbb{Q}, G) \neq 0$. But this issue is of no concern to us.

PROOF. The first three statements are proved almost the same way as we proved Lemma 57.17. We leave it to the reader to fill in the details. We turn to the final three statements.

(4) Let G be an abelian group and let $n \in \mathbb{N}$. We have
$$\operatorname{Ext}(\mathbb{Z}_n, G) \underset{\uparrow}{\overset{\substack{\text{Proposition 75.2 applied to the free resolution } 0\to\mathbb{Z}\xrightarrow{\cdot n}\mathbb{Z}\to\mathbb{Z}_n\to 0\\ \downarrow}}{=}} \operatorname{coker}\Big(\operatorname{Hom}(\mathbb{Z},G)\xrightarrow{(\cdot n)^*}\operatorname{Hom}(\mathbb{Z},G)\Big) \underset{\substack{\uparrow\\ \text{by the isomorphism } \operatorname{Hom}(\mathbb{Z},G)=G \text{ given by } f\mapsto f(1)}}{=} \operatorname{coker}\Big(G\xrightarrow{\cdot n}G\Big) = G/nG.$$

(5) This statement follows immediately from (1), (3) and (4) together with the classification of finitely generated abelian groups, see Theorem 19.4.

(6) This statement follows immediately from (4) and the calculation on page 1409. ■

We have one more general result for calculating Ext-groups. Before we can state the result we need one more definition.

Definition. An abelian group G is called *divisible* if for every $g \in G$ and $n \in \mathbb{N}$ there exists an $h \in G$ with $n \cdot h = g$.

Examples.

(1) The groups $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are divisible.

(2) Every quotient of a divisible group is easily seen to be divisible again. For example the quotient group $\mathbb{Q}/\mathbb{Z}$ is also divisible.[1092] This example also shows, using the

[1092] Odd as it may sound, this example will actually be important later in Section ??.

notation of the definition, that h is not necessarily unique. For example, for $g = [0]$ and $n = 3$ we could take $g = [0]$ or $g = [\frac{1}{3}]$.

(3) The group $\mathbb{Z}$ is not divisible, and the only finite group that is divisible is the trivial group.

Proposition 75.6. *Let G and H be abelian groups. If G is divisible, then* $\operatorname{Ext}(H, G) = 0$.

In the proof of Proposition 75.6 it is useful to introduce the following notion.

Definition. An abelian group G is called *injective* if for all abelian groups A and B, every injective homomorphism $\varphi\colon B \to A$ and every homomorphism $\beta\colon B \to G$ there exists a homomorphism $\alpha\colon A \to G$ such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, hook, "\varphi"] \arrow[d, "\beta"'] & A \arrow[dl, dashed, "\alpha"] \\ & G. & \end{tikzcd}$$

The following lemma shows why we are interested in injective groups.

Lemma 75.7. *Let G and H be abelian groups. If G is injective, then* $\operatorname{Ext}(H, G) = 0$.

Proof. Let H be an abelian group. As we pointed out in the proof of Lemma 57.16 the abelian group H admits a free resolution of length 2:

$$0 \;\to\; F_1 \;\xrightarrow{\varphi_1}\; F_0 \;\xrightarrow{\varphi_0}\; H \;\to\; 0.$$

By Proposition 75.2 the group $\operatorname{Ext}(H, G)$ is isomorphic to the cokernel of the map

$$\operatorname{Hom}\begin{pmatrix} F_1 \\ \downarrow \\ G \end{pmatrix} \;\xleftarrow{\varphi_1^*}\; \operatorname{Hom}\begin{pmatrix} F_0 \\ \downarrow \\ G \end{pmatrix}.$$

But by definition of G being injective this map φ_1^* is in fact an epimorphism. Thus we have shown that $\operatorname{Ext}(H, G) = 0$. ■

To provide a proof for Proposition 75.6 it remains to prove the following proposition.

Proposition 75.8. *Every abelian group that is divisible is also injective.*

Remark. It is straightforward to prove the converse to Proposition 75.8, namely every injective abelian group is also divisible.

In the proof of Proposition 75.8 we will need Zorn's Lemma which we now recall for the reader's convenience.

Lemma 75.9. (Zorn's Lemma) *Suppose a partially ordered set $(P, \leq)$ has the property that every chain has an upper bound in P. Then the set P contains at least one maximal element.*

We also recall the definitions in the statement of Zorn's Lemma:

(1) A *partially ordered set* $(P, \leq)$ is a set together with a preorder $\leq$ (see page 726) that has the property that $a \leq b$ and $b \leq a$ implies that $a = b$.

(2) A *chain* in P is a subset Q such that for any two elements $q, q' \in Q$ we have $q \leq q'$ or $q' \leq q$.

(3) An *upper bound* of a subset Q is an element $q \in Q$ such that $p \leq q$ for all $q \in Q$.

(4) A *maximal element* of a subset Q is an element $q \in Q$ such that there is no $r \in Q$ with $q < r$.[1093]

We refer to [**Cie97**, Theorem 4.3.4] for a proof of Zorn's Lemma. Perhaps it is more fun to look at Zorn's Lemma in "The Simpsons", see

http://www.simonsingh.net/Simpsons_Mathematics/zorns-lemma/

Now we can give the proof of Proposition 75.8.

Proof of Proposition 75.8 (*). Let G be an abelian group that is divisible. Suppose we are given a diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, hook, "\varphi"] \arrow[d, "\beta"'] & A \arrow[dl, "?\,\alpha\,?"] \\ & G. & \end{tikzcd}$$

Since φ is a monomorphism we can assume that B is in fact a subgroup of A and that φ is the inclusion map.

We denote by P the set of all pairs (C_i, γ_i) where C_i is a subgroup of A which contains B and where $\gamma_i\colon \mathrm{C}_i \to G$ is a homomorphism such that $\gamma_i|_B = \beta$. This family of pairs is non-empty since (B, β) satisfies the required conditions. We define $(\mathrm{C}_i, \gamma_i) < (\mathrm{C}_j, \gamma_j)$ if $\mathrm{C}_i \subset \mathrm{C}_j$ and if $\gamma_j|_{\mathrm{C}_i} = \gamma_i$. Given a chain $(\mathrm{C}_j, \gamma_j)_{j\in J}$ in P we get an upper bound by setting

$$C \;:=\; \bigcup_{j\in J} \mathrm{C}_j \quad\text{and}\quad \begin{array}{rcl} \gamma\colon C & \to & G \\ c & \mapsto & \gamma_j(c) \text{ if } j \in J \text{ with } c \in \mathrm{C}_j. \end{array}$$

Since $(\mathrm{C}_j, \gamma_j)_{j\in J}$ is a chain one can easily verify that C is in fact a subgroup of A which contains B and that γ is a homomorphism with $\gamma|_B = \beta$.

Therefore we can apply Zorn's Lemma to this family and we obtain a maximal pair (C_m, γ_m). It remains to prove the following claim.

Claim. We have $\mathrm{C}_m = A$.

Suppose that this is not the case. Then there exists an $a \in A \setminus \mathrm{C}_m$. We denote by $\langle\!\langle \mathrm{C}_m, a\rangle\!\rangle$ the subgroup of A generated by C_m and a, i.e. $\langle\!\langle \mathrm{C}_m, a\rangle\!\rangle$ is the smallest subgroup of A that contains C_m and a. To obtain a contradiction it suffices to show that there exists a homomorphism $\delta\colon \langle\!\langle \mathrm{C}_m, a\rangle\!\rangle \to G$ that extends γ_m. We distinguish two cases:

(1) If there exists no $n \in \mathbb{N}$ such that $n \cdot a \in \mathrm{C}_m$, then one can show easily that we have $\langle\!\langle \mathrm{C}_m, a\rangle\!\rangle = \mathrm{C}_m \oplus \langle a\rangle$. We extend γ_m to $\langle\!\langle \mathrm{C}_m, a\rangle\!\rangle = \mathrm{C}_m \oplus \langle a\rangle$ by setting $\delta(a) = 0$.
(2) Otherwise there exists a minimal $n \in \mathbb{N}$ such that $n \cdot a \in \mathrm{C}_m$. We denote by $r \in \mathbb{N}_0$ the order of $n \cdot a \in \mathrm{C}_m$. Here we say that the order is 0 if $n \cdot a \in \mathrm{C}_m$ has infinite order. We consider the maps

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}_r & \xrightarrow{\Phi} & \mathrm{C}_m \oplus \mathbb{Z}_{nr} & \xrightarrow{\Psi} & \langle\!\langle \mathrm{C}_m, a\rangle\!\rangle & \to & 0 \\ & & k & \mapsto & (k \cdot n \cdot a, -k \cdot n) & & & & \\ & & & & (c, l) & \mapsto & c + l \cdot a & & \end{array}$$

[1093] Aren't "upper bound" and "maximal element" the same?

It is clear that Φ is injective and that $\Psi \circ \Phi = 0$. Note that the image of Ψ lies a priori in A, but it is straightforward to see that the image of Ψ is precisely $\langle\langle \mathrm{C}_m, a\rangle\rangle$. By definition of n we have that $\ker(\Psi) \subset \operatorname{im}(\Phi)$. Put differently, the sequence is exact. Since G is divisible there exists an $h \in G$ such that $n \cdot h = \gamma(n \cdot a)$. We denote by $\alpha\colon \mathrm{C}_m \oplus \mathbb{Z}_{nr} \to G$ the homomorphism that is given by setting $\alpha(c) = \gamma(c)$ for $c \in \mathrm{C}_m$ and $\alpha(k) = k \cdot h$ for $k \in \mathbb{Z}$. One can easily verify that $\operatorname{im}(\Phi)$ is contained in $\ker(\alpha)$. It is now straightforward to show that the homomorphism

$$\begin{array}{rcccl} \delta\colon \langle\langle \mathrm{C}_m, a\rangle\rangle & \to & (\mathrm{C}_m \oplus \mathbb{Z}_{nr})/\operatorname{im}(\Phi) & \to & G \\ x & \mapsto & [\Psi^{-1}(x)] & & \\ & & [y] & \mapsto & \alpha(y) \end{array}$$

has the desired properties.

This concludes the proof of the claim and thus of the lemma. ■

The following proposition gives arguably slightly weird example.

Proposition 75.10. *The group* $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z})$ *is isomorphic to the additive group* $(\mathbb{R}, +)$.

Remark. Note that this proposition is not totally pointless, for example it follows from the existence of Moore spaces, see Proposition 47.11, that there exist topological spaces X with $\mathrm{H}_n(X) \cong (\mathbb{Q}, +)$. Thus even unexpected abelian groups can appear in topology.

Proof. This proposition is proved in [**Wie69**]. Alternatively the proposition is almost proved in [**Bre97**, Lemma 14.8]. We also refer to [**Boar10**] for the calculation of $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z})$. ■

This example also shows that Ext-groups can be quite large. In fact the following proposition holds.

Proposition 75.11. *Let G be an abelian group. If G is not a finitely generated group, then* $\operatorname{Hom}(G, \mathbb{Z})$ *is uncountable or* $\operatorname{Ext}(G, \mathbb{Z})$ *is uncountable.*

Proof. We will barely make use of this proposition, thus we refer to [**Hat02**, Proposition 3.F.12] instead for a proof. ■

75.2. **The Universal Coefficient Theorem for Cohomology Groups.** Finally we want to explore the connection between the cohomology groups of a chain complex C and the usual homology groups of the chain complex. The key to relating homology groups to cohomology groups is given by the following evaluation homomorphism.

Definition. Let $(\mathrm{C}_*, \partial_*)$ be a chain complex and let G be an abelian group. We refer to the map[1094]

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^n(C; G) & \to & \operatorname{Hom}(\mathrm{H}_n(C), G) \\ [\varphi\colon \mathrm{C}_n \to G] & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_n(C) & \to & G \\ [c] & \mapsto & \langle [\varphi], [c] \rangle = \varphi(c) \end{array} \right) \end{array}$$

as the *evaluation homomorphism.*

[1094] This map is well-defined by Lemma 74.6.

Remark. It follows immediately from Lemma 74.6 (3) that the evaluation homomorphisms in natural in the chain complex C_* in the sense that if $f\colon C_* \to D_*$ is a chain map, then for any $n \in \mathbb{N}_0$ and any abelian group G the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}^n(C;G) & \xrightarrow{\text{ev}} & \mathrm{Hom}(\mathrm{H}_n(C),G) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (f_*)^*} \\ \mathrm{H}^n(D;G) & \xrightarrow{\text{ev}} & \mathrm{Hom}(\mathrm{H}_n(D),G). \end{array}$$

Furthermore, it follows easily from the definitions that the evaluation map is natural in G, in the sense that if $\varphi\colon G_1 \to G_2$ is homomorphism of abelian groups, then for any $n \in \mathbb{N}_0$ and any chain complex C_* the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}^n(C;G_1) & \xrightarrow{\text{ev}} & \mathrm{Hom}(\mathrm{H}_n(C),G_1) \\ {\scriptstyle \varphi_*}\downarrow & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{H}^n(C;G_2) & \xrightarrow{\text{ev}} & \mathrm{Hom}(\mathrm{H}_n(C),G_2). \end{array}$$

Here the map on the left is the one we had introduced on page 1823.

Recall that in Lemma 46.9 we saw that $\mathrm{H}_2(\mathbb{RP}^2;\mathbb{Z}) = 0$ and that on page 1845 we showed that $\mathrm{H}^2(\mathbb{RP}^2;\mathbb{Z}) \cong \mathbb{Z}_2$. These calculations imply that the evaluation homomorphism $\text{ev}\colon \mathrm{H}^n(X;G) \to \mathrm{Hom}(\mathrm{H}_n(X);G)$ is in general not a monomorphism. The following theorem relates homology groups and cohomology groups via the evaluation homomorphism and the Ext-groups.

Theorem 75.12. (Universal Coefficient Theorem for Cohomology Groups) *Let $(\mathrm{C}_*,\partial_*)$ be a chain complex of free abelian groups and let G be an abelian group. Then for each $n \in \mathbb{N}_0$ there exists a short exact sequence*

$$0 \;\to\; \mathrm{Ext}(\mathrm{H}_{n-1}(C),G) \;\to\; \mathrm{H}^n(C;G) \;\xrightarrow{\text{ev}}\; \mathrm{Hom}(\mathrm{H}_n(C),G) \to 0$$

which is natural in the chain complex C_* and natural in the abelian group G. This short exact sequence splits and there exists therefore an isomorphism[1095]

$$\mathrm{H}^n(C;G) \;\cong\; \mathrm{Ext}(\mathrm{H}_{n-1}(C),G) \oplus \mathrm{Hom}(\mathrm{H}_n(C),G).$$

Before we provide the proof of the above Universal Coefficient Theorem for Cohomology Groups 75.12 let us formulate the most important application, namely let us apply it to the chain complex of a pair of topological spaces.

Theorem 75.13. (Universal Coefficient Theorem for Cohomology Groups) *Let (X,A) be a pair of topological spaces and let G be an abelian group. Then for each $n \in \mathbb{N}_0$ there exists a short exact sequence*

$$0 \;\to\; \mathrm{Ext}(\mathrm{H}_{n-1}(X,A),G) \;\to\; \mathrm{H}^n(X,A;G) \;\xrightarrow{\text{ev}}\; \mathrm{Hom}(\mathrm{H}_n(X,A),G) \to 0$$

[1095] Once again the short exact sequence is natural, whereas the isomorphism is not natural. More precisely, the discussion on page 1425 can easily be modified to show that there exists no natural isomorphism

$$\mathrm{H}^n(C;G) \;\cong\; \mathrm{Ext}(\mathrm{H}_{n-1}(C),G) \oplus \mathrm{Hom}(\mathrm{H}_n(C),G).$$

We will discuss this issue in Exercise 75.1.

which is natural in the chain complex C_* and natural in the abelian group G. This short exact sequence splits and there exists therefore an isomorphism

$$\mathrm{H}^n(X,A;G) \;\cong\; \mathrm{Ext}(\mathrm{H}_{n-1}(X,A),G) \oplus \mathrm{Hom}(\mathrm{H}_n(X,A),G).$$

Remark. The Universal Coefficient Theorem 75.13 for Cohomology Groups is the analogue of the Algebraic Universal Coefficient Theorem 57.18 together with Theorem 57.24.

Proof of Theorem 75.13 assuming Theorem 75.12. As the reader will have noticed there is really not much to say, except that we recall that we pointed out on page 1120 that the chain groups $\mathrm{C}_n(X,A)$, $n \in \mathbb{N}_0$ are free abelian groups. ■

Remark.

(1) Here is one straightforward but interesting consequence to the Universal Coefficient Theorem 75.13 for Cohomology Groups. Given any pair (X,A) of topological spaces we see that

$$\mathrm{H}^1(X;\mathbb{Z}) \underset{\substack{\uparrow\\ \text{Universal Coefficient}\\ \text{Theorem 75.13}}}{\cong} \mathrm{Ext}(\mathrm{H}_0(X,A),\mathbb{Z}) \oplus \mathrm{Hom}(\mathrm{H}_1(X,A),\mathbb{Z}) \underset{\substack{\uparrow\\ \text{by Lemma 75.5 and Corollary 41.15}}}{=} \mathrm{Hom}(\mathrm{H}_1(X,A),\mathbb{Z}) \underset{\substack{\uparrow\\ \text{since Hom}(G,\mathbb{Z})\text{ is}\\ \text{always torsion free}}}{=} \text{torsion-free.}$$

(2) The Universal Coefficient Theorem 75.13 for Cohomology Groups implies in particular that topological spaces with isomorphic homology groups also have isomorphic cohomology groups. In other words, if our only goal is to distinguish homeomorphism types of topological spaces, then cohomology groups will not be of much help.

(3) The fact that cohomology groups are determined by homology groups is also immediate consequence of the combination of Propositions 49.1 and 49.2 together with Lemma 73.5.

Now we turn to the proof of the purely algebraic Universal Coefficient Theorem 75.12 for Cohomology Groups. The proof is, perhaps not surprisingly, rather similar to the proofs of the Universal Coefficient Theorem 57.19 and Proposition 57.23.

Proof (∗). Let $(\mathrm{C}_*,\partial_*)$ be a chain complex of free abelian groups. For each $n \in \mathbb{N}_0$ we write as always $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$. Note that it follows from Lemma 19.2 that B_n and Z_n are free abelian groups.

As in the proof of the Universal Coefficient Theorem 57.19 we consider the following short exact sequence of vertical chain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "0"] & \mathrm{C}_n \arrow[r, "\partial_n"] \arrow[d, "\partial_n"] & B_{n-1} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] \arrow[d] & \mathrm{C}_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Now let G be an abelian group. We apply the functor $\operatorname{Hom}(-,G)$ to the above short exact sequence of chain complexes and we obtain the following diagram:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \operatorname{Hom}(B_{n-1},G) \arrow[u] \arrow[r, "\partial_n^*"] & \operatorname{Hom}(C_n,G) \arrow[u] \arrow[r] & \operatorname{Hom}(Z_n,G) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Hom}(B_{n-2},G) \arrow[u, "0"] \arrow[r, "\partial_{n-1}^*"] & \operatorname{Hom}(C_{n-1},G) \arrow[u, "\partial_n^*"] \arrow[r] & \operatorname{Hom}(Z_{n-1},G) \arrow[u, "0"] \arrow[r] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

This is again a sequence of cochain complexes. Since the Z_n are free abelian we obtain from Lemma 73.4 that the above horizontal sequences of cochain complexes are still exact. Therefore we can apply Proposition 73.6 to obtain the following long exact sequence

$$\to \operatorname{Hom}(Z_{n-1},G) \xrightarrow{d_n} \operatorname{Hom}(B_{n-1},G) \to \mathrm{H}^n(C;G) \xrightarrow{\partial_n} \operatorname{Hom}(Z_n,G) \xrightarrow{d_{n-1}} \operatorname{Hom}(B_n,G) \to \dots$$

where the maps d_n are the connecting homomorphisms of the long exact sequence. One can show easily that the connecting homomorphism $d_n\colon \operatorname{Hom}(Z_n,G) \to \operatorname{Hom}(B_n,G)$ is the map $d_n = i_n^*$ where $i_n\colon B_n \to Z_n$ is the inclusion map. By Lemma 46.4 we obtain the following short exact sequence

$$0 \to \operatorname{coker}(\operatorname{Hom}(Z_{n-1},G) \to \operatorname{Hom}(B_{n-1},G)) \to \mathrm{H}^n(C;G) \to \ker(\operatorname{Hom}(Z_n,G) \to \operatorname{Hom}(B_n,G)) \to 0.$$

We have to show that the expressions left and right coincide with the desired expressions. Both equalities follow immediately from Proposition 75.2 (together with Lemma 75.4 for the right-hand term) and the observation that for each $l \in \mathbb{N}_0$ a free resolution of $\mathrm{H}_l(C)$ is given by $0 \to B_l \xrightarrow{i_l} Z_l \to \mathrm{H}_l(C)$. We have thus shown the existence of the desired short exact sequence

$$0 \ \to \ \operatorname{Ext}(\mathrm{H}_{n-1}(C),G) \ \to \ \mathrm{H}^n(C;G) \ \xrightarrow{\mathrm{ev}} \ \operatorname{Hom}(\mathrm{H}_n(C),G) \to 0.$$

The statement that the short exact sequence is natural follows easily from the definitions and Lemma 57.13 (2). We leave the verification of the details to the reader.

It remains to show that the above short exact sequence splits. By definitions this means that we need to construct a homomorphism $\eta\colon \operatorname{Hom}(\mathrm{H}_n(C),G) \to \mathrm{H}^n(C;G)$ such that $\mathrm{ev} \circ \eta$ is the identity on $\operatorname{Hom}(\mathrm{H}_n(C),G)$.

We consider again the short exact sequence

$$0 \ \to \ Z_n \ \hookrightarrow \ \mathrm{C}_n \ \xrightarrow{\partial_n} \ B_{n-1} \ \to \ 0.$$

Since B_{n-1} is a free abelian group we obtain from Lemmas 46.1 and 46.2 that there exists a homomorphism $p_n\colon \mathrm{C}_n \to Z_n$ such that p_n is the identity on $Z_n \subset C_n$.

In the proof of Proposition 57.23 we showed that the following diagram commutes:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{C}_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift left, "p_{n+1}"] \arrow[dr, "\partial_{n+1}"'] & \mathrm{C}_n \arrow[r, "\partial_n"] \arrow[d, shift left, "p_n"] \arrow[dr, "\partial_n"'] & \mathrm{C}_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, shift left, "p_{n-1}"] & \dots \\
& Z_{n+1} \arrow[u, hook, shift left] \arrow[d, two heads, "q_{n+1}"] & Z_n \arrow[u, hook, shift left] \arrow[d, two heads, "q_n"] & Z_{n-1} \arrow[u, hook, shift left] \arrow[d, two heads, "q_{n-1}"] & \\
\dots \arrow[r] & \underbrace{\mathrm{H}_{n+1}(C)}_{=Z_{n+1}/B_{n+1}} \arrow[r, "0"] & \underbrace{\mathrm{H}_n(C)}_{=Z_n/B_n} \arrow[r, "0"] & \underbrace{\mathrm{H}_{n-1}(C)}_{=Z_{n-1}/B_{n-1}} \arrow[r, "0"] & \dots
\end{tikzcd}$$

where the maps q_n are just the obvious projection maps.

The top and the bottom horizontal sequences of the above diagram are chain complexes. Since the diagram commutes we see that the vertical maps from top to bottom are chain maps.

Now we apply the functor $\mathrm{Hom}(-,G)$ to the top and the bottom horizontal sequences and we obtain the following diagram of cochain complexes.

$$\begin{tikzcd}
\dots & \mathrm{Hom}(C_{n+1},G) \arrow[l] & \mathrm{Hom}(C_n,G) \arrow[l, "\partial_{n+1}^*"'] & \mathrm{Hom}(C_{n-1},G) \arrow[l, "\partial_n^*"'] & \dots \arrow[l] \\
\dots & \mathrm{Hom}(\mathrm{H}_{n+1}(C),G) \arrow[l] \arrow[u, "(q_{n+1}\circ p_{n+1})^*"'] & \mathrm{Hom}(\mathrm{H}_n(C),G) \arrow[l, "0"'] \arrow[u, "(q_n\circ p_n)^*"'] & \mathrm{Hom}(\mathrm{H}_{n-1}(C),G) \arrow[l, "0"'] \arrow[u, "(q_{n-1}\circ p_{n-1})^*"'] & \dots \arrow[l]
\end{tikzcd}$$

The induced maps on the cohomology groups of these two cochain complexes give us a homomorphism

$$\eta\colon\ \mathrm{Hom}(\mathrm{H}_n(C),G) = \mathrm{H}^n(\text{lower sequence}) \ \to\ \mathrm{H}^n(\text{upper sequence}) = \mathrm{H}^n(C;G).$$

It remains to prove the following claim.

Claim. The composition $\mathrm{ev}\circ\eta$ is the identity on $\mathrm{Hom}(\mathrm{H}_n(C),G)$.

Let $\varphi\colon\ \mathrm{H}_n(C)\to G$ be homomorphism. We need to show that $(\mathrm{ev}\circ\eta)(\varphi)=\varphi$ in $\mathrm{Hom}(\mathrm{H}_n(C),G)$. So let $[\sigma]\in\mathrm{H}_n(C)$ with $\sigma\in Z_n$. Then

$$\big((\mathrm{ev}\circ\eta)(\varphi)\big)([\sigma]) \underset{\substack{\uparrow\\ \text{definition of ev}}}{=} \langle\, \overbrace{\eta(\varphi)}^{\in\mathrm{H}^n(C;G)}, [\sigma]\rangle \underset{\substack{\uparrow\\ \text{definition of }\eta\text{ and }\langle\,,\,\rangle}}{=} (\overbrace{\varphi\circ q_n\circ p_n}^{\in\mathrm{Hom}(C_n,G)})(\sigma) \underset{\substack{\uparrow\\ \text{since }p_n=\mathrm{id}\text{ on }Z_n}}{=} (\varphi\circ q_n)(\sigma) \underset{\substack{\uparrow\\ \text{since }q_n(\sigma)=[\sigma]}}{=} \varphi([\sigma]). \quad \blacksquare$$

Example. We consider again the projective space $P:=\mathbb{R}\mathrm{P}^2$. As a reminder, we showed on page 1216 that $\mathrm{H}_2(P)=0$, $\mathrm{H}_1(P)\cong\mathbb{Z}_2$ and $\mathrm{H}_0(P)\cong\mathbb{Z}$. It follows from the Universal Coefficient Theorem 75.13 and from the properties of the Ext-group that we had listed in Lemma 75.5 that

$$\begin{array}{llll}
\mathrm{H}^2(P;\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_2(P),\mathbb{Z})\oplus\mathrm{Ext}(\mathrm{H}_1(P),\mathbb{Z}) & \cong \mathrm{Hom}(0,\mathbb{Z})\oplus\mathrm{Ext}(\mathbb{Z}_2,\mathbb{Z}) & = 0\oplus\mathbb{Z}_2=\mathbb{Z}_2 \\
\mathrm{H}^1(P;\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_1(P),\mathbb{Z})\oplus\mathrm{Ext}(\mathrm{H}_0(P),\mathbb{Z}) & \cong \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z})\oplus\mathrm{Ext}(\mathbb{Z},\mathbb{Z}) & = 0\oplus 0=0 \\
\mathrm{H}^0(P;\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_0(P),\mathbb{Z}) & \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) & \cong \mathbb{Z}
\end{array}$$

and

$$\begin{array}{llll} \mathrm{H}^2(P;\mathbb{Z}_2) \cong \mathrm{Hom}(\mathrm{H}_2(P),\mathbb{Z}_2)\oplus\mathrm{Ext}(\mathrm{H}_1(P),\mathbb{Z}_2) & \cong \mathrm{Hom}(0,\mathbb{Z}_2)\oplus\mathrm{Ext}(\mathbb{Z}_2,\mathbb{Z}_2) & = 0\oplus\mathbb{Z}_2 = \mathbb{Z}_2 \\ \mathrm{H}^1(P;\mathbb{Z}_2) \cong \mathrm{Hom}(\mathrm{H}_1(P),\mathbb{Z}_2)\oplus\mathrm{Ext}(\mathrm{H}_0(P),\mathbb{Z}_2) & \cong \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}_2)\oplus\mathrm{Ext}(\mathbb{Z},\mathbb{Z}_2) & = \mathbb{Z}_2\oplus 0 = \mathbb{Z}_2 \\ \mathrm{H}^0(P;\mathbb{Z}_2) \cong \mathrm{Hom}(\mathrm{H}_0(P),\mathbb{Z}_2) & \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_2) & = \mathbb{Z}_2. \end{array}$$

Fortunately this coincides with the calculations from page 1845.

Example. We return to the study of lens spaces. In Lemma 66.9 we saw that

$$\mathrm{H}_n(L(p,q);\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n=0,3, \\ 0, & \text{if } n=2 \text{ or } n>3, \\ \mathbb{Z}_p, & \text{if } n=1. \end{cases}$$

As in the previous example we can now use the Universal Coefficient Theorem 75.13 and Lemma 75.5 to compute the cohomology groups of $L(p,q)$. We leave it to the reader to verify that[1096]

$$\mathrm{H}^n(L(p,q);\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n=0,3 \\ 0, & \text{if } n=1 \text{ or } n>3, \\ \mathbb{Z}_p, & \text{if } n=2. \end{cases}$$

This shows, as was already foreshadowed by Corollary 73.19, that cohomology groups also cannot distinguish between two lens spaces of the form $L(p,q)$ and $L(p,r)$. In particular we have not made any progress on Question 16.7 since the introduction of the fundamental group. All the extra invariants, namely higher homotopy groups, homology groups and cohomology groups (with any coefficients) have proved useless.

We conclude this section with the following proposition that gives in particular a converse to Corollary 73.20.

Proposition 75.14.

(1) *Let C_* be a chain complex. The following two statements are equivalent:*
   (a) $\mathrm{H}_k(C_*)=0$ *for all* $k\in\mathbb{N}_0$.
   (b) $\mathrm{H}^k(C_*;\mathbb{Z})=0$ *for all* $k\in\mathbb{N}_0$.

(2) *Let $f\colon D_*\to E_*$ be a chain map. The following two statements are equivalent:*
   (a) *All the induced map* $f_*\colon\ \mathrm{H}_k(D_*)\to\mathrm{H}_k(E_*)$ *are isomorphisms.*
   (b) *All the induced map* $f^*\colon\ \mathrm{H}^k(E_*;\mathbb{Z})\to\mathrm{H}^k(D_*;\mathbb{Z})$ *are isomorphisms.*

PROOF.

(1) We start out with the following claim.

   Claim. Let A be an abelian group. If A is non-trivial, then $\mathrm{Hom}(A,\mathbb{Z})$ or $\mathrm{Ext}(A,\mathbb{Z})$ is non-trivial.

   Let A be a non-trivial abelian group. If A is finitely generated, then it follows from the Classification 19.4 of Finitely Generated Abelian Groups together with Lemma 75.5 that $\mathrm{Hom}(A,\mathbb{Z})$ or $\mathrm{Ext}(A,\mathbb{Z})$ is non-trivial. Furthermore, if A is infinitely generated, then the desired conclusion follows from Proposition 75.11. ⊞

[1096] Note that the $\mathbb{Z}_p$ shifted from $n=1$ in homology to $n=2$ in cohomology.

The equivalence of (a) and (b) follows immediately from the claim together with the Universal Coefficient Theorem 75.12 for Cohomology Groups.

(2) Let $f\colon D_* \to E_*$ be a chain map. We denote by $(\mathrm{M}(f), \partial_*)$ the corresponding algebraic mapping cone as defined on page 1284. By Lemma 49.7 there exists a long exact sequence of the form

$$\ldots \to \mathrm{H}_k(D_*) \xrightarrow{f_*} \mathrm{H}_k(E_*) \to \mathrm{H}_k(\mathrm{M}(f)) \to \mathrm{H}_{k-1}(D_*) \xrightarrow{f_*} \ldots$$

Basically the same argument also shows that there exists a long exact sequence of the form

$$\ldots \leftarrow \mathrm{H}^k(D_*;\mathbb{Z}) \xleftarrow{f^*} \mathrm{H}_k(E_*;\mathbb{Z}) \leftarrow \mathrm{H}_n(\mathrm{M}(f);\mathbb{Z}) \leftarrow \mathrm{H}_{n-1}(D_*;\mathbb{Z}) \xrightarrow{f^*} \ldots$$

Thus we see that the equivalence of (2a) and (2b) is an immediate consequence of the equivalence of (1a) and (1b) applied to $C_* = \mathrm{M}(f)$. ■

75.3. **Cohomology groups of topological manifolds.**

Proposition 75.15. *Let M be a connected n-dimensional topological manifold. Let G be an abelian group.*

(1) *Let $x \in M \setminus \partial M$. For $i \neq n$ we have $\mathrm{H}_i(M, M \setminus \{x\}; G) = 0$. Furthermore the map*

$$\mathrm{ev}\colon\ \mathrm{H}^n(M, M\setminus\{x\}; G) \ \to\ \mathrm{Hom}(\mathrm{H}_n(M, M\setminus\{x\};\mathbb{Z}), G)$$

is an isomorphism.

(2) *If M is compact but not closed, then $\mathrm{H}^n(M;G) = 0$.*
(3) *If M is closed orientable and non-empty, then $\mathrm{H}^n(M;G) \cong G$.*
(4) *If M is closed non-orientable and non-empty, then $\mathrm{H}^n(M;G) \cong G/2G$.*
(5) *For any $k > n$ we have $\mathrm{H}^k(M;G) = 0$.*

Proof. Let M be a compact connected n-dimensional topological manifold and let G be an abelian group.

(1) This statement follows immediately from Lemma 86.4 and the Universal Coefficient Theorem 75.13.
(2) We suppose that M has non-empty boundary. Then

$$\mathrm{H}^n(M;G) \underset{\substack{\uparrow \\ \text{Universal Coefficient} \\ \text{Theorem 75.13}}}{\cong} \mathrm{Hom}(\mathrm{H}_n(M;\mathbb{Z}),G) \oplus \mathrm{Ext}(\mathrm{H}_{n-1}(M;\mathbb{Z}),G) \underset{\substack{\uparrow \\ \text{by Theorem 87.3 we have } \mathrm{H}_n(M;\mathbb{Z})=0 \text{ and} \\ \text{by Proposition 87.22 (1) we have } \mathrm{H}_{n-1}(M;\mathbb{Z})\cong\mathbb{Z}^k \\ \text{for some } k\in\mathbb{N}_0}}{=} \mathrm{Ext}(\mathbb{Z}^k,G) \underset{\substack{\uparrow \\ \text{Lemma 75.5}}}{=} 0.$$

(3) Suppose that M is closed orientable and non-empty. The argument is very similar to the proof of (2). We just need to use that we know from Theorem 87.1 that $\mathrm{H}_n(M;\mathbb{Z}) \cong \mathbb{Z}$ and that from Proposition 87.22 (1) we know that $\mathrm{H}_{n-1}(M;\mathbb{Z}) \cong \mathbb{Z}^k$ for some k.
(4) Next suppose that M is closed non-orientable and non-empty. This time we use that we know from Theorem 87.1 and Proposition 87.22 (2) that $\mathrm{H}_n(M;\mathbb{Z}) \cong 0$ and that $\mathrm{H}_{n-1}(M;\mathbb{Z}) \cong \mathbb{Z}^k \oplus \mathbb{Z}_2$ for some k. Here we also use Lemma 75.5 (4) which says that $\mathrm{Ext}(\mathbb{Z}_2, G) \cong G/2G$.

(5) So let $k > n$. This time we use that by Theorem 87.3 we know that $\mathrm{H}_k(M;\mathbb{Z}) = 0$ and that we know from Theorems 87.1 and 87.3 that $\mathrm{H}_{k-1}(M;\mathbb{Z})$ is either zero or isomorphic to $\mathbb{Z}$. The desired statement follows again from the Universal Coefficient Theorem 75.13 and the fact that by Lemma 75.5 we have $\mathrm{Ext}(\mathbb{Z}, G) = 0$. ■

Lemma 75.16. *Let M be a compact oriented connected non-empty n-dimensional topological manifold. Then we have $\mathrm{H}^n(M, \partial M;\mathbb{Z}) \cong \mathbb{Z}$. Furthermore there exists a unique generator $[M]^* \in \mathrm{H}^n(M, \partial M;\mathbb{Z})$ with $\langle [M]^*, [M]\rangle_M = 1$.*[1097]

Proof (∗). Let M be a compact oriented connected n-dimensional non-empty topological manifold.

It follows from the Universal Coefficient Theorem 75.13 for Cohomology Groups together with Proposition 87.22 that the map

$$\mathrm{H}^n(M, \partial M;\mathbb{Z}) \xrightarrow{\text{ev}} \mathrm{Hom}(\underbrace{\mathrm{H}_n(M, \partial M;\mathbb{Z})}_{=\mathbb{Z}\cdot[M]}, \mathbb{Z})$$

is an isomorphism. By Corollary 87.4 (3) we have the equality $\mathrm{H}_n(M, \partial M;\mathbb{Z}) = \mathbb{Z}\cdot[M]$. Hence there exists a unique element $[M]^* \in \mathrm{H}^n(M, \partial M;\mathbb{Z})$ with $\mathrm{ev}([M]^*)([M]) = 1$, i.e. with $\langle [M]^*, [M]\rangle_M = 1$. ■

Definition.

(1) Given a compact oriented connected non-empty n-dimensional topological manifold M we refer to the uniquely determined element $[M]^* \in \mathrm{H}^n(M, \partial M;\mathbb{Z})$ from Lemma 75.16 as the *dual fundamental class* of M.
(2) Given a compact oriented n-dimensional non-empty topological manifold M with finitely many components $M_1, \dots, M_k$ we define its *dual fundamental class* to be

$$[M]^* := [M_1]^* + \dots + [M_k]^* \in \underbrace{\mathrm{H}^n(M_1, \partial M_1;\mathbb{Z}) \oplus \dots \oplus \mathrm{H}^n(M_k, \partial M_k;\mathbb{Z})}_{=\mathrm{H}^n(M,\partial M;\mathbb{Z})}.$$

Example. We consider the smooth manifold S^1 with the usual orientation. We saw on page 1714 that the singular simplex $\mu\colon \Delta^1 \to S^1$ that is given by $(1-t, t) \mapsto e^{2\pi i t}$ represents the fundamental class of S^1. On page 1826 we introduced a cocycle $\theta_{\mathbb{Z}} \in \mathrm{C}^1;\mathbb{Z})$ and we had explicitly calculated that $\langle [\theta_{\mathbb{Z}}], [S^1]\rangle = \theta_{\mathbb{Z}}(\mu) = 1$. This shows immediately that $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$ is the dual fundamental class of S^1.

Later on we will need the following lemma.

Lemma 75.17. *Let $f\colon M \to N$ be a map between closed oriented connected non-empty n-dimensional topological manifolds. Then*

$$f^*([N]^*) = \deg(f)\cdot[M]^*.$$

Proof. Let $f\colon M \to N$ be a map between closed, oriented connected n-dimensional non-empty topological manifolds. It follows from Lemma 75.16 that any class in $\varphi = \mathrm{H}^n(M;\mathbb{Z})$

[1097] Here $[M] \in \mathrm{H}_n(M, \partial M;\mathbb{Z})$ denotes the fundamental class provided by Corollary 87.4.

is of the form $\varphi = k \cdot [M]^*$ for some $k \in \mathbb{Z}$ and it follows from $\langle [M]^*, [M] \rangle = 1$ and the bilinearity of the Kronecker pairing that $k = \langle \varphi, [M] \rangle_M$. In our case we have

$$\langle f^*([N]^*), [M] \rangle \underset{\substack{\uparrow \\ \text{Lemma 74.6 (3)}}}{=} \langle [N]^*, f_*([M]) \rangle \underset{\substack{\uparrow \\ \text{by definition of } \deg(f)}}{=} \langle [N]^*, \deg(f) \cdot [N] \rangle \underset{\substack{\uparrow \\ \text{since } \langle\,,\,\rangle \text{ is bilinear}}}{=} \deg(f) \cdot \underbrace{\langle [N]^*, [N] \rangle}_{=1} = \deg(f).$$

It follows from the above discussion that $f^*([N]^*) = \deg(f) \cdot [M]^*$. ■

Next we record the following useful consequence of the Universal Coefficient Theorem 75.13 for Cohomology Groups:

Proposition 75.18. *Let X be a path-connected topological space, let $x_0 \in X$ and let G be an abelian group. Then the Hurewicz homomorphism $\pi_1(X, x_0) \to \mathrm{H}_1(X)$ from page 1314 and the evaluation homomorphism $\mathrm{ev}\colon \mathrm{H}^1(X; G) \to \mathrm{Hom}(\mathrm{H}_1(X); G)$ induce natural isomorphisms*

$$\mathrm{H}^1(X; G) \xrightarrow{\cong} \mathrm{Hom}(\mathrm{H}_1(X); G) \xrightarrow{\cong} \mathrm{Hom}(\pi_1(X, x_0), G).$$

One often uses the natural isomorphisms from Proposition 75.18 to identify these groups.

Proof. As we pointed out on page 1867 and Proposition 52.2 (4), both maps are natural. So it remains to show that both maps are isomorphisms.

We insert one extra map into the above sequence of maps to see that the maps are indeed isomorphisms":

$$\mathrm{H}^1(X; G) \underset{\substack{\uparrow \\ \text{isomorphism by Theorem 75.13} \\ \text{since } \mathrm{H}_0(X) \cong \mathbb{Z} \text{ by Proposition 41.5}}}{\xrightarrow{\mathrm{ev}}} \mathrm{Hom}(\mathrm{H}_1(X); G) \underset{\substack{\uparrow \\ \text{Hurewicz Theorem 52.5}}}{\xrightarrow{\cong}} \mathrm{Hom}(\pi_1(X, x_0)_{\mathrm{ab}}, G) \underset{\substack{\uparrow \\ \text{by Proposition 21.20 (2)} \\ \text{since } G \text{ is abelian}}}{\xleftarrow{\cong}} \mathrm{Hom}(\pi_1(X, x_0), G).$$

■

75.4. **Cohomology with field coefficients.** Before we continue we introduce a slight variation on the Kronecker pairing defined in Lemma 74.6. More precisely, let C_* be a chain complex and let R be a commutative ring. We consider the map

$$\begin{array}{rcl} \langle\,,\,\rangle\colon \mathrm{H}^n(C; R) \times \mathrm{H}_n(C; R) & \to & R \\ \left([\varphi], \left[\sum\limits_{i=1}^k c_i \otimes a_i\right]\right) & \mapsto & \sum\limits_{i=1}^k \varphi(c_i) \cdot a_i. \end{array}$$

We leave it to the reader to verify that this map is well-defined and bilinear. We refer to this pairing also as the *Kronecker pairing*.

The next proposition relates homology and cohomology with coefficients in a field $\mathbb{F}$.

Proposition 75.19. *Let (X, A) be a pair of topological spaces and let $\mathbb{F}$ be a field. The map*

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^n(X, A; \mathbb{F}) & \to & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_n(X, A; \mathbb{F}), \mathbb{F}) \\ \varphi & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_n(X, A; \mathbb{F}) & \to & \mathbb{F} \\ \sigma & \mapsto & \langle \varphi, \sigma \rangle \end{array} \right) \end{array}$$

is a natural isomorphism of $\mathbb{F}$-vector spaces.[1098]

We postpone the proof of Proposition 75.19 to the next section. In most applications we are only interested in the following corollary to Proposition 75.19.

Corollary 75.20. *Let (X, A) be a pair of topological space and let $k \in \mathbb{N}_0$ such that $\mathrm{H}_k(X, A; \mathbb{Z})$ and $\mathrm{H}_{k-1}(X, A; \mathbb{Z})$ are finitely generated. (By Proposition 85.13 (4) we know if X and A are compact topological manifolds, then this condition is satisfied.)*

(1) *Given any field $\mathbb{F}$ the homology and cohomology groups $\mathrm{H}_k(X, A; \mathbb{F})$ and $\mathrm{H}^k(X, A; \mathbb{F})$ are finite-dimensional $\mathbb{F}$-vector spaces with*
$$\dim_{\mathbb{F}}(\mathrm{H}_k(X, A; \mathbb{F})) \;=\; \dim_{\mathbb{F}}(\mathrm{H}^k(X, A; \mathbb{F})).$$
(2) *For any subfield $\mathbb{F}$ of $\mathbb{C}$ we have*
$$\dim_{\mathbb{F}}(\mathrm{H}_k(X, A; \mathbb{F})) \;=\; \dim_{\mathbb{F}}(\mathrm{H}^k(X, A; \mathbb{F})) \;=\; b_n(X, A).$$

Remark. Note that in Corollary 75.20 we only claim that the homology and cohomology groups are abstractly isomorphic. We do not claim that there is a natural isomorphism. In fact it does not make sense to ask for a natural isomorphism since the homology groups of a topological space X with $\mathbb{F}$-coefficients are covariant in X whereas the cohomology groups of X with $\mathbb{F}$-coefficients are contravariant in X.

PROOF. First note that Statement (2) is an immediate consequence of Statement (1) and the discussion on page 1422. So let turn to the proof of Statement (1). To simplify the notation we only consider the case $A = \varnothing$. Let X be a topological space and let $k \in \mathbb{N}_0$ such that $\mathrm{H}_k(X; \mathbb{Z})$ and $\mathrm{H}_{k-1}(X; \mathbb{Z})$ are finitely generated. Let p be a prime number. By the classification of finitely generated abelian groups, see Theorem 19.4, we can make the identifications
$$\mathrm{H}_{k-1}(X; \mathbb{Z}) \;=\; \mathbb{Z}^r \oplus \bigoplus_{i=1}^{\alpha} \mathbb{Z}_{a_i} \oplus \bigoplus_{i=1}^{\beta} \mathbb{Z}_{b_i} \quad \text{and} \quad \mathrm{H}_k(X; \mathbb{Z}) \;=\; \mathbb{Z}^s \oplus \bigoplus_{i=1}^{\gamma} \mathbb{Z}_{c_i} \oplus \bigoplus_{i=1}^{\delta} \mathbb{Z}_{d_i}$$
where all the a_i, b_i, c_i and d_i are non-zero, and where each a_i and each c_i is divisible by p and none of the b_i and d_i are divisible by p. We start out with the following claim:

Claim. We have
$$\dim_{\mathbb{F}}(\mathrm{H}_k(X, \mathbb{F})) \;=\; \begin{cases} s, & \text{if } \mathbb{F} \text{ is a field of characteristic zero,} \\ s + \alpha + \gamma, & \text{if } \mathbb{F} \text{ is a field of characteristic } p. \end{cases}$$

First assume that $\mathbb{F}$ is a field of characteristic zero. In this case we have
$$\mathrm{H}_k(X; \mathbb{F}) \underset{\substack{\uparrow \\ \text{by Theorem 57.19 and Lemma 57.17 (4) since } \mathbb{F} \text{ is torsion-free}}}{\cong} \mathrm{H}_k(X; \mathbb{Z}) \otimes \mathbb{F} \;=\; \Big(\mathbb{Z}^s \oplus \bigoplus_{i=1}^{\gamma} \mathbb{Z}_{c_i} \oplus \bigoplus_{i=1}^{\delta} \mathbb{Z}_{d_i}\Big) \otimes \mathbb{F} \underset{\substack{\uparrow \\ \text{see page 1396}}}{\cong} \mathbb{F}^s.$$

[1098]Note that "natural" makes sense since $(X, A) \mapsto \mathrm{H}^k(X, A; \mathbb{F})$ and $(X, A) \mapsto \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(X, A; \mathbb{F}), \mathbb{F})$ both define *contravariant* functors from the category of pairs of topological spaces to the category of $\mathbb{F}$-vector spaces.

Now suppose that $\mathbb{F}$ is a field of characteristic p. We make the following observation. For any $n \in \mathbb{N}$ we have

$$\mathbb{Z}_n \otimes \mathbb{F} \underset{\substack{\uparrow \\ \text{Lemma 57.3 (3)}}}{=} \operatorname{coker}\left(\mathbb{F} \xrightarrow{\cdot n} \mathbb{F}\right) = \begin{cases} \mathbb{F}, & \text{if } p|n, \\ 0, & \text{else} \end{cases} \quad \text{and} \quad \operatorname{Tor}(\mathbb{Z}_n, \mathbb{F}) \underset{\substack{\uparrow \\ \text{Lemma 57.17 (6)}}}{=} \ker\left(\mathbb{F} \xrightarrow{\cdot n} \mathbb{F}\right) = \begin{cases} \mathbb{F}, & \text{if } p|n, \\ 0, & \text{else.} \end{cases}$$

Then

$$\mathrm{H}_k(X;\mathbb{F}) \underset{\substack{\uparrow \\ \text{by Theorem 57.19}}}{\cong} \mathrm{H}_k(X;\mathbb{Z}) \otimes \mathbb{F} \oplus \operatorname{Tor}(\mathrm{H}_{k-1}(X;\mathbb{Z}), \mathbb{F}) \underset{\substack{\uparrow \\ \text{by Lemma 57.17, the discussion on page 1396} \\ \text{and the above observations}}}{\cong} \mathbb{F}^s \oplus \mathbb{F}^\gamma \oplus \mathbb{F}^\alpha.$$

⊞

Now we have

$$\dim_{\mathbb{F}}(\mathrm{H}^k(X;\mathbb{F})) \underset{\substack{\uparrow \\ \text{Proposition 75.19}}}{=} \dim_{\mathbb{F}}(\operatorname{Hom}(\mathrm{H}_k(X;\mathbb{F}), \mathbb{F})) \underset{\substack{\uparrow \\ \text{since by the claim } \mathrm{H}_k(X;\mathbb{F}) \text{ is finite-dimensional}}}{=} \dim_{\mathbb{F}}(\mathrm{H}_k(X;\mathbb{F}))$$

We could also prove the equality $\dim_{\mathbb{F}}(\mathrm{H}^k(X;\mathbb{F})) = \dim_{\mathbb{F}}(\mathrm{H}_k(X;\mathbb{F}))$ without appealing to Proposition 75.19. Indeed, using Theorem 75.13, the fact that for any abelian group G we have $\operatorname{Hom}(\mathbb{Z}_n, G) \cong \ker(G \xrightarrow{\cdot n} G)$ and Proposition 75.6 one can easily compute $\dim_{\mathbb{F}}(\mathrm{H}^k(X;\mathbb{F}))$ and one ends up with the same calculation as in the above claim. We leave it to the reader to fill in the details. ■

75.5. **Proof of Proposition 75.19** $(*)$**.** In this section we give the proof of Proposition 75.19. Before we can provide the proof we need to prepare the ground.

Let $\mathbb{F}$ be a field. We define a *chain complex over* $\mathbb{F}$ to be a chain complex (V_*, v_*) where all chain groups are $\mathbb{F}$-vector spaces and all boundary maps are $\mathbb{F}$-homomorphisms. For example, if $(\mathrm{C}_*, \partial_*)$ is a chain complex of abelian groups, then $(C_* \otimes \mathbb{F}, \partial_* \otimes \mathrm{id})$ is easily seen to be a chain complex over $\mathbb{F}$.

Given a chain complex (V_*, v_*) over the field $\mathbb{F}$ and given $n \in \mathbb{N}_0$ its n-homology

$$\mathrm{H}_n(V) := \frac{\ker(v_n \colon V_n \to V_{n-1})}{\operatorname{im}(v_{n+1} \colon V_{n+1} \to V_n)}.$$

is an $\mathbb{F}$-vector space in a natural way.

We also define cochain complexes and cochain maps over $\mathbb{F}$ in the obvious way. For example, if (C_*, c_*) is a chain complex, then the dual cochain complex $(\operatorname{Hom}(\mathrm{C}_*(X,\mathbb{F})), c^*)$ is a cochain complex over $\mathbb{F}$.

Given a chain complex (V_*, v_*) over $\mathbb{F}$ we obtain a cochain complex by considering $(\operatorname{Hom}_{\mathbb{F}}(V_*;\mathbb{F}), v^*)$ and we define its n-th cohomology group to be the $\mathbb{F}$-vector space

$$\mathrm{H}^n(V;\mathbb{F}) := \frac{\ker(v_{n+1}^* \colon \operatorname{Hom}_{\mathbb{F}}(V_n, \mathbb{F}) \to \operatorname{Hom}_{\mathbb{F}}(V_{n+1}, \mathbb{F}))}{\operatorname{im}(v_n^* \colon \operatorname{Hom}_{\mathbb{F}}(V_{n-1}, \mathbb{F}) \to \operatorname{Hom}_{\mathbb{F}}(V_n, \mathbb{F}))}.$$

The proof of the Universal Coefficient Theorem 75.12 for Cohomology Groups can be modified fairly easily to provide the following theorem.

Theorem 75.21. *Let $\mathbb{F}$ be a field and let (V_*, v_*) be a chain complex over $\mathbb{F}$. Then the map*

$$\begin{array}{rcl} \mathrm{ev}_{\mathbb{F}}\colon\ \mathrm{H}^k(V;\mathbb{F}) & \mapsto & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(V),\mathbb{F}). \\ (\varphi\colon V_k \to \mathbb{F}) & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_k(V) & \to & \mathbb{F} \\ {[c]} & \mapsto & \varphi(c) \end{array}\right) \end{array}$$

is well-defined and it is an isomorphism.

A proof of that theorem is provided by any self-respecting book on homological algebra, see e.g. [**DaK01**, Corollary 2.31].

We will also need the following lemma.

Lemma 75.22. *Let C be an abelian group and let $\mathbb{F}$ be a field. Then the map*[1099]

$$\begin{array}{rcl} \Phi_C\colon\ \mathrm{Hom}_{\mathbb{Z}}(C,\mathbb{F}) & \to & \mathrm{Hom}_{\mathbb{F}}(C\otimes\mathbb{F},\mathbb{F}) \\ (\varphi\colon C \to \mathbb{F}) & \mapsto & \left(\begin{array}{rcl} C\otimes\mathbb{F} & \to & \mathbb{F} \\ \sum\limits_{i=1}^{k} c_i\otimes a_i & \mapsto & \sum\limits_{i=1}^{k} \varphi(c_i)\cdot a_i \end{array}\right) \end{array}$$

is an isomorphism.

Proof. It is straightforward to verify that an inverse map to Φ_C is given by the following homomorphism of $\mathbb{F}$-vector spaces:

$$\begin{array}{rcl} \Psi_C\colon\ \mathrm{Hom}_{\mathbb{F}}(C\otimes\mathbb{F},\mathbb{F}) & \to & \mathrm{Hom}_{\mathbb{Z}}(C,\mathbb{F}) \\ (\varphi\colon C\otimes\mathbb{F} \to \mathbb{F}) & \mapsto & \big(C \xrightarrow{c\mapsto c\otimes 1} C\otimes\mathbb{F} \xrightarrow{\varphi} \mathbb{F}\big). \end{array}$$ ■

Corollary 75.23. *Let C_* be a chain complex and let $\mathbb{F}$ be a field. The maps*

$$\Phi_{\mathrm{C}_k}\colon\ \mathrm{Hom}_{\mathbb{Z}}(C_k,\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(\mathrm{C}_k\otimes\mathbb{F},\mathbb{F}) \quad \text{with } k\in\mathbb{N}_0$$

from the previous lemma define an isomorphism of cochain complexes of $\mathbb{F}$-vector spaces, in particular for each $k\in\mathbb{N}_0$ they induce an isomorphism

$$\Phi\colon\ \mathrm{H}^k(C;\mathbb{F}) \to \mathrm{H}^k(C\otimes\mathbb{F};\mathbb{F})$$

of $\mathbb{F}$-vector spaces.

Proof. By Lemma 75.22 the maps Φ_{C_k}, $k\in\mathbb{N}_0$, are isomorphisms of the cochain vector spaces. It is straightforward to see that these maps are also cochain maps. ■

Now we can finally provide the proof of Proposition 75.19.

Proof of Proposition 75.19. Let (X,A) be a pair of topological spaces and let $\mathbb{F}$ be a field. We write $\mathrm{C}_* = \mathrm{C}_*(X,A)$. Given $k\in\mathbb{N}_0$ we consider the diagram

$$\begin{tikzcd} & \mathrm{H}^k(C\otimes\mathbb{F};\mathbb{F}) \arrow[dr, "\mathrm{ev}_{\mathbb{F}}"] & \\ \mathrm{H}^k(C;\mathbb{F}) \arrow[ur, "\Phi"] \arrow[rr, "\mathrm{ev}"'] & & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_n(C\otimes\mathbb{F}),\mathbb{F}). \end{tikzcd}$$

[1099] Note that to the left we consider homomorphisms of $\mathbb{Z}$-modules, i.e. homomorphisms of abelian groups whereas to the right we consider homomorphisms of $\mathbb{F}$-vector spaces.

It follows easily from the definitions that for any representative in $\mathrm{H}^k(C;\mathbb{F})$ the image under the composition of the two diagonal maps agrees with the image under the map ev. This implies that ev is well-defined and that the diagram commutes.

Also note that the left diagonal map is an isomorphism by Corollary 75.23 whereas the right diagonal map is an isomorphism by Theorem 75.21. In particular the bottom map is an isomorphism. But that is exactly what we had wanted to show. ■

75.6. **Change of coefficients in cohomology** $(*)$**.** Let $(\mathrm{C}_*,\partial_*)$ be a chain complex of free abelian groups and let G be an abelian group. By the Algebraic Universal Coefficient Theorem 57.18 we can relate the homology groups of $C_*\otimes G$ to the homology groups of C_*. Furthermore, by the Universal Coefficient Theorem for Cohomology Groups 75.12 we can relate the cohomology groups $\mathrm{H}^*(\mathrm{C}_*;G)$ to the homology groups $\mathrm{H}_*(C)$. The following question arises.

Question 75.24. *Can we relate the cohomology groups* $\mathrm{H}^*(C;G)$ *to the cohomology groups* $\mathrm{H}^*(C;\mathbb{Z})$*?*

It turns out that the question is slightly subtle. To formulate the answer we need the following lemma.

Lemma 75.25. *Let G be an abelian group.*

(1) *For any abelian group H the map*

$$\begin{array}{rcl} \mu\colon\ \mathrm{Hom}(H,\mathbb{Z})\otimes G & \to & \mathrm{Hom}(H,G) \\ \sum\limits_{i=1}^{m}\varphi_i\otimes g_i & \mapsto & \left(\begin{array}{rcl} H & \to & G \\ h & \mapsto & \sum\limits_{i=1}^{m}\varphi_i(h)\cdot g_i \end{array}\right) \end{array}$$

is a well-defined natural *homomorphism.*

(2) *Given a chain complex* $(\mathrm{C}_*,\partial_*)$ *the map*

$$\begin{array}{rcl} \mathrm{H}^n(C;\mathbb{Z})\otimes G & \xrightarrow{\mu} & \mathrm{H}^n(C;G) \\ \sum\limits_{i=1}^{m}[\varphi_i]\otimes g_i & \mapsto & \sum\limits_{i=1}^{m}[\mu(\varphi_i\otimes g_i)] \end{array}$$

is well-defined.

Proof. The proof is elementary and is left to the reader. ■

The following theorem gives an affirmative answer to Question 75.24, albeit only under an extra hypothesis.

Theorem 75.26. (Universal Coefficient Theorem) *Let* $(\mathrm{C}_*,\partial_*)$ *be a chain complex of free abelian groups and let G be an abelian group. We suppose that all cohomology groups* $\mathrm{H}^n(\mathrm{C}_*;\mathbb{Z})$ *are finitely generated. Then for each* $n\in\mathbb{N}_0$ *there exists a short exact sequence of the form*

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}^n(C;\mathbb{Z})\otimes G & \xrightarrow{\mu} & \mathrm{H}^n(C;G) & \to & \mathrm{Tor}(\mathrm{H}^{n+1}(C;\mathbb{Z}),G) & \to & 0. \\ & & [\varphi]\otimes g & \mapsto & [\mu(\varphi\otimes g)] & & & & \end{array}$$

Furthermore the short exact sequence splits.[1100]

The key to proving the Universal Coefficient Theorem 75.26 is the following elementary but subtle lemma.

Lemma 75.27. *Let G be an abelian group. If H is a free abelian group of* finite rank, *then the map*

$$\mu\colon\ \operatorname{Hom}(H,\mathbb{Z})\otimes G \ \to\ \operatorname{Hom}(H,G)$$

is an isomorphism.

Proof. Let $v_1,\dots,v_n$ be a basis for the finitely generated free abelian group H. We denote by $\varphi_1,\dots,\varphi_n$ the dual basis of $\operatorname{Hom}(H,\mathbb{Z})$. In other words, these are the homomorphisms that are uniquely determined by $\varphi_i(v_j)=\delta_{ij}$. We leave it to the reader to verify that the map

$$\begin{array}{rrcl}\nu\colon & \operatorname{Hom}(H,G) & \to & \operatorname{Hom}(H,\mathbb{Z})\otimes G\\ & \psi & \mapsto & \sum\limits_{i=1}^{n}\varphi_i\otimes\psi(v_i)\end{array}$$

is an inverse to the homomorphism μ. ■

We will also need the following lemma.

Lemma 75.28. *Let $(\mathrm{C}_*,\partial_*)$ be a chain complex of free abelian groups. If all cohomology groups $\mathrm{H}^n(\mathrm{C}_*;\mathbb{Z})$ are finitely generated, then all homology groups $\mathrm{H}_n(C_*)$ are also finitely generated.*

Proof of Lemma 75.28. Let $(\mathrm{C}_*,\partial_*)$ be a chain complex of free abelian groups such that all cohomology groups $\mathrm{H}^n(\mathrm{C}_*;\mathbb{Z})$ are finitely generated. By the Universal Coefficient Theorem 75.12 for Cohomology Groups there exists an isomorphism

$$\mathrm{H}^n(C;\mathbb{Z})\ \cong\ \operatorname{Ext}(\mathrm{H}_{n-1}(C),\mathbb{Z})\oplus\operatorname{Hom}(\mathrm{H}_n(C),\mathbb{Z})\qquad\text{for every } n\in\mathbb{N}_0.$$

We see that all the groups $\operatorname{Ext}(\mathrm{H}_{n-1}(C),\mathbb{Z})$ and $\operatorname{Hom}(\mathrm{H}_n(C),\mathbb{Z})$ are finitely generated, in particular they are countable. But by Proposition 75.11 this implies that all the homology groups $\mathrm{H}_n(C_*)$ are finitely generated. ■

Proof of the Universal Coefficient Theorem 75.26. Let G be an abelian group. Furthermore let $(\mathrm{C}_*,\partial_*)$ be a chain complex of free abelian groups such that all cohomology groups $\mathrm{H}^n(\mathrm{C}_*;\mathbb{Z})$ are finitely generated. It follows from Lemma 75.28 that all homology groups $\mathrm{H}_n(C_*)$ are also finitely generated. Thus we obtain from Lemma 49.6 that there exists a chain homotopy equivalence $h\colon C_*\to D_*$ from $(\mathrm{C}_*,\partial_*)$ to a chain complex (D_*,∂_*) such that each chain group D_n is a finitely generated free abelian group. Now we consider

[1100]As always, the splitting is not natural. In fact in this theorem we do *not* even claim that the short exact sequence is natural. The reason is that the construction of the short exact sequence involves a choice of the chain complex D_* and we do not intend to discuss the question how that choice affects the map on the right hand side.

the following diagram:

$$\begin{array}{ccccccc}
0 \to \overbrace{\mathrm{H}_n(\mathrm{Hom}(D_*,\mathbb{Z}))}^{=\mathrm{H}^n(D;\mathbb{Z})\text{ by def.}}\otimes G & \xrightarrow{[\varphi]\otimes g\mapsto[\varphi\otimes g]} & \mathrm{H}_n(\mathrm{Hom}(D_*,\mathbb{Z});G) & \to & \mathrm{Tor}(\overbrace{\mathrm{H}_{n+1}(\mathrm{Hom}(D_*,\mathbb{Z}))}^{=\mathrm{H}^{n+1}(D;\mathbb{Z})\text{ by def.}},G) \to 0. \\
{=}\uparrow & & \cong\downarrow\mu & & {=}\uparrow \\
\mathrm{H}^n(D;\mathbb{Z})\otimes G & \xrightarrow{\mu} & \mathrm{H}_n(\mathrm{Hom}(D_*,G)) & & \mathrm{Tor}(\mathrm{H}^{n+1}(D;\mathbb{Z}),G) \\
{=}\uparrow & & {=}\uparrow & & {=}\uparrow \\
\mathrm{H}^n(D;\mathbb{Z})\otimes G & \xrightarrow{\mu} & \mathrm{H}^n(D_*;G) & & \mathrm{Tor}(\mathrm{H}^{n+1}(D;\mathbb{Z}),G) \\
\cong\downarrow h^* & & \cong\downarrow h^* & & \cong\downarrow h^* \\
\mathrm{H}^n(C;\mathbb{Z})\otimes G & \xrightarrow{\mu} & \mathrm{H}^n(C;G) & & \mathrm{Tor}(\mathrm{H}^{n+1}(C;\mathbb{Z}),G).
\end{array}$$

We make the following clarifications:

(1) The horizontal sequence on the top is the short exact sequence obtained from the Algebraic Universal Coefficient Theorem 57.18 which we applied to the chain complex $E_* = \mathrm{Hom}(D_{-*},\mathbb{Z})$. We need to invert the sign of the degrees twice, thus we end up with the degree $n+1 = -((-n)-1)$ on the right hand side. We know by Proposition 57.23 that the short exact sequence splits.

(2) It follows from Lemma 75.27 that the middle vertical map between the first two rows is an isomorphism. Here we use that each chain group of D_* is a free abelian group of finite rank.

(3) One can easily verify that the diagram commutes.

(4) By Lemma 73.8 (4) the bottom vertical maps are isomorphisms.

It is now clear that we obtain the desired short exact sequence. ■

The following theorem gives a practical "topological" version of the Universal Coefficient Theorem 75.26.

Theorem 75.29. (Universal Coefficient Theorem) *Let X be a topological space and let G be an abelian group. If all homology groups $\mathrm{H}_n(X)$ of X are finitely generated (e.g. X could be a compact topological manifold or X could be a CW-complex that has only finitely many cells in each dimension[1101]), then for each $n\in\mathbb{N}_0$ there exists a short exact sequence of the form*

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{H}^n(X;\mathbb{Z})\otimes G & \xrightarrow{\mu} & \mathrm{H}^n(X;G) & \to & \mathrm{Tor}(\mathrm{H}^{n+1}(X;\mathbb{Z}),G) & \to & 0. \\
 & & [\varphi]\otimes g & \mapsto & [\mu(\varphi\otimes g)] & & & &
\end{array}$$

Furthermore the short exact sequence splits.

Proof. This theorem follows immediately from applying the Universal Coefficient Theorem 75.26 to the singular chain complex of X. ■

Remark. In general the hypothesis on X that we put into the Universal Coefficient Theorem 75.29 cannot be dropped. For example consider the infinite string of circles X as shown in Figure 1131 and we take $G=\mathbb{Q}$. By Lemma 57.17 (4) we know that all Tor-groups $\mathrm{Tor}(H,\mathbb{Q})$ vanish. In Exercise 75.10 we will see that the groups $\mathrm{H}^1(X;\mathbb{Z})\otimes\mathbb{Q}$ and

[1101]By Proposition 85.13 (4) and Proposition 48.5 (1) such topological spaces qualify.

$\mathrm{H}^1(X;\mathbb{Q})$ are not isomorphic. This shows that in this setting the sequence stated in the Universal Coefficient Theorem 75.29 is not exact.

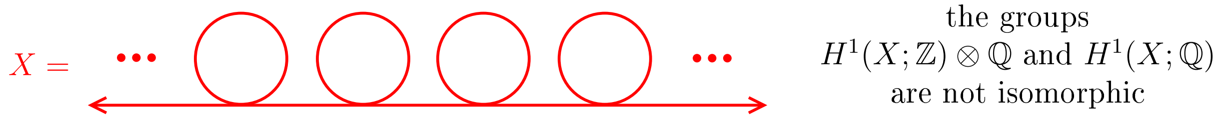


FIGURE 1131

The following lemma gives a convenient interpretation of the map μ appearing in the Universal Coefficient Theorem 75.29.

Lemma 75.30. *Let X be a topological space, let G be an abelian group and let $f\colon \mathbb{Z}\to G$ be a homomorphism. The following diagram commutes:*

$$\begin{tikzcd}
\mathrm{H}^n(X;\mathbb{Z}) \arrow[d, "a\mapsto a\otimes f(1)"'] \arrow[dr, "f_*"] & \\
\mathrm{H}^n(X;\mathbb{Z})\otimes G \arrow[r, "\mu"'] & \mathrm{H}^n(X;G).
\end{tikzcd}$$

PROOF. The statement follows immediately from explicitly writing down the homomorphisms. ■

Example. Let $n\in\mathbb{N}\cup\{\infty\}$. We denote by $f\colon\mathbb{Z}\to\mathbb{Z}_2$ the obvious group homomorphism. It follows from the Universal Coefficient Theorem 75.29, from Lemma 75.30, from the calculations of the cohomology groups $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ on page 1845, and from the properties of Tor-groups, see Lemma 57.17, that for any $k\in\mathbb{N}_0$ the map $f_*\colon\ \mathrm{H}_*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})\to\mathrm{H}_*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}_2)$ is an epimorphism.

We conclude this chapter with the following lemma which unwittingly gets used frequently.

Lemma 75.31. $(*)$ *We denote by $\varphi\colon\mathbb{Z}\to\mathbb{Q}$ the obvious inclusion. Let $n\in\mathbb{N}_0$.*

(1) *Let X be a topological space such that all homology groups are finitely generated. We suppose that $\mathrm{H}^n(X;\mathbb{Z})$ is a finitely generated free abelian group. If $v_1,\dots,v_m$ is a basis of $\mathrm{H}^n(X;\mathbb{Z})$, then $\varphi_*(v_1),\dots,\varphi_*(v_m)$ is a basis of $\mathrm{H}^n(X;\mathbb{Q})$.*

(2) *Let $f\colon X\to Y$ be a map between topological spaces. We suppose that all homology groups of X and Y are finitely generated and we suppose that $\mathrm{H}^n(X;\mathbb{Z})$ and $\mathrm{H}^n(Y;\mathbb{Z})$ are free abelian groups. We consider the following commutative diagram:*

$$\begin{tikzcd}
\mathrm{H}^n(Y;\mathbb{Z}) \arrow[r, "f^*"] \arrow[d, "\varphi_*"'] & \mathrm{H}^n(X;\mathbb{Z}) \arrow[d, "\varphi_*"] \\
\mathrm{H}^n(Y;\mathbb{Q}) \arrow[r, "f^*"] & \mathrm{H}^n(X;\mathbb{Q}).
\end{tikzcd}$$

Suppose $v_1,\dots,v_r$ is a basis for $\mathrm{H}^n(X;\mathbb{Z})$ and that $w_1,\dots,w_s$ is a basis for $\mathrm{H}^n(Y;\mathbb{Z})$. The matrix that represents $f^*\colon\ \mathrm{H}^n(Y;\mathbb{Z})\to\mathrm{H}^n(X;\mathbb{Z})$ with respect to the given bases also represents $f^*\colon\ \mathrm{H}^n(Y;\mathbb{Q})\to\mathrm{H}^n(X;\mathbb{Q})$ with respect to the bases $\varphi_*(v_1),\dots,\varphi_*(v_r)$ and $\varphi_*(w_1),\dots,\varphi_*(w_s)$.

Proof (*).

(1) This statement follows easily from the Universal Coefficient Theorem 75.29 together with Lemma 57.17 (4) and Lemma 75.30.
(2) The fact that the diagram commutes is an immediate consequence of Lemma 73.9. The remaining statement is just a psychologically helpful reinterpretation of the fact that the diagram commutes. ∎

Exercises for Chapter 75.

Exercise 75.1. Let C_* be a chain complex of free abelian groups and let $n \in \mathbb{N}$. Show that there does not exist a *natural* isomorphism

$$\mathrm{H}^n(C;G) \;\cong\; \mathrm{Ext}(\mathrm{H}_{n-1}(C),G) \oplus \mathrm{Hom}(\mathrm{H}_n(C),G).$$

Exercise 75.2. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of finitely generated abelian groups. We assume that A_0 and A_1 are free abelian groups. Show that there exists a connected CW-complex X such that for any $n \in \mathbb{N}$ we have $\mathrm{H}^n(X;\mathbb{Z}) \cong A_n$.
Hint. Use Proposition 47.10.
Remark. The statement is not true if we drop the hypothesis that the groups are finitely generated, in fact as is shown in [**Luc78**] (see also [**KaW61**]) there is no topological space X with $\mathrm{H}^n(X;\mathbb{Z}) \cong \mathbb{Q}$ for some $n \in \mathbb{N}_0$.

Exercise 75.3. Let $(P,\le)$ be a partially ordered set and let Q be a subset. Is an upper bound necessarily a maximal element? And conversely, is a maximal element necessarily an upper bound?

Exercise 75.4. Let X and Y are topological spaces. Suppose that $\mathrm{H}_n(X;\mathbb{Z}) \cong \mathrm{H}_n(Y;\mathbb{Z})$ for some $n \in \mathbb{N}_0$. Does it follow that $\mathrm{H}^n(X;\mathbb{Z})$ and $\mathrm{H}^n(Y;\mathbb{Z})$ are also isomorphic?

Exercise 75.5. Redo Exercise 74.3. More precisely, use the Universal Coefficient Theorem 75.13 for Cohomology Groups to determine the isomorphism types of the cohomology groups of the Klein bottle with $\mathbb{F}_2$-coefficients, with $\mathbb{F}_3$-coefficients and with $\mathbb{Z}$-coefficients.

Exercise 75.6. Give an explicit example of a cocycle $\varphi\colon \mathrm{C}_1(\mathbb{R}\mathrm{P}^2) \to \mathbb{F}_2$ that represents a generator of $\mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \cong \mathbb{F}_2$.

Exercise 75.7. Let M be a compact oriented connected n-dimensional topological manifold with non-empty connected boundary. Show that $\delta([\partial M]^*) = [M]^* \in \mathrm{H}^n(M,\partial M;\mathbb{Z})$ where δ denotes the connecting homomorphism in the long exact sequence of cohomology groups of the pair $(M,\partial M)$.
Hint. Use Proposition 87.27.

Exercise 75.8.

(a) Let G and H be $\mathbb{Q}$-vector spaces. Let $\varphi\colon G \to H$ be a homomorphism of abelian groups. Show that φ is also a homomorphism of $\mathbb{Q}$-vector spaces.
(b) Show that the abelian groups $\mathbb{Q}$ and $\mathbb{Q}^2$ are not isomorphic.
(c) Show that the abelian groups $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic.

(d) Use (a) and (b), together with the existence of Moore spaces, see Proposition 47.11, and together with the Universal Coefficient Theorem 75.13 for Cohomology Groups and Proposition 75.10 to conclude that there exist topological spaces X and Y such that $\mathrm{H}^n(X;\mathbb{Z}) \cong \mathrm{H}^n(Y;\mathbb{Z})$ for all $n \in \mathbb{N}_0$ but such that $\mathrm{H}_1(X;\mathbb{Z})$ is not isomorphic to $\mathrm{H}_1(Y;\mathbb{Z})$.

Remark. This exercise shows that in Corollary 73.19 the roles of homology and cohomology are not symmetric.

Exercise 75.9. Let H be a free abelian group and let G be an abelian group. As in Lemma 75.27 we consider the map

$$\begin{array}{rcl} \mathrm{Hom}(H,\mathbb{Z}) \otimes G & \to & \mathrm{Hom}(H,G) \\ \sum\limits_{i=1}^{m} \varphi_i \otimes g_i & \mapsto & \left(\begin{array}{rcl} H & \to & G \\ h & \mapsto & \sum\limits_{i=1}^{m} \varphi_i(h) \cdot g_i \end{array} \right). \end{array}$$

Show that this map is in general not an isomorphism.

Exercise 75.10. We consider the topological space X shown in Figure 1131. Show that the map

$$\mu\colon\ \mathrm{H}^1(X;\mathbb{Z}) \otimes \mathbb{Q} \ \to\ \mathrm{H}^1(X;\mathbb{Q})$$

from Lemma 75.25 is not an isomorphism.

Exercise 75.11. Show that there exists a topological space X such that the two groups $\mathrm{H}^1(X;\mathbb{Z}) \otimes \mathbb{Q}$ and $\mathrm{H}^1(X;\mathbb{Q})$ are not isomorphic.

Hint. You could make use of Proposition 47.10 to find unusual topological spaces.

Exercise 75.12. Let X be a topological space which is $\mathbb{Z}$-finite in the sense of the definition on page 1623, i.e. all homology groups $\mathrm{H}_i(X)$ are finitely generated abelian groups and all but finitely many homology groups $\mathrm{H}_i(X)$ are zero. Let $\varphi\colon X \to X$ be a map. Show that

$$\underbrace{\sum_{k\in\mathbb{N}_0} (-1)^k \cdot \mathrm{tr}(\varphi_*\colon\ \mathrm{H}_k(X;\mathbb{Z}) \to \mathrm{H}_k(X;\mathbb{Z}))}_{\text{Lefschetz number } \Lambda(\varphi) \text{ as defined on page 1623}} \ =\ \sum_{k\in\mathbb{N}_0} (-1)^k \cdot \mathrm{tr}(\varphi^*\colon\ \mathrm{H}^k(X;\mathbb{Z}) \to \mathrm{H}^k(X;\mathbb{Z})).$$

76. Cohomology groups and limits

76.1. **Direct limits of direct systems.** In the intermediate future direct limits of direct systems and inverse limits of inverse systems will play a large role. To fortify us for these discussions we recall the definitions and we prove some fairly elementary but useful lemmas regarding limits.

First we recall the following definition from page 726.

Definition. A *directed set* is a set I together with a relation $\leq$ on I which satisfies the following three conditions:

(1) the relation is reflexive, i.e. for any $i \in I$ we have $i \leq i$.
(2) the relation is transitive, i.e. for any $i, j, k \in I$ we have

$$i \leq j \quad \text{and} \quad j \leq k \quad \Longrightarrow \quad i \leq k.$$

(3) for any $i, j \in I$ there exists a $k \in I$ with $i \leq k$ and $j \leq k$.

Examples.

(a) Evidently the natural numbers $\mathbb{N}$ together with the usual "$\leq$" relation form a directed set $(\mathbb{N}, \leq)$.
(b) Let M be a set. We denote by $\mathcal{P}(M)$ the power set of M, i.e. the set of all subsets of M. Then the relation given by "$\subseteq$" turns $\mathcal{P}(M)$ into a directed set. To prove this claim it only remains to verify (3). But this also holds, since for any $X, Y \in \mathcal{P}(M)$, i.e. for any subsets X, Y of M we can take $Z = X \cup Y$ and we then have $X \subseteq Z$ and $Y \subseteq Z$.

Next we recall the following definition from page 728.

Definition. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. A *direct system over the directed set*[1102] $(I, \leq)$ is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ij} : X_i \to X_j\}$ for all $i, j \in I$ with $i \leq j$ such that the following two conditions are satisfied:

$$\begin{array}{lrcll} (1) & f_{ii} & = & \mathrm{id}_{X_i} & \text{for all } i \in I, \\ (2) & f_{ik} & = & f_{jk} \circ f_{ij} & \text{for all } i, j, k \in I \text{ with } i \leq j \leq k. \end{array}$$

Example. Let $\mathcal{C}$ be a category and let

$$X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} X_3 \xrightarrow{g_3} X_4 \xrightarrow{g_4} \dots$$

be a sequence of morphisms in the category $\mathcal{C}$. For $i \in \mathbb{N}_0$ we set $f_{ii} = \mathrm{id}$ and for $i \leq j$ we set $f_{ij} = g_{j-1} \circ \cdots \circ g_{i+1} \circ g_i : X_i \to X_j$. It is clear that $(\{X_i\}_{i \in \mathbb{N}}, \{f_{ij}\}_{i \leq j})$ is a direct system over the directed set $(\mathbb{N}, \leq)$. By a slight abuse of notation we refer to such a sequence $(\{X_i\}_{i \in \mathbb{N}}, \{g_i : X_i \to X_{i+1}\}_{i \in \mathbb{N}})$ also as a direct system.

Finally we also recall the definition of a direct limit from page 729.

[1102] On page 728 we had also considered direct systems that are defined over preordered sets, but for the remainder of this course we are only interested in direct systems over directed sets.

Definition. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. Suppose we are given a direct system $(\{X_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ in the category $\mathcal{C}$. A *direct limit* of the direct system is an object $\varinjlim X_i$ in $\mathcal{C}$ together with morphisms $h_i\colon X_i \to \varinjlim X_i, i \in I$ in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $k \leq l$ we have $h_l \circ f_{kl} = h_k\colon X_k \to \varinjlim X_i$, i.e. for all $k \leq l$ the following diagram commutes

$$\begin{tikzcd} X_l \arrow[dr, "h_l"] & \\ & \varinjlim X_i. \\ X_k \arrow[uu, "f_{kl}"] \arrow[ur, "h_k"'] & \end{tikzcd}$$

(2) If we are given another object Y and morphisms $h'_i\colon X_i \to Y, i \in I$ that satisfy (1), then there exists a unique morphism $F\colon \varinjlim X_i \to Y$ such that for all $k \leq l$ the following diagram commutes

$$\begin{tikzcd} X_l \arrow[dr, "h_l"] \arrow[drr, bend left, "h'_l"] & & \\ & \varinjlim X_k \arrow[r, dashed, "F", "\exists!"'] & Y. \\ X_i \arrow[uu, "f_{kl}"] \arrow[ur, "h_k"'] \arrow[urr, bend right, "h'_k"'] & & \end{tikzcd}$$

As we pointed out on page 730, the direct limit of direct system, if it exists, is in an appropriate sense unique up to isomorphism.

The following proposition is a variation on Proposition 25.1.

Proposition 76.1. *Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ be a direct system over one of the following categories:*

(1) *the category of abelian groups,*
(2) *the category of R-modules, where R is a commutative ring,*
(3) *the category of chain complexes,*
(4) *the category of cochain complexes.*

Then[1103]

$$\varinjlim X_i \ := \ \Big(\coprod_{i\in I} X_i\Big)/\sim \quad \text{where } x \sim f_{ij}(x) \text{ for all } i \leq j,\ x \in X_i$$

with the obvious maps $X_i \to \varinjlim X_i$ is a direct limit. Here we view $\varinjlim X_i$ as a group as follows: if we are given $a \in X_i$ and $b \in X_j$, then by definition of a directed set there exists a $k \in I$ with $i \leq k$ and $j \leq k$, and we define[1104]

$$[a] + [b] \ := \ \big[\underbrace{\underbrace{f_{ik}(a)}_{\in X_k} + \underbrace{f_{jk}(b)}_{\in X_k}}_{\text{addition in } X_k}\big] \ \in \ \varinjlim X_i.$$

Depending on the category we are dealing with we can also equip $\varinjlim X_i$ with an R-module structure or we can equip it with (co) boundary maps.

Proof. This explicit construction of the direct limit was already given on page 733. The proof of the proposition is a not overly exciting exercise in going through all the definitions. We leave this task to the reader. ■

Now we give three lemmas which will be helpful in getting direct limits "under control".

Lemma 76.2. *Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ij} \colon X_i \to X_j\}_{i\leq j})$ be a direct system over I in one of the four categories that we had considered in Proposition 76.1. Then the following four statements hold:*

(1) *Every element in $\varinjlim X_i$ lies in the image of one of the maps $X_j \to \varinjlim X_i$.*
(2) *Let $a_j \in X_j$ and let $a_k \in X_k$. The images of a_j and a_k in $\varinjlim X_i$ agree if and only if there exists an $l \in I$ with $l \geq j$ and $l \geq k$ such that $f_{jl}(a_j) = f_{kl}(a_k) \in X_l$.*
(3) *Let $a_j \in X_j$. The image of a_j in $\varinjlim X_i$ is zero if and only if there exists a $k \geq j$ such that $f_{jk}(a_j) = 0 \in X_k$.*
(4) *If all maps $f_{ij} \colon X_i \to X_j$ are isomorphisms, then for any $j \in I$ the obvious map $X_j \to \varinjlim X_i$ is an isomorphism.*

Proof.

(1) The first statement follows immediately from the explicit description of the direct limit given in Proposition 76.1.
(2) The second statement will be proved in Exercise 76.1. For the exercise it is useful to note that the symbol "$\sim$" in the definition of $\varinjlim X_i$ in Proposition 76.1 is only a relation and we really have to work with the equivalence relation generated by this relation, see page 197 for more information.
(3) The third statement is a special case of the second statement.
(4) The last statement will be proved in Exercise 76.1. ■

The second lemma is useful for simplifying calculations.

Lemma 76.3. *Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ij} \colon X_i \to X_j\}_{i\leq j})$ be a direct system over I in one of the four categories that we had considered in Proposition 76.1. Let J be a subset of I such that $(J, \leq)$ is again a directed set.*

(1) *There exists a unique map*

$$\varinjlim_{j\in J} X_j \longrightarrow \varinjlim_{i\in I} X_i$$

[1103] The construction of the direct limit that we gave in Proposition 25.1 was different, and arguably more complicated, since there we had considered direct systems over arbitrary preordered sets.

[1104] It is straightforward to verify that this definition does not depend on the choice of k.

such that for any $j \in J$ the following diagram commutes:

$$\begin{tikzcd} & X_j \arrow[dl] \arrow[dr] & \\ \varinjlim_{j\in J} X_j \arrow[rr] & & \varinjlim_{i\in I} X_i. \end{tikzcd}$$

(2) *If J has the property that given any $i \in I$ there exists a $j \in J$ with $i \le j$, then the above map*

$$\varinjlim_{j\in J} X_j \longrightarrow \varinjlim_{i\in I} X_i$$

is an isomorphism.

The first part of the lemma is just an immediate consequence of the definition of the direct limit over the directed set J. As we will see, the second statement is a fairly straightforward consequence of the next lemma. But it is also a good exercise to prove the above second statement "by hand".

The following definition is a generalization of the definition on page 1243.

Definition. Let $(I, \le)$ and $(J, \le)$ be directed sets.

(1) A *morphism* between the directed sets $(I, \le)$ and $(J, \le)$ is a map $\varphi\colon I \to J$ such that for any $i_1 \le i_2$ in I we have $\varphi(i_1) \le \varphi(i_2)$.
(2) A morphism $\varphi\colon I \to J$ is called *cofinal* if given any $j \in J$ there exists an $i \in I$ with $\varphi(i) \ge j$.
(3) Let $(\{K_i\}_{i\in I}, \{\chi_{i_1i_2}\}_{i_1\le i_2})$ and $(\{L_j\}_{j\in J}, \{\lambda_{j_1j_2}\}_{j_1\le j_2})$ be direct systems in one of the four categories that we had considered in Proposition 76.1. A *homomorphism between the direct systems* is a family $\{f_i\colon K_i \to L_{\varphi(i)}\}_{i\in I}$ of homomorphisms that has the property that for any $i_1 \le i_2$ the following diagram commutes:

$$\begin{tikzcd} K_{i_2} \arrow[r, "f_{i_2}"] & L_{\varphi(i_2)} \\ K_{i_1} \arrow[u, "\chi_{i_1i_2}"] \arrow[r, "f_{i_1}"] & L_{\varphi(i_1)} \arrow[u, "\lambda_{\varphi(i_1)\varphi(i_2)}"'] \end{tikzcd}$$

(4) The maps $K_i \xrightarrow{f_i} L_{\varphi(i)} \to \varinjlim L_j$ induce, by the universal property of a direct limit, a map

$$\varinjlim_{i\in I} K_i \;\to\; \varinjlim_{j\in J} L_j$$

which is uniquely determined by the property that for any $i \in I$ the following diagram commutes

$$\begin{tikzcd} K_i \arrow[r, "f_i"] \arrow[d] & L_{\varphi(i)} \arrow[d] \\ \varinjlim_{i\in I} K_i \arrow[r] & \varinjlim_{j\in J} L_j. \end{tikzcd}$$

Lemma 76.4. *Let $\{f_i\colon K_i \to L_{\varphi(i)}\}_{i\in I}$ be a homomorphism between two direct systems $(\{K_i\}_{i\in I}, \{\chi_{i_1 i_2}\}_{i_1\le i_2})$ and $(\{L_j\}_{j\in J}, \{\lambda_{j_1 j_2}\}_{j_1\le j_2})$ in one of the four categories that we had considered in Proposition 76.1. Suppose the following two conditions are satisfied:*

(1) *the morphism $\varphi\colon I \to J$ is cofinal, and*
(2) *for each $i \in I$ the map $f_i\colon K_i \to L_{\varphi(i)}$ is an isomorphism.*

Then the induced map

$$\varinjlim_{i\in I} K_i \;\to\; \varinjlim_{j\in J} L_j$$

is an isomorphism.

Sketch of proof. Let $j \in J$. Since $\varphi\colon I \to J$ is cofinal we can pick $i \in I$ with $f(i) \ge j$. We define

$$\begin{array}{rcl} g_j\colon L_j & \to & \varinjlim_{i\in I} K_i \\ x & \mapsto & [f_i^{-1}(\lambda_{j,f(i)}(x))]. \end{array}$$

It is straightforward to verify that this map does *not* depend on the choice of i. Using this observation it is elementary to verify that for any $j_1 \le j_2$ the following diagram commutes:

$$\begin{tikzcd} L_{j_2} \arrow[dr, "g_{j_2}"] & \\ & \varinjlim_{i\in I} K_i. \\ L_{j_1} \arrow[uu, "\lambda_{j_1 j_2}"] \arrow[ur, "g_{j_1}"'] & \end{tikzcd}$$

But by the universal property of the direct limit this implies that we get an induced

$$\varinjlim_{j\in J} L_j \;\to\; \varinjlim_{i\in I} K_i.$$

We leave it to the reader to verify that this map is an inverse to the map under consideration. ■

76.2. **Inverse limits of inverse systems.** We recall the following definition from page 745.

Definition. Let $(I, \le)$ be a directed set and let $\mathcal{C}$ be a category. An *inverse system in the category $\mathcal{C}$ over I* is a family of objects $\{X_i\}_{i\in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ji}\colon X_j \to X_i\}$ for all $i, j \in I$ with $i \le j$[1105] such that the following two conditions are satisfied:

(1)	$f_{ii} = \mathrm{id}_{X_i}$	for all $i \in I$,
(2)	$f_{ki} = f_{ji} \circ f_{kj}$	for all $i, j, k \in I$ with $i \le j \le k$.

Example. Let $\mathcal{C}$ be a category and let

$$\dots \xrightarrow{g_4} X_4 \xrightarrow{g_3} X_3 \xrightarrow{g_2} X_2 \xrightarrow{g_1} X_1$$

[1105] So in an inverse system, given $i \le j$ we have a morphism $X_j \to X_i$, whereas in a direct system we had a morphism $X_i \to X_j$.

be a sequence of morphisms in the category $\mathcal{C}$. For $i \in \mathbb{N}$ we set $f_{ii} = \mathrm{id}$ and for $i \leq j$ we set $f_{ji} = g_i \circ \cdots \circ g_{j-1} \colon X_j \to X_i$. It is clear that $(\{X_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ is an inverse system over the directed set $(\mathbb{N}, \leq)$. It is also clear that any inverse system over $(\mathbb{N}, \leq)$ arises that way. By a slight abuse of notation we refer to such a sequence $(\{X_i\}_{i\in\mathbb{N}}, \{g_i \colon X_{i+1} \to X_i\}_{i\in\mathbb{N}})$ also as an inverse system.

Using the convention from the previous example we can now reformulate in our context the original definition of an inverse limit from page 745.

Definition. Let $(\{X_i\}_{i\in\mathbb{N}}, \{g_i \colon X_{i+1} \to X_i\}_{i\in\mathbb{N}})$ be an inverse system in a category $\mathcal{C}$. An *inverse limit* of the inverse system is an object $\varprojlim X_i$ in $\mathcal{C}$ together with morphisms $h_i \colon \varprojlim X_i \to X_i$, $i \in I$, in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i \in \mathbb{N}$ we have $g_i \circ h_{i+1} = h_i \colon \varprojlim X_i \to X_i$, i.e. for all $i \in \mathbb{N}$ the following diagram commutes

$$\begin{tikzcd} & X_{i+1} \arrow[d, "g_i"] \\ \varprojlim X_i \arrow[ur, "h_{i+1}"] \arrow[r, "h_i"'] & X_i. \end{tikzcd}$$

(2) If we are given another object Y and morphisms $h_i' \colon Y \to X_i$, $i \in \mathbb{N}$, that satisfy (1), then there exists a unique morphism $F \colon Y \to \varprojlim X_i$ such that for all $i \in \mathbb{N}$ the following diagram commutes

$$\begin{tikzcd} & & X_{i+1} \arrow[dd, "g_i"] \\ Y \arrow[r, dashed, "F", "\exists!"'] \arrow[urr, bend left, "h_{i+1}'"] \arrow[drr, bend right, "h_i'"'] & \varprojlim X_i \arrow[ur, "h_{i+1}"] \arrow[dr, "h_i"] & \\ & & X_i. \end{tikzcd}$$

Throughout the remainder of this course we will only be interested in inverse systems over the directed set $(\mathbb{N}, \leq)$. In the following proposition we give an explicit description of inverse limits for this special case. The proposition can be viewed as a slight variation on Proposition 25.15.

Proposition 76.5. *Let $(\{X_i\}_{i\in\mathbb{N}}, \{g_i \colon X_{i+1} \to X_i\}_{i\in\mathbb{N}})$ be an inverse system over one of the following categories:*

(1) *the category of abelian groups,*
(2) *the category of R-modules, where R is a commutative ring,*
(3) *the category of chain complexes,*
(4) *the category of cochain complexes.*

Then

$$\varprojlim X_i \;:=\; \Big\{ (x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} X_i \,\Big|\, g_i(x_{i+1}) = x_i \text{ for all } i \in \mathbb{N} \Big\}$$

with the obvious projection maps $\varprojlim X_i \to X_i$ *is a direct limit.*

Proof. This proposition is implicitly proved on page 750. ■

Remark. Proposition 76.5 can be summarized as saying that, under the hypothesis of the proposition, we have

$$\varprojlim X_i \;=\; \text{all infinite sequences } (x_1, x_2, x_3, \dots) \text{ of the form } \begin{array}{ccc} x_3 & \in & X_3 \\ \downarrow & & \downarrow g_2 \\ x_2 & \in & X_2 \\ \downarrow & & \downarrow g_1 \\ x_1 & \in & X_1. \end{array}$$

Example. For each $i \in \mathbb{N}$ we define $X_i := \mathbb{Z}^i$ and we denote by

$$\begin{array}{rcl} g_i \colon \mathbb{Z}^{i+1} & \to & \mathbb{Z}^i \\ (x_1, \dots, x_{i+1}) & \mapsto & (x_1, \dots, x_i) \end{array}$$

the projection map. Then $(\{\mathbb{Z}^i\}_{i\in\mathbb{N}}, \{g_i\}_{i\in\mathbb{N}})$ forms an inverse system in the category of abelian groups. As we had already pointed out on page 751, using the description of the inverse limit provided in Proposition 76.5 one can show fairly easily that

$$\varprojlim_{i\in\mathbb{N}} \mathbb{Z}^i \;=\; \{(x_1, x_2, x_3, \dots) \mid x_j \in \mathbb{Z}\} \;\cong\; \mathbb{Z}^{\mathbb{N}}.$$

We conclude this section with the following analogue of Lemma 25.2.

Lemma 76.6. *Let* $f_n \colon X_n \to X_{n+1}$ *be a sequence of morphisms in any category. Suppose there exists an* $N \in \mathbb{N}$ *such that all* f_n *for* $n \geq N$ *are isomorphisms. Then* $\varprojlim X_n$ *exists and the natural map* $\varprojlim X_n \to X_N$ *is an isomorphism.*

Proof. We leave it to the reader to prove this elementary lemma. ■

76.3. **Cohomology groups and inverse limits.** In Propositions 25.7 and 47.4 we saw that in favorable situations the fundamental group and the homology groups of a union of topological spaces are the direct limit of the corresponding fundamental groups and homology groups. In this section we will try to extend these results to cohomology groups. The results will not only be useful for computing cohomology groups, but they will also play an essential role in many proofs later on. For example the proof that de Rham cohomology is isomorphic to singular cohomology with real coefficients will build on the somewhat technical results of this section.

We recall that Proposition 47.4 was a consequence of the following two lemmas. The first lemma is of topological nature.

Lemma 47.5. *Let* X *be a topological space and let* $X_1 \subset X_2 \subset X_3 \subset \dots$ *be a sequence of subsets such that every compact subset of* X *is already contained in one of the* X_i*. Then the chain complexes* $\mathrm{C}_*(X_i), i \in \mathbb{N}$ *together with the inclusion induced chain maps* $\mathrm{C}_*(X_i) \to \mathrm{C}_*(X_j)$ *for* $i \leq j$ *form a direct system of chain complexes. Furthermore the* *inclusion induced maps* $\mathrm{C}_*(X_i) \to \mathrm{C}_*(X)$ *induce an isomorphism*

$$\varinjlim \mathrm{C}_*(X_i) \xrightarrow{\cong} \mathrm{C}_*(X)$$

of chain complexes.

The second lemma is purely algebraic.

Lemma 47.6. *Let* $(\{\mathrm{C}_i\}_{i\in\mathbb{N}}, \{g_i\colon \mathrm{C}_i \to \mathrm{C}_{i+1}\}_{i\in\mathbb{N}})$ *be a direct system of chain complexes. Then for each* $n \in \mathbb{N}_0$ *we obtain an induced direct system* $(\{\mathrm{H}_n(\mathrm{C}_i)\}_{i\in\mathbb{N}}, \{g_{i*}\}_{i\in\mathbb{N}})$ *of abelian groups. Furthermore, given any* $n \in \mathbb{N}_0$ *the maps* $\mathrm{H}_n(\mathrm{C}_i) \to \mathrm{H}_n\big(\varinjlim \mathrm{C}_i\big)$ *induce an isomorphism*

$$\varinjlim \mathrm{H}_n(\mathrm{C}_i) \xrightarrow{\cong} \mathrm{H}_n\big(\varinjlim \mathrm{C}_i\big).$$

Let $(\{\mathrm{C}_i\}_{i\in\mathbb{N}}, \{g_i\colon \mathrm{C}_i \to \mathrm{C}_{i+1}\}_{i\in\mathbb{N}})$ be a direct system of chain complexes. The last lemma can be summarized as follows: for every $j \in \mathbb{N}$ and any $n \in \mathbb{N}_0$ we have the following diagram

$$\begin{tikzcd}
\mathrm{H}_n(\mathrm{C}_{j+1}) \arrow[dr] \arrow[drr, bend left] & & \\
& \varinjlim \mathrm{H}_n(\mathrm{C}_i) \arrow[r, "\cong"] & \mathrm{H}_n\big(\varinjlim \mathrm{C}_i\big) \\
\mathrm{H}_n(\mathrm{C}_j) \arrow[uu, "g_{j*}"] \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

Now we turn to cohomology groups. Let G be an abelian group. Whereas homology is a covariant functor, cohomology is a contravariant functor. This implies that from the above direct system of chain complexes we obtain for each $n \in \mathbb{N}_0$ an *inverse system* $(\{\mathrm{H}^n(\mathrm{C}_i;G)\}_{i\in\mathbb{N}}, \{g_i^*\colon \mathrm{H}^n(\mathrm{C}_{i+1};G) \to \mathrm{H}^n(\mathrm{C}_i;G)\}_{i\in\mathbb{N}})$ of abelian groups. Furthermore the maps $\mathrm{C}_i \to \varinjlim \mathrm{C}_i$ induce maps $\mathrm{H}^n(\varinjlim \mathrm{C}_i;G) \to \mathrm{H}^n(\mathrm{C}_i;G)$. Put differently, for any $j \in \mathbb{N}$ we are in the following situation

$$\begin{tikzcd}
& & \mathrm{H}^n(\mathrm{C}_{j+1};G) \arrow[dd, "g_j^*"] \\
\mathrm{H}^n\big(\varinjlim \mathrm{C}_i;G\big) \arrow[r, dashed] \arrow[urr, bend left] \arrow[drr, bend right] & \varprojlim \mathrm{H}^n(\mathrm{C}_i;G) \arrow[ur] \arrow[dr] & \\
& & \mathrm{H}^n(\mathrm{C}_j;G)
\end{tikzcd}$$

where the dotted arrow exists by the definition of the inverse limit of the groups $\mathrm{H}^n(\mathrm{C}_i;G)$. The question is whether the dotted arrow is in fact an isomorphism, put differently, the following question arises.

Question 76.7. *Let* $(\{\mathrm{C}_i\}_{i\in\mathbb{N}}, \{g_i\colon \mathrm{C}_i \to \mathrm{C}_{i+1}\}_{i\in\mathbb{N}})$ *be a direct system of chain complexes. Is the above map*

$$\mathrm{H}^n\big(\varinjlim \mathrm{C}_i;\mathbb{Z}\big) \to \varprojlim \mathrm{H}^n(\mathrm{C}_i;\mathbb{Z})$$

always an isomorphism?

It turns out that this is not necessarily the case. To formulate the relationship between the two sides of the above question it is useful to rewrite the inverse limit from Proposition 76.5. More precisely, given an inverse system $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1}\to D_i\}_{i\in\mathbb{N}})$ of abelian groups we consider the map

$$\begin{array}{rcl} \mu\colon \prod\limits_{i\in\mathbb{N}} D_i & \to & \prod\limits_{i\in\mathbb{N}} D_i \\ (x_i)_{i\in\mathbb{N}} & \mapsto & (x_i - g_i(x_{i+1}))_{i\in\mathbb{N}}. \end{array}$$

Proposition 76.5 can be reformulated as saying that

$$\varprojlim D_i \;=\; \ker(\mu).$$

This leads us to the following definition.

Definition. Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1}\to D_i\}_{i\in\mathbb{N}})$ be an inverse system of abelian groups. Using the above notation we define[1106]

$$\varprojlim{}^1 D_i \;:=\; \operatorname{coker}(\mu).$$

Example. We consider the inverse system

$$D_*\colon\; \dots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0.$$

In Exercise 76.2 we will see that $\varprojlim D_* = 0$ and $\varprojlim^1 D_* = 0$. Furthermore, we also consider the inverse system

$$E_*\colon\; \dots \xrightarrow{\cdot 5} \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0.$$

In Exercise 76.2 we will see that $\varprojlim^1 E_* \neq 0$. Furthermore in Exercise 76.3 we will see that this inverse system arises "in nature", i.e. we will see that there exists a nested sequence of topological spaces X_i such that for each $i\in\mathbb{N}$ we have $H^1(X_i;\mathbb{Z})\cong\mathbb{Z}$ and such that for each $i\in\mathbb{N}$ the map $H^1(X_{i+1};\mathbb{Z})\to H^1(X_i;\mathbb{Z})$ is given by multiplication by $i+1$.

We cannot proceed without introducing the following somewhat technical definition.

Definition. We say an inverse system of abelian groups $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1}\to D_i\}_{i\in\mathbb{N}})$ satisfies the *Mittag-Leffler condition* if given any $i\in\mathbb{N}$ there exists an $n\geq i$ such that for all $s\geq n$ we have

$$\operatorname{im}(g_i\circ\dots\circ g_{s-1}\colon D_s\to D_i) \;=\; \operatorname{im}(g_i\circ\dots\circ g_{n-1}\colon D_n\to D_i).$$

.

$$\begin{tikzcd} \dots & D_{s+2} \arrow[r, "g_{s+1}"] \arrow[rrrrrr, bend right] & D_{s+1} \arrow[r, "g_s"] \arrow[rrrrr, bend right] & D_s \arrow[r, "g_{s-1}"] \arrow[rrrr, bend right] & D_{s-1} \arrow[r, "g_{s-2}"] & \dots, & D_{i+1} \arrow[r, "g_i"] & D_i \arrow[r, "g_{i-1}"] & \dots \end{tikzcd}$$

same images

FIGURE 1132

[1106] If R is a commutative ring and if $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1}\to D_i\}_{i\in\mathbb{N}})$ is an inverse system of R-modules, then $\varprojlim^1 D_i$ is also an R-module in an obvious way.

Example. It is straightforward to verify that the above inverse system D_* satisfies the Mittag-Leffler condition whereas the above inverse system E_* does not satisfy the Mittag-Leffler condition.

In many situations one can ignore the precise definition of the Mittag-Leffler condition and instead one can appeal to the following lemma which in fact follows immediately from the definitions.

Lemma 76.8. *Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1}\to D_i\}_{i\in\mathbb{N}})$ be an inverse system of abelian groups. If there exists an $m\in\mathbb{N}$ such that for all $i\in\mathbb{N}$ the map $g_i\colon D_{i+1}\to D_i$ is an epimorphism, then the inverse system satisfies the Mittag-Leffler condition.*

Our main interest in the Mittag-Leffler condition arises from the fact that if a direct system satisfies this condition, then the pesky $\underleftarrow{\lim}^1$-limit vanishes.

Lemma 76.9. *Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1}\to D_i\}_{i\in\mathbb{N}})$ be an inverse system of abelian groups. If this inverse system satisfies the Mittag-Leffler condition, then*

$$\underleftarrow{\lim}^1 D_i \;=\; 0.$$

Proof (∗). Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1}\to D_i\}_{i\in\mathbb{N}})$ be an inverse system of abelian groups that satisfies the Mittag-Leffler condition. This means that given given any $i\in\mathbb{N}$ there exists an $n(i)\geq i$ such that for all $s\geq n(i)$ we have

$$\operatorname{im}(g_i\circ\cdots\circ g_{s-1}\colon D_s\to D_i) \;=\; \operatorname{im}(g_i\circ\cdots\circ g_{n(i)-1}\colon D_{n(i)}\to D_i).$$

Without loss of generality we can assume that $n(i)$ is monotonously increasing in i. We have to show that the map

$$\begin{array}{rcl}\mu\colon \prod\limits_{i\in\mathbb{N}} D_i & \to & \prod\limits_{i\in\mathbb{N}} D_i \\ (x_i)_{i\in\mathbb{N}} & \mapsto & (x_i - g_i(x_{i+1}))_{i\in\mathbb{N}}\end{array}$$

is an epimorphism.

Given $j\geq i$ we now write $f_{ji} := g_i\circ\cdots\circ g_{j-1}\colon D_j\to D_i$. First we consider $(a_i)_{i\in\mathbb{N}}\in\prod\limits_{i\in\mathbb{N}} D_i$ with the property that for all $i\in\mathbb{N}$ we have $a_i\in\operatorname{im}(f_{n(i),i}\colon D_{n(i)}\to D_i)$. To show that $\{a_i\}_{i\in\mathbb{N}}$ lies in the image of μ it suffices to prove the following claim.

Claim. There exist $b_i\in\operatorname{im}(f_{n(i),i}\colon D_{n(i)}\to D_i)$ with $i\in\mathbb{N}$ such that for each i we have $a_i = b_i - g_i(b_{i+1})$.

We define the b_i inductively. We set $b_0 = a_0$. Suppose $b_0,\dots,b_k$ are already defined with the desired properties. By our hypothesis we have $a_k\in\operatorname{im}(f_{n(k),k})$ and by our choice of b_k we have $b_k\in\operatorname{im}(f_{n(k),k})$. By the Mittag-Leffler condition and since we arranged that $n(k+1)\geq n(k)$ we have $\operatorname{im}(f_{n(k+1),k}) = \operatorname{im}(f_{n(k),k})$. Thus can we pick $c\in D_{n(k+1)}$ with $f_{n(k+1),k}(c) = a_k - b_k$. Now we put

$$b_{k+1} \;=\; -f_{n(k+1),k+1}(c).$$

Then

$$b_k - g_{k+1}(b_{k+1}) = b_k + g_{k+1}(f_{n(k+1),k+1}(c)) = b_k + (\underbrace{g_{k+1} \circ f_{n(k+1),k+1}}_{=f_{(n(k+1),k)}})(c) = b_k + (a_k - b_k) = a_k.$$

We have thus shown that $b_0, \ldots, b_{k+1}$ have the desired properties. ⊞

Now let $(b_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} D_i$ be an arbitrary element. For each $i \in \mathbb{N}$ we set

$$c_i \;=\; b_i + \sum_{j=i+1}^{n(i)} f_{j,i}(b_j).$$

Claim. For each $i \in \mathbb{N}$ we have $b_i - (c_i - g_i(c_{i+1})) \in \operatorname{im}(f_{n(i),i})$.

Let $i \in \mathbb{N}$. We have

$$\begin{aligned} b_i - (c_i - g_i(c_{i+1})) \;&=\; b_i - b_i - \sum_{j=i+1}^{n(i)} f_{j,i}(b_j) + g_i\Big(b_{i+1} + \sum_{j=i+2}^{n(i+1)} f_{j,i+1}(b_j)\Big) \\ &=\; -\sum_{j=i+1}^{n(i)} f_{j,i}(b_j) + f_{i+1,i}(b_{i+1}) + \sum_{j=i+2}^{n(i+1)} f_{j,i}(b_j) \;=\; \sum_{j=n(i)+1}^{n(i+1)} f_{j,i}(b_j). \end{aligned}$$

But by definition of $n(i)$ we know that for each $j \in \{n(i)+1, \ldots, n(i+1)\}$ we have $f_{j,i}(b_j) \in \operatorname{im}(f_{n(i),i})$. ⊞

It follows from the claim and the above discussion that $(b_i - (c_i - g_i(c_{i+1})))_{i\in\mathbb{N}}$ lies in the image of μ. Evidently $(c_i - g_i(c_{i+1}))_{i\in\mathbb{N}}$ lies in the image of μ. Since μ is a homomorphism we see that $(b_i)_{i\in\mathbb{N}}$ also lies in the image of μ. ■

We continue with the following definition.

Definition. Let R be a commutative ring and let $(X_i, f_i)_{i\in\mathbb{N}}$ and $(Y_i, g_i)_{i\in\mathbb{N}}$ be inverse systems of R-modules. A *map of inverse systems* from (X_i, f_i) to (Y_i, g_i) consists of a sequence of module homomorphisms $u_i\colon X_i \to Y_i$, $i \in \mathbb{N}$ such that for each i we have $g_i \circ u_i = u_{i-1} \circ f_i$.

Remark. A map $u\colon (X_i, f_i) \to (Y_i, g_i)$ of inverse systems induces a map $u_*\colon \varprojlim X_i \to \varprojlim Y_i$ and also a map $u_*\colon \varprojlim^1 X_i \to \varprojlim^1 Y_i$ in the obvious way.[1107]

Now we can formulate the following proposition.

Proposition 76.10. *Let $(\{C_i^*\}_{i\in\mathbb{N}}, \{f_i\colon C_{i+1}^* \to C_i^*\}_{i\in\mathbb{N}})$ be an inverse system of cochain complexes over $\mathbb{N}$. If for each $k \in \mathbb{N}_0$ the inverse system of cochain groups $\{C_i^k\}_{i\in\mathbb{N}}$ satisfies the Mittag-Leffler condition, then there exists a natural*[1108] *short exact sequence of the form*

$$0 \;\to\; \varprojlim_{i\in\mathbb{N}}{}^1\, H^{n-1}(C_i^*) \;\to\; H^n\Big(\varprojlim_{i\in\mathbb{N}} C_i^*\Big) \;\to\; \varprojlim_{i\in\mathbb{N}} H^n(C_i^*) \;\to\; 0$$

where the right-hand map is the obvious map.[1109] *If R is a commutative ring and if the C_i^* are cochain complexes of R-modules, then the maps in the short exact sequence are natural R-module homomorphisms.*

[1107] Whatever obvious means.

Lemma 76.11. *Given any short exact sequence*

$$0 \;\to\; (X_i, f_i) \;\xrightarrow{u_i}\; (Y_i, g_i) \;\xrightarrow{v_i}\; (Z_i, h_i) \;\to\; 0$$

of inverse systems of abelian groups there exists a natural exact sequence

$$0 \to \varprojlim X_i \xrightarrow{u_*} \varprojlim Y_i \xrightarrow{v_*} \varprojlim Z_i \xrightarrow{\delta} \varprojlim{}^1 X_i \xrightarrow{u_*} \varprojlim{}^1 Y_i \xrightarrow{v_*} \varprojlim{}^1 Z_i \to 0.$$

If R is a commutative ring and if all cochain complexes and all maps are in the category of R-modules, then the maps in short exact sequence are natural R-module homomorphisms.

Proof. We consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \prod\limits_{i\in\mathbb{N}} X_i \arrow[r, "u"] \arrow[d, "\mu"] & \prod\limits_{i\in\mathbb{N}} Y_i \arrow[r, "v"] \arrow[d, "\mu"] & \prod\limits_{i\in\mathbb{N}} Z_i \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & \prod\limits_{i\in\mathbb{N}} X_i \arrow[r, "u"] & \prod\limits_{i\in\mathbb{N}} Y_i \arrow[r, "v"] & \prod\limits_{i\in\mathbb{N}} Z_i \arrow[r] & 0.
\end{tikzcd}$$

The horizontal sequences are evidently exact. We can view the rows as chain complexes. The desired natural exact sequence now follows from the snake lemma that we proved in Exercise 43.9. ■

Now we can provide the proof of Proposition 76.10.

Proof of Proposition 76.10 (∗). Let $(\{\mathrm{C}_i^*\}_{i\in\mathbb{N}}, \{f_i\}_{i\in\mathbb{N}})$ be an inverse system of cochain complexes such that for each $k \in \mathbb{N}_0$ the inverse system of cochain groups $\{\mathrm{C}_i^k\}_{i\in\mathbb{N}}$ satisfies the Mittag-Leffler condition. In the following, given $m \in \mathbb{N}_0$ and $i \in \mathbb{N}$, we denote by $B_i^m = \mathrm{im}(\mathrm{C}_i^{m-1} \to \mathrm{C}_i^m)$ the image of the $(m-1)$-st coboundary map and we denote by $Z_i^m = \ker(\mathrm{C}_i^m \to \mathrm{C}_i^{m+1})$ the kernel of the m-th coboundary map. These give rise to the following two short exact sequences of inverse systems

$$\text{(i)} \qquad 0 \;\to\; B_i^m \;\to\; Z_i^m \;\to\; \mathrm{H}^m(\mathrm{C}_i) \;\to\; 0$$

and

$$\text{(ii)} \qquad 0 \;\to\; Z_i^m \;\to\; C_i^m \;\to\; B_i^{m+1} \;\to\; 0.$$

Let $m \in \mathbb{N}_0$. By our hypothesis the inverse systems $\{\mathrm{C}_i^m\}_{i\in\mathbb{N}}$ satisfy the Mittag-Leffler condition. It is straightforward to see that then the inverse system $\{B_i^m\}_{i\in\mathbb{N}}$ also satisfies the Mittag-Leffler condition. It follows from Lemma 76.9 that

$$\varprojlim{}^1 B_i^{m+1} \;=\; 0 \quad \text{and} \quad \varprojlim{}^1 C_i^m \;=\; 0.$$

[1108] As so often it requires a few moments of pondering to figure out what "natural" is actually supposed to mean.

[1109] Here it takes a little effort to figure out what the "obvious" map actually is supposed to be. By definition an inverse limit of cochain complexes comes with cochain maps $\varprojlim \mathrm{C}_i^* \to \mathrm{C}_i^*$. Since cohomology of cochain complexes is covariant in cochain maps we get induced maps $\mathrm{H}^n(\varprojlim \mathrm{C}_i) \to \mathrm{H}^n(\mathrm{C}_i^*)$ which then, by definition of the inverse limit, give rise to a map $\mathrm{H}^n(\varprojlim \mathrm{C}_i) \to \varprojlim \mathrm{H}^n(\mathrm{C}_i^*)$.

(Here and throughout the proof all limits are taken over $i \in \mathbb{N}$.) Using this data we obtain from Lemma 76.11, applied to the two short exact sequences (i) and (ii) of inverse systems above, the following two exact sequences

$$0 \to \varprojlim B_i^m \to \varprojlim Z_i^m \to \varprojlim \mathrm{H}^m(\mathrm{C}_i) \xrightarrow{\delta} \underbrace{\varprojlim{}^1 B_i^m}_{=0} \to \varprojlim{}^1 Z_i^m \to \varprojlim{}^1 \mathrm{H}^m(\mathrm{C}_i) \to 0$$

and

$$0 \to \varprojlim Z_i^m \to \varprojlim \mathrm{C}_i^m \to \varprojlim B_i^{m+1} \xrightarrow{\delta} \varprojlim{}^1 Z_i^m \to \underbrace{\varprojlim{}^1 \mathrm{C}_i^m}_{=0} \to \varprojlim{}^1 B_i^{m+1} \to 0$$

From these sequence we deduce that the maps

(i) $$\varprojlim{}^1 Z_i^m \to \varprojlim{}^1 \mathrm{H}^m(\mathrm{C}_i)$$

and

(ii) $$\varprojlim Z_i^m \to Z^m\big(\varprojlim \mathrm{C}_i\big) := \ker\big(\varprojlim \mathrm{C}_i^m \to \varprojlim B_i^{m+1}\big)$$

are isomorphisms and that the following sequences are exact

(i') $$0 \to \varprojlim B_i^m \to \varprojlim Z_i^m \to \varprojlim \mathrm{H}^m(\mathrm{C}_i) \to 0.$$

and

(ii') $$0 \to \underbrace{B^m\big(\varprojlim \mathrm{C}_i\big)}_{=:\ker(\varprojlim \mathrm{C}_i^m \to \varprojlim B_i^{m+1})} \to \varprojlim B_i^m \to \varprojlim{}^1 Z_i^{m-1} \to 0.$$

From (ii') and the isomorphism (ii) we obtain the inclusions

$$B^m\big(\varprojlim \mathrm{C}_i\big) \subset \varprojlim B_i^m \subset \varprojlim Z_i^m = Z^m\big(\varprojlim \mathrm{C}_i\big)$$

which gives rise to the short exact sequence[1110]

$$0 \to \underbrace{\frac{\varinjlim B_i^m}{B^m(\varinjlim \mathrm{C}_i)}}_{=\varinjlim{}^1 \mathrm{H}^{m-1}(\mathrm{C}_i) \text{ by (ii') and (i)}} \to \underbrace{\frac{Z^m(\varinjlim \mathrm{C}_i)}{B^m(\varinjlim \mathrm{C}_i)}}_{=\mathrm{H}^m(\varinjlim \mathrm{C}_i) \text{ by def.}} \to \underbrace{\frac{\varinjlim Z_i^m}{\varinjlim B_i^m}}_{=\varinjlim \mathrm{H}^m(\mathrm{C}_i) \text{ by (i')}} \to 0.$$

■

76.4. **The cohomology group of an ascending sequence of topological spaces.** The following proposition can be viewed as the analogue of Proposition 47.4 for cohomology groups.

Proposition 76.12. *Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subsets such that $X = \bigcup_{i\in\mathbb{N}} X_i$ and such that one the following three conditions holds:*

(1) *every compact subset of X is contained in one of the X_i, or*
(2) *each X_i is open in X, or*

[1110]Here we use that a filtration $P \subset Q \subset R$ of three abelian groups induces a short exact sequence $0 \to Q/P \to R/P \to R/Q \to 0$.

(3) *X is a* CW*-complex and each X_i is a subcomplex.*

Then for any $n \in \mathbb{N}_0$ and any abelian group R there exists a natural short exact sequence

$$0 \to \varprojlim_{i\in\mathbb{N}}{}^1 \mathrm{H}^{n-1}(X_i;R) \xrightarrow{\tau} \mathrm{H}^n(X;R) \to \varprojlim_{i\in\mathbb{N}} \mathrm{H}^n(X_i;R) \to 0$$

where the right-hand map is induced by the restriction maps $\mathrm{H}^n(X;R) \to \mathrm{H}^n(X_i;R)$. If R is a commutative ring, then all maps are R-module homomorphisms.

Remark. In many situations the term $\varprojlim^1 \mathrm{H}^{n-1}(X_i;R)$ will be zero. But it can be non-zero and one should view the proposition as a warning that in general it is not true that "limits commute with everything".

Proof. It follows easily from the Finiteness Theorem 36.14 and Lemma 25.8 we only have to prove the proposition under the hypothesis (1). Therefore suppose that we are given a topological space X and subspaces $X_1 \subset X_2 \subset X_3 \subset \dots$ with $X = \bigcup\limits_{i\in\mathbb{N}} X_i$ and such that the following condition holds:

(1) every compact subset of X is contained in one of the X_i.

Furthermore let R be a commutative ring.

For $i \in \mathbb{N}$ we now consider the cochain complex $D_i^* := \mathrm{Hom}(\mathrm{C}_*(X_i), R)$ and for $i \in \mathbb{N}$ we denote by $f_i \colon D_{i+1}^* \to D_i^*$ the map that is induced by the inclusion $X_i \to X_{i+1}$.

Claim. For every choice of $k \in \mathbb{N}$ the inverse system of cochain groups that is given by $(\{D_i^k\}_{i\in\mathbb{N}}, \{f_i \colon D_{i+1}^k \to D_i^k\}_{i\in\mathbb{N}})$ satisfies the Mittag-Leffler condition.

Let $k \in \mathbb{N}_0$. By Lemma 76.8 it suffices to show that for every $i \in \mathbb{N}$ the map

$$D_{i+1}^k = \mathrm{Hom}(\mathrm{C}_n(X_{i+1});R) \xrightarrow{f_i} \mathrm{Hom}(\mathrm{C}_n(X_i);R) = D_i^k$$

is an epimorphism. Since $X_i \subset X_{i+1}$ we know that the set of singular chains in X_i is a subset of the set of singular chains in X_{i+1}. It follows from the discussion on page 1816 that the map $\mathrm{Hom}(\mathrm{C}_n(X_{i+1});R) \to \mathrm{Hom}(\mathrm{C}_n(X_i);R)$ is indeed an epimorphism. ⊞

Let $n \in \mathbb{N}_0$. From Proposition 76.10 we obtain the short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{H}^{n-1}(D_i^*) \to \mathrm{H}^n\left(\varprojlim D_i^*\right) \to \varprojlim \mathrm{H}^n(D_i^*) \to 0$$

of R-modules. Let $i \in \mathbb{N}$. We make the following two observations:

(1) By definition we have $\mathrm{H}^k(D_i^*) = \mathrm{H}^k(X_i;R)$ for every $k \in \mathbb{N}_0$.

(2) We have natural isomorphisms

$$\varprojlim D_i^* \underset{\substack{\uparrow \\ \text{by definition of the } D_i}}{=} \varprojlim \mathrm{Hom}(\mathrm{C}_*(X_i);R) \underset{\substack{\uparrow \\ \text{Lemma 73.3}}}{=} \mathrm{Hom}\left(\varinjlim \mathrm{C}_*(X_i), R\right) \underset{\substack{\uparrow \\ \text{by Lemma 47.5 and hypothesis (1)}}}{=} \mathrm{Hom}(\mathrm{C}_*(X), R).$$

Combining all of the above we obtain our desired short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{H}^{n-1}(X_i;R) \to \mathrm{H}^n(X;R) \to \varprojlim \mathrm{H}^n(X_i;R) \to 0.$$

■

We provide a few examples for the above proposition.

Examples.

(1) As in Lemma 47.7 we consider the real line with infinitely many 2-dimensional spheres attached, i.e. we consider

$$X \;:=\; (\mathbb{R} \cup S^2 \times \mathbb{Z})/\sim$$

where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{R}$ with $((0,0,-1),i) \in S^2 \times \mathbb{Z}$. (See Figure 1133 on the left.) Furthermore, given $k \in \mathbb{N}$ we consider the open subset

$$X_k \;:=\; \big(\big(-k-\tfrac{1}{2}, k+\tfrac{1}{2}\big) \cup S^2 \times \{-k,\dots,k\}\big)\big)/\sim.$$

We leave it to the reader to verify that we can make the identifications[1111]

$$\mathrm{H}^n(X_k) \;=\; \begin{cases} \mathbb{Z}, & \text{if } n=0,\\ \mathbb{Z}^{2k+1}, & \text{if } n=2,\\ 0, & \text{else.}\end{cases}$$

such that for each $k \in \mathbb{N}$ the inclusion maps $X_k \to X_{k+1}$ induce the identity map on $\mathbb{Z} = \mathrm{H}^0(X_{k+1}) \to \mathrm{H}^0(X_k) = \mathbb{Z}$ and such that the inclusions induce the map

$$\begin{array}{rcl} \mathbb{Z}^{2k+3} = \mathrm{H}^2(X_{k+1}) & \to & \mathrm{H}^2(X_k) = \mathbb{Z}^{2k+1} \\ (a_1,\dots,a_{2k+3}) & \mapsto & (a_2,\dots,a_{2k+2}) \end{array}$$

It follows from Lemma 76.8 and the above calculations that for each $n \in \mathbb{N}_0$ the direct system of cohomology groups $\{\mathrm{H}^n(X_k)\}_{k\in\mathbb{N}}$ satisfies the Mittag-Leffler condition. We obtain that

$$\mathrm{H}^n(X) \underset{\substack{\uparrow\\ \text{Proposition 76.12 and Lemma 76.9}}}{=} \varprojlim \mathrm{H}^n(X_k) \;=\; \varprojlim \begin{cases} \mathbb{Z}, & \text{if } n=0,\\ \mathbb{Z}^{2k+1}, & \text{if } n=2,\\ 0, & \text{else}\end{cases} \underset{\substack{\uparrow\\ \text{see page 1890}}}{=} \begin{cases} \mathbb{Z}, & \text{if } n=0,\\ \mathbb{Z}^{\mathbb{N}}, & \text{if } n=2,\\ 0, & \text{else.}\end{cases}$$

Here we recall that $\mathbb{Z}^{\mathbb{N}} = \{(x_1,x_2,\dots) \mid x_i \in \mathbb{Z}\}$. Note that in Lemma 47.7 we saw that $\mathrm{H}_2(X) \cong \mathbb{Z}^{(\mathbb{N})}$. In this case we see that $\mathrm{H}_2(X) \not\cong \mathrm{H}^2(X)$ since the former group is countable whereas the latter group is uncountable.[1112]

(2) We consider again the surface Σ_∞ of infinite genus that we had already encountered on page 740 and in Lemma 48.4. (See also Figure 1133 on the right.) Using a variation on example (1) and the proof of Lemma 25.11 one can show that

$$\mathrm{H}^n(\Sigma_\infty) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n=0,\\ \mathbb{Z}^{\mathbb{N}}, & \text{if } n=1,\\ 0, & \text{if } n \geq 2.\end{cases}$$

We leave the task of providing full details to the valiant reader.

We conclude this chapter with the following variation on Proposition 76.12. It can be viewed as the cohomological analogue of Lemma 48.1 (3).

[1111] One possible approach to showing these statements is to use the Mayer–Vietoris Theorem 74.15 for Cohomology Groups.

[1112] Of course we could have obtained the same observation from the Universal Coefficient Theorem 75.13 for Cohomology Groups.

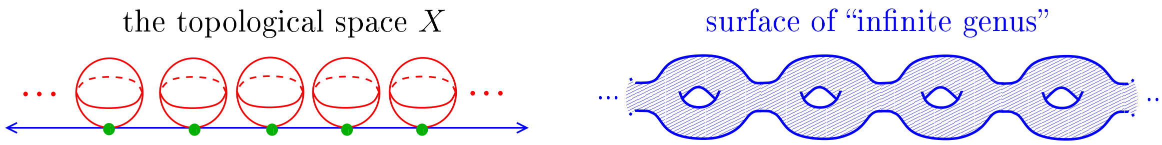


FIGURE 1133

Proposition 76.13. *Let X be a* CW*-complex and let $n \in \mathbb{N}$. For every $k > n$ and every abelian group G the inclusion $i\colon X^k \to X$ induces an isomorphism*

$$i^*\colon\ \mathrm{H}^n(X;G)\ \to\ \mathrm{H}^n(X^k;G).$$

Remark. We will give a very different proof of this proposition in Proposition **??**.

PROOF. By Proposition 76.12 there exists a natural short exact sequence

$$0\ \to\ \varprojlim_{j\in\mathbb{N}}{}^{1}\,\mathrm{H}^{n-1}(X_j;G)\ \xrightarrow{\tau}\ \mathrm{H}^n(X;G)\ \to\ \varprojlim_{j\in\mathbb{N}}\mathrm{H}^n(X_j;G)\ \to\ 0$$

where the right-hand map is induced by the restriction maps $\mathrm{H}^n(X;G)\ \to\ \mathrm{H}^n(X_i;G)$. It follows immediately from Lemmas 76.6 and Lemma 76.8 that is suffices to prove the following claim.

Claim. For any $j \geq k > n$ the inclusion induced map $\mathrm{H}^n(X^j;G)\ \to\ \mathrm{H}^n(X^k;G)$ is an isomorphism.

It follows from Lemma 48.1, the Universal Coefficient Theorem 75.13 for Cohomology Groups that for every $m \geq n$ we have $\mathrm{H}^n(X^{m+1},X^m;G)=0$. Using the long exact sequence in cohomology of the pairs $(X^{m+1},X^m),(X^{m+2},X^m),\dots$ one can now easily show that for any $j \geq k > n$ the inclusion induced map $\mathrm{H}^n(X^j;G)\to\mathrm{H}^n(X^k;G)$ is an isomorphism. ■

Exercises for Chapter 76.

Exercise 76.1. Let $(I,\leq)$ be a directed set and let $(\{X_i\}_{i\in I},\{f_{ij}\colon X_i\ \to\ X_j\}_{i\leq j})$ be a direct system of abelian groups over I.

(a) Let $a_j \in X_j$ and let $a_k \in X_k$. Prove the following statement: if the images of a_j and a_k in $\varinjlim X_i$ agree, then there exists an $l \in I$ with $l \geq j$ and $l \geq k$ such that $f_{jl}(a_j)=f_{kl}(a_k)\in X_l$.
(b) Show that if all maps $f_{ij}\colon X_i\to X_j$ are isomorphisms, then for any $j\in I$ the obvious map $X_j\to\varinjlim X_i$ is an isomorphism.

Exercise 76.2. We consider the inverse systems

$$D_*\colon\ \dots\ \xrightarrow{0}\ \mathbb{Z}\ \xrightarrow{\cdot 2}\ \mathbb{Z}\ \xrightarrow{0}\ \mathbb{Z}\ \xrightarrow{\cdot 2}\ \mathbb{Z}\ \to\ 0$$

and

$$E_*\colon\ \dots\ \xrightarrow{\cdot 5}\ \mathbb{Z}\ \xrightarrow{\cdot 4}\ \mathbb{Z}\ \xrightarrow{3}\ \mathbb{Z}\ \xrightarrow{\cdot 2}\ \mathbb{Z}\ \to\ 0.$$

(a) Show that $\varprojlim D_*=0$,

(b) show that $\varprojlim^1 D_* = 0$,

(c) show that $\varprojlim^1 E_* \neq 0$.

Exercise 76.3. Show that there exists a nested sequence of topological spaces X_i such that for each $i \in \mathbb{N}$ we have $\mathrm{H}^1(X_i;\mathbb{Z}) \cong \mathbb{Z}$ and such that for each $i \in \mathbb{N}$ the map $\mathrm{H}^1(X_{i+1};\mathbb{Z}) \to \mathrm{H}^1(X_i;\mathbb{Z})$ is given by multiplication by $i+1$.
Put differently, we want topological spaces and maps such that the inverse system

$$\mathrm{H}^1(X_4) \;\to\; \mathrm{H}^1(X_3) \;\to\; \mathrm{H}^1(X_2) \;\to\; \mathrm{H}^1(X_1)$$

is isomorphic to the inverse system E_* from Exercise 76.2.

77. Cohomology with compact support and bounded cohomology

77.1. **The definition of cohomology with compact support and basic properties.** In this section we introduce "cohomology with compact support", which is an interesting variation on the usual cohomology groups. Later on it will be a key tool in proving the Poincaré Duality Theorem, which is the most important result regarding homology and cohomology groups of topological manifolds.

We start out with the following definition.

Definition. Let X be a topological space and let G be an abelian group.

(1) Given a subset $A \subset X$ we say that a cochain $\varphi \in \mathrm{C}^i(X;G) = \mathrm{Hom}(\mathrm{C}_i(X), G)$ *vanishes on* A if $\varphi(c) = 0$ for all singular i-simplices $c\colon \Delta^i \to X$ in A, i.e. if φ vanishes on the image of $\mathrm{C}_i(A) \to \mathrm{C}_i(X)$.
(2) Given a subset $B \subset X$ we say that a cochain *vanishes outside* B if the cochain vanishes on $X \setminus B$.
(3) We say that a cochain $\varphi \in \mathrm{C}^i(X;G) = \mathrm{Hom}(\mathrm{C}_i(X), G)$ has *compact support* if there exists a compact set $K \subset X$ such that φ vanishes outside of K. We denote the set of all i-cochains with compact support by $\mathrm{C}^i_{\mathrm{c}}(X;G)$.

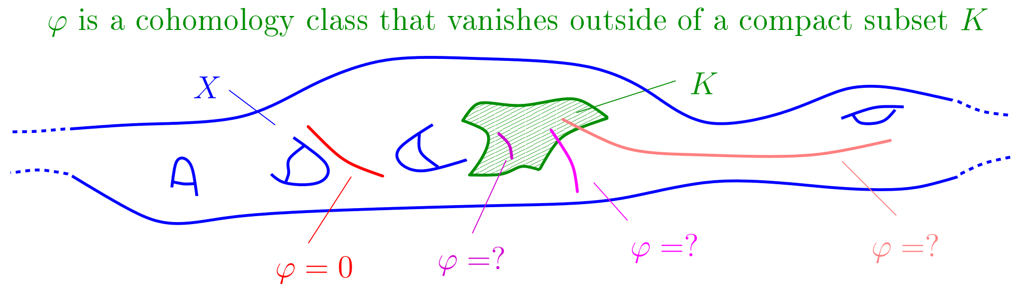


Figure 1134

Remark. Let X be a topological space and let G be an abelian group.

(1) Let A be a subset of X and let $n \in \mathbb{N}_0$. The inclusion $(X, \varnothing) \to (X, X \setminus A)$ induces an epimorphism $\mathrm{C}_n(X) \to \mathrm{C}_n(X, X \setminus A)$ and thus it also induces a monomorphism $\mathrm{C}^n(X, X \setminus A; G) \to \mathrm{C}^n(X;G)$. We have

$$\begin{aligned}\mathrm{im}(\mathrm{C}^n(X, X\setminus A; G) \to \mathrm{C}^n(X;G)) &= \text{all maps of the form } \mathrm{C}_n(X) \to \overbrace{\mathrm{C}_n(X, X\setminus A)}^{=\mathrm{C}_n(X)/\mathrm{C}_n(X\setminus A)} \to G\\ &= \{\varphi \in \mathrm{Hom}(\mathrm{C}_n(X), G) \,|\, \varphi \text{ vanishes on } X \setminus A\,\}\\ &= \{\varphi \in \mathrm{Hom}(\mathrm{C}_n(X), G) \,|\, \varphi \text{ vanishes outside } A\,\}.\end{aligned}$$

(2) On page 1824 we had made the identification

$$\mathrm{C}^0(X;G) \;=\; \{\text{maps from } X \text{ to } G\}.$$

With this identification we can now write

$$\mathrm{C}^0_{\mathrm{c}}(X;G) \;=\; \{\text{maps } f\colon X \to G \,|\, \begin{array}{c}\text{there exists a compact subset } K\\ \text{with } f|_{X\setminus K} \equiv 0\end{array}\}.$$

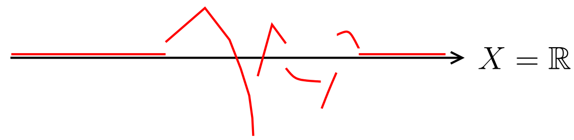


FIGURE 1135

The first question that arises is whether cochains with compact support give again rise to cohomology groups. The following lemma shows in particular that this is indeed the case.

Lemma 77.1. *Let X be a topological space and let G be an abelian group.*

(1) *A cochain $\varphi\in \mathrm{C}^i(X;G)$ has compact support if and only if there exists a compact subset K such that φ lies in the image of the map $\mathrm{C}^i(X,X\setminus K;G)\to \mathrm{C}^i(X;G)$.*
(2) *The set of cochains that have compact support form a subcomplex $(\mathrm{C}^*_{\mathrm{c}}(X;G);\delta)$ of the cochain complex $(\mathrm{C}^*(X;G);\delta)$.*
(3) *For any compact subset K of X the map $i_K\colon\ \mathrm{C}^*(X,X\setminus K;G)\to \mathrm{C}^*_{\mathrm{c}}(X;G)$ from (1) is a cochain map.*

PROOF. Let X be a topological space and let G be an abelian group. The first statement of the lemma follows immediately from the above remark.

We turn to the proof of the second statement.

Claim. Let $A,B\subset X$ be two subsets.

(a) If $\varphi\in \mathrm{C}^i(X;G)$ vanishes outside A and if $\psi\in \mathrm{C}^i(X;G)$ vanishes outside B, then for any $r,s\in\mathbb{Z}$ the linear combination $r\varphi+s\psi\in \mathrm{C}^i(X;G)$ vanishes outside the union $A\cup B$.
(b) If $\varphi\in \mathrm{C}^i(X;G)$ vanishes outside A, then $\delta\varphi\in \mathrm{C}^{i+1}(X;G)$ also vanishes outside A.

The first statement is obvious. Now we turn to the proof of the second statement. Suppose we are given a cochain $\varphi\in \mathrm{C}^i(X;G)$ that vanishes outside A. We need to show that $\delta\varphi\in \mathrm{C}^{i+1}(X;G)$ also vanishes outside A. So let c be a singular $(i+1)$-simplex in $X\setminus A$. Then

$$(\delta\varphi)(c) \;=\; \varphi(\partial c) \;=\; \sum_{j=0}^{n}(-1)^j\cdot\varphi(c\circ i_j^n) \underset{\uparrow}{=} 0.$$

the image of each $c\circ i_j^n$ also lies in $X\setminus A$ and φ vanishes outside A ⊞

The second statement is now an immediate consequence of the above claim and the observation that given two compact subsets $K,L\subset X$ the union $K\cup L$ is again a compact subset of X.

Finally the last statement is an immediate consequence of the definitions. ■

Definition. Let X be a topological space and let G be an abelian group. We define the *n-th cohomology group with compact support* $\mathrm{H}^n_{\mathrm{c}}(X;G)$ to be the cohomology of the cochain complex given by cochains with compact support, i.e. we define

$$\mathrm{H}^n_{\mathrm{c}}(X;G) \;:=\; \frac{\ker(\delta_n\colon\ \mathrm{C}^n_{\mathrm{c}}(X;G)\to \mathrm{C}^{n+1}_{\mathrm{c}}(X;G))}{\mathrm{im}(\delta_{n-1}\colon\ \mathrm{C}^{n-1}_{\mathrm{c}}(X;G)\to \mathrm{C}^{n}_{\mathrm{c}}(X;G))}.$$

Remark.

(1) If X is a compact topological space, then every cochain has compact support, in particular the cohomology groups with compact support are just the usual cohomology groups.

(2) Let X be a path-connected topological space. Then

$$\mathrm{H}^0_{\mathrm{c}}(X;G) \underset{\substack{\uparrow\\ \text{see proof of Proposition 73.11 and description of } \mathrm{C}^0_{\mathrm{c}}(X;G) \text{ on page 1901.}}}{=} \left\{ X\xrightarrow{f} G \,\middle|\, \begin{array}{l} f \text{ is constant and there exists a}\\ \text{compact subset } K \text{ with } \varphi|_{X\setminus K}\equiv 0\end{array}\right\} = \begin{cases} 0, & \text{if } X \text{ non-compact,}\\ G, & \text{if } X \text{ compact.}\end{cases}$$

Example. We consider the topological space $X=\mathbb{R}$. As on page 1827 we consider the singular 0-cochain

$$\varphi_0\colon\ \mathrm{C}_0(X) \;\to\; \mathbb{Z}$$

that is uniquely determined by the property that for any singular 0-simplex $x\in X$ we have

$$x \;\mapsto\; \begin{cases} 0, & \text{if } x\le 0,\\ 1, & \text{if } x>0.\end{cases}$$

It is clear that this 0-cochain does *not have* compact support. As on page 1827 we also consider the coboundary $\gamma_0 := \delta\varphi_0 \in \mathrm{C}^1(\mathbb{R};\mathbb{Z})$. Since γ_0 is a coboundary, it is in particular a cocycle. By definition $\gamma_0=\delta\varphi_0$ is uniquely determined by the property that for any singular 1-simplex $d\colon \Delta^1\to\mathbb{R}$ we have

$$(\delta\varphi_0)(d) \;=\; \varphi_0(\partial d) \underset{\substack{\uparrow\\ \text{see page 1079}}}{=} \varphi_0(d(0,1))-\varphi_0(d(1,0)).$$

We claim that $\gamma_0=\delta\varphi_0$ has compact support. In fact, we will now see that we can take K to be the compact set given by $K:=\{0\}$. Indeed, if $d\colon\Delta^1\to\mathbb{R}\setminus\{0\}$ is a singular 1-simplex, then it follows from the fact that Δ^1 is homeomorphic to the interval $[0,1]$ and the intermediate value theorem that either $d(0,1)$ and $d(1,0)$ both lie to the left of 0 or both lie to the right of 0. In both cases we obtain from the above formula that $\gamma_0(d)=(\delta\varphi_0)(d)=0$. Thus we see that $\gamma_0=\delta\varphi_0$ defines an element in $\mathrm{H}^1_{\mathrm{c}}(\mathbb{R})$.

Now we claim that $[\gamma_0]\in \mathrm{H}^1_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$ is non-zero. This means that we have to show that there is no $\psi\in \mathrm{C}^0_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$ with $\delta\psi=\gamma_0$. So let $\psi\in\mathrm{C}^0_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$. Following the discussion on page 1901 we view ψ as a function $\mathbb{R}\to\mathbb{Z}$ which is zero outside of a compact subset of $\mathbb{R}$. This implies that there exists a $C\ge 0$ such that $\psi(x)=0$ for all $x\notin[-C,C]$. Now we consider the singular 1-chain $d\colon\Delta^1\to\mathbb{R}$ that is given by $d(1-t,t)=-2C+4C\cdot t$ with $t\in[0,1]$. (See Figure 1136 for an illustration.) Then we have

$$\gamma_0(d) \;=\; (\delta\varphi_0)(d) \;=\; \varphi_0(d(0,1))-\varphi_0(d(1,0)) \;=\; \varphi_0(-2C)-\varphi_0(2C) \;=\; 0-1 \;=\; -1$$

but on the other hand we have

$$(\delta\psi)(d) \;=\; \psi(d(0,1))-\psi(d(1,0)) \;=\; \psi(-2C)-\psi(2C) \underset{\substack{\uparrow\\ \text{since } \psi \text{ is zero outside } [-C,C]}}{=} 0-0 \;=\; 0.$$

This concludes the proof that $\gamma_0 = \delta\varphi_0$ is non-trivial in $\mathrm{H}^1_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$.

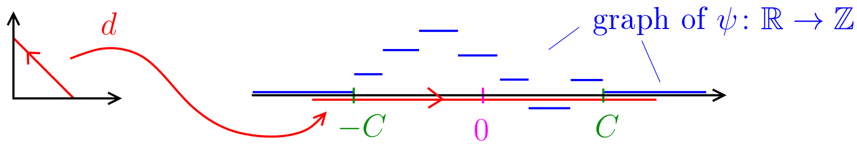


FIGURE 1136

The following lemma is an almost immediate consequence of Lemma 77.1 (1).

Lemma 77.2. *Let X be a topological space and let G be an abelian group. Given any $\varphi \in \mathrm{H}^i_{\mathrm{c}}(X;G)$ there exists a compact subset K such that φ lies in the image of the map $\mathrm{H}^i(X, X\setminus K;G) \to \mathrm{H}^i_{\mathrm{c}}(X;G)$.*

PROOF. Let $[\varphi] \in \mathrm{H}^i_{\mathrm{c}}(X;G)$. By Lemma 77.1 (1) there exists a compact subset K such that φ lies in the image of the map $\mathrm{C}^i(X, X\setminus K;G) \to \mathrm{C}^i_{\mathrm{c}}(X;G)$. Since $\mathrm{C}^*(X, X\setminus K;G)$ can be viewed as a subcomplex of $\mathrm{C}^*_{\mathrm{c}}(X;G)$ we see that φ is also a cocycle in $\mathrm{C}^*(X, X\setminus K;G)$, in particular φ lies in the image of the map $\mathrm{H}^i(X, X\setminus K;G) \to \mathrm{H}^i_{\mathrm{c}}(X;G)$. ■

We conclude this section with the discussion of induced maps on cohomology groups with compact support.

Let $f\colon X \to Y$ be a map between topological spaces and let G be an abelian group. We saw that f induces a map $f^*\colon \mathrm{C}^*(Y;G) \to \mathrm{C}^*(X;G)$. But in general this map does *not* restrict to a map $f^*\colon \mathrm{C}^*_{\mathrm{c}}(Y;G) \to \mathrm{C}^*_{\mathrm{c}}(X;G)$. For example, if $f\colon \mathbb{R} \to \{P\}$ is the map that sends every point in $\mathbb{R}$ to the point P and if $\varphi\colon \mathrm{C}_0(\{P\}) \to \mathbb{Z}$ is the map given by $P \mapsto 1$, then $f^*\varphi\colon \mathrm{C}_0(\mathbb{R}) \to \mathbb{Z}$ corresponds to the map that takes the value 1 on all of $\mathbb{R}$, in particular it does not have compact support.

A moment of reflection shows that if we want to get induced maps on cohomology groups with compact support, we have to restrict ourselves to "proper maps":

Definition. A map $f\colon X \to Y$ between topological spaces is called *proper* if the preimage of every compact subset of Y is a compact subset of X.

Examples.

(1) In Figure 1137 we show graphs of functions $\mathbb{R} \to \mathbb{R}$, some of the functions are proper and some are not proper.

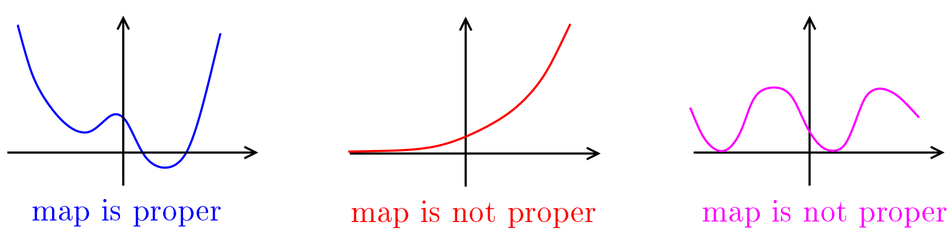


FIGURE 1137

(2) A map from a non-compact topological space to a compact topological space *cannot* be proper. In particular inclusion maps are not necessarily proper, for example the inclusion $i\colon (0,1) \to [-2,2]$ is not proper.

(3) Every homeomorphism is proper.
(4) Every map from a compact topological space to a Hausdorff space is proper. This statement follows immediately from Lemma 2.17 (1) and (2).

Remark. On page 342 we already introduced the adjective "proper" in a somewhat different way. Namely, given a *compact* topological manifold N and given any topological manifold M we defined on page 342 a map $\varphi\colon N \to M$ to be *proper* if $\varphi^{-1}(\partial M) = \partial N$. The good news is that, as we pointed out in the last of the above examples, any such map is also proper in the above sense. Unfortunately the converse does not hold. For example the inclusion map $f\colon N = [-1,1] \to \mathbb{R} = M$ is proper in the above sense, but not in the sense of the definition on page 342. Since both definitions of "proper" are commonly used in the literature we will have to live with this very unfortunate ambiguity. We will strive to make sure that, given a map between two topological manifolds, it is clear which of the two definitions of "proper" we are working with.

Lemma 77.3. *Let G be an abelian group.*

(1) *Let $f\colon X \to Y$ be a map between topological spaces. If f is a proper map, then the cochain map $f^*\colon \mathrm{C}^*(Y;G) \to \mathrm{C}^*(X;G)$ restricts to a cochain map*
$$f^*\colon \mathrm{C}^*_{\mathrm{c}}(Y;G) \;\to\; \mathrm{C}^*_{\mathrm{c}}(X;G).$$
In particular f^* induces a well-defined homomorphism
$$\mathrm{H}^*_{\mathrm{c}}(Y;G) \;\to\; \mathrm{H}^*_{\mathrm{c}}(X;G).$$
(2) *The maps $X \mapsto \mathrm{H}^n_{\mathrm{c}}(X;G)$ and $f \mapsto f^*$ define a contravariant functor*
$$\begin{array}{c}\textit{category } \mathcal{TopPropMap} \textit{ where objects are topological}\\ \textit{spaces and morphisms are proper maps}\end{array} \;\to\; \begin{array}{c}\textit{category } \mathcal{AbGr} \textit{ of}\\ \textit{abelian groups.}\end{array}$$
(3) *Homeomorphic topological spaces have isomorphic cohomology groups with compact support.*

Proof. Let $f\colon X \to Y$ be a proper map and let G be an abelian group. Let $\varphi\colon\ \mathrm{C}_n(Y) \to G$ be a cocycle with compact support. We pick a compact subset K such that φ vanishes on $Y \setminus K$. Then $f^*\varphi$ vanishes on $f^{-1}(Y \setminus K) = X \setminus f^{-1}(K)$, i.e. $f^*\varphi$ vanishes outside of the subset $f^{-1}(K)$. But $f^{-1}(K)$ is compact since f is proper. This concludes the proof of the first statement. The second statement and the third statement are obvious. ■

77.2. **Cohomology with compact support and direct limits.** Given a topological space X we consider the set $\mathcal{K}(X)$ of all compact subsets of X. The set $\mathcal{K}(X)$ together with the relation "$\subset$" defined by inclusion is a directed set since the union of two compact subsets is again compact. Furthermore, given any two compact subsets $K \subset L$ and any $i \in \mathbb{N}_0$ we get induced maps[1113]
$$\mathrm{H}^i(X, X \setminus K; G) \;\to\; \mathrm{H}^i(X, X \setminus L; G).$$

[1113]The most confusing part about cohomology is to figure out which way the maps go. In this case $K \subset L$ implies $X \setminus L \subset X \setminus K$ which means that we have a map of pairs $(X, X \setminus L) \to (X, X \setminus K)$ given by inclusion, which by the discussion on page 1830 implies that we do indeed get an induced map $\mathrm{H}^i(X, X \setminus K; G) \to \mathrm{H}^i(X, X \setminus L; G)$.

Since cohomology groups are contravariantly functorial we see that these cohomology groups form a *direct* system over the directed set $(\mathcal{K}(X), \subset)$ and we can form the direct limit

$$\varinjlim_{K \in \mathcal{K}(X)} \mathrm{H}^i(X, X \setminus K; G).$$

Recall that given any compact subset $K \subset X$ we have by Lemma 77.1 (3) a natural map $i_{K*}\colon \mathrm{H}^i(X, X \setminus K; G) \to \mathrm{H}^i_{\mathrm{c}}(X; G)$. Given compact subsets $L \subset M$ these maps give rise to the following commutative diagram:

$$\begin{tikzcd}
\mathrm{H}^i(X, X \setminus M; G) \arrow[dr, "\psi_M"'] \arrow[drr, bend left, "i_{M*}"] & & \\
& \varinjlim\limits_{K \in \mathcal{K}(X)} \mathrm{H}^i(X, X \setminus K; G) \arrow[r, dashed, "\exists!", "\Phi"'] & \mathrm{H}^i_{\mathrm{c}}(X; G). \\
\mathrm{H}^i(X, X \setminus L; G) \arrow[uu] \arrow[ur, "\psi_L"] \arrow[urr, bend right, "i_{L*}"'] & &
\end{tikzcd}$$

Since this diagram commutes for all compact subsets $L \subset M$ we get the induced dotted arrow by the universal property of the direct limit.

The following proposition says that the above dotted arrow is in fact an isomorphism. This gives in particular an often convenient reformulation of the concept of cohomology with compact support.

Proposition 77.4. *Let X be a topological space and let G be an abelian group. Then the above map*

$$\Phi\colon \varinjlim_{K \in \mathcal{K}(X)} \mathrm{H}^i(X, X \setminus K; G) \;\to\; \mathrm{H}^i_{\mathrm{c}}(X; G)$$

defines a natural isomorphism.[1114]

Proof. We need to show that the map Φ in the proposition is a bijection.

We first show that Φ is surjective. So let $\varphi \in \mathrm{H}^i_{\mathrm{c}}(X; G)$. By Lemma 77.2 there exists a compact subset L such that φ lies in the image of $\mathrm{H}^i(X, X \setminus L; G) \to \mathrm{H}^i(X; G)$. It follows immediately from the above commutative diagram that φ also lies in the image of $\Phi\colon \varinjlim \mathrm{H}^i(X, X \setminus K; G) \to \mathrm{H}^i_{\mathrm{c}}(X; G)$.

Now we show that the map Φ is injective. In the subsequent discussion, given a subset A of X, we identify the image of the monomorphism $\mathrm{C}^j(X, A; G) \to \mathrm{C}^j(X; G)$ with $\mathrm{C}^j(X, A; G)$, i.e. we view each $\mathrm{C}^j(X, A)$ as a subgroup of $\mathrm{C}^j(X)$.

Now suppose we are given an element c in the kernel of Φ. By Lemma 76.2 (1) we know that the element c lies in the image of some $\mathrm{H}^i(X, X \setminus L; G)$. We pick a representative $\varphi \in \mathrm{C}^i(X, X \setminus L; G)$. Since φ represents the zero class in $\mathrm{H}^i_{\mathrm{c}}(X; G)$ there exists a compactly supported $(i-1)$-cocycle η with $\delta\eta = \varphi$. We pick a compact subset M of X such that η

[1114] Here the term "natural" refers to the fact that both sides define a contravariant functor from the category where objects are topological spaces and morphisms are proper maps to the category of abelian groups.

vanishes on $X \setminus M$, i.e. according to the above notation and the discussion on page 1901 we have $\eta \in \mathrm{C}^{i-1}(X, X \setminus M)$. We consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{C}^{i-1}(X;G) \arrow[rrr, "\eta\to\varphi", "\delta"'] & & & \mathrm{C}^{i}(X;G) \\
& \mathrm{C}^{i-1}(X,X\setminus(M\cup L);G) \arrow[ul, hook] \arrow[r, "\delta"] & \mathrm{C}^{i}(X,X\setminus(M\cup L);G) \arrow[ur, hook] & \\
\mathrm{C}^{i-1}(X,X\setminus M;G) \arrow[uu, hook] \arrow[ur, hook] & & & \mathrm{C}^{i}(X,X\setminus L;G). \arrow[uu, hook] \arrow[ul, hook]
\end{tikzcd}$$

It follows immediately from the diagram that in $\mathrm{C}^i(X, X\setminus(M\cup L); G)$ we have $\delta\eta = \varphi$, which shows that $[\varphi]$ represents the zero class in $\mathrm{H}^i(X, X\setminus(M\cup L); G)$. It follows from Lemma 76.2 (2) or alternatively from the commutative diagram from page 1906 that $\psi_L([\varphi]) = c$ is zero. ■

Example. Now we turn to a key example, namely for each $k \in \mathbb{N}_0$ we consider the cohomology with compact support $\mathrm{H}^k_c(\mathbb{R}^n;\mathbb{Z})$ of $\mathbb{R}^n$. We have

$$\begin{array}{ccccc}
 & \text{Proposition 77.4} & & \text{Lemma 76.3} & \vdots \\
 & \downarrow & & \downarrow & \mathrm{H}^k(\mathbb{R}^n,\mathbb{R}^n\setminus\overline{B}_2(0);\mathbb{Z}) \\
 & & & & \uparrow \\
\mathrm{H}^k_c(\mathbb{R}^n;\mathbb{Z}) & = & \varinjlim\limits_{K\in\mathcal{K}(\mathbb{R}^n)} \mathrm{H}^k(\mathbb{R}^n,\mathbb{R}^n\setminus K;\mathbb{Z}) & = & \varinjlim\limits_{s\in\mathbb{N}} \mathrm{H}^k(\mathbb{R}^n,\mathbb{R}^n\setminus\overline{B}_1(0);\mathbb{Z}) \\
 & & & & \uparrow \\
 & & & & \mathrm{H}^k(\mathbb{R}^n,\mathbb{R}^n\setminus\overline{B}_0(0);\mathbb{Z}) \\
 & \cong & \mathrm{H}^k(\mathbb{R}^n,\mathbb{R}^n\setminus\overline{B}_0(0);\mathbb{Z}) & \cong & \begin{cases} \mathbb{Z}, & \text{if } k=n, \\ 0, & \text{otherwise.} \end{cases} \\
 & \uparrow & & \uparrow &
\end{array}$$

by Lemma 43.18 (2) all vertical maps in the direct system are isomorphisms, hence we can apply Lemma 76.2

same argument as in the proof of Lemma 86.4

We draw three interesting conclusions from this example:

(1) This example shows that cohomology with compact support can be used to show that the topological spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$.
(2) The example also shows that cohomology with compact support, in contrast to basically all other invariants we have studied so far, is not the same for homotopy equivalent topological spaces.
(3) Finally we consider the case $n = 1$ in more detail. We consider again the singular 1-cochain $\gamma_0 := \delta\varphi_0$ from page 1827. On pages 1835 and 1903 we saw that γ_0 defines an element in each $\mathrm{H}^1(\mathbb{R}, \mathbb{R}\setminus[-s,s];\mathbb{Z})$ and an element in $\mathrm{H}^1_c(\mathbb{R};\mathbb{Z})$. Furthermore on page 1835 we saw that for each $s \in N$ the class $[\gamma_0] \in \mathrm{H}^1(\mathbb{R}, \mathbb{R}\setminus[-s,s];\mathbb{Z})$ is a generator. The above argument now implies that $[\gamma_0] \in \mathrm{H}^1_c(\mathbb{R};\mathbb{Z})$ is also a generator.

Example. In Exercise 77.2 we will use cohomology with compact support to show that the topological spaces

$$X := \overline{B}_1(0) \cup x\text{-axis} \quad \text{and} \quad Y := \overline{B}_1(0) \cup x\text{-axis} \cup y\text{-axis}$$

that are illustrated in Figure 1138 are not homeomorphic.

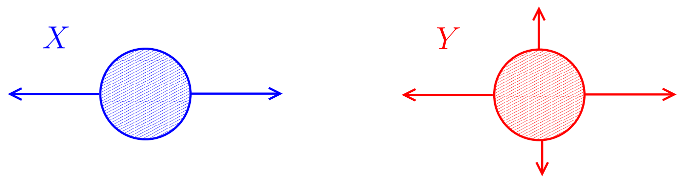


FIGURE 1138

In the previous examples cohomology with compact support turns out to be "bigger" than ordinary cohomology. On the other hand, Let us once again consider the topological space

$$X \;:=\; (\mathbb{R} \cup S^2 \times \mathbb{Z})/\sim$$

where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{Z} \subset \mathbb{R}$ with $((0,0,-1),i) \in S^2 \times \mathbb{Z}$. (See Figure 1139 for an illustration.) In Exercise 77.6 that

$$\mathrm{H}^2_{\mathrm{c}}(X) \;\cong\; \mathbb{Z}^{(\mathbb{N})},$$

in particular $\mathrm{H}^2_{\mathrm{c}}(X)$ is countable, whereas on page 1898 we saw that

$$\mathrm{H}^2(X) \;\cong\; \mathbb{Z}^{\mathbb{N}}$$

is uncountable.

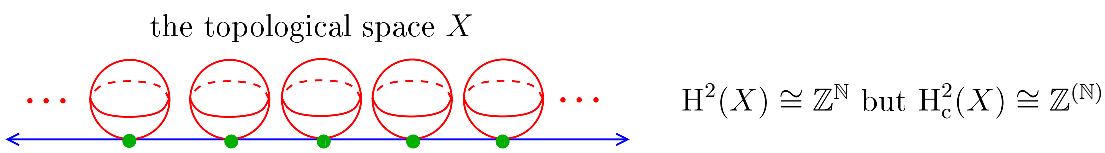


FIGURE 1139

In the following we will see the rather surprising fact that under fairly mild hypotheses an inclusion $U \to V$ of topological spaces induces for every $n \in \mathbb{N}_0$ a map $\mathrm{H}^n_{\mathrm{c}}(U) \to \mathrm{H}^n_{\mathrm{c}}(V)$ which "goes the same way".

We recall that given a topological space X we denote by $\mathcal{K}(X)$ the set of all compact subsets of X. Suppose U is an open subset of a topological space V and suppose that V is Hausdorff. We denote by $i_{UV*} = i_*\colon\ \mathrm{H}^n_{\mathrm{c}}(U) \to \mathrm{H}^n_{\mathrm{c}}(V)$ the composition of the following maps

$$\mathrm{H}^n_{\mathrm{c}}(U) \;\overset{\text{Proposition 77.4}}{=}\; \varinjlim_{K\in\mathcal{K}(U)} \mathrm{H}^n(U, U\setminus K) \;\overset{\cong}{\longleftarrow}\; \varinjlim_{K\in\mathcal{K}(U)} \mathrm{H}^n(V, V\setminus K) \;\to\; \varinjlim_{K\in\mathcal{K}(V)} \mathrm{H}^n(V, V\setminus K) \;\overset{\text{Proposition 77.4}}{=}\; \mathrm{H}^n_{\mathrm{c}}(V).$$

by the Excision Theorem 43.20 (2) the map $\mathrm{H}^n(V, V\setminus K) \to \mathrm{H}^n(U, U\setminus K)$ is an isomorphism for every compact subset K of U

given by Lemma 76.3 (1) and the fact that by Lemma 2.40 we have $\mathcal{K}(U) \subset \mathcal{K}(V)$

Note that, confusingly, the induced map on cohomology with compact support "goes the same direction" as the inclusion map $i\colon U \to V$.

Lemma 77.5. *Given a topological space X that is Hausdorff the above construction defines for each $n \in \mathbb{N}_0$ a* covariant *functor*

$$\begin{array}{c}\textit{category where the objects are open subsets}\\ \textit{of } X \textit{ and where the morphisms are inclusions}\end{array} \to \begin{array}{c}\textit{category of}\\ \textit{abelian groups.}\end{array}$$

Proof (∗). Let X be a topological space that is Hausdorff and let $U \subset V \subset W$ be open subsets. Let $n \in \mathbb{N}_0$. We have to show that $i_{UW*} = i_{VW*} \circ i_{UV*} \colon \mathrm{H}^n_{\mathrm{c}}(U) \to \mathrm{H}^n_{\mathrm{c}}(W)$. We consider the following diagram:

$$\begin{tikzcd}
\varinjlim\limits_{K\in\mathcal{K}(U)} \mathrm{H}^n(U,U\setminus K) \arrow[rr, bend left, "i_{UV*}"] \arrow[ddrr, bend right, "i_{UW*}"'] & \varinjlim\limits_{K\in\mathcal{K}(U)} \mathrm{H}^n(V,V\setminus K) \arrow[l, "\cong"'] \arrow[r] & \varinjlim\limits_{K\in\mathcal{K}(V)} \mathrm{H}^n(V,V\setminus K) \arrow[dd, bend left, "i_{VW*}"] \\
& \varinjlim\limits_{K\in\mathcal{K}(U)} \mathrm{H}^n(W,W\setminus K) \arrow[ul, "\cong"] \arrow[u, "\cong"'] \arrow[r] \arrow[dr] & \varinjlim\limits_{K\in\mathcal{K}(V)} \mathrm{H}^n(W,W\setminus K) \arrow[u, "\cong"'] \arrow[d] \\
& & \varinjlim\limits_{K\in\mathcal{K}(W)} \mathrm{H}^n(W,W\setminus K).
\end{tikzcd}$$

It follows from the explicit description of the direct limits in Proposition 76.1 and the uniqueness statement in the universal property of the direct limits that the two smaller triangles and the rectangle commute. Hence the diagram commutes and we get the desired statement. ■

We conclude this section with the following proposition.

Proposition 77.6. *Let X be a topological space that is Hausdorff. Let $X_i, i \in \mathbb{N}$ be a sequence of subsets such that the following holds:*

(1) *the sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$,*
(2) *each X_i is open in X,*
(3) *and $X = \bigcup\limits_{i\in\mathbb{N}} X_i$.*

Then for any $n \in \mathbb{N}_0$ the above inclusion induced maps $\mathrm{H}^n_{\mathrm{c}}(X_i) \to \mathrm{H}^n_{\mathrm{c}}(X)$ induce an isomorphism

$$\varinjlim \underbrace{\mathrm{H}^n_{\mathrm{c}}(X_i)}_{\substack{\uparrow\\ \textit{by Lemma 77.5 this is a direct system with the above inclusion induced maps}}} \xrightarrow{\cong} \mathrm{H}^n_{\mathrm{c}}(X).$$

Proof (∗). Using Proposition 77.4 we see that we have to show that the map

$$\Phi\colon \varinjlim_{i\in\mathbb{N}} \varinjlim_{K\in\mathcal{K}(X_i)} \mathrm{H}^n(X_i, X_i \setminus K) \xrightarrow{\cong} \varinjlim_{K\in\mathcal{K}(X)} \mathrm{H}^n(X, X \setminus K)$$

is an isomorphism. We need to show that Φ is injective and surjective. The key to the proof of these two statements are the following observations:

(a) it follows from (1) and (2), together with Lemma 2.41, that every compact subset L of X is already contained in some X_i,

(b) it follows from the Excision Theorem 43.20 (2) that for each compact subset $L \subset X_i$ we get an inclusion induced isomorphism $\mathrm{H}^n(X, X \setminus L) \to \mathrm{H}^n(X_i, X_i \setminus L)$.

Now we prove that Φ is surjective, the proof that Φ is injective is fairly similar and we leave that part of the proof to the reader.

So let c be an element of the limit on the right-hand side. By Lemma 76.2 there exists a compact subset L of X and some $\varphi \in \mathrm{H}^n(X, X \setminus L)$ such that c is the image of φ under the map $\mathrm{H}^n(X, X \setminus L) \to \varinjlim \mathrm{H}^n(X, X \setminus K)$. By (a) there exists an $i \in \mathbb{N}$ with $L \subset X_i$. We consider the following commutative diagram

$$\begin{tikzcd}
\varinjlim\limits_{i\in\mathbb{N}} \varinjlim\limits_{K\in\mathcal{K}(X_i)} \mathrm{H}^n(X_i, X_i \setminus K) \arrow[r, "\Phi"] & \varinjlim\limits_{K\in\mathcal{K}(X)} \mathrm{H}^n(X, X \setminus K) \\
\mathrm{H}^n(X_i, X_i \setminus L) \arrow[u, "\beta"] & \mathrm{H}^n(X, X \setminus L). \arrow[l, "\alpha", "\cong"'] \arrow[u]
\end{tikzcd}$$

The bottom horizontal map is an isomorphism by (b). Since the diagram is commutative we now see that $\Phi(\beta(\alpha(\varphi))) = c$. ■

77.3. **Bounded cohomology** $(*)$**.** We had just seen that with a modest modification of the original definition of cohomology we obtain a new invariant which for non-compact topological spaces behaves quite differently from the usual cohomology groups.

In this section we will introduce another variation on cohomology, namely bounded cohomology. We will not make use of it later on, but it does play an important role in topology. A much more detail discussion of bounded cohomology is for example provided in [**Lö10**].

Definition. Let X be a topological space.

(1) Let $C \in \mathbb{R}$. We say that a cochain $\rho \in \mathrm{C}_n(X;\mathbb{R}) = \mathrm{Hom}(\mathrm{C}_n(X), \mathbb{R})$ is *C-bounded* if for all singular n-simplices $\sigma\colon \Delta^n \to X$ we have $|\rho(\sigma)| \leq C$.

(2) We say that a cochain $\rho \in \mathrm{C}_n(X;\mathbb{R}) = \mathrm{Hom}(\mathrm{C}_n(X), \mathbb{R})$ is *bounded* if it is C-bounded for some $C \in \mathbb{R}$.

(3) Given $i \in \mathbb{N}_0$ we write

$$\mathrm{C}^n_{\mathrm{b}}(X;\mathbb{R}) \;:=\; \{\text{all bounded cochains in } \mathrm{C}^n(X;\mathbb{R})\}.$$

Examples.

(1) Let X be a topological space. The 0-cochain that assigns to each point $x \in X$ the same value $c \in \mathbb{R}$ is bounded.

(2) The singular 1-cochain $\theta_{\mathbb{R}} \in \mathrm{C}^1; \mathbb{R})$ defined on page 1825 is *not* bounded. Indeed, given $n \in \mathbb{N}$ we denote by $\mu_n\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $(1-t, t) \mapsto e^{2\pi i n t}$. It follows easily from the definition of $\theta_{\mathbb{R}}$ that $\theta_{\mathbb{R}}(\mu) = n$.

Lemma 77.7. *Let X be a topological space. The bounded cochains $\mathrm{C}^*_{\mathrm{b}}(X)$ form a subcomplex of the cochain complex $(\mathrm{C}^*(X), \delta)$.*

Proof. Suppose that $\rho \in \mathrm{C}^n(X;\mathbb{R})$ is a cochain that is C-bounded. For any $\varphi\colon \Delta^{n+1} \to X$ we obtain that

$$|(\delta\rho)(\varphi)| \;=\; |\rho(\partial\varphi)| \underset{\substack{\uparrow \\ \text{definition of } \partial \text{ and linearity of } \rho}}{=} \left|\sum_{j=0}^{n+1}(-1)^j \cdot \rho(\varphi \circ i_j^{n+1})\right| \underset{\substack{\uparrow \\ \text{triangle inequality}}}{\le} \sum_{j=0}^{n+1}|\rho(\varphi \circ i_j^{n+1})| \underset{\substack{\uparrow \\ \text{since } \rho \text{ is } C\text{-bounded}}}{\le} (n+2)\cdot C.$$

This shows that $\delta\rho$ is $C \cdot (n+2)$-bounded, in particular it is bounded. This shows that bounded cochains $\mathrm{C}^*_{\mathrm{b}}(X)$ form a subcomplex of the cochain complex $(\mathrm{C}^*(X), \delta)$. ■

This lemma leads us straight to the following definition.

Definition. Let X be a topological space and let $n \in \mathbb{N}_0$. The *n-th bounded cohomology group of X* is defined as

$$\mathrm{H}^n_{\mathrm{b}}(X;\mathbb{R}) \;:=\; \frac{\ker(\delta_n\colon\ \mathrm{C}^n_{\mathrm{b}}(X;\mathbb{R}) \to \mathrm{C}^{n+1}_{\mathrm{b}}(X;\mathbb{R}))}{\mathrm{im}(\delta_{n-1}\colon\ \mathrm{C}^{n-1}_{\mathrm{b}}(X;\mathbb{R}) \to \mathrm{C}^n_{\mathrm{b}}(X;\mathbb{R}))}.$$

Examples.

(1) The proof of Proposition 73.11 can be used, without any changes, to show that 0-th bounded cohomology of a path-connected topological space agrees with 0-th cohomology with real coefficients.[1115]

(2) Every cochain on a topological space with finitely many points is bounded. Thus it follows from Lemma 73.12 that for a topological space $\{x_0\}$ consisting of a single point x_0 we have

$$\mathrm{H}^n_{\mathrm{b}}(\{x_0\};\mathbb{R}) \;=\; \mathrm{H}^n(\{x_0\};\mathbb{R}) \;=\; \begin{cases} 0, & \text{if } n \ge 1, \\ \mathbb{R}, & \text{if } n = 0. \end{cases}$$

Proposition 77.8.

(1) *Let $f\colon X \to Y$ be a map between topological spaces. The map $f^*\colon \mathrm{C}^*(Y;\mathbb{R}) \to \mathrm{C}^*(X;\mathbb{R})$ on singular cochains restricts to a cochain map $f^*\colon\ \mathrm{C}^*_{\mathrm{b}}(Y;\mathbb{R}) \to \mathrm{C}^*_{\mathrm{b}}(X;\mathbb{R})$. In particular the map f^* induces a map*

$$f^*\colon\ \mathrm{H}^n_{\mathrm{b}}(Y;\mathbb{R}) \;\to\; \mathrm{H}^n_{\mathrm{b}}(X;\mathbb{R}).$$

(2) *For each $n \in \mathbb{N}_0$ the maps $X \mapsto \mathrm{H}^n_{\mathrm{b}}(X;\mathbb{R})$ and $f \mapsto f^*$ define a contravariant functor from the category of topological spaces to the category of real vector spaces.*

(3) *Homotopic maps induce the same maps on bounded cohomology.*

(4) *If $f\colon X \to Y$ is a homotopy equivalence, then the induced map*

$$f^*\colon\ \mathrm{H}^n_{\mathrm{b}}(Y;\mathbb{R}) \;\xrightarrow{\cong}\; \mathrm{H}^n_{\mathrm{b}}(X;\mathbb{R})$$

is in fact an isomorphism.

[1115] Is this statement also true for topological spaces that are not path-connected?

PROOF.

(1) It is obvious that the pullback of a bounded cohomology class is again bounded. The first statement is an immediate consequence of this observation.
(2) This statement is obvious.
(3) Let $f, g\colon X \to Y$ be two maps between topological spaces X and Y. Furthermore let $F\colon X \times [0,1] \to Y$ be a homotopy between the maps f and g. As on page 1102 we define
$$P_n\colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_{n+1}(Y)$$
to be the map that is given by
$$(\sigma\colon \Delta^n \to X) \;\mapsto\; \sum_{j=0}^{n} (-1)^j \cdot F \circ (\sigma \times \mathrm{id}_{[0,1]}) \circ \underbrace{[v_0, \dots, v_j, w_j, \dots, w_n]}_{\Delta^{n+1} \to \Delta^n \times [0,1]}.$$
In the proof of Proposition 42.5 we saw that the maps P_n define a chain homotopy between f_* and g_*. It follows from this explicit description of P_n that for any cochain $\varphi \in \mathrm{C}^{n+1}(Y;\mathbb{R})$ that is C-bounded the chain $P^*\varphi \in \mathrm{C}^n(X;\mathbb{R})$ is $(n+1)\cdot C$-bounded, the map $P^*\colon\ \mathrm{C}^{n+1}(Y;\mathbb{R}) \to \mathrm{C}^n(X;\mathbb{R})$ restricts to a map $\mathrm{C}_{\mathrm{b}}^{n+1}(Y;\mathbb{R}) \to \mathrm{C}_{\mathrm{b}}^n(X;\mathbb{R})$. As in the proof of Lemma 73.8 (3) one sees easily that the maps P^* form a cochain homotopy between $f^*, g^*\colon\ \mathrm{C}_{\mathrm{b}}^*(Y;\mathbb{R}) \to \mathrm{C}_{\mathrm{b}}^*(X;\mathbb{R})$. The third statement now follows from Lemma 73.5.
(4) This statement follows immediately from (2) and (3). ■

So far it seems like bounded cohomology might be quite similar to usual cohomology with real coefficients. It turns out that this guess is dramatically misguided.

For example the following theorem was proved by Mikhail Gromov [**Grom83**, p. 40], see also [**Iv87**, Theorem 4.3].[1116]

Theorem 77.9. *Let $f\colon X \to Y$ be a map between two connected* CW*-complexes. If the induced map $f_*\colon \pi_1(X) \to \pi_1(Y)$ is an epimorphism of fundamental groups such that the kernel* $\ker(f_*\colon \pi_1(X) \to \pi_1(Y))$ *is an amenable group, then for any $n \in \mathbb{N}_0$ the induced map*
$$f^*\colon\ \mathrm{H}_{\mathrm{b}}^n(Y;\mathbb{R}) \;\xrightarrow{\cong}\; \mathrm{H}_{\mathrm{b}}^n(X;\mathbb{R})$$
is an isomorphism.

Remark. In the formulation of Theorem 77.9 we used the notion of an "amenable group". We will not give the definition of an "amenable group", instead we refer to [**Pie84**] or [**CSC10**, Section 4] for the definition. What matters to us are the following two facts:

(1) Groups that admit a finite index subgroup that is solvable are amenable, see [**Pie84**, Propositions 13.3 and 13.4]. In particular finite groups and solvable groups are amenable.
(2) Groups that contain a free group on two generators are not amenable, see [**Pie84**, Proposition 14.1].

[1116] Mikhail Gromov (*1943) is a Russian-French mathematician who is easily one of the most influential geometers alive. He won the Abel prize in 2009.

Now we can formulate the following rather surprising corollary.

Corollary 77.10. *Let X be a connected* CW*-complex such that $\pi_1(X)$ is amenable, e.g. such that $\pi_1(X)$ is solvable. Then $\mathrm{H}_{\mathrm{b}}^n(X;\mathbb{R}) = 0$ for any $n \geq 1$.*

Example. If follows from Corollary 77.10 that the higher bounded cohomology groups of the n-tori $(S^1)^n$, the spheres S^n and lens spaces all vanish. In particular this new invariant is also not useful in our attempt to answer Question 16.7.

PROOF. We consider the map $f\colon X \to \{x_0\}$ that is given by sending every point in X to the point x_0. Then for any $n \geq 1$ we have

$$\mathrm{H}_{\mathrm{b}}^n(X;\mathbb{R}) \underset{\substack{\uparrow \\ \text{by Theorem 77.9 since} \\ \ker(f_*) = \pi_1(X) \text{ is amenable}}}{\xrightarrow[\cong]{f^*}} \mathrm{H}_{\mathrm{b}}^n(\{x_0\};\mathbb{R}) \underset{\substack{\uparrow \\ \text{see above example}}}{=} 0.$$

We have thus proved the desired result. ■

The following vanishing result is in fact fairly easy to show, see e.g. [**Frig17**, Chapter 2].

Lemma 77.11. *For any* CW*-complex X we have $\mathrm{H}_{\mathrm{b}}^1(X;\mathbb{R}) = 0$.*

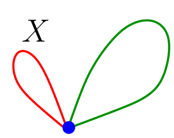
On the other hand, if the fundamental group is not amenable, then we also see some totally unexpected behavior as is shown in the following proposition.

Proposition 77.12. *Let X be the wedge of two circles. Then $\mathrm{H}_{\mathrm{b}}^2(X;\mathbb{R})$ and $\mathrm{H}_{\mathrm{b}}^3(X;\mathbb{R})$ are vector spaces of uncountable dimension.*

PROOF. The statement regarding $\mathrm{H}_{\mathrm{b}}^2(X;\mathbb{R})$ is proved in [**EF97**]. An alternative fairly straightforward proof, using quasi-morphisms as introduced on page 600, is given in [**Roll09**, Corollary 2.3]. Furthermore the statement regarding $\mathrm{H}_{\mathrm{b}}^3(X;\mathbb{R})$ is proved in [**So97**, Theorem 3]. ■

Somewhat shockingly the following question is still open:

Question 77.13. *Let X be the wedge of two circles. What are the bounded cohomology groups $\mathrm{H}_{\mathrm{b}}^k(X;\mathbb{R}) \neq 0$ for $k \geq 4$? Is any of the groups non-zero? Are all of them non-zero? Are all of them of uncountable dimension?*



$$\begin{aligned} &\dim(H_b^1(X;\mathbb{R})) = 0 \\ &\dim(H_b^2(X;\mathbb{R})) = \infty \\ &\dim(H_b^3(X;\mathbb{R})) = \infty \\ &\dim(H_b^4(X;\mathbb{R})) = ??? \end{aligned}$$

FIGURE 1140

Exercises for Chapter 77.

Exercise 77.1. We consider $X = \mathbb{R}$ and the singular 0-cochain
$$\varphi_0\colon\ \mathrm{C}_0(X) \ \to\ \mathbb{Z}$$
that is uniquely determined by the property that for any singular 0-simplex $x \in X = \mathbb{R}$ we have
$$x \ \mapsto\ \begin{cases} 1, & \text{if } x \leq 0, \\ 0, & \text{if } x > 0. \end{cases}$$
On page 1903 we saw that $\gamma_0 = \delta\varphi_0\colon\ \mathrm{C}_1(\mathbb{R}) \to \mathbb{Z}$ defines a class in $\mathrm{H}^1_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$. Prove, only using the definitions, that this class is non-zero.

Exercise 77.2. What is the cohomology with compact support of the topological spaces
$$X \ :=\ \overline{B}_1(0) \cup x\text{-axis} \quad \text{and} \quad Y \ :=\ \overline{B}_1(0) \cup x\text{-axis} \cup y\text{-axis}$$
that are illustrated in Figure 1138.

Exercise 77.3.

(a) Give an example of a path-connected topological space X such that $\mathrm{H}^1(X) = 0$ but such that $\mathrm{H}^1_{\mathrm{c}}(X) \cong \mathbb{Z}^{(\mathbb{N})}$. You do not have to justify why your example has the correct properties.

(b) Give an example of a path-connected topological space X such that $\mathrm{H}^2(X) = 0$ but such that $\mathrm{H}^2_{\mathrm{c}}(X) \cong \mathbb{Z}^3$. You do not have to justify why your example has the correct properties.

Exercise 77.4. Let X be a topological space. We consider the set $\mathcal{K}(X)$ of all compact subsets of X. The set $\mathcal{K}(X)$ together with the relation "$\subset$" defined by inclusion is a directed set since the union of two compact subsets is again compact. Given any two compact subsets $K \subset L$ we get an induced map $\mathrm{H}_n(X, X \setminus L; G) \to \mathrm{H}_n(X, X \setminus K; G)$. As in Exercise 87.5 we define
$$\overleftarrow{\mathrm{H}}_n(X;\mathbb{Z}) \ :=\ \varprojlim_{K \in \mathcal{K}(X)} \ \mathrm{H}_n(X, X \setminus K; \mathbb{Z}).$$

(a) Recall that on page 1903 we introduced cohomology group with compact support $\mathrm{H}^n_{\mathrm{c}}(X;\mathbb{Z}) = \mathrm{H}^n_{\mathrm{c}}(X;\mathbb{Z})$. Define a useful pairing $\mathrm{H}^n_{\mathrm{c}}(X) \times \overleftarrow{\mathrm{H}}_n(X;\mathbb{Z}) \to \mathbb{Z}$.
Remark. You could make use of Exercise 25.21.

(b) Use (a) to show that $\mathrm{H}^{\mathrm{lf}}_1(\mathbb{R}) \neq 0$.

Exercise 77.5. Let M be a *compact* topological manifold. We write $\mathring{M} = M \setminus \partial M$. The cohomology group $\mathrm{H}^n_{\mathrm{c}}(\mathring{M})$ is homeomorphic to some other cohomology group that might look more familiar. What is it? You do not have to justify your answer.

Exercise 77.6. Let us once again consider the topological space
$$X \ :=\ (\mathbb{R} \cup (S^2 \times \mathbb{Z}))/\sim$$
where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{Z} \subset \mathbb{R}$ with $((0,0,-1), i) \in S^2 \times \mathbb{Z}$. On page 1898 we saw that $\mathrm{H}^2(X) \cong \mathbb{Z}^{\mathbb{N}}$. Show that $\mathrm{H}^2_{\mathrm{c}}(X) \cong \mathbb{Z}^{(\mathbb{N})}$.

Exercise 77.7. Show that for every path-connected topological space X the boundary cohomology $\mathrm{H}^1_b(X;\mathbb{R}) = 0$.

78. De Rham cohomology of smooth manifolds

In this chapter we give a concise introduction to the de Rham cohomology groups of a smooth manifold. In particular we will prove several statements for de Rham cohomology which are reminiscent of properties that we had already proved for singular cohomology. In Chapter 79 we will explore the relationship between de Rham cohomology and singular cohomology.

78.1. **Alternating forms and the wedge-product.** An *alternating k-form* on a real vector space V is a map

$$\begin{array}{rcl}\omega\colon V^k & \to & \mathbb{R}\\ (v_1,\dots,v_k) & \mapsto & \omega(v_1,\dots,v_k)\end{array}$$

with the following properties:

(i) ω is linear in each argument, i.e. for all $v,v'\in V$ and $\lambda\in\mathbb{R}$ we have

$$\begin{array}{rcl}\omega(\dots,v+v',\dots) & = & \omega(\dots,v,\dots)+\omega(\dots,v',\dots)\\ \omega(\dots,\lambda v,\dots) & = & \lambda\cdot\omega(\dots,v,\dots),\end{array}$$

(ii) swapping two arguments flips the signs, i.e. for any $v,v'\in V$ we have

$$\omega(\dots,v,\dots,v',\dots) \;=\; -\omega(\dots,v',\dots,v,\dots).$$

We denote the vector space of alternating k-forms on V by $\wedge^k V^*$. A 0-form on V is by definition a real number, i.e. we define $\wedge^0 V^*=\mathbb{R}$.

Example.

(1) An alternating 1-form is by definition simply a linear map $V\to\mathbb{R}$. In particular $\wedge^1 V^*=V^*:=\operatorname{Hom}(V,\mathbb{R})$ is the dual space of V.

(2) For $V=\mathbb{R}^n$ we denote by dx_i the 1-form which is given by

$$dx_i(v_1,\dots,v_n) \;=\; v_i.$$

Put differently, $dx_1,\dots,dx_n$ is the basis of $(\mathbb{R}^n)^*$ that is dual to the standard basis $e_1,\dots,e_n$ of $\mathbb{R}^n$.

(3) If $V=\mathbb{R}^n$, then for each $\lambda\in\mathbb{R}$ the map

$$\begin{array}{rcl}(\mathbb{R}^n)^n & \to & \mathbb{R}\\ (v_1,\dots,v_n) & \mapsto & \lambda\cdot\det(v_1\,\dots\,v_n)\end{array}$$

is an alternating n-form. It follows easily from the uniqueness of the determinant, see e.g. [**Lan93**, Proposition XIII.4.6] that each alternating n-form on $\mathbb{R}^n$ is of this form, i.e. $\wedge^n(\mathbb{R}^n)^*\cong\mathbb{R}$.

Now we recall the wedge product of 1-forms. In the following let V be an n-dimensional real vector space and let $\varphi_1,\dots,\varphi_k\in V^*$ be 1-forms. We consider the map

$$\begin{array}{rcl}\varphi_1\wedge\dots\wedge\varphi_k\colon V^k & \to & \mathbb{R}\\ (v_1,\dots,v_k) & \mapsto & \det\begin{pmatrix}\varphi_1(v_1) & \dots & \varphi_1(v_k)\\ \vdots & & \vdots\\ \varphi_k(v_1) & \dots & \varphi_k(v_k)\end{pmatrix}.\end{array}$$

One can easily verify that this is an alternating k-form. We refer to this alternating k-form as the *wedge product* of $\varphi_1, \ldots, \varphi_k$.

Next let $\varphi_1, \ldots, \varphi_n$ be a basis for V^* and let $k \in \{0, \ldots, n\}$. It follows from [**Lee02**, Proposition 14.8] or [**Loe14**, Chapter 20.13] that the wedge products

$$\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} \quad \text{with} \quad i_1 < i_2 < \cdots < i_k$$

form a basis of $\wedge^k V^*$ and that $\wedge^k V^* = 0$ for $k > \dim(V)$.

Now we can introduce the wedge product of general alternating forms. Let $\omega \in \wedge^k V^*$ and $\sigma \in \wedge^l V^*$. Then by the above we can write

$$\omega \;=\; \sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k} \cdot \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$$

and

$$\sigma \;=\; \sum_{j_1 < \cdots < j_l} b_{j_1, \ldots, j_l} \cdot \varphi_{j_1} \wedge \cdots \wedge \varphi_{j_l}.$$

Finally we define

$$\omega \wedge \sigma \;:=\; \sum_{i_1 < \cdots < i_k} \sum_{j_1 < \cdots < j_l} a_{i_1, \ldots, i_k} b_{j_1, \ldots, j_l} \cdot \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} \wedge \varphi_{j_1} \wedge \cdots \wedge \varphi_{j_l}.$$

One can show easily that $\omega \wedge \sigma$ is an alternating $(k+l)$-form. The following lemma summarizes some of the key properties of the wedge product of alternating forms. The lemma is proved in [**Lee02**, Proposition 14.11], but it is also straightforward to deduce the statements of the lemma from the definitions.

Lemma 78.1. *Let V be a real vector space.*

(1) *For any $\omega \in \wedge^k V^*$ and $\sigma \in \wedge^l V^*$ we have*

$$\omega \wedge \sigma \;=\; (-1)^{kl} \cdot \sigma \wedge \omega.$$

(2) *The map*

$$\wedge^k V^* \times \wedge^l V^* \;\to\; \wedge^{k+l} V^*$$

is bilinear and distributive.

(3) *For any $\omega \in \wedge^k V^*$, $\sigma \in \wedge^l V^*$ and $\tau \in \wedge^m V^*$ we have*

$$(\omega \wedge \sigma) \wedge \tau \;=\; \omega \wedge (\sigma \wedge \tau).$$

78.2. **The de Rham cohomology of a smooth manifold.** What are cohomology groups good for? The universal coefficient theorem 75.13 for cohomology group says that the cohomology groups do not contain more information than the homology groups. Furthermore cohomology groups have the "disadvantage" that the induced maps go the "wrong" direction which often leads to total confusion.

In this chapter we want to sketch one answer: the cohomology groups with $\mathbb{R}$-coefficients of a smooth manifold are isomorphic to the de Rham cohomology groups which are defined using differential forms.

Before we can show that such an isomorphism exists we first need to give the definition of de Rham cohomology groups and we need to state and prove some properties of de Rham cohomology groups.

Let M be an n-dimensional smooth manifold.

(1) A *differential k-form on M* assigns to each point $P \in M$ an alternating k-form on the tangent space $\mathrm{T}_P M$. Using charts one can introduce the notions of continuity and smoothness of differential k-forms. We refer to [**Lee02**, Chapter 14], [**Tu11**, Chapter 18.3] or [**Frie16a**] for details. It follows from the convention on page 1915 that differential 0-forms are precisely the same as smooth maps $M \to \mathbb{R}$.

(2) One of the key reasons for studying differential forms on smooth manifolds is that these are the "right" objects on an oriented smooth manifold to be integrated. More precisely, let M be a compact oriented k-dimensional smooth manifold and let ω be a continuous differential k-form on M. We pick a finite atlas $\{\Phi_i : U_i \to V_i\}_{i=1,\dots,m}$ for M such that each Φ_i is orientation-preserving. By Theorem 6.57 there exists a partition of unity $f_1, \dots, f_m$ subordinate to the open covering $M = U_1 \cup \dots \cup U_m$. We set

$$\int_M \omega \;:=\; \sum_{i=1}^m \underbrace{\int_{V_i} f(\Phi_i^{-1}(x)) \cdot \omega(\mathrm{D}(\Phi_i^{-1})_x(e_1), \dots, \mathrm{D}(\Phi_i^{-1})_x(e_k))}_{\text{Lebesgue-integral for function on } V_i \subset \mathbb{R}^k} \;\in\; \mathbb{R}.$$

We refer to [**Lee02**, Chapter 16] or to [**Tu11**, Chapter 22.3] for details of the definition and for a summary of the key properties. The integral has the following simple, but nonetheless essential property: if M is a compact 0-dimensional smooth manifold, i.e. $M = \{P_1, \dots, P_s\}$, where all points are given the positive orientation, then the integral of a differential 0-form, i.e. of a function $f : M \to \mathbb{R}$, is just given by the sum $f(P_1) + \dots + f(P_s)$.

(3) Let $k \in \mathbb{N}_0$. We define the k-th de Rham group as the real vector space[1117][1118]

$$\mathrm{C}^k_{\mathrm{dR}}(M) \;:=\; \{\text{smooth differential } k\text{-forms on } M\}.$$

If $f : M \to N$ is a smooth map between smooth manifolds, then it follows from [**Lee02**, p. 360] that f induces a homomorphism[1119]

$$\begin{array}{rcl} f^* : \mathrm{C}^k_{\mathrm{dR}}(N) & \to & \mathrm{C}^k_{\mathrm{dR}}(M) \\ \omega & \mapsto & \text{the differential } k\text{-form } f^*\omega \text{ which at a point } P \in M \text{ is given by} \\ & & \begin{array}{rcl} (\mathrm{T}_P M)^k & \to & \mathbb{R} \\ (v_1, \dots, v_k) & \mapsto & \omega_{f(P)}(\mathrm{D} f_P(v_1), \dots, \mathrm{D} f_P(v_k)) \end{array} \end{array}$$

[1117]Georges de Rham (1903-1990) was a Swiss mathematician. The subscript "dR" in our notation $\mathrm{C}^k_{\mathrm{dR}}(M)$ stands for de Rham. We use the subscript do distinguish it from the singular cochains $\mathrm{C}^k(M;\mathbb{R})$.

[1118]In most references, see e.g. [**Lee02, BoT82, Frie16a**] one writes $\Omega^k(M)$ for the set of all smooth differential k-forms on M. Since in a second we will view $\mathrm{C}^k_{\mathrm{dR}}(M) = \Omega^k(M)$ as the groups in a cochain complex we prefer the notation $\mathrm{C}^k_{\mathrm{dR}}(M)$.

[1119]It is clear that $f^*\omega$ is a differential k-form on M, what one needs to show is that $f^*\omega$ is in fact a smooth differential k-form.

(4) We refer to [**Lee02**, p. 363] for the definition of the differential[1120]

$$d_k \colon \mathrm{C}^k_{\mathrm{dR}}(M) \;\to\; \mathrm{C}^{k+1}_{\mathrm{dR}}(M).$$

Often, when the subscript k is understood from the context, we just write d instead of d_k. A smooth form ω with $d\omega = 0$ is called *closed*. A smooth form ω for which there exists a smooth form μ with $d\mu = \omega$ is called *exact*.

The following proposition summarizes some of the key properties of the integral of forms and of the differential.

Proposition 78.2.

(1) *Let $f \colon M \to N$ be a diffeomorphism between compact oriented connected k-dimensional smooth manifolds and let ω be a smooth differential k-form on N. Then*

$$\int_M f^*\omega \;=\; \epsilon \cdot \int_N \omega \qquad \textit{where } \epsilon \;=\; \begin{cases} +1, & \textit{if } f \textit{ is orientation-preserving,} \\ -1, & \textit{if } f \textit{ is orientation-reversing.} \end{cases}$$

(2) *Let M be a compact oriented k-dimensional smooth manifold and let ω be a smooth differential k-form on M. If $M = A \cup B$ where A and B are compact k-dimensional submanifolds such that $A \cap B$ is a $(k-1)$-dimensional submanifold, then*

$$\int_M f^*\omega \;=\; \int_A f^*\omega + \int_B f^*\omega.$$

(3) *Let M be a smooth manifold. For every any $k \in \mathbb{N}_0$ the composition*

$$d \circ d \colon \mathrm{C}^k_{\mathrm{dR}}(M) \;\to\; \mathrm{C}^{k+2}_{\mathrm{dR}}(M)$$

is the zero map.

(4) *Let $\omega \in \mathrm{C}^0_{\mathrm{dR}}(M)$ be a smooth differential 0-form, in other words, let $\omega \colon M \to \mathbb{R}$ be a smooth map. In this case the differential $d\omega$ equals the differential $\mathrm{D}\,\omega$ from Proposition 6.37. More precisely, given any $P \in M$ and given any $v \in \mathrm{T}_P M$ we have $(d\omega(P))(v) = \mathrm{D}\,\omega_P(v) \in \mathbb{R} = \mathrm{T}_{\omega(P)}\,\mathbb{R}$.*

(5) *Given any $\mu \in \mathrm{C}^k_{\mathrm{dR}}(M)$ and given any $\nu \in \mathrm{C}^l_{\mathrm{dR}}(M)$ we have*

$$d(\mu \wedge \nu) \;=\; d\mu \wedge \nu + (-1)^k \cdot \mu \wedge d\nu.$$

(6) *Let $f \colon M \to N$ be a smooth map between two smooth manifolds. Then for any smooth differential k-form $\omega \in \mathrm{C}^k_{\mathrm{dr}}(N)$ on N we have*

$$d(f^*\omega) \;=\; f^*(d\omega).$$

[1120] Here is a quick summary of the definition. Using a partition of unity and charts it suffices to define the differential for differential k-forms on an open subset $U \subset \mathbb{R}^n$. We consider a differential k-form

$$\omega \;=\; \sum_{i_1 < \cdots < i_k} f_{i_1 \ldots i_k} \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where $f_{i_1 \ldots i_k}$ are real-valued smooth functions on U. The differential of ω is then defined as the differential $(k+1)$-form

$$d\omega \;:=\; \sum_{i_1 < \cdots < i_k} df_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

PROOF. The first statement is [**Lee02**, Proposition 16.6]. The second statement follows fairly easily from the definitions and basic properties of the Lebesgue integral. We leave it to the reader to fill in the details. Finally the last four statements are proved in [**Lee02**, Theorem 14.24 and Proposition 14.26]. ■

The next theorem is one of the key results on differential forms. We refer to [**Lee02**, Theorem 16.11] for a proof.

Theorem 78.3. (Stokes' Theorem) *If M is a compact oriented k-dimensional smooth manifold with boundary ∂M and if $\omega \in \mathrm{C}^{k-1}_{\mathrm{dR}}(M)$, then*[1121]

$$\int\limits_M d\omega \;=\; \int\limits_{\partial M} \omega.$$

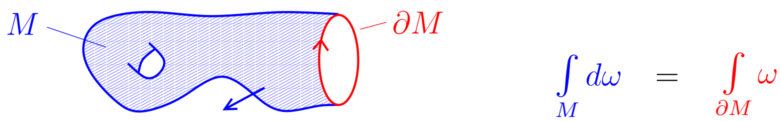


FIGURE 1141

Definition. Let M be a smooth manifold. As we had just pointed out in Proposition 78.2 (3), the differentials $d_k\colon \mathrm{C}^k_{\mathrm{dR}}(M) \to \mathrm{C}^{k+1}_{\mathrm{dR}}(M)$ have the property that $d_{k+1} \circ d_k = 0$. Thus we obtain a cochain complex

$$0 \;\to\; \mathrm{C}^0_{\mathrm{dR}}(M) \;\xrightarrow{d_0}\; \mathrm{C}^1_{\mathrm{dR}}(M) \;\xrightarrow{d_1}\; \mathrm{C}^2_{\mathrm{dR}}(M) \;\xrightarrow{d_2}\; \dots$$

in the sense of the discussion on page 1818. The *k–th de Rham cohomology group of M* is defined as the cohomology of this cochain complex, i.e. it is defined as

$$\mathrm{H}^k_{\mathrm{dR}}(M) \;:=\; \frac{\ker\left(d_k\colon\ \mathrm{C}^k_{\mathrm{dR}}(M) \to \mathrm{C}^{k+1}_{\mathrm{dR}}(M)\right)}{\mathrm{im}\left(d_{k-1}\colon\ \mathrm{C}^{k-1}_{\mathrm{dR}}(M) \to \mathrm{C}^k_{\mathrm{dR}}(M)\right)}.$$

Note that $\mathrm{H}^k_{\mathrm{dR}}(M)$ is the quotient of a real vector space by a subspace, in particular $\mathrm{H}^k_{\mathrm{dR}}(M)$ is itself a real vector space.

The following proposition summarizes some basic properties of de Rham cohomology groups.

Proposition 78.4. *Let M be an n-dimensional smooth manifold. Then the following hold:*

(1) *If $f\colon M \to N$ is a smooth map to another smooth manifold, then*

$$\begin{array}{rccc} f^*\colon & \mathrm{H}^k_{\mathrm{dR}}(N) & \to & \mathrm{H}^k_{\mathrm{dR}}(M) \\ & [\omega] & \mapsto & [f^*\omega] \end{array}$$

is a well-defined map.

(2) *If M is connected, then the map*

$$\begin{array}{ccc} \mathbb{R} & \to & \mathrm{H}^0_{\mathrm{dR}}(M) \\ r & \mapsto & [\textit{constant function } x \mapsto r] \end{array}$$

is an isomorphism. We use this map to make the identification $\mathrm{H}^0_{\mathrm{dR}}(M) = \mathbb{R}$.

[1121] Here the right-hand side is understood to be zero if $\partial M = \varnothing$.

(3) *If $f\colon M \to N$ is a smooth map between connected smooth manifolds, then the induced map $f^*\colon \mathrm{H}^0_{\mathrm{dR}}(N) \to \mathrm{H}^0_{\mathrm{dR}}(M)$ is an isomorphism.*
(4) $\mathrm{H}^k_{\mathrm{dR}}(M) = 0$ *for $k > n$.*
(5) *Suppose M consists of finitely many components $M_1, \dots, M_r$. Given $j \in \{1, \dots, r\}$ we denote by $i_j\colon M_i \to M$ the inclusion maps. Then for each $k \in \mathbb{N}_0$ the map*

$$\bigoplus_{j=1}^{r} i_j^*\colon\ \mathrm{H}^k_{\mathrm{dR}}(M) \ \to\ \bigoplus_{j=1}^{r} \mathrm{H}^k_{\mathrm{dR}}(M_j)$$

is an isomorphism.
(6) *If M has finitely many components, then $\dim_{\mathbb{R}}(\mathrm{H}^0_{\mathrm{dR}}(M))$ equals the number of components of M.*
(7) *If M is closed and oriented, then the map*

$$\begin{array}{rcl} \mathrm{H}^n_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ [\omega] & \mapsto & \int\limits_M \omega \end{array}$$

is well-defined. If M is non-empty, then the map is an epimorphism.
(8) *If M is closed orientable and non-empty, then $\mathrm{H}^n_{\mathrm{dR}}(M) \neq 0$.*

Sketch of proof.

(1) The first statement is an immediate consequence of Proposition 78.2 (6) together with Lemma 73.5 (1).
(2) The second statement follows fairly easily from the definitions and elementary arguments.
(3) The third statement follows immediately from the second statement.
(4) The fourth statement is an immediate consequence of the fact that for any vector space V we have $\wedge^k V^* = 0$ for $k > \dim(V)$, see page 1916.
(5) The fifth statement follows easily from the definition.
(6) The sixth statement is an immediate consequence of (2) and (5).
(7) The seventh statement is a consequence of Stokes' Theorem and a straightforward calculation.
(8) Finally the last statement follows immediately from the seventh statement. ■

Example. If $M = \{x\}$ is a smooth manifold consisting of a single point x, then it follows immediately from Proposition 78.4 (2) and (4) that

$$\mathrm{H}^k_{\mathrm{dR}}(\{x\}) \ = \ \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that we denote by $\mathcal{SmMfd}$ the category of smooth manifolds which is given by

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{SmMfd}) & := & \text{all smooth manifolds,} \\ \mathrm{Mor}(M, N) & := & \text{all smooth maps from } M \text{ to } N. \end{array}$$

It is straightforward to verify that for any $k \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} M & \mapsto & \mathrm{H}^k_{\mathrm{dR}}(M) \\ (f\colon M \to N) & \mapsto & (f^*\colon\ \mathrm{H}^k_{\mathrm{dR}}(N) \to \mathrm{H}^k_{\mathrm{dR}}(M)) \end{array}$$

define a contravariant functor from the category $\mathcal{SmMfd}$ of smooth manifolds to the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces.

78.3. **The Poincaré Lemma.** In the following two sections we will see that de Rham cohomology has properties that are quite similar to singular cohomology with real coefficients.

For example, in Lemma 73.13 we saw that homotopic maps $f, g\colon X \to Y$ between topological spaces induce the same map on cohomology groups. Now we will obtain a similar result for de Rham cohomology groups, but we will have to restrict ourselves to *smooth homotopies*.

Proposition 78.5. *Let M be a smooth manifold without boundary and let N be any smooth manifold. Let f and g be two smooth maps between M and N. If there exists a smooth homotopy between f and g, then for any $k \in \mathbb{N}_0$ we have*

$$f^* = g^*\colon \mathrm{H}^k_{\mathrm{dR}}(N) \to \mathrm{H}^k_{\mathrm{dR}}(M).$$

The key to the proof of the proposition is the following lemma.

Lemma 78.6. *Let M be a smooth manifold without boundary. For $r = 0, 1$ we denote by $i_r\colon M \to M \times [0,1]$ the inclusion maps given by $i_r(z) = (z, r)$. Then for any $k \in \mathbb{N}_0$ we have*

$$i_0^* = i_1^*\colon \mathrm{H}^k_{\mathrm{dR}}(M \times [0,1]) \to \mathrm{H}^k_{\mathrm{dR}}(M).$$

We will first show that Lemma 78.6 implies Proposition 78.5.

Proof of Proposition 78.5 assuming Lemma 78.6. So suppose that we are given a smooth homotopy $F\colon M \times [0,1] \to N$ between two maps $f, g\colon M \to N$. For $r = 0, 1$ let $i_r\colon M \to M \times [0,1]$ be the inclusion maps given by $i_r(z) = (z, r)$.[1122] Then

$$f^* = (F \circ i_0)^* \underset{\substack{\uparrow\\ \text{by functoriality}}}{=} i_0^* \circ F^* \underset{\substack{\uparrow\\ \text{by Lemma 78.6}}}{=} i_1^* \circ F^* \underset{\substack{\uparrow\\ \text{by functoriality}}}{=} (F \circ i_1)^* = g^*.$$

We have thus shown that the two maps f^* and g^* agree. ■

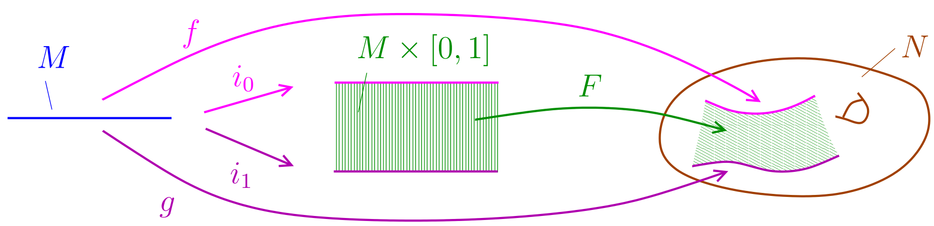


Figure 1142. Illustration for the proof of Proposition 78.5.

Proof of Lemma 78.6. Let M be an m-dimensional smooth manifold without boundary. By Lemma 73.5 it suffices to show that there exists a cochain homotopy

$$P_*\colon \mathrm{C}^*_{\mathrm{dR}}(M \times [0,1]) \to \mathrm{C}^{*-1}_{\mathrm{dR}}(M)$$

[1122]Note that the smooth structure of the smooth manifold $M \times [0,1]$ from Proposition 80.9 has the property that i_0 and i_1 are smooth maps. This implies that the maps i_0^* and i_1^* are actually defined.

from i_0^* to i_1^*, i.e. we have to find maps $\{P_n\colon \mathrm{C}^n_{\mathrm{dR}}(M\times[0,1]) \to \mathrm{C}^{n-1}_{\mathrm{dR}}(M)\}_{n\in\mathbb{N}_0}$ such that for each $n\in\mathbb{N}_0$ we have

$$d_{n-1}\circ P_n + P_{n+1}\circ d_n \;=\; i_0^* - i_1^*\colon\ \mathrm{C}^n_{\mathrm{dR}}(M\times[0,1]) \;\to\; \mathrm{C}^n_{\mathrm{dR}}(M).$$

For orientation we summarize in the following diagram all objects arising in the definition of a cochain homotopy:

$$\begin{tikzcd}
{} & \mathrm{C}^{n+1}_{\mathrm{dR}}(M\times[0,1]) \arrow[l, "d_{n+1}"'] \arrow[d, "i_0^*{,}i_1^*"] \arrow[dr, "P_{n+1}"] & \mathrm{C}^{n}_{\mathrm{dR}}(M\times[0,1]) \arrow[l, "d_n"'] \arrow[d, "i_0^*{,}i_1^*"] \arrow[dr, "P_n"] & \mathrm{C}^{n-1}_{\mathrm{dR}}(M\times[0,1]) \arrow[l, "d_{n-1}"'] \arrow[d, "i_0^*{,}i_1^*"] & {} \arrow[l, "d_{n-2}"'] \\
{} & \mathrm{C}^{n+1}_{\mathrm{dR}}(M) \arrow[l, "d_{n+1}"] & \mathrm{C}^{n}_{\mathrm{dR}}(M) \arrow[l, "d_n"] & \mathrm{C}^{n-1}_{\mathrm{dR}}(M) \arrow[l, "d_{n-1}"] & {} \arrow[l, "d_{n-2}"]
\end{tikzcd}$$

Given any $(x,t)\in M\times[0,1]$ we denote by $h\in \mathrm{T}_{(x,t)}(M\times[0,1])$ the tangent vector that is represented by the "horizontal" curve $s\mapsto (x,t+s)$. Given any $t\in[0,1]$ we have the obvious inclusion map $M\to M\times[0,1]$. Given any $x\in M$ this map induces an inclusion $\mathrm{T}_x M\to \mathrm{T}_{(x,t)}(M\times[0,1])$. We use this map to view $\mathrm{T}_x M$ as a subspace of $\mathrm{T}_{(x,t)}(M\times[0,1])$. Now we can write $\mathrm{T}_{(x,t)}(M\times[0,1]) = \mathrm{T}_x M\oplus\mathbb{R}\cdot h$.[1123] This decomposition is illustrated in Figure 1143. Given $n\in\mathbb{N}_0$ we consider the map

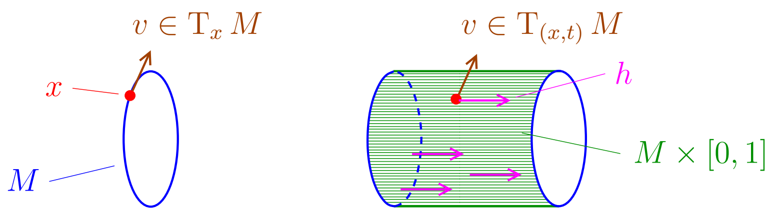


FIGURE 1143

$$\begin{array}{rcl}
P_n\colon\ \mathrm{C}^n_{\mathrm{dR}}(M\times[0,1]) & \to & \mathrm{C}^{n-1}_{\mathrm{dR}}(M) \\
 & & \text{the } (n-1)\text{-form that at a point } x\in M \text{ is given by} \\
\omega & \mapsto & \underbrace{(v_1,\ldots,v_{n-1})}_{(n\text{-}1)\text{ vectors in } \mathrm{T}_x M} \mapsto (-1)^{n+1}\cdot\displaystyle\int_{t=0}^{t=1}\omega_{(x,t)}(\underbrace{v_1,\ldots,v_{n-1},h}_{n\text{ vectors in } \mathrm{T}_{(x,t)}(M\times[0,1])})\,dt.
\end{array}$$

It is straightforward to verify that this map is well-defined, i.e. the right-hand side is an $(n-1)$-form on M, and that this map is $\mathbb{R}$-linear. It remains to show that these maps P_n define a cochain homotopy, i.e. we have to verify that for all $n\in\mathbb{N}_0$ we have

$$(*)\qquad d_{n-1}\circ P_n + P_{n+1}\circ d_n \;=\; i_1^* - i_0^*\colon\ \mathrm{C}^n_{\mathrm{dR}}(M\times[0,1]) \;\to\; \mathrm{C}^n_{\mathrm{dR}}(M).$$

The idea of the proof is to first prove the equality $(*)$ for $M=\mathbb{R}^m$, and then to reduce the case of a general m-dimensional smooth manifold to the case $M=\mathbb{R}^m$ via charts and a smooth partition of unity. So we start out with the following claim.

[1123] Why do we have this decomposition?

Claim. For any $n \in \mathbb{N}_0$ we have

$$d_{n-1} \circ P_n + P_{n+1} \circ d_n \ = \ i_1^* - i_0^* \colon \ \mathrm{C}^n_{\mathrm{dR}}(\mathbb{R}^m \times [0,1]) \ \to \ \mathrm{C}^n_{\mathrm{dR}}(\mathbb{R}^m).$$

We first prove the claim for $n \geq 1$. For any point in $\mathbb{R}^m \times [0,1]$ we can identify its tangent space with $\mathbb{R}^{m+1}$ and we equip its dual space with the basis given by $dx_1, \dots, dx_m, dt$. It follows from the discussion of the basis of $\wedge^k(\mathbb{R}^{m+1})^*$ on page 1916 that for $n \geq 1$ every differential n-form on $\mathbb{R}^m \times [0,1]$ is a real linear combination of forms of the form

$$\eta \ = \ \varphi \cdot dx_{i_1} \wedge \dots \wedge dx_{i_n} \qquad \text{and} \qquad \mu \ = \ \psi \cdot dx_{i_1} \wedge \dots \wedge dx_{i_{n-1}} \wedge dt$$

where $\varphi, \psi \colon \mathbb{R}^n \times [0,1] \to \mathbb{R}$ are smooth functions. By the linearity of the terms in the claim it suffices to prove the claim for each such η and each such μ. It is straightforward to see that at each point $(x_1, \dots, x_m) \in \mathbb{R}^m$ we have

$$P_n(\eta) \ = \ 0 \qquad \text{and} \qquad P_n(\mu) \ = \ (-1)^p \cdot \left(\int\limits_{t=0}^{t=1} \psi(x_1, \dots, x_m, t)\, dt \right) \cdot dx_{i_1} \wedge \dots \wedge dx_{i_{n-1}}.$$

From this we obtain that $(d_{n-1} \circ P_n)(\eta) = 0$ and that

$$\begin{aligned}
(P_{n+1} \circ d_n)(\eta) \ &= \ P_{n+1}(d_0(\varphi) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_n}) \\
&= \ P_{n+1}\Big(\Big(\sum_{i=1}^{p} \tfrac{\partial \varphi}{\partial x_i} \cdot dx_i + \tfrac{\partial \varphi}{\partial t} \cdot dt \Big) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_n} \Big) \\
&= \ (-1)^n \cdot P_{n+1}\Big(\tfrac{\partial \varphi}{\partial t} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_m} \wedge dt \Big) \\
&= \ \left(\int\limits_0^1 \tfrac{\partial \varphi}{\partial t} dt \right) \cdot dx_{i_1} \wedge \dots \wedge dx_{i_m} \underset{\substack{\uparrow \\ \text{fundamental theorem of calculus}}}{=} (i_1^*)(\eta) - (i_0^*)(\eta).
\end{aligned}$$

We have thus proved the desired equality for η.

Now we turn to μ. We have $i_0^*(\mu) = i_1^*(\mu) = 0$ since $i_0^*(dt) = i_1^*(dt) = 0$. We obtain that

$$\begin{aligned}
(d_{n-1} \circ P_n)(\mu) \ &= \ (-1)^n \cdot \sum_{i=1}^{m} \left(\int\limits_{t=0}^{t=1} \frac{\partial \psi}{\partial x_i} dx_i \right) \cdot dx_{i_1} \wedge \dots \wedge dx_{i_{n-1}}, \\
(P_{n+1} \circ d_n)(\mu) \ &= \ (-1)^{n+1} \cdot \sum_{i=1}^{m} \left(\int\limits_{t=0}^{t=1} \frac{\partial \psi}{\partial x_i} dx_i \right) \cdot dx_{i_1} \wedge \dots \wedge dx_{i_{n-1}}.
\end{aligned}$$

We have thus proved the claim for μ and $n \geq 1$. It remains to prove the claim for $n = 0$. This follows immediately from the calculation that

$$(d_{-1} \circ P_0)(\varphi) + (P_1 \circ d_0)(\varphi) = 0 + P_1\Big(\sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i} \cdot dx_i + \frac{\partial \varphi}{\partial t} \cdot dt \Big) = \int\limits_{t=0}^{t=1} \frac{\partial \varphi}{\partial t} dt \underset{\substack{\uparrow \\ \text{fundamental theorem of calculus}}}{=} (i_1^*)(\varphi) - (i_0^*)(\varphi).$$

⊞

Now we return to the case that M is any m-dimensional smooth manifold. We pick a smooth atlas $\{\Phi_j \colon U_j \to \mathbb{R}^m\}_{j \in J}$ for the smooth manifold M.[1124] By Theorem 6.57 we can

[1124] Here by a smooth atlas for M we mean an atlas $\{\Phi_j \colon U_j \to \mathbb{R}^n\}_{j \in J}$ such that all maps are smooth with respect to the given smooth structure on M. Using the fact that any open ball is diffeomorphic to $\mathbb{R}^n$ one can easily show that such an atlas exists.

pick a smooth partition of unity $\{f_j\colon M \to \mathbb{R}\}_{j\in J}$ that is subordinate to the open cover $\{U_j\}_{j\in J}$. We consider the diagram

$$\begin{tikzcd}
\mathrm{C}^n_{\mathrm{dR}}(M \times [0,1]) \arrow[r, "(d_{n-1}\circ P_n + P_{n+1}\circ d_n)-(i_1^*-i_0^*)"] \arrow[d, hook, "\omega\mapsto\prod f_j\omega|_{U_j}"] & \mathrm{C}^n_{\mathrm{dR}}(M) \arrow[d, hook, "\omega\mapsto\prod f_j\omega|_{U_j}"] \\
\prod\limits_{j\in J} \mathrm{C}^n_{\mathrm{dR}}(U_j \times [0,1]) \arrow[r, "\prod(d_{n-1}\circ P_n + P_{n+1}\circ d_n)-(i_1^*-i_0^*)"] \arrow[d, "\omega_j\mapsto(\Phi_j^{-1})^*(\omega_j)", "\cong"'] & \prod\limits_{j\in J} \mathrm{C}^n_{\mathrm{dR}}(U_j) \arrow[d, "\omega_j\mapsto(\Phi_j^{-1})^*(\omega_j)", "\cong"'] \\
\prod\limits_{j\in J} \mathrm{C}^n_{\mathrm{dR}}(\mathbb{R}^n \times [0,1]) \arrow[r, "\prod(d_{n-1}\circ P_n + P_{n+1}\circ d_n)-(i_1^*-i_0^*)"] & \prod\limits_{j\in J} \mathrm{C}^n_{\mathrm{dR}}(\mathbb{R}^n).
\end{tikzcd}$$

One can show easily that the maps P_n and i_0^* and i_1^* are natural, i.e. they commute with induced maps given by smooth maps between smooth manifolds. This observation, together with Proposition 78.2 (6) implies that the horizontal maps are natural, i.e. they commute with smooth maps between smooth manifolds. Therefore it follows that the diagram commutes. The top vertical maps are monomorphisms since the f_j form a smooth partition of unity. The bottom vertical maps are isomorphisms since the Φ_j are diffeomorphisms. We had just shown in the claim that the bottom map is the zero map. Hence the top map is also the zero map. But this is precisely what we had wanted to show. ■

The following result is known as the Poincaré Lemma. This humble name belies the fact that it is an extremely useful statement.

Lemma 78.7. (Poincaré Lemma) *If $U \subset \mathbb{R}^n$ is an open star-shaped set, then*

$$\mathrm{H}^k_{\mathrm{dR}}(U) \;=\; \begin{cases} \mathbb{R}, & \text{if } k=0, \\ 0, & \text{otherwise.} \end{cases}$$

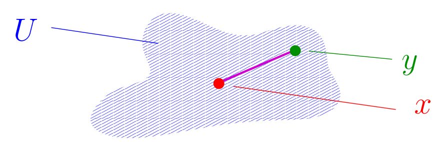


FIGURE 1144

PROOF. By definition of a star-shaped subset of $\mathbb{R}^n$ there exists an $x \in U$ such that for every $y \in U$ the segment $\{xt + y(1-t)\,|\,t \in [0,1]\}$ lies in U. Let $i\colon \{x\} \to U$ be the inclusion map and let $p\colon U \to \{x\}$ the unique map from U to $\{x\}$. Evidently $p \circ i = \mathrm{id}_{\{x\}}$. Furthermore $i \circ p$ and id_U are smoothly homotopic, in fact a smooth homotopy is given by $F(y,t) = xt + y(1-t)$. It follows immediately from Proposition 78.5 and the functoriality of de Rham cohomology that $i^*\colon \mathrm{H}^k_{\mathrm{dR}}(U) \to \mathrm{H}^k_{\mathrm{dR}}(\{x\})$ and $p^*\colon \mathrm{H}^k_{\mathrm{dR}}(\{x\}) \to \mathrm{H}^k_{\mathrm{dR}}(U)$ are isomorphisms. The corollary now follows from the calculation of the de Rham cohomology groups for a point that we gave on page 1920. ■

The following corollary is an immediate consequence of the Poincaré Lemma 78.7 and the functoriality of de Rham cohomology.

Corollary 78.8. *If M is an n-dimensional smooth manifold that is diffeomorphic to an open star-shaped subset of $\mathbb{R}^n$, e.g. that is diffeomorphic to $\mathbb{R}^n$ or the open n-ball B^n, then*

$$\mathrm{H}^k_{\mathrm{dR}}(M) \;=\; \begin{cases} \mathbb{R}, & \text{if } k=0,\\ 0, & \text{otherwise.}\end{cases}$$

78.4. **Applications of the Poincaré Lemma to classical vector calculus** $(*)$**.** In this section we apply the Poincaré Lemma to classical vector calculus. Let $U \subset \mathbb{R}^3$ be an open set. We denote by

$$C^\infty(U) \;:=\; \text{all smooth maps } U \to \mathbb{R} \quad\text{ and }\quad \mathcal{V}(U) \;:=\; \text{all smooth maps } U \to \mathbb{R}^3$$

the set of all smooth real-valued functions respectively smooth vector fields on U. We recall that there are three classical maps relating smooth functions and smooth vector fields:

$$\begin{array}{rcl} \operatorname{grad}\colon C^\infty(U) & \to & \mathcal{V}(U) \\ f & \mapsto & \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z}\end{pmatrix}\end{array} \qquad \begin{array}{rcl} \operatorname{curl}\colon \mathcal{V}(U) & \to & \mathcal{V}(U) \\ F=\begin{pmatrix}F_x\\F_y\\F_z\end{pmatrix} & \mapsto & \begin{pmatrix} \frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z} \\ -\frac{\partial F_z}{\partial x}+\frac{\partial F_x}{\partial z} \\ \frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\end{pmatrix}\end{array} \qquad \begin{array}{rcl} \operatorname{div}\colon \mathcal{V}(U) & \to & C^\infty(U) \\ F=\begin{pmatrix}F_x\\F_y\\F_z\end{pmatrix} & \mapsto & \frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}.\end{array}$$

Given $f \in C^\infty(U)$ we refer to $\operatorname{grad}(f)$ as the *gradient of f* and given $F \in \mathcal{V}(U)$ we refer to $\operatorname{curl}(F)$ as the *curl of f* and we refer to $\operatorname{div}(F)$ as the *divergence of F*. It is straightforward to verify that $\operatorname{curl}\circ\operatorname{grad}=0$ and $\operatorname{div}\circ\operatorname{curl}=0$. Put another way, the following sequence of maps is in fact a chain complex:

$$0 \;\to\; C^\infty(U) \;\xrightarrow{\operatorname{grad}}\; \mathcal{V}(U) \;\xrightarrow{\operatorname{curl}}\; \mathcal{V}(U) \;\xrightarrow{\operatorname{div}}\; C^\infty(U) \;\to\; 0.$$

Now we will relate the above discussion to forms. A smooth function is by definition the same as a differential 0-form. A smooth function also gives rise to a 3-form, more precisely we have the following simple-minded isomorphism

$$\begin{array}{rcl} \Theta\colon C^\infty(U) & \to & \mathrm{C}^3_{\mathrm{dR}}(U) \\ f & \mapsto & f\cdot dx\wedge dy\wedge dz.\end{array}$$

More interestingly, a smooth vector field gives rise to a differential 1-form on U and also a 2-form on U. More precisely we have the following maps

$$\begin{array}{rcl} \Phi\colon \mathcal{V}(U) & \to & \mathrm{C}^1_{\mathrm{dR}}(U) \\ & & \text{differential 1-form that} \\ & & \text{at } x \text{ is given by} \\ w & \mapsto & \mathrm{T}_x U=\mathbb{R}^3 \to \mathbb{R} \\ & & \qquad v \mapsto \underbrace{v\cdot w}_{\text{scalar product}}\end{array} \qquad \begin{array}{rcl} \Psi\colon \mathcal{V}(U) & \to & \mathrm{C}^2_{\mathrm{dR}}(U) \\ & & \text{differential 2-form that at a} \\ & & \text{point } x\in U \text{ is given by} \\ w & \mapsto & (\mathrm{T}_x U)^2=(\mathbb{R}^3)^2 \to \mathbb{R} \\ & & \qquad (v_1,v_2) \mapsto \underbrace{(v_1\times v_2)}_{\text{cross product}}\cdot w.\end{array}$$

It is straightforward to see that these maps are well-defined and that they are isomorphisms.

Summarizing we obtain the following diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & C^\infty(U) & \xrightarrow{\text{grad}} & \mathcal{V}(U) & \xrightarrow{\text{curl}} & \mathcal{V}(U) & \xrightarrow{\text{div}} & C^\infty(U) & \longrightarrow & 0 \\
 & & \cong\downarrow\text{id} & & \cong\downarrow\Phi & & \cong\downarrow\Psi & & \cong\downarrow\Theta & & \\
0 & \longrightarrow & \mathrm{C}^0_{\mathrm{dR}}(U) & \xrightarrow{d} & \mathrm{C}^1_{\mathrm{dR}}(U) & \xrightarrow{d} & \mathrm{C}^2_{\mathrm{dR}}(U) & \xrightarrow{d} & \mathrm{C}^3_{\mathrm{dR}}(U) & \longrightarrow & 0.
\end{array}$$

A heroic calculation, see e.g. [**Weint14**, Theorem 3.8], shows that the diagram commutes.

If U is star-shaped then we showed in the Poincaré Lemma 78.7 that the cohomology of the lower sequence vanishes for $k = 1, 2, 3$. Since the diagram commutes and since all vertical maps are isomorphisms we have thus proved the following classical theorem.

Theorem 78.9. *Let U be a star-shaped open subset of $\mathbb{R}^3$.*

(1) *Every smooth vector field on U with vanishing rotation is the gradient of a smooth function.*
(2) *Every smooth vector field on U with vanishing divergence is the curl of a smooth vector field.*
(3) *Every smooth function on U is the divergence of a smooth vector field.*

78.5. **More properties of the de Rham cohomology groups.** In this section we will continue the theme from the Section 78.3 and we will prove two more statements regarding de Rham cohomology groups that we had already proved for singular cohomology groups.

One of the key tools in computing singular homology and cohomology groups are the Mayer–Vietoris Theorems 46.5 and 74.15. The following theorem is the de Rham cohomology version of these two theorems.

Theorem 78.10. (Mayer–Vietoris Theorem for de Rham cohomology groups *Let M be a smooth manifold and let U, V be open subsets with $M = U \cup V$. Let $j_U\colon U \to M$, $j_V\colon V \to M$, $i_U\colon U \cap V \to U$ and $i_V\colon U \cap V \to V$ be the inclusion maps. Then the following two statements hold:*

(1) *The following sequence of cochain complexes is exact*
$$0 \;\to\; \mathrm{C}^*_{\mathrm{dR}}(M) \;\xrightarrow{j_U^*\oplus - j_V^*}\; \mathrm{C}^*_{\mathrm{dR}}(U)\oplus\mathrm{C}^*_{\mathrm{dR}}(V) \;\xrightarrow{i_U^*+i_V^*}\; \mathrm{C}^*_{\mathrm{dR}}(U\cap V) \;\to\; 0.$$
(2) *There exists a natural long exact sequence of the form*
$$\dots \to \mathrm{H}^k_{\mathrm{dR}}(M) \xrightarrow{j_U^*\oplus - j_V^*} \mathrm{H}^k_{\mathrm{dR}}(U)\oplus\mathrm{H}^k_{\mathrm{dR}}(V) \xrightarrow{i_U^*+i_V^*} \mathrm{H}^k_{\mathrm{dR}}(U\cap V) \xrightarrow{\delta} \mathrm{H}^{k+1}_{\mathrm{dR}}(M) \to \dots$$

In the proof of Theorem 78.10 we will require the following lemma.

Lemma 78.11. *Let W be a smooth manifold, let $U \subset W$ be an open subset and let φ be a smooth form on U. We define*
$$\epsilon(\varphi, W) \;:=\; \text{the form on } W \text{ that is given by } \varphi \text{ on } U \text{ and that is zero on } W\setminus U.$$
If $\operatorname{supp}(\varphi) \subset U$, *then $\epsilon(\varphi, M)$ is a smooth form on W.*

Proof. To show that a form μ on a smooth manifold W is smooth it suffices to show that there exists an open cover $\{W_i\}_{i\in I}$ such that the restriction of μ to each W_i is smooth. In our case we consider the open cover of U given by U and by $W \setminus \operatorname{supp}(\varphi)$. Note that this

FIGURE 1145

is indeed a covering since $\mathrm{supp}(\varphi) \subset U$. The restriction of the form $\epsilon(\varphi, W)$ to U equals φ, hence it is smooth. The restriction of $\epsilon(\varphi, W)$ to $W \setminus \mathrm{supp}(\varphi)$ is also smooth since it is by definition identically zero on $W \setminus \mathrm{supp}(\varphi)$. ■

Now we can provide the proof of Theorem 78.10.

PROOF OF THEOREM 78.10. Not surprisingly we first prove statement (1). Since $M = U \cup V$ it is clear that the left-hand map $j_U^* \oplus -j_V^*$ is a monomorphism. It is obvious that the composition of the two maps is zero and that the image of the left-hand map $j_U^* \oplus -j_V^*$ is the kernel of $i_U^* + i_V^*$. It remains to prove the following claim.

Claim. The map $i_U^* + i_V^* \colon \mathrm{C}^*_{\mathrm{dR}}(U) \oplus \mathrm{C}^*_{\mathrm{dR}}(V) \to \mathrm{C}^*_{\mathrm{dR}}(U \cap V)$ is in fact an epimorphism.

Let $\eta \in \mathrm{C}^*_{\mathrm{dR}}(U \cap V)$. By Theorem 6.57 there exists a smooth partition of unity φ_U, φ_V that is subordinate to the open cover of M given by U and V. We refer to Figure 1146 for an illustration.

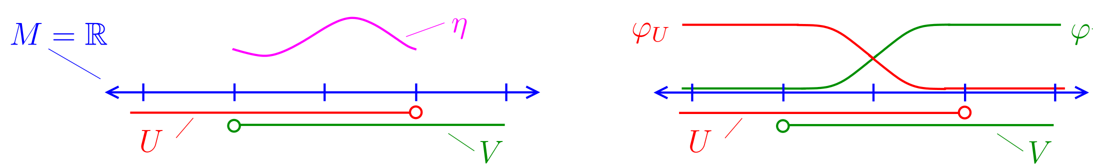


FIGURE 1146

We consider the forms $\epsilon(\eta \cdot \varphi_V, U)$ on U and $\epsilon(\eta \cdot \varphi_U, V)$. Both are smooth by Lemma 78.11. Now we have

$$(i_U^* + i_V^*)(\epsilon(\eta \cdot \varphi_V, U) + \epsilon(\eta \cdot \varphi_U, V)) \underset{\substack{\uparrow \\ \text{by definition of } i_U^* \text{ and } i_V^*}}{=} \epsilon(\eta \cdot \varphi_V, U)|_{U \cap V} + \epsilon(\eta \cdot \varphi_U, V)|_{U \cap V} \underset{\substack{\uparrow \\ \text{by definition of } \varphi_U \text{ and } \varphi_V}}{=} \eta \cdot \varphi_V + \eta \cdot \varphi_U \underset{\substack{\uparrow \\ \text{since } \varphi_U + \varphi_V \equiv 1}}{=} \eta.$$

This concludes the proof of the claim and thus also of statement (1).

The existence and naturality of the long exact sequence of statement (2) is an immediate consequence of statement (1) together with Proposition 73.6. ■

Using the Mayer–Vietoris Theorem 78.10 for de Rham cohomology groups and some of our earlier results one can now compute many de Rham cohomology groups. The following lemma gives a sample calculation.

Lemma 78.12. *For any $n \geq 1$ we have*

$$\mathrm{H}^k_{\mathrm{dR}}(S^n) \cong \begin{cases} \mathbb{R}, & \text{if } k = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. By induction we will prove the following statement: for any $n \geq 1$ we have

$$\mathrm{H}^k_{\mathrm{dR}}(S^n) \;\cong\; \begin{cases} \mathbb{R}, & \text{if } k=0, \\ 0, & \text{otherwise} \end{cases} \oplus \begin{cases} \mathbb{R}, & \text{if } k=n, \\ 0, & \text{otherwise.} \end{cases}$$

For $n = 0$ the statement holds by Proposition 78.4 (5) and the calculation of de Rham cohomology for a point that we gave on page 1920.

Given S^n with $n \geq 1$ we write

$$U \;:=\; \{(x_1,\dots,x_{n+1}) \in S^n \,|\, x_{n+1} \neq -1\} \quad \text{ and } V \;=\; \{(x_1,\dots,x_{n+1}) \in S^n \,|\, x_{n+1} \neq +1\}.$$

The stereographic projection that we defined in Lemma 2.44 gives diffeomorphisms from U and V to $\mathbb{R}^n$. We refer to Figure 1147 for an illustration. In particular we know from Corollary 78.8 that

$$\mathrm{H}^k_{\mathrm{dR}}(U) \;\cong\; \mathrm{H}^k_{\mathrm{dR}}(V) \;\cong\; \begin{cases} \mathbb{R}, & \text{if } k=0, \\ 0, & \text{otherwise.} \end{cases}$$

Using Proposition 78.5 it is also fairly straightforward to show that the inclusion map

$$\begin{array}{rcl} \iota\colon S^{n-1} & \to & U\cap V \\ P & \mapsto & (P,0) \end{array}$$

induces an isomorphism of de Rham cohomology groups.

Now we perform the induction step from $n = 0$ to $n = 1$. We consider the long exact Mayer–Vietoris sequence coming from Theorem 78.10 that corresponds to the decomposition $S^1 = U \cup V$:

$$0 \to \underbrace{\mathrm{H}^0_{\mathrm{dR}}(S^1)}_{\cong\mathbb{R}} \to \underbrace{\mathrm{H}^0_{\mathrm{dR}}(U)}_{\cong\mathbb{R}} \oplus \underbrace{\mathrm{H}^0_{\mathrm{dR}}(V)}_{\cong\mathbb{R}} \to \underbrace{\mathrm{H}^0_{\mathrm{dR}}(U\cap V)}_{\cong\mathbb{R}^2} \xrightarrow{\delta} \mathrm{H}^1_{\mathrm{dR}}(S^1) \to \underbrace{\mathrm{H}^1_{\mathrm{dR}}(U)}_{=0} \oplus \underbrace{\mathrm{H}^1_{\mathrm{dR}}(V)}_{=0} \to \dots$$

We recall that in Proposition 78.4 (3) we saw that inclusion induced maps between connected smooth manifolds induce isomorphisms on 0-th de Rham cohomology. Using this observation and using the long exact sequence it is now straightforward to show that the de Rham cohomology of S^1 is precisely as predicted.[1125]

The induction step for $n \geq 2$ from $n-1$ to n is even easier, since one does not have to worry about 0-th de Rham cohomology. More precisely, for $k \geq 1$ the above Mayer–Vietoris sequence now looks as follows:

$$\begin{tikzcd}[column sep=small] \dots \arrow[r] & \underbrace{\mathrm{H}^k_{\mathrm{dR}}(S^n)}_{\cong\mathbb{R}} \arrow[r] & \underbrace{\mathrm{H}^k_{\mathrm{dR}}(U)}_{=0} \oplus \underbrace{\mathrm{H}^k_{\mathrm{dR}}(V)}_{=0} \arrow[r] & \mathrm{H}^k_{\mathrm{dR}}(U\cap V) \arrow[r, "\delta"] & \mathrm{H}^{k+1}_{\mathrm{dR}}(S^n) \arrow[r] & \underbrace{\mathrm{H}^{k+1}_{\mathrm{dR}}(U)}_{=0} \oplus \underbrace{\mathrm{H}^{k+1}_{\mathrm{dR}}(V)}_{=0} \arrow[r] & \dots \\ & & & \mathrm{H}^k_{\mathrm{dR}}(S^{n-1}). \arrow[u, "\iota_*", "\cong"'] & & & \end{tikzcd}$$

It is now straightforward to prove the induction step. ■

Using the calculation from Lemma 78.12 and the Mayer–Vietoris Theorem 78.10 for de Rham cohomology groups one can now fairly easily modify the arguments in Lemmas 46.9, 46.12 and 46.13 to compute the de Rham cohomology of the real projective plane

[1125] Alternatively one can also use Lemma 55.4 to deduce that $\mathrm{H}^1_{\mathrm{dR}}(S^1) \cong \mathbb{R}$.

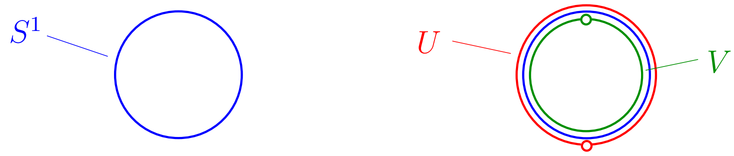


FIGURE 1147. Illustration for the proof of Lemma 78.12.

$\mathbb{R}\mathrm{P}^2$, the torus $T = S^1 \times S^1$ and the Klein bottle K:

$$\mathrm{H}^n_{\mathrm{dR}}(\mathbb{R}\mathrm{P}^2) \cong \begin{cases} 0, & \text{if } n \geq 1, \\ \mathbb{R}, & \text{if } n = 0 \end{cases} \quad \mathrm{H}^n_{\mathrm{dR}}(T) \cong \begin{cases} \mathbb{R}, & \text{if } n = 0, 2, \\ \mathbb{R}^2, & \text{if } n = 1, \\ 0, & \text{otherwise} \end{cases} \quad \mathrm{H}^n_{\mathrm{dR}}(K) \cong \begin{cases} \mathbb{R}, & \text{if } n = 0, \\ \mathbb{R}, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We conclude this discussion of de Rham cohomology with the following proposition which is very similar in nature to Proposition 76.12 for singular cohomology.

Proposition 78.13. *Let M be a smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets of M that satisfy the following three conditions:*

(1) *for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,*
(2) *for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$, and*
(3) *$M = \bigcup\limits_{i\in\mathbb{N}} U_i$.*

Then for every $n \in \mathbb{N}_0$ there exists a natural short exact sequence

$$0 \;\to\; \varprojlim{}^1\, \mathrm{H}^{n-1}_{\mathrm{dR}}(U_i) \;\xrightarrow{\tau}\; \mathrm{H}^n_{\mathrm{dR}}(M) \;\to\; \varprojlim\, \mathrm{H}^n_{\mathrm{dR}}(U_i) \;\to\; 0$$

where the right-hand map is induced by the obvious maps $\mathrm{H}^n_{\mathrm{dR}}(M) \to \mathrm{H}^n_{\mathrm{dR}}(U_i)$.

PROOF. For each $k \in \mathbb{N}_0$ we consider the inverse system

$$\mathrm{C}^k_{\mathrm{dR}}(U_1) \;\leftarrow\; \mathrm{C}^k_{\mathrm{dR}}(U_2) \;\leftarrow\; \mathrm{C}^k_{\mathrm{dR}}(U_3) \;\leftarrow\; \dots$$

where all the maps are induced by the inclusion, put differently, they are all restriction maps.

Claim. The inverse system satisfies the Mittag-Leffler condition from page 1892.

Given any $j \geq i$ we denote by $\rho_{ji}\colon \mathrm{C}^k_{\mathrm{dR}}(U_j) \to \mathrm{C}^k_{\mathrm{dR}}(U_i)$ the restriction map. Now let $i \in \mathbb{N}$. We have to show that there exists an $n \geq i$ such that for all $r \geq n$ we have

$$\mathrm{im}\left(\rho_{ri}\colon \mathrm{C}^k_{\mathrm{dR}}(U_r) \to \mathrm{C}^k_{\mathrm{dR}}(U_i)\right) \;=\; \mathrm{im}\left(\rho_{ni}\colon \mathrm{C}^k_{\mathrm{dR}}(U_n) \to \mathrm{C}^k_{\mathrm{dR}}(U_i)\right).$$

Since $\overline{U}_i$ is compact, since the U_j are nested and since $\overline{U}_i$ is contained in the union of all U_j there exists an n such that $\overline{U}_i \subset U_n$. We refer to Figure 1148 for an illustration. We claim that n has the desired property. So let $r \geq n$. We have to show that

$$\mathrm{im}(\rho_{ri}\colon \mathrm{C}^k_{\mathrm{dR}}(U_r) \to \mathrm{C}^k_{\mathrm{dR}}(U_i)) \;=\; \mathrm{im}(\rho_{ni}\colon \mathrm{C}^k_{\mathrm{dR}}(U_n) \to \mathrm{C}^k_{\mathrm{dR}}(U_i)).$$

Clearly we have the inclusion "$\subset$". It remains to prove the reverse inclusion. So suppose $\eta \in \mathrm{C}^k_{\mathrm{dR}}(U_i)$ is the restriction of some $\mu \in \mathrm{C}^k_{\mathrm{dR}}(U_n)$. Note that U_n and $M \setminus \overline{U}_i$ form an open cover for M. It follows immediately from Theorem 6.57 that there exists a smooth function $\varphi\colon M \to \mathbb{R}$ with $\varphi|_{\overline{U}_i} \equiv 1$ and such that $\operatorname{supp}(\varphi) \subset U_n$. Let $r \geq n$. We see that

$$\eta \;=\; \mu|_{U_i} \underset{\substack{\uparrow \\ \text{since } \varphi \equiv 1 \text{ on } U_i}}{=} (\varphi \cdot \mu)|_{U_i} \;=\; \overbrace{\epsilon(\varphi \cdot \mu, U_r)}^{\text{smooth by Lemma 78.11}}\;|_{U_i}.$$

We have thus shown that η is also the restriction of some smooth form on U_r. $\boxplus$

By the claim we can apply Proposition 76.10 to the inverse system $(\{\mathrm{C}^*_{\mathrm{dR}}(U_i)\}_{i\in\mathbb{N}}, \{\rho_{ij}\}_{i\geq j})$ of cochain complexes over $\mathbb{N}$ and we obtain the short exact sequence

$$0 \;\to\; \varprojlim_{i\in\mathbb{N}}{}^{1}\, \underbrace{\mathrm{H}^{n-1}(\mathrm{C}^*_{\mathrm{dR}}(U_i))}_{=\mathrm{H}^{n-1}_{\mathrm{dR}}(U_i)} \;\to\; \underbrace{\mathrm{H}^n\Bigg(\overbrace{\varprojlim_{i\in\mathbb{N}} \mathrm{C}^*_{\mathrm{dR}}(U_i)}^{=\mathrm{C}^*_{\mathrm{dR}}(M),\text{ see page 752}}\Bigg)}_{=\mathrm{H}^n_{\mathrm{dR}}(M)} \;\to\; \varprojlim_{i\in\mathbb{N}} \underbrace{\mathrm{H}^n(\mathrm{C}^*_{\mathrm{dR}}(U_i))}_{=\mathrm{H}^n_{\mathrm{dR}}(U_i)} \;\to\; 0.$$

We have thus obtained the desired short exact sequence. ■

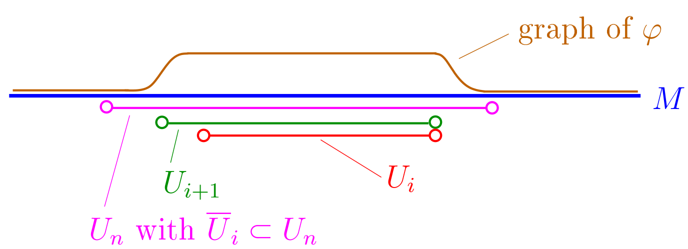


FIGURE 1148. Illustration for the proof of Proposition 78.13.

Exercises for Chapter 78.

Exercise 78.1. Let Σ be the surface of genus 2 equipped with some orientation. Furthermore let $\gamma\colon S^1 \to \Sigma$ be a smooth embedding. We write $C = \gamma(S^1)$ and we view C as an oriented curve on Σ. By the Tubular Neighborhood Theorem 8.24 we can pick a tubular neighborhood $[-2,2] \times C$ for C. We denote by dt the usual differential 1-form on $[-2,2] \subset \mathbb{R}$. Let $f\colon [-2,2] \to \mathbb{R}_{\geq 0}$ be a smooth function with $\operatorname{supp}(f) \subset [-1,1]$ and $\int\limits_{t=0}^{t=1} f(t)\,dt = 1$. We denote by $p\colon [-2,2] \times C \to [-2,2]$ the projection. Finally denote by ω the smooth closed differential 1-form on Σ that is given by $p^*(f \cdot dt)$ on $[-2,2] \times C$ and that is extended to all of Σ by setting it zero outside of $C \times [-2,2]$. (Some of the objects are illustrated in Figure 1149.)

Given a closed oriented curve $D \subset \Sigma$ we obtain the integral $\int\limits_D \omega$.

(a) Show that the integral is an integer.
(b) What is the geometric interpretation of this integer?

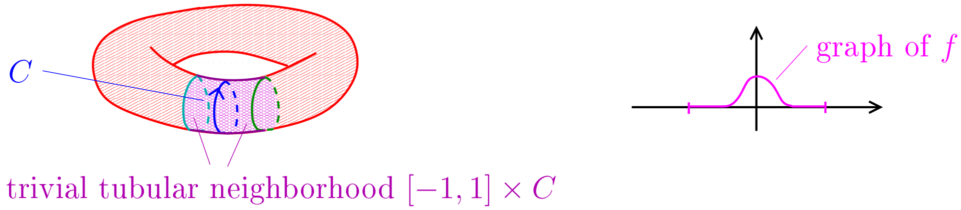


FIGURE 1149

Exercise 78.2. Let M be a smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets of M that satisfy the following three conditions:

(1) for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
(2) for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$, and
(3) $M = \bigcup\limits_{i\in\mathbb{N}} U_i$.

In the proof of Proposition 78.13 we show that for each $k \in \mathbb{N}_0$ the inverse system

$$\mathrm{C}^k_{\mathrm{dR}}(U_1) \;\leftarrow\; \mathrm{C}^k_{\mathrm{dR}}(U_2) \;\leftarrow\; \mathrm{C}^k_{\mathrm{dR}}(U_2) \;\leftarrow\; \dots$$

satisfies the Mittag-Leffler condition. Now let $i \in \mathbb{N}$. Does there exist an $n \geq i$ such that for all $r \geq n$ the map

$$\mathrm{C}^k_{\mathrm{dR}}(U_r) \;\to\; \mathrm{C}^k_{\mathrm{dR}}(U_i)$$

is an epimorphism?

Exercise 78.3. Let M be a closed connected smooth manifold. Use Proposition 63.36 to show that for any $k \in \mathbb{N}_0$ the de Rham cohomology group $\mathrm{H}^k_{\mathrm{dR}}(M)$ is finite-dimensional.

Exercise 78.4. Let $n \in \mathbb{N}$. As on page 1915 we consider the 1-forms $dx_1, \dots, dx_n$ on $\mathbb{R}^n$. We denote by $p\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ the obvious projection.

(a) Show that there exist closed 1-forms $\omega_1, \dots, \omega_n$ on the smooth manifold $\mathbb{R}^n/\mathbb{Z}^n$ such that for each $i \in \{1, \dots, n\}$ we have $dx_i = p^*(\omega_i)$.
(b) We equip $\mathbb{R}^n/\mathbb{Z}^n$ with the standard orientation induced by $\mathbb{R}^n$. Show that the integral of the n-form $\omega_1 \wedge \dots \wedge \omega_n$ over $\mathbb{R}^n/\mathbb{Z}^n$ equals $+1$.

Exercise 78.5. On S^2 we consider the 2-form

$$\begin{aligned} \omega\colon\; \mathrm{T}_P M \times \mathrm{T}_P M \;&\to\; \mathbb{R} \\ (v, w) \;&\mapsto\; \det(\underbrace{(P \;\; v \;\; w)}_{3\times 3\text{-matrix}}). \end{aligned}$$

We equip S^2 with the usual orientation introduced on page 299. Show that the integral of ω over S^2 equals 4π.

79. The relationship between de Rham cohomology and singular cohomology

In this chapter we want to find the connection between the de Rham cohomology groups from Chapter 78 and the singular cohomology groups that we defined earlier in Chapter 73.

To avoid a repetitive use of suffixes we will use the following convention.

Convention. In this chapter by a "manifold" we always mean a smooth manifold *without boundary*, unless we say explicitly otherwise.

79.1. **Smooth cohomology theories.** To a smooth manifold M we can associate the singular cohomology groups $\mathrm{H}^i(M;\mathbb{R})$ and also the de Rham cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(M)$. In the last chapter we had in particular seen that these two cohomologies have many properties in common, for example they both satisfy a Mayer–Vietoris type theorem. Now we want to put both cohomologies in a common framework.

Definition. A *smooth cohomology theory* is a contravariant functor[1126][1127]

$$\begin{array}{rcl} \mathcal{H}^*\colon \text{category of smooth manifolds} & \to & \text{category of } \mathbb{N}_0\text{-graded real vector spaces} \\ M & \mapsto & \mathcal{H}^*(M) = \bigoplus\limits_{n\in\mathbb{N}_0} \mathcal{H}^n(M) \\ (f\colon M\to N) & \mapsto & \{f^* = \mathcal{H}^n(f)\colon \mathcal{H}^n(N)\to\mathcal{H}^n(M)\}_{n\in\mathbb{N}_0} \end{array}$$

that satisfies the following three axioms:

(1) (Smooth homotopy invariance) Let $f,g\colon M\to N$ be smooth maps between smooth manifolds. If f and g are smoothly homotopic, then for all $n\in\mathbb{N}_0$ we have
$$\mathcal{H}^n(f) = \mathcal{H}^n(g)\colon \mathcal{H}^n(N) \;\to\; \mathcal{H}^n(M).$$

(2) (Mayer–Vietoris sequence) If M is a smooth manifold and U and V are open subsets such that $M=U\cup V$, then there exists a natural long exact sequence of the form
$$\ldots \to \mathcal{H}^k(M) \xrightarrow{j_U^*\oplus -j_V^*} \mathcal{H}^k(U)\oplus\mathcal{H}^k(V) \xrightarrow{i_U^*+i_V^*} \mathcal{H}^k(U\cap V) \xrightarrow{\delta} \mathcal{H}^{k+1}(M) \to \ldots$$
where $j_U\colon U\to M$, $j_V\colon V\to M$, $i_U\colon U\cap V\to U$ and $i_V\colon U\cap V\to V$ denote the inclusion maps.

(3) (Limit property) Let M be a smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets that satisfy the following three conditions:
  (a) for each $i\in\mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
  (b) for each $i\in\mathbb{N}$ we have $U_i\subset U_{i+1}$, and
  (c) $M=\bigcup\limits_{i\in\mathbb{N}} U_i$.

Then for every $n\in\mathbb{N}_0$ there exists a natural short exact sequence
$$0 \;\to\; \varprojlim{}^1\,\mathcal{H}^{n-1}(U_i) \;\xrightarrow{\tau}\; \mathcal{H}^n(M) \;\to\; \varprojlim \mathcal{H}^n(U_i) \;\to\; 0$$
where the right-hand map is induced by the maps $\mathcal{H}^n(M)\to\mathcal{H}^n(U_i)$ which in turn are induced by the inclusions $U_i\to M$.

This time it is easy to give examples for our new definition.

Proposition 79.1.

(1) *Singular cohomology with real coefficients is a smooth cohomology theory.*
(2) *De Rham cohomology is a smooth cohomology theory.*

PROOF.

(1) It follows from Lemma 73.13, Theorem 74.15, together with Proposition 74.12 and Proposition 76.12 (1), that singular cohomology with real coefficients is a smooth cohomology theory.
(2) It follows from Proposition 78.5, Theorem 78.10 and Proposition 78.13 that de Rham cohomology is a smooth cohomology theory. ■

One might now be optimistic and hope that smooth cohomology theories are necessarily isomorphic. As it stands this thought though is too optimistic. For example other smooth cohomology theories are given by $\mathcal{H}^n(M) = 0$ for all $n \in \mathbb{N}_0$ or by $\mathcal{H}^n(M) = \mathrm{H}^{n+5}(M;\mathbb{R})$ or by $\mathcal{H}^n(M) = \mathrm{H}^n(M;\mathbb{R}) \oplus \mathrm{H}^n(M;\mathbb{R})$. This shows that if we want to prove that smooth cohomology theories are isomorphic we have to add some extra hypotheses, for example we could demand that the 0-th cohomology groups of a point are isomorphic.

Notation. In the following $\star$ denotes the topological space consisting of a single point. Evidently $\star$ is a 0-dimensional smooth manifold.

On page 488 we introduced the notion of a natural transformation between two *covariant* functors. Almost the same definition can also be applied to contravariant functors. For completeness' sake we write down the formal definition.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two contravariant functors. A *natural transformation between the contravariant functors F and G* assigns to each object $X \in \mathrm{Ob}(\mathcal{C})$ a morphism $\Phi_X\colon F(X) \to G(X)$ in $\mathcal{D}$ such that for each morphism $f\colon X \to Y$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd} F(Y) \arrow[r, "F^*(f)"] \arrow[d, "\Phi_Y"'] & F(X) \arrow[d, "\Phi_X"] \\ G(Y) \arrow[r, "G^*(f)"] & G(X). \end{tikzcd}$$

If all the maps Φ_X are isomorphisms we also say it is a *natural isomorphism.*

Example. Given $n \in \mathbb{N}_0$ we consider the two contravariant functors $X \mapsto \mathrm{H}^n(X;\mathbb{Z})$ and $X \mapsto \mathrm{Hom}(\mathrm{H}_n(X),\mathbb{Z})$ from the category of topological spaces to the category of abelian groups. It follows from the discussion on page 1867 that the evaluation homomorphisms $\mathrm{ev}\colon \mathrm{H}^n(X;\mathbb{Z}) \to \mathrm{Hom}(\mathrm{H}_n(X),\mathbb{Z})$ define a natural transformation between the two contravariant functors, i.e. given any map $f\colon X \to Y$ between topological spaces the following

[1126]Given a smooth map $f\colon M \to N$ the contravariant functor provides for each $n \in \mathbb{N}_0$ a morphism $\mathcal{H}^n(f)\colon \mathcal{H}^n(N) \to \mathcal{H}^n(M)$. As usual, when there is no danger of confusion we just write f^* instead of $\mathcal{H}^n(f)$.

[1127]We refer to page 1427 for the definition of the category of $\mathbb{N}_0$-graded real vector spaces.

diagram commutes:

$$\begin{tikzcd}
\mathrm{H}^n(Y;\mathbb{Z}) \arrow[rr, "f^*"] \arrow[d, "\mathrm{ev}"'] & & \mathrm{H}^n(X;\mathbb{Z}) \arrow[d, "\mathrm{ev}"] \\
\mathrm{Hom}(\mathrm{H}_n(Y),\mathbb{Z}) \arrow[rr, "f^*"] & & \mathrm{Hom}(\mathrm{H}_n(X),\mathbb{Z}).
\end{tikzcd}$$

With this notation and this definition we can now formulate the following theorem.

Theorem 79.2. *Let $\mathcal{H}^*$ and $\mathcal{K}^*$ be smooth cohomology theories and let $t\colon \mathcal{H}^* \to \mathcal{K}^*$ be a natural transformation. Suppose t has the following three properties:*

(i) *the natural transformation t is compatible with the connecting homomorphism in the Mayer-Vietoris sequence, i.e. with the obvious notation we always have a commutative diagram*

$$\begin{tikzcd}
\dots \arrow[r] & \mathcal{H}^n(M) \arrow[r, "j_U^*\oplus -j_V^*"] \arrow[d, "t"] & \mathcal{H}^n(U)\oplus\mathcal{H}^n(V) \arrow[r, "i_U^*+i_V^*"] \arrow[d, "t\oplus t"] & \mathcal{H}^n(U\cap V) \arrow[r, "\delta"] \arrow[d, "t"] & \mathcal{H}^{n+1}(M) \arrow[r] \arrow[d, "t"] & \dots \\
\dots \arrow[r] & \mathcal{K}^n(M) \arrow[r, "j_U^*\oplus -j_V^*"] & \mathcal{K}^n(U)\oplus\mathcal{K}^n(V) \arrow[r, "i_U^*+i_V^*"] & \mathcal{K}^n(U\cap V) \arrow[r, "\delta"] & \mathcal{K}^{n+1}(M) \arrow[r] & \dots
\end{tikzcd}$$

(ii) *the natural transformation t is compatible with the map τ appearing in the limit property, i.e. with the obvious notation we always have a commutative diagram*[1128]

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim^1 \mathcal{H}^{n-1}(U_i) \arrow[r, "\tau"] \arrow[d, "t"] & \mathcal{H}^n(M) \arrow[r] \arrow[d, "t"] & \varprojlim \mathcal{H}^n(U_i) \arrow[r] \arrow[d, "t"] & 0 \\
0 \arrow[r] & \varprojlim^1 \mathcal{K}^{n-1}(U_i) \arrow[r, "\tau"] & \mathcal{K}^n(M) \arrow[r] & \varprojlim \mathcal{K}^n(U_i) \arrow[r] & 0,
\end{tikzcd}$$

(iii) *for all $n\in\mathbb{N}_0$ the map $t^n(\star)\colon \mathcal{H}^n(\star)\to\mathcal{K}^n(\star)$ is an isomorphism.*

Then for every smooth manifold M and every $n\in\mathbb{N}_0$ the map

$$t\colon \mathcal{H}^n(M) \;\to\; \mathcal{K}^n(M)$$

is an isomorphism.

Proof. Let $\mathcal{H}^*$ and $\mathcal{K}^*$ be smooth cohomology theories, which means they both satisfy

(α) smooth homotopy invariance,
(β) the Mayer-Vietoris sequence,
(γ) the limit property.

Furthermore let $t\colon \mathcal{H}^*\to\mathcal{K}^*$ be a natural transformation that has the above properties (i), (ii) and (iii). We need to prove the following statement:

Statement. Let $m\in\mathbb{N}_0$. For every m-dimensional smooth manifold M and every $n\in\mathbb{N}_0$ the map

$$t^n(M)\colon \mathcal{H}^n(M) \;\to\; \mathcal{K}^n(M)$$

is an isomorphism.

[1128] Note that the left and right vertical maps are given by the remark on page 1894.

We start out with the following two observations:

(A) Suppose M is an m-dimensional smooth manifold and U and V are open subsets with $M = U \cup V$. If all t^n are isomorphisms for U, V and $U \cap V$, then it follows from (β), (i) and the Five Lemma 43.12 that all the t^n are also isomorphisms for M.

(B) Let M be an m-dimensional smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets that satisfy the following three conditions:
   (a) for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
   (b) for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$, and
   (c) $M = \bigcup\limits_{i\in\mathbb{N}} U_i$.

   If all the t^n are isomorphisms for all U_i, then it follows from (γ) and the Five Lemma 43.12 that all the t^n are also isomorphisms for M.

After these preparations we will prove the desired statement in four surprisingly easy steps:

(1) Let M be an open convex subset of $\mathbb{R}^m$. We pick an inclusion $f\colon \star \to M$. As in the proof of the Poincaré Lemma 78.7 we see that f is a smooth homotopy equivalence. Given $n \in \mathbb{N}_0$ we consider the diagram

$$\begin{tikzcd} \mathcal{H}^n(M) \arrow[r, "f^*"] \arrow[d, "t"'] & \mathcal{H}^n(\star) \arrow[d, "t", "\cong"'] \\ \mathcal{K}^n(M) \arrow[r, "f^*"] & \mathcal{K}^n(\star). \end{tikzcd}$$

The diagram commutes since t is a natural transformation. It follows easily from (α) that the horizontal maps are isomorphisms. By (iii) the right vertical map is also an isomorphism. It follows that the left vertical map is, as desired, an isomorphism

(2) We claim that t is an isomorphism for every subset M of $\mathbb{R}^m$ that is the union of k open convex subsets of $\mathbb{R}^m$. We prove this by induction on the number k of open convex subsets:
   (a) If $k = 1$, then M itself is an open convex subset of $\mathbb{R}^m$, hence the statement follows from (1).
   (b) Suppose the statement holds whenever M is the union of $k-1$ open convex subsets. Now let M be the union of k open convex subsets $U_1, \dots, U_k$. We write $V = U_1 \cup \dots \cup U_{k-1}$. By induction the statement holds for V and also for U_k. Furthermore since the intersection of two convex open subsets is again convex and open we see that $V \cap U_k = (U_1 \cap U_k) \cup \dots \cup (U_{k-1} \cap U_k)$ is the union of $k-1$ open convex subsets, i.e. by induction the statement holds for $V \cap U_k$. But then it follows from (A) that the statement also holds for U.

(3) Let M be an open subset of $\mathbb{R}^m$. By Lemma 2.7 we can write M as the union of countably many open balls $B_i, i \in \mathbb{N}$. For $i \in \mathbb{N}$ we define $U_i := \bigcup\limits_{j\le i} B_j$. By (2) the statement holds for each U_i, but then it follows from (B) that the statement holds for M.

(4) Finally let M be any m-dimensional smooth manifold. We say a subset W of M is *small* if it is open, if $\overline{W}$ is compact and if W is contained in the domain of some chart for M. We make the following observations:
   (a) Since $\mathcal{H}$ and $\mathcal{K}$ are diffeomorphism invariants we obtain from (2) that the statement holds for every small subset of W (viewed as a smooth manifold in its own right).
   (b) The same induction argument as in (2) shows that the statement holds for finite unions of small subsets (here the key observation is that the intersection of two small subsets is again small).[1129]
   (c) Finally it follows easily from the fact that the smooth manifold M is second-countable, the existence of charts around every point and Lemma 2.25 that there exist small sets $W_i, i \in \mathbb{N}$ which cover M. As in (3) we consider $U_i := \bigcup\limits_{j \leq i} W_j$. We had just seen in (b) that the statement holds for all the U_i. But then it follows from (B) that the statement holds for M. ■

79.2. **Smooth singular cohomology.** One might think that it follows immediately from Proposition 79.1 and Theorem 79.2 (together with the calculations of the cohomology groups of a point in Lemma 73.12 and on page 1920) that the singular real valued cohomology and the de Rham cohomology groups of smooth manifolds are always isomorphic.

But recall that to apply Theorem 79.2 we need to find a natural transformation

$$t\colon \text{de Rham cohomology} \;\to\; \text{real valued singular cohomology}$$

that has the property that

$$\mathrm{H}^n_{\mathrm{dR}}(\star) \;\xrightarrow{\cong}\; \mathrm{H}^n(\star;\mathbb{R}) \qquad \text{for all } n \in \mathbb{N}_0.$$

Here is an idea: on page 1917 we had mentioned that given a compact oriented n-dimensional smooth manifold W and a smooth differential n-form ω on W there exists an

$$\int_W \omega \;\in\; \mathbb{R}$$

that has many good properties. So given some smooth manifold M (with no restrictions on the dimension) we could try to define a map

$$\begin{array}{rcl} t\colon\ \mathrm{C}^n_{\mathrm{dR}}(M) & \to & \mathrm{Hom}(\mathrm{C}_n(M);\mathbb{R}) \\ \omega & \mapsto & \left(\begin{array}{rcl} \mathrm{C}_n(M) & \to & \mathbb{R} \\ \sum\limits_{i=1}^{k} a_i \cdot (f_i\colon \Delta^n \to M) & \mapsto & \sum\limits_{i=1}^{k} a_i \cdot \int\limits_{\Delta^n} f_i^*\omega \end{array}\right). \end{array}$$

Unfortunately, there are several problems with this "definition":

(1) The first question that arises is the following: what does it mean to integrate a form over Δ^n? After all, it is not clear in what sense Δ^n is a smooth manifold. With some luck that should not be too much of an issue, since Δ^n is clearly pretty close to being a smooth manifold.

[1129] Here we implicitly use the following statement: if A and B are subsets of M such that $\overline{A}$ and $\overline{B}$ are compact, then $\overline{A \cap B}$ and $\overline{A \cup B}$ are also compact. We will prove this statement in Exercise 79.2.

(2) Let us suppose that we have resolved (1). Now let $f\colon \Delta^n \to M$ be a singular n-chain. If the map f is not smooth, then $f^*\omega$ just does not make any sense.[1130]

(3) After fixing the previous issues, why should t be a cochain map?

Arguably the most serious issue is (2). The only way around it is to only work with smooth singular simplices. This immediately leads us to the following definition:

Definition. Let $n \in \mathbb{N}_0$.

(1) As in Lemma 41.1 we consider the planar simplex

$$\Delta_n \;:=\; \Big\{(t_1,\dots,t_n)\in\mathbb{R}^n_{\geq 0}\;\Big|\;\sum_{i=1}^n t_i \in [0,1]\Big\}.$$

(2) In Lemma 41.1 we saw that the map[1131]

$$\begin{array}{rcl}\Theta_n\colon \Delta_n & \to & \Delta^n\\ (t_1,\dots,t_n) & \mapsto & \Big(t_1,\dots,t_n,1-\sum_{i=1}^n t_i\Big)\end{array}$$

is a homeomorphism.

Now let M be a smooth manifold.

(3) Recall that by the definition on page 278 a map $f\colon \Delta_n \to M$ is *smooth* if given any $a\in\Delta_n$ there exists an open neighborhood U in $\mathbb{R}^n$ of a and a smooth map $\widetilde{f}\colon U\to M$ which coincides with f on A. We define the differential of f at $a\in\Delta_n$ to be the differential of $\widetilde{f}$.[1132]

(4) We say a singular n-simplex $\sigma\colon\Delta^n\to M$ is *smooth* if the map $\sigma\circ\Theta_n\colon \Delta_n\to M$ is smooth. We illustrate the definition in Figure 1151.

(5) We write

$$\mathrm{C}_n^{\text{smooth}}(M) \;=\; \begin{array}{c}\text{the free abelian group generated by the}\\ \text{set of smooth singular } n\text{-simplices in } M.\end{array}$$

This is a subgroup of $\mathrm{C}_n(M)$ and it is straightforward to verify that the boundary map $\partial_n\colon \mathrm{C}_n(M)\to\mathrm{C}_{n-1}(M)$ restricts to a map $\partial_n\colon \mathrm{C}_n^{\text{smooth}}(M)\to \mathrm{C}_{n-1}^{\text{smooth}}(M)$.

(6) Given an abelian group G we write

$$\mathrm{C}^n_{\text{smooth}}(M;G) \;:=\; \mathrm{Hom}\big(\mathrm{C}_n^{\text{smooth}}(M),G\big)$$

and we define $\delta_n\colon \mathrm{C}^n_{\text{smooth}}(M;G)\to \mathrm{C}^{n+1}_{\text{smooth}}(M;G)$ as the map $\delta_n:=\partial^*_{n+1}$.

(7) We define the *n-th smooth singular cohomology group of a smooth manifold M with G-coefficients* as

$$\mathrm{H}^n_{\text{smooth}}(M;G) \;:=\; \frac{\ker\big(\delta_n\colon\ \mathrm{C}^n_{\text{smooth}}(M;G)\;\to\;\mathrm{C}^{n+1}_{\text{smooth}}(M;G)\big)}{\mathrm{im}\,\big(\delta_{n-1}\colon\ \mathrm{C}^{n-1}_{\text{smooth}}(M;G)\;\to\;\mathrm{C}^{n}_{\text{smooth}}(M;G)\big)}.$$

[1130] Recall that if $f\colon N\to M$ is a smooth map between smooth manifolds and ω is a smooth differential n-form on M, then $f^*\omega$ is defined to be the differential n-form that for $v_1,\dots,v_n\in \mathrm{T}_P N$ has the value

$$(f^*\omega)_P(v_1,\dots,v_n) \;=\; \omega_{f(P)}(\mathrm{D} f_P(v_1),\dots,\mathrm{D} f_P(v_n)).$$

But if f is not differentiable, then there is no differential $\mathrm{D} f_P\colon\ \mathrm{T}_P N\to \mathrm{T}_{f(P)} M$.

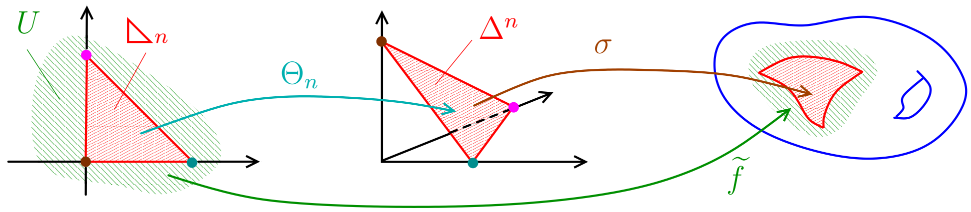


FIGURE 1151

Remark.

(1) Let $f\colon M \to N$ be a *smooth* map between two smooth manifolds. Evidently the induced map $f_*\colon \mathrm{C}^n(M) \to \mathrm{C}^n(N)$ restricts to a map $f_*\colon \mathrm{C}_n^{\mathrm{smooth}}(M) \to \mathrm{C}_n^{\mathrm{smooth}}(N)$. In particular we get an induced map $f^*\colon \mathrm{C}^n_{\mathrm{smooth}}(N,G) \to \mathrm{C}^n_{\mathrm{smooth}}(M,G)$. Thus we obtain for each $n \in \mathbb{N}_0$ an induced map

$$f^*\colon \mathrm{H}^n_{\mathrm{smooth}}(N;G) \;\to\; \mathrm{H}^n_{\mathrm{smooth}}(M;G).$$

(2) Every map $\Delta^n \to \star$ is smooth, so there is no difference between ordinary singular cohomology and smooth singular cohomology for smooth manifold $\star$.

Proposition 79.3. *Smooth singular cohomology with real coefficients is a smooth cohomology theory.*

SKETCH OF PROOF. We need to verify that smooth singular cohomology with real coefficients satisfies the properties of a smooth cohomology theory.

(0) It is clear that smooth singular cohomology with real coefficients is a contravariant functor from the category of smooth manifolds to the category of $\mathbb{N}_0$-graded real vector spaces.

(1) Suppose $f, g\colon M \to N$ are two smooth maps between two smooth manifolds and suppose we are given a *smooth* homotopy F between f and g. Some thought shows that the chain homotopy

$$P_n\colon \mathrm{C}_n(M) \;\to\; \mathrm{C}_{n+1}(N), \quad n \in \mathbb{N}_0$$

from the proof of Proposition 42.5 restricts to a chain homotopy

$$\mathrm{C}_n^{\mathrm{smooth}}(M) \;\to\; \mathrm{C}_{n+1}^{\mathrm{smooth}}(N), \quad n \in \mathbb{N}_0.$$

Now we can once again appeal to Lemma 73.8 to conclude that the induced maps on smooth singular cohomology are the same.

(2) Let M be a smooth manifold and let U and V be open subsets with $M = U \cup V$. The key to proving the existence of the Mayer–Vietoris sequence for ordinary singular cohomology was to show that the inclusion map

$$\mathrm{C}_*^{\{U,V\}}(M) \;\to\; \mathrm{C}_*(M)$$

[1131]Put differently, Θ_n is the inverse of the obvious projecton map $\Delta^n \to \llcorner_n$.

[1132]It is straightforward to verify that this definition does not depend on the choice of U and $\widetilde{f}$.

is a chain homotopy equivalence. This in turn we showed using Proposition 43.24. In the proof of Proposition 43.24 we introduced the subdivision maps

$$u_n\colon\ \mathrm{C}_n(M)\ \to\ \mathrm{C}_n(M),\quad n\in\mathbb{N}_0.$$

It follows immediately from the definitions that these maps restrict to maps

$$u_n\colon\ \mathrm{C}_n^{\text{smooth}}(M)\ \to\ \mathrm{C}_n^{\text{smooth}}(M),\quad n\in\mathbb{N}_0.$$

Using this observation the proof of the existence of a Mayer–Vietoris sequence for smooth singular cohomology is otherwise identical to the proof for ordinary singular cohomology.

(3) The limit property is proved the same way as Proposition 76.12 (1). We only have to replace the fact, proved in Lemma 47.5 that $\mathrm{C}_*(M)=\varinjlim \mathrm{C}_*(U_i)$ by the observation that the same argument as in Lemma 47.5 shows that $\mathrm{C}_*^{\text{smooth}}(M)=\varinjlim \mathrm{C}_*^{\text{smooth}}(U_i)$. ■

Now we can easily show that smooth singular cohomology with real coefficients and ordinary singular cohomology with real coefficients are naturally isomorphic. More precisely, we have the following proposition.

Proposition 79.4. *There exists a natural isomorphism from smooth singular cohomology with real coefficients to ordinary singular cohomology with real coefficients. In particular for any smooth manifold M and any $n\in\mathbb{N}_0$ we have a natural isomorphism*

$$\mathrm{H}^n(M;\mathbb{R})\ \xrightarrow{\cong}\ \mathrm{H}^n_{\text{smooth}}(M;\mathbb{R}).$$

Proof. For any smooth manifold M we have the natural inclusion

$$\mathrm{C}_*^{\text{smooth}}(M)\ \to\ \mathrm{C}_*(M)$$

which evidently respects the boundary maps. These maps thus give rise to natural maps

$$\mathrm{C}^*(M;\mathbb{R})\ \to\ \mathrm{C}^*_{\text{smooth}}(M;\mathbb{R})$$

which respect the coboundary maps. Thus we obtain a natural map

$$t(M)\colon\ \mathrm{H}^*(M;\mathbb{R})\ \to\ \mathrm{H}^*_{\text{smooth}}(M;\mathbb{R}).$$

As we pointed out above, for $\star$ there is no difference between smooth and ordinary singular cohomology, i.e.

$$t(\star)\colon\ \mathrm{H}^*(\star;\mathbb{R})\ \to\ \mathrm{H}^*_{\text{smooth}}(\star;\mathbb{R})$$

is an isomorphism. Since the natural transformation is induced by the inclusion map it follows easily that the natural transformation is compatible with the connecting homomorphism in the Mayer-Vietoris sequence and that it is compatible with the map τ appearing in the limit property. The proposition now follows immediately from Theorem 79.2. ■

Now we also want to relate smooth singular cohomology to de Rham cohomology. The key for doing so is the following definition and the subsequent lemma.

Definition. Given a smooth manifold M, given a smooth differential n-form ω on M and a smooth singular n-simplex $\sigma\colon \Delta^n \to M$ we define[1133][1134]

$$\int_\sigma \omega \;:=\; (-1)^n\cdot \int_{\text{◺}_n} (\sigma\circ\Theta_n)^*(\omega) \;:=\; (-1)^n\cdot \int_{x\in\text{◺}_n} \underbrace{\omega_{(\sigma\circ\Theta_n)(x)}(\mathrm{D}(\sigma\circ\Theta_n)_x(e_1),\ldots,\mathrm{D}(\sigma\circ\Theta_n)_x(e_n))}_{\text{continuous function in } x}.$$

We extend this definition linearly to define the integral of the smooth differential n-form ω over a smooth singular n-chain $\sum\limits_{i=1}^{k} a_i \circ \sigma_i$.

Lemma 79.5. *Let M be a smooth manifold, let ω be a smooth differential $(n-1)$-form on M and let $\sigma\colon \Delta^n \to M$ be a smooth singular n-simplex. Then*[1135]

$$\int_\sigma d\omega \;=\; \int_{\partial\sigma} \omega.$$

In the following we provide a sketch of the proof of Lemma 79.5. A detailed proof is given in [**War83**, p. 147].

Sketch of the proof. The idea of the proof is of course to use Stokes' Theorem 78.3. The fact that Δ^n, or equivalently ◺_n are not smooth manifolds in an obvious way is a slight nuisance. Here is one way how we can circumvent this problem. In [**Lee02**, Chapter 16] or alternatively [**Wall16**, p. 30] the theory of smooth manifolds with corner is developed. Prime examples are cubes $[0,1]^n$, half-balls $\{(x_1,\ldots,x_n)\in \overline{B}^n \,|\, x_n \geq 0\}$ and most interestingly for us, the standard n-simplex Δ^n and equivalently also ◺_n. Furthermore there is a version of Stokes' Theorem 78.3 for smooth manifolds with corner, see [**Lee02**, Theorem 16.25].

In the following we provide the proof of Lemma 79.5 with an intuitive understanding of the concepts. All arguments can be made rigorous using the techniques developed in [**Lee02**, Chapter 16].

[1133] The Lebesgue integral is defined since ◺_n is a compact subset of $\mathbb{R}^n$ and since the integrand is continuous.

[1134] Recall that on page 1075 we had equipped $\mathring{\Delta}^n$ with an orientation. The factor $(-1)^n$ is motivated by the fact that the restriction of the map $\Theta_n\colon \text{◺}_n \to \Delta^n$ to the interiors is orientation-preserving if and only if n is even.

[1135] Note that on the left we integrate a smooth differential n-form over a smooth singular n-simplex and that on the right we integrate a smooth $(n-1)$-form over a smooth singular $(n-1)$-chain.

Let M be a smooth manifold, let ω be a smooth $(n-1)$-form and let $\sigma\colon \Delta^n \to M$ be a smooth singular n-simplex. Then

$$\int_\sigma d\omega \underset{\substack{\uparrow\\ \text{by definition}}}{=} (-1)^n\cdot \int_{\triangle_n} (\sigma\circ\Theta)^*(d\omega) \underset{\substack{\uparrow\\ \text{Proposition 78.2 (6)}}}{=} (-1)^n\cdot \int_{\triangle_n} d(\sigma\circ\Theta)^*(\omega) \underset{\substack{\uparrow\\ \text{Stokes Theorem for smooth manifolds}\\ \text{with corner applied to } \triangle_n}}{=} (-1)^n\cdot \int_{\partial\triangle_n} (\sigma\circ\Theta_n)^*(\omega)$$

$$\underset{\substack{\uparrow\\ \text{since } \Theta_n^{-1}\circ i_j^n\circ\Theta_{n-1}\colon \triangle_{n-1}\to\partial\triangle_n \text{ is}\\ \text{orientation-reversing if and only if } j \text{ is even}\\ \text{(here we use Proposition 78.2 (1))}}}{=} \sum_{j=0}^{n}(-1)^j\cdot \int_{\triangle_{n-1}} (-1)^{n-1}\cdot(\sigma\circ i_j^n\circ\Theta_{n-1})^*(\omega) \underset{\substack{\uparrow\\ \text{by definition of the integral}\\ \text{of a smooth } (n-1)\text{-form over a}\\ \text{smooth singular } (n-1)\text{-simplex}}}{=} \sum_{j=0}^{n}(-1)^j\cdot\int_{\sigma\circ i_j^n}\omega \underset{\substack{\uparrow\\ \text{by definition of } \partial\sigma}}{=} \int_{\partial\sigma}\omega.$$

We refer to Figure 1152 for an illustration of the proof. ■

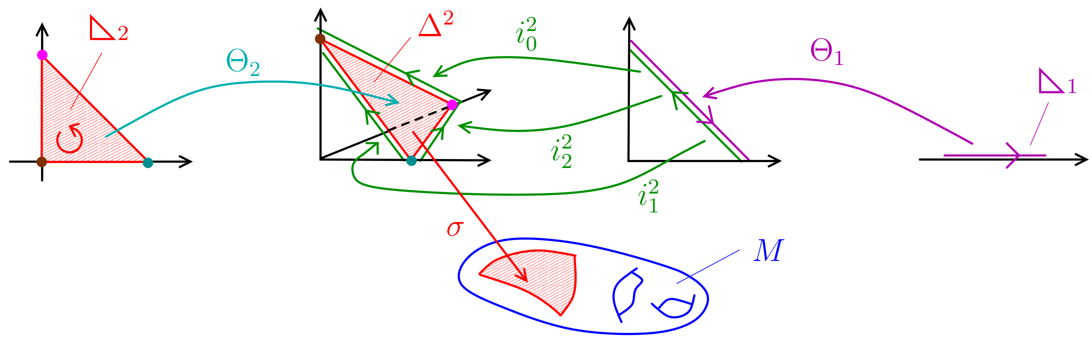


FIGURE 1152. Illustration for the proof of Lemma 79.5.

79.3. **The isomorphism between singular and de Rham cohomology.** After the preparations in the previous sections we can now easily prove the theorem that we had been aiming for.

Theorem 79.6. (De Rham Theorem) *There exists a natural isomorphism $u = u_M$ from de Rham cohomology to singular cohomology with real coefficients. In particular for any smooth manifold M and any $k\in\mathbb{N}_0$ we have a natural isomorphism*

$$u_M\colon\ \mathrm{H}^k_{\mathrm{dR}}(M) \xrightarrow{\cong} \mathrm{H}^k(M;\mathbb{R})$$

which has the property that for any closed differential k-form ω and any smooth singular k-chain σ that is a cycle we have

$$\Big\langle \underbrace{u_M([\omega])}_{\in \mathrm{H}^k(M;\mathbb{R})},\ \underbrace{[\sigma]}_{\in\mathrm{H}_k(M;\mathbb{Z})} \Big\rangle_M = \int_\sigma \omega.$$

(*Here we stress, that according to the convention from page 1932 it is understood that the smooth manifolds have empty boundary.*)

Remark. A rather different proof for Theorem 79.6 is given in [**War83**, p. 205] and [**Ram05**, Chapte 4.6]. More precisely, in [**War83, Ram05**] the equivalence of de Rham cohomology and real valued singular cohomology is proved by relating both to another type of cohomology theory, namely sheaf cohomology. We also refer to [**BoT82**, Theorems 8.9 and 15.8], [**Dup78**, Theorem 1.15], [**?**, Section 6.7.6] and [**Ful95**, Section 15C] for other proofs.

PROOF. We denote by s the natural isomorphism from real-valued singular cohomology to real-valued smooth singular cohomology that we had constructed in the proof of Proposition 79.4.

Now we construct a natural isomorphism t from de Rham cohomology to real-valued smooth singular cohomology. For any smooth manifold M and any $k \in \mathbb{N}_0$ we have, by the above discussion, a map

$$\begin{array}{rcl} \mathrm{C}^k_{\mathrm{dR}}(M) & \to & \mathrm{C}^k_{\mathrm{smooth}}(M;\mathbb{R}) \\ \omega & \mapsto & \left(\begin{array}{rcl} \mathrm{C}_k^{\mathrm{smooth}}(M) & \to & \mathbb{R} \\ \sigma & \mapsto & \int\limits_\sigma \omega \end{array} \right) \end{array}$$

Note that it follows basically immediately from the definitions that this map is natural in M. Also note that it is an immediate consequence of Lemma 79.5 that for given M all these maps together form a cochain map $\mathrm{C}^*_{\mathrm{dR}}(M) \to \mathrm{C}^*_{\mathrm{smooth}}(M;\mathbb{R})$. Summarizing we see that we have constructed a natural map

$$t(M)\colon\ \mathrm{H}^*_{\mathrm{dR}}(M) \ \to\ \mathrm{H}^*_{\mathrm{smooth}}(M;\mathbb{R}).$$

Furthermore, in the special case of $M = \star$ the map

$$t(\star)\colon\ \mathrm{H}^*_{\mathrm{dR}}(\star) \ \to\ \mathrm{H}^*_{\mathrm{smooth}}(\star;\mathbb{R})$$

is easily seen to be an isomorphism. Once one has unraveled all the definitions it becomes apparent that this natural transformation is compatible with the connecting homomorphism in the Mayer-Vietoris sequence and that it is compatible with the map τ appearing in the limit property. Now we obtain from Theorem 79.2 that t is a natural isomorphism from de Rham cohomology to real-valued smooth singular cohomology.

Finally we set $u := s^{-1} \circ t$. By the above u is a natural isomorphism from de Rham cohomology to singular cohomology with real coefficients. It follows easily from the definitions of s and t that for any closed differential k-form ω and any smooth singular k-chain σ that is a cycle we have

$$\langle\ \underbrace{u([\omega])}_{\in \mathrm{H}^k(M;\mathbb{R})}\ ,\ \underbrace{[\sigma]}_{\in \mathrm{H}_k(M;\mathbb{Z})}\ \rangle_M \ =\ \int\limits_\sigma \omega. \qquad \blacksquare$$

The following proposition summarizes a few useful properties of the natural isomorphism $u_M\colon\ \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k(M;\mathbb{R})$ from the above de Rham Theorem 79.6.

Proposition 79.7.

(1) *Let M be a closed oriented connected n-dimensional smooth manifold. Furthermore let $\omega \in \Omega_n(M)$ be a smooth differential n-form.*
   (a) *We have $\langle u_M([\omega]), [M]\rangle_M \ =\ \int_M \omega$.*
   (b) *If $\int_M \omega = 1$, then $u_M([\omega]) = [M]^* \in \mathrm{H}^n(M;\mathbb{R})$ where $[M]^*$ denotes the dual fundamental class that we introduced on page 1873.*

(2) *Let $f\colon M \to N$ be a smooth map between closed oriented connected n-dimensional smooth manifolds and let ω be a smooth differential n-form on N with $\int_N \omega = 1$. Then*

$$\int_M f^*\omega \;=\; \underbrace{\deg(f\colon M \to N)}_{\text{as defined on page 1737}}.$$

Proof.

(1) (a) Let M be a closed oriented connected n-dimensional smooth manifold and let $\omega \in \Omega_n(M)$ be a smooth differential n-form. As we explained in Algorithm 68.2, there exist smooth singular simplices $\varphi_1, \dots, \varphi_k\colon \Delta^n \to M$ with the following properties:

  (i) We have $\bigcup\limits_{i=1}^{k} \varphi_i(\Delta^n) = M$.

  (ii) For any $i, j \in \{1, \dots, k\}$ we have $\varphi_i(\Delta^n) \cap \varphi_j(\Delta^n) = \varphi_i(\partial\Delta^n) \cap \varphi_j(\partial\Delta^n)$.

  (iii) If for each $\in \{1, \dots, k\}$ we set $\epsilon_i = +1$ if φ_i is orientation-preserving and $\epsilon_i = -1$ otherwise, then the singular chain $\sigma := \epsilon_1 \cdot \varphi_1 + \cdots + \epsilon_k \cdot \varphi_k$ is a singular cycle that represents the fundamental class $[M]$.

As above we denote by $\Theta_n\colon \lrtriangle_n \to \Delta^n$ the inverse of the obvious projecton map. We calculate that

$$\int_M \omega \;\underset{\substack{\uparrow\\ \text{follows fairly easily from (i)– (ii)}\\ \text{and our orientation conventions}}}{=}\; \sum_{i=1}^{k} (-1)^n \cdot \epsilon_i \cdot \int_{\lrtriangle_n} (\varphi_i \circ \Theta_n)^*(\omega) \;\underset{\substack{\uparrow\\ \text{see definition}\\ \text{on page 1940}}}{=}\; \int_\sigma \omega \;\underset{\substack{\uparrow\\ \text{by Theorem 79.6}}}{=}\; \langle u_M([\omega]), [M]\rangle_M.$$

(b) This statement follows immediately from (a) and the definition of the dual fundamental class.

(2) Let $f\colon M \to N$ be a smooth map between closed oriented connected n-dimensional smooth manifolds and let ω be a smooth differential n-form on N with $\int_N \omega = 1$.

$$\begin{aligned}\int_M f^*\omega \;&\overset{\substack{\text{by (1a)}\\ \downarrow}}{=}\; \langle u_M(f^*([\omega])), [M]\rangle_M \;\overset{\substack{\text{since } u \text{ are } \textit{natural} \text{ isomorphisms}\\ \downarrow}}{=}\; \langle f^*(u_N([\omega])), [M]\rangle_M \;\overset{\substack{\text{Lemma 74.6 (3)}\\ \downarrow}}{=}\; \langle u_N([\omega]), f_*([M])\rangle_N\\ &\underset{\substack{\uparrow\\ \text{by definition of } \deg(f)}}{=}\; \deg(f) \cdot \langle u_N([\omega]), [N]\rangle_N \;\underset{\substack{\uparrow\\ \text{by (1a)}}}{=}\; \deg(f) \cdot \int_N \omega \;\underset{\substack{\uparrow\\ \text{by choice of } \omega}}{=}\; \deg(f).\end{aligned}$$

■

Theorem 79.6 allows us to transport information from de Rham cohomology to real-valued singular cohomology and vice-versa. For example the many computational techniques that we had developed, e.g. cellular cohomology, for singular cohomology can now be used to calculate isomorphism types of de Rham cohomology groups. In particular we can easily prove the following corollary.

Corollary 79.8. *If M is a closed, oriented connected, non-empty n-dimensional smooth manifold, then the map*

$$\begin{aligned} \mathrm{H}^n_{\mathrm{dR}}(M) &\to \mathbb{R} \\ [\omega] &\mapsto \int_M \omega \end{aligned}$$

is an isomorphism.

Proof. By Proposition 78.4 (7) the map is well-defined and it is an epimorphism. On the other hand, by Proposition 75.15 and Theorem 79.6 we know that $\mathrm{H}^n_{\mathrm{dR}}(M) \cong \mathbb{R}$, which implies that the map is an isomorphism. ■

79.4. **Realizing cohomology classes by maps to S^1.** We start out this section with the following definition.

Definition. Let G be a group, let Y be a topological space and let $y_0 \in Y$ be a point with an identification $\pi_1(Y, y_0) = G$. Given another topological space X and $x_0 \in X$ we say that a homomorphism $\varphi\colon \pi_1(X, x_0) \to G = \pi_1(Y, y_0)$ is *realized by a map* $g\colon X \to Y$ if $g(x_0) = y_0$ and if $g_* = \varphi$. If there is no danger of confusion we will drop the base points from the notation.

In the following we will mostly discuss the case $G = \mathbb{Z}$ and $Y = S^1$. Before we can continue we introduce the following conventions.

Convention.

(1) We use the diffeomorphism $\mathbb{R}/\mathbb{Z} \to S^1$ given by $[t] \mapsto e^{2\pi i t}$ to make the identification $\mathbb{R}/\mathbb{Z} = S^1$.
(2) By Proposition 16.17 we have an isomorphism $\mathbb{Z} \to \pi_1(S^1, 1)$ given by sending 1 to the equivalence class of the loop $t \mapsto e^{2\pi i t}$. We use this isomorphism to make the identification $\pi_1(\mathbb{R}/\mathbb{Z}, 0) = \pi_1(S^1, 1) = \mathbb{Z}$.
(3) Let X be a path-connected topological space and let $x_0 \in X$. As shown in Proposition 75.18 we can use the Hurewicz homomorphism and the evaluation homomorphism to make the identifications $\mathrm{H}^1(X; \mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(X); \mathbb{Z}) = \mathrm{Hom}(\pi_1(X, x_0), \mathbb{Z})$.

Now we can formulate the following question.

Question 79.9. *Let X be a path-connected topological space that is "reasonable", e.g. that is a smooth manifold or a* CW*-complex. Let $x_0 \in X$ be a base point. Can every homomorphism $\varphi\colon \pi_1(X, x_0) \to \mathbb{Z} = \pi_1(S^1, 1)$ be realized by a map $(X, x_0) \to (S^1, 1)$?*

The following proposition gives an affirmative answer if X is a smooth manifold.

Proposition 79.10. *Let M be a connected smooth manifold (here for once we allow M to have boundary) and let P be a base point.*

(1) *Every class $\phi \in \mathrm{H}^1(M; \mathbb{Z}) = \mathrm{Hom}(\pi_1(M, P), \mathbb{Z})$ can be realized by a smooth map $g\colon M \to S^1$ with $g(P) = 1$.*
(2) *We suppose that $\partial M = \varnothing$. If $f, g\colon M \to S^1$ are two smooth maps with $g(P) = 1$, such that $f_* = g_*\colon \pi_1(M, P) \to \pi_1(S^1, 1)$, then f and g are homotopic.*
(3) *Let $\phi \in \mathrm{H}^1(M; \mathbb{Z})$ and let $P \in \partial M$. If $f\colon \partial M \to S^1$ is a smooth map that realizes $\phi|_{\partial M}$, then there exists a smooth map $g\colon M \to S^1$ with $g|_{\partial M} = f$ that realizes ϕ.*

Remark.

(1) Sometimes the following reformulation of Proposition 79.10 is quite useful: Let M be a connected smooth manifold and let P be a base point. Given any $\phi \in \mathrm{H}^1(M;\mathbb{Z})$ there exists a smooth map $g\colon M \to S^1$ and a generator $\psi \in \mathrm{H}^1(S^1;\mathbb{Z})$ with $\phi = g^*(\psi)$.
(2) Proposition 79.10 (2) can be viewed as a partial converse to Proposition 78.5.
(3) Later on, in Proposition 120.15 we will prove an analogue of Proposition 79.10 for CW-complexes. This will allows in Proposition 120.20 to give a new proof of Proposition 79.10.

Examples.

(1) We consider the torus $T = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ and we make the usual identification $\mathrm{H}_1(T;\mathbb{Z}) = \mathbb{Z}^2$. We consider the cohomology classes in $\mathrm{H}^1(T;\mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(T),\mathbb{Z})$ that are given by

$$\begin{array}{rcl} \phi\colon\ \mathrm{H}_1(T;\mathbb{Z}) = \mathbb{Z}^2 & \to & \mathbb{Z} \\ (m,n) & \mapsto & m \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi\colon\ \mathrm{H}_1(T;\mathbb{Z}) = \mathbb{Z}^2 & \to & \mathbb{Z} \\ (m,n) & \mapsto & n. \end{array}$$

These two cohomology classes are evidently realized by the projection $S^1 \times S^1 \to S^1$ onto the first, respectively the second factor. In Exercise 79.3 the goal will be to give an example of an explicit map $g\colon T \to S^1$ that realizes the homomorphism $\varphi\colon\ \mathrm{H}_1(T;\mathbb{Z}) = \mathbb{Z}^2 \to \mathbb{Z}$ given by $\varphi(m,n) = 6m + 4n$.
(2) In Figure 1153 we consider the surface Σ of genus 2. As we saw on page 1270, the oriented curves x_1, y_1, x_2, y_2 represent a basis for $\mathrm{H}_1(\Sigma)$. We consider some random epimorphism $\phi\colon\ \mathrm{H}_1(\Sigma) \to \mathbb{Z}$, e.g. we could consider the epimorphism $\phi\colon\ \mathrm{H}_1(\Sigma) \to \mathbb{Z}$ that is given by $\phi(x_1) = 0$, $\phi(y_1) = 1$, $\phi(x_2) = -2$ and $\phi(y_2) = 1$. By Proposition 79.10 we know that ϕ can be realized by some map $g\colon \Sigma \to S^1$. But can one "visualize" such a map? We will return to this question on page 2370.

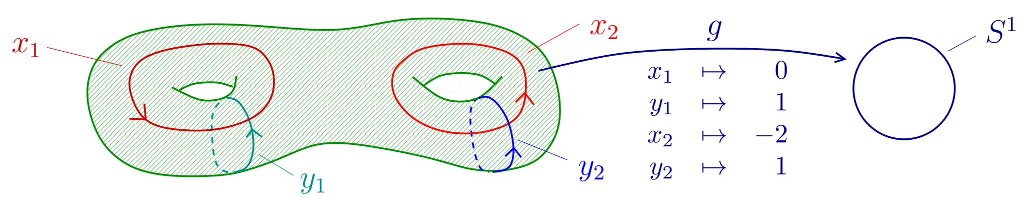


FIGURE 1153

We break the proof of Proposition 79.10 into three parts.

PROOF OF PROPOSITION 79.10 (1). Let M be a connected smooth manifold. We first assume that M has no boundary. We start out with the following definition. Let ω be a smooth differential 1-form on M and let $\gamma\colon [a,b] \to M$ be a piecewise smooth path. We pick $a = t_0 < t_1 < \cdots < t_n = b$ such that the restriction of γ to each subinterval $[t_i, t_{i+1}]$ is smooth. We define

$$\int_{[a,b]} \gamma^*\omega \;=\; \sum_{i=0}^{n-1} \int_{[t_i,t_{i+1}]} \gamma^*\omega.$$

It follows immediately from Proposition 78.2 (2) that this definition does not depend on the choice of the subdivision.

After this preparation we start with the actual proof of the proposition. We fix a base point $P \in M$. Furthermore let $\phi \in \mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M,P),\mathbb{Z})$. We denote by u the natural isomorphism from de Rham cohomology to singular cohomology with real coefficients from Theorem 79.6. We pick a smooth closed differential 1-form ω on M with $u([\omega]) = \phi$.

Claim. Let $\delta\colon [0,1] \to M$ be a piecewise smooth loop in P. Then

$$\int\limits_{[0,1]} \delta^*\omega \;=\; \phi(\underbrace{[\delta]}_{\in \pi_1(M,P)}) \;\in\; \mathbb{R}.$$

Let $\delta\colon [0,1] \to M$ be a piecewise smooth loop in P. One can show easily that after possibly performing an orientation-preserving reparametrization we can write $\delta = \delta_1 * \cdots * \delta_k$ as a product of smooth paths $\delta_i\colon [0,1] \to M$, $i = 1,\dots,k$. As in the definition of the Hurewicz homomorphism we use the map $t \mapsto (1-t,t)$ to make the identification $[0,1] = \Delta^1$. Using this identification we view each δ_i also as a smooth 1-simplex. In particular we have $\delta_1 + \cdots + \delta_k \in \mathrm{C}_1^{\mathrm{smooth}}(M)$. Then

$$\int\limits_{[0,1]} \delta^*\omega \underset{\substack{\uparrow \\ \text{by the definitions and} \\ \text{by Proposition 78.2} \\ \text{(1) and (2)}}}{=} \int\limits_{\delta_1+\cdots+\delta_k} \omega \underset{\substack{\uparrow \\ \text{by Theorem 79.6} \\ \text{since } u([\omega]) = \phi}}{=} \langle \phi, [\delta_1 + \cdots + \delta_k] \rangle \underset{\substack{\uparrow \\ \text{via the identification} \\ \mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M,P),\mathbb{Z})}}{=} \phi([\delta]).$$

⊞

We define

$$\begin{aligned} g\colon M &\to S^1 = \mathbb{R}/\mathbb{Z} \\ Q &\mapsto \Big[\int\limits_{[0,1]} \gamma^*\omega\Big] \quad \text{where } \gamma \text{ is a smooth path}^{1136} \text{ from } P \text{ to } Q. \end{aligned}$$

We claim that the map g has the required properties. It remains to prove the following four statements:

(1) The map g is well-defined.
(2) We have $g(P) = [0] \in \mathbb{R}/\mathbb{Z}$.
(3) The map g is smooth.
(4) We have $g_* = \phi\colon \pi_1(M,P) \to \pi_1(S^1,1) = \mathbb{Z}$.

Now we turn to the proofs of the statements.

(1) Let Q in M. Suppose we are given two piecewise smooth paths β and γ from the base point P to Q. (See the left of Figure 1154 for an illustration.) Then

$$\int\limits_{[0,1]} \beta^*\omega - \int\limits_{[0,1]} \gamma^*\omega \underset{\substack{\uparrow \\ \text{follows from the definitions} \\ \text{and Proposition 78.2 (1) and (2)}}}{=} \int\limits_{[0,1]} (\beta * \overline{\gamma})^*\omega \underset{\substack{\uparrow \\ \text{by the above claim, since } \beta*\overline{\gamma} \\ \text{is a piecewise smooth loop}}}{=} \phi([\beta * \overline{\gamma}]) \;\in\; \mathbb{Z}.$$

This shows that g is indeed well-defined.

[1136]By Lemma 8.30 (2) any two points on a connected smooth manifold, that do not lie on the boundary, can be connected by a smooth path. Here we again use that M has no boundary.

(2) We determine $g(P)$ by taking the constant path from P to P. It follows immediately from the definition of g that $g(P) = [0] \in \mathbb{R}/\mathbb{Z}$.

(3) Let $Q \in M$. We need to show that g is smooth in an open neighborhood of Q. We pick a path γ from P to Q. We pick a chart $\Phi \colon U \to B_r(0)$ around Q with $\Phi(Q) = 0$. Given a point $R \in U$ we denote by δ_R the piecewise smooth path that is given by the concatenation of the following two paths:
   (a) the path γ from P to Q,
   (b) the "radial path" from Q to R that corresponds, under the diffeomorphism Φ, to the radial path from 0 to $\Phi(R) \in B_r(0)$.

   (See the right of Figure 1154 for an illustration.) Using these paths δ_R it is now relatively straightforward to show that g is smooth in U. We leave the verification of the details to the reader.

(4) Suppose we are given an element in $\pi_1(M,P)$. By Lemma 8.30 (1) we can represent this element by a smooth loop $\delta \colon [0,1] \to M$. We have the following equality:

$$\begin{aligned} g_*([\delta]) &\underset{\text{by Theorem 16.16}}{=} \text{endpoint of the lift of } g\circ\delta \colon [0,1] \to \mathbb{R} \text{ to the universal covering } \mathbb{R} \text{ of } \mathbb{R}/\mathbb{Z} \\ &= \text{endpoint of the lift of the path } t \mapsto \Big[\int_{[0,t]} \delta^*\omega \Big] \text{ to the universal covering } \mathbb{R} \\ &= \text{endpoint of the path } t \mapsto \int_{[0,t]} \delta^*\omega \;=\; \text{the point } \int_{[0,1]} \delta^*\omega \underset{\text{by the above claim}}{=} \phi([\delta]). \end{aligned}$$

   We have thus shown that $g_* = \phi \colon \pi_1(M,P) \to \pi_1(S^1,1) = \mathbb{Z}$.

This concludes the proof of the theorem in the case that M has empty boundary.

Finally let M be a smooth manifold with non-empty boundary. To deal with this case we consider $W := M \cup (\partial M \times [0,1))$ where we make the obvious identification of $P = P \times \{0\}$ for $P \in \partial M$. Using the Collar Neighborhood Theorem 8.12 one can easily verify that W is a smooth manifold such that M is a submanifold and such that W has no boundary. Evidently M is a deformation retract of W. It follows from Lemma 18.14 and Proposition 18.16 (2) that the induced map $i_* \colon \pi_1(M,P) \to \pi_1(W,P)$ is an isomorphism. Now let $\phi \in \operatorname{Hom}(\pi_1(M,P),\mathbb{Z})$. By the previously considered case we can realize $\phi \circ (i_*)^{-1} \colon \pi_1(W,P) \to \mathbb{Z}$ by a smooth map $g \colon W \to S^1$. The restriction of g to M has all the desired properties. ■

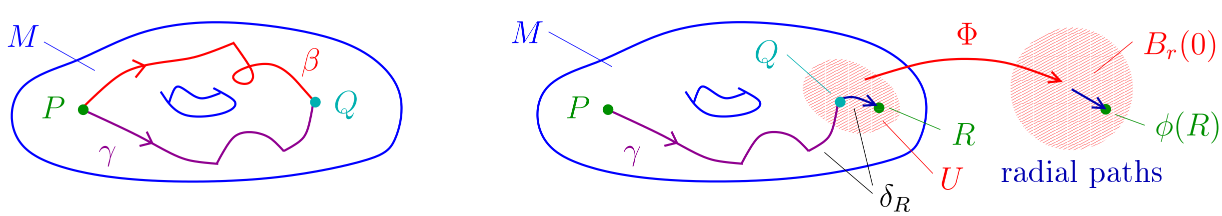


FIGURE 1154

PROOF OF PROPOSITION 79.10 (2). We suppose that $\partial M = \varnothing$. In the following let $f, g \colon M \to S^1 = \mathbb{R}/\mathbb{Z}$ be two smooth maps with $g(P) = 1$, such that we have the equality

$f_* = g_*\colon \pi_1(M,P) \to \pi_1(S^1,1)$. We need to show that f and g are homotopic. Let ω be the standard closed differential 1-form on $S^1 = \mathbb{R}/\mathbb{Z}$. We set $\alpha := f^*\omega$ and $\beta := g^*\omega$.

Claim. There exists a closed differential 1-form ψ on $M \times [0,1]$ such that the restriction to $M \times \{0\} = M$ equals α and such that the restriction to $M \times \{1\} = M$ equals β.

First note that it follows from the fact that $f_* = g_*$ and from the natural isomorphisms $\mathrm{H}^1_{\mathrm{dR}}(M) = \mathrm{H}^1(M;\mathbb{R}) = \mathrm{Hom}(\pi_1(M,P),\mathbb{R})$ that $[\alpha] = f^*([\omega]) = g^*([\omega]) = [\beta] \in \mathrm{H}^1_{\mathrm{dR}}(M)$. By definition this means that there exists a 0-form φ on M with $d\varphi = \alpha - \beta$. Next let $p\colon M \times [0,1] \to M$ and $q\colon M \times [0,1] \to [0,1]$ be the obvious projection. Furthermore let $h\colon [0,1] \to [0,1]$ be a map that is zero close to 0 and one close to 1. On $M \times [0,1]$ we consider the 1-form that given by $\psi := p^*\alpha - d(q^*h \cdot \varphi)$. It follows from Proposition 78.2 that ψ is closed. We calculate that

$$\psi \;=\; p^*\alpha - d(q^*h\cdot\varphi) \;\underset{\substack{\uparrow\\ \text{by Proposition 78.2}}}{=}\; p^*\alpha - (q^*(dh)\cdot\varphi + q^*h\cdot d\varphi) \;=\; p^*\alpha - \overbrace{q^*(dh)}^{\substack{=0 \text{ on } M\times\{0\} \text{ and } M\times\{1\}\\ \downarrow}}\cdot\varphi - \underbrace{q^*h\cdot d\varphi}_{\substack{\uparrow\\ =0 \text{ on } M\times\{0\} \text{ and } = p^*\alpha - p^*\beta \text{ on } M\times\{1\}}}. \quad \boxplus$$

Next note that, as in the proof of Statement (1), the closed 1-form ψ gives rise to a map $F\colon M \times [0,1] \to S^1$. It follows fairly easily from the definitions that $F_0 = f$ and $F_1 = g$. We leave it to the reader to fill in the details. ■

Proof of Proposition 79.10 (3). Let $\phi \in \mathrm{H}^1(M;\mathbb{Z})$, let $P \in \partial M$ and let $f\colon \partial M \to S^1$ be a smooth map that realizes $\phi|_{\partial M}$. By the Collar Neighborhood Theorem 8.12 we can pick a collar neighborhood $[0,1] \times \partial M$. We set $W := M \setminus ([0,1) \times \partial M)$ and we equip it with the base point $(P,1)$. By Corollary 8.14 we know that W is an n-dimensional smooth manifold with $\partial W = \{1\} \times M$. By Proposition 79.10 (1) there exists a smooth map $h\colon W \to S^1$ which realizes $\phi|_W$. It is now straightforward to see that under the obvious isomorphism $\{0\} \times M \to \{1\} \times M$ the maps $f_*\colon \mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(S^1;\mathbb{Z})$ and $h_*\colon \mathrm{H}_1(M;\mathbb{Z}) \to \mathrm{H}_1(S^1;\mathbb{Z})$ agree. At this point we can conclude from Proposition 79.10 (2) that there exists a smooth map $F\colon [0,1] \times \partial M \to S^1$ such that $F_0 = f$ and $F_1 = h|_{\{1\}\times M}$.

Note that by definition of a collar neighborhood and by Lemma 8.11 we know that $[0,1] \times \partial M$ and W are closed subsets of M. Thus we obtain from Lemma 2.35 (2) that the two maps F and g combine to a continuous map $h\colon M \to S^1$.

Since h is smooth on $[0,1] \times \partial M$ we obtain from the Whitney Approximation Theorem 9.3 that h is homotopic rel $\{0\} \times \partial M$ to a smooth map $g\colon M \to S^1$. This map now has all the desired properties. ■

Of course one can ask many variations on Question 79.9. Since arbitrary topological spaces can be quite weird we restrict ourselves in the following questions to CW-complexes.

Question 79.11. *Let X be a connected* CW*-complex.*

(1) *Let $g \geq 1$ and let Σ_g be the surface of genus $g \geq 1$. Can every homomorphism $\pi_1(X) \to \pi_1(\Sigma_g)$ be realized by a map $X \to \Sigma_g$?*

(2) *Can every homomorphism* $\pi_1(X) \to \pi_1\Big(\bigvee\limits_{i=1}^{k} S^1\Big) = \langle x_1, \dots, x_k \rangle$ *be realized by a map* $X \to \bigvee\limits_{i=1}^{k} S^1$*?*

(3) *Can every homomorphism* $\pi_1(X) \to \mathbb{Z}_2 = \pi_1(\mathbb{RP}^2)$ *be realized by a map* $X \to \mathbb{RP}^2$*?*

79.5. **Rewriting products** $M \times S^1$ $(*)$**.** Proposition 79.10 (1) has many consequences. For example in Proposition 98.4 we will use it to show that for a given compact orientable n-dimensional smooth manifold M any class in $\mathrm{H}_{n-1}(M;\mathbb{Z})$ can be represented by a submanifold. In this section we will give a different application, namely we will see that in many situations one can find interesting ways to rewrite a product $M \times S^1$ where M is a closed connected smooth manifold.

To formulate that application it is convenient to recall the following definition from page 711.

Definition. Let X be a topological space and let $f\colon X \to X$ be a map. We refer to

$$\mathrm{Tor}(X, f) \ := \ (X \times [0,1]) \,/ \sim \qquad \text{where } (x,0) \sim (f(x),1) \text{ for all } x \in X,$$

as the *mapping torus of* (X, f). We refer to the map

$$\begin{array}{rcl} q\colon \ \mathrm{Tor}(X,f) = (X \times [0,1])/(x,0) \sim (f(x),1) & \to & S^1 \\ {[(x,t)]} & \mapsto & e^{2\pi i t} \end{array}$$

as *the natural projection onto* S^1.

Throughout this section we also use the following notation.

Notation. We denote by $\Theta\colon \mathrm{H}_1(S^1) \to \mathbb{Z}$ the isomorphism given by $\Theta([S^1]) = 1$.

Now we can state the main result of this section.

Proposition 79.12. *Let* M *be a closed connected manifold and let* $\phi\colon \mathrm{H}_1(M \times S^1;\mathbb{Z}) \to \mathbb{Z}$ *be an epimorphism. We set* $k = k(\phi) := \phi([\{*\} \times S^1])$*. If* $k \neq 0$*, then there exists a* $|k|$*-fold covering* $p\colon \widetilde{M} \to M$ *and a deck transformation* $f\colon \widetilde{M} \to \widetilde{M}$ *together with a diffeomorphism*[1137] $\Xi\colon \mathrm{Tor}(\widetilde{M}, f) \to M \times S^1$ *such that the following diagram commutes:*

$$\begin{tikzcd} \mathrm{H}_1(\mathrm{Tor}(\widetilde{M}, f);\mathbb{Z}) \arrow[rr, "\Xi_*"] \arrow[dr, "\Theta \circ q_*"'] & & \mathrm{H}_1(M \times S^1;\mathbb{Z}) \arrow[dl, "\phi"] \\ & \mathbb{Z}. & \end{tikzcd}$$

Here the diagonal map to the left is induced by the natural projection.

Remark.

(1) Proposition 79.12 can be viewed as a "manifold analogue" of the group-theoretic statement of Exercise 24.17.
(2) One mathematician once called products "slippery". Indeed, as we will see in the examples below, Proposition 79.12 shows that many products have alternative descriptions.

[1137] By Lemma 24.22 the mapping torus $\mathrm{Tor}(\widetilde{M}, f)$ naturally has the structure of a smooth manifold.

Examples.

(1) Let M be a closed connected non-empty smooth manifold and let $\psi\colon \mathrm{H}_1(M;\mathbb{Z}) \to \mathbb{Z}$ be a homomorphism. We apply Proposition 79.12 to the epimorphism

$$\begin{array}{rcl} \phi\colon \mathrm{H}_1(M\times S^1;\mathbb{Z}) = \mathrm{H}_1(M;\mathbb{Z}) \oplus \mathrm{H}_1(S^1;\mathbb{Z}) & \to & \mathbb{Z} \\ (\sigma, n\cdot[S^1]) & \mapsto & \psi(\sigma)+n. \end{array}$$

Note that we have $k = k(\phi) = 1$. This implies that the map $p\colon \widetilde{M} \to M$ is a covering of degree 1. By Proposition 29.8 we can assume that $\widetilde{M} = M$ and that $p = \mathrm{id}$. Note that the only deck transformation of the 1-fold covering $\mathrm{id}\colon M \to M$ is the identity. Thus we obtain from Proposition 79.12 that there exists a diffeomorphism $\Xi\colon M\times S^1 \to M\times S^1$ such that the following diagram commutes

$$\begin{tikzcd} \mathrm{H}_1(M\times S^1;\mathbb{Z}) \arrow[rr, "\Xi_*"] \arrow[dr, "\Theta\circ q_*"'] & & \mathrm{H}_1(M\times S^1;\mathbb{Z}) \arrow[dl, "\phi"] \\ & \mathbb{Z}, & \end{tikzcd}$$

where $q\colon M\times S^1 \to S^1$ denotes the natural projection. Note that if $\psi\colon \mathrm{H}_1(M;\mathbb{Z}) \to \mathbb{Z}$ is non-zero, then Ξ_* is not the identity. Summarizing, we have now shown that if $\mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(M;\mathbb{Z}),\mathbb{Z}) \neq 0$, then $M\times S^1$ can be decomposed as a product in more than one way.

(2) Let Σ be the surface of some genus $g \in \mathbb{N}_{\geq 2}$ and let $\phi\colon \mathrm{H}_1(\Sigma\times S^1;\mathbb{Z}) \to \mathbb{Z}$ be an epimorphism such that $k = k(\phi) \neq 0$. It follows from Proposition 79.12, together with Proposition 31.6 (1) or alternatively with Lemma 55.9, that $\Sigma\times S^1$ is diffeomorphic to a mapping torus of the form $\mathrm{Tor}(\widetilde{\Sigma}, f)$ where $\widetilde{\Sigma}$ is a surface of genus $|k|\cdot(g-1)+1$. In particular we obtain the slightly mind boggling statement that the product $\Sigma\times S^1$ can also be viewed as a mapping torus of a self-diffeomorphism of a surface of arbitrarily high genus.

Proof. First we pick points $* \in M$ and $* \in S^1$ and we denote by $j_1\colon M \xrightarrow{x\,\mapsto\,(x,*)} M\times S^1$ and $j_2\colon S^1 \xrightarrow{z\,\mapsto\,(*,z)} M\times S^1$ the corresponding inclusions. Note that by Proposition 79.10 (1) there exists a smooth map $g\colon M \to S^1$ such that the map $\mathrm{H}_1(M) \xrightarrow{g_*} \mathrm{H}_1(S^1) \xrightarrow{\Theta} \mathbb{Z}$ equals $\phi\circ j_{1*}\colon \mathrm{H}_1(M) \to \mathbb{Z}$. Now let us consider the following pullback diagram:

$$\begin{tikzcd} \widetilde{M} := \{(x,z)\in M\times S^1 \,|\, g(x)=z^k\} \arrow[r, "(x{,}z)\mapsto z"] \arrow[d, "(x{,}z)\mapsto x"] & S^1 \arrow[d, "z\mapsto z^k"] \\ M \arrow[r, "g"] & S^1. \end{tikzcd}$$

By Lemma 25.16 we know that $p\colon \widetilde{M} \to M$ is a k-fold covering. The map

$$\begin{array}{rcl} f\colon \widetilde{M} & \to & \widetilde{M} \\ (x,z) & \mapsto & (x, e^{-2\pi\mathrm{i}/k}\cdot z) \end{array}$$

is evidently a deck transformation. Next we consider the following diagram:

$$\begin{tikzcd}
\widetilde{M}\times S^1 \arrow[r, "((x,z),w)\mapsto((x,z),z^{-1}\cdot w)"] \arrow[d, swap, "((x,z),e^{2\pi\,\mathrm{i}\,t})\mapsto[((x,z),k\cdot t \bmod \mathbb{Z})]"] & \widetilde{M}\times S^1 \arrow[d, "((x,z),w)\mapsto(x,w)"] \\
\operatorname{Tor}(\widetilde{M},f) \arrow[r, "{[((x,z),t)]\mapsto(x,z^{-1}\cdot e^{2\pi\,\mathrm{i}\,t/k})}", "\Xi"'] & M\times S^1.
\end{tikzcd}$$

We make the following observations:

(1) One can easily verify that the bottom map is in fact well-defined and that the diagram commutes.
(2) It is pretty clear that the top map is a diffeomorphism.[1138]
(3) It is straightforward to show that the vertical maps are local diffeomorphisms and they are degree k coverings.

It follows from the above together with Exercise 17.4 that the bottom map is also a diffeomorphism. It remains to prove the following claim.

Claim. We have $\phi\circ\Xi_*=\Theta\circ q_*\colon\ \mathrm{H}_1(\operatorname{Tor}(\widetilde{M},f))\to\mathbb{Z}$.

We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_1(\widetilde{M}) \arrow[dd, "i_*"] & \mathrm{H}_1(M)\oplus\mathrm{H}_1(S^1) \arrow[rr, "g_*\oplus h_*"] \arrow[dd, shift left, "\cong"', "j_{1*}\oplus j_{2*}"] & & \mathrm{H}_1(S^1)\oplus\mathrm{H}_1(S^1) \arrow[dd, "\varphi\oplus\psi\mapsto\varphi+\psi"] \\
& & \mathrm{H}_1(S^1\times S^1) \arrow[ur, "\pi_{1*}\oplus\pi_{2*}"] \arrow[dr, "\mu"] & \\
\mathrm{H}_1(\operatorname{Tor}(\widetilde{M},f)) \arrow[r, "\Xi_*"'] \arrow[d, "q_*"] & \mathrm{H}_1(M\times S^1) \arrow[uu, shift left, "\pi_{1*}\oplus\pi_{2*}", "\cong"'] \arrow[ur, "(g\times h)_*"] \arrow[r, "\phi"'] & \mathbb{Z} & \mathrm{H}_1(S^1) \arrow[l, "\Theta"] \\
\mathrm{H}_1(S^1) \arrow[rr, "\Theta"] & & \mathbb{Z}. \arrow[u, dashed, red, "\Omega"] &
\end{tikzcd}$$

We make the following clarifications and observations:

(1) The map $i\colon\widetilde{M}\to\operatorname{Tor}(\widetilde{M},f))$ is the natural inclusion given by $x\mapsto[(x,0)]$.
(2) The map $\mu\colon S^1\times S^1\to S^1$ is the multiplication map.
(3) In both settings we denote by π_1 and π_2 the projection onto the first respectively second factor.
(4) The map $h\colon S^1\to S^1$ is given by $z\mapsto z^k$. Note that by Lemma 45.10 we know that $h_*\colon\ \mathrm{H}_1(S^1)\to\mathrm{H}_1(S^1)$ is given by multiplication by k.
(5) It follows from Exercise 52.5, or alternatively from the discussion on page 1965, that the vertical maps in the middle are isomorphisms and inverses of one another.
(6) The upper left triangle commutes by the functoriality of homology groups.
(7) The triangle to the right commutes by Exercise 52.4.
(8) The big rectangle to the right commutes by the choice of g and h.

[1138]The reader might note that the top map is the restriction of the map $(M\times S^1)\times S^1\to(M\times S^1)\times S^1$ given by the same formula, which is clearly a diffeomorphism, to the submanifold $\widetilde{M}\times S^1$ of $(M\times S^1)\times S^1$.

(9) It follows from (5), (6), (7) and (8) that the triangle at the bottom commutes.

(10) Recall that any point $(x,z) \in \widetilde{M}$ satisfies by definition $g(x) = z^k$. It now follows immediately that the map $(g \times h) \circ \mu \circ \Xi \circ i \colon \widetilde{M} \to S^1$ is the constant map.

(11) It follows from (9) and (10) that $\phi \circ \Xi_* \circ i_* \colon \mathrm{H}_1(\widetilde{M}) \to \mathbb{Z}$ is the zero map.

(12) By Lemma 46.23 we know that the vertical sequence to the left is exact.

(13) It follows from (10) and (11) that the vertical map Ω exists.

(14) We pick $(x,z) \in \widetilde{M}$. We use the homeomorphism from Lemma 24.25 to make the identification $\mathrm{Tor}(\widetilde{M}, f) = (\widetilde{M} \times \mathbb{R})/(w,t) \sim (f(w), t+1)$. Now we consider the singular 1-cycle

$$\begin{array}{rcl} \sigma \colon \Delta^1 & \to & \mathrm{Tor}(\widetilde{M}, f) = (\widetilde{M} \times \mathbb{R})/(w,t) \sim (f(w), t+1) \\ (1-t, t) & \mapsto & [((x,z), kt)]. \end{array}$$

It follows immediately from the definitions that $(\phi \circ \Xi_*)([\sigma]) = k$ and $(\Theta \circ q_*)([\sigma]) = k$. Thus we have $\Omega(k) = k$. Since $k \neq 0$ we see that $\Omega = \mathrm{id}$. ■

79.6. **The wedge product of differential forms** $(*)$**.** For a real vector space V we had already introduced on page 1916 the bilinear wedge product

$$\wedge^k V^* \times \wedge^l V^* \;\to\; \wedge^{k+l} V^*.$$

Given a smooth manifold M we can define this wedge product for each tangent space $V = \mathrm{T}_P M$. Using charts one can show that this wedge product sends smooth forms to smooth forms. More precisely, we have a bilinear map

$$\wedge \colon \mathrm{C}^k_{\mathrm{dR}}(M) \times \mathrm{C}^l_{\mathrm{dR}}(M) \;\to\; \mathrm{C}^{k+l}_{\mathrm{dR}}(M).$$

We refer [**Tu11**, p. 205] for details.

Lemma 79.13. *Let M be a smooth manifold. For any $k, l \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl} \wedge : \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^l_{\mathrm{dR}}(M) & \to & \mathrm{H}^{k+l}_{\mathrm{dR}}(M) \\ ([\omega], [\sigma]) & \mapsto & [\omega] \wedge [\sigma] := [\omega \wedge \sigma] \end{array}$$

is well-defined.

Proof. By [**Tu11**, Proposition 4.7] we know that for any smooth differential k-form ω and any smooth differential l-form σ we have

$$(*) \qquad d(\omega \wedge \sigma) \;=\; d\omega \wedge \sigma + (-1)^k \cdot \omega \wedge d\sigma.$$

Now let $\omega \in \mathrm{C}^k_{\mathrm{dR}}(M)$ and $\sigma \in \mathrm{C}^l_{\mathrm{dR}}(M)$ be smooth forms.

(1) It follows immediately from $(*)$ that $\omega \wedge \sigma$ is again closed.

(2) If $\omega' = \omega + d\tau$ is another representative of $[\omega]$, then

$$\omega' \wedge \sigma - \omega \wedge \sigma \underset{\substack{\uparrow \\ \text{since “}\wedge\text{” is bilinear}}}{=} (\omega' - \omega) \wedge \sigma \;=\; d\tau \wedge \sigma \underset{\substack{\uparrow \\ \text{by } (*) \text{ and since } d\sigma = 0}}{=} d(\tau \wedge \sigma).$$

This shows that $\omega \wedge \sigma$ and $\omega' \wedge \sigma$ represent the same de Rham cohomology class, i.e. the wedge product does not depend on the choice of the representative of the de Rham cohomology class of the first entry.

(3) The proof that the wedge product does not depend on the choice of the representative of the second de Rham cohomology class is basically the same as the proof of (2). ■

At this point it is helpful to introduce the following definition.

Definition.

(1) A *graded ring* is a ring[1139] R together with a direct sum decomposition[1140] $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ into subgroups such that for any $m, n \in \mathbb{N}_0$ the multiplication map restricts to a map
$$R_m \times R_n \to R_{m+n}.$$
(2) A graded ring $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ such that for any two homogeneous elements a and b the equality $a \cdot b = (-1)^{\deg(a)\cdot\deg(b)} \cdot b \cdot a$ holds is called a *superalgebra.*[1141]
(3) Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ and $S = \bigoplus_{n\in\mathbb{N}_0} S_n$ be two graded rings. A *morphism between the two graded rings* is a ring homomorphism $\varphi\colon R \to S$ such that for any $n \in \mathbb{N}_0$ we have $\varphi(R_n) \subset S_n$.
(4) We define the *category $\mathcal{GrRing}$ of graded rings* to be the category where the objects are graded rings and the morphisms are morphisms of graded rings. Similarly we define the category of graded superalgebras.

Example. We consider the polynomial ring $R = \mathbb{Q}[x]$. We can view this ring as a graded ring by defining $R_n := \mathbb{Q} \cdot x^n$ for $n \in \mathbb{N}_0$.

Recall that for a given smooth manifold M we write
$$\mathrm{H}^*_{\mathrm{dR}}(M) = \bigoplus_{i=0}^{\infty} \mathrm{H}^i_{\mathrm{dR}}(M).$$
We consider the following map
$$\begin{array}{rcl} \wedge\colon \mathrm{H}^*_{\mathrm{dR}}(M) \times \mathrm{H}^*_{\mathrm{dR}}(M) & \to & \mathrm{H}^*_{\mathrm{dR}}(M) \\ \left(\sum\limits_{i=0}^{\infty} x_i, \sum\limits_{i=0}^{\infty} y_i\right) & \mapsto & \sum\limits_{i,j=0}^{\infty} x_i \wedge y_j. \end{array}$$
where it is understood that each x_i and y_i is homogeneous of degree i.

The following proposition summarizes the key features of the wedge product on de Rham cohomology.

Proposition 79.14. *The maps*
$$\begin{array}{rcl} M & \mapsto & (\mathrm{H}^*_{\mathrm{dR}}(M), \wedge) \\ (f\colon M \to N) & \mapsto & (f^*\colon \mathrm{H}^*_{\mathrm{dR}}(N) \to \mathrm{H}^*_{\mathrm{dR}}(M)) \end{array}$$

[1139] Here we mean by a ring an abelian group together with a multiplicative structure which is associative, distributive and that admits a multiplicatively neutral element. We do not demand that a ring is commutative.

[1140] In particular, R viewed as an abelian group is $\mathbb{N}_0$-graded.

[1141] The word "algebra" in our context is justified by the fact that any ring is a $\mathbb{Z}$-algebra. We could also have referred to such an object as a superring, but this terminology seems to be uncommon.

define a contravariant functor from the category of smooth manifolds to the category of graded superalgebras.

Proof.

(1) First note that it follows almost immediately from Lemma 78.1 that for a given smooth manifold M the wedge product turns $\mathrm{H}^*_{\mathrm{dR}}(M)$ into a graded superalgebra. It follows immediately from the definitions that the multiplicatively neutral element in de Rham cohomology is hereby the 0-dimensional cohomology class that is represented by the constant function $f\colon M \to \mathbb{R}$, $x \mapsto 1$.

(2) If $f\colon M \to N$ is a smooth map between smooth manifolds, then it follows immediately from the definitions that for any choice of $\omega \in \mathrm{C}^k_{\mathrm{dR}}(N)$ and $\sigma \in \mathrm{C}^l_{\mathrm{dR}}(N)$ the following equality holds:

$$(f^*\omega \wedge f^*\sigma) \;=\; f^*(\omega \wedge \sigma).$$

Therefore the smooth map f induces a morphism $f^*\colon \mathrm{H}^*_{\mathrm{dR}}(N) \to \mathrm{H}^*_{\mathrm{dR}}(M)$ of graded rings. ■

Example. We consider the differential 1-forms dx and dy on $\mathbb{R}^2$ which are given by[1142] $dx(v,w) = v$ and $dy(v,w) = w$. Furthermore we identify the 2-torus T with the quotient space $\mathbb{R}^2/\mathbb{Z}^2$ and we denote the projection $\mathbb{R}^2 \to T = \mathbb{R}^2/\mathbb{Z}^2$ by p. It is straightforward to see that dx and dy induce differential 1-forms on T.[1143] We denote these also by dx and dy.[1144] The 1-forms dx and dy are closed, i.e. $d(dx) = d(dy) = 0$.[1145] Thus they define cohomology classes $[dx], [dy] \in \mathrm{H}^1_{\mathrm{dR}}(T)$. A straightforward calculation, see e.g. [**Frie16a**, p. 248], shows that if we integrate the 2-form $dx \wedge dy$ on T with the standard orientation, then we obtain 1. It follows from Proposition 78.4 that $[dx \wedge dy] = [dx] \wedge [dy]$ is a non-trivial element in $\mathrm{H}^2_{\mathrm{dR}}(T)$.

From the fact that $\mathrm{H}^*_{\mathrm{dR}}(T)$ is a superalgebra we know that

$$[dx] \wedge [dy] \;=\; (-1)^{1\cdot 1} \cdot [dy] \wedge [dx] \;=\; -[dy] \wedge [dx].$$

Since this wedge product is non-zero we see that $(\mathrm{H}^*_{\mathrm{dR}}(T), \wedge)$ is a non-commutative ring.

This discovery of a ring structure on de Rham cohomology raises the question whether one can also define such a ring structure using singular cohomology and whether such a ring structure exists on singular cohomology for more general spaces. We will answer these questions in Chapter 80.

[1142]As always we identify $\mathrm{T}_P\,\mathbb{R}^2$ with $\mathbb{R}^2$.

[1143]Details are given in [**Frie16a**, p. 248].

[1144]More precisely, we showed that there exists a unique 1-form dx on T such that for each point $P \in \mathbb{R}^2$ and $v \in \mathrm{T}_P\,\mathbb{R}^2$ we have $dx_P(v) = dx_{p(P)}(p_*(v))$. Of course the same statement also holds for dy.

[1145]This is clear for dx and dy viewed as differential 1-forms on $\mathbb{R}^2$. But the projection map $p\colon \mathbb{R}^2 \to T = \mathbb{R}^2/\mathbb{Z}^2$ is a local diffeomorphism. Since the differential commutes with smooth maps, see Proposition 78.2 (6), we also see that $d(dx) = d(dy) = 0$ on $\mathbb{R}^2/\mathbb{Z}^2$.

Exercises for Chapter 79.

Exercise 79.1. Let $\mathcal{H}^*$ be a smooth cohomology theory with $\mathcal{H}^n(\star) = 0$ for $n \geq 1$. We denote by $f\colon S^1 \to S^1$ the map that is given by $(x, y) \mapsto (x, -y)$. Show, only using the axioms of a smooth cohomology theory, that $f^*\colon \mathcal{H}^1(S^1) \to \mathcal{H}^1(S^1)$ is given by multiplication by -1.

Exercise 79.2. Let M be a smooth manifold. Show that if A and B are subsets of M such that $\overline{A}$ and $\overline{B}$ are compact, then $\overline{A \cap B}$ and $\overline{A \cup B}$ are also compact.

Exercise 79.3. We make the usual identification $\pi_1(\mathbb{R}^2/\mathbb{Z}^2) = \mathbb{Z}^2$. We consider the homomorphism $\phi\colon \mathbb{Z}^2 = \pi_1(\mathbb{R}^2/\mathbb{Z}^2) \to \mathbb{Z}$ given by $\phi((1,0)) = 6$ and $\phi((0,1)) = 4$. Provide an *explicit* map $f\colon \mathbb{R}^2/\mathbb{Z}^2 \to S^1$ such that $f_* = \phi$.

Exercise 79.4. Let M be a smooth manifold and let $\phi \in \mathrm{H}^1(M;\mathbb{Z})$. Show that $\phi \cup \phi = 0$.

Exercise 79.5. Let M be a smooth manifold and let A be a union of components of ∂M. We define

$$\mathrm{C}^k_{\mathrm{dr}}(M, A) \;:=\; \{\varphi \in \Omega^k(M) \,|\, \text{there exists a neighborhood } U \text{ of } A \text{ such that } \varphi|_U \equiv 0\}.$$

(a) Show that the usual differential $d\colon \mathrm{C}^k_{\mathrm{dR}}(M) \to \mathrm{C}^{k+1}_{\mathrm{dR}}(M)$ restricts to a coboundary map $d\colon \mathrm{C}^k_{\mathrm{dR}}(M, A) \to \mathrm{C}^{k+1}_{\mathrm{dR}}(M, A)$. We refer to resulting cohomology groups by $\mathrm{H}^k_{\mathrm{dR}}(M, A)$.

(b) Show that there exists a natural connecting homomorphism $\delta\colon\ \mathrm{H}^k_{\mathrm{dR}}(A) \to \mathrm{H}^{k+1}_{\mathrm{dR}}(X, A)$ which gives rise to a long exact sequence

$$\ldots \longrightarrow \mathrm{H}^k_{\mathrm{dR}}(M, A) \longrightarrow \mathrm{H}^k_{\mathrm{dR}}(M) \longrightarrow \mathrm{H}^k_{\mathrm{dR}}(A) \xrightarrow{\delta} \mathrm{H}^{k+1}_{\mathrm{dR}}(M, A) \longrightarrow \ldots$$

(c) Show that there exists a natural isomorphism $u\colon\ \mathrm{H}_k(M, A) \to$ such that the following diagram commutes:

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{H}^k(M, A;\mathbb{R}) \arrow[r] \arrow[d, "\cong"', "u"] & \mathrm{H}^k(M;\mathbb{R}) \arrow[r] \arrow[d, "\cong"', "u"] & \mathrm{H}^k(A;\mathbb{R}) \arrow[r, "\delta"] \arrow[d, "\cong"', "u"] & \mathrm{H}^{k+1}(M, A;\mathbb{R}) \arrow[r] \arrow[d, "\cong"', "u"] & \ldots \\
\ldots \arrow[r] & \mathrm{H}^k_{\mathrm{dR}}(M, A) \arrow[r] & \mathrm{H}^k_{\mathrm{dR}}(M) \arrow[r] & \mathrm{H}^k_{\mathrm{dR}}(A) \arrow[r, "\delta"] & \mathrm{H}^{k+1}_{\mathrm{dR}}(M, A) \arrow[r] & \ldots
\end{tikzcd}$$

(d) Let $\varphi \in \mathrm{H}^1(M, A;\mathbb{Z})$. Show that there exists a smooth map $f\colon M \to S^1$ that vanishes on an open neighborhood of U and which has the property that the induced map $f_*\colon \mathrm{H}_1(M, A;\mathbb{Z}) \to \mathrm{H}_1(S^1) = \mathbb{Z}$ equals the given cohomology class $\varphi \in \mathrm{H}^1(M, A;\mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(M, A;\mathbb{Z}), \mathbb{Z})$.
Remark. This statement is the analogue of Proposition 79.10 (1).

80. The Eilenberg-Zilber theorem and the cross product

In Section 79.6 we saw that de Rham cohomology of a smooth manifold carries a product structure which turns de Rham cohomology into a ring. It is therefore natural to try to find such a product structure in singular homology and/or in singular cohomology.

The key to doing so will be the Eilenberg-Zilber Theorem 58.4 that we had stated before without providing a proof. In this chapter we will give a slightly more precise statement and we provide the proof. In this chapter and in Chapter 81 we will see how it can be used to find product structures on homology and cohomology groups.

80.1. **The Eilenberg-Zilber theorem.** We start out by recalling and introducing some definitions.

Definition.

(1) Let $\mathcal{C} = (\mathrm{C}_*, \partial_*)$ and $\mathcal{C} = (\mathrm{C}'_*, \partial'_*)$ be two chain complexes. On page 1433 we defined the tensor product $\mathcal{C} \otimes \mathcal{C}'$ of the chain complexes where the n-th chain group is defined as
$$(\mathcal{C} \otimes \mathcal{C}')_n \;:=\; \bigoplus_{p+q=n} \mathrm{C}_p \otimes \mathrm{C}'_q$$
and where the boundary homomorphisms are given by
$$\begin{array}{rcl} \partial^{\otimes} \colon (\mathcal{C} \otimes \mathcal{C}')_n & \to & (\mathcal{C} \otimes \mathcal{C}')_{n-1} \\ c_p \otimes c'_q & \mapsto & \partial c_p \otimes c'_q + (-1)^p \cdot c_p \otimes \partial' c'_q. \end{array}$$

(2) Given a point x in a topological space X we denote by x also the singular 0-simplex given by sending the unique point of Δ^0 to x.

(3) Given two topological spaces X and Y we refer to the maps
$$\begin{array}{rcl} \mathrm{C}_0(X) \otimes \mathrm{C}_0(Y) & \to & \mathrm{C}_0(X \times Y) \\ x \otimes y & \mapsto & (x, y) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{C}_0(X \times Y) & \to & \mathrm{C}_0(X) \otimes \mathrm{C}_0(Y) \\ (x, y) & \mapsto & x \otimes y \end{array}$$
as the *canonical maps*. We refer to the identity maps on $\mathrm{C}_0(X) \otimes \mathrm{C}_0(Y)$ and on $\mathrm{C}_0(X \times Y)$ also as the *canonical maps.*

The following theorem is a slightly more precise formulation of the original Eilenberg-Zilber Theorem 58.4.

Theorem 80.1. (Eilenberg-Zilber)

(1) *Given any topological spaces X and Y there exist natural chain maps*
$$\Upsilon \colon\; \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \;\to\; \mathrm{C}_*(X \times Y)$$
and
$$\Theta \colon\; \mathrm{C}_*(X \times Y) \;\to\; \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y),$$
which on the 0-level are the above canonical maps.

(2) *The maps Υ and Θ are unique up to natural chain homotopies.*[1146]

(3) *The maps Υ and Θ are naturally*[1147] *chain homotopy inverses of one another.*

We refer to the maps Υ and Θ as *Eilenberg-Zilber* maps. In the notation and our discussions we will usually ignore the fact that these maps are well-defined only up to a natural homotopy. We will only work with the properties of Υ and Θ stated in the theorem, so a different choice of maps Υ and Θ does not affect anything we say afterwards.

80.2. **Proof of the Eilenberg-Zilber Theorem 80.1.** The proof of the Eilenberg-Zilber Theorem 80.1 breaks up into three parts: we first construct Υ, then we construct Θ and finally we show that these maps are unique in the above sense and that they have the desired properties.

We start out with the following definition.

Definition. A topological space X is called *acyclic* if $\mathrm{H}_0(X;\mathbb{Z}) \cong \mathbb{Z}$ and if all higher homology groups are zero.

Example. In Corollary 42.8 we showed that every contractible topological space is acyclic. In particular every standard simplex Δ^n is contractible and thus acyclic. Furthermore, all products $\Delta^p \times \Delta^q$ of standard simplices are contractible and thus acyclic.[1148]

We also fix some notation.

Notation. Let X be a topological space and let $x \in X$. If $\sigma\colon \Delta^n \to Y$ is a singular n-simplex in a topological space Y, then we denote by $x \times \sigma\colon \Delta^n \to X \times Y$ the singular n-simplex given by $v \mapsto (x, \sigma(v))$. We use the analogous notation for the roles of X and Y swapped.

Now we can formulate the following theorem.

Theorem 80.2. *Given any two topological spaces X and Y and any $p, q \in \mathbb{N}_0$ there exists a homomorphism*

$$\Upsilon_{p,q}\colon \mathrm{C}_p(X) \otimes \mathrm{C}_q(Y) \ \to\ \mathrm{C}_{p+q}(X \times Y)$$

such that the following conditions are satisfied:

(1) *for $x \in X$ and $\sigma\colon \Delta^q \to Y$ we have $\Upsilon_{0,q}(x \otimes \sigma) = x \times \sigma$, the same statement also holds with the roles of X and Y swapped,*
(2) *the map $\Upsilon_{p,q}$ is natural, i.e. if $f\colon X' \to X$ and $g\colon Y' \to Y$ are maps between topological spaces, then for any $a \in \mathrm{C}_p(X')$ and any $b \in \mathrm{C}_q(Y')$ we have*

$$(f \times g)_*\big(\underbrace{\Upsilon_{p,q}(a \otimes b)}_{\in \mathrm{C}_{p+q}(X' \times Y')}\big) \ =\ \underbrace{\Upsilon_{p,q}(f_*(a) \otimes g_*(b))}_{\in \mathrm{C}_{p+q}(X \times Y)},$$

[1146]Here “natural chain homotopy” means that given two such maps Υ and Υ' there exists a choice of chain homotopies $(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_k \to \mathrm{C}_{k+1}(X \times Y)$, which is natural in X and Y, between Υ and Υ'. An analogous statement holds for maps Θ and Θ'.

[1147]Here “naturally” means that $\Upsilon \circ \Theta$ and $\Theta \circ \Upsilon$ are naturally homotopic to the identity maps, i.e. there exist natural chain homotopies from $\Upsilon \circ \Theta$ to the identity map and from $\Theta \circ \Upsilon$ to the identity map.

[1148]This can be seen in many ways, for example pick points $x \in \Delta^p$ and $y \in \Delta^q$, it is straightforward to show that there exists a deformation retraction r from Δ^p to $\{x\}$ and that there exists a deformation retraction s from Δ^q to $\{y\}$. But these two deformation retractions also define a deformation retraction from $\Delta^p \times \Delta^q$ to (x, y).

(3) *the maps $\Upsilon_{p,q}$ form a chain map, i.e. for any $a \in \mathrm{C}_p(X)$ and any $b \in \mathrm{C}_q(Y)$ we have*

$$\partial(\Upsilon_{p,q}(a \otimes b)) \;=\; \underbrace{\Upsilon_{p-1,q}(\partial a \otimes b) + (-1)^p \cdot \Upsilon_{p,q-1}(a \otimes \partial b)}_{=\Upsilon(\partial^{\otimes}(a \otimes b))}.$$

Proof. We prove the existence of the maps $\Upsilon_{p,q}$ by induction on $p \cdot q$. For clarity we henceforth suppress the subscript from $\Upsilon_{p,q}$.

Clearly for $p = 0$ or $q = 0$ the maps from (1) have the properties (2) and (3). So now suppose that we are given $p, q \in \mathbb{N}_0$ with $p \cdot q > 0$ such that the maps Υ have been defined for all p', q' with $p' \cdot q' < p \cdot q$.

Given any $n \in \mathbb{N}_0$ we denote by $\mathrm{id}_n \colon \Delta^n \to \Delta^n$ the identity map. We first want to come up with a reasonable definition of $\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q)$. If we want (3) to be satisfied we have to define $\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q) = z$ where $z \in \mathrm{C}_{p+q}(\Delta^p \times \Delta^q)$ satisfies

$$(*) \qquad \partial z \;=\; \Upsilon(\partial\,\mathrm{id}_p \otimes \mathrm{id}_q) + (-1)^p \cdot \Upsilon(\mathrm{id}_p \otimes \partial\,\mathrm{id}_q) \;\in\; \mathrm{C}_{p+q-1}(\Delta^p \times \Delta^q).$$

As we pointed out on page 1957, the product $\Delta^p \times \Delta^q$ is acyclic, in particular we have $\mathrm{H}_{p+q-1}(\Delta^p \times \Delta^q) = 0$, which means that every singular $(p+q-1)$-chain *that is a cycle* is in fact the boundary of some $(p+q)$-chain z. So we have to show that the right-hand side of $(*)$ is indeed a cycle. We compute that

$$\begin{aligned}
&\overbrace{\partial(\Upsilon(\partial\,\mathrm{id}_p \otimes \mathrm{id}_q) + (-1)^p \cdot \Upsilon(\mathrm{id}_p \otimes \partial\,\mathrm{id}_q))}^{\text{boundary of our hypothetical } \Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q)} \overset{\substack{\text{linearity of } \partial \\ \downarrow}}{=} \partial(\Upsilon(\partial\,\mathrm{id}_p \otimes \mathrm{id}_q)) + (-1)^p \cdot \partial(\Upsilon(\mathrm{id}_p \otimes \partial\,\mathrm{id}_q)) \\
&\underset{\substack{\uparrow \\ \text{by our induction hypothesis we can apply the boundary} \\ \text{formula (3) to } \Upsilon(\partial\,\mathrm{id}_p \otimes \mathrm{id}_q) \text{ and } \Upsilon(\mathrm{id}_p \otimes \partial\,\mathrm{id}_q)}}{=} \Upsilon(\underbrace{\partial\partial\,\mathrm{id}_p}_{=0} \otimes \mathrm{id}_q) + (-1)^{p-1} \cdot \Upsilon(\partial\,\mathrm{id}_p \otimes \partial\,\mathrm{id}_q) \\
&\qquad + (-1)^p \cdot \Upsilon(\partial\,\mathrm{id}_p \otimes \partial\,\mathrm{id}_q) + (-1)^p \cdot (-1)^p \cdot \Upsilon(\mathrm{id}_p \otimes \underbrace{\partial\partial\,\mathrm{id}_q}_{=0}) = 0.
\end{aligned}$$

This discussion gives us the definition of $\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q) := z$.

Now let X and Y be topological spaces and let $\varphi \colon \Delta^p \to X$ and $\psi \colon \Delta^q \to Y$ be singular simplices. If we want (2) to be satisfied then we have to define

$$\Upsilon(\underbrace{\varphi \otimes \psi}_{\in \mathrm{C}_p(X) \otimes \mathrm{C}_q(Y)}) \;:=\; (\varphi \times \psi)_*(\underbrace{\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q)}_{\in \mathrm{C}_{p+q}(\Delta^p \times \Delta^q)}) \;\in\; \mathrm{C}_{p+q}(X \times Y).$$

We extend this map linearly to all of $\mathrm{C}_p(X) \otimes \mathrm{C}_q(Y)$. It follows almost immediately from the definition of Υ that the map Υ is natural in X and Y.

It remains to prove (3). This property holds basically by definition. But for completeness' sake we carry out the proof. So let $\varphi \colon \Delta^p \to X$ and $\psi \colon \Delta^q \to Y$ be singular simplices.

We then have the following equality in $\mathrm{C}_{p+q-1}(X \times Y)$:

$$\begin{aligned}
\partial(\Upsilon(\varphi \otimes \psi)) &= \partial\big((\varphi \times \psi)_*(\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q))\big) \underset{\substack{\uparrow \\ \text{since } (\varphi\times\psi)_* \text{ is a chain map}}}{=} (\varphi \times \psi)_*(\partial(\overbrace{\Upsilon(\mathrm{id}_p \otimes \mathrm{id}_q)}^{=z})) \\
&\underset{\substack{\uparrow \\ \text{by definition of } z \text{ and the linearity of } (\varphi\times\psi)_*}}{=} (\varphi \times \psi)_*(\Upsilon(\partial\,\mathrm{id}_p \otimes \mathrm{id}_q)) + (-1)^p \cdot (\varphi \times \psi)_*(\Upsilon(\mathrm{id}_p \otimes \partial\,\mathrm{id}_q)) \\
&\underset{\substack{\uparrow \\ \text{by induction we can apply the naturality of } \Upsilon}}{=} (\Upsilon(\varphi_*(\partial\,\mathrm{id}_p) \otimes \psi_*(\mathrm{id}_q))) + (-1)^p \cdot \Upsilon(\varphi_*(\mathrm{id}_p) \otimes \psi_*(\partial\,\mathrm{id}_q)) \\
&\underset{\substack{\uparrow \\ \text{since } \varphi_* \text{ and } \psi_* \text{ are chain maps and since } \varphi_*(\mathrm{id}_p)=\varphi \text{ and } \psi_*(\mathrm{id}_q)=\psi}}{=} \Upsilon(\partial(\varphi) \otimes \psi) + (-1)^p \cdot \Upsilon(\varphi \otimes \partial\psi))
\end{aligned}$$

■

Now we turn to the construction of the second map in the Eilenberg-Zilber Theorem 80.1.

Theorem 80.3. *Given any two topological spaces X and Y and any $k \in \mathbb{N}_0$ there exists a homomorphism*

$$\Theta_k \colon \mathrm{C}_k(X \times Y) \;\to\; (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_k \;=\; \bigoplus_{p+q=k} \mathrm{C}_p(X) \otimes \mathrm{C}_q(Y)$$

such that the following conditions are satisfied:

(1) *the map Θ_0 is the canonical map on the 0-level, i.e. for every $x \in X$ and $y \in Y$ we have $\Theta_0((x,y)) = x \otimes y$,*

(2) *the map Θ_k is natural in X and Y, i.e. for any two maps $f \colon X' \to X$ and $g \colon Y' \to Y$ of topological spaces and any $c \in \mathrm{C}_k(X' \times Y')$ we have*

$$\Theta_k((f \times g)_*(c)) \;=\; (f_* \otimes g_*)(\Theta_k(c)),$$

(3) *the maps Θ_* form a chain map, i.e. for any $k \in \mathbb{N}_0$ we have*

$$\partial_k^{\otimes} \circ \Theta_k \;=\; \Theta_{k-1} \circ \partial_k.$$

In the proof of Theorem 80.3 we will need the following lemma.

Lemma 80.4. *Let X and Y be two acyclic topological spaces. For every $k \geq 1$ we have*

$$\mathrm{H}_k(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)) \;=\; 0.$$

Furthermore we have an isomorphism

$$\begin{aligned}
\epsilon \colon \mathrm{H}_0(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)) &\xrightarrow{\cong} \mathbb{Z} \\
\Big[\sum_{i=1}^{k} \underbrace{a_i}_{\in \mathbb{Z}} \cdot \underbrace{x_i}_{\in X} \otimes \underbrace{y_i}_{\in Y}\Big] &\mapsto \sum_{i=1}^{k} a_i.
\end{aligned}$$

Proof. The lemma is an immediate consequence of our hypotheses, Proposition 41.5 and the Künneth Theorem 58.7 for Chain Complexes. (Note that this Künneth Theorem is purely algebraic.) ■

Proof of Theorem 80.3. As we will see the idea behind the proof of Theorem 80.3 is very similar to the idea of the proof of Theorem 80.2. We start out with the outline of proof:

(a) We use (1) to define Θ_0,
(b) we suppose the map $\Theta_0, \dots, \Theta_{k-1}$ are defined, we then use (3) to make an educated guess for $\Theta_k(d_k)$ where we consider
  (i) the topological spaces $X = Y = \Delta^k$ and
  (ii) the singular k-simplex $d_k\colon \Delta^k \to \Delta^k \times \Delta^k$ that is given by the diagonal map $d_k(x) = (x, x)$.
(c) we then use the naturality statement of (2) to define Θ_k for all topological spaces X and Y and all singular simplices in $\mathrm{C}_*(X \times Y)$.

Now we carry out the above program. First of all, given a 0-simplex (x, y) in $X \times Y$ we define $\Theta_0((x, y)) = x \otimes y$. Then we extend this definition linearly to get the desired map $\Theta_0\colon \mathrm{C}_0(X \times Y) \to \mathrm{C}_0(X) \otimes \mathrm{C}_0(Y)$.

Now suppose we have already constructed the maps $\Theta_0, \dots, \Theta_{k-1}$. We want to define Θ_k and we start out by defining $\Theta_k(d_k\colon \Delta^k \to \Delta^k \times \Delta^k)$. Since we need (3) to be satisfied we need to define $\Theta_k(d_k) := c$ where $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_k$ satisfies $\partial_k^{\otimes}(c) = \Theta_{k-1}(\partial d_k)$. In the chain complex $\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)$ we calculate that

$$\partial^{\otimes}(\overbrace{\Theta_{k-1}(\partial_k(d_k))}^{\text{defined by induction}}) \underset{\substack{\uparrow \\ \text{by induction we can apply (3)}}}{=} \Theta_{k-2}(\underbrace{\partial_{k-1}\partial_k}_{=0} d_k) = 0,$$

i.e. the chain $\Theta_{k-1}(\partial_k d_k) \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_{k-1}$ is in fact a cycle. Since Δ^k is acyclic it follows immediately from Lemma 80.4 that this cycle is null-homologous. (For $k = 1$ one easily verifies, using the explicit description of Θ_0, that the cycle lies in the kernel of the isomorphism $\epsilon\colon \mathrm{H}_0(\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)) \to \mathbb{Z}$.) This means that there exists a singular chain $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_k$ with $\partial^{\otimes}(c) = \Theta_{k-1}(\partial_k d_k)$. We define $\Theta_k(d_k) := c$.

Now we define Θ_k for arbitrary topological spaces X and Y and an arbitrary singular k-simplex $\sigma\colon \Delta^k \to X \times Y$. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the obvious projection maps. Note that we have a commutative diagram

$$\begin{tikzcd} & \Delta^k \times \Delta^k \arrow[d, "(p\circ\sigma)\times(q\circ\sigma)"] \\ \Delta^k \arrow[ur, "d_k"] \arrow[r, "\sigma"'] & X \times Y, \end{tikzcd}$$

i.e. we have $\sigma = ((p \circ \sigma) \times (q \circ \sigma)) \circ d_k = ((p \circ \sigma) \times (q \circ \sigma))_*(d_k)$. For the naturality condition (2) to be satisfied we need to define

$$\Theta_k(\sigma) := ((p \circ \sigma)_* \otimes (q \circ \sigma)_*)(\Theta_k(d_k)) \in (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_k.$$

With this definition one readily verifies that the map Θ_k is natural in X and Y.

Finally we have to show that Θ_k satisfies the boundary formula (3). So as before let $\sigma\colon \Delta^k \to X \times Y$ be a singular k-simplex. Then

$$\begin{aligned}
\partial^{\otimes}\Theta_k(\sigma) &\overset{\text{by definition of }\Theta_k(\sigma)}{=} \partial^{\otimes}\big(((p\circ\sigma)_*\otimes(q\circ\sigma)_*)(\Theta_k(d_k))\big) \\
&\overset{\substack{\text{since }(p\circ\sigma)_*\otimes(q\circ\sigma)_*\colon\ \mathrm{C}_*(\Delta^k)\otimes\mathrm{C}_*(\Delta^k)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\\ \text{is a chain map by Lemma 58.3}}}{=} ((p\circ\sigma)_*\otimes(q\circ\sigma)_*)(\partial^{\otimes}\Theta_k(d_k)) \\
&\underset{\text{by definition of }\Theta_k(d_k)}{=} ((p\circ\sigma)_*\otimes(q\circ\sigma)_*)(\Theta_{k-1}(\partial d_k)) \\
&\underset{\text{by naturality of }\Theta_{k-1}}{=} \Theta_{k-1}\big(((p\circ\sigma)\times(q\circ\sigma))_*(\partial d_k)\big) \\
&\underset{\text{since }((p\circ\sigma)\times(q\circ\sigma))_*\text{ is a chain map}}{=} \Theta_{k-1}\partial\big(((p\circ\sigma)\times(q\circ\sigma))_*(d_k)\big) \\
&\underset{\text{since }\sigma=((p\circ\sigma)\times(q\circ\sigma))\circ d_k}{=} \Theta_{k-1}\partial\sigma.
\end{aligned}$$

We have now verified that Θ_k has all the desired properties. ■

As we saw in the proofs of Theorem 80.2 and Theorem 80.3, the natural chain maps $\Upsilon\colon\ \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\to\mathrm{C}_*(X\times Y)$ and $\Theta\colon\ \mathrm{C}_*(X\times Y)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)$ depend on many choices. The next theorem says that different choices in the proofs lead to natural chain maps that are naturally homotopy equivalent to one another.

Theorem 80.5. *Let X and Y be topological spaces. Any two natural chain maps*

$$\mathrm{C}_*(X\times Y) \ \to\ \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)$$

that are the canonical maps on the 0-level are in fact naturally chain homotopic. The same conclusion also holds for any two natural chain maps

$$\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y),\ \ \mathrm{C}_*(X\times Y)\to\mathrm{C}_*(X\times Y),\ \ \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\to\mathrm{C}_*(X\times Y)$$

that are the canonical maps on the 0-level.

Proof. The proof of this theorem is in many ways quite similar to the proofs of Theorem 80.2 and 80.3.

In the following we show the first statement. The proofs of the other three statements are very similar. So let

$$F,G\colon\ \mathrm{C}_*(X\times Y)\ \to\ \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)$$

be two natural chain maps that are the canonical maps on the 0-chains, i.e. which satisfy $F((x,y)) = x\otimes y = G((x,y))$ for all $x\in X, y\in Y$. Given any $k\in\mathbb{Z}_{\geq -1}$ we have to define a map

$$P_k\colon\ \mathrm{C}_k(X\times Y)\ \to\ (\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y))_{k+1}$$

that satisfies the following three properties:

(1) the map P_{-1} is the zero map,
(2) the map P_k is natural in X and Y,
(3) we have $\partial^{\otimes}\circ P_k + P_{k-1}\circ\partial = F - G$.

So suppose we have already defined the maps $P_{-1},\dots,P_{k-1}$ with $k\geq 0$. As before we denote by $d_k\colon \Delta^k\to\Delta^k\times\Delta^k$ the singular k-simplex that is given by the diagonal map

$d_k(x) := (x,x)$. We first want to define $P_k(d_k) \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)_{k+1}$. In the chain complex $\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)$ we calculate that

$$\begin{aligned}\partial^{\otimes}((F-G-P_{k-1}\partial)(d_k)) &= \partial^{\otimes}F(d_k) - \partial^{\otimes}G(d_k) - \partial^{\otimes}P_{k-1}(\partial d_k)\\ &\underset{\uparrow}{=} F(\partial d_k) - G(\partial d_k) - (F-G-P_{k-2}\partial)(\partial d_k) = P_{k-2}\partial\partial d_k = 0.\end{aligned}$$

since F and G are chain maps and since inductively we know that $\partial^{\otimes} \circ P_{k-1} + P_{k-1} \circ \partial = F - G$

It follows from Lemma 80.4 that there exists a $(k+1)$-chain $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_{k+1}$ with $\partial^{\otimes}(c) = (F-G-P_{k-1}\partial)(d_k)$. (For $k=0$ we need to show that $\epsilon((F-G)(d_0)) = 0$, to conclude this we need to use that F and G agree on 0-simplices.) We define $P_k(d_k) := c$.

Now let X and Y be two topological spaces and let $\sigma \in \mathrm{C}_k(X \times Y)$. As in the proof of Theorem 80.2 we denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the obvious projection maps and we set

$$P_k(\sigma) \;:=\; ((p\circ\sigma)_* \otimes (q\circ\sigma)_*)(P_k(d_k)) \;\in\; (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_{k+1}.$$

Basically by definition this map is natural in X and Y. As in the proof of Theorem 80.2 one easily verifies that condition (3) is satisfied. ■

Now we can easily provide the proof of the Eilenberg-Zilber Theorem 80.1.

Proof of the Eilenberg-Zilber Theorem 80.1. We pick natural chain maps

$$\Upsilon\colon\ \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \;\to\; \mathrm{C}_*(X \times Y)$$

and

$$\Theta\colon\ \mathrm{C}_*(X \times Y) \;\to\; \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$$

as provided by Theorems 80.2 and 80.3. These maps are the canonical maps on the 0-level. By Theorem 80.5 the maps Υ and Θ are thus unique up to a natural chain homotopy.

It remains to show that the maps Υ and Θ are naturally chain homotopy inverses of one another. The map $\Theta \circ \Upsilon\colon\ \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ is a natural chain map that is the canonical map on the 0-level. But this is also the case for the identity map. It follows from Theorem 80.5 that $\Theta \circ \Upsilon$ is naturally chain homotopic to the identity. The same argument shows that $\Upsilon \circ \Theta$ is naturally chain homotopic to the identity. ■

80.3. **The relative Eilenberg–Zilber map Υ.** In this short section we introduce a relative version of the Eilenberg–Zilber map Υ. We start out with the following elementary but essential remark.

Remark. Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets. We denote by $i\colon A \to X$ and $j\colon B \to Y$ the inclusion maps. Then the diagram

$$\begin{tikzcd} \mathrm{C}_*(A) \otimes \mathrm{C}_*(B) \arrow[r, hook, "i_*\otimes j_*"] \arrow[d, "\Upsilon"] & \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \arrow[d, "\Upsilon"] \\ \mathrm{C}_*(A \times B) \arrow[r, hook, "(i\times j)_*"] & \mathrm{C}_*(X \times Y) \end{tikzcd}$$

commutes by the naturality of Υ. In the following we do not explicitly mention the names of the various inclusion induced maps.

Proposition 80.6. *Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets.*

(1) *The Eilenberg-Zilber map Υ induces a natural chain homotopy equivalence*[1149]

$$\begin{array}{rcl} C_*(X,A) \otimes C_*(Y,B) & \to & C_*(X\times Y)/(C_*(A\times Y) + C_*(X\times B)) \\ \sum_{i=1}^n [c_i] \otimes [d_i] & \mapsto & \sum_{i=1}^n [\Upsilon(c_i \otimes d_i)]. \end{array}$$

(2) *The Eilenberg-Zilber map Υ descends to a well-defined chain map*

$$\begin{array}{rcl} C_*(X,A) \otimes C_*(Y,B) & \to & C_*(X \times Y, A \times Y \cup X \times B) \\ \sum_{i=1}^n [c_i] \otimes [d_i] & \mapsto & \sum_{i=1}^n \big[\Upsilon(c_i \otimes d_i)\big]. \end{array}$$

PROOF. Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets.

(1) We consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & C_*(A) \otimes C_*(Y) \arrow[r] \arrow[d, "\Upsilon"] & C_*(X) \otimes C_*(Y) \arrow[r] \arrow[d, "\Upsilon"] & C_*(X,A) \otimes C_*(Y) \arrow[r] \arrow[d, "\Upsilon"] & 0 \\ 0 \arrow[r] & C_*(A \times Y) \arrow[r] & C_*(X\times Y) \arrow[r] & C_*(X\times Y)/\,C_*(A\times Y) \arrow[r] & 0. \end{tikzcd}$$

The top sequence is given by tensoring a short exact sequence by the free abelian groups $C_*(Y)$, Lemma 57.9 implies that the resulting sequence is again exact. The bottom sequence is exact by definition. The two left vertical maps are homotopy equivalences by Theorem 80.1 and the diagram commutes by the naturality of Υ, as explained in the remark preceding the proposition. Finally note that it follows from Corollary 49.4 that the right vertical map is also a chain homotopy equivalence.

We consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & C_*(X,A) \otimes C_*(B) \arrow[r] \arrow[d, "\Upsilon"] & C_*(X,A) \otimes C_*(Y) \arrow[r] \arrow[d, "\Upsilon"] & C_*(X,A) \otimes C_*(Y,B) \arrow[r] \arrow[d, "\Upsilon"] & 0 \\ 0 \arrow[r] & C_*(X\times B)/\,C_*(A\times B) \arrow[r] & C_*(X\times Y)/\,C_*(A\times Y) \arrow[r] & \frac{C_*(X \times Y)}{C_*(A\times Y)+C_*(X\times B)} \arrow[r] & 0. \end{tikzcd}$$

As above, it follows from the fact that each $C_k(X,A)$ is a free abelian group and Lemma 57.9 that the top sequence is exact. It follows easily from[1150]

$$C_*(X \times B) \cap C_*(A \times Y) \;=\; C_*((X \times B) \cap (A \times Y)) \;=\; C_*(A \times B)$$

that the bottom sequence is also exact. The middle vertical map is a chain homotopy equivalence by the above discussion. Exactly the same argument also shows that the left vertical map is a chain homotopy equivalence. The diagram commutes by the naturality of Υ. It follows from Corollary 49.4 that the right vertical map is also a homotopy equivalence.

[1149] Note that $C_*(A\times Y) + C_*(X\times B)$ is in fact a subcomplex of the chain complex $C_*(X\times Y)$, so the right-hand side is indeed a chain complex.

[1150] Why does this equality of chain complexes hold?

(2) The second statement follows immediately from (1) together with the observation that $C_*(A \times Y) + C_*(X \times B)$ is a subcomplex of $C_*(A \times Y \cup X \times B)$. ■

80.4. **The cross product on homology.** Recall that one of our goals is to find some type of product structure in homology or cohomology. We start our search for a product structure in homology, since we are more comfortable with homology groups.

The following definition already gives us some type of product structure on homology.

Definition. Let (X, A) and (Y, B) be two pairs of topological spaces. We refer to the map[1151]

$$\begin{array}{rcl} \times\colon\ H_p(X,A) \otimes H_q(Y,B) & \xrightarrow{\times} & H_{p+q}(X \times Y, A \times Y \cup X \times B). \\ [c] \otimes [d] & \mapsto & [c] \times [d] := [\Upsilon(c \otimes d)] \end{array}$$

as the *cross product*.

In the following lemma we summarize some elementary properties of the cross product.

Lemma 80.7.

(1) *Let X and Y be two topological spaces and let $x \in X$ be a point. We denote by $f\colon Y \to X \times Y$ the map that is given by $y \mapsto (x, y)$ and we denote by $[x] \in H_0(X)$ the homology class defined by x. Then for any $q \in \mathbb{N}_0$ the maps*

$$H_q(Y) \underset{f_*}{\overset{\sigma \mapsto [x] \times \sigma}{\rightrightarrows}} H_q(X \times Y)$$

agree. The statement holds with the roles of X and Y reversed.

(2) *The cross product is natural, i.e. if $f\colon (X, A) \to (X', A')$ and $g\colon (Y, B) \to (Y', B')$ are maps between pairs of topological spaces, then for any $p, q \in \mathbb{N}_0$ the following diagram commutes:*

$$\begin{array}{ccc} H_p(X,A) \otimes H_q(Y,B) & \xrightarrow{\quad\times\quad} & H_{p+q}(X \times Y, A \times Y \cup X \times B) \\ \downarrow f_* \otimes g_* & & \downarrow (f \times g)_* \\ H_p(X',A') \otimes H_q(Y',B') & \xrightarrow{\quad\times\quad} & H_{p+q}(X' \times Y', A' \times Y' \cup X' \times B'). \end{array}$$

PROOF. The lemma follows immediately from the definitions and the properties of Υ given in Theorem 80.2 (1) and (2). ■

We have the following variation on the Künneth Theorem 58.8 for topological spaces.

Theorem 80.8. (The Künneth Theorem) *Let (X, A) and (Y, B) be pairs of topological spaces. We assume that one of the following two statements holds:*

(1) *A and B are open subsets (note that A and B can possibly be empty), or*
(2) *(X, A) and (Y, B) are pairs of CW-complexes.*

[1151]Note that the map is well-defined by Lemmas 58.6 and 80.6 (2) and it is independent of the choice of Υ by Theorem 80.1 (2).

Then for each $n \in \mathbb{N}$ there exists a natural short exact sequence

$$\begin{aligned} 0 \ \to \ \bigoplus_{p+q=n} \mathrm{H}_p(X,A) \otimes \mathrm{H}_q(Y,B) \ &\xrightarrow{\times}\ \mathrm{H}_n(X \times Y, A \times Y \cup X \times B) \\ &\to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X,A), \mathrm{H}_q(Y,B)) \ \to \ 0. \end{aligned}$$

Furthermore the above short exact sequence splits.

Examples.

(1) Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold. Let $x \in M \setminus \partial M$ and let $y \in N \setminus \partial N$. Then it follows from the relative Künneth Theorem 80.8 (2), Lemma 86.4 and Lemma 57.17 that the cross product defines an isomorphism

$$\mathrm{H}_m(M, M\setminus\{x\}) \otimes \mathrm{H}_n(N, N\setminus\{y\}) \xrightarrow{\times} \mathrm{H}_{m+n}(M \times N, \underbrace{(M\setminus\{x\}) \times N \cup M \times (N\setminus\{y\})}_{=(M\times N)\setminus\{(x,y)\}}).$$

(2) Let X and Y be path-connected topological spaces. Let $x \in X$ and let $y \in Y$. We denote by $i\colon X \to X \times Y$ the inclusion map that is given by $i(x) = (x,y)$ and similarly we denote by $j\colon Y \to X \times Y$ the inclusion map that is given by $j(y) = (x,y)$. It follows from Proposition 41.5, Lemma 80.7 (1) and the Künneth Theorem 80.8 that the map

$$i_* \oplus j_*\colon \ \mathrm{H}_1(X;\mathbb{Z}) \oplus \mathrm{H}_1(Y;\mathbb{Z}) \ \to \ \mathrm{H}_1(X \times Y;\mathbb{Z})$$

is an isomorphism.[1152] Let G be any abelian group. Furthermore note that it follows easily from the above, the naturality of the Universal Coefficient Theorem 57.19 and Lemma 57.17 (3) that the map

$$i_* \oplus j_*\colon \ \mathrm{H}_1(X;G) \oplus \mathrm{H}_1(Y;G) \ \to \ \mathrm{H}_1(X \times Y;G)$$

is also an isomorphism.

PROOF. We consider the following maps

$$\begin{array}{c} 0 \to \bigoplus\limits_{p+q=n} \mathrm{H}_p(X,A) \otimes \mathrm{H}_q(Y,B) \to \mathrm{H}_n(\mathrm{C}_*(X,A) \otimes \mathrm{C}_*(Y,B)) \to \bigoplus\limits_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X,A), \mathrm{H}_q(Y,B)) \to 0. \\ \downarrow \Upsilon \\ \mathrm{H}_n(\mathrm{C}_*(X \times Y)/(\mathrm{C}_*(A \times Y) + \mathrm{C}_*(X \times B))) \\ \downarrow \\ \mathrm{H}_n(\mathrm{C}_*(X \times Y)/\,\mathrm{C}_*(A \times Y \cup X \times B)) \\ \downarrow = \\ \mathrm{H}_n(X \times Y, A \times Y \cup X \times B) \end{array}$$

We first note that the top sequence is exact by the Künneth Theorem for Chain Complexes 58.7. (Here we use again that relative chain groups are free abelian groups.) Here

[1152] We had given an alternative proof of this fact in Exercise 52.5.

the top vertical map is an isomorphism by Proposition 80.6. Recall that we assume that one of the following two statements holds:

(1) A and B are open subsets, or
(2) (X,A) and (Y,B) are pairs of CW-complexes.

We then know from Proposition 74.12 (4) respectively (6) that the inclusion induced map

$$\underbrace{\mathrm{C}_*^{\{A\times Y, X\times B\}}(A\times Y\cup X\times B)}_{=\mathrm{C}_*(A\times Y)+\mathrm{C}_*(X\times B)} \;\to\; \mathrm{C}_*(A\times Y\cup X\times B)$$

is a chain homotopy equivalence. The same argument as in the proof of Corollary 43.18 shows that this piece of information implies that the second vertical map is indeed an isomorphism. It follows immediately from the definition of the cross product that the induced "diagonal map"

$$\bigoplus_{p+q=n} \mathrm{H}_p(X,A)\otimes \mathrm{H}_q(Y,B) \;\to\; \mathrm{H}_n(X\times Y, A\times Y\cup X\times B)$$

is precisely the cross product. ■

80.5. **Product orientations of topological manifolds.** In this second to last section we want to consider products of (topological) smooth manifolds. We start out with the following, arguably slightly surprising proposition. It can be viewed as a generalization of Propositions 6.5, 6.51 and Proposition 44.2 (7).

Proposition 80.9.

(1) *Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold. The topological space $M\times N$ is an $(m+n)$-dimensional topological manifold where the boundary is given by*

$$\partial(M\times N) \;=\; (\partial M\times N)\cup(M\times\partial N).$$

(2) *Given any two smooth manifolds M and N we can equip $M\times N$ with a natural smooth structure such that the following conditions are satisfied:*
   (a) *If $\partial M=\varnothing$ or $\partial N=\varnothing$, then the smooth structure on $M\times N$ is the obvious smooth structure given by taking products of charts.*
   (b) *For any two smooth manifolds M and N the following statements hold:*
      (i) *the inclusion map $\mathring{M}\times\mathring{N}\to M\times N$ is smooth,*[1153]
      (ii) *for any $P\in M$ and $Q\in N$ the inclusion maps*

$$\begin{array}{rcl} M & \to & M\times N \\ x & \mapsto & (x,Q)\end{array} \quad \textit{and} \quad \begin{array}{rcl} N & \to & M\times N \\ y & \mapsto & (P,y)\end{array}$$

      are smooth.

Example. Let $M=\overline{B}^2$ and $N=[0,1]$, then $M\times N=\overline{B}^2\times[0,1]$ is the solid cylinder which is a 3-dimensional topological manifold with boundary

$$\partial\big(\overline{B}^2\times[0,1]\big) \;=\; (\partial\overline{B}^2\times[0,1])\cup(\overline{B}^2\times\partial([0,1])) \;=\; (S^1\times[0,1])\cup(\overline{B}^2\times\{0,1\}).$$

[1153]Note that it follows from Proposition 44.2 that $\mathring{M}=M\setminus\partial M$ and $\mathring{N}=N\setminus\partial N$ do not have boundary. Hence the smooth structure on $\mathring{M}\times\mathring{N}$ is given by the products of charts.

We refer to Figure 1155 for an illustration.

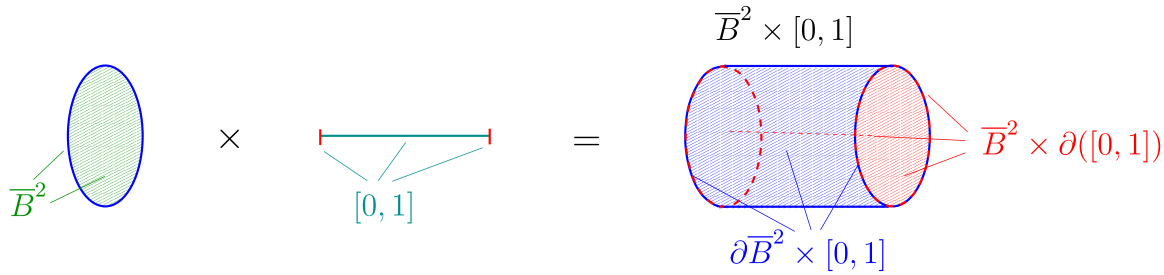


FIGURE 1155

PROOF. Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold. We consider the topological space $M \times N$. By Proposition 3.12 we remarked that the product of two Hausdorff spaces is again Hausdorff. Thus we see that $M \times N$ is Hausdorff. It follows from Lemma 6.1 that $M \times N$ is second-countable.

It remains to find charts for $M \times N$. More precisely, we have to find a chart of type (i) for each point in $(M \times N) \setminus (\partial M \times N \cup M \times \partial N)$ and we have to find a chart of type (ii) for each point in $\partial M \times N \cup M \times \partial N$. So let $(x, y) \in M \times N$. We distinguish three cases:

(1) If $x \in M \setminus \partial M$ and $y \in N \setminus \partial N$ then taking "product charts" it is straightforward to find a chart of type (i) for $(x, y) \in M \times N$.
(2) If $x \in M \setminus \partial M$ and $y \in \partial N$ or if $x \in \partial M$ and $y \in N \setminus \partial N$ then it is equally straightforward to find a chart of type (ii) for $(x, y) \in M \times N$.
(3) Finally suppose that $x \in \partial M$ and $y \in \partial N$. We have to find a chart of type (ii) for $(x, y) \in M \times N$. It is straightforward to see that there exists an open neighborhood U of $(x, y) \in M \times N$ and a homeomorphism $\Phi\colon U \to \mathbb{R}^{m+n-2} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. We then compose this homeomorphism with the homeomorphism

$$\begin{array}{rcl} \Psi\colon \mathbb{R}^{m+n-2} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} & \to & \mathbb{R}^{m+n-2} \times \mathbb{R} \times \mathbb{R}_{\geq 0} \\ (x_1, \dots, x_{m+n-2}, \underbrace{r\cos(\varphi), r\sin(\varphi)}_{=:z \in \mathbb{C}}) & \mapsto & (x_1, \dots, x_{m+n-2}, \underbrace{r\cos(2\varphi), r\sin(2\varphi)}_{=z^2}). \end{array}$$

It is straightforward to see that $\Psi \circ \Phi$ is a chart of type (ii) around (x, y).

If we start out with a smooth atlas for M and N then one can easily verify that the atlas that we have just constructed is also smooth and that it has the properties stated in (a) and (b). The statement that the smooth structure on $M \times N$ is natural is shown in [**Wall16**, Proposition 2.6.2]. ∎

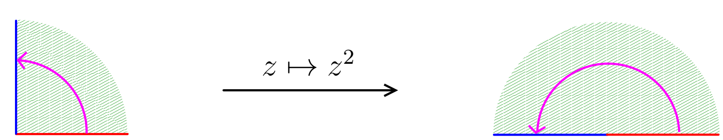


FIGURE 1156. Illustration of the proof of Proposition 80.9.

Remark. The approach in the proof of Proposition 80.9 for defining charts on $\partial M \times \partial N$ is sometimes called *rounding corners* or *smoothing corners*.

The cross product now allows us to formulate the following proposition.

Proposition 80.10. *Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold. Suppose M is equipped with an orientation $\{\mu_x\}_{x\in M\setminus\partial M}$ and that N is equipped with an orientation $\{\nu_y\}_{y\in N\setminus\partial N}$.*

(1) *Given $(x,y)\in(M\times N)\setminus\partial(M\times N)$ we denote by $\rho_{(x,y)}$ the image of $\mu_x\otimes\nu_y$ under the isomorphism*
$$\mathrm{H}_m(M,M\setminus\{x\})\otimes\mathrm{H}_n(N,N\setminus\{y\})\ \xrightarrow{\times}\ \mathrm{H}_{m+n}(M\times N,(M\times N)\setminus\{(x,y)\})$$
from page 1965. Then these classes $\rho_{(x,y)}$ define an orientation on $M\times N$.

(2) *Suppose both M and N are compact and connected, then the map*
$$\mathrm{H}_m(M,\partial M)\otimes\mathrm{H}_n(N,\partial N)\ \xrightarrow{\times}\ \mathrm{H}_{m+n}(M\times N,\partial(M\times N))$$
given by the cross product is an isomorphism and it has the property that
$$[M]\times[N]\ =\ [M\times N]\ \in\ \mathrm{H}_{m+n}(M\times M,\partial(M\times N))$$
where $[M\times N]$ denotes the fundamental class of the compact topological manifold $M\times N$ equipped with the orientation defined in (1).

Proof. Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold.

(1) The first statement of the proposition is a straightforward consequence of the definitions and the naturality of the cross product, see Lemma 80.7. For completeness' sake we sketch the proof. To simplify the notation we assume that $\partial M=\partial N=\emptyset$. Now let $(x,y)\in M\times N$. We pick an open neighborhood U of x and a class $\mu_U\in\mathrm{H}_m(M,M\setminus U)$ and an open neighborhood V of y and a class $\nu_V\in\mathrm{H}_n(N,N\setminus V)$ as given by the definition on page 2101. By the naturality of the cross product we obtain the following commutative diagram
$$\begin{tikzcd}
\mathrm{H}_m(M,M\setminus U)\times\mathrm{H}_n(N,N\setminus V) \arrow[r, "\times"] \arrow[d, "\mu_U\mapsto\mu_x"', "\nu_V\mapsto\nu_y"] & \mathrm{H}_{m+n}(M\times N,M\times N\setminus(U\times V)) \arrow[d] \\
\mathrm{H}_m(M,M\setminus\{x\})\times\mathrm{H}_n(N,N\setminus\{y\}) \arrow[r, "\times", "(\mu_x,\nu_y)\mapsto\rho_{(x,y)}"'] & \mathrm{H}_{m+n}(M\times N,M\times N\setminus\{(x,y)\}).
\end{tikzcd}$$
It is now straightforward to verify that $\phi_{U\times V}:=\mu_U\times\nu_V$ has the properties desired by the definition on page 2101.

(2) Now suppose that both M and N are compact. By Proposition 3.12, the product $M\times N$ is then also compact. Now let $x\in M\setminus\partial M$ and let $y\in N\setminus\partial N$. By Proposition 80.9 we have $(x,y)\in(M\times N)\setminus\partial(M\times N)$. Furthermore, by Lemma 80.7 we have the following commutative diagram
$$\begin{tikzcd}
\mathrm{H}_m(M,\partial M)\otimes\mathrm{H}_n(N,\partial N) \arrow[r, "\times"] \arrow[d] & \mathrm{H}_{m+n}(M\times N,\partial(M\times N)) \arrow[d] \\
\mathrm{H}_m(M,M\setminus\{x\})\otimes\mathrm{H}_n(N,N\setminus\{y\}) \arrow[r, "\times"] & \mathrm{H}_{m+n}(M\times N,M\times N\setminus\{(x,y)\}).
\end{tikzcd}$$

The vertical maps are isomorphisms by Theorem 87.1. On page 1965 we saw that the bottom horizontal map is an isomorphism. It follows that, as claimed, the top horizontal map is also an isomorphism. The statement regarding the fundamental classes is an immediate consequence of the commutativity of the diagram and the definitions. ■

Definition. Let M and N be oriented topological manifolds. We refer to the orientation on $M \times N$ defined in Proposition 80.10 (1) as the *product orientation.*

Remark. Recall that given any open subset $U \subset \mathbb{R}^k$, $k \in \mathbb{N}_0$, we introduced on page 2106 the standard orientation for U viewed as a topological manifold. (See also Figure 1157 for a reminder.) The following lemma says that the above definition of the product orientation gives us the "expected" answer when we deal with open subsets of $\mathbb{R}^k$'s.

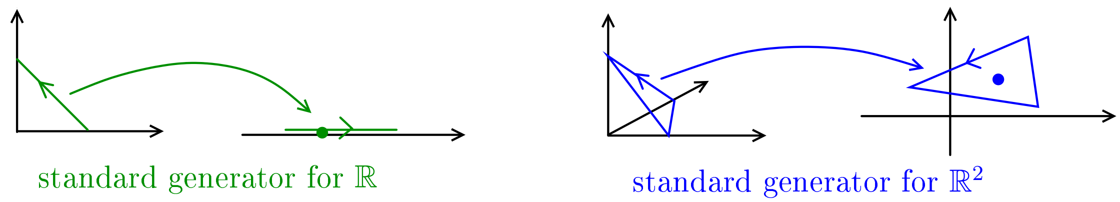


FIGURE 1157

Proposition 80.11. *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open non-empty subsets. We equip U and V with the standard orientation. Then the product orientation of $U \times V$, defined above, agrees with the standard orientation of $U \times V \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$.*

We postpone the proof of Proposition 80.11 to the next section.

Examples.

(1) Let M and N be two oriented smooth manifolds such that at most one of them has non-empty boundary. We have the following diagram

$$\begin{tikzcd}
\text{oriented smooth manifolds } M,N \arrow[r, squiggly, "\text{Proposition 86.11}"] \arrow[d, squiggly, "\text{page 304}"] & \text{oriented topological manifolds } M,N \arrow[d, squiggly, "\text{see above}"] \\
\text{oriented smooth manifold } M\times N \arrow[r, squiggly, "\text{Proposition 86.11}"] & \text{oriented topological manifold } M\times N.
\end{tikzcd}$$

It follows fairly easily from Proposition 80.11 that this diagram commutes.

(2) We write $I = [-1, 1]$. We view I as a 1-dimensional topological manifold and we equip I with the orientation coming from $\mathbb{R}$. Given $k \in \mathbb{N}_0$ we write

$$\partial I^k \;:=\; \{(x_1, \ldots, x_k) \,|\, \text{there exists an } i \text{ with } x_i = -1 \text{ or } x_i = 1\}.$$

Given any $m, n \in \mathbb{N}$ one sees easily that

$$\partial(I^m \times I^n) \;=\; (\partial I^m \times I^n) \cup (I^m \times \partial I^n) \;=\; \partial(I^{m+n}).$$

(See also Figure 1158 for an illustration.) It follows from Proposition 80.9 and the above discussion that we can view each $I^k = I \times \cdots \times I$ as a k-dimensional topological manifold with boundary given by the above subset ∂I^k. We equip I^k with the product orientation. It follows immediately from Proposition 80.11 that the orientation on the open subset $I^k \setminus \partial I^k \subset \mathbb{R}^k$ is the standard orientation.[1154] As usual we denote by $[I^k] \in \mathrm{H}_k(I^k, \partial I^k; \mathbb{Z})$ the fundamental class.

Now let $m, n \in \mathbb{N}_0$. We get a cross product

$$\overbrace{\mathrm{H}_m(I^m, \partial I^m)}^{=\mathbb{Z}\cdot[I^m]} \otimes \overbrace{\mathrm{H}_n(I^n, \partial I^n)}^{=\mathbb{Z}\cdot[I^n]} \;\to\; \overbrace{\mathrm{H}_{m+n}(I^{m+n}, \partial I^{m+n})}^{=\mathbb{Z}\cdot[I^{n+n}]}$$

which by Proposition 80.10 (2) has the property that

$$[I^m] \times [I^n] \;=\; [I^{m+n}].$$

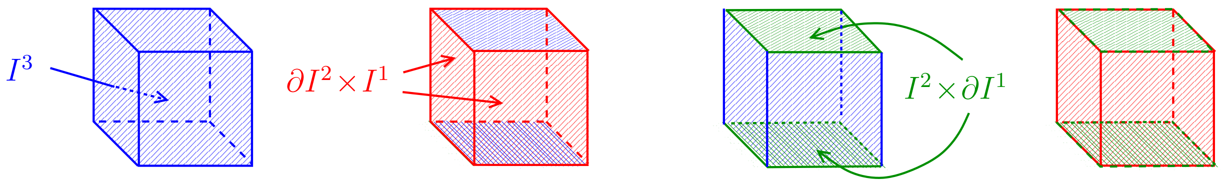


FIGURE 1158

80.6. **The shuffle product and the proof of Proposition 80.11** $(*)$**.** In this section we will provide the proof of Proposition 80.11. The difficulty in the proof is that the definition of the cross product relies on the map Υ from Theorem 80.2. We gave a non-constructive proof of the existence of the map Υ. In this section we give an *explicit* description of a map ∇ which has all the properties that we had asked for in Theorem 80.2. We will then use this map ∇ to prove Proposition 80.11.

First we recall some notation from page 1078.

Notation. For $i = 0, \dots, n$ we write

$$v_i \;:=\; (0, \dots, 1, 0, \dots, 0) \;\in\; \mathbb{R}^{n+1}.$$

Furthermore for $a_0, \dots, a_k \in \Delta^n$ we consider the map

$$\begin{aligned} [a_0, \dots, a_k] \colon \Delta^k \;&\to\; \Delta^n \\ \sum_{j=0}^{k} \lambda_j \cdot v_j \;&\mapsto\; \sum_{j=0}^{k} \lambda_j \cdot a_j. \end{aligned}$$

The map $[a_0, \dots, a_k]$ thus sends the standard k-simplex Δ^k to the simplex "spanned by $a_0, \dots, a_k$" in Δ^n.

Definition. Let $m, n \in \mathbb{N}_0$.

(1) An (m,n)-*shuffle* is a pair (μ, ν) of disjoint sets of integers

$$1 \le \mu_1 < \mu_2 < \dots < \mu_m \le m + n \quad \text{and} \quad 1 \le \nu_1 < \nu_2 < \dots < \nu_n \le m + n.$$

[1154] Note that the homeomorphism

$$f \colon (I^k, \partial I^k) \;\to\; (\overline{B}^k, S^{k-1})$$

from Proposition 2.52 (2) has the property that the restriction to a dense open subset of $I^k \setminus \partial I^k \to B^k$ is an orientation-preserving diffeomorphism. In particular this homeomorphism is thus an orientation-preserving homeomorphism from the oriented topological manifold I^k to the oriented topological manifold $\overline{B}^k$.

We denote by $\operatorname{sign}(\mu,\nu)$ the sign of the permutation $(\mu_1,\dots,\mu_m,\nu_1,\dots,\nu_n)$. We denote by $\eta^\mu\colon \Delta^{m+n}\to\Delta^m$ the unique affine linear map which satisfies $\eta^\mu(v_i)=v_j$ for $\mu_j\le i<\mu_{j+1}$.[1155] The same way we define $\eta^\nu\colon \Delta^{m+n}\to\Delta^n$.

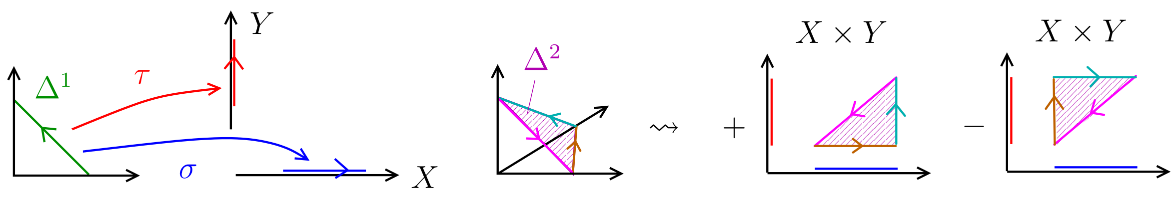
(2) Given two topological spaces X and Y we define

$$\begin{aligned}\nabla_{mn}\colon\ \mathrm{C}_m(X)\otimes\mathrm{C}_n(Y) &\ \to\ \mathrm{C}_{m+n}(X\times Y)\\ (\sigma\otimes\tau) &\ \mapsto\ \sum_{(\mu,\nu)}\operatorname{sign}(\mu,\nu)\cdot\underbrace{(\overbrace{\sigma\circ\eta^\mu}^{\Delta^{m+n}\to X}\times\overbrace{\tau\circ\eta^\nu}^{\Delta^{m+n}\to Y})}_{\text{map }\Delta^{m+n}\to X\times Y}\end{aligned}$$

where we take the sum over all (m,n)-shuffles.

Examples.

(1) We consider the case $X=Y=\mathbb{R}$ and $m=n=1$. There exist two $(1,1)$-shuffles, namely $\{\mu_1=1,\nu_1=2\}$ and $\{\mu_1=2,\nu_1=1\}$. Note that $\eta^1\colon\Delta^2\to\Delta^1$ equals the map $[v_0,v_1,v_1]$ and that $\eta^2\colon\Delta^2\to\Delta^1$ equals the map $[v_0,v_0,v_1]$. In Figure 1159 we illustrate $\nabla_{11}(\sigma\otimes\tau)$ for two affine linear singular 1-simplices $\sigma\colon\Delta^1\to X=\mathbb{R}$ and $\tau\colon\Delta^1\to Y=\mathbb{R}$.



$$\underbrace{\sigma\circ\eta^1\times\tau\circ\eta^2-\sigma\circ\eta^2\times\tau\circ\eta^1}_{=\nabla(\sigma\otimes\tau)}$$

FIGURE 1159. Illustration of the definition of the shuffle map.

(2) We consider the case $X=Y=S^1$ and $m=n=1$. Furthermore we denote by $\mu\colon\Delta^1\to S^1$ the singular 1-simplex that is given by $\mu(1-t,t)=e^{2\pi it}$. On page 1714 we saw that μ represents the fundamental class of S^1 with the standard orientation. Comparing Figures 1054 and 1159 we see that the shuffle $\nabla(\mu\otimes\mu)$ is precisely the usual representative for the fundamental class of the 2-dimensional torus $S^1\times S^1=([0,1]\times[0,1])/\sim$.

Proposition 80.12. *The maps ∇ that we had just defined have all the properties that we had mentioned in Theorem 80.2, i.e. the maps $\nabla\colon\ \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\to\mathrm{C}_*(X\times Y)$ are natural chain maps that on the 0-level equal the canonical maps that we introduced on page 1956.*

PROOF. It follows immediately from the definition that the maps ∇_{mn} are natural and that on the 0-level they agree with the canonical maps. It is elementary (albeit rather painful) to verify that these maps are chain maps. We refer to [**Dol56**, p. 184] and [**EM53**] for details and proofs. ■

Now we can finally provide the proof of Proposition 80.11.

[1155] Here we implicitly use that $\mu_0=0$ and $\mu_{m+1}=m+n+1$.

Proof of Proposition 80.11. It suffices to prove the statement for $U = \mathbb{R}^m$ and for $V = \mathbb{R}^n$. Given any $k \in \mathbb{N}_0$ we consider the singular chain $\sigma_k = (-1)^k \cdot [v_0, \dots, v_k, 0]$. (In plain English the singular simplex $[v_0, \dots, v_k, 0] \colon \Delta^k \to \mathbb{R}^k$ is just given by the obvious projection.) We make the following observation:

(∗) Let $z \in \mathbb{R}^k$ be any point that lies in the interior of the image of σ_k. It follows from the discussion on page 2094 that the chain σ_k represents the standard generator of $\mathrm{H}_k(\mathbb{R}^k, \mathbb{R}^k \setminus \{z\})$.

Next we consider the (m, n)-shuffle that is given by the sets $\mu_0 = \{1, \dots, m\}$ and by $\nu_0 = \{m+1, \dots, m+n\}$. It follows easily from the definitions that

$$\eta^{\mu_0} = [v_0, \dots, v_m, \dots, v_m] \colon \Delta^{m+n} \to \Delta^m \quad \text{and} \quad \eta^{\nu_0} = [v_m, \dots, v_m, \dots, v_{m+n}] \colon \Delta^{m+n} \to \Delta^n.$$

Also note that it is a straightforward consequence of the definitions that we have the following equality of singular $(m+n)$-chains in $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$:

$$(\sigma_m \circ \eta^{\mu_0}) \times (\sigma_n \circ \eta^{\nu_0}) \; = \; (-1)^m \cdot (-1)^n \cdot [v_0, \dots, v_{m-1}, v_m, \dots, v_{m+n-1}, 0] \; = \; \sigma_{m+n}.$$

We pick a point $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ that lies in the interior of the image of σ_{m+n}. Note that x lies in the interior of the image of σ_m and that y lies in the interior of the image of σ_n.

It follows from the observation (∗), the definition of the product orientation on page 1969 and Proposition 86.6 (1) that it suffices to prove the following claim.

Claim. The following equality holds in $\mathrm{H}_{m+n}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus \{(x, y)\})$:

$$[\sigma_m] \times [\sigma_n] \; = \; [\sigma_{m+n}].$$

We have

$$\begin{aligned} [\sigma_m] \times [\sigma_n] \; &\underset{\substack{\uparrow \\ \text{Proposition 80.12 and Theorem 80.5}}}{=} \; [\nabla_{mn}(\sigma_m \otimes \sigma_n)] \; \underset{\substack{\uparrow \\ \text{by definition}}}{=} \; \Big[\sum_{(\mu,\nu)} \operatorname{sign}(\mu, \nu) \cdot (\sigma_m \circ \eta^{\mu} \times \sigma_n \circ \eta^{\nu}) \Big] \\ &\underset{\substack{\uparrow \\ \text{for all other shuffles } (\mu,\nu) \text{ the point } (x,y) \text{ does not lie in the image of } \sigma_m \circ \eta^{\mu} \times \sigma_n \circ \eta^{\nu}}}{=} \; \Big[\underbrace{\operatorname{sign}(\mu_0, \nu_0)}_{=1} \cdot \underbrace{(\sigma_m \circ \eta^{\mu_0} \times \sigma_n \circ \eta^{\nu_0})}_{=\sigma_{m+n}} \Big] \; = \; [\sigma_{m+n}]. \end{aligned}$$

This concludes the proof of the claim and thus of the proposition. ■

80.7. **The Acyclic Model Theorems.** In this section we will prove two "Acyclic Model Theorems" which on many occasions will allow us to construct maps and homotopies. We start out with the following fairly friendly Acyclic Model Theorem. We will need this particular Acyclic Model Theorem later on in Chapter 109 when we construct Steenrod operations.

Theorem 80.13. (Acyclic Model Theorem) *Let $\mathcal{T}op$ be the category of topological spaces and let $\mathcal{C}$ be the category of generalized*[1156] *chain complexes. Let $G \colon \mathcal{T}op \to \mathcal{C}$ be a covariant functor. Furthermore let S be a natural transformation from the functor $X \mapsto \mathrm{C}_*(X)$ to the functor $X \mapsto G(X)$. If $\mathrm{H}_i(G(\Delta^k)) = 0$ for any $k \in \mathbb{N}_0$ and any $i \in \mathbb{N}_0$, then there exists a natural chain homotopy P from S to the zero map. In other words, for any topological space X and any $k \in \mathbb{N}$ there exists a natural map $P_X \colon \mathrm{C}_k(X) \to G_{k+1}(X)$,* *such that for any $k \in \mathbb{N}_0$ we have*

$$\partial \circ P_X + P_X \circ \partial \;=\; S_X\colon\; \mathrm{C}_k(X) \to G_k(X).$$

Sketch of proof. The proof of this theorem is very similar to the "naturally chain homotopic statement" of Theorem 80.5. First recall that the fact that S is a natural transformation means that for any topological space X we get a commutative diagram as follows:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{C}_2(X) \arrow[r, "\partial_2"] \arrow[d, "S"] & \mathrm{C}_1(X) \arrow[r, "\partial_1"] \arrow[d, "S"] & \mathrm{C}_0(X) \arrow[r] \arrow[d, "S"] & 0 \arrow[r] \arrow[d, "S"] & 0 \arrow[r] \arrow[d, "S"] & \dots \\
\dots \arrow[r] & G_2(X) \arrow[r, "\partial_2"] & G_1(X) \arrow[r, "\partial_1"] & G_0(X) \arrow[r, "\partial_0"] & G_{-1}(X) \arrow[r, "\partial_{-1}"] & G_{-2}(X) \arrow[r, "\partial_{-2}"] & \dots
\end{tikzcd}$$

Now we consider $\mathrm{id}_{\Delta^0} \in \mathrm{C}_0(\Delta^0)$. It is a cycle in the chain complex $\mathrm{C}_*(\Delta^0)$. Since the above diagram commutes we see that $S(\mathrm{id}_{\Delta^0})$ is also cycle in $G_0(\Delta^0)$. Since we assume that $\mathrm{H}_0(G(\Delta^0)) = 0$ we see that there exists some $\mu_1 \in G_1(\Delta^0)$ with $\partial_1(\mu_1) = S(\mathrm{id}_{\Delta^0})$. Now, given any topological space X we define

$$\begin{array}{rcl} P_0\colon\; \mathrm{C}_0(X) & \to & G_1(X) \\ \sum\limits_{i=1}^{m} a_i \cdot \underset{\substack{\uparrow \\ \text{map } \Delta^0 \to X}}{f_i} & \mapsto & \sum\limits_{i=1}^{m} a_i \cdot f_{i*}(\mu_1). \end{array}$$

We iterate this procedure. Namely suppose that for $i = 0, \dots, k-1$ we have already constructed natural maps $P_i\colon\; \mathrm{C}_i(X) \to G_{i+1}(X)$ such that $\partial \circ P_i + P_{i-1} \circ \partial = S_i$ for $i = 0, \dots, k-1$. For reference we consider the following diagram:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{C}_{k+1}(X) \arrow[r, "\partial_{k+1}"] \arrow[d, "S_{k+1}"] & \mathrm{C}_k(X) \arrow[r, "\partial_k"] \arrow[d, "S_k"] \arrow[dl, dashed, "P_k"] & \mathrm{C}_{k-1}(X) \arrow[r] \arrow[d, "S_{k-1}"] \arrow[dl, "P_k-1"] & \dots \\
\dots \arrow[r] & G_{k+1}(X) \arrow[r, "\partial_{k+1}"'] & G_k(X) \arrow[r, "\partial_k"'] & G_{k-1}(X) \arrow[r, "\partial_{k-1}"] & \dots
\end{tikzcd}$$

Next we need to construct a suitable map P_k. We calculate that

$$\begin{array}{rcl} \partial_k((S_k - P_{k-1} \circ \partial)(\mathrm{id}_{\Delta^k})) & = & \partial S_k(\mathrm{id}_{\Delta^k}) - \partial P_{k-1}(\partial\, \mathrm{id}_{\Delta^k}) \\ & \underset{\substack{\uparrow \\ \text{since } S \text{ is a chain map and since } \partial \circ P_{k-1} + P_{k-2} \circ \partial = S_{k-1}}}{=} & S_{k-1}(\partial\, \mathrm{id}_{\Delta^k}) - (S_{k-1} - P_{k-2}\partial)(\partial\, \mathrm{id}_{\Delta^k}) \;=\; (P_{k-2} \circ \overbrace{\partial \circ \partial}^{=0})(\mathrm{id}_{\Delta^k}) \;=\; 0\,. \end{array}$$

Since we assume that $\mathrm{H}_k(G(\Delta^k)) = 0$ we see that there exists some $\mu_{k+1} \in G_{k+1}(\Delta^k)$ with $\partial_{k+1}(\mu_{k+1}) = (S_k - P_{k-1} \circ \partial)(\mathrm{id}_{\Delta^k})$. Now, given any topological space X we define

$$\begin{array}{rcl} P_k\colon\; \mathrm{C}_k(X) & \to & G_{k+1}(X) \\ \sum\limits_{i=1}^{m} a_i \cdot \underset{\substack{\uparrow \\ \text{map } \Delta^k \to X}}{f_i} & \mapsto & \sum\limits_{i=1}^{m} a_i \cdot f_{i*}(\mu_{k+1}). \end{array}$$

Using the fact that S is natural one can easily verify that P_k has the desired properties. ■

[1156]Here "generalized" just means, see page 1086, that we also allow chain groups in negative degrees.

The above version of the Acyclic Model Theorem will not work for the situations that we need to deal with in the near future. There are two problems with the above Acyclic Model Theorem:

(1) On several occasions we will be interested in natural transformations of the form $\mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$, but for $X = \Delta^k$ the chain complexes to the right does not satisfy the condition that the homology groups for $X = \Delta^k$ vanish in dimensions ≥ 0.
(2) We will also need to deal with situations where to the left we have more complicated chain complexes, e.g. we might want to study natural transformations of the type $\mathrm{C}_*(X) \times \mathrm{C}_*(X) \to \mathrm{C}_*(X \times X)$.

To formulate a version of the Acyclic Model Theorem that deals with the above two issues we need a few more definitions.

Definition.
(1) An *augmented chain complex* is a chain complex

$$(\mathrm{C}_*, \partial_*) \;=\; \dots \xrightarrow{\partial_3} \mathrm{C}_2 \xrightarrow{\partial_2} \mathrm{C}_1 \xrightarrow{\partial_1} \mathrm{C}_0 \to 0$$

together with an epimorphism $\epsilon\colon \mathrm{C}_0 \to \mathbb{Z}$ such that $\epsilon \circ \partial_1\colon \mathrm{C}_1 \to \mathbb{Z}$ is the zero map.
(2) We say that an augmented chain complex $((\mathrm{C}_*, \partial_*), \epsilon\colon \mathrm{C}_0 \to \mathbb{Z})$ is *acyclic* if the generalized chain complex

$$\dots \xrightarrow{\partial_3} \mathrm{C}_2 \xrightarrow{\partial_2} \mathrm{C}_1 \xrightarrow{\partial_1} \mathrm{C}_0 \xrightarrow{\epsilon} \underset{\substack{\uparrow \\ \text{degree } -1}}{\mathbb{Z}} \to 0$$

is acyclic, i.e. if all homology groups are zero.
(3) Let $((\mathrm{C}_*, \partial_*), \epsilon)$ and $((\mathrm{C}'_*, \partial'_*), \epsilon')$ be two augmented chain complexes. A *chain map of augmented chain complexes* is a chain map $f_*\colon C_* \to \mathrm{C}'_*$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{C}_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"'] & \mathbb{Z} \arrow[d, "="] \\ \mathrm{C}'_0 \arrow[r, "\epsilon'"] & \mathbb{Z}. \end{tikzcd}$$

(4) We refer to the category $\mathcal{A}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{A}) &= \text{augmented chain complexes} \\ \mathrm{Mor}((C,\epsilon),(C',\epsilon')) &= \begin{array}{c}\text{set of chain maps of augmented chain complexes} \\ \text{from } (C,\epsilon) \text{ to } (C',\epsilon')\end{array} \end{aligned}$$

as the *category of augmented chain complexes.*

Examples.

(1) Let X be a non-empty topological space. We denote by $\epsilon_X\colon C_0(X) \to \mathbb{Z}$ the augmentation map that we introduced on page 1084, i.e. ϵ_X is the map that sends each singular 0-simplex to 1. It follows from Lemma 41.4 (1) that $(C_*(X), \epsilon_X)$ is an augmented chain complex. If X is a contractible topological space, e.g. if $X = \Delta^k$, then

it follows from Lemma 43.1 (7) that the corresponding augmented chain complex is acyclic.

(2) If (C_*, μ) and (D_*, ν) are two augmented chain complexes then the tensor product $C_* \otimes D_*$ of chain complexes, as defined in Lemma 58.2, together with the map

$$\begin{array}{rcl} \mu \otimes \nu \colon (C_* \otimes D_*)_0 = C_0 \otimes D_0 & \to & \mathbb{Z} \\ \sum\limits_{i=1}^{k} c_i \otimes d_i & \mapsto & \sum\limits_{i=1}^{k} \mu(c_i) \cdot \nu(d_i) \end{array}$$

is easily seen to be an augmented chain complex.[1157]

Definition. We say a chain complex C_* is *almost acyclic* if $\mathrm{H}_i(C_*) = 0$ for $i \geq 1$ and if $\mathrm{H}_0(C_*) \cong \mathbb{Z}$.

Example. If X is a contractible topological space, then it follows from Corollary 42.8 that $\mathrm{C}_*(X)$ is almost acyclic.

Proposition 80.14. *Let C_* and D_* be two chain complexes. If C_* and D_* are almost acyclic and if C_* is free, i.e. if each chain group C_n is a free abelian group, then $C_* \otimes D_*$ is also almost acyclic.*

Proof. Let $n \in \mathbb{N}_0$. Since C_* is free we can apply the Künneth Theorem for Chain Complexes 58.7 which gives us the following short exact sequence:

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(C_*) \otimes \mathrm{H}_q(D_*) \xrightarrow{\Theta} \mathrm{H}_n(C_* \otimes D_*) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(C_*), \mathrm{H}_q(D_*)) \to 0.$$

The proposition is an immediate consequence of this short exact sequence and of some very elementary statements regarding tensor products and torsion groups, see Lemma 57.3 and 57.17. ■

The following lemma gives a convenient criterion for showing that augmented chain complexes are acyclic.

Lemma 80.15. *Let $((\mathrm{C}_*, \partial_*), \epsilon)$ be an augmented chain complex. If the chain complex $(\mathrm{C}_*, \partial_*)$ is almost acyclic, then the augmented chain complex $((\mathrm{C}_*, \partial_*), \epsilon)$ is acyclic.*

Proof (*). We denote by $\widetilde{C}_*$ the generalized chain complex

$$\dots \xrightarrow{\partial_3} \mathrm{C}_2 \xrightarrow{\partial_2} \mathrm{C}_1 \xrightarrow{\partial_1} \mathrm{C}_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

We need to show that this chain complex is acyclic. It is clear that for $i \geq 1$ we have $\mathrm{H}_i(\widetilde{C}_*) = \mathrm{H}_i(C_*) = 0$. It remains to show that $\mathrm{H}_0(\widetilde{C}_*) = 0$. Since $\mathbb{Z}$ is a free abelian group

[1157] Indeed, by definition $(C_* \otimes D_*)_1$ is generated by elements $c_1 \otimes d_0$ with $c_1 \in C_1$ and $d_0 \in D_0$ and by elements $c_0 \otimes d_1$ with $c_0 \in C_0$ and $d_1 \in D_1$. We calculate that

$$((\mu\otimes\nu)\circ\partial_1)(c_1\otimes d_0) = (\mu\otimes\nu)(\partial c_1\otimes d_0+(-1)^1\cdot c_1\otimes\underbrace{\partial d_0}_{=0}) = (\mu\otimes\nu)(\partial c_1\otimes d_0) = \mu(\partial c_1)\cdot\nu(d_0) = 0\cdot\nu(d_0) = 0.$$

The same argument shows that $((\mu \otimes \nu) \circ \partial_1)(c_0 \otimes d_1) = 0$.

and since $\epsilon\colon \mathrm{C}_0 \to \mathbb{Z}$ is an epimorphism we can pick a map $s\colon \mathbb{Z} \to \mathrm{C}_0$ with $\epsilon \circ s = \mathrm{id}_{\mathbb{Z}}$. As in Lemma 43.1 (4) one can easily show that the map

$$\begin{array}{rcl} \mathrm{H}_0(\widetilde{C}_*) \oplus \mathbb{Z} & \to & \mathrm{H}_0(C_*) \\ [\sigma] \oplus n & \mapsto & [\sigma] + s(n) \end{array}$$

is an isomorphism. Since $\mathrm{H}_0(C_*) \cong \mathbb{Z}$ it follows from Exercise 19.3 (b) (see also more generally Proposition 19.10) that $\mathrm{H}_0(\widetilde{C}_*) = 0$. ■

Definition. Let $\mathcal{K}$ be a category (in all applications $\mathcal{K}$ will be the category of non-empty topological spaces or the category of tuples of non-empty topological spaces) and let $F\colon \mathcal{K} \to \mathcal{A}$ be a covariant functor from $\mathcal{K}$ to the category $\mathcal{A}$ of augmented chain complexes. Let $\mathcal{M}$ be a collection of objects of $\mathcal{K}$.

(1) We say the functor F is *acyclic relative to* $\mathcal{M}$ if given any $X \in \mathcal{M}$ the augmented chain complex $F(X)$ is acyclic.
(2) We say the functor F is *free relative to* $\mathcal{M}$ if for each $k \in \mathbb{N}_0$ there exists $\mathcal{M}_k \subset \mathcal{M}$ and if for each $M \in \mathcal{M}_k$ there exists an element $\sigma_M \in F_k(M)$[1158] such that given any $X \in \mathcal{K}$ the following two statements hold:
 (a) The set
 $$\{\underbrace{F(f)}_{\substack{\uparrow \\ \text{the morphism } f\colon M \to X \text{ induces a homomorphism } F(f)\colon F_k(M) \to F_k(X)}}(\sigma_M) \in F_k(X) \mid M \in \mathcal{M}_k \text{ and } f \in \mathrm{Mor}(M, X)\}$$
 is a basis for the abelian group $F_k(X)$.[1159]
 (b) The description of the set in (a) has no double elements, much more precisely, for $(M, f) \neq (M', f')$ we have $F(f)(\sigma_M) \neq F(f')(\sigma_{M'})$.

Remark. The objects in a collection $\mathcal{M}$ as in the previous definition are often called *models.*

The definition of “free relative to $\mathcal{M}$” is arguably somewhat intimidating. The following example should convince the reader that in practice it is not that bad.

Example. Let $\mathcal{K}$ be the category of non-empty topological spaces and let F be the covariant functor that assigns to each $X \in \mathcal{K}$ the augmented singular chain complex $(C_*(X), \epsilon_X)$. We consider

$$\mathcal{M} \;:=\; \{\Delta^k \mid k \in \mathbb{N}_0\} \;=\; \text{the set of all standard simplices.}$$

We claim that F is acyclic and free relative to $\mathcal{M}$:

(1) By Lemma 43.1 (7) we know that for every $k \in \mathbb{N}_0$ we have $\widetilde{\mathrm{H}}_*(\Delta^k) = 0$. But by Lemma 80.15 this just means that $\mathcal{M}$ is acyclic relative to F.

[1158]Here $F_k(M)$ is the k-chain group of the chain complex $F(M)$.
[1159]In particular this implies that each $F_k(X)$ is a free abelian group.

(2) Let $k \in \mathbb{N}_0$. We set $\mathcal{M}_k = \{\Delta^k\}$ and we consider $\sigma_k = \mathrm{id}_{\Delta^k} \in \mathrm{C}_k(\Delta^k)$. Now recall that given any topological space X and given any $k \in \mathbb{N}_0$ we have

$$\mathrm{C}_k(X) \underset{\substack{\uparrow \\ \text{by definition of a singular } k\text{-simplex, see page 1077}}}{\overset{\substack{\text{by definition of } \mathrm{C}_k(X)\text{, see page 1077} \\ \downarrow}}{=}} \text{free abelian group generated by the set of singular } k\text{-simplices in } X$$
$$= \text{free abelian group generated by the set of all maps } \underbrace{f\colon \Delta^k \to X}_{=f_*(\mathrm{id}_{\Delta^k})}.$$

But this means that the set $\{f_*(\mathrm{id}_{\Delta^k})\}_{f\colon \Delta^k \to X}$ is a basis for $\mathrm{C}_k(X)$.

Now we can formulate the main theorem of this section.

Theorem 80.16. (Acyclic Model Theorem) *Let $\mathcal{K}$ be a category with a collection of objects $\mathcal{M}$. Furthermore let $F, G\colon \mathcal{K} \to \mathcal{A}$ be two covariant functors from $\mathcal{K}$ to the category of augmented chain complexes. If F is free relative to $\mathcal{M}$ and if G is acyclic relative to $\mathcal{M}$, then the following statements hold:*

(1) *There exists a natural transformation T from F to G. In other words, given any object X of $\mathcal{K}$ we have a chain map $T_X\colon F(X) \to G(X)$ of augmented chain complexes such that for any morphism $f\colon X \to Y$ in $\mathcal{K}$ the following diagram commutes:*[1160]

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T_X"'] & F(Y) \arrow[d, "T_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y). \end{tikzcd}$$

(2) *Given two natural transformations S_X and T_X from F to G there is a natural chain homotopy P_X between them.*

Proof. The proof is, ultimately very much the same as the proof of Theorem 80.5. We decline to provide a proof and we follow the lead given by Munkres [**Mun84**, p. 185], namely like Munkres we assign the proof as an exercise to the reader. Alternatively the reader can find the proof in [**Spa95**, p. 169], [**Vic94**, Theorem 5.3] or [**Dol80**, p. 175]. ■

As an example for the Acyclic Model Theorem 80.16 we will prove a statement that we will anyway need shortly. To formulate the application we need the following definition.

Definition. A *diagonal approximation* is a natural chain map

$$\varphi\colon\ \mathrm{C}_*(X) \ \to\ \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

such that for each $x \in X$, viewed as a singular 0-simplex, we have

$$\varphi(x) \ = \ x \otimes x \ \in\ \underbrace{\mathrm{C}_0(X) \otimes \mathrm{C}_0(X)}_{=(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_0}.$$

Proposition 80.17. *Given any two diagonal approximations Φ and Ψ there exists a natural chain homotopy equivalence between Φ and Ψ.*

[1160] Note that the zero map is *not* a chain map of augmented chain complexes, since it does not respect the augmentation.

Proof. We let $\mathcal{K}$ be the category of non-empty topological spaces. As on page 1976 we consider the collection

$$\mathcal{M} := \{\Delta^k \mid k \in \mathbb{N}_0\} = \text{the set of all standard simplices.}$$

We consider the covariant functors[1161]

$$\begin{array}{rcl} F\colon \mathcal{K} & \to & \mathcal{A} \\ X & \mapsto & (\mathrm{C}_*(X), \epsilon_X) \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon \mathcal{K} & \to & \mathcal{A} \\ X & \mapsto & (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X), \epsilon_X \otimes \epsilon_X). \end{array}$$

Now let $\Phi, \Psi\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ be two diagonal approximations. By definition these are natural transformations from F to G. The hypothesis that Φ and Ψ are diagonal approximations implies that Φ and Ψ define chain maps of augmented chain complexes.

By the Acyclic Model Theorem 80.16 (2) it remains to prove the following claim.

Claim.

(1) The functor F is free relative to $\mathcal{M}$.
(2) The functor G is acyclic relative to $\mathcal{M}$.

We prove these two statements:

(1) Conveniently enough we showed on page 1977 that F is free relative to $\mathcal{M}$.
(2) We need to show that given any $k \in \mathbb{N}_0$ the corresponding augmented chain complex $(\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k), \epsilon_X \otimes \epsilon_X)$ is acyclic. By Lemma 80.15 it suffices to show that the chain complex $\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)$ is almost acyclic. But this is a consequence of the fact that $\mathrm{C}_*(\Delta^k)$ is almost acyclic, see page 1975, and Proposition 80.14. ■

Exercises for Chapter 80.

Exercise 80.1. Let M and N be topological manifolds and let $\varphi \in \mathrm{H}_k(M)$ and $\psi \in \mathrm{H}_l(N)$. If φ is realized by a closed oriented k-dimensional submanifold A and if ψ is realized by a closed oriented l-dimensional submanifold B, then the cross product $\varphi \times \psi \in \mathrm{H}_{k+l}(M \times N)$ is realized by the closed $(k+l)$-dimensional submanifold $A \times B$.

Exercise 80.2. Let X and Y be topological spaces. Show that for any $\sigma \in \mathrm{H}_0(X)$ and $\tau \in \mathrm{H}_0(X)$ we have

$$\epsilon_X(\sigma) \cdot \epsilon_Y(\tau) = \epsilon_{X \times Y}(\sigma \times \tau).$$

Here the maps ϵ are as always the various augmentation maps.

Exercise 80.3. Let M be the Möbius band. We consider the topological manifold $M \times [0,1]$.

(a) What is the homeomorphism type of $\partial(M \times [0,1])$?
(b) Is the topological manifold $M \times [0,1]$ orientable?

[1161] It follows from the discussion on page 1975 that both F and G are indeed functors to the category of augmented chain complexes.

81. The cup product

81.1. **A first definition of the cup product.** Let X be a topological space. The cross product from page 1964 applied to the same topological spaces $X = Y$ gives us in particular the cross product[1162]

$$\begin{array}{rcl} \times\colon \overbrace{\mathrm{H}_p(\mathrm{C}_*(X)) \otimes \mathrm{H}_q(\mathrm{C}_*(X))}^{=\mathrm{H}_p(X)\otimes \mathrm{H}_q(X)} & \to & \mathrm{H}_{p+q}(\mathrm{C}_*(X)\otimes \mathrm{C}_*(X)) \xrightarrow{\Upsilon_*} \overbrace{\mathrm{H}_{p+q}(\mathrm{C}_*(X\times X))}^{=\mathrm{H}_{p+q}(X\times X)}. \\ {[c]\otimes[d]} & \mapsto & [c\otimes d]. \end{array}$$

This is very nice, but this construction does not give us a product on the homology groups of X itself, since we end up in the homology of a different topological space.

Now we try our luck with cohomology groups. Before we can study maps on cohomology groups we need a little bit of algebraic preparation.

Definition. Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes.

(1) Given $p, q \in \mathbb{N}_0$ we denote by

$$\Pi_{p,q}\colon (\mathcal{C}_* \otimes \mathcal{D}_*)_{p+q} = \bigoplus_{i+j=p+q} \mathcal{C}_i \otimes \mathcal{D}_j \ \to\ \mathcal{C}_p \otimes \mathcal{D}_q$$

the obvious projection map.

(2) Let R be a commutative ring. We denote by Ξ the map

$$\begin{array}{rcl} \mathrm{Hom}(\mathcal{C}_p, R)\times \mathrm{Hom}(\mathcal{D}_q, R) & \to & \mathrm{Hom}((\mathcal{C}\otimes\mathcal{D})_{p+q}, R) \\ (\varphi,\psi) & \mapsto & \left(\begin{array}{rcl} (\mathcal{C}_*\otimes\mathcal{D}_*)_{p+q} \xrightarrow{\Pi_{p,q}} \mathcal{C}_p\otimes\mathcal{D}_q \xrightarrow{\varphi\otimes\psi} R\otimes R & \to & R \\ a\otimes b & \mapsto & a\cdot b \end{array}\right). \end{array}$$

Lemma 81.1. *Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes. For any commutative ring R the map*

$$\begin{array}{rcl} \mathrm{H}^p(\mathcal{C}_*; R) \times \mathrm{H}^q(\mathcal{D}; R) & \to & \mathrm{H}^{p+q}(\mathcal{C}_*\otimes\mathcal{D}_*; R) \\ ([\varphi],[\psi]) & \mapsto & [\Xi(\varphi,\psi)] \end{array}$$

is well-defined.

Proof. We leave the verification of the lemma to the reader. ■

We return to topology. So let X be a topological space. We consider the map[1163]

$$\begin{array}{rcl} \overbrace{\mathrm{H}^p(\mathrm{C}_*(X);\mathbb{Z}) \otimes \mathrm{H}^q(\mathrm{C}_*(X);\mathbb{Z})}^{=\mathrm{H}^p(X;\mathbb{Z})\otimes \mathrm{H}^q(X;\mathbb{Z})} & \xrightarrow{\Xi} & \mathrm{H}^{p+q}(\mathrm{C}_*(X)\otimes \mathrm{C}_*(X);\mathbb{Z}) \xrightarrow{\Theta^*} \overbrace{\mathrm{H}^{p+q}(\mathrm{C}_*(X\times X);\mathbb{Z})}^{=\mathrm{H}^{p+q}(X\times X;\mathbb{Z})}. \\ {[\varphi]\otimes[\psi]} & \mapsto & [\varphi\otimes\psi\circ\Pi_{p,q}]. \end{array}$$

Summarizing we get a map

$$\mathrm{H}^p(X;\mathbb{Z}) \otimes \mathrm{H}^q(X;\mathbb{Z}) \ \to\ \mathrm{H}^{p+q}(X\times X;\mathbb{Z})$$

[1162] Here $\Upsilon\colon \mathrm{C}_*(X)\otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X\times Y)$ denotes the Eilenberg-Zilber map from Theorem 80.1.

[1163] Here $\Theta\colon \mathrm{C}_*(X\times Y) \to \mathrm{C}_*(X)\otimes \mathrm{C}_*(Y)$ denotes the Eilenberg-Zilber map from Theorem 80.1 (1). It follows from Theorem 80.1 (2) and Lemma 73.8 (3) that the definition below does not depend on the choice of Θ.

which is sometimes called the *cross product on cohomology*. Again, this is nice, but exactly as in the case of the cross product on homology, we end up in the cohomology of $X \times X$, but we really would like to end up in the cohomology of X.

Now we have to find a way how to go back from the topological space $X \times X$ to the original space X. In general there does not exist an "interesting"[1164] map $X \times X \to X$. But, as we saw before, there does exist an interesting map in the opposite direction, namely the diagonal map

$$\begin{aligned} d\colon X &\to X \times X \\ x &\mapsto (x, x). \end{aligned}$$

This diagonal map d induces a map

$$d_*\colon \mathrm{H}_*(X;\mathbb{Z}) \to \mathrm{H}_*(X \times X;\mathbb{Z})$$

on homology, and, much more interesting in our context, it also induces a map

$$d^*\colon \mathrm{H}^*(X \times X;\mathbb{Z}) \to \mathrm{H}^*(X;\mathbb{Z}).$$

on cohomology groups that goes in the "opposite direction". This discussion leads us to the following definition.

Definition. Let X be a topological space. We define the *cup product*[1165] on the cohomology groups of X as the composition of the following maps:

$$\mathrm{H}^p(X;\mathbb{Z}) \times \mathrm{H}^q(X;\mathbb{Z}) \to \mathrm{H}^p(X;\mathbb{Z}) \otimes \mathrm{H}^q(X;\mathbb{Z}) \xrightarrow{\Theta^* \circ \Xi} \mathrm{H}^{p+q}(X \times X;\mathbb{Z}) \xrightarrow{d^*} \mathrm{H}^{p+q}(X;\mathbb{Z}).$$

For the resulting pairing we use the notation

$$\begin{aligned} \mathrm{H}^p(X;\mathbb{Z}) \times \mathrm{H}^q(X;\mathbb{Z}) &\to \mathrm{H}^{p+q}(X;\mathbb{Z}) \\ (a, b) &\mapsto a \widetilde{\cup} b \end{aligned}$$

and we refer to $a \widetilde{\cup} b$ as the *cup product of a and b.*

81.2. **An alternative definition of the cup product.** The definition of the cup product in the previous section was very natural and we could now develop the theory of the cup product using that definition. If one takes that route then one has to deal with the awkward fact that the Eilenberg-Zilber map Θ is not very explicit.

Now we give an alternative definition of the cup product which is more ad hoc, but which works surprisingly well in practice.

Definition. Let X be a topological space and let R be a commutative ring. Furthermore let $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ be cochains. We define the *cup product* of φ and ψ

[1164]The projection from $X \times X$ onto one of the two factors does not count as "interesting", since it does not "mix" the two factors. In fact in Exercise 81.1 we will see that the map

$$\mathrm{H}_*(X) \otimes \mathrm{H}_*(X) \xrightarrow{\times} \mathrm{H}_*(X \times X) \xrightarrow{p_*} \mathrm{H}_*(X),$$

where $p\colon X \times X \to X$ denotes the projection on the first factor, is the zero map.

[1165]The name "cup product" comes from the notation "$a \widetilde{\cup} b$" for the composition since $\cup$ looks like a cup.

to be the singular cochain[1166]

$$\begin{array}{rcl}\varphi \cup \psi\colon\ \mathrm{C}_{p+q}(X) & \to & R\end{array}$$

that is given by

$$\begin{array}{rcl}(\sigma\colon \Delta^{p+q} \to X) & \mapsto & \underbrace{\varphi(\sigma\circ[v_0,\dots,v_p])}_{\in R}\cdot\underbrace{\psi(\sigma\circ[v_p,\dots,v_{p+q}])}_{\in R}.\end{array}$$

The[1167] cup product thus defines a map

$$\cup\colon\ \mathrm{C}^p(X;R)\times \mathrm{C}^q(X;R)\ \to\ \mathrm{C}^{p+q}(X;R).$$

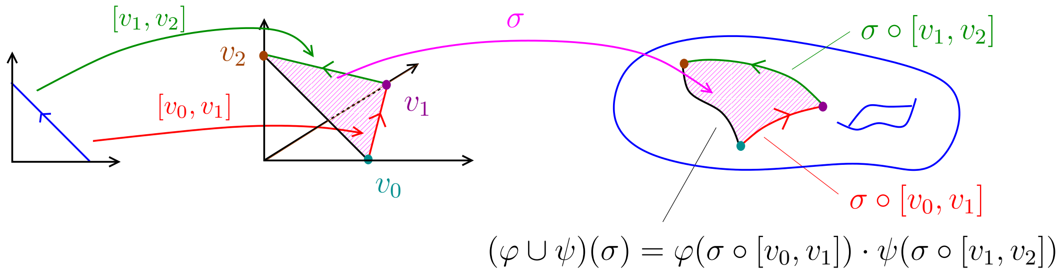


Figure 1160. Illustration of the definition of the cup product.

Remark. The sign conventions for the cup product for cohomology classes $\varphi\in \mathrm{H}^p(X;R)$ and $\psi\in\mathrm{H}^q(X;R)$ differ in the literature. More precisely, given a topological space X and cochains $\varphi\in\mathrm{C}^p(X;R)$ and $\psi\in\mathrm{C}^q(X;R)$ the cup product $\varphi\cup\psi$, applied to a singular $(p+q)$-simplex $\sigma\colon\Delta^{p+q}\to X$ is defined as follows:

		$(\varphi\cup\sigma)(\sigma)$
(a)	Hatcher [**Hat02**, p. 206]	$\varphi(\sigma\circ[v_0,\dots,v_p])\cdot\psi(\sigma\circ[v_p,\dots,v_{p+q}])$
(b)	Bredon [**Bre93**, p. 328]	$(-1)^{pq}\cdot\varphi(\sigma\circ[v_0,\dots,v_p])\cdot\psi(\sigma\circ[v_p,\dots,v_{p+q}])$
(c)	Dold [**Dol56**, p. 222]	$(-1)^{pq}\cdot\varphi(\sigma\circ[v_0,\dots,v_p])\cdot\psi(\sigma\circ[v_p,\dots,v_{p+q}])$
(d)	Spanier [**Spa95**, p. 251]	$\varphi(\sigma\circ[v_0,\dots,v_p])\cdot\psi(\sigma\circ[v_p,\dots,v_{p+q}])$
(e)	Munkres [**Mun84**, p. 288]	$\varphi(\sigma\circ[v_0,\dots,v_p])\cdot\psi(\sigma\circ[v_p,\dots,v_{p+q}])$
(f)	Greenberg-Harper [**GH81**, p. 195]	$\varphi(\sigma\circ[v_0,\dots,v_p])\cdot\psi(\sigma\circ[v_p,\dots,v_{p+q}])$

So we see that our sign convention agrees with the conventions of Hatcher, Spanier, Munkres and Greenberg-Harper, but they differ by the sign $(-1)^{pq}$ from the conventions used in Bredon and Dold.

Example. Since the second definition of the cup product is initially strange and not very intuitive we consider an example in slightly more detail. In the following we consider the torus $T=S^1\times S^1=([0,1]\times[0,1])/\sim$ with the two singular 2-simplices σ_1 and σ_2 illustrated in Figure 1161 which we already saw in Figure 1054. We recall the following notation:

(1) we denote by p respectively q the projection $S^1\times S^1\to S^1$ onto the first respectively second factor,

[1166] Here, when we write $[v_0,\dots,v_p]\colon\Delta^p\to\Delta^{p+q}$ and $[v_p,\dots,v_{p+q}]\colon\Delta^q\to\Delta^{p+q}$, we use the notation from page 1970.

[1167] In the literature, given a singular n-simplex $\sigma\colon\Delta^n\to X$ one often writes ${}_p\lfloor\sigma=\sigma\circ[v_0,\dots,v_p]$ and $\sigma\rfloor_p=\sigma\circ[v_{n-p},\dots,v_n]$.

(2) we denote by $\mu\colon \Delta^1 \to S^1$ the singular 1-simplex given by $\mu(1-t,t) = e^{2\pi \mathrm{i} t}$,
(3) furthermore we denote by $\nu\colon \Delta^1 \to S^1$ the constant singular 1-simplex that is given by $\nu(1-t,t) = 1$.
(4) finally we recall that on page 1825 we introduced a cochain $\theta := \theta_{\mathbb{Z}} \in \mathrm{C}^1; \mathbb{Z})$ and we showed that $\theta(\mu) = 1$, also note that $\theta(\nu) = 0$.

We want to determine the values of $p^*(\theta) \cup q^*(\theta) \in \mathrm{C}^2(T;\mathbb{Z}) = \mathrm{Hom}(\mathrm{C}_2(T),\mathbb{Z})$ on the singular 2-simplices σ_1 and σ_2. We have

$$\begin{array}{rcl} (p^*(\theta) \cup q^*(\theta))(\sigma_1) & = & p^*(\theta)(\sigma_1 \circ [v_0,v_1]) \cdot q^*(\theta)(\sigma_1 \circ [v_1,v_2]) \\ & = & \theta(\underbrace{p \circ \sigma_1 \circ [v_0,v_1]}_{=\nu}) \cdot \theta(\underbrace{q \circ \sigma_1 \circ [v_1,v_2]}_{=\nu}) \quad = \quad 0 \cdot 0 \quad = \quad 0. \end{array}$$

Much more interestingly we have

$$\begin{array}{rcl} (p^*(\theta) \cup q^*(\theta))(\sigma_2) & = & p^*(\theta)(\sigma_2 \circ [v_0,v_1]) \cdot q^*(\theta)(\sigma_2 \circ [v_1,v_2]) \\ & = & \theta(\underbrace{p \circ \sigma_2 \circ [v_0,v_1]}_{=\mu}) \cdot \theta(\underbrace{q \circ \sigma_2 \circ [v_1,v_2]}_{=\mu}) \quad = \quad 1 \cdot 1 \quad = \quad 1. \end{array}$$

To aid the reader we also recall in Figure 1162 the notation that we had initially introduced on page 1716 and that we use in Figure 1161.

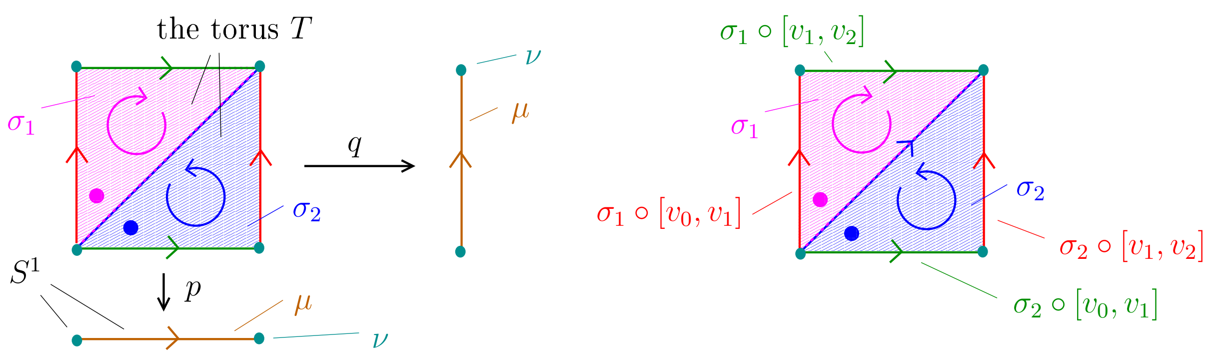


FIGURE 1161

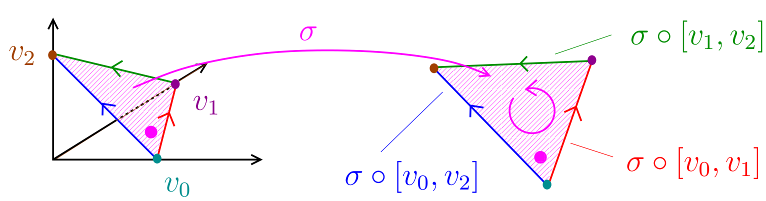


FIGURE 1162. Notation of the “dotted triangle”.

The definition of the cup product on cochains might look strange and arbitrary. But the following lemma shows that it behaves well with respect to coboundary maps.

Lemma 81.2. *Let X be a topological space and let R be a commutative ring. Furthermore let $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$. Then*

$$\delta(\varphi \cup \psi) \;=\; \delta\varphi \cup \psi + (-1)^p \cdot \varphi \cup \delta\psi \;\in\; \mathrm{C}^{p+q+1}(X;R).$$

Remark. Note that a similar looking statement holds for differential forms. More precisely, given a smooth manifold M and two differential forms $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$ it follows from Proposition 78.2 (5) that

$$d(\omega \wedge \eta) \;=\; d\omega \wedge \eta \,+\, (-1)^p \cdot \omega \wedge d\eta \;\in\; \Omega^{p+q+1}(M).$$

The (co-) boundary formula of Lemma 81.2 is of course also very similar to the definition of the boundary map in the tensor product of two chain complexes.

PROOF (*). The lemma follows from an elementary, albeit slightly confusing, calculation. Indeed, let $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ be cochains. We have to prove the following equality:

$$\delta(\varphi \cup \psi) \;=\; \delta\varphi \cup \psi \,+ (-1)^p \cdot \varphi \cup \delta\psi \;\in\; \mathrm{C}^{p+q+1}(X;R) = \mathrm{Hom}(\mathrm{C}_{p+q+1}(X), R).$$

So let $\sigma\colon \Delta^{p+q+1} \to X$ be a singular $(p+q+1)$-simplex. We have the following three equalities in R:[1168]

$$\begin{aligned}
\text{(a)}\quad (\delta(\varphi \cup \psi))(\sigma) &= (\varphi \cup \psi)\Big(\sum_{j=0}^{p+q+1}(-1)^j \cdot \sigma \circ i_j^{p+q+1}\Big)\\
&= (\varphi \cup \psi)\Big(\sum_{j=0}^{p}(-1)^j \cdot \sigma \circ i_j^{p+q+1}\Big) + (\varphi \cup \psi)\Big(\sum_{j=p+1}^{p+q+1}(-1)^j \cdot \sigma \circ i_j^{p+q+1}\Big)\\
&= \sum_{j=0}^{p}(-1)^j \cdot (\varphi \cup \psi)(\sigma \circ i_j^{p+q+1}) + (-1)^p \cdot \sum_{k=1}^{q+1}(-1)^k \cdot (\varphi \cup \psi)(\sigma \circ i_{k+p}^{p+q+1})\\
&= \quad \sum_{j=0}^{p}(-1)^j \cdot \varphi(\underbrace{\sigma \circ [v_0,\dots,v_{p+1}] \circ i_j^{p+1}}_{\text{map } \Delta^p \to X}) \cdot \psi(\underbrace{\sigma \circ [v_{p+1},\dots,v_{p+q+1}]}_{\text{map } \Delta^q \to X})\\
&\qquad + (-1)^p \cdot \sum_{k=1}^{q+1}(-1)^k \cdot \varphi(\underbrace{\sigma \circ [v_0,\dots,v_p]}_{\text{map } \Delta^p \to X}) \cdot \psi(\underbrace{\sigma \circ [v_p,\dots,v_{p+q+1}] \circ i_k^{q+1}}_{\text{map } \Delta^q \to X}).
\end{aligned}$$

$$\begin{aligned}
\text{(b)}\quad (\delta\varphi \cup \psi)(\sigma) &= (\delta\varphi)(\sigma \circ [v_0,\dots,v_{p+1}]) \cdot \psi(\sigma \circ [v_{p+1},\dots,v_{p+q+1}])\\
&= \varphi\Big(\sum_{j=0}^{p+1}(-1)^j \cdot \underbrace{\sigma \circ [v_0,\dots,v_{p+1}] \circ i_j^{p+1}}_{\text{map } \Delta^p \to X}\Big) \cdot \psi(\underbrace{\sigma \circ [v_{p+1},\dots,v_{p+q+1}]}_{\text{map } \Delta^q \to X})\\
&= \sum_{j=0}^{p+1}(-1)^j \cdot \varphi(\sigma \circ [v_0,\dots,v_{p+1}] \circ i_j^{p+1}) \cdot \psi(\sigma \circ [v_{p+1},\dots,v_{p+q+1}])
\end{aligned}$$

and

$$\begin{aligned}
\text{(c)}\quad (-1)^p \cdot (\varphi \cup \delta\psi)(\sigma) &= (-1)^p \cdot \varphi(\sigma \circ [v_0,\dots,v_p]) \cdot (\delta\psi)(\sigma \circ [v_p,\dots,v_{p+q+1}])\\
&= (-1)^p \cdot \varphi(\underbrace{\sigma \circ [v_0,\dots,v_p]}_{\text{map } \Delta^p \to X}) \cdot \psi\Big(\sum_{k=0}^{q+1}(-1)^k \cdot \underbrace{\sigma \circ [v_p,\dots,v_{p+q+1}] \circ i_k^{q+1}}_{\text{map } \Delta^q \to X}\Big)\\
&= (-1)^p \cdot \sum_{k=0}^{q+1}(-1)^k \cdot \varphi(\sigma \circ [v_0,\dots,v_p]) \cdot \psi(\sigma \circ [v_p,\dots,v_{p+q+1}] \circ i_k^{q+1}).
\end{aligned}$$

[1168] Here in the argument we use on several occasions that for a singular n-cochain ϕ and a singular $(n+1)$-simplex μ we have by definition

$$(\delta\phi)(\mu) \;=\; \phi(\partial\mu) \;=\; \phi\Big(\sum_{l=0}^{n+1}(-1)^l \cdot \mu \circ i_l^{n+1}\Big).$$

Now we observe that if we take the sum of (b) and (c), then the $(j = p+1)$-summand in (b) cancels with the $(k = 0)$-summand in (c). But after this observation it is obvious that the desired equality (b)+(c)=(a) holds. ■

Lemma 81.3. *Let X be a topological space and let R be a commutative ring. Then the map*

$$\begin{array}{rcl} \cup\colon \mathrm{H}^p(X;R) \times \mathrm{H}^q(X;R) & \to & \mathrm{H}^{p+q}(X;R) \\ ([\varphi],[\psi])] & \mapsto & [\varphi] \cup [\psi] := [\varphi \cup \psi] \end{array}$$

is well-defined.

Definition. We refer to the pairing of Lemma 81.3 also as the *cup product*.

As we will see, the proof of Lemma 81.3 is basically identical to the proof of Lemma 79.13.

Proof. Let $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ be cocycles.

(1) It follows immediately from Lemma 81.2 that $\varphi \cup \psi$ is again a cocycle.
(2) If $\varphi' = \varphi + \delta\tau$ is another representative of $[\varphi]$, then

$$\varphi' \cup \psi - \varphi \cup \psi \underset{\substack{\uparrow \\ \text{since ``}\cup\text{'' is clearly bilinear}}}{=} (\varphi' - \varphi) \cup \psi = \delta\tau \cup \psi \underset{\substack{\uparrow \\ \text{by Lemma 81.2 and since } \delta\psi = 0}}{=} \delta(\tau \cup \psi).$$

This shows that $\varphi \cup \psi$ and $\varphi' \cup \psi$ represent the same cohomology class, i.e. the cup product does not depend on the choice of the representative of the cohomology class of the first entry.
(3) The proof that the cup product does not depend on the choice of the representative of the second cohomology class is basically the same as the proof of (2). ■

The following proposition now says that the original definition of the cup product and the new definition actually agree.

Proposition 81.4. *Let X be a topological space. Then for any $\varphi \in \mathrm{H}^p(X;\mathbb{Z})$ and any $\psi \in \mathrm{H}^q(X;\mathbb{Z})$ we have*

$$\varphi \mathbin{\tilde{\cup}} \psi \;=\; \varphi \cup \psi \;\in\; \mathrm{H}^{p+q}(X;\mathbb{Z}).$$

In the proof of Proposition 81.4 we need the following definition and the subsequent lemma.

Definition. A *diagonal approximation* consists of a natural chain map

$$\Phi\colon\ \mathrm{C}_*(X) \;\to\; \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

for each topological space X, such that for each $x \in X$, viewed as a singular 0-simplex, the following equality holds in $(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_0 = \mathrm{C}_0(X) \otimes \mathrm{C}_0(X)$:

$$\Phi(x) \;=\; x \otimes x.$$

Example. Given any topological space X we denote by $d\colon X \to X \times X$ the diagonal map given by $d(x) = (x,x)$. Then the induced chain maps

$$\Theta \circ d_*\colon\ \mathrm{C}_*(X) \xrightarrow{d_*} \mathrm{C}_*(X \times X) \underset{\substack{\uparrow \\ \text{Eilenberg-Zilber map from Theorem 80.1}}}{\xrightarrow{\Theta}} \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

form a diagonal approximation in the above sense.

The following lemma gives another diagonal approximation which will be very useful for doing actual calculations.

Lemma 81.5. *Given any topological space X we consider the map*

$$\Delta\colon\ \mathrm{C}_*(X)\ \to\ \mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$$

which for each singular n-simplex $\sigma\colon\Delta^n\to X$ is given by

$$\sigma\ \mapsto\ \sum_{p=0}^{n}\sigma\circ[v_0,\dots,v_p]\otimes\sigma\circ[v_p,\dots,v_n].$$

These maps form a diagonal approximation.

The diagonal approximation from Lemma 81.5 is called the *Alexander-Whitney diagonal approximation.* [1169][1170]

Proof. It follows almost immediately from the definition that the maps Δ are natural. Furthermore it is obvious that on the 0-chains the maps are of the desired form. It remains to show that for a given topological space X the maps actually form a chain map.

This follows from an elementary and uneventful calculation. For completeness' sake we provide the argument. Let $\mu\colon\Delta^n\to X$ be a singular n-simplex. Given $i_1,\dots,i_r$ we write $[i_1,\dots,i_r]:=\mu\circ[v_{i_1},\dots,v_{i_r}]$. We also use the convention that the empty singular simplex $[\ ]$ is 0. We calculate that

$$\begin{aligned}\Delta(\partial[0,\dots,n])&=\Delta\Big(\sum_{i=0}^{n}(-1)^i\cdot[0,\dots,\widehat{i},\dots n]\Big)\\&\underset{\substack{\uparrow\\ \text{in the second sum we secretly used the substitution } q=p+1}}{=}\sum_{0\le p<i\le n}(-1)^i\cdot[0,..,p]\otimes[p,..,\widehat{i},..,n]+\sum_{0\le i<q\le n}(-1)^i\cdot[0,..,\widehat{i},..,q]\otimes[q,..,n]\end{aligned}$$

and

$$\begin{aligned}\partial(\Delta[0,\dots,n])&=\partial\Big(\sum_{p=0}^{n}[0,\dots,p]\otimes[p,\dots,n]\Big)\\&\underset{\substack{\uparrow\\ \text{here we use the sign convention for the boundary in a tensor product}}}{=}\sum_{0\le i\le p\le n}(-1)^i\cdot[0,..,\widehat{i},..,p]\otimes[p,..,n]+\sum_{0\le p\le i\le n}(-1)^i\cdot[0,..,p]\otimes[p,..,\widehat{i},..,n].\end{aligned}$$

We see that the difference between the two terms equals

$$\begin{aligned}&\sum_{i=p}(-1)^i\cdot[0,\dots,\widehat{i},\dots,p]\otimes[p,\dots,n]+\sum_{i=p}(-1)^i\cdot[0,\dots,p]\otimes[p,\dots,\widehat{i},\dots,n]\\&\underset{\substack{\uparrow\\ \text{here we use that } [\ ]=0}}{=}\sum_{i=1}^{n}(-1)^i\cdot[0,\dots,i-1]\otimes[i,\dots,n]+\sum_{i=0}^{n-1}(-1)^i\cdot[0,\dots,i]\otimes[i+1,\dots,n]\\&=\ \sum_{i=1}^{n}(-1)^i\cdot[0,\dots,i-1]\otimes[i,\dots,n]+\sum_{j=1}^{n}(-1)^{j-1}\cdot[0,\dots,j-1]\otimes[j,\dots,n]\ =\ 0.\end{aligned}$$

[1169] James Alexander (1888-1971) was an American mathematician whom we had already met on page 1297.

[1170] Hassler Whitney (1907–1989) was an American mathematician. We will encounter him again later on.

We have thus shown that Δ is indeed a chain map. ■

Proposition 81.6.

(1) *There exist diagonal approximations.*
(2) *Given any two diagonal approximations Φ and Ψ there exists a natural chain homotopy equivalence between Φ and Ψ.*

Proof.

(1) We had just given two examples of diagonal approximations.
(2) The uniqueness statement was proved in Proposition 81.6. ■

Now we are ready to provide the proof of Proposition 81.4.

Proof of Proposition 81.4. We start out with the following claim.

Claim. For any two cochains $\varphi \in \mathrm{C}^p(X;\mathbb{Z})$ and $\psi \in \mathrm{C}^q(X;\mathbb{Z})$ we have[1171]

$$\varphi \cup \psi \;=\; (\varphi \otimes \psi) \circ \Pi_{p,q} \circ \Delta \;\in\; \mathrm{C}^{p+q}(X;\mathbb{Z}) = \mathrm{Hom}(\mathrm{C}_{p+q}(X),\mathbb{Z}).$$

We have to show that the two sides are the same maps $\mathrm{C}_{p+q}(X) \to \mathbb{Z}$. Therefore let $\sigma\colon \Delta^{p+q} \to X$ be a singular $(p+q)$-simplex. We calculate that

$$\begin{array}{rcl} (\varphi \otimes \psi \circ \Pi_{p,q} \circ \Delta)(\sigma) & \overset{\substack{\text{definition of the Alexander-Whitney diagonalization}\\ \downarrow}}{=} & (\varphi \otimes \psi)\big(\Pi_{p,q}\big(\sum\limits_{i+j=p+q} \overbrace{\sigma \circ [v_0,\dots,v_i] \otimes \sigma \circ [v_i,\dots,v_{p+q}]}^{\in \mathrm{C}_i(X)\otimes \mathrm{C}_j(X)}\big)\big) \\ & = & (\varphi \otimes \psi)(\sigma \circ [v_0,\dots,v_p] \otimes \sigma \circ [v_q,\dots,v_{p+q}]) \\ & = & \varphi(\sigma \circ [v_0,\dots,v_p]) \cdot \psi(\sigma \circ [v_p,\dots,v_n]) \\ & = & (\varphi \cup \psi)(\sigma). \end{array}$$ ⊞

Now suppose that we are given cocycles $\varphi \in \mathrm{C}^p(X;\mathbb{Z})$ and $\psi \in \mathrm{C}^q(X;\mathbb{Z})$. We obtain the following equality in $\mathrm{H}^{p+q}(X;\mathbb{Z})$:[1172]

$$\begin{array}{rl} [\varphi \tilde{\cup} \psi] \overset{\substack{\text{by definition}\\ \downarrow}}{=} [(\varphi\otimes\psi)\circ\Pi_{p,q}\circ(\Theta\circ d_*)] & = (\Theta\circ d_*)^*\big(\overbrace{[(\varphi\otimes\psi)\circ\Pi_{p,q}]}^{\in \mathrm{H}^{p+q}(\mathrm{C}_*(X)\otimes \mathrm{C}_*(X))}\big) \\ & \underset{\uparrow}{=} \Delta^*\big([(\varphi\otimes\psi)\circ\Pi_{p,q}]\big) = [(\varphi\otimes\psi)\circ\Pi_{p,q}\circ\Delta] \underset{\uparrow}{=} [\varphi\cup\psi]. \end{array}$$

by Proposition 81.6 the maps $\Theta\circ d_*, \Delta\colon\ \mathrm{C}_*(X) \to \mathrm{C}_*(X)\otimes \mathrm{C}_*(X)$ are chain homotopic so by Lemma 73.8 (3) they induce the same map on cohomology; by the claim

We have thus obtained the desired equality of cohomology classes. ■

Henceforth we will mostly work with the second definition of the cup product. In the following proposition we summarize some of the key properties of the cup product.

Proposition 81.7. *Let X be a topological space and let R be a commutative ring.*

(1) *The cup product on $\mathrm{H}^*(X;R)$ is R-bilinear and associative.*

[1171]Here $\Delta\colon\ \mathrm{C}_{p+q}(X) \to (\mathrm{C}_*(X)\otimes \mathrm{C}_*(X))_{p+q}$ denotes the above Alexander-Whitney diagonalization.

[1172]Note that it follows from Lemma 81.1 that for two cocycles φ and ψ the composition $(\varphi\otimes\psi)\circ\Pi_{p,q}$ is indeed a cocycle, i.e. $(\varphi\otimes\psi)\circ\Pi_{p,q}$ defines an element in $\mathrm{H}^{p+q}(\mathrm{C}_*(X)\otimes \mathrm{C}_*(X);\mathbb{Z})$.

(2) *The abelian group* $\mathrm{H}^*(X;R) = \bigoplus\limits_{n\in\mathbb{N}_0} \mathrm{H}^n(X;R)$ *together with the map*

$$\cup\colon\ \mathrm{H}^*(X;R)\times \mathrm{H}^*(X;R)\to \mathrm{H}^*(X;R)$$

is a ring[1173] *where the multiplicatively neutral element is given by*

$$1_X \ :=\ \big[\text{constant map } X\to R \text{ given by } x\mapsto 1\big]\ \in\ \mathrm{H}^0(X;R).$$

Definition. Given a topological space X and a commutative ring R we refer to $(\mathrm{H}^*(X;R),\cup)$ as the *R-cohomology ring of X*.

Proof. Let X be a topological space.

(1) It is clear from the second definition of the cup product that it is R-bilinear. It is also almost obvious, using the second definition, that the cup product is associative, but for completeness' sake we carry out the argument. Thus let $\alpha\in \mathrm{C}^p(X;R)$, $\beta\in \mathrm{C}^q(X;R)$ and $\gamma\in \mathrm{C}^r(X;R)$ be cochains and let $\sigma\colon \Delta^{p+q+r}\to X$ be a singular simplex. Then

$$\begin{aligned}((\alpha\cup\beta)\cup\gamma)(\sigma) &= (\alpha\cup\beta)(\sigma\circ[v_0,\dots,v_{p+q}])\cdot\gamma(\sigma\circ[v_{p+q},\dots,v_{p+q+r}])\\ &= \alpha(\sigma\circ[v_0,\dots,v_p])\cdot\beta(\sigma\circ[v_p,\dots,v_{p+q}])\cdot\gamma(\sigma\circ[v_{p+q},\dots,v_{p+q+r}])\\ &\underset{\substack{\uparrow\\ \text{same argument backwards}}}{=} (\alpha\cup(\beta\cup\gamma))(\sigma).\end{aligned}$$

Thus we see that the cup product is already associative for cochains, so it is in particular associative for cohomology classes.

(2) Let $\eta\colon X\to R$ be the map that assigns to each point in X the value $1\in R$. As usual we view η as a singular 0-cochain. It follows from Proposition 73.11 that η is a 0-cocycle and that we obtain an element $1_X := [\eta]\in \mathrm{H}^0(X;R)$. It is clear from the definition that it is the neutral element with respect to the cup product. Together with (1) we have now shown that the cup product defines a ring structure on $\mathrm{H}^*(X;R)$. ■

Now that we have shown that $(\mathrm{H}^*(X;R),\cup)$ is a ring the question arises whether it is commutative or whether it is not commutative. It turns out that the answer, as for the wedge product on de Rham cohomology, lies somewhere in between. More precisely, we have the following proposition.

Proposition 81.8. *Let X be a topological space and let R be a commutative ring. For any $a\in \mathrm{H}^k(X;R)$ and $b\in \mathrm{H}^l(X;R)$ we have*

$$a\cup b \ =\ (-1)^{kl}\cdot b\cup a.$$

Remark.

(1) If $R=\mathbb{F}_2$, then we are in the pretty situation that $-1=1$, i.e. the cup product is commutative. Put differently, the $\mathbb{F}_2$-cohomology ring $\mathrm{H}^*(X;\mathbb{F}_2)$ of a topological space X is a commutative ring.

[1173] Recall that on page 81 we stated that in the lecture notes a ring is understood to be associative and it is understood to have a multiplicatively neutral element. We do not demand that a ring is commutative.

(2) The proof for Proposition 81.8 that we provide below is based on the second definition of the cup product. It is elementary but unfortunately unpleasant to read. Later, in Exercise 84.1, we will provide a different, more conceptual proof based on the first definition of the cup product.

Proof (∗). We give a proof of the proposition following [**Hat02**, p. 216-217]. In Hatcher's book more motivations for the steps in the proof are given.

So let $\varphi \in \mathrm{C}^k(X;R)$ and $\psi \in \mathrm{C}^l(X;R)$ be two cocycles. Given a singular n-simplex $\sigma\colon \Delta^n \to X$ we denote by $\overline{\sigma}$ the singular n-simplex obtained from σ by precomposing σ with the affine linear map $[v_n,\dots,v_0]\colon \Delta^n \to \Delta^n$ that reverses the order of the vertices. Furthermore we write $\epsilon_n = (-1)^{n(n+1)/2}$. We note that for any $m,n \in \mathbb{N}_0$ an elementary calculation shows that $\epsilon_{m+n} = (-1)^{mn}\cdot \epsilon_m\epsilon_n$ and that $\epsilon_1 = -1$.

Claim. The maps

$$\begin{array}{rccc} \rho\colon & \mathrm{C}_n(X) & \to & \mathrm{C}_n(X) \\ & \sigma & \mapsto & \epsilon_n\cdot\overline{\sigma} \end{array}$$

form a chain map that is chain homotopic to the identity.

We first show that the claim actually implies the statement of the proposition. Thus let $[\varphi] \in \mathrm{H}^k(X;R)$ and $[\psi] \in \mathrm{H}^l(X;R)$. Since by the claim ρ is chain homotopic to the identity it follows from Lemma 73.8 that the induced map $\rho^*\colon\ \mathrm{C}^*(X;R) \to \mathrm{C}^*(X;R)$ induces the identity on cohomology groups. So to deduce the statement of the proposition it suffices to show that

$$\rho^*(\psi\cup\varphi) \;=\; (-1)^{kl}\cdot\rho^*\varphi\cup\rho^*\psi \;\in\; \mathrm{C}^{k+l}(X;R).$$

Now let σ be a $(k+l)$-chain. Then

$$\begin{array}{rcl} (\rho^*(\psi\cup\varphi))(\sigma) & = & (\psi\cup\varphi)(\rho_*(\sigma)) \;=\; (\psi\cup\varphi)(\epsilon_{k+l}\cdot\sigma\circ[v_{k+l},\dots,v_0]) \\ & = & \epsilon_{k+l}\cdot\psi(\sigma\circ[v_{k+l},\dots,v_k])\cdot\varphi(\sigma\circ[v_k,\dots,v_0]) \\ & \underset{\substack{\uparrow \\ \text{since } \epsilon_{k+l}=(-1)^{kl}\cdot\epsilon_k\cdot\epsilon_l \text{ and since } R \text{ is a \textit{commutative} ring}}}{=} & (-1)^{kl}\cdot\varphi(\epsilon_k\cdot\sigma\circ[v_k,\dots,v_0])\cdot\psi(\epsilon_l\cdot\sigma\circ[v_{k+l},\dots,v_k]) \\ & = & (-1)^{kl}\cdot(\rho^*\varphi\cup\rho^*\psi)(\sigma). \end{array}$$

We have thus shown the statement of the proposition modulo the claim.

We turn to the proof of the claim. It is straightforward to verify that ρ is a chain map, indeed, for any singular n-simplex σ we have

$$\begin{array}{rcl} \partial\rho(\sigma) & = & \partial(\epsilon_n\cdot\sigma\circ[v_n,\dots,v_0]) \;=\; \epsilon_n\cdot\sum\limits_{i=0}^{n}(-1)^i\cdot\sigma\circ[v_n,\dots,\widehat{v_{n-i}},\dots,v_0], \\ \rho\partial(\sigma) & = & \rho\Big(\sum\limits_{i=0}^{n}(-1)^i\cdot\sigma\circ[v_0,\dots,\widehat{v_i},\dots,v_n]\Big) \underset{\substack{\uparrow \\ \text{definition of } \rho \text{ and substitution } j=n-i}}{=} \epsilon_{n-1}\cdot\sum\limits_{j=0}^{n}(-1)^{n-j}\cdot\sigma\circ[v_n,\dots,\widehat{v_{n-j}},\dots,v_0]. \end{array}$$

From $\epsilon_n = (-1)^n\cdot\epsilon_{n-1}$ we now obtain that $\partial\rho = \rho\partial$, i.e. we have shown that ρ is a chain map.

We still have to show that ρ is chain homotopic to the identity. We proceed with ideas similar to the ideas used in the proof of Proposition 42.5. As in that proof, given

$i \in \{0, \dots, n\}$ we write

$$v_i \;:=\; (0, \dots, 1, 0, \dots, 0) \times \{0\} \;\in\; \mathbb{R}^{n+1} \times \{0\},$$

and

$$w_i \;:=\; (0, \dots, 1, 0, \dots, 0) \times \{1\} \;\in\; \mathbb{R}^{n+1} \times \{1\}.$$

We justify the reassignment of v_i to a different point by saying that we identify Δ^n with $\Delta^n \times \{0\} \in \mathbb{R}^{n+1} \times \mathbb{R}$. We denote by $\pi\colon \Delta^n \times [0,1] \to \Delta^n$ the projection map. We consider the map

$$\begin{array}{rccl} P\colon & \mathrm{C}_n(X) & \to & \mathrm{C}_{n+1}(X) \\ & \sigma & \mapsto & \sum\limits_{i=0}^{n} (-1)^i \cdot \epsilon_{n-i} \cdot (\sigma \circ \pi) \circ [v_0, \dots, v_i, w_n, \dots, w_i]. \end{array}$$

Now we want to show that $\partial P + P\partial = \rho - \mathrm{id}$. We calculate

$$\begin{array}{rcl} (P\partial)(\sigma) & = & \sum\limits_{i<j} (-1)^i (-1)^j \cdot \epsilon_{n-i-1} \cdot (\sigma \circ \pi) \circ [v_0, \dots, v_i, w_n, \dots, \widehat{w_j}, \dots, w_i] \\ & & \quad + \sum\limits_{i>j} (-1)^{i-1} (-1)^j \cdot \epsilon_{n-i} \cdot (\sigma \circ \pi) \circ [v_0, \dots, \widehat{v_j}, \dots, v_i, w_n, \dots, w_i], \end{array}$$

On the other hand we have

$$\begin{array}{rcl} (\partial P)(\sigma) & = & \sum\limits_{j \leq i} (-1)^i (-1)^j \cdot \epsilon_{n-i} \cdot (\sigma \circ \pi) \circ [v_0, \dots, \widehat{v_j}, \dots, v_i, w_n, \dots, w_i] \\ & & \quad + \sum\limits_{j \geq i} (-1)^i (-1)^{i+1+n-j} \cdot \epsilon_{n-i} \cdot (\sigma \circ \pi) \circ [v_0, \dots, v_i, w_n, \dots, \widehat{w_j}, \dots, w_i]. \end{array}$$

Now using that $\epsilon_{n-i} = (-1)^{n-i} \epsilon_{n-i-1}$ we see that the sum over the $j \neq i$ terms agrees precisely with $-P\partial$ calculated above (with the roles of i and j reversed). It remains to show that the sum over the $i = j$ terms equals $(\rho - \mathrm{id})(\sigma)$. This is indeed the case since

$$\begin{array}{l} \text{sum of the } i = j \text{ summands in the above formula for } (\partial P)(\sigma) \\ \quad = \; \epsilon_n \cdot (\sigma \circ \pi) \circ [w_n, \dots, w_0] + \sum\limits_{i=1}^{n} \epsilon_{n-i} \cdot (\sigma \circ \pi) \circ [v_0, \dots, v_{i-1}, w_n, \dots, w_i] \\ \qquad + \sum\limits_{i=0}^{n-1} (-1)^{n+i+1} \epsilon_{n-i} \cdot (\sigma \circ \pi) \circ [v_0, \dots, v_i, w_n, \dots, w_{i+1}] - (\sigma \circ \pi) \circ [v_0, \dots, v_n] \\ \quad \underset{\uparrow}{=} \; \epsilon_n \cdot (\sigma \circ \pi) \circ [w_n, \dots, w_0] - (\sigma \circ \pi) \circ [v_0, \dots, v_n] \; = \; (\rho - \mathrm{id})(\sigma). \end{array}$$

in the second sum we substitute $j = i + 1$, we obtain a new sign $(-1)^{n+j} \epsilon_{n-j+1} = -\epsilon_{n-j}$, thus we see that the two sums over i cancel

We have now completed the proof that $\partial P + P\partial = \rho - \mathrm{id}$, in particular we have shown the claim and thus also the proposition. ■

Let $k \in \mathbb{N}$ be odd. It follows from Proposition 81.8 that given any topological space X and any $\varphi \in \mathrm{H}^k(X; \mathbb{Z})$ we have $\varphi \cup \varphi = -(1)^{k \cdot k} \cdot \varphi \cup \varphi$, put differently, we have $2 \cdot (\varphi \cup \varphi) = 0$. Thus we see that $\varphi \cup \varphi$ is 2-torsion. The next lemma makes the amusing observation that for $k = 1$ this "square" is in fact always zero.

Lemma 81.9. *Let X be a topological space. For any $\varphi \in \mathrm{H}^1(X; \mathbb{Z})$ we have $\varphi \cup \varphi = 0$.*

Remark. Note that there is no higher-dimensional analogue of Lemma 81.9. For example on [**Hat02**, p. 403] it is shown that given any odd k with $k \geq 3$ there exists a cohomology class $\gamma \in \mathrm{H}^k(\mathbb{RP}^\infty \times \mathbb{RP}^\infty; \mathbb{Z})$ with $\gamma^m \neq 0$ for all $m \in \mathbb{N}$.

Proof. Let X be a topological space and let $\varphi \in \mathrm{C}^1(X;\mathbb{Z})$ be a cocycle. We need to show that $\varphi \cup \varphi \in \mathrm{C}^2(X;\mathbb{Z})$ is a coboundary. We consider the 1-dimensional cochain

that is determined by
$$\begin{array}{rcl} \mu\colon\ \mathrm{C}_1(X) & \to & \mathbb{Z} \\ (\sigma\colon \Delta^1 \to X) & \mapsto & \tfrac{1}{2}(\underbrace{\varphi(\sigma)-\varphi(\sigma)^2}_{\in 2\mathbb{Z}}). \end{array}$$

It remains to prove the following claim.

Claim. We have $\varphi \cup \varphi = \delta\mu\colon\ \mathrm{C}_2(X) \to \mathbb{Z}$.

We only need to prove this equality for singular 2-simplices. So suppose we are given $\tau\colon \Delta^2 \to X$. Given $j \in \{0,1,2\}$ we write $i_j = i_j^2\colon \Delta^1 \to \Delta^2$. Note that

$$(*) \qquad \varphi(\tau\circ i_0) - \varphi(\tau\circ i_1) + \varphi(\tau\circ i_2) \;=\; \varphi(\partial\tau) \underset{\substack{\uparrow\\ \text{definition of } \delta\varphi}}{=} (\delta\varphi)(\tau) \underset{\substack{\uparrow\\ \text{since } \varphi \text{ is a cocycle}}}{=} 0.$$

Now we compute that

$$\begin{aligned} (\delta\mu)(\tau) &= \mu(\partial\tau) = \mu\big((\tau\circ i_0) - (\tau\circ i_1) + (\tau\circ i_0)\big) \\ &= \tfrac{1}{2}\big(\varphi(\tau\circ i_0) - \varphi(\tau\circ i_1) + \varphi(\tau\circ i_2) + \varphi(\tau\circ i_0)^2 - \varphi(\tau\circ i_1)^2 + \varphi(\tau\circ i_2)^2\big) \\ &\underset{\substack{\uparrow\\ \text{by } (*) \text{ we have } \varphi(\tau\circ i_1) = \varphi(\tau\circ i_0) + \varphi(\tau\circ i_2)}}{=} \tfrac{1}{2}\big(\varphi(\tau\circ i_0)^2 - (\varphi(\tau\circ i_0) + \varphi(\tau\circ i_2))^2 + \varphi(\tau\circ i_2)^2\big) \\ &\underset{\substack{\uparrow\\ \text{elementary algebra}}}{=} \varphi(\tau\circ i_2)\cdot\varphi(\tau\circ i_0) \underset{\substack{\uparrow\\ \text{definition of the cup-product}}}{=} (\varphi\cup\varphi)(\tau). \end{aligned}$$

This concludes the proof of the claim and thus of the lemma. ■

We continue with the following lemma that shows that the cup product behaves well under maps between topological spaces.

Lemma 81.10. *Let R be a commutative ring.*

(1) *Let $f\colon X \to Y$ be a map between topological spaces. For any $c \in \mathrm{H}^p(Y;R)$ and $d \in \mathrm{H}^q(Y;R)$ we have*
$$f^*(c) \cup f^*(d) \;=\; f^*(c \cup d).$$
In particular the map
$$f^*\colon\ \mathrm{H}^*(Y;R) \;\to\; \mathrm{H}^*(X;R)$$
is a homomorphism of graded rings[1174] *which satisfies $f^*(1_Y) = 1_X$.*

(2) *The maps*
$$\begin{array}{rcl} X & \mapsto & (\mathrm{H}^*(X;R), \cup) \\ (f\colon X \to Y) & \mapsto & (f^*\colon\ \mathrm{H}^*(Y;R) \to \mathrm{H}^*(X;R)) \end{array}$$
define a contravariant functor from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{GrRing}$ of graded rings.

PROOF. The lemma follows basically immediately from the definitions. But for completeness' sake we write down the proof. So let $c = [\varphi] \in \mathrm{H}^p(Y;R)$ and let $b = [\psi] \in \mathrm{H}^q(Y;R)$. Furthermore let $\sigma\colon \Delta^{p+q} \to X$ be a singular $(p+q)$-chain. Then

$$\begin{array}{rcl}(f^*\varphi \cup f^*\psi)(\sigma) & = & (f^*\varphi)(\sigma \circ [v_0,\dots,v_p]) \cdot (f^*\psi)(\sigma \circ [v_p,\dots,v_{p+q}]) \\ & = & \varphi(f \circ (\sigma \circ [v_0,\dots,v_p])) \cdot \psi(f \circ (\sigma \circ [v_p,\dots,v_{p+q}])) \\ & = & \varphi((f \circ \sigma) \circ [v_0,\dots,v_p)) \cdot \psi((f \circ \sigma) \circ [v_p,\dots,v_{p+q}]) \\ & = & (\varphi \cup \psi)(f \circ \sigma) \;=\; (f^*(\varphi \cup \psi))(\sigma).\end{array}$$

Finally note that it follows immediately from the definitions of 1_X, 1_Y and f^* that we have the equality $f^*(1_Y) = 1_X$. Statement (2) follows immediately from statement (1) and the fact that cohomology groups with R-coefficients form a contravariant functor of abelian groups. ■

We can use Lemma 81.10 to calculate the cup product of the wedge of finitely many spheres.

Lemma 81.11. *For any wedge of finitely many spheres all cup products in degrees ≥ 1 are zero.*

PROOF. Let $S^{r_1}\vee\dots\vee S^{r_k}$ be the wedge of finitely many spheres and let R be a commutative ring. For each $i \in \{1,\dots,k\}$ we denote by $p_i\colon S^{r_1} \vee \dots \vee S^{r_k} \to S^{r_i}$ the obvious projection map. By Proposition 74.17 we know that for every $n \in \mathbb{N}$ the map

$$p_1^* \oplus \dots \oplus p_k^*\colon\ \mathrm{H}^n(S^{r_1};R) \oplus \dots \oplus \mathrm{H}^n(S^{r_k};R) \ \xrightarrow{\cong}\ \mathrm{H}^n(S^{r_1} \vee \dots \vee S^{r_k};R)$$

is an isomorphism. It remains to show that for any choice of $i,j \in \{1\dots,k\}$ with $r_i > 0$ and $r_j > 0$ we have $p_i^*([S^{r_i}]^*) \cup p_j^*([S^{r_j}]^*) = 0$. We prove the statement for $i \neq j$. The proof of the statement for $i = j$ is basically the same. So suppose that $i \neq j$. We denote by $g\colon S^{r_1}\vee\dots\vee S^{r_k} \to S^{r_i}\vee S^{r_j}$ the obvious projection map and we denote by $q_i\colon S^{r_i}\vee S^{r_j} \to S^{r_i}$ and $q_j\colon S^{r_j} \vee S^{r_j} \to S^{r_j}$ the obvious projection maps. Then

$$p_i^*([S^{r_i}]^*) \cup p_j^*([S^{r_j}]^*) \underset{\substack{\uparrow \\ \text{since } q_i \circ g = p_i \text{ and } q_j \circ g = p_j}}{=} g^*(q_i^*([S^{r_i}]^*)) \cup g^*(q_j^*([S^{r_j}]^*)) \underset{\substack{\uparrow \\ \text{Lemma 81.10}}}{=} g^*(\overbrace{q_i^*([S^{r_i}]^*) \cup q_j^*([S^{r_j}]^*)}^{\in \mathrm{H}^{r_i+r_j}(S^{r_i}\vee S^{r_j})=0}) \;=\; 0.$$

We have thus proved that all cup products in degrees ≥ 1 are zero. ■

Later on, after we will have found methods for computing the cup product, we will often use the following corollary to show that certain topological spaces are not homotopy equivalent.

[1174]Recall that a *graded ring* is a ring R together with a decomposition $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ such that for any $m,n \in \mathbb{N}_0$ the multiplication map restricts to a map $R_m \times R_n \to R_{m+n}$. A homomorphism $f\colon R \to S$ between graded rings is a ring homomorphism with $f(R_n) \subset S_n$ for all $n \in \mathbb{N}_0$

Corollary 81.12. *Let X and Y be topological spaces. If X and Y are homotopy equivalent, then for any commutative ring R there exists an isomorphism*

$$(\mathrm{H}^*(Y;R),\cup) \xrightarrow{\cong} (\mathrm{H}^*(X;R),\cup)$$

of graded rings.

Proof. Let $f\colon X \to Y$ be a homotopy equivalence. In Lemma 73.13 we saw that the induced maps $f^*\colon \mathrm{H}^n(Y;R) \to \mathrm{H}^n(X;R)$ on cohomology *groups* are isomorphisms. But it follows from Lemma 81.10 that these isomorphisms preserve the ring structure, i.e. the map f^* is in fact an isomorphism of graded rings. ■

81.3. **The cup product of the torus.** Evidently our goal now is to determine the cup product of some of our favorite different topological spaces. We start out with two elementary observations regarding the cup product.

Observation 81.13. *Let R be a commutative ring.*

(1) *Let X be a path-connected topological space. It follows from Proposition 73.11 that $\mathrm{H}^0(X;R) = R\cdot 1_X$. For any $r\cdot 1_X \in \mathrm{H}^0(X;R)$ and $\varphi \in \mathrm{H}^k(X;R)$ we have*

$$(r\cdot 1_X)\cup\varphi \underset{\substack{\uparrow \\ \text{since } \cup \text{ is } R\text{-bilinear}}}{=} r\cdot(1_X\cup\varphi) \underset{\substack{\uparrow \\ \text{Proposition 81.7}}}{=} r\cdot\varphi \underset{\substack{\uparrow \\ \text{the same argument backwards}}}{=} \varphi\cup(r\cdot 1_X).$$

(2) *If M is an m-dimensional topological manifold, then for any p,q with $p+q>m$ the cup product $\mathrm{H}^p(M;R)\times\mathrm{H}^q(M;R)$ is zero, since it takes values in the group $\mathrm{H}^{p+q}(M;R)$, which is zero by Theorem 87.3.*

(3) *By Proposition 81.7 the cup product is R-bilinear, so it suffices to determine the cup product on a generating system.*

In this section we want to determine the cup product on the torus $T = S^1\times S^1$. By Observation 81.13 it remains to determine the cup product

$$\cup\colon\ \mathrm{H}^1(T;\mathbb{Z})\times\mathrm{H}^1(T;\mathbb{Z}) \ \to\ \mathrm{H}^2(T;\mathbb{Z}).$$

We first need to understand the cohomology groups $\mathrm{H}^1(T;\mathbb{Z})$ and $\mathrm{H}^2(T;\mathbb{Z})$ explicitly. We recall some notation and results from before:

(1) We denote by p respectively q the projection $S^1\times S^1\to S^1$ onto the first respectively second factor. Furthermore, we recall that on page 1825 we introduced a cochain $\theta := \theta_{\mathbb{Z}} \in \mathrm{C}^1;\mathbb{Z})$, we showed in Lemma 73.10 that θ is a cocycle and we showed on page 1848 that θ represents a generator of $\mathrm{H}^1(S^1;\mathbb{Z})$.

(2) In Lemma 74.9 we showed that $\mathrm{H}^1(T;\mathbb{Z})\cong\mathbb{Z}^2$ and that the two cohomology classes $\alpha := p^*([\theta])$ and $\beta := q^*([\theta])$ form a basis for $\mathrm{H}^1(S^1\times S^1;\mathbb{Z})$.

(3) On page 1717 we showed that the cycle $-\sigma_1+\sigma_2$, with the notation from Figure 1054, represents the fundamental class of T.

(4) In Lemma 74.9 and 75.16 we showed that

$$\begin{array}{rcl}\mathrm{H}^2(T;\mathbb{Z}) & \to & \mathbb{Z}\\ [\varphi] & \mapsto & \langle\varphi,[T]\rangle = \varphi(-\sigma_1+\sigma_2)\end{array}$$

is in fact an isomorphism. We will use this isomorphism to make the identification $\mathrm{H}^2(T;\mathbb{Z}) = \mathbb{Z}$.

By Oberservation 81.13 (3) and the above discussion it suffices to determine all entries in the following matrix

$$\begin{pmatrix} \langle \alpha \cup \alpha, [T] \rangle & \langle \alpha \cup \beta, [T] \rangle \\ \langle \beta \cup \alpha, [T] \rangle & \langle \beta \cup \beta, [T] \rangle \end{pmatrix}.$$

Now we determine these four entries separately.

(1) We first consider the entry $\langle \alpha \cup \beta, [T] \rangle$. Fortunately we already did all the work on page 1982, more precisely, we have

$$\langle \alpha \cup \beta, [T] \rangle = (p^*\theta \cup q^*\theta)(-\sigma_1 + \sigma_2) = (p^*\theta \cup q^*\theta)(-\sigma_1) + (p^*\theta \cup q^*\theta)(\sigma_2) \underset{\substack{\uparrow \\ \text{see calculations on page 1982}}}{=} 0 + 1 = 1.$$

(2) We could do the same argument as in (1) with α and β reversed. But it is easier to apply Proposition 81.8 which gives us that $\beta \cup \alpha = (-1)^{1\cdot 1} \cdot \alpha \cup \beta = -\alpha \cup \beta = -1$.

(3) There are many different ways for showing that $\alpha \cup \alpha = 0$:
   (a) We could do the calculation as in (1), but that is the least efficient approach.
   (b) We could use the symmetry from Proposition 81.8 to argue that $\alpha \cup \alpha = -\alpha \cup \alpha$, and hence $\alpha \cup \alpha = 0$ since $\mathrm{H}^2(T;\mathbb{Z})$ is torsion-free.
   (c) We could just use Lemma 81.9 and remove all the fun.
   (d) We can also argue that

$$\langle \alpha \cup \alpha, [T] \rangle = \langle [p^*\theta] \cup [p^*\theta], [T] \rangle \underset{\substack{\uparrow \\ \text{by Lemma 81.10}}}{=} \langle p^*([\theta] \cup [\theta]), [T] \rangle \underset{\substack{\uparrow \\ \text{since } [\theta]\cup[\theta] \in \mathrm{H}^2(S^1;\mathbb{Z}) = 0}}{=} 0.$$

(4) The same way as in (3) we see that $\beta \cup \beta = 0$.

Summarizing we have proved the following lemma.

Lemma 81.14. *Let T be the 2-dimensional torus. We use the above notation. The cup product $\mathrm{H}^1(T;\mathbb{Z}) \times \mathrm{H}^1(T;\mathbb{Z}) \to \mathrm{H}^2(T;\mathbb{Z})$ with respect to the basis α, β of $\mathrm{H}^1(T;\mathbb{Z})$ and the identification $\mathrm{H}^2(T;\mathbb{Z}) = \mathbb{Z} \cdot [T]^* = \mathbb{Z}$[1175] is given by the matrix*

$$\begin{pmatrix} \alpha \cup \alpha & \alpha \cup \beta \\ \beta \cup \alpha & \beta \cup \beta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus we have shown that the cup product on cohomology is in general non-zero and that it is in general not commutative.

Remark.

(1) We have just shown that the 2-cocycle $p^*(\theta) \cup q^*(\theta) \in \mathrm{C}^2(T;\mathbb{Z})$ represents a generator of $\mathrm{H}^2(T;\mathbb{Z})$. We have thus answered Question 74.10.

(2) We have now performed enough calculations to see that the cohomology rings contain more information than the homology or the cohomology groups. More precisely, it

[1175] Here $[T]^* \in \mathrm{H}^2(T;\mathbb{Z})$ denotes the dual fundamental class that we introduced on page 1873, it is uniquely determined by $\langle [T]^*, [T] \rangle = 1$.

follows from Lemma 46.12 and Propositions 43.4 and 47.8 that for the 2-dimensional torus $S^1 \times S^1$ and for $S^1 \vee S^1 \vee S^2$ we have

$$\mathrm{H}_i(S^1 \times S^1; \mathbb{Z}) \;\cong\; \mathrm{H}_i(S^1 \vee S^1 \vee S^2; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2, \\ \mathbb{Z}^2, & \text{if } i = 1, \\ 0, & \text{if } i \geq 3. \end{cases}$$

We deduce from the Universal Coefficient Theorem 75.13 for Cohomology Groups that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ also have isomorphic cohomology groups. In Lemma 81.14 we saw that the cup product on $\mathrm{H}^1(S^1 \times S^1; \mathbb{Z})$ is non-zero, whereas we showed in Lemma 81.11 that the cup product on $\mathrm{H}^1(S^1 \vee S^1 \vee S^2; \mathbb{Z})$ is zero.[1176]

In Exercise 81.2 we will use the above calculation of the cup product of the torus to prove the following slightly subtle lemma.

Lemma 81.15. *Let $f\colon T \to T$ be a homeomorphism of the torus $T = S^1 \times S^1$. If f is orientation-preserving, then*

$$\det\big(f_*\colon\ \mathrm{H}_1(T;\mathbb{Z}) \to \mathrm{H}_1(T;\mathbb{Z})\big) \;=\; \det\big(f^*\colon\ \mathrm{H}^1(T;\mathbb{Z}) \to \mathrm{H}^1(T;\mathbb{Z})\big) \;=\; +1.$$

Conversely, if f is orientation-reversing, then both determinants equal -1.

81.4. **The cup product of the surface of genus 2.** In this section we calculate the cup product for the surface Σ of genus 2. By Observation 81.13 it remains to determine the cup product

$$\mathrm{H}^1(\Sigma;\mathbb{Z}) \times \mathrm{H}^1(\Sigma;\mathbb{Z}) \;\to\; \mathrm{H}^2(\Sigma;\mathbb{Z})$$

on a basis of $\mathrm{H}^1(\Sigma;\mathbb{Z})$. We recall that on page 1850 we introduced for $i = 1, 2$ a map $p_i\colon \Sigma \to T = S^1 \times S^1$. We refer to Figure 1163 on the right for a reminder of the definition of the maps. We denote by $\alpha, \beta \in \mathrm{H}^1(T;\mathbb{Z})$ the basis from Lemma 74.9 that we had also

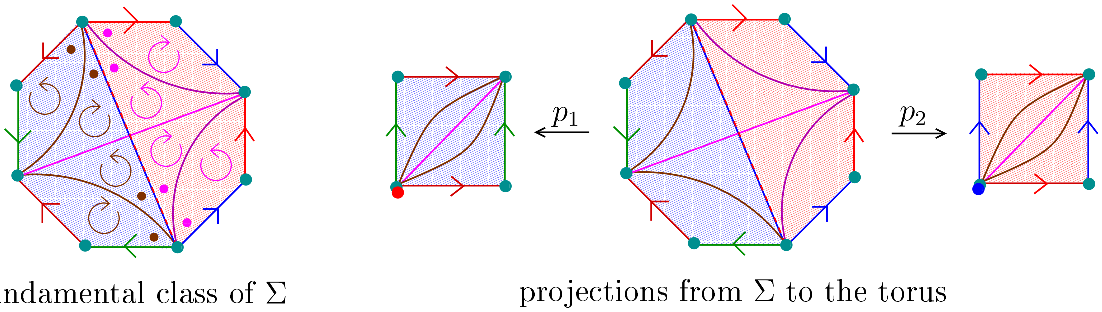


Figure 1163

used in the previous section.

In Lemma 74.11 we saw that the map

$$p_1^* \oplus p_2^*\colon\ \mathrm{H}^1(T;\mathbb{Z}) \oplus \mathrm{H}^1(T;\mathbb{Z}) \;\to\; \mathrm{H}^1(\Sigma;\mathbb{Z})$$

[1176] Of course we do not need to use cup products to distinguish the torus $S^1 \times S^1$ and the wedge $S^1 \vee S^1 \vee S^2$, this can already be done using fundamental groups. Nonetheless it is encouraging to see that we found a new way to prove this statement.

is an isomorphism. We deduce that a basis for $\mathrm{H}^1(\Sigma;\mathbb{Z}) \cong \mathbb{Z}^4$ is given by $\alpha_i := p_i^*(\alpha)$, $\beta_i := p_i^*(\beta)$, $i = 1, 2$. Now we can use the definition of the cup product and the explicit description of a representative of the fundamental class $[\Sigma] \in \mathrm{H}_2(\Sigma;\mathbb{Z})$ in Figure 1163 on the left (see also Figure 1055 for a more detailed discussion) to compute the cup product $\mathrm{H}^1(\Sigma;\mathbb{Z}) \times \mathrm{H}^1(\Sigma;\mathbb{Z}) \to \mathrm{H}^2(\Sigma;\mathbb{Z}) = \mathbb{Z}$.

A not particularly exciting calculation leads us eventually to the following lemma.[1177]

Lemma 81.16. *Let Σ be the surface of genus 2. We use the above notation. The cup product $\mathrm{H}^1(\Sigma;\mathbb{Z}) \times \mathrm{H}^1(\Sigma;\mathbb{Z}) \to \mathrm{H}^2(\Sigma;\mathbb{Z})$ with respect to the basis $\alpha_1, \beta_1, \alpha_2, \beta_2$ of $\mathrm{H}^1(\Sigma;\mathbb{Z})$ and the identification $\mathrm{H}^2(\Sigma;\mathbb{Z}) = \mathbb{Z} \cdot [\Sigma]^* = \mathbb{Z}$ is given by the matrix*

$$\begin{pmatrix} \alpha_1 \cup \alpha_1 & \alpha_1 \cup \beta_1 & \alpha_1 \cup \alpha_2 & \alpha_1 \cup \beta_2 \\ \beta_1 \cup \alpha_1 & \beta_1 \cup \beta_1 & \beta_1 \cup \alpha_2 & \beta_1 \cup \beta_2 \\ \alpha_2 \cup \alpha_1 & \alpha_2 \cup \beta_1 & \alpha_2 \cup \alpha_2 & \alpha_2 \cup \beta_2 \\ \beta_2 \cup \alpha_1 & \beta_2 \cup \beta_1 & \beta_2 \cup \alpha_2 & \beta_2 \cup \beta_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

An analogous statement holds for any surface of genus $g \geq 1$, i.e. the cup product on the first cohomology of the surface of genus $g \geq 1$ can be represented by a block diagonal matrix consisting of g copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Remark. In the next section we will obtain a more conceptual proof of Lemma 81.16.

Before we continue with more calculations of cup products we return to a question that we had asked earlier on.

Question 69.13.

(2) *Let $g \geq 2$ and let $k \geq 2$. Does there exist a self-map $f\colon \Sigma_g \to \Sigma_g$ of degree k?*[a]

[a]Recall that if $f\colon M \to N$ is a map between two closed, oriented connected, non-empty n-dimensional topological manifolds, then the degree $\deg(f) \in \mathbb{Z}$ is defined as the unique integer $\deg(f)$ that satisfies $f_*([M]) = \deg(f) \cdot [N]$.

Now we have the tools to answer this question. In this context it is useful to introduce the following definition.

Definition. A closed oriented connected non-empty topological manifold M is called *flexible* if given any $k \in \mathbb{Z}$ there exists a map $f\colon M \to M$ of degree k.

Example. In Lemma 45.10 we saw that S^1 is flexible. The combination of this fact with Lemma 45.12 shows that in fact any sphere S^n, $n \geq 1$ is flexible.

Now we can determine which closed oriented connected 2-dimensional smooth manifolds are flexible. More precisely, we have the following proposition which gives a negative answer to Question 69.13 (2).

[1177]We have to evaluate the cup products of the explicit 1-cochains on the eight singular 2-simplices illustrated in Figure 1163. It is straightforward to see that all 16 cup products evaluate to zero on the four central singular 2-simplices. For the remaining cup products the calculation is almost the same as for the torus on page 1982.

Proposition 81.17. *Let $g \geq 1$. We denote by Σ_g the surface of genus g.*

(1) *The torus $\Sigma_1 = S^1 \times S^1$ is flexible.*
(2) *For $g \geq 2$ the surface Σ_g is not flexible. In fact if $k \neq -1, 0, 1$, then there is no map $\Sigma_g \to \Sigma_g$ of degree k.*

Proof.

(1) We first show that the torus $S^1 \times S^1$ is flexible. So let $k \in \mathbb{Z}$. We consider the map $f\colon S^1 \to S^1$ that is given by $z \mapsto z^k$. We claim that $f \times \mathrm{id}\colon S^1 \times S^1 \to S^1 \times S^1$ has degree k. In fact by the naturality of the short exact sequence of the Künneth-Theorem 58.8 we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_1(S^1) \otimes \mathrm{H}_1(S^1) & \xrightarrow[\cong]{\Theta} & \mathrm{H}_2(S^1 \times S^1) \\ \downarrow{\scriptstyle f_* \otimes \mathrm{id}} & & \downarrow{\scriptstyle (f \times \mathrm{id})_*} \\ \mathrm{H}_1(S^1) \otimes \mathrm{H}_1(S^1) & \xrightarrow[\cong]{\Theta} & \mathrm{H}_2(S^1 \times S^1) \end{array}$$

where the horizontal maps are isomorphisms. By Lemma 45.10 the map on the left is given by multiplication by k. Hence the map on the right is also given by multiplication by k.

(2) Now let $g \geq 2$ and let $f\colon \Sigma_g \to \Sigma_g$ be a map of degree $k \neq 0$. We have to show that $k \in \{\pm 1\}$. We write $\pi := \pi_1(\Sigma_g)$. We start out with the following claim.

Claim. The map $f_*\colon \pi \to \pi$ is an epimorphism.

First note that the argument in the proof of Proposition 69.9 shows that $f_*(\pi)$ is a finite-index subgroup of π. (Here we used that $k \neq 0$.) We denote by $n \in \mathbb{N}$ the index of the subgroup $f_*(\pi) \subset \pi$. In the proof of Proposition 31.17 we used the multiplicativity of the Euler characteristic under finite covers to show that the abelianization of $f_*(\pi)$ is isomorphic to $\mathbb{Z}^{2n\cdot(g-1)+2}$. On the other hand the epimorphism $\pi \to f_*(\pi)$ induces evidently an epimorphism of the abelianization of π onto the abelianization of $f_*(\pi)$, i.e. we have an epimorphism from $\mathbb{Z}^{2g}$ onto $\mathbb{Z}^{2n\cdot(g-1)+2}$. Since $g \geq 2$ this is only possible if $n = 1$. ⊞

Claim. The map $f^*\colon \mathrm{H}^1(\Sigma_g;\mathbb{Z}) \to \mathrm{H}^1(\Sigma_g;\mathbb{Z})$ is an isomorphism.

By the first claim we know that the map $f_*\colon \pi \to \pi$ is an epimorphism. It follows from the naturality of the Hurewicz Isomorphism, see Propositions 52.3 and 52.2 (4), that the map $f_*\colon \mathrm{H}_1(\Sigma_g) \to \mathrm{H}_1(\Sigma_g)$ is also an epimorphism. But any epimorphism from a finitely generated abelian group to itself is in fact an isomorphism[1178], so we see that $f_*\colon \mathrm{H}_1(\Sigma_g;\mathbb{Z}) \to \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ is in fact an isomorphism. By Proposition 75.18 the evaluation homomorphism $\mathrm{ev}\colon \mathrm{H}^1(\Sigma_g;\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_1(\Sigma_g;\mathbb{Z}),\mathbb{Z})$ is a

[1178] Why is that?

natural isomorphism, i.e. we have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(\Sigma_g;\mathbb{Z}) \arrow[r, "\mathrm{ev}", "\cong"'] \arrow[d, "f^*"] & \mathrm{Hom}(\mathrm{H}_1(\Sigma_g;\mathbb{Z}),\mathbb{Z}) \arrow[d, "\cong"', "(f_*)^*"] \\
\mathrm{H}^1(\Sigma_g;\mathbb{Z}) \arrow[r, "\mathrm{ev}", "\cong"'] & \mathrm{Hom}(\mathrm{H}_1(\Sigma_g;\mathbb{Z}),\mathbb{Z}).
\end{tikzcd}$$

The right-hand map is an isomorphism, it follows from the commutative diagram that the left-hand map $f^*\colon \mathrm{H}^1(\Sigma_g;\mathbb{Z}) \to \mathrm{H}^1(\Sigma_g;\mathbb{Z})$ is also an isomorphism. ⊞

By Lemma 81.16 we know that there exist $a, b \in \mathrm{H}^1(\Sigma_g;\mathbb{Z})$ with $a \cup b = [\Sigma_g]^*$. Since $f^*\colon \mathrm{H}^1(\Sigma_g;\mathbb{Z}) \to \mathrm{H}^1(\Sigma_g;\mathbb{Z})$ is an isomorphism we can in particular find cohomology classes $c, d \in \mathrm{H}^1(\Sigma_g;\mathbb{Z})$ with $f^*(c) = a$ and $f^*(d) = b$. Now we have

$$[\Sigma_g]^* = a \cup b = f^*(c) \cup f^*(d) \underset{\substack{\uparrow \\ \text{Lemma 81.10}}}{=} f^*(c \cup d) \overset{\substack{\text{since } c\cup d \in \mathrm{H}^2(\Sigma_g;\mathbb{Z}) \text{ and since } [\Sigma_g]^* \in \mathrm{H}^2(\Sigma_g;\mathbb{Z}) \text{ is a generator such an } m \text{ exists} \\ \downarrow}}{=} f^*(m \cdot [\Sigma_g]^*) \underset{\substack{\uparrow \\ \text{by Lemma 75.17 we have } f^*([\Sigma_g]^*) = \deg(f)\cdot[\Sigma_g]^*}}{=} \deg(f) \cdot m \cdot [\Sigma_g]^*.$$

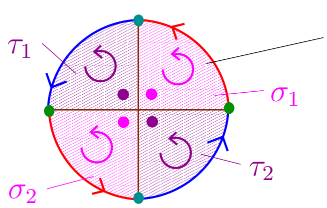
Evidently this is only possible if $\deg(f) \in \{\pm 1\}$. ■

81.5. **The cup product of the real projective plane** $(*)$**.** Now we want to understand the cup product in the $\mathbb{F}_2$-cohomology of the real projective plane $\mathbb{RP}^2$. Recall that on page 1845 we saw that $\mathrm{H}^i(\mathbb{RP}^2;\mathbb{F}_2) = \mathbb{F}_2$[1179] for $i = 0, 1, 2$ and that the $\mathbb{F}_2$-cohomology vanishes in all other dimensions. By Observation 81.13 it remains to determine the cup product on the first cohomology.

To compute the cup product we need to get a good understanding of the cohomology groups. Recall that on page 1731 we had seen that $(\sigma_1 + \sigma_2 + \tau_1 + \tau_2) \otimes 1 \in \mathrm{C}_2(\mathbb{RP}^2;\mathbb{F}_2)$ is a representative of the $\mathbb{F}_2$-fundamental class $[\mathbb{RP}^2]$ of $\mathbb{RP}^2 = \overline{B}^2/\sim$.[1180] (See Figure 1164 for the definition of $\sigma_1, \sigma_2, \tau_1$ and τ_2.) It follows from Proposition 75.19 and the above that the map

$$\begin{array}{rcl}
\eta\colon\ \mathrm{H}^2(\mathbb{RP}^2;\mathbb{F}_2) & \to & \mathbb{F}_2 \\
{[\varphi \in \mathrm{C}^2(\mathbb{RP}^2;\mathbb{F}_2)]} & \mapsto & [\varphi(\sigma_1) + \varphi(\sigma_2) + \varphi(\tau_1) + \varphi(\tau_2)]
\end{array}$$

is an isomorphism.



projective plane $\mathbb{RP}^2 = \overline{B}^2/\sim$

the 2-cycle $(\sigma_1 + \sigma_2 + \tau_1 + \tau_2) \otimes 1 \in \mathrm{C}_2(\mathbb{RP}^2;\mathbb{F}_2)$
represents the $\mathbb{F}_2$-fundamental class
of the real projective plane

FIGURE 1164

[1179] Why is it safe to write $\mathrm{H}^i(\mathbb{RP}^2;\mathbb{F}_2) = \mathbb{F}_2$? Would it not be much more appropriate to write $\mathrm{H}^i(\mathbb{RP}^2;\mathbb{F}_2) \cong \mathbb{F}_2$?

[1180] Here we use the identification $\mathbb{RP}^2 = \overline{B}^2/\sim$ from page 196.

Now we also need to find an explicit representative of $x \in \mathrm{H}^1(\mathbb{RP}^2;\mathbb{F}_2)$. We will find such a representative in a similar fashion to the construction of the singular 1-cocycle $\theta_{\mathbb{Z}} \in \mathrm{H}^1(S^1;\mathbb{Z})$ on page 1826. More precisely, we consider the map

$$\begin{array}{rcl} f\colon S^2 & \to & \mathbb{F}_2 \\ (x,y,z) & \mapsto & \begin{cases} 1, & \text{if } (z>0) \text{ or } (z=0 \text{ and } y>0) \text{ or } (z=y=0 \text{ and } x=1), \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

Note that for any point $P \in S^2$ we have $f(-P) = 1 - f(P)$.[1181] For a singular 1-simplex $\sigma\colon \Delta^1 \to \mathbb{RP}^2$ we pick a lift $\widetilde{\sigma}\colon \Delta^1 \to S^2$ of σ and we define

$$\varphi(\sigma) \;:=\; f\big(\widetilde{\sigma}(0,1)\big) - f\big(\widetilde{\sigma}(1,0)\big) \;\in\; \mathbb{F}_2.$$

We refer to Figure 1165.

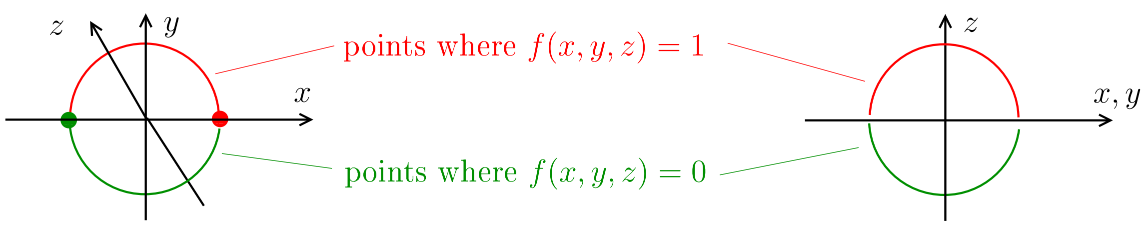


FIGURE 1165

Note that φ is well-defined since the only other lift of σ is given by multiplying $\widetilde{\sigma}$ by -1,[1182] but then it follows from $f(-P) = 1 - f(P)$ that the difference of the evaluations at the endpoints does not change. (Here we use that we work with $\mathbb{F}_2$-coefficients instead of $\mathbb{Z}$-coefficients.) The argument of Lemma 73.10 can easily be adapted to show that the singular 1-cochain $\varphi\colon \mathrm{C}_1(\mathbb{RP}^2) \to \mathbb{F}_2$ is a cocycle.

Example. We consider the map $\mu\colon \Delta^1 \to \mathbb{RP}^2 = S^2/\sim$ given by $\mu(1-t,t) = [e^{\pi \mathrm{i} t}]$. Then $\mu \in \mathrm{C}_1(\mathbb{RP}^2)$. Thus we can apply the Kronecker pairing $\mathrm{H}^1(\mathbb{RP}^2;\mathbb{F}_2) \times \mathrm{H}_1(\mathbb{RP}^2) \to \mathbb{F}_2$ from Lemma 74.6 to $[\varphi] \in \mathrm{H}^1(\mathbb{RP}^2;\mathbb{F}_2)$ and $[\mu] \in \mathrm{H}_1(\mathbb{RP}^2)$ and we calculate that

$$\langle [\varphi],[\mu]\rangle \;=\; \varphi(\mu) \;\underset{\substack{\uparrow \\ \text{since a lift of } \mu(1-t,t)=[e^{\pi \mathrm{i} t}] \text{ to } S^2 \text{ is given by } \widetilde{\mu}(1-t,t)=e^{\pi \mathrm{i} t}}}{=}\; f(e^{\pi \mathrm{i}}) - f(e^0) \;=\; 1 \in \mathbb{F}_2.$$

Since this is non-zero we have now actually shown two statements at once: the cohomology class $[\varphi] \in \mathrm{H}^1(\mathbb{RP}^2;\mathbb{F}_2)$ is non-zero and we have also certified that the homology class $[\mu] \in \mathrm{H}_1(\mathbb{RP}^2)$ is non-zero.[1183]

Now we have the tools to prove the following lemma.

[1181]In fact for the subsequent discussion any map $f\colon S^2 \to \mathbb{F}_2$ with this property would work. We just picked this example for concreteness.

[1182]Why is that true?

[1183]Do you know any other methods for showing that $[\mu] \neq 0 \in \mathrm{H}_1(\mathbb{RP}^2)$?

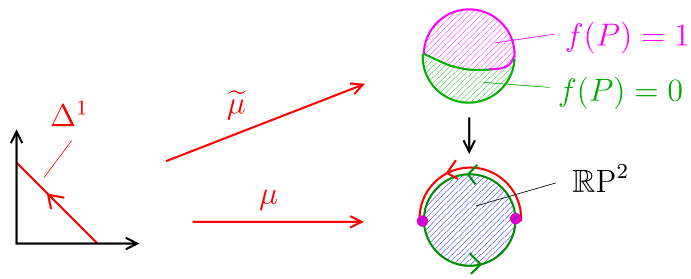


FIGURE 1166

Lemma 81.18. *Under the identifications* $\mathrm{H}^i(\mathbb{RP}^2;\mathbb{F}_2)=\mathbb{F}_2$ *the cup product*

$$\cup\colon\ \mathrm{H}^1(\mathbb{RP}^2;\mathbb{F}_2)\times\mathrm{H}^1(\mathbb{RP}^2;\mathbb{F}_2)\ \to\ \mathrm{H}^2(\mathbb{RP}^2;\mathbb{F}_2)$$

is given by

$$\begin{array}{rcl}\mathbb{F}_2\times\mathbb{F}_2 & \to & \mathbb{F}_2\\ (a,b) & \mapsto & a\cdot b.\end{array}$$

Remark. Note that the calculation in Lemma 81.18 that the statement of Lemma 81.9 does not hold for $\mathbb{F}_2$-coefficients.

PROOF. By the discussion above it suffices to show that $\eta([\varphi]\cup[\varphi])\neq 0$. Put differently, by definition of η we need to show that

$$(\varphi\cup\varphi)(\sigma_1)+(\varphi\cup\varphi)(\sigma_2)+(\varphi\cup\varphi)(\tau_1)+(\varphi\cup\varphi)(\tau_2)\ =\ 1\ \in\mathbb{F}_2.$$

In Exercise 81.4 we will determine all four summands and we will see that the sum is indeed 1. ■

The above calculation of the cup product of the real projective plane $\mathbb{RP}^2$ is very explicit and hands on, but not very conceptual. Later on, in Proposition 90.16, we will give a very different calculation of the cup product on $\mathbb{F}_2$-cohomology of all real projective spaces $\mathbb{RP}^n$.

81.6. **The cup product and the wedge product** $(*)$**.** As a reminder, we denote by *SmMfd* the category of smooth manifolds. As we already saw, the maps

$$\begin{array}{rcl}M & \mapsto & (\mathrm{H}^k_{\mathrm{dR}}(M),\cup)\\ (f\colon M\to N) & \mapsto & (f^*\colon\ \mathrm{H}^k_{\mathrm{dR}}(N)\to\mathrm{H}^k_{\mathrm{dR}}(M))\end{array}$$

define a contravariant functor from the category *SmMfd* of smooth manifolds to the category of graded rings *GrRing*.

If we assign to a smooth manifold its k-th real cohomology groups, then we saw in Theorem 79.6 that it is naturally isomorphic to the k-th de Rham cohomology group. Now we can generalize Theorem 79.6.

Theorem 81.19. *The natural isomorphisms u from Theorem 79.6 are in fact ring homomorphisms, i.e. they define a natural isomorphism from the functor*

$$\begin{array}{rcl}\mathit{SmMfd} & \to & \mathit{GrRing}\\ M & \mapsto & (\mathrm{H}^*_{\mathrm{dR}}(M),\wedge)\end{array}\quad\textit{to the functor}\quad\begin{array}{rcl}\mathit{SmMfd} & \to & \mathit{GrRing}\\ M & \mapsto & (\mathrm{H}^*(M;\mathbb{R}),\cup).\end{array}$$

Example. As usual we denote by dt on $\mathbb{R}$ the 1-form that for every $v \in T_P\mathbb{R} = \mathbb{R}$ is given by $dt(v) = v$. We denote by $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ the obvious projection map. The closed 1-form dt descends to a closed 1-form on S^1, i.e. there exists a unique 1-form $\overline{dt}$ on $\mathbb{R}/\mathbb{Z} = S^1$ whose pullback equals dt. Let $\sigma\colon \Delta^1 \to \mathbb{R}/\mathbb{Z} = S^1$ be a smooth singular 1-simplex. Since Δ^1 is simply connected we can lift this singular 1-simplex, i.e. there exists a map (which is in fact smooth) $\widetilde{\sigma}\colon \Delta^1 \to \mathbb{R}$ with $\pi \circ \widetilde{\sigma} = \sigma$. We have

$$\int_\sigma \overline{dt} \underset{\substack{\uparrow\\ \text{see definition on page 1940}}}{=} -\int_{s=0}^{s=1} \overline{dt}(\sigma'(t,1-t))ds \underset{\substack{\uparrow\\ \text{definition of } \overline{dt}}}{=} -\int_{s=0}^{s=1} \widetilde{\sigma}'(t,1-t)ds \underset{\substack{\uparrow\\ \text{Fundamental Theorem}\\ \text{of Calculus}}}{=} -\widetilde{\sigma}(1,0)+\widetilde{\sigma}(0,1) \underset{\substack{\uparrow\\ \text{see page 1825}}}{=} \theta_{\mathbb{R}}(\sigma).$$

This little calculation shows that under the natural isomorphisms from Theorem 79.6 the class $[\overline{dt}] \in H^1_{\mathrm{dR}}(M)$ gets sent to our favorite generator $[\theta_{\mathbb{R}}] \in H^1(M;\mathbb{R})$, i.e. we have $u([\overline{dt}]) = [\theta_{\mathbb{R}}]$.

Next we consider the torus $M = S^1 \times S^1$. As before we denote $p\colon S^1 \times S^1 \to S^1$ respectively $q\colon S^1 \times S^1 \to S^1$ the projection onto the first, respectively the second factor. A direct calculation shows that, with the usual product orientation on the torus $M = S^1 \times S^1$ we have

$$\int_{S^1\times S^1} p^*(\overline{dt}) \wedge q^*(\overline{dt}) \;=\; +1.$$

On the other hand, by the naturality of u we have $u(p^*([\overline{dt}])) = p^*(u([\overline{dt}])) = p^*([\theta_{\mathbb{R}}])$ and similarly we also have $u(q^*([\overline{dt}]) = q^*([\theta_{\mathbb{R}}])$. The calculation of the above integral is now consistent with the calculation in Lemma 81.14 where we had, implicitly, shown that $\langle p^*([\theta_{\mathbb{R}}]) \cup q^*([\theta_{\mathbb{R}}]), [S^1 \times S^1]\rangle = +1$.

Proof. It is shown in [**Dup78**, Theorem 2.4], through a rather lengthy calculation, that the natural isomorphisms from Theorem 79.6 are ring homomorphisms. Unfortunately it is rather difficult to make sure that all the different sign conventions match.[1184] The above example gives us at least some confidence that the signs are right. We also refer to [**War83**, Theorem 5.45] for a proof of the theorem. ■

Exercises for Chapter 81.

Exercise 81.1. Let X be a topological space. We consider the map

$$\Phi\colon\ H_*(X) \otimes H_*(X) \xrightarrow{\times} H_*(X \times X) \xrightarrow{p_*} H_*(X)$$

where $p\colon X \times X \to X$ denotes the projection on the first factor. Show that Φ is the trivial map.

Exercise 81.2. Let $f\colon T \to T$ be a homeomorphism of the torus $T = S^1 \times S^1$. Show that if f orientation-preserving, then the following two statements hold:

(a) $f_*\colon H_1(T;\mathbb{Z}) \to H_1(T;\mathbb{Z})$ has determinant one,

[1184] For example, on [**Dup78**, p. 20] it says "let $\cup$" be the usual cup-product", but as we had discussed on page 1981, it is not so clear what "usual cup-product" is supposed to mean.

(b) $f^*\colon \mathrm{H}^1(T;\mathbb{Z}) \to \mathrm{H}^1(T;\mathbb{Z})$ has determinant one.

Exercise 81.3.

(a) Use cup products to show that there is no map $f\colon S^1 \times S^1 \to S^2$ of degree one, i.e. there is no map that induces an isomorphism on second homology.
(b) Given $g \in \mathbb{N}_0$ we denote by Σ_g the surface of genus g. Let $g < k$. Use cup product to show that there is no degree-one map from Σ_g to Σ_k.

Remark. This exercise shows that cup products can also be used to answer Question 69.6. Note that in Corollary 69.10 we used the fundamental class to prove the above statements.

Exercise 81.4. We consider $\mathbb{R}\mathrm{P}^2 = S^2/\sim$ which we identify with $\overline{B}^2/\sim$ via the homeomorphism that is induced by the projection map $S^2 \to \overline{B}^2$. Let $\varphi \in \mathrm{C}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ be the cocycle that we defined on page 1998. Determine *each of the four summands* in the sum

$$(\varphi \cup \varphi)(\sigma_1) + (\varphi \cup \varphi)(\sigma_2) + (\varphi \cup \varphi)(\tau_1) + (\varphi \cup \varphi)(\tau_2).$$

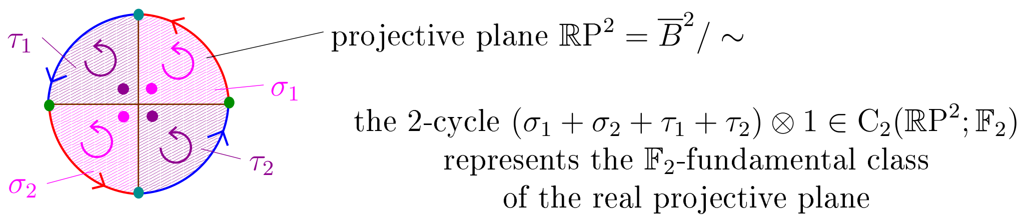


FIGURE 1167

Exercise 81.5. Let K be the Klein bottle.

(a) Is the cup product $\mathrm{H}^1(K;\mathbb{Z}) \times \mathrm{H}^1(K;\mathbb{Z}) \to \mathrm{H}^2(K;\mathbb{Z})$ is trivial?
(b) Show that the cup product $\mathrm{H}^1(K;\mathbb{F}_2) \times \mathrm{H}^1(K;\mathbb{F}_2) \to \mathrm{H}^2(K;\mathbb{F}_2)$ is non-trivial.

Exercise 81.6. Let $k \in \mathbb{N}$. We denote by N_k the non-orientable surface of genus k as defined on page 206. For $i = 1, \dots, k$ we denote by $p_i\colon N_k \to \mathbb{R}\mathrm{P}^2$ the map that is illustrated in Figure 1168.

(a) Show that the map $p_1^* \oplus \dots \oplus p_k^*\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \oplus \dots \oplus \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \to \mathrm{H}^1(N_k;\mathbb{F}_2)$ is an isomorphism.
Hint. You could use cellular cohomology.
(b) Let $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ be the unique non-trivial element. For $i = 1, \dots, k$ we write $y_i = p_i^*(x)$. Furthermore we denote by $[N_k]^* \in \mathrm{H}^2(N_k;\mathbb{F}_2)$ the unique non-trivial element. Show that for any $i, j \in \{1, \dots, k\}$ we have $y_i \cup y_j = \delta_{ij} \cdot [N_k]^*$.
(c) Let $g < k$. Use cup product to show that there is no map $f\colon N_g \to N_k$ that induces an isomorphism $f_*\colon \mathrm{H}_2(N_g;\mathbb{F}_2) \to \mathrm{H}_2(N_k;\mathbb{F}_2)$.
Remark. This statement is the non-orientable analogue of Exercise 81.3. Note that we had proved this statement in Exercise 69.7 using an alternative approach. Both are fun!

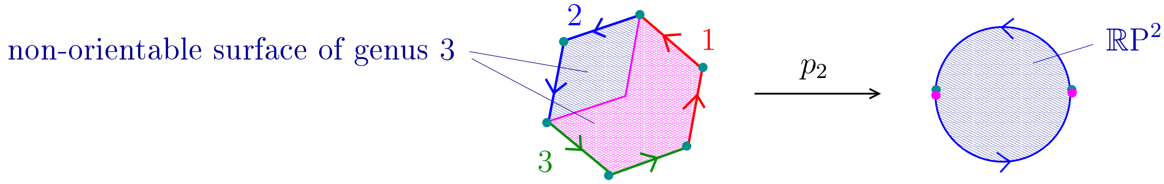


FIGURE 1168

Exercise 81.7. Given a matrix $A \in \mathrm{M}(n \times n, \mathbb{F}_2)$ we define the bilinear form

$$\begin{array}{rcl}\lambda(A)\colon \mathbb{F}_2^n \times \mathbb{F}_2^n & \to & \mathbb{F}_2 \\ (v,w) & \mapsto & v^T A w.\end{array}$$

We consider the two matrices $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ over $\mathbb{F}_2$. Next we consider the torus $T = S^1 \times S^1$ and the Klein bottle $K = N_2$. By a slight variation on Lemma 81.14 and by Exercise 81.6 we know that the cup products $\mathrm{H}^1(T;\mathbb{F}_2) \times \mathrm{H}^1(T;\mathbb{F}_2) \to \mathrm{H}^2(T;\mathbb{F}_2)$ and $\mathrm{H}^1(K;\mathbb{F}_2) \times \mathrm{H}^1(K;\mathbb{F}_2) \to \mathrm{H}^2(K;\mathbb{F}_2)$ are isometric to $\lambda(P)$ respectively $\lambda(Q)$.

(a) Show that the forms $\lambda(P)$ and $\lambda(Q)$ are not isometric.

(b) Show that there is no map $f\colon T \to K$ such that the map $f_*\colon \mathrm{H}_2(T;\mathbb{F}_2) \to \mathrm{H}_2(K;\mathbb{F}_2)$ is an isomorphism.

Remark. This approach gives an alternative, and arguably more systematic solution, to Exercise 81.7.

82. The relative cup product

Our goal in this chapter is to introduce a relative version of the cup product. More precisely, given an excisive triad (X, A, B) and a commutative ring R we will introduce a cup product

$$\mathrm{H}^p(X, A; R) \times \mathrm{H}^q(X, B; R) \xrightarrow{\quad \cup \quad} \mathrm{H}^{p+q}(X, A \cup B; R)$$

that is functorial in an appropriate sense and that equals the previous definition of the cup product for the special case that $A = B = \varnothing$. This relative cup product will help us in determining the cup products of wedges of topological spaces and of the suspension of a topological space. Also it will help us to determine the mysterious sounding Lusternik-Schnirelmann category.

82.1. **Definition of the relative cup product.** In this section we will work a lot with relative cochains. Before we do so we recall the following convention from page 1829.

Convention.

(1) Given an abelian group C, a subgroup D and an abelian group G we have a natural isomorphism

$$\begin{array}{rcl} \mathrm{Hom}(C/D, G) & \xrightarrow{\cong} & \{f\colon C \to G \,|\, f \text{ vanishes on } D\} \\ (\varphi\colon C/D \to G) & \mapsto & (C \to C/D \xrightarrow{\varphi} G). \end{array}$$

We will use this isomorphism to identify the groups left and right.

(2) In particular, given a pair of topological spaces (X, A), a commutative ring R and $n \in \mathbb{N}_0$ we make the identification

$$\mathrm{C}^n(X, A; R) \xrightarrow{\cong} \{f\colon \mathrm{C}_n(X) \to R \,|\, f(\sigma) = 0 \text{ for all } \sigma\colon \Delta^n \to A\}.$$

The definition of the relative cup product requires some preparation. We start out with the following definition.

Definition. Let (X, A, B) be a triad of topological spaces. As in the definition on page 1851 we write

$$\mathrm{C}_n^{\{A,B\}}(A \cup B) \;:=\; \left\{ \sum_{j=1}^{k} n_j \sigma_j \in \mathrm{C}_n(A \cup B) \,\middle|\, \begin{array}{c} \text{for each } j \text{ the image of } \sigma_j \\ \text{lies in } A \text{ or it lies in } B \end{array} \right\}$$

The usual boundary map on the groups $\mathrm{C}_*(A \cup B)$ restricts to a boundary map on the groups $\mathrm{C}_*^{\{A,B\}}(A \cup B)$. Given a commutative ring R we consider

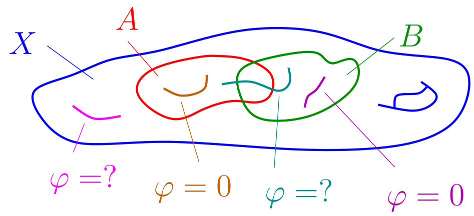
$$\mathrm{C}^n(X, \{A, B\}; R) \;:=\; \ker\Big(\mathrm{C}^n(X; R) \to \mathrm{Hom}\big(\underbrace{\mathrm{C}_n^{\{A,B\}}(A \cup B)}_{\text{subgroup of } \mathrm{C}_n(X)}, R\big)\Big).$$

This is again a cochain complex and we denote the corresponding cohomology groups by $\mathrm{H}^*(X, \{A, B\}; R)$.

Remark. With the above notation we could write that

$$\mathrm{C}^n(X, \{A, B\}; R) \;=\; \{\varphi\colon \mathrm{C}_n(X) \to R \,|\, \varphi(\sigma) = 0 \text{ for any } \sigma\colon \Delta^n \to A \text{ and any } \sigma\colon \Delta^n \to B\}.$$

The definition of $\mathrm{C}^n(X, \{A, B\}; R)$ is illustrated in Figure 1169.



$\varphi \in \mathrm{H}^1(X, \{A, B\}; R)$

FIGURE 1169

Lemma 82.1. *Let (X, A, B) be an excisive triad of topological spaces. Given any commutative ring R the natural*[1185] *map*

$$\mathrm{C}^*(X, A \cup B; R) \quad \to \quad \mathrm{C}^*(X, \{A, B\}; R)$$

induces for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}^n(X, A \cup B; R) \quad \xrightarrow{\cong} \quad \mathrm{H}^n(X, \{A, B\}; R).$$

PROOF. We have the following commutative diagram of short exact sequences of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{C}_*^{\{A,B\}}(A \cup B) \arrow[r] \arrow[d] & \mathrm{C}_*(X) \arrow[r] \arrow[d, "=\mathrm{id}"] & \frac{\mathrm{C}_*(X)}{\mathrm{C}_*^{\{A,B\}}(A \cup B)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{C}_*(A \cup B) \arrow[r] & \mathrm{C}_*(X) \arrow[r] & \mathrm{C}_*(X, A \cup B) \arrow[r] & 0.
\end{tikzcd}$$

By our hypothesis the triad (X, A, B) is excisive. By definition this means that the left vertical map is a chain homotopy equivalence. The middle vertical map is the identity. It now follows from Corollary 49.4 that the right vertical map is also a chain homotopy equivalence.

Recall that by the convention on page 2003 we have an identification

$$\mathrm{Hom}\left(\frac{\mathrm{C}_n(X)}{\mathrm{C}_n^{\{A,B\}}(A \cup B)}, R\right) \;=\; \mathrm{C}^n(X, \{A, B\}; R).$$

The lemma follows from the above vertical chain homotopy equivalence, the above observation and Lemma 73.8. ■

The following lemma is the motivation for introducing the rather technical cohomology modules $\mathrm{H}^*(X; \{A, B\}; R)$.

Lemma 82.2. *Let (X, A, B) be a triad of topological spaces and let R be a commutative ring. Then the map*

$$\begin{array}{rcl} \mathrm{C}^p(X, A; R) \times \mathrm{C}^q(X, B; R) & \to & \mathrm{C}^{p+q}(X, \{A, B\}; R) \\ (\varphi, \psi) & \mapsto & \left[\begin{array}{rcl} \mathrm{C}_{p+q}(X) & \to & R \\ \sigma & \mapsto & \varphi(\sigma \circ [v_0, \dots, v_p]) \cdot \psi(\sigma \circ [v_p, \dots, v_{p+q}]) \end{array}\right]. \end{array}$$

is well-defined and it induces an R-bilinear map

$$\cup\colon\ \mathrm{H}^p(X, A; R) \times \mathrm{H}^q(X, B; R) \;\to\; \mathrm{H}^{p+q}(X, \{A, B\}; R).$$

[1185] What is the "natural" map?

PROOF. The lemma follows easily from the following claim.

Claim. Given $\varphi \in \mathrm{C}^p(X, A; R)$ and $\psi \in \mathrm{C}^q(X, B; R)$ the homomorphism

$$\begin{array}{rcl} \mathrm{C}_{p+q}(X) & \to & R \\ (\sigma\colon \Delta^{p+q} \to X) & \mapsto & \varphi(\sigma \circ [v_0, \dots, v_p]) \cdot \psi(\sigma \circ [v_p, \dots, v_{p+q}]) \end{array}$$

lies actually in $\mathrm{C}^{p+q}(X, \{A, B\})$.

By the remark on page 2003 we have to show that the above cochain vanishes on all singular $(p+q)$-simplices that lie entirely in A or that lie entirely in B. But this is clear:

(1) given $\sigma\colon \Delta^{p+q} \to A$ the first factor $\varphi(\sigma \circ [v_0, \dots, v_p])$ is necessarily zero since we have $\varphi \in \mathrm{C}^p(X, A; R)$, and similarly,
(2) given $\sigma\colon \Delta^{p+q} \to B$ the second factor $\psi(\sigma \circ [v_p, \dots, v_{p+q}])$ is necessarily zero since we have $\psi \in \mathrm{C}^q(X, B; R)$.

This concludes the proof of the claim and thus also of the lemma. ■

Definition. Let (X, A, B) be an excisive triad of topological spaces and let R be a commutative ring. We refer to the composition

$$\mathrm{H}^p(X, A; R) \times \mathrm{H}^q(X, B; R) \xrightarrow{(c,d) \mapsto c \cup d} \mathrm{H}^{p+q}(X, \{A, B\}; R) \underset{\substack{\uparrow \\ \text{isomorphism by Lemma 82.1}}}{\xleftarrow{\cong}} \mathrm{H}^{p+q}(X, A \cup B; R)$$

again as a *cup product* (or sometimes as a *relative cup product*) and we denote it as usual by the "$\cup$"-symbol.

Remark. Let X be a topological space, let $A, B \subset X$ be subsets and let R be a commutative ring. Suppose that $\mathrm{C}_*^{\{A,B\}}(A \cup B) = \mathrm{C}_*(A \cup B)$. (This is for example the case if $A = B = \varnothing$ or more generally, if A is a subset of B or if B is a subset of A.) Then the above definition of the cup product is just the "naive" definition given by

$$\begin{array}{rcl} \mathrm{C}^p(X, A; R) \times \mathrm{C}^q(X, B; R) & \to & \mathrm{C}^{p+q}(X, A \cup B; R) \\ (\varphi, \psi) & \mapsto & \left(\begin{array}{rcl} \mathrm{C}_{p+q}(X, A \cup B) & \to & R \\ \sigma & \mapsto & \underbrace{\varphi(\sigma \circ [v_0, \dots, v_p])}_{\in R} \cdot \underbrace{\psi(\sigma \circ [v_p, \dots, v_{p+q}])}_{\in R} \end{array} \right). \end{array}$$

In the following we summarize a few properties of the relative cup product. We start out with the obvious generalization of Proposition 81.8.

Proposition 82.3. *Let (X, A, B) be an excisive triad of topological spaces and let R be a commutative ring. For every $\varphi \in \mathrm{H}^k(X, A; R)$ and $\psi \in \mathrm{H}^l(X, B; R)$ we have*

$$\varphi \cup \psi \;=\; (-1)^{kl} \cdot \psi \cup \varphi \;\in\; \mathrm{H}^{k+l}(X, A \cup B; R).$$

PROOF. In Exercise 82.1 we will see that a modification of the proof of Proposition 81.8 provides us with a proof of Proposition 82.3. ■

Note that the first part of following proposition is a generalization of Lemma 81.10.

Proposition 82.4.

(1) *Let (X,A,B) and (Y,C,D) be two excisive triads and let R be a commutative ring. Let $f\colon (X,A,B)\to(Y,C,D)$ be a map. Then for all $p,q\in\mathbb{N}_0$ the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}^p(Y,C;R)\times \mathrm{H}^q(Y,D;R) & \xrightarrow{\ \cup\ } & \mathrm{H}^{p+q}(Y,C\cup D;R) \\ \downarrow{\scriptstyle f^*\times f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^p(X,A;R)\times \mathrm{H}^q(X,B;R) & \xrightarrow{\ \cup\ } & \mathrm{H}^{p+q}(X,A\cup B;R). \end{array}$$

(2) *Let (X,A,B) be an excisive triad. If $\varphi\colon R\to S$ is a homomorphism between commutative rings, then for all $p,q\in\mathbb{N}_0$ the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}^p(X,A;R)\times \mathrm{H}^q(X,B;R) & \xrightarrow{\ \cup\ } & \mathrm{H}^{p+q}(X,A\cup B;R) \\ \downarrow{\scriptstyle \varphi_*\times \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{H}^p(X,A;S)\times \mathrm{H}^q(X,B;S) & \xrightarrow{\ \cup\ } & \mathrm{H}^{p+q}(X,A\cup B;S). \end{array}$$

Here the vertical maps to the left are the maps defined on page 1823.

Proof.

(1) It follows easily from the definitions that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}^p(Y,C;R)\times \mathrm{H}^q(Y,D;R) & \xrightarrow{\ \cup\ } & \mathrm{H}^{p+q}(Y,\{C,D\};R) & \xleftarrow{\ \cong\ } & \mathrm{H}^{p+q}(Y,C\cup D;R) \\ \downarrow{\scriptstyle f^*\times f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^p(X,A;R)\times \mathrm{H}^q(X,B;R) & \xrightarrow{\ \cup\ } & \mathrm{H}^{p+q}(X,\{A,B\};R) & \xleftarrow{\ \cong\ } & \mathrm{H}^{p+q}(X,A\cup B;R). \end{array}$$

The desired statement follows immediately from this observation.

(2) This statement follows, similar to (1), basically immediately from the definitions. ■

We record a particularly useful special case of Proposition 82.4 (1) as a corollary.

Corollary 82.5. *Let (X,A,B) be an excisive triad and let R be a commutative ring. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}^p(X,A;R)\times \mathrm{H}^q(X,B;R) & \xrightarrow{\ \cup\ } & \mathrm{H}^{p+q}(X,A\cup B;R) \\ \downarrow & & \downarrow \\ \mathrm{H}^p(X;R)\times \mathrm{H}^q(X;R) & \xrightarrow{\ \cup\ } & \mathrm{H}^{p+q}(X;R). \end{array}$$

In particular, if $\alpha\in\mathrm{H}^p(X;R)$ lies in the image of $\mathrm{H}^p(X,A;R)$ and if $\beta\in\mathrm{H}^q(X;R)$ lies in the image of $\mathrm{H}^q(X,B;R)$ and if $A\cup B=X$, then $\alpha\cup\beta=0$.

Proof. The first part of the corollary follows immediately from Proposition 82.4 (1) applied to the obvious map $f\colon (X,\emptyset,\emptyset)\to(X,A,B)$ of excisive triads.

We turn to the proof of the second part. We denote by $\sigma\colon \mathrm{H}^p(X,A;R)\to\mathrm{H}^p(X;R)$, $\tau\colon \mathrm{H}^p(X,B;R)\to\mathrm{H}^p(X;R)$ and by $\upsilon\colon \mathrm{H}^{p+q}(X,A\cup B;R)\to\mathrm{H}^{p+q}(X;R)$ the obvious maps. Let $\widehat{\alpha}\in\mathrm{H}^p(X,A;R)$ and $\widehat{\beta}\in\mathrm{H}^q(X,B;R)$. We write $\alpha=\sigma(\widehat{\alpha})$ and $\beta=\tau(\widehat{\beta})$. We suppose that $A \cup B = X$. Then

$$\alpha \cup \beta \;=\; \sigma(\widehat{\alpha}) \cup \tau(\widehat{\beta}) \underset{\substack{\uparrow \\ \text{by the commutative diagram}}}{=} \upsilon\big(\widehat{\alpha} \cup \widehat{\beta}\big) \underset{\substack{\uparrow \\ \text{since } \widehat{\alpha} \cup \widehat{\beta} \text{ lies in the group} \\ \mathrm{H}^{p+q}(X, A \cup B; R) = \mathrm{H}^{p+q}(X, X; R) = 0}}{=} \upsilon(0) \;=\; 0.$$

■

Before we state the next lemma we introduce one more convention.

Convention. Let A, B, C, D, X and Y be sets, let $f\colon C \to A$, $g\colon B \to D$ and $\mu\colon X \to Y$ be maps and let $\varphi\colon A \times B \to X$ and $\psi\colon C \times D \to Y$ be maps. We say that the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\quad\varphi\quad} & X \\ f\uparrow \quad \downarrow g & & \mu\downarrow \\ C \times D & \xrightarrow{\quad\psi\quad} & Y \end{array}$$

commutes, if for all $c \in C$ and $b \in B$ we have

$$\psi(c, g(b)) \;=\; \mu(\varphi(f(c), b)).$$

A similar convention holds with the directions of f and g reversed. Furthermore, if we are dealing with group homomorphisms, then there is an obvious notion of "commutes up to a given sign".

Using this convention we can now formulate our next lemma.

Lemma 82.6. *Let (X, A) be a pair of topological spaces and let R be a commutative ring. We denote by $i\colon A \to X$ the inclusion map and we denote by δ the connecting homomorphisms in the long exact sequence in cohomology corresponding to the pair (X, A). For any $p, q \in \mathbb{N}_0$ the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}^q(A;R) \times \mathrm{H}^p(A;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q}(A;R) \\ \delta\downarrow \qquad\qquad \uparrow i^* & & \downarrow\delta \\ \mathrm{H}^{q+1}(X,A;R) \times \mathrm{H}^p(X;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q+1}(X,A;R) \end{array}$$

and the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}^p(A;R) \times \mathrm{H}^q(A;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q}(A;R) \\ i^*\uparrow \qquad\qquad \downarrow\delta & & \downarrow\delta \\ \mathrm{H}^p(X;R) \times \mathrm{H}^{q+1}(X,A;R) & \xrightarrow{\quad\cup\quad} & \mathrm{H}^{p+q+1}(X,A;R) \end{array}$$

up to the sign $(-1)^p$.

Remark. In Lemma 100.18 we will formulate a generalization of Lemma 82.6.

PROOF (∗). For no particular reason we prove the second statement. The first statement can either be proved the same way or it can be deduced from the second statement using Proposition 82.3. We pick cocycles $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(A;R)$. In the following we denote by $\widetilde{\psi}\colon \mathrm{C}^q(X) \to R$ the extension of ψ that is defined to be zero on all singular

k-simplices that do not lie in the subset A. We have the following equalities:

$$\begin{aligned}
\delta([i^*\varphi]\cup[\psi]) \;&=\; \delta([i^*\varphi\cup\psi]) \overset{\substack{\text{by the discussion of the connecting homomorphism in the remark on page 1833 and}\\ \text{since } \varphi\cup\widetilde{\psi} \text{ is the trivial extension of } i^*\varphi\cup\psi\colon\, \mathrm{C}_{p+q}(A)\to R \text{ to a homomorphism } \mathrm{C}_{p+q}(X)\to R\\ \downarrow}}{=} [\delta(\varphi\cup\widetilde{\psi})] \underset{\substack{\uparrow\\ \text{Lemma 81.2}}}{=} [\delta\varphi\cup\widetilde{\psi}+(-1)^p\cdot(\varphi\cup\delta\widetilde{\psi})]\\
&\underset{\substack{\uparrow\\ \text{since } \varphi \text{ is a cocycle}}}{=} (-1)^p\cdot[\varphi\cup\delta\widetilde{\psi}] \;=\; (-1)^p\cdot([\varphi]\cup[\delta\widetilde{\psi}]) \underset{\substack{\uparrow\\ \text{again by the remark on page 1833}}}{=} (-1)^p\cdot([\varphi]\cup\delta[\psi]).
\end{aligned}$$

This equality implies that the diagram commutes up to the sign $(-1)^p$. ■

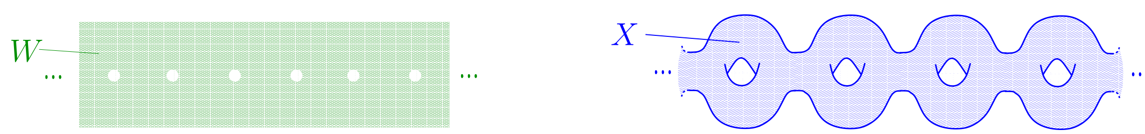
Finally, recall that on page 1903 we introduced cohomology with compact support $\mathrm{H}^*_c(X;\mathbb{Z})$. For the insatiable reader we also provide the definition of the cup product on cohomology with compact support.

Definition. Let X be a topological space, let $\varphi\in\mathrm{H}^k_c(X;\mathbb{Z})$ and let $\psi\in\mathrm{H}^l_c(X;\mathbb{Z})$. Recall that by Lemma 77.2 there exists a compact subset $K\subset X$ and a cohomology class $\widetilde{\varphi}\in\mathrm{H}^k(X,X\setminus K;\mathbb{Z})$ such that the image of $\widetilde{\varphi}$ in $\mathrm{H}^k_c(X;\mathbb{Z})$ equals φ. Analogously we find a compact set L and $\widetilde{\psi}\in\mathrm{H}^l(X,X\setminus L;\mathbb{Z})$. We define

$$\varphi\cup\psi \;:=\; \text{the image of } \varphi\cup\psi\in\mathrm{H}^{k+l}(X,X\setminus(K\cup L)) \text{ in } \mathrm{H}^{k+l}_c(X;\mathbb{Z}).$$

It follows easily from Corollary 82.5 that this definition does not depend on any of the choices. We refer to $\varphi\cup\psi$ as the *cup product in cohomology with compact support.*

The cup product on cohomology with compact support can be very useful for tackling non-compact topological spaces. For example in Exercise 82.5 we will make of use of the cup product on cohomology with compact support to show that the topological manifold $W=\mathbb{R}^2\setminus\{(n,0)\,|\,n\in\mathbb{Z}\}$ is not homeomorphic to the surface X of infinite genus that is shown in Figure 1170. On a more theoretical side we will make of the above definition in Theorem 95.9 when we will learn how to compute cup products on topological manifolds by "intersecting cycles".



cup product on $\mathrm{H}^*_c(W;\mathbb{Z})$ is trivial — cup product on $\mathrm{H}^*_c(X;\mathbb{Z})$ is non-trivial

FIGURE 1170

82.2. **Calculations using the relative cup product.** In this section we will use Proposition 82.4 (1) to calculate cup products of several topological spaces. We start out with the suspension $\Sigma(X)=(X\times[-1,1])/\sim$ of a topological space X. We refer to page 694 for the precise definition and we refer to Figure 1171 for a reminder.

The following lemma shows that the cup product of the suspension of a topological space is quite dull.

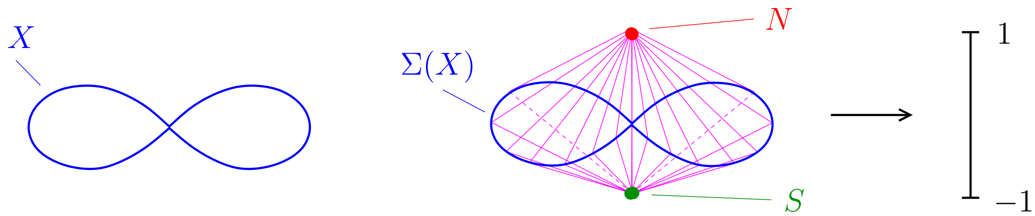


FIGURE 1171

Lemma 82.7. *Let X be a path-connected topological space and let R be a commutative ring.*

(1) *We have*

$$\mathrm{H}^k(\Sigma(X);R) \;\cong\; \begin{cases} R, & \text{if } k=0, \\ 0, & \text{if } k=1, \\ \mathrm{H}^{k-1}(X;R), & \text{if } k\geq 2. \end{cases}$$

(2) *For any $k,l\geq 1$ the cup product*

$$\mathrm{H}^k(\Sigma(X);R)\times \mathrm{H}^l(\Sigma(X);R) \;\to\; \mathrm{H}^{k+l}(\Sigma(X);R)$$

is the zero map.

PROOF. We denote by $p\colon X\times[-1,1]\to\Sigma(X)$ the obvious projection map. We denote by $N:=[X\times\{1\}]$ the "North Pole" and we denote by $S:=[X\times\{-1\}]$ the South Pole. We refer to Figure 1171 for an illustration. We write $U=\Sigma(X)\setminus\{S\}$ and $V=\Sigma(X)\setminus\{N\}$. By the same argument as in Lemma 24.1 that U admits deformation retraction from U to the North Pole N. This shows that U is contractible. The same argument shows that V is contractible.

(1) Note that U and V are open subsets of $\Sigma(X)$.[1186] It follows from Proposition 74.12 (4) that $(\Sigma(X),U,V)$ is excisive. Therefore we can apply the Mayer–Vietoris Theorem 74.15 for Cohomology Groups and we obtain the following long exact sequence

$$\ldots\to\mathrm{H}^i(\Sigma(X);R)\to\mathrm{H}^i(U;R)\oplus\mathrm{H}^i(V;R)\to\mathrm{H}^i(U\cap V;R)\to\mathrm{H}^{i+1}(\Sigma(X);R)\to\ldots$$

The desired isomorphism now follows easily from the above long exact sequence and the following:
   (a) above we had observed that U and V are contractible,
   (b) the intersection $U\cap V$ is homeomorphic to $X\times(-1,1)$, in particular it is homotopy equivalent to X,
   (c) Lemmas 73.14 and 73.12 and the discussion of induced maps on 0-th cohomology on page 1830.

(2) Now let $k,l\geq 1$. It follows immediately from the long exact sequence of cohomology groups corresponding to the pair $(\Sigma(X),U)$, the fact that U is homotopy equivalent to a point (together with Lemmas 73.12 and 73.13) that the inclusion induced map

[1186] Why is that?

$\mathrm{H}^k(\Sigma(X), U; R) \to \mathrm{H}^k(\Sigma(X); R)$ is an isomorphism. Evidently the same also holds if we replace U by V.

By Corollary 82.5, and the fact that $(\Sigma(X), U, V)$ is excisive, we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^k(\Sigma(X), U; R) \times \mathrm{H}^l(\Sigma(X), V; R) & \xrightarrow{\ \cup\ } & \mathrm{H}^{k+l}(\Sigma(X), \overbrace{U \cup V}^{=\Sigma(X)}; R) \\
\downarrow \cong & & \downarrow \\
\mathrm{H}^k(\Sigma(X); R) \times \mathrm{H}^l(\Sigma(X); R) & \xrightarrow{\ \cup\ } & \mathrm{H}^{k+l}(\Sigma(X); R).
\end{array}$$

Since $U \cup V = \Sigma(X)$ we see that the cohomology group on the top-right is zero. Furthermore, since the vertical map on the left is an isomorphism we see that the cup product on the bottom is zero. ■

Let X and Y be two topological spaces and let $x \in X$ and $y \in Y$ be good points. (See page 604 for the definition of a good point.) We use these points to define the wedge $X \vee Y$ of X and Y. Furthermore let R be a commutative ring. We denote by $p\colon X \vee Y \to X$ and $q\colon X \vee Y \to Y$ the obvious projection maps. By Proposition 74.17 we know that for every $n \in \mathbb{N}$ the map

$$p^* \oplus q^*\colon\ \mathrm{H}^n(X; R) \oplus \mathrm{H}^n(Y; R) \xrightarrow{\cong} \mathrm{H}^n(X \vee Y; R)$$

is an isomorphism. The following lemma determines the cup product of $X \vee Y$ in terms of the cup products of X and Y. It can be viewed as a generalization of Lemma 81.11.

Lemma 82.8. *We continue with the above notation. Then for any $\alpha, \beta \in \mathrm{H}^*(X; R)$ and $\varphi, \psi \in \mathrm{H}^*(Y; R)$ in degrees ≥ 1 the following statements hold in $\mathrm{H}^*(X \vee Y; R)$:*

$\cup$	$p^*(\beta)$	$q^*(\psi)$
$p^*(\alpha)$	$p^*(\alpha \cup \beta)$	0
$q^*(\varphi)$	0	$q^*(\varphi \cup \psi)$

Proof $(*)$. The statement regarding the "diagonal" terms of the table follows immediately from Lemma 81.10. It remains to prove that the off-diagonal terms are zero. We show that the top-right entry is zero. The proof that the bottom-left entry is zero is basically identical.

Thus let $\alpha \in \mathrm{H}^k(X; R)$ and $\psi \in \mathrm{H}^l(Y; R)$ with $k, l \geq 1$. We view X and Y as subsets of $X \vee Y$. We denote by $f\colon (X \vee Y, \varnothing) \to (X \vee Y, X)$ and $g\colon (X \vee Y, \varnothing) \to (X \vee Y, Y)$ the two inclusion maps of pairs of topological spaces.

Claim. There exist $\widetilde{\alpha} \in \mathrm{H}^k(X \vee Y, Y; R)$ with $g^*(\widetilde{\alpha}) = p^*(\alpha)$ and $\widetilde{\psi} \in \mathrm{H}^l(X \vee Y, X; R)$ with $f^*(\widetilde{\psi}) = q^*(\psi)$.

To prove the claim we consider the following two diagrams[1187]

$$\begin{tikzcd} \mathrm{H}^k(X \vee Y; R) & \mathrm{H}^k(X \vee Y, Y; R) \arrow[l, "g^*"'] \arrow[d] \\ \mathrm{H}^k(X; R) \arrow[u, "p^*"] & \mathrm{H}^k(X, \{x\}; R) \arrow[l] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{H}^l(X \vee Y; R) & \mathrm{H}^l(X \vee Y, X; R) \arrow[l, "f^*"'] \arrow[d] \\ \mathrm{H}^l(Y; R) \arrow[u, "q^*"] & \mathrm{H}^l(Y, \{y\}; R). \arrow[l] \end{tikzcd}$$

Here the maps that are unlabeled are induced by the obvious inclusions. It is easy to verify that the diagrams commute. It follows easily from the long exact sequence in cohomology of a pair of topological spaces that the two bottom horizontal maps are isomorphisms. Recall that in Lemma 74.16 we showed that the triad $(X \vee Y, X, Y)$ is excisive. Therefore it follows from Lemma 74.13 that the vertical maps on the right are isomorphisms. The claim is an immediate consequence of these observations.

Since, as we had just seen, the triad $(X \vee Y, X, Y)$ is excisive we can apply Corollary 82.5 and we obtain that

$$p^*(\alpha) \cup q^*(\psi) \;=\; g^*(\widetilde{\alpha}) \cup f^*(\widetilde{\psi}) \underset{\substack{\uparrow \\ \text{by Corollary 82.5 since } X \cup Y = X \vee Y}}{=} 0. \qquad \blacksquare$$

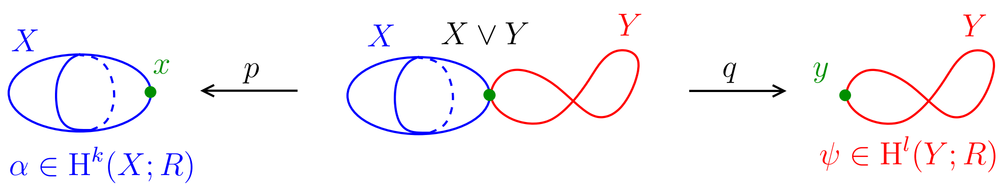


FIGURE 1172. Illustration for the proof of Lemma 82.8.

Example. Now we can also give a new proof of Lemma 81.16, more precisely, we can give a new and somewhat more conceptual proof of the calculation of the cup product of the surface Σ of genus 2. First we recall that on page 1850 we introduced two projection maps $p_i\colon \Sigma \to T_i = S^1 \times S^1$ for $i = 1, 2$. It follows easily from Lemma 72.3 that these projection maps are degree-one maps, i.e. they send the fundamental class $[\Sigma]$ to the fundamental class $[T_i]$. In Lemma 74.11 we saw that the map

$$p_1^* \oplus p_2^*\colon\ \mathrm{H}^1(T_1; \mathbb{Z}) \oplus \mathrm{H}^1(T_2; \mathbb{Z}) \;\to\; \mathrm{H}^1(\Sigma; \mathbb{Z})$$

is an isomorphism. We make the following observations:

(1) Let $\varphi, \psi \in \mathrm{H}^1(T_i; \mathbb{Z})$, $i = 1, 2$. Then

$$\langle \varphi \cup \psi, [T_i] \rangle_{T_i} \underset{\substack{\uparrow \\ \text{since } p_i \text{ is of degree one}}}{=} \langle \varphi \cup \psi, p_{i*}([\Sigma]) \rangle_{T_i} \underset{\substack{\uparrow \\ \text{Lemma 74.6 (3)}}}{=} \langle p_i^*(\varphi \cup \psi), [\Sigma] \rangle_\Sigma \underset{\substack{\uparrow \\ \text{Lemma 81.10}}}{=} \langle p_i^*(\varphi) \cup p_i^*(\psi), [\Sigma] \rangle_\Sigma.$$

(2) We denote by $q_i\colon T_1 \vee T_2 \to T_i$, $i = 1, 2$ the obvious projections. The two projection maps p_1, p_2 give rise to a map $p_1 \vee p_2\colon \Sigma \to T_1 \vee T_2$ which has the property that

[1187] Recall that in $X \vee Y$ the points x and y get identified, so we can view x as an element in Y and we can view y as an element in X.

$q_i \circ (p_1 \vee p_2) = p_i$, $i = 1, 2$. (See Figure 1173 for an illustration.) Now let $\varphi_1 \in \mathrm{H}^1(T_1; \mathbb{Z})$ and $\varphi_2 \in \mathrm{H}^1(T_2; \mathbb{Z})$. Then

$$p_1^*(\varphi_1) \cup p_2^*(\varphi_2) \underset{\substack{\uparrow \\ \text{by Lemma 81.10, since } q_i \circ (p_1 \vee p_2) = p_i}}{=} (p_1 \vee p_2)^*(q_1^*(\varphi_1) \cup q_2^*(\varphi_2)) \underset{\substack{\uparrow \\ \text{Lemma 82.8}}}{=} (p_1 \vee p_2)^*(0) = 0.$$

Precisely the same proof shows also that $p_2^*(\varphi_2) \cup p_1^*(\varphi_1) = 0$.

These two claims, together with the previous calculation of the cup product on the torus $T = S^1 \times S^1$, see Lemma 81.14, give a new proof of Lemma 81.16.

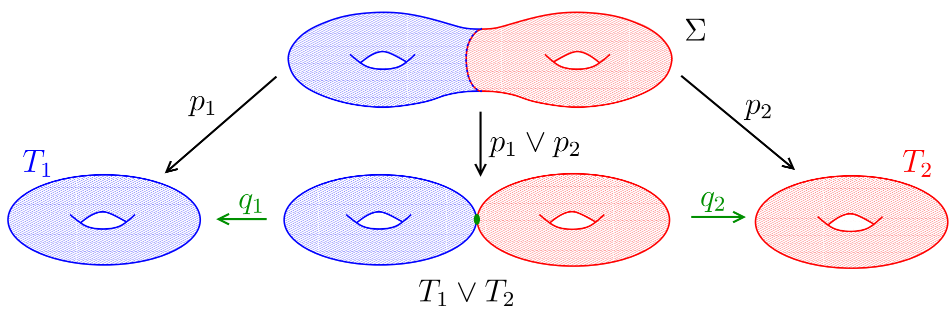


FIGURE 1173

82.3. **The Lusternik-Schnirelmann category.** We start out with the following definition.

Definition. Let M be an m-dimensional smooth manifold.

(1) A *parametrization for M* is a smooth map $\Psi\colon V \to M$ with the following properties:
   (a) the domain V is either an open ball in $\mathbb{R}^m$, or an open rectangle, or all of $\mathbb{R}^m$,[1188]
   (b) the map Ψ is a diffeomorphism onto its image.
(2) We say a family $\{\Psi_i\colon V_i \to M\}_{i\in I}$ of parametrizations *covers* M if $\bigcup\limits_{i\in I} \Psi_i(V_i) = M$.

The following slightly vague question arises:

Question 82.9. *For a given smooth manifold M, how many parametrizations do we need to cover all of M?*

Examples.

(1) Let $N = (0, \ldots, 0, 1)$ be the North Pole and let $S = (0, \ldots, 0, -1)$ be the South Pole of the n-dimensional sphere S^n. We can cover S^n by the two open subsets $S^n \setminus \{N\}$ and $S^n \setminus \{S\}$. Stereographic projection, see Figure 30, gives rise to two diffeomorphisms $\Phi_1\colon S^n \setminus \{N\} \to \mathbb{R}^n$ and $\Phi_2\colon S^n \setminus \{S\} \to \mathbb{R}^n$. The inverses of these two maps define parametrizations that cover S^n.

   Since S^n is not diffeomorphic to any of the admissible domains of a parametrization we see that the above collection of parametrizations is optimal in the sense that we cannot parametrize S^n with fewer parametrizations.

[1188]The condition on V sounds slightly arbitrary, and it is, but it is reasonable that for practical purposes one prefers domains that are "easy to describe".

(2) We consider the n-torus $(S^1)^n = \mathbb{R}^n/\mathbb{Z}^n$. We denote by $p\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ the projection map. We pick real numbers $0 < s_1 < s_2 < \dots < s_{n+1} < 1$. For $i = 1, \dots, n+1$ we consider the map

$$\begin{array}{rcl} \Psi_i\colon \overbrace{(s_i, s_i+1)\times\ldots\times(s_i, s_i+1)}^{=:V_i} & \to & \mathbb{R}^n/\mathbb{Z}^n \\ (t_1, \dots, t_n) & \mapsto & p(t_1, \dots, t_n). \end{array}$$

For $i = 1, \dots, n+1$ we write $U_i = \Psi_i(V_i)$. It is straightforward to see that the maps $\Psi_1, \dots, \Psi_{n+1}$ are parametrizations that cover $\mathbb{R}^n/\mathbb{Z}^n$.[1189] We illustrate the images of the parametrizations for the 2-torus in Figure 1174.

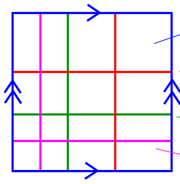


FIGURE 1174

(3) For $i = 1, \dots, n+1$ we consider the map

$$\begin{array}{rcl} \Psi_i\colon \mathbb{R}^n & \to & \mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim \\ (x_1, \dots, x_n) & \mapsto & \underbrace{[x_1 : \ldots : x_{i-1} : 1 : x_i : \ldots : x_n]}_{\text{point in } \mathbb{RP}^n \text{ represented by } (x_1, \dots, x_{i-1}, 1, x_i, \dots, x_n)}. \end{array}$$

It is straightforward to see that the maps $\Psi_1, \dots, \Psi_{n+1}$ are parametrizations that cover the real projective space $\mathbb{RP}^n$.

(4) Almost the same construction as in (3) shows that we can cover the complex projective space $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$ by $n+1$ parametrizations such that the domain of each parametrization is diffeomorphic to $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

The following question is now a more precise version of Question 82.9.

Question 82.10.

(1) *Can we cover $(S^1)^n$ by fewer than $n+1$ parametrizations?*
(2) *Can we cover $\mathbb{RP}^n$ by fewer than $n+1$ parametrizations?*
(3) *Can we cover $\mathbb{CP}^n$ by fewer than $n+1$ parametrizations?*

In the remainder of this chapter we will study a generalization of these questions and we will see how cup products can be used to approach this question and its generalizations. We will eventually answer Question 82.10 in Corollary 84.23 and Proposition 90.18.

Now we will generalize this discussion to arbitrary topological spaces by introducing the following definition.

Definition.

(1) We say that a map $f\colon Y \to X$ between topological spaces is *null-homotopic* if it is homotopic to a constant map.

[1189] Hmm, why is it true that $U_1 \cup \dots \cup U_{n+1} = \mathbb{R}^n/\mathbb{Z}^n$?

(2) The *Lusternik-Schnirelmann category* $\mathrm{cat}(X)$[1190] of a topological space X is defined as the smallest $n \in \mathbb{N}_0$ such that there exists an open cover $U_1, \ldots, U_{n+1}$ of X such that each inclusion map $U_i \to X$ is null-homotopic.[1191] If no such n exists, then we define $\mathrm{cat}(X) = \infty$.

Remark.

(1) Suppose that a given smooth manifold M can be covered by $n+1$ parametrizations. Since the domain of a parametrization is contractible and since a parametrization is a diffeomorphism onto its image we see that $\mathrm{cat}(M) \leq n$.[1192]
(2) The sets U_i in the definition of the Lusternik-Schnirelmann category do not have to be contractible, for example if X itself is contractible, then for any subset $U \subset X$ the inclusion map $U \to X$ is null-homotopic.
(3) The Lusternik-Schnirelmann category plays an important role in topological robotics, see e.g. [**Far08**] for more details.

Examples.

(1) For a non-empty topological space X we have $\mathrm{cat}(X) = 0$ if and only if $\mathrm{id}\colon X \to X$ is null-homotopic, i.e. if X is contractible.
(2) The above discussion of parametrizations of S^n together with statement (1) shows that $\mathrm{cat}(S^n) = 1$.
(3) (a) In Figure 1175 we consider the wedge X of $n \geq 1$ circles together with two open covers $X = U_1 \cup U_2$ and also $X = V_1 \cup V_2$. Note that all the inclusion maps $U_i \to X$ and also $V_i \to X$ are null-homotopic. So this gives us two separate proofs that $\mathrm{cat}(X) \leq 1$. Together with Statement (1) this shows that we have $\mathrm{cat}(X) = 1$.
(b) Using the obvious generalization of the sets V_1 and V_2 shown in Figure 1175 one can easily show that the Lusternik-Schnirelmann category of any non-empty wedge of spheres has Lusternik-Schnirelmann category equal to 1.

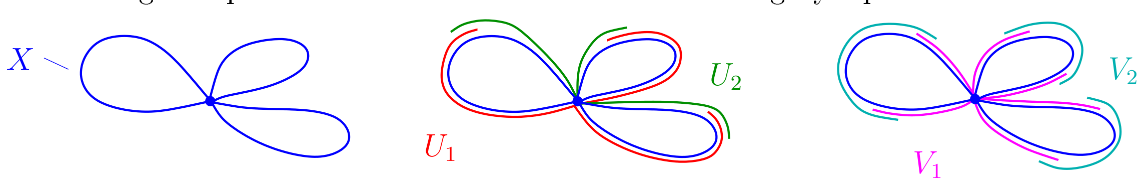


FIGURE 1175

(4) The discussion preceding Question 82.10 gives us the inequalities $\mathrm{cat}((S^1)^n) \leq n$, $\mathrm{cat}(\mathbb{R}\mathrm{P}^n) \leq n$ and also $\mathrm{cat}(\mathbb{C}\mathrm{P}^n) \leq n$.
(5) For any $g \geq 2$ the surface Σ of genus g satisfies $\mathrm{cat}(\Sigma) \leq 2$, see Figure 1176 for an illustration in the case $g = 2$.

[1190] The word "category" with its modern meaning was first used by Eilenberg-Maclane [**EM45**] in 1945. On the other hand the term "Lusternik-Schnirelmann category" was already established by 1941, see [**Fox41**].

[1191] Note the fairly well-hidden "+1" in the subscript of U_{n+1}. Sometimes, especially in the older literature the Lusternik-Schnirelmann category is defined without the adjustment given by the "+1".

[1192] Hmm, why is the image of a parametrization an open subset of the smooth manifold?

(6) In [**Hau14**, p. 147] or alternatively in [**CLOT03**, Chapter 1.2] it is shown that for any n-dimensional connected CW-complex X the inequality $\mathrm{cat}(X) \leq n$ holds. This general statement generalizes all of the above statements (2) to (5).

(7) It follows from statement (6), together with Lemma 61.24 and Theorem 64.2 that for any compact connected n-dimensional smooth manifold M we have $\mathrm{cat}(M) \leq n$.

(8) It follows from (6), together with the discussion in Section 85.6, that the same statement also holds for closed connected n-dimensional topological manifolds, as long as $n \neq 4$.[1193]

(9) If X and Y are two connected CW-complexes with finite Lusternik-Schnirelmann categories, then $\mathrm{cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y)$, see e.g. [**Fox41**, Theorem 9] or [**CLOT03**, Theorem 1.37]. The Ganea Conjecture from 1971 predicted that for any CW-complex X one has $\mathrm{cat}(X \times S^n) = \mathrm{cat}(X) + \mathrm{cat}(S^n) = \mathrm{cat}(X) + 1$. But this conjecture was eventually disproved by Norio Iwase [**Iw98, Iw02**] in 1998.

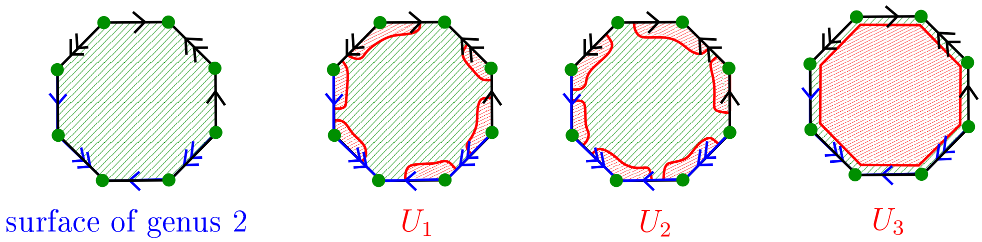


FIGURE 1176

As we saw in the above examples, one can obtain an upper bound on the Lusternik-Schnirelmann category of a given topological space by writing down explicit open subsets. To obtain lower bounds on the Lusternik-Schnirelmann category we introduce the following definition.

Definition. Given a topological space X we define its *cup length* $\mathrm{cl}(X) \in \mathbb{N}_0 \cup \{\infty\}$ as follows:

$$\mathrm{cl}(X) \;:=\; \max\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{c} \text{there exists a commutative ring } R \text{ and } R\text{-cohomology} \\ \text{classes } x_1, \dots, x_n \text{ in degrees} \geq 1 \text{ with } x_1 \cup \dots \cup x_n \neq 0 \end{array} \right\}.$$

Examples.

(1) If X is an n-dimensional CW-complex, then it follows immediately from Proposition 74.4 that $\mathrm{cl}(X) \leq n$.

(2) In Lemma 81.14 and Lemma 81.16 we saw that for any surface Σ of genus $g \geq 1$ there exist $x, y \in \mathrm{H}^1(\Sigma; \mathbb{Z})$ with $x \cup y \neq 0$, which together with (1) shows that $\mathrm{cl}(\Sigma) = 2$.

(3) In Lemma 81.18 we saw that there exists an $x \in \mathrm{H}_1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$ with $x \cup x \neq 0$, which shows that $\mathrm{cl}(\mathbb{R}\mathrm{P}^2) = 2$.

The following proposition shows that the cup length is a lower bound on the Lusternik-Schnirelmann category.

[1193] I do not know whether the case of 4-dimensional topological manifolds is known.

Proposition 82.11. *For every topological space X we have*

$$\mathrm{cl}(X) \;\le\; \mathrm{cat}(X).$$

Example. It follows immediately from Proposition 82.11 and the above examples that

$$\mathrm{cat}(S^1 \times S^1) \;=\; \mathrm{cat}(\text{surface of genus } g \ge 2) \;=\; \mathrm{cat}(\mathbb{RP}^2) \;=\; 2.$$

Proof. Let X be a topological space, let R be a commutative ring and let $x_j \in \mathrm{H}^{s_j}(X;R)$, $j = 1, \dots, n$ be cohomology classes in dimensions $s_j \ge 1$ such that $x_1 \cup x_2 \cup \dots \cup x_n \ne 0$. We have to show that $\mathrm{cat}(X) \ge n$.

Thus let $k \le n$ and let $U_1, \dots, U_k$ be open subsets in X with $U_1 \cup \dots \cup U_k = X$ such that each inclusion map $U_i \to X$ is null-homotopic. We have to show that $U_1 \cup \dots \cup U_k$ is a *proper* subset of X.

Claim. For any $j \in \{1, \dots, k\}$ the obvious map $\rho_j \colon \mathrm{H}^{s_j}(X, U_j; R) \to \mathrm{H}^{s_j}(X;R)$ is in fact an isomorphism.

By our hypothesis the inclusion map $i_j \colon U_j \to X$ is null-homotopic, it follows that the restriction map $i_j^* \colon \mathrm{H}^k(X;R) \to \mathrm{H}^k(U_j;R)$ is the zero map for $k \ge 1$ and an epimorphism for $k = 0$.[1194] We deduce from this observation, the long exact sequence of cohomology groups corresponding to the pair (X, U_j) and the fact that $s_j \ge 1$ that the obvious map $\rho_j \colon \mathrm{H}^{s_j}(X, U_j; R) \to \mathrm{H}^{s_j}(X;R)$ is in fact an isomorphism. ⊞

Given $j \in \{1 \dots, k\}$ we write $y_j := \rho_j^{-1}(x_j) \in \mathrm{H}^{s_j}(X, U_j; R)$ and we set $s = s_1 + \dots + s_k$. By Corollary 82.5[1195] we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^{s_1}(X,U_1;R) \times \dots \times \mathrm{H}^{s_k}(X,U_k;R) \arrow[r, "\cup"] \arrow[d, "{\substack{(y_1,\dots,y_k)\\ \downarrow \\ (x_1,\dots,x_k)}}"] & \mathrm{H}^s(X, U_1 \cup \dots \cup U_k; R) \arrow[d] \\
\mathrm{H}^{s_1}(X;R) \times \dots \times \mathrm{H}^{s_k}(X;R) \arrow[r, "\cup"] & \mathrm{H}^s(X;R).
\end{tikzcd}$$

By our hypotheses the cup product $x_1 \cup \dots \cup x_k$ is non-zero, which then implies, by the commutativity of the diagram, that the cup product $y_1 \cup \dots \cup y_k$ is also non-zero. But this implies that the group $\mathrm{H}^s(X, U_1 \cup \dots \cup U_k; R)$ is non-zero, which in turn implies that $U_1 \cup \dots \cup U_k$ is a proper subset of X. ■

Proposition 82.11 gives us a tool to attack Question 82.10. In fact we saw right after Proposition 82.11 that we can now answer the question for the 2-dimensional torus $(S^1)^2$ and the real projective plane $\mathbb{RP}^2$.

The following questions naturally arise:

Question 82.12.

[1194] This can be seen as follows: the map $i_j \colon U_j \to X$ is homotopic to a constant map that takes the value $P_j \in X$. We denote by $c_j \colon U_j \to \{P_j\}$ the constant map and we denote by $k_j \colon \{P_j\} \to X$ the inclusion map. We statement now follows from $i_j^* = (k_j \circ c_j)^* = c_j^* \circ k_j^*$ and Lemmas 73.13 and 73.12.

[1195] Strictly speaking Corollary 82.5 was formulated for the cup product of two terms, but one can easily reduce the general case of k terms to the special case of 2 terms.

(1) *What is the $\mathbb{Z}$-cohomology ring of $(S^1)^n$?*
(2) *What is the $\mathbb{F}_2$-cohomology ring of $\mathbb{R}\mathrm{P}^n$?*
(3) *What is the $\mathbb{Z}$-cohomology ring of $\mathbb{C}\mathrm{P}^n$?*

We will answer these questions in Propositions 84.22, 90.16 and 90.7.

There are many other questions that we cannot answer in these notes, in fact which are still open. As an example we conclude this introduction to the Lusternik-Schnirelmann category of a topological space with the following conjecture which is formulated as a question in [**Rudy17**, Question 1.1].

Conjecture 82.13. (Rudyak Conjecture) *Let M and N be two closed oriented connected n-dimensional topological manifolds. If there exists a map $f\colon M \to N$ of degree one, then* $\mathrm{cat}(M) \geq \mathrm{cat}(N)$.

Exercises for Chapter 82.

Exercise 82.1. Let (X, A, B) be an excisive triad of topological spaces and let R be a commutative ring. Show that for every $\varphi \in \mathrm{H}^k(X, A; R)$ and $\psi \in \mathrm{H}^l(X, B; R)$ the following equality holds:

$$\varphi \cup \psi \;=\; (-1)^{kl} \cdot \psi \cup \varphi \;\in\; \mathrm{H}^{k+l}(X, A \cup B; R).$$

Hint. Evidently you should study the proof of Proposition 81.8 carefully.

Exercise 82.2. Let M be the torus minus one open disk. Show that the cup product $\mathrm{H}^1(M, \partial M; \mathbb{Z}) \times \mathrm{H}^1(M, \partial M; \mathbb{Z}) \to \mathrm{H}^2(M, \partial M; \mathbb{Z})$ is non-trivial.
Hint. Reduce this statement to the calculation of the cup product of the torus.

Exercise 82.3. Let N be the disk $\overline{B}^2$ minus two open disks.

(a) Show that the two homology classes $\alpha, \beta \in \mathrm{H}_1(N, \partial N; \mathbb{Z})$ shown in Figure 1177 are a basis for $\mathrm{H}_1(N, \partial N; \mathbb{Z}) \cong \mathbb{Z}^2$.
(b) We denote by $\varphi, \psi \in \mathrm{H}^1(N, \partial N; \mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(N, \partial N; \mathbb{Z}), \mathbb{Z})$ the basis dual to α, β. Show that $\alpha \cup \beta = 0 \in \mathrm{H}^2(N, \partial N; \mathbb{Z})$.
Hint. Show that there exist open subsets A and B with the following properties:
(i) $\partial N \subset A$ and $\partial N \subset B$.
(ii) $A \cup B = N$.
(iii) φ lies in the image of $\mathrm{H}^1(N, A; \mathbb{Z}) \to \mathrm{H}^1(N, \partial N; \mathbb{Z})$ and ψ lies in the image of $\mathrm{H}^1(N, B; \mathbb{Z}) \to \mathrm{H}^1(N, \partial N; \mathbb{Z})$.
(c) Show that the cup product $\mathrm{H}^1(N, \partial N; \mathbb{Z}) \times \mathrm{H}^1(N, \partial N; \mathbb{Z}) \to \mathrm{H}^2(N, \partial N; \mathbb{Z})$ is the zero map.

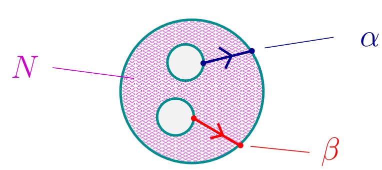


FIGURE 1177. Illustration for Exercise 82.3.

Exercise 82.4. Let M be the torus minus one open disk and let N be the disk $\overline{B}^2$ minus two open disks. Use the calculations from Exercise 82.2 and 82.3 to show that there is no degree one map $g\colon N \to M$.
Remark. This gives an alternative solution to Exercise 69.6 (b).

Exercise 82.5. As in Exercise 82.5 we consider $W = \mathbb{R}^2 \setminus \{(n,0)\,|\,n \in \mathbb{Z}\}$ and we consider the surface X of infinite genus that is shown in Figure 1178.

(a) Show that the cup product $\mathrm{H}^1_{\mathrm{c}}(W;\mathbb{Z}) \times \mathrm{H}^1_{\mathrm{c}}(W;\mathbb{Z}) \to \mathrm{H}^2_{\mathrm{c}}(W;\mathbb{Z})$, that we defined on page 2008, is the zero map.
(b) Show that the cup product $\mathrm{H}^1_{\mathrm{c}}(X;\mathbb{Z}) \times \mathrm{H}^1_{\mathrm{c}}(X;\mathbb{Z}) \to \mathrm{H}^2_{\mathrm{c}}(X;\mathbb{Z})$ is non-zero.
Hint. You could use the pairing $\mathrm{H}^2_{\mathrm{c}}(X;\mathbb{Z}) \times \mathrm{H}^{\mathrm{lf}}_2(X) \to \mathbb{Z}$ from Exercise 87.5 to detect that an element in $\mathrm{H}^2_{\mathrm{c}}(X;\mathbb{Z})$ is non-zero.
(c) Show that X and W are not homeomorphic.
(d) Does this argument imply that X and W are not homotopy equivalent?

Remark. In Exercise 23.14 we showed the weaker statement that W and X, viewed as 2-dimensional smooth manifolds, are not diffeomorphic.

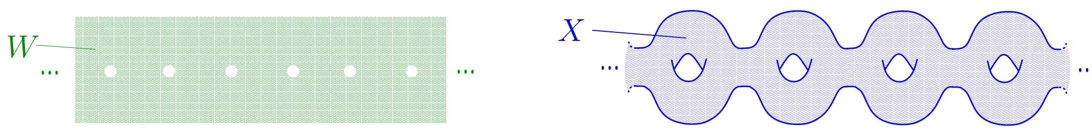


FIGURE 1178. Illustration of Exercise 82.5.

Exercise 82.6. Let X and Y be non-empty topological spaces. Show that we have the following lower bound on the cup length of $X \times Y$:

$$\mathrm{cl}(X \times Y) \;\geq\; \max\{\mathrm{cl}(X), \mathrm{cl}(Y)\}.$$

Exercise 82.7. Let X and Y be topological spaces. Show that if X and Y are homotopy equivalent, then the cup lengths of X and Y agree, i.e. we have $\mathrm{cl}(X) = \mathrm{cl}(Y)$.

Exercise 82.8. Determine the Lusternik-Schnirelmann category for all compact (not necessarily connected) 2-dimensional smooth manifolds.

83. The cap product

In this chapter we introduce the cap product that relates the homology and the cohomology groups. It can be viewed as a generalization of the Kronecker pairing. Initially it might not be clear that the cap product is good for, but later on we will see that it plays an essential role in many calculations and in the formulation of the Poincaré Duality Theorem.

83.1. **The definition of the cap product and calculations.**

We start out by introducing a convention that simplifies our notation later on.

Convention. Let R be a commutative ring and let S be a non-empty set. We consider

$$R^{(S)} \;:=\; \text{all maps from } S \text{ to } R \text{ which are non-zero for only finitely many } s\in S$$

with the obvious R-module structure. We refer to $R^{(S)}$ as the *free R-module generated by S*. For example, the free $\mathbb{Z}$-module generated by S is precisely the same as the free abelian group generated by S. We use the same convention as on page 580 to identify elements in $R^{(S)}$ with finite formal linear combinations $r_1s_1+\dots+r_ks_k$ where $r_1,\dots,r_k\in R$ and $s_1,\dots,s_k\in S$. By Lemma 57.3 the map

$$\begin{array}{rcl} \mathbb{Z}^{(S)}\otimes R & \to & R^{(S)} \\ \sum\limits_{i=1}^{n} n_i\cdot s_i\otimes r_i & \mapsto & \sum\limits_{i=1}^{n} \underbrace{n_ir_i}_{\in R}\cdot s_i \end{array}$$

is an isomorphism. We use this isomorphism to identify the left-hand side with the right-hand side. In particular, given a topological space X and given $k\in N_0$ we write

$$\mathrm{C}_k(X;R) \;:=\; \mathbb{Z}^{S_k(X)}\otimes R \;=\; R^{S_k(X)} \;=\; \Big\{\sum_{i=1}^{n} r_i\cdot\sigma_i \,\Big|\, r_1,\dots,r_n\in R \text{ and } \sigma_i\colon \Delta^k\to X\Big\},$$

where $S_k(X)$ denotes the set of singular k simplices in X.

The following definition is inspired by the second definition of the cup product that we gave on page 1981.

Definition. Let X be a topological space and let R be a commutative ring. Given a singular cochain $\varphi\in\mathrm{C}^k(X;R)$ and a singular simplex $\sigma\colon\Delta^l\to X$ with $k\le l$ we define the *cap product* as[1196]

$$\varphi\cap\sigma \;:=\; \underbrace{\varphi(\sigma\circ[v_0,\dots,v_k])}_{\in R}\cdot\underbrace{\sigma\circ[v_k,\dots,v_l]}_{\text{map } \Delta^{l\text{-}k}\to X}.$$

We extend this definition to singular R-chains and we obtain the cap product

$$\begin{array}{rrcl} \cap\colon & \mathrm{C}^k(X;R)\times\mathrm{C}_l(X;R) & \to & \mathrm{C}_{l-k}(X;R) \\ & \varphi\cap r\cdot\sigma & \mapsto & r\cdot\underbrace{(\varphi\cap\sigma)}_{\in R}. \end{array}$$

For $k>l$ we define the cap product to be the zero map.

[1196] Here we use, as usual, the notation from page 1970.

As for the cup product, the first thing we want to verify is that this map on the cochain and chain level defines a map on the level of cohomology groups and homology groups. The key to doing so is the following elementary lemma.

Lemma 83.1. *Let X be a topological space and let R be a commutative ring. Furthermore let $\varphi \in \mathrm{C}^k(X;R)$ and let $\sigma \in \mathrm{C}_l(X;R)$. Then*

$$\partial(\varphi \cap \sigma) \;=\; (-1)^k \cdot (-\delta\varphi \cap \sigma \,+\, \varphi \cap \partial\sigma) \;\in\; \mathrm{C}_{l-k-1}(X;R).$$

In particular this implies that $\varphi \cap \partial\sigma$ and $\delta\varphi \cap \sigma$ are homologous.

Proof. For $k > l$ the statement is trivial. So assume that $k \leq l$. Clearly it suffices to prove the equality for the case that $\sigma\colon \Delta^l \to X$ is a singular l-simplex. This case follows from an elementary calculation. Indeed, we have

$$\begin{array}{llcl}
\text{(a)} & \varphi \cap \partial\sigma & = & \sum\limits_{i=0}^{k}(-1)^i \cdot \varphi(\sigma \circ [v_0,\dots,\widehat{v}_i,\dots,v_{k+1}]) \cdot \sigma \circ [v_{k+1},\dots,v_l] \\
 & & & \quad + \sum\limits_{i=k+1}^{l} (-1)^i \cdot \varphi(\sigma \circ [v_0,\dots,v_k]) \cdot \sigma \circ [v_k,\dots,\widehat{v}_i,\dots,v_l] \\
\text{(b)} & \delta\varphi \cap \sigma & = & \sum\limits_{i=0}^{k+1}(-1)^i \cdot \varphi(\sigma \circ [v_0,\dots,\widehat{v}_i,\dots,v_{k+1}]) \cdot \sigma \circ [v_{k+1},\dots,v_l] \\
\text{(c)} & (-1)^k \cdot \partial(\varphi \cap \sigma) & = & \sum\limits_{i=k}^{l}(-1)^i \cdot \varphi(\sigma \circ [v_0,\dots,v_k]) \cdot \sigma \circ [v_k,\dots,\widehat{v}_i,\dots,v_l].
\end{array}$$

Note that the summand of (b) corresponding to $i = k+1$ equals *minus* the summand of (c) corresponding to $i = k$. Adding (b) and (c) and comparing it to (a) we see that

$$\varphi \cap \partial\sigma \;=\; \delta\varphi \cap \sigma + (-1)^k \cdot \partial(\varphi \cap \sigma)$$

which gives the desired equality. ■

The following lemma says that "$\cap$" descends to a map on homology and cohomology.

Lemma 83.2. *Let X be a topological space and let R be a commutative ring. Then for any $k, l \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl}
\mathrm{H}^k(X;R) \times \mathrm{H}_l(X;R) & \to & \mathrm{H}_{l-k}(X;R) \\
([\varphi],[\sigma]) & \mapsto & [\varphi \cap \sigma]
\end{array}$$

is well-defined and R-bilinear.

We refer to the pairing from Lemma 83.2 as the *cap product.*[1197]

Proof. For $k > l$ the statement is trivial. So assume that $k \leq l$. The statement now follows easily from Lemma 83.1. Indeed, let $\varphi \in \mathrm{C}^k(X;R)$ be a cocycle and let $\sigma \in \mathrm{C}_l(X;R)$ be a cycle.

(1) It follows from Lemma 83.1 that $\varphi \cap \sigma$ is a cycle, i.e. it represents a homology class.

[1197]The name "cap product" comes from the notation "$a \cap b$" for the composition since $\cap$ looks like, just a little bit, like the "a" in cap. The other reason is that the cup product was first introduced, and mathematicians needed a different name for a somewhat related object. Also mathematicians are well-known for their great sense of humor.

(2) If $\sigma' = \sigma + \partial\tau$ is another representative of $[\sigma]$, then

$$\varphi \cap \sigma - \varphi \cap \sigma' \;=\; \varphi \cap (\sigma - \sigma') \;=\; \varphi \cap \partial\tau \underset{\substack{\uparrow \\ \text{by Lemma 83.1 and since } \delta\varphi = 0}}{=} \pm\partial(\varphi \cap \tau).$$

This shows that $\varphi \cap \sigma$ and $\varphi \cap \sigma'$ represent the same homology class, i.e. the cap product does not depend on the choice of the representative of the homology class.

(3) The proof that the cap product does not depend on the choice of representative of the cohomology class is basically the same as the proof of (2).

This shows that the map is well-defined. It is clear that the map is indeed R-bilinear. ■

Remark. We already saw on page 1981 that the definition of the cup product in the literature is not uniform. Rather depressingly, the situation is even worse for the cap product, there is an endless variety of definitions of the cap product in the literature. More precisely, given a cochain $\varphi \in C^k(X;R)$ and a singular simplex $\sigma\colon \Delta^l \to X$ the following definitions of $\varphi \cap \sigma$ can be found in the literature:[1198]

		$\varphi \cap \sigma$
(a)	Hatcher [**Hat02**, p. 239]	$\varphi(\sigma \circ [v_0,\dots,v_k]) \cdot \sigma \circ [v_k,\dots,v_l]$
(b)	Bredon [**Bre93**, p. 335]	$(-1)^{k(l-k)} \cdot \varphi(\sigma \circ [v_0,\dots,v_k]) \cdot \sigma \circ [v_k,\dots,v_l]$
(c)	Dold [**Dol56**, p. 335]	$(-1)^{k(l-k)} \cdot \sigma \circ [v_0,\dots,v_{l-k}] \cdot \varphi(\sigma \circ [v_{l-k},\dots,v_l])$
(d)	Spanier [**Spa95**, p. 254]	$\sigma \circ [v_0,\dots,v_{l-k}] \cdot \varphi(\sigma \circ [v_{l-k},\dots,v_l])$
(e)	Munkres [**Mun84**, p. 389]	$\sigma \circ [v_0,\dots,v_{l-k}] \cdot \varphi(\sigma \circ [v_{l-k},\dots,v_l])$

To the best of my understanding, see [**Spa95**, p. 250], the singular chains

$$\varphi(\sigma \circ [v_0,\dots,v_k]) \cdot \sigma \circ [v_k,\dots,v_l] \quad \text{and } (-1)^{k(l-k)} \cdot \sigma \circ [v_0,\dots,v_{l-k}] \cdot \varphi(\sigma \circ [v_{l-k},\dots,v_l])$$

are homologous. This shows that on the level of homology and cohomology groups the definition (a), which is also our definition, agrees with (c), but that the sign differs for (b), (d) and (e). For the most part this is not an issue, but one needs to be careful when one cites formulas from the literature. For example, as we had already mentioned on page 1821, Bredon [**Bre93**] defined the coboundary map as

$$\delta_k^{\text{Bre}} \;:=\; (-1)^{k+1} \cdot \partial_{k+1}^* \;=\; (-1)^{k+1} \cdot \delta_k$$

and as we saw above, the cap product is also defined differently. Thus our Lemma 83.1, which says that for $\varphi \in C^k(X;R)$ and $\sigma\colon \Delta^l \to X$ we have

$$\partial(\varphi \cap \sigma) \;=\; (-1)^k \cdot (\varphi \cap \partial\sigma \,-\, \delta\varphi \cap \sigma)$$

becomes the following equality in [**Bre93**, Proposition VI.5.1]:[1199]

$$\partial(\varphi \cap \sigma) \;=\; (-1)^k \cdot \varphi \cap \partial\sigma \,+\, \delta^{\text{Bre}}(\varphi) \cap \sigma.$$

[1198] The definition of the cap product given by Greenberg-Harper in [**GH81**, p. 205] agrees with our definition, *except*, that they use a different notation, namely they cap a homology class with a cohomology class instead of capping a cohomology class with a homology class. In other words, in our setting they write $\sigma \cap \varphi$ instead of $\varphi \cap \sigma$.

[1199] The lesson you should draw is that if you ever have to write algebraic topology lectures notes, stick to a single book and do not try to mix books.

The following lemma follows immediately from the definitions.

Lemma 83.3. *Let X be a topological space and let R be a commutative ring. Then for any $\sigma \in \mathrm{H}_k(X;R)$ we have*
$$1_X \cap \sigma \;=\; \sigma.$$

For future applications we state the special case $k=l$ in the cap product in the following lemma.

Lemma 83.4. *Let X be a topological space and let R be a commutative ring.*
(1) *The composition*
$$\mathrm{H}^k(X;R) \times \mathrm{H}_k(X;R) \xrightarrow{\cap} \mathrm{H}_0(X;R) \xrightarrow[\text{from page 1401}]{\text{augmentation } \epsilon} R$$
is precisely the Kronecker pairing from pages 1847 and 1874.
(2) *If X is path-connected, and if we make the usual identification $\mathrm{H}_0(X;R) = R$ given by the augmentation map, see Proposition 41.5, then for any $\varphi \in \mathrm{H}^k(X;R)$ and $\sigma \in \mathrm{H}_k(X;R)$ we have*
$$\varphi \cap \sigma \;=\; \langle \varphi, \sigma \rangle \;\in\; R.$$

Proof. Both statements follow immediately from recalling all the relevant definitions. ■

Example. We consider some examples of the cap product in detail to get some feeling for what it does.

(1) Let $T = S^1 \times S^1 = ([0,1] \times [0,1])/\sim$ be the torus. We recall the following notations and results that we have by now used on many occasions. (We refer to Figure 1179 for some of the notation.)
  (a) We denote by σ_1, σ_2 the singular 2-simplices from Figure 1179. As we pointed out on page 1717, the singular 2-chain $-\sigma_1 + \sigma_2$ is a representative of the fundamental class $[T]$.
  (b) We denote by p respectively q the projection $S^1 \times S^1 \to S^1$ onto the first respectively second factor.
  (c) We denote by $\theta := \theta_{\mathbb{Z}} \in \mathrm{C}^1; \mathbb{Z})$ the singular 1-cocycle that we introduced on page 1825.
  (d) As before we denote by $\mu\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $\mu(1-t,t) = e^{2\pi i t}$. By the discussion on page 1714 we know that μ represents the fundamental class $[S^1]$ of S^1 equipped with the standard orientation. By a slight abuse of notation we also denote by $[1 \times S^1]$ the image of the fundamental class under the monomorphism $\mathrm{H}_1(1 \times S^1; \mathbb{Z}) \to \mathrm{H}_1(S^1 \times S^1; \mathbb{Z})$.

  We calculate that
$$p^*(\theta) \cap \sigma_1 \;=\; \underbrace{p^*(\theta)(\sigma_1 \circ [v_0,v_1])}_{=\theta(p \circ \sigma_1 \circ [v_0,v_1]) = 0} \cdot \sigma_1 \circ [v_1,v_2] \;=\; 0$$
  and similarly we calculate that
$$p^*(\theta) \cap \sigma_2 \;=\; \underbrace{p^*(\theta)(\sigma_2 \circ [v_0,v_1])}_{=\theta(p \circ \sigma_2 \circ [v_0,v_1]) = 1} \cdot \sigma_2 \circ [v_1,v_2] \;=\sigma_2 \circ [v_1,v_2] = 1 \times \mu.$$

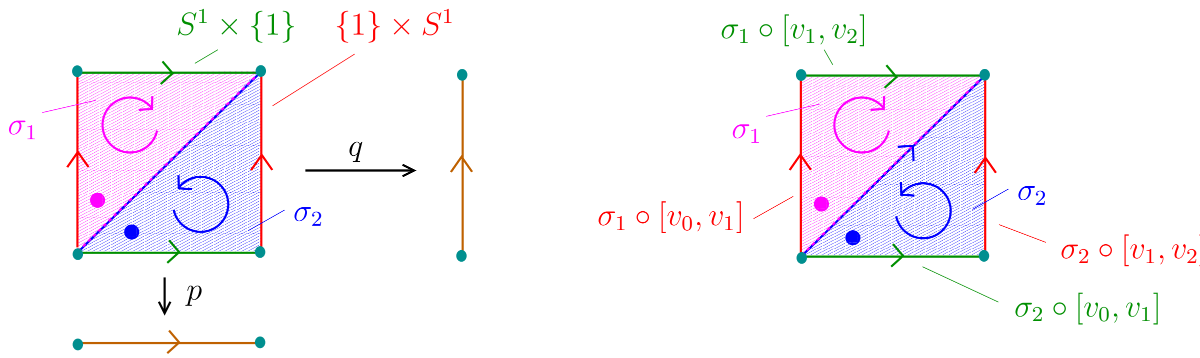


FIGURE 1179

Putting these calculations together we obtain that

$$[p^*(\theta)] \cap [T] = [p^*(\theta) \cap (-\sigma_1 + \sigma_2)] = [1 \times S^1].$$

Basically the same calculation shows that

$$[q^*(\theta)] \cap [T] = [q^*(\theta) \cap (-\sigma_1 + \sigma_2)] = -[S^1 \times 1].$$

So we see that the map

$$\cap[T]\colon \mathrm{H}^1(T;\mathbb{Z}) \to \mathrm{H}_1(T;\mathbb{Z})$$

is non-trivial, in fact we have just shown that this map satisfies

$$\begin{array}{rcl} [p^*(\theta)] & \mapsto & [1 \times S^1] \\ [q^*(\theta)] & \mapsto & -[S^1 \times 1] \end{array}$$

which shows that "capping with the fundamental class" is in this case an isomorphism.

(2) In Exercise 83.2 we will deal with the surface Σ of genus 2 and there we will also see that the map

$$\cap[\Sigma]\colon \mathrm{H}^1(\Sigma;\mathbb{Z}) \to \mathrm{H}_1(\Sigma;\mathbb{Z})$$

is an isomorphism.

(3) Finally we consider again the real projective plane $\mathbb{RP}^2 = \overline{B}^2/\sim$. We introduce the following notation:
   (a) We denote by $\varphi \in \mathrm{C}^1(\mathbb{RP}^2;\mathbb{F}_2)$ the cocycle that we introduced on page 1998.
   (b) We denote by $\gamma\colon \Delta^1 \to \mathbb{RP}^2 = \overline{B}^2/\sim$ the singular 1-simplex that is given by $\gamma(t, 1-t) = [e^{\pi i t}]$. Note that γ is a cycle in $\mathrm{C}_1(\mathbb{RP}^2)$ and note that $\varphi(\gamma) = 1$.
   (c) In Figure 1064 we already saw a representative for the $\mathbb{F}_2$-fundamental class, but we will work with a different one that is "smaller" and that is shown in Figure 1180. In Figure 1180 we see the singular 2-simplex σ that sends the face $[v_0, v_2]$ to a point P and the singular 2-simplex τ that sends all points on Δ^2 to P. It is straightforward to see that $(\sigma + \tau) \otimes 1 \in \mathrm{C}_2(\mathbb{RP}^2;\mathbb{F}_2)$ is a cycle. Using Proposition 68.18 we see that it represents the $\mathbb{F}_2$-fundamental class of $\mathbb{RP}^2$.

It is straightforward to see that

$$[\varphi] \cap [\mathbb{RP}^2] = [\varphi] \cap [(\sigma + \tau) \otimes 1] = [\underbrace{\varphi \cap \sigma}_{=\gamma} \otimes 1 + \underbrace{\varphi \cap \tau}_{=0} \otimes 1] = [\gamma \otimes 1]$$

We saw on page 1998 that $[\gamma \otimes 1]$ is the unique non-trivial element of $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$. Thus in this case we see that the map

$$\cap[\mathbb{R}\mathrm{P}^2]\colon\ \mathrm{H}^1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2) \ \to\ \mathrm{H}_1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$$

is also an isomorphism.

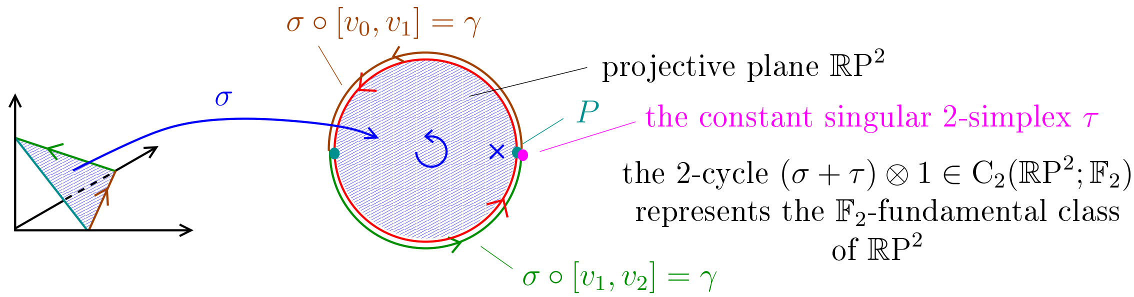


Figure 1180

Remark. In the last three examples we saw for the closed 2-dimensional smooth manifolds $F = S^1 \times S^1$, $F = \Sigma$, $F = \mathbb{R}\mathrm{P}^2$ and for $R = \mathbb{Z}$ respectively $R = \mathbb{F}_2$ that the map

$$- \cap [F]_R\colon\ \mathrm{H}^1(F; R) \ \to\ \mathrm{H}_1(F; R)$$

is an isomorphism. In Chapter 88 we will see that it is not a coincidence. More precisely we will see that this observation can be generalized to all closed topological manifolds.

83.2. **The relative cap product and more properties of the cap product.** After considering the various examples of the cap product we continue with developing the theory.

Similar to the case of the cup product, we intend to extend the notion of cap product to relative homology and cohomology groups. As in the case of the cup product it requires some delicate footwork to get to the right notion of a relative cap product. We start out with the following rather technical definition.

Definition. Let (X, A, B) be a triad of topological spaces and let R be a commutative ring. We write

$$\mathrm{C}_n(X, \{A, B\}; R) \ :=\ \frac{\mathrm{C}_n(X) \otimes R}{\mathrm{C}_n^{\{A,B\}}(A \cup B) \otimes R}.$$

The boundary map on $\mathrm{C}_*(X; R)$ descends to a boundary map on $\mathrm{C}_*(X, \{A, B\}; R)$ which we can thus view as a chain complex. We denote the corresponding homology groups by $\mathrm{H}_*(X, \{A, B\}; R)$.

This technical definition allows us to define a first version of a relative cap product.

Lemma 83.5. *Let (X, A, B) be a triad of topological spaces, let $k \leq l$ and let R be a commutative ring. Then the map*

$$\begin{array}{rcl} \cap\colon\ \mathrm{C}^k(X, A; R) \times \mathrm{C}_l(X, \{A, B\}; R) & \to & \mathrm{C}_{l-k}(X, B; R) \\ (\varphi, [\sigma\colon \Delta^l \to X]) & \mapsto & \Big[\underbrace{\varphi(\sigma \circ [v_0, \dots, v_k])}_{\in R} \cdot \underbrace{\sigma \circ [v_k, \dots, v_l]}_{\text{map } \Delta^{l\text{-}k} \to X}\Big] \end{array}$$

is well-defined and it descends to a map

$$\cap\colon\ \mathrm{H}^k(X,A;R)\times\mathrm{H}_l(X,\{A,B\};R)\ \to\ \mathrm{H}_{l-k}(X,B;R)$$

that is R-bilinear.

Proof. We first show that the cap product on (co-) chains is well-defined. By definition of

$$\mathrm{C}_n(X,\{A,B\};R)\ :=\ \frac{\mathrm{C}_n(X)\otimes R}{\mathrm{C}_n^{\{A,B\}}(A\cup B)\otimes R}\ =\ \frac{\mathrm{C}_n(X)\otimes R}{\mathrm{C}_n(A)\otimes R+\mathrm{C}_n(B)\otimes R}$$

we have to show that if we are given a singular l-simplex $\sigma\colon\Delta^l\to X$ such that

(1) the image of σ lies in A, or
(2) the image of σ lies in B,

then the product $\varphi(\sigma\circ[v_0,\dots,v_k])\cdot\sigma\circ[v_0,\dots,v_k]$ is zero in $\mathrm{C}_{l-k}(X,B;R)$. But this is indeed the case:

(1) first, if the image of σ lies in A, then it follows immediately from the hypothesis that φ lies in $\mathrm{C}^k(X,A;R)$ that $\varphi(\sigma\circ[v_0,\dots,v_k])=0$, and
(2) if the image of σ lies in B, then the image of $\sigma\circ[v_0,\dots,v_k]$ also lies in B. Hence the term is zero in $\mathrm{C}_{l-k}(X,B;R)$.

This concludes the proof of the first part of the lemma.

The argument that the map on (co-) chain complexes descends to a map on the level of homology and cohomology groups is the same as in the proof of Lemma 83.2. It is obvious that the map $\cap$ is R-bilinear. ■

Evidently we are not very interested in the group $\mathrm{H}_n(X,\{A,B\};R)$. The following lemma allows us to replace this group by the much more agreeable group $\mathrm{H}_n(X,A\cup B;R)$, at least for reasonable triads.

Lemma 83.6. *If (X,A,B) is an excisive triad of topological spaces, then given any commutative ring R the obvious map*

$$\mathrm{C}_*(X,\{A,B\};R)\quad\to\quad\mathrm{C}_*(X,A\cup B;R)$$

induces for every $n\in\mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X,\{A,B\};R)\quad\xrightarrow{\cong}\quad\mathrm{H}_n(X,A\cup B;R).$$

Proof. In the proof of Lemma 82.1 we already showed that the map

$$\mathrm{C}_*(X,\{A,B\};\mathbb{Z})\ \to\ \mathrm{C}_*(X,A\cup B;\mathbb{Z})$$

on chain complexes is a chain homotopy equivalence. By tensoring all maps and chain homotopies with R we see that the map

$$\mathrm{C}_*(X,\{A,B\};R)\ \to\ \mathrm{C}_*(X,A\cup B;R)$$

is also a chain homotopy equivalence. The second statement is then an immediate consequence of Corollary 42.3. ■

Lemma 83.6 leads to the following definition.

Definition. Let (X, A, B) be an excisive triad of topological spaces, let $k \leq l$ and let R again be a commutative ring. We refer to the map

$$\mathrm{H}^k(X,A;R)\times \mathrm{H}_l(X,A\cup B;R) \;\underset{\substack{\uparrow\\ \text{isomorphism given by Lemma 83.6}}}{\xleftarrow{\cong}}\; \mathrm{H}^k(X,A;R)\times \mathrm{H}_l(X,\{A,B\};R) \;\underset{\substack{\uparrow\\ \text{map from Lemma 83.5}}}{\xrightarrow{\cap}}\; \mathrm{H}_{l-k}(X,B;R)$$

as the *cap product* (sometimes we refer to it as the *relative cap product*) that we also denote by "$\cap$". For $k > l$ we define the cap product to be the zero map.

Remark. In many examples, e.g. if $A = \varnothing$, or $B = \varnothing$ or $A = B$, then the slightly mysterious first map in the definition of the cap product is just the identity. In particular, for any subset A of X we obtain cap products

$$\mathrm{H}^k(X;R) \;\times\; \mathrm{H}_l(X,A;R) \;\xrightarrow{\cap}\; \mathrm{H}_{l-k}(X,A;R)$$

and

$$\mathrm{H}^k(X,A;R) \;\times\; \mathrm{H}_l(X,A;R) \;\xrightarrow{\cap}\; \mathrm{H}_{l-k}(X;R)$$

and both of them are induced, on the (co-) chain level, by the formula

$$(\varphi, \sigma\colon \Delta^l \to X) \quad\mapsto\quad \varphi(\sigma\circ[v_0,\dots,v_k])\cdot\sigma\circ[v_k,\dots,v_l].$$

This observation also shows that we can easily generalize Lemma 83.4 to the relative case in the obvious way. For example, if X is path-connected, and if we make the usual identification $\mathrm{H}_0(X;\mathbb{Z}) = \mathbb{Z}$ given by the augmentation map, see Proposition 41.5, then for any $\varphi \in \mathrm{H}^k(X,A;\mathbb{Z})$ and $\sigma \in \mathrm{H}_k(X,A;\mathbb{Z})$ we have

$$\varphi \cap \sigma \;=\; \langle \varphi, \sigma\rangle \;\in\; \mathbb{Z}.$$

For the record we point out that it follows from this observation and Lemma 75.16 that if M is a compact oriented path-connected n-dimensional topological manifold, then for any point $P \in M$ we have

$$[M]^* \cap [M] \;=\; [P]$$

and $[M]^*$ is the only element in $\mathrm{H}^n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$ that has that property.

Example. Let $X = S^1 \times [0,1]$ be the annulus. In Figure 1181 we show two singular 2-simplices σ_1, σ_2. It follows easily from Proposition 68.4 that $-\sigma_1 + \sigma_2$ represents the fundamental class $[X] \in \mathrm{H}_2(X,\partial X;\mathbb{Z})$ of the annulus X with the usual orientation. We denote by $p\colon X \to [0,1]$ the projection map. As on page 1830 we consider the function

$$\begin{array}{rcl} f\colon [0,1] & \to & \mathbb{Z} \\ x & \mapsto & \begin{cases} 0, & \text{if } x \in [0,\tfrac{1}{2}) \\ 1, & \text{otherwise} \end{cases} \end{array}$$

which we use to define the singular 1-cochain

$$\begin{array}{rcl} \rho\colon\ \mathrm{C}_1([0,1]) & \to & \mathbb{Z} \\ (\sigma\colon \Delta^1 \to [0,1]) & \mapsto & f(\sigma(0,1)) - f(\sigma(1,0)). \end{array}$$

It follows from the discussion on page 1830 that $p^*(\rho) \in \mathrm{C}^1(X, \partial X; \mathbb{Z})$ is a cochain, thus it represents an element in $\mathrm{H}^1(X, \partial X; \mathbb{Z})$. We calculate that

$$\begin{aligned} [p^*(\rho)] \cap \sigma_1 &= \underbrace{p^*(\sigma_1 \circ [v_0, v_1])}_{=1} \cdot \underbrace{\sigma_1 \circ [v_1, v_2]}_{=[S^1 \times 1]} &= [S^1 \times 1] \\ [p^*(\rho)] \cap \sigma_2 &= \underbrace{p^*(\sigma_2 \circ [v_0, v_1])}_{=0} \cdot \sigma_2 \circ [v_1, v_2] &= 0. \end{aligned}$$

Adding up we see that

$$[p^*(\rho)] \cap [X] \;=\; [p^*(\rho) \cap (-\sigma_1 + \sigma_2)] \;=\; -[S^1 \times 1].$$

Since the inclusion $S^1 \times 1 \to X$ is a homotopy equivalence we see that $\mathrm{H}_1(X; \mathbb{Z}) \cong \mathbb{Z}$ and that $[S^1 \times 1]$ is a generator. A straightforward argument, e.g. using the long exact sequence of the pair $(X, \partial X)$, shows that $\mathrm{H}^1(X, \partial X; \mathbb{Z}) \cong \mathbb{Z}$. The above calculation thus shows that the map

$$\begin{aligned} \mathrm{H}^1(X, \partial X; \mathbb{Z}) &\to \mathrm{H}_1(X; \mathbb{Z}) \\ \varphi &\mapsto \varphi \cap [X] \end{aligned}$$

is an epimorphism. Since both groups are isomorphic to $\mathbb{Z}$ we see that the map is in fact an isomorphism and as a bonus we obtain that $[\rho] \in \mathrm{H}^1(X, \partial X; \mathbb{Z})$ is a generator. In Exercise 83.3 we will show that the map

$$\begin{aligned} \mathrm{H}^1(X; \mathbb{Z}) &\to \mathrm{H}_1(X, \partial X; \mathbb{Z}) \\ \varphi &\mapsto \varphi \cap [X] \end{aligned}$$

is also an isomorphism.

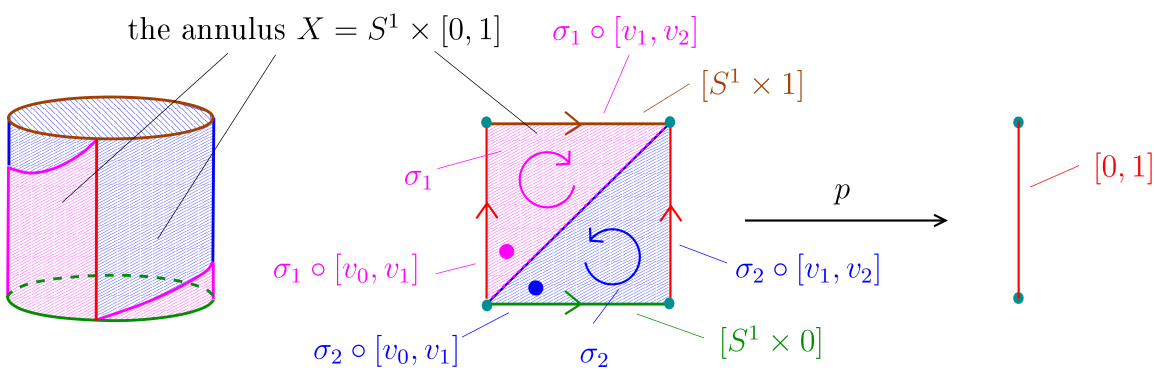


FIGURE 1181

The following lemma relates the cup product to the cap product.

Lemma 83.7. *Let X be a topological space and let R be a commutative ring.*

(1) *Let $\varphi \in \mathrm{C}^k(X; R)$, $\psi \in \mathrm{C}^l(X; R)$ and let $\sigma \in \mathrm{C}_n(X; R)$. Then*[1200]

$$\varphi \cap (\psi \cap \sigma) \;=\; (\psi \cup \varphi) \cap \sigma \;\in\; \mathrm{C}_{n-k-l}(X; R).$$

(2) *If $A \subset X$ is a subset and $C, D \in \{\varnothing, A\}$ we have for any $\varphi \in \mathrm{H}^k(X, C; R)$, any $\psi \in \mathrm{H}^l(X, D; R)$ and any $\sigma \in \mathrm{H}_n(X, C \cup D; R)$ that*

$$\varphi \cap (\psi \cap \sigma) \;=\; (\psi \cup \varphi) \cap \sigma \;\in\; \mathrm{H}_{n-k-l}(X; R).$$

(3) *If $A \subset X$ is a subset and $C, D \in \{\varnothing, A\}$ we have for any $\varphi \in \mathrm{H}^k(X, C; R)$, any $\psi \in \mathrm{H}^l(X, D; R)$ and any $\sigma \in \mathrm{H}_{k+l}(X, C \cup D; R)$ that*

$$\langle \varphi, \psi \cap \sigma \rangle \;=\; \langle \psi \cup \varphi, \sigma \rangle \;\in\; R.$$

PROOF. The case $k + l > n$ is trivial. So suppose that $k + l \le n$.

(1) So let $\varphi \in \mathrm{C}^k(X; R)$, $\psi \in \mathrm{C}^l(X; R)$. It suffices to consider the case that $\sigma\colon \Delta^n \to X$ is a singular n-simplex. We have

$$\begin{aligned} \varphi \cap (\psi \cap \sigma) &= \varphi \cap \big(\psi(\sigma \circ [v_0, \dots, v_l]) \cdot \sigma \circ [v_l, \dots, v_n]\big) \\ &= \psi(\sigma \circ [v_0, \dots, v_l]) \cdot \varphi \cap (\sigma \circ [v_l, \dots, v_n]) \\ &= \psi(\sigma \circ [v_0, \dots, v_l]) \cdot \varphi(\sigma \circ [v_l, \dots, v_{l+k}]) \cdot \sigma \circ [v_{l+k}, \dots, v_n] \\ &= (\psi \cup \varphi)(\sigma \circ [v_0, \dots, v_{l+k}]) \cdot \sigma \circ [v_{l+k}, \dots, v_n] \\ &= (\psi \cup \varphi) \cap \sigma. \end{aligned}$$

(2) The statement regarding the (co-) homology classes follows almost immediately from the above result on (co-) chains obtained in (1).

(3) This statement follows immediately from (2) together with Lemma 83.4. ∎

Now suppose we are given a map of excisive triads of topological spaces. We get induced maps on homology and cohomology and the following lemma tells us the relationship of these induced maps and the cap product.

Lemma 83.8. *Let R be a commutative ring.*

(1) *If $f\colon (X, A, B) \to (\widetilde{X}, \widetilde{A}, \widetilde{B})$ is a map between two excisive triads, then for any $k, l \in \mathbb{N}_0$ the following diagram commutes*[1201]

$$\begin{array}{ccccc} \mathrm{H}^k(X, A; R) & \times & \mathrm{H}_l(X, A \cup B; R) & \xrightarrow{\cap} & \mathrm{H}_{l-k}(X, B; R) \\ {\scriptstyle f^*}\uparrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{H}^k(\widetilde{X}, \widetilde{A}; R) & \times & \mathrm{H}_l(\widetilde{X}, \widetilde{A} \cup \widetilde{B}; R) & \xrightarrow{\cap} & \mathrm{H}_{l-k}(\widetilde{X}, \widetilde{B}; R), \end{array}$$

i.e. for any $\varphi \in \mathrm{H}^k(\widetilde{X}, \widetilde{A}; R)$ and any $\sigma \in \mathrm{H}_l(X, A \cup B; R)$ we have

$$\varphi \cap f_*(\sigma) \;=\; f_*(\underbrace{f^*(\varphi) \cap \sigma}_{\in \mathrm{H}_{l-k}(X,B;R)}) \;\in\; \mathrm{H}_{l-k}(\widetilde{X}, \widetilde{B}; R).$$

(2) *Let X and $\widetilde{X}$ be path-connected topological spaces and let $B \subset X$ and $\widetilde{B} \subset \widetilde{X}$ be subsets. If $f\colon (X, B) \to (\widetilde{X}, \widetilde{B})$ is a map, then for any $\varphi \in \mathrm{H}^k(\widetilde{X}, \widetilde{B}; R)$ and any*

[1200] Note that on the right-hand side it says $(\psi \cup \varphi) \cap \sigma$ instead of $(\varphi \cup \psi) \cap \sigma$. In fact if one picks a different sign convention for the cap product this sign goes away. For example with the sign conventions used by Bredon, see [**Bre93**, Chapter 5], the equation becomes

$$\varphi \cap (\psi \cap \sigma) \;=\; (\varphi \cup \psi) \cap \sigma$$

which looks much more appealing. Once we consider (co-) homology classes we obtain from Proposition 81.8 that the difference between these two expressions $(\psi \cup \varphi) \cap \sigma$ and $(\varphi \cup \psi) \cap \sigma$ is just a factor of $(-1)^{kl}$.

$\sigma \in \mathrm{H}_k(X,B;R)$ we have

$$\varphi \cap f_*(\sigma) \;=\; f^*(\varphi)\cap\sigma \;\in\; \underbrace{\mathrm{H}_0(X;R) = R = \mathrm{H}_0(\widetilde{X};R)}_{\substack{\text{identification from}\\ \text{Proposition 41.5}}}.$$

Remark. Note that by Lemma 83.4 (1) we can view the above as a generalization of Lemma 74.6 (3).

PROOF. The first statement follows immediately from the definitions. We leave the details to the skeptical reader. The second statement is an immediate consequence of the first statement. ■

The next lemma says that cup and cap products behave in the obvious way with respect to the path-components. The statements in the lemma follow easily from the definitions. Therefore we leave the proof of the lemma to the reader.

Lemma 83.9. (∗) *Let X be a topological space and let R be a commutative ring. We denote by $\{X_j\}_{j\in J}$ the path-components of X. By Lemmas 41.14 and 73.14 the inclusion maps $\iota_j\colon X_j \to X$, $j\in J$, induce for any $n\in\mathbb{N}_0$ isomorphisms*

$$\Phi := \bigoplus_{j\in J}(\iota_j)_*\colon \bigoplus_{j\in J}\mathrm{H}_n(X_j;G) \;\xrightarrow{\cong}\; \mathrm{H}_n(X;G)$$

and

$$\Psi := \prod_{j\in J}\iota_j^*\colon\; \mathrm{H}^n(X;G) \;\xrightarrow{\cong}\; \prod_{j\in J}\mathrm{H}^n(X_j;G).$$

Then the following two statements hold:

(1) *For any $i\neq j$ and any $\alpha\in\mathrm{H}^m(X_i;R)$ and $\beta\in\mathrm{H}^n(X_j;R)$ we have*
$$\Psi^{-1}(\alpha)\,\cup\,\Psi^{-1}(\beta) \;=\; 0.$$

(2) *For any $i\neq j$ and any $\alpha\in\mathrm{H}^m(X_i;R)$ and $\beta\in\mathrm{H}_n(X_j;R)$ we have*
$$\Psi^{-1}(\alpha)\,\cap\,\Phi(\beta) \;=\; 0.$$

The same type of statements hold for cup and cap products on relative (co-) homology groups.

In the next lemma of the chapter we relate the relative cap product to (co-) boundary maps on (co-) homology. More precisely, we have the following lemma which is similar in spirit to Lemma 82.6.

[1201] Here "the diagram commutes" is understood in the sense of the convention from page 2007.

Lemma 83.10. *Let R be a commutative ring.*

(1) *Let (X, B, A) be a triple of topological spaces. For any $n, k \in \mathbb{N}_0$ the following diagram commutes:*

$$\begin{tikzcd} \mathrm{H}^k(B, A; R) \times \mathrm{H}_{n-1}(B, A; R) \arrow[r, "\cap"] \arrow[d, "\delta"'] & \mathrm{H}_{n-k-1}(B; R) \arrow[d, "i_*"] \\ \mathrm{H}^{k+1}(X, B; R) \times \mathrm{H}_n(X, B; R) \arrow[u, "\partial"'] \arrow[r, "\cap"] & \mathrm{H}_{n-k-1}(X; R). \end{tikzcd}$$

Here $i\colon B \to X$ denotes the inclusion map and δ and ∂ denote the connecting homomorphisms in the long exact sequences in (co-) homology corresponding to the triple (X, B, A).

(2) *Let (X, A, B) be an excisive triad and let $n \in \mathbb{N}_0$. Note that by Lemma 74.13 the map $j_*\colon \mathrm{H}_{n-1}(B, A \cap B; R) \to \mathrm{H}_{n-1}(A \cup B, A; R)$ is an isomorphism. For any $k \in \mathbb{N}_0$ the following diagram commutes up to the sign $(-1)^{k+1}$:*

$$\begin{tikzcd} \mathrm{H}^k(X, A; R) \times \mathrm{H}_n(X, A \cup B; R) \arrow[rr, "\cap"] \arrow[d, "i^* \times \partial"] & & \mathrm{H}_{n-k}(X, B; R) \arrow[dd, "\partial"] \\ \mathrm{H}^k(A \cup B, A; R) \times \mathrm{H}_{n-1}(A \cup B, A; R) \arrow[d, "j^* \times (j_*)^{-1}", "\cong"'] & & \\ \mathrm{H}^k(B, A \cap B; R) \times \mathrm{H}_{n-1}(B, A \cap B; R) \arrow[rr, "\cap"] & & \mathrm{H}_{n-k-1}(B; R). \end{tikzcd}$$

Here $i\colon (A \cup B, A) \to (X, A)$ and $j\colon (B, A \cap B) \to (A \cup B, A)$ denote the inclusion maps and ∂ denotes the connecting homomorphisms in the long exact sequences in homology corresponding to the triple $(X, A \cup B, A)$ respectively the pair (X, B).

Proof (∗).

(1) We pick $\psi \in \mathrm{H}^k(B, A; R)$ and $\tau \in \mathrm{H}_n(B, A; R)$. We make the following observations:
 (a) We can find $\sigma \in \mathrm{C}_n(B; R)$ such that the image of σ in $\mathrm{C}_n(B, A; R)$ is a cycle that represents τ.
 (b) We can pick a singular cochain $\varphi \in \mathrm{C}^k(B; R) = \mathrm{Hom}(\mathrm{C}_k(B), R)$ that vanishes on $\mathrm{C}_k(A)$ and such that the corresponding element in $\mathrm{C}^k(B, A; R)$ represents ψ. We denote by $\widetilde{\varphi}\colon \mathrm{C}_k(X) \to R$ the extension of φ that is defined to be zero on all singular k-simplices that do not lie in B.

 We have the following equalities:

$$\delta\psi \cap \tau \underset{\substack{\uparrow \\ \text{see Lemma 73.15 and the} \\ \text{discussion on page 1833}}}{=} \left[\delta\widetilde{\varphi}\right] \cap [\sigma] \overset{\substack{\text{by the discussion of relative cap products on page 2026} \\ \downarrow}}{=} \left[\delta\widetilde{\varphi} \cap \sigma\right] \underset{\substack{\uparrow \\ \text{by Lemma 83.1 the classes } \delta\varphi \cap \sigma \\ \text{and } \varphi \cap \partial\sigma \text{ are homologous}}}{=} [\widetilde{\varphi} \cap \partial\sigma] = [\varphi \cap \partial\sigma] \overset{\downarrow}{=} [\varphi] \cap [\partial\sigma] \underset{\substack{\uparrow \\ \text{Proposition 43.15}}}{=} \psi \cap \partial\tau.$$

 This equality implies that the diagram commutes.[1202]

[1202] With the various quotient groups, lifts, extensions, inclusions and so on it is admittedly quite tricky to figure out in between where any of the equalities hold.

(2) We pick $\psi \in \mathrm{H}^k(X,A;R)$ and $\tau \in \mathrm{H}_n(X,A\cup B;R)$. We make the following observations:
   (a) It follows immediately from our hypothesis that the triad (X,A,B) is excisive together with Lemma 83.6 that we can find $\sigma \in \mathrm{C}_n(X;R)$ such that $\partial\sigma = \alpha+\beta$ with $\alpha \in \mathrm{C}_{n-1}(A;R)$ and $\beta \in \mathrm{C}_{n-1}(B;R)$ and such that the image of σ in $\mathrm{C}_n(X,A\cup B;R)$ is a cycle that represents τ.
   (b) As on page 2003 we make the identification

$$\mathrm{C}^k(X,A;R) \xrightarrow{=} \text{ all homomorphisms } \mathrm{C}_k(X)\to R \text{ that vanish on } \mathrm{C}_k(A).$$

   We can pick a singular cochain $\varphi \in \mathrm{C}^k(X;R) = \mathrm{Hom}(\mathrm{C}_k(X),R)$ that vanishes on $\mathrm{C}_k(A)$ and such that the corresponding element in $\mathrm{C}^k(X,A;R)$ is a cocycle that represents ψ. Note that $\delta\varphi = 0$.

We have the following equalities:

choice of σ and definition of relative cap product — Proposition 43.15 — Lemma 83.1

$$\begin{aligned} \partial(\psi\cap\tau) &\overset{\downarrow}{=} \partial([\varphi\cap\sigma]) \overset{\downarrow}{=} [\partial(\varphi\cap\sigma)] \overset{\downarrow}{=} (-1)^k\cdot[-\delta\varphi\cap\sigma+\varphi\cap\partial\sigma] \\ &\underset{\uparrow}{=} (-1)^k\cdot[\varphi\cap\alpha+\varphi\cap\beta]) \underset{\uparrow}{=} (-1)^k\cdot[\varphi\cap\beta]) \end{aligned}$$

since $\delta\varphi=0$ and $\partial\sigma=\alpha+\beta$ — since φ vanishes on $\mathrm{C}_k(A)$ and since $\alpha\circ[v_0,\dots,v_k]\in\mathrm{C}_k(A)$

$$\underset{\uparrow}{=} (-1)^k\cdot([j^*i^*\varphi]\cap j_*^{-1}(\partial[\sigma])).$$

by Proposition 43.15 we have $j_*([\beta]) = \partial[\sigma]$

This equality implies that the diagram commutes. ■

As we had mentioned before, a frequently used trick in the study of topological manifolds is to replace a topological manifold with non-empty boundary by its double. When doing this trick the following lemma can at times be useful.

Lemma 83.11. *Let M be a topological manifold. We denote by $r\colon \mathrm{D}\,M \to M$ the folding map from page 1164 and given $l \in \mathbb{N}_0$ we denote by $D\colon \mathrm{H}_l(M,\partial M;\mathbb{Z}) \to \mathrm{H}_l(\mathrm{D}\,M;\mathbb{Z})$ the doubling homomorphism that we introduced in Lemma 44.16. Then for any $\phi \in \mathrm{H}^k(M;\mathbb{Z})$ and any $\sigma \in \mathrm{H}_l(M,\partial M;\mathbb{Z})$ we have*

$$r^*\phi \cap D(\sigma) \;=\; D(\phi\cap\sigma) \;\in\; \mathrm{H}_{l-k}(\mathrm{D}\,M;\mathbb{Z}).$$

Proof. We will show that the desired equality actually holds already on the (co-) chain level. So let $\phi \in \mathrm{C}^k(M;\mathbb{Z})$ and let $\sigma\colon \Delta^l \to M$ be a singular l-simplex. If $k > l$, then both sides of the lemma are trivially zero. Hence we can assume that $k \le l$. For $i = 1,2$ we denote by $j_i\colon M \to M \times \{i\} \subset \mathrm{D}\,M$ the obvious map. Then

$$\begin{array}{rcl} & \text{by definition of } D(\sigma) & \quad\text{since } \cap \text{ is bilinear} \\ & \downarrow & \quad\downarrow \\ r^*\phi \cap D(\sigma) & = & r^*\phi \cap (j_1 \circ \sigma - j_2 \circ \sigma) \;=\; r^*\phi \cap j_1 \circ \sigma - r^*\phi \cap j_2 \circ \sigma \\ & = & \phi(\underbrace{r \circ j_1 \circ \sigma}_{=\sigma} \circ [v_0, \dots, v_k]) \cdot j_1 \circ \sigma \circ [v_k, \dots, v_l] - \phi(\underbrace{r \circ j_2 \circ \sigma}_{=\sigma} \circ [v_0, \dots, v_k]) \cdot j_2 \circ \sigma \circ [v_k, \dots, v_l] \\ & \uparrow & \\ & \multicolumn{2}{l}{\text{definition of cap product and definition of } r^*\phi} \\ & = & \phi(\sigma \circ [v_0, \dots, v_k]) \cdot j_1 \circ (\sigma \circ [v_k, \dots, v_l]) - \phi(\sigma \circ [v_0, \dots, v_k]) \cdot j_2 \circ (\sigma \circ [v_k, \dots, v_l]) \\ & = & j_{1*}(\phi \cap \sigma) - j_{2*}(\phi \cap \sigma) \;=\; D(\phi \cap \sigma). \\ & \uparrow & \\ & \multicolumn{2}{l}{\text{definition of } j_{1*},\, j_{2*} \text{ and the cap product}} \end{array}$$

■

Exercises for Chapter 83.

Exercise 83.1. Let $X = S^1 \vee S^2$. We denote by $p\colon X \to S^1$ and $q\colon X \to S^2$ the obvious projection maps and we denote by $i\colon S^1 \to X$ and $j\colon S^2 \to X$ the obvious inclusion maps. Determine $p^*([\theta_{\mathbb{Z}}]) \cap j_*([S^2])$.

Exercise 83.2. We denote by Σ the surface of genus two. Furthermore we denote by $p_1, p_2\colon \Sigma \to T = S^1 \times S^1$ the two projection maps and we denote by $p, q\colon S^1 \times S^1 \to S^1$ the projection onto the first respectively the second factor. As usual we write $\alpha = p^*(\theta)$ and $\beta = q^*(\theta)$. We saw that $[p_1^*(\alpha)], [p_1^*(\beta)], [p_2^*(\alpha)], [p_2^*(\beta)]$ form a basis for $\mathrm{H}^1(\Sigma;\mathbb{Z})$. Furthermore we denote by $x_1, y_1, x_2, y_2 \in \mathrm{H}_1(\Sigma;\mathbb{Z})$ the standard basis. For each of the four basis elements of $\mathrm{H}^1(\Sigma;\mathbb{Z})$ determine the cap product with $[\Sigma]$ in terms of the basis for $\mathrm{H}_1(\Sigma;\mathbb{Z})$.
Remark. This exercise is not as bad as it might sound initially.

Exercise 83.3. We consider the annulus $X = S^1 \times [0,1]$. Determine the following map:

$$\begin{array}{rcl} \mathrm{H}^1(X;\mathbb{Z}) & \to & \mathrm{H}_1(X, \partial X;\mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [X] \end{array}$$

More precisely, give a basis for $\mathrm{H}^1(X;\mathbb{Z})$ and $\mathrm{H}_1(X, \partial X;\mathbb{Z})$ and express the map $- \cap [X]$ in terms of that basis.

Exercise 83.4. Let X be a topological space and let $k, l \in \mathbb{N}_0$. We consider the cohomology with compact support group $\mathrm{H}^k_{\mathrm{c}}(X;\mathbb{Z})$ and as in Exercise 87.5 we consider the group $\overleftarrow{\mathrm{H}}_l(X;\mathbb{Z}) := \varprojlim_{K \in \mathcal{K}(X)} \mathrm{H}_n(X, X \setminus K; G)$. Show that there exists a natural cap product

$$\cap\colon\ \mathrm{H}^k_{\mathrm{c}}(X;\mathbb{Z}) \times \overleftarrow{\mathrm{H}}_l(X;\mathbb{Z}) \;\to\; \mathrm{H}_{l-k}(X;\mathbb{Z})$$

that equals the usual cap product if X is compact.
Remark. Use the description of cohomology with compact support that is provided by Proposition 77.4.

84. The Product Theorem and the Künneth Theorem for cohomology

In this chapter we will deal with the cup and cap product of product topological spaces. In particular at the end of the chapter we will have determined the cup product of the product $S^m \times S^n$ of two spheres and of the n-dimensional torus $(S^1)^n$. Furthermore we will have determined the Lusternik-Schnirelmann category of the n-dimensional torus.

84.1. **The Product Theorem.** The following theorem relates the cross product to the cap product and the cup product.

Theorem 84.1. (Product Theorem) *Let (X, A) and (Y, B) be two pairs of topological spaces such that the triad $(X\times Y, A\times Y, X\times B)$ is excisive. We denote by $p\colon X\times Y \to X$ and $q\colon X \times Y \to Y$ the obvious projection maps. Furthermore let*[1203] *$A_1, A_2 \in \{\varnothing, A\}$ and let $B_1, B_2 \in \{\varnothing, B\}$. Then for any $\alpha \in \mathrm{H}^k(X, A_1; \mathbb{Z})$, $\mu \in \mathrm{H}_m(X, A_1 \cup A_2; \mathbb{Z})$, $\beta \in \mathrm{H}^l(Y, B_1; \mathbb{Z})$ and $\nu \in \mathrm{H}_n(Y, B_1 \cup B_2; \mathbb{Z})$ we have the equality*[1204]

$$\big(p^*(\alpha) \,\cup\, q^*(\beta)\big) \cap (\mu \times \nu) \;=\; (-1)^{\deg(\alpha)\cdot\deg(\beta)+\deg(\beta)\cdot\deg(\mu)} \cdot (\alpha \cap \mu) \times (\beta \cap \nu)$$

in $\mathrm{H}_{m+n-k-l}(X \times Y, X \times B_2 \cup A_2 \times Y; \mathbb{Z})$.

Remark. Even though the equality in the Product Theorem 84.1 is easy to write down, it is much harder to figure out in which (co-) homology groups the various objects actually live. In fact the equality is understood as follows:

$$\underbrace{\Big(\underbrace{p^*(\alpha)}_{\in \mathrm{H}^k(X\times Y, A_1\times Y)} \;\cup\; \underbrace{q^*(\beta)}_{\in \mathrm{H}^l(X\times Y, X\times B_1)}\Big)}_{\in \mathrm{H}^{k+l}(X\times Y, A_1\times Y\cup X\times B_2)} \cap \overbrace{(\mu\times\nu)}^{\text{lies in } \mathrm{H}_{m+n}(X\times Y, (A_1\cup A_2)\times Y\cup X\times(B_1\cup B_2))} = (-1)^{\deg(\alpha)\cdot\deg(\beta)+\deg(\beta)\cdot\deg(\mu)}\cdot \underbrace{\underbrace{(\alpha\cap\mu)}_{\in \mathrm{H}_{m-k}(X,A_2)} \;\times\; \underbrace{(\beta\cap\nu)}_{\in \mathrm{H}_{n-l}(Y,B_2)}}_{\in \mathrm{H}_{m+n-k-l}(X\times Y, X\times B_2\cup A_2\times Y)}.$$

In this formula we have to ensure that all products are in fact defined. This follows easily from Proposition 74.12, our hypothesis that $A_1, A_2 \in \{\varnothing, A\}$ and $B_1, B_2 \in \{\varnothing, B\}$ and our hypothesis that the triad $(X \times Y, A \times Y, X \times B)$ is excisive.

Remark.

(1) The condition in the Product Theorem 84.1 that the triad $(X \times Y, A \times Y, X \times B)$ needs to be excisive is arguably a little awkward. By Proposition 74.12 this condition is satisfied if we are in one of the following situations:
   (a) $A = \varnothing$ or $B = \varnothing$,
   (b) (X, A) and (Y, B) are pairs of CW-complexes,

[1203] By $A_1, A_2 \in \{\varnothing, A\}$ we mean that $A_1 = \varnothing$ or $A_1 = A$ and that $A_2 = \varnothing$ or $A_2 = A$.

[1204] Here, and throughout this section, we denote by p also the map of pairs $(X \times Y, A \times Y) \to (X, A)$ given by projection onto the first factor and we denote by q the map of pairs $(X \times Y, X \times B) \to (Y, B)$ given by projection onto the second factor.

(c) $(X, A) = (X, \partial X)$ and $(Y, B) = (Y, \partial Y)$ where X and Y are topological manifolds,[1205]

(d) A and B are open subsets of X respectively Y.

(2) In fact a more general statement of the Product Theorem 84.1 holds if we deal with excisive triads (X, A_1, A_2) and (Y, B_1, B_2) and (co-) homology classes $\alpha \in \mathrm{H}^k(X, A_1; \mathbb{Z})$, $\mu \in \mathrm{H}_m(X, A_2; \mathbb{Z})$, $\beta \in \mathrm{H}^l(Y, B_1; \mathbb{Z})$ and $\nu \in \mathrm{H}_n(Y, B_2; \mathbb{Z})$. But some care needs to be taken to see which other triads need to be excisive to ensure that all the cap and cup products are defined. We will not attempt to formulate this more general statement and we will not make use of it. (See also [**Dol56**, p. 240 and p. 244] for a few more details.)

(3) The cross product is defined the same way in all of the usual text books in algebraic topology. But we saw on pages 1981 and 2021 that in the literature different sign conventions are used for the cup product and the cap product. These also lead to different formulations of the Product Theorem 84.1. More precisely, given topological spaces X and Y and $\alpha \in \mathrm{H}^k(X; \mathbb{Z})$, $\mu \in \mathrm{H}_m(X; \mathbb{Z})$, $\beta \in \mathrm{H}^l(Y; \mathbb{Z})$ and $\nu \in \mathrm{H}_n(Y; \mathbb{Z})$ the following equations can be found in the literature:

$$\begin{array}{rrcl} [\mathbf{Bre93}, \text{ p. } 337] & (p^*(\alpha) \cup q^*(\beta)) \cap (\mu \times \nu) & = & (-1)^{\deg(\beta)\cdot\deg(\mu)} \cdot (\alpha \cap \mu) \times (\beta \cap \nu) \\ [\mathbf{Dol56}, \text{ p. } 240] & (p^*(\alpha) \cup q^*(\beta)) \cap (\mu \times \nu) & = & (-1)^{\deg(\beta)\cdot\deg(\mu)} \cdot (\alpha \cap \mu) \times (\beta \cap \nu) \\ [\mathbf{Spa95}, \text{ p. } 255] & (p^*(\alpha) \cup q^*(\beta)) \cap (\mu \times \nu) & = & (-1)^{\deg(\alpha)\cdot(\deg(\nu)-\deg(\beta))} \cdot (\alpha \cap \mu) \times (\beta \cap \nu). \end{array}$$

With the aforementioned sign conventions the formulas of Dold and Spanier are consistent with our formula, whereas the formula of Bredon differs from the formula of Dold and Spanier by the term $(-1)^{\deg(\alpha)\cdot\deg(\nu)+\deg(\beta)\cdot\deg(\mu)}$.[1206][1207]

(4) The conclusion of the Product Theorem 84.1 also holds, with verbatim the same proof, if throughout we replace $\mathbb{Z}$-coefficients by $\mathbb{F}_2$-coefficients.

The proof of the Product Theorem 84.1 is somewhat lengthy and technical. Therefore we first discuss some consequences of Theorem 84.1 which will then hopefully convince us that it is worth the effort to prove Theorem 84.1.

The following is one of the key calculations of the cup product and the cap product. As we will see later, many other calculations can be reduced to this calculation.

Proposition 84.2. *Let X be a compact oriented connected m-dimensional topological manifold and let Y be a compact oriented connected n-dimensional topological manifold.*

[1205] Let X be an m-dimensional topological manifold and let Y be an n-dimensional smooth manifold. It follows from Propositions 80.9 and 44.2 that $W := \partial(X \times Y) = \partial X \times Y \cup X \times \partial Y$ is an $(m+n-1)$-dimensional topological manifold. It is straightforward to verify that $\partial X \times Y$ and $X \times \partial Y$ are submanifolds of W such that

$$(\partial X \times Y) \cap (X \times \partial Y) \;=\; \partial X \times \partial Y = \partial(\partial X \times Y) \;=\; \partial(X \times \partial Y).$$

Therefore it follows from Proposition 74.12 (5) that the triad $(X \times Y, \partial X \times Y, X \times \partial Y)$ is excisive.

[1206] Mathematicians with a long experience of teaching cohomology groups sidestep all issues with signs by working with $\mathbb{F}_2$-coefficients, see [**Hau14**].

[1207] We leave it to the never-flagging enthusiasm of the reader to figure out whether the formulas agree with the one given by Greenberg-Harper [**GH81**, p. 262].

We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the obvious projection maps. Then the following two statements hold:

(1)
$$\underbrace{p^*([X]^*)}_{\in \mathrm{H}^m(X\times Y,\partial X\times Y)} \cup \underbrace{q^*([Y]^*)}_{\in \mathrm{H}^n(X\times Y,X\times\partial Y)} = \underbrace{[X\times Y]^*}_{\in \mathrm{H}^{m+n}(X\times Y,\partial(X\times Y))}.$$

(2) *If $P \in X$ and $Q \in Y$ are points, then*[1208]
$$\overbrace{p^*([X]^*)}^{\in \mathrm{H}^m(X\times Y,\partial X\times Y)} \cap \overbrace{[X\times Y]}^{\in \mathrm{H}_{m+n}(X\times Y,\partial(X\times Y))} = \overbrace{[\{P\}\times Y]}^{\in \mathrm{H}_n(X\times Y,X\times\partial Y)}$$
and similarly
$$\underbrace{q^*([Y]^*)}_{\in \mathrm{H}^n(X\times Y,X\times\partial Y)} \cap \underbrace{[X\times Y]}_{\in \mathrm{H}_{m+n}(X\times Y,\partial(X\times Y))} = (-1)^{mn}\cdot \underbrace{[X\times\{Q\}]}_{\in \mathrm{H}_m(X\times Y,\partial X\times Y)}.$$

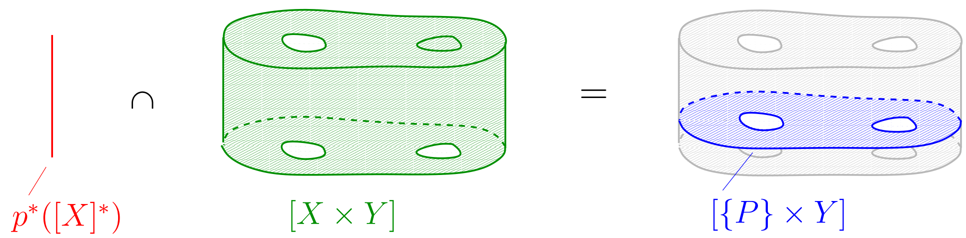


FIGURE 1182

Remark. As we pointed out above, the conclusion of the Product Theorem 84.1 also holds if we work with $\mathbb{F}_2$-coefficients. It follows easily that there is also an $\mathbb{F}_2$-coefficient analogue of Proposition 84.2.

Example. If we apply Proposition 84.2 (2) to $X = S^1$ and $Y = S^1$ we see that
$$p^*([S^1]^*) \cap [S^1\times S^1] = [1\times S^1] \quad \text{and} \quad q^*([S^1]^*) \cap [S^1\times S^1] = -[S^1\times 1].$$
This matches our down-to-earth calculation from page 2023.

PROOF. Let X be a compact oriented connected m-dimensional topological manifold and let Y be a compact oriented connected n-dimensional topological manifold. Furthermore let P be a point in X and let Q be a point in Y. We start out with the following two observations:

(a) For any compact oriented connected k-dimensional topological manifold M and any point $R \in M$ we obtain from the discussion on page 2026 that $[M]^* \cap [M] = [R]$ and that $[M]^*$ is the unique element in $\mathrm{H}^k(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$ with $[M]^*\cap[M] = [R]$ in $\mathrm{H}_0(M;\mathbb{Z}) = \mathbb{Z}\cdot[R]$.
(b) By Proposition 80.10 we have $[X\times Y] = [X]\times[Y]$.

[1208] Here we use the following suggestive notation: We can view $\{P\}\times Y$ as a submanifold of $X\times Y$. It is canonically homeomorphic to Y, thus can view $\{P\}\times Y$ as a compact oriented topological manifold. We denote by $[\{P\}\times Y]$ its fundamental class, and by a slight abuse of notation we also denote by $[\{P\}\times Y]$ the image of the fundamental class under the inclusion $\{P\}\times Y \to X\times Y$.

Now we turn to the actual proof of the two statements.

(1) We perform the following calculation in $\mathrm{H}_0(X \times Y; \mathbb{Z})$:

$$\begin{aligned}
(p^*([X]^*) \cup q^*([Y]^*)) \cap ([X \times Y]) &\overset{\text{observation (b)}}{=} (p^*([X]^*) \cup q^*([Y]^*)) \cap ([X] \times [Y]) \\
\underset{\substack{\text{by Theorem 84.1 applied}\\ \text{to } (X, A_1, A_2) = (X, \partial X, \partial X)\\ \text{and } (Y, B_1, B_2) = (Y, \partial Y, \partial Y)}}{=} (-1)^{mn+mn} \cdot ([X]^* \cap [X]) \times ([Y]^* \cap [Y]) &\underset{\text{observation (a)}}{=} [P] \times [Q] \underset{\text{see Lemma 80.7 (2)}}{=} [(P, Q)].
\end{aligned}$$

This implies, by observation (a), that $p^*([X]^*) \cup q^*([Y]^*) = [X \times Y]^*$.

(2) Let $P \in X$. Then

$$p^*([X]^*) \cap ([X \times Y]) \underset{\substack{\text{by Proposition 82.4 (1) we have } q^*(1_Y) = 1_{X\times Y} \text{ and by observation (b) we have } [X\times Y] = [X]\times[Y]}}{=} (p^*([X]^*) \cup q^*(1_Y)) \cap ([X] \times [Y])$$

$$\underset{\substack{\text{by Theorem 84.1 applied to } (Y, B_1, B_2) = (Y, \varnothing, \partial Y)\\ \text{note that here we use that } 1_Y \text{ lies in } \mathrm{H}^0(Y; \mathbb{Z})}}{=} ([X]^* \cap [X]) \times (1_Y \cap [Y]) \underset{\substack{\text{by Lemma 83.3}\\ \text{and observation (a)}}}{=} [P] \times [Y] \underset{\text{Lemma 80.7 (2)}}{=} [\{P\} \times Y].$$

This proves the first equality of (2). Now let $Q \in Y$. Then, very similar to the above, we have

$$\begin{aligned}
q^*([Y]^*) \cap ([X \times Y]) &= (p^*(1_X) \cup q^*([Y]^*)) \cap ([X] \times [Y]) \\
&= (-1)^{mn} \cdot (1_X \cap [X]) \times ([Y]^* \cap [Y]) \\
&= (-1)^{mn} \cdot [X] \times [Q] \ = \ (-1)^{mn} \cdot [X \times \{Q\}]. \qquad \blacksquare
\end{aligned}$$

Using Proposition 84.2 (1) we can now easily compute the cup product of the product $S^m \times S^n$ of two spheres.

Lemma 84.3. *Let $m, n \in \mathbb{N}$. We denote by $p\colon S^m \times S^n \to S^m$ and by $q\colon S^m \times S^n \to S^n$ the two obvious projection maps. Let $1 \in \mathrm{H}^0(S^m \times S^n; \mathbb{Z})$ be the neutral element from Proposition 81.7.*

(1) *The cohomology of $S^m \times S^n$ is given as follows:*

$$\mathrm{H}^*(S^m \times S^n; \mathbb{Z}) \ = \ \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot p^*([S^m]^*) \oplus \mathbb{Z} \cdot q^*([S^n]^*) \oplus \mathbb{Z} \cdot [S^m \times S^n]^*.$$

(2) *The multiplication table for the cup product is given by*

$\cup$	1	$p^*([S^m]^*)$	$q^*([S^n]^*)$	$[S^m \times S^n]^*$
1	1	$p^*([S^m]^*)$	$q^*([S^n]^*)$	$[S^m \times S^n]^*$
$p^*([S^m]^*)$	$p^*([S^m]^*)$	0	$[S^m \times S^n]^*$	0
$q^*([S^n]^*)$	$q^*([S^n]^*)$	$(-1)^{mn} \cdot [S^m \times S^n]^*$	0	0
$[S^m \times S^n]^*$	$[S^m \times S^n]^*$	0	0	$0.$

Here the table shows the cup product of the term in the first column with the term in the first row (and not the other way round).

Remark.

(1) For $m = n = 1$, i.e. for the torus $S^1 \times S^1$ we had calculated the cup product in Lemma 81.14.
(2) If we use Proposition 84.2 (1) with $\mathbb{F}_2$-coefficients then we can also determine the cup product on $\mathrm{H}^*(S^m \times S^n; \mathbb{F}_2)$ and we obtain basically the same result.

PROOF.

(1) Using Proposition 43.4, the Künneth Theorem 58.8 for topological spaces and the Universal Coefficient Theorem 75.13 for Cohomology Groups one can easily determine the isomorphism type of $\mathrm{H}^*(S^m \times S^n; \mathbb{Z})$. Furthermore, using Lemma 74.7 and the homology classes given by the fundamental classes of S^m, S^n and $S^m \times S^n$ it is straightforward to verify that the given cohomology classes do indeed form a basis for $\mathrm{H}^*(S^m \times S^n; \mathbb{Z})$.
(2) By Observation 81.13 it remains to show that the red-colored and the blue-colored entries are correct.
   (a) First we consider the red-colored entries. We have
$$p^*([S^m]^*) \cup p^*([S^m]^*) \underset{\substack{\uparrow \\ \text{Lemma 81.10}}}{=} p^*([S^m]^* \cup [S^m]^*) \underset{\substack{\uparrow \\ \text{since } [S^m]^* \cup [S^m]^* \in \mathrm{H}^{2m}(S^m;\mathbb{Z}) = 0}}{=} p^*(0) = 0.$$
   The same argument also shows that $q^*([S^n]^*) \cup q^*([S^n]^*) = 0$.
   (b) We turn to the blue-colored entries. We calculate that
$$(-1)^{mn} \cdot q^*([S^n]^*) \cup p^*([S^m]^*) \underset{\substack{\uparrow \\ \text{Proposition 81.8}}}{=} p^*([S^m]^*) \cup q^*([S^n]^*) \underset{\substack{\uparrow \\ \text{Proposition 84.2 (1)}}}{=} [S^m \times S^n]^*.$$
■

In the following we consider the wedge $X := S^2 \vee S^2 \vee S^4$ and we consider the product $Y := S^2 \times S^2$. It follows from Lemma 46.12 and Propositions 43.4 and 47.8 and the Universal Coefficient Theorem 75.13 for Cohomology Groups that $X = S^2 \vee S^2 \vee S^4$ and $Y = S^2 \times S^2$ have isomorphic homology and cohomology groups. Furthermore one obtains easily from Propositions 14.14, 16.20 and 20.3 that both X and Y are simply connected.

It is quite easy to show that $X = S^2 \vee S^2 \vee S^4$ and $Y = S^2 \times S^2$ are not homeomorphic. Indeed, suppose there exists a homeomorphism $\phi\colon S^2 \vee S^2 \vee S^4 \to S^2 \times S^2$. We denote by P we wedge point of $X = S^2 \vee S^2 \vee S^4$ and we write $Q = f(P)$. Then $X \setminus \{P\}$ and $Y \setminus \{Q\}$ would also be homeomorphic. But the former topological space has three path-components whereas the latter has just one path-component.[1209] We refer to Figure 1183 for a dimensionally reduced illustration.

FIGURE 1183

The following lemma is much deeper than the above discussions.

[1209] How can you prove rigorously that given any point $Q \in S^2 \times S^2$ the complement $S^2 \times S^2 \setminus \{Q\}$ is still path-connected?

Lemma 84.4. *The topological spaces $S^2 \vee S^2 \vee S^4$ and $S^2 \times S^2$ are not even homotopy equivalent.*

Proof. We had just seen in Lemma 84.3 that there exist cohomology classes α and β in $\mathrm{H}^2(S^2 \times S^2; \mathbb{Z})$ with $\alpha \cup \beta \neq 0$. On the other hand in Lemma 81.11 we saw that the cup product

$$\cup \colon \mathrm{H}^2(S^2 \vee S^2 \vee S^4; \mathbb{Z}) \times \mathrm{H}^2(S^2 \vee S^2 \vee S^4; \mathbb{Z}) \ \to \ \mathrm{H}^4(S^2 \vee S^2 \vee S^4; \mathbb{Z})$$

is the zero-map. It follows from Corollary 81.12 that the topological spaces $S^2 \vee S^2 \vee S^4$ and $S^2 \times S^2$ are not homotopy equivalent. ■

In Proposition 69.3 we saw that there exists a degree-one map from $S^2 \times S^2$ onto the 4-dimensional sphere S^4. In Question 69.11 we had asked whether there exists a degree-one map from the 4-dimensional sphere S^4 to $S^2 \times S^2$. Now we can answer this question in the negative.

Lemma 84.5. *There is no map $f \colon S^4 \to S^2 \times S^2$ of non-zero degree.*

Proof. Let $f \colon S^4 \to S^2 \times S^2$ be a map. We have to show that $\deg(f) = 0$. By Lemma 75.17 it suffices to show that $f^*([S^2 \times S^2]^*) = 0$. In Lemma 84.3 we showed that there exist cohomology classes $a, b \in \mathrm{H}^2(S^2 \times S^2; \mathbb{Z})$ such that $a \cup b = [S^2 \times S^2]^* \in \mathrm{H}^4(S^2 \times S^2; \mathbb{Z})$. It follows that

$$f^*([S^2 \times S^2]^*) \underset{\substack{\uparrow \\ \text{since } [S^2 \times S^2]^* = a \cup b}}{=} f^*(a \cup b) \underset{\substack{\uparrow \\ \text{Lemma 81.10}}}{=} f^*(a) \cup f^*(b) \underset{\substack{\uparrow \\ \text{since } f^*(a), f^*(b) \in \mathrm{H}^2(S^4; \mathbb{Z}) = 0}}{=} 0 \cup 0 = 0.$$ ■

84.2. **Proof of the Product Theorem 84.1.** Now that we are fully convinced of the value of Theorem 84.1 it is time to actually provide a proof thereof.

Initially we introduced only the tensor product $C_* \otimes D_*$ of two chain complexes C_* and D_*. The definition of the boundary map in the tensor product $C_* \otimes D_*$ is *not* symmetric in the two terms. But it is straightforward to see that the tensor product of chain complexes is associative. More precisely, given chain complexes $\mathrm{C}^1_*, \dots, \mathrm{C}^k_*$ the boundary map on the tensor product $\mathrm{C}^1_* \otimes \dots \otimes \mathrm{C}^k_*$ is always determined by

$$\partial(c_1 \otimes c_2 \otimes \dots \otimes c_k) \ = \ \sum_{i=1}^{k} (-1)^{\deg(c_1) + \dots + \deg(c_{i-1})} \cdot c_1 \otimes \dots \otimes \partial c_i \otimes \dots \otimes c_k,$$

regardless of the way how one breaks the tensor product into the tensor product of pairs of two chain complexes.

The following lemma is basically the same as Lemma 58.3, but we state it here for the convenience of the reader.

Lemma 84.6. *Let $f_i\colon C^i_* \to D^i_*$, $i = 1,\ldots,k$ be chain maps between chain complexes. Then the map*

$$f_1 \otimes \cdots \otimes f_k\colon C^1_* \otimes \cdots \otimes C^k_* \;\to\; D^1_* \otimes \cdots \otimes D^k_*$$

that is determined by

$$c_1 \otimes \cdots \otimes c_k \;\mapsto\; f(c_1) \otimes \cdots \otimes f(c_k)$$

is also a chain map.

Remark. Even the definition of $f_1 \otimes \cdots \otimes f_k$ is, oddly enough, not the same everywhere. For example [**Bre93**, p. 315] adds a sign to the map.

Proof. Since in the proof of the Product Theorem 84.1 there are lots of steps later on which can make us nervous, let us convince ourselves that the statement is indeed correct. We have

$$\begin{aligned}
\partial\big((f_1\otimes\ldots\otimes f_k)(c_1\otimes\ldots\otimes c_k)\big) &= \partial(f_1(c_1)\otimes\ldots\otimes f_k(c_k)) \\
&= \sum_{i=1}^{k}(-1)^{\deg(f_1(c_1))+\cdots+\deg(f_{i-1}(c_{i-1}))}\cdot f_1(c_1)\otimes\ldots\otimes\overbrace{f_i(\partial c_i)}^{=\partial f_i(c_i)}\otimes\ldots\otimes f_k(c_k) \\
&\underset{\substack{\uparrow \\ \text{since } \deg(f_i(c_i)) = \deg(c_i)}}{=} (f_1\otimes\ldots\otimes f_k)\Big(\sum_{i=1}^{k}(-1)^{\deg(c_1)+\cdots+\deg(c_{i-1})}\cdot c_1\otimes\ldots\partial c_i\otimes\ldots\otimes c_k\Big) \\
&= (f_1\otimes\ldots\otimes f_k)(\partial(c_1\otimes\ldots\otimes c_k)).
\end{aligned}$$

■

Note though that the naive swapping of two factors in the tensor product of two chain complexes is not necessarily a chain map. But we have the following lemma.

Lemma 84.7. *Let A_* and B_* be chain complexes. The unique map*

$$\tau\colon A_* \otimes B_* \;\to\; B_* \otimes A_*$$

that for $a \in A_k$ and $b \in B_l$ is given by

$$a \otimes b \;\mapsto\; (-1)^{kl}\cdot b \otimes a$$

is a chain map.

Remark. In Exercise 84.1 we will see that we can use Lemma 84.7 to give a new proof of Proposition 81.8, i.e. we can give a new proof that the cup product is (anti-) commutative.

Proof. The difficult bit about the lemma is coming up with the statement. The verification is now almost trivial. So let $a \in A_k$ and $b \in B_l$. Then we have

$$\begin{aligned}
\tau(\partial(a\otimes b)) &= \tau\big(\partial a\otimes b + (-1)^k\cdot a\otimes\partial b\big) \qquad\qquad = \tau(\partial a\otimes b) + (-1)^k\cdot\tau(a\otimes\partial b) \\
&= (-1)^{(k-1)l}\cdot b\otimes\partial a + (-1)^{k+k(l-1)}\cdot\partial b\otimes a = (-1)^{kl}\cdot\partial b\otimes a + (-1)^{l+kl}\cdot b\otimes\partial a \\
&= \partial((-1)^{kl}\cdot b\otimes a) \;=\; \partial(\tau(a\otimes b)).
\end{aligned}$$

■

Now we have to introduce a plethora of new definitions and notations.

Notation.

(1) We denote by $\mathbb{Z}$ also the chain complex given by the group $\mathbb{Z}$ in degree zero and where all other chain groups are zero.
(2) Given a chain complex C_* and $k \in \mathbb{N}_0$ we denote by $C_*[k]$ the chain complex which is the result of "shifting C_* by k to the left", i.e. it is the chain complex for which the

m-th chain groups $(\mathrm{C}_*[k])_m$ is given by C_{m-k}.[1210] For example, $\mathbb{Z}[3]$ is the following chain complex:

$$\begin{array}{ccccccccccc} \dots & \to & 0 & \to & \mathbb{Z} & \to & 0 & \to & 0 & \to & 0. \\ & & & & \uparrow & & & & & & \\ & & & & \text{degree } 3 & & & & & & \end{array}$$

(3) If $\alpha\colon C_* \to D_*$ is a chain map, then for any $k \in \mathbb{N}_0$ we denote by $\alpha[k]\colon \mathrm{C}_*[k] \to D_*[k]$ the obvious shift of the chain map. Evidently this is again a chain map.

(4) Let C_* be a chain complex and let $k \in \mathbb{N}_0$.
  (a) Given a cocycle $\varphi\colon \mathrm{C}_k \to \mathbb{Z}$ we denote by $\Pi_k(\varphi)\colon C_* \to \mathbb{Z}[k]$ the map that is given by φ on C_n and zero on all other chain groups.
  (b) Conversely, given a chain map $\psi\colon C_* \to \mathbb{Z}[k]$ we denote by $\psi_k\colon \mathrm{C}_k \to \mathbb{Z}$ the map that is the k-th map of the chain map.

Lemma 84.8. *Let C_* be a chain complex and let $k \in \mathbb{N}_0$. The following statements hold:*
(1) *Let $\varphi\colon \mathrm{C}_k \to \mathbb{Z}$ be a cocycle. The map $\Pi_k(\varphi)\colon C_* \to \mathbb{Z}[k]$ is a chain map.*
(2) *Let $\psi\colon C_* \to \mathbb{Z}[k]$ be a chain map. The map $\psi_k\colon \mathrm{C}_k \to \mathbb{Z}$ is a cocycle.*
(3) *Let $\alpha, \beta\colon C_* \to \mathbb{Z}[k]$ be chain maps. If α and β are chain homotopic, then α_k and β_k are cohomologous.*

PROOF.

(1) Let $\varphi\colon \mathrm{C}_k \to \mathbb{Z}$ be a cocycle. We consider the following diagram:

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{C}_{k+1} \arrow[r, "\partial"] \arrow[d] & \mathrm{C}_k \arrow[r, "\partial"] \arrow[d, "\varphi"] & \mathrm{C}_{k-1} \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & \dots. \end{tikzcd}$$

The hypothesis that φ is a cocycle means that $\delta\varphi = \varphi \circ \partial = 0$, which implies immediately that $\Pi_k(\varphi)$ is a chain map.

(2) The proof of statement (2) is almost the same as the proof of statement (1).

(3) Let $\alpha, \beta\colon C_* \to \mathbb{Z}[k]$ be chain maps and let $\{P_m\colon \mathrm{C}_m \to \mathbb{Z}[k]_{m+1}\}_{m\in\mathbb{N}_0}$ be a chain homotopy between α and β. We summarize all relevant maps in the following diagram:

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{C}_{k+1} \arrow[r, "\partial"] \arrow[d] & \mathrm{C}_k \arrow[r, "\partial"] \arrow[d, "\alpha_k-\beta_k"] \arrow[dl, "P_k"] & \mathrm{C}_{k-1} \arrow[r] \arrow[d] \arrow[dl, "P_{k-1}"] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & \dots. \end{tikzcd}$$

By definition of a chain homotopy we obtain that $\alpha_k - \beta_k = P_{k-1} \circ \partial$. But this means that $\alpha_k - \beta_k = \delta P_{k-1}$, i.e. α_k and β_k are cohomologous. ■

[1210] Once again, conventions in the literature differ. It is fairly common to define the boundary maps on $\mathrm{C}_*[k]$ to be the boundary maps of C_* multiplied by $(-1)^k$, see e.g. [**Weib94**, p. 10].

Lemma 84.9. *Let C_* be a chain complex and let $k \in \mathbb{Z}$. The maps*

$$\begin{array}{rcl} \eta_m\colon (\mathbb{Z}[k] \otimes \mathrm{C}_*)_m = \mathbb{Z} \otimes \mathrm{C}_{m-k} & \to & (\mathrm{C}_*[k])_m = \mathrm{C}_{m-k} \\ a \otimes c_{m-k} & \mapsto & (-1)^{k(m-k)} \cdot a \cdot c_{m-k} \end{array}$$

form an isomorphism

$$\eta\colon \mathbb{Z}[k] \otimes \mathrm{C}_* \xrightarrow{\cong} \mathrm{C}_*[k]$$

of chain complexes.

Example. Let $k, l \in \mathbb{N}_0$. The chain map

$$\eta\colon \mathbb{Z}[k] \otimes \mathbb{Z}[l] \ \to \ \mathbb{Z}[k+l]$$

is determined by the fact that for $a \in (\mathbb{Z}[k])_k = \mathbb{Z}$ and $b \in (\mathbb{Z}[l])_l = \mathbb{Z}$ we have

$$\eta(a \otimes b) \ = \ (-1)^{kl} \cdot a \cdot b.$$

Proof. It follows from Lemma 57.3 (3) that each η_m is an isomorphism. But we still need to verify that the maps η_m form a chain map $\eta\colon \mathbb{Z}[k] \otimes C_* \to \mathrm{C}_*[k]$. Thus let $k \in \mathbb{N}_0$, let $a \in (\mathbb{Z}[k])_k = \mathbb{Z}$ and let $c_{m-k} \in \mathrm{C}_{m-k}$. Then

$$\begin{array}{rcl} \eta_{m-1}(\partial(a \otimes c_{m-k})) & \underset{\text{since } a \text{ is of degree } k \text{ and since } \partial a = 0}{=} & \eta_{m-1}((-1)^k \cdot a \otimes \partial c_{m-k}) \underset{\text{since } \partial c_{m-k} \text{ is of degree } m-k-1}{=} (-1)^k \cdot (-1)^{k(m-1-k)} \cdot a \cdot \partial c_{m-k} \\ & = & \partial((-1)^{k(m-k)} \cdot a \cdot c_{m-k}) \ = \ \partial(\eta_m(a \otimes c_{m-k})). \end{array}$$

It follows immediately that the maps η_m form a chain map. ■

Notation.

(1) Given topological spaces X and Y we denote by $\Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$ and $\Theta\colon \mathrm{C}_*(X \times Y) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ the chain maps from the Eilenberg-Zilber Theorem 80.1.
(2) Given a topological space W we denote by $\Delta_W\colon \mathrm{C}_*(W) \to \mathrm{C}_*(W) \otimes \mathrm{C}_*(W)$ the Alexander-Whitney diagonalization from page 1985. Recall that Δ_W is also a chain map.

In the following two lemmas we will reinterpret the cup product and the cap product as chain maps.

Lemma 84.10. *Let W be a topological space and let $\alpha \in \mathrm{C}^k(W;\mathbb{Z})$ and $\beta \in \mathrm{C}^l(W;\mathbb{Z})$ be cocycles. Then we have the following equality of chain maps:*

$$(-1)^{kl} \cdot \Pi_{k+l}\big(\underbrace{\alpha \cup \beta}_{\text{cocycle in } \mathrm{C}^{k+l}(W;\mathbb{Z})}\big) \ = \ \eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Delta_W\colon \ \mathrm{C}_*(W) \to \mathbb{Z}[k+l].$$

Proof. We introduce the following notation:

(1) We denote by $\mu\colon \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ the obvious isomorphism of abelian groups given by $\mu(a \otimes b) = a \cdot b$.
(2) We denote by $\Pi_{k,l}\colon (\mathrm{C}_*(W) \otimes \mathrm{C}_*(W))_{k+l} \to \mathrm{C}_k(W) \otimes \mathrm{C}_l(W)$ the obvious projection map.

In the proof of Proposition 81.4 on page 1986 we showed that

$$\alpha \cup \beta \;=\; \mu \circ (\alpha \otimes \beta) \circ \Pi_{k,l} \circ \Delta_W \colon\; \mathrm{C}_{k+l}(W) \;\to\; \mathbb{Z}.$$

The lemma follows immediately from this equality and the following claim.

Claim. [1211]

$$(-1)^{kl} \cdot \mu \circ (\alpha \otimes \beta) \circ \Pi_{k,l} \;=\; \underbrace{(\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)))_{k+l}}_{\substack{\text{the restriction to the } (k+l)\text{-th chain group of the} \\ \text{chain map } \mathrm{C}_*(W) \otimes \mathrm{C}_*(W) \to \mathbb{Z}[k] \otimes \mathbb{Z}[l] \to \mathbb{Z}[k+l] \\ a \otimes b \mapsto (-1)^{kl} \cdot ab}} \;\colon (\mathrm{C}_*(W) \otimes \mathrm{C}_*(W))_{k+l} \;\to\; \mathbb{Z}.$$

Let $\sigma_i \in \mathrm{C}_i(W)$ and $\sigma_j \in \mathrm{C}_j(W)$ with $i+j = k+l$. If $(i,j) \neq (k,l)$, then both maps are zero. So now suppose that $(i,j) = (k,l)$. Then it is clear that both maps send $\sigma_k \otimes \sigma_l$ to $(-1)^{kl} \cdot \alpha(\sigma_k) \cdot \beta(\sigma_l)$. (Note that the sign $(-1)^{kl}$ for the map on the right-hand side stems from the definition of the chain map η.) ■

Given a topological space W and given $\varphi \in \mathrm{C}^k(W;\mathbb{Z})$ we obtain for each $m \in \mathbb{N}_0$ a map

$$\begin{array}{rcl} \varphi \cap -\colon\; \mathrm{C}_m(W) & \to & \mathrm{C}_{m-k}(W) = (\mathrm{C}_*(W)[k])_m \\ \sigma & \mapsto & \varphi \cap \sigma. \end{array}$$

Unfortunately in general these maps do *not* form a chain map from the chain complex $\mathrm{C}_*(W)$ to the chain complex $\mathrm{C}_*(W)[k]$. This motivates introducing the following slight variation on the cap product.

Definition. Let W be a topological space. Given $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ and given $\sigma \in \mathrm{C}_n(W)$ we define[1212]

$$\varphi \,\widetilde{\cap}\, \sigma \;:=\; (-1)^{m(n-m)} \cdot \varphi \cap \sigma.$$

In particular for $m = n$ we have $\varphi \,\widetilde{\cap}\, \sigma = \varphi \cap \sigma$.

Lemma 84.11. *Let W be a topological space.*

(1) *If $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ is a cocycle, then the map*

$$\begin{array}{rcl} \varphi \,\widetilde{\cap}\, -\colon\; \mathrm{C}_*(W) & \to & \mathrm{C}_*(W)[m] \\ \sigma & \mapsto & \varphi \,\widetilde{\cap}\, \sigma \end{array}$$

is a chain map.

(2) *If $\varphi, \psi \in \mathrm{C}^m(W;\mathbb{Z})$ are cohomologous cocycles, then the chain maps $\varphi \,\widetilde{\cap}\, -$ and $\psi \,\widetilde{\cap}\, -$ are chain homotopic.*

Proof. Let W be a topological space. We start out with the following claim.

Claim. For any cochain $\tau \in \mathrm{C}^l(W;\mathbb{Z})$ and any chain $\mu \in \mathrm{C}_n(W;\mathbb{Z})$ we have

$$\partial(\tau \,\widetilde{\cap}\, \mu) \;=\; (-1)^{l+n} \cdot \delta\tau \,\widetilde{\cap}\, \mu + \tau \,\widetilde{\cap}\, \partial\mu.$$

[1211] Here, given two chain complexes $\mathcal{C}_*$ and $\mathcal{D}_*$ the map $\Pi_{k,l}\colon \mathcal{C}_* \otimes \mathcal{D}_* \to \mathcal{C}_k \otimes \mathcal{B}_l$ is the obvious projection map that we introduced on page 1979.

[1212] If we compare this definition with the discussion on page 2021 we see that the definition of $\widetilde{\cap}$ is precisely the definition of the cap product in Bredon [**Bre93**] and Dold [**Dol56**].

The claim is an immediate consequence of Lemma 83.1. Since we are worried about signs we write down the detailed calculation. We have

$$\begin{aligned}\partial(\tau \,\widetilde{\cap}\, \mu) &\underset{\substack{\uparrow\\ \text{definition of } \widetilde{\cap}}}{=} (-1)^{l(n-l)} \cdot \partial(\tau \cap \mu) \underset{\substack{\uparrow\\ \text{Lemma 83.1}}}{=} (-1)^{l(n-l)} \cdot (-1)^{l} \cdot (-\delta\tau \cap \mu + \tau \cap \partial\mu)\\ &= (-1)^{l(n-l)+l} \cdot \big(-(-1)^{(l+1)\cdot(n-l-1)} \cdot \delta\tau \,\widetilde{\cap}\, \mu + (-1)^{l\cdot(n-l-1)} \cdot \tau \,\widetilde{\cap}\, \partial\mu\big)\\ &= (-1)^{l+n} \cdot \delta\tau \,\widetilde{\cap}\, \mu + \tau \,\widetilde{\cap}\, \partial\mu.\end{aligned}$$

⊞

Now we turn to the actual proof of the two statements.

(1) Let $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ be a cocycle. Furthermore let $\sigma \in \mathrm{C}_n(W)$. By the definition of the cap product and by definition of the shifting of chain complexes we have $\varphi \cap \sigma \in \mathrm{C}_{n-m}(W) = (\mathrm{C}_*(W)[m])_n$. This shows that $\varphi \,\widetilde{\cap}\, -$ preserves degrees. It remains to show that the map $\varphi \,\widetilde{\cap}\, -$ commutes with the boundary maps. But this is an immediate consequence of the above claim and the fact that φ is a cocycle.

(2) So let $\varphi, \psi \in \mathrm{C}^m(W;\mathbb{Z})$ be cocycles. We assume they are cohomologous which means that there exists a $\tau \in \mathrm{C}^{m-1}(W;\mathbb{Z})$ such that $\varphi = \psi + \delta\tau$. For any $k \in \mathbb{N}_0$ we consider the map

$$\begin{aligned} P_k\colon\ \mathrm{C}_k(W) &\to (\mathrm{C}_*(W)[m])_{k+1} = \mathrm{C}_{k-m+1}(W)\\ \sigma &\mapsto (-1)^{m+k} \cdot \tau \,\widetilde{\cap}\, \sigma.\end{aligned}$$

We calculate that for any $\sigma \in \mathrm{C}_k(W)$ we have

$$\begin{aligned}\varphi \,\widetilde{\cap}\, \sigma - \psi \,\widetilde{\cap}\, \sigma = \delta\tau \,\widetilde{\cap}\, \sigma &\overset{\substack{\text{by the claim}\\ \downarrow}}{=} (-1)^{m+k} \cdot \partial(\tau \,\widetilde{\cap}\, \sigma) + (-1)^{m+k-1} \cdot \tau \,\widetilde{\cap}\, \partial\sigma\\ &= (\partial \circ P_k)(\sigma) + (P_{k-1} \circ \partial)(\sigma).\end{aligned}$$

This shows that the maps $\{P_k\}_{k\in\mathbb{N}_0}$ form a chain homotopy between the chain maps $\varphi \,\widetilde{\cap}\, -$ and $\psi \,\widetilde{\cap}\, -$. ■

In the proof of Theorem 84.1 we will also need the following reformulation of the cap product.

Proposition 84.12. *Let W be a topological space and let $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ be a cocycle. Then we have the following equality of chain maps*

$$\varphi \,\widetilde{\cap}\, - \;=\; \eta \circ (\Pi_m(\varphi) \otimes \mathrm{id}) \circ \Delta_W\colon\ \mathrm{C}_*(W) \;\to\; \mathrm{C}_*(W)[m].$$

Proof. Let W be a topological space, let $\varphi \in \mathrm{C}^m(W)$ be a cocycle and let $\sigma\colon \Delta^n \to W$ be a singular n-simplex. We have the following equalities in $\mathrm{C}_*(W)[m]_n = \mathrm{C}_{n-m}(W)$:

$$\begin{array}{rcl}
\big(\eta \circ (\Pi_m(\varphi) \otimes \mathrm{id}) \circ \Delta_W\big)(\sigma) & \underset{\text{definition of the Alexander-Whitney diagonalization}}{=} & \eta\Big((\Pi_m(\varphi) \otimes \mathrm{id})\Big(\sum_{j=0}^{n} \sigma \circ [v_0,\dots,v_j] \otimes \sigma \circ [v_j,\dots,v_n]\Big)\Big) \\
& = & \eta\big(\underbrace{\varphi(\sigma \circ [v_0,\dots,v_m])}_{\in \mathbb{Z}[m]_m} \otimes \underbrace{\sigma \circ [v_m,\dots,v_n]}_{\in \mathrm{C}_{n-m}}\big) \\
& & \uparrow \\
& & \text{by definition of the chain map } \Pi_m(\varphi) \otimes \mathrm{id}\colon \mathrm{C}_*(W) \otimes \mathrm{C}_*(W) \to \mathbb{Z}[m] \otimes \mathrm{C}_*(W) \\
& = & (-1)^{m(n-m)} \cdot \underbrace{\varphi(\sigma \circ [v_0,\dots,v_m]) \cdot \sigma \circ [v_m,\dots,v_n]}_{=\varphi \cap \sigma} \;\underset{\text{definition of } \widetilde{\cap} \text{ on page 2042}}{=}\; \varphi \,\widetilde{\cap}\, \sigma. \\
& \uparrow & \\
& \text{definition of } \eta \text{ in Lemma 84.9} &
\end{array}$$

We have thus verified that the desired equality holds. ■

The following statement is just the definition of the cross product.

Tautology 84.13. *Let X and Y two topological spaces. For any $[\mu] \in \mathrm{H}_m(X;\mathbb{Z})$ and $[\nu] \in \mathrm{H}_n(Y;\mathbb{Z})$ we have*

$$[\mu] \times [\nu] \;=\; [\Upsilon(\mu \otimes \nu)].$$

We need one more slightly delicate chain map before we can finally turn to the proof of the Product Theorem 84.1.

Lemma 84.14. *Let C_* and D_* be chain complexes and let $k, l \in \mathbb{N}_0$. The map*

$$\Psi_{k,l}\colon\ \mathrm{C}_*[k] \otimes D_*[l] \;\to\; (C_* \otimes D_*)[k+l]$$

which is defined by the property that for $c_m \in \mathrm{C}_m$ and $d_n \in D_n$ we have

$$c_m \otimes d_n \;\mapsto\; (-1)^{k(l+n)} \cdot c_m \otimes d_n.$$

is an isomorphism of chain complexes.

Proof. First note that it is clear that $\Psi_{k,l}$ preserves degrees and that on each degree it is an isomorphism. It remains to show that the map $\Psi_{k,l}$ is a chain map. Let $c_m \in \mathrm{C}_m$ and $d_n \in D_n$. We calculate that

$$\begin{array}{rcl}
\Psi_{k,l}(\partial(c_m \otimes d_n)) & \underset{\text{since } c_m \text{ has degree } m+k \text{ in } \mathrm{C}_*[k]}{=} & \Psi_{k,l}(\partial c_m \otimes d_n + (-1)^{m+k} \cdot c_m \otimes \partial d_n) \\
& = & (-1)^{k(l+n)} \cdot \partial c_m \otimes d_n + (-1)^{m+k} \cdot (-1)^{k(l+n-1)} \cdot c_m \otimes \partial d_n \\
& = & (-1)^{k(l+n)} \cdot (\partial c_m \otimes d_n + (-1)^m \cdot c_m \otimes \partial d_n) \underset{\text{since } c_m \text{ has degree } m \text{ in } C_*}{=} (-1)^{k(l+n)} \cdot \partial(c_m \otimes d_n) \\
& = & \partial((-1)^{k(l+n)} \cdot c_m \otimes d_n)) \;=\; \partial(\Psi_{k,l}(c_m \otimes d_n)).
\end{array}$$

This calculation implies that $\Psi_{k,l}$ is a chain map. ■

The following theorem is the main technical result of this section. As we will see shortly, the Product Theorem 84.1 is a straightforward consequence of Theorem 84.15.

Theorem 84.15. *Let X and Y be topological spaces. We denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the projection maps. Let $\alpha\in \mathrm{C}^k(X;\mathbb{Z})$ and $\beta\in \mathrm{C}^l(Y;\mathbb{Z})$ be cocycles. Then the chain maps*

$$(-1)^{kl}\cdot((p^*\alpha\cup q^*\beta)\,\widetilde{\frown}\,-)\circ\Upsilon \qquad \textit{and} \qquad \Upsilon[k+l]\circ\Psi_{k,l}\circ(\alpha\,\widetilde{\frown}\,-\ \otimes\ \beta\,\widetilde{\frown}\,-)$$

from $\mathrm{C}_*(X)\otimes \mathrm{C}_*(Y)$ to $\mathrm{C}_*(X\times Y)[k+l]$ are chain homotopic.

PROOF. Let X and Y be topological spaces and let $\alpha\in \mathrm{C}^k(X;\mathbb{Z})$ and $\beta\in\mathrm{C}^l(Y;\mathbb{Z})$ be cocycles. We consider the following diagram of chain maps

$$\begin{tikzcd}
\mathrm{C}_* X\otimes \mathrm{C}_* Y \arrow[rr, "\Upsilon"] \arrow[d, "\Delta_X\otimes\Delta_Y"] & & \mathrm{C}_*(X\times Y) \arrow[rr, "\Delta_{X\times Y}"] & & \mathrm{C}_*(X\times Y)\otimes \mathrm{C}_*(X\times Y) \arrow[d, "\Theta\otimes\mathrm{id}"] \\
\mathrm{C}_* X\otimes \mathrm{C}_* X\otimes \mathrm{C}_* Y\otimes \mathrm{C}_* Y \arrow[rr, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"] \arrow[d, "\Pi_k(\alpha)\otimes\mathrm{id}\otimes\Pi_l(\beta)\otimes\mathrm{id}"'] & & \mathrm{C}_* X\otimes \mathrm{C}_* Y\otimes \mathrm{C}_* X\otimes \mathrm{C}_* Y \arrow[rr, "\mathrm{id}\otimes\mathrm{id}\otimes\Upsilon"] \arrow[d, "\Pi_k(\alpha)\otimes\Pi_l(\beta)\otimes\mathrm{id}\otimes\mathrm{id}"] & & \mathrm{C}_* X\otimes \mathrm{C}_* Y\otimes \mathrm{C}_*(X\times Y) \arrow[d, "\Pi_k(\alpha)\otimes\Pi_l(\beta)\otimes\mathrm{id}"] \\
\mathbb{Z}[k]\otimes \mathrm{C}_* X\otimes \mathbb{Z}[l]\otimes \mathrm{C}_* Y \arrow[rr, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"] \arrow[dd, "\eta\otimes\eta"'] & & \mathbb{Z}[k]\otimes \mathbb{Z}[l]\otimes \mathrm{C}_* X\otimes \mathrm{C}_* Y \arrow[rr, "\mathrm{id}\otimes\mathrm{id}\otimes\Upsilon"] \arrow[d, "\eta\otimes\mathrm{id}\otimes\mathrm{id}"] & & \mathbb{Z}[k]\otimes\mathbb{Z}[l]\otimes \mathrm{C}_*(X\times Y) \arrow[d, "\eta\otimes\mathrm{id}"] \\
& & \mathbb{Z}[k+l]\otimes(\mathrm{C}_* X\otimes \mathrm{C}_* Y) \arrow[rr, "\mathrm{id}\otimes\Upsilon"] \arrow[d, "\eta"] & & \mathbb{Z}[k+l]\otimes \mathrm{C}_*(X\times Y) \arrow[d, "\eta"] \\
\mathrm{C}_* X[k]\otimes \mathrm{C}_* Y[l] \arrow[rr, "\Psi_{k,l}"] & & (\mathrm{C}_* X\otimes \mathrm{C}_* Y)[k+l] \arrow[rr, "\Upsilon[k+l]"] & & \mathrm{C}_*(X\times Y)[k+l].
\end{tikzcd}$$

Claim. The above diagram commutes up to chain homotopies.

To prove the claim we consider the various smaller squares and rectangles.

(1) The two paths along the top rectangle from the top-left corner to the bottom-right corner are both natural chain maps. Furthermore, both have the property that for $x\in X$ and $y\in Y$ the element $x\otimes y\in \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)$ gets sent to $x\otimes y\otimes(x,y)$ in $\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\otimes\mathrm{C}_*(X\times Y)$. In particular both paths agree on the 0-level.

   It follows from Acyclic Model Theorem 80.16 (2) that the rectangle commutes up to a chain homotopy. More precisely, we apply the Acyclic Model Theorem 80.16 as follows:
   (a) We take $\mathcal{C}$ to be the category of tuples (X,Y) of topological spaces.
   (b) We consider the two functors $F\colon (X,Y)\mapsto (\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y),\epsilon_X\otimes\epsilon_Y)$ and $G\colon (X,Y)\mapsto(\mathrm{C}_*(X\times Y),\epsilon_{X\times Y})$.
   (c) We take $\mathcal{M}=\{\Delta^k\times\Delta^l\}_{k,l\in\mathbb{N}_0}$.
   (d) We consider the two natural transformations that are given by the two extremal paths in the diagram (shifted downward in degrees).

(2) It is clear that the two squares between the second and the third row commute.
(3) It is straightforward to verify that the two squares at the bottom right commute.
(4) It remains to consider the big square to the bottom left of the diagram. Starting with $a\in(\mathbb{Z}[k])_k=\mathbb{Z}$, $b\in\mathrm{C}_m(X)$, $c\in(\mathbb{Z}[l])_l=\mathbb{Z}$ and $d\in\mathrm{C}_n(Y)$ and keeping in mind

the sign conventions of Lemmas 84.7, 84.9 and 84.14 we get the following results for $a \otimes b \otimes c \otimes d$:
(a) going down-right gives $(-1)^{km} \cdot (-1)^{ln} \cdot (-1)^{k(l+n)} \cdot ac \cdot (b \otimes d)$,
(b) going right-down-down gives $(-1)^{ml} \cdot (-1)^{kl} \cdot (-1)^{(k+l)(m+n)} \cdot ac \cdot (b \otimes d)$.
One easily verifies that the signs agree. So we see that the bottom left square also commutes. ⊞

In light of the previous claim it suffices to prove the following claim.

Claim.

(1) the "left-bottom route" equals the chain map $\Upsilon[k+l] \circ \Psi_{k,l} \circ ((\alpha \,\widetilde{\frown}\, -) \otimes (\beta \,\widetilde{\frown}\, -))$,
(2) the "top-right route" is chain homotopic to $(-1)^{kl} \cdot ((p^*\alpha \cup q^*\beta) \,\widetilde{\frown}\, -) \circ \Upsilon$.

We start out with the proof of statement (1). First we note that

$$(\alpha \,\widetilde{\frown}\, -) \otimes (\beta \,\widetilde{\frown}\, -) \underset{\uparrow}{=} (\eta \circ (\Pi_k(\alpha) \otimes \mathrm{id}) \circ \Delta_X) \otimes (\eta \circ (\Pi_l(\beta) \otimes \mathrm{id}) \circ \Delta_Y)$$

by Proposition 84.12

$$\underset{\uparrow}{=} (\eta \otimes \eta) \circ ((\Pi_k(\alpha) \otimes \mathrm{id} \otimes \Pi_l(\beta) \otimes \mathrm{id})) \circ (\Delta_X \otimes \Delta_Y).$$

follows from the observation that for chain maps a, b, c and d we have $(a \circ b) \otimes (c \circ d) = (a \otimes b) \circ (c \otimes d)$

If we compose this equality with $\Upsilon[k+l] \circ \Psi_{k,l}$ then we obtain the "bottom left route" in the above diagram. This concludes the proof of statement (1).

We continue with the proof of statement (2). In that proof we will make use of the following subclaim.

Subclaim. The two cocycles[1213]

$$(\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta)_{k+l} \quad \text{and} \quad (-1)^{kl} \cdot p^*(\alpha) \cup q^*(\beta)$$

in $\mathrm{C}^{k+l}(X \times Y; \mathbb{Z})$ are cohomologous.

We consider the following diagram

$$\begin{tikzcd}
\mathrm{C}_{k+l}(X \times Y) \arrow[r, hook] & \mathrm{C}_*(X \times Y) \arrow[rr, "\Theta"] \arrow[d, "\Delta_{X \times Y}"] & & \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \arrow[d, "\Pi_k(\alpha) \otimes \Pi_l(\beta)"] & \\
& \mathrm{C}_*(X \times Y) \otimes \mathrm{C}_*(X \times Y) \arrow[urr, "p_* \otimes q_*"] \arrow[rr, "\Pi_k(p^*(\alpha)) \otimes \Pi_l(q^*(\beta))"'] & & \mathbb{Z}[k] \otimes \mathbb{Z}[l] \arrow[r, "\eta"'] & \mathbb{Z}[k+l].
\end{tikzcd}$$

Note that by the above discussion all maps except for the top left map are chain maps. It follows immediately from the definitions that the bottom triangle of chain maps commutes "on the nose". The top triangle of chain maps commutes, up to a chain homotopy, by Theorem 80.5.[1214] It follows from Lemma 84.8 (3) that the restrictions of these two chain

[1213] Recall that $\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta$ is a chain map $\mathrm{C}_*(X \times Y) \to \mathbb{Z}[k+l]$. Furthermore recall that the map $(\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta)_{k+l} \colon \mathrm{C}_{k+l}(X \times Y) \to (\mathbb{Z}[k+l])_{k+l} = \mathbb{Z}$ is the $(k+l)$-th map of the chain map.

[1214] More precisely, given $x \in X$ and $y \in Y$ both routes along the triangle send (x, y) to $x \otimes y$. Furthermore both routes are natural chain maps. Thus we can apply Theorem 80.5.

maps to maps $C_{k+l}(X\times Y)\to\mathbb{Z}=\mathbb{Z}[k+l]$ are cohomologous cochains. But the restriction of the "top route" equals evidently the left-hand side of the subclaim. On the other hand, by Lemma 84.10 the restriction of the "bottom route" equals the right-hand side of the subclaim. This concludes the proof of the subclaim.

Now we see that

$$(-1)^{kl}\cdot((p^*\alpha\cup q^*\beta)\,\widetilde{\cap}\,-)\circ\Upsilon \overset{\substack{\text{the above subclaim and Lemma 84.11 (2)}\\ \downarrow}}{\simeq} \Big((\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\circ\Theta)_{k+l}\,\widetilde{\cap}\,-\Big)\circ\Upsilon$$

$$\underset{\uparrow}{=}\ \eta\circ\Big(\Pi_{k+l}((\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\circ\Theta)_{k+l})\otimes\mathrm{id}\Big)\circ\Delta_{X\times Y}\circ\Upsilon$$

Proposition 84.12 applied to the $(k+l)$-cocycle $(\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\circ\Theta)_{k+l}$

$$\underset{\uparrow}{=}\ \eta\circ\Big(\Pi_{k+l}((\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta)))_{k+l})\circ\Theta\otimes\mathrm{id}\Big)\circ\Delta_{X\times Y}\circ\Upsilon$$

since for every chain map $\Omega\colon C_*(X)\times C_*(Y)\to\mathbb{Z}[k+l]$ we have $\Pi_{k+l}((\Omega\circ\Theta)_{k+l})=\Pi_{k+l}(\Omega_{k+l})\circ\Theta$ as maps $C_*(X\times Y)\to\mathbb{Z}[k+l]$, applies this observationc to $\Omega=\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))$

$$\underset{\uparrow}{=}\ \eta\circ((\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\circ\Theta)\otimes\mathrm{id})\circ\Delta_{X\times Y}\circ\Upsilon$$

here we use the equality of chain maps

$$\Pi_{k+l}((\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta)))_{k+l})=\eta\circ(\Pi_k(\alpha)\otimes\Pi_l(\beta))\colon\ C_*(X)\otimes C_*(Y)\to\mathbb{Z}[k+l]$$

which is obvious once one unravels the definitions

$$\underset{\uparrow}{=}\ \text{the "top right route" in the above diagram.}$$

follows from $(f\otimes\mathrm{id})\circ(g\otimes\mathrm{id})=(g\circ f)\otimes\mathrm{id}$

This concludes the proof of the claim and thus also of the theorem. ■

Now we can finally provide the proof of Theorem 84.1.

Proof of Theorem 84.1. To simplify the notation we only consider the special case that $A=B=\varnothing$. For the general case one needs to take more care, we refer to [**Dol56**, p. 240 and p. 244] for details. So let X and Y be two topological spaces. We denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the projection maps. Let $[\alpha]\in H^k(X;\mathbb{Z})$, $[\beta]\in H^l(Y;\mathbb{Z})$ and let $[\mu]\in H_m(X;\mathbb{Z})$ and $[\nu]\in H_n(Y;\mathbb{Z})$. We have to show that

$$\Big(\underbrace{[p^*(\alpha)]}_{\in H^k(X\times Y)}\cup\underbrace{[q^*(\beta)]}_{\in H^l(X\times Y)}\Big)\cap\underbrace{\Big([\mu]\times[\nu]\Big)}_{\in H_{m+n}(X\times Y)} = (-1)^{kl+lm}\cdot\underbrace{([\alpha]\cap[\mu])}_{\in H_{m-k}(X)}\times\underbrace{([\beta]\cap[\nu])}_{\in H_{n-l}(Y)}$$

We have the following equality in $\mathrm{H}_{m+n-k-l}(X \times Y)$:

$$
\begin{aligned}
(p^*([\alpha]) \cup q^*([\beta])) \frown ([\mu] \times [\nu]) &\underset{\substack{\uparrow \\ \text{Tautology 84.13 and the definition of } \widetilde{\frown}\text{, see page 2042}}}{=} (-1)^{(k+l)(m+n-k-l)} \cdot ([p^*\alpha \cup q^*\beta]) \,\widetilde{\frown}\, [\Upsilon(\mu \otimes \nu)] \\
&= (-1)^{(k+l)(m+n-k-l)} \cdot (((p^*\alpha \cup q^*\beta) \,\widetilde{\frown}\, -) \circ \Upsilon)_*([\mu \otimes \nu]) \\
&\underset{\substack{\uparrow \\ \text{by Theorem 84.15 and the fact that chain homotopic maps} \\ \text{induce the same map on homology groups}}}{=} (-1)^{(k+l)(m+n-k-l)} \cdot ((-1)^{kl} \cdot \Upsilon[k+l] \circ \Psi_{k,l} \circ (\alpha \,\widetilde{\frown}\, - \otimes \beta \,\widetilde{\frown}\, -))_*([\mu \otimes \nu]) \\
&= (-1)^{(k+l)(m+n-k-l)} \cdot (-1)^{kl} \cdot \big[\Upsilon[k+l] \circ \Psi_{k,l} \circ \big(\underbrace{\underbrace{\alpha \,\widetilde{\frown}\, \mu}_{=(-1)^{k(m-k)} \cdot \alpha \frown \mu}}_{\in \mathrm{C}_{m-k}(X)[k]} \otimes \underbrace{\underbrace{\beta \,\widetilde{\frown}\, \nu}_{=(-1)^{l(n-l)} \cdot \beta \frown \nu}}_{\in \mathrm{C}_{n-l}(Y)[l]} \big)\big] \\
&\underset{\substack{\uparrow \\ \text{definition of } \Psi_{k,l}\text{, see page 2044, and of the shifting from } \Upsilon \text{ to } \Upsilon[k+l]}}{=} (-1)^{(k+l)(m+n-k-l)} \cdot (-1)^{kl} \cdot (-1)^{k(l+n-l)} \cdot (-1)^{k(m-k)} \cdot (-1)^{l(n-l)} \cdot \big[\Upsilon(\alpha \frown \mu \otimes \beta \frown \nu)\big] \\
&\underset{\substack{\uparrow \\ \text{cleaning up signs and Tautology 84.13}}}{=} (-1)^{kl+lm} \cdot [\alpha \frown \mu \,\times\, \beta \frown \nu] = (-1)^{kl+lm} \cdot ([\alpha] \frown [\mu] \,\times\, [\beta] \frown [\nu]).
\end{aligned}
$$

■

84.3. **The Künneth Theorem for cohomology groups.** Given two topological spaces X and Y the Künneth Theorem 80.8 for topological spaces determines the homology groups of $X \times Y$ in terms of the homology groups of X and Y. The following proposition gives us, under some mild hypothesis, a similar answer for cohomology groups.

Theorem 84.16. (The Künneth Theorem for cohomology groups) *Let X and Y be two topological spaces. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the obvious projection maps. If all the homology groups of X are finitely generated*[1215]*, then given any $n \in \mathbb{N}_0$ there exists a short exact sequence*

$$
0 \to \bigoplus_{k+l=n} \mathrm{H}^k(X;\mathbb{Z}) \otimes \mathrm{H}^l(Y;\mathbb{Z}) \xrightarrow{\varphi \otimes \psi \,\mapsto\, p^*\varphi \cup q^*\psi} \mathrm{H}^n(X \times Y;\mathbb{Z}) \to \bigoplus_{k+l=n+1} \mathrm{Tor}(\mathrm{H}^k(X;\mathbb{Z}), \mathrm{H}^l(Y;\mathbb{Z})) \to 0
$$

which is natural with respect to the topological spaces X and Y. If all the homology groups of Y are also finitely generated, then the short exact sequence splits (albeit not naturally).

Remark. Let X and Y be two topological spaces. In the literature the map

$$
\begin{array}{rcl}
\mathrm{H}^k(X;\mathbb{Z}) \times \mathrm{H}^l(Y;\mathbb{Z}) & \to & \mathrm{H}^{k+l}(X \times Y;\mathbb{Z}) \\
(\varphi, \psi) & \mapsto & p^*\varphi \cup q^*\psi
\end{array}
$$

is often referred to as the *cohomology cross product* and it is written as $\varphi \times \psi := p^*\varphi \cup q^*\psi$. We will not make use of this notation.

The proof of the theorem rests on a purely algebraic theorem that, given two chain complexes $\mathcal{C}$ and $\mathcal{C}'$, relates the cohomology groups of $\mathcal{C}$ and $\mathcal{C}'$ to the cohomology groups of $\mathcal{C} \otimes \mathcal{C}'$. Before we can state the theorem we need to introduce one more definition, which is closely related to the definitions introduced on page 1979.

[1215]By this we mean that for every $k \in \mathbb{N}_0$ the abelian group $\mathrm{H}_k(X;\mathbb{Z})$ is finitely generated.

Definition. Let $\mathcal{C}$ and $\mathcal{C}'$ be two chain complexes and let $p, q \in \mathbb{N}_0$.

(1) We denote by

$$\Pi_{p,q}\colon (\mathcal{C}_* \otimes \mathcal{C}'_*)_{p+q} = \bigoplus_{i+j=p+q} \mathrm{C}_i \otimes \mathrm{C}'_j \;\to\; \mathrm{C}_p \otimes \mathrm{C}'_q$$

the obvious projection map.

(2) We denote by $\mu\colon \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ the obvious isomorphism given by $\mu(a \otimes b) = ab$.

(3) We consider the map:

$$\begin{aligned} \Xi_{p,q}\colon \mathrm{Hom}(\mathrm{C}_p, \mathbb{Z}) \otimes \mathrm{Hom}(\mathrm{C}'_q, \mathbb{Z}) \;&\to\; \mathrm{Hom}\big((\mathcal{C} \otimes \mathcal{C}')_{p+q}, \mathbb{Z}\big) \\ \varphi_p \otimes \varphi'_q \;&\mapsto \Big(\bigoplus_{i+j=p+q} \mathrm{C}_i \otimes \mathrm{C}'_j \xrightarrow{\Pi_{p,q}} \mathrm{C}_p \otimes \mathrm{C}'_q \xrightarrow{\varphi_p \otimes \varphi'_q} \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{\mu} \Big). \end{aligned}$$

(4) It is straightforward to verify that the map $\Xi_{p,q}$ induces a well-defined map on cohomology groups, i.e. that the map

$$\begin{array}{rccc} \Xi_{p,q}\colon & \mathrm{H}^p(\mathcal{C}; \mathbb{Z}) \otimes \mathrm{H}^q(\mathcal{C}'; \mathbb{Z}) & \to & \mathrm{H}^{p+q}(\mathcal{C} \otimes \mathcal{C}'; \mathbb{Z}) \\ & ([\varphi_p] \otimes [\varphi'_q]) & \mapsto & [\Xi_{p,q}(\varphi_p \otimes \varphi'_q)] \end{array}$$

is well-defined. Furthermore it follows immediately from the definition that $\Xi_{p,q}$ is natural[1216] with respect to $\mathcal{C}$ and $\mathcal{C}'$.

Now we can state the promised purely algebraic theorem.

Theorem 84.17. (The Künneth Theorem for cochain complexes) *Let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes. If $\mathcal{C}$ is a free chain complex and if all homology groups of $\mathcal{C}$ are finitely generated, then for every $n \in \mathbb{N}_0$ there exists a short exact sequence*

$$0 \to \bigoplus_{p+q=n} \mathrm{H}^p(\mathcal{C}; \mathbb{Z}) \otimes \mathrm{H}^q(\mathcal{C}'; \mathbb{Z}) \xrightarrow{\Xi_{p,q}} \mathrm{H}^n(\mathcal{C} \otimes \mathcal{C}'; \mathbb{Z}) \to \bigoplus_{p+q=n+1} \mathrm{Tor}(\mathrm{H}^p(\mathcal{C}; \mathbb{Z}), \mathrm{H}^q(\mathcal{C}; \mathbb{Z})) \to 0$$

which is natural with respect to $\mathcal{C}$ and $\mathcal{C}'$. If all the chain groups of $\mathcal{C}'$ are also free abelian and all the homology groups of $\mathcal{C}'$ are also finitely generated, then the short exact sequence splits (albeit not naturally).

In the proof of the Künneth Theorem 84.17 for cochain complexes we need one technical lemma. To state the lemma we recall that on page 1433 we introduced the tensor product of two chain complexes. Basically the same definition also allows us to define the tensor product of two cochain complexes. With this preamble we can formulate the next lemma.

[1216] In this case this means that for chain maps $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(\mathcal{D}; \mathbb{Z}) \otimes \mathrm{H}^q(\mathcal{D}'; \mathbb{Z}) & \xrightarrow{\Xi_{p,q}} & \mathrm{H}^{p+q}(\mathcal{D} \otimes \mathcal{D}'; \mathbb{Z}) \\ \downarrow{\scriptstyle f^* \otimes (f')^*} & & \downarrow{\scriptstyle (f \otimes f')^*} \\ \mathrm{H}^p(\mathcal{C}; \mathbb{Z}) \otimes \mathrm{H}^q(\mathcal{C}'; \mathbb{Z}) & \xrightarrow{\Xi_{p,q}} & \mathrm{H}^{p+q}(\mathcal{C} \otimes \mathcal{C}'; \mathbb{Z}). \end{array}$$

Lemma 84.18. *Let $\mathcal{C}_*$ and $\mathcal{C}'_*$ be two chain complexes. We assume that all chain groups of $\mathcal{C}$ are finitely generated free abelian groups. Then the map*

$$\begin{array}{rcl} \Omega\colon \operatorname{Hom}(\mathcal{C}_*,\mathbb{Z})\otimes\operatorname{Hom}(\mathcal{C}'_*,\mathbb{Z}) & \to & \operatorname{Hom}(\mathcal{C}_*\otimes\mathcal{C}'_*,\mathbb{Z}) \\ (\varphi_p\colon\, \mathrm{C}_p\to\mathbb{Z})\otimes(\varphi'_q\colon C'_q\to\mathbb{Z}) & \mapsto & (\mu\circ(\varphi_p\otimes\varphi'_q)\circ\Pi_{p,q}\colon (\mathcal{C}_*\otimes\mathcal{C}'_*)_{p+q}\to\mathbb{Z}) \end{array}$$

is an isomorphism of cochain complexes.

Proof (∗). We start out the proof of the lemma with the following elementary well-known observation:

(∗) For finitely many abelian groups $A_1,\dots,A_k$ we have a canonical isomorphism

$$\operatorname{Hom}(A_1\oplus\cdots\oplus A_k,\mathbb{Z}) \;=\; \operatorname{Hom}(A_1,\mathbb{Z})\oplus\cdots\oplus\operatorname{Hom}(A_k,\mathbb{Z}).$$

Now we turn to the actual proof of the lemma. It is elementary to verify that the map Ω is in fact a cochain map. So it remains to show that for every $n\in\mathbb{N}_0$ the map

$$\begin{array}{rcl} \bigoplus\limits_{p+q=n}\operatorname{Hom}(\mathrm{C}_p,\mathbb{Z})\otimes\operatorname{Hom}(C'_q,\mathbb{Z}) & \to & \operatorname{Hom}\Big(\bigoplus\limits_{p+q=n}\mathrm{C}_p\otimes\mathrm{C}'_q,\mathbb{Z}\Big) \\ \varphi_p\otimes\varphi'_q & \mapsto & \mu\circ(\varphi_p\otimes\varphi'_q)\circ\Pi_{p,q} \end{array}$$

is an isomorphism. Applying (∗) to the right-hand side we see that it suffices to prove the following claim.

Claim. For every choice of $p,q\in\mathbb{N}_0$ the map

$$\begin{array}{rcl} \operatorname{Hom}(\mathrm{C}_p,\mathbb{Z})\otimes\operatorname{Hom}(C'_q,\mathbb{Z}) & \to & \operatorname{Hom}(\mathrm{C}_p\otimes C'_q,\mathbb{Z}) \\ \varphi_p\otimes\varphi'_q & \mapsto & \mu\circ(\varphi_p\otimes\varphi'_q) \end{array}$$

is an isomorphism.

Let $p,q\in\mathbb{N}_0$. By our hypothesis on $\mathcal{C}_*$ the chain group C_p is a finitely generated free abelian group. Therefore we can find a basis $a_1,\dots,a_k$ for C_p. We denote by $\widehat{a}_1,\dots,\widehat{a}_k$ the dual basis of $\operatorname{Hom}(\mathrm{C}_p,\mathbb{Z})$. We can thus write

$$\operatorname{Hom}(\mathrm{C}_p,\mathbb{Z})=\bigoplus_{i=1}^{k}\mathbb{Z}\cdot\widehat{a}_i \quad\text{ and }\quad \mathrm{C}_p\otimes\mathrm{C}'_q=\bigoplus_{i=1}^{k}\mathbb{Z}\cdot a_i\otimes\mathrm{C}'_q\,.$$

Applying (∗) to the above decompositions and both sides of the claim one sees easily that the given homomorphism is an isomorphism. This concludes the proof. ■

Now we can finally prove the Künneth Theorem 84.17 for cochain complexes.

Proof of the Künneth Theorem 84.17 for cochain complexes. We only provide an outline of the proof. Let $\mathcal{C}=(\mathrm{C}_*,c_*)$ and $\mathcal{C}'=(\mathrm{C}'_*,c'_*)$ be chain complexes such that all the chain groups C_n are free abelian groups and such that all the homology groups $\mathrm{H}_k(\mathcal{C};\mathbb{Z})$ are finitely generated.

The idea is that one can turn any cochain complex into a chain complex by reversing the signs in the degrees. Thus one can try to reduce the theorem to the Künneth Theorem 58.7. Towards the end of the proof we need to insert Lemma 84.18 to make everything work.

The proof proceeds in three steps:

(1) Since $\mathcal{C}$ is a free chain complex and since all homology groups of $\mathcal{C}$ are finitely generated we can use Lemma 49.6 to replace $\mathcal{C}$ by a chain complex such that all chain groups are finitely generated free abelian groups.

(2) Given $n \in \mathbb{Z}_{\leq 0}$ we write $E_n := \operatorname{Hom}(\mathrm{C}_{-n}, \mathbb{Z})$, $e_n := \partial^*_{-n+1} \colon E_n \to E_{n-1}$ and similarly we write $E'_n := \operatorname{Hom}(\mathcal{C}'_{-n}, \mathbb{Z})$ and $e'_n := \partial^*_{-n+1} \colon E'_n \to E'_{n-1}$. It follows immediately from the definitions that $\mathcal{E}_* := (E_*, e_*)$ and $\mathcal{E}'_* := (E'_*, e'_*)^*)$ form chain complexes in negative degrees. (We defined chain complexes to be given by chain groups indexed by $n \in \mathbb{N}_0$ but evidently the same theory also works for chain groups indexed by $\mathbb{Z}$ and $\mathbb{Z}_{\leq 0}$.) Since the groups E_n are free abelian we can apply the Künneth Theorem 58.7 (which also applies to chain complexes in non-positive degrees) to obtain for each $n \in \mathbb{Z}_{\leq 0}$ a natural short exact sequence

$$0 \to \bigoplus_{p+q=n} \underset{[\varphi]\otimes[\psi]}{\mathrm{H}_p(\mathcal{E}) \otimes \mathrm{H}_q(\mathcal{E}')} \;\to\; \underset{:}{\mathrm{H}_n(\mathcal{E} \otimes \mathcal{E}')} \to \bigoplus_{p+q=n-1} \operatorname{Tor}(\mathrm{H}_p(\mathcal{E}), \mathrm{H}_q(\mathcal{E}')) \to 0.$$

Put differently, for each $n \in \mathbb{N}_0$ we have a natural short exact sequence

$$0 \to \bigoplus_{-p-q=n} \underbrace{\mathrm{H}_{-p}(\mathcal{E})}_{=\mathrm{H}^p(\mathcal{C};\mathbb{Z})} \otimes \underbrace{\mathrm{H}_{-q}(\mathcal{E}')}_{=\mathrm{H}^q(\mathcal{E}';\mathbb{Z})} \to \mathrm{H}_{-n}(\mathcal{E}\otimes\mathcal{E}') \to \bigoplus_{-p-q=n+1} \operatorname{Tor}(\underbrace{\mathrm{H}_{-p}(\mathcal{E})}_{=\mathrm{H}^p(\mathcal{C};\mathbb{Z})}, \underbrace{\mathrm{H}_{-q}(\mathcal{E}')}_{=\mathrm{H}^p(\mathcal{C};\mathbb{Z})}) \to 0.$$

(3) For each $n \in \mathbb{N}_0$ we have isomorphisms

$$\mathrm{H}_{-n}(\mathcal{E}_*\otimes\mathcal{E}'_*) \underset{\substack{\uparrow\\ \text{by definition}}}{=} \mathrm{H}^n(\mathcal{E}_{-*}\otimes\mathcal{E}'_{-*}) \underset{\substack{\uparrow\\ \text{by definition}}}{=} \mathrm{H}^n\Big(\overbrace{\operatorname{Hom}(\mathcal{C}_*, \mathbb{Z})}^{=\mathcal{E}_{-*}} \otimes \overbrace{\operatorname{Hom}(\mathcal{C}'_*, \mathbb{Z})}^{=\mathcal{E}'_{-*}}\Big) \underset{\substack{\uparrow\\ \text{by Lemma 84.18 and our hypothesis on } \mathcal{C}_*}}{\xrightarrow{\Omega}} \mathrm{H}^n(\mathcal{C}_*\otimes\mathcal{C}'_*; \mathbb{Z}).$$

We use this isomorphism to replace the middle term of the second short exact sequence in (2) and we obtain the desired short exact sequence.

Finally suppose that all the chain groups of $\mathcal{C}'$ are also free abelian and that all the homology groups of $\mathcal{C}'$ are also finitely generated. As in (1) we can use Lemma 49.6 to replace $\mathcal{C}$ by a chain complex such that all chain groups are finitely generated free abelian groups. It follows that $\mathcal{E}_*$ is a free chain complex which implies, by the last statement of the Künneth Theorem 58.7, that the short exact sequence splits. ■

Now we are in a position to prove the Künneth Theorem 84.16 for cohomology groups.

PROOF OF THE KÜNNETH THEOREM 84.16 FOR COHOMOLOGY GROUPS (∗). Let X and Y be two topological spaces such that all the homology groups of X are finitely generated. We denote by $p \colon X \times Y \to X$ and $q \colon X \times Y \to Y$ the obvious projection maps. Throughout the proof we use all the definitions that we introduced in Section 84.2.

It follows from the Eilenberg-Zilber Theorem 80.1, together with comparing the conclusion of the Künneth Theorem 84.17 for cochain complexes with the desired statement of the Künneth Theorem 84.16 for cohomology groups, that it remains to prove the following claim.

Claim. For any $k, l \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{H}^k(X;\mathbb{Z}) \otimes \mathrm{H}^l(Y;\mathbb{Z}) & \to & \mathrm{H}^{k+l}(X \times Y;\mathbb{Z}) \\ [\alpha] \otimes [\beta] & \mapsto & [p^*\alpha \cup q^*\beta] \end{array}$$

agrees with the map that is given by

$$[\alpha] \otimes [\beta] \ \mapsto \ [(\alpha \otimes \beta) \circ \Pi_{k,l} \circ \Theta].$$

Let $\alpha \in \mathrm{C}^k(X;\mathbb{Z})$ and $\beta \in \mathrm{C}^l(Y;\mathbb{Z})$ be cocycles. We have to show that the two cocycles

$$\underbrace{p^*\alpha \cup q^*\beta}_{=:\aleph} \quad \text{and} \quad \underbrace{(\alpha \otimes \beta) \circ \Pi_{k,l}}_{=:\beth} \circ \Theta$$

are cohomologous.

As usual, given a topological space W we denote by $\epsilon_W \colon \mathrm{C}_0(W) \to \mathbb{Z}$ the augmentation map. Clearly ϵ_W is a cocycle so by Lemma 84.8 (2) we can extend ϵ_W to a chain map $\mathrm{C}_*(W) \to \mathbb{Z}$ which, by a slight abuse of notation, we also denote by ϵ_W. It follows from the Eilenberg-Zilber Theorem 80.1, Theorem 84.15 and Lemma 84.8 (2) that the maps

$$\underbrace{(-1)^{kl} \cdot \epsilon_{X\times Y}[k+l] \ \circ \ ((p^*\alpha \cup q^*\beta) \,\widetilde{\frown}\, -)}_{=:\aleph'} \colon \ \mathrm{C}_{k+l}(X \times Y) \to \mathbb{Z}[k+l] = \mathbb{Z}$$

and

$$\underbrace{\epsilon_{X\times Y}[k+l] \circ \Upsilon[k+l] \circ \Psi_{k,l} \circ (\alpha \,\widetilde{\frown}\, - \otimes \beta \,\widetilde{\frown}\, -)}_{=:\beth'} \circ \Theta \colon \ \mathrm{C}_{k+l}(X \times Y) \to \mathbb{Z}[k+l] = \mathbb{Z}$$

are cocycles that are cohomologous. It remains to prove the following two subclaims.

Subclaim. We have

$$\aleph' = (-1)^{kl} \cdot \aleph \colon \ \mathrm{C}_{k+l}(X \times Y) \ \to \ \mathbb{Z}.$$

Let $\sigma \in \mathrm{C}_{k+l}(X \times Y)$. Then

$$\aleph'(\sigma) \underset{\substack{\uparrow \\ \text{under the identification } \mathbb{Z}[k+l]=\mathbb{Z} \text{ we} \\ \text{have } \epsilon_{X\times Y}[k+l] = \epsilon_{X\times Y}}}{=} (-1)^{kl} \cdot \epsilon_{X\times Y}((p^*\alpha \cup q^*\beta) \,\widetilde{\frown}\, \sigma) \underset{\substack{\uparrow \\ \text{by definition of } \widetilde{\frown}\text{, see page 2042,} \\ \text{together with Lemma 83.4 (1)}}}{=} (-1)^{kl} \cdot (p^*\alpha \cup q^*\beta)(\sigma) \ = \ (-1)^{kl} \cdot \aleph(\sigma).$$

This concludes the proof of the subclaim.

Subclaim. We have

$$\beth' = (-1)^{kl} \cdot \beth \colon \ (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_{k+l} \ \to \ \mathbb{Z}.$$

We have to show that both sides take the same value on any tensor product of the form $\sigma_i \otimes \sigma_j$ with $i + j = k + l$ and $\sigma_i \in \mathrm{C}_i(X)$ and let $\sigma_j \in \mathrm{C}_j(Y)$. It follows easily from the definition that both sides are zero for $\sigma_i \otimes \sigma_j$ with $(i, j) \neq (k, l)$. So let $\sigma_k \in \mathrm{C}_k(X)$ and let $\sigma_l \in \mathrm{C}_l(Y)$. Then

$$\begin{aligned}
\beth'(\sigma_k \otimes \sigma_l) &\underset{\substack{\uparrow \\ \text{since } \epsilon_{X\times Y}[k+l]\circ\Upsilon[k+l] = (\epsilon_{X\times Y}\circ\Upsilon)[k+l]}}{=} (\epsilon_{X\times Y}\circ\Upsilon)[k+l]\circ\Psi_{k,l}\circ(\alpha\,\widetilde{\frown}\,\sigma_k\otimes\beta\,\widetilde{\frown}\,\sigma_l) \\
&\underset{\substack{\uparrow \\ \text{we use } \epsilon_{X\times Y}\circ\Upsilon = \epsilon_X\otimes\epsilon_Y\colon (\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y))_0\to\mathbb{Z}\otimes\mathbb{Z}=\mathbb{Z}, \\ \text{the definition of } \Psi_{k,l} \text{ in Lemma 84.14 and the definition of } \widetilde{\frown} \text{ on page 2042}}}{=} (-1)^{kl}\cdot(\epsilon_X\otimes\epsilon_Y)\circ(\alpha\frown\sigma_k\otimes\beta\frown\sigma_l) \underset{\substack{\uparrow \\ \text{by Lemma 83.4}}}{=} (-1)^{kl}\cdot\alpha(\sigma_k)\cdot\beta(\sigma_l) \\
&= (-1)^{kl}\cdot((\alpha\otimes\beta)\circ\Pi_{k,l})(\sigma_k\otimes\sigma_l) = (-1)^{kl}\cdot\beth(\sigma_k\otimes\sigma_l).
\end{aligned}$$

This concludes the proof of the second subclaim, thus also of the claim and therefore also of the theorem. ■

84.4. **The cohomology ring of product topological spaces.** In this section we will determine the cohomology ring of the m-dimensional torus $(S^1)^m$ and more generally, under some mild technical hypothesis, we will determine the cohomology ring of a product $X\times Y$ of topological spaces in terms of the cohomology rings of X and Y.

We recall the following definition.

Definition.

(1) A *graded ring* is a ring R together with a decomposition $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ such that for any $m,n\in\mathbb{N}_0$ the multiplication map restricts to a map $R_m\times R_n\to R_{m+n}$. We say that an element r of R that lies in some R_n is *homogeneous* and we refer to $\deg(r) := n$ as the degree of r.
(2) A homomorphism $\varphi\colon A\to B$ between graded rings is a ring homomorphism that preserves the gradings, i.e. that satisfies $\varphi(A_n)\subset B_n$ for all $n\in\mathbb{N}_0$.
(3) A graded ring $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ such that for any two homogeneous elements a and b the equality $a\cdot b = (-1)^{\deg(a)\cdot\deg(b)}\cdot b\cdot a$ holds is called a *superalgebra*.

Examples.

(1) Proposition 81.8 says in particular that for any topological space X the cohomology ring $(\mathrm{H}^*(X;\mathbb{Z}),\cup)$ is a superalgebra. Furthermore Lemma 81.10 implies that for any map $f\colon X\to Y$ the induced map $f^*\colon \mathrm{H}^*(Y;\mathbb{Z})\to\mathrm{H}^*(X;\mathbb{Z})$ is a homomorphism of superalgebras.
(2) The polynomial ring $\mathbb{F}_2[a] = \bigoplus_{i\in\mathbb{N}_0}\mathbb{F}_2\cdot a^i$ is a superalgebra if we equip the ring with the grading that is determined by setting $\deg(a^i) := i$.
(3) The polynomial ring $\mathbb{Z}[x] = \bigoplus_{i\in\mathbb{N}_0}\mathbb{Z}\cdot x^i$ is a superalgebra if we equip the ring with the grading that is determined by setting $\deg(x^i) := 2i$.
(4) More generally, the polynomial ring $\mathbb{Z}[x_1,\dots,x_k]$ in k variables $x_1,\dots,x_k$ can be viewed as a superalgebra by setting $\deg(x_1^{r_1}\cdot\dots\cdot x_k^{r_k}) = 2(r_1+\dots+r_k)$.
(5) The ring $\mathbb{Z}[x]/(x^2) = \mathbb{Z}\oplus\mathbb{Z}\cdot x$ is a superalgebra by viewing elements in $\mathbb{Z}$ as elements of degree 0 and by viewing elements in $\mathbb{Z}\cdot x$ as elements of degree 1. Note that the

map
$$\begin{array}{rcl}\mathbb{Z}[x]/(x^2) & \to & \mathrm{H}^*(S^1;\mathbb{Z})\\ a\cdot 1+b\cdot x & \mapsto & a\cdot 1_{S^1}+b\cdot [S^1]^*\end{array}$$
is an isomorphism of superalgebras.

(6) We consider the ring
$$\Lambda[x_1,\dots,x_k] \;:=\; \mathbb{Z}[x_1,\dots,x_k]/\text{ideal generated by } x_1^2,\dots,x_k^2,\{x_ix_j+x_jx_i\}_{i,j=1,\dots,k}.$$
In this ring we use the symbol "$\wedge$" for multiplication instead of "$\cdot$". For example for any $i,j\in\{1,\dots,k\}$ we have $x_i\wedge x_i=0$ and $x_i\wedge x_j=-x_j\wedge x_i$. This ring can be viewed in a unique way as a superalgebra such that $\deg(x_j)=1$ for $j=1,\dots,k$. The ring $\Lambda[x_1,\dots,x_k]$ is often referred to as the *exterior algebra on generators* $x_1,\dots,x_k$, see e.g. [**Hat02**, p. 217] and [**Mac75**, p. 183].[1217] We leave it as an amusing exercise to verify that $\Lambda[x_1,\dots,x_k]$ is a free abelian group of rank 2^k where a basis is given by
$$\{x_{n_1}\wedge\dots\wedge x_{n_l}\,|\,n_1<\dots<n_l\}.$$

In Exercise 75.2 we saw that any sequence $\{A_n\}_{n\geq 1}$ of finitely generated abelian groups, with A_1 torsion-free, can appear as the sequence of cohomology groups $\mathrm{H}^*(X;\mathbb{Z})$ of some CW-complex. It is natural to ask whether any superalgebra structure can be realized as the cohomology ring of a topological space. More precisely, we have the following question, to which we will return later with Proposition 110.6.

Question 84.19. *Let $R=\bigoplus\limits_{n\in\mathbb{N}_0}R_n$ be superalgebra such that $_0\cong\mathbb{Z}$, R_1 is torsion-free and such that all R_n are finitely generated. Does there exist a topological space X such that $(\mathrm{H}^*(X;\mathbb{Z}),\cup)$ is isomorphic to R?*

Since there is not much to say right now about this question we continue with the next definition.

Definition. The tensor product of two superalgebras $A=\bigoplus\limits_{n\in\mathbb{N}_0}A_n$ and $B=\bigoplus\limits_{n\in\mathbb{N}_0}B_n$ is defined as the graded abelian group
$$A\otimes B \;:=\; \bigoplus_{n\in\mathbb{N}_0}\underbrace{\bigoplus_{p+q=n}A_p\otimes B_q}_{=:(A\otimes B)_n}$$
where for $a_k\in A_k, a_l\in A_l, b_m\in B_m$ and $b_n\in B_n$ the multiplication is given by
$$(a_k\otimes b_l)\cdot(a_m\otimes b_n) \;=\; (-1)^{lm}\cdot a_k\cdot a_m\otimes b_l\cdot b_n.$$

[1217] If we replace the $\mathbb{Z}$-coefficients by $\mathbb{R}$-coefficients we obtain a ring that we denote by $\Lambda_{\mathbb{R}}[x_1,\dots,x_k]$. This ring is isomorphic to the ring $\wedge^*(\mathbb{R}^k)^*=\bigoplus\limits_{j=0}^{k}\wedge^j(\mathbb{R}^k)^*$ of alternating forms on $\mathbb{R}^k$, where the ring structure on $\wedge^*(\mathbb{R}^k)^*$ is given by the wedge product of alternating forms. More precisely a ring isomorphism is given by
$$\begin{array}{rcl}\Lambda_{\mathbb{R}}[x_1,\dots,x_k] & \to & \wedge^*(\mathbb{R}^k)^*\\ x_{i_1}\wedge\dots\wedge x_{i_r} & \to & dy_{i_1}\wedge\dots\wedge dy_{i_r}\end{array}$$
where $dy_i\colon\mathbb{R}^k\to\mathbb{R}$ is the projection onto the i-th coordinate.

Lemma 84.20.
(1) *The tensor product of two superalgebras is again a superalgebra.*
(2) *The tensor product of superalgebras is associative, i.e. for any three superalgebras A, B and C the map*
$$(A \otimes B) \otimes C \;\to\; A \otimes (B \otimes C)$$
that is induced by
$$(a \otimes b) \otimes c \;\mapsto\; a \otimes (b \otimes c)$$
is an isomorphism of superalgebras.

It follows immediately from Lemma 84.20 (2) that we can ignore parentheses when we consider the tensor product of several superalgebras.

PROOF (∗). Let A, B and C be superalgebras and let $a_k \in A_k, a_l \in A_l, b_m \in B_m$ and $b_n \in B_n$.

(1) We have the following equalities:
$$\begin{aligned}(a_k \otimes b_l)\cdot(a_m \otimes b_n) = (-1)^{lm}\cdot a_k\cdot a_m \otimes b_l\cdot b_n &= (-1)^{lm+km+ln}\cdot a_m\cdot a_k \otimes b_n\cdot b_l \\ &= (-1)^{lm+km+ln+kn}\cdot(a_m \otimes b_n)\cdot(a_k \otimes b_l) \\ &= (-1)^{(m+n)(k+l)}\cdot(a_m \otimes b_n)\cdot(a_k \otimes b_l) \\ &= (-1)^{\deg(a_m\otimes b_n)\cdot\deg(a_k\otimes b_l)}\cdot(a_m \otimes b_n)\cdot(a_k \otimes b_l).\end{aligned}$$
It follows immediately that $A \otimes B$ is again a superalgebra.

(2) It suffices to show that the indicated map is a homomorphism of superalgebras. Thus let $c_r \in \mathrm{C}_r$ and $c_s \in \mathrm{C}_s$. Then
$$\begin{aligned}((a_k \otimes b_l) \otimes c_r)\cdot((a_m \otimes b_n) \otimes c_s) &= (-1)^{r(m+n)}\cdot((a_k \otimes b_l)\cdot(a_m \otimes b_n)) \otimes (c_r\cdot c_s) \\ &= (-1)^{r(m+n)+lm}\cdot(a_k\cdot a_m \otimes b_l\cdot b_n \otimes c_r\cdot c_s) \\ &= (-1)^{m(l+r)}\cdot(a_k\cdot a_m \otimes (b_l \otimes c_r)\cdot(b_n \otimes c_s)) \\ &= (a_k \otimes (b_l \otimes c_r))\cdot(a_m \otimes (b_n \otimes c_s)).\end{aligned}$$
■

Examples.

(1) As above we view the polynomial rings $\mathbb{F}_2[a]$ and $\mathbb{F}_2[b]$ as superalgebras by setting $\deg(a^i) = \deg(b^i) = i$. We consider the map
$$\begin{aligned}\mathbb{F}_2[a] \otimes \mathbb{F}_2[b] &\to \mathbb{F}_2[a, b] \\ a^r \otimes b^s &\mapsto a^r \cdot b^s.\end{aligned}$$
Using Lemma 57.3 (1) it is straightforward to show that this map is an isomorphism of abelian groups. One can easily verify that the map is a homomorphism of superalgebras. Summarizing we have convinced ourselves that this map is in fact an isomorphism of superalgebras.

(2) As before we view the polynomial rings $\mathbb{Z}[x_j]$, $j = 1, \ldots, k$ as superalgebras with $\deg(x_j^i) = 2i$. We consider the map
$$\begin{aligned}\mathbb{Z}[x_1] \otimes \cdots \otimes \mathbb{Z}[x_k] &\to \mathbb{Z}[x_1, \ldots, x_k] \\ x_1^{r_1} \otimes \cdots \otimes x_k^{r_k} &\mapsto x_1^{r_1} \cdot \cdots \cdot x_k^{r_k}.\end{aligned}$$
The same argument as in (1) shows that this map is an isomorphism of superalgebras.

(3) As above we view the rings $\mathbb{Z}[x_j]/(x_j^2)$ as superalgebras with $\deg(x_j)=1$. We leave it as an amusing exercise to verify that the map

$$\begin{array}{rcl}\mathbb{Z}[x_1]/(x_1^2)\otimes\cdots\otimes\mathbb{Z}[x_k]/(x_k^2) & \to & \Lambda[x_1,\dots,x_k]\\ x_1^{r_1}\otimes\cdots\otimes x_k^{r_k} & \mapsto & x_1^{r_1}\wedge\cdots\wedge x_k^{r_k}\end{array}$$

is an isomorphism of superalgebras.

The following proposition explains our sudden interest in tensor products of superalgebras.

Proposition 84.21. *Let X and Y be two topological spaces. We denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the obvious projection maps.*

(1) *The map*

$$\begin{array}{rrcl}\Phi\colon & \mathrm{H}^*(X;\mathbb{Z})\otimes\mathrm{H}^*(Y;\mathbb{Z}) & \to & \mathrm{H}^*(X\times Y;\mathbb{Z})\\ & \varphi\otimes\psi & \mapsto & p^*\varphi\cup q^*\psi\end{array}$$

is a homomorphism of superalgebras.

(2) *If all the homology groups of X are finitely generated and torsion-free, then the homomorphism from (1) is in fact an isomorphism.*

Proof. Let X and Y be two topological spaces.

(1) Let $\varphi_k\in\mathrm{H}^k(X;\mathbb{Z})$, $\psi_l\in\mathrm{H}^l(Y;\mathbb{Z})$, $\varphi_m\in\mathrm{H}^m(X;\mathbb{Z})$ and $\psi_n\in\mathrm{H}^n(Y;\mathbb{Z})$. Then

$$\begin{array}{rl}\overbrace{\Phi\big((-1)^{lm}\cdot(\varphi_k\cup\varphi_m)\otimes(\psi_l\cup\psi_n)\big)}^{=\Phi((\varphi_k\otimes\psi_l)\cdot(\varphi_m\otimes\psi_n))} & = (-1)^{lm}\cdot p^*(\varphi_k\cup\varphi_m)\cup q^*(\psi_l\cup\psi_n)\\ \underset{\text{Proposition 81.8}}{\overset{\uparrow}{=}} (-1)^{lm}\cdot(p^*(\varphi_k)\cup p^*(\varphi_m)\cup q^*(\psi_l)\cup q^*(\psi_n)) & \underset{\text{Lemma 81.10}}{\overset{\uparrow}{=}} \underbrace{p^*(\varphi_k)\cup q^*(\psi_l)\cup p^*(\varphi_m)\cup q^*(\psi_n)}_{=\Phi(\varphi_k\otimes\psi_l)\cup\Phi(\varphi_m\otimes\psi_n))}.\end{array}$$

It follows immediately that Φ is a homomorphism of superalgebras.

(2) Now suppose that all the homology groups of X are finitely generated and torsion-free. This hypothesis allows us to apply the Künneth Theorem 84.16 for cohomology groups and we obtain the following short exact sequence:

$$0\to\bigoplus_{k+l=n}\mathrm{H}^k(X;\mathbb{Z})\otimes\mathrm{H}^l(Y;\mathbb{Z})\xrightarrow[\varphi\otimes\psi\mapsto p^*\varphi\cup q^*\psi]{}\mathrm{H}^n(X\times Y;\mathbb{Z})\to\bigoplus_{k+l=n+1}\mathrm{Tor}(\mathrm{H}^k(X;\mathbb{Z}),\mathrm{H}^l(Y;\mathbb{Z}))\to 0.$$

Since the homology groups of X are torsion-free we obtain from the Universal Coefficient Theorem 75.13 for Cohomology Groups that the cohomology groups $\mathrm{H}^k(X;\mathbb{Z})$ are also torsion-free. But by Lemma 57.17 this implies that all the Tor-terms on the right-hand side of the short exact sequence vanish. ■

The following proposition in particular provides an answer to Question 82.12 (1).

Proposition 84.22. *Let $n\in\mathbb{N}_0$. We denote by $T=(S^1)^n$ the n-dimensional torus. For $i=1,\dots,n$ we denote by $p_i\colon T\to S^1$ the projection onto the i-th factor. Then there exists* *a unique isomorphism*

$$\psi\colon \Lambda[x_1,\dots,x_n] \;\to\; \mathrm{H}^*(T;\mathbb{Z})$$

of superalgebras which has the property that for each $i\in\{1,\dots,n\}$ we have

$$\psi(x_i) \;=\; p_i^*([S^1]^*).$$

PROOF. We consider the following isomorphism of superalgebras

$$\Lambda[x_1,\dots,x_n] \underset{\substack{\uparrow\\ \text{isomorphism from page 2056}}}{\xleftarrow{\cong}} \mathbb{Z}[x_1]/(x_1^2)\otimes\ldots\otimes\mathbb{Z}[x_k]/(x_k^2) \underset{\substack{\uparrow\\ \text{isomorphism from page 2054}}}{\xrightarrow{\cong}} \mathrm{H}^*(S^1;\mathbb{Z})\otimes\ldots\otimes\mathrm{H}^*(S^1;\mathbb{Z}) \underset{\substack{\uparrow\\ \text{Proposition 84.21}}}{\xrightarrow{\cong}} \mathrm{H}^*((S^1)^n;\mathbb{Z}).$$

It is clear from the definitions of the various maps that the map from left to right sends each x_i to $p_i^*([S^1]^*)$. The composition of the isomorphisms thus gives us the desired isomorphism. The uniqueness follows immediately from the fact that the x_i generate $\Lambda[x_1,\dots,x_n]$ as a ring. ■

The following corollary gives in particular a negative answer to Question 82.10.

Corollary 84.23. *For the n-dimensional torus $T=(S^1)^n$ the Lusternik-Schnirelmann category $\mathrm{cat}(T)$ equals n.*

PROOF. Given any $n\in\mathbb{N}_0$ we have

$$n \overset{\substack{\text{see page 2014}\\ \downarrow}}{\geq} \text{Lusternik-Schnirelmann category } \mathrm{cat}((S^1)^n) \overset{\substack{\text{by Proposition 82.11}\\ \downarrow}}{\geq} \text{cup length } \mathrm{cl}((S^1)^n) \underset{\uparrow}{\geq} n.$$

by Proposition 84.22 there exists an isomorphism $\psi\colon\Lambda[x_1,\dots,x_n]\to\mathrm{H}^*((S^1)^n;\mathbb{Z})$ of graded rings,
furthermore by the discussion on page 2054 we know that $x_1\wedge\cdots\wedge x_n\neq 0$,
since ψ is an isomorphism we deduce that $\psi(x_1)\cup\cdots\cup\psi(x_n)\neq 0$, i.e. $\mathrm{cl}((S^1)^n)\geq n$ ■

We conclude this chapter with the following variation on Proposition 84.21.

Proposition 84.24. *Let X and Y be two topological spaces. We denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the obvious projection maps. If all the homology groups of X are finitely generated, then the map*

$$\begin{aligned}\Phi\colon\ \mathrm{H}^*(X;\mathbb{F}_2)\otimes\mathrm{H}^*(Y;\mathbb{F}_2) &\;\to\; \mathrm{H}^*(X\times Y;\mathbb{F}_2)\\ \varphi\otimes\psi &\;\mapsto\; p^*\varphi\cup q^*\psi\end{aligned}$$

is an isomorphism of superalgebras.

In the proof of Proposition 84.24 we will need the following theorem which is a variation on the Künneth Theorem 84.17 for cochain complexes.

Theorem 84.25. *Let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes. If $\mathcal{C}$ is a free chain complex and if all homology groups of $\mathcal{C}$ are finitely generated, then for every $n\in\mathbb{N}_0$ there exists a natural isomorphism*

$$\bigoplus_{k+l=n}\mathrm{H}^k(\mathcal{C};\mathbb{F}_2)\otimes\mathrm{H}^l(\mathcal{C}';\mathbb{F}_2) \xrightarrow{\ \Xi\ } \mathrm{H}^n(\mathcal{C}\otimes\mathcal{C}';\mathbb{F}_2).$$

The proof of Theorem 84.25 is evidently very similar to the proof of the Künneth Theorem 84.17 for cochain complexes. We refer to [**Mun84**, Theorem 60.6] for a proof.

Proof of Proposition 84.24. The proof is almost the same as the proof of Proposition 84.21. More precisely, we first note that if in the proof of the Künneth Theorem 84.16 for cohomology groups we replace the Künneth Theorem 84.17 for cochain complexes by Theorem 84.25 we obtain that the map

$$\begin{array}{rcl} \bigoplus\limits_{k+l=n} \mathrm{H}^k(X;\mathbb{F}_2)\otimes \mathrm{H}^l(Y;\mathbb{F}_2) & \to & \mathrm{H}^n(X\times Y;\mathbb{F}_2) \\ \varphi\otimes\psi & \mapsto & p^*\varphi\cup q^*\psi \end{array}$$

is an isomorphism. But the same argument as in the proof of Proposition 84.21 shows that this map is in fact a homomorphism of superalgebras. ■

Exercises for Chapter 84.

Exercise 84.1. Let X be a topological space. Recall that Proposition 81.8 says that for any $a\in \mathrm{H}^k(X;\mathbb{Z})$ and $b\in\mathrm{H}^l(X;\mathbb{Z})$ we have

$$a\cup b \;=\; (-1)^{kl}\cdot b\cup a.$$

Provide a proof of this statement using the original definition of the cup product from page 1980 and using Lemma 84.7.

Exercise 84.2. Let $g\in\mathbb{N}$ and denote by Σ_g the surface of genus g. Show that the Lusternik-Schnirelmann category of $S^1\times\Sigma_g$ is greater or equal than three.

Exercise 84.3. Let Σ be the surface of genus 2, let $q\colon \Sigma\times S^1\to\Sigma$ be the projection map and let $p\colon\Sigma\to S^1$ be the map shown in Figure 1184. Determine the cap product

$$q(p^*([S^1]^*))\cap[\Sigma\times S^1] \;\in\; \mathrm{H}_2(\Sigma\times S^1;\mathbb{Z})$$

up to sign.

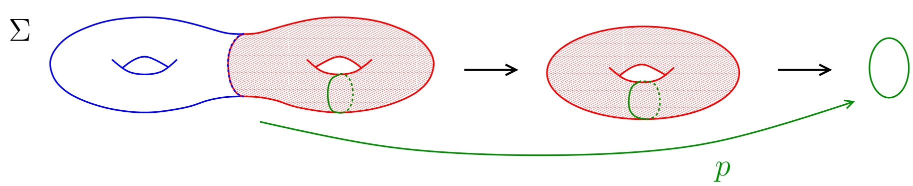


Figure 1184

Exercise 84.4. Let $n\in\mathbb{N}_0$. We write $T:=(S^1)^n$. Let $\varphi\colon T\to T$ be map. We pick a basis for $\mathrm{H}_1(T)\cong\mathbb{Z}^n$ and we denote by A the $(n\times n)$-matrix that represents the map $\varphi_*\colon\ \mathrm{H}_1(T)\to\mathrm{H}_1(T)$ with respect to the given basis. Show that

$$\underbrace{\sum_{k=0}^{n}(-1)^k\cdot\operatorname{tr}(\varphi_*\colon\ \mathrm{H}_k(T)\to\mathrm{H}_k(T))}_{\text{Lefschetz number }\Lambda(\varphi)} \;=\; \det(\mathrm{id}-A).$$

Hint. Use the cohomological reformulation of the Lefschetz number given in Exercise 75.12 and use Proposition 84.22.
Remark. It follows from this calculation together with the Lefschetz Fixed Point Theorem 65.9 that unless 1 is an eigenvalue of A the map φ has a fixed point.

Exercise 84.5. Let $n \in \mathbb{N}$ and let W be a compact orientable $(n+1)$-dimensional topological manifold such that ∂W is homeomorphic to the n-torus $(S^1)^n$. Show that the inclusion induced map $\mathrm{H}_1(\partial W;\mathbb{Z}) \to \mathrm{H}_1(W;\mathbb{Z})$ is not an injection.
Hint. Consider the cup product on $(S^1)^n$ and consider the interplay of the fundamental classes of ∂W and W.

Exercise 84.6. Given $n \in \mathbb{N}$ we denote by $p\colon S^n \times S^n \to S^n$ and $q\colon S^n \times S^n \to S^n$ the two obvious projections. Let $\varphi \in \mathrm{H}^n(S^n;\mathbb{Z})$ be a generator. We know, for example by Proposition 84.21, that $p^*(\varphi)$ and $q^*(\varphi)$ form a basis for $\mathrm{H}^n(S^n \times S^n;\mathbb{Z})$. Given a map $f\colon S^n \times S^n \to S^n \times S^n$ we denote by $A(f)$ the matrix representing the automorphism f^* of $\mathrm{H}^n(S^n \times S^n;\mathbb{Z})$ with respect to the basis $p^*(\varphi)$ and $q^*(\varphi)$.

(a) Let $B \in \mathrm{SL}(2,\mathbb{Z})$. Show that there exists a homeomorphism $f\colon S^1 \times S^1 \to S^1 \times S^1$ with $A(f) = B$.
Hint. Use Lemma 52.7.

(b) Now let $f\colon S^n \times S^n \to S^n \times S^n$ be a homeomorphism. Show that if n is even, then $A(f)$ is given by one of the following eight matrices
$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}.$$

Part IX

Poincaré Duality

In this chapter we will state and prove Borsuk's Theorem 85.3 which will give us many goodies:

(1) We can prove the Lefschetz Fixed Point Theorem also for CW-complexes and topological manifolds.[1218]
(2) We can get some control over the invariants of compact topological manifolds, for example we will be able to show that the homology groups of a compact topological manifold are finitely generated and that the fundamental groups are finitely presented.

85.1. **Neighborhood retracts.** We recall two definitions from pages 547 and 138 and we introduce two related definitions.

Definition. Let X be a topological space.

(1) We say X is *contractible* if X is homotopy equivalent to a point.
(2) We say X is *locally contractible* if given any $Q \in X$ and given any open neighborhood U of Q there exists an open neighborhood V of Q that is contained in U and that is contractible.
(3) We say that a subset U of X is *contractible in* X if the inclusion map $U \to X$ is homotopic to a constant map.[1219] We illustrate this definition in Figure 1185.
(4) We say X is *weakly locally contractible* if given any $Q \in X$ and given any open neighborhood U of Q there exists an open neighborhood V of Q that is contained in U and that is contractible in U. We illustrate this definition in Figure 1186.[1220]

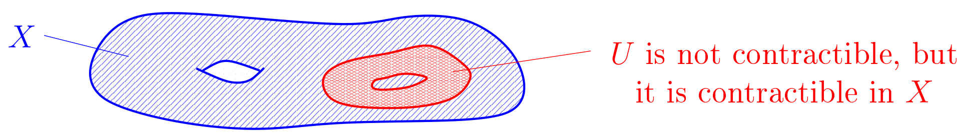


Figure 1185

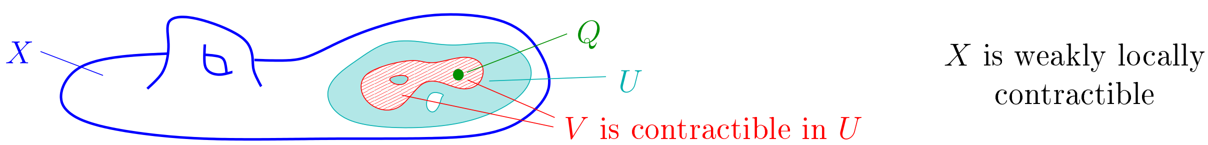


Figure 1186

Examples.

(1) It follows almost immediately from the definitions that a topological space that is locally contractible is also weakly locally contractible.

[1218]So far, in Chapter 65 we only proved the Lefschetz Fixed Point Theorem for simplicial complexes and smooth manifolds.

[1219]Note that Lemma 18.12 says that a topological space X is contractible if and only if the identity map is homotopic to a constant map.

[1220]What we call a "weakly locally contractible" topological space is referred to in many books, see e.g. [**Bor67, Saka13**], as a "locally contractible" topological space. Since on page 138 we defined once and for all what it means that a topological space is "locally P" we stick with our more cumbersome name.

(2) By Lemma 6.9 (1) every topological manifold is locally homeomorphic to some non-empty convex subset of $\mathbb{R}^n$. Since non-empty convex subsets are contractible we see that every topological manifold is locally contractible.
(3) In Proposition 36.10 (6) we showed that CW-complexes are locally contractible. In particular topological graphs are locally contractible. (Note that we had also shown in Exercise 29.1 that topological graphs are locally contractible.)
(4) It is easy to show that $\mathbb{R}^2 \setminus \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ is not weakly locally contractible.
(5) It is an amusing exercise to show that the topologist's sine curve

$$\big\{(0,y)\,\big|\,y\in[-1,1]\big\}\,\cup\,\big\{(x,\sin(\tfrac{1}{x}))\,\big|\,x\in(0,\pi]\big\}\;\subset\;\mathbb{R}^2,$$

which we had initially introduced on page 131, is not weakly locally contractible.
(6) It is an equally amusing (but not entirely trivial) exercise to come up with a topological space that is weakly locally contractible but not locally contractible.

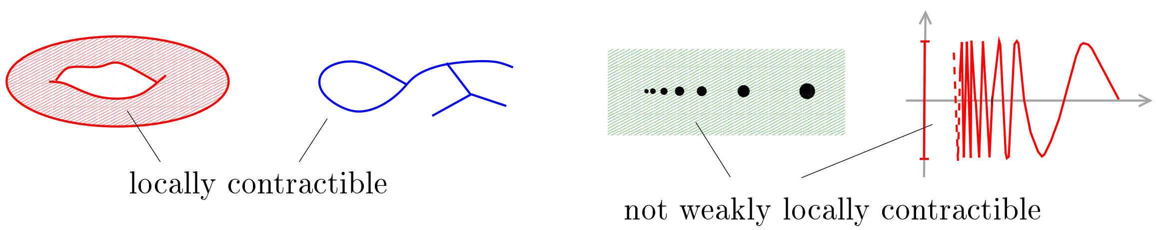


FIGURE 1187

Next we recall the following definition from page 484.

Definition. Let X be a topological space. We say a subset $A \subset X$ is a *retract of* X if there exists a map $r\colon X \to A$ with $r(a) = a$ for all $a \in A$.

In our discussions of retracts the following two basically trivial observations will at times be helpful.

Observation 85.1. *Let Z be a topological space and let K be a subset.*

(1) *Let $K \subset U \subset V \subset Z$ be subsets. If K is a retract of V, then K is also a retract of U.*
(2) *If K is a retract of a neighborhood, then it is also a retract of an open neighborhood.*

PROOF. The first statement follows from the observation that the restriction of any retraction $r\colon V \to K$ to U is again a retraction. The second statement follows immediately from the first statement. ■

Definition. Let Y be a topological space and let X be a subset. We say X is a *neighborhood retract in* Y if there exists an open neighborhood N of X in Y such that X is a retract of N.[1221]

[1221]By Observation 85.1 (2) this definition is equivalent to asking that there exists any neighborhood (not necessarily closed) that admits a retract to X. This formulation is often the definition of a neighborhood retract that is used in the literature. For our discussion it is convenient to already assume that the neighborhood is open.

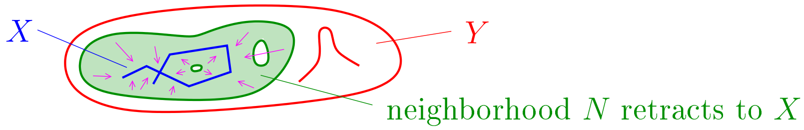


FIGURE 1188

Examples.

(1) Let $K \subset S^3$ be a knot. By the Tubular Neighborhood Theorem 8.24 we know that there exists a tubular neighborhood $\overline{B}^2 \times K$ of K in S^3. Since $K = \{0\} \times K$ is evidently a retract of $\overline{B}^2 \times K$ we see that K is a neighborhood retract in S^3.

(2) Later on we will generalize (1) as follows. It is a consequence of the Regular Neighborhood Theorem 10.3, which is a fairly immediate consequence of the General Tubular Neighborhood Theorem 10.5, that every compact proper submanifold X of a given smooth manifold M is a neighborhood retract in M.

(3) We will see in Exercise 85.2 that $X = \{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$, viewed as a subset of $Y = \mathbb{R}$, is not a neighborhood retract.

We continue with the following fairly elementary lemma.

Lemma 85.2. *Let $n \in \mathbb{N}_0$ and let X be a subset of $\mathbb{R}^n$. If X is a neighborhood retract in $\mathbb{R}^n$, then the topological space X is weakly locally contractible.*

PROOF $(*)$. Let W be a neighborhood of X that admits a retraction $r\colon W \to X$. Let $Q \in X$ and let U be a neighborhood of Q in X. Since W is a neighborhood of X there exists an $s > 0$ such that $B^n_s(Q) \subset r^{-1}(U)$. We denote by $F\colon B^n_s(Q) \times [0,1] \to B^n_s(Q)$ the obvious homotopy from the identity to the constant map. Finally we consider the map

$$\begin{array}{rcl} (B^n_s(Q) \cap X) \times [0,1] & \to & U \\ (x,t) & \mapsto & r\big(\underbrace{F(x,t)}_{\in B^n_s(Q) \subset r^{-1}(U)}\big). \end{array}$$

Evidently this map is a homotopy from the identity to a constant map. In particular $V := B^n_s(Q) \cap X$ is contractible in U. ∎

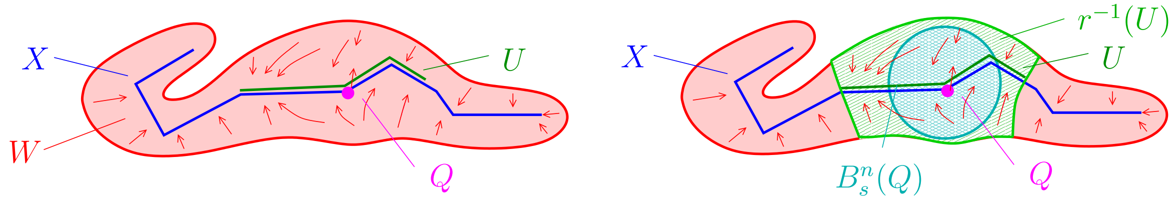


FIGURE 1189. Illustration of the proof of Lemma 85.2.

It is rather stunning that for closed subsets of $\mathbb{R}^n$ the converse to Lemma 85.2 holds. More precisely, the following theorem was proved by Karol Borsuk [**Bor32**] in 1932.

Theorem 85.3. (Borsuk's Theorem) *Let $n \in \mathbb{N}_0$ and let X be a closed subset of $\mathbb{R}^n$. If the topological space X is weakly locally contractible, then X is a neighborhood retract in $\mathbb{R}^n$.*[a]

[a] Is is perhaps worth stressing that the proposition does *not* claim that X is a *deformation retract* of a neighborhood of X.

Remark. Proofs of Borsuk's Theorem 85.3 are also given in [**Bre93**, Theorem E.3] and [**Hat02**, Theorem A.7].

Example. We define a *wild knot* to be the image of an embedding $\gamma\colon S^1 \to \mathbb{R}^3$. If the embedding is smooth, then we obtain the usual notion of a knot in $\mathbb{R}^3$. But once we drop smoothness one can get weird examples. For example using Proposition 2.43 (3) it is not difficult to show that the example shown in Figure 1190 is indeed a wild knot.[1222]

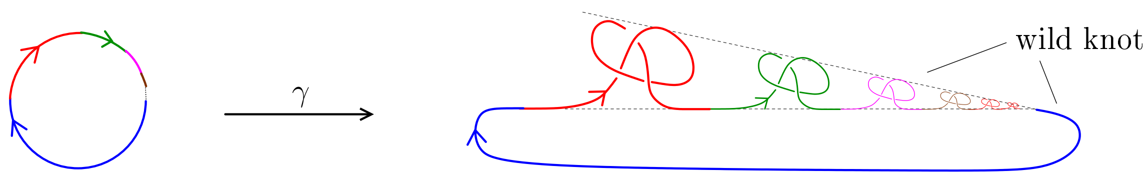


FIGURE 1190

Evidently S^1 is locally contractible and by Lemma 2.40 together with Lemma 2.17 we know that $\gamma(S^1)$ is a closed subset of $\mathbb{R}^3$. Thus it follows from Borsuk's Theorem 85.3 that every wild knot is a neighborhood retract in $\mathbb{R}^3$. It is a fantastic exercise to try to visualize a neighborhood of the above wild knot together with a retraction to $\gamma(S^1)$.

85.2. **Proof of the Borsuk's Theorem 85.3.** Before we provide the proof of Borsuk's Theorem 85.3 it is convenient to introduce one definition and to prove one lemma.

Definition. Let $k \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. A $\frac{1}{2^k}$-*cube* is a subset C of $\mathbb{R}^n$ of the form

$$C \;=\; \left[\tfrac{a_1}{2^k}, \tfrac{a_1+1}{2^k}\right] \times \cdots \times \left[\tfrac{a_n}{2^k}, \tfrac{a_n+1}{2^k}\right] \quad \text{with } a_1, \dots, a_n \in \mathbb{Z}.$$

Put differently, a $\frac{1}{2^k}$-cube is a cube in $\mathbb{R}^n$ with side length $\frac{1}{2^k}$ and where the coordinates of all vertices lie in $\frac{1}{2^k} \cdot \mathbb{Z}$.

Lemma 85.4. *Let $x \in \mathbb{R}^n$ and let $r > 0$. If $k \in \mathbb{N}_0$ is chosen such that $\frac{1}{2^k} < \frac{1}{3\sqrt{n}} \cdot r$, then there exist finitely many $\frac{1}{2^k}$-cubes $W_1, \dots, W_m$ such that*

$$x \;\in\; \text{interior of } (W_1 \cup \cdots \cup W_m) \;\subset\; W_1 \cup \cdots \cup W_m \;\subset\; B^n_r(x).$$

PROOF ($*$). Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and let $r > 0$. Furthermore let $k \in \mathbb{N}_0$ such that $\frac{1}{2^k} < \frac{1}{3\sqrt{n}} \cdot r$. We write $\epsilon = \frac{1}{2^k}$. For $i = 1, \dots, n$ we denote by y_i the result of rounding x_i down to the closest number in $\mathbb{Z} \cdot \epsilon$. Clearly we have

$$(x_1, \dots, x_n) \in [y_1, y_1+\epsilon] \times \cdots \times [y_n, y_n+\epsilon] \subset \text{interior of } \underbrace{[y_1-\epsilon, y_1+2\epsilon] \times \ldots \times [y_n-\epsilon, y_n+2\epsilon]}_{=:V}.$$

[1222] It is also not difficult, but rather pointless, to write down such an embedding explicitly.

Since V is the union of finitely many ϵ-cubes it suffices to show that V is contained in $B^n_r(x)$. Thus let $y \in V$. We have

$$\underset{\text{since } x\in V}{d(y,x) \underset{\uparrow}{\leq}} \text{ diameter of } V \underset{\substack{\uparrow\\ \text{see page 141}}}{=} \sqrt{n}\cdot 3\epsilon = \sqrt{n}\cdot\frac{3}{2^k} \underset{\substack{\uparrow\\ \text{choice of } k}}{<} r$$

which means by definition that $y \in B^n_r(x)$. Thus we have shown that $V \subset B^n_r(x)$. ■

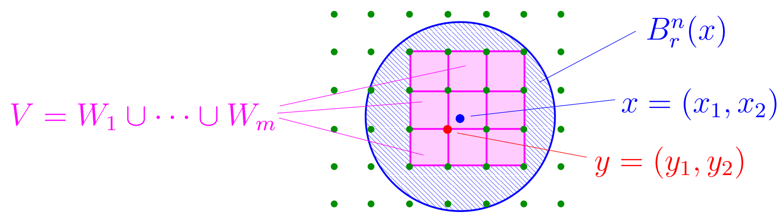


FIGURE 1191. Illustration of the proof of Lemma 85.4.

Now we turn to the actual proof of Borsuk's Theorem 85.3

PROOF OF BORSUK'S THEOREM 85.3. Let X be a closed, weakly locally contractible subset of $\mathbb{R}^n$. By Observation 85.1 (2) it suffices to show that there exists a neighborhood W of X in $\mathbb{R}^n$ that admits a retraction $W \to X$. Evidently we only have to consider the case that X is non-empty. We introduce the following notation.

(1) We define C_1 to be the union of all 1-cubes that do not intersect X.
(2) We define C_2 to be the union of all $\frac{1}{2}$-cubes that do not intersect $\mathring{C}_1$ and that do not intersect X.
(3) We iterate this process, i.e. for any $k \geq 3$ we define C_n to be the union of all $\frac{1}{2^k}$-cubes that do not intersect the interior of $C_1 \cup \cdots \cup C_{k-1}$ and that do not intersect X.

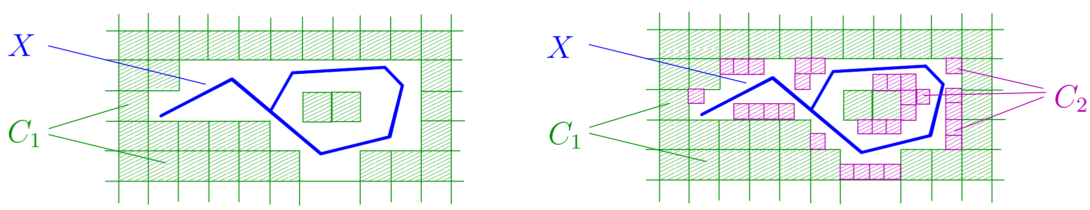


FIGURE 1192

Claim.

(1) We have $\bigcup\limits_{k\in\mathbb{N}} C_k = \mathbb{R}^n \setminus X$.
(2) Given any $x_0 \in \mathbb{R} \setminus X$ there exists an open neighborhood U that intersects only finitely many cubes of $C_1, C_2, \ldots$.

To prove the claim, let $x_0 \in \mathbb{R}^n \setminus X$. Since X is closed we know that $\mathbb{R}^n \setminus X$ is open. This means that there exists an $r > 0$ such that $B^n_r(x_0) \subset \mathbb{R}^n \setminus X$. We pick $k \in \mathbb{N}_0$ such that $\frac{\sqrt{n}}{3\cdot 2^k} < r$. By Lemma 85.4 there exist finitely many $\frac{1}{2^k}$-cubes $W_1, \dots, W_m$ such that

$$x_0 \;\in\; \underbrace{\text{interior of } W_1 \cup \dots \cup W_m}_{:=U} \;\subset\; B^n_r(x).$$

This implies in particular that x_0 is contained in C_n or it is contained already in one of $C_1, \dots, C_{k-1}$. Furthermore it is fairly easy to see that if U intersects a $\frac{1}{2^m}$-cube with $m \geq k$ it already intersects a $\frac{1}{2^k}$-cube. It follows from this observation and the iterative definition of the C_i that U intersects only finitely many cubes of $C_1, C_2, \dots$. ⊞

We view each C_i as a CW-complex in the obvious way: the n-cells correspond precisely to the cubes and the lower dimensional cells correspond precisely to the faces of the cubes. For each k the union $C_1 \cup \dots \cup C_n$ is also a CW-complex in an obvious way and finally we can also view the union $C := \bigcup\limits_{k\in\mathbb{N}} C_k = \mathbb{R}^n \setminus X$ as a CW-complex. We leave it to the reader to verify, using statement (2) of the claim and using Lemma 2.6 (2), that the topology on this CW-complex, given by the weak topology defined on page 941, agrees with the usual subspace topology on $\mathbb{R}^n \setminus X$.

We continue with the following definition: given a cell σ of C and a map $f\colon \sigma \to X$ we define[1223][1224]

$$\rho(f) \;:=\; \max\{d(z, f(z)) \,|\, z \in \sigma\}.$$

We start with the iterative definition of a subset of $\mathbb{R}^n$ together with a map to X.

(a) We denote by A_0 the 0-skeleton of C. Next we define a map $r_0\colon A_0 \to X$. Let $a \in A_0$. Since X is a closed subset of $\mathbb{R}^n$ it follows from Corollary 3.19 (1) that we can find a point $r_0(a) \in X$ with $d(a, r_0(a)) = d(a, X)$.

(b) Now suppose inductively that we have defined a map $r_i\colon A_i \to X$ on a union A_i of some of the i-cells of C. We define the next set A_{i+1} and the corresponding map $r_{i+1}\colon A_{i+1} \to X$ as follows:

(b$_1$) Let σ be an $(i+1)$-cell of C for which r_i is defined on $\partial\sigma$. Suppose that r_i can be extended to a map $g\colon \sigma \to X$ that agrees with r_i on $\partial\sigma$. Then we pick a map $r_{i+1}\colon \sigma \to X$ with $r_{i+1}|_{\partial\sigma} = r_i$ such that

$$\rho(r_{i+1}) \;\leq\; 2 \cdot \underbrace{\inf\{\rho(g) \,|\, g\colon \sigma \to X \text{ is a map with } g|_{\partial\sigma} = r_i|_{\partial\sigma}\}}_{\substack{\geq d(\sigma, X) \text{ which in turn is } > 0 \text{ by Corollary 3.19 (2),} \\ \text{here we use again that } X \subset \mathbb{R}^n \text{ is closed}}}.$$

(b$_2$) We denote by A_{i+1} the union of all $(i+1)$-cells as in (b$_1$) and we define r_{i+1} on A_{i+1} as in (b$_1$).

(c) We set $A := \bigcup\limits_{i\in\mathbb{N}} A_i \cup X$ and we define $r\colon A \to X$ to be the unique map that equals r_i on each A_i and that is the identity on X.

We refer to Figure 1193 for an illustration. It remains to show that the set A is a neighborhood of X and that the map $r\colon A \to X$ is continuous. Both statements follow from the following claim.

[1223] Given $a, b \in \mathbb{R}^n$ we denote by $d(a, b) = \|a - b\|$ the usual Euclidean distance.

[1224] Note that σ is compact, hence the maximum exists by Lemma 2.40 (2).

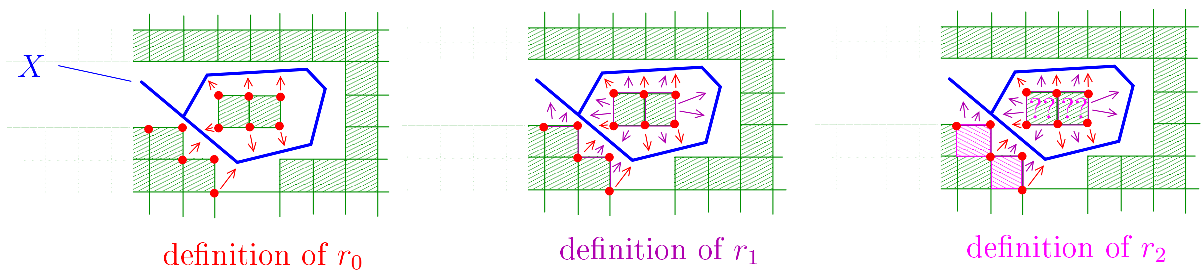


FIGURE 1193

Claim. Given any $x \in X$ and any $\epsilon_0 > 0$ there exists an open neighborhood U of x that is contained in A and such that $\|\underbrace{r(x)}_{=x} - r(y)\| < \epsilon_0$ for all $y \in U$, i.e. such that $r(U) \subset B^n_{\epsilon_0}(x)$.

So let $x \in X$ and let $\epsilon_0 > 0$. We adopt the following notation: given $\mu > 0$ we write $W(\mu) = X \cap B^n_\mu(x)$.

Subclaim. We can find $\epsilon_1, \dots, \epsilon_{2n}$ with the following two properties:

(1) for each $i \in \{0, \dots, 2n-1\}$ we have $\epsilon_{i+1} < \frac{1}{3} \cdot \epsilon_i$,
(2) for every $i = 1, \dots, 2n$ the subset $W(\epsilon_i)$ is contractible in $W(\epsilon_{i-1})$.

So suppose that we have already found $\epsilon_0, \dots, \epsilon_i$ for some $i \in \{0, \dots, 2n-1\}$ with the desired properties. Since X is weakly locally contractible we can find an open neighborhood U of x in X that is contained in $W(\epsilon_i)$ and that is contractible in $W(\epsilon_i)$. Since U is open there exists an $\epsilon_{i+1} < \frac{1}{3}\epsilon_i$ such that $W(\epsilon_{i+1}) \subset U$. Since U is contractible in $W(\epsilon_i)$ the subset $W(\epsilon_{i+1})$ is also contractible in $W(\epsilon_i)$. We refer to Figure 1194 for an illustration. This concludes the proof of the subclaim.

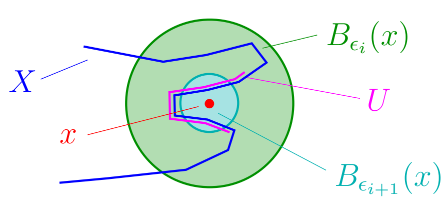


FIGURE 1194

We set $U := B^n_{\frac{1}{4}\epsilon_{2n}}(x)$. It suffices to prove the following two subclaims.

Subclaim 1. For each $i = 0, \dots, n$ the map r is defined for each i-cell σ of the CW-complex C that lies in U and for each such σ we have $r(\sigma) \subset W(\epsilon_{2n-2i})$.

Subclaim 2. The union of X with all cells of C that lie in U is a neighborhood of x.

First we prove Subclaim 1. We prove Subclaim 1 iteratively for $i = 0, 1, \dots, n$. We start out with $i = 0$. Thus we have to show that $r(U \cap A_0) \subset W(\epsilon_{2n})$. Therefore let $y \in U \cap A_0$.

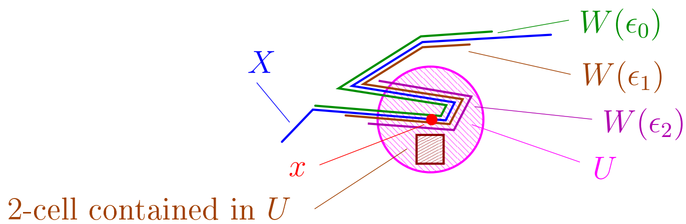


FIGURE 1195

Then we have

$$d(x,r(y)) \underset{\substack{\uparrow\\ \text{triangle inequality and } r(y)=r_0(y)}}{\leq} d(x,y)+d(y,r_0(y)) \underset{\substack{\uparrow\\ \text{by (a) since } x\in X}}{\leq} d(x,y)+d(y,x) = 2d(x,y) \underset{\substack{\uparrow\\ \text{since } y\in U=B^n_{\frac14\epsilon_{2n}}(x)}}{<} 2\cdot\frac14\cdot\epsilon_{2n} = \frac12\cdot\epsilon_{2n}.$$

But since $r(y)\in X$ this means precisely that $r(y)\in W(\epsilon_{2n}) = X\cap B^n_{\epsilon_{2n}}(x)$. This shows that Subclaim 1 holds for $i=0$.

Now suppose that Subclaim 1 holds for some $i\in\{0,\dots,n-1\}$. Let σ be an $(i+1)$-cell of the CW-complex C inside U. Since $r(\partial\sigma)\subset W(\epsilon_{2n-2i})$ and since $W(\epsilon_{2n-2i})$ is contractible in $W_{2n-2i-1}$ it follows from Proposition 2.52 and Proposition 40.7 (3) that we can extend $r|_{\partial\sigma}=r_i|_{\partial\sigma}$ to a map $s\colon\sigma\to W(\epsilon_{2n-2i-1})$. By definition of A_{i+1} this implies that $\sigma\subset A_{i+1}$. In particular r is defined on σ. It remains to show that $r(\sigma)\subset W(\epsilon_{2n-2i-2})$. Thus let $y\in\sigma$.

We have

$$\begin{aligned}
d(x,r(y)) &\underset{\substack{\uparrow\\ \text{by the triangle inequality}\\ \text{and since } r(y)=r_{i+1}(y)}}{\leq} d(x,y)+d(y,r_{i+1}(y)) \underset{\substack{\uparrow\\ \text{since } y\in\sigma\subset U=B^n_{\frac14\epsilon_{2n}}(x)\\ \text{and by definition of } \rho(r_{i+1})}}{\leq} \frac14\epsilon_{2n}+\rho(r_{i+1}) \underset{\substack{\uparrow\\ \text{by (b}_1\text{)}}}{\leq} \frac14\epsilon_{2n}+2\cdot\inf\left\{\rho(g)\,\middle|\, \begin{array}{c} g\colon\sigma\to X \text{ with}\\ g|_{\partial\sigma}=r_i|_{\partial\sigma}\end{array}\right\}.\\
&\underset{\substack{\uparrow\\ \text{since } s \text{ is such a map}}}{\leq} \frac14\epsilon_{2n}+2\cdot\underbrace{\max\{d(z,s(z))\,|\,z\in\sigma\}}_{=\rho(s)} \underset{\substack{\uparrow\\ \text{triangle inequality}}}{\leq} \frac14\epsilon_{2n}+2\cdot\max\{d(z,x)+d(x,s(z))\,|\,z\in\sigma\}\\
&\underset{\substack{\uparrow\\ \text{since } \sigma\subset U=B^n_{\frac14\epsilon_{2n}}(x) \text{ and since for each}\\ z\in\sigma \text{ we have } s(z)\in W(\epsilon_{2n-2i-1})\subset B^n_{\epsilon_{2n-2i-1}}(x)}}{\leq} \frac14\epsilon_{2n}+2\left(\frac14\epsilon_{2n}+\epsilon_{2n-2i-1}\right) = \frac34\epsilon_{2n}+2\epsilon_{2n-2i-1} \underset{\substack{\uparrow\\ \text{since for each } j \text{ we have } \epsilon_{j+1}\leq\frac13\epsilon_j}}{\leq} \frac{1}{12}\epsilon_{2n-2i-2}+\frac23\epsilon_{2n-2i-2}\leq\epsilon_{2n-2i-2}.
\end{aligned}$$

This concludes the proof that $r(\sigma)\subset W(\epsilon_{2n-2i-2})$. In particular this concludes the proof of Subclaim 1.

Finally it remains to prove Subclaim 2. We set $s=\frac14\epsilon_{2n}$. As a reminder, we have to show that the union of X with all cells of C that lie in $B^n_s(x)$ is a neighborhood of x. Let W be a $\frac{1}{2^k}$-cube with $W\cap B^n_{\frac{s}{2}}(x)\neq\varnothing$ and $W\cap(\mathbb{R}^n\setminus B^n_s(x))\neq\varnothing$. It follows that the diameter of W is at least $\frac{s}{2}$ which in turn implies that $\frac{\sqrt{n}}{2^k}\geq\frac{s}{2}$. We set $k_0=\lfloor\ln_2(\frac{2\sqrt{n}}{s})\rfloor$.

We see that

$$\begin{aligned}
&\text{union of } X \text{ with all cells of } C \text{ that lie in } B^n_s(x)\\
&\supset\ \text{union of } X \text{ with all cubes of } C \text{ that lie in } B^n_s(x)\\
&\supset\ B^n_s(x) \setminus \text{union of all cubes of } C \text{ that are not completely contained in } B^n_s(x)\\
&\overset{\text{since } \mathbb{R}^n \setminus X = C}{\uparrow}\\
&=\ B^n_s(x) \setminus \text{union of all cubes of } C \text{ that are not completely contained in } B^n_s(x)\\
&\qquad\qquad\quad \text{which do not contain the point } x \text{ but which intersect } B^n_{\frac{s}{s}}(x)\\
&\supset\ B^n_s(x) \setminus \underbrace{\text{union of all } \tfrac{1}{2^k}\text{-cubes with } k \le k_0 \text{ of } C \text{ that are not contained in } B^n_s(x) \text{ which do not contain the original point } x \text{ but which do intersect } B^n_{\frac{s}{s}}(x).}_{\text{finite union of cubes, hence a closed subset}}\\
&\underset{\text{definition of } k_0}{\uparrow}
\end{aligned}$$

The last set is clearly an (open) neighborhood of x. But then the first set is also a neighborhood of x. This concludes the proof of Subclaim 2. ■

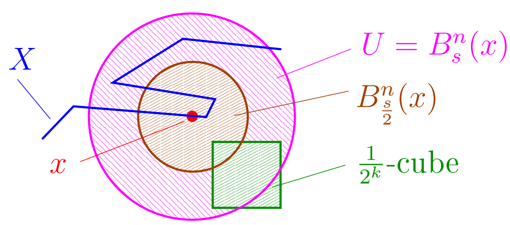


FIGURE 1196. Illustration for the proof of Theorem 85.3.

85.3. **Euclidean Neighborhood Retracts.** Borsuk's Theorem 85.3 has many applications. For example, much later, it will be a key ingredient in the proof of the Generalized Alexander Duality Theorem 92.10.

In this section we will discuss several fairly direct consequences of Borsuk's Theorem 85.3. For the following discussion it is convenient to introduce the following definition.

Definition. A topological space X is called a *euclidean neighborhood retract* (or short *ENR*) if there exists an embedding $f\colon X \to \mathbb{R}^n$ and an open neighborhood U of $f(X)$ such that $f(X)$ is a retract of U.

Remark. Let $i\colon X \to \mathbb{R}^n$ be an embedding of a topological space. The property of X being a retract of an open neighborhood, rather surprisingly, does not depend on the choice of the embedding. More precisely, [**tD08**, Proposition 18.4.1] says the following. If X is an ENR, then given any embedding $f\colon X \to Y$ into a metric space Y there exists an open neighborhood U of $f(X)$ such that X is a retract of U. We will not make use of this pretty statement.

In the following we will mostly work with the following slight reformulation of Borsuk's Theorem.

Theorem 85.5. (Borsuk's Theorem) *Let X be a topological space which is weakly locally contractible and which admits an embedding $f\colon X \to \mathbb{R}^n$.*

(1) *If $f(X)$ is closed, then X is an* ENR.

(2) *If X is regionally compact (e.g. if X is compact), then X is an* ENR.

Proof.

(1) This statement follows immediately from Borsuk's Theorem 85.5.
(2) Now assume that X is regionally compact. If X is compact, then we know of course by Lemma 2.17 (2) that $f(X)$ is compact and we can apply (1). Slightly more interestingly, if X is regionally compact, then we know by Exercise 85.6 that there exists an embedding $g\colon X \to \mathbb{R}^{n+1}$ such that $g(X)$ is a *closed* subset of $\mathbb{R}^{n+1}$. Thus we obtain again from (1) that X is an ENR. ■

Definition. Let X be a topological space and let $\mathcal{C}$ be a class of topological spaces (e.g. $\mathcal{C}$ could be the class of CW-complexes). We say *X is a retract of an element in $\mathcal{C}$* if there exists a topological space $Y \in \mathcal{C}$, a map $i\colon X \to Y$ and a map $r\colon Y \to X$ with $r \circ i = \mathrm{id}_X$.[1225]

In this chapter our key interest in ENR and thus in Borsuk's Theorem 85.3 stems from the following fairly simple proposition.

Proposition 85.6.

(1) *If $K \subset \mathbb{R}^n$ is a compact neighborhood retract, then K is the retract of a finite n-dimensional simplicial complex.*
(2) *Every compact* ENR *is a retract of a finite simplicial complex.*

Proof.

(1) Let K be a compact subset of $\mathbb{R}^n$ which is a retract of some open neighborhood V. The idea is to define a simplicial complex X to be the union of finitely many sufficiently small cubes. To do so we set $r := d(K, \mathbb{R}^n \setminus V)$. Since K is compact and since $\mathbb{R}^n \setminus V$ is closed it follows from Corollary 3.19 that $r > 0$. We pick $k \in \mathbb{N}_0$ such that $\frac{1}{2^k} < \frac{1}{\sqrt{n}} \cdot r$. We denote by X the union of all $\frac{1}{2^k}$-cubes that intersect K. (See page 2065 for the definition of a $\frac{1}{2^k}$-cube.) Since K is compact we see that X is the union of finitely[1226] many $\frac{1}{2^k}$ cubes. The same argument as in the proof of Lemma 85.4 shows, by our choice of k, that each $\frac{1}{2^k}$-cube is contained in V.

   We equip each cube with the canonical simplicial structure that we introduced on page 1503. As in Lemma 61.17 we see that the union of the cubes X also admits a simplicial structure. It follows from Observation 85.1 (1) that K is a retract of the finite simplicial complex X.
(2) This statement follows immediately from (1). ■

The statement of the following proposition is quite similar to the statement of Proposition 85.6, it just replaces simplicial complexes by smooth manifolds. We will not make much use of Proposition 85.7, except that it will come in handy at one occasion later on.

[1225]Note that Exercise 2.42 implies that such a map $i\colon X \to Y$ is necessarily an embedding.
[1226]Why is that? Why do we need only finitely many such cubes?

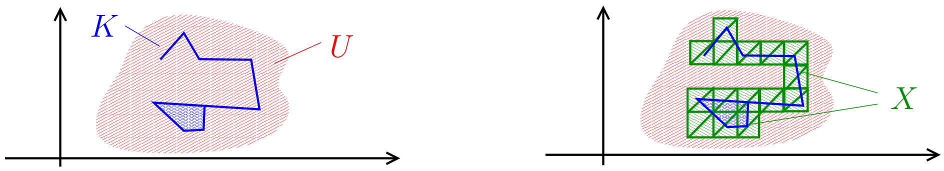


FIGURE 1197. Illustration of the proof of Proposition 85.6.

Proposition 85.7. (∗)

(1) *If $K \subset \mathbb{R}^n$ is a compact neighborhood retract, then K is the retract of a compact n-dimensional smooth submanifold of $\mathbb{R}^n$.*
(2) *Every compact* ENR *is a retract of a compact smooth manifold.*

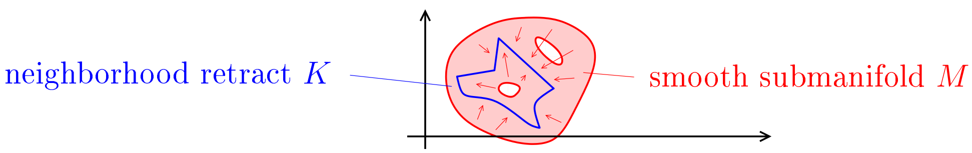


FIGURE 1198. Illustration of Proposition 85.7.

PROOF (∗).

(1) Let K be a compact subset of $\mathbb{R}^n$ that is a retract of some open neighborhood V. Since V is an open subset of $\mathbb{R}^n$ it is in particular an n-dimensional smooth manifold. Therefore we can apply the Proposition 6.64 and we obtain a sequence $M_1, M_2, \ldots$ of subsets of V with the following three properties:
   (a) the sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $M_i \subset M_{i+1}$,
   (b) each M_i is a compact n-dimensional smooth submanifold of V,
   (c) we have $\bigcup\limits_{i\in\mathbb{N}} \mathring{M}_i = V$ where $\mathring{M}_i := M_i \setminus \partial M_i$.

   We make the following observations:
   (i) By Lemma 89.1 (3) each $\mathring{M}_i$ is an open subset of V and thus also of $\mathbb{R}^n$.
   (ii) It follows from (a), (c) and (i), the hypothesis that K is compact and Lemma 2.41 that there exists an $i \in \mathbb{N}$ such that $f(K) \subset M_i$.
   (iii) Since V is an open subset of $\mathbb{R}^n$ every smooth submanifold of V is in fact also a smooth submanifold of $\mathbb{R}^n$.

   It follows from this discussion and property (2) and Observation 85.1 (1) that the smooth manifold $M := M_i$ has all the desired properties.
(2) This statement follows immediately from (1). ■

The following proposition shows that many of our favorite topological spaces are ENRs.

Proposition 85.8.

(1) *Every topological manifold is an* ENR.
(2) *Every countable* CW*-complex that is regionally compact and finite-dimensional is an* ENR.

Remark.

(1) By Corollary 36.15 we know that a finite CW-complex is compact, in particular regionally compact. Thus Proposition 85.8 (2) implies that every finite CW-complex is an ENR.
(2) By Exercise 36.8 we know that a CW-complex is regionally compact if and only if it is locally finite.

The proof of Proposition 85.8 (2) relies very much on the following lemma which is perhaps interesting in its own right.

Lemma 85.9. *Let X be a CW-complex. If X is countable, regionally compact and finite-dimensional, then X admits an embedding into some $\mathbb{R}^n$.*

Remark. The special case of Lemma 85.9 provided by finite CW-complexes is precisely the content of Exercise 36.11.

Proof of Lemma 85.9. The theorem in its full generality is proved in [**FrPi90b**, Theorem A]. In the discussion below we will only deal with the much simpler case that X is actually a finite CW-complex. We prove this case by induction on the dimension of the CW-complex X. The case $\dim(X) = -1$ is of course trivial. So suppose we already know the statement for all finite CW-complexes of dimension $\leq m-1$. Now let X be a finite m-dimensional CW-complex. As usual we denote by X^{m-1} the m-skeleton of X. We denote by $\varphi_1, \dots, \varphi_k \colon S^{m-1} \to X^{m-1}$ the attaching maps of the m-cells of X. By induction there exists an $n \in \mathbb{N}_0$ and an embedding $f \colon X^{m-1} \to \mathbb{R}^n$. We consider the map

$$\begin{aligned}
\overbrace{\Big(X^{m-1} \sqcup \coprod_{i=1}^{k} \overline{B}_i^m\Big)/\sim}^{=X} &\to \mathbb{R}^n \times \overbrace{\mathbb{R}^{m+1} \times \dots \times \mathbb{R}^{m+1}}^{k \text{ copies}} \\
[x] &\mapsto \begin{cases} (f(x), 0, \dots, 0), & \text{if } x \in X^{m-1}, \\ (r \cdot f(\varphi_i(v)), 0, \dots, \underbrace{rv \cdot (1-r), 1-r}_{\text{in the } i\text{-th } \mathbb{R}^{m+1}\text{-factor}}, \dots, 0), & \text{if } x = r \cdot v \in \overline{B}_i^m \text{ where} \\ & r \in [0,1] \text{ and } v \in S^{m-1}. \end{cases}
\end{aligned}$$

It follows from Lemma 3.25 (2) that the somewhat risqué notation for points in $\overline{B}_i^m$ actually leads to a meaningful and continuous map on each $\overline{B}_i^m$. Note that from Lemma 36.7 (4) we obtain that the overall map is continuous. Our hypothesis that X is a finite CW-complex implies by Corollary 36.15 that X is compact. Finally it is elementary to verify that the map is actually injective. Thus, in summary, we can appeal to everybody's best friend, i.e. Proposition 2.43 (2), to conclude the proof that this map is in fact an embedding. ∎

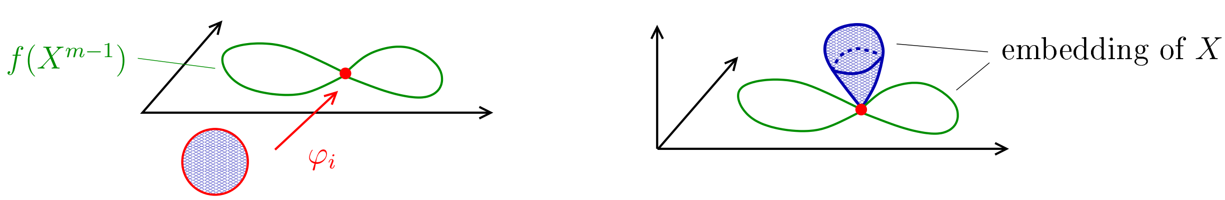


Figure 1199. Illustration for the proof of Lemma 85.9.

Now we can provide the proof of Proposition 85.8.

Proof of Proposition 85.8.

(1) Just for fun we first give a separate, more down to earth argument, for the case of a closed smooth manifold M. By Proposition 9.1 (1) and Proposition 8.1 we can view M as a closed submanifold of some $\mathbb{R}^n$. Now we apply Proposition 8.25 to conclude that $M \subset \mathbb{R}^n$ admits an open neighborhood that retracts to M.

Now we turn to the general case. Thus let M be a topological manifold. By Proposition 9.1 we know that there exists an embedding $f\colon M \to \mathbb{R}^n$ such that $f(M)$ is a closed subset of $\mathbb{R}^n$. As we discussed on page 2063, every topological manifold is weakly locally contractible. Thus we can apply Borsuk's Theorem 85.5 to deduce that M is indeed an ENR.

(2) Let X be a CW-complex that is countable, regionally compact and finite-dimensional. By Lemma 85.9 we know that there exists an embedding $f\colon X \to \mathbb{R}^n$. Furthermore, as we discussed on page 2063, it follows almost immediately from Proposition 36.10 (6) that the CW-complex X is weakly locally contractible. Thus we can once again apply Borsuk's Theorem 85.5 (2) to conclude that X is an ENR. ∎

Recall that in Theorem 65.9 and 65.10 we formulated and proved the Lefschetz Fixed Point Theorem in the realm of finite simplicial complexes and compact smooth manifolds. Our aim now is to extend the Lefschetz Fixed Point Theorem to compact ENRs. To do so we first need to convince ourselves that in this setting the Lefschetz number is actually defined. This follows from the following lemma.

Lemma 85.10. *If X is a compact* ENR*, then for every field $\mathbb{F}$ the homology of X is $\mathbb{F}$-finite.*

Proof. Let X be a compact ENR and let $\mathbb{F}$ be a field. It follows from Propositions 85.6 that X is a retract of a finite simplicial complex Y. Recall that this means that there exists a map $i\colon X \to Y$ and a map $r\colon Y \to X$ with $r \circ i = \mathrm{id}_X$.

(1) It follows from Lemma 61.24 together with Proposition 48.5 and the discussion on page 1402 that the homology of Y is $\mathbb{F}$-finite.
(2) It follows from the functoriality of homology groups and the fact that $r \circ i = \mathrm{id}_X$ that for every $k \in \mathbb{N}$ the map $r_*\colon \mathrm{H}_k(Y;\mathbb{F}) \to \mathrm{H}_k(X;\mathbb{F})$ is an epimorphism.

It follows immediately from (1) and (2) that the homology of X is $\mathbb{F}$-finite. ∎

Now that we know that the Lefschetz number of every self-map of a compact ENR is defined we can now formulate the following theorem.

Theorem 85.11. (Lefschetz Fixed Point Theorem III) *Let X be a compact* ENR *(e.g. X could be a compact topological manifold or a finite* CW*-complex) and let $\varphi\colon X \to X$ be a map. If the $\mathbb{F}$-Lefschetz number $\Lambda(\varphi,\mathbb{F})$ is non-zero for some field $\mathbb{F}$, then φ has a fixed point.*

Remark.

(1) The above Lefschetz Fixed Point Theorem 85.11 together with Proposition 85.8 also gives a different, now totally self-contained, proof of the Lefschetz Fixed Point Theorem 65.10 for compact smooth manifolds.

(2) For compact orientable topological manifolds we will provide a generalization of the above Lefschetz Fixed Point Theorem 85.11 in the Lefschetz-Hopf Theorem 96.9.

PROOF. Let X be a compact ENR. By Proposition 85.6 we know that X is a retract of a finite simplicial complex Y. Recall that this means that there exists a map $i\colon X \to Y$ and a map $r\colon Y \to X$ with $r \circ i = \mathrm{id}_X$. By the very harmless Exercise 2.42 we know that i is an embedding. Thus in the following we will view X as a subspace of Y. Now let $\varphi\colon X \to X$ be a map. We consider the map $i \circ \varphi \circ r\colon Y \to Y$.

Claim. We have $\Lambda(i \circ \varphi \circ r, \mathbb{F}) = \Lambda(\varphi, \mathbb{F})$.

Let $k \in \mathbb{N}_0$. We consider the following sequence of maps

$$\begin{tikzcd} 0 \arrow[r] & \ker(r_*) \arrow[r, hook] & \mathrm{H}_k(Y;\mathbb{F}) \arrow[r, "r_*"] & \mathrm{H}_k(X;\mathbb{F}) \arrow[r] \arrow[l, bend left, "i_*"] & 0. \end{tikzcd}$$

Note that we have the equality $r_* \circ i_* = (r \circ i)_* = \mathrm{id}_* = \mathrm{id}_{\mathrm{H}_k(X;\mathbb{F})}$. This observation implies that the horizontal sequence is exact and that it splits. As in Splitting Lemma 46.2 we see that this implies that we can write $\mathrm{H}_k(Y;\mathbb{F}) = i_*(\mathrm{H}_k(X;\mathbb{F})) \oplus \ker(r_*)$. With slightly suggestive notation we see that

$$\mathrm{tr}\left(\mathrm{H}_k(Y;\mathbb{F}) \xrightarrow{i_*\circ\varphi_*\circ r_*} \mathrm{H}_k(Y;\mathbb{F})\right) = \mathrm{tr}\Bigg(\underbrace{\begin{pmatrix}\mathrm{id}\\0\end{pmatrix}(\varphi_*)\begin{pmatrix}\mathrm{id} & 0\end{pmatrix}}_{=\begin{pmatrix}\varphi_* & 0\\ 0 & 0\end{pmatrix}}\Bigg) = \mathrm{tr}\left(\mathrm{H}_k(X;\mathbb{F}) \xrightarrow{\varphi_*} \mathrm{H}_k(X;\mathbb{F})\right).$$

The desired formula for the Lefschetz numbers follows immediately from this claim. ⊞

From our hypothesis, the claim and the Lefschetz Fixed Point Theorem 65.9 for simplicial complexes we obtain that $i \circ \varphi \circ r\colon Y \to Y$ admits a fixed point $y \in Y$. This fixed point y lies necessarily in the image of the last map, i.e. it lies in $i(X) = X$. It is clear that the point y is also a fixed point for φ. ■

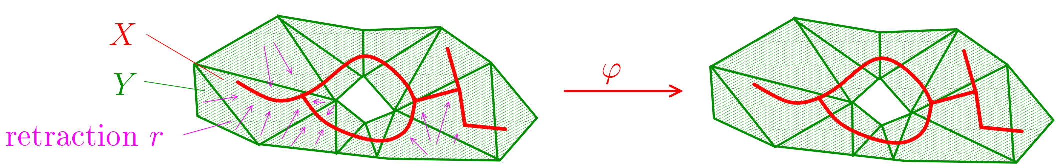


FIGURE 1200. Illustration for the proof of Lefschetz Fixed Point Theorem III 85.11.

Remark. In the possibly unlikely event that the reader wants to know more about ENRs we refer to [**Dol80**, Chapter IV.8] for more information.

85.4. **Invariants of compact topological manifolds.** There are several ways how one can get a grip on the topology of a (compact) *smooth* manifold M:

(a) In Corollary 63.37 we used the fact that (compact) smooth manifolds admit good (finite) covers together with the Nerve Theorem 63.32 to show that M is weakly homotopy equivalent to a finite simplicial complex.

(b) In Theorem 64.5 we used the fact that M is diffeomorphic to a smooth submanifold of some $\mathbb{R}^n$ to show that M actually admits a (finite) smooth simplicial structure.
(c) Later on, in Propositions 104.12 and 105.8, we will use "Morse theory" to show that M is homotopy equivalent to a (finite) CW-complex X with $\dim(X) \leq \dim(M)$.

These results, together with basic facts about invariants of finite CW-complexes, see Proposition 37.13 (3) and Proposition 48.5 (2), imply the following excerpt of Proposition 64.6.

Proposition 64.6. *Let M be a 0-connected smooth manifold.*

(1) *For any $k > n$ we have $\mathrm{H}_k(M) = 0$ and for every abelian group G we also have $\mathrm{H}_k(M; G) = 0$.*
(2) *The higher homotopy groups of M are countable.*

Furthermore, if M is compact, then the following statements hold:

(3) *The fundamental group of M is finitely presented.*
(4) *All homology groups of M are finitely generated abelian groups. Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(M; \mathbb{F})$ are finite-dimensional.*

To the best of the author's understanding none of the above approaches (a), (b) and (c) can be carried over to topological manifolds.

It seems like the only tool we have for handling "the size" of invariants of compact topological manifolds is the following theorem, which is an immediate consequence of Proposition 85.8 together with Proposition 85.6.

Theorem 85.12. *Every compact topological manifold is a retract of a finite simplicial complex.*

Fortunately this is enough to extend most of the results of Proposition 64.6:

Proposition 85.13. *Let M be a compact 0-connected topological manifold. The following statement holds:*

(1) *There exists an $l \in \mathbb{N}_0$ such that for every $k > l$ we have $\mathrm{H}_k(M) = 0$ and such that for every abelian group G we have $\mathrm{H}_k(M; G) = 0$.*
(2) *The higher homotopy groups of M are countable.*
(3) *The fundamental group of M is finitely presented.*
(4) *All homology groups $\mathrm{H}_k(M)$ are finitely generated abelian groups. Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(M; \mathbb{F})$ are finite-dimensional.*

Remark.

(1) Let M be an n-dimensional topological manifold.
    (a) In Theorem 87.3 we will see that in (1) we can take $l = \dim(M)$.
    (b) If M is in fact a smooth manifold, then Theorem 66.6 allows us to compute the homology group $\mathrm{H}_n(M)$. In Theorem 87.3 we will prove the analogous result without the hypothesis that M is smooth.
(2) Proposition 85.13 together with Lemma 64.7 often allows us to control the "size" of relative homology groups of a compact topological manifold. For example for a compact topological manifold M all relative homology groups $\mathrm{H}_k(M, \partial M)$ are finitely generated.
(3) We will deal with non-compact topological manifolds in Theorem 85.17 below.

PROOF OF PROPOSITION 85.13. [1227] Let M be a compact 0-connected topological manifold, let $\mathbb{F}$ be a field, let G be an abelian group and let $n \in \mathbb{N}$. We pick a base point $x_0 \in M$. It follows from Propositions 85.6 and 85.8 (1) that M is a retract of a finite simplicial complex Y. Recall that this means that there exists a map $i\colon M \to Y$ and a map $r\colon Y \to M$ with $r \circ i = \mathrm{id}_M$. We write $y_0 = i(x_0)$. Furthermore we denote the dimension of the simplicial complex by l. We make the following observations:

(a) By Proposition 62.9 we know that $\pi_n(Y, y_0)$ is countable.
(b) It follows from Lemma 61.24 together with Proposition 37.13 that $\pi_1(Y, y_0)$ is finitely presented.
(c) It follows from Lemma 61.24 together with Proposition 48.5 and the discussion on page 1402 that the homology groups $\mathrm{H}_k(Y)$ are finitely generated, that the homology groups $\mathrm{H}_k(Y;\mathbb{F})$ are finite dimensional and that $\mathrm{H}_k(Y) = 0$ and $\mathrm{H}_k(Y;G) = 0$ for $k > l$.

Now we turn to the invariants of M.

(1) Let $k \in \mathbb{N}_0$. It follows from the functoriality of homology groups and the fact that $r \circ i = \mathrm{id}_M$ that the maps $r_*\colon \mathrm{H}_k(Y) \to \mathrm{H}_k(M)$ and $\mathrm{H}_k(Y;G) \to \mathrm{H}_k(M;G)$ are epimorphisms. In particular if the groups to the left vanish, then so do the groups to the right. This observation together with (c) implies (1).
(4) The argument in (1) also shows that if $\mathrm{H}_k(Y)$ is finitely generated, then so is $\mathrm{H}_k(M)$. Similarly, if $\mathrm{H}_k(Y;\mathbb{F})$ is finite-dimensional, then so is $\mathrm{H}_k(M;\mathbb{F})$. Thus we see that (c) implies (4).
(2) As in (1) we see $r_*\colon \pi_n(Y, y_0) \to \pi_n(M, x_0)$ is an epimorphism. It follows from (a) and Lemma 1.7 that $\pi_n(M, x_0)$ is also countable.
(3) The logic of (1), (2) and (4) shows immediately that $\pi_1(M, x_0)$ is *finitely generated*. The statement that $\pi_1(M, x_0)$ is *finitely presented* requires a little more thought. More precisely, by the functoriality of fundamental groups we know that $\pi_1(M, x_0)$ is a *retract* of $\pi_1(Y, y_0)$. By (b) we know that the group $\pi_1(Y, y_0)$ is finitely presented. Thus we obtain from the purely group-theoretic Lemma 21.16 that $\pi_1(M, x_0)$ is also finitely presented. ■

85.5. **ANRs and the homotopy type of topological manifolds.** In this section we will see in particular that all topological manifolds are homotopy equivalent to locally finite simplicial complexes, in particular, they are homotopy equivalent to CW-complexes. In contrast to the previous sections we will not attempt to prove the statements of this section, instead we try our best to give a good overview of the literature.

We start out with the following definition.

Definition. We say that a topological space X is *dominated by a topological space* Y if there exist maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to the identity id_X.

Example.

[1227] The argument below is quite similar to the proof of Lemma 85.10.

(1) If a topological space X is a retract of some topological space Y, then evidently X is dominated by Y.
(2) It follows from Proposition 85.6 together with Proposition 85.8 that every compact topological manifold and every finite CW-complex is dominated by a finite simplicial complex.

The following proposition is the most powerful result on dominations that we actually prove in these notes.

Proposition 119.12. *Let X be a topological space. If X is dominated by a* CW*-complex, then X is homotopy equivalent to a* CW*-complex.*

Remark. Suppose a topological space X is a retract of some CW-complex Y. By Proposition 119.12 we now know that X is homotopy equivalent to a CW-complex. One might hope that X should actually admit the structure of a CW-complex. But in Exercise 85.5 we will see that in general this is *not* the case.

Proof. We will prove Proposition 119.12 in Section 119.4, using the Whitehead Theorem 119.9. Alternatively see the proofs of Theorems 85.17, 85.20 and 85.21 for alternative proofs. ■

It turns out that there exist several powerful generalizations of Proposition 119.12. To formulate these generalizations it is helpful to introduce the following definition.

Definition. A topological space X is called an *absolute neighborhood retract* (or short *ANR*) if X is metrizable and if the following condition is satisfied: whenever $f\colon X \to Y$ is an embedding into a metric space Y such that $f(X)$ is a closed subset of Y, then $f(X)$ is a neighborhood retract in Y.

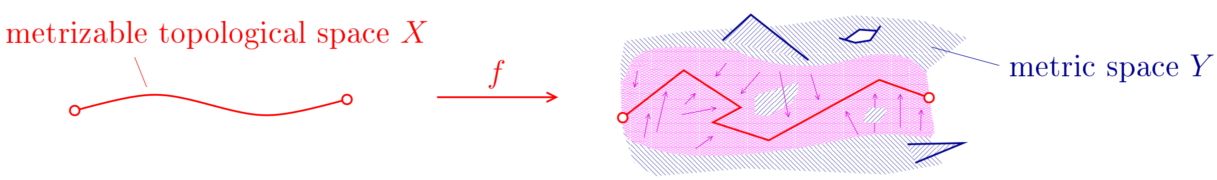


Figure 1201

Remark.
(1) Like in many other settings in the literature there exist various equivalent approaches to defining an ANR, see e.g. [**MaS82**, Chaper I.3.1(ii)] and [**FrPi90a**, Proposition A.6.2]. The definition above is the one given in [**FrPi90a**, p. 281].
(2) Let X be an ANR. If X admits a closed embedding into some $\mathbb{R}^n$, then it follows immediately from the definitions that X is an ENR. Nonetheless, not every ANR is an ENR. For example we will see shortly that the disjoint union $X = \bigsqcup_{k\in\mathbb{N}} \overline{B}^k$ is an ANR, but it follows easily from the Topological Invariance of Domain Theorem 50.6 that X does not admit an embedding into any $\mathbb{R}^n$.
(3) Some references use a definition of an ANR which differs slightly from ours. More precisely, in [**Han51**, p. 390] and [**Miln59**, p. 272] an ANR is supposed to satisfy the extra condition that it is separable, i.e. that it contains a countable subset that

is dense. Note that in Exercise 6.4 we saw that a metrizable topological space is separable if and only if it is second-countable. Thus, since in these notes we are more familiar with the concept of second-countable topological spaces we will use that notion in the remainder of our discussion of ANRs.

(4) The theory of ANRs is developed in great detail in [**Hu65, Bor67, Chig96**].

We will soon see what ANRs are good for. But first let us quote two theorems which give us many examples of ANRs.

Theorem 85.14. (Dugundji Extension Theorem) *Let V be a real vector space and $\|-\|\colon V \to \mathbb{R}_{\geq 0}$ a norm on V. We equip V with the corresponding metric and we use this metric to view V as a topological space. With respect to this topology every convex subset of V is an* ANR.[1228]

Example. The only consequence of the Dugundji Extension Theorem 85.14 that we will use is that every convex subset of some $\mathbb{R}^n$ is an ANR.

PROOF. This theorem was proved by James Dugundji [**Dug51**, Corollary 4.2] in 1952. A proof is also given in [**MaS82**, Theorem I.3.1], [**Maye89**, Theorem I.5.7] and [**Saka13**, Theorem 6.1.1]. ■

Theorem 85.15. (Hanner's Theorem) *Let X be a topological space which is second-countable and metrizable. If X is the union of a family $\{U_i\}_{i\in I}$ of open subsets such that each U_i is an* ANR*, then X itself is an* ANR.

PROOF. This theorem was proved by Olof Hanner [**Han51**, Theorem 3.3] in 1950. An exposition of the proof is also given in [**Fera04**, Theorem 2.77] and in [**Hu65**, Theorem III.8.1]. ■

Corollary 85.16. *Every topological manifold is an* ANR.

PROOF. Let X be an n-dimensional topological manifold. By definition X is second-countable. By Corollary 9.2 we know that X is metrizable. In Lemma 6.9 (1) we showed that X is locally homeomorphic to some non-empty convex subset of $\mathbb{R}^n$. By the Dugundji Extension Theorem 85.14 every non-empty convex subset of $\mathbb{R}^n$ is an ANR. In other words, we have shown that X admits an open cover by ANRs. Thus we obtain from Hanner's Theorem 85.15 that X is indeed an ANR. ■

The following amazing theorem now tells us what ANRs are good for.

Theorem 85.17. *Let X be a topological space. The following statements are equivalent:*

(1) *X is dominated by a countable* CW*-complex.*
(2) *X is homotopy equivalent to a countable* CW*-complex.*
(3) *X is homotopy equivalent to a countable simplicial complex.*
(3') *X is homotopy equivalent to a countable simplicial complex that is locally finite.*

[1228] Note that there are no other hidden conditions on the subset except for convexity, in particular we do not demand that the subset is open, closed or compact.

(4) *X is homotopy equivalent to an* ANR *which is second-countable.*

Furthermore the following refinement of the equivalence of (2) *and* (3) *holds: If X is homotopy equivalent to an n-dimensional countable* CW*-complex, then it is also homotopy equivalent to an n-dimensional countable simplicial complex, and vice versa.*

Proof. The fact that all the statements are equivalent goes well beyond the scope of these notes. Thus we just give the necessary references. The implications (3')⇒(3)⇒(2)⇒(1) are of course trivial. Next note that the equivalence of (2) and (3) is a consequence of Lemma 61.24 and Proposition 62.12. As is explained in [**Miln59**, Theorem 1], the implication (1)⇒(3') follows from work of James Whitehead [**WhdJ50**, Theorem 24] and the implication (3')⇒(4)⇒(1) follows from the work of Olof Hanner [**Han51**, Corollary 3.5 and Theorem 6.1]. Alternatively we refer to [**FrPi90a**, Theorem 5.2.1] or [**LW69**, p. 137] for a proof that the five statements are equivalent. ■

The above results now give us several corollaries on topological manifolds.

Corollary 85.18.
1. *Every topological manifold is homotopy equivalent to a countable locally finite simplicial complex.*
2. *The homology groups, the fundamental group and the higher homotopy groups of a 0-connected topological manifold are countable.*

Remark. Let M be a 0-connected topological manifold. By Corollary 85.18 we now know that $\pi_1(M)$ is countable. A significantly more down-to-earth proof for this statement is given in [**Lee00**, Proposition 7.21].

Proof.

1. This statement follows from Corollary 85.16 together with Theorem 85.17 (4)⇒ (3') and the fact that topological manifolds are by definition second-countable.
2. Let X be a 0-connected topological manifold. By (1) we know that X is homotopy equivalent to a 0-connected countable simplicial complex Y. It follows from Proposition 62.9 and Lemma 61.24 together with Proposition 48.5 that all homotopy groups[1229] and all homology groups of Y are countable. Finally note that it is a consequence of Proposition 40.7 and Corollary 42.8 that the same statements hold for X. ■

By Proposition 54.15 we know that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\mathbb{Z}^\infty$, in particular it is infinitely generated. This result, foreshadowed by our discussion on page 1069, shows that homotopy groups of finite CW-complexes can be infinitely generated. In Question 40.16 we asked, somewhat apprehensively, whether the homotopy groups of a countable CW-complex are necessarily countable. With a slight sigh of relief we can now give an affirmative answer to this question.

Proposition 85.19. *If X is a* CW*-complex with countably many cells, then for any $n \in \mathbb{N}$ and any $x_0 \in X$ the group $\pi_n(X, x_0)$ is countable.*

[1229]For the fundamental group we could alternatively use the argument given on page 1032.

Proof. This follows from the combination of Theorem 85.17 (1)⇒(4) together with Proposition 40.7 and Proposition 62.9. ■

Theorem 85.17 deals with (second-) countable objects. The following theorem is the analogue of Theorem 85.17 without any cardinality restrictions and, on the flip side, without any cardinality information.

Theorem 85.20. *If X is a topological space, then the following statements are equivalent:*

1. *X is dominated by a* CW*-complex.*
2. *X is homotopy equivalent to a* CW*-complex.*
3. *X is homotopy equivalent to a simplicial complex.*
4. *X is homotopy equivalent to an* ANR*.*

Proof. The implication "(1)⇒(2)" is precisely the statement of Proposition 119.12. The fact that (2) and (3) are equivalent is a consequence of Lemma 61.24 and Proposition 62.12. The statement that in fact all four statements are equivalent is proved in [**FrPi90a**, Theorem 5.2.1]. ■

Theorem 85.20, which gives no control of the "size" of the objects is not all that helpful. It is much more interesting to ask whether the analogue of Theorem 85.17 holds if we replace "countable" by "finite". In fact the following theorem holds.

Theorem 85.21. *Let X be a topological space. The following statements are equivalent:*

(2) *X is homotopy equivalent to a finite* CW*-complex.*
(3) *X is homotopy equivalent to a finite simplicial complex.*
(4) *X is homotopy equivalent to a compact* ANR*.*

Furthermore the following refinement of the equivalence of (2) and (3) holds: If X is homotopy equivalent to a finite n-dimensional CW*-complex, then it is also homotopy equivalent to a finite n-dimensional simplicial complex, and vice versa.*

Proof. The fact that (2) and (3) are equivalent is a consequence of Lemma 61.24 and Proposition 62.12. Next note that one can show, with some effort, that it is a consequence of the Dugundji Extension Theorem 85.14 that every finite simplicial complex is an ANR, we refer to [**LW69**, p. 210] and [**HNV04**, p. 472] for details. This observation gives us the implication (3)⇒(4). Finally the implication (4)⇒(2) was shown by James West [**Wes77**] in 1977. ■

The attentive reader will have noticed that the dog that did not bark is the (1)⇒(2) implication. It turns out that there is a good reason why it did not bark. The following theorem, proved by C.T.C. Wall shows that in general the statement one would hope for is actually incorrect.

Theorem 85.22. *There exists a topological space that is dominated by a finite* CW*-complex but that is not homotopy equivalent to a finite* CW*-complex.*

Proof. The statement is proved in [**Wall65a**, p. 66]. Alternatively see [**Var89**, p. 163] and [**FR01**, Theorem 3.1]. More precisely, in [**Wall65a, Var89**] it is shown that to a topological space X that is dominated by a finite CW-complex one can associate an invariant in the

"K-group $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$" which vanishes if and only if X is homotopy equivalent to a finite CW-complex. It is shown in the above references that there exist examples where the invariant is indeed non-zero. Interestingly it is is conjectured, see e.g. [**KrL05**, p. 179], that the group $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ is zero, whenever $\pi_1(X)$ is a torsion-free group. ■

From this discussion we obtain the following corollary.

Corollary 85.23. *Every compact topological manifold is homotopy equivalent to a finite simplicial complex.*

PROOF. This statement follows immediately from Corollary 85.16 together with Theorem 85.21 (4)⇒(3). A very different proof for the corollary was given by Rob Kirby and Larry Siebenmann in [**KSi69**, Chapter III.2] and [**KSi77**, p. 744]. ■

Let M be a (compact) n-dimensional topological manifold. By the above we now know that M is homotopy equivalent to a (finite) simplicial complex X. But the above results do not say anything about the dimension of the simplicial complex X. But it turns out, that at least for compact topological manifolds this can mostly be fixed.

Theorem 85.24.

(1) *Every closed n-dimensional topological manifold is homotopy equivalent to a finite n-dimensional simplicial complex.*
(2) *Every compact connected n-dimensional topological manifold with non-empty boundary is homotopy equivalent to a finite $(n-1)$-dimensional simplicial complex.*

Remark. It seems like a nice project to figure out whether Theorem 85.24 (2) can be generalized to non-compact topological manifolds.

SKETCH OF PROOF. First note that the 1-dimensional case follows immediately from the classification result proved in Theorem 7.1. Furthermore the 2-dimensional case and the 3-dimensional case follow from Radó's Theorem 85.27 and Moise's Theorem 85.28 below. Thus we can now assume that $n \geq 4$.

First note that by Theorem 85.20 (2)⇔(3) it suffices to prove the statement with simplicial complexes replaced by CW-complexes.

Next we cite the following theorem of C.T.C. Wall [**Wall66a**, Corollary 5.1] (see also [**Wall65a**, Theorem E] and [**Var89**, Chapter 5.2]):

> **Wall's Finiteness Theorem.** *Let Y be a connected finite* CW*-complex such that there exists an $r \in \mathbb{N}_0$ such that all twisted cohomology groups $\mathrm{H}^s(Y;V)$ vanish for $s > r$.* (*The precise definition of twisted cohomology is irrelevant to our discussion, so stay with us.*) *Now* [**Wall66a**, Corollary 5.1] *says that if these conditions are satisfied and if $r \geq 3$, then Y is homotopy equivalent to a finite r-*dimensional CW-*complex.*[1230]

Theorem 85.24 is now a consequence of Wall's Theorem and some reasonably standard facts in algebraic topology. In the discussion below we will outline the argument, the detailed proof for reducing Theorem 85.24 to Wall's Theorem is given in [**FNOP19**, Theorem 4.1].

[1230] To the best of my knowledge it seems to be unknown whether the statement holds for $r = 2$.

Now let X be a compact orientable n-dimensional topological manifold. It follows from Theorem 85.21 that X is homotopy equivalent to a finite CW-complex Y. It follows from "Poincaré duality with twisted coefficients", see [**FNOP19**, Theorem A.15], that for any $r \in \mathbb{N}_0$ and any $\mathbb{Z}[\pi_1(X, x_0)]$-left module V we have $\mathrm{H}^r(Y;V) \cong \mathrm{H}^r(X;V) \cong \mathrm{H}_{n-r}(X, \partial X; V)$. If $r > n$, i.e. if $n-r < 0$, then we have $\mathrm{H}_{n-r}(X, \partial X; V) = 0$. Furthermore, if X is connected and if $\partial X \neq 0$ and if $r = n$, i.e. if $n - r = 0$ we have, by an argument similar to the one given on page 1120, that $\mathrm{H}_{n-r}(X, \partial X; V) = 0$. The desired statement now follows from these calculations together with Wall's Theorem.

Finally we consider the case of a compact non-orientable n-dimensional topological manifold. The argument is basically the same, except that one needs Poincaré Duality for non-orientable manifolds. For untwisted coefficients this can be found in [**DaK01**, Theorem 5.7]. For twisted coefficients this follows from an argument as in [**FNOP19**, Theorem A.15], but it is conceivable that nobody ever bothered to work out the details. ■

We conclude this section with the following proposition which can be viewed as an analogue of Proposition 66.16.

Proposition 85.25. *Every closed connected non-empty n-dimensional topological manifold is homotopy equivalent to a finite n-dimensional* CW*-complex which has a single 0-cell and which has a single n-cell.*

PROOF. Let X be a closed connected non-empty n-dimensional topological manifold. We pick an embedding $\varphi\colon \overline{B}^n \to X$. It follows from Proposition 44.3 that $Y := X \setminus \varphi(B^n)$ is a compact connected non-empty n-dimensional topological manifold with $\partial Y = \varphi(S^{n-1})$. By Theorem 85.24 we know that there exists a finite $(n-1)$-dimensional CW-complex Z and a homotopy equivalence $h\colon Y \to Z$. We consider the following diagram:

$$\begin{tikzcd}
\overline{B}^n \arrow[d, "\mathrm{id}"] & S^{n-1} \arrow[l, hook', "i"'] \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & Y \arrow[d, "h"] \\
\overline{B}^n & S^{n-1} \arrow[l, hook', "i"'] \arrow[r, "\varphi\circ f"] & Z.
\end{tikzcd}$$

Note that the diagram commutes, note that the vertical maps are homotopy equivalences and note that it follows from Proposition 39.4 that the inclusion map $i\colon S^{n-1} \to \overline{B}^n$ is a closed cofibration. Thus it follows from the Homotopy Pushout Theorem 39.18 that the pushouts $\overline{B}^n \cup_{S^{n-1}} Y$ and $\overline{B}^n \cup_{S^{n-1}} Z$ are homotopy equivalent. By Lemma 3.45 the former pushout is homeomorphic to X. Furthermore by Lemma 36.32 the latter pushout admits a CW-structure with a single n-cell and no cells of dimension $> n$. Finally note that it follows from Proposition 39.11 that we can in fact find a CW-complex that is homotopy equivalent to X that contains not only a single n-cell but that also contains only a single 0-cell. ■

85.6. **Topological manifolds and CW-structures.** Recall that by Theorem 64.5 we know that every smooth manifold admits a simplicial structure. By Corollaries 85.18 and 85.23 we now know that every (compact) topological manifold is homotopy equivalent to a (finite) CW-complex. The following question naturally arises.

Question 85.26.

(1) *Does every topological manifold admit a simplicial structure?*
(2) *Does every topological manifold admit a* CW*-structure?*

In the following we will survey what is known regarding Question 85.26. Let us start out with some positive statements.

Theorem 85.27. (Radó's Theorem) *Every 2-dimensional topological manifold admits a PL-structure, in particular it admits a simplicial and a* CW*-structure.*

Proof. In 1926 it was shown by Tibor Radó [**Rad26**] that every 2-dimensional topological manifold admits a simplicial structure. It is elementary to see that the simplicial structure of any 1-dimensional topological manifold is in fact a PL-structure. It follows from Proposition 64.11 that the simplicial structure of any 2-dimensional topological manifold is in fact a PL-structure.

Note that we already mentioned the above result of Tibor Radó in the proof of Theorem 23.5. We refer to that proof to alternative references. ■

Theorem 85.28. (Moise's Theorem) *Every 3-dimensional topological manifold admits a PL-structure, in particular it admits a simplicial and a* CW*-structure.*

Proof. This theorem was first proved in 1952 by Edwin Moise [**Moi52**]. Alternative proofs are given in [**Moi77**, p. 252 and 253], [**Moi52**, Theorem 6], [**Bin83**, Theorem XVIII.3.B], [**Ham76**, Theorem 2] and [**Shal84**]. ■

Now we turn to the higher dimensional setting. In this setting the stories between CW-structures and simplicial structures diverge. In the following we first consider CW-structures on high-dimensional topological manifolds. It turns out that the 4-dimensional case is a no-show, in the sense that the following question is still open.

Question 85.29. *Does every closed 4-dimensional topological manifold admit a* CW*-structure?*

Once we move beyond dimension 4, the situation becomes somewhat clearer again.

Theorem 85.30. (Kirby-Quinn-Siebenmann Theorem) *Every* closed *topological manifold of dimension ≥ 5 has a* CW*-structure.*

Proof. For dimensions ≥ 6 this theorem was proved in the 1970s by Rob Kirby and Larry Siebenmann [**KSi77**, Essay III.2]. In 1982 this result was extended to the 5-dimensional case by Frank Quinn [**Qu82**, Theorem 2.3.1]. ■

Note that the Kirby-Quinn-Siebenmann Theorem 85.30 only deals with *closed* topological manifolds. The following question is still open.

Question 85.31. *Let $n \in \mathbb{N}_{\geq 5}$. Does every* compact *$n$-dimensional topological manifold admit a* CW*-structure?*

Now what about simplicial structures?

Theorem 85.32. *There exists a closed 4-dimensional topological manifold that does not admit a simplicial structure.*

PROOF. This theorem was proved by Andrew Casson, see [**AkM90**, page xvi], in the 1980s building on the work of Mike Freedman [**Fre82**]. An account of the proof is also given in [**Sav12**, Theorem 18.3]. ■

The question whether topological manifolds of dimension ≥ 5 admit a simplicial structure had been open for a very long time and it was finally resolved in 2013 by Ciprian Manolescu [**Man16a**].

Theorem 85.33. (Manolescu's Theorem) *For every $n \in \mathbb{N}_{\geq 5}$ there exists a closed n-dimensional topological manifold that does not admit a simplicial structure.*

FIGURE 1202. Ciprian Manolescu (on the right).

PROOF. The theorem is proved in [**Man16a**]. A more relaxed exposition of the key ideas of the proof is also given in [**Man16b**]. Note that, as is explained in [**Man16a, Man16b**], in dimension 5 the examples are necessarily non-orientable, whereas for every $n \geq 5$ there exists a closed orientable n-dimensional topological manifold that does not admit a simplicial structure.[1231] ■

In the unlikely event that the reader got slightly confused by the various theorems and counterexamples we now summarize some of the key statements about the topology of compact topological manifolds in the following table:

	$n=1,2,3$	$n=4$	$n\geq 5$
each closed top. n-mfd. is homotopy equiv. to a finite CW-cplx.	✓	✓	✓
each closed top. n-mfd. admits a CW-structure	✓	?	✓
each compact top. n-mfd. is homotopy equ. to a finite CW-cplx.	✓	✓	✓
each compact top. n-manifold admits a CW-structure	✓	?	?
each compact top. n-manifold admits a simplicial structure	✓	✗	✗

Finally we note that we will discuss the existence and uniqueness of PL-structures and smooth structures on topological manifolds in greater detail in Chapter **??**.

85.7. **The Euler characteristic of compact topological manifolds.** The Euler characteristic of a finite CW-complex is one of the most convenient invariants and often it gives a convenient approach to calculating invariants of CW-complexes. As we just saw, it is not clear whether (compact) topological manifolds admit CW-structures. But fortunately we

[1231]We will explain later on in Chapter 86 what it means for a topological manifold to be orientable.

assembled just enough results on compact topological manifolds to be able to develop the theory of Euler characteristics of compact topological manifolds.

The first part of the following definition was already introduced on page 1422.

Definition.

(1) Given a pair (X, A) of topological spaces and $n \in \mathbb{N}_0$ we refer to
$$b_n(X, A) \ := \ \operatorname{rank}(\mathrm{H}_n(X, A))$$
as the *n-th Betti number of* (X, A). For $A = \varnothing$ we write of course $b_n(X) := b_n(X, \varnothing)$.

(2) Let (X, A) be a pair of topological spaces. If all $b_n(X, A)$ are finite and if the sum $\sum\limits_{n\in\mathbb{N}_0} b_n(X, A)$ is finite, then we define the Euler characteristic of (X, A) to be
$$\chi(X, A) \ := \ \sum_{n\in\mathbb{N}_0} (-1)^n \cdot b_n(X, A).$$
For $A = \varnothing$ we write $\chi(X) := \chi(X, \varnothing)$.

Remark.

(1) It follows from Proposition 55.1 that for a finite CW-complex the above definition agrees with the original definition of the Euler characteristic of a finite CW-complex.

(2) Let X be a compact topological manifold and let A be a compact submanifold or let A be a union of components of ∂X. It follows from Propositions 44.2 (3) and (4) and Proposition 85.13 (0) and (4), together with Lemma 64.7, that the Euler characteristic $\chi(X, A)$ is defined.

It turns out that the formulas we proved before for the Euler characteristic of CW-complexes also hold for topological manifolds. More precisely, we have the following proposition, of which the first three statements are the analogue of Lemmas 55.8 and 55.5 and of Proposition 37.4 in our new context.

Proposition 85.34.

(1) *Let X and Y be homotopy equivalent topological spaces. If the Euler characteristic is defined for one of the two topological spaces, then it is defined for the other topological space and we have*
$$\chi(X) \ = \ \chi(Y).$$

(2) *Let X be and Y be two compact topological manifolds, then*
$$\chi(X \times Y) \ = \ \chi(X) \cdot \chi(Y).$$

(3) *Let $X = Y \cup Z$ be a decomposition of a compact topological manifold into two compact submanifolds Y and Z such that $Y \cap Z$ is also a compact submanifold of X. Then the following equality holds*
$$\chi(X) \ = \ \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

(4) *Let X be a compact topological manifold and let $A \subset X$ be a subspace that is a compact topological manifold in its own right.*[1232] *Then*
$$\chi(X, A) \ = \ \chi(X) \ - \ \chi(A).$$

Furthermore, for any field $\mathbb{F}$ we have

$$\chi(X,A) \;=\; \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(X,A;\mathbb{F})).$$

(5) *Let X be a compact topological manifold and let $p\colon \widetilde{X} \to X$ be a finite covering. Then*

$$\chi(\widetilde{X}) \;=\; \big[\widetilde{X} : X\big] \cdot \chi(X).$$

Sketch of proof.

(1) This statement follows immediately from Corollary 42.8.
(2) This equality can be proved using the Künneth Theorem 58.7 for Chain Complexes. We will fill in the details in Exercise 85.9.
(3) We will not make use of this statement. Therefore we hide, with a good conscience, the proof in Exercise 85.10.
(4) Let X be a compact topological manifold and let $A \subset X$ be a subspace that is a compact topological manifold in its own right.
   (a) It follows immediately from considering the long exact sequence in homology of the pair (X,A) together with Lemma 55.4 that $\chi(X,A) = \chi(X) - \chi(A)$.
   (b) Let $\mathbb{F}$ be a field. Let Y be a compact topological manifold. From Corollary 85.23 we know that Y is homotopy equivalent to a finite CW-complex Z. We observe that

$$\chi(Y) \underset{\substack{\uparrow\\ \text{Corollary 42.8}}}{=} \chi(Z) \underset{\substack{\uparrow\\ \text{see page 1402}}}{=} \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(Z;\mathbb{F})) \underset{\substack{\uparrow\\ \text{by the generalization of Corollary 42.8}\\ \text{to }\mathbb{F}\text{-coefficients, see page 1401}}}{=} \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(Y;\mathbb{F})).$$

   Using this observation we can now prove that

$$\begin{aligned}\chi(X,A) &\underset{\substack{\uparrow\\ \text{by (a)}}}{=} \chi(X) - \chi(A) \underset{\substack{\uparrow\\ \text{by the above observation}}}{=} \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(X;\mathbb{F})) - \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(A;\mathbb{F})) \\ &\underset{\substack{\uparrow\\ \text{by the long exact sequence of the pair }(X,A)\text{ with }\mathbb{F}\text{-coefficients and Lemma 55.3}}}{=} \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(X,A;\mathbb{F})).\end{aligned}$$

(5) Let X be a compact topological manifold and let $p\colon \widetilde{X} \to X$ be a finite covering. Without loss of generality we can assume that X and $\widetilde{X}$ are connected. By Corollary 85.23 there exists a homotopy equivalence $f\colon X \to Y$, where Y is a finite CW-complex. By Proposition 29.5 there exists a connected covering $q\colon \widetilde{Y} \to Y$ such that $q_*(\pi_1(\widetilde{Y})) = f_*(p_*(\pi_1(\widetilde{X})))$. Using Proposition 29.2 one can now easily show that $f\colon X \to Y$ lifts to map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ and that this map is again a homotopy

[1232] For example A could be compact submanifold or A could be a union of boundary components.

equivalence[1233]. Now we obtain that

$$\chi(\widetilde{X}) \underset{\substack{\uparrow\\ \text{by (1) since } \widetilde{X} \text{ and } \widetilde{Y}\\ \text{are homotopy equivalent}}}{=} \chi(\widetilde{Y}) \underset{\substack{\uparrow\\ \text{Proposition 37.4}\\ \text{and Lemma 16.15}}}{=} \big[\pi_1(Y):q_*(\pi_1(\widetilde{Y}))\big]\cdot\chi(Y) \overset{\substack{\text{we have } [\pi_1(\widetilde{Y}):\pi_1(Y)]=[\pi_1(\widetilde{X}):\pi_1(X)] \text{ since } f_* \text{ is an isomorphism}\\ \downarrow}}{\underset{\substack{\uparrow\\ \text{by (1) we have } \chi(Y)=\chi(X) \text{ since } X\\ \text{and } Y \text{ are homotopy equivalent}}}{=}} \big[\pi_1(X):p_*(\pi_1(\widetilde{X}))\big]\cdot\chi(X) \underset{\substack{\uparrow\\ \text{Lemma 16.15}}}{=} \big[\widetilde{X}:X\big]\cdot\chi(X).$$

■

Exercises for Chapter 85.

Exercise 85.1.

(1) Show that there are only countably many homotopy types of finite CW-complexes. *Hint.* Use Theorem 85.21.
(2) Show that there are uncountably many homotopy types of countable CW-complexes.
(3) Let $k, n \in \mathbb{N}$.
   (a) Are there only finitely many homotopy types of k-dimensional CW-complexes with at most n cells?
   (b) Are there only finitely many homotopy types of k-dimensional simplicial complexes with at most n simplices?

Exercise 85.2. We consider $X = \{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ as a subset of $Y = \mathbb{R}$. Show that there is no open neighborhood of X in Y that admits a retraction onto X.

Exercise 85.3. We consider the embedding $\gamma\colon S^1 \to \mathbb{R}^3$ that is shown in Figure 1203. We set $K := \gamma(S^1)$.

(a) Show that $\pi_1(\mathbb{R}^3 \setminus K)$ is an infinitely generated group.
   Remark. The rigor of the solution is bounded from above by the precision of the statement of the problem.
(b) What do you think are the homology groups of $\mathbb{R}^3 \setminus K$?
(c) Show that there is no neighborhood U of K that admits a deformation retraction to K.
   Remark. Write to the author if you have a solution.

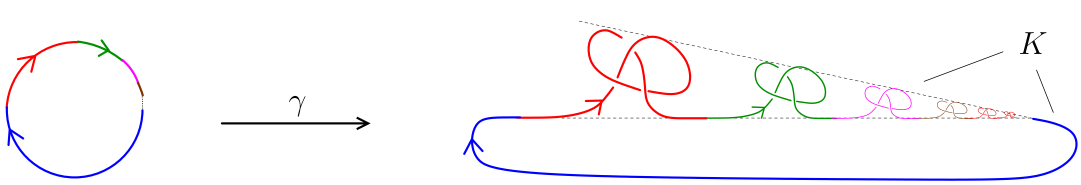


FIGURE 1203

[1233] Here the point is that not only the homotopy equivalences but also the homotopies lift to the covering spaces. We leave the details to the reader.

Exercise 85.4. Show that there exist embeddings $f\colon S^1 \to S^3$ which do not admit a tubular neighborhood, i.e. for which there is no map $\Phi\colon \overline{B}^2 \times S^1 \to S^3$ with $f(P) = \Phi(P, 0)$ for all $P \in S^1$.

Exercise 85.5. Let C be the Cantor set as defined on page 100. Note that basically by construction the complement of C in $[0,1]$ is the disjoint union of countably many open intervals $\{(a_n - \epsilon_n, a_n + \epsilon_n)\}_{n\in\mathbb{N}}$. We set

$$X \;:=\; \underbrace{\{(c,0) \in \mathbb{R}^2 \,|\, c \in C\}}_{=C\times\{0\}\subset\mathbb{R}^2} \cup \bigcup_{n\in\mathbb{N}} \overline{B}^2_{\epsilon_n}((a_n,0)) \;\subset\; \mathbb{R}^2.$$

(We refer to Figure 1204 for an illustration.) In the following we will show that X is a retract of a finite CW-complex but that X itself is not a CW-complex.

(a) Show that X is a deformation retract of the CW-complex $[0,1] \times [-1,1] \subset \mathbb{R}^2$.
(b) Show that X is compact.

Given a path-connected topological space Y we define

$$S(Y) := \{y \in Y \,|\, Y \setminus \{y\} \text{ is not path-connected}\}.$$

(c) Show that $S(X) = \{(c,0) \,|\, c \in C \setminus \{0,1\}\}$.
(d) Let Y be a finite CW-complex. Show that $S(Y)$ has only finitely many path-components.
(e) Show that X does not admit the structure of a CW-complex.

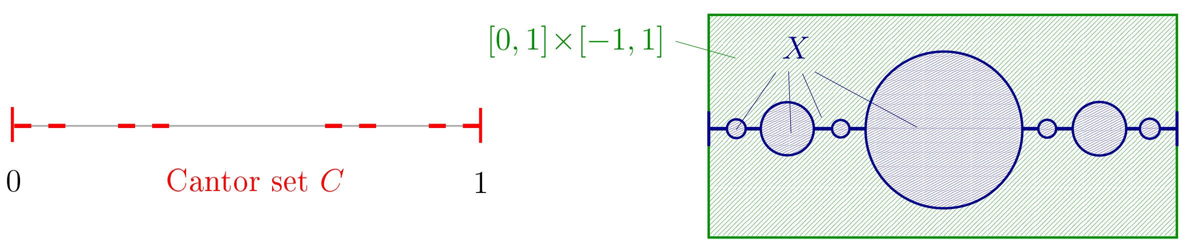


FIGURE 1204. Illustration of Exercise 85.5.

Exercise 85.6. We say that a topological space X is *weakly regionally compact* if every point $x \in X$ admits a compact neighborhood. Now let $X \subset \mathbb{R}^n$ be a subset that is weakly regionally compact. In the following we will show that there exists an embedding $g\colon X \to \mathbb{R}^{n+1}$ such that $g(X)$ is a *closed* subset of $\mathbb{R}^{n+1}$.

(a) Show that there exists an open subset $U \subset \mathbb{R}^n$ with $U \cap \overline{X} = X$.
(b) We set $C := \overline{X} \setminus X = \overline{X} \setminus U$. Note that it follows from Lemma 2.8 that C is a closed subset of $\mathbb{R}^n$ and note that it follows from Corollary 3.19 that $d(x, C) > 0$ for every $x \in X$. Now we consider the map $f\colon X \to \mathbb{R}$ given by $x \mapsto d(x, C)$. Show that the map $g\colon X \to \mathbb{R}^{n+1}$ given by $x \mapsto (x, \frac{1}{f(x)})$ is an embedding such that $g(X)$ is a closed subset of $\mathbb{R}^{n+1}$.

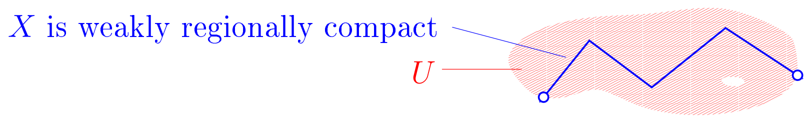


FIGURE 1205. Illustration of Exercise 85.6.

Exercise 85.7. Let X be a compact topological space that is a retract of a finite simplicial complex X. Show that X is an ENR.
Remark. This is the, very easy, converse to Proposition 85.6.

Exercise 85.8. Let X be a compact topological space. Is $\mathrm{H}_1(X)$ necessarily finitely generated?

Exercise 85.9. Let X be and Y be two compact topological manifolds. Show that

$$\chi(X \times Y) \;=\; \chi(X) \cdot \chi(Y).$$

Exercise 85.10. Let $X = Y \cup Z$ be a decomposition of a compact topological manifold into two compact submanifolds Y and Z such that $\partial Y = \partial Z$ is a submanifold of X. Show that

$$\chi(X) \;=\; \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

86. Orientations of topological manifolds

In Section 6.11 we introduced the notion of an orientation of a smooth manifold. The basic idea is that an orientation of a smooth manifold is a "continuous" choice of orientations for each tangent space. Since topological manifolds do not come with tangent spaces there is no obvious extension of this definition of an orientation to the context of topological manifolds. In this chapter we will use "orientations of local homology groups" to introduce the concept of an orientation on a topological manifold.

86.1. **Relative homology and convex subsets of $\mathbb{R}^n$.** The next lemma is at first glance a little dull, but i will come in handy on several occasions later on in this chapter. In all likelihood it is best to have a quick glance at the statement and to move on to the next section.

Lemma 86.1. *Let $n \in \mathbb{N}$, let U be an open subset of $\mathbb{R}^n$, let A be a convex bounded subset of U with $\overline{A} \subset U$ and let $x \in A$. Then for every abelian group G and every $i \in \mathbb{N}_0$ the following inclusion induced maps are isomorphisms:*

$$\begin{array}{lrcl} (1) & \mathrm{H}_i(\mathbb{R}^n \setminus A; G) & \to & \mathrm{H}_i(\mathbb{R}^n \setminus \{x\}; G) \\ (2) & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus A; G) & \to & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}; G) \\ (3) & \mathrm{H}_i(U, U \setminus A; G) & \to & \mathrm{H}_i(U, U \setminus \{x\}; G). \end{array}$$

If $\mathring{A}$ is non-empty and if $x \in \mathring{A}$ is a point, then also the following inclusion induced maps are isomorphisms

$$\begin{array}{lrcl} (4) & \mathrm{H}_i(\partial\overline{A}; G) & \to & \mathrm{H}_i(\mathbb{R}^n \setminus \{x\}; G) \\ (5) & \mathrm{H}_i(\mathbb{R}^n, \partial\overline{A}; G) & \to & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}; G) \end{array}$$

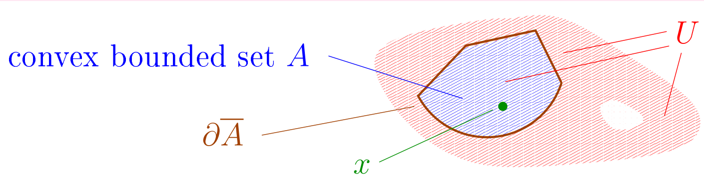


Figure 1206. Illustration of Lemma 86.1.

Proof. Let U be an open subset of $\mathbb{R}^n$, let A be a convex bounded subset of U with $\overline{A} \subset U$ and let $x \in A$. To simplify the notation we drop the coefficients G from the notation and without loss of generality we assume that $x = 0$. Now let $i \in \mathbb{N}_0$.

(1) We have to show that the inclusion induced map $\mathrm{H}_i(\mathbb{R}^n \setminus A) \to \mathrm{H}_i(\mathbb{R}^n \setminus \{0\})$ is an isomorphism. We had already stated this fact in Exercise 42.2. Now that we actually need the statement we give the full details.

Since A is bounded we can pick an $r > 0$ such that $A \subset B^n_r(0)$. We consider the following map:

$$\begin{array}{rcl} F\colon (\mathbb{R}^n \setminus A) \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x \in \mathbb{R}^n \setminus B^n_r(0), \\ x \cdot \underbrace{\left((1-t) + t \cdot \frac{r}{\|x\|}\right)}_{\geq 1 \text{ since } r > \|x\|}, & \text{if } x \in \overline{B}^n_r(0) \setminus A. \end{cases} \end{array}$$

Note that if $y \notin A$, then it follows from the convexity of A and the fact that $0 \in A$, that for any $s \geq 1$ the point sy cannot lie in A either. This shows that the above map actually takes values in $\mathbb{R}^n \setminus A$.

Now it is obvious that the map F defines a deformation retraction from $\mathbb{R}^n \setminus A$ to $\mathbb{R}^n \setminus B^n_r(0)$.[1234] [1235] Clearly $\mathbb{R}^n \setminus B^n_r(0)$ is also a deformation retract of $\mathbb{R}^n \setminus \{0\}$. Now we consider the following commutative diagram of inclusion induced maps:

$$\begin{tikzcd}
\mathrm{H}_i(\mathbb{R}^n \setminus A) \arrow[rr] & & \mathrm{H}_i(\mathbb{R}^n \setminus \{0\}) \\
& \mathrm{H}_i(\mathbb{R}^n \setminus B^n_r(0)). \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

It follows from Corollary 42.8 and the above discussion that the two diagonal maps are isomorphisms. It follows that the horizontal map is also an isomorphism.

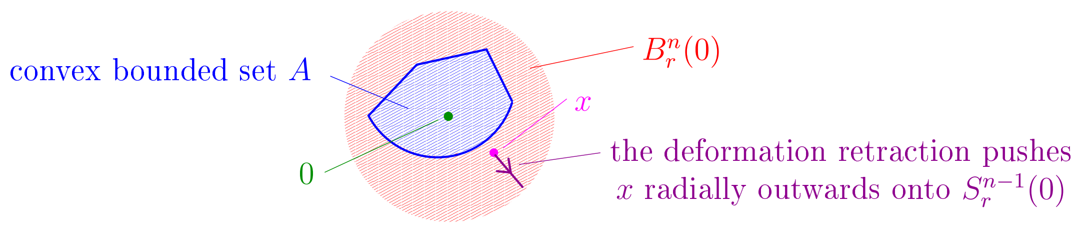


FIGURE 1207

(2) Now we consider the following commutative diagram of long exact sequences

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_i(\mathbb{R}^n \setminus A) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_i(\mathbb{R}^n) \arrow[r] \arrow[d, "="] & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus A) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_{i-1}(\mathbb{R}^n \setminus A) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_{i-1}(\mathbb{R}^n) \arrow[r] \arrow[d, "="] & \dots \\
\dots \arrow[r] & \mathrm{H}_i(\mathbb{R}^n \setminus \{x\}) \arrow[r] & \mathrm{H}_i(\mathbb{R}^n) \arrow[r] & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \arrow[r] & \mathrm{H}_{i-1}(\mathbb{R}^n \setminus \{x\}) \arrow[r] & \mathrm{H}_{i-1}(\mathbb{R}^n) \arrow[r] & \dots
\end{tikzcd}$$

By (1) the first and the fourth vertical maps are isomorphisms. The second and the fifth vertical map are the identity. It follows from the Five Lemma 43.12 that the middle vertical map is also an isomorphism.

(3) We consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}_i(U, U \setminus A) \arrow[r] \arrow[d] & \mathrm{H}_i(U, U \setminus \{x\}) \arrow[d] \\
\mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus A) \arrow[r] & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})
\end{tikzcd}$$

of inclusion induced homomorphisms. Since $\overline{A} \subset U$ we deduce from the Excision Theorem 43.20 that the two vertical maps are isomorphisms. By (2) we know that the bottom horizontal map is an isomorphism. Hence the top horizontal map is also an isomorphism.

[1234] Note that the fact that $0 \in A$ and the convexity of A imply that for any $x \notin A$ and any $s \geq 1$ we have $s \cdot x \notin A$, in particular the map F does indeed take values in $\mathbb{R}^n \setminus A$.

[1235] As always, checking that a map is a deformation retraction also requires showing that the map is continuous. In this case the map is continuous by Lemma 2.74 and the fact that $B^n_r(0) \setminus A$ is a closed subset of $\mathbb{R}^n \setminus A$.

Now we suppose that $\mathring{A}$ is non-empty and we suppose that x is a point in $\mathring{A}$. It follows from Exercise 2.32 that without loss of generality we can assume that $x = 0$ and that $A = \overline{A}$.

(4) Since A is compact, convex with non-empty boundary we can apply Lemma 18.2. Thus we now know that there exists an isotopy

$$F\colon A \times [0,1] \;\to\; \mathbb{R}^n$$

from the inclusion map $i\colon A \to \mathbb{R}^n$ to a homeomorphism $\varphi\colon A \to \overline{B}^n$ that has the following two properties:
(a) $F_1 = \varphi$ restricts to a homeomorphism $\partial A \to S^{n-1}$, and
(b) F has the property that $F(0,t) = 0$ for all $t \in [0,1]$.
Now we have the following diagram

$$\begin{tikzcd} & \mathrm{H}_i(\partial A) \arrow[dl, "\cong"', "\varphi_*"] \arrow[d, "\varphi_*"] \arrow[dr, "(i|_{\partial A})_*"] & \\ \mathrm{H}_i(S^{n-1}) \arrow[r, "\cong"'] & \mathrm{H}_i(\mathbb{R}^n \setminus \{0\}) \arrow[r, "="'] & \mathrm{H}_i(\mathbb{R}^n \setminus \{0\}). \end{tikzcd}$$

Here all the unmarked maps are induced by inclusions. The right-hand triangle commutes by Corollary 42.8 since, by the above property (b), the isotopy F restricts to an isotopy $\partial A \times [0,1] \to \mathbb{R}^n \setminus \{0\}$. The left horizontal map is an isomorphism since S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$. It is now clear that the right-hand diagonal map is also an isomorphism.

(5) With the same argument as in (2) we can reduce the proof of this statement to (4). ■

86.2. **The local homology of topological manifolds.** We recall the following definition from page 1151.

Definition. Given a topological space X and a point $x_0 \in X$ the homology groups $\mathrm{H}_n(X, X \setminus \{x_0\}; R)$ is called the *n-local homology group* of X at the point x_0.

In this section we will study the local homology of topological manifolds, in particular we want to gain a good understanding of generators of these groups.

Some of the following definitions and conventions we had already introduced earlier.

Definition and notation. Let $n \in \mathbb{N}$.

(1) We equip $\mathbb{R}^n$ with the orientation given by the standard basis $e_1, \ldots, e_n$. With this orientation we can view any open subset of $\mathbb{R}^n$ as an oriented smooth manifold.

(2) We write

$$\partial\Delta^n \;=\; \bigcup_{i=0}^{n} \{(t_0, \ldots, t_n) \in \Delta^n \,|\, t_i = 0\}.$$

and we refer to $\mathring{\Delta}^n := \Delta^n \setminus \partial\Delta^n$ as the *interior* of Δ^n.

(3) As on page 1075 we say that a basis $\{v_1, \ldots, v_n\}$ for $\mathrm{T}_P \mathring{\Delta}^n$ is *positive*, if the ordered set of vectors $\{(1, \ldots, 1), v_1, \ldots, v_n\}$ is a positive basis for $\mathbb{R}^{n+1}$. This equips $\mathring{\Delta}^n$ with the structure of an oriented smooth manifold.

(4) Given a subset V of $\mathbb{R}^n$ we say that a map $f\colon V \to \mathbb{R}^k$ is *affine linear* if there exists a $P \in \mathbb{R}^k$ and a $k \times n$–matrix A such that $f(v) = P + Av$ for all $v \in V$.

orientations of the interiors of the standard simplices

FIGURE 1208

Example. The projection map

$$\begin{array}{rcl} p\colon \Delta^n & \to & \mathbb{R}^n \\ (x_1, \dots, x_{n+1}) & \mapsto & (x_1, \dots, x_n) \end{array}$$

is an injective affine linear map. One can easily verify that the restriction of this map to the interior $\mathring{\Delta}^n$ is orientation-preserving if and only if n is even. We refer to Figure 1209 for an illustration.

projection is orientation-reversing projection is orientation-preserving

FIGURE 1209

We continue with the following lengthy lemma.

Lemma 86.2. *Let R be a commutative ring.*

(1) *For any $k \in \mathbb{N}_0$ we have*

$$\mathrm{H}_k(\Delta^n, \partial\Delta^n; R) \;=\; \begin{cases} 0, & \text{if } k \neq n, \\ R \cdot [\mathrm{id}_{\Delta^n} \otimes 1], & \text{if } k = n. \end{cases}$$

(2) *Let $\Phi\colon \Delta^n \to \mathbb{R}^n$ be an injective affine linear map such that 0 lies in the image of $\mathring{\Delta}^n$ and such that the restriction of Φ to $\mathring{\Delta}^n$ is orientation-preserving. Then for any $k \in \mathbb{N}_0$ the induced map*

$$\Phi_*\colon\; \mathrm{H}_k(\Delta^n, \partial\Delta^n; R) \;\to\; \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R)$$

is an isomorphism and this map does not depend on the choice of Φ.

(3) *Let $P \in \mathbb{R}^n$. We pick $r > 0$ such that $B^n_r(0)$ contains P. Then for any $k \in \mathbb{N}_0$ the inclusion induced maps*

$$\mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) \;\leftarrow\; \mathrm{H}_k\left(\mathbb{R}^n, \mathbb{R}^n \setminus B^n_r(0); R\right) \;\leftarrow\; \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R)$$

are isomorphisms and the resulting isomorphism

$$\mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) \;\xrightarrow{\cong}\; \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R)$$

does not depend on the choice of r.

(4) *Let $P \in \mathbb{R}^n$ be a point. If $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ is the reflection in an affine hyperplane through P, then the induced map*

$$\rho_*\colon \ \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R) \to \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R)$$

is given by multiplication by minus -1. The same conclusion also holds if we replace $\mathbb{R}^n$ by any open ball centered at P.

(5) *Let $U \subset \mathbb{R}^n$ be an open set and let $\overline{B}^n_r(Q) \subset U$ be a closed ball. Then for any $P \in B^n_r(Q)$ the inclusion induced maps*

$$\begin{array}{ccc} \mathrm{H}_k(U, U \setminus B^n_r(Q); R) & \longrightarrow & \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus B^n_r(Q); R) \\ \downarrow & & \downarrow \\ \mathrm{H}_k(U, U \setminus \{P\}; R) & \longrightarrow & \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R) \end{array}$$

are isomorphisms and they form a commutative diagram.

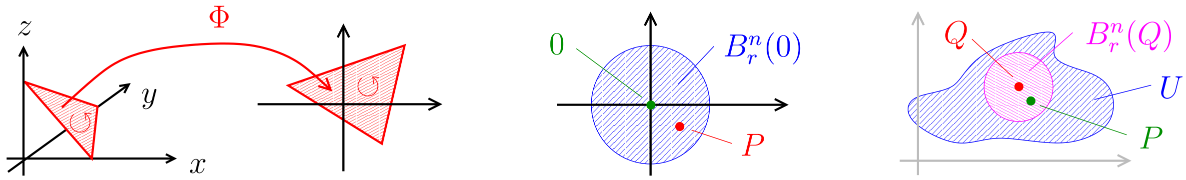


FIGURE 1210. Illustration for Lemma 86.2 (2), (3) and (5)

PROOF (∗).

(1) For $k \neq n$ the statement is proved the same way as Lemma 45.1. Next recall that in Lemma 45.2 (2) we saw that the identity map $\mathrm{id}\colon \Delta^n \to \Delta^n$ represents a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n; \mathbb{Z}) \cong \mathbb{Z}$. Basically the same proof shows that for any commutative ring R the element $\mathrm{id}_{\Delta^n} \otimes 1 \in \mathrm{C}_n(\Delta^n, \partial\Delta^n; R)$ represents a generator of the homology group $\mathrm{H}_n(\Delta^n, \partial\Delta^n; R) \cong R$.

(2) Let $\Phi\colon \Delta^n \to \mathbb{R}^n$ be an injective affine linear map such that 0 lies in the image of $\mathring{\Delta}^n$ and such that the restriction of Φ to $\mathring{\Delta}^n$ is orientation-preserving. Note that it follows from Proposition 2.43 (2) that Φ is an embedding.

Then we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}_k(\Delta^n, \partial\Delta^n; R) & \xrightarrow{\ \Phi_*\ } & \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R). \\ {\scriptstyle \Phi_*}\downarrow & & \uparrow \\ \mathrm{H}_k(\Phi(\Delta^n), \Phi(\partial\Delta^n); R) & \longrightarrow & \mathrm{H}_k(\mathbb{R}^n, \Phi(\partial\Delta^n); R) \end{array}$$

where the bottom and the right vertical map are induced by inclusions. Here the left vertical map is an isomorphism since Φ is an embedding.

Since Φ is an injective affine linear map we see that $\Phi(\Delta^n)$ is a convex subset of $\mathbb{R}^n$ with non-empty interior. By our hypothesis the point 0 lies in the interior of $\Phi(\Delta^n)$. It follows from Lemma 86.1 (5) that the right vertical map is an isomorphism. Clearly $\Phi(\Delta^n) \to \mathbb{R}^n$ is a homotopy equivalence, hence it follows from Corollary 43.18

that the bottom horizontal map is an isomorphism. Since the diagram commutes we have thus now shown, as desired, that the top horizontal map is also an isomorphism.

If $\Psi\colon \Delta^n \to \mathbb{R}^n$ is another such injective affine linear orientation-preserving map, then it follows from elementary linear algebra that there exists some $P \in \mathbb{R}^n$ and a matrix A with *positive* determinant such that

$$\Phi \;=\; (v \mapsto P + A\cdot v)\circ \Psi\colon \Delta^n \to \mathbb{R}^n.$$

We write $\Omega = A \circ \Psi$. It suffices to show that Ψ and Ω and also Ω and Φ induce the same maps $\mathrm{H}_k(\Delta^n, \partial\Delta^n; R) \to \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R)$.

(a) By Lemma 2.65 (1) we know that $\{A \in \mathrm{GL}(n,\mathbb{R}) \,|\, \det(A) > 0\}$ is path-connected. This implies in particular that there exists a path $\gamma\colon [0,1] \to \mathrm{GL}(n,\mathbb{R})$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = A$. We consider the map

$$\begin{array}{rcl} F\colon \Delta^n \times [0,1] & \to & \mathbb{R}^n \\ (Q,t) & \mapsto & \gamma(t)\cdot\Psi(Q). \end{array}$$

Since by hypothesis we have $0 \notin \Phi(\partial\Delta^n)$ and since $\gamma(t)(0) = 0$ we see that $\gamma(t)(\partial\Delta^n) \subset \mathbb{R}^n \setminus \{0\}$ for all t. In particular we deduce that F defines a homotopy between Ψ and Ω as maps $(\Delta^n, \partial\Delta^n) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. But by Proposition 43.17 this means that $\Psi_* = \Omega_*$ as maps on relative homology.

(b) Now we compare Ω and $\Phi = (w \mapsto P + w) \circ \Omega$. We consider the map

$$\begin{array}{rcl} G\colon \Delta^n \times [0,1] & \to & \mathbb{R}^n \\ (Q,t) & \mapsto & (P\cdot t + \Omega(Q)). \end{array}$$

Note that $0 \in \Omega(\mathring{\Delta}^n) = A(\Psi(\mathring{\Delta}^n))$ and that $0 \in \Phi(\mathring{\Delta}^n)$ which implies by $\Phi = P + \Omega$ that $-P \in \Omega(\mathring{\Delta}^n)$. Since $\Omega(\mathring{\Delta}^n)$ is convex we see that G defines a homotopy between Ω and Φ as maps $(\Delta^n, \partial\Delta^n) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. But again by Proposition 43.17 this means that $\Omega_* = \Phi_*$ as maps on relative homology.

(3) The fact that both inclusion induced maps are isomorphisms is an immediate consequence of Lemma 86.1 (2). Now suppose we are given another such $s > 0$. Without loss of generality we can assume that $r \geq s$. We obtain the following commutative diagram:

$$\begin{tikzcd} & \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus B^n_r(0); R) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] \arrow[dd] & \\ \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) & & \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R). \\ & \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus B^n_s(0); R) \arrow[ul, "\cong"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

All maps are induced by inclusions, so the diagram commutes. By Lemma 86.1 the diagonal maps are isomorphisms. It is now obvious that the composition of the top maps and the composition of the bottom maps agree.

(4) Let $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ be the reflection in an affine hyperplane V through a point P. We denote by $f\colon \mathbb{R}^n \to \mathbb{R}^n$ the map that is given by addition by P and furthermore we write $W = f^{-1}(V)$. Note that W goes through the origin. Finally we denote by

$\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ the reflection in the hyperplane W. We consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_{n-1}(S^{n-1}) \otimes R \arrow[r, "\mu"', "\cong"] \arrow[d, "\sigma_* \otimes \mathrm{id}"] & \mathrm{H}_{n-1}(S^{n-1}; R) \arrow[r, "f_*"] \arrow[d, "\sigma_*"] & \mathrm{H}_{n-1}(\mathbb{R}^n \setminus \{P\}; R) \arrow[d, "\rho_*"] & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R) \arrow[l, "\partial"', "\cong"] \arrow[d, "\rho_*"] \\
\mathrm{H}_{n-1}(S^{n-1}) \otimes R \arrow[r, "\mu"', "\cong"] & \mathrm{H}_{n-1}(S^{n-1}; R) \arrow[r, "f_*"] & \mathrm{H}_{n-1}(\mathbb{R}^n \setminus \{P\}; R) & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R). \arrow[l, "\partial"', "\cong"]
\end{tikzcd}$$

where μ is the natural map from page 1417 and where the right-hand horizontal maps are the connecting homomorphisms from Corollary 43.16.

It follows from Lemma 45.11 that the vertical map on the left is given by multiplication by -1. It is straightforward to verify that the diagram commutes and that all horizontal maps are isomorphisms. Furthermore the horizontal maps on the top and bottom are identical. It thus follows that the vertical map on the right is also given by multiplication by -1.

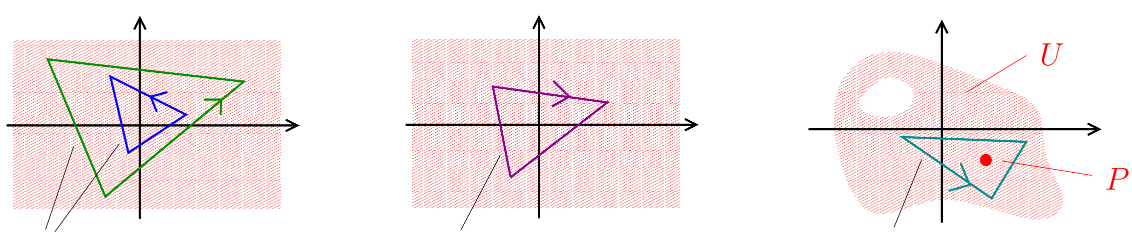
(5) It is clear that the diagram commutes since all maps are induced by inclusions. The horizontal maps are isomorphisms by the Excision Theorem 43.20 and the vertical maps are isomorphisms by Lemma 86.1. ■

Convention. We use the isomorphisms from Lemma 86.2 to identify the various relative homology modules. In particular, for any open set U of $\mathbb{R}^n$, any open ball $B_r(P)$ with $\overline{B}_r(P) \subset U$, any point $Q \in U$ and any commutative ring R we make the identifications

$$R = \mathrm{H}_n(\Delta^n, \partial\Delta^n; R) = \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) = \mathrm{H}_n(U, U \setminus B_r(P); R) = \mathrm{H}_n(U, U \setminus \{Q\}; R).$$

Furthermore, we refer to the image of the generator $[\mathrm{id}_{\Delta^n} \otimes 1] \in \mathrm{H}_n(\Delta^n, \partial\Delta^n; R)$ as the *standard generator* of the various relative homology modules.

Example. We refer to Figure 1211 for an illustration of representatives of the standard generators.[1236] Here, for simplicity, given a singular 2-simplex $\Delta^2 \to \mathbb{R}^2$ we only sketch the image of $\partial\Delta^2$ with the given orientation. Finally note that it follows from Lemma 86.2 (4) that the singular simplex illustrated in the middle of Figure 1211 represents *minus* the standard generator.



standard generator for $\mathrm{H}_2(\mathbb{R}^2, \mathbb{R}^2 \setminus \{0\}; \mathbb{Z})$ | minus the standard generator | standard generator for $\mathrm{H}_2(U, U \setminus \{P\}; \mathbb{Z})$

FIGURE 1211

[1236] At this occasion it is good to remember that if $\sigma\colon \Delta^k \to X$ is a map with $\sigma(\partial\Delta^n) \subset A$, then σ is a cycle in $\mathrm{C}_k(X, A)$, i.e. it represents a class in $\mathrm{H}_k(X, A)$.

Lemma 86.3. *Let R be a commutative ring, let U be an open subset of $\mathbb{R}^n$, let $P \in U$ and let $\sigma\colon \Delta^n \to U$ be a singular n-simplex with the following properties:*

(1) *the point P lies in the image of $\mathring{\Delta}^n$,*

(2) *the restriction of σ to a map $\mathring{\Delta}^n \to U$ is a diffeomorphism onto its image.*

Then $\sigma \otimes 1 \in \mathrm{C}_n(U) \otimes R$ represents a generator of $\mathrm{H}_n(U, U \setminus \{P\}; R)$.[1237] *Furthermore, if*

(3) *the restriction of σ to a map $\mathring{\Delta}^n \to U$ is orientation-preserving,*

then $\sigma \otimes 1$ represents the standard generator of $\mathrm{H}_n(U, U \setminus \{P\}; R)$.

We refer to Figure 1212 for an illustration of Lemma 86.3.

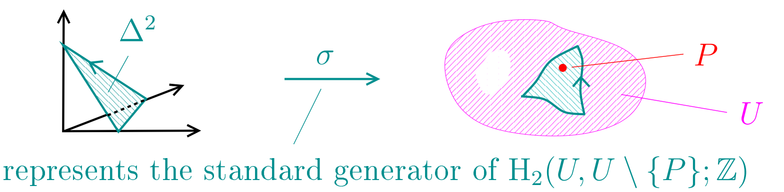


FIGURE 1212

PROOF. We denote by $Q \in \mathring{\Delta}^n$ the point with $\sigma(Q) = P$. To simplify the notation we work throughout the proof with the coefficients $R = \mathbb{Z}$. Using the fact that the inclusion induced map

$$\mathrm{H}_n(U, U \setminus \{P\}) \xrightarrow{\cong} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\})$$

is an isomorphism by the Excision Theorem 43.20 we can without loss of generality assume that $U = \mathbb{R}^n$. As always we denote by $\mathrm{D}\,\sigma_Q\colon\ \mathrm{T}_Q\,\mathring{\Delta}^n \to \mathrm{T}_P\,U = \mathbb{R}^n$ the differential of σ at the point Q. We denote by

$$\begin{array}{rcl} f\colon \Delta^n & \to & \mathbb{R}^n \\ x & \mapsto & (\mathrm{D}\,\sigma_Q)\underbrace{(x - Q)}_{\in \mathrm{T}_Q\,\Delta^n} + P \end{array}$$

the linear approximation of σ at the point Q. We make the following two observations:

(1) Since $\sigma\colon \mathring{\Delta}^n \to U$ is a diffeomorphism onto its image we obtain that $\mathrm{D}\,\sigma_Q$ is an isomorphism of vector spaces, in particular f is an injective affine linear map.

(2) Basically the same argument as in the proof of Lemma 45.17 shows that there exists an open neighborhood V of Q in $\mathring{\Delta}^n$ such that the restrictions of σ and f to maps

$$\sigma, f\colon (V, V \setminus \{Q\}) \ \to \ (\mathbb{R}^n, \mathbb{R}^n \setminus \{P\})$$

are homotopic.

[1237] Note that we do not make any assumptions on the behavior of σ restricted to $\partial\Delta^n$. If one looks very carefully, then one realizes that one has to convince oneself that $\sigma(\partial\Delta^n)$ is a subset of $U \setminus \{P\}$, i.e. one has to show that it follows from (1) and (2), that $P \notin \sigma(\partial\Delta^n)$. How can one prove this statement?

Now consider the following diagram

$$\begin{tikzcd}[column sep=small]
\mathrm{H}_n(\Delta^n,\partial\Delta^n) \arrow[r] \arrow[d, "\sigma_*"] & \mathrm{H}_n(\Delta^n,\Delta^n\setminus\{Q\}) \arrow[d, "\sigma_*"] & \mathrm{H}_n(V,V\setminus\{Q\}) \arrow[l] \arrow[r, "\cong"] \arrow[d, "\sigma_*"] & \mathrm{H}_n(V,V\setminus\{Q\}) \arrow[r] \arrow[d, "f_*"] & \mathrm{H}_n(\Delta^n,\Delta^n\setminus\{Q\}) \arrow[d, "f_*"] \\
\mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{P\}) \arrow[r, "\cong"] & \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{P\}) \arrow[r, "\cong"] & \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{P\}) \arrow[r, "\cong"] & \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{P\}) \arrow[r, "\cong"] & \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{P\}).
\end{tikzcd}$$

It is clear that the first two squares and the last square commute. Furthermore the third square commutes by the above observation (2) together with Proposition 43.17. As in the proof of Lemma 86.2 (2) we see that the last vertical map is an isomorphism. It now follows easily that $\sigma \in \mathrm{C}_n(\mathbb{R}^n)$ represents a generator of $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\})$.

Finally suppose that the restriction of σ to a map $\mathring{\Delta}^n \to U$ is orientation-preserving. In this case, by definition of orientation-preserving, the map f is also orientation-preserving, therefore by the above commutative diagram the singular n-simplex σ does indeed represent the standard generator of $\mathrm{H}_n(U, U \setminus \{P\})$. ■

We continue with the following lemma which is almost identical to Lemma 44.1.

Lemma 86.4. *Let M be an n-dimensional topological manifold and let R be a commutative ring. Then for every $x \in M \setminus \partial M$ and any $k \in \mathbb{N}_0$ we have*

$$\mathrm{H}_k(M, M \setminus \{x\}; R) \;\cong\; \begin{cases} R, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if $\Phi\colon U \to V$ is a chart around x and if $\sigma\colon \Delta^n \to V$ represents a generator of $\mathrm{H}_n(V, V \setminus \{\Phi(x)\}; R)$, then $\Phi^{-1} \circ \sigma$ represents a generator of $\mathrm{H}_n(M, M \setminus \{x\}; R)$.

We refer to Figure 1213 for an illustration of Lemma 86.4.

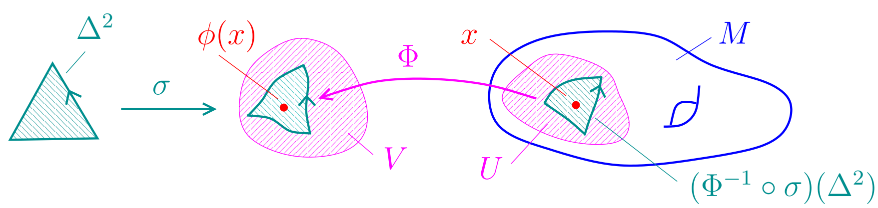


FIGURE 1213

Remark. Let $n \in \mathbb{N}_0$. A topological space X that has the property that for each $x \in X$ we have

$$\mathrm{H}_k(X, X \setminus \{x\}; \mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise} \end{cases}$$

is often called a *k-dimensional homology manifold*. By Lemma 86.4 such topological spaces can be viewed as generalizations of closed topological manifolds.

Let Y be an homology n-sphere.[1238] By Proposition 45.7 we know that the suspension $\Sigma(Y)$ of Y is an $(n+1)$-dimensional homology manifold. On the other hand we saw

[1238] Recall that by the definition on page 1670 an homology n-sphere is an n-dimensional topological manifold X such that for any $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X; \mathbb{Z}) \cong \mathrm{H}_k(S^n; \mathbb{Z})$. On page 1672 we saw that there exists a smooth homology 3-sphere with non-trivial fundamental group, namely the Poincaré Homology Sphere.

in Exercise 45.9 that the suspension $\Sigma(Y)$ is *not* a topological manifold.[1239] Recall that the Double Suspension Theorem 64.17 says that the *double* suspension $\Sigma(\Sigma(Y))$ of any homology n-sphere is actually homeomorphic to S^{n+2}.

The proof of Lemma 86.4 is basically identical to the proof of Lemma 44.1. We recall the argument just in case some readers might have forgotten.

Proof. Let M be an n-dimensional topological manifold, let R be a commutative ring and let $x \in M \setminus \partial M$. Since $x \notin \partial M$ we can pick a chart $\Phi\colon U \to V$ such that V is an open subset of $\mathbb{R}^n$. We write $y = \Phi(x)$. For any $k \in \mathbb{N}_0$ we then have

$$\underset{\substack{\uparrow\\ \text{Excision Theorem 43.20}}}{\mathrm{H}_k(M, M \setminus \{x\}; R) \xleftarrow{\cong} \mathrm{H}_k(U, U \setminus \{x\}; R)} \underset{\substack{\uparrow\\ \text{since } \Phi \text{ is a homeomorphism}}}{\xrightarrow[\cong]{\Phi_*}} \mathrm{H}_k(V, V \setminus \{y\}; R) \underset{\substack{\uparrow\\ \text{see Lemma 86.2 (5)}}}{\xrightarrow{\cong}} \begin{cases} R, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

The last statement regarding generators now follows immediately from the above isomorphisms. ■

86.3. **R-Sections on topological manifolds.** In Section 6.11 we defined the notion of an oriented smooth manifold via orientations of the tangent spaces. The notion of a tangent space requires the smooth structure on a smooth manifold, put differently, topological manifolds do not come with tangent spaces. In the next section we will see that we can still define the notion of an orientation of a topological manifold. Furthermore we will see that for smooth manifolds this new notion of an orientation will be equivalent to the previous notion of an orientation.

The key to defining the new notion of orientation is the following, initially rather obscure definition.

Definition. Let M be an n-dimensional topological manifold and let R be a commutative ring.

(1) An *R-section for M at $x \in M \setminus \partial M$* is an element $\mu_x \in \mathrm{H}_n(M, M \setminus \{x\}; R) \cong R$.
(2) Let $W \subset M$ and let $x \in W$. Given any $k \in \mathbb{N}_0$ we denote by ρ_x the natural map
$$\rho_x\colon\ \mathrm{H}_k(M, M \setminus W; R) \ \to\ \mathrm{H}_k(M, M \setminus \{x\}; R).$$
Given $\varphi \in \mathrm{H}_k(M, M \setminus W; R)$ we often write $\varphi_x := \rho_x(\varphi)$, i.e. the above map is also written as
$$\varphi \ \mapsto\ \varphi_x.$$
(3) Let A be a subset of $M \setminus \partial M$. (In the following we will mostly be interested in the case $A = M \setminus \partial M$). An *$R$-section for M along A* is a choice for each $x \in A$ of an R-section $\mu_x \in \mathrm{H}_n(M, M \setminus \{x\}; R)$ at x such that the following "continuity condition" is satisfied: for each $x \in A$ there exists an open neighborhood U of x in A and a homology class $\mu_U \in \mathrm{H}_n(M, M \setminus U; R)$ such that for each $y \in U$ the image of

[1239] To be super-precise we showed that it is not an $(n+1)$-dimensional topological manifold, but it follows easily from Proposition 44.4 that if $\Sigma(Y)$ was a topological manifold it would need to be $(n+1)$-dimensional.

μ_U under the natural map

$$\rho_y\colon\ \mathrm{H}_n(M, M\setminus U; R) \;\to\; \mathrm{H}_n(M, M\setminus\{y\}; R)$$

equals μ_y.

(4) We denote by $\Gamma_R^A(M)$ the set of all R-sections of M along A.

For $A = M\setminus\partial M$ we say "section" instead of "section along A" and we write $\Gamma_R(M)$ instead of $\Gamma_R^{M\setminus\partial M}(M)$.

Example. In Figure 1214 we attempt to illustrate the definition of a $\mathbb{Z}$-section on a surface M. Here, for simplicity, given a singular 2-simplex $\Delta^2\to M$ we only sketch the image of $\partial\Delta^2$ with the given orientation.

(1) On the left of Figure 1214 we show two points x and y with singular 2-simplices representing $\mu_x\in\mathrm{H}_2(M,M\setminus\{x\};\mathbb{Z})$ and $\mu_y\in\mathrm{H}_2(M,M\setminus\{y\};\mathbb{Z})$. Furthermore we sketch an open set U that contains x and y and we sketch a homology class $\mu_U\in\mathrm{H}_2(M,M\setminus U;\mathbb{Z})$ that gets sent to μ_x and μ_y via the natural maps[1240]

$$\mathrm{H}_n(M,M\setminus U;\mathbb{Z})\xrightarrow{\rho_x}\mathrm{H}_n(M,M\setminus\{x\};\mathbb{Z})\text{ and }\mathrm{H}_n(M,M\setminus U;\mathbb{Z})\xrightarrow{\rho_y}\mathrm{H}_n(M,M\setminus\{y\};\mathbb{Z}).$$

In fact these $\mathbb{Z}$-sections at x and y form part of a $\mathbb{Z}$-section of M.

(2) To the right of Figure 1214 we show two $\mathbb{Z}$-sections ν_x and ν_y that cannot be part of a $\mathbb{Z}$-section of M.

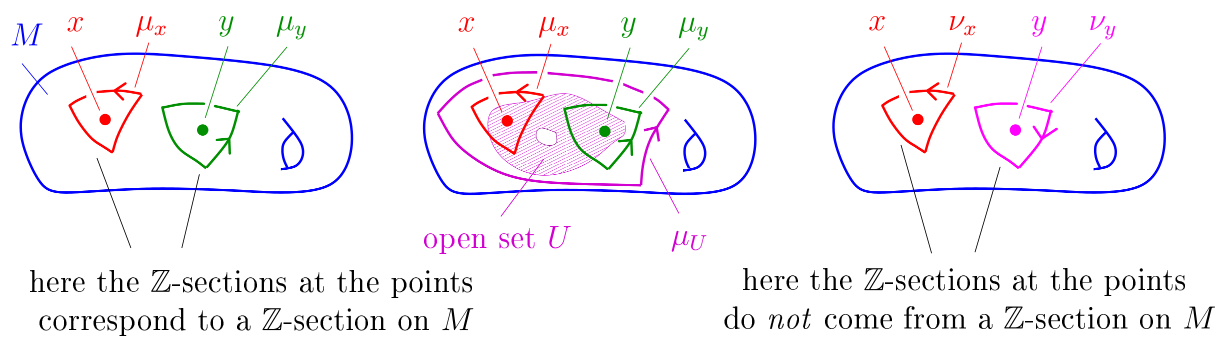


FIGURE 1214

Lemma 86.5. *Let M be an n-dimensional topological manifold and let R be a commutative ring.*

(1) *Given any subset $A\subset M\setminus\partial M$ the set $\Gamma_R^A(M)$ admits a unique R-module structure such that for each $y\in A$ the map $\Gamma_R^A(M)\to\mathrm{H}_n(M,M\setminus\{y\};R)$ that is given by $\{\alpha_x\}_{x\in A}\mapsto\alpha_y$ is an R-module homomorphism.*

(2) *For every subset $A\subset M\setminus\partial M$ (often we will take $A=M\setminus\partial M$) the map*

$$\begin{array}{rcl}\mathrm{H}_n(M,M\setminus A;R) & \to & \Gamma_R^A(M)\\ \alpha & \mapsto & \{\alpha_x\}_{x\in A}\end{array}$$

is a well-defined homomorphism of R-modules.

[1240] Note that in this example the group $\mathrm{H}_2(M,M\setminus U;\mathbb{Z})$ is not isomorphic to $\mathbb{Z}$, i.e. the two maps are not isomorphisms.

Remark. In Theorem 87.10 we will see that in many settings the converse to Lemma 86.5 holds.

Proof (∗).

(1) We will prove this statement in Exercise 86.2.
(2) Let $A \subset M \setminus \partial M$ be a subset and let $\alpha \in \mathrm{H}_n(M, M \setminus A; R)$. Let $x \in A$. We need to show that there exists an open neighborhood U of x in A and a homology class $\varphi \in \mathrm{H}_n(M, M \setminus U; R)$ such that for each $y \in U$ we have $\rho_x(\varphi) = \alpha_x$. But this is trivial, we just take $U = A$ and $\varphi = \alpha$. ■

The following proposition contains much more interesting results on $\Gamma_R^A(M)$.

Proposition 86.6. *Let M be a topological manifold and let A be a* connected *subset of $M \setminus \partial M$.*[1241]

(1) *Let R be a commutative ring and let $\{\alpha_x\}_{x\in A}$ and $\{\beta_x\}_{x\in A}$ be two R-sections for M along A. If there exists an $x \in A$ with $\alpha_x = \beta_x$, then the two R-sections agree.*
(2) *Let $\{\alpha_x\}_{x\in A}$ be a $\mathbb{Z}$-section for M along M. Suppose there exists a $y \in A$, an $m \in \mathbb{Z}$ and a $\beta_y \in \mathrm{H}_n(M, M \setminus \{y\}; \mathbb{Z})$ such that $\alpha_y = m \cdot \beta_y$, then there exists a $\mathbb{Z}$-section $\{\beta_x\}_{x\in A}$ along A such that $\alpha_x = m \cdot \beta_x$ for every $x \in A$.*
(3) *Let $\{\alpha_x\}_{x\in A}$ be a $\mathbb{Z}$-section along A for M. If there exists a $y \in A$ such that α_y is a generator of $\mathrm{H}_n(M, M\setminus\{y\}; \mathbb{Z}) \cong \mathbb{Z}$, then α_x is a generator of $\mathrm{H}_n(M, M\setminus\{x\}; \mathbb{Z}) \cong \mathbb{Z}$ for every $x \in A$.*

Before we provide the proof of Proposition 86.6 we need to prove one little lemma which in turn requires the following definition.

Definition. Let M be an n-dimensional topological manifold. We say that $Y \subset M$ is a *small ball* in M if there exists a chart $\Phi\colon U \to V$ such that $B_2^n(0) \subset V$ and such that $\Phi(Y) = B^n$. We illustrate the definition of a small ball in Figure 1215.

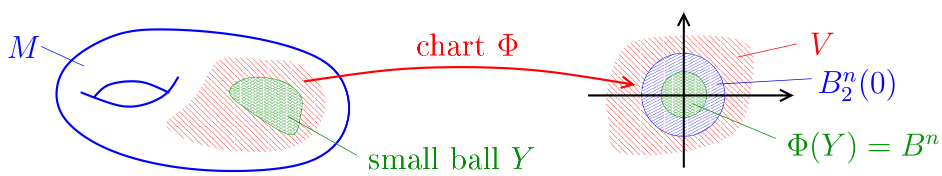


Figure 1215

Lemma 86.7. *Let M be an n-dimensional topological manifold.*

(1) *Given any point $x \in M \setminus \partial M$ and any neighborhood U of x there exists a small ball contained in U that contains x.*

Let R be a commutative ring.

(2) *For any point y in a small ball Y and any $k \in \mathbb{N}_0$ the map*

$$\rho_y\colon\ \mathrm{H}_k(M, M \setminus Y; R) \ \to\ \mathrm{H}_k(M, M \setminus \{y\}; R)$$

[1241] In some applications we will use that by Exercise 44.4 we know that if M is a connected topological manifold, then $M \setminus \partial M$ is also connected.

is an isomorphism.

(3) *Let A be a subset of $M \setminus \partial M$, let $\{\alpha_x\}_{x\in A}$ be an R-section along A and let $x \in A$. There exists a small ball Y in $M \setminus \partial M$ that contains x and a $\varphi \in \mathrm{H}_k(M, M \setminus Y; R)$ such that for every $y \in Y \cap A$ we have $\rho_y(\varphi) = \alpha_y \in \mathrm{H}_n(M, M \setminus \{y\}; R)$.*

Proof. Let M be an n-dimensional topological manifold.

(1) The proof of the first statement is elementary.

(2) Now let y be a point that is contained in a small ball Y. To simplify the notation we drop the commutative ring R from the notation. We consider the long exact sequence of the triple $(M, M \setminus \{y\}, M \setminus Y)$ given by Corollary 43.16:

$$\to \mathrm{H}_k(M\setminus\{y\}, M\setminus Y) \to \mathrm{H}_k(M, M\setminus Y) \to \mathrm{H}_k(M, M\setminus\{y\}) \to \mathrm{H}_{k-1}(M\setminus\{y\}, M\setminus Y) \to$$

We see immediately that it suffices to show that all the relative homology groups $\mathrm{H}_k(M \setminus \{y\}, M \setminus Y)$ vanish. Since Y is a small ball we can pick a chart $\Phi\colon U \to V$ such that $B^n_2(0) \subset V$ and such that $\Phi(Y) = B^n$. We pick $s \in (1, 2)$ and we write $Z = \Phi^{-1}(B^n_s(0))$. Then

$$\mathrm{H}_k(M\setminus\{y\}, M\setminus Y) \underset{\uparrow}{\xleftarrow{\cong}} \mathrm{H}_k(Z\setminus\{y\}, Z\setminus Y) \xrightarrow[\cong]{\Phi_*} \mathrm{H}_k(B^n_s(0)\setminus\{\Phi(y)\}, B^n_s(0)\setminus B^n) \underset{\uparrow}{=} 0.$$

Excision Theorem 43.20

by Corollary 43.18, since $B^n_s(0) \setminus B^n$ is a deformation retract of $B^n_s(0) \setminus \{\Phi(y)\}$

We refer to Figure 1216 for an illustration of the proof.

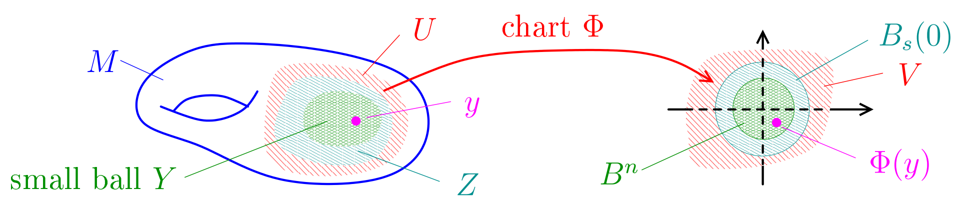


Figure 1216. Illustration for the proof of Lemma 86.7 (2).

(3) Let A be a subset of M, let $\{\alpha_x\}_{x\in A}$ be an R-section along A and let $x \in A$. By definition of an R-section along A and by definition of the subspace topology there exists an open neighborhood V of $x \in M \setminus \partial M$ and a singular chain $\sigma \in \mathrm{C}_n(M; R)$ that represents a cycle in $\mathrm{C}_n(M, M \setminus (V \cap A); R)$ such that for each $y \in V \cap A$ we have $\alpha_y = \rho_y([\sigma]) \in \mathrm{H}_n(M, M \setminus \{y\}; R)$.

Since σ represents a cycle in the relative chain complex $\mathrm{C}_*(M, M\setminus(V\cap A); R)$ we know that $\partial\sigma \in \mathrm{C}_{n-1}(M \setminus (V \cap A); R)$. It follows from the fact that Δ^{n-1} is compact together with Lemmas 2.40 and 2.17 that there exists an open neighborhood $U \subset M$ of x with $U \subset V$ such that $\partial\sigma \in \mathrm{C}_{n-1}(M \setminus U; R)$. In other words, σ represents a cycle in $\mathrm{C}_n(M, M \setminus U)$. By (1) there exists a small ball Y with $x \in Y \subset U$. It follows immediately from the functoriality of homology groups that the homology class $[\sigma] \in \mathrm{H}_n(M, M \setminus Y; R)$ has all the desired properties. ■

Now we can provide the proof of Proposition 86.6.

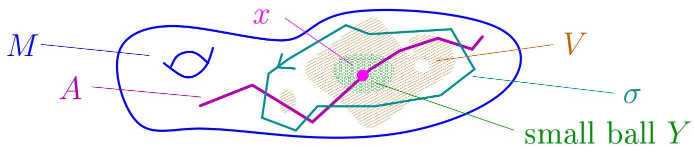


FIGURE 1217. Illustration for the proof of Lemma 86.7 (3).

PROOF OF PROPOSITION 86.6. Let M be a n-dimensional topological manifold, let A be a connected subset of $M \setminus \partial M$ and let R be a commutative ring.

(1) Let $\{\alpha_x\}_{x\in A}$ and $\{\beta_x\}_{x\in A}$ be two R-sections for M along A. We assume that there exists a $z \in A$ with $\alpha_z = \beta_z$. We consider the R-section along A that is given by $\gamma_x = \alpha_x - \beta_x$ for $x \in A$. Next we consider the following two subsets of A:

$$V \;:=\; \{y \in A \,|\, \gamma_y = 0\} \quad \text{and} \quad W \;:=\; \{y \in A \,|\, \gamma_y \neq 0\}.$$

We need to show that $V = A$. Since V is by hypothesis non-empty and since A is connected it suffices to show that V and W are both open. In other words, it suffices to prove the following claim.

Claim. Let $x \in A$. There exists an open neighborhood U of x in A such that either γ_y is zero for every $z \in U$ or γ_y is non-zero for every $z \in U$.

Let $x \in A$. It follows immediately from Lemma 86.7 (3) that there exists a small ball Y in $M \setminus \partial M$ that contains x and a $\varphi \in \mathrm{H}_k(M, M \setminus Y; R)$ such that for every $y \in Y \cap A$ we have $\rho_y(\varphi) = \alpha_y \in \mathrm{H}_n(M, M \setminus \{y\}; R)$. Furthermore, note that by Lemma 86.7 (2) we know that for every $y \in Y$ the map ρ_y is an isomorphism. But this implies that $\gamma_x = 0$ if and only if $\gamma_y = 0$ for all $y \in Y$. ⊞

(2) Let $n \in \mathbb{N}$. We start out with the following definition: Given an element g of a group G with $G \cong \mathbb{Z}$ we say that g is divisible by n if there exists an $h \in G$ with $n \cdot h = g$. In this case we write $g/n := h$. Note that h is well-defined since $G \cong \mathbb{Z}$.

Let $\{\alpha_x\}_{x\in A}$ be a $\mathbb{Z}$-section for M along A. Given $n \in \mathbb{N}$ we consider the two sets

$$V \;:=\; \{y \in A \,|\, \alpha_y \text{ is divisible by } n\}$$

and

$$W \;:=\; \{y \in A \,|\, \alpha_y \text{ is not divisible by } n\}.$$

The same argument as in (1) one shows that V and W are open.

If V is non-empty, then $V = M \setminus \partial M$ and it is straightforward to verify that $\beta_y := \alpha_y/n$, $y \in A$, is in fact a $\mathbb{Z}$-section along A. We leave it to the reader to fill in the details.

(3) Let $\{\alpha_x\}_{x\in A}$ be a $\mathbb{Z}$-section for M along A. Let $y \in A$ be a point such that α_y is a generator of $\mathrm{H}_n(M, M\setminus\{y\}; \mathbb{Z}) \cong \mathbb{Z}$. Now let $z \in A$. If α_z is not a generator, then there exists a $\beta_z \in \mathrm{H}_n(M, M\setminus\{z\}; \mathbb{Z}) \cong \mathbb{Z}$ and some $n \geq 2$ such that $\alpha_z = n \cdot \beta_z$. By (2) we can extend β_z to a $\mathbb{Z}$-section $\{\beta_x\}_{x\in A}$ such that $\alpha_x = n \cdot \beta_x$ for every $x \in A$. But this implies that α_y is not a generator. So we have obtained a contradiction. ■

86.4. **Orientations of topological manifolds.** After the technical preparations from the last section we can now finally introduce the notion of an orientation of a topological manifold.

Definition. Let M be an n-dimensional topological manifold and let R be a commutative ring.

(1) An R-section for M at a point $x \in M \setminus \partial M$ that is in fact a generator of the R-module $\mathrm{H}_n(M, M \setminus \{x\}; R) \cong R$ is called an *R-orientation for M at $x \in M$.*
(2) An *R-section for M* that is an R-orientation at each point $x \in M \setminus \partial M$ is called an *R-orientation of M.*
(3) The topological manifold M is called *R-orientable* if it admits an R-orientation.
(4) Let M and N be two n-dimensional topological manifolds that are equipped with R-orientations $\{\alpha_x\}_{x \in M \setminus \partial M}$ and $\{\beta_x\}_{x \in N \setminus \partial N}$. Let $f \colon M \to N$ be a map that is a local homeomorphism. We say the map f is *orientation-preserving* if the following holds: for any open set $U \subset M$, such that $f \colon U \to f(U)$ is a homeomorphism and such that $f(U)$ is open, and any $x \in U \setminus \partial M$ the image of α_x under the map[1242][1243][1244]

these maps are isomorphisms by the Excision Theorem 43.20

$$\mathrm{H}_n(M, M \setminus \{x\}) \xleftarrow{\cong} \mathrm{H}_n(U, U \setminus \{x\}) \xrightarrow[\cong]{f_*} \mathrm{H}_n(f(U), f(U) \setminus \{f(x)\}) \xrightarrow{\cong} \mathrm{H}_n(N, N \setminus \{f(x)\}).$$

equals $\beta_{f(x)}$. Otherwise we call f *orientation-reversing.*

Remark. Note that the above definition of an orientation of an n-dimensional topological manifold also makes perfect sense if $n = 0$. In particular for topological manifolds we do not have to make the slightly awkward distinction between 0-dimensional case and the higher dimensional case that we had to make for smooth manifolds, see pages 297 and page 1718.

We leave the proof of the following lemma as an amusing exercise to the reader.

Lemma 86.8. *Let M and N be two n-dimensional topological manifolds and furthermore let $f, g \colon M \to N$ be two homeomorphisms. If f and g are isotopic, then f is orientation-preserving if and only if g is orientation-preserving.*

Before we start with the discussions of orientations in detail we get the following technical lemma out of the way.

Lemma 86.9. *Let M be an-dimensional topological manifold.*

(1) *Let R be a commutative ring and let $A \subset M \setminus \partial M$ be a* connected *subset. If M is R-orientable, then for every $x \in M \setminus \partial M$ the map*

$$\begin{array}{rcl} \Phi \colon \Gamma_R^A(M) & \to & \overbrace{\mathrm{H}_n(M, M \setminus \{x\}; R)}^{\cong R} \\ \{\alpha_y\}_{y \in A} & \mapsto & \alpha_x \end{array}$$

[1242]Since $f \colon U \to f(U)$ is a homeomorphism we have $f(U \setminus \{x\}) = f(U) \setminus \{f(x)\}$.
[1243]It follows from the fact that $f \colon U \to f(U)$ is a homeomorphism, the fact that $x \in U \setminus \partial U$ and Proposition 44.2 that $f(x) \notin \partial N$.
[1244]We suppress the R-coefficients from the notation for space reasons.

is an isomorphism of R-modules.

(2) *If M is non-orientable and connected, then $\Gamma_{\mathbb{Z}}(M) = 0$.*

PROOF. Let M be a connected n-dimensional topological manifold.

(1) Let R be a commutative ring such that M is R-orientable. We pick an R-orientation $\{\mu_y\}_{y\in M\setminus\partial M}$ for M. Next let $A \subset M\setminus\partial M$ be a connected subset and let $x \in M\setminus\partial M$. By definition μ_x is a generator of the R-module $\mathrm{H}_n(M, M\setminus\{x\}; R) \cong R$, i.e. every element in $\mathrm{H}_n(M, M\setminus\{x\}; R)$ is of the form $r\cdot\mu_x$ for some unique $r \in R$. We consider the map

$$\begin{array}{rcl}\Psi\colon\ \mathrm{H}_n(M, M\setminus\{x\}; R) & \to & \Gamma^A_R(M) \\ r\cdot\mu_x & \mapsto & \{r\cdot\mu_y\}_{y\in A}.\end{array}$$

Evidently we have $\Phi\circ\Psi = \mathrm{id}$. It follows from our hypothesis that A is connected and from Proposition 86.6 (1) that $\Psi\circ\Phi$ is also the identity.

(2) Now we suppose that M admits a non-zero $\mathbb{Z}$-section $\{\alpha_x\}_{x\in M\setminus\partial M}$. We need to show that M is orientable. We pick $y \in M\setminus\partial M$ such that $\alpha_y \neq 0$. We pick a generator $\beta_y \in \mathrm{H}_n(M, M\setminus\{y\}; \mathbb{Z}) \cong \mathbb{Z}$. Then $\alpha_y = n\cdot\beta_y$ for some $n \in \mathbb{Z}$. It follows again from our hypothesis that M is connected and from Proposition 86.6 (2) that there exists a $\mathbb{Z}$-section $\{\beta_x\}_{x\in M\setminus\partial M}$ such that $\alpha_x = n\cdot\beta_x$ for all $x \in M\setminus\partial M$. It follows immediately from Proposition 86.6 (3) that $\{\beta_x\}_{x\in M\setminus\partial M}$ is an orientation for M. ■

In the following lemma we collect some basic facts about $\mathbb{Z}$-orientations.

Lemma 86.10. *Let M be a topological manifold.*

(1) *For any point $x \in M\setminus\partial M$ there exist precisely two $\mathbb{Z}$-orientations for M at x.*

(2) *If M is connected and orientable, then M admits precisely two $\mathbb{Z}$-orientations.*

PROOF. The first statement is an immediate consequence of the fact, proved in Lemma 86.4, that for any $x \in M\setminus\partial M$ we have $\mathrm{H}_n(M, M\setminus\{x\}; \mathbb{Z}) \cong \mathbb{Z}$. The second statement follows easily from Proposition 86.6 and the definitions. ■

Examples.

(1) Let V be a connected open subset of $\mathbb{R}^n$. For any $x \in V$ we define

$$\mu_x \ := \ \text{the standard generator of } \mathrm{H}_n(V, V\setminus\{x\}; \mathbb{Z}) \text{ as defined on page 2097.}$$

We claim that the μ_x define a $\mathbb{Z}$-orientation for V. Indeed, let $x \in V$. We pick an $r > 0$ such that $\overline{B}_r(x) \subset V$. We set $U := B_r(x)$. We denote by μ_U the standard generator of $\mathrm{H}_n(V, V\setminus U; \mathbb{Z})$ as defined on page 2097. It follows from Lemma 86.2 (5) that μ_U has the desired property. We refer to this $\mathbb{Z}$-orientation as the *standard $\mathbb{Z}$-orientation* of V.

By Lemma 86.10 (2) the topological manifold V admits precisely two different $\mathbb{Z}$-orientations. We have already given one orientation above. By Lemma 86.2 the other one is given by a reflection of the standard generators in hyperplanes. In Figure 1218 we sketch the two different orientations of V.

(2) In Figure 1219 we sketch the two different orientations of the torus.

standard generators for $H_2(V, V \setminus \{x\}; \mathbb{Z})$ — the negative of the standard generators

FIGURE 1218

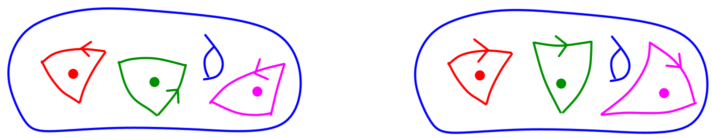


FIGURE 1219

(3) Let R be a commutative ring. Recall that for $n \geq 1$ we denote by $[S^n] \in H_n(S^n; \mathbb{Z})$ the standard generator that we introduced on page 1174. It follows from the Universal Coefficient Theorem 57.19 together with Lemma 57.17 (3) that the natural map

$$H_n(S^n) \otimes R \ \to \ H_n(S^n; R)$$

is an isomorphism. By a slight abuse of notation we denote by $[S^n]$ also the image of $[S^n] \otimes 1 \in H_n(S^n; R)$ under the above map. Note that $[S^n] \in H_n(S^n; R) \cong R$ is a generator. We refer to it as the *standard generator of* $H_n(S^n; R)$. For any $x \in S^n$ it follows easily from the long exact sequence of homology groups with R-coefficients of the pair $(S^n, S^n \setminus \{x\})$ that the natural map

$$H_n(S^n; R) \ \to \ H_n(S^n, S^n \setminus \{x\}; R)$$

is an isomorphism. We denote by $\mu_x \in H_n(S^n, S^n \setminus \{x\}; R)$ the image of $[S^n]$. By the above μ_x is an R-orientation at x. In fact all these R-orientations $\{\mu_x\}_{x \in S^n}$ define an R-orientation of S^n. Indeed, in the definition on page 2105 we just need to set $U = S^n$ and $\mu_U = [S^n] \in H_n(S^n, \underbrace{S^n \setminus S^n}_{=\emptyset}; \mathbb{Z})$. We refer to this orientation of S^n as the *standard R-orientation of S^n.*

The following proposition shows in particular that for smooth manifolds the notion of orientability from Section agrees with the above notion of orientability.

Proposition 86.11.

(1) *Give any smooth manifold M there exists a natural bijection*

$$\left\{ \begin{array}{c} \textit{orientations on } M \textit{ in} \\ \textit{the sense of Section} \end{array} \right\} \xrightarrow{\cong} \{ \mathbb{Z}\textit{-orientations on } M \}$$

such that the following two statements hold:

(a) *For any open subsets of some $\mathbb{R}^n$ the above bijections sends the standard orientation from page 297 to the above standard $\mathbb{Z}$-orientation.*

(b) *If $f\colon M \to N$ is a local diffeomorphism between oriented connected smooth manifolds, then the following two statements are equivalent:*

(i) *f preserves the orientations on M and N in the sense of Section ,*

(ii) *f preserves the $\mathbb{Z}$-orientations on M and N.*

(2) *Given an n-dimensional smooth manifold M we have*

$$M \text{ is orientable in the sense of Section } \iff M \text{ is } \mathbb{Z}\text{-orientable.}$$

In the proof of Proposition 86.11 we will use the following lemma.

Lemma 86.12. *Let $\Psi\colon V \to V'$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. Then for any $x \in V$ we have*

$$\begin{array}{c}\Psi \text{ preserves the standard}\\ \mathbb{Z}\text{-orientation of } V \text{ at the point } x\end{array} \iff \begin{array}{c}\Psi \text{ preserves the orientation of } V \text{ in the}\\ \text{sense of Section at the point } x.\end{array}$$

Proof ($*$). Let $\Psi\colon V \to V'$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$ and let $x \in V$. We write $x' = \Psi(x)$. Given $w \in \mathbb{R}^n$ we denote by t_w the translation $\mathbb{R}^n \to \mathbb{R}^n$ that is given by $v \mapsto v + w$. We set $\epsilon := 1$ if Ψ preserves the orientation of V in the sense of Section at the point x. Otherwise we set $\epsilon := -1$. We consider the following diagram

$$\begin{tikzcd}
H_n(V, V\setminus\{x\};\mathbb{Z}) \arrow[d, "\Psi_*"'] & H_n(t_{-x}(V), t_{-x}(V)\setminus\{0\}) \arrow[l, "t_{x*}"'] \arrow[r] \arrow[d, "(t_x\circ\Psi\circ t_{-x'})_*"] & H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \arrow[d, "\cdot\epsilon"] \\
H_n(V', V'\setminus\{x'\};\mathbb{Z}) & H_n(t_{-x}(V), t_{-x}(V)\setminus\{0\}) \arrow[l, "t_{x'*}"'] \arrow[r] & H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}).
\end{tikzcd}$$

We make the following observations:

(1) It follows easily from the definitions that each translation t_v is orientation-preserving in both senses of an orientation.

(2) It follows from Proposition 45.16 together with (1) that the square at the right commutes.

(3) It is clear that the square at the left commutes.

(4) The statement of the lemma follows from (2) and (3). ■

Now we can provide the proof of Proposition 86.11.

Sketch of the proof of Proposition 86.11. First we note that the case of 0-dimensional smooth manifolds is basically a tautology, at least as long as all statements are interpreted in a reasonable way. Thus in the following we assume that we are dealing with smooth manifolds of dimension ≥ 1.

Next note that statement (2) is just a reader-friendly condensation of statement (1). Thus we need to prove statement (1). We define the promised natural bijections. Thus let M be an n-dimensional smooth manifold where $n \geq 1$. We first suppose that M is oriented in the sense of Section . Let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to V$ around x

that is orientation-preserving in the sense of Section . We define

$$\mu_x \;:=\; \Phi_*^{-1}\big(\text{standard generator of } \mathrm{H}_n(V, V \setminus \{\Phi(x)\}; \mathbb{Z}) \text{ as defined on page 2097}\big).$$

It follows from Lemma 86.12 that this definition does not depend on the choice of the orientation-preserving chart Φ. We claim that the classes $\{\mu_x\}_{x \in M \setminus \partial M}$ form a $\mathbb{Z}$-orientation for M. We only have to show that these classes satisfy the "continuity condition". We claim that for any $x \in M \setminus \partial$ a sufficiently small ball U containing x has the desired property. The proof of this fact is very similar to the arguments used in the proof of Proposition 86.6. We leave it to the reader to fill in the details.

Now we define the inverse map. So suppose we are given a $\mathbb{Z}$-orientation $\{\mu_x\}_{x \in M \setminus \partial M}$ for M. Note that by Lemma 6.46 we only need to find an orientation for $M \setminus \partial M$. Thus let $x \in M \setminus \partial M$. We pick a chart $\Phi \colon U \to V$ around x such that V is connected. It follows from Lemma 86.2 that, after possibly reflecting V in a hyperplane, we can assume that Φ preserves the $\mathbb{Z}$-orientation at x, i.e. it sends the given $\mathbb{Z}$-orientation μ_x to the standard orientation of the open subset V of $\mathbb{R}^n$ that we introduced on page 2106. We define

$$\text{orientation of } \mathrm{T}_x M \;:=\; \Phi^*(\text{standard orientation of } \mathrm{T}_{\Phi(x)} V = \mathbb{R}^n).$$

It follows from Lemma 86.12 that this definition does not depend on the choice of Φ. As before, using small balls, one can show that this defines an orientation of M in the sense of Section . We leave the verification of the details once again to the surely highly motivated reader.

It is basically clear, that by construction, the bijections have the desired properties (a) and (b). This concludes the proof of statement (1). ■

Convention. Proposition 86.11 says in particular that for smooth manifolds the notion of orientability that we introduced in Section agrees with the notion of $\mathbb{Z}$-orientability that we had just introduced. For $R = \mathbb{Z}$ we will therefore often drop the "R" from the definitions on page 2105, i.e. often we just say *orientation at a point*, *orientation* and *orientable*.

Remark. It follows basically from the definitions that under the equivalence of Proposition 86.11 the orientations of S^n introduced on page 299 and page 2107 agree.

The following proposition shows that the notion of R-orientability is more flexible than the previous notion of orientability.

Proposition 86.13. *Every topological manifold is $\mathbb{F}_2$-orientable.*

Example. Proposition 86.13 says in particular that the real projective plane, the Klein bottle and the Möbius band are $\mathbb{F}_2$-orientable.

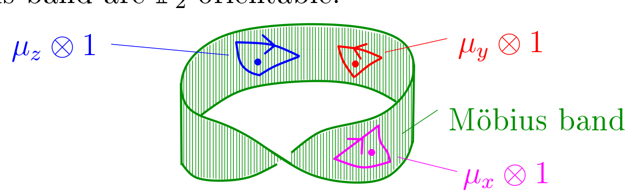


FIGURE 1220

Proof. Let M be an n-dimensional topological manifold. Let $x \in M \setminus \partial M$. By Lemma 86.4 we know that $\mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2) \cong \mathbb{F}_2$. We define μ_x to be the unique non-trivial element in $\mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2) \cong \mathbb{F}_2$. Using Lemma 86.7 it is straightforward to show that the continuity condition is satisfied, in particular we have defined an $\mathbb{F}_2$-orientation on M. ■

Lemma 86.14. *Let M be a topological manifold. If M is orientable, then it is R-orientable for any commutative ring R.*

Proof. Let M be an n-dimensional orientable topological manifold and let R be a commutative ring. Let $x \in M \setminus \partial M$. It follows from the Universal Coefficient Theorem 57.19, together with the fact that $\mathrm{H}_{n-1}(M, M \setminus \{x\}; \mathbb{Z}) = 0$ obtained in Lemma 86.4, that the natural map

$$\begin{aligned} \mu\colon\ \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z}) \otimes R &\to \mathrm{H}_n(M, M \setminus \{x\}; R) \\ \sum_{i=1}^{k} [\sigma_i] \otimes r_i &\mapsto \Big[\sum_{i=1}^{k} \sigma_i \otimes r_i \Big] \end{aligned}$$

is an isomorphism.

Now we pick an orientation $\{\alpha_x\}_{x \in M \setminus \partial M}$ for M. For each $x \in M \setminus \partial M$ we denote by $\widetilde{\alpha}_x \in \mathrm{H}_n(M, M \setminus \{x\}; R)$ the image of $\alpha_x \otimes 1$ under the above isomorphism. It is straightforward to verify that these generators $\{\widetilde{\alpha}_x\}_{x \in M \setminus \partial M}$ define an R-orientation for the topological manifold M. ■

We continue with the following proposition which is a variation on Proposition 17.3.

Proposition 86.15.

(1) *To each topological manifold M we can canonically associate an oriented topological manifold $\widetilde{M}$ together with a 2-fold covering $p\colon \widetilde{M} \to M$ and to each local homeomorphism $f\colon M \to N$ we can associate an orientation-preserving local homeomorphism $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$ such that the following diagram commutes:*[1245]

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \widetilde{N} \arrow[d, "p"] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

(2) *Let M be a connected topological manifold and let $p\colon \widetilde{M} \to M$ be the 2-fold covering constructed in* (1)*.*

(a) *If M is orientable, then there exists an orientation-preserving homeomorphism to $\Theta\colon \widetilde{M} \to M \sqcup -M$ such that the following diagram commutes:*

$$\begin{tikzcd} \widetilde{M} \arrow[rr, "\Theta", "\cong"'] \arrow[dr, "p"'] & & M \sqcup -M \arrow[dl, "q"] \\ & M & \end{tikzcd}$$

where q is the obvious map given by the identity on each component of $M \sqcup -M$.

(b) *If M is non-orientable, then $\widetilde{M}$ is connected.*

(c) *In either case, the oriented opological manifold $\widetilde{M}$ admits an orientation-reversing self-homeomorphism $\Xi_M\colon \widetilde{M} \to \widetilde{M}$ such that the following diagram commutes:*

$$\begin{tikzcd} \widetilde{M} \arrow[rr, "\Xi_M", "\cong"'] \arrow[dr, "p"'] & & \widetilde{M} \arrow[dl, "p"] \\ & M. & \end{tikzcd}$$

Furthermore, given an orientation-preserving local homeomorphism $f\colon M \to N$ we have $\widetilde{f} \circ \Xi_M = \Xi_N \circ \widetilde{f}$.

(3) *Let M be a connected non-orientable topological manifold. Suppose that $p\colon \widetilde{M} \to M$ and $q\colon \widehat{M} \to M$ are two connected 2-fold coverings such that $\widetilde{M}$ and $\widehat{M}$ are orientable. There exists a homeomorphism $\Phi\colon \widetilde{M} \to \widehat{M}$ such that the following diagram commutes:*

$$\begin{tikzcd} \widetilde{M} \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & \widehat{M} \arrow[dl, "q"] \\ & M. & \end{tikzcd}$$

If $\widetilde{M}$ and $\widehat{M}$ are oriented, then we can arrange that Φ is orientation-preserving.

Definition. Let M be a connected non-orientable topological manifold without boundary. We refer to the connected 2-fold covering $p\colon \widetilde{M} \to M$ from Proposition 86.15, which is unique up to equivalence, as the *orientation covering* of M.

Sketch of proof. The proof is very similar to the proof of Proposition 17.3.

(1) Let M be a topological manifold.
   (a) For $x \in M \setminus \partial M$ we define an orientation at x to be a $\mathbb{Z}$-orientation for M at the point x.
   (b) Let $x \in \partial M$. An orientation-neighborhood for x is a triple (U, μ, x) where U is an open neighborhood of $x \in M$ and μ is a $\mathbb{Z}$-orientation on U. We say that two orientation-neighborhoods (U, μ, x) and (V, ν, x) are equivalent if the orientations μ and ν agree on $U \cap V$. For the purpose of this proof we define an orientation for x to be an equivalence class of orientation-neighborhoods for x.

   We consider the set

$$\widetilde{M} \;:=\; \{\mu_x \mid x \in M \text{ and } \mu_x \text{ is an orientation for } M \text{ at the point } x\}$$

   and the projection map $p\colon \widetilde{M} \to M$ that is given by $\mu_x \mapsto x$ for $x \in M \setminus \partial M$ and $[(U, \mu, x)] \mapsto x$ for $x \in \partial M$. It follows from Lemma 86.10 (1) that for every $x \in M \setminus \partial M$ the preimage $p^{-1}(x)$ has precisely two elements. We leave it to the reader to verify that for every $x \in \partial M$ the preimage $p^{-1}(x)$ also has precisely two elements.

[1245] We leave it to the reader to express this statement in categorical language.

We equip $\widetilde{M}$ with a topology similar to the proof of Proposition 17.3. It is then straightforward to see that $p\colon \widetilde{M} \to M$ is a 2-fold covering. Similar to the proof of Proposition 17.3 one can show that $\widetilde{M}$ admits a canonical orientation.

Now let $f\colon M \to N$ be a local homeomorphism. We leave it to the reader to define the map $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$ that has the promised properties.[1246]

(2) The proof of this statement is almost identical to the proof of Proposition 17.3 (2).
(3) Not surprisingly the proof of this statement is almost identical to the proof of Proposition 17.3 (2).

We leave it to the reader to flesh out all the details. ■

Finally we conclude this section with the following corollary which is closely related to Corollary 17.4.

Corollary 86.16. *Let M be a connected topological manifold without boundary. If M is non-orientable, then there exists an epimorphism $\pi_1(M) \to \mathbb{Z}_2$.*

Proof. The proof of the corollary is basically identical to the proof of Corollary 17.4, we just need to replace Proposition 17.3 by Proposition 86.15. ■

86.5. **Induced orientations.** (∗) In this final section we will show that various constructions of new topological manifolds out of a given oriented topological manifold are naturally equipped with orientations. For the most part the statements sound reasonable and familiar from the setting of smooth manifolds. Nonetheless, the constructions of the induced representations for topological manifolds are at times quite different from the constructions of induced orientations for smooth manifolds. Truth be told, the reader will not miss out on much by just moving on to the next chapter.

Lemma 86.17. *Let R be a commutative ring and let M be a topological manifold that is equipped with an R-orientation.*

(1) *Let N be a topological manifold and let $f\colon N \to M$ be a map. If $f\colon N \setminus \partial N \to M$ is a local homeomorphism, then there exists a unique R-orientation on N such that f is orientation-preserving.*
(2) *If N is a submanifold of codimension zero (for example N could be an open subset of M), then N admits a unique R-orientation such that the inclusion map $N \to M$ is orientation-preserving.*
(3) *If $p\colon \widetilde{M} \to M$ is a covering, then there exists a unique orientation on $\widetilde{M}$ that turns p into an orientation-preserving map.*

Proof. To simplify the notation we suppress the ring R from the notation. Thus let M be an n-dimensional topological manifold that is equipped with an orientation $\{\alpha_x\}_{x\in M\setminus \partial M}$.

(1) Let N be a topological manifold and let $f\colon N \to M$ be a local homeomorphism. Let $x \in N \setminus \partial N$. We pick an open neighborhood U of x such that $f\colon U \to f(U)$ is a

[1246] It is obvious how to define the map if $\partial M = \partial N = \varnothing$. Dealing with the boundaries is slightly awkward.

homeomorphism and such that the image $f(U)$ is an open subset of M. We define $\beta_x \in \mathrm{H}_n(N, N \setminus \{x\}; R)$ to be the image of α_x under the map[1247]

$$\mathrm{H}_n(M, M\setminus\{f(x)\}) \overset{\cong}{\underset{\uparrow}{\longleftarrow}} \mathrm{H}_n(f(U), f(U)\setminus\{f(x)\}) \xrightarrow[\cong]{f_*^{-1}} \mathrm{H}_n(U, U\setminus\{x\}) \overset{\cong}{\underset{\uparrow}{\longrightarrow}} \mathrm{H}_n(N, N\setminus\{x\}).$$

(these maps are isomorphisms by the Excision Theorem 43.20)

We leave it to the reader to verify that these classes β_x define an orientation on N. By definition the map f is orientation-preserving with respect to the orientations on M and N. It is elementary to verify that the orientation on N is uniquely determined.

(2) This statement follows from (1) since the inclusion map of a codimension-zero submanifold is, basically by definition, a local homeomorphism.
(3) This statement also follows from (1) since a covering map is a local homeomorphism. ■

The following lemma, which can be viewed as an analogue of Lemma 44.12 (5), says in particular that the double of an R-orientable topological manifold is again R-orientable.

Lemma 86.18. *Let M be a connected topological manifold with non-empty boundary and let R be a commutative ring. Then*

$$M \text{ is } R\text{-orientable} \quad \Longleftrightarrow \quad \text{the double } \mathrm{D}\,M \text{ is } R\text{-orientable.}$$

Furthermore, if M is equipped with an R-orientation, then $\mathrm{D}\,M$ admits a unique R-orientation such that the inclusion map $M \to \mathrm{D}\,M$ is orientation-preserving.

Sketch of proof (*). Let M be a connected topological manifold. In the following we write $\mathrm{D}\,M = (M \sqcup M')/\sim$ where M' is another copy of M. We apply the Topological Collar Neighborhood Theorem 44.5 to obtain a collar neighborhood $[-2, 0] \times \partial M$ of ∂M in M. To simplify the notation we henceforth drop the coefficient ring R from the notation.

If $\mathrm{D}\,M$ is orientable, then it follows immediately from Lemma 86.17 that the submanifold M is also orientable.

Now suppose that M is equipped with an orientation $\{\mu_x\}_{x \in M\setminus\partial M}$. We define orientations for the points of $\mathrm{D}\,M$ in $M \setminus \partial M$, $M' \setminus \partial M'$ and in $\partial M = \partial M'$ separately.

(1) Given $x \in M \setminus \partial M$ we denote by α_x the image of μ_x under the inclusion induced isomorphism $\mathrm{H}_n(M, M \setminus \{x\}) \to \mathrm{H}_n(\mathrm{D}\,M, \mathrm{D}\,M \setminus \{x\})$ given by the Excision Theorem 44.10.
(2) We denote by $r\colon \mathrm{D}\,M \to \mathrm{D}\,M$ the obvious homeomorphism given by interchanging the two copies of M in $\mathrm{D}\,M$. Given $x \in M' \setminus \partial M'$ we define $\alpha_x := -r_*^{-1}(\alpha_{r(x)})$.
(3) Finally let $x \in \partial M$. We define $\alpha_x \in \mathrm{H}_n(\mathrm{D}\,M, \mathrm{D}\,M \setminus \{x\})$ to be the image of the class $\alpha_{\{-1\}\times x} \in \mathrm{H}_n(\mathrm{D}\,M, \mathrm{D}\,M \setminus (\{-1\} \times x))$, which we defined in (1), under the

[1247]Since $f\colon U \to f(U)$ is a homeomorphism we have $f(U \setminus \{x\}) = f(U) \setminus \{f(x)\}$. Furthermore it follows from the fact that $x \in U \setminus \partial U$ and Proposition 44.2 that $f(x) \notin \partial N$.

isomorphism

$$\mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus \{(-1,x)\} \xleftarrow{\cong} \mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus ([-1,0] \times \{x\})) \xrightarrow{\cong} \mathrm{H}_n(\mathrm{D}M, \mathrm{D}M \setminus \overbrace{\{(0,x)\}}^{=x}).$$

these maps are isomorphisms by an excision argument together with Lemma 86.1

We leave it as a slightly annoying exercise to the indefatigable reader to show that $\{\alpha_x\}_{x \in \mathrm{D}M}$ does indeed define an orientation on $\mathrm{D}M$. By construction of the orientation on $\mathrm{D}M$ the inclusion $M \to \mathrm{D}M$ is orientation-preserving. The uniqueness of the orientation on $\mathrm{D}M$ is a consequence of Proposition 86.6 (1) and of our hypothesis that M is connected. ■

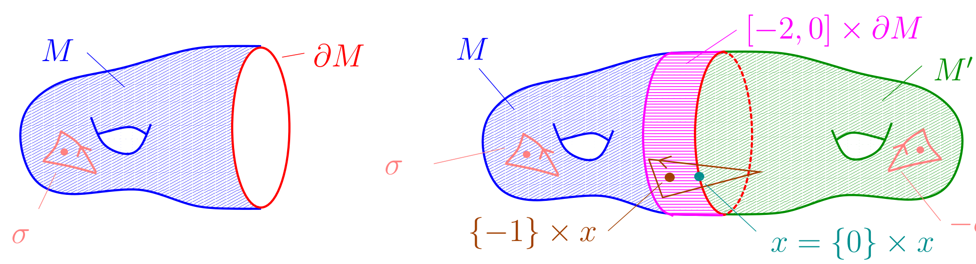


FIGURE 1221

Finally let M be an n-dimensional topological manifold equipped with an orientation $\{\mu_x\}_{x \in M \setminus \partial M}$. Now we will see that the given orientation induces an orientation on ∂M.

Proposition 86.19. *Let R be a commutative ring. Given any topological manifold M of dimension ≥ 2 there exists a natural map*

$$\{R\text{-orientations of } M\} \;\to\; \{R\text{-orientations of } \partial M\}.$$

Here natural means that if $f\colon M \to N$ is a homeomorphism, then the following diagram commutes:

$$\begin{array}{ccc} \{R\text{-orientations of } M\} & \longrightarrow & \{R\text{-orientations of } \partial M\} \\ \downarrow f_* & & \downarrow f_* \\ \{R\text{-orientations of } N\} & \longrightarrow & \{R\text{-orientations of } \partial N\}. \end{array}$$

Convention. If M is an R-oriented topological manifold of dimension ≥ 2, then invariably we will equip ∂M with the R-orientation provided by the construction in the proof of Proposition 86.19.

PROOF (∗). To simplify the notation we will only deal with case $R = \mathbb{Z}$, which is anyway the only case we care about. We suppose we are given an orientation $\{\mu_y\}_{y \in M \setminus \partial M}$ for M. In the following we will define a natural orientation for ∂M.

We set $n := \dim(M)$ and we write $A = [0,1] \times [-1,1]^n$, we write $A_0 := \{0\} \times [-1,1]^{n-1}$ and we write $A' = (0,1) \times (-1,1)^{n-1}$. Now let $x \in \partial M$. Since M is a topological manifold and since $x \in \partial M$ we can pick a chart $\Phi\colon U \to [0,2) \times (-2,2)^{n-1}$ with $\Phi(x) = (0,0)$. We write $\Psi := \Phi^{-1}$. Furthermore we write $V = \Psi(A)$, $V_0 = \Psi(A_0)$ and we write $V' = \Psi(A')$. Finally we pick a point $y \in V'$. (We refer to Figure 1222 for an illustration.)

We consider the following sequence of maps

$$\begin{tikzcd} \mathrm{H}_n(M, M\setminus\{y\}) & \mathrm{H}_n\left(M, M\setminus V'\right) \arrow[l, "(1)"'] \arrow[r, "\partial", "(2)"'] & \mathrm{H}_{n-1}(M\setminus V', (M\setminus V')\setminus\{x\}) & \\ & & \mathrm{H}_{n-1}(V_0, V_0\setminus\{x\}) \arrow[u, "(3)"] \arrow[r, "(4)"'] & \mathrm{H}_{n-1}(\partial M, \partial M\setminus\{x\}). \arrow[ul, "(5)"'] \end{tikzcd}$$

Here all the maps are induced by the obvious inclusions of pairs of topological spaces, except for the second map which is the connecting homomorphism of the long exact sequence of the triple $(M, M \setminus V', (M\setminus V')\setminus\{x\})$ that is provided by Proposition 43.15.

Claim. All the maps in the above diagram are isomorphisms.

In the following we treat the four maps separately.

(1) It is elementary to see that $A \setminus \Psi^{-1}(y)$ is a deformation retract of ∂A. It follows that $M \setminus V'$ is a deformation retract of $M \setminus \{y\}$.[1248] We obtain from Corollary 43.18 that (1) an isomorphism.
(2) We consider the long exact sequence of the triple $(M, M \setminus V', (M\setminus V')\setminus\{x\})$ that is provided by Proposition 43.15:
$$\begin{array}{rl} \mathrm{H}_n(M, (M \setminus V') \setminus \{x\}) & \to \mathrm{H}_n(M, M \setminus V') \xrightarrow{\partial} \mathrm{H}_{n-1}(M \setminus V', M \setminus V) \to \\ \to \mathrm{H}_{n-1}(M, (M \setminus V') \setminus \{x\}). & \end{array}$$
Similarly to (1) one sees that $(M \setminus V')\setminus\{x\}$ is a deformation retract of M. Therefore it follows from Corollary 43.18 that the groups at the two extremes are zero. In other words, the connecting homomorphism is an isomorphism.
(3) It follows quite easily from the Excision Theorem 43.20 that (3) is an isomorphism.
(4) Replace (3) by (4) in the previous sentence.
(5) The triangle is given by inclusion induced maps, hence it commutes. It now follows from (3) and (4) that (5) is also an isomorphism. ⊞

Given $x \in \partial M$ we now define $\nu_x \in \mathrm{H}_{n-1}(\partial M, \partial M\setminus\{x\})$ to be the image of the generator $\mu_y \in \mathrm{H}_n(M, M\setminus\{y\})$ under the above sequence of isomorphisms. We continue with our next claim.

Claim.

(1) The definition of ν_x is independent of the choice of y.
(2) The definition of ν_x is independent of the choice of Φ.

[1248] Looking at Figure 1222 this statement sounds reasonable. But the statement nonetheless requires a proof. So let r be a deformation retraction from $A \setminus \Psi^{-1}(y)$ to ∂A. We consider the map

$$\begin{array}{rcl} s\colon (M \setminus \{y\}) \times [0,1] & \to & M \setminus \{y\} \\ (P,t) & \mapsto & \begin{cases} P, & \text{if } P \notin V \setminus \{y\}, \\ \Psi(r(\Psi^{-1}(P),t)), & \text{if } P \in V \setminus \{y\}. \end{cases} \end{array}$$

Note that $V \setminus \{y\}$ and $M \setminus \Psi(A')$ are closed subsets of $M \setminus \{y\}$. (Hmm, why is that?). It follows from Lemma 3.10 that the map s is continuous.

We turn to the proof of the two statements.

(1) We need to show that for any $z \in \Psi(V')$ we have

$$\begin{array}{c}\text{preimage of } \mu_y \text{ under the map}\\ \mathrm{H}_n(M, M\setminus V') \xrightarrow{\cong} \mathrm{H}_n(M, M\setminus\{y\})\end{array} = \begin{array}{c}\text{preimage of } \mu_z \text{ under the map}\\ \mathrm{H}_n(M, M\setminus V') \xrightarrow{\cong} \mathrm{H}_n(M, M\setminus\{z\}).\end{array}$$

But this statement can be shown in a similar fashion as Proposition 86.6 (1). We leave it to the reader to fill in the details.

(2) Suppose we are given a different chart than Φ. We define W, W_0, W' in the obvious way. By possibly taking a third chart whose image lies in the intersection of V and W we can assume, without loss of generality, that $W \subset V$. (We refer to Figure 1223 for an illustration.) Furthermore by (1) we can work with a point $y \in W$ for both charts. We consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_n(M, M\setminus\{y\}) & \mathrm{H}_n\big(M, M\setminus V'\big) \arrow[l, "\cong"', "(1)"] \arrow[d, "(2)"', "\partial"] \arrow[dl, dashed] \\
\mathrm{H}_n\big(M, M\setminus W'\big) \arrow[u, "\cong", "(1)"'] \arrow[d, "(2)"', "\partial"] & \mathrm{H}_{n-1}(M\setminus V', (M\setminus V')\setminus\{x\}) \arrow[dl, dashed] \\
\mathrm{H}_{n-1}(M\setminus W', (M\setminus W')\setminus\{x\}) & \mathrm{H}_{n-1}(V_0, V_0\setminus\{x\}) \arrow[u, "(3)", "\cong"'] \arrow[d, "(4)"'] \\
\mathrm{H}_{n-1}(W_0, W_0\setminus\{x\}) \arrow[u, "(3)", "\cong"'] \arrow[ur, dashed] \arrow[r, "(4)"'] & \mathrm{H}_{n-1}(\partial M, \partial M\setminus\{x\}).
\end{tikzcd}$$

Here the two "outer paths" from the top left to the lower right correspond to the two "competing" definitions of ν_x. The dashed arrows are given by inclusions. We make the following observations:

(a) The triangle on top commutes since all maps are induced by inclusions.

(b) The upper parallelogram commutes since the connecting homomorphism is natural, see Proposition 43.15 (3) for details.

(c) The lower parallelogram commutes since all the maps are induced by inclusions. (Note that the reversal of the direction of the arrow is not a typo.)

(d) The triangle on the bottom commutes since all maps are induced by inclusions.

We saw above that all maps (1), (2), (3) and (4) are isomorphisms. By going from the top to the bottom we see that all the dashed arrows are isomorphisms.

Since all pieces of the diagram commute and since all maps are isomorphisms we see that the total diagram commutes. This shows that the two "'competing" definitions of ν_x agree. ⊞

Using the previous claim it is now straightforward to show that the $\{\nu_x\}_{x\in\partial M}$ form an orientation for ∂M and that this orientation is natural. ■

Remark. Let M be an n-dimensional smooth manifold that is equipped with an orientation in the sense of Section 6.11. On page 303 we had equipped the $(n-1)$-dimensional smooth manifold ∂M with an orientation in the sense of Section 6.11. If one takes the

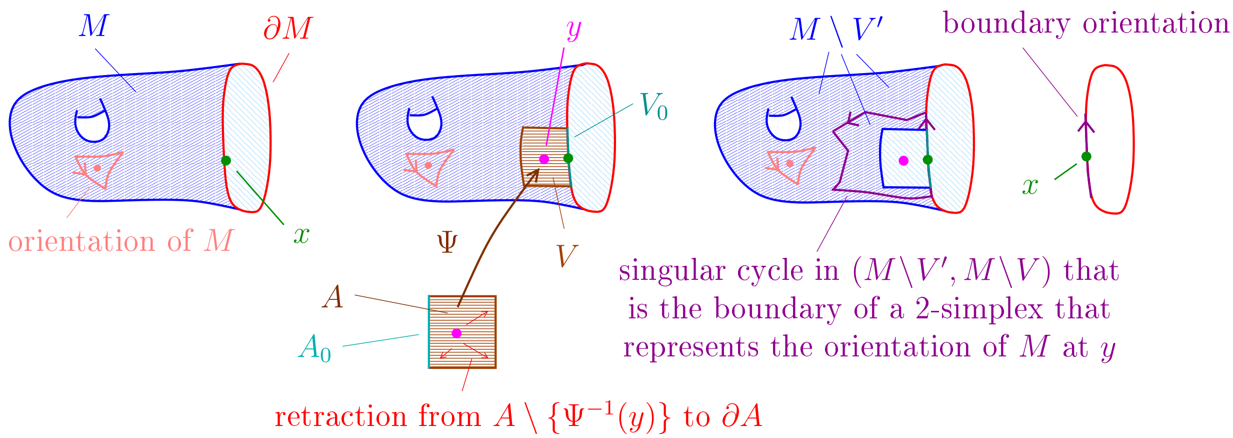


FIGURE 1222

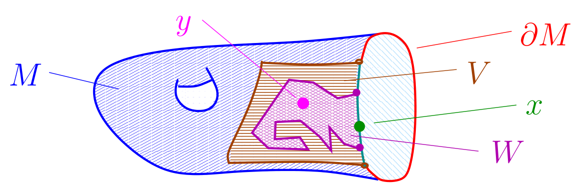


FIGURE 1223

strenuous journey through all conventions and definitions one sees that the boundary orientations from page 303 and the proof of Proposition 86.19 are compatible. More precisely, for any oriented n-dimensional M we have the following commutative "diagram"

$$\begin{tikzcd}
\text{oriented smooth manifold } M \arrow[r, mapsto, "\text{take boundary}"] \arrow[d, mapsto, "\text{Proposition 86.11}"] & \text{oriented smooth manifold } \partial M \arrow[d, mapsto, "\text{Proposition 86.11}"] \\
\mathbb{Z}\text{-oriented smooth manifold } M \arrow[r, mapsto, "\text{take boundary}"] & \mathbb{Z}\text{-oriented smooth manifold } \partial M.
\end{tikzcd}$$

Remark. Given topological manifolds M and N we can form the product $M \times N$. If M and N are oriented, then one would expect that $M \times N$ comes with a natural orientation. This is indeed the case, but the definition of this natural orientation requires more tools than we have at the moment. Thus we postpone the definition of the product orientation to Section 80.5.

Exercises for Chapter 86.

Exercise 86.1. We view $\mathbb{R}$ as a topological manifold and we equip it with the standard $\mathbb{Z}$-orientation that we had introduced on page 2106. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a map that is a local homeomorphism and that is orientation-preserving. Show that f is strictly monotonously increasing.

Exercise 86.2. Let M be an n-dimensional topological manifold, let R be a commutative ring and let $A \subset M \setminus \partial M$ be a subset.

(a) Show that for R-sections $\{\alpha_x\}_{x\in A}$ and $\{\beta_x\}_{x\in A}$ along A the collection $\{\alpha_x+\beta_x\}_{x\in A}$ is also an R-section along A.

(b) Show that the set $\Gamma_R^A(M)$ admits a unique R-module structure such that for each $y\in A$ the map $\Gamma_R^A(M)\to \mathrm{H}_n(M,M\setminus\{y\};R)$ given by $\{\alpha_x\}_{x\in A}\mapsto\alpha_y$ is an R-module homomorphism.

Exercise 86.3. Let M be an oriented n-dimensional topological manifold. Furthermore let $f,g\colon M\to M$ be two homeomorphisms. Show that if f is orientation-preserving and if g is homeotopic to f, then g is also orientation-preserving.
Remark. In Lemma 8.6 we proved the analogue in the setting of smooth manifolds.

Exercise 86.4. Let M be an oriented n-dimensional topological manifold, let A and B be disjoint unions of boundary components of M and let $f\colon A\to B$ be a homeomorphism. We equip A and B with the boundary orientations, as defined in Proposition 86.19.

(a) If f is orientation-reversing, then $M/a\sim f(a)$ admits an orientation such that the inclusion $M\setminus(A\cup B)\to M/a\sim f(a)$ is orientation-preserving.

(b) If M is connected and if f is not orientation-reversing, then $M/a\sim f(a)$ is non-orientable.

Remark. This is the long overdue proof of Proposition 44.8 (6).

87. The fundamental class of topological manifolds

In several previous chapters, e.g. in Chapter 64 and 66, we have assembled lots of information on homology groups of smooth manifolds. Let us recall some of the results. Thus let M be a connected n-dimensional smooth manifold.

(1) In Proposition 64.6 (2) we saw, using Theorem 64.5, that $\mathrm{H}_i(M;\mathbb{Z}) = 0$ for $i > n$.
(2) Furthermore, in Theorem 66.6 and Theorem 66.8 we showed that

$$\mathrm{H}_n(M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ closed and orientable,} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathrm{H}_n(M,\partial M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ compact and orientable,} \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if M is compact and orientable, then we used (2) to define the corresponding fundamental class $[M] \in \mathrm{H}_n(M,\partial M;\mathbb{Z})$ which turned out to be very useful tool.

The key tool in proving the above statements was the fact, established in Theorem 64.2, that smooth manifolds admit simplicial structures. As we discussed in Section 85.6, in general topological manifolds do not admit simplicial structures, so if we want to replicate the above results for topological manifolds we will have to use a very different approach.

In this chapter we will indeed manage to reprove all of the above statements for topological manifolds. The advantage of the new approach is that it is more general, it covers topological manifolds, and that it does not require any extra topological input, i.e. it does not require the existence of a simplicial or cellular structure. The flip side of this chapters approach is that the proofs are arguably less intuitive.

87.1. **The fundamental class of an oriented topological manifold.** We recall the following notation that we introduced on page 2119.

Notation. Let M be a topological manifold, let R be a commutative ring and finally let $x \in M \setminus \partial M$. Given any $k \in \mathbb{N}_0$ we denote by ρ_x the obvious map

$$\rho_x\colon\ \mathrm{H}_k(M,\partial M;R) \ \to\ \mathrm{H}_k(M,M\setminus\{x\};R).$$

Given $\alpha \in \mathrm{H}_k(M,\partial M;R)$ we often write $\alpha_x := \rho_x(\alpha)$, i.e. the above map is also written as

$$\alpha \ \mapsto\ \alpha_x.$$

The following three theorems contain most of the main results of this chapter.

Theorem 87.1. *Let M be a compact connected non-empty n-dimensional topological manifold. Then the following statements are equivalent:*

(1) *the topological manifold M is orientable,*
(2) $\mathrm{H}_n(M,\partial M;\mathbb{Z}) \neq 0$,
(3) $\mathrm{H}_n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$,
(4) *for every $x \in M \setminus \partial M$ the map*

$$\begin{aligned} \rho_x\colon\ \mathrm{H}_n(M,\partial M;\mathbb{Z}) &\ \to\ \mathrm{H}_n(M,M\setminus\{x\};\mathbb{Z}) \\ \alpha &\ \mapsto\ \alpha_x \end{aligned}$$

is an isomorphism.

Theorem 87.2. *Let M be a compact connected non-empty n-dimensional topological manifold M. Then the following two statements hold:*

(1) $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$, *and*

(2) *for every $x \in M \setminus \partial M$ the map*

$$\begin{array}{rcl} \rho_x \colon \mathrm{H}_n(M, \partial M; \mathbb{F}_2) & \to & \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2) \\ \alpha & \mapsto & \alpha_x \end{array}$$

is an isomorphism.

Theorem 87.3. *Let M be a connected n-dimensional topological manifold and let R be a commutative ring. Then the following two statements hold:*

(1) *If M is not closed, then $\mathrm{H}_i(M; R) = 0$ for $i \geq n$.*

(2) *If M is closed, then we have $\mathrm{H}_i(M; R) = 0$ for $i > n$.*

Remark.

(1) As mentioned before, the above theorems generalize most of the results of Proposition 64.6 (2), Theorem 66.6 and Theorem 66.8 from smooth manifolds to topological manifolds.

(2) One might be tempted to consider Theorem 87.3 (2) as "intuitively clear". After all, how could an n-dimensional object possibly have homology groups in higher dimensions? To shatter this belief, fix some $n \geq 2$. We consider the following subset of $\mathbb{R}^n$:[1249]

$$\begin{aligned} X &:= \bigcup_{n \in \mathbb{N}} n\text{-dimensional sphere of radius } \tfrac{1}{n} \text{ around } (\tfrac{1}{n}, 0, \dots, 0) \\ &= \bigcup_{n \in \mathbb{N}} \{x \in \mathbb{R}^n \mid \|x - (\tfrac{1}{n}, 0, \dots, 0)\| = \tfrac{1}{n^2}\}. \end{aligned}$$

In **[BaM62]** it is shown, rather shockingly, that for every q with $q \equiv 1 \bmod (r-1)$ the group $\mathrm{H}_q(X; \mathbb{Q})$ is not only non-zero, it is even uncountable.

We will not immediately prove these three theorems, but we will first draw many conclusions, before we head towards the somewhat intricate proofs of the above three theorems.

Definition. Let M be a compact n-dimensional topological manifold and let R be a commutative ring. An *R-fundamental class of M* is an element $\beta \in \mathrm{H}_n(M, \partial M; R)$ with the property that for any $x \in M \setminus \partial M$ the element $\beta_x \in \mathrm{H}_n(M, M \setminus \{x\}; R)$ is a generator. For $R = \mathbb{Z}$ we just say *fundamental class*.

Remark. Let M be a compact n-dimensional topological manifold and let R be a commutative ring.

(1) Let $[M]$ be a $\mathbb{Z}$-fundamental class for M. The image of $[M] \otimes 1$ under the natural map $\mu \colon \mathrm{H}_n(M; \mathbb{Z}) \otimes R \to \mathrm{H}_n(M; R)$ is an R-fundamental class. Indeed, let $x \in M \setminus \partial M$.

[1249] As the reader will surely have noticed, this example can be viewed as a high-dimensional analogue of the Hawaiian Earings that we introduced in Exercise 25.1.

From Lemma 57.7 (4) we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{H}_n(M;\mathbb{Z}) \otimes R \arrow[r, "\mu"] \arrow[d] & \mathrm{H}_n(M;R) \arrow[d] \\
\mathrm{H}_n(M, M \setminus \{x\};\mathbb{Z}) \otimes R \arrow[r, "\mu"] & \mathrm{H}_n(M, M\setminus\{x\};R)
\end{tikzcd}$$

By Lemma 86.4 and the Universal Coefficient Theorem 57.19 the bottom horizontal map is an isomorphism. Since the image of the $\mathbb{Z}$-fundamental class $[M]$ under the obvious map $\mathrm{H}_n(M;\mathbb{Z}) \to \mathrm{H}_n(M, M\setminus\{x\};\mathbb{Z})$ is a generator it follows from the commutative diagram that the image of $\mu([M]\otimes 1)$ under the obvious map $\mathrm{H}_n(M;R) \to \mathrm{H}_n(M, M\setminus\{x\};R)$ is indeed a generator.

(2) If M has connected components $M_1, \dots, M_k$ and if for $i = 1, \dots, k$ we are given R-fundamental classes $[M_i]_R \in \mathrm{H}_n(M_i, \partial M_i; R)$, then

$$\sum_{i=1}^{k} [M_i]_R \in \mathrm{H}_n(M, \partial M; R) = \bigoplus_{i=1}^{k} \mathrm{H}_n(M_i, \partial M_i; R)$$

is easily seen to be a R-fundamental class of M. The fundamental classes for the M_i are unique if and only if the fundamental class for M is unique.

Example. Let M be the empty topological space. As we mentioned on page 264, basically by definition the empty set is a topological manifold of any dimension. Note that for the empty manifold $\varnothing$ we have the *equality* $\mathrm{H}_0(\varnothing) = \mathrm{H}_1(\varnothing) = \mathrm{H}_2(\varnothing) = \cdots = \{0\}$. By definition this element 0 is a fundamental class of the empty manifold and it is evidently unique. We write $[\varnothing] := 0$. This is consistent with the discussion on page 1719.

The following corollary to Theorem 87.1 says in particular that an orientation determines and is in turn uniquely determined by a fundamental class.

Corollary 87.4. *Let M be a compact n-dimensional topological manifold.*

(1) *If $[M]$ is a fundamental class, then $\{[M]_x\}_{x\in M\setminus\partial M}$ defines an orientation for M.*
(2) *If $\{\alpha_x\}_{x\in M\setminus\partial M}$ is an orientation for M, then there exists a unique fundamental class $[M] \in \mathrm{H}_n(M,\partial M;\mathbb{Z})$ with $[M]_x = \alpha_x$ for all $x \in M\setminus\partial M$.*

Furthermore, if M is connected, non-empty and orientable, then the following two statements hold:

(3) *Every fundamental class $[M] \in \mathrm{H}_n(M,\partial M;\mathbb{Z})$ is a generator of $\mathrm{H}_n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$.*
(4) *Every generator of $\mathrm{H}_n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$[1250] is a fundamental class.*

Proof. Let M be a compact non-empty n-dimensional topological manifold. By considering the components of M separately we see that without loss of generality we can assume that M is connected. It is also clear that we only have to study the case that M is non-empty.

(1) Let $[M]$ be a fundamental class for the topological manifold M. It follows from Lemma 86.5 (2) that the homology classes $\{\rho_x([M])\}_{x\in M\setminus\partial M}$ form a section on M.

[1250] Here we use that in Theorem 87.1 we saw that for a compact orientable, non-empty n-dimensional topological manifold we have $\mathrm{H}_n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$.

By definition of a fundamental class each $\rho_x([M])$ is a generator. Therefore the section $\{\rho_x([M])\}_{x\in M\setminus\partial M}$ is indeed an orientation.

(2) Suppose we are given an orientation $\{\alpha_x\}_{x\in M\setminus\partial M}$ for M. By Theorem 87.1 (1) $\Rightarrow$ (4) we know that for every $x\in M\setminus\partial M$ the map

$$\rho_x\colon\ \mathrm{H}_n(M,\partial M;\mathbb{Z})\ \to\ \mathrm{H}_n(M,M\setminus\{x\};\mathbb{Z})$$

is an isomorphism. Now pick a $y\in M$. We write $[M]:=\rho_y^{-1}(\alpha_y)$. Since ρ_y is an isomorphism the class $[M]$ is uniquely determined. So it remains to show that $[M]_x=\alpha_x$ for all $x\in M\setminus\partial M$. By Lemma 86.5 (2) the classes $\{[M]_x\}_{x\in M\setminus\partial M}$ form a $\mathbb{Z}$-section. It follows from Proposition 86.6 that $[M]_x=\alpha_x$ for all $x\in M\setminus\partial M$.

(3) This statement follows immediately from the definition of a fundamental class and Theorem 87.1 (1) $\Rightarrow$ (4).

(4) The last statement is an immediate consequence of Lemma 86.5 (2) and Theorem 87.1 (1) $\Rightarrow$ (4). ■

We also have the following corollary to Theorem 87.2 which deals with $\mathbb{F}_2$-coefficients.

Corollary 87.5. *Let M be a compact, non-empty n-dimensional topological manifold.*

(1) *If M is connected, then*[1251]

$$[M]_{\mathbb{F}_2}\ :=\ \textit{the unique non-zero element of } \mathrm{H}_n(M,\partial M;\mathbb{F}_2)\cong\mathbb{F}_2$$

is the unique $\mathbb{F}_2$-fundamental class of M.

(2) *If M has connected components $M_1,\dots,M_k$, then*

$$[M]_{\mathbb{F}_2}\ :=\ \sum_{i=1}^{k}[M_i]_{\mathbb{F}_2}\ \in\ \mathrm{H}_n(M,\partial M;\mathbb{F}_2)=\bigoplus_{i=1}^{k}\mathrm{H}_n(M_i,\partial M_i;\mathbb{F}_2)$$

is the unique $\mathbb{F}_2$-fundamental class of M.

Proof. It follows immediately from Theorem 87.2 (2) that the class $[M]_{\mathbb{F}_2}$ that we had just defined in (1) is indeed an $\mathbb{F}_2$-fundamental class for M and that it is unique. As we pointed out on page 2121, the second statement is an immediate consequence of the first statement. ■

Remark. Let R be a commutative ring. We consider the empty topological manifold $\varnothing$. Note that we have the equality of sets $\mathrm{H}_0(\varnothing;R)=\mathrm{H}_1(\varnothing;R)=\mathrm{H}_2(\varnothing;R)=\cdots=\{0\}$. As on page 1719 we now define the fundamental class $[\varnothing]$ of the empty topological manifold $\varnothing$ to be the unique element in the group $\mathrm{H}_0(\varnothing;R)=\mathrm{H}_1(\varnothing;R)=\mathrm{H}_2(\varnothing;R)=\cdots=\{0\}$.

Let M be a compact non-empty n-dimensional *smooth* manifold. On page 1730 we introduced the corresponding $\mathbb{F}_2$-fundamental class $[M]_{\mathbb{F}_2}\in\mathrm{H}_n(M,\partial M;\mathbb{F}_2)$. Furthermore, if M is in fact oriented, then on page 1712 we introduced the corresponding fundamental class $[M]\in\mathrm{H}_n(M,\partial M;\mathbb{Z})$. We will now see that the previous definitions are consistent with the definitions from this chapter.

First let us consider the case of $\mathbb{F}_2$-fundamental classes. This case is easy: it follows basically immediately from Corollary 87.5 (1) that the two definitions actually agree.

[1251] Here in the formulation we already use Theorem 87.2 (1) to guarantee that $\mathrm{H}_n(M,\partial M;\mathbb{F}_2)\cong\mathbb{F}_2$.

Now let us assume that the smooth manifold M is oriented in the sense of Section 6.11. In Theorem 68.1 we defined the fundamental class $[M]$ in $\mathrm{H}_n(M, \partial M; \mathbb{Z})$. On the other hand, in Proposition 86.11 we saw that the orientation of M also determines a $\mathbb{Z}$-orientation and thus we obtain a fundamental class $[M] \in \mathrm{H}_n(M, \partial M; \mathbb{Z})$ by Corollary 87.4. Not surprisingly we will now see that these two definitions agree.

Lemma 87.6. *The two definitions of the fundamental class of a compact oriented non-empty smooth manifold agree.*

Remark. Let M be a compact oriented non-empty smooth manifold. In Algorithm 68.2 and Proposition 68.4 we developed some techniques for finding explicit cycles representing the fundamental class of the oriented smooth manifold M. By Lemma 87.6 these approaches also work for finding explicit cycles representing the fundamental class of a smooth manifold that is equipped with a $\mathbb{Z}$-orientation.

PROOF ($*$). Let M be a compact n-dimensional smooth manifold that is oriented in the sense of Section 6.11. The case $n = 0$ is basically trivial. Hence we now only consider the case $n \geq 1$. As we will see, the lemma follows almost immediately from the definitions.

Now let us first recall the construction, in the proof of Proposition 86.11, of the corresponding $\mathbb{Z}$-orientation of M. Let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to V$ around x that is orientation-preserving in the sense of Section . We define

$$\mu_x \;:=\; \Phi_*^{-1}\big(\text{standard generator of } \mathrm{H}_n(V, V \setminus \{\Phi(x)\}; \mathbb{Z}), \text{ as defined on page 2097}\big).$$

In the proof of Proposition 86.11 we saw this definition of μ_x does not depend on the choice of the orientation-preserving chart. Furthermore we saw these classes $\{\mu_x\}_{x \in M \setminus \partial M}$ define a $\mathbb{Z}$-orientation for M.

We denote by $\alpha \in \mathrm{H}_n(M; \mathbb{Z})$ the fundamental class of Theorem 68.1. By definition of the fundamental class from Corollary 87.4 it now remains to prove the following claim.

Claim. Given any $x \in M \setminus \partial M$ we have $\alpha_x = \mu_x \in \mathrm{H}_n(M, M \setminus \{x\})$.

Let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to V$ around x with $\Phi(x) = 0$ that is orientation-preserving. Furthermore let $\Theta\colon \Delta^n \to V$ be an affine linear map such that 0 lies in the image of $\Theta(\mathring{\Delta}^n)$ and which has the property that $\Theta\colon \mathring{\Delta}^n \to V$ is orientation-preserving. We see that

$$\alpha_x \underset{\substack{\uparrow \\ \text{follows from Theorem 68.1 and the fact that } \Phi^{-1}\circ\Theta \text{ is orientation-preserving}}}{=} [\Phi^{-1} \circ \Theta\colon \Delta^n \to M] \overset{\substack{[\Theta\colon \Delta^n \to V] \text{ is by definition a standard generator of } \mathrm{H}_n(V, V\setminus\{0\};\mathbb{Z}) \\ \downarrow}}{=} \Phi_*^{-1}([\text{standard generator of } \mathrm{H}_n(V, V \setminus \{0\}; \mathbb{Z}))] \;=\; \mu_x.$$

■

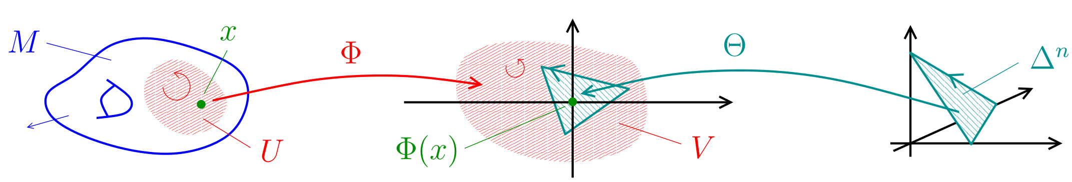


FIGURE 1224. Illustration for the proof of Lemma 87.6.

We continue with the following two propositions which can be viewed as a generalizations of Propositions 68.4 and 68.18.

Proposition 87.7. *Let M be a compact oriented connected n-dimensional topological manifold. Let $r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m \in \mathrm{C}_n(M, \partial M)$ be a cycle. Suppose there exists a $j \in \{1, \ldots, m\}$ such that the following three conditions are satisfied:*

(1) *The map $\sigma_j \colon \mathring{\Delta}^n \to M$ is an orientation-preserving embedding.*[1252]
(2) *There exists an $x \in \sigma_j(\Delta^n) \setminus \sigma_j(\partial \Delta^n)$ which is not contained in the image of any other $\sigma_i \colon \Delta^n \to M$.*
(3) *$r_j = 1$.*

Then the cycle $r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m$ represents the fundamental class of the oriented topological manifold.

Proposition 87.8. *Let M be a compact connected n-dimensional topological manifold. Let $(\sigma_1 + \cdots + \sigma_m) \otimes 1 \in \mathrm{C}_n(M, \partial M; \mathbb{F}_2) = \mathrm{C}_n(M, \partial M) \otimes \mathbb{F}_2$ be a cycle. Suppose there exists a $j \in \{1, \ldots, m\}$ such that the following two conditions are satisfied:*

(1) *The map $\sigma_j \colon \mathring{\Delta}^n \to M$ is an embedding.*
(2) *There exists an $x \in \sigma_j(\Delta^n) \setminus \sigma_j(\partial \Delta^n)$ which is not contained in the image of any other $\sigma_i \colon \Delta^n \to M$.*

Then the cycle $(\sigma_1 + \cdots + \sigma_m) \otimes 1$ represents the $\mathbb{F}_2$-fundamental class of M.

Proof. We leave it to the reader to modify the proofs of Propositions 68.4 and 68.18 to obtain the above result. ■

For the readers convenience we recall the discussion of cycles representing fundamental classes of the torus and the surface of genus 2 which we already gave in Figures 1054 and 1055. There is always the outside chance that a reader decided to give Chapter 68 a pass.

Example. We will give some examples of fundamental classes of 2-dimensional topological manifolds. We use the notation from Figure 1225 to indicate a singular simplex. In Figure 1226 we now show cycles representing the fundamental classes of the torus

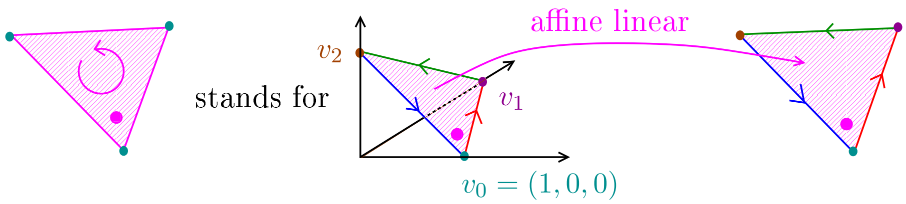


Figure 1225

and the surface of genus 2 and we show a cycle representing the $\mathbb{F}_2$-fundamental class of $\mathbb{RP}^2 = \overline{B}^2 / z \sim -z$.

[1252] We equip Δ^n, and thus also its submanifold $\mathring{\Delta}^n$, with the $\mathbb{Z}$-orientation which is given by considering the orientation given by $[\mathrm{id}_{\Delta^n}] \in \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{x\})$ for each $x \in \mathring{\Delta}^n$.

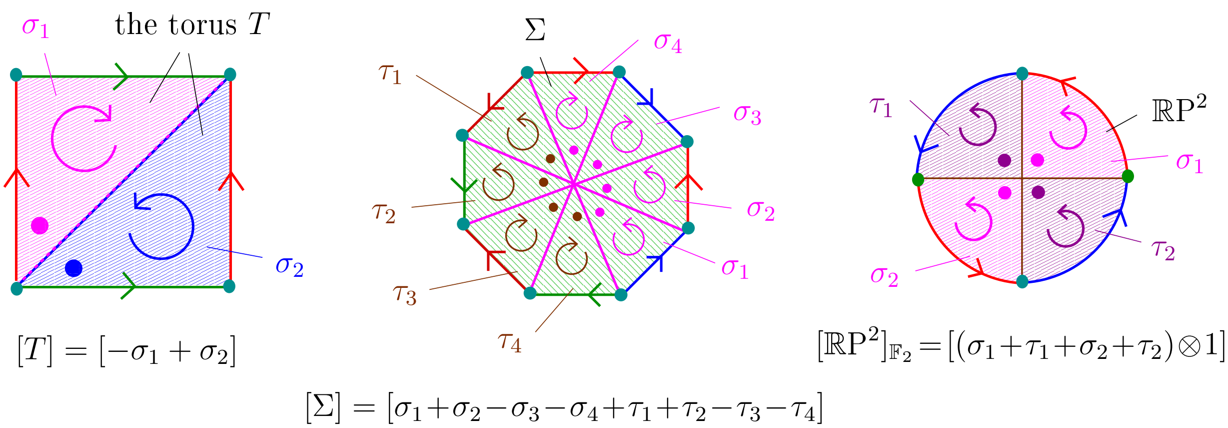


FIGURE 1226

87.2. **Lifts of sections on topological manifolds.** In this section we will prove the fundamental theorem of topological manifolds which lies at the heart of the proofs of Theorems 87.1, 87.2 and 87.3. Before we can state the lemma we introduce the following helpful notation.

Notation. Let M be a topological manifold without boundary, let R be a commutative ring and let $A \subset M$ be a subset.

(1) We write $\mathrm{H}_n(M|A;R) := \mathrm{H}_n(M, M\setminus A; R)$. Given a point $x \in M$ we use the obvious abbreviation $\mathrm{H}_n(M|x;R) := \mathrm{H}_n(M|\{x\};R)$.
(2) Given a homology class $\alpha \in \mathrm{H}_n(M|A;R)$ and a point $x \in A$ we denote by α_x the image of α under the obvious map $\mathrm{H}_n(M|A;R) \to \mathrm{H}_n(M|x;R)$.
(3) Let $\{\alpha_x\}_{x\in M}$ be an R-section on M and let $A \subset M$ be a subset. We call a class $\beta \in \mathrm{H}_n(M|A;R)$ a *lift to* A if for all $x \in A$ we have $\beta_x = \alpha_x$.
(4) As on page 2101 we denote by $\Gamma_R^A(M)$ the set of all R-sections of M along A.

The following lemma follows immediately from the definitions. But since we will refer to it on several occasions it is good to write it down.

Lemma 87.9. *Let M be a topological manifold without boundary, let R be a commutative ring and let $A \subset B \subset M$ be subsets:*

(1) *Since $M\setminus B \subset M\setminus A$ we have an inclusion induced map* $\mathrm{H}_n(M|B;R) \to \mathrm{H}_n(M|A;R)$.
(2) *Let $\{\alpha_x\}_{x\in M}$ be an R-section on M. If $\mu_B \in \mathrm{H}_n(M|B;R)$ is a lift to B, then the image of μ_B under the above map $\mathrm{H}_n(M|B;R) \to \mathrm{H}_n(M|A;R)$ is a lift of the R-section to the subset A.*

The following theorem is the main technical result of this chapter.

Theorem 87.10. *Let M be an n-dimensional topological manifold with empty boundary and let $A \subset M$ be a compact subset. Let R be a commutative ring. Then the following hold:*

(i) *The map*
$$\begin{array}{rcl} \mathrm{H}_n(M|A;R) & \to & \Gamma_M^A(R) \\ \varphi & \mapsto & \{\varphi_x\}_{x\in A} \end{array}$$
is an isomorphism. In other words, if $\{\alpha_x\}_{x\in A}$ is an R-section for M along A, then there exists a unique lift $\alpha_A^M = \alpha_A$ to A, i.e. there exists a unique homology class $\alpha_A^M = \alpha_A \in \mathrm{H}_n(M|A;R)$ such that for each $x \in A$ we have $\alpha_{A,x} = \alpha_x$.

(ii) *For $i > n$ we have $\mathrm{H}_i(M|A;R) = 0$.*

Remark. To avoid any nervousness later on we remark that the statements (i) and (ii) hold trivially for $A = \varnothing$ since $\mathrm{H}_n(M|\varnothing;R) = \mathrm{H}_n(M,M;R) = 0$.

Corollary 87.11. *Let M be a n-dimensional topological manifold with empty boundary, let R be a commutative ring and let $A \subset M$ be a compact connected subset. If M is R-orientable, then for every $x \in A$ the map*
$$\mathrm{H}_n(M, M\setminus A;R) \ \to\ \mathrm{H}_n(M, M\setminus\{x\};R) \ \underset{\substack{\uparrow \\ \text{isomorphic by Lemma 86.4}}}{\cong}\ R$$
is an isomorphism.

Proof. We consider the following diagram:

$$\begin{tikzcd} \mathrm{H}_n(M,M\setminus A;R) \arrow[rr, "\varphi\mapsto\varphi_x"] \arrow[dr, "\varphi\mapsto\{\varphi_x\}_{x\in A}"'] & & \mathrm{H}_n(M,M\setminus\{x\};R) \\ & \Gamma_M^A(R) \arrow[ur, "\{\varphi_x\}_{x\in A}\mapsto\varphi_x"'] & \end{tikzcd}$$

The diagram commutes by definition. Since A is compact we know by Theorem 87.10 (i) that the left diagonal map is an isomorphism. Furthermore, since A is connected we obtain from Lemma 86.9 that the right diagonal map is an isomorphism. Thus we see that the horizontal map is an isomorphism, as promised. ■

For later on we also record the following corollary to Theorem 87.10.

Corollary 87.12. *Let M be an oriented n-dimensional topological manifold with empty boundary and let R be a commutative ring. We suppose that M is equipped with an R-orientation. We use the notation from Theorem 87.10.*

(1) *Let $K \subset L \subset M$ be compact subsets. We denote by $f_{KL}\colon (M, M\setminus L) \to (M, M\setminus K)$ the inclusion map of pairs. The map $f_{KL*}\colon \mathrm{H}_n(M, M\setminus L) \to \mathrm{H}_n(M, M\setminus K)$ sends α_L^M to α_K^M.*

(2) *Let U be an open subset of M and let $A \subset U$ be a compact subset. We use Lemma 86.17 to view U as an oriented topological manifold. If $i\colon U \to M$ denotes the inclusion map, then $i_*(\alpha_A^U) = \alpha_A^M$.*

PROOF. The lemma follows easily from the functoriality of homology groups and the uniqueness statement of Theorem 87.10. ■

Now we turn to the proof of Theorem 87.10. The very rough idea behind the proof of the fundamental theorem of topological manifolds can be summarized as follows:

(1) Using Mayer–Vietoris sequences we want to show that if the compact subsets A, B and $A \cap B$ have the desired properties, then so does $A \cup B$.
(2) Using charts we want to reduce the proof of the theorem to the case that $M = \mathbb{R}^n$.
(3) Finally for $M = \mathbb{R}^n$ we want to reduce the proof of the theorem to compact convex subsets A of $\mathbb{R}^n$.

The following lemmas give us the tools to put this idea into practice.

Lemma 87.13. *Let M be an n-dimensional topological manifold, let R be a commutative ring and let A and B be two compact subsets of M. If the statements* (i) *and* (ii) *from Theorem 87.10 hold for A, B and $A \cap B$, then they also hold for $A \cup B$.*

PROOF. Let M be an n-dimensional topological manifold, let R be a commutative ring and let A and B be two compact subsets of M such that A, B and $A \cap B$ satisfy statements (i) and (ii). To simplify the notation we drop R from the notation. We consider the inverted Mayer–Vietoris Sequence from Theorem 46.7

$$\cdots \to \overbrace{\mathrm{H}_{n+1}(M|A\cap B)}^{=0,\text{ by hypothesis on } A\cap B} \to \mathrm{H}_n(M|A\cup B) \xrightarrow{\Phi} \mathrm{H}_n(M|A)\oplus \mathrm{H}_n(M|B) \xrightarrow{\Psi} \mathrm{H}_n(M|A\cap B) \to \dots$$
$$\alpha \mapsto (\alpha, -\alpha) \qquad\qquad (\alpha,\beta)\mapsto \alpha+\beta$$

Now we can prove that $A \cup B$ also satisfies statements (i) and (ii).

(ii) For $i > n$ the group $\mathrm{H}_i(M|A \cup B)$ is stuck, by our hypothesis on A, B and $A \cap B$, between groups that are zero, thus we see that $\mathrm{H}_i(M|A \cup B) = 0$ for $i > n$.

(i) Now suppose we are given a section $\{\alpha_x\}_{x\in M}$. We denote by α_A, α_B and $\alpha_{A\cap B}$ the unique lifts to A, B and $A \cap B$ given by our hypothesis. By Lemma 87.9 the images of α_A and α_B in $\mathrm{H}_n(M|A \cap B)$ are lifts to $A \cap B$. By the uniqueness of the lift $\alpha_{A\cap B}$ these images thus equal $\alpha_{A\cap B}$. By the exactness of the above sequence we see that $(\alpha_A, -\alpha_B) = \Phi(\alpha_{A\cup B})$ for some $\alpha_{A\cup B} \in \mathrm{H}_n(M|A\cup B)$. We claim that $\alpha_{A\cup B}$ is indeed a lift to $A \cup B$. So let $x \in A \cup B$. First we consider the case that $x \in A$. We have the following commutative diagram

$$\begin{tikzcd} \mathrm{H}_n(M|A\cup B) \arrow[r, "{[\alpha_{A\cup B}\mapsto\alpha_A]}"] \arrow[dr] & \mathrm{H}_n(M|A) \arrow[d] \\ & \mathrm{H}_n(M|x). \end{tikzcd}$$

We see that the image of $\alpha_{A\cup B}$ in $\mathrm{H}_n(M|x)$ is the image of α_A in $\mathrm{H}_n(M|x)$, hence it equals α_x. The same argument applies to the case that $x \in B$.

It remains to prove the uniqueness of the lift $\alpha_{A\cup B}$. If $\beta \in \mathrm{H}_n(M|A \cup B)$ is another lift, then it follows immediately from Lemma 87.9 and the uniqueness of α_A and α_B that $\Phi(\beta) = (\alpha_A, -\alpha_B)$. By the exactness of the sequence we see that we already have the equality $\alpha_{A\cup B} = \beta$ in $\mathrm{H}_n(M|A \cup B)$. ■

Later on we will need the following variation on Lemma 87.13.

Lemma 87.14. *Let M be an n-dimensional topological manifold, let R be a commutative ring. Let $\mathcal{A}$ be a family of compact subsets of M with the following two properties:*

(1) *Statements* (i) *and* (ii) *from Theorem 87.10 hold for each $A \in \mathcal{A}$.*
(2) *The family $\mathcal{A}$ is closed under intersections, i.e. whenever $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.*

Then the statements hold for all finite unions of elements in $\mathcal{A}$.

PROOF. Given $k \in \mathbb{N}$ we write

$$\mathcal{U}_k := \{\text{all subsets of } M \text{ that are the union of at most } k \text{ subsets in } \mathcal{A}\}.$$

We claim that for every $k \in \mathbb{N}$ statements (i) and (ii) hold for all elements in $\mathcal{U}_k$. We prove this claim by induction on k.

(1) If $k = 1$, then the claim holds by hypothesis (1).
(2) Suppose that the claim holds for some k. Let $A_1, \dots, A_k, A_{k+1} \in \mathcal{A}$. We write $B = A_2 \cup \dots \cup A_{k+1}$. Note that

$$A_1 \cap B \;=\; A_1 \cap (A_2 \cup \dots \cup A_{k+1}) \;=\; \bigcup_{i=2}^{k+1} \underbrace{(A_1 \cap A_i)}_{\in \mathcal{A} \text{ by } (2)} \;\in \mathcal{U}_k.$$

Thus by induction hypothesis we see that the statements hold for A_1, B and also for $A_1 \cap B$. So the statements also hold for $A_1 \cup B = A_1 \cup \dots \cup A_{k+1}$ by Lemma 87.13. ■

Our next goal is to prove Theorem 87.10 for $M = \mathbb{R}^n$. We will proceed in several steps.

Lemma 87.15. *Let R be a commutative ring and let $\{\mu_x\}_{x \in \mathbb{R}^n}$ be an R-section on $\mathbb{R}^n$, let $A \subset \mathbb{R}^n$ be a convex bounded subset and let $\beta \in \mathrm{H}_n(\mathbb{R}^n|A; R)$. If $\mu_y = \beta_y$ for some $y \in A$, then $\mu_x = \beta_x$ for all $x \in A$, i.e. β is a lift to A.*

PROOF. The proof is very similar to the proof of Proposition 86.6. The following are the two key observations:

(1) a convex set is path-connected and thus connected,
(2) since A is convex, for any $x \in A$ and any $\epsilon > 0$ the intersection $A \cap B_\epsilon(x)$ is the intersection of two convex bounded sets, hence convex and bounded.

In the proof of Proposition 86.6 we now need to replace Lemma 86.7 by Lemma 86.1. We leave it to the reader to modify the proof of Proposition 86.6 to obtain a proof of Lemma 87.15. ■

Lemma 87.16. *Statements* (i) *and* (ii) *from Theorem 87.10 hold for $M = \mathbb{R}^n$, any commutative ring R and any convex compact subset A of $\mathbb{R}^n$.*

PROOF. Let A be a convex compact subset of $\mathbb{R}^n$ and let R be a commutative ring. As pointed out above, if $A = \varnothing$, then there is nothing to prove. Thus we can assume that $A \neq \varnothing$.

Now we can prove that A satisfies the statements (i) and (ii).

(i) Let $\{\alpha_x\}_{x\in\mathbb{R}^n}$ be a section for $\mathbb{R}^n$. Since $A \neq \emptyset$ we can pick a $y \in A$. We denote by $\alpha_A \in \mathrm{H}_n(\mathbb{R}^n|A)$ the preimage of α_y under the isomorphism
$$\mathrm{H}_n(\mathbb{R}^n|A;R) \;\to\; \mathrm{H}_n(\mathbb{R}^n|\{y\};R)$$
given by Lemma 86.1. It follows from Lemma 87.15 that $\alpha_{A,x} = \alpha_x$ for all $x \in A$, i.e. we have shown that α_A is a lift to A. The uniqueness of the lift α_A follows again from the above isomorphism.

(ii) It follows Lemma 86.1 and 44.1 that $\mathrm{H}_i(\mathbb{R}^n|A;R) = \mathrm{H}_i(\mathbb{R}^n|\{y\};R) = 0$ for every $i > n$. ■

Lemma 87.17. *Every subset of $\mathbb{R}^n$ that is the union of finitely many convex compact sets satisfies the statements* (i) *and* (ii) *from Theorem 87.10.*

Proof. We consider
$$\mathcal{A} := \{\text{all convex compact subsets of } \mathbb{R}^n\}.$$
This family of subsets of $\mathbb{R}^n$ is closed under intersections.[1253] By Lemma 87.16 the two statements hold for each $A \in \mathcal{A}$. The lemma is now an immediate consequence of Lemma 87.14. ■

The following lemma is arguably the hardest step towards proving the fundamental theorem of topological manifolds.

Lemma 87.18. *Statements* (i) *and* (ii) *from Theorem 87.10 hold for $M = \mathbb{R}^n$, any commutative ring R and any compact subset A of $\mathbb{R}^n$.*

Proof. To simplify the notation we again drop the R from the notation. Let $\{\alpha_x\}_{x\in\mathbb{R}^n}$ be a section for $\mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be an arbitrary compact set.

We first prove statement (i). We pick a closed ball B that contains A. We denote by $\alpha_B \in \mathrm{H}_n(\mathbb{R}^n|B)$ the lift constructed in Lemma 87.16. By Lemma 87.9 the image α_A of the class α_B under the map $\mathrm{H}_n(\mathbb{R}^n|B) \to \mathrm{H}_n(\mathbb{R}^n|A)$ is a lift to A. To complete the proof of statement (i) we still need to prove the uniqueness of α_A.

This uniqueness statement is equivalent to showing that if $\beta \in \mathrm{H}_n(\mathbb{R}^n|A)$ is a class such that $\beta_x = 0$ for all $x \in A$, then β is already zero. So let $\beta \in \mathrm{H}_n(\mathbb{R}^n|A)$. We pick a singular n-chain $z \in \mathrm{C}_n(\mathbb{R}^n)$ that represents β. We denote by $C \subset \mathbb{R}^n \setminus A$ the union of the images of the singular simplices appearing in $\partial z \subset \mathrm{C}_{n-1}(\mathbb{R}^n \setminus A)$. Note that C is the union of finitely many compact sets, hence C itself is compact.

Claim. There exist finitely many closed balls $B_1, \dots, B_m$ with the following three properties:

(1) each B_i is contained in $\mathbb{R}^n \setminus C$,
(2) each B_i contains a point $x_i \in A$,
(3) we have $A \subset B_1 \cup \dots \cup B_m$.

[1253] Here we use one of the most convenient properties of convex sets: the intersection of two convex sets is again convex.

Since C is compact and since $A \cap C = \varnothing$ we can find for each $x \in A$ an $r_x > 0$ with $B_{2r_x}(x) \subset \mathbb{R}^n \setminus C$. Since A is compact there exist finitely many points $x_1, \dots, x_m$ in A such that A is contained in $B_{r_{x_1}}(x_1) \cup \dots \cup B_{r_{x_m}}(x_m)$. It is now clear that $B_i := \overline{B}_{r_{x_i}}(x_i)$, $i = 1, \dots, m$ have the desired property. We refer to Figure 1227 for an illustration. ⊞

We write $K := B_1 \cup \dots \cup B_m$. By (1) the singular n-chain $z \in \mathrm{C}_n(\mathbb{R}^n)$ also represents an element $\gamma \in \mathrm{H}_n(\mathbb{R}^n|K)$. Note that for each $x \in A$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{H}_n(\mathbb{R}^n|K) \arrow[rr, "\gamma=[z]\mapsto\beta=[z]"] \arrow[dr, "\gamma\mapsto\gamma_x"'] & & \mathrm{H}_n(\mathbb{R}^n|A) \arrow[dl, "\beta\mapsto\beta_x"] \\ & \mathrm{H}_n(\mathbb{R}^n|x). & \end{tikzcd}$$

It follows in particular from the commutative diagram that it suffices to show that we have $\gamma = 0 \in \mathrm{H}_n(\mathbb{R}^n|K)$. We do so as follows:

(1) By our hypothesis on β and by the above commutative diagram we know that for each $i \in \{1, \dots, m\}$ we have $\gamma_{x_i} = \beta_{x_i} = 0$.
(2) Let $i \in \{1, \dots, m\}$. Since $\gamma_{x_i} = 0$ and since B_i is convex and bounded it follows from Lemma 87.15 (applied to the zero section and to the image of $\gamma \in \mathrm{H}_n(\mathbb{R}^n|B_i)$) that $\gamma_x = 0$ for all $x \in B_i$.
(3) By (2) we know that $\gamma_x = 0$ for all $x \in K = B_1 \cup \dots \cup B_m$.
(4) But now it follows from Lemma 87.17 that $\gamma = 0$.

This concludes the proof of statement (i).

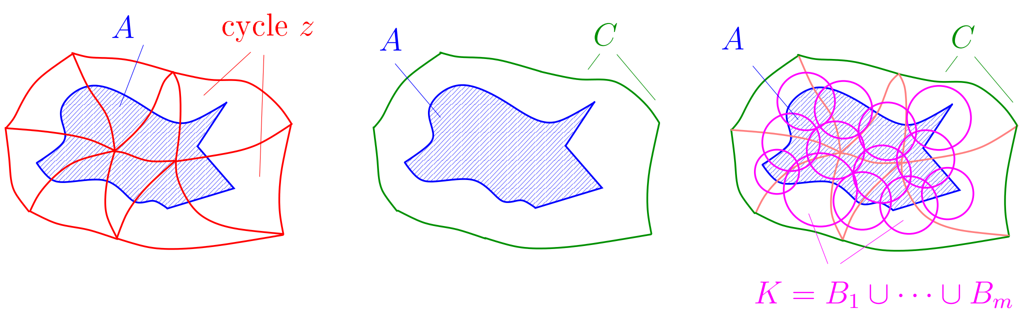


FIGURE 1227. Illustration for the proof of Theorem 87.10.

Finally we need to show statement (ii) for A, i.e. we need to show that $\mathrm{H}_i(\mathbb{R}^n|A) = 0$ for $i > n$. So let $\beta \in \mathrm{H}_i(\mathbb{R}^n|A)$. As above we pick a singular n-chain $z \in \mathrm{C}_n(\mathbb{R}^n)$ that represents β. We define C, K and γ as above. By Lemma 87.17 we have $\gamma = 0 \in \mathrm{H}_i(\mathbb{R}^n|K)$. But as before this implies that also $\beta = 0 \in \mathrm{H}_i(\mathbb{R}^n|A)$. ■

Lemma 87.19. *Let M be an n-dimensional topological manifold with empty boundary and let $A \subset M$ be a compact subset. If there exists a chart $\Phi\colon U \to \mathbb{R}^n$ such that A is contained in U, then statements* (i) *and* (ii) *hold for A.*

PROOF. The lemma is a fairly straightforward consequence of Lemma 87.18 and the fact that we have isomorphisms

$$\mathrm{H}_n(M, M\setminus A;R) \underset{\substack{\uparrow\\ \text{Excision Theorem 43.20}}}{\xleftarrow{\cong}} \mathrm{H}_n(U,U\setminus A;R) \underset{\substack{\uparrow \\ \text{isomorphism since }\Phi\text{ is a homeomorphism}}}{\xrightarrow{\Phi}} \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\Phi(A);R)$$

We leave it to the reader to fill in the details. ■

Now we can finally complete the proof of Theorem 87.10.

PROOF OF THEOREM 87.10. Let M be an n-dimensional topological manifold with empty boundary. We consider

$\mathcal{A}$:= all compact subsets A of M for which there exists a chart $\Phi\colon U \to \mathbb{R}^n$ with $A \subset U$.

By Lemma 87.19 statements (i) and (ii) hold for all subsets in $\mathcal{A}$. It is clear that $\mathcal{A}$ is closed under intersections. Thus it follows from Lemma 87.14 that the statements (i) and (ii) hold for all finite unions of sets in $\mathcal{A}$. Therefore it suffices to prove the following claim.

Claim. Every compact subset A of M is the union of finitely many sets in $\mathcal{A}$.

Let A be a compact subset of M. It is clear that given any point $x \in A$ there exists a chart $\Phi_x\colon U_x \to V_x$ around x such that the image is some open ball in $\mathbb{R}^n$. But since each open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ we can find around each $x \in A$ a chart around x such that the image V_x equals $\mathbb{R}^n$. Now we write $A_x := A \cap \Phi_x^{-1}(\overline{B}_1^n(0))$ and $U_x' := \Phi_x^{-1}(B_1^n(0))$. Since A is compact we can find finitely many $x_1, \dots, x_m$ in A such that A is contained in $U_{x_1}' \cup \dots \cup U_{x_m}'$. By construction the corresponding charts $\Phi_i := \Phi_{x_i}\colon U_{x_i} \to \mathbb{R}^n$ and the sets $A_i := A_{x_i}$ have the desired properties. ⊞

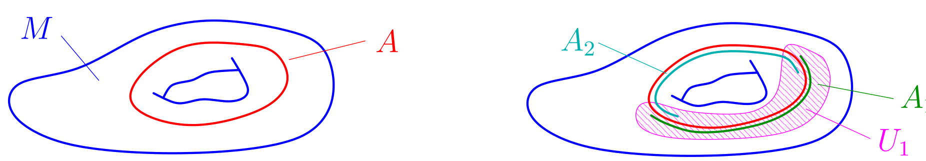


FIGURE 1228. Illustration for the proof of Lemma 87.19.

It follows from Lemma 87.19 that statements (i) and (ii) hold for all the A_i and for all arbitrary intersections of the A_i. But then Lemma 87.13 implies that they also hold for $A = A_1 \cup \dots \cup A_m$. ■

87.3. **Proof of Theorem 87.3.** Now we are in a position that we can provide the long-delayed proofs of Theorems 87.1, 87.2 and 87.3. We start out with the proof of Theorem 87.3 since we will use that theorem in the proof of Theorem 87.1.

The key extra ingredient in the proof of Theorem 87.3 is the following proposition.

Proposition 87.20. *Let M be a connected n-dimensional topological manifold and let R be a commutative ring. If M is non-compact and if $\partial M = \varnothing$, then $\mathrm{H}_k(M;R) = 0$ for every $k \geq n$.*

Proof. Let M be a connected non-compact n-dimensional topological manifold such that $\partial M = \varnothing$, let R be a commutative ring and let $k \geq n$. Let $\alpha \in \mathrm{H}_k(M;R)$. We need to show that $\alpha = 0$. We represent α by a cycle $z = \sum\limits_{i=1}^{m} \sigma_i \otimes r_i \in \mathrm{C}_k(M;R) = \mathrm{C}_k(M) \otimes R$. The union of the images of $\sigma_1, \ldots, \sigma_m$ is a compact subset K of M. Since K is compact it follows from Exercise 87.1 that there exists an open subset $U \subset M$ that contains K and such that the closure $\overline{U}$ is compact. We consider the open subset $V = M \setminus \overline{U}$ and we consider the long exact sequence of the triple $(M, U \cup V, V)$, see Corollary 43.16. It fits into the following commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{H}_{k+1}(M, U \cup V; R) \arrow[r, "\partial"] & \mathrm{H}_k(U \cup V, V; R) \arrow[r] & \mathrm{H}_k(M, V; R) \arrow[r] & \ldots \\
& & \mathrm{H}_k(U;R) \arrow[u, "\cong"] \arrow[r, "{[z] \mapsto [z] = \alpha}"] & \mathrm{H}_k(M;R). \arrow[u] &
\end{tikzcd}$$

Here the left vertical map is an isomorphism since $U \cup V$ is the disjoint union of the open sets U and V.[1254]

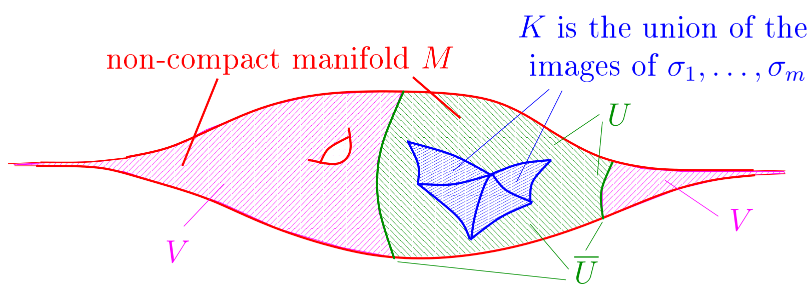


Figure 1229. Illustration for the proof of Proposition 87.20.

Now we consider the modules to the top-left and top-right of the diagram. We have

$$\begin{aligned}
\mathrm{H}_{k+1}(M, U \cup V; R) \;&=\; \mathrm{H}_{k+1}(M|\overbrace{M \setminus (U \cup V)}^{=\overline{U}\setminus U}; R) \;\overset{\uparrow}{=}\; 0 \quad \text{if } k > n-1 \\
&\text{by Theorem 87.10 (ii) since } M \text{ has no boundary and since } \overline{U} \text{ and } \overline{U} \setminus U \text{ are compact} \\
\mathrm{H}_k(M, V; R) \;&=\; \mathrm{H}_k(M|\underbrace{M \setminus V}_{=\overline{U}}; R) \;\overset{\downarrow}{=}\; 0 \quad \text{if } k > n.
\end{aligned}$$

After these preparations we come to the actual proof that $\alpha = [z] = 0 \in \mathrm{H}_k(M;R)$. We first consider the case $k > n$. We deduce from the above exact sequence and the above calculations that $\mathrm{H}_k(U;R) = 0$. But z represents an element in $\mathrm{H}_k(U;R)$. It follows that $[z] = 0$ in $\mathrm{H}_k(M;R)$.

[1254] One could at this point refer to the Excision Theorem 43.19 but this isomorphism follows already from the observation that $\mathrm{C}_*(U \cup V) = \mathrm{C}_*(U) \oplus \mathrm{C}_*(V)$ as chain complexes, since U and V are both open subsets of $U \cup V$.

Now we consider the case $k = n$. It follows from the above discussion that we have a commutative diagram

$$\begin{tikzcd} \mathrm{H}_n(U\cup V, V; R) \arrow[r, hook] & \mathrm{H}_n(M, V; R) \\ \mathrm{H}_n(U; R) \arrow[u, "\cong"] \arrow[r, "{[z]\mapsto[z]=\alpha}"] & \mathrm{H}_n(M; R). \arrow[u] \end{tikzcd}$$

Thus it suffices to show that the image of α in $\mathrm{H}_n(M, V; R)$ is zero. We showed in Lemma 86.5 (2) that $\{\alpha_x\}_{x\in M}$ is an R-section of M.[1255] Since M is non-compact we can find a point x outside of the compact set K. The class $\alpha_x \in \mathrm{H}_n(M, M\setminus\{x\}; R)$ is again represented by z which lies in $K \subset M\setminus\{x\}$. It is thus clear that $\alpha_x = 0 \in \mathrm{H}_n(M, M\setminus\{x\}; R)$. But since M is connected it follows from Proposition 86.6 (1) that $\{\alpha_x\}_{x\in M}$ is the zero section. Both the image of α in $\mathrm{H}_n(M, V; R)$ and the zero-class are lifts of the zero section to $\mathrm{H}_n(M, V; R)$. By Theorem 87.10 (i) both lifts agree, i.e. the image of α in $\mathrm{H}_n(M, V; R)$ is zero. ∎

Now we can easily provide the proof of Theorem 87.3.

Proof of Theorem 87.3. Let M be a connected n-dimensional topological manifold and let R be a commutative ring. Recall that we need to prove the following two statements.

(1) If M is not closed, then $\mathrm{H}_i(M; R) = 0$ for $i \geq n$.
(2) If M is closed, then we have $\mathrm{H}_i(M; R) = 0$ for $i > n$.

First suppose that M is not closed. The hypothesis that M is not closed means that M is not compact or that it has a non-empty boundary. If M is not compact without boundary, then we had just proved in Proposition 87.20 that $\mathrm{H}_i(M; R) = 0$ for $i \geq n$. If M has non-empty boundary, then we saw in Corollary 44.7 that M is homotopy equivalent to a non-compact n-dimensional topological manifold N without boundary. It follows from Corollary 42.8 and Proposition 87.20 that for any $i \geq n$ we have $\mathrm{H}_i(M; R) \cong \mathrm{H}_i(N; R) = 0$.

Now suppose that M is closed. Let $i > n$. We have

$$\mathrm{H}_i(M; R) \;=\; \mathrm{H}_i(M, \overbrace{M\setminus M}^{=\emptyset}; R) \;=\; \mathrm{H}_i(M|M; R) \underset{\substack{\uparrow \\ \text{by Theorem 87.10 (ii) since } M \text{ is compact and since } M \text{ has no boundary}}}{=} 0.$$ ∎

87.4. **Proof of Theorems 87.1 and 87.2.** The key to proving Theorems 87.1 and 87.2 is the following proposition which later on we will apply to $R = \mathbb{Z}$ and $R = \mathbb{F}_2$.

Proposition 87.21. *Let M be a connected, compact n-dimensional topological manifold and let R be a commutative ring. Then the following hold:*

(1) *If M is R-orientable, then for any $x \in M\setminus\partial M$ the map*

$$\mathrm{H}_n(M, \partial M; R) \;\to\; \mathrm{H}_n(M, M\setminus\{x\}; R) \underset{\substack{\uparrow \\ \textit{isomorphic by Lemma 86.4}}}{\cong} R$$

[1255] Here we used a second time that M has no boundary.

is an isomorphism.

(2) *If M is non-orientable, then* $\mathrm{H}_n(M, \partial M; \mathbb{Z}) = 0$.

Proof. Let M be a connected, compact n-dimensional topological manifold and let R be a commutative ring.

We first consider the case that M is actually closed. Recall that we denote by $\Gamma_R(M)$ the set of all R-sections of M. By Lemma 86.5 (2) we have a well-defined R-module homomorphism

$$\begin{array}{rcl} \mathrm{H}_n(M;R) & \mapsto & \Gamma_R(M) \\ \alpha & \mapsto & \{\alpha_x\}_{x\in M}. \end{array}$$

Since M is closed we can apply Theorem 87.10 (i) to the special case of the compact subset $A = M$ and we obtain that the above map is an isomorphism. The desired statements (1) and (2) now both follow from Lemma 86.9.

Now we consider the case that M is not closed, i.e. we suppose that $\partial M \neq \varnothing$. As usual we denote by $\mathrm{D}\,M$ the double of M. The idea now is to prove the desired statements for M by reducing them to the equivalent statements for the closed topological manifold $\mathrm{D}\,M$ that we had just proved.

As before we think of the double as $\mathrm{D}\,M = (M \sqcup M')/\sim$ where M' is another copy of M. We consider the maps

$$\begin{tikzcd}
\underbrace{\mathrm{H}_n(M';R)}_{=0 \text{ by Thm. 87.3}} \arrow[r] & \mathrm{H}_n(\mathrm{D}\,M;R) \arrow[r] \arrow[dr, "\Phi"'] & \mathrm{H}_n(\mathrm{D}\,M, M';R) \arrow[r] & \mathrm{H}_{n-1}(M';R) \arrow[r, "\text{injective}", "\text{by 44.13}"'] & \mathrm{H}_{n-1}(\mathrm{D}\,M;R) \\
& & \mathrm{H}_n(M, \partial M;R), \arrow[u, "\cong"'] & &
\end{tikzcd}$$

where on the top we have written down the long exact sequence of the pair $(\mathrm{D}\,M, M')$. Note that the vertical map is an isomorphism by the Excision Theorem 44.10. The map Φ is defined to be the unique homomorphism that makes the triangle commute.

It follows immediately from the information provided in the above diagram that the map $\Phi\colon \mathrm{H}_n(\mathrm{D}\,M;R) \to \mathrm{H}_n(M, \partial M;R)$ is an isomorphism. Now we turn to the proof of the two statements of the proposition.

(1) Now suppose that M is R-orientable. Let $x \in M \setminus \partial M$. We consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_n(\mathrm{D}\,M;R) \arrow[rr, "{[\mathrm{D}\,M] \mapsto \sigma_x}", "\text{isomorphism since } \mathrm{D}\,M \text{ closed}"'] \arrow[dr, "\cong"'] & & \mathrm{H}_n(\mathrm{D}\,M, \mathrm{D}\,M \setminus \{x\};R) \\
& \mathrm{H}_n(\mathrm{D}\,M, M';R) \arrow[ur] & \\
\mathrm{H}_n(M, \partial M;R) \arrow[ur, "\cong", "\text{Excision Thm. 44.10}"'] \arrow[rr] & & \mathrm{H}_n(M, M \setminus \{x\};R). \arrow[uu, "\mu_x \mapsto \sigma_x", "\text{Excision Thm. 44.10}"']
\end{tikzcd}$$

Here all maps are the obvious inclusion induced maps. This implies that the diagram commutes. Since $\mathrm{D}\,M$ is closed and since it is R-orientable by Lemma 86.18 we know from the first part of the proof that the top horizontal map is an isomorphism. Furthermore by the above discussion we know that the top-left diagonal map is an isomorphism. Finally two of the maps are isomorphisms by the Excision Theorem 44.10. It now follows immediately that the bottom horizontal map is also an isomorphism. But that is exactly what we were supposed to show.

(2) If M is non-orientable, then it follows from Lemma 86.18 that the double $\mathrm{D}\,M$ is non-orientable either. We obtain that

$$\mathrm{H}_n(M,\partial M;\mathbb{Z}) \underset{\substack{\uparrow \\ \text{see above}}}{\cong} \mathrm{H}_n(\mathrm{D}\,M;\mathbb{Z}) \underset{\substack{\uparrow \\ \text{since } \mathrm{D}\,M \text{ is closed and non-orientable and} \\ \text{since we have proved (1) in the closed case}}}{=} 0.$$

■

Remark. Note that the second statement of Proposition 87.21 only deals with integer coefficients. A more general statement is given by [**Hat02**, Theorem 3.26]. This theorem says that if M is a closed connected n-dimensional topological manifold that is *not* R-orientable, then for any $x \in M$ the map $\rho_x\colon \mathrm{H}_n(M;R) \to \mathrm{H}_n(M, M\setminus\{x\};R) \cong R$ is injective with image $\{r \in R \mid 2r = 0\}$. For example for $R = \mathbb{Z}$ we have $\{r \in \mathbb{Z} \mid 2r = 0\} = \{0\}$, i.e. the map ρ_x is the zero map and by injectivity we have $\mathrm{H}_n(M;\mathbb{Z}) = 0$. In the case of closed topological manifolds we can thus view [**Hat02**, Theorem 3.26] as a generalization of Proposition 87.21 (2). We will not make use of [**Hat02**, Theorem 3.26].

Now we can finally provide the proofs of Theorems 87.1 and 87.2.

Proof of Theorem 87.1. Let M be a compact connected non-empty n-dimensional topological manifold. We have to show that the following statements are equivalent:

(1) the topological manifold M is orientable,
(2) $\mathrm{H}_n(M,\partial M;\mathbb{Z}) \neq 0$,
(3) $\mathrm{H}_n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$,
(4) for every $x \in M\setminus\partial M$ the map

$$\rho_x\colon\ \mathrm{H}_n(M,\partial M;\mathbb{Z}) \ \to\ \mathrm{H}_n(M, M\setminus\{x\};\mathbb{Z})$$

is an isomorphism.

We have (1) $\Rightarrow$ (4) by Proposition 87.21 (1) applied with $R = \mathbb{Z}$. Since M is non-empty we obtain from Proposition 44.2 (5) that there exists a point $x \in M\setminus\partial M$. Therefore we have (4) $\Rightarrow$ (3) by Lemma 86.4. Evidently we have (3) $\Rightarrow$ (2). Furthermore we have (2) $\Rightarrow$ (1) by Proposition 87.21 (2). ■

Proof of Theorem 87.2. Let M be a compact connected non-empty n-dimensional topological manifold. We have to show the following two statements:

(1) $\mathrm{H}_n(M,\partial M;\mathbb{F}_2) \cong \mathbb{F}_2$, and
(2) for every $x \in M\setminus\partial M$ the map

$$\rho_x\colon\ \mathrm{H}_n(M,\partial M;\mathbb{F}_2) \ \to\ \mathrm{H}_n(M, M\setminus\{x\};\mathbb{F}_2)$$

is an isomorphism.

By Proposition 86.13 we know that M admits an $\mathbb{F}_2$-orientation. Statement (2) follows immediately from this fact and Proposition 87.21 (1) applied to $R = \mathbb{F}_2$. As in the proof of Theorem 87.1, the first statement is an immediate consequence of the second statement, together with the fact that there exists a point in $M\setminus\partial M$ and Lemma 86.4. ■

87.5. **Further properties of topological manifolds.** Let M be a compact n-dimensional topological manifold. With Theorem 87.1 we now completely understand the n-dimensional homology of M. In this section we will obtain some results on the $(n-1)$-dimensional homology of M which generalize the results of Theorem 66.6 (2) and Theorem 66.8 (2) from smooth manifolds to topological manifolds.

Proposition 87.22. *Let M be a connected compact n-dimensional topological manifold.*

(1) $$\mathrm{H}_{n-1}(M;\mathbb{Z}) \cong \begin{cases} \textit{free abelian group}, & \textit{if } M \textit{ is orientable or if } \partial M \neq \emptyset, \\ \textit{free abelian group} \oplus \mathbb{Z}_2, & \textit{if } M \textit{ is non-orientable and closed.} \end{cases}$$

(2) $$\mathrm{H}_{n-1}(M,\partial M;\mathbb{Z}) \cong \begin{cases} \textit{free abelian group}, & \textit{if } M \textit{ is orientable}, \\ \textit{free abelian group} \oplus \mathbb{Z}_2, & \textit{if } M \textit{ is non-orientable.} \end{cases}$$

In all cases the free abelian group is finitely generated.

Remark.

(1) As a reality check it is good to recall the calculation of the homology groups of closed 2-dimensional topological manifolds that we gave in Proposition 48.9. There we showed for any $g \in \mathbb{N}_0$ and any $k \in \mathbb{N}_0$ we have
$$\mathrm{H}_n(\Sigma_g) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^{2g}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0 \end{cases} \quad \text{and} \quad \mathrm{H}_n(k \cdot \mathbb{R}\mathrm{P}^2) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}^{k-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0, \end{cases}$$
where, as always, we denote by Σ_g the surface of genus g. Fortunately these calculations are coherent with Proposition 87.22.

(2) It follows immediately from the Universal Coefficient Theorem 57.19, Theorem 87.1, Proposition 87.22 and Lemma 57.17 that for any closed orientable n-dimensional topological manifold and any abelian group G we have $\mathrm{H}_n(M;G) \cong G$.

In the proof of Proposition 87.22 we will use the following lemma which is an immediate consequence of the classification of finitely generated abelian groups, see Theorem 19.4, and of Lemma 57.17.

Lemma 87.23. *Let A be a finitely generated abelian group. Then the following holds:*
$$A \textit{ is a free abelian group} \quad \Longleftrightarrow \quad \mathrm{Tor}(A,\mathbb{Z}_p) = 0 \textit{ for all primes } p.$$

Now we can provide the proof of Proposition 87.22. First we consider Statement (1). For that statement we consider the two cases that the topological manifold is closed and that it has non-empty boundary separately.

Proof of Proposition 87.22 (1) I. Let M be a closed connected n-dimensional topological manifold. By Proposition 85.13 (4) we know that $\mathrm{H}_{n-1}(M;\mathbb{Z})$ is a finitely generated abelian group.

We first consider the case that M is orientable. We want to show that $\mathrm{H}_{n-1}(M;\mathbb{Z})$ is a free abelian group. By Lemma 87.23 it suffices to show that $\mathrm{Tor}(\mathrm{H}_{n-1}(M;\mathbb{Z}),\mathbb{F}_p) = 0$ for all primes p.

So let p be a prime. We have

$$\mathbb{F}_p \overset{\substack{\text{by Proposition 87.21 (1), since by Lemma 86.14 we know that } M \text{ is also } \mathbb{F}_p\text{-orientable}\\ \downarrow}}{\cong} \mathrm{H}_n(M;\mathbb{F}_p) \underset{\substack{\uparrow\\ \text{Universal Coefficient Theorem 57.24}}}{\cong} \mathrm{H}_n(M;\mathbb{Z})\otimes\mathbb{F}_p \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(M;\mathbb{Z}),\mathbb{F}_p) \underset{\substack{\uparrow\\ \text{by Theorem 87.1 (3) we have } \mathrm{H}_n(M;\mathbb{Z})\cong\mathbb{Z}}}{\cong} \mathbb{F}_p\oplus\mathrm{Tor}(\mathrm{H}_{n-1}(M;\mathbb{Z}),\mathbb{F}_p).$$

It follows from Lemma 19.8 that $\mathrm{Tor}(\mathrm{H}_{n-1}(M;\mathbb{Z}),\mathbb{F}_p)=0$.

Now we turn to the case that M is non-orientable. By Proposition 86.15 there exists a connected 2-fold covering $q\colon \widetilde{M}\to M$ such that $\widetilde{M}$ is orientable. We consider the maps

$$\begin{tikzcd} \mathrm{H}_{n-1}(M;\mathbb{Z}) \arrow[r, "q^*"] \arrow[rr, bend right, "\cdot 2"'] & \mathrm{H}_{n-1}(\widetilde{M};\mathbb{Z}) \arrow[r, "q_*"] & \mathrm{H}_{n-1}(M;\mathbb{Z}). \end{tikzcd}$$

Here q^* denotes the transfer map that we introduced on page 1445 and q_* denotes of course the map induced by the projection q. By Proposition 59.1 the composition of q^* and q_* is given by multiplication by 2. Put differently, the above diagram commutes.

Since $\widetilde{M}$ is orientable we obtain from the case that we had just proved that $\mathrm{H}_{n-1}(\widetilde{M};\mathbb{Z})$ is torsion-free. Now we see that

$$\text{torsion subgroup of } \mathrm{H}_{n-1}(M;\mathbb{Z}) \underset{\substack{\uparrow\\ \text{since } \mathrm{H}_{n-1}(\widetilde{M};\mathbb{Z}) \text{ is torsion-free}}}{\subset} \ker(q^*) \subset \ker(q_*\circ q^*) \underset{\substack{\uparrow\\ \text{by the above commutative diagram}}}{\subset} \{a\in \mathrm{H}_{n-1}(M;\mathbb{Z})\,|\,2a=0\}.$$

From these inclusions and the classification of finitely generated abelian groups we obtain easily that $\mathrm{H}_{n-1}(M;\mathbb{Z})\cong(\mathbb{Z}_2)^r\oplus\mathbb{Z}^k$ for some $r\in\mathbb{N}_0$ and some $k\in\mathbb{N}_0$.

It remains to show that $r=1$. In light of Lemma 57.17 it suffices to show that we have $\mathrm{Tor}(\mathrm{H}_{n-1}(M;\mathbb{Z}),\mathbb{F}_2)\cong\mathbb{F}_2$. The argument is very similar to the argument in the orientable case. More precisely, we have

$$\mathbb{F}_2 \underset{\substack{\uparrow\\ \text{Proposition 86.13}\\ \text{and Theorem 87.2}}}{\cong} \mathrm{H}_n(M;\mathbb{F}_2) \underset{\substack{\uparrow\\ \text{Universal Coefficient Theorem 57.24}}}{\cong} \mathrm{H}_n(M;\mathbb{Z})\otimes\mathbb{F}_2\oplus\mathrm{Tor}(\mathrm{H}_{n-1}(M;\mathbb{Z}),\mathbb{F}_2) \underset{\substack{\uparrow\\ \text{by Theorem 87.1 (3)}\\ \text{we have } \mathrm{H}_n(M;\mathbb{Z})=0}}{\cong} 0\oplus\mathrm{Tor}(\mathrm{H}_{n-1}(M;\mathbb{Z}),\mathbb{F}_2).$$

■

Now we deal with topological manifolds with non-empty boundary.

Proof of Proposition 87.22 (1) II (∗). Let M be a connected, compact n-dimensional topological manifold with non-empty boundary. We need to show that $\mathrm{H}_{n-1}(M;\mathbb{Z})$ is a free abelian group. There are two approaches to the proof of this statement.

(1) We can imitate the proof of the closed case, or
(2) we reduce the proof of the proposition to the case of closed topological manifolds that we had just dealt with.

We will pursue the second approach. First recall that by Lemma 86.18 the double $\mathrm{D}\,M$ is orientable if and only if M is orientable.

We first consider the case that M is orientable. By Proposition 87.22 we know that $\mathrm{H}_{n-1}(\mathrm{D}\,M;\mathbb{Z})$ is a finitely generated free abelian group. It follows from Lemma 44.13 that the group $\mathrm{H}_{n-1}(M;\mathbb{Z})$ is isomorphic to a subgroup of $\mathrm{H}_{n-1}(\mathrm{D}\,M;\mathbb{Z})$, hence by Lemma 19.8 (3) it is also a finitely generated free abelian group.

Now we consider the case that M is non-orientable. We denote by M' a copy of M and as usual we denote by $\mathrm{D}\,M = M \cup_{\partial M=\partial M'} M'$ the double. From the Mayer–Vietoris Theorem 46.10 for Manifolds we obtain the following long exact sequence:

$$\cdots \to \mathrm{H}_n(\mathrm{D}\,M;\mathbb{Z}) \to \mathrm{H}_{n-1}(\partial M;\mathbb{Z}) \to \mathrm{H}_{n-1}(M;\mathbb{Z}) \oplus \mathrm{H}_{n-1}(M';\mathbb{Z}) \to \mathrm{H}_{n-1}(\mathrm{D}\,M;\mathbb{Z}) \to \dots$$

We make the following observations:

(1) By Proposition 44.2 (3) and (4) we know that ∂M is a closed $(n-1)$-dimensional topological manifold. Thus we obtain from Theorem 87.1 that $\mathrm{H}_{n-1}(\partial M;\mathbb{Z})$ is isomorphic to a free abelian group $\mathbb{Z}^r$ for some $r \in \mathbb{N}_0$. (We do not have any control over the orientability of ∂M and the number of components of ∂M but this does not matter.)
(2) Since $\mathrm{D}\,M$ is non-orientable we know from Theorem 87.1 that $\mathrm{H}_n(\mathrm{D}\,M;\mathbb{Z}) = 0$ and we know from Proposition 87.22 that $\mathrm{H}_{n-1}(\mathrm{D}\,M;\mathbb{Z}) \cong \mathbb{Z}^s \oplus \mathbb{Z}_2$ for some $s \in \mathbb{N}_0$.
(3) Since $M = M'$ we have $\mathrm{H}_{n-1}(M;\mathbb{Z}) = \mathrm{H}_{n-1}(M';\mathbb{Z})$. By the classification of finitely generated abelian groups we can pick an isomorphism $\mathrm{H}_{n-1}(M;\mathbb{Z}) \cong F \oplus T$ where F is a free abelian group and T is a torsion group.

We can thus rewrite the above sequence as follows:

$$0 \;\to\; \mathbb{Z}^r \;\xrightarrow{\varphi}\; F \oplus F \oplus T \oplus T \;\to\; \mathbb{Z}^s \oplus \mathbb{Z}_2 \;\to\; \dots$$

The map $\varphi\colon \mathbb{Z}^r \to F \oplus F \oplus T \oplus T$ is a monomorphism, but since $\mathbb{Z}^r$ is torsion-free and since T is finite we see that $\operatorname{im}(\varphi) \cap (T \oplus T)$ is trivial. But this implies that $T \oplus T$ is isomorphic to a subgroup of $\mathbb{Z}^s \oplus \mathbb{Z}_2$. But this is only possible if $T = 0$. This concludes the proof that $\mathrm{H}_{n-1}(M;\mathbb{Z})$ is a free abelian group. ■

Proof of Proposition 87.22 (2) $(*)$. First we note that by Propositions 44.2 (3) and (4) and 85.13 (4) together with Lemma 64.7 (1) we know that $\mathrm{H}_{n-1}(M,\partial M;\mathbb{Z})$ is finitely generated. Now there are two approaches to proving Statement (2):

(1) We can use basically exactly the argument as in the proof of Theorem 66.8 (2) to reduce, via a doubling construction, the study of relative homology groups to the study of absolute homology groups. The only difference is that we need to replace Theorem 66.6 by Statement (1) of the present proposition.
(2) Alternatively we can adopt the argument of Statement (1). First we assume that M is orientable. By Lemma 87.23 it suffices to show that $\operatorname{Tor}(\mathrm{H}_{n-1}(M,\partial M;\mathbb{Z}),\mathbb{F}_p) = 0$ for all primes p. So let p be a prime. We have

$$\begin{array}{rcl} & & \text{Universal Coefficient Theorem 57.24} \\ & & \downarrow \\ \mathbb{F}_p \;\cong\; \mathrm{H}_n(M,\partial M;\mathbb{F}_p) & \cong & \mathrm{H}_n(M,\partial M;\mathbb{Z}) \otimes \mathbb{F}_p \oplus \operatorname{Tor}(\mathrm{H}_{n-1}(M,\partial M;\mathbb{Z}),\mathbb{F}_p) \\ \uparrow \qquad\qquad & & \cong\; \mathbb{F}_p \oplus \operatorname{Tor}(\mathrm{H}_{n-1}(M,\partial M;\mathbb{Z}),\mathbb{F}_p). \\ \text{by Proposition 87.21 (1),} & & \uparrow \\ \text{since by Lemma 86.14} & & \text{by Theorem 87.1 (3) we} \\ M \text{ is also } \mathbb{F}_p\text{-orientable} & & \text{have } \mathrm{H}_n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z} \end{array}$$

It follows from Lemma 19.8 that $\operatorname{Tor}(\mathrm{H}_{n-1}(M,\partial M;\mathbb{Z}),\mathbb{F}_p)=0$. The case that M is non-orientable is very similar, we leave it to the reader to make the necessary modifications. ■

87.6. **The fundamental class and induced orientations.** In Section 86.5 we saw that an orientation on a topological manifold M induces orientations on codimension-zero submanifolds of M, on the boundary ∂M and on covering spaces of M. If M is compact, then M admits a fundamental class and we will see in this section how it is related to the fundamental classes of the various corresponding oriented topological manifolds.

First of all, in Lemma 86.17 we saw that a codimension-zero submanifold W of an oriented topological manifold M inherits an orientation. For compact topological manifolds the following lemma, which is the analogue of Lemma 68.12, shows the relationship between the corresponding fundamental classes.

Lemma 87.24. *Let M be a compact oriented n-dimensional topological manifold. Furthermore let $W\subset M$ be a compact non-empty codimension-zero submanifold. We write $\mathring{W}=W\setminus\partial W$. The following two statements hold:*

1. *The inclusion induced map $\mathrm{H}_n(W,\partial W;\mathbb{Z})\to\mathrm{H}_n(M,M\setminus\mathring{W};\mathbb{Z})$ is an isomorphism.*
2. *The image of the fundamental classes $[M]\in\mathrm{H}_n(M,\partial M;\mathbb{Z})$ and $[W]\in\mathrm{H}_n(W,\partial W;\mathbb{Z})$ under the maps*

$$\mathrm{H}_n(M,\partial M;\mathbb{Z})\;\longrightarrow\;\mathrm{H}_n(M,M\setminus\mathring{W};\mathbb{Z})\;\longleftarrow\;\mathrm{H}_n(W,\partial W;\mathbb{Z})$$

agree.

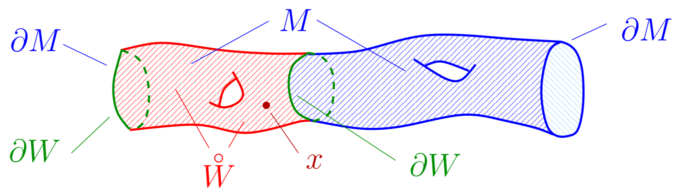


FIGURE 1230. Illustration for the proof of Lemma 87.24.

Remark. In Lemma 89.4 we will generalize Lemma 87.24 to the case that W is a codimension-zero submanifold *with corner*.

PROOF (∗).

(1) The first statement is an immediate consequence of the Excision Theorem 44.10.
(2) We denote by $\Phi\colon\,\mathrm{H}_n(M,\partial M;\mathbb{Z})\to\mathrm{H}_n(W,\partial W;\mathbb{Z})$ the map given in statement (2). We have to show that $\Phi([M])$ is the fundamental class for W. But this follows immediately from Corollary 87.4 and from considering for each $x\in W\setminus\partial W$ the

following commutative diagram

$$\begin{tikzcd}
\mathrm{H}_n(M,\partial M;\mathbb{Z}) \arrow[r] \arrow[d] \arrow[rr, bend left, "\Phi"] & \mathrm{H}_n(M,M\setminus \mathring{W};\mathbb{Z}) \arrow[dl] & \mathrm{H}_n(W,\partial W;\mathbb{Z}) \arrow[l, "\cong"'] \arrow[d] \\
\mathrm{H}_n(M,M\setminus\{x\};\mathbb{Z}) & & \mathrm{H}_n(W,W\setminus\{x\};\mathbb{Z}). \arrow[ll, "\cong"', "\text{isomorphism by the Excision Theorem 43.20}"]
\end{tikzcd}$$

We leave the few remaining details to the reader. ■

In Lemma 86.17 (3) we showed that a covering space $\widetilde{M}$ of an oriented topological manifold M inherits an orientation from M. The following proposition can now be viewed as a generalization of Exercise 68.10 and Proposition 69.8.

Proposition 87.25. *Let M be a compact oriented non-empty n-dimensional topological manifold and let $p\colon \widetilde{M} \to M$ be a finite covering. We denote by $[M]$ the fundamental class of M and we denote by $p^*\colon \mathrm{H}_n(M;\mathbb{Z}) \to \mathrm{H}_n(\widetilde{M};\mathbb{Z})$ the transfer map that we introduced on page 1445. $p^*([M])$ equals the fundamental class $\big[\widetilde{M}\big]$ of the oriented topological manifold $\widetilde{M}$ and we have $p_*\big(\big[\widetilde{M}\big]\big) = \big[\widetilde{M} : M\big] \cdot [M]$.*

Proof ($*$). Let M be a compact n-dimensional topological manifold. To simplify the notation we assume that M is closed. Furthermore let $p\colon \widetilde{M} \to M$ be a finite covering and let $\{\mu_x\}_{x\in M}$ be an orientation for M. We denote by $\{\nu_x\}_{x\in\widetilde{M}}$ the orientation on $\widetilde{M}$ that we defined in Lemma 86.17 (3).

We want to show that $p^*([M])$ equals the fundamental class $\big[\widetilde{M}\big]$. By Corollary 87.4 it suffices to show that for all $y \in \widetilde{M}$ we have $p^*([M])_y = \nu_y \in \mathrm{H}_n(\widetilde{M},\widetilde{M}\setminus\{y\};\mathbb{Z})$. So let $y \in \widetilde{M}$. We pick an open neighborhood V around y such that $U := p(V)$ is uniformly covered. We write $x = p(y)$ and we denote by $q\colon U \to V$ the inverse of the homeomorphism $p\colon V \to U$.

We pick a representative $z = a_1\cdot\sigma_1 + \dots + a_m\cdot\sigma_m \in \mathrm{C}_n(M)$ for $[M]$. By applying Proposition 43.24 to the *open* covering of M given by $\{U, M\setminus\{x\}\}$ we can arrange that the image of each σ_i lies in U or in $M\setminus\{x\}$. We denote by z' the sum of all summands in z such that the image of σ_i lies in U.

A moment's thought shows that in $\mathrm{H}_n(M,M\setminus\{x\};\mathbb{Z})$ we have $[z]=[z']$ and that in $\mathrm{H}_n(\widetilde{M},\widetilde{M}\setminus\{y\})$ we have $[p^*(z)] = [q_*(z')]$. From these observations we deduce that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}_n(M;\mathbb{Z}) \arrow[rrr, "p^*"] \arrow[d, "{[M]=[z]\mapsto[z]=[z']=[\mu_x]}"] & & & \mathrm{H}_n(\widetilde{M};\mathbb{Z}) \arrow[d, "{[p^*z]\mapsto[p^*z]=[q_*(z')]}"'] \\
\mathrm{H}_n(M,M\setminus\{x\};\mathbb{Z}) \arrow[rrr, bend right, "\mu_x\mapsto\nu_y"'] & \mathrm{H}_n(U,U\setminus\{x\};\mathbb{Z}) \arrow[l, "\cong"'] \arrow[r, "\cong", "q_*"'] & \mathrm{H}_n(V,V\setminus\{y\};\mathbb{Z}) \arrow[r, "\cong"] & \mathrm{H}_n(\widetilde{M},\widetilde{M}\setminus\{y\};\mathbb{Z}).
\end{tikzcd}$$

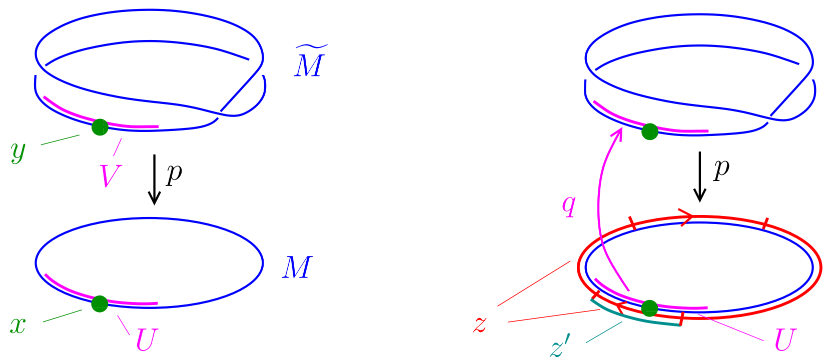


FIGURE 1231

Note that the bottom horizontal maps to the left and right are isomorphisms by the Excision Theorem 43.20 and the middle horizontal map is an isomorphism since q is a homeomorphism. It follows from this diagram that we indeed have $p^*([M])_y = \nu_y$.

Finally we have

$$p_*\big([\widetilde{M}]\big) \underset{\substack{\uparrow \\ \text{by the above}}}{=} p_*(p^*([M])) \underset{\substack{\uparrow \\ \text{Proposition 59.1}}}{=} [\widetilde{M} : M] \cdot [M].$$

■

In Lemma 86.18 we saw that the double of an oriented topological manifold with non-empty boundary comes with a natural orientation. The following lemma following relates the various fundamental classes in this construction.

Lemma 87.26. *Let M be a compact oriented n-dimensional topological manifold with non-empty boundary. If we equip $\mathrm{D}\,M$ with the orientation coming from Lemma 86.18, then*

$$[\mathrm{D}\,M] \;=\; D([M]) \;\in\; \mathrm{H}_n(\mathrm{D}\,M;\mathbb{Z}),$$

where $D\colon \mathrm{H}_n(M,\partial M;\mathbb{Z}) \to \mathrm{H}_n(\mathrm{D}\,M;\mathbb{Z})$ denotes the doubling homomorphism that we introduced in Lemma 44.16.

PROOF. Let $\{\mu_x\}_{x\in M\setminus\partial M}$ be an orientation for M. We denote by $\{\alpha_x\}_{x\in \mathrm{D}\,M}$ the corresponding orientation for $\mathrm{D}\,M$ that we had constructed in Lemma 86.18. To simplify the notation we write $\mathrm{D}\,M = (M \sqcup M')/\sim$ where M' is a copy of M and we view M as a submanifold of the double. By Proposition 44.2 (5) there exists an $x \in M\setminus\partial M$. We consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_k(M,\partial M;\mathbb{Z}) \arrow[rr, "D"] \arrow[dr] \arrow[dd] & & \mathrm{H}_k(\mathrm{D}\,M;\mathbb{Z}) \arrow[dl] \arrow[dddl, "\cong"] \\
& \mathrm{H}_k(\mathrm{D}\,M, M';\mathbb{Z}) \arrow[dd] & \\
\mathrm{H}_n(M, M\setminus\{x\};\mathbb{Z}) \arrow[dr, "\cong", "\mu_x\mapsto\alpha_x"'] & & \\
& \mathrm{H}_n(\mathrm{D}\,M, \mathrm{D}\,M\setminus\{x\};\mathbb{Z}). &
\end{tikzcd}$$

Here all maps are the obvious maps for pairs of topological spaces. The lemma follows from the following observations:

(1) The top triangle commutes by Lemma 44.16 (3). The parallelogram on the left and the triangle on the right commute by the functoriality of homology.
(2) The homomorphism to the bottom left is an isomorphism by the Excision Theorem 44.10 for topological Manifolds.
(3) The right diagonal map is an isomorphism by Theorem 87.1.
(4) By definition $[M]$ gets sent to $\mu_x \in \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{Z})$ and by definition μ_x gets sent to α_x. It follows from the commutativity of the diagram that $D([M])$ gets sent to α_x. But since the right diagonal map is an isomorphism it follows, from the definition of $[\mathrm{D}\, M]$, that $[\mathrm{D}\, M] = D([M])$. ■

In Proposition 86.19 we saw that the boundary of an oriented topological manifold inherits a natural orientation. The following proposition can now be seen as a generalization of Proposition 68.9 from smooth manifolds to topological manifolds.

Proposition 87.27. *Let M be a compact n-dimensional topological manifold with non-empty boundary.*

(1) *We denote by $\partial_n\colon \mathrm{H}_n(M, \partial M; \mathbb{Z}) \to \mathrm{H}_{n-1}(\partial M; \mathbb{Z})$ the connecting homomorphism of the long exact sequence of the pair $(M, \partial M)$. If M is oriented, then*[1256]
$$\partial_n([M]) \;=\; [\partial M] \;\in\; \mathrm{H}_{n-1}(\partial M; \mathbb{Z}).$$
(2) *Similarly, for the connecting homomorphism $\partial_n\colon \mathrm{H}_n(M, \partial M; \mathbb{F}_2) \to \mathrm{H}_{n-1}(\partial M; \mathbb{F}_2)$ we have*
$$\partial_n([M]_{\mathbb{F}_2}) \;=\; [\partial M]_{\mathbb{F}_2} \;\in\; \mathrm{H}_{n-1}(\partial M; \mathbb{F}_2).$$

Proof (∗). As a health warning we point out that in this proof we make heavy use of the notation introduced in Proposition 86.19. We only provide the proof of Statement (1); the proof of Statement (2) is almost identical.

Let $\{\mu_y\}_{y \in M \setminus \partial M}$ be the orientation of for M. Next let $x \in \partial M$. We pick a chart $\Phi\colon U \to [0,2) \times (-2,2)^{n-1}$ with $\Phi(x) = (0,0)$. We write $V' = \Phi^{-1}((0,1) \times (-1,1)^{n-1})$. Finally we pick a point $y \in V'$. We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_n(M, M\setminus\{y\}) \arrow[rrr, bend left, "\mu_y \mapsto \nu_x"] & \mathrm{H}_n\left(M, M\setminus V'\right) \arrow[l, "\cong"'] \arrow[r, "\partial_n", "\cong"'] & \mathrm{H}_{n-1}(M\setminus V', (M\setminus V')\setminus\{x\}) & \mathrm{H}_{n-1}(\partial M, \partial M\setminus\{x\}) \arrow[l, "\cong"'] \\
\mathrm{H}_n(M, \partial M) \arrow[u, "{[M] \mapsto \mu_y}"] \arrow[ur] \arrow[rrr, "\partial_n"] & & & \mathrm{H}_{n-1}(\partial M). \arrow[u, "{[\partial M] \mapsto \nu_x}"'] \arrow[ul]
\end{tikzcd}$$

We make the following observations and clarifications:

(1) All the unmarked maps are induced by the obvious inclusions of pairs of topological space.

[1256] Note that ∂M is a closed $(n-1)$-dimensional topological manifold by Proposition 44.2 (3) and (4).

(2) In the proof of Proposition 86.19 we showed that the blue maps on the top, connecting $\mathrm{H}_n(M, M \setminus \{y\})$ to $\mathrm{H}_{n-1}(\partial M, \partial M \setminus \{x\})$, are isomorphisms.
(3) By definition of the orientation $\{\nu_x\}_{x\in\partial M}$ of ∂M the composition of the top horizontal maps sends μ_x to ν_y.
(4) The maps ∂_n are the connecting homomorphism of the long exact sequence of the triple $(M, M \setminus V', (M \setminus V') \setminus \{x\})$ respectively of the pair $(M, \partial M)$.
(5) The triangles are given by inclusion induced maps, hence they both commute.
(6) Finally note that it follows from the naturality of the connecting homomorphisms, see Proposition 43.15 (3), that the quadrilateral commutes.

It follows from the above and the defining property of $[M]$ that for any $x \in \partial M$ the image of $\partial_n([M]) \in \mathrm{H}_{n-1}(M)$ under the map $\mathrm{H}_{n-1}(\partial M) \to \mathrm{H}_{n-1}(\partial M, \partial M \setminus \{x\})$ equals ν_x. But that means that $\partial_n([M]) = [\partial M]$. ■

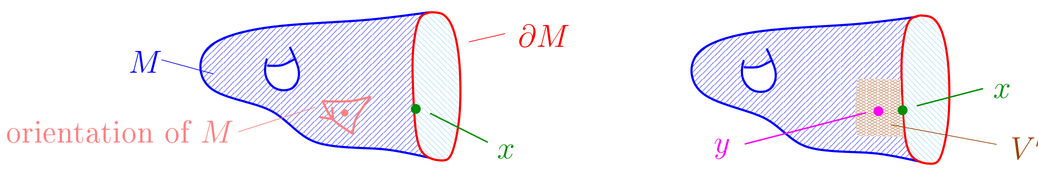


FIGURE 1232. Illustration for the proof of Proposition 87.27.

Remark. Given a compact oriented n-dimensional topological manifold the conclusion of Proposition 87.27 can also be summarized in the following commutative "diagram"

$$\begin{tikzcd}
\mathbb{Z}\text{-oriented topological manifold } M \arrow[r, mapsto, "\text{take boundary}"] \arrow[d, mapsto, "\text{take fundamental class}"] & \mathbb{Z}\text{-oriented topological manifold } \partial M \arrow[d, mapsto, "\text{take fundamental class}"] \\
{[M]} \arrow[r, mapsto, "\text{apply connecting homomorphism}", "\mathrm{H}_n(M,\partial M;\mathbb{Z})\to\mathrm{H}_{n-1}(\partial M;\mathbb{Z})"'] & \partial[M] = [\partial M].
\end{tikzcd}$$

Example. We will illustrate Proposition 87.27 with the Möbius band M. By Proposition 86.13 and Theorem 87.2 it has an $\mathbb{F}_2$-fundamental class $[M] \in \mathrm{H}_2(M, \partial M; \mathbb{F}_2)$. In Figure 1233 we consider the Möbius band with two singular 1-simplices α_1, α_2 and two singular 2-simplices σ_1, σ_2. In this case we have

$$\partial\big(\underbrace{\sigma_1 + \sigma_2}_{=:\mu}\big) \;=\; \underbrace{\alpha_1 + \alpha_2}_{\in \mathrm{C}_1(\partial M)}.$$

In particular, $\mu \otimes 1$ is a cycle in $\mathrm{C}_2(M, \partial M; \mathbb{F}_2)$. It easily follows from Proposition 68.18 that $\mu \otimes 1$ represents the $\mathbb{F}_2$-fundamental class in $\mathrm{H}_2(M, \partial M; \mathbb{F}_2)$. Furthermore, if we denote by $\partial\colon \mathrm{H}_2(M, \partial M; \mathbb{F}_2) \to \mathrm{H}_1(\partial M; \mathbb{F}_2)$ the connecting homomorphism of the long exact sequence in $\mathbb{F}_2$-homology of the pair $(M, \partial M)$, then

$$\partial[M] \;=\; \partial[\mu \otimes 1] \underset{\substack{\uparrow \\ \text{Proposition 43.15}}}{=} [\partial\mu \otimes 1] \underset{\substack{\uparrow \\ \text{by the above}}}{=} \big[(\alpha_1 + \alpha_2) \otimes 1\big],$$

but the last class is clearly[1257] an $\mathbb{F}_2$-fundamental class for the circle ∂M.

[1257]Why is this clear?

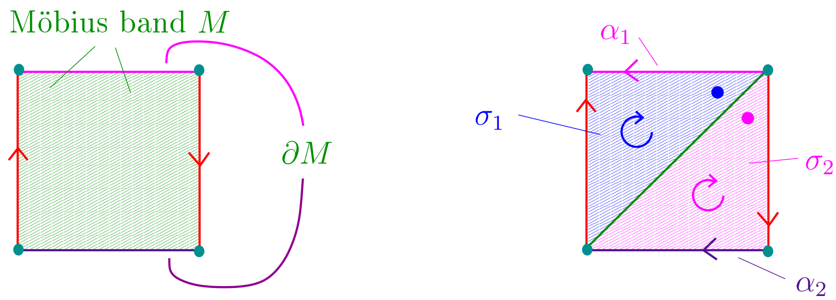


FIGURE 1233

The following corollary is the generalization of Corollary 68.11 to the case of topological manifolds.

Corollary 87.28. *Let M be a compact n-dimensional topological manifold. Then the following two statements hold:*

(1) *We have*
$$\ker\big(\mathrm{H}_{n-1}(\partial M;\mathbb{F}_2)\to \mathrm{H}_{n-1}(M;\mathbb{F}_2)\big) \;=\; \mathbb{F}_2\cdot[\partial M]_{\mathbb{F}_2} \;\in\; \mathrm{H}_{n-1}(\partial M;\mathbb{F}_2).$$
Furthermore, if M is oriented, then
$$\ker\big(\mathrm{H}_{n-1}(\partial M;\mathbb{Z})\to \mathrm{H}_{n-1}(M;\mathbb{Z})\big) \;=\; \mathbb{Z}\cdot[\partial M] \;\in\; \mathrm{H}_{n-1}(\partial M;\mathbb{Z}).$$

(2) *Let M be a compact n-dimensional topological manifold with non-trivial boundary. Then the boundary ∂M is not a retract of M.*

PROOF. The proof is verbatim the same as the proof of Corollary 68.11, the only difference in the proof is that we need to replace Proposition 68.9 by Proposition 87.27.

As a reminder, the first statement is proved using the long exact sequence in homology of the pair $(M,\partial M)$ together with Proposition 87.27. The second statement follows from the first statement and the usual functoriality argument, this time we employ the functoriality of homology with $\mathbb{F}_2$-coefficients and the fact that $[\partial M]_{\mathbb{F}_2}$ is non-zero. ■

Example. It follows from Corollary 87.28 that the boundary of the Möbius band M is not a retract of the Möbius band. A more down-to-earth proof is given as follows: we pick a base point $x_0\in\partial M$ and we denote by α and β the loops shown in Figure 1234. Note that α represents a generator of $\pi_1(\partial M,x_0)\cong\mathbb{Z}$ and that β represents a generator of $\pi_1(M,x_0)\cong\mathbb{Z}$. We denote by $i\colon \partial M\to M$ the inclusion map. Note that $i_*([\alpha])=2\cdot[\beta]$. If there was a retraction $r\colon M\to\partial M$, then we would obtain a commutative diagram

$$\begin{tikzcd} & \pi_1(M,x_0) \arrow[dr, "r_*"] & \\ \pi_1(\partial M,x_0) \arrow[ur, "i_*"] \arrow[rr, "(r\circ i)_*"'] & & \pi_1(\partial M,x_0) \end{tikzcd}$$

which by the above gives rise to the commutative diagram

$$\begin{tikzcd} & \mathbb{Z} \arrow[dr, "??"] & \\ \mathbb{Z} \arrow[ur, "\cdot 2"] \arrow[rr, "\mathrm{id}"'] & & \mathbb{Z}, \end{tikzcd}$$

but such a commutative diagram cannot exist. Looking more carefully one sees that the two proofs that ∂M is not a retract of the Möbius band M are in some sense the same proof. More precisely, we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{Z} = \pi_1(\partial M, x_0) \arrow[r, "i_*"] \arrow[d] & \pi_1(M, x_0) = \mathbb{Z} \arrow[d] \\
\mathbb{Z} = \mathrm{H}_1(\partial M; \mathbb{Z}) \arrow[r, "i_*"] \arrow[d] & \mathrm{H}_1(M; \mathbb{Z}) = \mathbb{Z} \arrow[d] \\
\mathbb{F}_2 = \mathrm{H}_1(\partial M; \mathbb{F}_2) \arrow[r, "i_*"] & \mathrm{H}_1(M; \mathbb{F}_2) = \mathbb{F}_2.
\end{tikzcd}$$

Here the top vertical maps are given by the Hurewicz homomorphism from page 1314 and the bottom vertical maps are given by the map $[\alpha] \mapsto [\alpha \otimes 1]$ from page 1417. Since both maps are natural, see Proposition 52.2 (4) and page 1417, we see that the diagram commutes. The above observation that the top horizontal map is given by multiplication by 2 translates into the fact that the bottom horizontal map is the zero map.

FIGURE 1234

87.7. **Lifts of homology orientations for topological manifolds with boundary** $(*)$**.** In this very technical section we generalize Theorem 87.10 (i) to the case of topological manifolds with boundary. Unless the reader is blessed with a surplus of energy it is best to skip this section.

Definition. Let M be a topological manifold. We say a compact subset $A \subset M$ is a *product close to the boundary* if there exists a collar neighborhood $[0,1] \times \partial M$ for the boundary ∂M and a compact subset $K \subset \partial M$ such that $A \cap ([0,1] \times \partial M) = [0,1] \times K$.

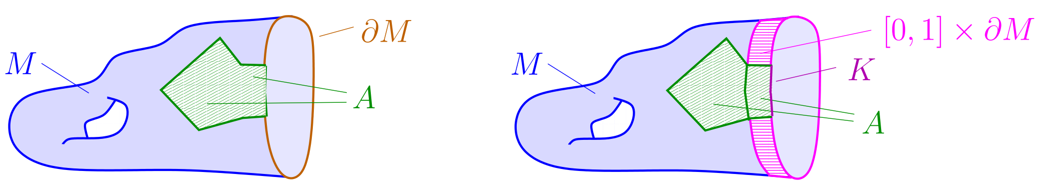


FIGURE 1235

Remark. If M is a topological manifold with *empty boundary*, then every compact subset is in fact a product close to the boundary.

We leave the task of providing a proof of the following elementary lemma to the reader.

Lemma 87.29. *Let M be a topological manifold and let $[0,1] \times \partial M$ be a collar neighborhood (which always exists by the topological Collar Neighborhood Theorem 44.5). Given any compact subset K of M there exists a compact subset A that contains K and that is a product close to the boundary.*

Throughout this section we will need the following notation.

Notation. Given a subset A of a topological manifold M we write $A_\partial := A \cap (M \setminus \partial M)$. In other words, we have $A_\partial = A \setminus \partial M$. Note that $(M \setminus A) \cup \partial M = M \setminus A_\partial$.

Now we can formulate the main result of this section.

Theorem 87.30. *Let M be an n-dimensional topological manifold and let $A \subset M$ be a compact subset that is a product close to the boundary. Let R be a commutative ring. If $\{\alpha_x\}_{x \in M \setminus \partial M}$ is an R-section for M, then there exists a unique lift $\alpha_A^M = \alpha_A$ to A_∂, i.e. there exists a unique class $\alpha_A^M = \alpha_A \in \mathrm{H}_n(M, M \setminus A_\partial; R)$ such that for each $x \in A_\partial$ we have $\alpha_{A,x} = \alpha_x$.*

Proof. Let M be an n-dimensional topological manifold and let R be a commutative ring. Furthermore let $A \subset M$ be a compact subset that is a product close to the boundary. We introduce the following notation and conventions:

(1) As usual we denote by $\mathrm{D}\,M$ the double of M. As before we think of the double as $\mathrm{D}\,M = (M \sqcup M')/\sim$ where M' is another copy of M. We recall that by Lemma 44.12 we know that $\mathrm{D}\,M$ is an n-dimensional topological manifold with empty boundary such that $M \subset \mathrm{D}\,M$ is a submanifold.
(2) By definition there exists a collar neighborhood $[0,1] \times \partial M$ for the boundary ∂M and a compact subset $K \subset \partial M$ such that $A \cap ([0,1] \times \partial M) = [0,1] \times K$.
(3) We denote by $DA = A \cup A' \subset \mathrm{D}\,M$ the double of A. As DA is the union of two compact sets we see that DA itself is compact.
(4) Given $t \in (0,1]$ we write $A_t := A \setminus ([0,t) \times K) = A \cap (M \setminus ([0,t) \times \partial M))$. It follows easily from Lemma 2.17 that A_t is again compact.
(5) It follows from the Excision Theorem 44.10 for topological manifolds that for any $P \in M \setminus \partial M$ the inclusion induced map
$$\mathrm{H}_n(M, M \setminus \{P\}; R) \;\to\; \mathrm{H}_n(\mathrm{D}\,M, \mathrm{D}\,M \setminus \{P\}; R)$$
is an isomorphism. We will use this isomorphism to identify these two homology groups.

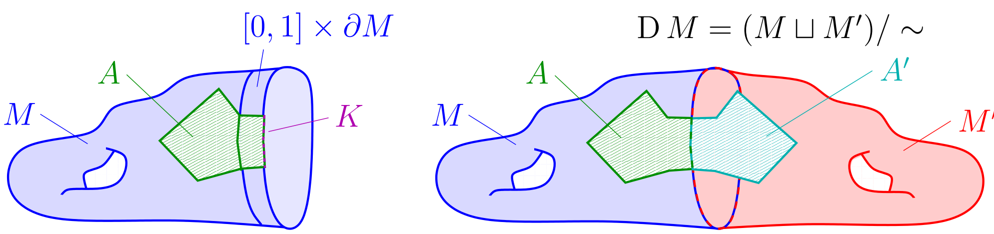


Figure 1236. Illustration for the proof of Theorem 87.30.

The idea now is the same as in the proof of Proposition 87.21. Namely we want to prove the desired statement for M by reducing it to a corresponding statement for the topological manifold $\mathrm{D}\,M$ that has no boundary.

Claim. The inclusion induced map
$$\beta\colon\ \mathrm{H}_n(M, M \setminus A_\partial; R) \;\to\; \mathrm{H}_n(\mathrm{D}\,M, (M \setminus A) \cup M'; R)$$

is an isomorphism.

Recall that A is a product close to the boundary. Using this fact and using a gentle generalization of the Excision Theorem 44.10 for topological manifolds one can easily show that the inclusion induced map is an isomorphism. We leave it to the reader to fill in the details. ⊞

We denote by Φ the composition of the following two maps

$$\mathrm{H}_n(\mathrm{D}\,M, \mathrm{D}\,M \setminus DA; R) \;\to\; \mathrm{H}_n(\mathrm{D}\,M, (M \setminus A) \cup M'; R) \xrightarrow{\beta^{-1}} \mathrm{H}_n(M, M \setminus A_\partial; R).$$

After all these preparations we now turn to the actual proof of the theorem. Thus let $\{\alpha_x\}_{x\in M\setminus\partial M}$ be an R-section for M. As in the proof of Lemma 86.18 we see that there exists an R-section $\{\alpha_x\}_{x\in \mathrm{D}\,M}$ for $\mathrm{D}\,M$ that restricts to the given R-section on M. By Theorem 87.10 (i) there exists a lift $\mu_{DA} \in \mathrm{H}_n(\mathrm{D}\,M, \mathrm{D}\,M \setminus DA; R)$ of the R-section on $\mathrm{D}\,M$. We claim that $\Phi(\mu_{DA})$ is a lift for M. So let $P \in A_\partial$. We have the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}_n(\mathrm{D}\,M, \mathrm{D}\,M \setminus DA; R) \arrow[r] \arrow[d] \arrow[rr, bend left, "\Phi"] & \mathrm{H}_n(\mathrm{D}\,M, (M \setminus A) \cup M'; R) \arrow[d] & \mathrm{H}_n(M, M \setminus A_\partial; R) \arrow[l, "\beta", "\cong"'] \arrow[d] \\
\mathrm{H}_n(\mathrm{D}\,M, \mathrm{D}\,M \setminus \{P\}; R) \arrow[r, "="] & \mathrm{H}_n(\mathrm{D}\,M, \mathrm{D}\,M \setminus \{P\}; R) & \mathrm{H}_n(M, M \setminus \{P\}; R). \arrow[l, "\cong"]
\end{tikzcd}$$

Since all maps are induced by inclusions of pairs of topological spaces we see that the diagram commutes. By definition of μ_{DA} the left-hand vertical map sends μ_{DA} to α_P. But this implies that the right-hand vertical map sends $\Phi(\mu_{DA})$ to $\alpha_P \in \mathrm{H}_n(M, M \setminus \{P\}; R)$.

Now we turn to the proof of the uniqueness of a lift. The following claim is the key to proving uniqueness.

Claim. Let $t \in (0, 1)$. Note that $A_t \subset A_\partial$. The two inclusion induced maps

$$\begin{array}{rcl}
\gamma\colon\ \mathrm{H}_n(M, M \setminus A_\partial; R) & \to & \mathrm{H}_n(M, M \setminus A_t; R) \\
\delta\colon\ \mathrm{H}_n(M, M \setminus A_t; R) & \to & \mathrm{H}_n(\mathrm{D}\,M, \mathrm{D}\,M \setminus A_t; R)
\end{array}$$

are isomorphisms.

It follows immediately from the Excision Theorem 44.10 for topological manifolds that the map δ is an isomorphism. Regarding the map γ, note that it follows from the long exact sequence of the triple $(M, M \setminus A_t, M \setminus A_\partial)$, see Proposition 43.15, that it suffices to show that $\mathrm{H}_i(M \setminus A_t, M \setminus A_\partial; R) = 0$ for all i. We pick $s \in (t, 1)$. We see that indeed we have

$$\mathrm{H}_i(M\setminus A_t, M\setminus A_\partial; R) \overset{\substack{\text{by the Excision Theorem 43.19 where we excise everything in the complement of } ([0,s)\times\partial M)\cap A\\ \downarrow}}{\xleftarrow{\cong}} \mathrm{H}_i(([0,s)\times\partial M)\setminus(K\times[t,s)), ([0,s)\times\partial M)\setminus((0,s)\times K); R) \underset{\substack{\uparrow\\ \text{follows from Corollary 43.18 since } ([0,s)\times\partial M)\setminus((0,s)\times K) \text{ is a}\\ \text{deformation retract of } ([0,s)\times\partial M)\setminus([t,s)\times K)}}{=} 0.$$

⊞

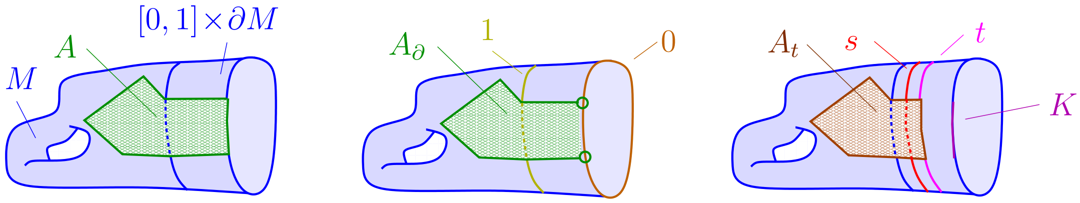


FIGURE 1237. Second illustration for the proof of Theorem 87.30.

Now let $\mu, \nu \in \mathrm{H}_n(M, M \setminus A_\partial; R)$ be two lifts of the R-section $\{\alpha_x\}_{x \in M \setminus \partial M}$. With the same arguments as before one can convince oneself easily of the fact that $\delta^{-1}(\gamma(\mu))$ and $\delta^{-1}(\gamma(\nu))$ are lifts of the R-section $\{\alpha_x\}_{x \in \mathrm{D}\, M}$. By Theorem 87.10 these two lifts are identical. But then it follows from the claim that also $\mu = \nu$. ■

We conclude this long and tiring chapter with the following lemma that is a generalization of Corollary 87.12.

Lemma 87.31. *Let M be an oriented n-dimensional topological manifold and let R be a commutative ring. We suppose that M is equipped with an R-orientation. We use the notation from Theorem 87.30.*

(1) *Let $K \subset L \subset M$ be compact subsets that are products close to the boundary. We denote by $f_{KL} \colon (M, M \setminus L) \to (M, M \setminus K)$ the inclusion map of pairs. The map $f_{KL*} \colon \mathrm{H}_n(M, M \setminus L) \to \mathrm{H}_n(M, M \setminus K)$ sends α_L^M to α_K^M.*
(2) *Let U be an open subset of M and let $A \subset U$ be a compact subset that is a product close to the boundary. We use Lemma 86.17 to view U as an oriented topological manifold. If $i \colon U \to M$ denotes the inclusion map, then $i_*(\alpha_A^U) = \alpha_A^M$.*

PROOF. The lemma follows easily from the functoriality of homology groups and the uniqueness statement of Theorem 87.30. ■

87.8. **The degree of a map between oriented topological manifolds** (∗)**.** Recall that on page 1737 we used the fundamental class to define the degree of a map between smooth manifolds. Now that we also managed to introduce the fundamental class for topological manifolds we can generalize the earlier definition to the setting of topological manifolds. More precisely, we have the following definition.

Definition. Let $f \colon M \to N$ be a map between two compact oriented connected non-empty n-dimensional topological manifolds. We assume that $f(\partial M) \subset \partial N$. We denote by $\deg(f) \in \mathbb{Z}$ the unique integer with

$$\underset{\substack{\uparrow \\ \text{since } f(\partial M) \subset \partial N \text{ we have an induced map } f_* \colon \mathrm{H}_n(M, \partial M) \to \mathrm{H}_n(N, \partial N)}}{f_*}([M]) \;=\; \deg(f) \cdot [N] \;\in\; \mathrm{H}_n(N, \partial N) \;\overset{\substack{\text{by Corollary 87.4 (3)} \\ \downarrow}}{=}\; \mathbb{Z} \cdot [N].$$

We refer to $\deg(f)$ as the *degree of f*.

Remark. For a map $f\colon S^n \to S^n$ the above degree is, by definition, exactly the same as the degree that we defined on page 1181. Furthermore, given a map between *smooth* manifolds it follows from Lemma 87.6 that the above definition of the degree agrees with the definition given on page 1737.

In Propositions 45.23 and 69.7 we already saw that in some settings we can calculate the "global" degree of a map by "local" degrees. To formulate the analogue in our present setting we need the following straightforward definition.

Definition. Let $f\colon M \to N$ be a map between two oriented n-dimensional topological manifolds and let $x \in M \setminus \partial M$ such that $f(x) \in N \setminus \partial N$. We set $y := f(x)$. We suppose that there exists an open neighborhood U of x such that $f(z) \neq f(x)$ for every $z \in U \setminus \{x\}$. We pick an embedding $\Psi\colon \overline{B}^n \to U \subset M$ with $\Phi(0) = x$ and we pick an embedding $\Theta\colon \overline{B}^n \to M$ with $\Psi(0) = y$ and with $f(\Psi(\overline{B}^n)) \subset \Theta(\overline{B}^n)$. We assume that Ψ and Θ are either both orientation-preserving or they are both orientation-reversing. We define the local degree $\deg(f,x)$ as follows:

$$\deg(f,x) := \deg\big((\Theta^{-1}\circ f \circ \Psi)_*\colon \mathrm{H}_n(\overline{B}^n, \overline{B}^n\setminus\{0\}) \to \mathrm{H}_n(\overline{B}^n, \overline{B}^n\setminus\{0\})\big).$$

We leave it to the reader to verify that $\deg(f,x)$ is well-defined.

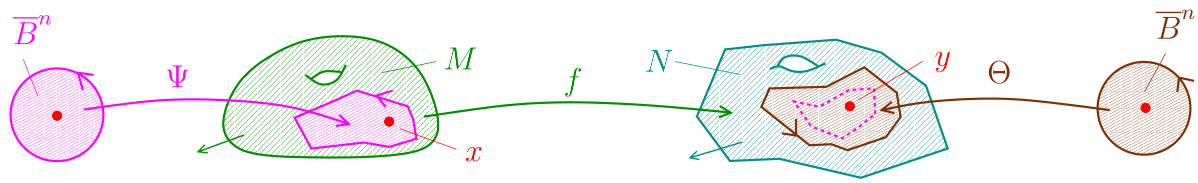


FIGURE 1238

Remark.

(1) It is fairly straightforward to show that for $M = N = S^n$ the above local degree agrees with the definition given on page 1191.
(2) Let $f\colon M \to N$ be a map between two oriented n-dimensional topological manifolds. Let $x \in M \setminus \partial M$ be a point such that f is a local homeomorphism around x. It follows easily from the definitions that

$$\deg(f,x) = \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

It follows from the above remark that the following proposition is a generalization of Propositions 45.23 and 69.7.

Proposition 87.32. *Let $f\colon M \to N$ be a map between two closed oriented connected n-dimensional topological manifolds M and N and let $x \in M$. Let $y \in N$ such that $f^{-1}(\{y\})$ consists of finitely many points $x_1, \dots, x_m$. Then*

$$\deg(f\colon M \to N) = \sum_{i=1}^m \deg(f, x_i).$$

PROOF OF PROPOSITION 87.32. The proof of the proposition is similar to the proofs of Propositions 45.23 and 69.7. Thus, in a vein attempt to keep these notes concise, we will not provide a proof. Instead we hope that the reader will take the baton and carry out the proof. ■

87.9. **Applications.** The results from this chapter makes it possible to read several sections which are placed in the part on simplicial complexes. For example, with the results from this chapter one can now read the following sections:

(1) Section 66.3, in which we calculate the homology groups of lens spaces.
(2) Section 66.4, in which we construct the Poincaré Homology Sphere.
(3) Section 68.5, in which we calculate the homology groups of the connected sum of two closed oriented smooth manifolds.
(4) Section 68.6, in which we calculate the homology groups of a knot complement.

Furthermore, using Lemma 87.6 and a little bit of mental agility one can also read Chapter 69, where we introduce and study the degree of a map between closed oriented connected non-empty smooth manifolds. Finally, Chapter 70, in which we discuss the question, which homology classes can be represented by (sub-) manifolds, is now also accessible to the reader.

Exercises for Chapter 87.

Exercise 87.1. Let M be a connected n-dimensional topological manifold without boundary and let K be a compact subset. Show that there exists an open subset $U \subset M$ that contains K and such that the closure $\overline{U}$ is compact.

Exercise 87.2. Let M be a closed orientable connected non-empty topological manifold of dimension $n \geq 3$. Now we know that $\mathrm{H}_n(M;\mathbb{Z}) \cong \mathbb{Z}$ and that $\mathrm{H}_{n-1}(M;\mathbb{Z})$ is torsion-free. Furthermore we know that $\mathrm{H}_0(M;\mathbb{Z}) \cong \mathbb{Z}$. Can we also say something about the other homology groups? More precisely, are there any restrictions on $\mathrm{H}_{n-2}(M;\mathbb{Z})$ or can any finitely generated abelian group be isomorphic to $\mathrm{H}_{n-2}(M;\mathbb{Z})$ for some such smooth manifold?

Exercise 87.3.

(a) Let M be a compact, connected n-dimensional topological manifold with non-empty boundary. Show that $\mathrm{H}_{n-1}(M;\mathbb{Z}) \cong \mathbb{Z}^k$ for some $k \in \mathbb{N}_0$.
Hint. You can use that we already obtained some results for closed smooth manifolds, in particular we have a good understanding of $\mathrm{H}_{n-1}(\mathrm{D}\,M;\mathbb{Z})$. You could start with the case that M is orientable and afterwards you could consider the case that M is non-orientable.
(b) Let M be a connected n-dimensional topological manifold. Suppose that M is non-compact. Does it follow that $\mathrm{H}_{n-1}(M;\mathbb{Z})$ is a free abelian group? It suffices to give a short justification for your answer.

Exercise 87.4. Let $X = S^1 \cup \{*\}$ be the topological space that is given by the circle "with two 1's", where the topology is defined in the same way as we defined the topology on the "line with two 0's" on page 95. It is straightforward to see that X is locally homeomorphic to open subsets of $\mathbb{R}$, in fact one can easily find a finite atlas for X. Furthermore we know by Lemma 6.1 (4) that X is second-countable. But X is not a topological manifold since it is not Hausdorff. The argument of Exercise 46.5 shows that $\mathrm{H}_1(X;\mathbb{F}_2) \cong \mathbb{F}_2^2$. This shows that in the proof of Theorem 87.2 at some point we must have used that topological manifolds are Hausdorff. Where did we make use of this fact?

Exercise 87.5. Let X be a topological space, let $n \in \mathbb{N}_0$ and let G be an abelian group.

(1) A *locally finite singular n-chain* is a (possibly infinite) sum $\sum\limits_{i\in I} a_i \cdot (\sigma_i\colon \Delta^n \to X)$ with $a_i \in \mathbb{Z}$, such that for every $x \in X$ there exists an open neighborhood U of x such that $\{i \in I \,|\, \sigma_i(\Delta^n) \cap U \neq \varnothing\}$ is finite. We define $\mathrm{C}_n^{\mathrm{lf}}(X)$ as the abelian group given by all locally finite singular n-chains. Note that the usual boundary map defines in an obvious way a boundary map $\mathrm{C}_n^{\mathrm{lf}}(X) \otimes G \to \mathrm{C}_{n-1}^{\mathrm{lf}}(X) \otimes G$. We define the *$n$-th locally finite homology group* $\mathrm{H}_n^{\mathrm{lf}}(X;G)$ to be the n-th homology of the corresponding chain complex.

(2) We consider the set $\mathcal{K}(X)$ of all compact subsets of X. The set $\mathcal{K}(X)$ together with the relation "$\subset$" defined by inclusion is a directed set since the union of two compact subsets is again compact. Given any two compact subsets $K \subset L$ we get an induced map $\mathrm{H}_n(X, X \setminus L; G) \to \mathrm{H}_n(X, X \setminus K; G)$. These homology groups form an *inverse* system over the directed set $(\mathcal{K}(X), \subset)$ and we can form the inverse limit[1258]

$$\overleftarrow{\mathrm{H}}_n(X;G) \;:=\; \varprojlim_{K\in\mathcal{K}(X)} \;\mathrm{H}_n(X, X \setminus K; G).$$

In the following we will see that these two types of homology groups are often naturally isomorphic.

(a) Let *TopPropMap* be the category of topological spaces with proper maps, i.e. the objects are all topological spaces and the morphisms are proper continuous maps, i.e. maps such that preimage of compact sets are compact. Show that both homology groups define a covariant functor from *TopPropMap* to the category *AbGr* of abelian groups.

(b) Give an explicit natural map $\mathrm{H}_n^{\mathrm{lf}}(X;G) \to \overleftarrow{\mathrm{H}}_n(X;G)$ which is the identity for all compact topological spaces.

(c) We assume that there exists a nested sequence $K_1 \subset K_2 \subset \ldots$ of compact subsets of X such that $X = \bigcup\limits_{i\in\mathbb{N}} \mathring{K}_n$. Show that the natural map $\mathrm{H}_n^{\mathrm{lf}}(X;G) \to \overleftarrow{\mathrm{H}}_n(X;G)$ from (b) is an isomorphism.

(d) Let $k \in \mathbb{N}$. Compute $\mathrm{H}_n^{\mathrm{lf}}(\mathbb{R}^k;\mathbb{Z})$.

Remark. Locally finite homology is discussed in detail in[**HR96**, p. 30-34] and [**Spa93**]. Note that statement (c) is [**Spa93**, Theorem 10.1], also see [**HR96**, p 34] for a related result. For completeness sake we note that it follows with some effort from [**Bre97**, Corollary V.12.21 and pages 288-292] that for suitable topological spaces these two homology groups are naturally isomorphic to Borel-Moore homology.

[1258] To the best of my knowledge there is no established name for this natural object.

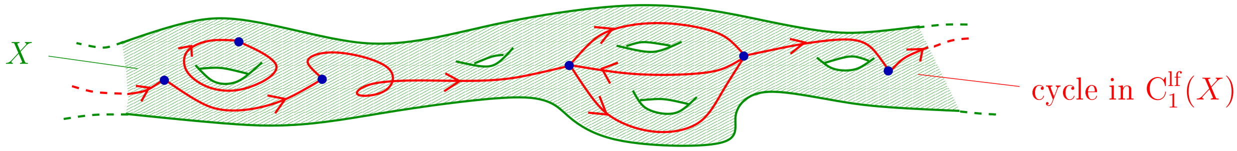


FIGURE 1239. Illustration of Exercise 87.5.

Exercise 87.6. Let M be a connected n-dimensional topological manifold with $\partial M = \varnothing$, let $K \subset M$ be a compact subset and let $x_0 \in K$ be a point.

(a) Show that there exists a *connected* compact subset $L \subset M$ with $K \subset L$.

(b) Show that the map

$$\begin{array}{rcl} \mathrm{Im}(\mathrm{H}_n(M, M \setminus L; \mathbb{Z}) \to \mathrm{H}_n(M, M \setminus K; \mathbb{Z})) & \to & \mathrm{H}_n(M, M \setminus \{x_0\}; \mathbb{Z}) \\ [\sigma] & \mapsto & [\sigma] \end{array}$$

is an isomorphism.

Hint. Use Corollary 87.11.

Exercise 87.7. Let M be a connected orientable non-empty n-dimensional topological manifold such that $\partial M = \varnothing$. As in the above Exercise 87.5 we now consider the group $\overleftarrow{\mathrm{H}}_n(M; \mathbb{Z}) := \varprojlim_{K \in \mathcal{K}(M)} \mathrm{H}_n(M, M \setminus K; \mathbb{Z})$. Show that in our situation this group is naturally isomorphic to $\mathbb{Z}$. We refer to the corresponding natural generator as the fundamental class $[M] \in \overleftarrow{\mathrm{H}}_n(M; \mathbb{Z})$.
Hint. Use Exercise 87.6.

88. The Topological Poincaré Duality Theorem

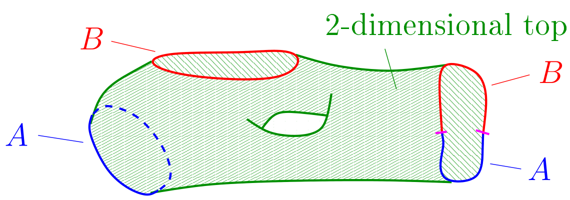
In Chapter 71 we stated and proved the Simplicial Poincaré Duality Theorem 71.4 and we saw that it gives rise to the Poincaré Duality Theorem 71.5 for closed smooth manifolds. As discussed on page 1783, that approach has its limitations. We will rectify this by proving in this chapter the Poincaré Duality Theorem 88.1, which in particular has the advantage that it also applies to topological manifolds.

88.1. **The Poincaré Duality Theorem.** The following theorem is by far the most important theorem regarding homology and cohomology groups of topological manifolds.

Theorem 88.1. (Poincaré Duality Theorem) *Let M be a compact, non-empty n-dimensional topological manifold and let R be a commutative ring. We suppose that M is R-oriented. We denote by $[M] \in \mathrm{H}_n(M, \partial M; R)$ the R-fundamental class. Furthermore suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Then for each $k \in \mathbb{N}_0$ the map*[1259]

$$\begin{array}{rcl} \mathrm{H}^k(M, A; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M, B; R) \\ \sigma & \mapsto & \sigma \cap [M] \end{array}$$

is an isomorphism.



for all commutative rings R we have
$\mathrm{H}^k(M, A; R) \xrightarrow{\cong} \mathrm{H}_{2-k}(M, B; R)$

Figure 1240. Illustration of the Poincaré Duality Theorem 88.1.

We proceed as follows: in this section and the next section we discuss several special cases and applications of the Poincaré Duality Theorem 88.1. In the final two sections of the chapter we will then provide a rigorous, but arguably less intuitive proof, of the theorem. In the coming chapters we will then use the Poincaré Duality Theorem 88.1 to prove many more statements.

Notation. The isomorphism of the Poincaré Duality Theorem 88.1 is called the *Poincaré Duality isomorphism*. We denote it and its inverse both by PD_M, i.e. with the notation of

[1259] Recall that Proposition 74.12 (5) implies that (M, A, B) is an excisive triad. In particular, following the definition of the relative cap product on page 2026, we obtain a relative cap product

$$\mathrm{H}^k(M, A; R) \times \underbrace{\mathrm{H}_n(M, \overbrace{A \cup B}^{=\partial M}; R)}_{\text{contains } [M]} \xrightarrow{\cap} \mathrm{H}_{n-k}(X, B; R).$$

This shows that the map in the Poincaré Duality Theorem is in fact defined.

the Poincaré Duality Theorem 88.1 we have the isomorphism[1260]

$$\begin{array}{rrcl} \mathrm{PD}_M\colon & \mathrm{H}^k(M,A;R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M,B;R) \\ & \sigma & \mapsto & \sigma \cap [M] \end{array}$$

together with the inverse

$$\mathrm{PD}_M\colon\ \mathrm{H}_{n-k}(M,B;R) \xrightarrow{\cong} \mathrm{H}^k(M,A;R).$$

The isomorphisms PD_M thus allow us to go back and forth between homology and cohomology of an R-oriented topological manifold.

Remark. Let M be a compact n-dimensional topological manifold.

(1) Proposition 86.13 says that M is $\mathbb{F}_2$-orientable, i.e. the duality statement holds with $\mathbb{F}_2$-coefficients without any further assumptions on M.
(2) If M is in fact orientable, then it follows from Lemma 86.14 that M is R-orientable for any commutative ring R. In particular the duality statement holds for any commutative ring.
(3) The two most important instances of A and B are that we have either $A = \varnothing$ or $A = \partial M$. Under the hypothesis of the Poincaré Duality Theorem 88.1 we then get isomorphisms

$$\begin{array}{rcl} \mathrm{H}^k(M;R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M,\partial M;R) \\ \sigma & \mapsto & \sigma\cap[M] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{H}^k(M,\partial M;R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M;R) \\ \sigma & \mapsto & \sigma\cap[M]. \end{array}$$

(4) For topological manifolds with non-empty boundary the Poincaré Duality Theorem is sometimes also called the *Poincaré-Lefschetz Duality Theorem*.

Examples.

(1) Let P be a point in M and suppose that M is connected. As we pointed out on page 2026, we have $[M]^* \cap [M] = [P] \in \mathrm{H}_0(M;\mathbb{Z})$. Therefore we obtain that $\mathrm{PD}_M([P]) = [M]^*$.
(2) Earlier we had already performed several explicit calculations which can be viewed as special case of the Poincaré Duality Theorem 88.1:
   (a) On page 2023 we saw explicitly that for any surface Σ the map

$$-\cap[\Sigma]\colon\ \mathrm{H}^1(\Sigma;\mathbb{Z}) \to \mathrm{H}_1(\Sigma;\mathbb{Z})$$

   is an isomorphism. On page 2024 we had obtained similar results for $\mathbb{RP}^2$ with $\mathbb{F}_2$-coefficients.
   (b) On page 2027 we saw explicitly that for the annulus $X = S^1 \times [0,1]$ the map

$$-\cap[X]\colon\ \mathrm{H}^1(X;\mathbb{Z}) \to \mathrm{H}_1(X,\partial X;\mathbb{Z})$$

   is an isomorphism.

[1260] When M is understood, then we drop it from the notation.

(3) It is instructive to consider the case $k = 0$ of the Poincaré Duality Theorem 88.1. We obtain for a compact connected non-empty n-dimensional topological manifold that

$$\mathrm{H}_n(M,\partial M;\mathbb{Z}) \cong \mathrm{H}^0(M;\mathbb{Z}) \underset{\substack{\uparrow \\ \text{Proposition 73.11}}}{\cong} \mathbb{Z} \quad \& \quad \mathrm{H}_n(M;\mathbb{Z}) \cong \mathrm{H}^0(M,\partial M;\mathbb{Z}) \underset{\substack{\uparrow \\ \text{see Proposition 73.11 and page 1830}}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } \partial M = \varnothing, \\ 0, & \text{if } \partial M \neq \varnothing. \end{cases}$$

This calculation is consistent with the results from Theorem 87.1 and Theorem 87.3 (1).

We formulate our last example as a proposition.

Proposition 88.2. *Let M and N be two compact oriented connected topological manifolds. Furthermore let $x_0 \in M$ and $y_0 \in N$.*

(a) *We denote by $p\colon M \times N \to M$ and $q\colon M \times N \to N$ the obvious projections.*
(b) *We denote by $i\colon M \xrightarrow{x\mapsto(x,y_0)} M\times N$ and $j\colon N \xrightarrow{y\mapsto(x_0,y)} M\times N$ the obvious inclusions.*
(c) *We equip $M \times N$ with the product orientation introduced in Proposition 80.10.*
(d) *We work with the decomposition $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$.*
(e) *We write $m = \dim(M)$ and $n = \dim(N)$.*

The following equalities hold:

$$\begin{aligned} \mathrm{PD}_{M\times N}(i_*([M])) &= (-1)^{mn}\cdot q^*([N]^*) &&\in \mathrm{H}^m(M\times N, M\times\partial N;\mathbb{Z}) \\ \mathrm{PD}_{M\times N}(j_*([N])) &= p^*([M]^*) &&\in \mathrm{H}^n(M\times N, M\times\partial N;\mathbb{Z}) \\ \mathrm{PD}_{M\times N}(i_*([M]))\cup\mathrm{PD}_{M\times N}(j_*([N])) &= [M\times N]^* &&\in \mathrm{H}^{m+n}(M\times N,\partial(M\times N);\mathbb{Z}). \end{aligned}$$

PROOF. By Proposition 84.2 (2) we have

$$p^*([M]^*) \cap [M\times N] \,=\, i_*([M]) \quad \text{and} \quad q^*([N]^*) \cap [M\times N] \,=\, (-1)^{mn}\cdot j_*([N]).$$

The first two promised equalities follow from these two calculations and the definition of the Poincaré Duality isomorphisms. The third equality is verified in the following little calculation:

$$\begin{aligned} \mathrm{PD}_{M\times N}(i_*([M]))\cup\mathrm{PD}_{M\times N}(j_*([N])) &\overset{\substack{\text{by the first two equalities} \\ \downarrow}}{=} (-1)^{mn}\cdot q^*([N]^*)\cup p^*([M]^*) \\ &\underset{\substack{\uparrow \\ \text{by Proposition 82.3}}}{=} (-1)^{mn}\cdot(-1)^{mn}\cdot p^*([M]^*)\cup q^*([N]^*) \underset{\substack{\uparrow \\ \text{by Proposition 84.2 (2)}}}{=} [M\times N]^*. \end{aligned}$$

■

88.2. **Consequences of the Poincaré Duality Theorem 88.1.** In this section we give several more or less immediate corollaries to the Poincaré Duality Theorem 88.1. If we restrict ourselves to closed smooth manifolds, then the statements below can also be deduced from the Poincaré Duality Theorem 71.5 that we proved in the last chapter.

We start out with the following proposition.

Proposition 88.3. *Let M be a compact orientable n-dimensional topological manifold and suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. For every $k \in \mathbb{N}_0$*

there exists an isomorphism[1261]

$$\mathrm{H}_{n-k}(M,A;\mathbb{Z}) \;\cong\; \underset{\substack{\uparrow \\ \text{given an abelian group } G \text{ we denote by } \mathrm{Tor}(G) \text{ its torsion-subgroup} \\ \text{and we denote by } \mathrm{F}G := G/\mathrm{Tor}(G) \text{ the maximal torsion-free quotient}}}{\mathrm{FH}_k(M,B;\mathbb{Z}) \oplus \mathrm{Tor}(\mathrm{H}_{k-1}(M,B;\mathbb{Z}))}.$$

Proof. We have the following isomorphisms

$$\begin{array}{rcl} \mathrm{H}_{n-k}(M,A;\mathbb{Z}) & \overset{\substack{\text{Poincaré Duality Theorem 88.1} \\ \downarrow}}{\cong} \mathrm{H}^k(M,B;\mathbb{Z}) & \overset{\substack{\text{Universal Coefficient Theorem 75.13} \\ \downarrow}}{\cong} \mathrm{Hom}(\mathrm{H}_1(M,B),\mathbb{Z}) \oplus \mathrm{Ext}(\mathrm{H}_{k-1}(M,B;\mathbb{Z}),\mathbb{Z}) \\ & & \underset{\substack{\uparrow \\ \text{follows from Lemma 75.5 (5) and the fact that by Lemma 64.7 together with Proposition 85.13 (4)} \\ \text{we know that the homology groups } \mathrm{H}_i(M,B;\mathbb{Z}) \text{ are finitely generated abelian groups}}}{\cong} \mathrm{FH}_k(M,B;\mathbb{Z}) \oplus \mathrm{Tor}(\mathrm{H}_{k-1}(M,B;\mathbb{Z})). \end{array}$$

■

Example. Let $p,q \in \mathbb{N}$ be coprime. In this example we consider the corresponding lens space $L(p,q) = S^3/\mathbb{Z}_p$ as defined and discussed on page 502. Note that $L(p,q)$ is a closed orientable 3-dimensional smooth manifold. In Lemma 66.9 we computed the following homology groups:

$$\begin{array}{rcl} \mathrm{H}_3(L(p,q);\mathbb{Z}) & \cong & \mathbb{Z} \\ \mathrm{H}_2(L(p,q);\mathbb{Z}) & \cong & 0 \\ \mathrm{H}_1(L(p,q);\mathbb{Z}) & \cong & \mathbb{Z}_p \\ \mathrm{H}_0(L(p,q);\mathbb{Z}) & \cong & \mathbb{Z}. \end{array}$$

Fortunately this data is indeed consistent with Proposition 88.3.

The following straightforward corollary to the previous proposition gives in particular a new proof to Theorem 66.6 (2) and Proposition 87.22.

Corollary 88.4. *Let M be a compact orientable n-dimensional topological manifold and suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Then the homology groups $\mathrm{H}_{n-1}(M,A;\mathbb{Z})$ and $\mathrm{H}_{n-1}(M,B;\mathbb{Z})$ are free abelian groups.*

Proof. We have

$$\mathrm{H}_{n-1}(M,A;\mathbb{Z}) \overset{\substack{\text{Proposition 88.3} \\ \downarrow}}{\cong} \underbrace{\mathrm{FH}_1(M,B)}_{\text{free abelian group}} \oplus \underbrace{\mathrm{Ext}(\overbrace{\mathrm{H}_0(M,B)}^{\text{free abelian group}},\mathbb{Z})}_{=0 \text{ by Lemma 75.5}}$$

■

Let X be a topological space X and let $n \in \mathbb{N}_0$. Recall that on page 1366 we defined the n-th Betti number of X as $b_n(X) := \mathrm{rank}(\mathrm{H}_n(X))$. With this definition we have the following immediate corollary to Proposition 88.3.

Corollary 88.5. *Let M be a closed orientable n-dimensional topological manifold. Given any $k \in \mathbb{N}_0$ we have $b_{n-k}(M) = b_k(M)$.*

[1261] It is worth stressing that this isomorphism is not natural.

Remark. Henri Poincaré [**Poi1895, Poi2010, PGL53**, Chapter 9], formulated the conclusion of Corollary 88.5 in a distinctly more elegant way: in his seminal paper "analysis situs", published in 1895, he wrote

"[...] pour une variété fermé les nombres de Betti
également distants des extrêmes sont égaux".

In fact when Henri Poincaré first developed what became known as Poincaré Duality he thought that for any closed orientable connected n-dimensional smooth manifold the groups $\mathrm{H}_k(M;\mathbb{Z})$ and $\mathrm{H}_{n-k}(M;\mathbb{Z})$ are isomorphic. A Danish mathematician, Poul Heegaard [**Hee1898**], pointed out to him that this is not true in general, for example as we just saw, for the lens space $L(p,q)$ we have by Lemma 66.9 that $\mathrm{H}_1(L(p,q);\mathbb{Z}) \cong \mathbb{Z}_p$ but $\mathrm{H}_2(L(p,q);\mathbb{Z}) = 0$. Shortly afterwards Henri Poincaré [**Poi1900**] came up with the correct formulation of Poincaré Duality, namely the one given in Proposition 88.3.

Next let us consider (co-) homology groups with field coefficients. In this context we have the following almost immediate consequence of the Poincaré Duality Theorem 88.1.

Theorem 88.6. *Let M be a compact n-dimensional topological manifold and let $k \in \mathbb{N}_0$.*

(1) *We have*[1262]

$$\begin{array}{rrcl} & \dim_{\mathbb{F}_2}(\mathrm{H}_k(M;\mathbb{F}_2)) & = & \dim_{\mathbb{F}_2}(\mathrm{H}_{n-k}(M,\partial M;\mathbb{F}_2)) \\ \textit{and} & \dim_{\mathbb{F}_2}(\mathrm{H}^k(M;\mathbb{F}_2)) & = & \dim_{\mathbb{F}_2}(\mathrm{H}^{n-k}(M,\partial M;\mathbb{F}_2)). \end{array}$$

(2) *If M is orientable, then for any field $\mathbb{F}$ we have*

$$\begin{array}{rrcl} & \dim_{\mathbb{F}}(\mathrm{H}_k(M;\mathbb{F})) & = & \dim_{\mathbb{F}}(\mathrm{H}_{n-k}(M,\partial M;\mathbb{F})) \\ \textit{and} & \dim_{\mathbb{F}}(\mathrm{H}^k(M;\mathbb{F})) & = & \dim_{\mathbb{F}}(\mathrm{H}^{n-k}(M,\partial M;\mathbb{F})). \end{array}$$

Proof. Let M be a compact n-dimensional topological manifold. We have

$$\begin{aligned} \dim_{\mathbb{F}_2}(\mathrm{H}_k(M;\mathbb{F}_2)) &\underset{\substack{\uparrow \\ \text{Theorem 88.1}}}{=} \dim_{\mathbb{F}_2}(\mathrm{H}^{n-k}(M,\partial M;\mathbb{F}_2)) \underset{\substack{\uparrow \\ \text{Proposition 75.19}}}{=} \dim_{\mathbb{F}_2}(\mathrm{Hom}(\mathrm{H}_{n-k}(M,\partial M;\mathbb{F}_2),\mathbb{F}_2)) \\ &\underset{\substack{\uparrow \\ \text{since Propositions 85.13 (4) and 44.2 (3) and (4) together with Lemma 64.7} \\ \text{imply that } \mathrm{H}_{n-k}(M,\partial M;\mathbb{F}_2) \text{ is finite-dimensional}}}{=} \dim_{\mathbb{F}_2}(\mathrm{H}_{n-k}(M,\partial M;\mathbb{F}_2)). \end{aligned}$$

The proofs of all other statements are basically the same. ■

Example. We consider the real projective space $\mathbb{RP}^n$. On page 1420 we had determined that

$$\mathrm{H}_k(\mathbb{RP}^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0,\dots,n, \\ 0, & \text{otherwise.} \end{cases}$$

So the homology groups with $\mathbb{F}_2$-coefficients are in particular symmetric, which is exactly what the above Poincaré Duality Theorem 88.6 with $\mathbb{F}_2$-coefficients predicts.

[1262]The statement implies of course that $\mathrm{H}^k(M;\mathbb{F})$ and $\mathrm{H}^{n-k}(M,\partial M;\mathbb{F})$ are isomorphic. We prefer to state the result in terms of dimensions to stress that the isomorphism is not natural.

Remark. Let M be a compact n-dimensional topological manifold and let $\mathbb{F}$ be a field. In Proposition 85.13 (4) we had mentioned that all homology groups $\mathrm{H}_k(M;\mathbb{F})$ are finite-dimensional. But this proof relied on the slightly mysterious Theorem 85.12. Now we can give an alternative argument if M is closed and orientable. Indeed, we have

$$\mathrm{H}_k(M;\mathbb{F}) \underset{\substack{\uparrow \\ \text{Theorem 88.1}}}{\cong} \mathrm{H}^{n-k}(M;\mathbb{F}) \underset{\substack{\uparrow \\ \text{Proposition 75.19}}}{\cong} \mathrm{H}_{n-k}(M;\mathbb{F})^* \underset{\substack{\uparrow \\ \text{Theorem 88.1}}}{\cong} \mathrm{H}^k(M;\mathbb{F})^* \underset{\substack{\uparrow \\ \text{Proposition 75.19}}}{\cong} (\mathrm{H}_k(M;\mathbb{F})^*)^*.$$

(Here, as always given an $\mathbb{F}$-vector space V we denote by $V^* = \mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$ its dual vector space.) But by [**Jacb53**, p. 244-248] (see also [**DF04**, Chapter 11.4]) a vector space V is isomorphic to its double dual $(V^*)^*$ if and only if the vector space is finite-dimensional.

Using Theorem 88.6 we obtain the following pretty proposition.

Proposition 88.7. *For every compact odd-dimensional topological manifold M we have*

$$\chi(M) \;=\; \tfrac{1}{2}\chi(\partial M).$$

In particular if M is closed, then $\chi(M) = 0$.

Remark. In Corollary 106.5 we will use handle decompositions to give an alternative proof of Proposition 88.7 in the setting of closed smooth manifolds.

Proof. Let M be a compact topological manifold of dimension $2n+1$. Then

$$\begin{aligned}
\chi(M) &\underset{\substack{\uparrow \\ \text{Proposition 85.34 (4)}}}{=} \sum_{i=0}^{2n+1}(-1)^i\cdot\dim_{\mathbb{F}_2}(\mathrm{H}_i(M;\mathbb{F}_2)) \underset{\substack{\uparrow \\ \text{by Theorem 88.6}}}{=} \sum_{i=0}^{2n+1}(-1)^i\cdot\dim_{\mathbb{F}_2}(\mathrm{H}_{2n+1-i}(M,\partial M;\mathbb{F}_2)) \\
&\underset{\substack{\uparrow \\ \text{follows from the substitution } j=2n+1-i \\ \text{together with the fact that } (-1)^{2n+1}=-1}}{=} -\sum_{j=0}^{2n+1}(-1)^j\cdot\dim_{\mathbb{F}_2}(\mathrm{H}_j(M,\partial M;\mathbb{F}_2)) \underset{\substack{\uparrow \\ \text{Proposition 85.34 (4)}}}{=} -\chi(M,\partial M) \underset{\substack{\uparrow \\ \text{Proposition 85.34 (4)}}}{=} -\chi(M)+\chi(\partial M).
\end{aligned}$$

Summarizing we have shown that $\chi(M) = -\chi(M) + \chi(\partial M)$. The desired equality follows from solving for $\chi(M)$. ■

Proposition 88.7 raises the following question: what can we say about Euler characteristics of closed even-dimensional topological manifolds. By the classification of closed oriented connected 2-dimensional topological manifolds, see the Surface Classification Theorem 23.4, we know that the Euler characteristic of such a topological manifold is even. In higher dimensions there is evidently much more freedom for constructing topological manifolds.

Question 88.8. *Let $n \geq 4$ be even. Is every integer the Euler characteristic of a closed orientable connected n-dimensional topological manifold?*

Using Proposition 88.7 we easily obtain the following surprising (?) corollary.

Corollary 88.9. *The 2-dimensional topological manifold $\mathbb{RP}^2$ is not the boundary of a compact 3-dimensional topological manifold.*

PROOF. Suppose there was a compact 3-dimensional topological manifold M with $\partial M = \mathbb{R}\mathrm{P}^2$. We would obtain that

$$\chi(M) \underset{\substack{\uparrow \\ \text{Proposition 88.7}}}{=} \tfrac{1}{2}\chi(\partial M) = \tfrac{1}{2}\chi(\mathbb{R}\mathrm{P}^2) \underset{\substack{\uparrow \\ \text{see page 1364}}}{=} \tfrac{1}{2}.$$

But the Euler characteristic of a compact topological manifold is by definition an integer. ■

Remark. In Exercise 88.5 we will study the question, whether the Klein bottle is the boundary of a compact 3-dimensional topological manifold M. It follows from the CW-structure on page 934 that the Euler characteristic of the Klein bottle K is zero, therefore Proposition 88.7 does not obstruct the existence of such an M.

Clearly non-orientable topological manifolds exhibit some unusual behavior. So the following question arises.

Question 88.10. *Is every closed* orientable *topological manifold the boundary of a compact topological manifold?*

Note that in Corollary 17.4 we showed that if N is a non-orientable topological manifold, then $\pi_1(N)$ admits an epimorphism onto $\mathbb{Z}_2$.[1263] It follows from the Hurewicz Theorem 52.5 that there exists also an epimorphism $\mathrm{H}_1(N;\mathbb{Z}) \to \mathbb{Z}_2$. For closed 3-dimensional topological manifolds the following Lemma 88.11 is thus a considerable strengthening of Corollary 17.4.

Lemma 88.11. *Let M be a connected closed 3-dimensional topological manifold. If M is non-orientable, then $\mathrm{H}_1(M;\mathbb{Z})$ is infinite.*

PROOF. Let M be a connected closed 3-dimensional topological manifold. We have

$$0 \underset{\substack{\uparrow \\ \text{Proposition 88.7}}}{=} \chi(M) \underset{\substack{\uparrow \\ \text{see page 2086}}}{=} \sum_{n\in\mathbb{N}_0} (-1)^n \cdot b_n(M) \underset{\substack{\uparrow \\ \text{Theorems 87.1 and 87.3}}}{=} b_0(M) - b_1(M) + b_2(M) \underset{\substack{\uparrow \\ \text{by Proposition 41.5}}}{=} 1 - b_1(M) + b_2(M).$$

This equality can only hold if $b_1(M) > 0$, i.e. if $\mathrm{H}_1(M;\mathbb{Z})$ is infinite. ■

Definition. Let $f\colon M \to N$ be a map between closed, oriented connected non-empty n-dimensional topological manifolds. We consider the following diagram of maps:

$$\begin{tikzcd}
\mathrm{H}_k(M;\mathbb{Z}) \arrow[r, shift left, "\mathrm{PD}_M"] \arrow[d, "f_*"'] & \mathrm{H}^{n-k}(M;\mathbb{Z}) \arrow[l, shift left, "\cap[M]"] \\
\mathrm{H}_k(N;\mathbb{Z}) \arrow[r, shift left, "\mathrm{PD}_N"] & \mathrm{H}^{n-k}(N;\mathbb{Z}). \arrow[l, shift left, "\cap[N]"] \arrow[u, "f^*"']
\end{tikzcd}$$

We refer to

$$\begin{array}{rrcl} f^!\colon & \mathrm{H}_k(N;\mathbb{Z}) & \to & \mathrm{H}_k(M;\mathbb{Z}) \\ & \sigma & \mapsto & f^*(\mathrm{PD}_N(\sigma)) \cap [M] \end{array}$$

[1263] If in the proof of Corollary 17.4 we replace Proposition 17.3 by Proposition 86.15 we see that the statement also holds for non-orientable topological manifolds.

as the *Umkehr map* of f.[1264][1265]

The following lemma shows that in some circumstances the Umkehr map is just the transfer map that we introduced on page 1445.

Lemma 88.12. *Let $p\colon \widetilde{N} \to N$ be a finite cover of a closed, oriented, non-empty n-dimensional topological manifold. If we equip $\widetilde{N}$ with the orientation given by Lemma 86.17 (3), then for any $k \in \mathbb{N}_0$ the Umkehr map $p^!\colon \mathrm{H}_k(N;\mathbb{Z}) \to \mathrm{H}_k(\widetilde{N};\mathbb{Z})$ equals the transfer map p^*.*

PROOF. We start our proof with the following claim.

Claim. Let $r \leq s$. Given any singular simplex $\sigma\colon \Delta^s \to N$ and any cochain $\phi \in \mathrm{C}^r(N;\mathbb{Z})$ we have

$$p^*(\phi) \cap p^*(\sigma) \;=\; p^*(\phi \cap \sigma).$$

We write $m = [\widetilde{N} : N]$. We denote by $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_m$ the lifts of σ. Then

$$\begin{aligned} p^*(\phi) \cap p^*(\sigma) \;&=\; p^*(\phi) \cap \sum_{i=1}^m \widetilde{\sigma}_i \;=\; \sum_{i=1}^m p^*(\phi)(\widetilde{\sigma}_i \circ [v_0, \dots, v_r]) \cdot \widetilde{\sigma}_i \circ [v_r, \dots, v_s] \\ &=\; \sum_{i=1}^m \phi((p_* \circ \widetilde{\sigma}_i) \circ [v_0, \dots, v_r]) \cdot \widetilde{\sigma}_i \circ [v_r, \dots, v_s] \\ &\underset{\uparrow}{=}\; p^*\big(\phi(\sigma \circ [v_0, \dots, v_r]) \cdot \sigma \circ [v_r, \dots, v_s]\big) \;=\; p^*(\phi \cap \sigma) \end{aligned}$$

since $p_* \circ \widetilde{\sigma}_i = \sigma$ and since $\widetilde{\sigma}_i \circ [v_0, \dots, v_r]$, $i = 1, \dots, m$ are the lifts of $\sigma \circ [v_0, \dots, v_r]$ ⊞

Now we turn to the actual proof of the lemma. We pick an n-cycle τ that is a representative for the fundamental class $[N] \in \mathrm{H}_n(N;\mathbb{Z})$ of N. By Proposition 87.25 we know that $p^*(\tau)$ represents the fundamental class of $\widetilde{N}$. By definition of the Umkehr map it suffices to prove the following claim.

Claim. For any cocycle $\phi \in \mathrm{C}^l(N;\mathbb{Z})$ we have

$$f^*(\phi) \cap p^*(\tau) \;=\; p^*(\phi \cap \tau).$$

The claim is an immediate consequence of the first claim. ■

We continue with the following useful and typical application of the Poincaré Duality Theorem 88.1.

Proposition 88.13. *Let $n \geq 2$. Give a closed connected n-dimensional topological manifold X the following statements are equivalent:*

(1) *X is $\lfloor \frac{n}{2} \rfloor$-connected,*
(2) *X is $(n-1)$-connected,*
(3) *$\pi_1(X)$ is trivial and $\mathrm{H}_i(X;\mathbb{Z}) = 0$ for $i = 2, \dots, n-1$,*
(4) *$\pi_1(X)$ is trivial and $\mathrm{H}_i(X;\mathbb{Z}) = 0$ for $i = 2, \dots, \lfloor \frac{n}{2} \rfloor$.*

[1264]This map is indeed called “Umkehr map” in English. The name stems from the fact that it is a map on homology that “goes the wrong way”.

[1265]Wikipedia writes: “In category theory [...] certain unusual functors are denoted $f^!$ and $f_!$, with the exclamation mark used to indicate that they are exceptional in some way. They are thus accordingly sometimes called shriek maps, with “shriek” being slang for an exclamation mark [...].”

Definition. A closed connected n-dimensional topological manifold M that satisfies any of the four equivalent statements in Proposition **??** is called an *homotopy n-sphere.* If M is actually a smooth manifold, then we refer to M as a *smooth homotopy n-sphere.*

Remark. In Corollaries 111.15 and Proposition **??** we will give alternative characterizations of homotopy spheres.

PROOF. We have (1) $\Rightarrow$ (4) by Corollary 53.7. We can easily show that (4) $\Rightarrow$ (3) by applying the Poincaré Duality Theorem 88.1 (here we use that X is orientable by Corollary 86.16) and the Universal Coefficient Theorem 75.13 for Cohomology Groups.[1266] The conclusion (3) $\Rightarrow$ (2) is again Corollary 53.7 and finally (2) $\Rightarrow$ (1) is trivial. ■

Finally we want to study degree-one maps between topological manifolds. We recall that on page 1737 we defined the degree of a map $f\colon M \to N$ between closed, oriented connected non-empty n-dimensional topological manifolds to be the unique integer $\deg(f) \in \mathbb{Z}$ with $f_*([M]) = \deg(f) \cdot [N]$. We had obtained the following results:

(1) In Proposition 69.3 we showed that for any closed oriented non-empty n-dimensional topological manifold M there exists a degree-one map $M \to S^n$.[1267]
(2) In Proposition 69.9 we saw that if $\deg(f) = \pm 1$, then $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism.
(3) In Lemma 84.5 we had used cup products to show that there does not exist a map $f\colon S^4 \to S^2 \times S^2$ of non-zero degree.

Now we add the following proposition to our list on results on the non-existence of degree-one maps.

Proposition 88.14. *Let $f\colon M \to N$ be a map between closed oriented non-empty n-dimensional topological manifolds. For every $k \in \mathbb{N}_0$ the following statements hold:*

(1) *The map $f_* \circ f^!$ is multiplication by $\deg(f)$.*
(2) *Suppose that f is a map of degree ± 1.*
    (a) *The map $f^!\colon \mathrm{H}_k(N;\mathbb{Z}) \to \mathrm{H}_k(M;\mathbb{Z})$ splits.*
    (b) *The image $f^!(\mathrm{H}_k(N;\mathbb{Z}))$ is a subsummand of $\mathrm{H}_k(M;\mathbb{Z})$, i.e. there exists a subgroup Q of $\mathrm{H}_k(M;\mathbb{Z})$ such that $\mathrm{H}_k(M;\mathbb{Z}) = f^!(\mathrm{H}_k(N;\mathbb{Z})) \oplus Q$.*
    (c) *There exists an isomorphism $\mathrm{H}_k(M;\mathbb{Z}) \cong \mathrm{H}_k(N;\mathbb{Z}) \oplus Q$ for some finitely generated abelian group Q.*

PROOF. Let $f\colon M \to N$ be a between closed oriented non-empty n-dimensional topological manifolds. We write $d = \deg(f)$. Let $\sigma \in \mathrm{H}_k(N;\mathbb{Z})$. We have

$$(f_* \circ f^!)(\sigma) \underset{\substack{\uparrow \\ \text{definition of } f^!}}{=} f_*\big(f^*(\mathrm{PD}_N(\sigma)) \cap [M]\big) \underset{\substack{\uparrow \\ \text{Lemma 83.8}}}{=} \mathrm{PD}_N(\sigma) \cap f_*([M]) \underset{\substack{\uparrow \\ \text{by definition of } d=\deg(f)}}{=} d\cdot\mathrm{PD}_N(\sigma) \cap [N] \underset{\substack{\uparrow \\ \text{definition of } \mathrm{PD}_N}}{=} d\cdot\sigma.$$

The second statement of the lemma follows from combining the first statement with the Splitting Lemma 46.2. ■

[1266]Note that the proof of (4) $\Rightarrow$ (3) is almost identical to the proof of Lemma **??**.

[1267]Basically the same proof, together with say Theorem 87.1 (4) and an excision argument, shows that the same statement also holds for topological manifolds. We leave it to the reader to fill in the details.

It follows from remark (2) above that there is no degree-one map from $\mathbb{RP}^3$ to $S^1 \times S^2$. In Question 69.12 we had asked whether there exists a degree-one map from $S^1 \times S^2$ to $\mathbb{RP}^3$. Now we can give a negative answer.

Corollary 88.15. *There is no degree-one map from $S^1 \times S^2$ to $\mathbb{RP}^3$.*

Proof. The corollary follows immediately from Proposition 88.14 and the observation that the group $\mathrm{H}_1(\mathbb{RP}^3;\mathbb{Z}) \cong \mathbb{Z}_2$ is not a subsummand of $\mathrm{H}_1(S^1 \times S^2;\mathbb{Z}) \cong \mathbb{Z}$. ■

We conclude this long list of applications of the Poincaré Duality Theorem 88.1 with the following result.

Corollary 88.16. *Let $n \in \mathbb{N}_{\geq 2}$ and let M be a closed oriented connected n-dimensional topological manifold. If there exists a degree one map $f\colon S^n \to M$, then M is a homotopy n-sphere.*

Proof. Let $f\colon S^n \to M$ be a degree one map. Recall that by Proposition 69.9 we know that $f_*\colon \pi_1(S^n) \to \pi_1(M)$ is an epimorphism. Thus we see that M is simply connected. Furthermore it follows from Proposition 88.14 that for $i = 2, \dots, n-1$ the homology group $\mathrm{H}_i(M;\mathbb{Z})$ is isomorphic to a subgroup of $\mathrm{H}_i(S^n;\mathbb{Z})$. Since the latter is zero we obtain from Proposition 88.13 that M is indeed a homotopy sphere. ■

88.3. **The proof of the Poincaré Duality Theorem 88.1 for topological manifolds without boundary.** In this section we will prove the Poincaré Duality Theorem 88.1 for closed topological manifolds. More precisely, we will prove the following theorem.

Theorem 88.17. *Let M be a closed, non-empty n-dimensional topological manifold and let R be a commutative ring. We suppose that M is R-oriented. We denote by $[M] \in \mathrm{H}_n(M;R)$ the R-fundamental class. Then for each $k \in \mathbb{N}_0$ the map*

$$\begin{array}{rccc} \mathrm{PD}_M\colon & \mathrm{H}^k(M;R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M;R) \\ & \sigma & \mapsto & \sigma \cap [M] \end{array}$$

is an isomorphism.

In order to simplify the notation a little bit we will work throughout the section with $R = \mathbb{Z}$. The proof for other commutative rings is verbatim the same.

The key idea is somewhat similar to the strategy of the proof of Theorem 87.10. More precisely, using a Mayer–Vietoris argument and a limit argument we want to reduce the proof eventually to the fairly trivial case of an open ball in $\mathbb{R}^n$.

To carry out the above strategy we need a version of Poincaré Duality for every topological manifold (not necessarily compact) with empty boundary. Formulating this generalized Poincaré Duality Theorem requires some preparations. First we recall some definitions and results from earlier on:

(1) Given a topological space X we denote by $\mathcal{K}(X)$ the set of all compact subsets of X. Let $k \in \mathbb{N}_0$. By Proposition 77.4 we have an identification

$$\varinjlim_{K \in \mathcal{K}(X)} \mathrm{H}^k(X, X \setminus K) \;=\; \mathrm{H}^k_{\mathrm{c}}(X).$$

(2) Let M be an n-dimensional topological manifold with empty boundary. If M is equipped with an orientation $\{\mu_x\}_{x\in M}$, then we saw in Theorem 87.10 (i) that given any compact subset K of M there exists a unique class $\mu_K^M \in \mathrm{H}_n(M, M\setminus K)$ such that for any $x \in K$ the image of μ_K^M under the map

$$\mathrm{H}_n(M, M\setminus K) \;\to\; \mathrm{H}_n(M, M\setminus \{x\})$$

equals the given orientation μ_x at x. If M is understood, then we drop it from the notation, i.e. we just write μ_K instead of μ_K^M.

Lemma 88.18. *Let M be an oriented n-dimensional topological manifold with empty boundary.*

(1) *Let $K \subset L \subset M$ be compact subsets. We denote by $f_{KL}\colon (M, M\setminus L) \to (M, M\setminus K)$ the inclusion map of pairs.*

(a) *The map $f_{KL*}\colon \mathrm{H}_n(M, M\setminus L) \to \mathrm{H}_n(M, M\setminus K)$ sends μ_L to μ_K.*

(b) *For any $k \in \mathbb{N}_0$ the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{H}^k(M, M\setminus K) \arrow[d, "f_{KL}^*"] \arrow[dr, "\cap\mu_K"] & \\
\mathrm{H}^k(M, M\setminus L) \arrow[r, "\cap\mu_L"'] & \mathrm{H}_{n-k}(M).
\end{tikzcd}$$

(2) *For any $k \in \mathbb{N}_0$ there exists a unique map*

$$\mathrm{PD}_M\colon \mathrm{H}^k_{\mathrm{c}}(M) \to \mathrm{H}_{n-k}(M)$$

such that for each compact subset $K \subset M$ the following diagram commutes[1268]

$$\begin{tikzcd}
\mathrm{H}^k(M, M\setminus K) \arrow[rr, "\cap\mu_K"] \arrow[dr] & & \mathrm{H}_{n-k}(M). \\
& \mathrm{H}^k_{\mathrm{c}}(M) \arrow[ur, "\mathrm{PD}_M"'] &
\end{tikzcd}$$

Proof. Let M be an oriented n-dimensional topological manifold with empty boundary.

(1) (a) This is precisely Corollary 87.12 (1).

(b) Let $\varphi \in \mathrm{H}^k(M, M\setminus K)$. We see that

$$\varphi \cap \mu_K \underset{\substack{\uparrow \\ \text{part (1a)}}}{=} \varphi \cap f_{KL*}(\mu_L) \underset{\substack{\uparrow \\ \text{Lemma 83.8}}}{=} f_{KL}^*(\varphi) \cap \mu_L.$$

But this means precisely that the diagram under consideration commutes.

(2) This statement follows immediately from (1b) and the universal property of the direct limit. ■

Throughout the remainder of this section we use the following notation.

[1268] Note that for a class $\varphi \in \mathrm{H}^k(M, M\setminus K)$ and $\mu_K \in \mathrm{H}_n(M, M\setminus K)$ the cap product $\varphi \cap \mu_K$ does indeed lie in $\mathrm{H}_{n-k}(M)$.

Notation. Let $k \in \mathbb{N}_0$ and let X be a topological space.

(1) Given any subset A of X we denote by $i_*\colon\ \mathrm{H}_k(A) \to \mathrm{H}_k(X)$ and $i^*\colon\ \mathrm{H}_k(X) \to \mathrm{H}_k(A)$ the inclusion induced map.

(2) Suppose that X is Hausdorff and that U is an open subset of X. Then we denote by $i\colon\ \mathrm{H}^k_\mathrm{c}(U) \to \mathrm{H}^k_\mathrm{c}(X)$ the map defined by

$$\mathrm{H}^n_\mathrm{c}(U) = \varinjlim_{K\in\mathcal{K}(U)} \mathrm{H}^n(U, U\setminus K) \xleftarrow{\cong} \varinjlim_{K\in\mathcal{K}(U)} \mathrm{H}^n(X, X\setminus K) \to \varinjlim_{K\in\mathcal{K}(X)} \mathrm{H}^n(X, X\setminus K) = \mathrm{H}^n_\mathrm{c}(X).$$

We refer to Lemma 77.5 for the proof that these inclusion maps are covariantly functorial with respect to compositions of inclusions.

We will prove the following generalization of Theorem 88.17.[1269] This more general version of Poincaré Duality will not only be useful for the divide-et-impera approach to the proof of Theorem 88.17, but it will also be of interest in its own right later on.

Theorem 88.19. *Let M be an oriented n-dimensional topological manifold with empty boundary. Then for each $k \in \mathbb{N}_0$ the map*

$$\mathrm{PD}_M\colon\ \mathrm{H}^k_\mathrm{c}(M) \xrightarrow{\cong} \mathrm{H}_{n-k}(M)$$

from Lemma 88.18 is an isomorphism.

Example. Let M be an oriented connected, non-empty n-dimensional topological manifold with empty boundary. Then

$$\mathrm{H}_n(M;\mathbb{Z}) \underset{\substack{\uparrow \\ \text{Theorem 88.19}}}{\cong} \mathrm{H}^0_\mathrm{c}(M;\mathbb{Z}) \underset{\substack{\uparrow \\ \text{see page 1903}}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } M \text{ is compact,} \\ 0, & \text{if } M \text{ is non-compact.} \end{cases}$$

This calculation is consistent with the results obtained in Theorems 87.1 and 87.3.

Remark.

(1) Let M be an oriented connected, non-empty n-dimensional topological manifold with empty boundary. Following the convention introduced on page 2153 we denote the inverse of the isomorphism $\mathrm{PD}_M\colon\ \mathrm{H}^k_\mathrm{c}(M) \xrightarrow{\cong} \mathrm{H}_{n-k}(M)$ also by PD_M.

(2) Given a possibly non-compact oriented topological manifold without boundary Theorem 88.19 allows us to relate singular homology to cohomology with compact support. There is also a different version of Poincaré Duality that relates singular cohomology to a different flavor of homology, namely Borel-Moore homology. We refer to [**Greenb67**], [**Bre97**, Chapter V.9] and [**PS08**, p. 332] for details.

As a warm up for the proof of Theorem 88.19 we will deal with the case $M = B^n$ by hand.[1270] More precisely, we will prove the following lemma.

[1269] Why is it a generalization of Theorem 88.17?

[1270] We use the convention that any n-dimensional submanifold of $\mathbb{R}^n$ is equipped with the orientation induced from the standard orientation of $\mathbb{R}^n$, see Lemma 86.17.

Lemma 88.20. *For each $k \in \mathbb{N}_0$ the map*

$$\mathrm{PD}_{B^n}\colon\ \mathrm{H}^k_{\mathrm{c}}(B^n) \xrightarrow{\cong} \mathrm{H}_{n-k}(B^n)$$

is an isomorphism.

PROOF. Let $k \in \mathbb{N}_0$. Given any $s \in (0,1)$ we write $\overline{B}^n_s := \{x \in \mathbb{R}^n \mid \|x\| \leq s\}$ and we also write $S^{n-1}_s := \partial \overline{B}^n_s$. For the remainder of the proof we fix $0 < r < s < 1$. We denote by $i\colon (\overline{B}^n_s, S^n_s) \to (B^n, B^n \setminus \overline{B}^n_r)$ the inclusion map of pairs.

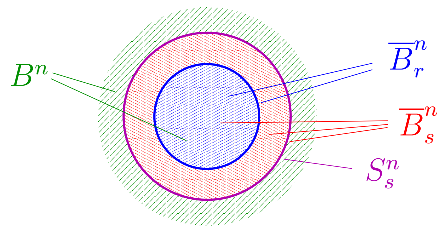


FIGURE 1241. Illustration for the proof of Lemma 88.20.

We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}^k\left(\overline{B}^n_s, S^n_s\right) \arrow[d, "\cap[\overline{B}^n_s]"] & \mathrm{H}^k\left(B^n, B^n \setminus \overline{B}^n_r\right) \arrow[l, "i^*"'] \arrow[r] \arrow[d, "\cap\mu_{\overline{B}^n_r}"] & \mathrm{H}^k_{\mathrm{c}}(B^n) \arrow[d, "\mathrm{PD}_{B^n}"] \\
\mathrm{H}_{n-k}\left(\overline{B}^n_s\right) \arrow[r, "i_*"] & \mathrm{H}_{n-k}(B^n) \arrow[r, "="] & \mathrm{H}_{n-k}(B^n).
\end{tikzcd}$$

The lemma is an immediate consequence of the following claim.

Claim.

(1) The diagram commutes,
(2) the left vertical map is an isomorphism, and
(3) all horizontal maps are isomorphisms.

We turn to the proof of the three statements.

(1) The square on the right commutes by definition of PD_{B^n}. So it remains to show that the square on the left commutes. Thus let $\varphi \in \mathrm{H}^k(B^n, B^n \setminus \overline{B}^n_r)$. We have

$$i_*\big(i^*\varphi \cap [\overline{B}^n_s]\big) \underset{\substack{\uparrow \\ \text{Lemma 83.8}}}{=} \varphi \cap i_*([\overline{B}^n_s]) \underset{\substack{\uparrow \\ \text{since the images of } [\overline{B}^n_s] \text{ and } \mu_{\overline{B}^n_r} \\ \text{in } \mathrm{H}_n(B^n, B^n \setminus \{0\}) \text{ agree}}}{=} \varphi \cap \mu_{\overline{B}^n_r} \ \in\ \mathrm{H}_{n-k}(B^n).$$

(2) We consider the left vertical map of the diagram. Clearly both groups are zero for $k \neq n$. Evidently a generator for $\mathrm{H}^n(\overline{B}^n_s, S^n_s)$ is given by the dual fundamental class $[\overline{B}^n_s]^*$. Since $\overline{B}^n_s$ is path-connected we see, following the discussion on page 2026, that $[\overline{B}^n_s]^* \cap [\overline{B}^n_s] = \langle [\overline{B}^n_s]^*, [\overline{B}^n_s] \rangle = 1 \in \mathrm{H}_0(\overline{B}^n_s) = \mathbb{Z}$. In particular the left vertical map is an isomorphism.

(3) It follows easily from Corollary 43.18 and Lemma 73.13 that the left horizontal maps are isomorphisms. It remains to show that the top right horizontal map is an isomorphism. We consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^k(B^n, B^n \setminus \overline{B}^n_r) \arrow[rr] \arrow[dr] & & \varinjlim\limits_{K \in \mathcal{K}(B^n)} \mathrm{H}^k(B^n, B^n \setminus K) = \mathrm{H}^k_{\mathrm{c}}(B^n). \\
& \varinjlim\limits_{t \in (0,1)} \mathrm{H}^k(B^n, B^n \setminus \overline{B}^n_t) \arrow[ur] &
\end{tikzcd}$$

We make the following two observations:

(a) Every compact subset of B^n is contained in a closed ball $\overline{B}^n_t$ with $t \in (0,1)$. Therefore it follows from Lemma 76.3 that the right diagonal map is an isomorphism.

(b) It follows easily from Lemmas 73.13 and 76.2 (4) that the left diagonal map is an isomorphism.

These two observations together imply that the horizontal map is also an isomorphism. ■

In the proof of Theorem 88.19 we will need the following two lemmas.

Lemma 88.21. *Let M be a topological manifold and let U, V be open subsets such that $M = U \cup V$. Then given any compact subset $C \subset M$ there exist compact subsets $K \subset U$ and $L \subset V$ such that $C = K \cup L$.*

Figure 1242. Illustration of Lemma 88.21.

Proof (∗). Let M be an n-dimensional topological manifold and let U, V be open subsets with $M = U \cup V$. Furthermore let C be a compact subset of M. Using charts one can easily show that given any point $x \in C$ there exists a compact set A_x with the following properties:

(1) the point x lies in the interior of A_x,
(2) if $x \in U$ then the set A_x is contained in U,
(3) if $x \in V$ then the set A_x is contained in V.

It follows from (1), the fact that the sets A_x cover all of C together with the hypothesis that C is compact that there exist finitely many $y_1, \ldots, y_r \in C$ with $C \subset \mathring{A}_{y_1} \cup \cdots \cup \mathring{A}_{y_r}$. One can now easily verify that $K := \bigcup\limits_{y_i \in U} (A_{y_i} \cap C)$ and $L := \bigcup\limits_{y_i \in V} (A_{y_i} \cap C)$ have the desired properties. ■

Lemma 88.22. *Let M be a compact oriented n-dimensional topological manifold and let U, V be open subsets such that $M = U \cup V$ and let $K \subset U$ and $L \subset V$ be compact subsets. Then there exist singular n-chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U \setminus L$, $U \cap V$ and $V \setminus K$ such that the following three equalities hold:*

$$\begin{array}{llll} \text{(a)} & \mu^M_{K\cup L} = [\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}] & \textit{in } \mathrm{H}_n(M, M \setminus (K \cup L)) \\ \text{(b)} & \mu^{U\cap V}_{K\cap L} = [\alpha_{U\cap V}] & \textit{in } \mathrm{H}_n(U\cap V, U \cap V \setminus (K \cap L)) \\ \text{(c)} & \mu^U_K = [\alpha_{U\setminus L} + \alpha_{U\cap V}] & \textit{in } \mathrm{H}_n(U, U \setminus K). \end{array}$$

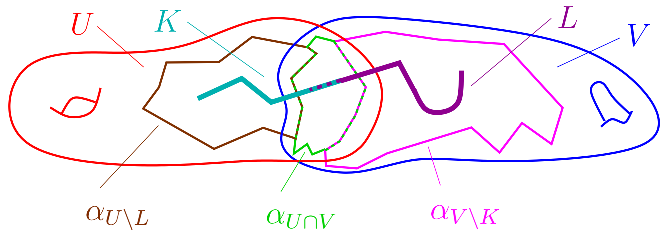


FIGURE 1243. Illustration of Lemma 88.22.

PROOF. We pick a representative $\alpha \in \mathrm{C}_n(M)$ for $\mu^M_{K\cup L}$. Note that the open sets $U\setminus L, U\cap V$ and $V\setminus K$ cover M. It follows from Lemmas 43.28 and 43.31 that there exist singular chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U \setminus L$, $U \cap V$ and $V \setminus K$ such that α is homologous in $\mathrm{C}_k(M)$ to $\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}$. The chains are in particular homologous in $\mathrm{C}_k(M, M \setminus (K \cup L))$ which implies that (a) holds.[1271]

Next note that we have

$$\mu^M_{K\cap L} \underset{\substack{\uparrow \\ \text{Lemma 88.18}}}{=} \text{image of } \mu^M_{K\cup L} = [\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}] \underset{\substack{\uparrow \\ \text{since } \alpha_{U\setminus L}, \alpha_{V\setminus K} \text{ lie in } \mathrm{C}_n(M\setminus(K\cap L))}}{=} [\alpha_{U\cap V}] \in \mathrm{H}_n(M, M \setminus (K \cap L)).$$

But $\alpha_{U\cap V}$ lies in $U \cap V$, therefore it follows from Corollary 87.12 (2) that $\mu^{U\cap V}_{K\cap L} = [\alpha_{U\cap V}]$ in $\mathrm{H}_n(U \cap V, U \cap V \setminus (K \cap L))$. We have thus verified (b).

Finally the proof of equality (c) is basically the same as the proof of equality (b). ■

We recall the following notation and convention:

Convention. Let X be a topological space, let A be a subset and let $k \in \mathbb{N}_0$. As on page 2003 we make the identification

$$\mathrm{C}^k(X, A) \xrightarrow{=} \text{all homomorphisms } \mathrm{C}_k(X) \to \mathbb{Z} \text{ that vanish on } \mathrm{C}_k(A).$$

Furthermore, given another subset B of X we write

$$\mathrm{C}^k(X, \{A, B\}) := \text{all homomorphisms } \mathrm{C}_k(X) \to \mathbb{Z} \text{ that vanish on } \mathrm{C}_k(A) \text{ and on } \mathrm{C}_k(B).$$

If A and B are open subsets of X then we use the isomorphism

$$\mathrm{H}^k(X, A \cup B) \xrightarrow{\cong} \mathrm{H}^k(X, \{A, B\})$$

from Lemma 82.1 to identify the two groups.

[1271] Why is $\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}$ a cycle in $\mathrm{C}_*(M, M \setminus (K \cup L))$?

In the proof of Theorem 88.19 we will also need the following "inverted" Mayer–Vietoris Theorem, which is very similar to Theorem 46.7.

Theorem 88.23. (The inverted Mayer–Vietoris Theorem for Cohomology groups) *Let X be a topological space and let A and B be open subsets.*

(1) *For every $n \in \mathbb{N}_0$ and every cochain $\varphi \in \mathrm{C}^n(X, A \cup B)$ there exist $\varphi_A \in \mathrm{C}^n(X, A)$ and $\varphi_B \in \mathrm{C}^n(X, B)$ such that $\varphi_A + \varphi_B = \varphi$.*

(2) *For all $n \in \mathbb{N}_0$ there exists a unique homomorphism*

$$\delta\colon \mathrm{H}^n(X, A \cap B) \ \to \ \mathrm{H}^{n+1}(X, A \cup B)$$

which has the following property: for every $\varphi_A \in \mathrm{C}^n(X, A)$ and $\varphi_B \in \mathrm{C}^n(X, B)$ such that $\varphi_A + \varphi_B$ is a cocycle in $\mathrm{C}^n(X, A \cap B)$ we have

$$\delta([\varphi_A + \varphi_B]) \ = \ [\delta\varphi_A] \ = \ -[\delta\varphi_B] \ \in \mathrm{H}^n(X, \{A, B\}) = \mathrm{H}^n(X, A \cup B).$$

(3) *The sequence*

$$\begin{aligned} \dots \to \mathrm{H}^n(X, A \cup B) \ &\xrightarrow{i^* \oplus -i^*} \mathrm{H}^n(X, A) \oplus \mathrm{H}_n(X, B) \xrightarrow{i^* \oplus i^*} \mathrm{H}^n(X, A \cap B) \\ \xrightarrow{\delta} \mathrm{H}^{n+1}(X, A \cup B) \ &\to \dots \end{aligned}$$

is exact.

Proof. We start out with the following two observations:

(1) It is straightforward to verify that the following sequence of cochain maps between cochain complexes is exact:

$$0 \ \to \ \mathrm{C}^*(X, \{A, B\}) \xrightarrow{i^* \oplus -i^*} \mathrm{C}^*(X, A) \oplus \mathrm{C}^*(X, B) \xrightarrow{i^* \oplus i^*} \mathrm{C}^*(X, A \cap B) \ \to \ 0.$$

(2) Since the subsets A and B are open we know from Proposition 74.12 (4) that the triad (X, A, B) is excisive. Therefore it follows from Lemma 82.1 that the obvious map $\mathrm{C}^*(X, A \cup B) \to \mathrm{C}^*(X, \{A, B\})$ induces for every $n \in \mathbb{N}_0$ an isomorphism $\mathrm{H}^n(X, A \cup B) \xrightarrow{\cong} \mathrm{H}^n(X, \{A, B\})$.

With these two observations it is straightforward to modify the proof of Theorem 46.5 to obtain the theorem. We leave it to the reader to fill in the details. ■

The key to the proof of Theorem 88.19 is the following lemma.

Lemma 88.24. *Let M be an oriented n-dimensional topological manifold with empty boundary that is the union of two open sets U and V. Then there exists a diagram*

$$\begin{array}{ccccccccc} \dots \longrightarrow & \mathrm{H}^k_{\mathrm{c}}(U \cap V) & \xrightarrow{i \oplus -i} & \mathrm{H}^k_{\mathrm{c}}(U) \oplus \mathrm{H}^k_{\mathrm{c}}(V) & \xrightarrow{i+i} & \mathrm{H}^k_{\mathrm{c}}(M) & \longrightarrow & \mathrm{H}^{k+1}_{\mathrm{c}}(U \cap V) & \longrightarrow \dots \\ & \downarrow{\scriptstyle \mathrm{PD}_{U \cap V}} & & \downarrow{\scriptstyle \mathrm{PD}_U \oplus \mathrm{PD}_V} & & \downarrow{\scriptstyle \mathrm{PD}_M} & & \downarrow{\scriptstyle \mathrm{PD}_{U \cap V}} & \\ \dots \to & \mathrm{H}_{n-k}(U \cap V) & \xrightarrow{i \oplus -i} & \mathrm{H}_{n-k}(U) \oplus \mathrm{H}_{n-k}(V) & \xrightarrow{i+i} & \mathrm{H}_{n-k}(M) & \to & \mathrm{H}_{n-k-1}(U \cap V) & \to \dots \end{array}$$

with the following properties:

(1) *the first two squares commute, the third square commutes up to the sign $(-1)^{k+1}$,*
(2) *the lower exact sequence is the long exact Mayer–Vietoris sequence for $M = U \cup V$,*
(3) *the upper horizontal sequence is also exact.*

PROOF. Let M be an oriented n-dimensional topological manifold with empty boundary. Throughout the proof, given any two subsets $X \subset Y$ and $i \in \mathbb{N}_0$ we write as before $\mathrm{H}^i(Y|X) := \mathrm{H}^i(Y, Y \setminus X)$. We note that if U is open and $K \subset U$ is compact we obtain a map

$$- \cap \mu_K^U \colon \ \mathrm{H}^k(U|K) \ \to \ \mathrm{H}_{n-k}(U).$$

Now let let U and V be open subsets with $M = U \cup V$. Furthermore let $K \subset U$ and $L \subset V$ be compact subsets. We consider the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}^k(M|K\cap L) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}^k(M|K)\oplus\mathrm{H}^k(M|L) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}^k(M|K\cup L) \arrow[r, "\delta"] \arrow[dd, "\cap\mu^M_{K\cup L}"] & \mathrm{H}^{k+1}(M|K\cap L) \arrow[r] \arrow[d, "\cong"] & \dots \\
& \mathrm{H}^k(U\cap V|K\cap L) \arrow[d, "\cap\mu^{U\cap V}_{K\cap L}"] & \mathrm{H}^k(U|K)\oplus\mathrm{H}^k(V|L) \arrow[d, "\cap\mu^U_K\oplus\cap\mu^V_L"] & & \mathrm{H}^{k+1}(U\cap V|K\cap L) \arrow[d, "\cap\mu^{U\cap V}_{K\cap L}"] & \\
\dots \arrow[r] & \mathrm{H}_{n-k}(U\cap V) \arrow[r] & \mathrm{H}_{n-k}(U)\oplus\mathrm{H}_{n-k}(V) \arrow[r] & \mathrm{H}_{n-k}(M) \arrow[r, "\partial"] & \mathrm{H}_{n-k-1}(U\cap V) \arrow[r] & \dots
\end{tikzcd}$$

which we now explain in more detail:

(1) We write $A = M \setminus K$ and $B = M \setminus L$. The upper horizontal sequence is the inverted Mayer–Vietoris Sequence for cohomology groups from Theorem 88.23.
(2) The lower horizontal sequence is the Mayer-Vietoris sequence coming from Theorem 46.5, corresponding to the decomposition $M = U \cup V$.
(3) The three vertical maps on the top are induced by inclusions of pairs of topological spaces, they are isomorphisms by the Excision Theorem 43.20 (2).

The following is the key claim of the proof.

Claim. The first two rectangles commute and the third rectangle commutes up to the sign $(-1)^{k+1}$.

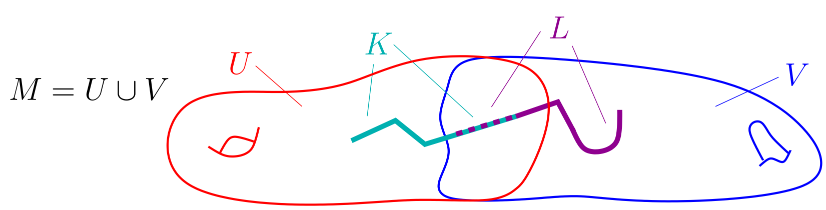


FIGURE 1244

Before we prove the claim we show that it implies Theorem 88.19. For each group appearing in the above diagram we take the direct limit over the directed set[1272]

$$I \ = \ \{(K, L) \mid K \text{ compact subset of } U \text{ and } L \text{ compact subset of } V\}$$

[1272]We view I as a directed set given by the relation $(K, L) \leq (K', L')$ if $K \subset K'$ and $L \subset L'$. It is clear that each of the groups in the diagram forms a direct system with respect to this directed set.

and we consider the induced maps of direct limits that were introduced on page 1243. We make the following remarks regarding this new diagram:

(1) Using Lemma 76.4 one can show that the groups that we obtain taking the direct limit over I are naturally isomorphic to the corresponding cohomology groups with compact support. We carry out the argument for two groups:

$$\varinjlim_{(K,L)\in I} \mathrm{H}^k(U\cap V|K\cap L) \underset{\substack{\uparrow \\ \text{we apply Lemma 76.4 to the cofinal morphism of directed sets given by } (K,L)\mapsto K\cap L}}{=} \varinjlim_{C\in\mathcal{K}(U\cap V)} \mathrm{H}^k(U\cap V|C) = \mathrm{H}^k_{\mathrm{c}}(U\cap V).$$

and

$$\varinjlim_{(K,L)\in I} \mathrm{H}^k(M|K\cup L) \underset{\substack{\uparrow \\ \text{we apply Lemma 76.4 to the morphism of directed sets given} \\ \text{by } (K,L)\mapsto K\cup L\text{, this morphism is cofinal by Lemma 88.21}}}{=} \varinjlim_{C\in\mathcal{K}(M)} \mathrm{H}^k(M|C) = \mathrm{H}^k_{\mathrm{c}}(M).$$

All other cases are treated in a very similar fashion.

(2) By the functoriality of direct limits, and since the signs in the commutativity do not depend on K and L, the new diagram also commutes (up to the same signs).

(3) Using (2) and the definition of the "PD-maps" in Lemma 88.18 one can easily verify that the direct limit of the diagram does indeed give rise to the diagram specified in the statement of the lemma.

(4) Proposition 47.2 says that the direct limit of exact sequences is again exact.

All these remarks put together show that the direct limit of the above diagram gives rise precisely to the diagram that we claimed exists and that all claimed statements are satisfied. This concludes the proof that the claim implies the lemma.

We turn to the proof of the claim. We show in three separate subclaims that the three rectangles commute. The proof of the first two subclaims consists of a clever usage of Lemma 83.8. The proof of the third subclaim is much more sophisticated since it relies on understanding the connecting homomorphisms in the Mayer-Vietoris sequences in (co-) homology.

Subclaim. The first rectangle of the diagram on page 2169 commutes.

So we need to show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}^k(M, M\setminus K\cap L) \arrow[rr, "i\oplus -i"] \arrow[d, "\cong"] \arrow[drr, "i\oplus -i"] & & \mathrm{H}^k(M, M\setminus K)\oplus \mathrm{H}^k(M, M\setminus L) \arrow[d, "\cong"] \\
\mathrm{H}^k(U\cap V, U\cap V\setminus K\cap L) \arrow[d, "\cap\mu^{U\cap V}_{K\cap L}"] & & \mathrm{H}^k(U, U\setminus K)\oplus \mathrm{H}^k(V, V\setminus L) \arrow[d, "\cap\mu^U_K\oplus\cap\mu^V_L"] \\
\mathrm{H}_{n-k}(U\cap V) \arrow[rr, "i\oplus -i"] & & \mathrm{H}_{n-k}(U)\oplus \mathrm{H}_{n-k}(V).
\end{tikzcd}$$

Evidently the triangle commutes. We note that the remaining pentagon breaks up into the "direct sum" of two diagrams. By symmetry reasons it suffices to consider the left-hand summands. So let $\phi \in \mathrm{H}^k(M, M \setminus K \cap L)$. We consider the following diagram of maps between pairs of topological spaces

$$\begin{tikzcd}
(U\cap V, U\cap V \setminus K\cap L) \arrow[r, "\alpha"] \arrow[d, "f\circ\alpha"] & (U, U\setminus K\cap L) \arrow[dl, "f"'] \\
(M, M\setminus K\cap L) & (U, U\setminus K). \arrow[l, "f\circ\beta"] \arrow[u, "\beta"']
\end{tikzcd}$$

With this notation we need to show that

$$\alpha_*\big((f\circ\alpha)^*(\phi) \cap \mu_{K\cap L}^{U\cap V}\big) \;=\; (f\circ\beta)^*(\phi) \cap \mu_K^U.$$

But this is indeed the case, since

$$\begin{aligned}
\alpha_*\big(\underbrace{(f\circ\alpha)^*}_{=\alpha^*\circ f^*}(\phi) \cap \mu_{K\cap L}^{U\cap V}\big) \;&\underset{\substack{\uparrow\\ \text{Lemma 83.8}}}{=}\; f^*(\phi)\cap \alpha_*(\mu_{K\cap L}^{U\cap V}) \;\underset{\substack{\uparrow\\ \text{by Corollary 87.12 we have } \alpha_*(\mu_{K\cap L}^{U\cap V}) = \mu_{K\cap L}^U}}{=}\; f^*(\phi)\cap\mu_{K\cap L}^U \\
&\underset{\substack{\uparrow\\ \text{same argument backwards}\\ \text{with the roles of } \alpha \text{ and } \beta \text{ exchanged}}}{=}\; \beta_*\big((f\circ\beta)^*(\phi)\cap\mu_K^U\big) \;\underset{\substack{\uparrow\\ \text{since } \beta \text{ on the first coordinate is the identity}}}{=}\; (f\circ\beta)^*(\phi)\cap\mu_K^U
\end{aligned}$$

This concludes the proof of the subclaim.

Subclaim. The second rectangle of the diagram on page 2169 commutes.

The proof of that subclaim is very similar to the proof of the previous subclaim. Thus we leave it to the reader to fill in the details.

Subclaim. The third rectangle of the diagram on page 2169 commutes up to the sign $(-1)^{k+1}$.

As a reminder, we need to show that the following diagram commutes up to the sign $(-1)^{k+1}$:

$$\begin{tikzcd}
\mathrm{H}^k(M, M\setminus K\cup L) \arrow[r, "\delta"] \arrow[d, "\cap\mu_{K\cup L}"] & \mathrm{H}^{k+1}(M, M\setminus K\cap L) \arrow[r, "\cong"] & \mathrm{H}^{k+1}(U\cap V, U\cap V\setminus K\cap L) \arrow[d, "\cap\mu_{K\cap L}"] \\
\mathrm{H}_{n-k}(M) \arrow[rr, "\partial"] & & \mathrm{H}_{n-k-1}(U\cap V).
\end{tikzcd}$$

So let $[\varphi] \in \mathrm{H}^k(M, M\setminus K\cup L) = \mathrm{H}^k(M, A\cap B)$. We start out with the following preparations:

(1) By Theorem 88.23 (1) we can find $\varphi_A \in \mathrm{C}^k(M, A)$ and $\varphi_B \in \mathrm{C}^k(M, B)$ such that $\varphi = \varphi_A + \varphi_B$.
(2) By Lemma 88.22 there exist singular chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U\setminus L$, $U\cap V$ and $V\setminus K$ such that the following three equalities hold:

$$\begin{array}{lrcll}
\text{(2a)} & \mu_{K\cup L} & = & [\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}] & \text{in } \mathrm{H}_n(M, M\setminus(K\cup L)) \\
\text{(2b)} & \mu_{K\cap L}^{U\cap V} & = & [\alpha_{U\cap V}] & \text{in } \mathrm{H}_n(U\cap V, U\cap V\setminus(K\cap L)) \\
\text{(2c)} & \mu_K^U & = & [\alpha_{U\setminus L} + \alpha_{U\cap V}] & \text{in } \mathrm{H}_n(U, U\setminus K).
\end{array}$$

After these preparations we obtain the following equalities in $\mathrm{H}_{n-k-1}(U\cap V)$:[1273]

$$\text{maps in the rectangle applied clockwise to } [\varphi] \;=\; \delta[\varphi]\cap\mu^{U\cap V}_{K\cap L} \underset{\substack{\uparrow\\ \text{Theorem 88.23 and (2b)}}}{=} [\delta\varphi_A\cap\alpha_{U\cap V}]$$

$$\underset{\substack{\uparrow\\ \text{Lemma 83.1}}}{=} [\varphi_A\cap\partial\alpha_{U\cap V}] \underset{\substack{\uparrow\\ \text{note that (2c) implies that } \partial(\alpha_{U\cap V}+\alpha_{U\setminus L}) \text{ is a singular chain in } U\setminus K,\\ \text{but } \varphi_A \text{ vanishes on singular chains in } A=M\setminus K}}{=} [\varphi_A\cap\partial\alpha_{U\cap V}-\underbrace{\varphi_A\cap\partial(\alpha_{U\cap V}+\alpha_{U\setminus L})}_{=0}] = [-\varphi_A\cap\partial\alpha_{U\setminus L}]$$

$$\underset{\substack{\uparrow\\ \text{since } \varphi_B \text{ vanishes on singular chains in } B=M\setminus L}}{=} [-\varphi_A\cap\partial\alpha_{U\setminus L}-\underbrace{\varphi_B\cap\partial\alpha_{U\setminus L}}_{=0}] = [-(\varphi_A+\varphi_B)\cap\partial\alpha_{U\setminus L}] \underset{\substack{\uparrow\\ \text{since } \varphi=\varphi_A+\varphi_B}}{=} [-\varphi\cap\partial\alpha_{U\setminus L}]$$

$$\underset{\substack{\uparrow\\ \text{by Lemma 83.1 and}\\ \text{since } \delta\varphi=0}}{=} [(-1)^{k+1}\cdot\partial(\varphi\cap\alpha_{U\setminus L})] \underset{\substack{\uparrow\\ \text{by the definition of the connecting homomorphism } \partial \text{ in the}\\ \text{Mayer–Vietoris sequence, see Theorem 46.5 (1)}}}{=} (-1)^{k+1}\cdot\partial[\underbrace{(\varphi\cap\alpha_{U\setminus L})}_{\in \mathrm{C}_{n-k}(U)}+\underbrace{(\varphi\cap\alpha_{U\cap V}+\varphi\cap\alpha_{V\setminus K})}_{\in\mathrm{C}_{n-k}(V)}]$$

$$\underset{\substack{\uparrow\\ \text{by equality (2a)}}}{=} (-1)^{k+1}\cdot\partial\big([\varphi]\cap\mu^{M}_{K\cup L}\big) = (-1)^{k+1}\cdot\text{maps in the rectangle applied counterclockwise to } [\varphi].$$

This concludes the proof of the claim and thus also of the lemma. ■

Lemma 88.24 actually contains an interesting and useful statement which is worth jotting down separately.

Lemma 88.25. *Let M be an oriented n-dimensional topological manifold with empty boundary and let U be an open subset of M. We denote by $i\colon U\to M$ the inclusion. For any $k\in\mathbb{N}_0$ the following two diagrams commute:*

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{c}}(U) & \xrightarrow{i_*} & \mathrm{H}^k_{\mathrm{c}}(M)\\ \downarrow{\scriptstyle \mathrm{PD}_U} & & \downarrow{\scriptstyle \mathrm{PD}_M}\\ \mathrm{H}_{n-k}(U) & \xrightarrow{i_*} & \mathrm{H}_{n-k}(M)\end{array} \qquad \textit{and} \qquad \begin{array}{ccc} \mathrm{H}_k(U) & \xrightarrow{i_*} & \mathrm{H}_k(M)\\ \downarrow{\scriptstyle \mathrm{PD}_U} & & \downarrow{\scriptstyle \mathrm{PD}_M}\\ \mathrm{H}^{n-k}_{\mathrm{c}}(U) & \xrightarrow{i_*} & \mathrm{H}^{n-k}_{\mathrm{c}}(M).\end{array}$$

Proof. The statement that the diagram on the left commutes for any k is contained in Lemma 88.24 (1). Since the maps PD_U and PD_M are isomorphisms the statement that the diagram on the right commutes is just a reformulation of the fact that the diagrams on the left commutes. ■

Now we turn to the actual proof of Theorem 88.19.

Proof of Theorem 88.19. We fix a dimension n. Throughout this proof by a "manifold" we mean an oriented n-dimensional topological manifold without boundary. We say *Poincaré Duality holds for a topological manifold M* if for each $k\in\mathbb{N}_0$ the map

$$\mathrm{PD}_M\colon\ \mathrm{H}^k_{\mathrm{c}}(M)\ \to\ \mathrm{H}_{n-k}(M)$$

[1273] The maps δ and ∂ written in blue are the connecting homomorphisms in the Mayer–Vietoris sequences.

from Lemma 88.18 is an isomorphism. Our goal is to show that Poincaré Duality holds for all topological manifolds.

We start out with the following claim.

Claim. Let W be a topological manifold.

(A) If Poincaré Duality holds for W equipped with an orientation, then it also holds for any other orientation of W.
(B) If W is homeomorphic to a topological manifold for which Poincaré Duality holds, then Poincaré Duality also holds for W.
(C) Suppose that W is the union of two open subsets U and V. If Poincaré Duality holds for U, V and $U \cap V$, then it also holds for W.
(D) If W is the union of a nested sequence $U_1 \subset U_2 \subset \ldots$ of open subsets such that Poincaré Duality holds for each U_i, then it also holds for W.

So let W be a topological manifold.

(A) Clearly it suffices to consider the case that W is path-connected. By Lemma 86.10 the path-oriented connected topological manifold W admits only one other orientation, namely the reverse orientation. Evidently Poincaré Duality also holds for that orientation.
(B) This statement is obvious.
(C) If all maps $\mathrm{PD}_U, \mathrm{PD}_V$ and $\mathrm{PD}_{U\cap V}$ are isomorphisms, then it follows immediately from Lemma 88.24 and the Five Lemma 43.12 that also each PD_W is an isomorphism, i.e. Poincaré Duality holds for $W = U \cup V$.
(D) Let $k \in \mathbb{N}_0$. We suppose that for every $i \in \mathbb{N}$ the map $\mathrm{PD}_{U_i}\colon \mathrm{H}^k_{\mathrm{c}}(U_i) \to \mathrm{H}_{n-k}(U_i)$ is an isomorphism. Using Lemmas 83.8 and 88.18 it is straightforward to verify that for any $i \leq j$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{c}}(U_i) & \xrightarrow{\mathrm{PD}_{U_i}} & \mathrm{H}_{n-k}(U_i) \\ \downarrow & & \downarrow \\ \mathrm{H}^k_{\mathrm{c}}(U_j) & \xrightarrow{\mathrm{PD}_{U_j}} & \mathrm{H}_{n-k}(U_j). \end{array}$$

We obtain isomorphisms

$$\begin{array}{ccccccc} \mathrm{H}^k_{\mathrm{c}}(W) & = & \varinjlim \mathrm{H}^k_{\mathrm{c}}(U_i) & \xrightarrow{\varinjlim \mathrm{PD}_{U_i}} & \varinjlim \mathrm{H}_{n-k}(U_i) & = & \mathrm{H}_{n-k}(W). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Proposition 77.6} & & \substack{\text{isomorphism by our hypothesis and} \\ \text{since the above diagram commutes}} & & \text{Proposition 47.4} & \end{array}$$

We leave it to the reader to verify that the map from left to right is indeed given by PD_W. ⊞

The following claim completes the proof of the theorem.

Claim.

(1) Poincaré Duality holds for every open bounded convex subset of $\mathbb{R}^n$.

(2) For every $m \in \mathbb{N}$ Poincaré Duality holds for the union of m open bounded convex subsets of $\mathbb{R}^n$.
(3) Poincaré Duality holds for every open subset of $\mathbb{R}^n$.
(4) For every $m \in \mathbb{N}$ Poincaré Duality holds for all topological manifolds that admit an atlas with m charts.
(5) Poincaré Duality holds for all topological manifolds.

Now we provide the proof of the claim.

(1) Let M be an open bounded convex subset of $\mathbb{R}^n$. By Proposition 2.52 (1) there exists a homeomorphism $M \to B^n$. It follows from Lemma 88.20 and (B) that Poincaré Duality also holds for M.
(2) We prove the claim by induction on m. By (1) we know that the statement holds for $m = 1$. Now suppose it holds for all unions of $m-1$ open bounded convex subsets of $\mathbb{R}^n$. Let M be the union of m open bounded convex subsets $U_1, \dots, U_m$. We set $V = U_1 \cup \dots \cup U_{m-1}$. By our hypothesis and by induction the statement holds for U_m, for V and also for $V \cap U_m = (U_1 \cap U_m) \cup \dots \cup (U_{m-1} \cap U_m)$, which is the union of $(m-1)$ open bounded convex subsets.[1274] Hence by (C) the statement also holds for
$$M \;=\; V \cup U_m \;=\; (U_1 \cup \dots \cup U_{m-1}) \cup U_m.$$
(3) Let M be an arbitrary open subset of $\mathbb{R}^n$. By Lemma 2.7 we can write M as the union of countably many open balls $B_i, i \in \mathbb{N}$. For $i \in \mathbb{N}$ we set $U_i := \bigcup\limits_{j \leq i} B_j$. By (2) Poincaré Duality holds for each $U_i \in \mathbb{N}$. But then it follows from (D) that Poincaré Duality also holds for M.
(4) We prove this statement by induction on m. The case $m = 1$ is taken care of by (3) and (B). Now suppose Poincaré Duality holds for all topological manifolds that admit an atlas with $m-1$ charts. Let M be a topological manifold that admits an atlas $\{\phi_i \colon U_i \to V_i\}_{i=1,\dots,m}$ with m charts. We write $U = U_1 \cup \dots \cup U_{m-1}$ and $V = U_m$. Both are open subsets of a topological manifold, hence they are also topological manifolds. Furthermore U and V are topological manifolds that admit an atlas with $m-1$ charts respectively one chart. Similarly $U \cap V = (U_1 \cap U_m) \cup \dots \cup (U_{m-1} \cap U_m)$ is a topological manifold that admits an atlas with $m-1$ charts. By induction Poincaré Duality holds for U, V and $U \cap V$. Thus by (C) Poincaré Duality also holds for M itself.
(5) Finally let M be any topological manifold. We saw in Exercise 6.19 (1), using the fact that M is second-countable and using Lemma 2.25, that M admits a countable atlas $\{\Phi_j \colon U_j \to V_j\}_{j \in \mathbb{N}}$. For $i \in \mathbb{N}$ we set $X_i := \bigcup\limits_{j \leq i} U_j$. By (4) Poincaré Duality holds for each X_i. It follows from (D) that Poincaré Duality also holds for M. ■

88.4. **The proof of Poincaré Duality for topological manifolds with boundary.** Now we will finally prove the general statement of the Poincaré Duality Theorem 88.1. For the reader's convenience we recall the statement.

[1274] Here we once again use the convenient fact that the intersection of two convex sets is convex.

Theorem 88.1. (Poincaré Duality Theorem) *Let M be a compact, non-empty n-dimensional topological manifold and let R be a commutative ring. We suppose that M is R-oriented. We denote by $[M] \in \mathrm{H}_n(M, \partial M; R)$ the R-fundamental class. Furthermore suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Then for each $k \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl} \mathrm{H}^k(M, A; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M, B; R) \\ \sigma & \mapsto & \sigma \cap [M] \end{array}$$

is an isomorphism.

Proof of Theorem 88.1 for $A = \partial M$ and $B = \varnothing$. In this proof we consider the case that $A = \partial M$ and $B = \varnothing$. To simplify the notation we will once again only consider the case that $R = \mathbb{Z}$. The proof of the general case is basically the same. We introduce the following objects:

(1) From the Topological Collar Neighborhood Theorem 44.5 we obtain a collar neighborhood $[-1, 0] \times \partial M$ of $\partial M = \{0\} \times M$. We refer to it as the "internal collar".
(2) We define

$$W \ := \ M \cup ([0, 1) \times \partial M)/\sim \ \text{ with } x \sim (x, 0) \text{ for every } x \in \partial M.$$

Sometimes $[0, 1) \times \partial M$ is called an *external collar of M*. Furthermore, given any $t \in [-1, 1)$ we write

$$K_t \ := \ (M \setminus [-1, 0] \times \partial M) \cup ([-1, t] \times \partial M)/\sim .$$

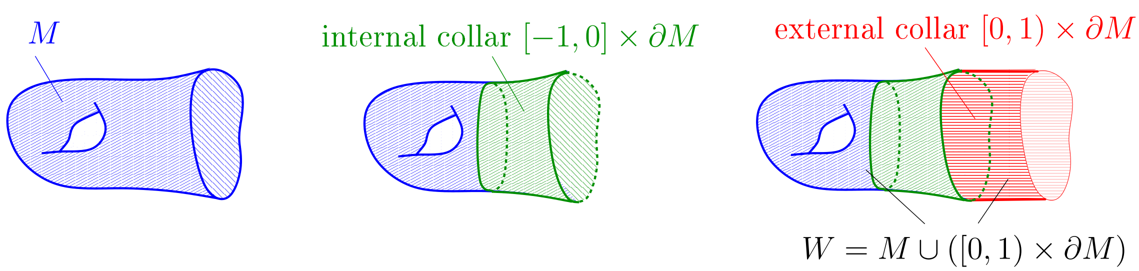


Figure 1245

By elementary arguments and using Proposition 44.2 (1) one can easily show the following statements:

(A) W is an n-dimensional topological manifold with empty boundary,
(B) W admits an orientation such that M is a submanifold and such that $M \to W$ is orientation-preserving,
(C) every K_t is a compact subset of W and given any compact subset K of W there exists a $t \in [-1, 1)$ with $K \subset K_t$.

We denote by $i\colon (M,\partial M) \to (W, W\setminus K_{-1})$ the obvious inclusion map. We obtain the following diagram:

$$\begin{tikzcd}
\mathrm{H}^k(M,\partial M) \arrow[d, "\cap[M]"] & \mathrm{H}^k(W,W\setminus K_{-1}) \arrow[l, "i^*"'] \arrow[r] \arrow[d, "\cap\mu_{K_t}"] & \mathrm{H}^k_{\mathrm{c}}(W) \arrow[d, "\mathrm{PD}_W"] \\
\mathrm{H}_{n-k}(M) \arrow[r, "i_*"] & \mathrm{H}_{n-k}(W) \arrow[r, "="] & \mathrm{H}_{n-k}(W).
\end{tikzcd}$$

It remains to prove the following claim which resembles closely the claim in the proof of Lemma 88.20.

Claim. The following three statements hold:

(1) the diagram commutes,
(2) the right vertical map is an isomorphism, and
(3) all horizontal maps are isomorphisms.

We turn to the proof of the claim.

(1) The right-hand square commutes by definition of PD_W. Using (B) we can apply the same argument as in the proof of the claim in Lemma 88.20 to show that the left-hand square commutes.
(2) Theorem 88.19 implies that the right vertical map is an isomorphism.
(3) It follows easily from Corollary 43.18 and Lemma 73.13 that the left horizontal maps are isomorphisms. It remains to show that the top right horizontal map is an isomorphism. Similar to the proof of Lemma 88.20 we consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^k(W,W\setminus K_{-1}) \arrow[rr] \arrow[dr] & & \varinjlim\limits_{K\in\mathcal{K}(W)} \mathrm{H}^k(W,W\setminus K) = \mathrm{H}^k_{\mathrm{c}}(W). \\
& \varinjlim\limits_{t\in[-1,1)} \mathrm{H}^k(W,W\setminus K_t) \arrow[ur] &
\end{tikzcd}$$

We make the following two observations:
(a) By (C) every compact subset of W is contained in some K_t. Therefore it follows from Lemma 76.3 that the right diagonal map is an isomorphism.
(b) It follows easily from Lemmas 73.13 and 76.2 (4) that the left diagonal map is an isomorphism.

These two observations together imply that the horizontal map is also an isomorphism.

This concludes the proof of the claim and it concludes the proof of Theorem 88.1 in the special case that $A = \partial M$ and $B = \emptyset$. ■

Proof of Theorem 88.1 for the general case. Now we finally consider the general case of Theorem 88.1. So we suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. We recall that by Proposition 74.12 (5) we know that the triad (M, A, B)

is excisive. In this proof, once again we simplify the notation and we only work with $\mathbb{Z}$-coefficients.

We equip ∂M and then also B with the orientations coming from Proposition 86.19 and Lemma 86.17. We consider the following diagram

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{H}^k(M,\partial M) \arrow[r] \arrow[dd, "\cong"', "-\cap[M]"] & \mathrm{H}^k(M,A) \arrow[r] \arrow[dd, "-\cap[M]"] & \mathrm{H}^k(\partial M,A) \arrow[r, "\delta"] \arrow[d, "\cong"', "i^*"] & \mathrm{H}^{k+1}(M,\partial M) \arrow[r] \arrow[dd, "\cong"', "-\cap[M]"] & \ldots \\
& & & \mathrm{H}^k(B,\partial B) \arrow[d, "\cong"', "-\cap[B]"] & & \\
\ldots \arrow[r] & \mathrm{H}_{n-k}(M) \arrow[r] & \mathrm{H}_{n-k}(M,B) \arrow[r, "\partial"] & \mathrm{H}_{n-k-1}(B) \arrow[r] & \mathrm{H}_{n-k-1}(M) \arrow[r] & \ldots
\end{tikzcd}$$

We make the following observations:

(1) The top horizontal sequence is the long exact sequence in cohomology of the triple $(M,\partial M,A)$ provided by Lemma 73.15.
(2) The bottom horizontal sequence is the long exact sequence in homology of the pair (M,B) provided by Corollary 43.16.
(3) The above vertical map $i^*\colon \mathrm{H}^k(\partial M,A) \to \mathrm{H}^k(B,\partial B)$ is induced by the inclusion $i\colon (B,\partial B) \to (\partial M,A)$. It is an isomorphism by Proposition 74.12 (5) together with Lemma 74.13 (3).
(4) The other maps decorated with "$\cong$" are isomorphisms by the special case of the Poincaré Duality Theorem 88.1 that we have already dealt with.

By the Five Lemma 43.12 it remains to prove the following claim.

Claim. The above diagram commutes up to signs, more precisely:

(1) the first rectangle commutes,
(2) the second rectangle commutes up to the sign $(-1)^k$,
(3) the third rectangle commutes.

We consider the three rectangles separately.

(1) We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}^k(M,\partial M) \times \mathrm{H}_n(M,\partial M) \arrow[r] & \mathrm{H}_{n-k}(M) \arrow[d] \\
\mathrm{H}^k(M,\partial M) \times \mathrm{H}_n(M,\partial M) \arrow[r] \arrow[u, "="] \arrow[d] & \mathrm{H}_{n-k}(M,B) \\
\mathrm{H}^k(M,A) \times \mathrm{H}_n(M,\partial M) \arrow[r] & \mathrm{H}_{n-k}(M,B). \arrow[u, "="]
\end{tikzcd}$$

We can convince ourselves that the diagram commutes by applying Lemma 83.8 to the maps $(M,\partial M,\varnothing) \to (M,\partial M,B)$ and $(M,A,B) \to (M,\partial M,B)$ of excisive triads. It follows easily from this observation that the first rectangle of the claim commutes.

Before we continue with the discussion of the other two rectangles we need to make one remark. We denote by $j\colon (B,\partial B) \to (\partial M,A)$ the inclusion map and we denote by $\partial\colon \mathrm{H}_n(M,\partial M) \to \mathrm{H}_{n-1}(\partial M,A)$ the connecting homomorphism in the long exact sequence of the triple $(M, \partial M, A)$. It follows from Proposition 87.27 (1), Lemma 87.24 and the remark on page 1125 that the following equality holds

$$(*) \qquad i_*([B]) \;=\; \partial[M] \in \mathrm{H}_{n-1}(\partial M, A).$$

Now we continue with the proof of the claim.

(2) We apply Lemma 83.10 (2) to the excisive triad (M, A, B). (Recall that $A \cup B = \partial M$ and $A \cap B = \partial B$.) We obtain that for any $k \in \mathbb{N}_0$ the following diagram commutes up to the sign $(-1)^k$:

$$\begin{tikzcd}
\mathrm{H}^k(M,A) \times \mathrm{H}_n(M,\partial M) \arrow[r, "\cap"] \arrow[d, "i^* \quad \partial"'] & \mathrm{H}_{n-k}(M,B) \arrow[dd, "\partial"] \\
\mathrm{H}^k(\partial M,A) \times \mathrm{H}_{n-1}(\partial M,A) \arrow[d, "j^* \cong \quad (j_*)^{-1} \cong"'] & \\
\mathrm{H}^k(B,\partial B) \times \mathrm{H}_{n-1}(B,\partial B) \arrow[r, "\cap"] & \mathrm{H}_{n-k-1}(B).
\end{tikzcd}$$

Together with $(*)$ we obtain that the second rectangle of the claim does indeed commute up to the sign $(-1)^k$.

(3) We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}^k(B,\partial B) \times \mathrm{H}_{n-1}(B,\partial B) \arrow[r, "\cap"] & \mathrm{H}_{n-k-1}(B) \arrow[d, "i_*"] \\
\mathrm{H}^k(\partial M,A) \times \mathrm{H}_{n-1}(\partial M,A) \arrow[r, "\cap"] & \mathrm{H}_{n-k-1}(\partial M) \arrow[d, "i_*"] \\
\mathrm{H}^{k+1}(M,\partial M) \times \mathrm{H}_n(M,\partial M) \arrow[r, "\cap"] & \mathrm{H}_{n-k-1}(M).
\end{tikzcd}$$

The upper part of this diagram commutes by Lemma 83.8 applied to the map $(B, \partial B, \varnothing) \to (\partial M, A, \varnothing)$ between excisive triads. The lower part commutes by Lemma 83.10 (1) applied to the triple $(M, \partial M, A)$. This discussion, together with $(*)$, implies that the third rectangle of the diagram in the claim commutes.

This concludes the proof of the claim and thus also of the Poincaré Duality Theorem 88.1. ■

Remark. It follows from the Poincaré Duality Theorem 88.1 and Corollary 81.12 that if X is a topological space that is homotopy equivalent to a closed orientable n-dimensional topological manifold, then there exists a $\phi \in \mathrm{H}_n(X; \mathbb{Z})$ such that for any $k \in \mathbb{N}_0$ the map

$$- \cap \phi \colon\; \mathrm{H}^k(X; \mathbb{Z}) \;\to\; \mathrm{H}_{n-k}(X; \mathbb{Z})$$

is an isomorphism. One can ask to what degree the converse holds. Namely, if say a CW-complex X admits such a ϕ, is X homotopy equivalent to a closed orientable topological manifold? We will discuss this question on page **??**.

We conclude this section with two propositions that relate Poincaré duality on a manifold to Poincaré duality on its boundary.

Proposition 88.26. *Let M be a compact n-dimensional topological manifold with non-empty boundary. Let R be a commutative ring. We suppose that M is R-oriented. We equip the boundary ∂M with the R-orientation given by Proposition 86.19. As usual we denote by $[M] \in \mathrm{H}_n(M, \partial M; R)$ and $[\partial M] \in \mathrm{H}_{n-1}(\partial M; R)$ the corresponding R-fundamental classes. We consider the following diagram:*

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{H}^k(M,\partial M;R) \arrow[r] \arrow[d, "\cong"', "-\cap[M]"] & \mathrm{H}^k(M;R) \arrow[r] \arrow[d, "\cong"', "-\cap[M]"] & \mathrm{H}^k(\partial M;R) \arrow[r, "\delta"] \arrow[d, "\cong"', "-\cap[\partial M]"] & \mathrm{H}^{k+1}(M,\partial M;R) \arrow[r] \arrow[d, "\cong"', "-\cap[M]"] & \ldots \\
\ldots \arrow[r] & \mathrm{H}_{n-k}(M;R) \arrow[r] & \mathrm{H}_{n-k}(M,\partial M;R) \arrow[r] & \mathrm{H}_{n-k-1}(\partial M;R) \arrow[r] & \mathrm{H}_{n-k-1}(M;R) \arrow[r, "\partial"] & \ldots
\end{tikzcd}$$

where the horizontal sequences are the long exact sequences in (co-) homology of the pair $(M, \partial M)$. (Note that the vertical maps are isomorphisms by Theorem 88.1.) The first and the third square commute, furthermore the second square commutes up to the sign $(-1)^k$.

Proof. We had implicitly proved the proposition in the above proof of the Poincaré Duality Theorem 88.1 applied to the special case that $A = \emptyset$ and $B = \partial M$. ■

Finally we have the following proposition.

Proposition 88.27. *Let M be a compact, non-empty n-dimensional topological manifold and let R be a commutative ring. We suppose that M is R-oriented. We denote by $[M] \in \mathrm{H}_n(M, \partial M; R)$ the R-fundamental class. Furthermore suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. We consider the following two diagrams:*

$$\begin{tikzcd}
\mathrm{H}_k(M,A;R) \arrow[r, "\partial"] \arrow[d, "\mathrm{PD}_M"'] & \mathrm{H}_k(A;R) \arrow[d, "\mathrm{PD}_A"] \\
\mathrm{H}^{n-k}(M,B;R) \arrow[r, "w^*"] & \mathrm{H}^{n+1-k}(A,\partial A;R)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{H}_k(A,\partial A;R) \arrow[r, "w_*"] \arrow[d, "\mathrm{PD}_A"'] & \mathrm{H}_k(M,B;R) \arrow[d, "\mathrm{PD}_M"] \\
\mathrm{H}^{n-k}(A;R) \arrow[r, "\delta"] & \mathrm{H}^{n+1-k}(M,A;R).
\end{tikzcd}$$

The first diagram commutes up to multiplication by $(-1)^k$ whereas the second commutes as it is.

Proof (∗). We already showed on page 2178 that the first diagram commutes up to multiplication by $(-1)^k$. The fact that the second diagram commutes is a fairly straightforward consequence of Lemma 83.10. We leave it to the reader to fill in the details. ■

Exercises for Chapter 88.

Exercise 88.1. Let M be a closed 0-connected n-dimensional topological manifold. We suppose there exists an $(n+1)$-dimensional topological manifold N such that for every $i \in \{1, \ldots, n\}$ we have $\widetilde{\mathrm{H}}_i(N) \cong \mathrm{H}_{i-1}(M)$. Show that M is a homology n-sphere, i.e. show that $\mathrm{H}_i(M) \cong \mathrm{H}_i(S^n)$ for all $i \in \mathbb{N}_0$.
Remark. In view of Proposition 45.7 this exercise can be viewed as a variation on Exercise 45.8.

Exercise 88.2. Let M and N be two closed orientable connected non-empty 3-dimensional topological manifolds such that $\pi_1(M)$ is isomorphic to $\pi_1(N)$. Show that for all $i \in \mathbb{N}_0$ there exists an isomorphism $\mathrm{H}_i(M;\mathbb{Z}) \cong \mathrm{H}_i(N;\mathbb{Z})$.
Remark. This exercise is almost identical to Exercise 71.4.

Exercise 88.3.

(a) Let M be an n-dimensional closed orientable connected topological manifold such that $\pi_1(M)$ is isomorphic to the permutation group S_5. What can you say about $\mathrm{H}^{n-1}(M;\mathbb{Z})$? Does the answer depend on whether or not M has boundary?
(b) Let M be a non-compact orientable topological manifold. Show that the group $\mathrm{H}_{n-1}(M,\partial M;\mathbb{F}_2)$ is non-zero.

Exercise 88.4. Let $p\colon \widetilde{N} \to N$ be a finite covering of a closed, oriented n-dimensional topological manifold. We equip $\widetilde{N}$ with the orientation given by Lemma 86.17

(a) Let $r \leq s$. Show that given any singular simplex $\sigma\colon \Delta^s \to N$ and any cochain $\phi \in \mathrm{C}^r(N;\mathbb{Z})$ we have
$$p^*(\phi) \cap p^*(\sigma) \;=\; p^*(\phi \cap \sigma).$$
Here p^* for homology classes denotes the transfer map.
(b) Show that for any $k \in \mathbb{N}_0$ the Umkehr map $p^!\colon \mathrm{H}_k(N;\mathbb{Z}) \to \mathrm{H}_k(\widetilde{N};\mathbb{Z})$ equals the transfer map p^*.
Hint. Proposition 87.25 might be useful.

Exercise 88.5. Is the Klein bottle the boundary of a compact 3-dimensional topological manifold?

Exercise 88.6.

(a) Let M be a topological manifold and let U, V be open subsets such that $M = U \cup V$. Show that given any compact subset $C \subset M$ there exist compact subsets $K \subset U$ and $L \subset V$ such that $C = K \cup L$.
(b) Does statement (a) hold for any topological space?

Exercise 88.7. Let M be a closed connected 4-dimensional topological manifold that is simply connected and with $\chi(M) = 2$. What can you say about the homeomorphism type of M?

Exercise 88.8.

(a) Let W be a closed orientable connected 4-dimensional topological manifold with $\chi(W) = -3$. Show that W admits a finite connected covering $\widetilde{W}$ with $\chi(\widetilde{W}) < -27$.
(b) Is the hypothesis in (a) that W is orientable necessary?

Remark. This exercise is almost identical to Exercise 71.5.

Exercise 88.9. Let M be a closed 4-dimensional topological manifold.

(a) Show that if M is simply connected, then there exist at most finitely many isomorphism classes of finite groups which can act discretely on M.
(b) Is the hypothesis in (a) that M is simply connected necessary?

Exercise 88.10. Let M be a compact orientable 4-dimensional topological manifold with $\partial M = A \sqcup B$ where A and B are two connected 3-dimensional topological manifolds with $\mathrm{H}_1(A;\mathbb{Z})$ finite. We suppose that the inclusion induced map $\mathrm{H}_1(A;\mathbb{Z}) \to \mathrm{H}_1(M;\mathbb{Z})$ is an epimorphism. Show that the inclusion induced map $\mathrm{H}_1(B;\mathbb{Z}) \to \mathrm{H}_1(M;\mathbb{Z})$ is a monomorphism.

Exercise 88.11. Let M be a closed orientable connected odd-dimensional topological manifold and let $f\colon M \to M$ be a map such that the induced map $f_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)$ is an isomorphism. Show that the Lefschetz number

$$\Lambda(\varphi) \;=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \mathrm{tr}(\varphi_*\colon \mathrm{H}_n(M) \to \mathrm{H}_n(M)),$$

equals zero.
Remark. This exercise is almost identical to Exercise 71.6.

Exercise 88.12. Let M be a closed oriented connected n-dimensional topological manifold. Furthermore let $i \in \{0,\dots,n\}$ and let $v_1,\dots,v_m$ be a basis of $\mathrm{FH}^i(M;\mathbb{Z})$. Show that there exists a basis $\widetilde{v}_1,\dots,\widetilde{v}_m$ of $\mathrm{FH}^{n-i}(M;\mathbb{Z})$ such that for any $k,l \in \{1,\dots,m\}$ we have $\langle v_k \cup \widetilde{v}_l, [M]\rangle = \delta_{kl}$.

Exercise 88.13. Let M be a compact oriented connected n-dimensional topological manifold. We say M is an *n-dimensional homotopy ball* if $\pi_i(M) = 0$ for $i \geq 1$ and if $\pi_1(\partial M) = \{e\}$. Show that if M is an n-dimensional homotopy ball, then ∂M is an homotopy $(n-1)$-sphere, as defined in Exercise 88.13.

Exercise 88.14. Let $n \in \mathbb{N}_{\geq 2}$ and let M be a compact oriented connected n-dimensional topological manifold. Suppose that there exists a degree one map $f\colon (\overline{B}^n, S^{n-1}) \to (M,\partial M)$. Show that M is an n-dimensional homotopy ball.

Exercise 88.15. Let M be a compact oriented connected n-dimensional topological manifold. Let Y be a boundary component of M which is homeomorphic to S^{n-1}. Show that either $[Y] \neq 0 \in \pi_{n-1}(M)$ or M is a homotopy ball, as defined in Exercise 88.13.
Hint. Show that if $[Y]=0$, then there exists a degree one map $(\overline{B}^n, S^{n-1}) \to (M,Y)$. Now use Exercise 88.14.

Exercise 88.16. Let $f\colon M \to N$ be a homeomorphism between two compact oriented connected n-dimensional topological manifolds and let A, B be two compact $(n-1)$-dimensional submanifolds of ∂M with $A \cup B = \partial M$, $A \cap B = \partial A = \partial B$. Show that for any $\sigma \in \mathrm{H}_k(M,A)$ we have

$$f^*(\mathrm{PD}_N(f_*(\sigma))) \;=\; \begin{cases} \ \ \mathrm{PD}_M(\sigma), & \text{if } \sigma \text{ is orientation-preserving,} \\ -\mathrm{PD}_M(\sigma), & \text{if } \sigma \text{ is orientation-reversing.} \end{cases}$$

Remark. This exercise consists mostly of collecting fairly standard facts.

Exercise 88.17. Let M be an oriented connected non-empty n-dimensional topological manifold. As in Exercise 87.5 we consider the group $\overleftarrow{\mathrm{H}}_n(M;\mathbb{Z}) := \varprojlim \mathrm{H}_n(M, M\setminus K; G)$. In Exercise 87.7 we showed that $\overleftarrow{\mathrm{H}}_n(M;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and we showed that there exists a natural generator $[M]$. Let $k \in \mathbb{N}_0$.

(a) Let $\cap\colon \mathrm{H}_c^{n-k}(M;\mathbb{Z}) \times \overleftarrow{\mathrm{H}}_n(M;\mathbb{Z}) \to \mathrm{H}_k(M;\mathbb{Z})$ be the a natural cap product that we introduced in Exercise 83.4. Show that the map $\mathrm{H}_c^{n-k}(M;\mathbb{Z}) \to \mathrm{H}_k(M;\mathbb{Z})$ given by capping with $[M]$ defines an isomorphism.
Remark. Use Lemma 88.18.

(b) Now let $f\colon M \to N$ be a proper map between two connected orientable non-empty n-dimensional topological manifolds. Since f is proper we obtain by Exercise 87.5 an induced map $f_*\colon \overleftarrow{\mathrm{H}}_k(M;\mathbb{Z}) \to \overleftarrow{\mathrm{H}}_k(N;\mathbb{Z})$. We define $\deg(f) \in \mathbb{Z}$ via the equality $f_*([M]) = \deg(f) \cdot [N]$. Furthermore, using (a), we can define the Umkehr map $f^!\colon \overleftarrow{\mathrm{H}}_k(N;\mathbb{Z}) \to \overleftarrow{\mathrm{H}}_k(M;\mathbb{Z})$. Show that the map $f^! \circ f_*\colon \mathrm{H}_k(M;\mathbb{Z}) \to \mathrm{H}_k(M;\mathbb{Z})$ is given by multiplication by $\deg(f)$.

Remark. The goal of this exercise is to generalize the ideas and concepts behind Proposition 88.14 to the setting of non-compact topological manifolds.

Exercise 88.18. Recall that given a topological space X we defined on page 2015 its *cup length* $\mathrm{cl}(X) \in \mathbb{N}_0 \cup \{\infty\}$ as follows:

$$\mathrm{cl}(X) \;:=\; \max\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{c} \text{there exists a commutative ring } R \text{ and } R\text{-cohomology} \\ \text{classes } x_1, \dots, x_n \text{ in degrees} \geq 1 \text{ with } x_1 \cup \dots \cup x_n \neq 0 \end{array} \right\}.$$

Let M and N be two closed oriented connected n-dimensional topological manifolds. Show that if there exists a map $f\colon M \to N$ of degree one, then $\mathrm{cl}(M) \geq \mathrm{cl}(N)$.
Remark. At least in spirit this exercise is related to the Rudyak Conjecture 82.13.

Exercise 88.19. Let M be a closed simply connected 4-dimensional topological manifold. Show that $\mathrm{H}_1(M;\mathbb{Z}) = 0$, $\mathrm{H}_3(M;\mathbb{Z}) = 0$ and show that that $\mathrm{H}_2(M;\mathbb{Z})$ is torsion-free.

Exercise 88.20. Let X and Y be compact oriented connected non-empty n-dimensional topological manifolds and let $f\colon X \to Y$ be a map of degree one. Does it follow that the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}_k(X;\mathbb{Z}) \arrow[r, "f_*"] \arrow[d, "\mathrm{PD}_X"', "\cong"] & \mathrm{H}_k(Y;\mathbb{Z}) \arrow[d, "\cong"', "\mathrm{PD}_Y"] \\ \mathrm{H}^{n-k}(X;\mathbb{Z}) & \mathrm{H}^{n-k}(Y;\mathbb{Z}) \arrow[l, "f^*"'] \, ? \end{tikzcd}$$

Exercise 88.21. Let M be a compact orientable n-dimensional topological manifold. Determine the homology groups of the double $\mathrm{D}\,M$ in terms of the homology groups of M.
Remark. It is tempting to use Mayer–Vietoris sequences, but this will most likely not lead very far. At some stage in the calculation it will be helpful to use the Poincaré Duality Theorem 88.1.

89. Poincaré duality and codimension-zero submanifolds (∗)

In this short chapter we will study the interplay between Poincaré Duality on a given compact topological manifold M and on a given codimension-zero submanifold W of M. Later on, in the applications of this chapter's results, we will also deal with submanifolds with corner, so we will use this opportunity to finally introduce the notion of a submanifold with corner. The results and concepts of this chapter we will be used on several occasions. Nonetheless it is safe to skip this chapter and to move on to the next, significantly more entertaining, chapter.

89.1. **Submanifolds with corner.** Recall that on page 270 we defined what it means for a subset W of a manifold M to be a submanifold, namely it means that given any $P \in W$ there is a chart for M of one of the three types (α), (β) and (γ) that are defined properly on page 270 and that are sketched in Figure 1246.

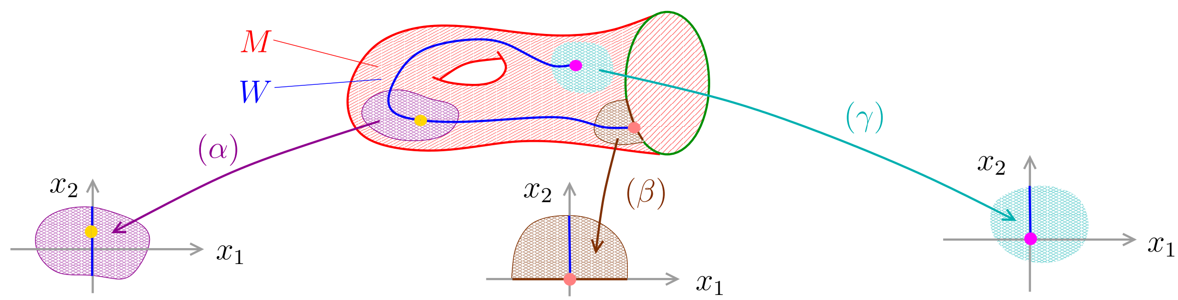


Figure 1246

Now we introduce the more general notion of a submanifold with corner.

Definition. Let M be an n-dimensional topological manifold. We say a subset $W \subset M$ is a *k-dimensional submanifold with corner*[1275] if for every $P \in W$ there exists a chart of the type (α), (β) and (γ) from page 270 or if

(δ) there exists a chart $\Phi\colon U \to V$ of type (ii)[1276] for M such that

$$\Phi(U \cap W) \;\subset\; \{(0,\dots,0,x_1,\dots,x_k)\,|\,x_1,\dots,x_k \in \mathbb{R} \text{ with } x_{k-1} \geq 0 \text{ and } x_k \geq 0\}$$

and with $\Phi(P) \in \{(0,\dots,0,x_1,\dots,x_{k-2},0,0)\,|\,x_1,\dots,x_{k-2} \in \mathbb{R}\}$.

If M is an n-dimensional smooth manifold, then we demand that the charts come from the maximal smooth atlas. We refer to the set of points in W that admit a chart of type (δ) as the *corner set of* W and we denote it by $\partial_c W$.

Remark. Throughout these lecture notes we will only consider submanifolds with corner of codimension *zero*. But the concept can of course also play a role for other codimensions.

The following lemma also introduces a notation that we will use throughout these notes.

[1275] Often in the literature such objects are called "submanifold with corners". Following [**Wall16**, p. 30] we write "submanifold with corner" instead.

[1276] See page 261 for the definition of a chart of type (ii).

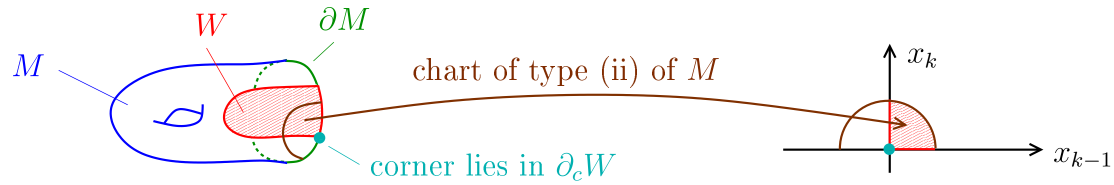


FIGURE 1247

Lemma 89.1. *Let M be an n-dimensional topological manifold and let W be a compact codimension-zero submanifold with corner. We write*

$$\partial_0 W \; := \; W \cap \overline{M \setminus W} \quad \text{ and } \quad \partial_1 W \; := \; W \cap \partial M.$$

The following five statements hold:

(1) *The subset W itself is an n-dimensional topological manifold.*
(2) *The boundary of the topological manifold W equals $\partial_0 W \cup \partial_1 W$. Furthermore $\partial_0 W$ and $\partial_1 W$ are $(n-1)$-dimensional submanifolds of the boundary ∂W such that*

$$\partial_0 W \cap \partial_1 W \; = \; \partial(\partial_0 W) \; = \; \partial(\partial_1 W) \; = \; \partial_c W.$$

(3) *The set $\mathring{W} := W \setminus \partial_0 W$ is the interior of W in M, in particular $\mathring{W}$ is an open subset of M.*
(4) *The subset $M \setminus \mathring{W}$ is an codimension-zero submanifold of M with corner given by $\partial_c(M \setminus \mathring{W}) = \partial_c W$. Furthermore $\partial_0(M \setminus \mathring{W}) = \partial_0 W$ and $\partial_1(M \setminus \mathring{W}) = \overline{\partial M \setminus \partial_1 W}$.*
(5) *$\partial_0 W$ is a proper submanifold of M, in the sense of the definition on page 270.*

The analogous statements also hold if we replace "topological" by "smooth".

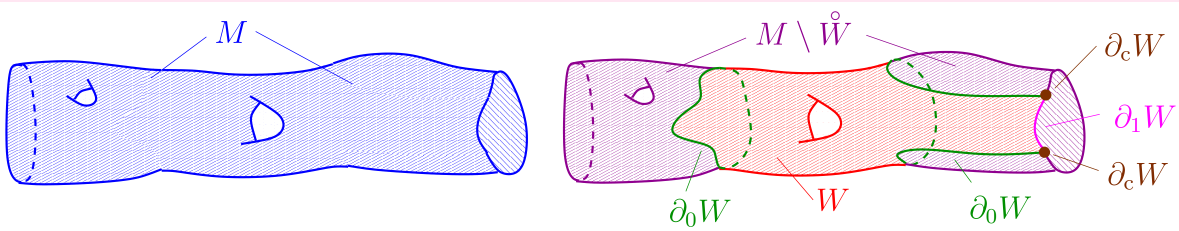


FIGURE 1248. Illustration of Lemma 89.1.

Remark. Let M be a topological manifold and let W be a submanifold in the sense of the definition on page 270. Evidently it is also a submanifold with corner. Furthermore note that it follows from Proposition 44.3 (1) together with Lemma 89.1 that $\partial_c W = \varnothing$.

SKETCH OF PROOF. In the proof that W itself is a (topological) manifold we only have to worry about the corners. But for these we can use the approach taken in Proposition 80.9 to find a (smooth) atlas. The remaining statements follow reasonably easily from Proposition 44.2 and the definitions. We leave the slightly annoying task of filling in the details to the reader. ■

In the following we will generalize a few results on codimension-zero submanifolds to codimension-zero submanifolds with corner. It is unlikely that the reader's pulse will quicken while reading the remainder of this section.

First we have the following variation on the Excision Theorem 44.10 for Topological Manifolds.

Theorem 89.2. (Excision Theorem for Submanifolds with Corner) *Let M be an n-dimensional topological manifold and let W be a compact codimension-zero submanifold of M with corner. We write $\mathring{W} := W \setminus \partial_0 W$. For every $k \in \mathbb{N}_0$ and every abelian group G the inclusion induced maps*

$$
\begin{array}{lrcl}
\text{(a)} & \mathrm{H}_k(W, \partial_0 W; G) & \xrightarrow{\cong} & \mathrm{H}_k(M, M \setminus \mathring{W}; G) \\
\text{(b)} & \mathrm{H}_k(W, \partial W; G) & \xrightarrow{\cong} & \mathrm{H}_k(M, \underbrace{(M \setminus \mathring{W}) \cup \partial M}_{(M\setminus\mathring{W})\cup\partial_1 W}; G) \\
\text{(c)} & \mathrm{H}_k(W, \partial_1 W; G) & \xrightarrow{\cong} & \mathrm{H}_k(W \cup \partial M, \partial M; G)
\end{array}
$$

are isomorphisms. Similarly, all maps in cohomology "in the other direction" are also isomorphisms.

Proof. We will prove the statements on homology groups with $\mathbb{Z}$-coefficients in Exercise 89.2. The statement for homology with G-coefficients and the statement for cohomology with G-coefficients are an immediate consequence of the above together with Corollaries 57.20 and 73.20. ■

We turn to the topic of orientations and fundamental classes. First we have the following lemma which generalizes Lemma 86.17 (2).

Lemma 89.3. *Let R be a commutative ring and let M be a topological manifold that is equipped with an R-orientation. If W is a codimension-zero submanifold of M with corner, then W admits a unique R-orientation such that the inclusion map $N \to M$ is orientation-preserving.*

Proof. This statement follows immediately from Lemma 86.17 (1). ■

Convention. Let M be a topological manifold and let W be a codimension-zero submanifold with corner. If M is oriented, then we equip W with the orientation coming from Lemma 89.3.

The following lemma is a generalization of Lemma 87.24.

Lemma 89.4. *Let M be a compact oriented n-dimensional topological manifold. Furthermore let $W \subset M$ be a compact non-empty codimension-zero submanifold* with corner. *We write $\mathring{W} = W \setminus \partial W$. The following two statements hold:*

1. *The inclusion induced map $\mathrm{H}_n(W, \partial W; \mathbb{Z}) \to \mathrm{H}_n(M, (M \setminus \mathring{W}) \cup \partial_1 W; \mathbb{Z})$ is an isomorphism.*
2. *The image of the fundamental classes $[M] \in \mathrm{H}_n(M, \partial M; \mathbb{Z})$ and $[W] \in \mathrm{H}_n(W, \partial W; \mathbb{Z})$ under the maps*

$$\mathrm{H}_n(M, \partial M; \mathbb{Z}) \longrightarrow \mathrm{H}_n(M, (M \setminus \mathring{W}) \cup \partial_1 W; \mathbb{Z}) \underset{\substack{\uparrow \\ \text{isomorphism by the Excision Theorem 89.2 (2)}}}{\longleftarrow} \mathrm{H}_n(W, \partial W; \mathbb{Z})$$

agree.

PROOF. The proof of Lemma 89.4 is basically identical to the proof of Lemma 87.24. ■

89.2. **Poincaré duality and codimension-zero submanifolds.** The following lemma says that given a homology class in a codimension-zero submanifold W of some topological manifold M we can apply Poincaré duality in M or W and essentially we end up with the same result.

Lemma 89.5. *Let M be a compact oriented n-dimensional topological manifold and let W be a compact codimension-zero with corner. We equip W with the induced orientation from Lemma 86.17. We write $\mathring{W} = W \setminus \partial_0 W$.*[1277] *Furthermore let A and B be disjoint unions of components of ∂M with $\partial M = A \cup B$ and $\partial_1 W \subset A$. We fix the following notation:*

(1) *We denote by w the obvious map $(W, \partial_1 W) \to (M, A)$ and also the obvious map $(W, \partial_0 W) \to (M, M \setminus \mathring{W})$,*
(2) *we denote by $p\colon (M, B) \to (M, M \setminus \mathring{W})$ the obvious map.*

Then for every $k \in \mathbb{N}_0$ the following diagram commutes[1278][1279]

$$\begin{array}{ccccc}
\mathrm{H}_k(W, \partial_1 W; \mathbb{Z}) & \xrightarrow{\mathrm{PD}_W} & \mathrm{H}^{n-k}(W, \partial_0 W; \mathbb{Z}) & \xrightarrow[\cong]{(w^*)^{-1}} & \mathrm{H}^{n-k}(M, M \setminus \mathring{W}; \mathbb{Z}) \\
\downarrow{\scriptstyle w_*} & & & & \downarrow{\scriptstyle p^*} \\
\mathrm{H}_k(M, A; \mathbb{Z}) & & \xrightarrow{\mathrm{PD}_M} & & \mathrm{H}^{n-k}(M, B; \mathbb{Z}).
\end{array}$$

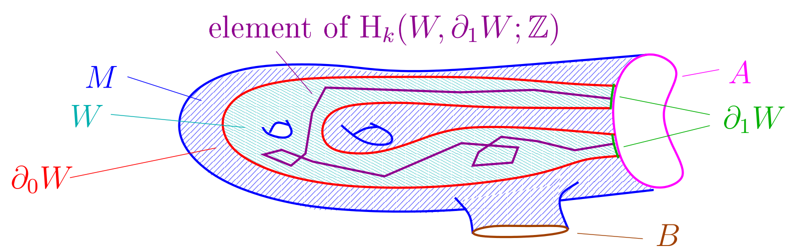


FIGURE 1249. Illustration of Lemma 89.5.

In the proof of Lemmas 89.5 and 89.6 will use the full force of Lemma 83.8 applied to excisive triads. Since this can be seriously confusing we recall the general statement of the lemma, before we provide the proof of Lemma 89.5.

[1277]Recall that by Proposition 44.3 the set $\mathring{W} = W \setminus \partial_0 W$ is indeed the interior of W in M.

[1278]Here we use that by the Excision Theorem 89.2 we know that $w\colon (W, \partial_0 W) \to (M, M \setminus \mathring{W})$ induces an isomorphism on cohomology groups.

[1279]Note that by Lemma 89.1 we can apply the Poincaré Duality Theorem 88.1 to obtain that $\cap[W]\colon \mathrm{H}^{n-k}(W, \partial_1 W; \mathbb{Z}) \to \mathrm{H}_k(W, \partial_0 W; \mathbb{Z})$ is an isomorphism. We denote its inverse by PD_W.

Lemma 83.8. *Let $f\colon (X,A,B)\to(\widetilde{X},\widetilde{A},\widetilde{B})$ be a map between two excisive triads. Then for any choice of $\varphi\in \mathrm{H}^{n-k}(\widetilde{X},\widetilde{A};\mathbb{Z})$ and $\sigma\in \mathrm{H}_m(X,A\cup B;\mathbb{Z})$ we have*

$$\varphi\cap \overset{\substack{f_*\colon\ \mathrm{H}_m(X,A\cup B)\to \mathrm{H}_m(\widetilde{X},\widetilde{A}\cup\widetilde{B})\\ \downarrow}}{f_*}(\sigma) \;=\; \underset{\substack{\uparrow\\ f_*\colon\ \mathrm{H}_k(X,B)\to \mathrm{H}_k(\widetilde{X},\widetilde{B})}}{f_*}(\overset{\substack{f^*\colon\ \mathrm{H}^{n-k}(\widetilde{X},\widetilde{A})\to \mathrm{H}^{n-k}(X,A)\\ \downarrow}}{f^*}(\varphi)\cap\sigma)\;\in\; \mathrm{H}_k(\widetilde{X},\widetilde{B};\mathbb{Z}).$$

PROOF OF LEMMA 89.5. By definition of the isomorphisms PD_W and PD_M, proving the lemma is equivalent to proving that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}_k(W,\partial_1 W;\mathbb{Z}) \arrow[d, "w_*"] & \mathrm{H}^{n-k}(W,\partial_0 W;\mathbb{Z}) \arrow[l, "\cap[W]"', "\cong"] & \mathrm{H}^{n-k}(M,M\setminus \mathring{W};\mathbb{Z}) \arrow[l, "\cong"', "w^*"] \arrow[d, "p^*"] \\
\mathrm{H}_k(M,A;\mathbb{Z}) & & \mathrm{H}^{n-k}(M,B;\mathbb{Z}). \arrow[ll, "\cap[M]"', "\cong"]
\end{tikzcd}$$

In the following we use the following notation:

(1) We write $\widehat{W}:=W\setminus\partial W$.
(2) We denote the map $(W,\partial W)\to(M,M\setminus\widehat{W})$ by w as well
(3) We denote the map $(M,\partial M)\to(M,M\setminus\widehat{W})$ by p as well.

Now let $\beta\in \mathrm{H}^{n-k}(M,M\setminus\mathring{W};\mathbb{Z})$. We have the following equality in $\mathrm{H}_k(M,A;\mathbb{Z})$:

$$w_*(w^*(\beta)\cap[W]) \underset{\substack{\uparrow\\ \text{Lemma 83.8 applied to the map}\\ w\colon (W,\partial_0 W,\partial W)\to(M,M\setminus\mathring{W},M\setminus\widehat{W})}}{=} \beta\cap w_*([W]) \overset{\substack{\text{by Lemma 89.4 we have } p_*([M])=w_*([W])\in \mathrm{H}_m(M,M\setminus\widehat{W})\\ \downarrow}}{=} \beta\cap p_*([M]) \underset{\substack{\uparrow\\ \text{Lemma 83.8 applied to the map}\\ p\colon (M,B,A)\to(M,M\setminus\mathring{W},A)}}{=} p_*(\overbrace{p^*(\beta)\cap[M]}^{\in \mathrm{H}_k(M,A)}) \underset{\substack{\uparrow\\ \text{since } p\colon (M,A)\to(M,A)\\ \text{is id we have } p_*=\mathrm{id}_*=\mathrm{id}}}{=} p^*(\beta)\cap[M].$$

We have thus shown that the above diagram does indeed commute. ■

We conclude this short introduction to submanifolds with corner with the following lemma that also relates Poincaré Duality in a topological manifold M to Poincaré Duality in a compact submanifold W with corner.

Lemma 89.6. *Let M be a compact oriented n-dimensional topological manifold and let W be a compact codimension-zero submanifold with corner. Furthermore let A and B be disjoint unions of components of ∂M with $\partial M=A\cup B$. We fix the following notation:*

(1) *we set $C:=\partial_1 W\cap A$ and $D:=(\partial_1 W\cap B)\cup\partial_0 W$,*
(2) *we set $\mathring{W}:=W\setminus\partial_0 W$,*
(3) *the obvious maps $(W,C)\to(M,A)$ and $(W,D)\to(M,(M\setminus\mathring{W})\cup B)$ are denoted by w, and*
(4) *we denote by $p\colon (M,B)\to(M,M\setminus\mathring{W})$ the obvious map.*

Then for every $k \in \mathbb{N}_0$ the following diagram commutes [1280]

$$\begin{tikzcd}
\mathrm{H}_k(M,B;\mathbb{Z}) \arrow[rr, "\mathrm{PD}_M"] \arrow[d, "p_*"] & & \mathrm{H}^{n-k}(M,A;\mathbb{Z}) \arrow[d, "w^*"] \\
\mathrm{H}_k(M,(M\setminus \mathring{W})\cup B;\mathbb{Z}) \arrow[r, "(w_*)^{-1}", "\cong"'] & \mathrm{H}_k(W,D;\mathbb{Z}) \arrow[r, "\mathrm{PD}_W"] & \mathrm{H}^{n-k}(W,C;\mathbb{Z}).
\end{tikzcd}$$

Example. We continue with the notation from the previous lemma. If we set $A := \partial M$ and $B := \varnothing$, then $C = \partial_1 W$ and $D = \partial_0 W$ and we obtain that the following diagram commutes:

$$\text{(a)} \qquad \begin{tikzcd}
\mathrm{H}_k(M;\mathbb{Z}) \arrow[rr, "\mathrm{PD}_M"] \arrow[d, "p_*"] & & \mathrm{H}^{n-k}(M,\partial M;\mathbb{Z}) \arrow[d, "w^*"] \\
\mathrm{H}_k(M,M\setminus \mathring{W};\mathbb{Z}) \arrow[r, "(w_*)^{-1}", "\cong"'] & \mathrm{H}_k(W,\partial_0 W;\mathbb{Z}) \arrow[r, "\mathrm{PD}_W"] & \mathrm{H}^{n-k}(W,\partial_1 W;\mathbb{Z}).
\end{tikzcd}$$

Furthermore, if we set $A := \varnothing$ and $B := \partial M$, then $C = \varnothing$ and $D = \partial W$, and we see that the following diagram commutes:

$$\text{(b)} \qquad \begin{tikzcd}
\mathrm{H}_k(M,\partial M;\mathbb{Z}) \arrow[rr, "\mathrm{PD}_M"] \arrow[d, "p_*"] & & \mathrm{H}^{n-k}(M;\mathbb{Z}) \arrow[d, "w^*"] \\
\mathrm{H}_k(M,(M\setminus \mathring{W})\cup \partial M;\mathbb{Z}) \arrow[r, "(w_*)^{-1}", "\cong"'] & \mathrm{H}_k(W,\partial W;\mathbb{Z}) \arrow[r, "\mathrm{PD}_W"] & \mathrm{H}^{n-k}(W;\mathbb{Z}).
\end{tikzcd}$$

These two important special cases are illustrated in Figure 1250.

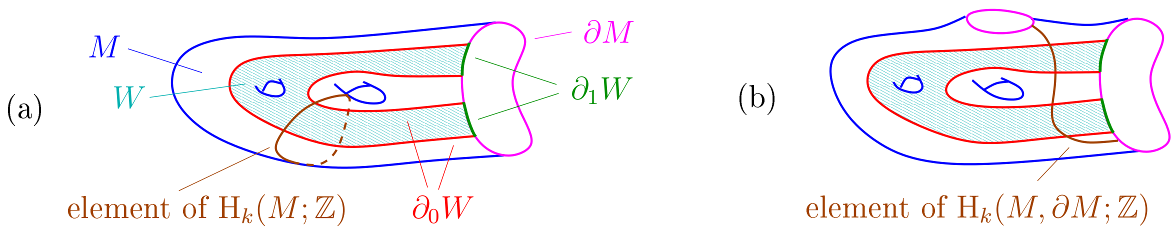


FIGURE 1250. Illustration of Lemma 89.6.

PROOF. We set $\widehat{W} := W \setminus (\partial_0 W \cup C)$. Note that $M \setminus \widehat{W} = (M \setminus \mathring{W}) \cup B$. By definition of the isomorphisms PD_W and PD_M, proving the lemma is equivalent to proving that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}_k(M,B;\mathbb{Z}) \arrow[d, "p_*"] & & \mathrm{H}^{n-k}(M,A;\mathbb{Z}) \arrow[ll, "\cap[M]"', "\cong"] \arrow[d, "w^*"] \\
\mathrm{H}_k(M,M\setminus \widehat{W};\mathbb{Z}) & \mathrm{H}_k(W,D;\mathbb{Z}) \arrow[l, "w_*"', "\cong"] & \mathrm{H}^{n-k}(W,C;\mathbb{Z}). \arrow[l, "\cap[W]"']
\end{tikzcd}$$

[1280] We use that $w\colon (W,D) \to (M,(M\setminus \mathring{W})\cup B)$ induces an isomorphism on cohomology. This follows from a slight generalization of the Excision Theorem 89.2.

So let $\beta \in H^{n-k}(M, A; \mathbb{Z})$. We have the following equalities in $H_k(M, M \setminus \mathring{W}; \mathbb{Z})$:

$$p_*(\beta \cap [M]) \underset{\substack{\uparrow \\ \text{Lemma 83.8 applied to the map} \\ p\colon (M,A,B) \to (M,A,M\setminus\widehat{W}) \\ \text{of excisive triads}}}{=} \beta \cap p_*([M]) \underset{\substack{\uparrow \\ \text{by Lemma 89.4 we} \\ \text{have } p_*([M]) = w_*([W]) \\ \text{in } H_n(M,(M\setminus\mathring{W})\cup\partial M)}}{=} \beta \cap w_*([W]) \underset{\substack{\uparrow \\ \text{Lemma 83.8 applied to the map} \\ w\colon (W,C,D) \to (M,A,M\setminus\widehat{W}) \\ \text{of excisive triads}}}{=} w_*(w^*(\beta) \cap [W]).$$

■

Exercises for Chapter 89.

Exercise 89.1. Let M be an n-dimensional topological manifold and let W be a compact codimension-zero submanifold of M with corner. Show that there exists an embedding $\varphi\colon [0,1] \times \partial_0 W \to W$ with the following two properties:

(1) For every $x \in \partial_0 W$ we have $\varphi(0, x) = x$.
(2) For every $t \in [0,1]$ and every $x \in \partial_c W = \partial(\partial_0 W)$ we have $\varphi(t, x) \in \partial_1 W$.

Hint. Adapt the proof of the Topological Collar Neighborhood Theorem 44.5.

Exercise 89.2. Let M be an n-dimensional topological manifold and let W be a compact codimension-zero submanifold of M with corner. We write $\mathring{W} := W \setminus \partial_0 W$. Show that for every $k \in \mathbb{N}_0$ the inclusion induced maps

(a) $$H_k(W, \partial_0 W) \xrightarrow{\cong} H_k(M, M \setminus \mathring{W})$$
(b) $$H_k(W, \partial W) \xrightarrow{\cong} H_k(M, (M \setminus \mathring{W}) \cup \partial M)$$
(c) $$H_k(W, \partial_1 W) \xrightarrow{\cong} H_k(W \cup \partial M, \partial M)$$

are isomorphisms.
Remark. This exercise is a variation on the Excision Theorem 44.10 for Topological Manifolds. You might want to use Exercise 89.1.

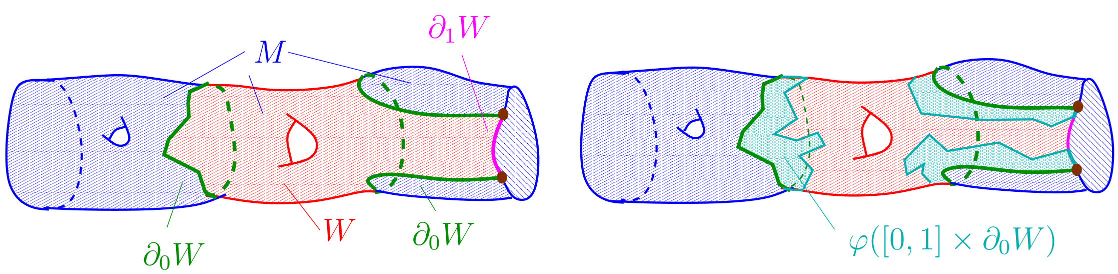


FIGURE 1251. Illustration for Exercises 89.1 and 89.2.

Part X

Applications of Poincaré Duality

In this chapter we will use the Poincaré Duality Theorem 88.1 to determine the cup product on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$. These calculations will allow us to prove several interesting consequences which at first glance have very little to do with cup products.

90.1. **Non-singular pairings and the cup product.** Before we start drawing more conclusions from the Poincaré Duality Theorem 88.1 we need to introduce some purely algebraic definitions.

Definition. Let R be a commutative ring and let V, W and S be R-modules.

(1) A *pairing* on $V \times W$ is a map
$$V \times W \;\to\; S$$
that is R-bilinear, i.e. that is R-linear in V and that is R-linear in W.[1281]

(2) We say that a pairing $\langle\ ,\ \rangle \colon V \times W \to S$ is *non-singular* if the induced maps
$$\begin{array}{rcl} V & \to & \mathrm{Hom}_R(W,S) \\ v & \mapsto & \left(\begin{array}{rcl} W & \to & S \\ w & \mapsto & \langle v,w\rangle \end{array}\right) \end{array} \quad\text{and}\quad \begin{array}{rcl} W & \to & \mathrm{Hom}_R(V,S) \\ w & \mapsto & \left(\begin{array}{rcl} V & \to & S \\ v & \mapsto & \langle v,w\rangle \end{array}\right) \end{array}$$
are isomorphisms.

We start out with the following elementary lemma.

Lemma 90.1. *Let R be a commutative ring. Let V and W be free R-modules of the same finite rank and let*
$$\langle\ ,\ \rangle \colon V \times W \to R$$
be a pairing.

(1) *We denote by n the rank of V. Let $v_1, \dots, v_n$ be a basis for V and let $w_1, \dots, w_n$ be a basis for W. Then the following statement holds:*
$$\text{the pairing } \langle\ ,\ \rangle \text{ is non-singular} \quad\Longleftrightarrow\quad \det(\langle v_i, w_j\rangle)_{i,j=1,\dots,n}) \text{ is a unit of } R.$$

(2) *Suppose V and W are of rank one. If v and w are generators of V respectively W, then the following statement holds:*
$$\text{the pairing } \langle\ ,\ \rangle \text{ is non-singular} \quad\Longleftrightarrow\quad \langle v,w\rangle \text{ is a unit of the ring } R.$$

Proof. The second statement is evidently just a special case of the first statement. So it suffices to prove the first statement. Let $v_1, \dots, v_n$ be a basis for V and let $w_1, \dots, w_n$ be a basis for W. We denote by $v_1^*, \dots, v_n^* \in \mathrm{Hom}(V,R)$ and $w_1^*, \dots, w_n^* \in \mathrm{Hom}(W,R)$ the dual bases. We consider the maps
$$\begin{array}{rcl} V & \to & \mathrm{Hom}_R(W,R) \\ v & \mapsto & \left(\begin{array}{rcl} W & \to & R \\ w & \mapsto & \langle v,w\rangle \end{array}\right) \end{array} \quad\text{and}\quad \begin{array}{rcl} W & \to & \mathrm{Hom}_R(V,R) \\ w & \mapsto & \left(\begin{array}{rcl} V & \to & R \\ v & \mapsto & \langle v,w\rangle \end{array}\right) \end{array}.$$

[1281] In most examples we will be interested in $S = R$, but later on we will, oddly enough, also develop a fondness for $R = \mathbb{Z}$ and $S = \mathbb{Q}/\mathbb{Z}$.

Since $\langle\, , \,\rangle$ is R-bilinear both maps are R-homomorphisms. Furthermore, it is straightforward to show that with respect to these bases both the R-homomorphisms are represented by the transpose of the matrix $(\langle v_i, w_j\rangle_{i,j=1,\dots,n})$. The lemma is an immediate consequence of this observation. ■

Remark. The proof of Lemma 90.1 shows that if V and W are free R-modules of finite rank, then the pairing is non-singular if and only if either of the two maps $V \to \operatorname{Hom}(W,R)$ or $W \to \operatorname{Hom}(V,R)$ is an isomorphism.[1282]

Now we return to topology.

Proposition 90.2. *Let M be a closed n-dimensional topological manifold and let $k \in \mathbb{N}_0$.*

(1) *Suppose M is oriented with fundamental class $[M]$. If $\mathrm{H}_{k-1}(M;\mathbb{Z})$ and $\mathrm{H}_{n-k-1}(M;\mathbb{Z})$ are torsion-free, then the cup product pairing*
$$\begin{array}{rcl} \mathrm{H}^k(M;\mathbb{Z}) \times \mathrm{H}^{n-k}(M;\mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi,\psi) & \mapsto & \langle \varphi \cup \psi, [M]\rangle \end{array}$$
is non-singular.

(2) *Let $\mathbb{F}$ be a field. If M is $\mathbb{F}$-oriented with fundamental class $[M]$, then for any $k \in \mathbb{N}_0$ the cup product pairing*
$$\begin{array}{rcl} \mathrm{H}^k(M;\mathbb{F}) \times \mathrm{H}^{n-k}(M;\mathbb{F}) & \to & \mathbb{F} \\ (\varphi,\psi) & \mapsto & \langle \varphi \cup \psi, [M]\rangle \end{array}$$
is non-singular.

Proof. Let M be a closed n-dimensional topological manifold and let $k \in \mathbb{N}_0$.

(1) We suppose that M is oriented with fundamental class $[M]$ and we suppose that $\mathrm{H}_{k-1}(M;\mathbb{Z})$ and $\mathrm{H}_{n-k-1}(M;\mathbb{Z})$ are torsion-free. It follows from Proposition 85.13 (4), our hypothesis on $\mathrm{H}_{k-1}(M;\mathbb{Z})$ and $\mathrm{H}_{n-k-1}(M;\mathbb{Z})$, the classification of finitely generated abelian groups, Lemma 75.5 (3) and the Universal Coefficient Theorem 75.13 that $\mathrm{H}_{k-1}(M;\mathbb{Z})$, $\mathrm{H}_{n-k-1}(M;\mathbb{Z})$, $\mathrm{H}^k(M;\mathbb{Z})$ and $\mathrm{H}^{n-k}(M;\mathbb{Z})$ are finitely generated free abelian groups. Therefore by the remark on page 2193 it suffices to show that the map
$$\begin{array}{rrcl} \Omega\colon & \mathrm{H}^{n-k}(M;\mathbb{Z}) & \to & \operatorname{Hom}(\mathrm{H}^k(M;\mathbb{Z}),\mathbb{Z}) \\ & \psi & \mapsto & (\varphi \mapsto \langle \varphi\cup\psi,[M]\rangle) \end{array}$$
is an isomorphism. We start out with the following claim.

Claim. The following diagram commutes:[1283]
$$\begin{tikzcd} \mathrm{H}^{n-k}(M;\mathbb{Z}) \arrow[rr, "\Omega"] \arrow[dr, "\mathrm{ev}"'] & & \operatorname{Hom}(\mathrm{H}^k(M;\mathbb{Z}),\mathbb{Z}). \\ & \operatorname{Hom}_{\mathbb{Z}}(\mathrm{H}_{n-k}(M;\mathbb{Z}),\mathbb{Z}) \arrow[ur, "(\cap[M])^*"'] & \end{tikzcd}$$

[1282] Is the hypothesis on V and W necessary to draw this conclusion?

[1283] The right diagonal is the dual map corresponding to the map $\mathrm{H}^k(M;\mathbb{Z}) \xrightarrow{\cap[M]} \mathrm{H}_{n-k}(M;\mathbb{Z})$.

Note that for any $\psi \in \mathrm{H}^{n-k}(M;\mathbb{Z})$ and any $\varphi \in \mathrm{H}^k(M;\mathbb{Z})$ the following equalities hold in $\mathbb{Z}$:

$$\Omega(\psi)(\varphi) \underset{\substack{\uparrow \\ \text{by definition of } \Omega \\ \text{and Lemma 83.4}}}{=} (\varphi \cup \psi) \cap [M] \underset{\substack{\uparrow \\ \text{Lemma 83.7}}}{=} \psi \cap (\varphi \cap [M]) \underset{\substack{\uparrow \\ \text{Lemma 83.4}}}{=} \langle \psi, \varphi \cap [M] \rangle = \mathrm{ev}(\psi)(\varphi \cap [M]).$$

But this observation immediately shows that the diagram commutes. ⊞

We need to show that in the above diagram the horizontal map is an isomorphism. By the claim it suffices to show that the two diagonal maps are isomorphisms. The left diagonal map is an isomorphism by the Universal Coefficient Theorem 75.13 for Cohomology Groups together with Lemma 75.5 (3) and the fact, pointed out above, that $\mathrm{H}_{n-k-1}(M;\mathbb{Z})$ is free abelian. Furthermore the second map is an isomorphism by the Poincaré Duality Theorem 88.1.

(2) Now let $\mathbb{F}$ be a field. We suppose that M is $\mathbb{F}$-oriented with corresponding fundamental class $[M]$. By Proposition 75.19 the map

$$\begin{array}{rcl} \mathrm{ev}\colon\ \mathrm{H}^k(M;\mathbb{F}) & \to & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(M;\mathbb{F}),\mathbb{F}) \\ \varphi & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_k(M;\mathbb{F}) & \to & \mathbb{F} \\ \sigma & \mapsto & \langle \varphi, \sigma \rangle \end{array} \right) \end{array}$$

is a natural isomorphism of $\mathbb{F}$-vector spaces. The proof of (2) is now basically the same as the proof of (1). ■

The following corollary will be surprisingly effective in determining cup products on topological manifolds.

Corollary 90.3. *Let M be a closed connected n-dimensional topological manifold and let $k \in \mathbb{N}_0$.*

(1) *Suppose that M is oriented and that $\mathrm{H}_{k-1}(M;\mathbb{Z})$ and $\mathrm{H}_{n-k-1}(M;\mathbb{Z})$ are torsion-free. Furthermore suppose that $\mathrm{H}^k(M;\mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{H}^{n-k}(M;\mathbb{Z}) \cong \mathbb{Z}$. If $\varphi \in \mathrm{H}^k(M;\mathbb{Z})$ and $\psi \in \mathrm{H}^{n-k}(M;\mathbb{Z})$ are generators, then*

$$\varphi \cup \psi \;=\; \epsilon \cdot [M]^* \;\in\; \mathrm{H}^n(M;\mathbb{Z})$$

for some $\epsilon \in \{-1,1\}$.

(2) *Let $\mathbb{F}$ be a field such that M is $\mathbb{F}$-orientable. Suppose that $\mathrm{H}^k(M;\mathbb{F}) \cong \mathbb{F}$ and $\mathrm{H}^{n-k}(M;\mathbb{F}) \cong \mathbb{F}$. If $\varphi \in \mathrm{H}^k(M;\mathbb{F})$ and $\psi \in \mathrm{H}^{n-k}(M;\mathbb{F})$ are non-zero, then $\varphi \cup \psi$ is non-zero in $\mathrm{H}^n(M;\mathbb{F}) \cong \mathbb{F}$.*

Proof. Let M be a closed connected n-dimensional topological manifold and let $k \in \mathbb{N}_0$.

(1) By our hypothesis and by Proposition 90.2 (1) the cup product pairing

$$\begin{array}{rcl} \mathrm{H}^k(M;\mathbb{Z}) \times \mathrm{H}^{n-k}(M;\mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi,\psi) & \mapsto & \langle \varphi \cup \psi, [M] \rangle \end{array}$$

is non-singular. It follows from this fact and Lemma 90.1 (2) that $\langle \varphi \cup \psi, [M] \rangle = \pm 1$. But by Lemma 75.16 this just means that $\varphi \cup \psi \in \mathrm{H}^n(M;\mathbb{Z})$ is a generator.

(2) This proof is almost identical to the proof of (1), we just need to replace Proposition 90.2 (1) by Proposition 90.2 (2). ■

Example. Let $m \neq n \in \mathbb{N}$. As on many earlier occasions we denote by $p\colon S^m \times S^n \to S^m$ and by $q\colon S^m \times S^n \to S^n$ the two obvious projection maps. Let $1 \in \mathrm{H}^0(S^m \times S^n;\mathbb{Z})$ be the neutral element from Proposition 81.7. In Lemma 84.3 we showed that the cohomology of $S^m \times S^n$ is given as follows:

$$\mathrm{H}^*(S^m \times S^n;\mathbb{Z}) \;=\; \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot p^*([S^m]^*) \oplus \mathbb{Z}\cdot q^*([S^n]^*) \oplus \mathbb{Z}\cdot [S^m \times S^n]^*.$$

It follows from Corollary 90.3 (1)[1284] that $p^*([S^m]^*) \cup q^*([S^n]^*) = \pm[S^m \times S^n]^*$. This result recovers (up to the sign indeterminacy) the calculation of the non-trivial cup product on $S^m \times S^n$ that we had initially performed in Lemma 84.3 using the somewhat mysterious Product Theorem 84.1. In Exercise 90.3 we will see, that with slightly more effort, one can also prove the same type of result for $m = n$.

Next we consider the topological spaces $S^2 \vee S^4$ and $\mathbb{C}\mathrm{P}^2$. It follows from Propositions 14.14 and 20.3 and the discussion on page 997 that $S^2 \vee S^4$ and $\mathbb{C}\mathrm{P}^2$ are both simply connected. Furthermore it follows from Propositions 43.4 and 47.8, the calculation on page 1262 and the Universal Coefficient Theorem 75.13 for Cohomology Groups that for any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(S^2 \vee S^4;\mathbb{Z}) \;\cong\; \mathrm{H}^k(S^2 \vee S^4;\mathbb{Z}) \;\cong\; \mathrm{H}^k(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \;\cong\; \mathrm{H}_k(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0,2,4, \\ 0, & \text{otherwise.} \end{cases}$$

We can thus view $S^2 \vee S^4$ as a (co-) homological doppelgänger to $\mathbb{C}\mathrm{P}^2$.

Using Corollary 90.3 (1) we can easily prove the following lemma which implies in particular that $S^2 \vee S^4$ is not homotopy equivalent to $\mathbb{C}\mathrm{P}^2$.

Lemma 90.4. *The topological space $S^2 \vee S^4$ is not homotopy equivalent to any closed topological manifold.*

PROOF. Let X be a closed topological manifold whose homology and cohomology groups are isomorphic to the corresponding groups of $S^2 \vee S^4$. It follows from the above calculation of homology groups, our hypothesis that X is compact, from Propositions 41.5 and 87.22 and Theorems 87.1 and 87.3 that X is a closed oriented closed connected 4-dimensional topological manifold. But then Corollary 90.3 (1) implies that the cup product

$$\mathrm{H}^2(X;\mathbb{Z}) \times \mathrm{H}^2(X;\mathbb{Z}) \;\to\; \mathrm{H}^4(X;\mathbb{Z})$$

is non-zero.

On the other hand in Lemma 81.11 we saw that the cup product

$$\cup\colon\; \mathrm{H}^2(S^2 \vee S^4;\mathbb{Z}) \times \mathrm{H}^2(S^2 \vee S^4;\mathbb{Z}) \;\to\; \mathrm{H}^4(S^2 \vee S^4;\mathbb{Z})$$

is the zero-map. By Corollary 81.12 and the above calculation of the cup product of X this implies that $S^2 \vee S^4$ and X are not homotopy equivalent. ■

[1284] Of course here we also use the fact, with which we are very familiar by now, that the homology groups of $S^m \times S^n$ are torsion-free.

Remark. One can ask whether the adjective "closed" in Lemma 90.4 is necessary. It turns out that this is the case. In fact every wedge of finitely many spheres can be viewed as a deformation retract of an open subset of a sufficiently high-dimensional $\mathbb{R}^n$. For the wedge $S^1 \vee S^1$ we sketch the argument in Figure 1252. With slightly more care one can also show that this wedge of finitely many spheres is also a deformation retract of some compact n-dimensional smooth manifold of $\mathbb{R}^n$. We leave it to the reader to provide a formal proof for these statements.

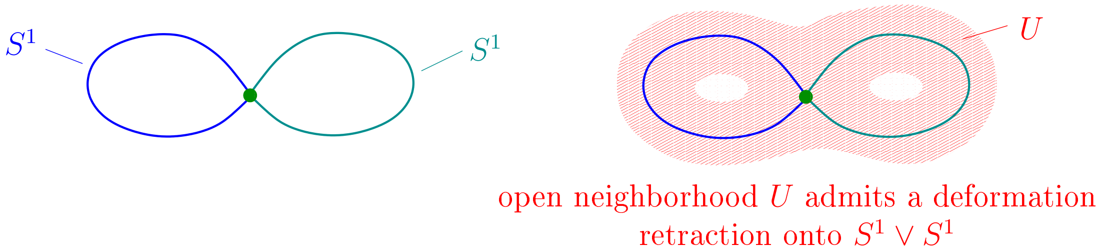


FIGURE 1252

We conclude this section with the following application to de Rham cohomology. We will not make use of it later on.

Corollary 90.5. *If M is a closed oriented n-dimensional smooth manifold, then for any $k \in \mathbb{N}_0$ the pairing*

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^{n-k}_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ ([\varphi],[\psi]) & \mapsto & \int\limits_M \varphi \wedge \psi \end{array}$$

is non-singular.

Since there is no "de Rham homology" the above corollary is usually called the Poincaré Duality Theorem for de Rham cohomology.

Remark. Evidently there are also more direct proofs of Corollary 90.5 that take place within the setting of de Rham cohomology. We refer to [**BoT82**, p. 44] or alternatively [**War83**, Theorem 6.13] for such a proof.

PROOF. It is straightforward to see that it suffices to deal with the case that M is connected. Theorem 81.19 and Proposition 79.7 give us the following commutative diagram with vertical isomorphisms:

$$\begin{tikzcd} \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^{n-k}_{\mathrm{dR}}(M) \arrow[rr, "(\varphi,\psi)\mapsto \int_M \varphi\wedge\psi"] \arrow[d, "\cong"'] & & \mathbb{R} \arrow[d, "="] \\ \mathrm{H}^k(M;\mathbb{R}) \times \mathrm{H}^{n-k}(M;\mathbb{R}) \arrow[rr, "(\varphi,\psi)\mapsto\langle\varphi\cup\psi,[M]\rangle"] & & \mathbb{R}. \end{tikzcd}$$

By Proposition 90.2 (2) the pairing on the bottom is non-singular. Thus the pairing on the top is also non-singular. ■

90.2. **The cup product of $\mathbb{C}\mathbf{P}^n$ and applications.** In this section we apply our acquired wisdom to the complex projective spaces $\mathbb{C}\mathrm{P}^n$. Recall that in Lemma 12.5 we showed that

$\mathbb{C}\mathrm{P}^n$ is a closed $2n$-dimensional smooth manifold that comes with a canonical orientation. Thus we will always view $\mathbb{C}\mathrm{P}^n$ as an *oriented* smooth manifold. In particular $\mathbb{C}\mathrm{P}^n$ comes with a canonical fundamental class $[\mathbb{C}\mathrm{P}^n] \in \mathrm{H}_{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and a canonical dual fundamental class $[\mathbb{C}\mathrm{P}^n]^* \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$.

An astute observer might already have noticed that Corollary 90.3 (1) can be used to compute the cup product on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$. In fact in this section we will use Corollary 90.3 (1) and the naturality of the cup product to compute the cohomology ring of any complex projective space $\mathbb{C}\mathrm{P}^n$. Throughout this section, given $m \leq n$ we will view $\mathbb{C}\mathrm{P}^m$ as a submanifold of $\mathbb{C}\mathrm{P}^n$ via the smooth embedding

$$\begin{array}{rcl} i\colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [z_0:\dots:z_m] & \mapsto & [z_0:\dots:z_m:0:\dots:0]. \end{array}$$

We prepare the ground with the following elementary lemma.

Lemma 90.6.

(1) *For every $n \in \mathbb{N}$ and every $k \in \mathbb{N}_0$ we have*

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \;\cong\; \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \textit{if } k = 0,2,4,\dots,2n, \\ 0, & \textit{otherwise.} \end{cases}$$

(2) *Let $m \leq n$. We denote by $i\colon \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ the inclusion map. For any $k \leq 2m$ the induced maps $i_*\colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and $i^*\colon \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to \mathrm{H}^k(\mathbb{C}\mathrm{P}^m;\mathbb{Z})$ are isomorphisms.*

Proof (∗). First we recall that in Lemma 36.1 we showed that we can view $\mathbb{C}\mathrm{P}^n$ as a CW-complex which admits exactly one cell in dimensions $0,2,\dots,2n$ and which admits no cells in any other dimension. Furthermore we saw that the CW-structure can be chosen in such a way that for $k \in \{0,\dots,n\}$ the $(2k+1)$-skeleton on $\mathbb{C}\mathrm{P}^n$ equals $\mathbb{C}\mathrm{P}^k$.

Furthermore recall that by Propositions 48.4 we know that singular and cellular homology are *naturally* isomorphic. Together with the above discussion of the CW-structures it follows that for every $n \in \mathbb{N}$ and every $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0,2,4,\dots,2n, \\ 0, & \text{otherwise} \end{cases}$$

and that for every $m \leq n$ the inclusion induced maps $i_*\colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ are isomorphisms for $k \leq 2m$.

In Proposition 74.4 we had also seen that singular and cellular cohomology are naturally isomorphic. So the same logic applies to cohomology groups. ■

We also introduce the following notation.

Notation.

(1) Let X be a topological space, let R be a commutative ring and let $\varphi \in \mathrm{H}^k(X;R)$. We define $\varphi^0 = 1_X$ and given any $n \in \mathbb{N}$ we define

$$\varphi^n \;=\; \underbrace{\varphi \cup \dots \cup \varphi}_{n \text{ times}} \;\in\; \mathrm{H}^{kn}(X;R).$$

(2) Recall that we denote by $[\mathbb{C}\mathrm{P}^1]^* \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^1;\mathbb{Z})$ the dual fundamental class of $\mathbb{C}\mathrm{P}^1$. For any $n \geq 1$ we denote by $i\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ the above inclusion map. By Lemma 90.6 we can define $x := (i^*)^{-1}([\mathbb{C}\mathrm{P}^1]^*) \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$. We refer to $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ as the *standard generator.*

Now we have the tools to determine the cup product of all complex projective spaces.

Proposition 90.7. *Let $n \in \mathbb{N}$.*

(1) *For every $k \in \{0,\dots,n\}$ the k-th power $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ is a generator.*
(2) *If we view $\mathbb{Z}[x]/(x^{n+1})$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 2$, then the obvious map*
$$\mathbb{Z}[x]/(x^{n+1}) \ \to \ (\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}),\cup)$$
is an isomorphism of superalgebras.

Proof. It follows immediately from the definitions and Proposition 81.8 that the map in (2) is a homomorphism of superalgebras. This observation, together with Lemma 90.6, shows that (2) is a consequence of (1). Thus it remains to prove (1)

We prove (1) by induction on n. Evidently the statement holds for $n = 1$. Now suppose it holds for some $n-1$ with $n \geq 2$. We denote by $i\colon \mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^n$ the inclusion map. Note that $i^*(x) = x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^{n-1};\mathbb{Z})$. It follows easily from Lemma 90.6 together with Lemma 81.10 and our induction hypothesis that for each $k \in \{1,\dots,n-1\}$ the cohomology class $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is a generator. It remains to show that $x^n \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is also a generator. This is an immediate consequence of Corollary 90.3 (1), applied to the generator $\varphi := x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ and the generator $\psi := x^{n-1} \in \mathrm{H}^{2(n-1)}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$. ■

The previous proof leaves it open whether nor not $x^n \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ equals the dual fundamental class $[\mathbb{C}\mathrm{P}^n]^*$. We formulate this as a question which we will answer later in Proposition 94.11, when we will have obtained more tools for calculating cup products.

Question 90.8. *Let $n \in \mathbb{N}$. Does the n-th power $x^n \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ equal the dual fundamental class $[\mathbb{C}\mathrm{P}^n]^*$?*

In Lemma 90.4 we had in particular shown that $\mathbb{C}\mathrm{P}^2$ is not homotopy equivalent to its (co-) homological doppelgänger $S^2 \vee S^4$. On page 1442 we pointed out that $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ is another example of a pair of topological spaces that cannot be distinguished using the fundamental group and using homology groups. This motivated us to ask in Question 58.9 whether the topological spaces $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ are homeomorphic. Now we can answer the question in the negative, in fact we can prove an even stronger result.

Corollary 90.9. *The topological spaces $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ are not homotopy equivalent.*

Proof. In Lemma 84.3 we saw that the cup product
$$\mathrm{H}^2(S^2 \times S^4;\mathbb{Z}) \times \mathrm{H}^2(S^2 \times S^4;\mathbb{Z}) \ \to \ \mathrm{H}^4(S^2 \times S^4;\mathbb{Z})$$
is zero, whereas we saw in Proposition 90.7 that the cup product
$$\mathrm{H}^2(\mathbb{C}\mathrm{P}^3;\mathbb{Z}) \times \mathrm{H}^2(\mathbb{C}\mathrm{P}^3;\mathbb{Z}) \ \to \ \mathrm{H}^4(\mathbb{C}\mathrm{P}^3;\mathbb{Z})$$
is non-zero. It follows from Corollary 81.12 that $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ are not homotopy equivalent. ■

We recall the following definition from page 1808.

Definition. Let M be a compact oriented topological manifold. We say that M is *amphichiral* if M admits an orientation-reversing self-homeomorphism, otherwise we say that M is *chiral*.

On page 1811 we saw that many smooth manifolds are amphichiral, for example we showed that all compact oriented 2-dimensional topological manifolds are amphichiral.

In Question 72.5 we had asked whether there exist compact oriented topological manifolds that are chiral. Now we will see that the complex projective space $\mathbb{C}\mathrm{P}^2$ is chiral. This gives an affirmative answer to Question 72.5. Note that the following corollary also gives a negative answer to Question 69.13 (3).

Corollary 90.10. *The smooth manifold $\mathbb{C}\mathrm{P}^2$ is chiral, i.e. there does not exist a self-homeomorphism $f\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ with $f_*([\mathbb{C}\mathrm{P}^2]) = -[\mathbb{C}\mathrm{P}^2]$.*

Remark. In Exercise 90.4 we will discuss for which $n \in \mathbb{N}$ the complex projective space $\mathbb{C}\mathrm{P}^n$ is chiral.

Proof. As usual we denote by $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ the standard generator. By Proposition 90.7 we know that $x \cup x \in \mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ is a generator. This implies that $x \cup x = \epsilon \cdot [\mathbb{C}\mathrm{P}^2]^*$ for some $\epsilon \in \{-1,1\}$.

Now let $f\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ be a self-homeomorphism. Since we want to show that $\mathbb{C}\mathrm{P}^2$ is chiral we need to show that $f_*([\mathbb{C}\mathrm{P}^2]) = [\mathbb{C}\mathrm{P}^2] \in \mathrm{H}_4(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$, or equivalently, that $f^*([\mathbb{C}\mathrm{P}^2]^*) = [\mathbb{C}\mathrm{P}^2]^* \in \mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$.

Now we see that

$$\begin{aligned} f^*([\mathbb{C}\mathrm{P}^2]^*) &= \epsilon \cdot f^*(x \cup x) \underset{\substack{\uparrow \\ \text{Lemma 81.10}}}{=} \epsilon \cdot (f^*(x) \cup f^*(x)) \underset{\substack{\uparrow \\ f^*(x) \text{ is also a generator of } \mathrm{H}^2(\mathbb{C}\mathrm{P}^4;\mathbb{Z}), \\ \text{hence } f^*(x) = \eta \cdot x \text{ for some } \eta \in \{-1,1\}}}{=} \epsilon \cdot (\eta \cdot x \cup \eta \cdot x) \\ &\underset{\substack{\uparrow \\ \cup \text{ is bilinear}}}{=} \epsilon \cdot \eta^2 \cdot (x \cup x) \underset{\substack{\uparrow \\ \text{since } \eta \in \{-1,1\}}}{=} \epsilon \cdot (x \cup x) = [\mathbb{C}\mathrm{P}^2]^*. \end{aligned}$$

■

Next we turn to the calculation of the cohomology ring of the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty = \varinjlim \mathbb{C}\mathrm{P}^n$.

Lemma 90.11.

(1) *For every $k \in \mathbb{N}$ we have*

$$\mathrm{H}^k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong \mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

(2) *Let $n \in \mathbb{N}$. We denote by $i\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^\infty$ the inclusion map. For any $k \le 2n$ the induced maps $i_*\colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ and $i^*\colon \mathrm{H}^k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \to \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ are isomorphisms.*

Proof. The statements in (1) and (2) regarding homology groups follow from the discussion on page 1263. The statements in (1) and (2) regarding cohomology groups then follow from the Universal Coefficient Theorem 75.13 for Cohomology Groups. ■

Remark. One could also prove the lemma using Lemma 90.6 and a suitable generalization of Proposition 76.12.

Notation. We denote by $i\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^\infty$ the inclusion map. By Lemma 90.11 we can set $x := (i^*)^{-1}([\mathbb{C}\mathrm{P}^1]^*) \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$. We refer to x as the *standard generator* of $\mathrm{H}^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$.

Proposition 90.12.
(1) *For every $k \in \mathbb{N}$ the k-th power $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong \mathbb{Z}$ is a generator.*
(2) *If we view $\mathbb{Z}[x]$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 2$, then the obvious map*

$$\mathbb{Z}[x] \;\to\; (\mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}),\cup)$$

is an isomorphism of superalgebras.

Proof. Exactly as in the proof of Proposition 90.7 we see that (2) is a consequence of (1). Thus it remains to prove (1) So let $k \in \mathbb{N}$. We consider the inclusion map $j\colon \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^\infty$. By Proposition 90.7 and Lemma 81.10 we know that $j^*(x^k) = j^*(x)^k = x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^k;\mathbb{Z})$ is a generator. It follows from Lemma 90.11 that $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ is also a generator. ■

Remark. It follows from Proposition 84.21 and the discussion on page 2055 that for each $k \in \mathbb{N}$ we also have an isomorphism

$$(\mathrm{H}^*(\underbrace{\mathbb{C}\mathrm{P}^\infty \times \ldots \times \mathbb{C}\mathrm{P}^\infty}_{k \text{ copies}};\mathbb{Z}),\cup) \;\cong\; \mathbb{Z}[x_1,\ldots,x_k]$$

of superalgebras. Here $x_1,\ldots,x_k$ are given by preimages of the standard generator of the group $\mathrm{H}^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ under the k obvious projection maps.

The fact that so far we worked with $\mathbb{Z}$-coefficients was mostly irrelevant. In fact most of the results carry over, with obvious modifications, to the cohomology groups $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;R)$ where R is an arbitrary commutative ring. For future reference we state the following proposition.

Proposition 90.13. *Let $n \in \mathbb{N}$.*
(1) *For every every $k \in \mathbb{N}_0$ we have*

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}_2) \;\cong\; \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}_2) \;\cong\; \begin{cases} \mathbb{Z}_2, & \text{if } k = 0,2,4,\ldots,2n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) *We denote by $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}_2)$ the unique non-trivial element. If we view the ring $\mathbb{Z}_2[x]/(x^{n+1})$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 2$, then the obvious map*

$$\mathbb{Z}_2[x]/(x^{n+1}) \;\to\; (\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}_2),\cup)$$

is an isomorphism of superalgebras.

The obvious analogues for $\mathbb{C}\mathrm{P}^\infty$ also hold.

PROOF. This proposition can be proved in an almost identical way as Proposition 90.12. Alternatively we can deduce the proposition from Proposition 90.12, the Universal Coefficient Theorem 75.26, Lemma 75.30 and Proposition 82.4 (2). We leave it to the impeccably conscientious reader to fill in the details. ■

Towards the end of this section we recall the definition of a retract. Thus let A be a subset of a topological space X. We denote by $i\colon A \to X$ the inclusion map. Recall that we say that A is a *retract* of X if there exists a map $r\colon X \to A$ such that $r \circ i = \mathrm{id}_A$. It follows immediately from the functoriality of homology groups and Splitting Lemma 46.2 that for each k the group $\mathrm{H}_k(A;\mathbb{Z})$ is a subsummand of $\mathrm{H}_k(X;\mathbb{Z})$.

Let $m < n$. In Question 53.4 we had asked whether $\mathbb{C}\mathrm{P}^m \subset \mathbb{C}\mathrm{P}^n$ is a retract of $\mathbb{C}\mathrm{P}^n$. Note that Lemma 90.6 shows that a study of homology groups does not preclude that for $m < n$ the complex projective space $\mathbb{C}\mathrm{P}^m$ is a retract of $\mathbb{C}\mathrm{P}^n$. Nonetheless, the final result of the section gives a negative answer to Question 53.4, i.e. we show that $\mathbb{C}\mathrm{P}^m$ is not a retract of $\mathbb{C}\mathrm{P}^n$.

Proposition 90.14. *If $1 \leq m < n$, then $\mathbb{C}\mathrm{P}^m \subset \mathbb{C}\mathrm{P}^n$ is not a retract of $\mathbb{C}\mathrm{P}^n$.*

PROOF. Let $1 \leq m < n$. We denote by $i\colon \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ the inclusion map. Suppose there exists a retraction $r\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^m$. We denote by $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ the standard generator. We have

$$x^n \underset{\substack{\uparrow \\ \text{by Lemma 90.6 the map} \\ i^*\colon \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to \mathrm{H}^2(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) \\ \text{is an isomorphism, together} \\ \text{with } i^*\circ r^* = (r\circ i)^* = \mathrm{id} \\ \text{we obtain that } r^*\circ i^* = \mathrm{id}}}{=} (r^*(i^*(x)))^n \underset{\substack{\uparrow \\ \text{Lemma 81.10}}}{=} r^*(i^*(x)^n) \underset{\substack{\uparrow \\ \text{note that } n>m \text{ implies that } i^*(x)^n \text{ lies} \\ \text{in the trivial group } \mathrm{H}^n(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) = 0}}{=} r^*(0) = 0 \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}).$$

But this contradicts the fact shown in Proposition 90.7 that $x^n \neq 0$. ■

90.3. **The cup product of $\mathbb{R}\mathrm{P}^n$ and applications.** In this section we intend to determine the cup product of the real projective space $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients. Evidently the approach will be very similar to the one taken in the previous section. First we recall that on page 287 we saw that the real projective space $\mathbb{R}\mathrm{P}^n$ is a closed n-dimensional smooth manifold. Given $m \leq n$ we will view $\mathbb{R}\mathrm{P}^m$ as a submanifold of $\mathbb{R}\mathrm{P}^n$ via the smooth embedding

$$\begin{aligned} i\colon \mathbb{R}\mathrm{P}^m &\to \mathbb{R}\mathrm{P}^n \\ [x_0 : \dots : x_m] &\mapsto [x_0 : \dots : x_m : 0 : \dots : 0]. \end{aligned}$$

The following lemma plays the role of Lemma 90.6.

Lemma 90.15.

(1) *For every $n \in \mathbb{N}$ we have*

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0,1,\dots,n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) *Let $m \leq n$. For any $k \leq m$ the induced maps $i_*\colon \mathrm{H}_k(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2) \to \mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ and $i^*\colon \mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \to \mathrm{H}^k(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2)$ are isomorphisms.*

Proof. The calculation of the isomorphism types of the (co-) homology groups and of the induced maps on homology is basically the same as in the proof of Lemma 90.15. The argument that determines the induced map on $\mathbb{F}_2$-cohomology is slightly different. This time we use Proposition 75.19 which gives us the following commutative diagram:

$$\begin{tikzcd} \mathrm{H}^k(\mathbb{RP}^n;\mathbb{F}_2) \arrow[r, "\mathrm{ev}", "\cong"'] \arrow[d, "i^*"] & \mathrm{Hom}(\mathrm{H}_k(\mathbb{RP}^n;\mathbb{F}_2),\mathbb{F}_2) \arrow[d, "i^*"] \\ \mathrm{H}^k(\mathbb{RP}^m;\mathbb{F}_2) \arrow[r, "\mathrm{ev}", "\cong"'] & \mathrm{Hom}(\mathrm{H}_k(\mathbb{RP}^m;\mathbb{F}_2),\mathbb{F}_2). \end{tikzcd}$$

As we had just pointed out, the maps on the right are isomorphisms for $k \leq m$. It follows that the maps on the left are also isomorphisms for $k \leq m$. ∎

Notation. Given $n \in \mathbb{N}$ we denote by $x \in \mathrm{H}^1(\mathbb{RP}^n;\mathbb{F}_2)$ the unique non-trivial element.

The following proposition is the analogue of Propositions 90.7 and 90.12.

Proposition 90.16. *Let $n \in \mathbb{N}$.*

(1) *For every $k \in \{0,\dots,n\}$ the k-th power $x^k \in \mathrm{H}^k(\mathbb{RP}^n;\mathbb{F}_2) \cong \mathbb{F}_2$ is non-trivial.*
(2) *If we view $\mathbb{F}_2[x]/(x^{n+1})$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 1$, then the obvious map*
$$\mathbb{F}_2[x]/(x^{n+1}) \;\to\; (\mathrm{H}^*(\mathbb{RP}^n;\mathbb{F}_2),\cup)$$
is an isomorphism of superalgebras.
(3) *If we view $\mathbb{F}_2[x]$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 1$, then the obvious map*
$$\mathbb{F}_2[x] \;\to\; (\mathrm{H}^*(\mathbb{RP}^\infty;\mathbb{F}_2),\cup)$$
is an isomorphism of superalgebras.

Proof. First recall that Proposition 86.13 implies that any topological manifold, in particular $\mathbb{RP}^n$, is $\mathbb{F}_2$-orientable. With this observation the proof of the proposition is basically identical to the proofs of Proposition 90.7 and 90.12. We just need to replace Lemma 90.6 by Lemma 90.15 and we need to replace Corollary 90.3 (1) by Corollary 90.3 (2). ∎

Remark.

(1) In principle one could have tried to compute the cup product on $\mathbb{RP}^n$ using the same approach that we had taken in Section 81.5. More precisely, suppose one has found an explicit $\mathbb{F}_2$-fundamental class σ for $\mathbb{RP}^n$. We denote by $\varphi \in C^1(\mathbb{RP}^2;\mathbb{F}_2)$ the 1-cocycle from page 1998. Now we know from Proposition 90.16 that $\langle \varphi^n, \sigma\rangle = 1$. But more interestingly, if one shows with an explicit calculation that $\langle \varphi^n, \sigma\rangle = 1$, then this gives a new and elementary proof of Proposition 90.16.
(2) We have just computed the cup product on $\mathrm{H}^*(\mathbb{RP}^n;\mathbb{F}_2)$. We leave it as a highly instructive exercise to use the same approach to determine the cup product on $\mathrm{H}^*(\mathbb{RP}^n;\mathbb{Z})$. This cup product is also determined, using a somewhat different approach, in [**Hat02**, p. 214].

(3) In Corollary 88.15 we showed that there is no map $S^1 \times S^2$ to $\mathbb{R}\mathrm{P}^3$ of degree one, thus giving a negative answer to Question 69.12. On page 2037 and in Proposition 90.16 we had determined the cup product of $S^1 \times S^2$ and $\mathbb{R}\mathrm{P}^3$ on cohomology with $\mathbb{F}_2$-coefficients. Using an argument similar to the proof of Lemma 84.5 one can easily give an alternative proof of Corollary 88.15.

For completeness' sake we mention the following analogue of Proposition 90.14.

Proposition 90.17. *If $1 \leq m < n$, then $\mathbb{R}\mathrm{P}^m \subset \mathbb{R}\mathrm{P}^n$ is not a retract of $\mathbb{R}\mathrm{P}^n$.*

PROOF. The proof is totally analogous to the proof of Proposition 90.14. In the proof we just need to replace Lemma 90.6 by Lemma 90.15. ■

Remark. We can also give an alternative proof of Proposition 90.17 using higher homotopy groups. More precisely, let $1 \leq m < n$. By Proposition 40.10 together with Proposition 40.13 we know that $\pi_m(S^n) = 0$. Furthermore by Lemma 53.2 together with Proposition 40.13 we know that $\pi_m(S^m) \neq 0$. Since higher homotopy groups are functorial, see Proposition 40.6, we can use the standard argument, see e.g. the proof of Lemma 15.7, to show that $\mathbb{R}\mathrm{P}^m$ is not a retract of $\mathbb{R}\mathrm{P}^n$.

One of the many consequences of our calculations from this chapter is that we can now calculate the Lusternik-Schnirelmann category for $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$. More precisely, we have the following proposition that gives a complete answer to Question 82.10.

Proposition 90.18. *For any $n \in \mathbb{N}_0$ we have $\operatorname{cat}(\mathbb{R}\mathrm{P}^n) = \operatorname{cat}(\mathbb{C}\mathrm{P}^n) = n$.*

PROOF. Given any $n \in \mathbb{N}_0$ we have

$$n \underset{\substack{\uparrow \\ \text{see page 2013}}}{\geq} \operatorname{cat}(\mathbb{R}\mathrm{P}^n) \underset{\substack{\uparrow \\ \text{by Proposition 82.11}}}{\geq} \overbrace{\text{cup length}\,\operatorname{cl}(\mathbb{R}\mathrm{P}^n)}^{\text{see page 2015 for the definition}} \underset{\substack{\uparrow \\ \text{by Proposition 90.16}}}{\geq} n.$$

The same argument also works for $\mathbb{C}\mathrm{P}^n$, we just need to replace Propositions 90.16, by Propositions 90.7. ■

We conclude this section with an alternative proof for the Borsuk-Ulam Theorem. First we recall the statement of the Borsuk-Ulam Theorem.

Theorem 59.3. (Borsuk-Ulam) *For every map $f\colon S^n \to \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $f(x) = f(-x)$.*

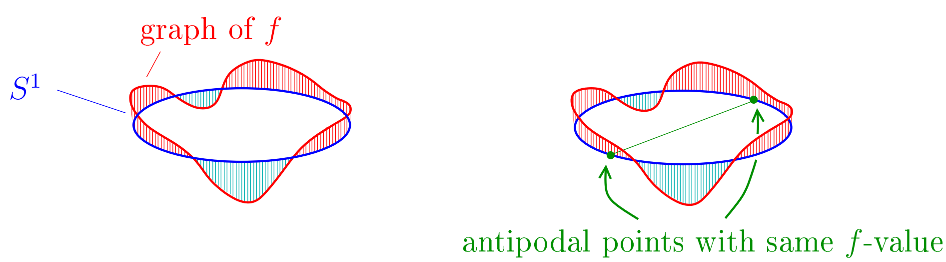


FIGURE 1253. Illustration of the Borsuk-Ulam Theorem.

Remark. We remind that the reader that in Section 59.4 and Section 59.5 we collected several interesting consequences of the Borsuk-Ulam Theorem 59.3.

In the new proof of the Borsuk-Ulam Theorem below we will need the following Lemma.

Lemma 90.19.

(1) *Let $k \geq 2$. As usual we denote by $p\colon S^k \to \mathbb{R}\mathrm{P}^k$ the obvious projection map. If $\gamma\colon [0,1] \to S^k$ is a path with $\gamma(1) = -\gamma(0)$, then $p \circ \gamma$ represents the unique non-trivial element in $\pi_1(\mathbb{R}\mathrm{P}^k)$.*

(2) *Let $k, l \geq 2$. Furthermore let $h\colon \mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^l$ be a map. If $h_*\colon \pi_1(\mathbb{R}\mathrm{P}^k) \to \pi_1(\mathbb{R}\mathrm{P}^l)$ has non-trivial image, then the induced maps $h_*\colon \mathrm{H}_1(\mathbb{R}\mathrm{P}^k;\mathbb{F}_2) \to \mathrm{H}_1(\mathbb{R}\mathrm{P}^l;\mathbb{F}_2)$ and $h^*\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^l;\mathbb{F}_2) \to \mathrm{H}^1(\mathbb{R}\mathrm{P}^k;\mathbb{F}_2)$ are also isomorphisms.*

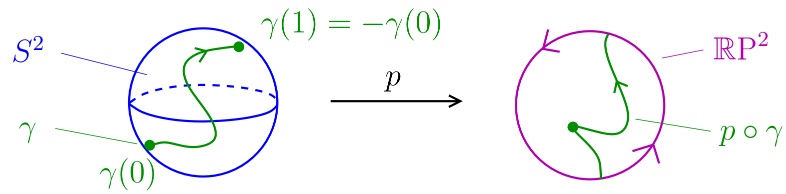


FIGURE 1254. Illustration of Lemma 90.19.

PROOF OF LEMMA 90.19. First we recall that for $k \geq 2$ we know from Corollary 16.18 that $\pi_1(\mathbb{R}\mathrm{P}^k) \cong \mathbb{Z}_2$.

(1) The first statement is an immediate consequence of Lemma 16.15 (1) and the fact that $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}_2$.

(2) Since $k, l \geq 2$ both fundamental groups are isomorphic to $\mathbb{Z}_2$. In particular if $h_*\colon \pi_1(\mathbb{R}\mathrm{P}^k) \to \pi_1(\mathbb{R}\mathrm{P}^l)$ has non-trivial image, then this map is already an isomorphism. It follows from the naturality of the Hurewicz homomorphism, see Proposition 52.2 (4), together with the Hurewicz Theorem 52.5 that the induced map $h_*\colon \mathrm{H}_1(\mathbb{R}\mathrm{P}^k;\mathbb{F}_2) \to \mathrm{H}_1(\mathbb{R}\mathrm{P}^l;\mathbb{F}_2)$ is also an isomorphism. Finally note that it follows from the naturality of the isomorphisms in Proposition 75.18 that the induced map $h^*\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^l;\mathbb{F}_2) \to \mathrm{H}^1(\mathbb{R}\mathrm{P}^k;\mathbb{F}_2)$ is also an isomorphism. ■

PROOF OF THE BORSUK-ULAM 59.3 FOR $n \geq 2$. Let us suppose that there exists a map $f\colon S^n \to \mathbb{R}^n$ such that for any $x \in S^n$ we have that $f(x) \neq f(-x)$. We consider the map

$$\begin{array}{rcl} g\colon S^n & \to & S^{n-1} \\ x & \mapsto & \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}. \end{array}$$

By our hypothesis this map is defined. It is clear that $g(-x) = -g(x)$ for all $x \in S^n$. Therefore this map g descends to a map

$$\begin{array}{rcl} h\colon \mathbb{R}\mathrm{P}^n & \to & \mathbb{R}\mathrm{P}^{n-1} \\ {[x]} & \mapsto & [g(x)] \end{array}$$

i.e. we get the following commutative diagram

$$\begin{tikzcd} S^n \arrow[r, "g"] \arrow[d, "p"] & S^{n-1} \arrow[d, "q"'] \\ \mathbb{RP}^n \arrow[r, "{[x]\mapsto h(x):=[g(x)]}"] & \mathbb{RP}^{n-1} \end{tikzcd}$$

where p and q denote the obvious projection maps.[1285]

Claim. The map $h^*\colon\ \mathrm{H}^1(\mathbb{RP}^{n-1};\mathbb{F}_2)\to \mathrm{H}^1(\mathbb{RP}^n;\mathbb{F}_2)$ is an isomorphism.

By Lemma 90.19 (2) we only have to show that $h_*\colon \pi_1(\mathbb{RP}^n)\to\pi_1(\mathbb{RP}^{n-1})$ has non-trivial image. Let $\gamma\colon [0,1]\to S^n$ be a path with $\gamma(1)=-\gamma(0)$. Note that $p\circ\gamma\colon [0,1]\to\mathbb{RP}^n$ is a loop. We calculate that

$$h_*([p\circ\gamma]) \;=\; [h\circ p\circ\gamma] \;\overset{\substack{\text{by the above commutative diagram}\\ \downarrow}}{=}\; [q\circ(g\circ\gamma)] \;\underset{\substack{\uparrow\\ \text{by Lemma 90.19 (1) since } (g\circ\gamma)(1)=g(\gamma(1))=g(-\gamma(0))=-g(\gamma(0))=-(g\circ\gamma)(0)}}{\neq}\; e\;\in\pi_1(\mathbb{RP}^{n-1}).$$

⊞

We denote by $y\in\mathrm{H}^1(\mathbb{RP}^{n-1};\mathbb{F}_2)$ and $x\in\mathrm{H}^1(\mathbb{RP}^n;\mathbb{F}_2)$ the unique non-trivial elements. We have the following equality in $\mathrm{H}^n(\mathbb{RP}^n;\mathbb{F}_2)$:[1286]

$$0\;\underset{\substack{\uparrow\\ \text{Proposition 90.16}}}{\neq}\;x^n\;\underset{\substack{\uparrow\\ \text{by the claim}}}{=}\;(h^*(y))^n\;\underset{\substack{\uparrow\\ \text{Lemma 81.10}}}{=}\;h^*(y^n)\;\underset{\substack{\uparrow\\ \text{Proposition 90.16}}}{=}\;h^*(0)\;=\;0.$$

We have thus obtained the desired contradiction. ■

90.4. **The cup product of quaternionic and octonionic projective spaces (∗).** In this very short section we discuss the cup product of the quaternionic projective spaces $\mathbb{HP}^n$ that we introduced on page 1472. We will also calculate the cup product of the octonionic projective plane $\mathbb{OP}^2$ that we defined on page 1473. These calculations might not sound all that interesting, but later in Chapter 91 we will use them to show that certain homotopy groups of spheres are non-trivial.

The following proposition is the only, and thus also the most interesting, result of this section. The statement is of course, in light of Proposition 90.7, entirely predictable.

Proposition 90.20.

(1) *Given any $n\in\mathbb{N}$ there exists an isomorphism*
$$\mathbb{Z}[z]/(z^{n+1}) \;\to\; (\mathrm{H}^*(\mathbb{HP}^n;\mathbb{Z}),\cup)$$
of superalgebras, where z is an element of degree 4.

(2) *There exists an isomorphism*
$$\mathbb{Z}[z]/(z^3) \;\to\; (\mathrm{H}^*(\mathbb{OP}^2;\mathbb{Z}),\cup)$$
of superalgebras, where z is an element of degree 8.

[1285]The map $q\circ g\colon S^n\to\mathbb{RP}^{n-1}$ is continuous. It follows from Lemma 3.22 that the map h is also continuous.

[1286]The argument is very similar to the argument in the proof of Proposition 90.14.

In both cases the obvious analogues with $\mathbb{Z}$ replaced by $\mathbb{Z}_2$ also hold.

PROOF.

(1) Let $n \in \mathbb{N}_0$. By Lemma 60.9 we know that $\mathbb{H}\mathrm{P}^n$ is a $4n$-dimensional closed orientable smooth manifold which admits a CW-structure that has exactly one cell in the dimensions $0, 4, 8, \ldots, 4n$. The proof of Proposition 90.7 can be used, with very minor modifications, to prove the desired result.

(2) In Proposition 60.10 (3) we showed that the octonionic projective plane $\mathbb{O}\mathrm{P}^2$ is a closed orientable 16-dimensional smooth manifold which admits a CW-structure with precisely one cell in dimensions 0, 8 and 16, and which has no other cells. Once again the proof of Proposition 90.7 clinches the deal. ■

Propositions 90.7 and 90.20 do raise a few questions, which can be viewed as a more specific versions of Question 84.19.

The first question asks whether we can realize polynomial rings in other degrees.

Question 90.21. *Let m be an even number and $n \in \mathbb{N}$. Does there exist a topological space X such that $(\mathrm{H}^*(X;\mathbb{Z}), \cup)$ is isomorphic to $\mathbb{Z}[z]/(z^{n+1})$ where z is now an element of degree m?*

The following question is even more specific, it asks whether there exists a "cohomological $\mathbb{O}\mathrm{P}^3$.

Question 90.22. *Does there exist a topological space X such that $(\mathrm{H}^*(X;\mathbb{Z}), \cup)$ is isomorphic to $\mathbb{Z}[z]/(z^4)$ where z is now an element of degree 8?*

It will not take the reader very long to see that a negative answer to Question 90.22 would also tragically give a negative answer to Question 60.11.

90.5. **Division algebras.** We recall that on page 1461 we defined a *real division algebra* to be an algebra over $\mathbb{R}$ such that for all $a \neq 0$ and $b \in A$ there exists an $x \in A$ with $a \cdot x = b$ and a $y \in A$ with $y \cdot a = b$. In Section 60.1 we gave several examples of real division algebras, namely $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ (which are not commutative) and the Cayley Octonions $\mathbb{O}$ (which are neither commutative nor associative). These real division algebras are defined over $\mathbb{R}^n$ with $n = 1, 2, 4$ and 8.

These dimensions are powers of 2 and the following theorem, which was first proved by Heinz Hopf [**Hopf40a**] in 1940, shows that this is not a coincidence.

Theorem 90.23. (Hopf 1940) *If $\mathbb{R}^n$ admits the structure of a real division algebra, then n is a power of 2.*

In the proof of Theorem 90.23 we will need the following lemma which is interesting in its own right.

Lemma 90.24. *Let $k \in \mathbb{N}$. We denote by $\varphi \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^k;\mathbb{F}_2)$ the unique non-trivial element. We denote by p and q the two obvious projection maps $\mathbb{R}\mathrm{P}^k \times \mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^k$ and we write $\alpha = p^*(\varphi)$ and $\beta = q^*(\varphi)$. Then the obvious homomorphism*

$$\mathbb{F}_2[\alpha,\beta]/(\alpha^{k+1},\beta^{k+1}) \ \to \ \mathrm{H}^*(\mathbb{R}\mathrm{P}^k \times \mathbb{R}\mathrm{P}^k;\mathbb{F}_2)$$

of superalgebras is an isomorphism.

PROOF. The lemma is an immediate consequence of Propositions 90.16 and 84.24 and the discussion on page 2055. ■

Now we can provide the proof of Theorem 90.23.

PROOF OF THEOREM 90.23. Let "$*$" be a real division algebra structure on $\mathbb{R}^n$. We want to show that n is a power of 2. Since 1 and 2 are powers of 2 we can assume that $n \geq 3$. As in the proof of Theorem 60.4 we consider the map

$$\begin{array}{rcl} \widetilde{f}\colon S^{n-1}\times S^{n-1} & \to & S^{n-1} \\ (x,y) & \mapsto & \widetilde{f}(x,y) := \dfrac{x*y}{\|x*y\|}. \end{array}$$

We make the following observations:

(1) In the proof of Theorem 60.4 we already showed, using the hypothesis that "$*$" defines a division algebra, that this map is well-defined, i.e. that we never divide by zero.
(2) In the proof of Theorem 60.4 we had also shown that the map $\widetilde{f}$ is continuous.
(3) Since "$*$" defines in particular an algebra we have

$$\widetilde{f}(-x,y) = -\widetilde{f}(x,y) \quad \text{ and } \quad \widetilde{f}(x,-y) = -\widetilde{f}(x,y) \quad \text{ for any } x,y\in S^{n-1}.$$

From (3) it follows that the above map $\widetilde{f}$ descends to a map[1287]

$$\begin{array}{rcl} f\colon \mathbb{RP}^{n-1}\times \mathbb{RP}^{n-1} & \to & \mathbb{RP}^{n-1} \\ [\widetilde{x}],[\widetilde{y}] & \mapsto & [\widetilde{f}(x,y)]. \end{array}$$

We pick a point $z\in S^{n-1}$. We consider the inclusion maps

$$\begin{array}{rcl} i\colon \mathbb{RP}^{n-1} & \to & \mathbb{RP}^{n-1}\times\mathbb{RP}^{n-1} \\ x & \mapsto & (x,[z]) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon \mathbb{RP}^{n-1} & \to & \mathbb{RP}^{n-1}\times\mathbb{RP}^{n-1} \\ x & \mapsto & ([z],x). \end{array}$$

Our first goal is to prove the following claim.

Claim 1. The map

$$(f\circ i)_*\colon\ \mathrm{H}_1(\mathbb{RP}^{n-1};\mathbb{F}_2) \ \to\ \mathrm{H}_1(\mathbb{RP}^{n-1};\mathbb{F}_2)$$

is an isomorphism. The same conclusion holds with i_* replaced by j_*.

We provide the proof for the first statement. The proof with i_* replaced by j_* is basically the same. By Lemma 90.19 (2) it suffices to show that the map

$$(f\circ i)_*\colon \pi_1(\mathbb{RP}^{n-1}) \ \to\ \pi_1(\mathbb{RP}^{n-1})$$

has non-trivial image. We introduce the following notation:

(1) We denote by $\pi\colon S^{n-1}\to\mathbb{RP}^{n-1}$ the obvious projection map.
(2) We pick a path $\gamma\colon[0,1]\to S^{n-1}$ with $\gamma(1)=-\gamma(0)$. Note that $\pi\circ\gamma\colon[0,1]\to\mathbb{RP}^{n-1}$ is a loop.
(3) We denote by $\widetilde{i}$ the inclusion map $S^{n-1}\to S^{n-1}\times S^{n-1}$ that is given by $\widetilde{i}(x)=(x,z)$.

[1287]The argument of Footnote 1285 shows that f is continuous.

Note that we get the following commutative diagram

$$\begin{tikzcd}
{[0,1]} \arrow[r, "\gamma"] & S^{n-1} \arrow[r, "\widetilde{i}"] \arrow[d, "\pi"'] & S^{n-1}\times S^{n-1} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi\times\pi"] & S^{n-1} \arrow[d, "\pi"] \\
& \mathbb{RP}^{n-1} \arrow[r, "i"] & \mathbb{RP}^{n-1}\times\mathbb{RP}^{n-1} \arrow[r, "f"] & \mathbb{RP}^{n-1}.
\end{tikzcd}$$

Finally we note that

$$(f\circ i)_*([\pi\circ\gamma]) \;=\; [f\circ i\circ\pi\circ\gamma] \;\underset{\text{by the above commutative diagram}}{=}\; \big[\pi\circ(\widetilde{f}\circ\widetilde{i}\circ\gamma)\big] \;\underset{\substack{\text{this follows from Lemma 90.19 (1) since the following equality holds:}\\ (\widetilde{f}\circ\widetilde{i}\circ\gamma)(1)=\widetilde{f}(\gamma(1),z)=\widetilde{f}(-\gamma(0),z)=-\widetilde{f}(\gamma(0),z)=-(\widetilde{f}\circ\widetilde{i}\circ\gamma)(0)}}{\neq}\; e \;\in\pi_1(\mathbb{RP}^{n-1}).$$

⊞

We denote by $\varphi\in \mathrm{H}^1(\mathbb{RP}^{n-1};\mathbb{F}_2)$ the unique non-trivial element. Furthermore, as so often we denote by p and q the two obvious projection maps $\mathbb{RP}^{n-1}\times\mathbb{RP}^{n-1}\to\mathbb{RP}^{n-1}$. Finally we write $\alpha=p^*(\varphi)$ and $\beta=q^*(\varphi)\in \mathrm{H}^1(\mathbb{RP}^{n-1};\mathbb{F}_2)$.

Claim 2. The following equality holds

$$f^*(\varphi) \;=\; \alpha+\beta \;\in\; \mathrm{H}^1(\mathbb{RP}^{n-1}\times\mathbb{RP}^{n-1};\mathbb{F}_2).$$

We start out the proof of the claim with the following three observations:

(1) Proposition 75.19 says that for any topological space W the map

$$\begin{array}{rcl}\mathrm{ev}\colon\ \mathrm{H}^1(W;\mathbb{F}_2) &\to& \mathrm{Hom}(\mathrm{H}_1(W;\mathbb{F}_2),\mathbb{F}_2)\\ \psi &\mapsto& (\sigma\mapsto\langle\psi,\sigma\rangle)\end{array}$$

is an isomorphism.

(2) Let $x\in \mathrm{H}_1(\mathbb{RP}^{n-1};\mathbb{F}_2)$ be the unique non-trivial element. By the discussion on page 1965 the map

$$\begin{array}{rcl}\mathrm{H}_1(\mathbb{RP}^{n-1};\mathbb{F}_2)\oplus\mathrm{H}_1(\mathbb{RP}^{n-1};\mathbb{F}_2) &\to& \mathrm{H}_1(\mathbb{RP}^{n-1}\times\mathbb{RP}^{n-1};\mathbb{F}_2)\\ a\oplus b &\mapsto& i_*(a)\oplus j_*(b)\end{array}$$

is an isomorphism. In particular we see that $\mathrm{H}_1(\mathbb{RP}^{n-1}\times\mathbb{RP}^{n-1};\mathbb{F}_2)$ is generated by $i_*(x)$ and $j_*(x)$.

(3) Since $\mathrm{H}^1(\mathbb{RP}^{n-1};\mathbb{F}_2)$ and $\mathrm{H}_1(\mathbb{RP}^{n-1};\mathbb{F}_2)$ both contain a unique non-trivial element, namely φ and x, we obtain from Observation (1) that $\mathrm{ev}(\varphi)(x)=\langle\varphi,x\rangle=1$.

It follows from (1) and (2) that it suffices to show that $\mathrm{ev}(f^*(\varphi))$ and $\mathrm{ev}(\alpha+\beta)$ agree on $i_*(x)$ and $j_*(y)$. In fact we have

$$\mathrm{ev}(f^*(\varphi))(i_*(x)) \;\underset{\text{by definition}}{=}\; \langle f^*(\varphi),i_*(x)\rangle \;\underset{\text{Lemma 74.6 (3)}}{=}\; \langle\varphi,f_*(i_*(x))\rangle \;\underset{\substack{\text{by Claim 1 since } x \text{ is}\\ \text{the unique non-zero}\\ \text{element of } \mathrm{H}_1(\mathbb{RP}^{n-1};\mathbb{F}_2)}}{=}\; \langle\varphi,x\rangle \;\underset{\text{observation (3)}}{=}\; 1.$$

We also calculate that

$$\operatorname{ev}(\alpha+\beta)(i_*(x)) \underset{\substack{\uparrow\\ \text{by definition}}}{=} \langle p^*\varphi+q^*\varphi, i_*(x)\rangle \underset{\substack{\uparrow\\ \text{Lemma 74.6 (3)}}}{=} \langle\varphi,p_*(i_*(x))\rangle+\langle\varphi,q_*(i_*(x))\rangle \underset{\substack{\uparrow\\ \text{since } p\circ i=\mathrm{id} \text{ and } q\circ i \text{ is a constant map}}}{=} \langle\varphi,x\rangle = 1.$$

We have thus shown that $\operatorname{ev}(f^*(\varphi))$ and $\operatorname{ev}(\alpha+\beta)$ agree on $i_*(x)$. The same argument also works with $i_*(x)$ replaced by $j_*(y)$. ⊞

Using Lemma 90.24 we can and will make the identification

$$\mathrm{H}^*(\mathbb{RP}^{n-1}\times\mathbb{RP}^{n-1};\mathbb{F}_2) \;=\; \mathbb{F}_2[\alpha,\beta]/(\alpha^n,\beta^n).$$

Using this identification we see that

$$0 \;=\; f^*(0) \underset{\substack{\uparrow\\ \text{by Proposition 90.16}\\ \text{since } \varphi\in \mathrm{H}^1(\mathbb{RP}^{n-1};\mathbb{F}_2)}}{=} f^*(\varphi^n) \underset{\substack{\uparrow\\ \text{Lemma 81.10}}}{=} (f^*(\varphi))^n \underset{\substack{\uparrow\\ \text{by the previous claim}}}{=} (\alpha+\beta)^n \;=\; \sum_{k=0}^{n}\binom{n}{k}\cdot\alpha^k\beta^{n-k} \underset{\substack{\uparrow\\ \text{since } \alpha^n=\beta^n=0}}{=} \sum_{k=1}^{n-1}\binom{n}{k}\cdot\alpha^k\beta^{n-k}.$$

This equality holds in the polynomial ring $\mathbb{F}_2[\alpha,\beta]/(\alpha^n,\beta^n)$. In this ring the $n-1$ monomials $\alpha\beta^{n-1},\alpha^2\beta^{n-2},\dots,\alpha^{n-1}\beta$ are linearly independent over the field $\mathbb{F}_2$. This implies that for $k=1,\dots,n-1$ we have $\binom{n}{k}\equiv 0 \bmod 2$. Thus it remains to prove the following purely algebraic claim.

Claim. If $n\in\mathbb{N}$ satisfies that $\binom{n}{k}\equiv 0 \bmod 2$ for $k=1,\dots,n-1$, then n is a power of 2.

We can and will write n as a sum $n=n_1+\dots+n_r$ where each n_i is a power of 2 and with $n_1<\dots<n_r$. In the polynomial ring $\mathbb{F}_2[x]$ we obtain that

$$\begin{aligned} 1+x^n &\overset{\substack{\text{by our hypothesis}\\ \downarrow}}{=} \sum_{k=0}^{n}\binom{n}{k}x^k \;=\; (1+x)^n \overset{\substack{\text{since } n=n_1+\dots+n_r\\ \downarrow}}{=} (1+x)^{n_1}\cdot\dots\cdot(1+x)^{n_r}\\ &\underset{\substack{\uparrow\\ \text{since each } n_i \text{ is a power of 2 and}\\ \text{since in } \mathbb{F}_2[x] \text{ we have } (a+b)^2=a^2+b^2}}{=} (1+x^{n_1})\cdot\dots\cdot(1+x^{n_r}) \underset{\substack{\uparrow\\ \text{since } n_1<\dots<n_r}}{=} 1+x^{n_1}+\text{higher powers.}\end{aligned}$$

But this is only possible if $n=n_1$, i.e. if n itself is a power of 2. ■

In fact, with more fancy topological methods one can actually strengthen the previous theorem to obtain the following theorem which gives a complete answer to Question 60.7. Unfortunately the proof is well beyond the scope of these modest lecture notes.

Theorem 90.25. (Bott-Kervaire-Milnor 1958) *Every finite-dimensional division algebra over $\mathbb{R}$ is of dimension 1,2, 4 or 8.*

Proof. The theorem was first proved using topological methods by Michel Kervaire [**Kerv58**] and John Milnor [**Miln58a**, Corollary 1], building on the "Bott Periodicity Theorem" proved by Raoul Bott [**Bot57, Bot59**].[1288] Full details for the proof can also be found

[1288]We will discuss the statement of the Bott Periodicity Theorem in Section 114.4.

in [**Hat2**, Theorem 2.16]. As of right now there is no purely algebraic proof for the statement. ■

Remark. One might ask whether any finite-dimensional division algebra over $\mathbb{R}$ is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. The statement holds under the stronger hypothesis that the algebra is "normed" or "alternative". We refer to [**BD73**, p. 73], [**EHHKMNPR91**, p. 262] and [**Bae01**] for results in this direction. As is shown in [**Hat02**, p. 59], [**Bruc44, Osb62**], without any extra hypotheses one cannot expect a uniqueness statement.

Exercises for Chapter 90.

Exercise 90.1. Let V and W be free $\mathbb{Z}$-modules of finite rank, i.e. V and W are free abelian groups. Furthermore let $\varphi\colon V \times W \to \mathbb{Z}$ be a pairing.

(a) Let $v_1, \dots, v_n$ be a basis for V. Show that the following two statements are equivalent:
   (i) There exists some basis $w_1, \dots, w_n$ for W such that $\lambda(v_i, w_j) = \delta_{ij}$ for every choice of $i, j \in \{1, \dots, n\}$.[1289]
   (ii) The pairing λ is non-singular.

(b) Suppose that λ is non-degenerate. Show that V and W have the same rank.

(c) Suppose that V and W are of rank n and suppose we are given $v_1, \dots, v_n \in V$ and $w_1, \dots, w_n \in W$ such that $\det(\lambda(v_i, w_j)) = \pm 1$. Show that $v_1, \dots, v_n$ is a basis for V and that $w_1, \dots, w_n$ is a basis for W.

Exercise 90.2. Let V be a free $\mathbb{Z}$-module of finite rank, i.e. V is a free abelian group. Let $\varphi\colon V \times V \to \mathbb{Z}$ be a form. Given a subgroup $U \subset V$ we write

$$U^\perp \;=\; \{v \in V \,|\, \varphi(u, v) = 0 \text{ for all } u \in U\}.$$

We suppose that the form φ is non-singular. Show that for any subgroup $U \subset V$ we have $\operatorname{rank}(U^\perp) = \operatorname{rank}(V) - \operatorname{rank}(U)$.

Exercise 90.3. Let $m, n \in \mathbb{N}$. We denote by $p\colon S^m \times S^n \to S^m$ and by $q\colon S^m \times S^n \to S^n$ the two obvious projection maps. Use Poincaré Duality to show that

$$p^*([S^m]^*) \cup q^*([S^n]^*) \;=\; \pm[S^m \times S^n]^*.$$

Put differently, you should show the equality without using the Product Theorem 84.1.

Exercise 90.4. For which $n \in \mathbb{N}$ is the complex projective space $\mathbb{C}\mathrm{P}^n$ chiral, i.e. for which $n \in \mathbb{N}$ does $\mathbb{C}\mathrm{P}^n$ admit an orientation-reversing self-homeomorphism?

Exercise 90.5. Does there exist a degree one map from $\mathbb{C}\mathrm{P}^2$ to $S^2 \times S^2$?

Exercise 90.6. Is $S^3 \vee \mathbb{R}\mathrm{P}^2$ homotopy equivalent to a smooth manifold?

Exercise 90.7. Show that there exists a connected CW-complex X with base point x_0 and a homomorphism $\varphi\colon \pi_1(X, x_0) \to \mathbb{Z}_2 = \pi_1(\mathbb{R}\mathrm{P}^2, *)$ which cannot be realized by a map $g\colon X \to \mathbb{R}\mathrm{P}^2$, i.e. such that there is no map $f\colon X \to \mathbb{R}\mathrm{P}^2$ with $f(x_0) = *$ such that $f_* = \varphi$.
Remark. This gives a negative answer to Question 79.11 (3).

[1289] The basis $w_1, \dots, w_n$ is in fact unique.

Exercise 90.8.

(a) Let $n \in \mathbb{N}$ and let M be a closed orientable $2n$-dimensional smooth manifold. Suppose that M admits a CW-structure with precisely three cells. Show that M has precisely one cell in dimensions 0, n and $2n$.

(b) Show that there is no closed orientable 6-dimensional smooth manifold that has only three cells.

(c) In (a) and (b), do we really need to assume that M is orientable?

Exercise 90.9. Let $n \in \mathbb{N}_0$ and let $\varphi\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ be map.

(a) Show that there exists a $d \in \mathbb{Z}$ such that

$$\underbrace{\sum_{k=0}^{2n}(-1)^k \cdot \operatorname{tr}(\varphi_*\colon\ \mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^n))}_{\text{Lefschetz number } \Lambda(\varphi)} \;=\; 1 + \sum_{l=1}^{n} d^l.$$

Hint. Use the cohomological reformulation of the Lefschetz number given in Exercise 75.12.

(b) Suppose n is odd. Use the Lefschetz Fixed Point Theorem 65.9 to show that φ admits a fixed point.

Remark. In Exercise 65.8 we showed that for any even $n \in \mathbb{N}_0$ there exists a self-map of $\mathbb{C}\mathrm{P}^n$ with no fixed points.

Exercise 90.10. Let $n \in \mathbb{N}_{\geq 2}$ and let $\varphi\colon S^2 \to S^2$ be a map of degree n. We consider $X := \mathrm{Cone}(S^2 \xrightarrow{\varphi} S^2 = \mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^2)$ and $Y := \mathrm{Cone}(S^2 \xrightarrow{\varphi} S^2) \vee S^4$.

(a) Show that the cohomology rings of X and Y with $\mathbb{Z}$-coefficients are isomorphic.

(b) Show that the cohomology rings of X and Y with $\mathbb{Z}$-coefficients are not isomorphic.

91. The Hopf invariant

In previous chapters we have seen that the introduction and the calculation of the cup product allowed us to answer many questions on topological spaces that we could not deal with before. In this section we want to use the cup product to address the following question that we formulated a long time ago.

Question 40.18. *Does the Hopf map $H\colon S^3 \to S^2$ represent a non-trivial element in the homotopy group $\pi_3(S^2)$?*

In this chapter we will use the calculation of the cup product of $\mathbb{C}\mathrm{P}^2$ to give an affirmative answer to this question. Furthermore we will use the quaternions and octonions to define higher dimensional analogues of the Hopf map which also represent non-trivial elements in homotopy groups of spheres.

91.1. The Hopf map is not null-homotopic.

First we recall the definition of the Hopf map. On page 197 we gave an explicit identification $\mathbb{C}\mathrm{P}^1 = S^2$. Furthermore on page 1071 we made the following definition.

Definition. We refer to the map[a]

$$\begin{array}{rcl} H\colon S^3 = \{(z_1,z_2)\in\mathbb{C}^2\,|\,|z_1|^2+|z_2|^2=1\} & \to & S^2=\mathbb{C}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1:z_2] \end{array}$$

as the *Hopf map.*

[a]See also Figure 692 for an illustration of the map.

Since the Hopf map is so much fun we will introduce two higher-dimensional analogues of the Hopf map. More precisely, we will repeat the trick with $\mathbb{C}$ replaced by the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. First we point out that we can use the homeomorphisms from Lemma 60.9 (2) and Proposition 60.10 (2) to make the identifications $\mathbb{H}\mathrm{P}^1 = S^4$ and $\mathbb{O}\mathrm{P}^1 = S^8$. Now we can introduce the quaternionic and octonionic Hopf maps.

Definition. We refer to

$$\begin{array}{rcl} \mathrm{H}_{\mathbb{H}}\colon S^7 = \{(z_1,z_2)\in\mathbb{H}^2\,|\,|z_1|^2+|z_2|^2=1\} & \to & S^4=\mathbb{H}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1:z_2] \end{array}$$

as the *quaternionic Hopf map* and we refer to

$$\begin{array}{rcl} \mathrm{H}_{\mathbb{O}}\colon S^{15} = \{(z_1,z_2)\in\mathbb{O}^2\,|\,|z_1|^2+|z_2|^2=1\} & \to & S^8=\mathbb{O}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1:z_2] \end{array}$$

as the *octonionic Hopf map.*

The idea for answering Question 40.18 is to translate this question about maps to a question about topological spaces for which we can then bring our new invariants to bear. To carry out this approach we recall the following definitions from earlier on:

Definition.

(1) On page 692 we defined the *cone* of a topological space A to be the topological space $\mathrm{Cone}(A) := (A\times[0,1])/\sim$ where $(a,0)\sim(a',0)$ for all $a,a'\in A$.

(2) On page 701 we defined the *mapping cone* of a map $f\colon A \to X$ between topological spaces as

$$\mathrm{Cone}(f\colon A \to X) \;:=\; (\mathrm{Cone}(A) \cup X)/\sim$$

where $(a,1) \sim f(a)$ for all $a \in A$.

Both definitions are illustrated in Figure 1255.

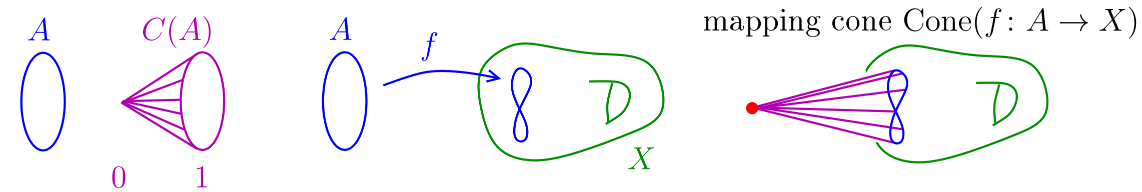


FIGURE 1255

Examples.

(1) It follows from Proposition 2.43 (3) and Lemma 3.22 that the map

$$\begin{array}{rcl} \mathrm{Cone}(S^{n-1}) & \to & \overline{B}^n \\ [(P,t)] & \mapsto & t \cdot P \end{array}$$

is a homeomorphism. We use this homeomorphism to identify $\mathrm{Cone}(S^{n-1})$ with $\overline{B}^n$. In particular, if X is a topological space and if $f\colon S^{n-1} \to X$ is a map, then we get an identification

$$\mathrm{Cone}(f\colon S^{n-1} \to X) \;=\; \overline{B}^n \cup_f X.$$

It follows immediately from this observation that any finite CW-complex can be viewed as an iterated mapping cone (where we start out with a finite discrete set).

(2) In Exercise 91.1 we will see that the mapping cone of the map $f\colon S^1 \to S^1$ that is given by $f(z) = z^2$ is a topological space that we are very familiar with.

The next lemma gives us another elementary but oddly useful example.

Lemma 91.1. *If $f\colon S^k \to X$ is a constant map, then there exists a homeomorphism*

$$\mathrm{Cone}(f\colon S^k \to X) \;\xrightarrow{\cong}\; S^{k+1} \vee X,$$

where we perform the wedge for suitable points in S^{k+1} and X.

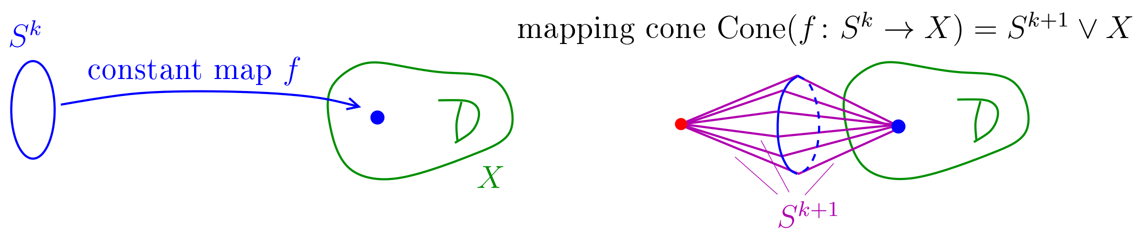


FIGURE 1256. Illustration of Lemma 91.1.

PROOF. Suppose $f\colon S^k \to X$ is a constant map. It follows easily from the example on page 2213 and the homeomorphism $\overline{B}^{k+1}/S^k \cong S^{k+1}$ from page 182 that $\mathrm{Cone}(f\colon S^k \to X)$ is homeomorphic to a wedge $S^{k+1} \vee X$. We leave it to the reader to fill in the details. ■

The following lemma gives a particularly interesting example of a mapping cone.

Lemma 91.2. *There exist homeomorphisms*

$$\begin{array}{lrcl} (1) & \mathrm{Cone}(H\colon S^3 \to S^2) & \cong & \mathbb{CP}^2 \\ (2) & \mathrm{Cone}(H_{\mathbb{H}}\colon S^7 \to S^4) & \cong & \mathbb{HP}^2 \\ (3) & \mathrm{Cone}(\mathrm{H}_{\mathbb{O}}\colon S^{15} \to S^8) & \cong & \mathbb{OP}^2. \end{array}$$

Proof.

(1) In Lemma 36.1 we saw that $\mathbb{CP}^2$ is a CW-complex that is obtained from the CW-complex $\mathbb{CP}^1 = S^2$ by attaching a 4-cell. The attaching map $S^3 \to \mathbb{CP}^1$ of the 4-cell is precisely the Hopf map. This observation, together with the discussion on page 2213, already implies the statement. For completeness' sake we give the argument more explicitly and in more detail. We consider the maps[1290]

$$\begin{array}{rcl} f\colon \mathrm{Cone}(S^3) & \to & \mathbb{CP}^2 \\ [(z_0,z_1),r] & \mapsto & [r\cdot z_0 : r\cdot z_1 : 1-r] \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{CP}^1 & \to & \mathbb{CP}^2 \\ [z_0 : z_1] & \mapsto & [z_0 : z_1 : 0]. \end{array}$$

It follows immediately from the definitions together with Lemma 3.22 that these two maps define a continuous map

$$\underbrace{\mathrm{Cone}(H\colon S^3 \to S^2)}_{=(\mathrm{Cone}(S^3)\cup \mathbb{CP}^1)/\sim} \to \mathbb{CP}^2.$$

One can easily verify that this map is a bijection. Furthermore, using Proposition 2.43 (3) and Proposition 3.40 one sees that the map is in fact a homeomorphism.

(2) The proof of (2) is basically identical to the proof of (1).

(3) Implicitly we mostly proved this statement in the proof of Proposition 60.10. Alternatively we also refer to [**Lackm12**, p. 4] for details. ■

Now we can finally answer Question 40.18 in the affirmative. More precisely we have the following theorem. The statement regarding the "classical" Hopf map was first proved by Heinz Hopf [**Hopf31**] in 1931.

Theorem 91.3.

(1) *The Hopf map $H\colon S^3 \to S^2$ represents a non-trivial element in the third homotopy group $\pi_3(S^2)$.*

(2) *We have $[H_{\mathbb{H}}] \neq 0 \in \pi_7(S^4)$.*

(3) *We have $[H_{\mathbb{O}}] \neq 0 \in \pi_{15}(S^8)$.*

Proof. First we prove (1). Recall that for two topological spaces X and Y we write $X \simeq Y$ if they are homotopy equivalent. We have

$$\mathrm{Cone}(\text{Hopf map } H) \underset{\substack{\uparrow \\ \text{Lemma 91.2 (1)}}}{\cong} \mathbb{CP}^2 \underset{\substack{\uparrow \\ \text{Lemma 90.4}}}{\not\simeq} S^2 \vee S^4 \underset{\substack{\uparrow \\ \text{Lemma 91.1}}}{\cong} \mathrm{Cone}(\text{constant map}).$$

[1290] It follows from Lemma 3.22 and Lemma 3.21 (3) that these two maps are continuous.

But by Lemma 24.12 this means that the Hopf map is not homotopic to the constant map. In particular the Hopf map $H\colon S^3 \to S^2$ represents a non-trivial element in the homotopy group $\pi_3(S^2)$.

The proof of the other two statements is quite similar. We again use Lemma 91.2 and we use the calculations of the cup products of $\mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$, see Proposition 90.20, to conclude that the mapping cones of the quaternionic and octonionic Hopf maps are not homotopy equivalent to a wedge of spheres. ■

91.2. **The Hopf invariant.** It is natural to ask how far we can push the methods of Theorem 91.3 to gain more insight into homotopy groups of spheres. In this section, given $n \geq 2$ we will introduce the Hopf invariant

$$\mathrm{Hopf}\colon \pi_{2n-1}(S^n) \;\to\; \mathbb{Z}$$

which satisfies $\mathrm{Hopf}(H) = \pm 1$, $\mathrm{Hopf}(H_{\mathbb{H}}) = \pm 1$ and $\mathrm{Hopf}(\mathrm{H}_{\mathbb{O}}) = \pm 1$.

The idea behind the definition of the Hopf invariant is straightforward. Namely, given $n \in \mathbb{N}_{\geq 2}$ and given a map $f\colon S^{2n-1} \to S^n$ we consider again the corresponding mapping cone $\mathrm{Cone}(f\colon S^{2n-1} \to S^n)$. We will see in Lemma 91.5 that it follows from the fact that $n \geq 2$ that we have $\mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z}) \cong \mathbb{Z}$. We will equip $\mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z})$ with a "preferred generator". We pick a generator $\varphi \in \mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z})$ and we define $\mathrm{Hopf}(f)$ to be the unique integer that satisfies

$$\varphi \cup \varphi \;=\; \mathrm{Hopf}(f)\cdot \text{preferred generator} \;\in\; \mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z}).$$

To carry out this program we need to carefully study the cohomology groups of a mapping cone. Recall that in Lemma 46.18 we saw that the homology groups of a mapping cone fit naturally into a long exact sequence of homology groups. The following lemma shows that the analogous statement also holds for the cohomology groups of a mapping cone.

Lemma 91.4. *Let R be a commutative ring.*

(1) *Let $f\colon A \to X$ be a map between topological spaces.*
   (a) *We denote by $j(f)\colon X \to \mathrm{Cone}(f) = (\mathrm{Cone}(A) \sqcup X)/\sim$ the obvious inclusion map.*
   (b) *We denote by U the image of $A \times [0, \frac{3}{4})$ in $\mathrm{Cone}(f)$ and we denote by V the complement of the image of $A \times [0, \frac{1}{4}]$. (See also Figure 1257.) We consider the following diagram*

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}^{k-1}(U\cap V;R) \arrow[r, "\delta"] \arrow[d, "="] & \mathrm{H}^k(\mathrm{Cone}(f);R) \arrow[r] & \mathrm{H}^k(U;R)\oplus \mathrm{H}^k(V;R) \arrow[r] & \dots \\ & \mathrm{H}^{k-1}(A;R). \arrow[ur, "=:\delta^{k-1}(f)"'] & & & \end{tikzcd}$$

Here the top sequence is the long exact sequence arising from the Mayer–Vietoris Theorem 74.15 for Cohomology Groups[1291] *and the vertical map comes from the obvious identifications $\mathrm{H}^{k-1}(U \cap V;R) = \mathrm{H}^{k-1}(A \times \frac{1}{2};R) = \mathrm{H}^{k-1}(A;R)$. From* *the above diagram we obtain the map*

$$\delta^{k-1}(f)\colon\ \mathrm{H}^{k-1}(A;R)\ \to\ \mathrm{H}^k(\mathrm{Cone}(f);R).$$

The following sequence is exact

$$\ldots \to \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f);R) \xrightarrow{j(f)^*} \mathrm{H}^k(X;R) \xrightarrow{f^*} \mathrm{H}^k(A;R) \xrightarrow{\delta^k(f)} \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(f);R) \to \ldots$$

(2) *The connecting homomorphism in* (1) *is natural. In particular given any commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi"'] & X \arrow[d, "\psi"] \\ \widetilde{A} \arrow[r, "\widetilde{f}"] & \widetilde{X} \end{tikzcd}$$

of maps between topological spaces the following diagram commutes:

$$\begin{tikzcd} \ldots \arrow[r] & \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f);R) \arrow[r, "j(f)^*"] & \mathrm{H}^k(X;R) \arrow[r, "f^*"] & \mathrm{H}^k(A;R) \arrow[r, "\delta^k(f)"] & \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(f);R) \arrow[r] & \ldots \\ \ldots \arrow[r] & \widetilde{\mathrm{H}}^k(\mathrm{Cone}(\widetilde{f});R) \arrow[r, "j(\widetilde{f})^*"] \arrow[u] & \mathrm{H}^k(\widetilde{X};R) \arrow[r, "\widetilde{f}^*"] \arrow[u, "\psi^*"] & \mathrm{H}^k(\widetilde{A};R) \arrow[r, "\delta^k(\widetilde{f})"] \arrow[u, "\varphi^*"] & \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(\widetilde{f});R) \arrow[r] \arrow[u] & \ldots \end{tikzcd}$$

Here the vertical maps between the cohomology groups of the mapping cones are induced by the maps from Lemma 24.10 (3).

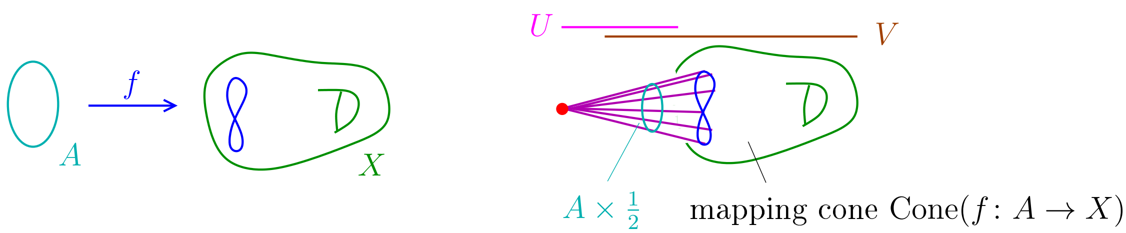


FIGURE 1257. Illustration of Lemma 91.4.

PROOF. The proof is basically identical to the proof of Lemma 46.18. The meticulous reader who filled in the details of the proof of Lemma 46.18 will have no troubles working out the details of the present proof. ■

We continue with the following lemma.

Lemma 91.5. *Let $f\colon S^m \to S^n$ be a map. If $m \geq n+1$ and $n \geq 1$, then the maps*

$$\mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z}) \quad \xrightarrow{j(f)^*} \quad \mathrm{H}^n(S^n;\mathbb{Z})$$

and

$$\mathrm{H}^m(S^m;\mathbb{Z}) \quad \xrightarrow{\delta^m(f)} \quad \mathrm{H}^{m+1}(\mathrm{Cone}(f);\mathbb{Z})$$

[1291]Note that U and V are open subsets of $\mathrm{Cone}(f)$. Thus it follows from Proposition 74.12 that we can in fact apply the theorem.

are isomorphisms and all other reduced cohomology groups of $\operatorname{Cone}(f)$ *are zero. An analogous statement holds for homology groups. In particular we have isomorphisms*

$$\mathrm{H}_k(\operatorname{Cone}(f\colon S^m \to S^n);\mathbb{Z}) \;\cong\; \mathrm{H}^k(\operatorname{Cone}(f\colon S^m \to S^n);\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, n, m+1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The lemma is an almost immediate consequence of Lemma 91.4 and the calculation of the cohomology groups of spheres, see page 1844. For $n = 1$ one also needs to invoke Exercise 73.6. ■

Throughout this section we use the notation that we had already introduced in the formulation of Lemma 91.4. More precisely, given a map $f\colon A \to X$ between two topological spaces we denote by $j(f)\colon X \to \operatorname{Cone}(f) = (\operatorname{Cone}(A) \cup X)/\sim$ the obvious inclusion map and furthermore we denote by $\delta^{k-1}(f)\colon \mathrm{H}^{k-1}(A;\mathbb{Z}) \to \mathrm{H}^k(\operatorname{Cone}(f);\mathbb{Z})$ the coboundary map in the long exact sequence in cohomology of Lemma 91.4.

Now we are finally in a position to turn the idea from page 2215 into a proper definition.

Definition. Let $f\colon S^{2n-1} \to S^n$ be a map with $n \geq 2$. The *Hopf invariant* $\operatorname{Hopf}(f)$ is defined to be the unique integer that satisfies[1292]

$$\underbrace{(j(f)^*)^{-1}([S^n]^*)}_{\substack{\text{generator of}\\ \mathrm{H}^n(\operatorname{Cone}(f);\mathbb{Z})}} \cup \underbrace{(j(f)^*)^{-1}([S^n]^*)}_{\substack{\text{generator of}\\ \mathrm{H}^n(\operatorname{Cone}(f);\mathbb{Z})}} \;=\; \operatorname{Hopf}(f)\cdot \underbrace{\delta^{2n-1}(f)([S^{2n-1}]^*)}_{\substack{\text{generator of}\\ \mathrm{H}^{2n}(\operatorname{Cone}(f);\mathbb{Z})}}$$

in $\mathrm{H}^{2n}(\operatorname{Cone}(f);\mathbb{Z}) \cong \mathbb{Z}$.

Remark.

(1) In the definition of the Hopf invariant we could have squared any of the two generators of $\mathrm{H}^n(\operatorname{Cone}(f);\mathbb{Z}) \cong \mathbb{Z}$, since the sign ambiguity goes away with the square.

(2) It follows immediately from the (anti-) commutativity of the cup product, see Proposition 81.8, that the Hopf invariant is zero for all odd n.

Example.

(1) As usual we denote by $H\colon S^3 \to S^2$ the Hopf map. In Lemma 91.2 we showed that the mapping cone $\operatorname{Cone}(H)$ is homeomorphic to $\mathbb{C}\mathrm{P}^2$. By Lemma 91.5 we know that $(j(H)^*)^{-1}([S^2]^*) \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ and $\delta^3(f)([S^3]^*) \in \mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ are generators. It follows from the calculation of the cup product on $\mathbb{C}\mathrm{P}^2$, see Proposition 90.7, that $\operatorname{Hopf}(H) = -1$ or $\operatorname{Hopf}(H) = 1$.[1293]

(2) Using Proposition 90.20 one can show, by the same argument as in (1), that the Hopf invariants of the quaternionic Hopf map $H_{\mathbb{H}}\colon S^7 \to S^4$ and the octonionic Hopf map $H_{\mathbb{O}}\colon S^{15} \to S^8$ are equal to ± 1.

Lemma 91.6. *Let $n \geq 2$ and let $f_0, f_1\colon S^{2n-1} \to S^n$ be two maps. If f_0 and f_1 are homotopic, then* $\operatorname{Hopf}(f_0) = \operatorname{Hopf}(f_1)$.

[1292]Here we implicitly use that in Lemma 91.5 we showed that $j(f)^*$ and $\delta^{2n-1}(f)$ are isomorphisms.

[1293]With more care one can show that in fact $\operatorname{Hopf}(H) = 1$, but this is irrelevant for our purpose.

Proof (∗). Let $H\colon S^{2n-1}\times[0,1]\to S^n$ be a homotopy between f_0 and f_1. We denote by $i_0,i_1\colon S^{2n-1}\to S^{2n-1}\times[0,1]$ the two obvious inclusion maps. We obtain the following two commutative diagrams

$$\begin{tikzcd} S^{2n-1} \arrow[r, "f_0"] \arrow[d, "i_0"'] & S^n \arrow[d, "="] \\ S^{2n-1}\times[0,1] \arrow[r, "H"] & S^n \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} S^{2n-1} \arrow[r, "f_1"] \arrow[d, "i_1"'] & S^n \arrow[d, "="] \\ S^{2n-1}\times[0,1] \arrow[r, "H"] & S^n. \end{tikzcd}$$

By Lemma 24.10 (3) the diagrams give rise to inclusion maps $\Xi_0\colon \operatorname{Cone}(f_0)\to\operatorname{Cone}(H)$ and $\Xi_1\colon \operatorname{Cone}(f_1)\to\operatorname{Cone}(H)$. From Lemma 91.4 we obtain the following commutative diagram of long exact sequences:

$$\begin{tikzcd} \dots \arrow[r] & \widetilde{\mathrm{H}}^k(\operatorname{Cone}(f_0)) \arrow[r, "j(f_0)^*"] & \mathrm{H}^k(S^n) \arrow[r, "f_0^*"] & \mathrm{H}^k(S^{2n-1}) \arrow[r, "\delta"] & \widetilde{\mathrm{H}}^{k+1}(\operatorname{Cone}(f_0)) \arrow[r] & \dots \\ \dots \arrow[r] & \widetilde{\mathrm{H}}^k(\operatorname{Cone}(H)) \arrow[r, "j(H)^*"] \arrow[u, "\Xi_0^*"] \arrow[d, "\Xi_1^*"'] & \mathrm{H}^k(S^n) \arrow[r, "\mathrm{H}^*"] \arrow[u, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \mathrm{H}^k(S^{2n-1}\times[0,1]) \arrow[r, "\delta"] \arrow[u, "\cong", "i_0^*"'] \arrow[d, "\cong"', "i_1^*"] & \widetilde{\mathrm{H}}^{k+1}(\operatorname{Cone}(H)) \arrow[r] \arrow[u, "\Xi_0^*"] \arrow[d, "\Xi_1^*"'] & \dots \\ \dots \arrow[r] & \widetilde{\mathrm{H}}^k(\operatorname{Cone}(f_1)) \arrow[r, "j(f_1)^*"] & \mathrm{H}^k(S^n) \arrow[r, "f_1^*"] & \mathrm{H}^k(S^{2n-1}) \arrow[r, "\delta"] & \widetilde{\mathrm{H}}^{k+1}(\operatorname{Cone}(f_1)) \arrow[r] & \dots \end{tikzcd}$$

Note that the vertical maps between the cohomology groups of the mapping cones are isomorphisms. This can be shown using the Five Lemma 43.12, or alternatively, in a more fancy way using Lemma 24.12.

Next we consider the generator $\Theta := (i_0^*)^{-1}([S^{2n-1}]^*) = (i_1^*)^{-1}([S^{2n-1}]^*)$ of the group $\mathrm{H}^{2n-1}(S^{2n-1}\times[0,1];\mathbb{Z})$. We define a number $\operatorname{Hopf}(H)$ via the equality

$$(j(H)^*)^{-1}([S^n]^*)\cup(j(H)^*)^{-1}([S^n]^*) \;=\; \operatorname{Hopf}(H)\cdot\delta^{2n-1}(H)(\Theta)$$

in $\mathrm{H}^{2n}(\operatorname{Cone}(H);\mathbb{Z})\cong\mathbb{Z}$. Finally we apply the above commutative diagram and we apply Lemma 81.10 to the maps $\Xi_0\colon\operatorname{Cone}(f_0)\to\operatorname{Cone}(H)$ and $\Xi_1\colon\operatorname{Cone}(f_1)\to\operatorname{Cone}(H)$ and we see that $\operatorname{Hopf}(f_0)=\operatorname{Hopf}(H)=\operatorname{Hopf}(f_1)$. ■

It follows from Lemma 91.6 that the Hopf invariant descends to a well-defined map on homotopy groups:

Definition. It follows from Lemma 91.6 that for each $n\geq 2$ the map

$$\begin{array}{rcl} \pi_{2n-1}(S^n,*) & \to & \mathbb{Z} \\ {[f]} & \mapsto & \operatorname{Hopf}(f) \end{array}$$

is well-defined. This map is called the *Hopf invariant.*

Our next proposition says that the Hopf invariant is a homomorphism. This is arguably somewhat surprising since at first glance the definition of the Hopf variant, given by squaring an element, does not look particularly like a linear map.

Proposition 91.7. *For any $n\geq 2$ the Hopf invariant* $\operatorname{Hopf}\colon\pi_{2n-1}(S^n,*)\to\mathbb{Z}$ *is a homomorphism.*

Together with the calculation of the Hopf invariant for the Hopf maps H, $\mathrm{H}_{\mathbb{H}}$ and $\mathrm{H}_{\mathbb{O}}$ on page 2217 we obtain the following immediate corollary to Proposition 91.7.

Corollary 91.8. *The Hopf invariants*

$$\begin{array}{lrcl} (1) & \mathrm{Hopf}\colon \pi_3(S^2) & \to & \mathbb{Z} \\ (2) & \mathrm{Hopf}\colon \pi_7(S^4) & \to & \mathbb{Z} \\ (3) & \mathrm{Hopf}\colon \pi_{15}(S^8) & \to & \mathbb{Z} \end{array}$$

are epimorphisms.

The technically slightly dicey proof of Proposition 91.7 will require the remainder of this section. The reader is cordially invited to skip the argument and to jump to the much more entertaining content of the next section.

Our proof[1294] of Proposition 91.7 will require a few preparations. First of all, on several occasions we will need the following construction.

Construction 91.9. *Suppose we are given the following commutative diagram of topological spaces:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ \widetilde{X} \arrow[r, "\widetilde{f}"] & \widetilde{Y}. \end{tikzcd}$$

Then, as discussed in Lemma 24.10 (3), we obtain an induced map

$$\begin{array}{rcl} \mathrm{Cone}(f\colon X \to Y) & \to & \mathrm{Cone}(f\colon \widetilde{X} \to \widetilde{Y}) \\ {[P]} & \mapsto & \begin{cases} [(\varphi(x), t)], & \text{if } P = (x,t) \text{ with } x \in X,\ t \in [0,1], \\ [\psi(y)], & \text{if } P = y \text{ with } y \in Y \end{cases} \end{array}$$

By Lemma 24.10 (4) the map fits into the following commutative diagram:

$$\begin{tikzcd} Y \arrow[r, hook, "j"] \arrow[d, "\psi"'] & \mathrm{Cone}(f\colon X \to Y) \arrow[r, "p"] \arrow[d] & \Sigma(X) \arrow[d, "\Sigma(\varphi)"] \\ \widetilde{Y} \arrow[r, hook, "j"] & \mathrm{Cone}(\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}) \arrow[r, "p"] & \Sigma(\widetilde{X}) \end{tikzcd}$$

where the horizontal maps to the left are the inclusions maps i that we introduced in Lemma 24.9 (2) and where the horizontal maps p on the right are given by the projection and the homeomorphism from Lemma 24.9 (4).

Finally we will need the following slight reinterpretation of the Hopf invariant.

Lemma 91.10. *Let $f\colon S^{2n-1} \to S^n$ be a map. We denote by $j\colon S^n \to \mathrm{Cone}(f)$ the inclusion map and we denote by $p\colon \mathrm{Cone}(f) \to \mathrm{Cone}(f)/S^n = \Sigma(S^{2n-1})$ the map given by the projection and homeomorphism from Lemma 24.9 (4). Then*

$$(j^*)^{-1}([S^n]^*) \cup (j^*)^{-1}([S^n]^*) \;=\; \mathrm{Hopf}(f) \cdot p^*(\Sigma([S^{2n-1}]^*)) \;\in\; \mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z}).$$

Proof of Lemma 91.10. The statement follows basically immediately from the discussion on page 1857[1295] and the cohomological analogue of Lemma 46.20. ■

[1294]Our proof is based on the proof in [**MTa68**, p. 35]. An alternative proof is sketched in [**Stro11**, p. 557].

[1295]Note that here we use the convenient fact that $(-1)^{(2n-1)+1} = +1$. In reality the sign is largely irrelevant, as long as we can be sure that the sign depends only on n.

Proof of Proposition 91.7. We start out with a few preparations.

(1) Given a pointed topological space (X, x_0) we define $X \vee X$ by identifying two copies of X along the base point. We refer to the map $\varphi\colon X \vee X \to X$ which is given by the identity on either of the two copies as the *folding map*.

(2) Let $k \in \mathbb{N}_{\geq 2}$. On page 1060 we introduced a pinching map $\pi\colon S^k \to S^k \vee S^k$. The precise definition is rather irrelevant to us right now. What matters is that we saw on page 1060 that given any pointed topological space (X, x_0) and given any two maps $f_1, f_2\colon (S^k, *) \to (X, x_0)$ we have

$$[f_1 * f_2] \;=\; \left[S^k \;\xrightarrow{\pi}\; S^k \vee S^k \;\xrightarrow{f_1 \vee f_2}\; X \vee X \;\xrightarrow{\varphi}\; X\right] \;\in\; \pi_k(X, x_0).$$

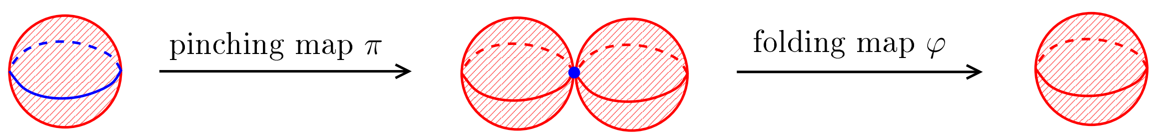


Figure 1258

Now we turn to our actual situation. Let $n \in \mathbb{N}_{\geq 2}$. Suppose we are given two maps $f_1, f_2\colon (S^{2n-1}, *) \to (S^n, *)$. For $i = 1, 2$ we consider the following diagram

$$\begin{tikzcd}
S^n \arrow[r, "j"] & \operatorname{Cone}(\varphi \circ (f_1 \vee f_2) \circ \pi\colon S^{2n-1} \to S^n) \arrow[r, "p"] & \Sigma S^{2n-1} \\
S^n \vee S^n \arrow[u, "\varphi"] \arrow[r, "j"] \arrow[d, "="'] & \operatorname{Cone}((f_1 \vee f_2) \circ \pi\colon S^{2n-1} \to S^n \vee S^n) \arrow[u, "\varphi"'] \arrow[r, "p"] \arrow[d, "\psi"] & \Sigma S^{2n-1} \arrow[u, "="'] \arrow[d, "\Sigma(\pi)"] \\
S^n \vee S^n \arrow[r, "j"] & \operatorname{Cone}(f_1 \vee f_2\colon S^{2n-1} \vee S^{2n-1} \to S^n \vee S^n) \arrow[r, "p"] & \Sigma(S^{2n-1} \vee S^{2n-1}) \\
S^n \vee S^n \arrow[u, "="] \arrow[r, "j \vee j"] \arrow[d, "q_i"'] & \operatorname{Cone}(f_1\colon S^{2n-1} \to S^n) \vee \operatorname{Cone}(f_2\colon S^{2n-1} \to S^n) \arrow[u, "\Xi"'] \arrow[r, "p \vee p"] \arrow[d, "q_i"] & \Sigma S^{2n-1} \vee \Sigma S^{2n-1} \arrow[u, "\Omega"'] \arrow[d, "q_i"] \\
S^n \arrow[r, "j"] & \operatorname{Cone}(f_i\colon S^{2n-1} \to S^n) \arrow[r, "p"] & \Sigma S^{2n-1}.
\end{tikzcd}$$

The top three vertical maps are illustrated in Figure 1259. We make the following clarifi-

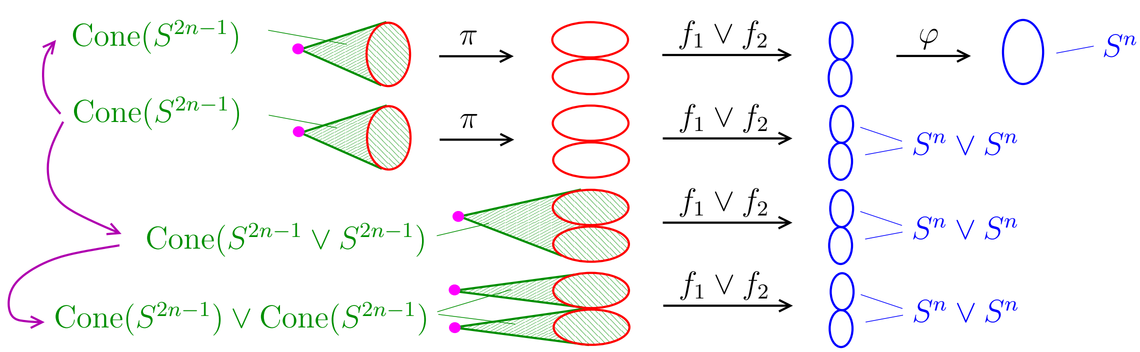


Figure 1259. Illustration for the proof of Proposition 91.7.

cations:

(1) The horizontal maps to the left are the inclusions maps j introduced in Lemma 24.9 (2) and the horizontal maps p on the right are given by the projection and the homeomorphism from Lemma 24.9 (4).
(2) The vertical maps between the first and the second and between the second and the third row are obtained from Construction 91.9 applied to the following two commutative diagrams

$$\begin{tikzcd} S^{2n-1} \arrow[r, "(f_1\vee f_2)\circ\pi"] \arrow[d, "="'] & S^n\vee S^n \arrow[d, "\varphi"] \\ S^{2n-1} \arrow[r, "\varphi\circ(f_1\vee f_2)\circ\pi"] & S^n \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} S^{2n-1} \arrow[r, "(f_1\vee f_2)\circ\pi"] \arrow[d, "\pi"'] & S^n\vee S^n \arrow[d, "="] \\ S^{2n-1}\vee S^{2n-1} \arrow[r, "f_1\vee f_2"] & S^n\vee S^n. \end{tikzcd}$$

In particular we see that the four upper squares commute.
(3) Next we consider the two diagrams

$$\begin{tikzcd} S^{2n-1} \arrow[r, "f_i"] \arrow[d] & S^n \arrow[d] \\ S^{2n-1}\vee S^{2n-1} \arrow[r, "f_1\vee f_2"] & S^n\vee S^n \end{tikzcd}$$

which are given by the injections into the first respectively second factor. By Construction 91.9 these diagrams give rise to two sequences of maps. We “wedge” the two sequences and we obtain the vertical maps from the fourth to the third row.
(4) At the bottom, in each column the map q_i is the projections onto the i-th factor.

Next we apply the cohomology functor $X\mapsto \mathrm{H}^k(X)=\mathrm{H}^k(X;\mathbb{Z})$ with $k=n$ and $k=2n$ to the above diagram (except that at the bottom we consider both maps q_1 and q_2). We obtain the following commutative diagram:[1296]

$$\begin{tikzcd} \mathrm{H}^k(S^n) & \mathrm{H}^k(\mathrm{Cone}(\varphi\circ(f_1\vee f_2)\circ\pi\colon S^{2n-1}\to S^n)) \arrow[l, "j^*"', "\cong"] \arrow[d, "\varphi^*"] & \mathrm{H}^k(\Sigma S^{2n-1}) \arrow[l, "p^*"', "\cong"] \arrow[d, "="] \\ \mathrm{H}^k(S^n\vee S^n) \arrow[u, hook, "\varphi^*"] & \mathrm{H}^k(\mathrm{Cone}((f_1\vee f_2)\circ\pi\colon S^{2n-1}\to S^n\vee S^n)) \arrow[l, "j^*"', "\cong"] & \mathrm{H}^k(\Sigma S^{2n-1}) \arrow[l, "p^*"', "\cong"] \\ \mathrm{H}^k(S^n\vee S^n) \arrow[u, "="] \arrow[d, "="] & \mathrm{H}^k(\mathrm{Cone}(f_1\vee f_2\colon S^{2n-1}\vee S^{2n-1}\to S^n\vee S^n)) \arrow[l, "j^*"', "\cong"] \arrow[u, "\psi^*"] \arrow[d, "\cong", "\Xi^*"'] & \mathrm{H}^k(\Sigma(S^{2n-1}\vee S^{2n-1})) \arrow[l, "p^*"', "\cong"] \arrow[u, "\Sigma(\pi)^*"] \arrow[d, "\cong", "\Omega^*"'] \\ \mathrm{H}^k(S^n\vee S^n) & \mathrm{H}^k(\mathrm{Cone}(f_1)\vee\mathrm{Cone}(f_2)) \arrow[l, "(j\vee j)^*"', "\cong"] & \mathrm{H}^k(\Sigma S^{2n-1}\vee\Sigma S^{2n-1}) \arrow[l, "(p\vee p)^*"', "\cong"] \\ \mathrm{H}^k(S^n)\oplus\mathrm{H}^k(S^n) \arrow[u, "q_1^*\oplus q_2^*", "\cong"'] & \mathrm{H}^k(\mathrm{Cone}(f_1))\oplus\mathrm{H}^k(\mathrm{Cone}(f_2)) \arrow[l, "j_1^*\oplus j_2^*"', "\cong"] \arrow[u, "\cong", "q_1^*\oplus q_2^*"'] & \mathrm{H}^k(\Sigma S^{2n-1})\oplus\mathrm{H}^k(\Sigma S^{2n-1}). \arrow[l, "p_1^*\oplus p_2^*"', "\cong"] \arrow[u, "\cong", "q_1^*\oplus q_2^*"'] \end{tikzcd}$$

[1296] Any reader who knows of a program which can automatically turn the latex-code for a diagram of topological spaces into the latex-code for the corresponding diagram of cohomology groups should contact me urgently.

We make the following observations:

(1) It follows almost immediately from Lemma 91.4 that the horizontal maps to the left are isomorphisms in degree n. Since these isomorphisms are only in degree n they are colored in red.
(2) It follows easily from Lemma 91.4 and the cohomological analogue of Lemma 46.20 that the horizontal maps to the right are isomorphisms for $k = 2n$. Since these isomorphisms are only in degree $2n$ they are colored in green.
(3) Similar to the calculation in Exercise 47.1 we see that the map Ω^* is an isomorphism.
(4) It follows from (1), (2) and (3) that the map Ξ^* is an isomorphism.
(5) The vertical maps at the very bottom are induced by the obvious projection maps. The maps are isomorphisms by Proposition 74.17.

Next we denote by $\alpha = [S^n]^* \in \mathrm{H}^n(S^n)$ the dual fundamental class. Furthermore we write $\beta = ([\Sigma([S^{2n-1}]^*)]) \in \mathrm{H}^{2n}(\Sigma S^{2n-1})$.

Claim 1. We have $\varphi^*(\alpha) = q_1^*(\alpha) + q_2^*(\alpha)$ and $((\Sigma(\pi)^* \circ (\Omega^*)^{-1})(q_1^*(\beta) + q_2^*(\beta)) = \beta$.

We leave it to the reader to verify this claim.

The key calculation takes place in the fourth row. More precisely, we have the following claim.

Claim 2. The following equality holds in $\mathrm{H}^{2n}(\mathrm{Cone}(f_1) \vee \mathrm{Cone}(f_2))$:

$$\big((j \vee j)^*)^{-1}(q_1^*(\alpha) + q_2^*(\alpha))\big)^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot (p \vee p)^*(q_1^*(\beta) + q_2^*(\beta)).$$

We calculate that

$$\begin{aligned}
\big((j \vee j)^{*-1}(q_1^*\alpha + q_2^*\alpha)\big)^2 &\underset{\text{since } j_i \circ q_i = q_i \circ (j \vee j)\colon S^n \vee S^n \to \mathrm{Cone}(f_i\colon S^{2n-1} \to S^n)}{=} \big(q_1^*(j_1^{*-1}\alpha) + q_2^*(j_2^{*-1}\alpha)\big) \cup \big(q_1^*(j_1^{*-1}\alpha) + q_2^*(j_2^{*-1}\alpha)\big) \\
&\underset{\text{by Lemma 82.7 the “mixed” terms are zero}}{=} q_1^*(j_1^{*-1}\alpha) \cup q_1^*(j_1^{*-1}\alpha) + q_2^*(j_2^{*-1}\alpha) \cup q_2^*(j_2^{*-1}\alpha) \\
&\underset{\text{since the cup product is natural, see Lemma 81.10}}{=} q_1^*(j_1^{*-1}\alpha \cup j_1^{*-1}\alpha) + q_2^*(j_2^{*-1}\alpha \cup j_2^{*-1}\alpha) \\
&\underset{\text{by the reinterpretation of Hopf}(f_1)\text{ and Hopf}(f_2)\text{ given in Lemma 91.10}}{=} q_1^*(\mathrm{Hopf}(f_1) \cdot p_1^*(\beta)) + q_2^*(\mathrm{Hopf}(f_2) \cdot p_2^*(\beta)) \\
&\underset{\text{since } p_i \circ q_i = q_i \circ (p \vee p)\colon\ \mathrm{Cone}(f_1) \vee \mathrm{Cone}(f_2) \to \Sigma S^{2n-1}}{=} (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot (p \vee p)^*(q_1^*(\beta) + q_2^*(\beta)).
\end{aligned}$$

⊞

It follows from $[f_1 * f_2] = [\varphi \circ (f_1 \vee f_2) \circ \pi]$, the definition of α and β and the reinterpretation of the Hopf invariant given in Lemma 91.10, that it remains to prove the following claim.

Claim 3. The following equality holds in the first row:

$$((j^*)^{-1}(\alpha))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*(\beta).$$

The idea is of course to push the equality from Claim 2 through the above commutative diagram towards the top. The key tool hereby is the naturality of the cup product, see Lemma 81.10. First we note that it follows from Lemma 81.10 and the fact that φ^* is an isomorphism in degree $2n$ that it remains to prove that the following equality holds in the second row:

$$(*) \qquad (j^*)^{-1}(\varphi^*(\alpha))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*(\beta).$$

To prove this equality we note that it follows from Claims 1 and 2 together with Lemma 81.10 that the following equality holds in the third row:

$$((j^*)^{-1}(\varphi^*(\alpha)))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*((\Omega^*)^{-1}(q_1^*(\beta) + q_2^*(\beta))).$$

Again, using Lemma 81.10 we see that the following equality holds in the second row:

$$((j^*)^{-1}(\varphi^*(\alpha)))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*(\underbrace{\Sigma(\pi)^*((\Omega^*)^{-1}(q_1^*(\beta) + q_2^*(\beta)))}_{=\beta \text{ by Claim 1}}).$$

But using Claim 1 we see that this is precisely the desired equality that we formulated in $(*)$. ■

91.3. **The Hopf invariant and multiplication maps.** By a *multiplication map* we mean any map of the form $f\colon S^k \times S^k \to S^k$. In this section we will see that such multiplication maps give rise to potentially interesting maps $\Theta_f\colon S^{2k+1} \to S^k$. In the key calculation of this section we show how we can determine the Hopf invariant $\mathrm{Hopf}(\Theta_f)$ directly in terms of homological data of the original map f.

We start out with recalling the following not terribly inspired definition from Exercise 45.20.

Definition. Let $n \in \mathbb{N}$ and let $f\colon S^n \times S^n \to S^n$ be a map. We pick $x_0, y_0 \in S^n$. We define the *bidegree* of f as[1297]

$$\mathrm{bideg}(f) := (p, q) \;\text{ where } p := \deg\left(\begin{array}{rcl} S^n & \to & S^n \\ x & \mapsto & f(x, y_0) \end{array}\right) \text{ and } q := \deg\left(\begin{array}{rcl} S^n & \to & S^n \\ y & \mapsto & f(x_0, y) \end{array}\right).$$

Examples.

(1) The map $f\colon S^1 \times S^1 \to S^1$ that is given by complex multiplication has bidegree $(1,1)$. This can be seen as follows: taking $x_0 = y_0 = 1$, we see that the two maps $S^1 \to S^1$ given by $z \mapsto f(z,1)$ and $z \mapsto f(1,z)$ are in fact the identity.
(2) The same argument as in (1) shows that the quaternion multiplication $S^3 \times S^3 \to S^3$ is a map of bidegree $(1,1)$ and that the octonion multiplication $S^7 \times S^7 \to S^7$ is also a map of bidegree $(1,1)$.
(3) Let $n \in \mathbb{N}$ and let $f\colon S^n \times S^n \to S^n$ be a map of bidegree (p,q). Furthermore let $\varphi, \psi\colon S^n \to S^n$ two maps. It follows almost immediately from the definitions that the bidegree of the map $(x,y) \mapsto f(\varphi(x), \psi(y))$ equals $(p \cdot \deg(\varphi), q \cdot \deg(\psi))$.

The previous example shows that the division algebras $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ give rise to maps of bidegree $(1,1)$. This result can be generalized to all division algebras.

Lemma 91.11. *Let $n \in \mathbb{N}_{\geq 2}$. If $\mathbb{R}^n$ admits the structure of a division algebra*[1298]*, then there exists a map $S^{n-1} \times S^{n-1} \to S^{n-1}$ of bidegree (p,q) where both p and q are odd.*

[1297] In Exercise 45.20 (a) we saw that the definition not depend on the choice of x_0 and y_0?

Proof. Let $*$ be the multiplication map of a division algebra on $\mathbb{R}^n$. As in the proof of Theorem 60.4 we consider the map

$$\begin{array}{rcl} f\colon S^{n-1}\times S^{n-1} & \to & S^{n-1} \\ (x,y) & \mapsto & f(x,y) := \dfrac{x*y}{\|x*y\|}. \end{array}$$

In the proof of Theorem 60.4 we already showed, using the hypothesis that "$*$" defines a division algebra, that this map is well-defined and that it is continuous. Now we pick a point $\star\in S^{n-1}$ and we consider the maps

$$\begin{array}{rcl} \varphi\colon S^{n-1} & \to & S^{n-1} \\ x & \mapsto & f(x,\star) \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi\colon S^{n-1} & \to & S^{n-1} \\ y & \mapsto & f(\star,y). \end{array}$$

It follows easily from the fact that $*$ is the multiplication of a real algebra that for all $z\in S^{n-1}$ we have $\varphi(-z)=-\varphi(z)$ and $\psi(-z)=-\psi(z)$. By Proposition 59.4 this implies that $\deg(\varphi)$ and $\deg(\psi)$ are odd. Conveniently enough that is exactly what he had set out to show. ■

Before we continue we recall a few conventions that we had introduced along the way. Most of them are also illustrated in Figure 1260.

(1) Given a topological space we consider the cone $\operatorname{Cone}(X)$, the suspension $\Sigma(X)$ and we consider the two copies $\operatorname{Cone}_\pm(X)$ of the cone in the suspension.[1299]

(2) Let $m\in\mathbb{N}_0$. Note that by the discussion on page 692 we can make the identification $\operatorname{Cone}(S^m)=\overline{B}^{m+1}$ and by the discussion on page 695 we can make the identification $\Sigma(S^m)=S^{m+1}$.

(3) Let $m,n\in\mathbb{N}_0$. By the discussion on page 203 we can make the identification

$$(S^m\times\overline{B}^{n+1})\cup_{\mathrm{id}_{S^m\times S^n}}(\overline{B}^{m+1}\times S^n) \xrightarrow{=} S^{m+n+1}.$$

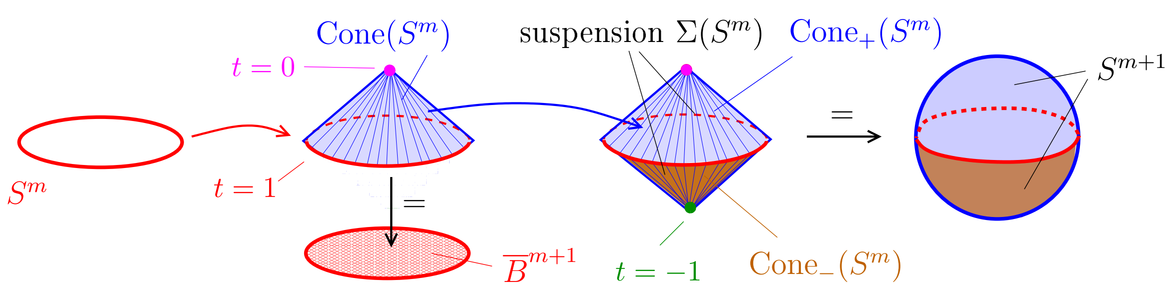


Figure 1260

Now we can move on to the promised construction of potentially interesting maps between spheres.

[1298] Recall that by the definition on page 1461 a division algebra is an algebra that has left and right inverses. We do not assume that division algebras are associative and we do not assume that they preserve the scalar product.

[1299] Here we implicitly use Lemma 24.2.

Definition. Let $f\colon S^p \times S^q \to S^r$ be a map. We consider the maps

$$\begin{array}{rcl} S^p \times \mathrm{Cone}(S^q) & \to & \mathrm{Cone}_+(S^r) \\ (P,[(Q,t)]) & \mapsto & [(f(P,Q),1-t)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{Cone}(S^p) \times S^q & \to & \mathrm{Cone}_-(S^r) \\ ([(P,t)],Q) & \mapsto & [(f(P,Q),t-1)]. \end{array}$$

We will see in Exercise 91.2 that these two maps are continuous. Evidently the two maps agree on the overlap $S^p \times S^q$. Thus by Lemma 3.44 (5) these two maps define a continuous map

$$\Theta_f\colon \underbrace{(S^p \times \mathrm{Cone}(S^q)) \cup_{S^p \times S^q} (\mathrm{Cone}(S^p) \times S^q) = S^{p+q+1}}_{=(S^p \times \overline{B}^{q+1}) \cup_{S^p \times S^q} (\overline{B}^{p+1} \times S^q)} \to \underbrace{\Sigma(S^r)}_{=S^{r+1}}.$$

We refer to Figure 1261 for a moderately successful attempt at illustrating the map Θ_f.

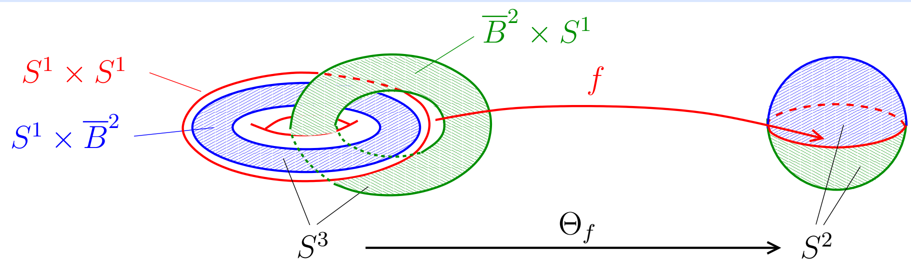


FIGURE 1261

Example.

(1) Again we consider the map $f\colon S^1 \times S^1 \to S^1$ that is given by complex multiplication. We obtain by the above construction a map $\Theta_f\colon S^3 \to S^2$. The heroically inclined reader will surely rise to the challenge of showing that this map is homotopic to the Hopf map.[1300]
(2) Similarly to (1) one can show that the quaternion multiplication $S^3 \times S^3 \to S^3$ gives rise to the quaternionic Hopf map $S^7 \to S^4$ and that the octonion multiplication $S^7 \times S^7 \to S^7$ gives rise to the quaternionic Hopf map $S^{15} \to S^8$.

Given a map $f\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ the next proposition computes the Hopf invariant of Θ_f in terms of the bidegree of f.

Proposition 91.12. *Let $n \in \mathbb{N}_{\geq 2}$. If $f\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ is a map of bidegree (r,s), then the induced map $\Theta_f\colon S^{2n-1} \to S^n$ has Hopf invariant $\pm rs$.*[1301]

PROOF. We make the following preparations:

(1) Throughout the proof all cohomology groups are understood to be with $\mathbb{Z}$-coefficients.
(2) To simplify the notation we write $\mathrm{C}(\Theta_f)$ instead of $\mathrm{Cone}(\Theta_f)$.
(3) By Lemma 24.6 we know that the inclusion $S^n \to \mathrm{C}(\Theta_f\colon S^{2n-1} \to S^n)$ is actually an embedding, thus we can identify S^n with its image in $\mathrm{C}(\Theta_f)$.
(4) We denote by $\overline{B}^n_+$ the upper hemisphere of S^n and we denote by $\overline{B}^n_-$ the lower hemisphere of S^n.

[1300]To be on the safe side, let us more cautiously say that the map Θ_f is either homotopic to the Hopf map or that it is homotopic to the composition of the Hopf map with a reflection in a hyperplane.

[1301]We leave it to the sign aficionados to determine the correct sign.

(5) By the discussion on page 692 we can make the identification

$$\overline{B}^n \times \overline{B}^n \;=\; \overline{B}^{2n} \;=\; \operatorname{Cone}\big(\underbrace{(S^{n-1}\times \overline{B}^n)\cup_{S^{n-1}\times S^{n-1}} (\overline{B}^n\times S^{n-1})}_{=S^{2n-1}}\big).$$

We denote by

$$F\colon \overline{B}^n \times \overline{B}^n \;\to\; \mathrm{C}(\Theta_f\colon S^{2n-1}\to S^n) \;=\; \underbrace{\operatorname{Cone}(S^{2n-1})}_{=\overline{B}^n\times\overline{B}^n=\overline{B}^{2n}} \cup_{\Theta_f} S^n$$

the obvious map which is given by the identification $\overline{B}^n\times\overline{B}^n = \operatorname{Cone}(S^{2n-1})$ followed by the projection.

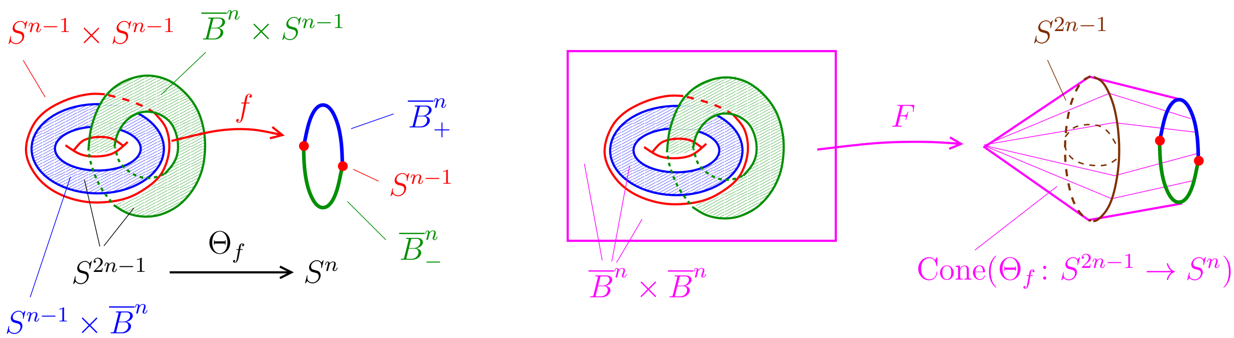


FIGURE 1262. Illustration for the proof of Proposition 91.12.

Next we pick generators for various cohomology groups:

(1) We denote by $x \in \mathrm{H}^n(\mathrm{C}(\Theta_f))$ and $z\in \mathrm{H}^{2n}(\mathrm{C}(\Theta_f))$ the generators that are given in the definition of the Hopf invariant, see page 2217.
(2) The long exact sequence in cohomology of the pair $(\mathrm{C}(\Theta_f),\overline{B}^n_\pm)$ shows that the maps $\mathrm{H}^n(\mathrm{C}(\Theta_f),\overline{B}^n_\pm)\to \mathrm{H}^n(\mathrm{C}(\Theta_f))$ are isomorphisms. Let $x_\pm\in \mathrm{H}^n(\mathrm{C}(\Theta_f),\overline{B}^n_\pm)$ be the generators that correspond to x under the given isomorphisms.
(3) We denote by $[S^{n-1}]^*\in \mathrm{H}^{n-1}(S^{n-1})$ and $[\overline{B}^n]^*\in \mathrm{H}^n(\overline{B}^n,S^{n-1})$ the standard dual generators. For $i=1,2$ we denote by $p_i\colon \overline{B}^n\times\overline{B}^n\to\overline{B}^n$ the projection onto the first respectively second coordinate. We write

$$\begin{aligned} y_1 &= p_1^*([\overline{B}^n]^*)\in \mathrm{H}^n(\overline{B}^n\times\overline{B}^n, S^{n-1}\times\overline{B}^n) \quad\text{and}\\ y_2 &= p_2^*([\overline{B}^n]^*)\in \mathrm{H}^n(\overline{B}^n\times\overline{B}^n, \overline{B}^n\times S^{n-1}).\end{aligned}$$

The following claim contains the key calculation of the proof.

Claim. The map F induces the following maps on relative cohomology:

$$\begin{array}{lrcl} \text{(i)} & \mathrm{H}^n(\mathrm{C}(\Theta_f),\overline{B}^n_+) & \xrightarrow{F^*} & \mathrm{H}^n(\overline{B}^n\times\overline{B}^n, S^{n-1}\times\overline{B}^n)\\ & x_+ & \mapsto & \pm r\cdot y_1\\ \text{(ii)} & \mathrm{H}^n(\mathrm{C}(\Theta_f),\overline{B}^n_-) & \xrightarrow{F^*} & \mathrm{H}^n(\overline{B}^n\times\overline{B}^n, \overline{B}^n\times S^{n-1})\\ & x_- & \mapsto & \pm s\cdot y_2.\end{array}$$

Before we turn to the actual calculation we make the following two preparations:

(a) We point out that the restrictions of F to various subsets gives us the following commutative diagram

$$\begin{tikzcd}
\overline{B}^n \times \overline{B}^n \arrow[r, "F"] & \mathrm{C}(\Theta_f) \\
\overline{B}^n \times S^{n-1} \arrow[u, hook] \arrow[r, "\Theta_f"] & \overline{B}^n_- \arrow[u, hook] \\
S^{n-1} \times S^{n-1} \arrow[u, hook] \arrow[r, "f"] & S^{n-1}. \arrow[u, hook]
\end{tikzcd}$$

(b) We pick a point $* \in S^{n-1}$.

To provide the actual proof of the claim we consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}^n(\mathrm{C}(\Theta_f)) \arrow[d, "\cong"] & & \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_+) \arrow[ll, "\cong"'] \arrow[r, "F^*"] \arrow[d] \arrow[dl] & \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n) \arrow[d, "\cong"] \\
\mathrm{H}^n(S^n) & \mathrm{H}^n(S^n, \overline{B}^n_+) \arrow[l, "\cong"'] \arrow[r, "\cong", "\text{exc.}"'] & \mathrm{H}^n(\overline{B}^n_-, S^{n-1}) \arrow[r, "\Theta_f^*"] & \mathrm{H}^n(\overline{B}^n \times *, S^{n-1} \times *) \\
& & \mathrm{H}^{n-1}(S^{n-1}) \arrow[u, "\cong", "\delta"'] \arrow[r, "f^*", "{[S^{n-1}]^* \mapsto r \cdot [S^{n-1} \times *]^*}"'] & \mathrm{H}^{n-1}(S^{n-1} \times *). \arrow[u, "\delta", "\cong"']
\end{tikzcd}$$

We make the following clarifications and observations:

(1) The maps that are not specified are induced by the obvious inclusions of (pairs) of topological spaces.
(2) The inclusion induced maps that are decorated by "$\cong$" are isomorphisms for the following reasons:
   (a) by the long exact sequence in cohomology for the corresponding pair,
   (b) the maps are induced by a homotopy equivalence,
   (c) by the long exact sequence provided by Lemma 91.4,
   (d) by an appropriate version of the Excision Theorem, see the remark on page 1840.
(3) The quadrilateral to the left commutes since all maps are induced by inclusions.
(4) It follows from (2) and (3) that the diagonal map is an isomorphism.
(5) The triangle towards the center commutes since all maps are induced by inclusions.
(6) It follows from (2), (4) and (5) that the middle vertical map is an isomorphism.
(7) The squares to the right commute by the commutative diagram shown in (a) and by the naturality of the connecting homomorphisms.
(8) The connecting homomorphisms δ are easily seen to be isomorphisms since the other groups in the corresponding long exact sequence are zero.
(9) Basically by the definition of the bidegree of the map f the bottom horizontal map is given by $[S^{n-1}]^* \mapsto r \cdot [S^{n-1} \times *]^*$.

The above arguments show that the vertical maps of the blue part of the diagram are isomorphisms. Since the bottom map is given by multiplication by r we see that the top map, with whatever choice of generators is also given by multiplication by $\pm r$. This concludes the proof of part (i) of the claim. The proof of part (ii) is evidently basically the same. ⊞

Finally we consider the following diagram

$$\begin{tikzcd}
\mathrm{H}^n(\mathrm{C}(\Theta_f)) \times \mathrm{H}^n(\mathrm{C}(\Theta_f)) \arrow[d, "\cup"', "(x,x)\mapsto x^2=\mathrm{Hopf}(\Theta_f)\cdot z"] & \mathrm{H}^n(\mathrm{C}(\Theta_f),\overline{B}^n_-) \times \mathrm{H}^n(\mathrm{C}(\Theta_f),\overline{B}^n_+) \arrow[l, "{x \leftarrow\!\shortmid x_-,\ x \leftarrow\!\shortmid x_+}"', "\cong"] \arrow[r, "{x_- \mapsto \pm r\cdot y_1,\ x_+ \mapsto \pm s\cdot y_2}", "F^*"'] \arrow[d, "\cup"] & \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n) \times \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1}) \arrow[d, "\cup"', "(y_1,y_2)\mapsto y_1\cup y_2=[\overline{B}^n\times\overline{B}^n]^*"] \\
\mathrm{H}^{2n}(\mathrm{C}(\Theta_f)) & \mathrm{H}^{2n}(\mathrm{C}(\Theta_f),S^n) \arrow[l, "\cong"'] \arrow[r, "F^*", "\cong"'] & \mathrm{H}^{2n}(\overline{B}^n\times\overline{B}^n, \overline{B}^n\times S^{n-1}\cup S^{n-1}\times\overline{B}^n). \arrow[ll, bend left, "\pm z \leftarrow\!\shortmid [\overline{B}^n\times\overline{B}^n]^*"]
\end{tikzcd}$$

We make the following comments:

(1) It follows almost immediately from Proposition 74.12 (5) that $(\mathrm{C}(\Theta_f), \overline{B}^n_-, \overline{B}^n_+)$ and $(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n, \overline{B}^n \times S^{n-1})$ are excisive triads. Thus we see that the relative cup product in the center and to the right are actually defined.

(2) It follows from the naturality of the cup product, see Proposition 82.4, that the two squares commute.

(3) By Proposition 84.2 (1) we have $y_1 \cup y_2 = p_1^*([\overline{B}^n]) \cup p_2^*([\overline{B}^n]) = [\overline{B}^n \times \overline{B}^n]^*$.

(4) By the above claim we know that the top right horizontal map is in fact given by $x_- \mapsto \pm r \cdot y_1$ and $x_+ \mapsto \pm s \mapsto s \cdot y_2$.

(5) Since the bottom horizontal maps are isomorphisms we see that going right to left the generator $[\overline{B}^n \times \overline{B}^n]^*$ gets sent to $\pm z$.

It follows immediately from the above comments that $\mathrm{Hopf}(\Theta_f) = \pm r \cdot s$. ■

Since the Hopf invariant gives us a homomorphism $\pi_{2n-1}(S^n) \to \mathbb{Z}$ we are now interested in finding interesting maps $S^{n-1} \times S^{n-1} \to S^{n-1}$. Perhaps somewhat surprisingly it is not that hard to find interesting maps.

Proposition 91.13. *Given any* even $n \in \mathbb{N}$ *the map*

$$\begin{array}{rcll} f\colon S^{n-1} \times S^{n-1} & \to & S^{n-1} & \\ (x, y) & \mapsto & y - 2 \cdot \langle x, y\rangle \cdot x & = \textit{reflection of } y \textit{ in the hyperplane } x^\perp \end{array}$$

has bidegree $(2, -1)$.

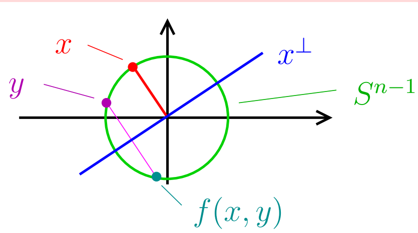


FIGURE 1263. Illustration of Proposition 91.13.

PROOF. We need to show that the given map $f\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ has bidegree $(2, -1)$. The proof of this statement naturally breaks up into two parts.

(1) We take $y_0 = (1, 0, \ldots, 0)$. We consider the map $\varphi\colon S^{n-1} \to S^{n-1}$ that is given by $x \mapsto (x, y_0)$. Note that $\varphi^{-1}((1, 0, \ldots, 0)) = (\pm 1, 0, \ldots, 0)$. Using this observation

together with Proposition 45.23 and Proposition 45.22 one can easily show that the degree of the map $x \mapsto (x, y_0)$ equals $1 + (-1)^{n-2}$. Since n is by hypothesis even we see that the degree equals two.

(2) We take $x_0 = (1, 0, \ldots, 0)$. The corresponding $S^{n-1} \to S^{n-1}$, $y \mapsto (x_0, y)$ is the reflection in a hyperplane, hence it follows from Lemma 45.11 that the map has degree -1. ■

The combination of Propositions 91.7 and 91.13 gives us the following interesting corollary.

Corollary 91.14. *Given any even $n \in \mathbb{N}$ there exists an epimorphism $\pi_{2n-1}(S^n) \to \mathbb{Z}$.*

As always in mathematics, every result opens up lots of new questions. Here are a few:

Question 91.15.

(1) *We have now seen that there exists an epimorphism $\pi_3(S^2) \to \mathbb{Z}$. Is this map an isomorphism?*

(2) *We have now seen that there exists an epimorphism $\pi_7(S^4) \to \mathbb{Z}$. Is this map an isomorphism?*

We will provide the answer to Question 91.15 (1) in Theorem 113.10 and we will see the answer to Question 91.15 (2) in Theorem 117.19 (1).

Question 91.15 points to a problem: we have now developed some tools for showing that certain homotopy groups of spheres are non-trivial, but we have no tools yet for computing any interesting homotopy groups. We will partially rectify this problem in Chapter 117.

Here is another question. We delay the answer to Theorem 118.1.

Question 91.16. *Given any $n \in \mathbb{N}$ we know by Lemma 53.2 and Corollary 91.14 that $\pi_n(S^n)$ and $\pi_{4n-1}(S^{2n})$ admit an epimorphism onto $\mathbb{Z}$. Are there are any other homotopy groups of spheres that admit an epimorphism onto $\mathbb{Z}$?*

In light of Lemma 91.11 and Proposition 91.13 the following question becomes interesting:

Question 91.17. *For which $n \in \mathbb{N}_{\geq 2}$ does there exist a map $f\colon S^{2n-1} \to S^n$ such that $\mathrm{Hopf}(f) = 1$?*[1302]

We will see the complete answer to Question 91.17 in Theorems 110.10 and 110.11.

Initially one might not appreciate Question 91.17. But rather mysteriously it is not that hard to come up with maps of Hopf degree 2, but it is strikingly difficult to come up with maps of Hopf degree 1. To underline our point we conclude this chapter with another construction of a map of Hopf degree 2.

Example. Let $n \in \mathbb{N}$. We start out with the following two reminders:

(1) We can view S^n as a CW-complex with one 0-cell and one n-cell. In particular, if we denote by S^n_a and S^n_b two copies of S^n then we view S^n_a as a CW-complex with one 0-cell P and one n-cell that we denote by S^n_a. Similarly we view S^n_b as a CW-complex with one 0-cell Q and one n-cell S^n_b.

[1302] Note that Propositions 91.7 and 91.13 tell us that this question is equivalent to the question, for which $n \in \mathbb{N}$ does there exist a map $f\colon S^{2n-1} \to S^n$ such that $\mathrm{Hopf}(f)$ is odd?

(2) Given two finite CW-complexes X and Y we introduced on page 960 the product CW-structure on $X \times Y$. Recall that the n-cells of $X \times Y$ correspond precisely to products $e_x \times f_y$ where e_x is a p-cell of X and f_y is an $(n-p)$-cell of Y.

It follows from (1) and (2) that we can view $S_a^n \times S_b^n$ as a CW-complex with one 0-cell (P,Q), two n-cells $S_a^n \times \{Q\}$ and $\{P\} \times S_b^n$ and one $2n$-cell $S_a^n \times S_b^n$. We consider the maps

$$S^{2n-1} \xrightarrow{\varphi} \overbrace{S_a^n \times \{Q\} \cup \{P\} \times S_b^n}^{=S_a^n \vee S_b^n} \xrightarrow{p} S^n.$$

attaching map of the $2n$-cell to the $(2n-1)$-skeleton of $S_a^n \times S_b^n$ which is precisely $S_a^n \times \{Q\} \cup \{P\} \times S_b^n$

the obvious projection on $S_a^n \times \{Q\}$ and $\{P\} \times S_b^n$

The composition of the maps φ and p defines a map $f\colon S^{2n-1} \to S^n$. It is clear from

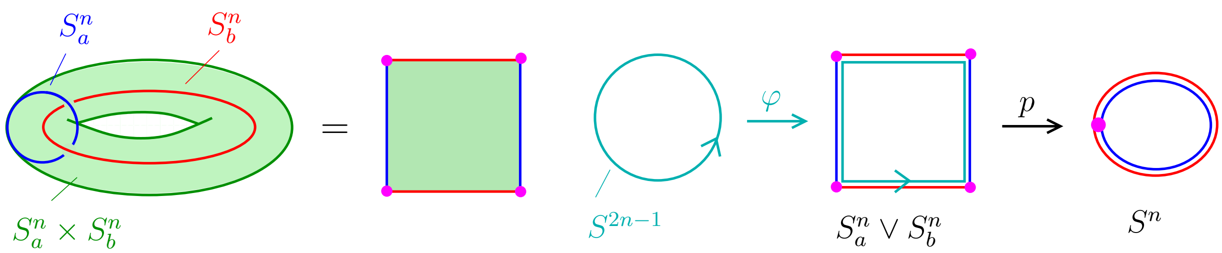


FIGURE 1264

Figure 1264 that this map is not very interesting for $n = 1$. But using fairly elementary arguments one can show that for all even n we have $\mathrm{Hopf}(f) = \pm 2$. We refer to [**Hat02**, p. 428] or [**Vic94**, p. 138] for details. Alternatively we leave it as a brain teaser for the reader to perform this calculation.

Exercises for Chapter 91.

Exercise 91.1. What is the homeomorphism type of the mapping cone of the map $S^1 \to S^1$ given by $z \mapsto z^2$?
Hint. It is a topological space that we are quite familiar with.

Exercise 91.2. Let $f\colon S^p \times S^q \to S^r$ be a map. (As always, f is understood to be continuous.) Show that the map

$$\begin{array}{rcl} S^p \times \mathrm{Cone}(S^q) & \to & \mathrm{Cone}(S^r) \\ (P, [(Q,t)]) & \mapsto & [(f(P,Q), t)] \end{array}$$

is continuous.
Hint. Use Theorem 5.16 and Lemma 5.17 (2).

Exercise 91.3. Let $f\colon S^k \times S^k \to S^k$ be the projection onto the first factor. Show that the corresponding map $\Theta_f\colon S^{2k+1} \to S^{k+1}$, that is defined on page 2225, is homotopic to a constant map.
Hint. The exercise is very easy.

Exercise 91.4. Let $n \in \mathbb{N}$ be even and let $a, b \in \mathbb{Z}$. Show that if at least one of a or b is even, then there exists a map $S^{n-1} \times S^{n-1} \to S^{n-1}$ of bidegree (a, b).

92. Alexander Duality

In this chapter we will study the Alexander Duality Theorem which, given a "reasonable" subset X of S^n relates the homology groups of $S^n \setminus X$ to the cohomology groups of X.

92.1. **The basic Alexander Duality Theorem.** We start out with the basic Alexander Duality Theorem which will be surprisingly easy to prove and which is already very helpful.

Theorem 92.1. (Alexander Duality Theorem) *Let K be a compact non-empty subset of S^n that admits a regular neighborhood. For any $i \in \mathbb{N}_0$ there exist isomorphisms*

$$\text{(a)} \qquad \widetilde{\mathrm{H}}_i(S^n \setminus K;\mathbb{Z}) \;\cong\; \widetilde{\mathrm{H}}^{n-i-1}(K;\mathbb{Z})$$

and

$$\text{(b)} \qquad \widetilde{\mathrm{H}}^i(S^n \setminus K;\mathbb{Z}) \;\cong\; \widetilde{\mathrm{H}}_{n-i-1}(K;\mathbb{Z}).$$

By the Regular Neighborhood Theorem 10.3 this conclusion holds for any closed non-empty submanifold of S^n. In this particular case the isomorphisms are natural.

Proof. Let K be a compact non-empty subset of S^n that admits a regular neighborhood N. We first consider statement (a) and the case that $i > 0$. In this case we have the following isomorphisms:

Poincaré Duality Theorem 88.1 applied to $S^n \setminus \mathring{N}$, which is a compact smooth manifold by Lemma 89.1 (4)

$$\widetilde{\mathrm{H}}_i(S^n \setminus K;\mathbb{Z}) \xleftarrow{\cong} \widetilde{\mathrm{H}}_i(S^n \setminus \mathring{N};\mathbb{Z}) \xleftarrow{\cong} \mathrm{H}^{n-i}(S^n \setminus \mathring{N}, \partial N;\mathbb{Z}) \xleftarrow{\cong} \mathrm{H}^{n-i}(S^n, N;\mathbb{Z})$$

Lemmas 10.1 and 73.16 (6) — Excision Theorem 43.20

$$\xleftarrow{\cong} \widetilde{\mathrm{H}}^{n-i-1}(N;\mathbb{Z}) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-i-1}(K;\mathbb{Z}).$$

by the long exact sequence in reduced cohomology of the pair (S^n, N), see Lemma 73.17 here we use that $i > 0$ — by Lemma 73.16 (6) since K is a deformation retract of N

The case $i = 0$ is proved almost the same way. Thus we leave it to the reader to verify this case. This concludes the proof of (a). The proof of the existence of the isomorphism (b) is basically identical.

Finally suppose that K is a closed non-empty submanifold of S^n. It follows from the General Tubular Neighborhood Theorem 10.5 and Proposition 10.11 that we can take a tubular neighborhood as a regular neighborhood. By Proposition 10.7 tubular neighborhoods are unique in an appropriate sense which then allows us to get a natural isomorphism. ■

Remark.

(1) The argument in the proof of the Alexander Duality Theorem 92.1 implies in particular that given any compact, non-empty n-dimensional submanifold W of S^n there exists an isomorphism

$$\widetilde{\mathrm{H}}_i(S^n \setminus \mathring{W};\mathbb{Z}) \;\cong\; \widetilde{\mathrm{H}}^{n-i-1}(W;\mathbb{Z}).$$

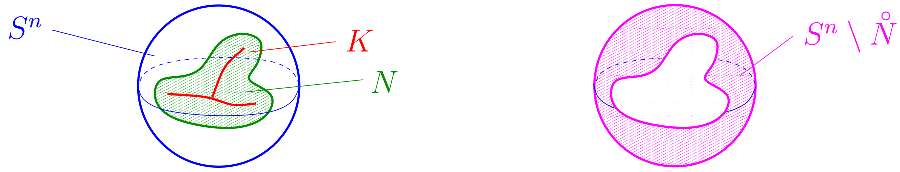


FIGURE 1265. Illustration for the proof of the Alexander Duality Theorem 92.1.

(2) In Alexander Duality Theorem 92.1 we explicitly do not write "natural" isomorphism. The isomorphism that we construct in the proof depends on the choice of a regular neighborhood. It is not clear to me whether two regular neighborhoods are necessarily in any way "equivalent".

In practice we often like to consider objects that lie in $\mathbb{R}^n$ instead of S^n. The following variation on the Alexander Duality Theorem 92.1 deals with this situation.

Corollary 92.2. *Let K be a compact non-empty subset of $\mathbb{R}^n$. If K admits a regular neighborhood, then for any $i \in \mathbb{N}_0$ we have an isomorphism*

$$\widetilde{\mathrm{H}}_i(\mathbb{R}^n \setminus K) \;\cong\; \mathrm{H}^{n-i-1}(K).$$

By the Regular Neighborhood Theorem 10.3 this conclusion holds for any compact submanifold of $\mathbb{R}^n$.

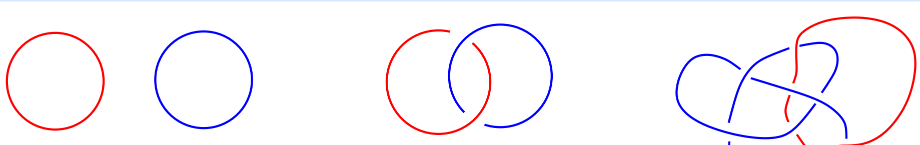
PROOF. Let K be a compact non-empty subset of $\mathbb{R}^n$ that admits a regular neighborhood N in $\mathbb{R}^n$. Recall that N is by definition compact. It follows easily that N is also a regular neighborhood for K in S^n. Now let $i \in \mathbb{N}_0$. Then

$$\widetilde{\mathrm{H}}^{n-i-1}(K) \underset{\substack{\uparrow \\ \text{Alexander Duality Theorem 92.1} \\ \text{which we can apply since } K \\ \text{admits a regular neighborhood in } S^n}}{\cong} \widetilde{\mathrm{H}}_i(S^n \setminus K) \underset{\substack{\uparrow \\ \text{see Exercise 46.12}}}{\cong} \begin{cases} \widetilde{\mathrm{H}}_i(\mathbb{R}^n \setminus K), & \text{if } i \neq n-1, \\ \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus K) \oplus \mathbb{Z}, & \text{if } i = n-1. \end{cases}$$

The corollary now follows from Lemmas 73.16 and 19.8 (1). ■

In the remainder of this section we discuss a few applications of the Alexander Duality Theorem 92.1. The following definition generalizes the notion of a knot.

Definition. An *m-component link* is a 1-dimensional submanifold of S^3 that is diffeomorphic to the disjoint union of m circles.[1303]



2-component trivial link Hopf link

FIGURE 1266. Examples of 2-component links.

[1303] Sometimes, to distinguish this notion from the higher-dimensional analogue that we define on page **??** we refer to such a link in S^3 as a *classical link*.

The following lemma computes the homology groups of a link complement. It can be viewed as a generalization of Lemma 68.16 (4).

Lemma 92.3. *For any m-component link L in S^3 we have*

$$\mathrm{H}_i(S^3 \setminus L;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}^{m-1}, & \text{if } i=2,\\ \mathbb{Z}^{m}, & \text{if } i=1,\\ \mathbb{Z}, & \text{if } i=0,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Let L be an m-component link. We calculate that

$$\widetilde{\mathrm{H}}_i(S^3 \setminus L;\mathbb{Z}) \;\underset{\substack{\uparrow\\ \text{Alexander Duality Theorem 92.1}}}{\cong}\; \widetilde{\mathrm{H}}^{2-i}(L;\mathbb{Z}) \;\cong\; \widetilde{\mathrm{H}}^{2-i}(\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{m\text{-copies}};\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}^{m-1}, & \text{if } i=2,\\ \mathbb{Z}^{m}, & \text{if } i=1,\\ 0, & \text{otherwise.}\end{cases}$$

The lemma follows from this calculation together with Lemma 73.16 (3). ■

For any 2-component link $L = K_1 \cup K_2$, for example for any of the links shown in Figure 1266, we see that

$$\mathrm{H}_1(S^3 \setminus L;\mathbb{Z}) \;\cong\; \mathbb{Z}^2 \quad\text{ and }\quad \mathrm{H}_2(S^3 \setminus L;\mathbb{Z}) \;\cong\; \mathbb{Z}.$$

The problem with this calculation is that we do not obtain any understanding of what the homology classes “look like”. It seems reasonable to guess, along the lines of Lemma 68.16, that the first homology group is generated by the two meridians. But (a), we do not know how to prove it, and (b), we have no idea what a generator of the second homology group looks like. So the following question arises:

Question 92.4. *Given a link L, what are explicit bases for the homology groups of $S^3 \setminus L$?*

We will return to this question in Lemma 99.18. We continue with another example.

Examples.

(1) We consider the connected topological graph G with two vertices and three edges that is illustrated in Figure 1267 on the left. On the right of Figure 1267 we show a spatial graph G' in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is homeomorphic to G. We have the following isomorphisms:

$$\widetilde{\mathrm{H}}_i(S^3 \setminus G';\mathbb{Z}) \;\underset{\substack{\uparrow\\ \text{by Proposition 10.13 we can apply}\\ \text{the Alexander Duality Theorem 92.1}}}{\cong}\; \widetilde{\mathrm{H}}^{2-i}(G';\mathbb{Z}) \;\cong\; \widetilde{\mathrm{H}}^{2-i}(G;\mathbb{Z}) \;\underset{\substack{\uparrow\\ \text{similar to Lemma 55.12}}}{\cong}\; \begin{cases} \mathbb{Z}^{2}, & \text{if } i=1,\\ 0, & \text{otherwise.}\end{cases}$$

In this case it is also rather unclear what a basis of $\mathrm{H}_1(S^3 \setminus G;\mathbb{Z})$ should look like. Later, on page 2405, we will find an explicit basis for $\mathrm{H}_1(S^3 \setminus G;\mathbb{Z})$.

(2) Let G be the spatial graph that we consider in Exercise 10.6. The same argument as in (1) shows that $\mathrm{H}_1(S^3 \setminus G;\mathbb{Z}) \cong \mathbb{Z}^2$. Together with the Hurewicz Theorem 52.5 and Exercise 46.12 we obtain that the answer to Exercise 10.6 (b) is $\mathbb{Z}^2$.

graph G the spatial graph G' in $S^3 = \mathbb{R}^3 \cup \{\infty\}$

FIGURE 1267

We conclude this section with the following variation on the above Alexander Duality Theorem 92.1. It will come in handy at times. But arguably it is best to study it only when it is actually needed.

Theorem 92.5. (Alexander Duality Theorem) $(*)$ *Let $n \in \mathbb{N}_0$ and let $j \in \{0, \dots, n\}$. Given any compact oriented n-dimensional smooth manifold X which has the property that $\mathrm{H}_j(X;\mathbb{Z}) = \mathrm{H}_{j+1}(X;\mathbb{Z}) = 0$ and given any compact proper submanifold G of X there exist natural isomorphisms*

$$\alpha_G\colon \mathrm{H}_j(X \setminus G;\mathbb{Z}) \xrightarrow{\cong} \mathrm{H}^{n-j-1}(G, \partial G;\mathbb{Z}) \quad \textit{and} \quad \alpha_G\colon \mathrm{H}^j(X \setminus G;\mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_{n-j-1}(G, \partial G;\mathbb{Z})$$

such that the following conditions are satisfied:

(1) *If $X = S^n$, then the isomorphisms are precisely the isomorphisms from the Alexander Duality Theorem 92.1.*
(2) *The isomorphisms decompose in the obvious way if X is disconnected.*
(3) *If W is a compact oriented n-dimensional smooth manifold with*
$$\mathrm{H}_j(W;\mathbb{Z}) = \mathrm{H}_{j+1}(W;\mathbb{Z}) = \mathrm{H}_j(\partial W;\mathbb{Z}) = \mathrm{H}_{j+1}(\partial W;\mathbb{Z}) = 0,$$
and if F is a compact proper submanifold of W, then we have commutative diagrams
$$\begin{tikzcd} \mathrm{H}_j(W\setminus F;\mathbb{Z}) \arrow[r, "\cong", "\alpha_F"'] & \mathrm{H}^{n-j-1}(F,\partial F;\mathbb{Z}) \\ \mathrm{H}_j(\partial W\setminus \partial F;\mathbb{Z}) \arrow[u, "i_*"] \arrow[r, "\cong", "\alpha_{\partial F}"'] & \mathrm{H}^{n-j-2}(\partial F;\mathbb{Z}) \arrow[u, "\delta"'] \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathrm{H}^j(W\setminus F;\mathbb{Z}) \arrow[r, "\cong", "\alpha_F"'] \arrow[d, "i^*"'] & \mathrm{H}_{n-j-1}(F,\partial F;\mathbb{Z}) \arrow[d, "\partial"] \\ \mathrm{H}^j(\partial W\setminus \partial F;\mathbb{Z}) \arrow[r, "\cong", "\alpha_{\partial F}"'] & \mathrm{H}_{n-j-2}(\partial F;\mathbb{Z}) \end{tikzcd}$$
where we denote by $i\colon \partial W\setminus\partial F \to W\setminus F$ the inclusion map and where the right-hand vertical maps are the connecting homomorphisms of the long exact sequences in (co-)homology of the pair $(F, \partial F)$.

Example. The above Alexander Duality Theorem 92.5 will be applied later on to the smooth manifolds $W = [0,1] \times S^n$ and to $W = \overline{B}^n$. For example, suppose that F is a connected properly embedded surface in $\overline{B}^4$ such that $\partial F = K$ is a knot. We can apply the Alexander Duality Theorem 92.5 for $j = 1$. Together with Proposition 88.26 we obtain the following commutative diagram:

$$\begin{tikzcd} \mathrm{H}_1(\overline{B}^4 \setminus F;\mathbb{Z}) \arrow[r, "\alpha_F", "\cong"'] & \mathrm{H}^2(F,K;\mathbb{Z}) \arrow[r, "-\cap[F]", "\cong"'] & \mathrm{H}_0(F;\mathbb{Z}) \arrow[d, "i_*"] \\ \mathrm{H}_1(S^3 \setminus K;\mathbb{Z}) \arrow[u, "i_*"] \arrow[r, "\alpha_K", "\cong"'] & \mathrm{H}^1(K;\mathbb{Z}) \arrow[u, "\delta"'] \arrow[r, "-\cap[K]", "\cong"'] & \mathrm{H}_0(K;\mathbb{Z}). \end{tikzcd}$$

The right hand vertical map is of course an isomorphism and both groups are isomorphic to $\mathbb{Z}$. Now we see that both groups on the left are isomorphic to $\mathbb{Z}$ and that the inclusion induced map is an isomorphism.

Sketch of proof. Let $n \in \mathbb{N}_0$ and let $j \in \{0, \ldots, n\}$. Let X be a compact orientable n-dimensional smooth manifold with $\mathrm{H}_j(X;\mathbb{Z}) = \mathrm{H}_{j+1}(X;\mathbb{Z}) = 0$ and let G be a compact proper submanifold of X. We pick a tubular neighborhood N for G as provided by the General Tubular Neighborhood Theorem 10.5. Recall that N is a submanifold with corner. We define $\mathring{N}$ as in Lemma 89.1. We write $M := N \cap \partial X$. Note that M is a tubular neighborhood for $\partial F \subset \partial X$. We refer to Figure 1268 for an illustration. We consider the

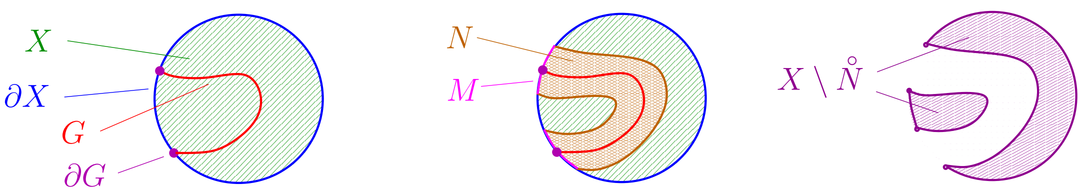


Figure 1268

following diagram

$$\begin{tikzcd}
\mathrm{H}_j(X\setminus\mathring{N}) \arrow[d, "\cong"'] & \mathrm{H}^{n-j}(X\setminus\mathring{N}, \partial(X\setminus\mathring{N})) \arrow[l, "\cong"', "\mathrm{PD}"] & \mathrm{H}^{n-j}(X, N\cup\partial X) \arrow[l, "\cong"', "\mathrm{exc}"] & \mathrm{H}^{n-j-1}(N\cup\partial X, \partial X) \arrow[l, "\cong"', "\delta"] \\
 & & & \mathrm{H}^{n-j-1}(N, M) \arrow[u, "\mathrm{exc}", "\cong"'] \arrow[d, "\cong"] \\
\mathrm{H}_j(X\setminus G) \arrow[rrr, dashed, "=:\alpha_G"] & & & \mathrm{H}^{n-j-1}(G, \partial G).
\end{tikzcd}$$

We make the following comments:

(a) All maps where no map is specified are induced by inclusions. It follows from Proposition 10.11, together with Corollary 42.8 and Lemma 73.13 (4) that the vertical inclusion maps to the left and bottom right are isomorphisms.

(b) The two maps that are decorated with "exc" are inclusion induced maps that are isomorphisms by "excision". We leave it to the reader to verify that these inclusion maps are indeed isomorphisms. As an example, if $\partial X = \varnothing$, then it follows from the Excision Theorem 43.20 that the top "exc"-map is an isomorphism.

(c) The map on the top right indicated by "δ" is the connecting homomorphism of the long exact sequence in cohomology of the triple $(X, N\cup\partial X, \partial X)$. By our hypothesis we have $\mathrm{H}_j(X;\mathbb{Z}) = \mathrm{H}_{j+1}(X;\mathbb{Z}) = 0$. By the Poincaré Duality Theorem 88.1 this implies that $\mathrm{H}^{n-j}(X,\partial X;\mathbb{Z}) = \mathrm{H}^{n-j-1}(X,\partial X;\mathbb{Z}) = 0$, which in turn implies that the connecting homomorphism $\delta\colon\ \mathrm{H}^{n-j-1}(N\cup\partial X, \partial X;\mathbb{Z}) \to \mathrm{H}^{n-j}(X, N\cup\partial X;\mathbb{Z})$ is an isomorphism.

We have now seen that all maps in the diagram are isomorphisms. Thus we can define α_G to be the map on the bottom. A priori the definition of α_G depends on the choice of the tubular neighborhood N. But by Proposition 10.7 tubular neighborhoods are unique in an appropriate sense. This implies that isomorphisms do not depend on the choice of the tubular neighborhood.

First note that Statement (1) is basically clear once one compares the definitions. Next note that Statement (2) follows from all the usual results for the behavior of invariants under the disjoint union. Finally we turn to the proof of Statement (3). Let W be a compact oriented n-dimensional smooth manifold with

$$(*) \qquad \mathrm{H}_j(W;\mathbb{Z}) = \mathrm{H}_{j+1}(W;\mathbb{Z}) = \mathrm{H}_j(\partial W;\mathbb{Z}) = \mathrm{H}_{j+1}(\partial W;\mathbb{Z}) = 0.$$

Furthermore let F be a compact proper submanifold of W. We pick a tubular neighborhood N for F. We define $\mathring{N}$ as above. We write $M := N \cap \partial W$. Note that M is a tubular neighborhood for ∂F. We consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_j(W\setminus\mathring{N}) \arrow[dd, "\cong"'] & \mathrm{H}^{n-j}(W\setminus\mathring{N}, \partial(W\setminus\mathring{N})) \arrow[l, "\cong", "\mathrm{PD}"'] & \mathrm{H}^{n-j}(W, N\cup\partial W) \arrow[l, "\cong", "\mathrm{exc}"'] & \mathrm{H}^{n-j-1}(N\cup\partial W, \partial W) \arrow[l, "\cong", "\delta"'] \\
 & & & \mathrm{H}^{n-j-1}(N, M) \arrow[u, "\mathrm{exc}", "\cong"'] \arrow[d, "\cong"] \\
\mathrm{H}_j(W\setminus F) \arrow[rrr, "\alpha_F"] & & & \mathrm{H}^{n-j-1}(F, \partial F) \\
\mathrm{H}_j(\partial W\setminus\partial F) \arrow[u, "i_*"] \arrow[rrr, "\alpha_{\partial F}"] & & & \mathrm{H}^{n-j-2}(\partial F) \arrow[u, "\delta"] \\
\mathrm{H}_j(\partial W\setminus\mathring{M}) \arrow[u, "\cong"] & \mathrm{H}^{n-j}(\partial W\setminus\mathring{M}, \partial M) \arrow[l, "\cong", "\mathrm{PD}"'] & \mathrm{H}^{n-j-1}(\partial W, M) \arrow[l, "\cong", "\mathrm{exc}"'] & \mathrm{H}^{n-j-2}(M). \arrow[l, "\cong", "\delta"'] \arrow[u, "\cong"']
\end{tikzcd}$$

We make the following observations:

(1) By our hypothesis $(*)$ and the discussion above the horizontal maps α_G and $\alpha_{\partial G}$ are defined.
(2) One can easily see that the bottom "$\cup$" of the diagram is indeed the map $\alpha_{\partial F}$.
(3) If one ignores the blue arrows, then the brave reader can verify that the diagram does indeed commute.

This concludes the proof of Statement (3) for the diagram to the left. Basically the same argument also takes care of the proof of the diagram to the right. ■

92.2. **Immersions and smooth embeddings of smooth manifolds into $\mathbb{R}^n$.** In Question 3.46 we had asked whether the projective space $\mathbb{RP}^2$ or the Klein bottle can be viewed as subsets of $\mathbb{R}^3$. We can reinterpret this question in three different ways:

Question 92.6.

(1) *Does $\mathbb{RP}^2$ (respectively the Klein bottle) admit an immersion into $\mathbb{R}^3$?*
(2) *Does $\mathbb{RP}^2$ (respectively the Klein bottle) admit a smooth embedding into $\mathbb{R}^3$?*
(3) *Does $\mathbb{RP}^2$ (respectively the Klein bottle) admit an embedding into $\mathbb{R}^3$?*

We start out with Question 92.6 (1).

(a) We saw in Figure 87 that there exists an immersion of the Klein bottle K into $\mathbb{R}^3$. In Figure 1269 we show the images of two other immersions of the Klein bottle into $\mathbb{R}^3$. This gives a positive answer to Question 92.6 (1) for the Klein bottle.

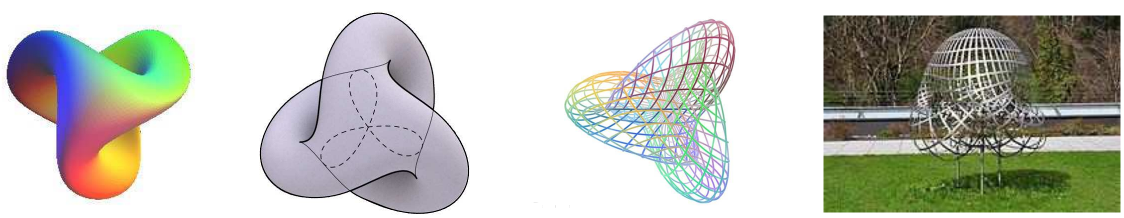
FIGURE 1269

(b) The case of the real projective plane is much more interesting. In 1901 David Hilbert, arguably the greatest German mathematician of his generation, gave his PhD student Werner Boy [**Boy1903**][1304] the task to prove that there is no immersion of $\mathbb{R}\mathrm{P}^2$ into $\mathbb{R}^3$. To everyone's surprise Boy did the opposite, he showed that it was possible to immerse $\mathbb{R}\mathrm{P}^2$ into $\mathbb{R}^3$. In Figure 1270 we give several illustrations of the image of the immersion. A detailed description of the immersion is given in [**Kir07, Ap87**]. But perhaps it is more helpful to just watch the following video:

`https://www.youtube.com/watch?v=uiq-EcQz_uU`

A fun description of Boy's surface, and of many other concepts from topology, can also be found here:

`http://www.savoir-sans-frontieres.com/JPP/telechargeables/English/Topo_the_world_eng.pdf`



Boy's surface at Oberwolfach

FIGURE 1270

Lemma 6.48 says that the projective plane $\mathbb{R}\mathrm{P}^2$ and the Klein bottle are non-orientable. Therefore the following proposition gives in particular a negative answer to Question 92.6 (2).

Proposition 92.7. *Let N be a closed n-dimensional connected smooth manifold. If N is non-orientable, then there is no smooth embedding of N into $\mathbb{R}^{n+1}$.*

PROOF. Let N be a closed n-dimensional connected non-orientable smooth manifold. Suppose that N can be embedded into $\mathbb{R}^{n+1}$. We can thus view N as a submanifold of $\mathbb{R}^{n+1}$.

[1304] Werner Boy (1879-1914) was a German mathematician who died in the first weeks of World War I.

We obtain that

$$\underset{\text{Proposition 75.15}}{\mathbb{Z}_2 \cong} \mathrm{H}^n(N;\mathbb{Z}) \underset{\substack{\text{Alexander Duality}\\ \text{Theorem 92.2}}}{\cong} \widetilde{\mathrm{H}}_0(\mathbb{R}^{n+1}\setminus N;\mathbb{Z}) \underset{\text{Proposition 41.5}}{\cong} \mathbb{Z}^{\text{number of path-components of } \mathbb{R}^{n+1}\setminus N \text{ minus } 1}.$$

But this is absurd. We have thus obtained the desired contradiction. ■

We recall the following related question.

Question 11.9. *Given $n \in \mathbb{N}$, what is the minimal $k \in \mathbb{N}$ such that $\mathbb{RP}^n$ admits a (topological) smooth embedding into $\mathbb{R}^k$?*

Recall that it follows from Theorem 87.1 and the calculation of the homology groups of $\mathbb{RP}^n$ in Proposition 48.10 that $\mathbb{RP}^n$ is orientable if and only if n is odd. In particular it follows from Proposition 92.7 that for n even it is not possible to embed $\mathbb{RP}^n$ into S^{n+1}. But this raises the question, whether for n odd one can embed $\mathbb{RP}^n$ into S^{n+1}. It turns out that this is also not the case:

Proposition 92.8. *If $n \in \mathbb{N}_{\geq 2}$, then there is no smooth embedding from $\mathbb{RP}^n$ into S^{n+1}.*

In the proof of Proposition 92.8 we will need the following lemma which is interesting in its own right.

Lemma 92.9. *If K is a closed orientable connected $(n-1)$-dimensional submanifold of the smooth manifold S^n, then there exist two compact n-dimensional submanifolds A and B of S^n such that $A \cap B = \partial A = \partial B = K$.*

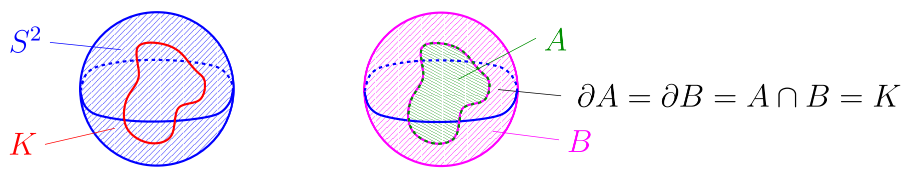


FIGURE 1271

PROOF (∗). First we note that by Proposition 10.9 the submanifold K admits a tubular neighborhood $[-1,1] \times K$. Our hypothesis that K is connected implies that $[-1,0] \times K$ and $[0,1] \times K$ are also connected.

We first calculate that

$$\mathrm{H}_0(S^n \setminus K;\mathbb{Z}) \underset{\text{Lemma 43.1 (4a)}}{\cong} \widetilde{\mathrm{H}}_0(S^n \setminus K;\mathbb{Z}) \oplus \mathbb{Z} \underset{\text{Theorem 92.1}}{\cong} \mathrm{H}^{n-1}(K;\mathbb{Z}) \oplus \mathbb{Z} \underset{\text{Theorem 87.1}}{\cong} \mathbb{Z} \oplus \mathbb{Z}.$$

It follows from Proposition 41.5 that $S^n \setminus K$ has precisely two path-components A' and B'. Since $[-1,0] \times K$ is connected and since it is contained in $S^n \setminus K = A' \cup B'$ it follows that $[-1,0] \times K$ is either contained in A' or it is contained in B'. Without loss of generality we can assume that $[-1,0] \times K$ is contained in A'.

Claim. The subset $[0,1] \times K$ is contained in B'.

We consider the set $A'' := A' \cup [-1,1] \times K$. Note that $S^n = A'' \cup B'$. Since both A'' and B' are open and since S^n is connected it follows that $A'' \cap B' \neq \varnothing$. But this implies that

$[0,1]\times K\cap B'\neq\emptyset$. But since $[0,1]\times K$ is connected we see that $[0,1]\times K$ is contained in B'. ⊞

We set $A:=A'\cup K=A'\cup[-1,0]\times K$ and $B:=B'\cup K=B'\cup[0,1]\times K$. It is easy to show that A and B are compact submanifolds of S^n such that $A\cap B=\partial A=\partial B=K$. ■

Now we can provide the proof of Proposition 92.8.

Proof of Proposition 92.8. As we had discussed just before the proposition, it remains to deal with the case that n is odd. So suppose that $n\geq 3$ is odd and that there exists a smooth embedding of $\mathbb{RP}^n$ into S^{n+1}. We identify $\mathbb{RP}^n$ with its image under the smooth embedding, i.e. we view $\mathbb{RP}^n$ as a submanifold of S^{n+1}. Note it follows from $n\geq 3$ and the discussion on page 1841 that $\mathrm{H}^1(S^n;\mathbb{F}_2)=\mathrm{H}^2(S^n;\mathbb{F}_2)=0$.

Since n is odd we know that $\mathbb{RP}^n$ is orientable. Thus we can apply Lemma 92.9 which tells us that there exist two compact $(n+1)$-dimensional submanifolds A and B of S^{n+1} with $A\cap B=\partial A=\partial B=K$. We denote by $i\colon\mathbb{RP}^n\to A$ and by $j\colon\mathbb{RP}^n\to B$ the inclusion maps. As usual we denote by $x\in\mathrm{H}^1(\mathbb{RP}^n;\mathbb{F}_2)\cong\mathbb{F}_2$ the unique generator.

By Proposition 74.12 (5) we can apply the Mayer–Vietoris Theorem 74.15 for Cohomology Groups to $S^{n+1}=A\cup B$ and we obtain the long exact sequence

$$\cdots\to\underbrace{\mathrm{H}^1(S^{n+1};\mathbb{F}_2)}_{=0\text{ since }n\geq 3}\to\mathrm{H}^1(A;\mathbb{F}_2)\oplus\mathrm{H}^1(B;\mathbb{F}_2)\xrightarrow{i^*+j^*}\underbrace{\mathrm{H}^1(\mathbb{RP}^n;\mathbb{F}_2)}_{=\mathbb{F}_2\cdot x}\to\underbrace{\mathrm{H}^2(S^{n+1};\mathbb{F}_2)}_{=0\text{ since }n\geq 3}\to\dots$$

Thus we see that the map $i^*+j^*\colon\ \mathrm{H}^1(A;\mathbb{F}_2)\oplus\mathrm{H}^1(B;\mathbb{F}_2)\to\mathrm{H}^1(\mathbb{RP}^n;\mathbb{F}_2)\cong\mathbb{F}_2\cdot x$ is an isomorphism. Without loss of generality we can assume that i^* is non-trivial which means that there exists an $a\in\mathrm{H}^1(A;\mathbb{F}_2)$ with $i^*(a)=x$. We have

$$i^*(a^n)\ \underset{\substack{\uparrow\\\text{Lemma 81.10}}}{=}\ (i^*(a))^n\ =\ x^n\ \underset{\substack{\uparrow\\\text{Proposition 90.16}}}{\neq}\ 0\ \in\mathrm{H}^n(\mathbb{RP}^n;\mathbb{F}_2).$$

In particular the map $i^*\colon\ \mathrm{H}^n(A;\mathbb{F}_2)\to\mathrm{H}^n(\mathbb{RP}^n;\mathbb{F}_2)$ is non-zero. Now we consider the other end of the above Mayer-Vietoris sequence and we obtain the exact sequence

$$\begin{array}{ccccccc}\mathrm{H}^n(A;\mathbb{F}_2)\oplus\mathrm{H}^n(B;|!\mathbb{F}_2)&\xrightarrow{i^*+j^*}&\overbrace{\mathrm{H}^n(\mathbb{RP}^n;\mathbb{F}_2)}^{=\mathbb{F}_2\cdot x^n}&\to&\overbrace{\mathrm{H}^{n+1}(S^{n+1};\mathbb{F}_2)}^{\cong\mathbb{F}_2}&\to&\underbrace{\mathrm{H}^{n+1}(A;\mathbb{F}_2)\oplus\mathrm{H}^{n+1}(B;\mathbb{F}_2)}_{\substack{\text{both are zero by Proposition 75.15,}\\\text{since }A\text{ and }B\text{ are compact, connected}\\(n+1)\text{-dimensional smooth manifolds}\\\text{with non-empty boundary}}}.\\a^n\oplus 0&\mapsto&x^n&&&&\end{array}$$

Thus we see that the map on the left is an epimorphism and that the group on the right is trivial. But then the fact that $\mathrm{H}^{n+1}(S^{n+1};\mathbb{F}_2)\neq 0$ shows that the sequence cannot possibly be exact. We have thus obtained the desired contradiction. ■

Remark. The above proposition does not give a complete answer to Question 11.9, we have only shown that $\mathbb{RP}^n$ cannot embed into S^{n+1}. More lower bounds on the minimal dimension of some $\mathbb{R}^k$ into which we can embed $\mathbb{RP}^n$ have been obtained by Michael Atiyah [**At61**].

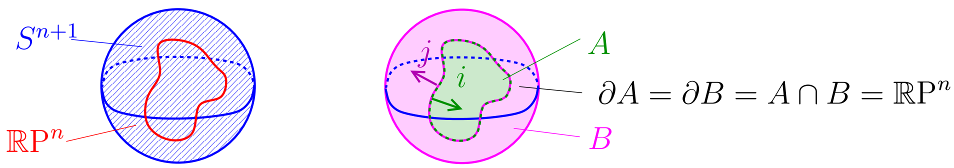


FIGURE 1272. Illustration for the Proof of Proposition 92.8.

92.3. **The generalized Alexander Duality Theorem.** In a bout of optimism one might hope that the statement of Alexander Duality Theorem 92.1 holds for any (compact) subset of S^n. But this is not the case. More precisely, once again we consider the topologist's sine curve

$$X \;=\; \big\{(0,y)\,\big|\,y\in[-1,1]\big\}\;\cup\;\big\{(x,\sin(\tfrac{1}{x}))\,\big|\,x\in(0,\pi]\big\}\;\subset\;\mathbb{R}^2$$

that we had initially introduced on page 131. It is fairly straightforward to see that X is a closed subset of $\mathbb{R}^2$. Evidently X is bounded. Thus the Heine-Borel Theorem 2.20 implies that X is in fact a compact subset of $\mathbb{R}^2$ and thus of $S^2 = \mathbb{R}^2 \cup \{\infty\}$. Back on page 131 we saw that X is not path-connected, which by Proposition 73.11 and Lemma 73.14 implies that $\widetilde{\mathrm{H}}^0(X;\mathbb{Z}) \neq 0$. On the other hand, with some effort one can show that there exists a deformation retraction from $S^2 \setminus X = (\mathbb{R}^2 \setminus X) \cup \{\infty\}$ to ∞, which implies that $\mathrm{H}_1(S^2 \setminus X;\mathbb{Z}) = 0$. But this shows that $\widetilde{\mathrm{H}}_1(S^2 \setminus X;\mathbb{Z}) \not\cong \widetilde{\mathrm{H}}^0(X;\mathbb{Z})$.

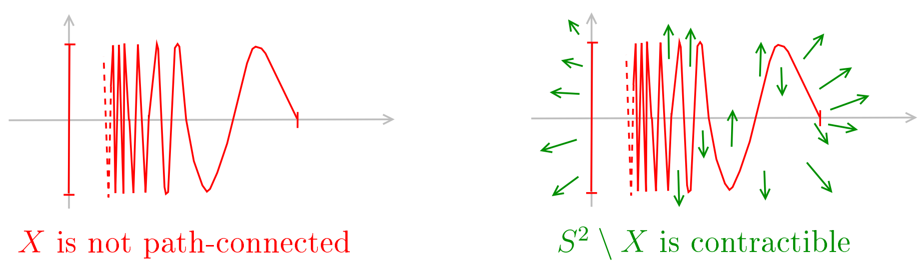


FIGURE 1273

Perhaps rather surprisingly one only needs a fairly modest hypothesis to ensure that Alexander Duality holds after all. More precisely, we have the following theorem which, by Lemma 85.2, is a significant generalization of the earlier Alexander Duality Theorem 92.1 (a).

Theorem 92.10. (Generalized Alexander Duality Theorem) *Let $n \geq 2$ and let $X \subsetneq S^n$ be a weakly locally contractible, non-empty compact subset of S^n. Given any $j \in \mathbb{N}_0$ we have a natural[1305] isomorphism*

$$\widetilde{\mathrm{H}}_j(S^n \setminus X;\mathbb{Z}) \;\xrightarrow{\cong}\; \widetilde{\mathrm{H}}^{n-j-1}(X;\mathbb{Z}).$$

Example. In Figure 1274 we show a subset X of $\mathbb{R}^3$ that is homeomorphic to the compact interval $[0,1]$. (We had already encountered this subset in Figure 805.) We leave it as a somewhat challenging exercise to convince oneself that X does not admit a regular

[1305] What does "natural" mean in this context? What are the categories? What are the functors?

neighborhood. In particular this is a situation where the Generalized Alexander Duality Theorem 92.10 applies, even though our initial Alexander Duality Theorem 92.1 fails.

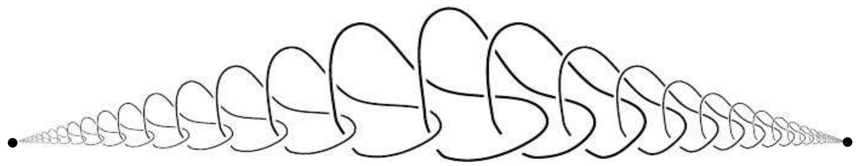


FIGURE 1274

Remark. We leave it to the reader to determine whether the obvious generalization of Alexander Duality Theorem 92.1 (b) holds.

For most applications of Alexander Duality we will actually need the following straightforward consequence of the above theorem.

Theorem 92.11. (Alexander Duality Theorem for Topological Submanifolds) *Let $n \geq 2$ and let $k < n$. Furthermore let X be a compact, non-empty k-dimensional topological manifold. If $f\colon X \to S^n$ is an injective map, then for any $j \in \mathbb{N}_0$ we have a natural isomorphism*

$$\widetilde{\mathrm{H}}_j(S^n \setminus f(X);\mathbb{Z}) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X;\mathbb{Z}).$$

PROOF OF THE ALEXANDER DUALITY THEOREM 92.11. Let $f\colon X \to S^n$ be an injective map from a compact, non-empty k-dimensional topological manifold to S^n where $k < n$. It is a consequence of Proposition 2.43 (3) that $f\colon X \to f(X)$ is a homeomorphism. From $k < n$ and Proposition 44.4 it follows that $f(X) \neq S^n$. We have the following isomorphisms:

$$\widetilde{\mathrm{H}}_j(S^n \setminus f(X);\mathbb{Z}) \underset{\substack{\uparrow \\ \text{we can apply the Generalized Alexander Duality Theorem 92.10} \\ \text{since } f(X) \text{ is weakly locally contractible and since } f(X)\neq S^n}}{\xrightarrow{\cong}} \widetilde{\mathrm{H}}^{n-j-1}(f(X);\mathbb{Z}) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X;\mathbb{Z}).$$

■

Example. Let $n \geq 2$ and let $k < n$. Furthermore let $f\colon S^k \to S^n$ be an injective map. We obtain that

$$\widetilde{\mathrm{H}}_j\big(S^n \setminus f(S^k);\mathbb{Z}\big) \underset{\substack{\uparrow \\ \text{Theorem 92.11}}}{\cong} \widetilde{\mathrm{H}}^{n-j-1}(S^k;\mathbb{Z}) \underset{\substack{\uparrow \\ \text{see page 1844 and Lemma 73.16}}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } j = n-k-1, \\ 0, & \text{otherwise.} \end{cases}$$

This calculation gives a new proof for Proposition 50.1 (2). Basically the same approach also gives a new proof of Proposition 50.1 (1). Finally note that this calculation gives an answer to Exercise 85.3 (b).

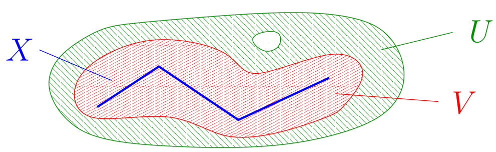
Remark. The Generalized Alexander Duality Theorem 92.10 can be generalized even further. More precisely, an analogous statement also holds for any compact subset K of S^n, e.g. the topologist's sine curve. But hereby one needs to replace the usual singular cohomology on the right-hand side by the *Čech cohomology* of K. We refer to [**Hat02**, p. 256] or [**Pra07**, p. 271] for more details.

Proof of the Generalized Alexander Duality Theorem 92.10. Let $n \geq 2$. Furthermore let $X \neq S^n$ be a weakly locally contractible, non-empty compact subset of the sphere $S^n = \mathbb{R}^n \cup \{\infty\}$. Without loss of generality we can assume that $X \subset \mathbb{R}^n \subset S^n$.[1306] We recall the following definition:

(1) Given a topological space Y we denote by $\mathcal{K}(Y)$ the set of all compact subsets of Y. Given $K, L \in \mathcal{K}(Y)$ we write $K \leq L$ if $K \subset L$. The set $\mathcal{K}(X)$ together with the relation "$\leq$" is a directed set since the union of two compact subsets is again compact.

We also introduce a new definition and notation:

(2) We denote by $\mathcal{N}(X)$ the set of all open neighborhoods of X. Given $U, V \in \mathcal{N}(X)$ we write $U \leq V$ if $V \subset U$. (Note the reversal of V and U.) The set $\mathcal{N}(X)$ together with the relation "$\leq$" is a directed set since the intersection of two open neighborhoods of X is again an open neighborhood of X.[1307]
(3) Given any subsets $A \subset B$ of S^n we denote by $i_{A,B}\colon A \to B$ the inclusion map.
(4) It follows from our definition of "$\leq$" on $\mathcal{N}(X)$ that for any $m \in \mathbb{N}_0$ the cohomology groups $\{\widetilde{\mathrm{H}}^m(U)\}_{U \in \mathcal{N}(X)}$ together with the inclusion induced maps form a *direct* system of abelian groups.



we have $U \leq V$ and $i^*_{V,U}\colon \widetilde{\mathrm{H}}^m(U) \to \widetilde{\mathrm{H}}^m(V)$

Figure 1275

We continue with the key technical part of the theorem.

Claim. For any $m \in \mathbb{N}_0$ the map

$$\Phi\colon \varinjlim_{U \in \mathcal{N}(X)} \widetilde{\mathrm{H}}^m(U) \;\to\; \widetilde{\mathrm{H}}^m(X),$$

that is induced by the restriction maps $i^*_{X,U}\colon \widetilde{\mathrm{H}}^m(U) \to \widetilde{\mathrm{H}}^m(X)$, is an isomorphism.

By our hypothesis we know that X is compact and weakly locally contractible. Thus it follows from Borsuk's Theorem 85.3 that there exists an open neighborhood $W \subset \mathbb{R}^n$ of $X \subset \mathbb{R}^n$ for which there exists a retraction $r\colon W \to X$. (We refer to Figure 1276 for an

[1306]Indeed, by our hypothesis that $X \neq S^n$ there exists a point $P \in S^n$ that does not lie in X. By Lemma 3.32 (3) we know that the group $\mathrm{O}(n+1)$ acts transitively on S^n. In particular there exists a homeomorphism h of S^n with $h(P) = \infty$. Thus we can replace X by $h(X)$ which evidently lies in $\mathbb{R}^n$.

[1307]Here the idea is that the neighborhoods "approximate" X, in particular $U \leq V$ means that V is a "better approximation of X", which just means that $V \subset U$.

illustration.) We consider the following commutative diagram

$$\begin{tikzcd}
& \widetilde{\mathrm{H}}^m(W) \arrow[dl] \arrow[dr, "i^*_{X,W}"'] & \\
\varinjlim\limits_{U\in\mathcal{N}(X)} \widetilde{\mathrm{H}}^m(U) \arrow[rr, "\Phi"'] & & \widetilde{\mathrm{H}}^m(X). \arrow[ul, bend right, "r^*"']
\end{tikzcd}$$

From $(i^*_{X,W} \circ r^*) = (r \circ i_{X,W})^* = \mathrm{id}^*_X$ we deduce that the map $i^*_{X,W}$ is an epimorphism. It follows that the horizontal map Φ is also an epimorphism. It remains to show that Φ is a monomorphism.

By Lemma 76.2 (3) it remains to show that for any $U \in \mathcal{N}(X)$ there exists a $V \in \mathcal{N}(X)$ with $U \leq V$, i.e. with $V \subset U$, such that

$$\ker\left(i^*_{X,U} \colon \widetilde{\mathrm{H}}^m(U) \to \widetilde{\mathrm{H}}^m(X)\right\} \;\subset\; \ker\left(i^*_{V,U} \colon \widetilde{\mathrm{H}}^m(U) \to \widetilde{\mathrm{H}}^m(V)\right).$$

So let $U \in \mathcal{N}(X)$. After possibly replacing U by $U \cap W$ we can without loss of generality assume that $U \subset W$. We continue with the following subclaim.

Subclaim. There exists an open neighborhood V with $X \subset V \subset U$ such that the inclusion $i_{V,U} \colon V \to U$ and the retraction $r \colon V \to X \subset U$ are homotopic maps from V to U.

We consider the homotopy

$$\begin{array}{rcl} F \colon U \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & (1-t)\cdot x + t \cdot r(x) \end{array}$$

from the identity id_U to the retraction $r \colon U \to X$. Note that $F(X \times [0,1]) = X$. Since X is compact and since F is continuous there exists an open neighborhood V of X, contained in U, such that $F(V \times [0,1]) \subset U$.[1308] But then the restriction of F to $F \colon V \times [0,1] \to U$ defines a homotopy from the inclusion $V \to U$ to the retraction $r \colon V \to X \subset U$. (We refer to Figure 1276 for an illustration.) This concludes the proof of the subclaim.

We consider the following diagram

$$\begin{tikzcd}
\widetilde{\mathrm{H}}^m(U) \arrow[rr, "i^*_{X,U}"] \arrow[dr, "i^*_{V,U}"'] \arrow[dr, bend left, "r^*"] & & \widetilde{\mathrm{H}}^m(X) \arrow[dl, "r^*"'] \\
& \widetilde{\mathrm{H}}^m(V). &
\end{tikzcd}$$

The above subclaim says that the two maps $i_{V,U} \colon V \to U$ and $r \colon V \to X \subset U$ are homotopic. It follows from Lemma 73.13 that the two maps $i^*_{V,U}$ and r^* from $\widetilde{\mathrm{H}}^m(U)$ to $\widetilde{\mathrm{H}}^m(V)$ agree. Since r is the identity on X we see that $r \colon U \to X \subset V$ equals the map

[1308] This can be seen as follows. The set $W := F^{-1}(U) \subset U \times [0,1]$ is open and it evidently contains $X \times [0,1]$. Since X and $[0,1]$ are compact and since W is open one sees easily that there exists an $r > 0$ such that for any $z \in X$ we have $B_r(z) \times [0,1] \subset W$. But then $V := \bigcup\limits_{x\in X} B_r(x)$ has the property that $V \times [0,1] \subset W$.

$i_{X,U} \circ r \colon U \to X$. This shows that the diagram commutes. It follows immediately from these observations that V has the desired property. ⊞

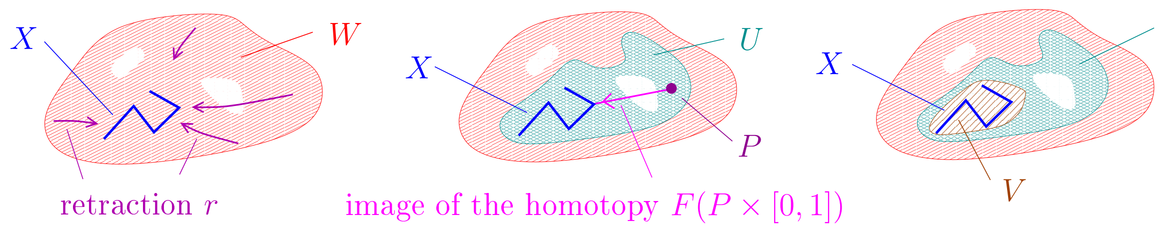


FIGURE 1276

We turn to the actual proof of the theorem. First let $j \neq 0$. The desired isomorphism will be given by a sequence of natural isomorphisms as follows:

$$\begin{aligned}
\mathrm{H}_j(S^n \setminus X) &\underset{(1)}{\overset{\cong}{\longleftarrow}} \mathrm{H}_c^{n-j}(S^n \setminus X) \underset{(2)}{\overset{\cong}{\longleftarrow}} \varinjlim_{K \in \mathcal{K}(S^n \setminus X)} \mathrm{H}^{n-j}(S^n \setminus X, \overbrace{(S^n \setminus X) \setminus K}^{=(S^n \setminus K) \setminus X}) \underset{(3)}{\overset{\cong}{\longleftarrow}} \varinjlim_{U \in \mathcal{N}(X)} \mathrm{H}^{n-j}(S^n \setminus X, U \setminus X) \\
&\underset{(4)}{\overset{\cong}{\longleftarrow}} \varinjlim_{U \in \mathcal{N}(X)} \mathrm{H}^{n-j}(S^n, U) \underset{(5)}{\overset{\cong}{\longleftarrow}} \varinjlim_{U \in \mathcal{N}(X)} \widetilde{\mathrm{H}}^{n-j-1}(U) \underset{(6)}{\overset{\cong}{\longrightarrow}} \widetilde{\mathrm{H}}^{n-j-1}(X).
\end{aligned}$$

We explain the natural isomorphisms in more detail:

(1) Since X is compact we know from Lemma 2.17 (2) that $S^n \setminus X$ is an open subset of $S^n \setminus X$. In particular it is also an n-dimensional smooth manifold and via Lemma 86.17 it inherits an orientation from S^n. Also note that it follows from Proposition 44.2 that $S^n \setminus X$ has no boundary. After this preamble the first isomorphism is the Poincaré Duality isomorphism from Theorem 88.19 corresponding to the oriented smooth manifold $S^n \setminus X$. By Lemma 88.25 this isomorphism is natural.

(2) The second isomorphism is provided by Proposition 77.4.

(3) The third isomorphisms comes from the straightforward observation that the map $\mathcal{N}(X) \to \mathcal{K}(S^n \setminus X)$ given by $U \mapsto S^n \setminus U$ is an isomorphism of directed sets which gives us an isomorphism of direct systems.

(4) For any open neighborhoods U and V of X with $U \leq V$, i.e. with $V \subset U$, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^{n-j}(S^n \setminus X, V \setminus X) & \longleftarrow & \mathrm{H}^{n-j}(S^n, V) \\
\uparrow & & \uparrow \\
\mathrm{H}^{n-j}(S^n \setminus X, U \setminus X) & \longleftarrow & \mathrm{H}^{n-j}(S^n, U)
\end{array}$$

where all the maps are induced by the obvious inclusions of pairs of topological spaces. It follows from the Excision Theorem 74.1 for Cohomology Groups that the horizontal maps are isomorphisms. The desired isomorphism of direct limits is now an immediate consequence of Lemma 76.4.

(5) For any open neighborhoods U and V of X with $U \leq V$, i.e. with $V \subset U$, we have a commutative diagram

$$\begin{tikzcd} \mathrm{H}^{n-j}(S^n, V) & \widetilde{\mathrm{H}}^{n-j-1}(V) \arrow[l, "\delta"'] \\ \mathrm{H}^{n-j}(S^n, U) \arrow[u] & \mathrm{H}^{n-j-1}(U) \arrow[l, "\delta"'] \arrow[u] \end{tikzcd}$$

where the vertical maps are induced by the obvious inclusions and where the horizontal maps are the connecting homomorphisms of the long exact sequences in reduced cohomology provided by Lemma 73.17. Since $j \geq 1$ we obtain from these long exact sequences and the calculation of the cohomology groups of S^n that the horizontal maps are in fact isomorphisms. As in (4) we appeal to Lemma 76.4 to obtain the desired isomorphism of direct limits.

(6) The last map is the isomorphism of the above claim.

It still remains to deal with the case $j = 0$. In this case all of the above maps are still isomorphisms, except for map (5). To address this issue we consider the following diagram:

$$\begin{tikzcd} \mathrm{H}_0(S^n \setminus X) \arrow[r, "(1)", "\cong"'] \arrow[d] & \mathrm{H}^n_c(S^n \setminus X) \arrow[r, "(2),(3)", "\cong"'] \arrow[d] & \varinjlim_{U \in \mathcal{N}(X)} \mathrm{H}^n(S^n \setminus X, U \setminus X) \arrow[r, "(4)", "\cong"'] \arrow[d, "\cong"'] & \varinjlim_{U \in \mathcal{N}(X)} \mathrm{H}^n(S^n, U) \arrow[d] \\ \mathrm{H}_0(S^n) \arrow[r, "(1)", "\cong"'] & \mathrm{H}^n_c(S^n) \arrow[r, "(2),(3)", "\cong"'] & \varinjlim_{U \in \mathcal{N}(X)} \mathrm{H}^n(S^n, U) \arrow[r, "(4)", "\cong"'] & \varinjlim_{U \in \mathcal{N}(X)} \mathrm{H}^n(S^n). \end{tikzcd}$$

Here the second vertical map is the inclusion induced map defined by Lemma 77.5. The first and the fourth vertical maps are the obvious inclusion induced maps. Furthermore the third vertical map is the inverse of the isomorphism which is induced by the isomorphisms $\mathrm{H}^n(S^n, U) \to \mathrm{H}^n(S^n \setminus X, U \setminus X)$ that are given by the Excision Theorem 74.1 for Cohomology Groups. The top horizontal maps are precisely the above maps (1), (2), (3) and (4). The bottom horizontal maps are defined completely analogously with $X = \varnothing$. Since the maps defined in (1), (2), (3) and (4) are natural we see that the diagram commutes. It follows that

$$\begin{aligned} \widetilde{\mathrm{H}}_0(S^n \setminus X) &\overset{\text{follows easily from } \mathrm{H}_0(S^n) \cong \mathbb{Z}}{\cong} \ker(\mathrm{H}_0(S^n \setminus X) \to \mathrm{H}_0(S^n)) \overset{\text{by the above commutative diagram}}{\cong} \ker\left(\varinjlim \mathrm{H}^n(S^n, U) \to \varinjlim \mathrm{H}^n(S^n)\right) \\ &\underset{\text{Corollary 47.3}}{\cong} \varinjlim \ker(\mathrm{H}^n(S^n, U) \to \mathrm{H}^n(S^n)) \underset{\substack{\text{by the long exact sequence} \\ \text{of Lemma 73.17 and since } n \geq 2}}{\cong} \varinjlim \widetilde{\mathrm{H}}^{n-1}(U) \underset{\text{by the claim}}{\cong} \widetilde{\mathrm{H}}^{n-1}(X). \end{aligned}$$

■

The following proposition finally gives a negative answer to Question 92.6 (3).

Proposition 92.12. *Let M be a closed connected n-dimensional topological manifold. If M is non-orientable, then there is no* topological *embedding of M into $\mathbb{R}^{n+1}$.*

PROOF. The proof of Proposition 92.12 is verbatim the same as the proof of Proposition 92.7, we only need to replace the original Alexander Duality Theorem 92.1 by the Alexander Duality Theorem 92.11 for Topological Submanifolds.

We also give an alternative argument that does not require the slightly obscure Generalized Alexander Duality Theorem 92.10. Thus let M be a closed connected n-dimensional topological manifold which admits an embedding $f\colon M \to \mathbb{R}^{n+1}$. We have to show that M is orientable. As usual we view $\mathbb{R}^{n+1}$ as a subset of S^{n+1}. We make the following observations:

(1) It follows from Borsuk's Theorem 85.5 together with Proposition 85.7 (1) that there exists a compact $(n+1)$-dimensional smooth submanifold X of $\mathbb{R}^{n+1}$ that contains $f(M)$ and such that there exists a retraction $r\colon X \to f(M)$. In the following we denote by $i\colon f(M) \to X$ the inclusion map.
(2) Since $r \circ i = \mathrm{id}_{f(M)}$ we see that the map $r^*\colon \mathrm{H}^n(f(M);\mathbb{Z}) \to \mathrm{H}^n(X;\mathbb{Z})$, induced by the retraction, is a monomorphism.
(3) By the remark on page 2232 there is an isomorphism $\mathrm{H}^n(X;\mathbb{Z}) \cong \mathrm{H}_0(S^{n+1} \setminus X;\mathbb{Z})$. It follows in particular from Proposition 41.5 that $\mathrm{H}^n(X;\mathbb{Z})$ is a free abelian group.
(4) It follows from (2) and (3) that $\mathrm{H}^n(f(M);\mathbb{Z})$ is a torsion-free abelian group.
(5) Since $f\colon M \to f(M)$ is a homeomorphism we obtain from (4) that $\mathrm{H}^n(M;\mathbb{Z})$ is a torsion-free abelian group.
(6) We deduce from (5) and Proposition 75.15 that the closed connected n-dimensional topological manifold M is orientable. ■

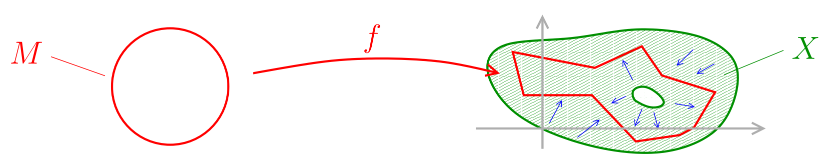


FIGURE 1277. Illustration of the proof of Proposition 92.12.

The following proposition is a strengthening of Proposition 92.8.

Proposition 92.13. *If $n \in \mathbb{N}_{\geq 2}$, then there is no embedding of $\mathbb{RP}^n$ into S^{n+1}.*

PROOF. First let n be even. In this case $\mathbb{RP}^n$ is non-orientable and the result is an immediate consequence of Proposition 92.12.

Now let n be odd. This case is significantly harder and it was first proved by René Thom [**Tho52**, p. 180] in 1952. We cannot provide the argument. But it seems like the argument of Thom is somewhat reminiscent of the argument in the proof of Proposition 92.8. So suppose there exists an embedding $f\colon \mathbb{RP}^n \to S^{n+1}$. It follows from Proposition 41.5, the Generalized Alexander Duality Theorem 92.10 and the fact that $\mathrm{H}^n(\mathbb{RP}^n;\mathbb{Z}) \cong \mathbb{Z}$ (here we use that n is odd), that $S^{n+1} \setminus f(\mathbb{RP}^n)$ consists of two path components. We denote the closures of the path components by A and B. The problem is that the triad (S^{n+1}, A, B) is in general not excisive so we cannot apply the Mayer–Vietoris Theorem 74.15 for Cohomology Groups. The idea in the proof is to replace usual singular cohomology by Alexander-Spanier cohomology for which the above triad is excisive. We refer to [**Mas78a**, Chapter 8] or [**Mas78b**] for an introduction to Alexander-Spanier cohomology and we refer to [**Mas78a**, Theorem II.8.15] for the proof that in this setting the above triad is excisive. ■

We conclude this chapter with the following convenient variation on the Generalized Alexander Duality Theorem 92.10.

Theorem 92.14. (Alexander Duality Theorem for Balls) *Let $n \geq 2$ and let $X \neq \overline{B}^n$ be a weakly locally contractible, non-empty compact subset of $\overline{B}^n$. We write $\partial X := X \cap \overline{S}^{n-1}$. Given any $j \in \mathbb{N}_0$ we have a natural isomorphism*

$$\widetilde{\mathrm{H}}_j(\overline{B}^n \setminus X;\mathbb{Z}) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X,\partial X;\mathbb{Z}).$$

such that the following diagram commutes

$$\begin{tikzcd}
\widetilde{\mathrm{H}}_j(\overline{B}^n \setminus X;\mathbb{Z}) \arrow[r, "\cong"] \arrow[d, "\partial"] & \mathrm{H}^{n-j-1}(X,\partial X;\mathbb{Z}) \arrow[d, "i^*"] \\
\widetilde{\mathrm{H}}_{j-1}(S^{n-1} \setminus \partial X;\mathbb{Z}) \arrow[r, "\cong"] & \widetilde{\mathrm{H}}^{n-j-1}(\partial X;\mathbb{Z}),
\end{tikzcd}$$

up to a sign that depends only on j and n.[1309] *Here the bottom map is the isomorphism given in the proof of the Generalized Alexander Duality Theorem 92.10.*

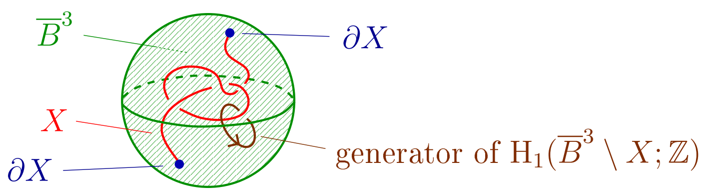


FIGURE 1278. Illustration of the Alexander Duality Theorem 92.14 for Balls.

SKETCH OF PROOF. One could try to imitate the proof of the Generalized Alexander Duality Theorem 92.10, but instead we will do a little trick to reduce most of the statement to the Generalized Alexander Duality Theorem 92.10.

Let $X \subsetneq \overline{B}^n$ be a weakly locally contractible compact non-empty subset. We consider

$$K := (X \times \{0\}) \cup \underbrace{(\partial X \times [0,1]) \cup (\overline{B}^n \times \{1\})}_{=:Y} \subset \underbrace{(\overline{B}^n \times \{0\}) \cup (\partial K \times [0,1]) \cup (\overline{B}^n \times \{1\})}_{=\partial(\overline{B}^n \times [0,1])}.$$

Now we consider the following maps

$$\begin{tikzcd}
\widetilde{\mathrm{H}}_j(\overline{B}^n \setminus X) \arrow[r, "(a)", "\cong"'] \arrow[d, "\cong"] & \widetilde{\mathrm{H}}_j((\overline{B}^n \times 0) \setminus (X \times \{0\})) \arrow[r, "(b)", "\cong"'] \arrow[d, "\cong"] & \widetilde{\mathrm{H}}_{j-1}(\partial(\overline{B}^n \times [0,1]) \setminus K) \arrow[d, "(c)"', "\cong"] \\
\mathrm{H}^{n-j-1}(X,\partial X) \arrow[r, "\cong", "(e)"'] & \mathrm{H}^{n-j-1}(K,Y) & \widetilde{\mathrm{H}}^{n-j-1}(K). \arrow[l, "\cong", "(d)"']
\end{tikzcd}$$

We make the following observations regarding the various maps:

(a) This map is induced by the obvious homeomorphism.

[1309] I will not embarrass myself by trying to guess the sign.

(b) This map is an isomorphism since $(\overline{B}^n \times 0) \setminus (X \times \{0\})$ is a deformation retract of $\partial(\overline{B}^n \times [0,1]) \setminus K$.
(c) Note that $\partial(\overline{B}^n \times [0,1])$ is homeomorphic to S^n. Also note that one can easily verify that K is again weakly locally contractible.[1310] Thus we can appeal to the Generalized Alexander Duality Theorem 92.10 to obtain a natural isomorphism.
(d) This map is an isomorphism since Y is contractible.
(e) This map is an isomorphism by a suitable variation on the Excision Theorem 74.1.

The combination of the above gives us a natural vertical isomorphism to the left. Unfortunately this neat argument does not elucidate why the diagram in the statement of the theorem commutes. We leave the verification of this statement as a slightly challenging exercise to the reader. ■

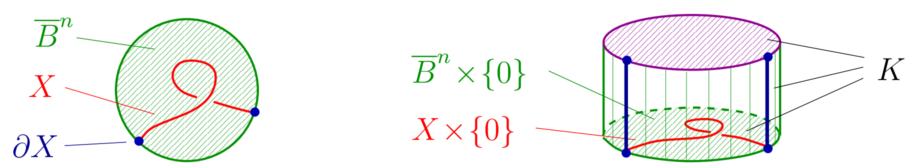


FIGURE 1279. Illustration for the proof of Theorem 92.14.

Remark. More information on Alexander dualities in manifolds can be found in [**Mas81b**].

Exercises for Chapter 92.

Exercise 92.1. Let $X \subset S^3$ be a closed orientable connected 2-dimensional submanifold. We view S^3 as a subset of S^4 in the usual way.
(a) Determine the homology groups of $S^4 \setminus X$.
(b) Now let $X \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be the surface shown in Figure 1280. Give an example of a closed curve in $S^4 \setminus X$ that is not null-homotopic.

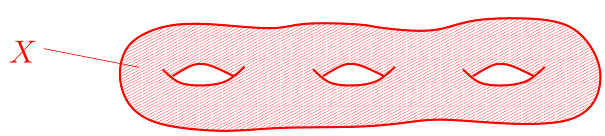


FIGURE 1280

Exercise 92.2. This exercise is a continuation of Exercise 92.1.
(a) Let $X \subset S^n$ be a closed submanifold. We view S^n as a subset of S^{n+1} in the usual way. Show that $S^{n+1} \setminus X$ is homotopy equivalent to $\Sigma(S^n \setminus X)$. (Here $\Sigma(S^n \setminus X)$

[1310]The only points one needs to worry about for a second are the points $(x, 0)$, with $x \in \partial X$. But given any neighborhood R of $x \in X$ there exists a neighborhood S of $(x, 0) \in K$ which admits a deformation retraction to R. Using this observation it is now straightforward to see that K is indeed weakly locally contractible.

denotes the suspension of $S^n \setminus X$.)

Remark. You will need to use at some point that X is a "reasonable subset", e.g. that X is a closed submanifold. The statement does not hold for all subsets of S^n.

(b) Again let $X \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be the surface shown in Figure 1280. As above we view it as a submanifold of S^4. Determine $\pi_1(S^4 \setminus X)$.

Exercise 92.3.

(a) Let F be an orientable connected 2-dimensional smooth manifold with non-empty boundary and let $i\colon F \to \mathbb{R}^3$ be an injective map. Is the complement $\mathbb{R}^3 \setminus F$ necessarily path-connected?

(b) Let F be an orientable connected surface non-compact 2-dimensional smooth manifold with empty boundary and let $i\colon F \to S^3$ be an injective map. Is the complement $S^3 \setminus F$ necessarily path-connected?

93. Applications of cohomology groups

In this chapter we give two applications of cohomology groups, namely we prove the table theorem and we address two special cases of the square peg problem. Neither of these applications can count as a "real life" application, but at least in both cases cohomology, Poincaré duality and Alexander duality get used to prove statements which initially have absolutely no connection to cohomology theory at all.

93.1. **The Table Theorem ❀.**

As a warm-up we start out with the fairly elementary but charming Wobbly Table Theorem. In the vernacular it says that given a wobbly table we can rotate it around its center, by an angle of at most $\frac{\pi}{2}$, such that eventually all four legs touch the ground. Slightly more precisely, it can be formulated as follows:

Theorem 93.1. (Wobbly Table Theorem) *Let $f\colon \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}$ be a function, let $x \in \mathbb{C}$ and let $z \in \mathbb{C} \setminus \{0\}$. There exists a $\theta \in [0, \frac{\pi}{2}]$ such that*

$$f\big(x + e^{\mathrm{i}\theta} z\big) \;=\; f\big(x + e^{\mathrm{i}(\theta+\frac{\pi}{2})} z\big) \;=\; f\big(x + e^{\mathrm{i}(\theta+\pi)} z\big) \;=\; f\big(x + e^{\mathrm{i}(\theta+\frac{3\pi}{2})} z\big).$$

Proof. The theorem is a fairly elementary consequence of the Intermediate Value Theorem from real analysis. Filling in the details is your task in Exercise 93.1. Alternatively the reader can look at [**Loy17**, p. 289] for a proof. ■

Our goal in this section is to prove a much more fancy theorem. More precisely we will prove the following "Table Theorem" which was first formulated and proved by Roger Fenn [**Fen70**] in 1970.

Theorem 93.2. (Table Theorem) *Let $D \subset \mathbb{R}^2$ be a compact convex non-empty subset and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a map such that $f(x) \geq 0$ for all $x \in D$ and such that $f(x) = 0$ for all $x \notin D$. Given any $d \in \mathbb{R}_{>0}$ there exist four points $a_1, a_2, a_3, a_4 \in \mathbb{R}^2$ with the following properties:*

(1) *the points form a square of side length d,*
(2) *the center of the square lies in D,*
(3) *we have $f(a_1) = f(a_2) = f(a_3) = f(a_4)$.*

In other words, the Table Theorem 93.2 says that any given square table can be placed on the ground defined by the graph of f, such that all four legs lie on the ground, such that the table is horizontal and such that the center of the table is located in D.

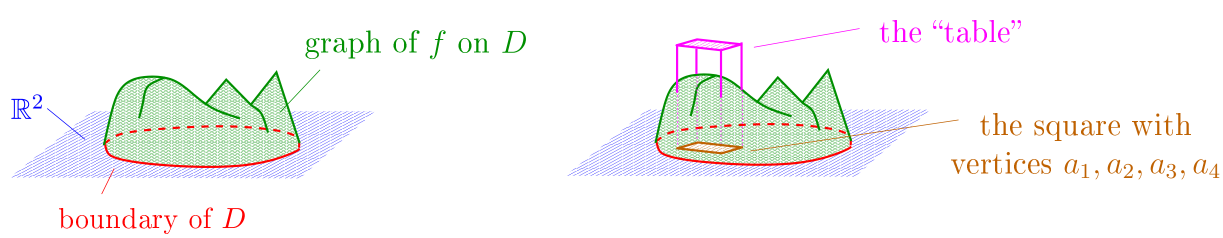


Figure 1281. Illustration of the Table Theorem 93.2.

Remark. In [**Meye81**] it is shown that in the formulation of the Table Theorem 93.2 one cannot drop the hypothesis that D is convex.

The proof of the Table Theorem 93.2 will require the remainder of this section. Throughout the section we fix a compact convex non-empty subset $D \subset \mathbb{R}^2$ and we fix a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in D$ and such that $f(x) = 0$ for all $x \notin D$. Let $d \in \mathbb{R}_{>0}$. We make the following elementary observations and preparations:

(i) Since we can rescale D we can and will without loss of generality assume that $d = 1$.
(ii) If the interior of D is the empty set, then the function f is necessarily the zero function[1311]. Thus the conclusion of the Table Theorem 93.2 holds trivially.
(iii) If follows from (ii) that henceforth we can assume that the interior of D is non-empty. Note that Proposition 2.52 implies that there exists a homeomorphism of pairs $(D, \partial D) \xrightarrow{\cong} (\overline{B}^2, S^1)$.
(iv) After a translation we can and will assume that the origin $(0,0)$ is contained in the interior of D.
(v) As always we will make the identification $\mathbb{R}^2 = \mathbb{C}$, in particular we will work a lot with the multiplication "$\cdot\, i$", i.e. with "rotation by $\frac{\pi}{2}$".
(vi) If there exists a square in $\mathbb{R}^2$ of side length $d = 1$ with center in D, such that the f-value at all of the four vertices is zero, then we have evidently found the desired table. Thus we can henceforth assume that for any square of side length $d = 1$ with center in D the f-value of at least one vertex is non-zero.

In the following discussion we will at times take a few liberties with mathematical language, but these liberties will not infringe on the rigor of the argument.

We start out with introducing the following notation, which gives us a convenient way for parameterizing the set of all squares with side length $d = 1$ and whose center lies in D.

Notation.

(1) We consider the map[1312]

$$\begin{array}{rcl} a\colon D \times S^1 & \to & \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \\ (x,\gamma) & \mapsto & \begin{array}{c}\text{the four vertices } a_1, a_2, a_3, a_4 \text{ of the unique square} \\ \text{with center } x \text{ such that } a_2 - a_1 = \gamma \text{ and} \\ \text{such that } a_1, a_2, a_3, a_4 \text{ are ordered in the} \\ \text{counterclockwise direction.}\end{array} \end{array}$$

This definition is illustrated in Figure 1282 on the left.

(2) We also consider the closely related map

$$\begin{array}{rcl} Q\colon D \times S^1 & \to & \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \\ (x,\gamma) & \mapsto & \text{the points } Q_1, Q_2, Q_3, Q_4 \text{ on the graph corresponding to } a(x,\gamma). \end{array}$$

When $(x,\gamma) \in D \times S^1$ is understood from the context, then we drop it at times from the notation. In other words, often we write a_i instead of $a_i(x,\gamma)$ and we write Q_i instead of $Q_i(x,\gamma)$.

[1311] Why is that?

[1312] We leave the task of writing down an explicit formula to the reader. Once the formula is written down it is clear that a is actually continuous.

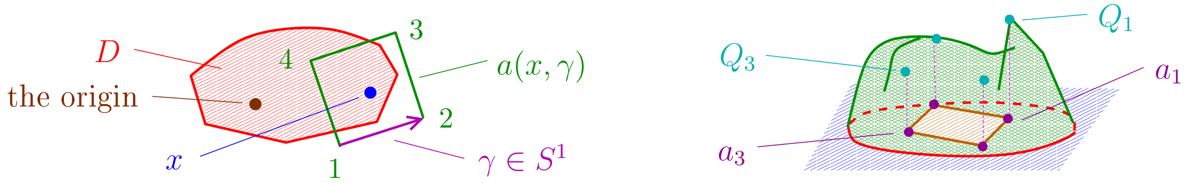


FIGURE 1282

One of the key ideas behind the proof of the Table Theorem 93.2 is to split Condition (3) of the Table Theorem 93.2 into two separate statements. Namely we note that the condition

(3) $f(a_1) = f(a_2) = f(a_3) = f(a_4)$,

is satisfied precisely if the following two conditions are satisfied:

(3a) the points $\underbrace{(a_1, f(a_1))}_{=Q_1}, \ldots, \underbrace{(a_4, f(a_4))}_{=Q_4}$ lie on a plane,

(3b) the plane from (3a) is horizontal.

The trick now is to encode these two statements in a clever way. We will do so by introducing two maps.

Definition.

(1) Given $(x, \gamma) \in D^2 \times S^1$ we denote by $\pi(x, \gamma)$ the plane through the three points $Q_1(x, \gamma), Q_2(x, \gamma)$ and $Q_3(x, \gamma)$.

(2) We consider the map[1313]

$$\begin{array}{rcl} \phi\colon D \times S^1 & \to & \mathbb{R} \\ (x, \gamma) & \mapsto & \underbrace{x_3\text{-value of the plane } \pi(x,\gamma) \text{ at } a_4(x,\gamma) \text{ minus } f(a_4(x,\gamma))}_{=f(a_1)+f(a_3)-f(a_2)-f(a_4)} \end{array}$$

The definition of this map is illustrated in Figure 1283. Note that $\phi(x, \gamma) = 0$ if and only if the corresponding points Q_1, Q_2, Q_3, Q_4 lie on a plane.

(3) We set $A := \{(x, \gamma) \in D \times S^1 \,|\, \phi(x, \gamma) = 0\}$.

(4) Given $(x, \gamma) \in D^2 \times S^1$ we denote by $n(x, \gamma) \in \mathbb{R}^3$ the unique vector that is normal to $\pi(x, \gamma)$, that has unit length and that has positive x_3-coordinate. This definition is illustrated in Figure 1284 on the left.

(5) We consider the map

$$\begin{array}{rcl} \psi\colon D \times S^1 & \to & \mathbb{R}^2 \\ (x, \gamma) & \mapsto & \text{projection of } n(x,\gamma) \text{ onto the } x_1x_2\text{-plane.} \end{array}$$

The definition of this map is illustrated on the right hand side of Figure 1286. Note that $\psi(x, \gamma) = 0$ if and only if the plane $\pi(x, \gamma)$ is horizontal.

(6) We set $B := \{(x, \gamma) \in D \times S^1 \,|\, \psi(x, \gamma) = 0\}$.

[1313]Note that the equality

$$a_4 = a_2 + (a_3 - a_2) + \underbrace{(a_4 - a_3)}_{=a_1 - a_2} = a_2 + (a_1 - a_2) + (a_3 - a_2) = a_1 + a_3 - a_2,$$

FIGURE 1283

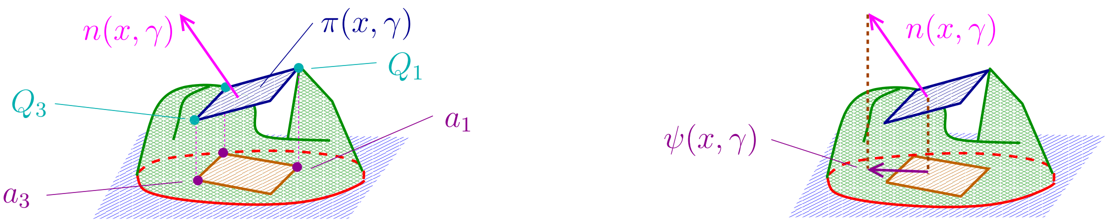


FIGURE 1284

Remark. The Wobbly Table Theorem 93.1 implies that given any $x \in D$ there exist at least four $\gamma \in S^1$ with $(x,\gamma) \in A$. In particular the set A is more complicated than in our subsequent illustrations.

We make the following elementary observation.

Observation 93.3. *To prove the Table Theorem 93.2 it suffices to show that $A \cap B \neq \varnothing$.*

In the following two lemmas we collect some properties of the two maps ϕ and ψ.

Lemma 93.4. *The map $\phi\colon D \times S^1 \to \mathbb{R}$ has the following properties:*

(1) *The map $\phi\colon D \times S^1 \to \mathbb{R}$ is continuous.*
(2) *For any $(x,\gamma) \in D \times S^1$ we have $\phi(x, i\cdot\gamma) = -\phi(x,\gamma)$.*
(3) *Given $(x,\gamma) \in D \times S^1$ the following statements are equivalent:*
   (a) $\phi(x,\gamma) = 0$,
   (b) $\phi(x, i\cdot\gamma) = 0$,
   (c) *(x,γ) lies in A,*
   (d) *the corresponding points Q_1, Q_2, Q_3, Q_4 lie on a plane.*

PROOF.

(1) As we pointed out above, we have

$$\phi(x,\gamma) \;=\; f(\underbrace{a_1(x,\gamma)}_{\text{odd vertex}}) + f(\underbrace{a_3(x,\gamma)}_{\text{odd vertex}}) - f(\underbrace{a_2(x,\gamma)}_{\text{even vertex}}) - f(\underbrace{a_4(x,\gamma)}_{\text{even vertex}}).$$

It follows from the continuity of f and the continuity of $(x,\gamma) \mapsto a_i(x,\gamma)$ that ϕ is indeed continuous.

---

together with the fact that the x_3-coordinate of the plane $\pi(x,\gamma)$ is an affine function on $\mathbb{R}^2$, implies that

$$x_3\text{-coordinate of the plane } \pi(x,\gamma) \text{ at } a_4 \;=\; f(a_1) + f(a_3) - f(a_2).$$

(2) Note that for any $(x,y) \in D \times S^1$ and any $j \in \{1,2,3,4\}$ we have the equality $a_j(x,i\cdot\gamma) = a_{j+1}(x,\gamma)$. In particular the odd/even vertices corresponding to $(x,i\cdot\gamma)$ are precisely the even/odd vertices corresponding to (x,γ). It follows immediately from this observation and the explicit formula for ϕ given in (1) that the equality $\phi(x,i\cdot\gamma) = -\phi(x,\gamma)$ holds. We refer to Figure 1285 for an illustration.

(3) This statement follows immediately from (2) and the definition of A. ■

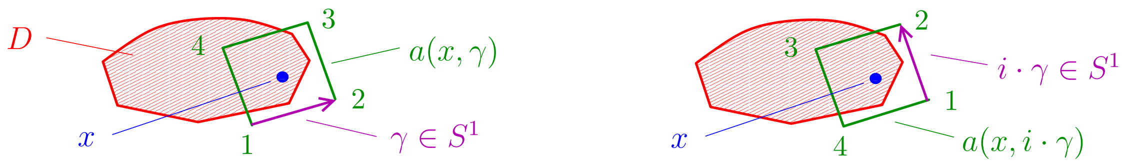


FIGURE 1285

Next we turn to the study of the map $\psi\colon D \times S^1 \to \mathbb{R}^2$. In the formulation of the next lemma we need the following definition.

Definition. We say that $v, w \in \mathbb{R}^2$ are *parallel* if one of v, w is zero or if there exists an $\lambda \in \mathbb{R}_{>0}$ such that $v = \lambda w$.

Lemma 93.5. *The map $\psi\colon D \times S^1 \to \mathbb{R}^2$ has the following properties:*

(1) *The map $\psi\colon D \times S^1 \to \mathbb{R}$ is continuous.*
(2) *If $(x,\gamma) \in A$, then $\psi(x,\gamma) = \psi(x, i^k \cdot \gamma)$ for any $k \in \mathbb{Z}$.*
(3) *Let $x \in \partial D$ and let $\gamma \in S^1$. If $(x,\gamma) \in A$, then the vector $\psi(x,\gamma)$ is not parallel to $-x$.*

SKETCH OF PROOF.

(1) Let $(x,\gamma) \in D \times S^1$. Basic linear algebra shows that

$$\psi(x,\gamma) = \begin{array}{l} \text{the vector in } \mathbb{R}^3 \text{ that is obtained as follows:} \\ \text{(1) perform the cross product } (Q_3(x,\gamma)-Q_2(x,\gamma))\times(Q_1(x,\gamma)-Q_2(x,\gamma)) \\ \text{(2) normalize the vector from (1) to length one}^{1314} \\ \text{(3) project the vector from (2) onto the } xy\text{-plane.} \end{array}$$

It follows immediately from this observation that ψ is continuous.

(2) Let $(x,\gamma) \in A$. As mentioned in Lemma 93.4 (3), this means that the four points $Q_1(x,\gamma)$, $Q_2(x,\gamma)$, $Q_3(x,\gamma)$ and $Q_4(x,\gamma)$ lie on a plane. But this immediately implies that for any $k \in \mathbb{Z}$ we have $\pi(x,\gamma) = \pi(x, i^k\cdot\gamma)$ which then gives us the equality $\psi(x,\gamma) = \psi(x, i^k \cdot \gamma)$.

(3) The final statement is a little more interesting. So let $x \in \partial D$ and let $\gamma \in S^1$. We assume that $(x,\gamma) \in A$. We write $a_i = a_i(x,\gamma)$, $i = 1,2,3,4$.

Claim. There exist two adjacent vertices which lie outside of $\mathring{D}$ and there are two adjacent vertices which have non-zero f-value.

[1314] Since $a_3(x,\gamma)-a_2(x,\gamma)$ and $a_1(x,\gamma)-a_2(x,\gamma)$ are linearly independent we see that the two vectors in the cross product are linearly independent, in particular the cross product is indeed non-zero. Thus it makes sense to normalize the vector to length one.

We make the following observations:
(a) Since $(x,\gamma) \in A$ we have $f(a_1)+f(a_3)-f(a_2)-f(a_4)=0$.
(b) By choice of f we know that all f-values are non-negative.
(c) By the preliminary remark (vi) on page 2252 we know that not all of the f-values are zero.
(d) Since x lies on ∂D and since D is convex we see that at most two f-values can be non-zero.[1315]

The claim follows easily from the above considerations and the elementary observation that every odd vertex is adjacent to every even vertex. ⊞

To simplify the notation let us now assume that the adjacent vertices a_1 and a_2 do *not* lie in the interior of D. Consequently $f(a_3)>0$ and $f(a_4)>0$. A straightforward calculation shows that

$$\psi(x,\gamma) \;=\; \begin{array}{c}\text{projection of}\\ \text{normalization of}\end{array} \begin{pmatrix} a_3-a_2 \\ f(a_3)\end{pmatrix} \times \begin{pmatrix} a_1-a_2 \\ 0 \end{pmatrix} \text{ into the } xy\text{-plane} \;=\; \lambda\cdot(a_2-a_3)$$

for some $\lambda>0$.

Now suppose that $\psi(x,\gamma)$ is parallel to $-x$. This means, by the above calculation, that $a_3-a_2=-\mu\cdot x$ for some $\mu>0$. It is now straightforward to verify that x lies in the interior of the convex hull of 0, a_3 and a_4. Since 0, a_3 and a_4 lie in the interior of D we see that x also lies in the interior of D. But that is a contradiction to our hypothesis that $x \in \partial D$. ■

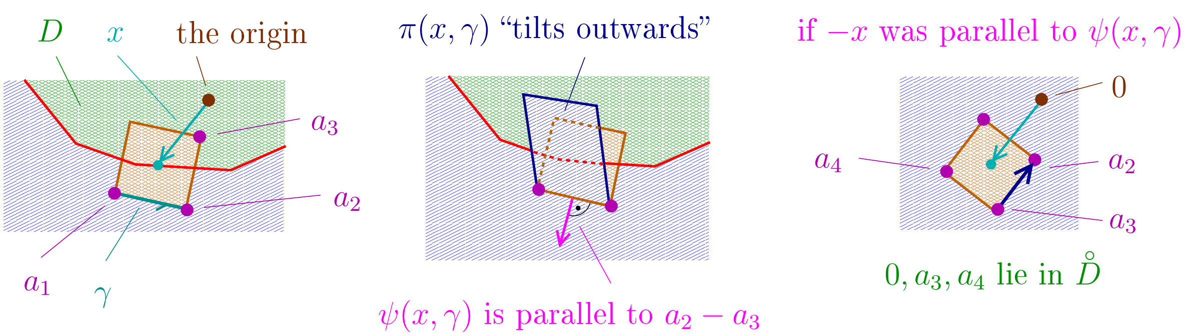


FIGURE 1286. Illustration for the proof of Lemma 93.5 (2).

Lemma 93.6. *We set*

$$T \;:=\; (D\times S^1)/\sim \quad \textit{where } (x,\gamma)\sim(x,i\cdot\gamma)$$

[1315] Any vertex with non-zero f-value lies in the interior of D. So if three vertices have non-zero f, then the triangle spanned by these vertices lies in the interior, in particular x would lie in the interior of D.

and we consider the following diagram

$$\begin{tikzcd} D \times S^1 \arrow[rr, "\psi"] \arrow[d, "=:p"', "(x,\gamma)\mapsto[(x,\gamma)]"] & & \mathbb{R}^2 \arrow[d, dashed] \\ T \arrow[rr, "=:\psi'", "{[(x,\gamma)]\mapsto \frac{1}{4}\sum_{k=0}^{3}\psi(x,i^k\gamma)}"'] & & \mathbb{R}^2. \end{tikzcd}$$

The following statements hold:

1. *The map $p\colon D \times S^1 \to T$ is a 4-fold covering and the topological space T is homeomorphic to the solid torus $D^2 \times S^1$.*
2. *The map ψ' is continuous.*
3. *The restriction of the diagram to $A \subset D \times S^1$ is commutative.*
4. *If we set*
$$A' := p(A) \quad \text{and} \quad B' := \{[(x,\gamma)] \in T \mid \psi'([x,\gamma]) = 0\}$$
then we have $p(A \cap B) = A' \cap B'$.

Proof.

1. For the experienced reader it should be clear that $p\colon D \times S^1 \to T = (D \times S^1)/\sim$ is a 4-fold covering and that $T \cong D \times (S^1/\sim) \cong D \times S^1$. On page 2252 we remarked that D is homeomorphic to $\overline{B}^2$, thus we see that T is indeed homeomorphic to the solid torus $\overline{B}^2 \times S^1$.
2. The map $D^2 \times S^1 \to \mathbb{R}^2$ given by $(x,\gamma) \to \frac{1}{4}\sum_{k=0}^{3}\psi(x, i^k\gamma)$ is continuous. It descends to our map $\psi'\colon T = (D \times S^1)/\sim \to \mathbb{R}^2$. Thus it follows from Lemma 3.22 that ψ' is continuous.
3. It follows immediately from Lemma 93.5 (2) that for any $(x,\gamma) \in A$ we have the equality $\psi'([x,\gamma]) = \psi(x,\gamma)$.
4. The equality $p(A \cap B) = A' \cap B'$ follows easily from (3). ■

From Lemma 93.6 (3) and Observation 93.3 we obtain the following observation.

Observation 93.7. *To prove the Table Theorem 93.2 it suffices to show that $A' \cap B' \neq \varnothing$.*

After spending so much time on preparations we can now finally apply the machinery from algebraic topology to tackle the problem. We introduce the following convention.

Convention. By Lemma 93.6 (1) we know that T is a solid torus. Given any $x_0 \in T$ we make an identification $\pi_1(T, x_0) = \mathbb{Z}$.[1316]

The following lemma turns our data into algebraic-topological information.

Lemma 93.8. *We continue with the above notation. Let $x_0 \in T \setminus A'$. The image of $\pi_1(T \setminus A', x_0)$ in $\pi_1(T, x_0) = \mathbb{Z}$ is contained in the subgroup $2 \cdot \mathbb{Z}$.*

[1316] In the following we do not care about the precise isomorphism, in particular we do not care about any sign ambiguity.

Proof. Let $x_0 = [(x, \gamma)] \in T \setminus A'$. Note that by definition of A' we have $\phi(x, \gamma) \neq 0$.

($*$) Without loss of generality we can assume that $\phi(x, \gamma) > 0$.

Now let $[f] \in \pi_1(X \setminus A', x_0)$. In other words, $f\colon [0,1] \to T \setminus A'$ is a loop in $x_0 = [(x, \gamma)]$. We need to show that $[f] \in 2 \cdot \mathbb{Z}$. By Proposition 16.11 we know that there exists a unique lift $\widetilde{f}\colon [0,1] \to (D \times S^1) \setminus A$ with $\widetilde{f}(0) = (x, \gamma)$. We consider the following diagram:

$$\begin{tikzcd}
 & & \mathbb{R} \setminus \{0\} \arrow[r, hook] & \mathbb{R} \\
 & (D \times S^1) \setminus A \arrow[ur, "\phi"] \arrow[d, "p"] \arrow[rr, hook] & & D \times S^1 \arrow[u, "\phi"'] \arrow[d, "p"] \\
{[0,1]} \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & T \setminus A' \arrow[rr, hook] & & T.
\end{tikzcd}$$

We make the following observations:

(1) By definition of a lift we know that $\widetilde{f}(1)$ lies in the subset
$$p^{-1}(f(1)) \;=\; p^{-1}(f(0)) \;=\; p^{-1}(x_0) \;=\; \{(x, \gamma), (x, i \cdot \gamma), (x, -\gamma), (x, -i \cdot \gamma)\}.$$
(2) By Lemma 93.4 (2) and ($*$) we know that $\phi(x, \gamma) = \phi(x, -\gamma) > 0$ and we know that $\phi(x, i \cdot \gamma) = \phi(x, -i \cdot \gamma) < 0$.
(3) Recall that by definition $A' = p(A)$. Therefore it follows from Lemma 93.4 (3) that $A = p^{-1}(A')$. In particular we have $(D \times S^1) \setminus A = p^{-1}(T \setminus A')$.
(4) It follows from (3) and the above diagram that the image of $\phi \circ \widetilde{f}\colon [0,1] \to \mathbb{R}$ is contained in $\mathbb{R} \setminus \{0\}$.
(5) Since $[0,1]$ is connected and since $\phi(\widetilde{f}(0)) = \phi(x, \gamma) > 0$ we obtain from (4) that $\phi(\widetilde{f}(1)) > 0$.
(6) It follows from (1), (2) and (5) that $\widetilde{f}(1) = (x, \gamma)$ or $\widetilde{f}(1) = (x, -\gamma)$.
(7) It follows easily[1317] from (6) that the loop $f * f$ lifts to a loop in $(D \times S^1, (x, \gamma))$.
(8) It follows from (7) and Lemma 16.15 (1) that $2 \cdot [f] = [f] \cdot [f] = [f * f]$ actually lies in the subgroup $p_*(\pi_1(D \times S^1, (x, \gamma)))$.
(9) Since $p\colon D \times S^1 \to T$ is a 4-fold connected covering we obtain from Lemma 16.15 (3) that $p_*(\pi_1(D \times S^1, (x, \gamma))) = 4 \cdot \mathbb{Z}$.
(10) The combination of (8) and (9) shows that $[f]$ lies in $2 \cdot \mathbb{Z}$. ■

The next convention is basically self-explanatory.

Convention. Any map denoted by i is given by an inclusion of the corresponding (pairs) of topological spaces.

The following lemma consists mostly of repackaging the result of Lemma 93.8.

Lemma 93.9. *We continue with the above notation. Suppose that we are given a compact submanifold with corner $L \subset T$ such that $A' \subset \mathring{L} := L \setminus \partial_0 L$.*[1318] *The map*
$$i_*\colon \ \mathrm{H}_2(L, L \cap \partial T; \mathbb{Z}_2) \;\to\; \mathrm{H}_2(T, \partial T; \mathbb{Z}_2)$$
is an epimorphism.

[1317] How does this work?

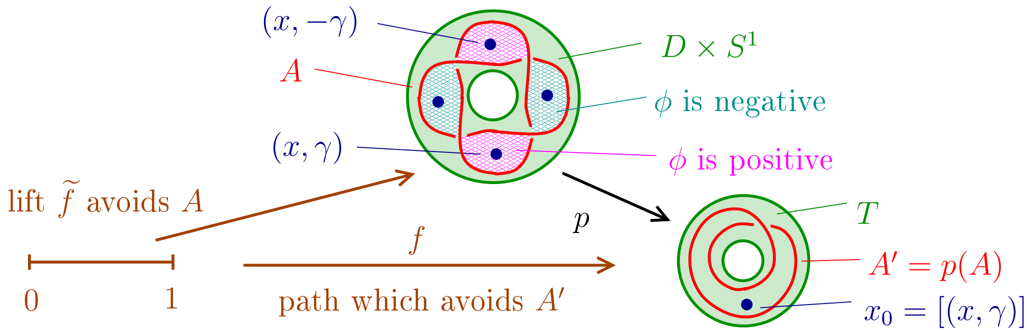


FIGURE 1287. Illustration for the proof of Lemma 93.8.

FIGURE 1288. Illustration for Lemma 93.9.

PROOF. Let $L\subset T$ be a compact submanifold with corner such that $A'\subset \mathring{L}:=L\setminus\partial_0 L$. If $L=T$ then there is nothing to show. So we assume that $L\neq T$. This implies in particular that $T\setminus\mathring{L}\neq\varnothing$.

We start out with the following claim which is basically a rewriting of Lemma 93.8.

Claim. For any homomorphism $\varphi\colon\ \mathrm{H}_1(T;\mathbb{Z})\to\mathbb{Z}_2$ the composition

$$\mathrm{H}_1(T\setminus\mathring{L};\mathbb{Z})\ \xrightarrow{i_*}\ \mathrm{H}_1(T;\mathbb{Z})\ \xrightarrow{\varphi}\ \mathbb{Z}_2$$

is the zero map.

To prove the claim it suffices to show the analogous statement for any path-component Y of $T\setminus\mathring{L}$. So let Y be a path-component of $T\setminus\mathring{L}$. We pick a point $x_0\in Y$. We consider the following commutative diagram

$$\begin{tikzcd}
\pi_1(Y,x_0) \arrow[r] \arrow[d, two heads] & \pi_1(T\setminus\mathring{L},x_0) \arrow[r] \arrow[d] & \pi_1(T\setminus A',x_0) \arrow[r] \arrow[d] \arrow[rr, bend left, "=0\text{ by }(4)"] & \pi_1(T,x_0) \arrow[r, two heads, "\eta"'] \arrow[d, "\cong"] & \mathbb{Z}_2 \\
\mathrm{H}_1(Y;\mathbb{Z}) \arrow[r] & \mathrm{H}_1(T\setminus\mathring{L};\mathbb{Z}) \arrow[r] & \mathrm{H}_1(T\setminus A';\mathbb{Z}) \arrow[r] & \mathrm{H}_1(T;\mathbb{Z}). &
\end{tikzcd}$$

We make the following clarifications and observations:

(1) The map $\eta\colon\pi_1(T,x_0)\to\mathbb{Z}_2$ is the unique epimorphism.
(2) All the vertical maps are given by the corresponding Hurewicz homomorphism. The diagram commutes by the naturality of the Hurewicz homomorphisms.
(3) The Hurewicz homomorphism to the left is an epimorphism by the Hurewicz Theorem 52.5 since Y is path-connected. The Hurewicz homomorphism to the right is

[1318] Here we use the notation for submanifolds with corner that we introduced in Lemma 89.1.

an isomorphism by the Hurewicz Theorem 52.5 since T is path-connected and since $\pi_1(T, x_0) \cong \mathbb{Z}$ is abelian.

(4) It follows from Lemma 93.8 that the map $\pi_1(T \setminus A', x_0) \to \pi_1(T, x_0) \xrightarrow{\eta} \mathbb{Z}_2$ is the trivial map.

(5) The claim follows immediately from combining (1) to (4). $\boxplus$

Now we turn to the actual proof of the desired statement. We write $W := T \setminus \mathring{L}$. We consider the following diagram

$$\begin{tikzcd}
\operatorname{Hom}(\mathrm{H}_1(W;\mathbb{Z}),\mathbb{Z}_2) & \mathrm{H}^1(W;\mathbb{Z}_2) \arrow[l, "\mathrm{ev}"', "\cong"] \arrow[r, "\cap[W]", "\cong"'] & \mathrm{H}_2(W,\partial W;\mathbb{Z}_2) \arrow[r, "i_*", "\cong"'] & \mathrm{H}_2(T,\partial T \cup L;\mathbb{Z}_2) \\
\operatorname{Hom}(\mathrm{H}_1(T;\mathbb{Z}),\mathbb{Z}_2) \arrow[u, "i^*", "=0"'] & \mathrm{H}^1(T;\mathbb{Z}_2) \arrow[l, "\mathrm{ev}"', "\cong"] \arrow[u, "i^*"] \arrow[rr, "\cap[T]", "\cong"'] & & \mathrm{H}_2(T,\partial T;\mathbb{Z}_2) \arrow[u, "i_*"] \\
& & \mathrm{H}_2(L,L\cap\partial T;\mathbb{Z}_2) \arrow[r, "i_*", "\cong"'] \arrow[ur, "i_*"] & \mathrm{H}_2(\partial T \cup L,\partial T;\mathbb{Z}_2). \arrow[u, "i_*"]
\end{tikzcd}$$

We make the following comments and clarifications:

(1) The vertical maps are induced by the various inclusions.

(2) The square to the left commutes by the naturality of the evaluation homomorphism. Furthermore, the horizontal maps to the left are isomorphisms by Proposition 75.18.

(3) By Lemma 89.1 we know that W is a compact 3-dimensional smooth manifold. We equip it with the orientation coming from T.

(4) The horizontal maps "$\cap[W]$" and "$\cap[T]$" in the center are given by capping with the fundamental classes $[W]$ and $[T]$. These maps are isomorphisms by the Poincaré Duality Theorem 88.1.

(5) The horizontal map to the top right is an isomorphism by the Excision Theorem 89.2 (b).

(6) The vertical maps to the right are a piece of the long exact sequence of the triple $(T, \partial T \cup L, \partial T)$, see Proposition 43.15.

(7) The rectangle to the right commutes by Lemma 89.6, see in particular the example (a) on page 2188.

(8) By the above claim we know that the vertical map to the left is zero. Since all horizontal maps are isomorphisms and since the upper part of the diagram commutes we see that the vertical map to the top right is also the zero map.

(9) It follows from (6) and (8) that the vertical map to the bottom right is an epimorphism.

(10) The bottom triangle commutes since all maps are inclusion induced maps.

(11) The horizontal map at the very bottom is an isomorphism by the Excision Theorem 89.2 (c).

(12) Finally we note that it follows immediately from (9), (10) and (11) that the diagonal map $i_*\colon \mathrm{H}_2(L, L \cap \partial T; \mathbb{Z}_2) \to \mathrm{H}_2(T, \partial T; \mathbb{Z}_2)$ is an epimorphism. Fortunately that is precisely what we wanted to show. $\blacksquare$

Lemma 93.10. *Let X be a topological space and let $f, g\colon X \to S^1$ be maps. If for every $x \in X$ we have $f(x) \neq -g(x) \in S^1$, then f and g are homotopic.*

SKETCH OF PROOF. We consider the map

$$\begin{array}{rcccl} X & \xrightarrow{f\times g} & S^1 \times S^1 & \mapsto & S^1 \\ & & (z,w) & \mapsto & z \cdot w. \end{array}$$

It follows from our hypothesis that this map takes values in $S^1 \setminus \{-1\} \cong (-\pi, \pi)$. Since $(-\pi, \pi)$ admits a deformation retraction to 0 one can now easily show that f and g are homotopic. We are fully confident that the reader did not fail to fill in all details when solving Exercise 18.1. ■

Lemma 93.11. *Let $f, g\colon X \to Y$ be two maps between topological spaces. Suppose that there exists a compact subset $K \subset X$ such that $f(x) \neq g(x)$ for all $x \in K$. If Y is Hausdorff and normal (by Lemma 2.46 these conditions are satisfied if Y is Hausdorff and compact), then there exists an open neighborhood U of K such that $f(x) \neq g(x)$ for all $x \in U$.*

PROOF (∗). We consider the map

$$\begin{array}{rcl} f \times g\colon X & \to & Y \times Y \\ x & \mapsto & (f(x), g(x)). \end{array}$$

By Lemma 3.6 (1) we know that $f \times g$ is continuous. Since K is compact we obtain from Lemma 2.40 (1) that $(f \times g)(K)$ is a compact subset of $Y \times Y$. Let $D := \{(y, y) \,|\, y \in Y\}$ be the "diagonal" in $Y \times Y$. Since Y is Hausdorff we obtain from Exercise 3.21 that D is a closed subset of $Y \times Y$. Now, using our hypothesis that Y is normal, we obtain from Lemma 2.46 an open neighborhood V of $(f \times g)(K)$ with $V \cap D = \emptyset$. We set $U := (f \times g)^{-1}(V)$. Since $f \times g$ is continuous we see that U is an open neighborhood of K. Evidently for any $x \in U$ we have $(f \times g)(x) \notin D$, i.e. we have $f(x) \neq g(x)$. ■

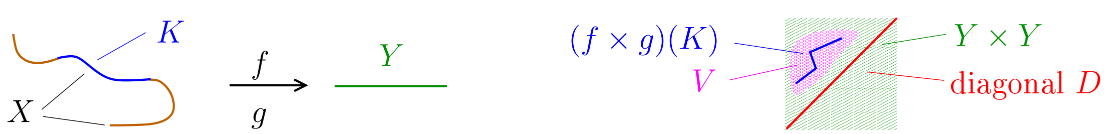


FIGURE 1289. Illustration for the proof of Lemma 93.11.

Now we head towards the denouement.

PROOF OF THE TABLE THEOREM 93.2. First recall that by Observation 93.7 it suffices to show that $A' \cap B' \neq \emptyset$. We do a proof by contradiction. So we make the following assumption:

(∗) We assume that $A' \cap B' = \emptyset$.

In other words, we assume that ψ' never vanishes on A'. Throughout the proof the following two maps will play a major role:

$$\begin{array}{rcl} \widehat{\psi}\colon \{z \in T \,|\, \psi'(z) \neq 0\} & \to & S^1 \\ z & \mapsto & \frac{1}{\|\psi'(z)\|} \cdot \psi'(z) \end{array} \quad \text{and} \quad \begin{array}{rcl} \theta\colon \partial T = \partial D \times (S^1/\sim) \to \partial D & \to & S^1 \\ z & \mapsto & \frac{1}{\|z\|} \cdot z. \end{array}$$

Note that A' has the following two properties:

(α) ψ' is non-zero on A' (in particular $\widehat{\psi}$ is defined on A'), and
(β) By Lemma 93.5 (3) we know that for any $z \in A' \cap \partial T$ we have $\widehat{\psi}(z) \neq -\theta(z)$.

The following claim says that we can enlarge A' to a submanifold with corner which still has properties (α) and (β). More precisely, we have the following claim.

Claim. The smooth manifold T admits a compact submanifold with corner L with the following properties:

(i) $A' \subset \mathring{L}$,
(ii) ψ' is non-zero on L (in particular $\widehat{\psi}$ is defined on L), and
(iii) for any $z \in L \cap \partial T$ we have $\widehat{\psi}(z) \neq -\theta(z)$.

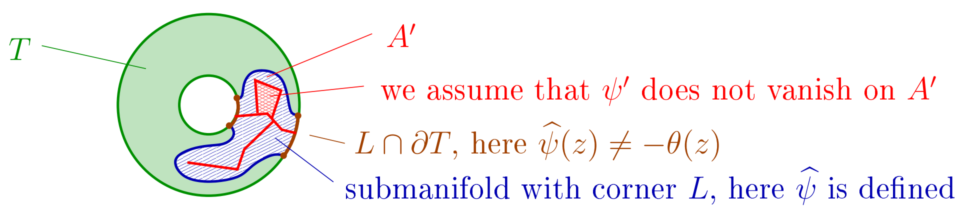


FIGURE 1290. Illustration for the proof of the Table Theorem 93.2.

Recall that T is a solid torus, in particular T is compact. Since A is a closed subset of $D \times S^1$ we obtain from Lemma 2.17 (1) and Lemma 2.40 that $A' = p(A)$ is compact.

(1) It follows from (α), the fact that A' is compact and Lemma 93.11 that there exists an open neighborhood U of A' such that ψ' vanishes nowhere on U.
(2) It follows from (β), Exercise 2.28 and Lemma 93.11 that there exists an open neighborhood V of $A' \cap \partial T$ such that $\widehat{\psi}(z) \neq -\theta(z)$ for all $z \in V$.

It follows from Lemma 11.5 (2) that there exists a submanifold L with corner of T such that $A' \subset \mathring{L} \subset L \subset U$ and such that $L \cap \partial T \subset V$. In particular we see that ψ' is non-zero on L and that for any $z \in L \cap \partial T$ we have $\widehat{\psi}(z) \neq -\theta(z)$. $\boxplus$

We consider the following diagram:

$$
\begin{tikzcd}
\underbrace{\mathrm{H}_2(T;\mathbb{Z}_2)}_{=0} \arrow[r] & \mathrm{H}_2(T,\partial T;\mathbb{Z}_2) \arrow[r, hook, "\partial"] & \mathrm{H}_1(\partial T;\mathbb{Z}_2) \arrow[r, "j_*"] \arrow[dr, "\pi_*"] \arrow[ddr, "\theta_*\oplus\pi_*", "\cong"'] \arrow[dddr, bend right, "\theta_*"] & \mathrm{H}_1(T;\mathbb{Z}_2) \\
 & & & \mathrm{H}_1(S^1/\sim;\mathbb{Z}_2) \arrow[u, "m_*", "\cong"'] \\
 & & & \mathrm{H}_1(S^1;\mathbb{Z}_2)\oplus\mathrm{H}_1(S^1/\sim;\mathbb{Z}_2) \arrow[u, two heads] \arrow[d, two heads] \\
 & & & \mathrm{H}_1(S^1;\mathbb{Z}_2) \\
 & \mathrm{H}_2(L,L\cap\partial T;\mathbb{Z}_2) \arrow[r, "\partial"'] \arrow[uuuu, two heads, "i_*", "\text{epimorphism by Lemma 93.9}"'] & \mathrm{H}_1(L\cap\partial T;\mathbb{Z}_2) \arrow[r, "l_*"'] \arrow[uuuu, "k_*"'] \arrow[ur, "(\theta|_{L\cap\partial T})_*", "=(\widehat{\psi}|_{L\cap\partial T})_*"'] & \mathrm{H}_1(L;\mathbb{Z}_2). \arrow[u, "\widehat{\psi}_*"']
\end{tikzcd}
$$

We make the following comments, clarifications and observations:

(1) The horizontal maps at the top respectively the bottom are extracts of the long exact sequence in homology of the pair $(T,\partial T)$ respectively of the pair $(L, L\cap\partial T)$.

(2) The maps i,j,k,l indicate the obvious inclusion maps.

(3) We denote by $m\colon S^1/\sim \xrightarrow{\cong} \{0\}\times S^1/\sim \to D\times(S^1/\sim) = T$ the obvious map. Since D is convex we see that m is a homotopy equivalence, hence we obtain from Corollary 42.8 that m_* is an isomorphism.

(4) The map $\pi\colon \partial T = \partial D\times(S^1/\sim)\to S^1/\sim$ is the obvious projection. It is elementary to show that the maps $m\circ\pi$ and j from ∂T to T are homotopic. Thus we obtain from Proposition 42.5 that they induce the same map on homology. In other words, the triangle to the top right commutes.

(5) It follows from the naturality of the connecting homomorphism, see Proposition 43.15, that the rectangle to the left commutes. All the remaining triangles to the right commute basically by definition or by the functoriality of homology groups.

(6) By the discussion on page 126 the map $\theta\colon\partial D\to S^1$ is a homeomorphism. It follows easily from this observation, together with the fact that $\partial T = \partial D\times(S^1/\sim)$, that the map $\theta_*\oplus\pi_*\colon \mathrm{H}_1(\partial T;\mathbb{Z}_2)\to\mathrm{H}_1(S^1;\mathbb{Z}_2)\oplus\mathrm{H}_1(S^1/\sim;\mathbb{Z}_2)$ is an isomorphism.

(7) It follows from (ii) and Lemma 93.10 that the restrictions of $\widehat{\psi}$ and θ to $L\cap\partial T$ are homotopic. We deduce from Proposition 42.5 that the induced maps on homology are the same. Put differently, the two maps indicated at the upward-pointing diagonal arrow are actually the same.

(8) By Lemma 93.9 the left vertical map is an epimorphism. Since $\mathrm{H}_2(T,\partial T;\mathbb{Z}_2)\neq 0$ there exists a $\xi\in\mathrm{H}_1(L,L\cap\partial T;\mathbb{Z}_2)$ with $i_*(\xi)\neq 0$.

(9) It follows from the long exact sequence on top that $\partial i_*(\xi)\neq 0$ and that $j_*\partial i_*(\xi)=0$.

(10) It follows from (3), (4) and (9) that $\pi_*\partial i_*(\xi)=0$.

(11) By (6) we know that the map $\theta_* \oplus \pi_*$ is an isomorphism. Therefore we obtain from (8) and (10) that $\theta_* \partial i_*(\xi) \neq 0 \in \mathrm{H}_1(S^1; \mathbb{Z}_2)$.
(12) Since the diagram commutes we obtain from (11) that $l_* \partial \xi \neq 0$ in $\mathrm{H}_1(L; \mathbb{Z}_2)$. But that is absurd, since the horizontal sequence at the bottom is exact. ■

93.2. **Trivial and non-trivial solutions to the table theorem ❀.** In this short section we want to discuss the (non-) existence of trivial solutions to the table theorem. More precisely, we have the following definition.

Definition. Let $D \subset \mathbb{R}^2$ be a compact convex non-empty subset of $\mathbb{R}^2$.
(1) Let $s \in \mathbb{R}_{>0}$. We say D is *s-trivial* if there exists a square in $\mathbb{R}^2$ such that the vertices lie outside of $\mathring{D}$ and such that the center of the square lies in D. Otherwise we say that D is *s-non trivial*.
(2) We say D is *trivial* if D is s-trivial for every $s \in \mathbb{R}_{>0}$.

It is basically clear that any compact convex subset with empty interior is trivial. One might initially hope that every compact convex subset with non-empty interior is non-trivial. But that is not the case.

Example. We consider the convex subset D that is the convex hull of the points $(0,0)$, $(2,1)$ and $(2,-1)$. We claim that it is trivial. Indeed, let $s \in \mathbb{R}_{>0}$. We write $t = \frac{s}{2}$. The square with vertices $a_1 = (-t + \frac{t}{4}, t)$, $a_2 = (-t + \frac{t}{4}, -t)$, $a_3 = (t + \frac{t}{4}, -t)$ and $a_4 = (t + \frac{t}{4}, t)$, whose center equals $(\frac{t}{4}, 0)$, shows that D is s-trivial.

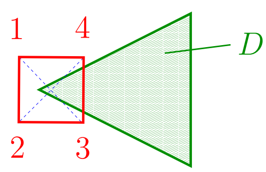


Figure 1291

This example leads us to the following definition.

Definition. We say that a subset D of $\mathbb{R}^2 = \mathbb{C}$ is *obtuse* if given any $x \in \partial D$ there exists a $v \in \mathbb{R}^2 \setminus \{(0,0)\} = \mathbb{C} \setminus \{0\}$ and a $\theta > \frac{\pi}{4}$ such that

$$\{x + r \cdot v \cdot e^{\mathrm{i}\varphi} \mid \varphi \in [-\theta, \theta] \text{ and } r \in [0,1]\} \subset D.$$

The definition is illustrated in Figure 1292.

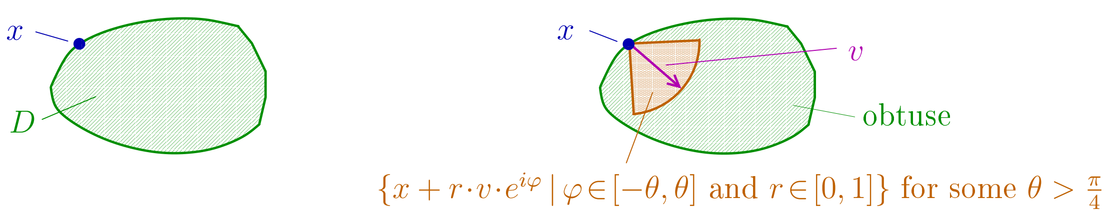


Figure 1292

We conclude this short section with the following lemma.

Lemma 93.12. *Let D be a compact convex non-empty subset of $\mathbb{R}^2$. If D is obtuse, then there exists an $t > 0$ such that D is s-non trivial for every $s \in (0, t)$.*

PROOF. We leave the elementary proof of this lemma to the reader. ■

93.3. **The square-peg problem ❀.** In this section we will give another amusing, but unfortunately even less practical application of cohomology theory. To introduce the topic we need the following definition.

Definition.

(1) The image of an injective map $S^1 \to \mathbb{R}^2$ is calleda *Jordan curve.*[1319]
(2) Given a Jordan curve C we say that a rectangle R is *inscribed in* C if all vertices of R lie on C.

The following question was asked by Otto Toeplitz [**Toep1911**] in 1911.[1320]

Question 93.13. *Does every Jordan curve admit an inscribed square?*

This question is known as the "inscribed square problem" or the "square-peg problem". Lev Schnirelmann [**Schni44**][1321] gave a positive answer if the curve is smooth, in fact he gave a proof if the curve is C^2, i.e. if the map $S^1 \to \mathbb{R}^2$ is twice continuously differentiable. But the general case is still open.[1322] A detailed discussion of the question and its history is given in [**Matsc14**].

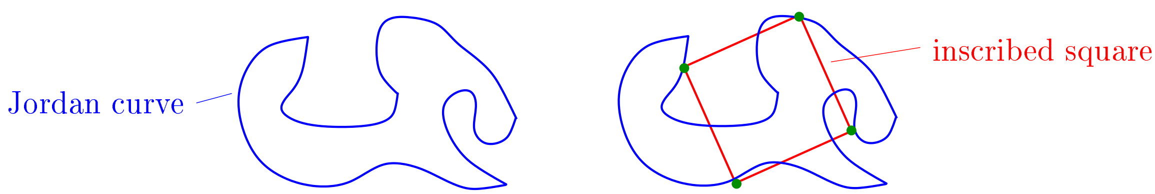


FIGURE 1293

FIGURE 1294. The more common square peg problem.

In this section we will provide two partial answers to Question 93.13.

(1) We will use the Table Theorem 93.2 to deal with the case that the Jordan curve is the boundary of a compact convex obtuse subset of $\mathbb{R}^2$.

[1319]Note that this definition is slightly different from the definition we gave on page 452 where we defined a Jordan curve to be an injective map $S^1 \to \overline{\mathbb{C}} = S^2$.

[1320]Otto Toeplitz (1881-1940) was a Jewish German mathematician who was forced to emigrate in 1939. He is mainly known for his contributions to analysis.

[1321]We had already encountered Lev Schnirelmann in Theorem 59.5.

[1322]Recently the Fields medalist Terence Tao [**Tao17**] wrote a long paper in which he outlined an approach to the square-peg problem. See also the following video:

`https://www.youtube.com/watch?v=RGxnWgy5i80`

(2) We will show that we can inscribe a rectangle into every Jordan curve.

Let us start with the first result.

Proposition 93.14. *Let J be a Jordan curve which is the boundary of a compact convex subset D. If D is obtuse, then J admits an inscribed square.*

Remark. This proposition also admits proofs that do not rely on topology, see for example [**Zi21, Chr50**].

Proof. Let $D \subset \mathbb{R}^2$ be a compact convex obtuse subset. We need to show that there exist four points $a_1, a_2, a_3, a_4 \in \partial D$ which form a square. After a translation we can assume that the origin 0 is contained in the interior of D. Given $x \in D \setminus \{0\}$ we set

$$\rho(x) \;:=\; \sup\left\{\|rx\| \,\middle|\, r \in \mathbb{R}_{>0} \text{ and } r \cdot x \in D\right\} \;\in\; \mathbb{R}_{>0}.$$

It follows almost immediately from Lemma 2.54[1323] that the map

$$\begin{aligned} f\colon \mathbb{R}^2 &\;\to\; [0,1] \\ x &\;\mapsto\; \begin{cases} 1 - \|x\| \cdot \frac{1}{\rho(x)}, & \text{if } x \in D \setminus \{0\}, \\ 1, & \text{if } x = 0, \\ 0, & \text{if } x \notin D \end{cases} \end{aligned}$$

is continuous. Since D is obtuse we obtain from Lemma 93.12 that there exists a $d > 0$ such that D is d-non trivial. By the Table Theorem 93.2 there exist four points $b_1, b_2, b_3, b_4 \in \mathbb{R}^2$ with the following properties:

(1) the points form a square of side length d,
(2) the center of the square lies in D,
(3) we have $f(b_1) = f(b_2) = f(b_3) = f(b_4)$.

Since D is d-non trivial we see that at least one vertex lies inside of $\mathring{D}$. But since f is non-zero on $\mathring{D}$ we see that the common f-value, let's call it y, lies in the open interval $(0,1)$. It follows immediately from the definition of f that the four points b_1, b_2, b_3, b_4 lie on the subset $(1-y) \cdot \partial D$. In other words, if we multiply these four points by $\frac{1}{1-y}$, then we obtain the desired four points a_1, a_2, a_3, a_4 on ∂D. ■

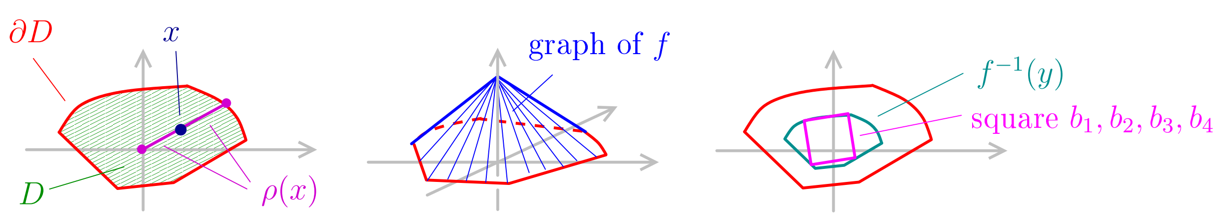


Figure 1295. Illustration for the proof of Proposition 93.14.

We conclude this chapter with the second contribution to the square-peg problem.

[1323] Well-trained eyes will not fail to spot that implicitly we also make use of Lemmas 2.8, 2.35 and 2.17 to show that f is indeed continuous.

Proposition 93.15. *Every Jordan curve admits an inscribed rectangle.*

In [**Meye82**] the subsequent ingenious proof of Proposition 93.15 is attributed to Herbert Vaughan. In preparation for the proof we need the following elementary lemma.

Lemma 93.16. *Let $T = S^1 \times S^1$ be the torus. Let $\sim$ be the equivalence relation on T generated by $(z,w) \sim (w,z)$. The quotient space $T/\sim$ is homeomorphic to the Möbius band and*

$$\partial(T/\sim) \;=\; \{[(z,z)] \,|\, z \in S^1\}.$$

PROOF. The homeomorphism is illustrated in Figure 1296. ∎

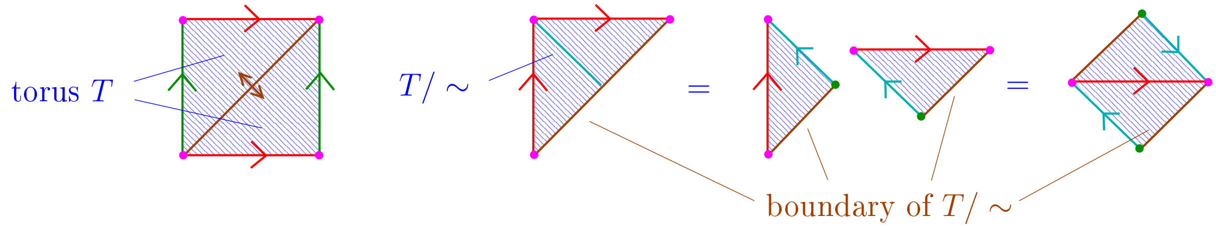


FIGURE 1296

Now we are already ready to prove Proposition 93.15.

PROOF OF PROPOSITION 93.15. Let $\gamma\colon S^1 \to \mathbb{R}^2$ be an injective map. We have to show that $\gamma(S^1)$ admits an inscribed rectangle. We consider the map

$$\begin{aligned} \Phi\colon T := S^1 \times S^1 &\;\to\; \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \\ (z,w) &\;\mapsto\; \Big(\tfrac{\gamma(z)+\gamma(w)}{2}, |\gamma(z) - \gamma(w)|\Big). \end{aligned}$$

Since for all $z, w \in S^1$ we have $\Phi(z,w) = \Phi(w,z)$ we see that Φ descends to a map

$$\Psi\colon T/\sim \;\to\; \mathbb{R}^3$$

where $\sim$ denotes of course the equivalence relation generated by $(z,w) \sim (w,z)$.

Claim. The map $\Psi\colon T/\sim \to \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ is not injective.

We first show that the claim implies the proposition. So now that we know that Ψ is not injective we know that there exist distinct pairs of points $\{z,w\}$ and $\{u,v\}$ with $\Phi(z,w) = \Phi(u,v)$. We consider the quadrilateral given by the points z, u, v, w. It follows from $\Phi(z,w) = \Phi(u,v)$ that the diagonals bisect each other and that the diagonals have the same length. But such a quadrilateral is necessarily a rectangle.[1324] We refer to Figure 1297 for an illustration.

It remains to prove the claim. Let us assume that $\Psi\colon T/\sim \to H := \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ is injective. By Lemma 93.16 the quotient $M := T/\sim$ is homeomorphic to the Möbius band and the boundary ∂M is given by $\{[(z,z)] \,|\, z \in S^1\}$. We see immediately from the definition of Ψ that for a point $[(z,w)] \in M = T/\sim$ we have $\Psi([(z,w)]) \in \partial H = \mathbb{R} \times \mathbb{R} \times \{0\}$ if and only if $[(z,w)] \in \partial M$. We denote by M_1 and M_2 two copies of M and we denote by $\mathrm{D}\,M = M_1 \cup_{\partial M_1 = \partial M_2} M_2$ the double of M. It follows from the proof of Lemma 8.33

[1324] Hmm, why is that?

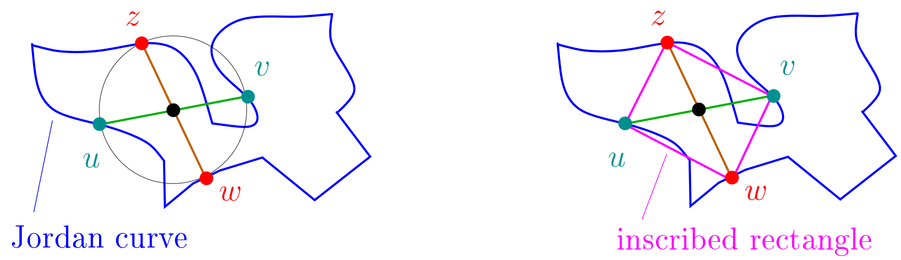


FIGURE 1297

(1) that $\mathrm{D}\,M$ is in fact homeomorphic to the Klein bottle, in particular $\mathrm{D}\,M$ is a closed, non-orientable 2-dimensional smooth manifold.

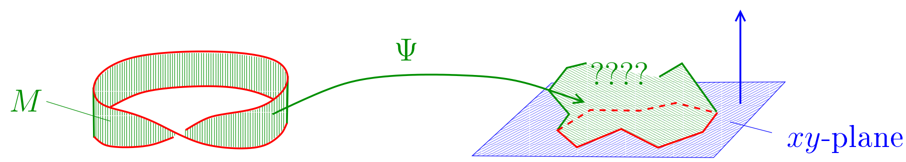


FIGURE 1298

We denote by $\rho\colon \mathbb{R}^3 \to \mathbb{R}^3$ the reflection in the xy-plane, i.e. ρ is the map that is given by $\rho(x, y, z) = (x, y, -z)$. We consider the map

$$\mathrm{D}\,M = M_1 \cup_{\partial M_1 = \partial M_2} M_2 \quad \to \quad \mathbb{R}^3$$

that is given by

$$\begin{array}{rcll} P & \mapsto & \Psi(P), & \text{if } P \in M_1, \\ P & \mapsto & \rho(\Psi(P)), & \text{if } P \in M_2. \end{array}$$

It follows immediately from the above discussion that this map is well-defined and injective. Furthermore it follows from Lemma 3.22 that the map is in fact continuous. But now we have constructed an injective map from the Klein bottle to $\mathbb{R}^3$. On page 433 we pointed out that such a map is in fact an embedding. But we had just proved in Proposition 92.12 that such an embedding cannot exist. ■

Remark.

(1) The above proof of Proposition 93.15 is also explained in the following video:

`https://www.youtube.com/watch?v=AmgkSdhK4K8`

(2) There are many variations on Question 93.13. For example, given $r \neq 1$ one can ask whether every Jordan curve admits an inscribed rectangle with aspect ratio r. Even for smooth curves this had been an open problem for a long time. The first real progress since the work by Schnirelmann [**Schni44**] happened in 2018 when Cole Hugelmeyer [**Hug18**] showed that a deep result in knot theory, due to Josh Batson [**Bats14**], implies that every smooth Jordan curve admits an inscribed rectangle of aspect ratio $\sqrt{3}$. The key idea of [**Hug18**] is rather similar to the proof of Proposition 93.15. Even more impressively, an affirmative answer for smooth curves and any aspect ratio was given in 2020 by Josh Greene and Andrew Lobb [**GrL20**].

Exercises for Chapter 93.

Exercise 93.1. Let $f\colon \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}$ be a function, let $x \in \mathbb{C}$ and let $z \in \mathbb{C} \setminus \{0\}$. Show that there exists a $\theta \in [0, \frac{\pi}{2}]$ such that

$$f\big(x + e^{\mathrm{i}\theta} z\big) \;=\; f\big(x + e^{\mathrm{i}(\theta+\frac{\pi}{2})} z\big) \;=\; f\big(x + e^{\mathrm{i}(\theta+\pi)} z\big) \;=\; f\big(x + e^{\mathrm{i}(\theta+\frac{3\pi}{2})} z\big).$$

Exercise 93.2. Let X be a non-empty compact subset of $\mathbb{R}^2$ of diameter d. Does there exist a $P \in \mathbb{R}^2$ such that $X \subset \overline{B}^2_{\frac{d}{2}}(P)$?

Exercise 93.3. We consider the solid torus $T = \overline{B}^2 \times S^1$. Let $f\colon T \to \mathbb{R}$ be a map such that for any $(x, \gamma) \in T$ we have $f(x, -\gamma) = -f(x, \gamma)$. We write $A := \{(x, \gamma) \in T \mid f(x, \gamma) = 0\}$.

(a) Show that the inclusion induced map $\mathrm{H}_1(T \setminus A; \mathbb{Z}_2) \to \mathrm{H}_1(T; \mathbb{Z}_2)$ is the zero map.
(b) Give an example of a map $f\colon T \to \mathbb{R}$ as above such that the inclusion induced map $\mathrm{H}_1(T \setminus A; \mathbb{Z}) \to \mathrm{H}_1(T; \mathbb{Z})$ is *not* the zero map.

In this chapter we will introduce the algebraic intersection number of two oriented submanifolds of complementary dimensions of a given smooth manifold M. In Theorem 94.5 we relate this algebraic intersection number to the cup product of suitable cohomology classes of M. This result can be used to compute cup products in a geometric and often entertaining way. Conversely this result can also be used to prove deep statements about intersections of submanifolds.

94.1. **Algebraic intersection numbers of submanifolds.** Before we introduce the algebraic intersection number of submanifolds we want to recall a definition from page 350 and we want to give a motivating question.

Definition. Let M be a smooth manifold.

(1) We say that two oriented proper submanifolds Z_0 and Z_1 of M are *properly smoothly isotopic* if there exists a proper smooth isotopy between Z_0 and Z_1, i.e. if there exists a proper smooth isotopy $F\colon Z_0 \times [0,1] \to M$ with $F_0 = \mathrm{id}_{Z_0}$ and such that $F_1\colon Z_0 \to Z_1$ is an orientation-preserving diffeomorphism.
(2) Let X and Y be two oriented proper submanifolds of M. We say that X *and* Y *can be made disjoint* if there exist oriented proper *disjoint* submanifolds $\widetilde{X}$ and $\widetilde{Y}$ of M such that X and $\widetilde{X}$ and also Y and $\widetilde{Y}$ are properly smoothly isotopic.

Example. We consider the three submanifolds X, Y and Z of the torus that are illustrated in Figure 1299 on the left. As is shown in the same figure on the right, one can make X and Y disjoint.

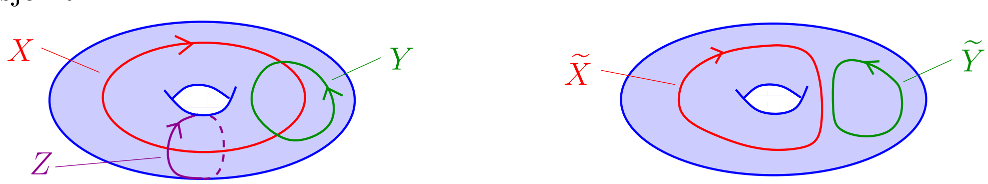


Figure 1299

It does not look like we can make the submanifolds X and Z disjoint. But how can we prove it? We will record this question for later.

Question 94.1. *Can we make the submanifolds X and Z, that are illustrated in Figure 1299, disjoint?*

Our approach to addressing Question 94.1 is to introduce the "algebraic intersection number". Before we can introduce this notion we need to recall the following definition from page 306.

Definition. Let M be an m-dimensional smooth manifold. Let X and Y be proper submanifolds of M.

(1) Let $P \in X \cap Y$. We say P is a *transverse intersection point* if $P \notin \partial M$ and if X and Y *intersect transversally in* P, i.e. if the equality $\mathrm{T}_P X + \mathrm{T}_P Y = \mathrm{T}_P M$ holds.

(2) We say that X and Y *intersect transversally* if each intersection point is a transverse intersection point.

Lemma 94.2. *Let M be an m-dimensional smooth manifold, let $X \subset M$ be a proper submanifold of dimension k and let $Y \subset M$ be a proper submanifold of the complementary dimension $m-k$. If X and Y intersect transversally and if X and Y are both compact, then the intersection $X \cap Y$ consists of finitely many points.*

Proof (∗). By Lemma 6.52 we know that $X \cap Y$ is a 0-dimensional submanifold of M. It follows easily from the definition of a 0-dimensional submanifold that $X \cap Y$ is a discrete subset of M. Since X and Y are compact we know by Lemma 2.16 (2) that $X \cap Y$ is also compact. It now follows from Lemma 2.18 that $X \cap Y$ is finite. ■

Definition. Let $m \in \mathbb{N}$ and let $k \in \{1, \dots, m-1\}$. Furthermore let M be an m-dimensional oriented smooth manifold, let $X \subset M$ be a compact oriented proper submanifold of dimension k and let $Y \subset M$ be a compact oriented proper submanifold of the complementary dimension $m-k$. Suppose that X and Y intersect transversally.

(1) Let P be an intersection point of X and Y. We set $\operatorname{sign}(P) = 1$, if

$$(\text{positive basis of } \mathrm{T}_P X, \text{positive basis of } \mathrm{T}_P Y) \;=\; \text{positive basis of } \mathrm{T}_P M.$$

Otherwise we se $\operatorname{sign}(P) = -1$.[1325]

(2) We define[1326]

$$\begin{aligned} \textit{geometric intersection number} \text{ of } X \text{ and } Y \;&:=\; \#(X \cap Y) \\ \text{and} \\ \textit{algebraic intersection number} \text{ of } X \text{ and } Y \;&:=\; X \cdot Y \;:=\; \sum_{P \in X \cap Y} \operatorname{sign}(P) \in \mathbb{Z}. \end{aligned}$$

If X and Y do not intersect, then the algebraic intersection number of X and Y is of course zero.

Examples.

(1) In Figure 1300 we show pieces of oriented submanifolds of the oriented smooth manifold $\mathbb{R}^2$. On the left the sign of the intersection point is positive, on the right it is negative.

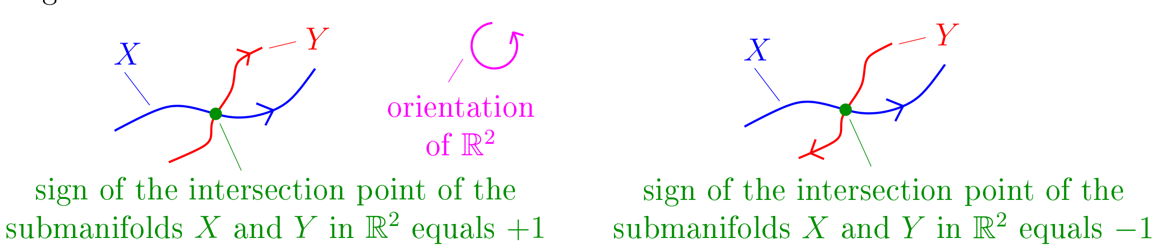


Figure 1300

[1325] It is a straightforward exercise in linear algebra to show that the definition of $\operatorname{sign}(P)$ does not depend on the choice of the positive bases of $\mathrm{T}_P X$ and $\mathrm{T}_P Y$.

[1326] It follows from Lemma 94.2 that the geometric intersection number is finite and that the algebraic intersection number is given by a finite sum.

(2) We consider the surface of genus 2 with the orientation where the normal vector "sticks outward". Furthermore we consider the oriented submanifolds A, B, C, D and E that are illustrated in Figure 1301. For the submanifolds A and B we have

$$\text{geometric intersection number of } A \text{ and } B \;=\; 1,$$

the sign of the intersection point is negative, hence

$$\text{algebraic intersection number of } A \text{ and } B \;=\; -1.$$

For the submanifolds A and C we have

$$\text{geometric intersection number of } A \text{ and } C \;=\; 2,$$

the intersection points have opposite signs, therefore

$$\text{algebraic intersection number of } A \text{ and } C \;=\; 0.$$

Finally for the submanifolds D and E we have

$$\text{geometric intersection number of } D \text{ and } E \;=\; 2,$$

here both intersection points have negative sign, thus

$$\text{algebraic intersection number of } D \text{ and } E \;=\; -2.$$

It is particularly instructive to consider the submanifolds A and C. They intersect geometrically in two points, but the signs are opposite. Note that these two intersection points also "vanish" if we "pull A and C apart" by a smooth isotopy.

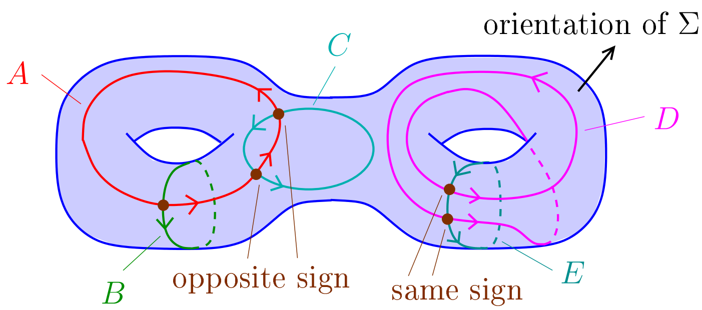


FIGURE 1301

(3) Now suppose that M and N are oriented smooth manifolds of dimensions m respectively n. We saw in Propositions 6.51 (1) and 80.9 that $M \times N$ is naturally[1327] an $(m+n)$-dimensional smooth manifold. It is straightforward to show that for each $P \in M \setminus \partial M$ and $Q \in N \setminus \partial N$ the natural map[1328]

$$\mathrm{T}_P\, M \oplus \mathrm{T}_Q\, N \;\to\; \mathrm{T}_{(P,Q)}(M \times N)$$

is an isomorphism. If $\{v_1, \ldots, v_m\}$ is a positive basis for $\mathrm{T}_P\, M$ and $\{w_1, \ldots, w_n\}$ is a positive basis for $\mathrm{T}_Q\, N$, then we endow $\mathrm{T}_{(P,Q)}(M \times N)$ with the orientation that makes the above isomorphism orientation-preserving.[1329]

Given $P \in M$ and $Q \in N$ it follows easily from the definitions that

$$\text{algebraic intersection number of } M \times \{Q\} \text{ and } \{P\} \times N \;=\; 1.$$

[1327] What does he mean by "naturally"?

[1328] What is the natural map?

[1329] One can easily verify that these orientations on the tangent spaces of $M \times N$ define an orientation on $M \times N$. In fact it follows fairly easily from Proposition 80.11 that this definition of the product orientation agrees with the definition we gave on page 1969.

In Figure 1302 we illustrate the statement that in the torus $S^1 \times S^1$ we have

$$\text{algebraic intersection number of } S^1 \times \{1\} \text{ and } \{1\} \times S^1 \; = \; 1.$$

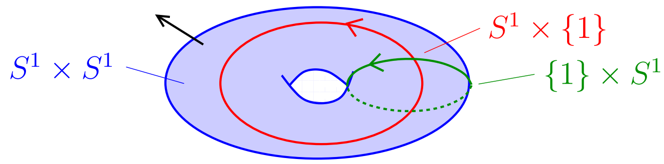


FIGURE 1302

(4) In Section 94.4 we will study the algebraic intersection number of complex submanifolds of a given complex manifold.
(5) The above example (2) suggests that if two submanifolds have non-zero algebraic intersection number, then they cannot be made disjoint. As we will see in Proposition 94.6, this hunch is basically correct. But if we deal with submanifolds with boundary one needs to think carefully about the correct setting. For example in Figure 1303 on the left we show a smooth manifold M and we show two proper submanifolds X and Y such that the boundaries ∂X and ∂Y lie on the same component of ∂M. The algebraic intersection number is $+1$, but as we see in Figure 1303 on the right, these two submanifolds can be made disjoint. Therefore in the following discussion we will always assume that the boundary components of X and Y lie in different boundary components of M.

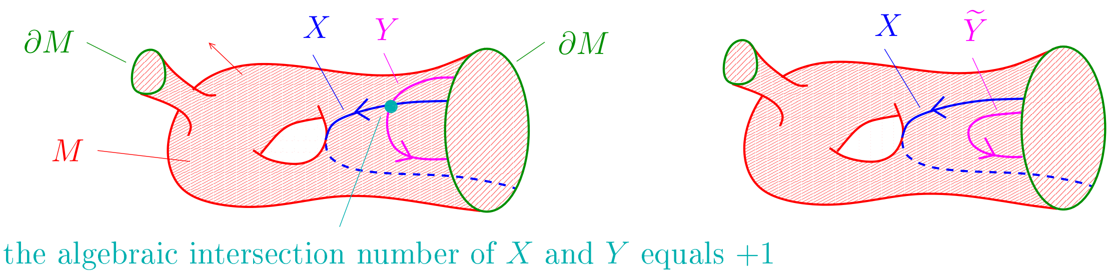


FIGURE 1303

Before we continue, and in light of the last example, we introduce the following definition.

Definition. Let M be a compact oriented smooth manifold.

(1) A *boundary decomposition for M* is a decomposition $\partial M = A \cup B$ where A and B are unions of components of ∂M and $A \cap B = \varnothing$.
(2) Let X and Y be two submanifolds of M. We say *X and Y are complementary* if the following five conditions are satisfied:
   (a) X and Y are proper non-empty submanifolds,
   (b) there exists a boundary decomposition $\partial M = A \cup B$ with $\partial X \subset A$ and $\partial Y \subset B$,
   (c) both X and Y are oriented,
   (d) we have $\dim(X) + \dim(Y) = \dim(M)$ and we have $\dim(X), \dim(Y) \geq 1$,
   (e) the submanifolds X and Y intersect transversally.

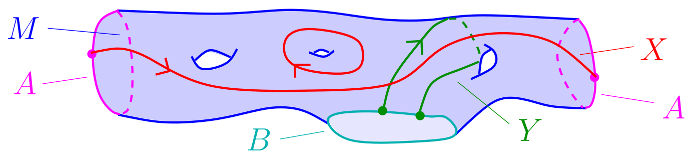


FIGURE 1304

We would be able to answer Question 94.1 if we could prove the following statement.

Statement 94.3. *Let M be a compact oriented smooth manifold and let $\{X, Y\}$ and $\{X', Y'\}$ be two pairs of complementary submanifolds of M. Then the following holds:*

$$\begin{array}{c} X, X' \text{ are properly smoothly isotopic and} \\ Y, Y' \text{ are properly smoothly isotopic} \end{array} \implies X \cdot Y = X' \cdot Y'.$$

But how can we prove such a statement? In Figure 1305 we sketch an idea: We break the smooth isotopies from X to X' and from Y to Y' into "smaller isotopies" such that each smaller isotopy either does not change the type of intersection points, or two intersection points of opposite sign get created or erased. Even though the idea is fairly convincing, it is very hard to turn this idea into a proper proof.[1330][1331] So for the time being we will not attempt to provide a proof for Statement 94.3.

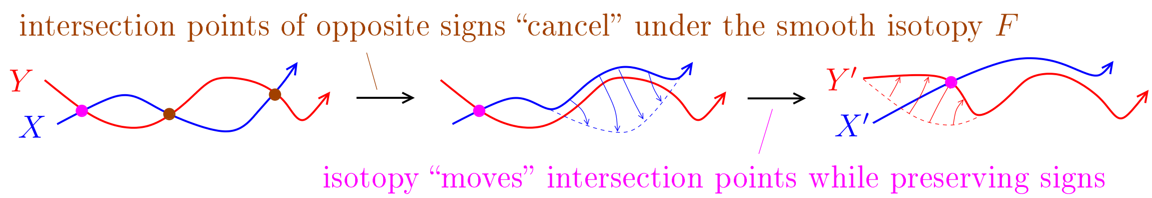


FIGURE 1305

We conclude this introduction to algebraic intersection numbers with the following elementary lemma.

Lemma 94.4. *Let M be a compact oriented smooth manifold. If X and Y are complementary submanifolds of M, then*

$$Y \cdot X \;=\; (-1)^{\dim(X)\cdot\dim(Y)} \cdot X \cdot Y.$$

SKETCH OF PROOF. The statement follows easily from the definitions and the observation that the matrix that represents the linear homeomorphism that swaps the first k vectors with the last $(m-k)$ vectors of a basis of an m-dimensional vector space is represented by the matrix $\begin{pmatrix} 0 & \mathrm{id}_k \\ \mathrm{id}_{m-k} & 0 \end{pmatrix}$. One easily verifies that the determinant of this matrix equals $(-1)^{k(m-k)}$. ■

[1330]You will see it is already hard to prove the statement in dimension 2, let alone in higher dimensions. In fact I am not aware of a direct geometric proof of Statement 94.3.

[1331]Even though we will not follow this approach, it was certainly this type of picture mathematicians had in mind when they developed the concept of algebraic intersection numbers in the early 20th century.

94.2. **The cup product and algebraic intersection numbers.** We recall the following notation.

Notation.

(1) Given a compact oriented n-dimensional smooth manifold W we denote as usual by $[W] \in \mathrm{H}_n(W, \partial W)$ the fundamental class that we introduced on page 1712 or alternatively on page 2120. If W is a proper submanifold of some topological manifold M, then sometimes we denote by $[W] \in \mathrm{H}_n(M, \partial M)$ the image of $[W]$ under the inclusion induced map $\mathrm{H}_n(W, \partial W) \to \mathrm{H}_n(M, \partial M)$.

(2) Let M be a compact oriented n-dimensional smooth manifold and suppose that we are given a boundary decomposition $\partial M = A \cup B$. By the Poincaré Duality Theorem 88.1 we know that for each $k \in \mathbb{N}_0$ the map

$$\mathrm{H}^k(M, A) \xrightarrow{\cap [M]} \mathrm{H}_{n-k}(M, B)$$

is an isomorphism. We denote by

$$\mathrm{PD}_M \colon \mathrm{H}_{n-k}(M, B) \longrightarrow \mathrm{H}^k(M, A)$$

its inverse.[1332] If M is understood from the context then often we just write PD instead of PD_M.

The following theorem relates the cup product on a smooth manifold to algebraic intersection numbers. Or conversely, it expresses the algebraic intersection number of submanifolds as a cup product.

Theorem 94.5. *Let M be a compact oriented m-dimensional smooth manifold together with a boundary decomposition $\partial M = A \cup B$. Let X and Y be a complementary pair of submanifolds of M with $\partial X \subset A$ and $\partial Y \subset B$. We write $k = \dim(X)$ and we denote by $i \colon X \to M$ and $j \colon Y \to M$ the obvious inclusion maps. Furthermore we denote by $[X] \in \mathrm{H}_k(X, \partial X)$ and $[Y] \in \mathrm{H}_{m-k}(Y, \partial Y)$ the fundamental classes of X and Y. Then*

$$X \cdot Y = \Big\langle \overbrace{\underbrace{\mathrm{PD}_M(\underbrace{i_*([X])}_{\in \mathrm{H}_k(M,A)})}_{\in \mathrm{H}^{m-k}(M,B)} \cup \underbrace{\mathrm{PD}_M(\underbrace{j_*([Y])}_{\in \mathrm{H}_{m-k}(M,B)})}_{\in \mathrm{H}^k(M,A)}}^{\in \mathrm{H}^m(M,\partial M)}, \overbrace{[M]}^{\in \mathrm{H}_m(M,\partial M)} \Big\rangle \in \mathbb{Z}.$$

In particular with the abbreviation $[X] = i_*([X])$ and $[Y] := j_*([Y])$ we have the slightly more readable equality

$$X \cdot Y = \langle \mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]), [M] \rangle.$$

Remark.

(1) As a reality check, note that a short moment's thought shows that the (anti-) commutativity of the algebraic intersection number of Lemma 94.4 does indeed correspond precisely to the (anti-) commutativity of the cup product from Proposition 82.3.

[1332]If M is closed this statement is also contained in the Poincaré Duality Theorem 71.5.

(2) The following citation from [**Kir89**, p. 21] encapsulates the roles of algebraic intersection numbers and cup products:

"Think with [algebraic] intersection [numbers], prove with cup products."

More precisely, algebraic intersection numbers are very intuitive and often they can be used to guess a statement. But when it comes to proving statements, e.g. why does the intersection number only depend on the smooth isotopy type of a submanifold?, it is usually much easier to work with cup products.

(3) Alternative proofs of Theorem 94.5 are given in [**GH81**, Proposition 31.7], [**Bre93**, Theorem VI.11.9] and [**DaK01**, Theorem 10.32]. Furthermore a quick outline of a proof is given in [**GoS99**, Proposition 1.2.5].

(4) One could use the right-hand side of Theorem 94.5 as the definition of the algebraic intersection number. We will discuss this idea in Exercise 94.3.

(5) If we drop the condition that M, X and Y are orientable, then we can still define an algebraic intersection number $X \cdot_{\mathbb{F}_2} Y$ and we can formulate Poincaré Duality with $\mathbb{F}_2$-coefficients. This allows us to formulate a variation on Theorem 94.5. To save the trees we will not spell out the details at this point, instead we refer to the more general Theorem 95.15 for details.

Before we move on to the somewhat lengthy proof of Theorem 94.5 we want to convince ourselves that proving the theorem is worth the effort.

One application of Theorem 94.5 is that now we can provide a proof of a strengthened version of Statement 94.3. More precisely, we have the following proposition.

Proposition 94.6. *Let M be a compact oriented smooth manifold.*

(1) *Let $\{X, Y\}$ and $\{X', Y'\}$ be pairs of complementary submanifolds of M. If X, X' are properly smoothly isotopic and if Y, Y' are also properly smoothly isotopic, then $X \cdot Y = X' \cdot Y'$.*

(2) *If X and Y are two complementary submanifolds of M that can be made disjoint, then $X \cdot Y = 0$.*

Example. Now we can answer Question 94.1. It follows from Proposition 94.6 that the submanifolds X and Z, that are illustrated in Figure 1299, cannot be made disjoint since their algebraic intersection number is non-zero.

Proof. Let M be a compact oriented smooth manifold. We start out with the following claim.

Claim. Let Z and Z' be oriented proper s-dimensional submanifolds of M. If Z and Z' are properly smoothly isotopic, then $[Z] = [Z'] \in \mathrm{H}_s(M, \partial M)$.

We denote by $i\colon Z \to M$ and $i'\colon Z' \to M$ the inclusion maps. Since Z and Z' are properly smoothly isotopic there exists in particular an orientation-preserving diffeomorphism $f\colon Z \to Z'$ such that the maps $i, i' \circ f\colon (Z, \partial Z) \to (M, \partial M)$ are homotopic. We obtain that

$$[Z] \;=\; i_*([Z]) \underset{\substack{\uparrow \\ \text{Proposition 43.17}}}{=} (i' \circ f)_*([Z]) = i'_*(f_*([Z])) \underset{\substack{\uparrow \\ \text{by Proposition 72.2 since } f \text{ is an} \\ \text{orientation-preserving homeomorphism}}}{=} i'_*([Z']) \;=\; [Z']$$

⊞

Now let $\{X, Y\}$ and $\{X', Y'\}$ be two pairs of complementary submanifolds of M. We assume that the submanifolds X, X' are properly smoothly isotopic and that the smooth manifolds Y, Y' are smoothly isotopic. We obtain that

$$X \cdot Y \underset{\substack{\uparrow \\ \text{Theorem 94.5}}}{=} \langle \mathrm{PD}_M([X]), \mathrm{PD}_M([Y]), [M] \rangle \underset{\substack{\uparrow \\ \text{by the claim}}}{=} \langle \mathrm{PD}_M([X']), \mathrm{PD}_M([Y']), [M] \rangle \underset{\substack{\uparrow \\ \text{Theorem 94.5}}}{=} X' \cdot Y'.$$

This concludes the proof of the first statement. The second statement is an immediate consequence of the first statement. ■

Now we will use Theorem 94.5 "the other way", i.e. we will apply the theorem to use the algebraic intersection number to draw conclusions on (co-) homology classes and to determine cup products using our geometric understanding of submanifolds.

Corollary 94.7. *Let M be a compact oriented connected m-dimensional smooth manifold together with a boundary decomposition $\partial M = A \cup B$. Let X be a k-dimensional proper submanifold of M with $\partial X \subset A$. If there exists a proper submanifold Y with $\partial Y \subset B$ such that X and Y form a complementary pair and if $X \cdot Y = \pm 1$, then we can write $\mathrm{H}_k(M, A) = \mathbb{Z} \cdot [X] \oplus K$ for some abelian group K. In particular we see that $[X]$ is a primitive element of $\mathrm{H}_k(M, A)$.*

Proof. We consider the homomorphism

$$\begin{array}{rccl} \Theta\colon & \mathrm{H}_k(M, A) & \to & \mathbb{Z} \\ & \varphi & \mapsto & \langle \mathrm{PD}_M(\varphi) \cup \mathrm{PD}_M([Y]), [M] \rangle. \end{array}$$

It follows from our hypothesis and Theorem 94.5 that $\Theta([X]) = \pm 1$. The corollary now follows from the elementary discussion on page 1208. ■

Examples.

(1) Let M be the complement in S^3 of the two open 3-dimensional tori shown in Figure 1306 on the left. On the right of that figure we show an annulus X that is properly embedded in M. On the right we also see a closed 1-dimensional submanifold Y of M. The two submanifolds X and Y are clearly transverse and they intersect in one point. So with whatever orientation, we have $X \cdot Y = \pm 1$. By Corollary 94.7 this implies in particular that $[X] \in \mathrm{H}_2(M, \partial M)$ is a primitive element.

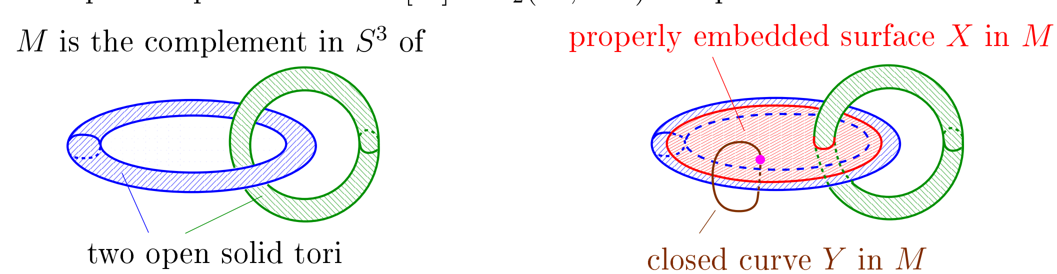


Figure 1306

(2) This time let M be the complement in S^3 of the two open 3-dimensional tori shown in Figure 1307 on the left. Almost the same argument as in (1) shows that the surface X defines a primitive element in $\mathrm{H}_2(M)$.

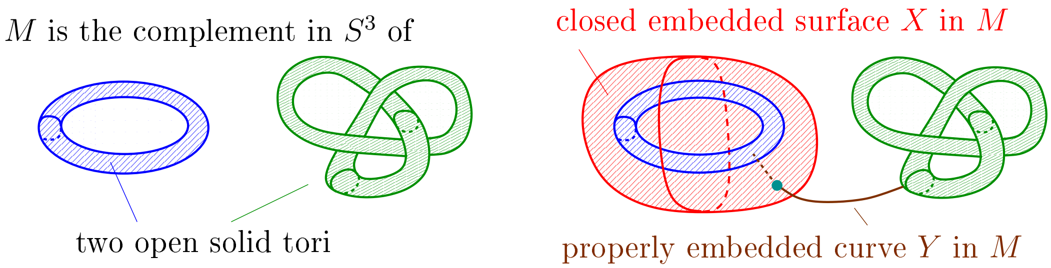


FIGURE 1307

In the next example we will show how to use Theorem 94.5 to compute cup products.

Example. We consider the torus $T = S^1 \times S^1$ together with the submanifolds $X = S^1 \times \{1\}$ and $Y = \{1\} \times S^1$ which we endow with the usual orientations. By now we have many ways[1333] for showing that $\mathrm{PD}([X])$ and $\mathrm{PD}([Y])$ form a basis for $\mathrm{H}^1(T)$. Thus if we want to understand the cup product on the torus, it suffices to understand the cup products involving these two cohomology classes. We calculate that

$$-\langle \mathrm{PD}([Y]) \cup \mathrm{PD}([X]), [S^1 \times S^1]\rangle \underset{\substack{\uparrow \\ \text{Proposition 81.8}}}{=} \langle \mathrm{PD}([X]) \cup \mathrm{PD}([Y]), [S^1 \times S^1]\rangle \underset{\substack{\uparrow \\ \text{Theorem 94.5}}}{=} X \cdot Y \underset{\substack{\uparrow \\ \text{Figure 1308}}}{=} +1.$$

Next we want to calculate the "square" $\mathrm{PD}([X]) \cup \mathrm{PD}([X])$. This requires a little trick since evidently X is not transverse to itself. To deal with this issue we consider a "parallel copy" of X, namely we consider $X' = S^1 \times \{z\}$ with $z \neq 1$ We calculate that

$$\langle \mathrm{PD}([X]) \cup \mathrm{PD}([X]), [S^1 \times S^1]\rangle \underset{\substack{\uparrow \\ \text{by the claim in the proof of Proposition 94.6}}}{=} \langle \mathrm{PD}([X]) \cup \mathrm{PD}([X']), [S^1 \times S^1]\rangle \underset{\substack{\uparrow \\ \text{by Theorem 94.5, since } X \text{ and } X' \text{ are} \\ \text{disjoint, in particular transverse}}}{=} X \cdot X' \underset{\uparrow}{=} 0.$$

and with the same argument we show that $\langle \mathrm{PD}([Y]) \cup \mathrm{PD}([Y]), [S^1 \times S^1]\rangle = 0$. We will now convince ourselves that these calculations match our previous calculations. To do so we denote by p_1 respectively p_2 the projection $S^1 \times S^1 \to S^1$ onto the first respectively the second factor and we denote by $\theta := \theta_{\mathbb{Z}} \in \mathrm{C}^1; \mathbb{Z})$ the singular 1-cocycle that we introduced on page 1825. Now we see that

$$\begin{pmatrix} \mathrm{PD}([X]) \cup \mathrm{PD}([X]) & \mathrm{PD}([X]) \cup \mathrm{PD}([Y]) \\ \mathrm{PD}([Y]) \cup \mathrm{PD}([X]) & \mathrm{PD}([Y]) \cup \mathrm{PD}([Y]) \end{pmatrix} \underset{\substack{\uparrow \\ \text{on page 2022 we already showed that} \\ [p_1^*\theta] \cap [T] = [1 \times S^1] = [Y] \text{ and } [p_2^*\theta] \cap [T] = -[S^1 \times 1] = -[X]}}{=} \begin{pmatrix} [p_2^*\theta] \cup [p_2^*\theta] & -[p_2^*\theta] \cup [p_1^*\theta] \\ -[p_1^*\theta] \cup [p_2^*\theta] & [p_1^*\theta] \cup [p_1^*\theta] \end{pmatrix} \underset{\substack{\uparrow \\ \text{see page 2022}}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since this was so much fun, let us consider one more example.

[1333] So, how many ways do you know?

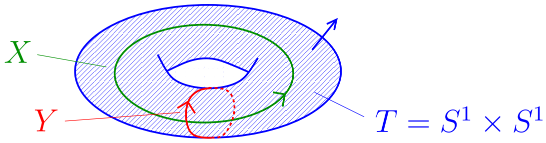


FIGURE 1308

Example. Let Σ be the surface of genus two. We consider the following bilinear pairing:[1334]

$$\begin{array}{rcl} Q\colon\ \mathrm{H}_1(\Sigma)\times \mathrm{H}_1(\Sigma) & \to & \mathbb{Z} \\ (\sigma,\tau) & \mapsto & \langle \mathrm{PD}_\Sigma(\sigma)\cup \mathrm{PD}_\Sigma(\tau),[\Sigma]\rangle. \end{array}$$

Recall that by Proposition 48.9 we know that $\mathrm{H}_1(\Sigma)\cong\mathbb{Z}^4$. Next we consider the closed oriented curves a_1,b_1,a_2,b_2 that are shown in Figure 1309. It follows easily from Theorem 94.5 that[1335]

$$\begin{pmatrix} Q([a_1],[a_1]) & Q([a_1],[b_1]) & Q([a_1],[a_2]) & Q([a_1],[b_2]) \\ Q([b_1],[a_1]) & Q([b_1],[b_1]) & Q([b_1],[a_2]) & Q([b_1],[b_2]) \\ Q([a_2],[a_1]) & Q([a_2],[b_1]) & Q([a_2],[a_2]) & Q([a_2],[b_2]) \\ Q([b_2],[a_1]) & Q([b_2],[b_1]) & Q([b_2],[a_2]) & Q([b_2],[b_2]) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Note that the determinant of the matrix to the right is one. Since Q is bilinear we obtain from Lemma 74.8 a new proof that $[a_1]$, $[b_1]$, $[a_2]$, $[b_2]$ is a basis for $H_1(\Sigma)$.[1336] By the Poincaré Duality Theorem 88.1 together with Theorem 94.5 we have now also computed the cup product $\mathrm{H}^1(\Sigma;\mathbb{Z})\times\mathrm{H}^1(\Sigma;\mathbb{Z})\to\mathrm{H}^2(\Sigma;\mathbb{Z})$.

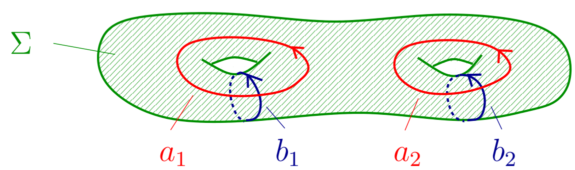


FIGURE 1309

We will do one more calculation of the cup product right after the proof of Theorem 94.5.

We conclude this lengthy discussion of the merits of Theorem 94.5 with the following definition that generalizes the concept introduced on page

Definition. Let $m\in\mathbb{N}$ and let M be a compact oriented m-dimensional smooth manifold together with a boundary decomposition $\partial M = A\cup B$. Furthermore let $k\in\{1,\dots,m-1\}$, let X be a compact oriented proper k-dimensional submanifold of M with $\partial X\subset A$ and let Y be a compact oriented proper $(m-k)$-dimensional submanifold of M with $\partial Y\subset B$.

[1334]Later on, on page 2413 we will refer to this map as the *intersection form of* Σ and we will introduce the intersection form of any closed oriented even-dimensional topological manifold.

[1335]For the diagonal terms we use the "little trick" from the previous example, namely we use a "parallel copy" of the curves a_1,b_1,a_2,b_2.

[1336]We gave a different argument on page 1270.

We denote by $i\colon X \to M$ and $j\colon Y \to M$ the inclusion maps. Then we refer to

$$\Big\langle \overbrace{\underbrace{\mathrm{PD}_M(\underbrace{i_*([X])}_{\in \mathrm{H}_k(M,A)})}_{\in \mathrm{H}^{m-k}(M,B)} \cup \underbrace{\mathrm{PD}_M(\underbrace{j_*([Y])}_{\in \mathrm{H}_{m-k}(M,B)})}_{\in \mathrm{H}^{k}(M,A)}}^{\in \mathrm{H}^m(M,\partial M)},\ \overbrace{[M]}^{\in \mathrm{H}_m(M,\partial M)} \Big\rangle \in \mathbb{Z}$$

as the *algebraic intersection number of X and Y*. By Theorem 94.5 this definition is consistent with the earlier definition on page 2271.

94.3. **Proof of Theorem 94.5.** In this section we will prove Theorem 94.5. Our proof of Theorem 94.5 relies heavily on Lemmas 89.5 and 89.6 that relate Poincaré duality in an oriented topological manifold M to Poincaré duality in a compact codimension 0 submanifold W. For the reader's convenience we recall Lemma 89.5. This lemma says that given a homology class in W we can apply Poincaré duality in M or W and essentially we end up with the same result:

Lemma 89.5. *Let M be a compact oriented m-dimensional topological manifold and let W be a compact m-dimensional submanifold with $\partial M \cap \partial W = \varnothing$. We write $\mathring{W} = W \setminus \partial W$. We fix the following notation:*

1. *We denote by w the inclusion map $W \to M$ and also the inclusion map of the pairs $(W, \partial W) \to (M, M \setminus \mathring{W})$.*
2. *We denote by $p\colon (M, \partial M) \to (M, M \setminus \mathring{W})$ the inclusion map of pairs.*

For every $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}_k(W;\mathbb{Z}) \arrow[r, "\mathrm{PD}_W"] \arrow[d, "w_*"] & \mathrm{H}^{m-k}(W,\partial W;\mathbb{Z}) \arrow[r, "(w^*)^{-1}", "\cong"'] & \mathrm{H}^{m-k}(M, M\setminus \mathring{W};\mathbb{Z}) \arrow[d, "p^*"] \\
\mathrm{H}_k(M;\mathbb{Z}) \arrow[rr, "\mathrm{PD}_M"] & & \mathrm{H}^{m-k}(M,\partial M;\mathbb{Z}).
\end{tikzcd}$$

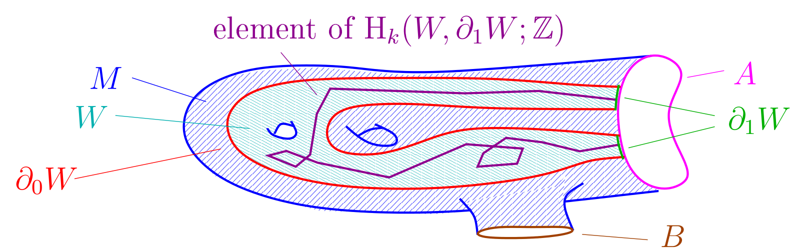


Figure 1310. Illustration of Lemma 89.5.

We take a short rest from more technicalities and enjoy the fact that we can already prove Theorem 94.5 in the special case that M is closed and that the submanifolds *do not intersect*.

Proof of Theorem 94.5 for disjoint submanifolds. Let M be a closed oriented m-dimensional smooth manifold and let X and Y be a complementary pair of *disjoint* submanifolds of M. Since the submanifolds are disjoint it suffices to show that the cup product of the Poincaré duals of $[X]$ and $[Y]$ is also zero.

We write $k = \dim(X)$. We apply the General Tubular Neighborhood Theorem 10.5 to the submanifold $X \cup Y$ of M. We obtain disjoint tubular neighborhoods A for X and B for Y. We denote by $p\colon (M, \varnothing) \to (M, M \setminus \mathring{A})$ and $p\colon (M, \varnothing) \to (M, M \setminus \mathring{B})$ the obvious inclusion maps of pairs of topological spaces.

We apply Lemma 89.5 to the codimension-zero submanifold A and the homology class $[X] \in \mathrm{H}_k(A; \mathbb{Z})$. We obtain an $\alpha \in \mathrm{H}^{m-k}(M, M \setminus \mathring{A}; \mathbb{Z})$ with $p^*(\alpha) = \mathrm{PD}_M([X])$. Similarly we see that there exists a $\beta \in \mathrm{H}^k(M, M \setminus \mathring{B}; \mathbb{Z})$ with $p^*(\beta) = \mathrm{PD}_M([Y])$.

We consider the following diagram

$$\begin{tikzcd}
\mathrm{H}^{m-k}(M;\mathbb{Z}) \times \mathrm{H}^k(M;\mathbb{Z}) \arrow[r, "\cup"] & \mathrm{H}^m(M;\mathbb{Z}) \\
\mathrm{H}^{m-k}(M, M\setminus\mathring{A};\mathbb{Z}) \times \mathrm{H}^k(M, M\setminus\mathring{B};\mathbb{Z}) \arrow[u, "p^*\times p^*"] \arrow[r, "\cup"] \arrow[u, phantom, "(\alpha,\beta) \mapsto (\mathrm{PD}_M([X]), \mathrm{PD}_M([Y]))"] & \mathrm{H}^m(M, \underbrace{(M\setminus\mathring{A}) \cup (M\setminus\mathring{B})}_{=M};\mathbb{Z}). \arrow[u]
\end{tikzcd}$$

The diagram commutes by Proposition 82.4 (1). It follows from the fact that the lower right group is zero that the cup product of $\mathrm{PD}_M([X])$ and $\mathrm{PD}_M([Y])$ is zero. ■

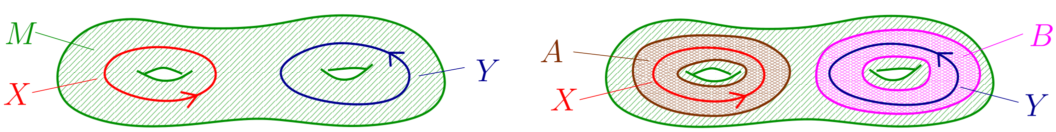


FIGURE 1311. Illustration for the proof of Theorem 94.5.

After this refreshing proof of the special case we continue with trying to prove the general case of Theorem 94.5. We will need Lemma 89.6. Since the statement is rather technical, we recall the statement of the special case that is discussed in example (a) on page 2188.

Lemma 89.6. *Let M be a compact oriented m-dimensional smooth manifold and let W be a compact m-dimensional submanifold with corner. We fix the following notation:*

1. *We write $\partial_1 W = W \cap \partial M$ and $\partial_0 W = \overline{\partial W \setminus \partial_1 W}$,*
2. *We write $\mathring{W} = W \setminus \partial_0 W$.*
3. *The inclusion maps $(W, \partial_0 W) \to (M, M\setminus\mathring{W})$ and $(W, \partial_1 W) \to (M, \partial M)$ are denoted by w.*
4. *Finally we denote by $p\colon (M, \varnothing) \to (M, M \setminus \mathring{W})$ the obvious map.*

For every $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}_k(M;\mathbb{Z}) \arrow[rr, "\mathrm{PD}_M"] \arrow[d, "p_*"] & & \mathrm{H}^{m-k}(M, \partial M;\mathbb{Z}) \arrow[d, "w^*"] \\
\mathrm{H}_k(M, M \setminus \mathring{W};\mathbb{Z}) \arrow[r, "(w_*)^{-1}", "\cong"'] & \mathrm{H}_k(W, \partial_0 W;\mathbb{Z}) \arrow[r, "\mathrm{PD}_W"] & \mathrm{H}^{m-k}(W, \partial_1 W;\mathbb{Z}).
\end{tikzcd}$$

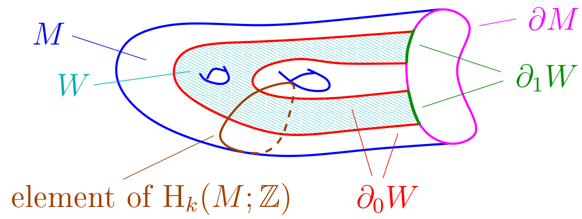


FIGURE 1312. Illustration of Lemma 89.6.

In the proof of Theorem 94.5 we will also need the following proposition.

Proposition 94.8. *Let M be a compact oriented smooth manifold and let X and Y be two complementary proper submanifolds of dimensions k and l. We denote by $P_1, \ldots, P_s$ the intersection points of X and Y. Then there exist*

(1) *compact submanifolds with corner A and B such that A is a neighborhood for X and B is a neighborhood for Y, and*
(2) *injective maps $\Psi_i \colon \overline{B}^k \times \overline{B}^l \to M$, $i = 1, \ldots, s$*

such that the following statements hold:

(a) *For each $i \in \{1, \ldots, s\}$ we have $\Psi_i((0,0)) = P_i$.*
(b) *The images $Q_i := \Psi_i\big(\overline{B}^k \times \overline{B}^l\big)$, $i = 1, \ldots, s$ are disjoint.*
(c) *We have $Q_1 \cup \cdots \cup Q_s = A \cap B$.*
(d) *For each $i \in \{1, \ldots, s\}$ the following statements hold:*
  (d1) *the restriction of Ψ_i to $B^k \times B^l$ is a diffeomorphism onto its image,*
  (d2) *we have $Q_i \cap X = \Psi_i(\overline{B}^k \times \{0\})$ and the map $\Psi_i \colon \overline{B}^k \times \{0\} \to X$ is an orientation-preserving smooth embedding,*
  (d3) *we have $Q_i \cap Y = \Psi_i(\{0\} \times \overline{B}^l)$ and the map $\Psi_i \colon \{0\} \times \overline{B}^l \to Y$ is an orientation-preserving smooth embedding.*

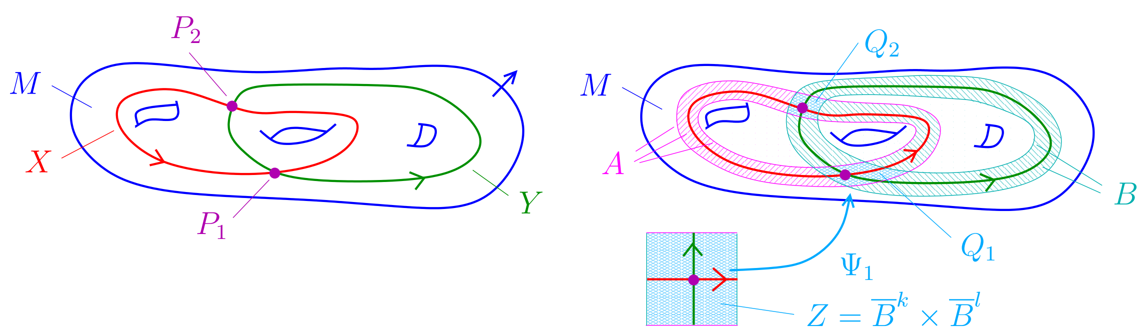


FIGURE 1313. Illustration of Proposition 94.8.

SKETCH OF A PROOF. Let M be a compact oriented n-dimensional smooth manifold and let X and Y be two complementary proper submanifolds of dimensions k and l. We denote by $P_1, \ldots, P_s$ the intersection points of X and Y. For $i = 1, \ldots, s$ we can and will pick smooth embeddings $\Psi_i \colon \overline{B}^n \to M \setminus \partial M$ such that the following conditions are satisfies:

(1) the images are disjoint,

(2) for each $i \in \{1, \dots, s\}$ the map Ψ_i restricts to orientation-preserving diffeomorphisms $B^k \times \{0\} \to \Psi_i(B^n) \cap X$ and $\{0\} \times B^l \to \Psi_i(B^n) \cap Y$.

We consider $W := M \setminus \bigcup_{i=1}^{s} \Psi_i(B^n)$. Note that $X' := W \cap X$ and $Y' := W \cap Y$ are proper submanifolds of the compact smooth manifold W. Thus we can apply the General Tubular Neighborhood Theorem 10.5 to obtain tubular neighborhoods for X' and Y'. Using Proposition 8.35 we can arrange the intersections of the tubular neighborhoods with $\Psi_i(S^{n-1})$ to be "standard $(n-1)$-dimensional balls". At this stage it is fairly straightforward to extend the tubular neighborhoods to the interiors of the $\Psi_i(B^n)$ and to obtain all the desired objects. We have faith in the unwavering enthusiasm of our readers for filling in the details. ■

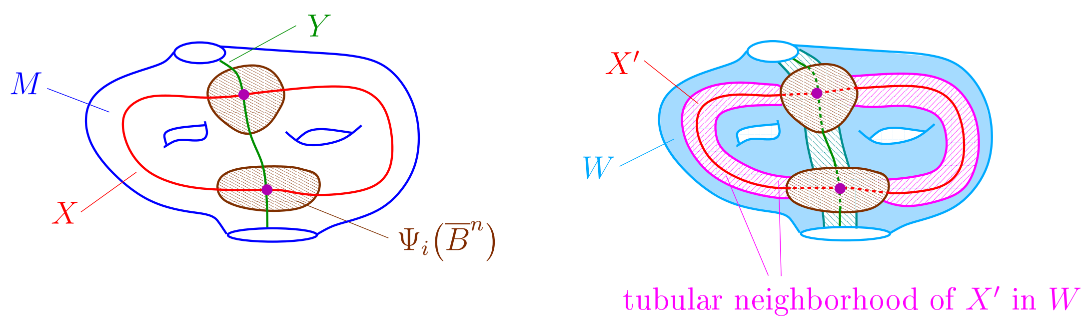


FIGURE 1314. Illustration of the proof of Proposition 94.8.

Remark. An alternative approach to Proposition 94.8 would be to use tubular neighborhoods provided by Lemma 10.12.

Given the situation of Proposition 94.8 we can make one more statement.

Lemma 94.9. *We use the notation of Proposition 94.8. Let $i \in \{1, \dots, s\}$. We equip the topological manifold Q_i, which is a codimension-zero submanifold of M, with the induced orientation coming from Lemma 86.17. Furthermore we equip the topological manifold $\overline{B}^k \times \overline{B}^l$ with the orientation coming from the standard orientation of $\mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l}$. With these choices the following statement holds:*

(f) *If we denote by σ_i the sign of the intersection point P_i, then*

$$\Psi_{i*}\big([\overline{B}^k \times \overline{B}^l]\big) \;=\; \sigma_i \cdot [Q_i].$$

PROOF. Let $i \in \{1, \dots, s\}$. If follows from (d2) and (d3) that at the point $(0,0)$ the diffeomorphism $\Psi_i \colon B^k \times B^l \to \mathring{Q}_i$ is orientation-preserving if and only if $\sigma_i = 1$. Now the statement follows easily from the definition of σ_i, Proposition 72.2 and Lemma 86.12. ■

The last preparation we need before we can provide the actual proof of Theorem 94.5 is Proposition 88.2, which for the reader's convenience we recall using our present notation.

Proposition 88.2. *Let X be a compact oriented connected m-dimensional topological manifold and let Y be a compact oriented connected n-dimensional topological manifold. We denote by $p \colon X \times Y \to X$ and $q \colon X \times Y \to Y$ the obvious projection maps. We equip*

$X \times Y$ with the product orientation introduced in Proposition 80.10. Given any points $P \in X$ and $Q \in Y$ we have the following equality in $\mathrm{H}^{m+n}(X \times Y, \partial(X \times Y); \mathbb{Z})$*:*

$$\mathrm{PD}_{X\times Y}([X \times \{Q\}]) \cup \mathrm{PD}_{X\times Y}([\{P\} \times Y]) \;=\; [X \times Y]^*.$$

We turn to the actual proof of Theorem 94.5.

Proof of Theorem 94.5. Let M be a compact oriented connected m-dimensional smooth manifold. To simplify the notation we assume that M is closed. The case that M has boundary is proved in a rather similar fashion.[1337]

Let X and Y be a complementary pair of submanifolds of M. We introduce the following notation:

(1) We write $k = \dim(X)$ and $l = \dim(Y)$. Note that $l = m - k$.
(2) As usual we denote by $[X] \in \mathrm{H}_k(X)$ and $[Y] \in \mathrm{H}_l(Y)$ the fundamental classes of X and Y. By a slight abuse of notation we denote by $[X]$ respectively $[Y]$ also the images of these homology classes in any subset of M that contains X respectively Y.
(3) We denote the intersection points of X and Y by $P_1, \dots, P_s$ and we denote the corresponding signs by $\sigma_1, \dots, \sigma_s$.
(4) As usual, given a topological space X we denote by $\epsilon_X = \epsilon\colon \mathrm{H}_0(X) \to \mathbb{Z}$ the augmentation map.

With the notation that we just introduced, and keeping in mind Lemma 83.4 (1), the statement of Theorem 94.5 can be reformulated as the following statement.

Statement.

$$\epsilon_M\big((\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y])) \cap [M]\big) \;=\; \sum_{i=1}^{s} \sigma_i.$$

We introduce many more objects, notations and conventions:

(5) We apply Proposition 94.8 to the submanifolds X and Y. In the remainder of the proof we use the notation that we introduced in that proposition. As a reminder, most of the relevant objects are illustrated in Figure 1313.
(6) We write $\mathring{A} = A \setminus \partial A$ and $\mathring{B} = B \setminus \partial B$.
(7) We denote all of the inclusion maps $(M, \varnothing) \to (M, M \setminus \mathring{A})$, $(M, \varnothing) \to (M, M \setminus \mathring{B})$ and $(M, \varnothing) \to (M, M \setminus (\mathring{A} \cap \mathring{B}))$ by p.
(8) We write $C = A \cap B = Q_1 \cup \dots \cup Q_s$. Furthermore we write $\partial_A C = \partial A \cap B$ and $\partial_B C = A \cap \partial B$. Note that $\partial C = \partial_A C \cup \partial_B C$. Also note that $(M \setminus \mathring{A}) \cup (M \setminus \mathring{B}) = M \setminus \mathring{C}$.
(9) We denote by $c\colon C \to M$ the inclusion map and we denote by c also all the corresponding inclusion maps of pairs where the first terms are C and M.
(10) Given $i \in \{1, \dots, s\}$ we write $\partial_A Q_i = \partial_A C \cap Q_i$ and $\partial_B Q_i = \partial_B C \cap Q_i$. Note that $\partial Q_i = \partial_A Q_i \cup \partial_B Q_i$.
(11) Given $i \in \{1, \dots, s\}$ we denote by $q_i\colon Q_i \to C$ the inclusion map and we denote by q_i also all the corresponding inclusion maps of pairs where the first terms are Q_i and C.
(12) Note that C and each Q_i are codimension-zero submanifolds of M. We equip these submanifolds with the corresponding orientation introduced in Lemma 86.17.

[1337] Admittedly, the case that M has boundary also requires the generalization of Lemma 89.5 to the case that W is a codimension-zero submanifold with corner. But that is not really a problem.

We illustrate some of the definitions in Figure 1315.

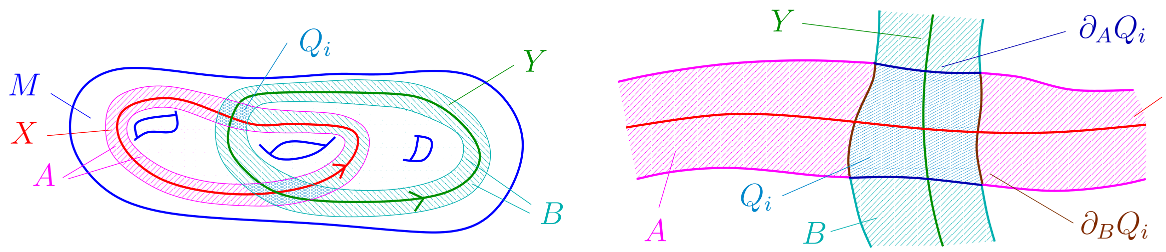


FIGURE 1315

Now we consider the following diagram:[1338]

$$\begin{tikzcd}
\mathrm{H}^l(M) \times \mathrm{H}^k(M) \arrow[r, "\cup"] & \mathrm{H}^m(M) \arrow[r, "\cap[M]"] & \mathrm{H}_0(M) \arrow[r, "\epsilon_M"] \arrow[d, "="] & \mathbb{Z} \arrow[d, "="'] \\
\mathrm{H}^l(M, M\setminus \mathring{A}) \times \mathrm{H}^k(M, M\setminus \mathring{B}) \arrow[r, "\cup"] \arrow[u, "p^*\times p^*"] \arrow[d, "c^*\times c^*"] & \mathrm{H}^m(M, M\setminus \mathring{C}) \arrow[r, "\cap p_*([M])"] \arrow[u, "p^*"] \arrow[d, "c^*"] & \mathrm{H}_0(M) \arrow[r, "\epsilon_M"] & \mathbb{Z} \\
\mathrm{H}^l(C, \partial_A C) \times \mathrm{H}^k(C, \partial_B C) \arrow[r, "\cup"] \arrow[d, "\bigoplus_{i=1}^{s} q_i^*\times q_i^*"] & \mathrm{H}^m(C, \partial C) \arrow[r, "\cap[C]"] \arrow[d, "\bigoplus_{i=1}^{s} q_i^*"] & \mathrm{H}_0(C) \arrow[r, "\epsilon_C"] \arrow[u, "c_*"] & \mathbb{Z} \arrow[u, "="'] \\
\bigoplus_{i=1}^{s} \mathrm{H}^l(Q_i, \partial_A Q_i) \times \mathrm{H}^k(Q_i, \partial_B Q_i) \arrow[r, "\oplus\cup"] & \bigoplus_{i=1}^{s} \mathrm{H}^m(Q_i, \partial Q_i) \arrow[r, "\bigoplus_{i=1}^{s} \cap[Q_i]"] & \bigoplus_{i=1}^{s} \mathrm{H}_0(Q_i) \arrow[r, "\sum \epsilon_{Q_i}"] \arrow[u, "\sum_{i=1}^{s} q_{i*}"] & \mathbb{Z}. \arrow[u, "="']
\end{tikzcd}$$

Claim 1. The above diagram commutes.

In the following we deal with all of the nine squares of the diagram.

(a) The left squares in the first and second row commute by Proposition 82.4 (1).
(b) The left square in the third row commutes by Lemma 83.9 (1) together with the aforementioned Proposition 82.4 (1).
(c) The middle square in the first row commutes by Lemma 83.8 (1).
(d) It follows from our orientation conventions and Lemma 87.24 that $c_*([C]) = p_*([M])$. It follows from Lemma 83.8 (1) that the middle square in the second row commutes.
(e) It follows from the remark on page 2121 that $\sum_{i=1}^{s} q_{i*}([Q_i]) = [C]$. Now we obtain from Lemma 83.8 (1) and Lemma 83.9 (2) that the middle square in the third row commutes.
(f) The squares that involve the augmentation maps commute by the discussion on page 1088. ⊞

We introduce a few last pieces of notation:

(13) We denote by $a\colon (A, \partial A) \to (M, M \setminus \mathring{A})$ and $b\colon (B, \partial B) \to (M, M \setminus \mathring{B})$ the inclusion maps, note that a and b induce isomorphisms on cohomology by excision, see page 1840 for details.

[1338] Many of the vertical maps in the diagram are isomorphisms by excision, see page 1840 for details. But this is in fact irrelevant for our purpose.

(14) We write

$$\alpha = (a^*)^{-1}(\underbrace{\mathrm{PD}_A([X])}_{\in \mathrm{H}^l(A,\partial A)}) \in \mathrm{H}^l(M, M\setminus\mathring{A}) \text{ and } \beta = (b^*)^{-1}(\underbrace{\mathrm{PD}_B([Y])}_{\in \mathrm{H}^k(B,\partial B)}) \in \mathrm{H}^k(M, M\setminus\mathring{B}).$$

(15) Given $i \in \{1,\dots,s\}$ we write $X_i = X \cap Q_i$ and $Y_i = Y \cap Q_i$. It follows from Proposition 94.8 (e) and (f) that X_i and Y_i are codimension 0 submanifolds of X and Y. Since X and Y are oriented we can view X_i and Y_i also as oriented topological manifolds. As usual we denote by $[X_i] \in \mathrm{H}_k(X_i, \partial X_i)$ and $[Y_i] \in \mathrm{H}_k(Y_i, \partial Y_i)$ the corresponding fundamental classes. With the same convention as before we denote by $[X_i]$ also the image under the inclusion induced map $\mathrm{H}_k(X_i, \partial X_i) \to \mathrm{H}_k(Q_i, \partial_B Q_i)$ and similarly we denote by $[Y_i]$ also the image under the inclusion induced map $\mathrm{H}_l(Y_i, \partial Y_i) \to \mathrm{H}_l(Q_i, \partial_A Q_i)$.

Claim 2. We have

(a) We have $p^*(\alpha) = \mathrm{PD}_M([X]) \in \mathrm{H}^l(M)$ and $p^*(\beta) = \mathrm{PD}_M([Y]) \in \mathrm{H}^k(M)$.

(b) For any $i \in \{1,\dots,s\}$ we have

$$(q_i^* \circ c^*)(\alpha) = \mathrm{PD}_{Q_i}([X_i]) \in \mathrm{H}^l(Q_i, \partial_A Q_i) \text{ and } (q_i^* \circ c^*)(\beta) = \mathrm{PD}_{Q_i}([Y_i]) \in \mathrm{H}^k(Q_i, \partial_B Q_i).$$

(c) For any $i \in \{1,\dots,s\}$ we have

$$\epsilon_{Q_i}\big(\big(\mathrm{PD}_{Q_i}([X_i]) \cup \mathrm{PD}_{Q_i}([Y_i])\big) \cap [Q_i]\big) \;=\; \sigma_i.$$

We provide the proofs for the three statements of the claim.

(a) By Lemma 89.5 the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}_k(A) \arrow[r, "\mathrm{PD}_A"'] \arrow[rr, bend left, "{[X]\mapsto\alpha}"] \arrow[d, "{[X]\,\mapsto\,[X]}"'] & \mathrm{H}^l(A,\partial A) \arrow[r, "\cong", "(a^*)^{-1}"'] & \mathrm{H}^l(M, M\setminus\mathring{A}) \arrow[d, "p^*"] \\ \mathrm{H}_k(M) \arrow[rr, "\mathrm{PD}_M"] & & \mathrm{H}^l(M). \end{tikzcd}$$

We apply this commutative diagram to $[X] \in \mathrm{H}_k(A)$ and we obtain immediately that $p^*(\alpha) = \mathrm{PD}_M([X])$. The same way one shows that $p^*(\beta) = \mathrm{PD}_M([Y])$.

(b) Let $i \in \{1,\dots,s\}$. We denote by $r_i\colon Q_i \to A$ the inclusion map. We consider the following diagram:

$$\begin{tikzcd} \mathrm{H}_k(A) \arrow[rr, "\mathrm{PD}_A"] \arrow[d] & & \mathrm{H}^l(A,\partial A) \arrow[d, "r_i^*"] & \mathrm{H}^l(M, M\setminus\mathring{A}) \arrow[l, "a^*"', "\cong"] \arrow[dl, "(q_i\circ c)^*"] \\ \mathrm{H}_k(A, A\setminus\mathring{Q}_i) & \mathrm{H}_k(Q_i, \partial_B Q_i) \arrow[l, "r_{i*}"', "\cong"] \arrow[r, "\mathrm{PD}_{Q_i}"] & \mathrm{H}^l(Q_i, \partial_A Q_i). & \end{tikzcd}$$

We make the following observations:

(α) The triangle on the right commutes since $a \circ r_i = c \circ q_i\colon Q_i \to M$ are both the inclusion map.

(β) It follows from Lemma 89.6, applied to the codimension-zero submanifold Q_i of A, that the rectangle commutes.[1339]

(γ) By Lemma 87.24 we have $r_{i*}([X_i]) = [X] \in \mathrm{H}_k(A, A \setminus \mathring{Q}_i)$.

(δ) By the Excision Theorem 89.2 the map $r_{i*}\colon\, \mathrm{H}_k(Q_i, \partial_B Q_i) \to \mathrm{H}_k(A, A \setminus \mathring{Q}_i)$ is an isomorphism.

The desired equality $(q_i^* \circ c^*)(\alpha) = \mathrm{PD}_{Q_i}([X_i])$ follows from the above commutative diagram applied to $[X] \in \mathrm{H}_k(A)$. Exactly the same way we can also show that $(q_i^* \circ c^*)(\beta) = \mathrm{PD}_{Q_i}([Y_i])$.

(c) Let $i \in \{1, \dots, s\}$. We set $Z = \overline{B}^k \times \overline{B}^l$. We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}^l(Q_i, \partial_A Q_i) \times \mathrm{H}^k(Q_i, \partial_B Q_i) \arrow[r, "\cup"] \arrow[d, "\Psi_i^* \times \Psi_i^*"] & \mathrm{H}^m(Q_i, \partial Q_i) \arrow[r, "\cap [Q_i]"] \arrow[d, "\Psi_i^*"] & \mathrm{H}_0(Q_i) \arrow[r, "\epsilon_{Q_i}"] & \mathbb{Z} \\
\mathrm{H}^l(Z, \overline{B}^k \times \partial \overline{B}^l) \times \mathrm{H}^k(Z, \partial \overline{B}^k \times \overline{B}^l) \arrow[r, "\cup"] & \mathrm{H}^m(Z, \partial Z) \arrow[r, "\cap \sigma_i \cdot [Z]"] & \mathrm{H}_0(Z) \arrow[r, "\epsilon_Z"] \arrow[u, "\Psi_i^*"] & \mathbb{Z}. \arrow[u, "="]
\end{tikzcd}$$

As in the previous diagram, the squares to the left and right commute. The middle square is slightly more interesting. Lemma 94.9 says that $\Psi_{i*}([Q_i]) = \sigma_i \cdot [Z]$. Using this observation we obtain once again from Lemma 83.8 (1) that the middle square commutes.

We calculate that

$$\begin{aligned}
\epsilon_{Q_i}\big(\big(\mathrm{PD}_{Q_i}([X_i]) \cup \mathrm{PD}_{Q_i}([Y_i])\big) \cap [Q_i]\big) &\underset{\text{by the above commutative diagram}}{=} \sigma_i \cdot \epsilon_Z\big(\big(\Psi_i^*(\mathrm{PD}_{Q_i}([X_i])) \cup \Psi_i^*(\mathrm{PD}_{Q_i}([Y_i]))\big) \cap [Z]\big) \\
&\underset{\text{by Proposition 94.8 (d)}}{=} \sigma_i \cdot \epsilon_Z\big(\big(\mathrm{PD}_Z([\overline{B}^k \times \{0\}]) \cup \mathrm{PD}_Z([\{0\} \times \overline{B}^l])\big) \cap [Z]\big) \\
&\underset{\text{Proposition 88.2}}{=} \sigma_i \cdot \epsilon_{\overline{B}^k \times \overline{B}^l}\big([\overline{B}^k \times \overline{B}^l]^* \cap [\overline{B}^k \times \overline{B}^l]\big) \underset{\text{definition of the dual fundamental class}}{=} \sigma_i.
\end{aligned}$$

⊞

After all these preparations we can prove the desired equality:

$$\begin{aligned}
\epsilon_M\big((\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y])) \cap [M]\big) &\underset{\text{by the first row of the commutative diagram from Claim 1 and by Claim 2 (a)}}{=} \epsilon_M\big((\alpha \cup \beta) \cap p_*([M])\big) \\
&\underset{\text{by the second and third row of commutative diagram from Claim 1 and by Claim 2 (b)}}{=} \sum_{i=1}^{s} \epsilon_{Q_i}\big((\mathrm{PD}_{Q_i}([X_i]) \cup \mathrm{PD}_{Q_i}([Y_i])) \cap [Q_i]\big) \underset{\text{by Claim 2 (c)}}{=} \sum_{i=1}^{s} \sigma_i.
\end{aligned}$$

This concludes the proof of the theorem. ■

Remark. Towards the end of the proof of the theorem we marked in red color that we used Proposition 88.2. Most of the other statements that we used in the proof were fairly straightforward lemmas regarding cup and cap products. But arguably the key ingredient

[1339] One can easily convince oneself that $\partial_A Q_i = \partial_1 Q_i$ and $\partial_B Q_i = \partial_0 Q_i$ for Q_i viewed as a submanifold with corner of A.

in the proof is Proposition 88.2 which was a variation on Proposition 84.2 and which in turn was a fairly straightforward consequence of the Product Theorem 84.1.

94.4. **Algebraic intersection numbers of complex submanifolds.** In the following section we will apply Theorem 94.5 to the study of the complex projective spaces $\mathbb{C}\mathrm{P}^n$. Before we do so it is convenient to study the algebraic intersection numbers of complex submanifolds of any complex manifold.

First let us recall the following definition that goes all the way back to pages 447 and 450.

Definition.

(1) An *n-dimensional complex manifold* is a $2n$-dimensional smooth manifold M without boundary together with a smooth atlas $\{\phi_j\colon U_j \to V_j\}_{j\in J}$ such that all transition maps are biholomorphisms.
(2) We define the *maximal holomorphic atlas* of a complex manifold the same way as we defined the maximal smooth atlas of a smooth manifold, see page 279.
(3) We say $W \subset M$ is a *k-dimensional complex submanifold* if for every point $P \in W$ there exists a chart $\phi_j\colon U_j \to V_j$ from the maximal holomorphic atlas with $P \in U_j$ and which satisfies that $\phi_j(U_j \cap W) \subset \mathbb{C}^k \times \{0\}$.
(4) Let V be an n-dimensional complex vector space. We say that a set of vectors $\{v_1,\dots,v_n\}$ in V is *totally real* if the vectors $v_1, iv_1, \dots, v_n, iv_n$ form a basis of V, viewed as a $2n$-dimensional real vector space.

Remark.

(1) Let M be a complex manifold. It follows easily from the definitions that every complex submanifold of M is a complex manifold in its own right.
(2) Let V be an n-dimensional complex vector space. In Lemma 12.6 we showed that V admits a set of totally real vectors and that the ordered basis $v_1, iv_1, \dots, v_n, iv_n$ give a well-defined basis of the $2n$-dimension real vector space V, called the *canonical basis*.
(3) Let M be an n-dimensional complex manifold. In Proposition 12.7 we showed that $\mathrm{T}_P\, M$ is canonically a complex vector space and we showed that the corresponding canonical orientations define a canonical orientation of M, viewed as a $2n$-dimensional smooth manifold.

Now we can formulate this elementary but rather useful lemma.

Lemma 94.10. *Let M be a closed complex manifold and let X and Y be two closed complex submanifolds of M with* $\dim(X) + \dim(Y) = \dim(M)$. *Then the following hold:*

(1) *The sign at every transverse intersection point of X and Y is equal to $+1$.*
(2) *If X and Y intersect transversally, then the algebraic intersection number equals the geometric intersection number.*

Proof.

(1) We write $k = \dim(X)$ and $l = \dim(Y)$. Suppose that P is a transverse intersection point of X and Y. We pick a totally real basis $v_1,\dots,v_k$ for the complex vector space $\mathrm{T}_P\, X$ and we pick a totally real basis $w_1,\dots,w_l$ for the complex vector space

$\mathrm{T}_P\,Y$. Then $v_1,\dots,v_k,w_1,\dots,w_l$ is evidently a totally real basis for the complex vector space $\mathrm{T}_P\,X\oplus\mathrm{T}_P\,Y=\mathrm{T}_P\,M$. We have

$$\underbrace{v_1,iv_1,\dots,v_k,iv_k}_{\text{positive basis for }\mathrm{T}_P\,X},\underbrace{w_1,iw_1,\dots,w_l,iw_l}_{\text{positive basis for }\mathrm{T}_P\,Y} \;=\; \underbrace{v_1,iv_1,\dots,v_k,iv_k,w_1,iw_1,\dots,w_l,iw_l}_{\text{positive basis for }\mathrm{T}_P\,M}.$$

This shows that $\operatorname{sign}(P)=+1$. We have concluded the proof of the first statement.

(2) The second statement is an immediate consequence of the first statement. ■

Example. We consider the complex projective space $\mathbb{C}\mathrm{P}^n$. In Lemma 12.5 we saw that $\mathbb{C}\mathrm{P}^n$ is a complex manifold of complex dimension n. It follows easily from the definitions that for any $k\le n$ the subset

$$X_k \;:=\; \{[z_0:\ldots:z_k:0:\ldots:0]\,|\,(z_0,\dots,z_k)\in\mathbb{C}^{k+1}\setminus\{0\}\}$$

is a k-dimensional complex submanifold. Similarly we see that for any $l\le n$ the subset

$$Y_l \;:=\; \{[0:\ldots:0:z_{n-l}:\ldots:z_n]\,|\,(z_{n-l},\dots,z_n)\in\mathbb{C}^{l+1}\setminus\{0\}\}$$

is an l-dimensional complex submanifold. Evidently X_k and Y_l are biholomorphic to $\mathbb{C}\mathrm{P}^k$ respectively $\mathbb{C}\mathrm{P}^l$. Furthermore given any $k\le n$ we have

$$X_k\cap Y_{n-k} \;=\; [0:\ldots:0:\underset{\substack{\uparrow\\ \text{the }(k+1)\text{-st entry}}}{1}:0:\ldots:0]$$

consists of a single point P. We leave it as an illuminating exercise to verify that X_k and Y_{n-k} intersect transversally at P. It follows from Lemma 94.10 that $X_k\cdot Y_{n-k}=1$. The same way we also see that $Y_k\cdot X_{n-k}=1$.

94.5. **Applications of Theorem 94.5 to complex projective spaces.** In this section we will give two more applications of Theorem 94.5.

Notation. Let $n\in\mathbb{N}$ and let $m\in\{0,\dots,n\}$.

(1) We consider the inclusion map

$$\begin{aligned} i_{m,n}\colon \mathbb{C}\mathrm{P}^m &\;\to\; \mathbb{C}\mathrm{P}^n\\ [z_0:\ldots:z_m] &\;\mapsto\; [z_0:\ldots:z_m:0:\cdots:0].\end{aligned}$$

When m and n are understood we write i instead of $i_{m,n}$. Recall that in Lemma 90.6 we showed that for any $k\le 2m$ the induced maps $i_*\colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^m;\mathbb{Z})\to\mathrm{H}_k(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and $i^*\colon \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z})\to\mathrm{H}^k(\mathbb{C}\mathrm{P}^m;\mathbb{Z})$ are isomorphisms.

(2) By a slight abuse of notation we abbreviate $(i_{m,n})_*([\mathbb{C}\mathrm{P}^m])$ to $[\mathbb{C}\mathrm{P}^m]\in\mathrm{H}_{2m}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$. Furthermore, again by a slight abuse of notation, we write $[\mathbb{C}\mathrm{P}^m]^*\in\mathrm{H}^{2m}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ for the unique element with $\langle[\mathbb{C}\mathrm{P}^m]^*,[\mathbb{C}\mathrm{P}^m]\rangle=1$. Finally we write $x=[\mathbb{C}\mathrm{P}^1]^*$.

Now we can formulate the main result of this section.

Proposition 94.11. *Let $n\in\mathbb{N}$. For every $k\in\{0,\dots,n\}$ we have*

$$(\underbrace{[\mathbb{C}\mathrm{P}^1]^*}_{=x})^k \;=\; [\mathbb{C}\mathrm{P}^k]^* \;\in\; \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}).$$

Remark. In Proposition 90.7 we had used the Poincaré Duality Theorem 88.1 to show that for given $n \in \mathbb{N}$ and $k \in \{0,\ldots,n\}$ the k-th power $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ is a generator. In other words, we had proved that $x^k = \pm[\mathbb{C}\mathrm{P}^k]^*$. Proposition 94.11 is a much better result since it allows us to nail down the cohomology class x^k. In particular we see that Proposition 94.11 gives an affirmative answer to Question 90.8.

The key to the proof of Proposition 94.11 is the following lemma.

Lemma 94.12. *Let $n \in \mathbb{N}$. For any $k \in \{0,\ldots,n\}$ we have*

$$\mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]) \cup \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^{n-k}]) \;=\; [\mathbb{C}\mathrm{P}^n]^* \;\in\; \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}).$$

Proof. Let $n \in \mathbb{N}$. For $k \in \{0,\ldots,n\}$ we consider the inclusion map

$$\begin{array}{rcl} t_{k,n}\colon \mathbb{C}\mathrm{P}^k & \to & \mathbb{C}\mathrm{P}^n \\ [z_0:\ldots:z_k] & \mapsto & [0:\ldots:0:z_0:\ldots:z_k]. \end{array}$$

We also consider the map[1340]

$$\begin{array}{rcl} \Phi\colon \mathbb{C}\mathrm{P}^k \times [0,1] & \to & \mathbb{C}\mathrm{P}^n \\ ([z],t) & \mapsto & \Big[\underbrace{\big(z\cdot\cos^2\big(t\cdot\tfrac{\pi}{2}\big),0\big)}_{\in\mathbb{C}^{n+1}} + \underbrace{\big(0,z\cdot\sin^2\big(t\cdot\tfrac{\pi}{2}\big)\big)}_{\in\mathbb{C}^{n+1}}\Big]. \end{array}$$

Then

$$\begin{array}{rcl} \langle \mathrm{PD}([\mathbb{C}\mathrm{P}^k]) \cup \mathrm{PD}([\mathbb{C}\mathrm{P}^{n-k}]),[\mathbb{C}\mathrm{P}^n]\rangle & = & \langle \mathrm{PD}(i_{k,n*}([\mathbb{C}\mathrm{P}^k])) \,\cup\, \mathrm{PD}(i_{n-k,n*}([\mathbb{C}\mathrm{P}^{n-k}])),[\mathbb{C}\mathrm{P}^n]\rangle \\ & \underset{\substack{\uparrow \\ \text{by Proposition 42.5, since } \Phi \text{ is a homotopy from } i_{k,n} \text{ to } t_{k,n}}}{=} & \langle \mathrm{PD}(t_{k,n*}([\mathbb{C}\mathrm{P}^k])) \,\cup\, \mathrm{PD}(i_{n-k,n*}([\mathbb{C}\mathrm{P}^{n-k}])),[\mathbb{C}\mathrm{P}^n]\rangle \\ & = & \langle \mathrm{PD}([t_{k,n}(\mathbb{C}\mathrm{P}^k)]) \,\cup\, \mathrm{PD}([i_{n-k,n}(\mathbb{C}\mathrm{P}^{n-k})]),[\mathbb{C}\mathrm{P}^n]\rangle \\ & \underset{\substack{\uparrow \\ \text{by Theorem 94.5 since the submanifolds} \\ t_{k,n}(\mathbb{C}\mathrm{P}^k) \text{ and } i_{n-k,n}(\mathbb{C}\mathrm{P}^{n-k}) \text{ are transverse}}}{=} & t_{k,n}(\mathbb{C}\mathrm{P}^k)\cdot i_{n-k,n}(\mathbb{C}\mathrm{P}^{n-k}) \underset{\substack{\uparrow \\ \text{see page 2289 for details}}}{=} 1. \end{array}$$

By definition of the dual fundamental class this implies the desired result. ■

We record the following fairly immediate corollary.

Corollary 94.13. *Let $n \in \mathbb{N}$. For any $k \in \{0,\ldots,n\}$ we have*

$$\mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]) \;=\; [\mathbb{C}\mathrm{P}^{n-k}]^* \in \mathrm{H}^{2n-2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}).$$

[1340] It is quite straightforward to verify that this map is well-defined, i.e. it does not depend on the choice of the representative $z \in \mathbb{C}^{k+1}\setminus\{0\}$ and it does indeed take values in $\mathbb{C}\mathrm{P}^n$ since for any $t \in [0,1]$ we have $\big(z\cdot\cos^2\big(t\cdot\frac{\pi}{2}\big),0\big)+\big(0,z\cdot\sin^2\big(t\cdot\frac{\pi}{2}\big)\big) \neq 0$. Also note that it follows fairly easily from Proposition 18.20 that the map is continuous.

Proof. Let $n \in \mathbb{N}$ and let $k \in \{0, \ldots, n\}$. Then

$$\begin{aligned}
\langle \mathrm{PD}_{\mathbb{CP}^n}([\mathbb{CP}^k]), [\mathbb{CP}^{n-k}] \rangle &\underset{\downarrow}{\overset{\text{by definition of } \mathrm{PD}_{\mathbb{CP}^n}([\mathbb{CP}^{n-k}])}{=}} \langle \mathrm{PD}_{\mathbb{CP}^n}([\mathbb{CP}^k]), \mathrm{PD}_{\mathbb{CP}^n}([\mathbb{CP}^{n-k}]) \cap [\mathbb{CP}^n] \rangle \\
&\underset{\substack{\uparrow \\ \text{Lemma 83.7 and Proposition 81.8}}}{=} \langle \mathrm{PD}_{\mathbb{CP}^n}([\mathbb{CP}^k]) \cup \mathrm{PD}_{\mathbb{CP}^n}([\mathbb{CP}^{n-k}]), [\mathbb{CP}^n] \rangle \underset{\substack{\uparrow \\ \text{Lemma 94.12}}}{=} 1.
\end{aligned}$$

By definition of the dual fundamental class this implies the desired result. ■

Now we can provide the proof of Proposition 94.11.

Proof of Proposition 94.11. Recall that we need to show that for every $n \in \mathbb{N}$ and every $k \in \{0, \ldots, n\}$ we have $([\mathbb{CP}^1]^*)^k = [\mathbb{CP}^k]^* \in \mathrm{H}^{2k}(\mathbb{CP}^n; \mathbb{Z})$. We start out with the following observation that follows immediately from the discussion on page 2289 and Lemma 81.10.

Observation. Let $m \in \mathbb{N}$ and let $k \in \{1, \ldots, m\}$. If $([\mathbb{CP}^1]^*)^k = [\mathbb{CP}^k]^* \in \mathrm{H}^{2k}(\mathbb{CP}^m; \mathbb{Z})$, then for any $n \geq m$ we also have $([\mathbb{CP}^1]^*)^k = [\mathbb{CP}^k]^* \in \mathrm{H}^{2k}(\mathbb{CP}^n; \mathbb{Z})$

By the observation it suffices to prove the following claim.

Claim. For any $n \in \mathbb{N}$ we have $([\mathbb{CP}^1]^*)^n = [\mathbb{CP}^n]^* \in \mathrm{H}^{2n}(\mathbb{CP}^n; \mathbb{Z})$.

We prove the claim by induction on n. For $n = 1$ the statement holds by definition. So suppose that the statement holds for some $n - 1$ with $n \geq 2$. Then the following equality holds in $\mathrm{H}^{2n}(\mathbb{CP}^n; \mathbb{Z})$:

$$([\mathbb{CP}^1]^*)^n = ([\mathbb{CP}^1]^*)^{n-1} \cup [\mathbb{CP}^1]^* \underset{\substack{\uparrow \\ \text{by induction and the observation}}}{=} [\mathbb{CP}^{n-1}]^* \cup [\mathbb{CP}^1]^* \underset{\substack{\uparrow \\ \text{Corollary 94.13}}}{=} \mathrm{PD}([\mathbb{CP}^1]) \cup \mathrm{PD}([\mathbb{CP}^{n-1}]) \underset{\substack{\uparrow \\ \text{Lemma 94.12}}}{=} [\mathbb{CP}^n]^*.$$

This concludes the proof of the claim and thus of the proposition. ■

Now we can answer Question 10.10.

Proposition 94.14. *Every tubular neighborhood of $\mathbb{CP}^1 = i_{1,2}(\mathbb{CP}^1)$ in $\mathbb{CP}^2$ is non-trivial.*[1341]

In Figure 147 we had implicitly seen that the tubular neighborhood of $\mathbb{RP}^1$ in $\mathbb{RP}^2$ is a Möbius band, and thus non-trivial. The above proposition is thus an analogue of this simpler, and more visual, observation.

Proof. Suppose that $\mathbb{CP}^1$ admits a trivial tubular neighborhood in $\mathbb{CP}^2$. By definition this would mean that there exists a smooth embedding $\Phi\colon \mathbb{CP}^1 \times \overline{B}^2 \to \mathbb{CP}^2$ with $\Phi(P, 0) = P$ for all $P \in \mathbb{CP}^1$. We denote by $i = i_{1,2}\colon \mathbb{CP}^1 \to \mathbb{CP}^2$ the usual inclusion map and we denote

[1341] We refer to page 421 for the definition of a trivial tubular neighborhood.

by $f\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ the smooth embedding that is given by $P \mapsto \Phi(P,1)$. Then

$$\pm 1 \underset{\substack{\uparrow\\ \text{Proposition 94.11 or alternatively}\\ \text{Proposition 90.7}}}{=} \langle \mathrm{PD}(i_*([\mathbb{C}\mathrm{P}^1]) \cup \mathrm{PD}(i_*([\mathbb{C}\mathrm{P}^1])), [\mathbb{C}\mathrm{P}^2]\rangle \underset{\substack{\uparrow\\ \text{by Proposition 42.5 since } i,f\colon \mathbb{C}\mathrm{P}^1\to\mathbb{C}\mathrm{P}^2 \text{ are homotopic}\\ \text{via the homotopy } (P,t)\mapsto \Phi(P,t)}}{=} \langle \mathrm{PD}(i_*([\mathbb{C}\mathrm{P}^1]) \cup \mathrm{PD}(f_*([\mathbb{C}\mathrm{P}^1])), [\mathbb{C}\mathrm{P}^2]\rangle$$

$$\underset{\substack{\uparrow\\ \text{by Theorem 94.5}}}{=} i(\mathbb{C}\mathrm{P}^1)\cdot f(\mathbb{C}\mathrm{P}^1) \underset{\substack{\uparrow\\ \text{since } i(\mathbb{C}\mathrm{P}^1) \text{ and } f(\mathbb{C}\mathrm{P}^1) \text{ are disjoint}}}{=} 0.$$

We have thus obtained a contradiction. ■

94.6. **Is the algebraic intersection number a complete invariant?** It is natural to ask whether the converse to Proposition 94.6 (2) holds. Playing around with curves on a surface shows that this looks dubious:

Example. We consider the oriented curves X and Y on the surface of genus 2 that are illustrated in Figure 1316. Their algebraic intersection number is zero. There are several

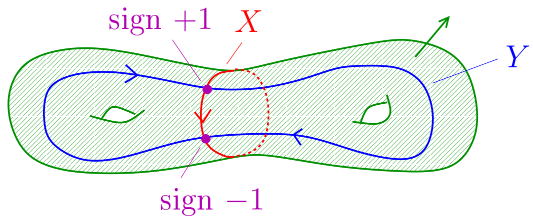


FIGURE 1316

ways to see that the curves cannot be made disjoint:

(1) In Exercise 94.12 we will use covering spaces to see that the curves cannot be made disjoint.
(2) The so-called *bigon criterion* says that if M is a closed 2-dimensional smooth manifold and if C and D are closed oriented curves on M that intersect transversally, then C and D have the minimal number of geometric intersection points within their smooth isotopy classes if and only if C and D do not form a bigon, see Figure 1317. We refer to [**FaM11**, Proposition 1.7] for details and a proof.

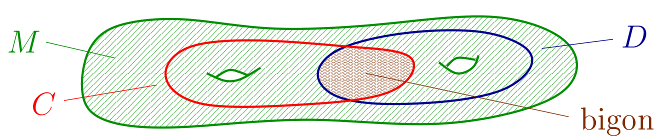


FIGURE 1317

If one thinks for a while about the previous example, then one realizes that the issue in the example is that the smooth manifold M is not simply connected. So it is more reasonable to demand some condition on the fundamental groups

Question 94.15. *Let M be an oriented m-dimensional smooth manifold and let X and Y be complementary connected submanifolds of M. Suppose that $X \cdot Y = 0$ and that M is simply connected, does it follow that X and Y can be made disjoint?*

It turns out that the answer depends on the dimension of M.

Theorem 94.16. (Whitney) *Let M be an oriented simply connected smooth manifold and let X and Y be complementary connected submanifolds of M.*

(1) *The answer to Question 94.15 is yes if* $\dim(X) \geq 3$ *and* $\dim(Y) \geq 3$.
(2) *The answer to Question 94.15 is yes if* $\dim(X) = 2$, $\dim(Y) \geq 3$ *and* $\pi_1(M \setminus Y) = 0$.

Remark. The fact that we have a positive answer to Question 94.15 in dimensions ≥ 6 turns out to be extremely useful, since, together with Theorem 94.5, it makes it possible to translate algebraic topological data, namely cup products, directly into geometric statements. This fact was a key ingredient in Hassler Whitney's proof that every n-dimensional smooth manifold without boundary can be embedded into $\mathbb{R}^{2n}$, see Theorem 11.14. Furthermore, as explained in [**Miln65b**], this fact played a crucial role in Steven Smale's proof of the Poincaré Conjecture in dimensions ≥ 5, see Theorem **??**.

For dimensions ≥ 5 the theorem was proved by Hassler Whitney [**Why44a**, Theorem 4] in 1944. An exposition of the proof is also given in [**Kir13**], [**Scor05**, Chapter 1.5], [**Miln65b**, Theorem 6.6], [**GoS99**, Theorem 9.2.7] or [**RS72**, Theorem 5.12]. We only give a quick outline of the proof. Our main goal is to show where the restrictions on the dimensions enter.

Sketch of a proof of Theorem 94.16 (1). Let M be an oriented simply connected m-dimensional smooth manifold and let X and Y be complementary connected submanifolds of M with $X \cdot Y = 0$ and with $\dim(X) \geq 3$ and $\dim(Y) \geq 3$. We do induction on the geometric intersection number $\#(X \cap Y)$. If $X \cap Y = \emptyset$, then there is nothing to prove. Now suppose that $X \cap Y \neq \emptyset$. There exist two intersection points P and Q of opposite signs. Since X is connected we obtain from Corollary 8.38 an embedded curve α on X from P to Q and since Y is connected we can similarly find an embedded curve β on Y from Q to P. Note that $\varphi := \alpha * \beta$ is in fact a loop. Since M is simply connected the curve $\varphi = \alpha * \beta \colon S^1 \to M$ is null-homotopic.

By Lemma 14.1 there exists a map $\Phi \colon \overline{B}^2 \to M$ with $\Phi|_{S^1} = \varphi$. Theorem 9.15 suggests[1342] that we can find a homotopy from Φ to a map $\Psi \colon \overline{B}^2 \to M$ with $\Psi(S^1) = \Phi(S^1)$ and such that $\Psi \colon \overline{B}^2 \to M$ is a smooth embedding.[1343] It follows from $\dim(X) \geq 3$ and $\dim(Y) \geq 3$ that $\dim(\Phi(\overline{B}^2)) + \dim(X) < \dim(M)$ and $\dim(\Phi(\overline{B}^2)) + \dim(Y) < \dim(M)$. Thus the Transversality Theorem 9.10 suggests that we can find a further smooth isotopy so that we end up with an embedded disk D with $\partial D = |\varphi| = |\alpha| \cup |\beta|$ and such that the interior of the disk intersects neither X nor Y. One can now use this disk to find a smooth isotopy of X that removes the two intersection points P and Q without creating new intersection points. The precise construction of the smooth isotopy is rather delicate. In fact, as the perspicacious reader might have noticed, so far we have not yet used that P and Q have opposite signs. We need it in finding this last smooth isotopy. We refer to the aforementioned references for the details. ■

[1342]Here the exposition starts becoming rather vague.

[1343]Here there are some issues since the map $\alpha * \beta \colon S^1 \to M$ is not smooth at two points. In particular it is not quite clear what "smooth embedding" is supposed to mean.

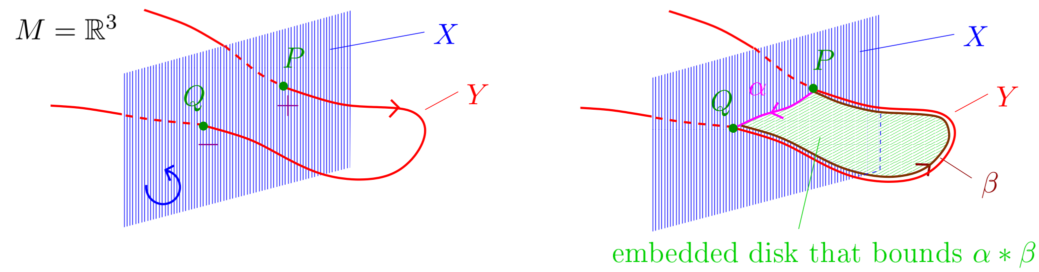


FIGURE 1318. Illustration for the sketch of the proof of Theorem 94.16 (1).

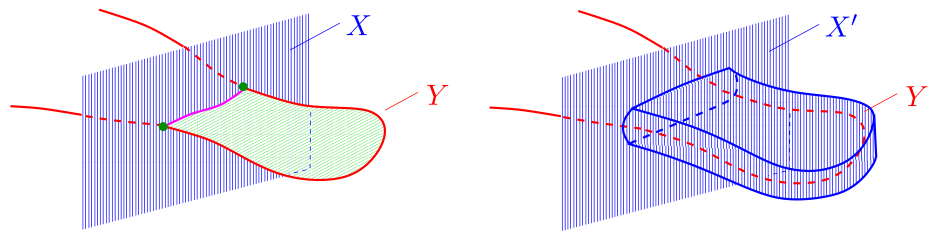


FIGURE 1319. Second illustration for the sketch of the proof of Theorem 94.16 (1).

Remark. The approach taken in the proof of making submanifolds disjoint using embedded disks is often referred to as the "Whitney trick".

It turns out that we also have an affirmative answer to Question 94.15 in very low dimensions.

Theorem 94.17. *The answer to Question 94.15 is also yes if the dimension of the smooth manifold is* ≤ 3.

PROOF. The theorem is trivial for 1-dimensional smooth manifolds. For 2-dimensional smooth manifolds the proof is an amusing exercise using the fact, proved in the Surface Classification Theorem 23.4, that every simply connected 2-dimensional smooth manifold is diffeomorphic to S^2 or to $\overline{B}^2$.

For 3-dimensional smooth manifolds the proof is similar to the proof in dimensions ≥ 5 except that one also needs to make use of the Loop Theorem (see **[Pap57a]** and **[Hem76]**) to find the embedded disk needed for the "Whitney trick". We leave it as a slightly challenging exercise to fill in the details in the 3-dimensional setting. ■

So we are still missing 4-dimensional smooth manifolds. Here the situation becomes tricky. It was known by 1961, see **[KeM61]**, that the approach of the proof of Theorem 94.16, i.e. the Whitney trick, does not work for 4-dimensional smooth manifolds. Later, see **[Lacke96]**, it was also shown that the Whitney trick does not work for 4-dimensional topological manifolds either.

In the 4-dimensional setting it is not only the case that the Whitney trick does not work. But more fundamentally, Question 94.15 has in fact a negative answer. For example it is known that there exist closed simply connected 4-dimensional smooth manifolds and embedded spheres X, Y with zero intersection number that cannot be made disjoint by a smooth isotopy, in fact they cannot even be made disjoint by an isotopy that is not necessarily smooth, see e.g. [**As98**, Theorem 1.2] for details.[1344] As is pointed out in [**As98**, Proposition 3.9], it is a consequence of the Freedman disk smooth embedding theorem [**FQ90**, Theorem 5.1.A] that under even stronger hypotheses the algebraic intersection number is in fact a complete invariant in the setting of 4-dimensional *topological* manifolds.

Later, in Chapter **??** we will see that 4-dimensional smooth manifolds exhibit many peculiar properties. Arguably the main reason why 4-dimensional smooth manifolds differ dramatically from higher-dimensional smooth manifolds is the failure of the Whitney trick.

94.7. **Higher dimensional analogue of Theorem 94.5.** Let M be a compact oriented m-dimensional smooth manifold, let $X \subset M$ be a closed oriented submanifold of dimension k and let Y be a compact oriented proper submanifold of the complementary dimension $m-k$. We suppose that X and Y are transverse. By assigning to each intersection point in $X \cap Y$ the sign introduced on page 2271 we can view $X \cap Y$ as a 0-dimensional oriented submanifold of M.[1345] With this convention we can reformulate the conclusion of Theorem 94.5 as follows:

Theorem 94.18. *With the above notation we have*

$$\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]) \;=\; \mathrm{PD}_M([X \cap Y]) \;\in\; \mathrm{H}^m(M, \partial M; \mathbb{Z}).$$

PROOF. Using Lemma 83.9 one can easily verify that it suffices to prove the theorem for each connected component of M separately. Put differently, without loss of generality we can assume that M is connected. We denote by $P_1, \dots, P_n$ the points in $X \cap Y$ and we denote by $\sigma_1, \dots, \sigma_n \in \{\pm 1\}$ the corresponding signs. Now we calculate that

$$\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]) \underset{\substack{\uparrow \\ \text{by Theorem 94.5 and since } M \text{ is connected}}}{=} (X \cdot Y) \cdot [M]^* = \sum_{i=1}^n \sigma_i \cdot [M]^* \underset{\substack{\uparrow \\ \text{see page 2154}}}{=} \sum_{i=1}^n \mathrm{PD}_M(\sigma_i \cdot [\{P_i\}]) = \mathrm{PD}_M([X \cap Y]).$$

■

The formulation of Theorem 94.18 has the nice feature that with some effort one can also generalize it to submanifolds that are not of complementary dimensions. Before we can formulate this generalization we need to formulate the following lemma.

Lemma 94.19. *Let M be an oriented m-dimensional smooth manifold, let $X \subset M$ be an oriented proper submanifold and let Y be an oriented proper submanifold. Suppose that X and Y intersect transversally. By Lemma 6.52 we know that $X \cap Y$ is a submanifold. Given $P \in X \cap Y$ we have a natural short exact sequence induced by the various inclusion maps:*

$$0 \;\to\; \mathrm{T}_P(X \cap Y) \;\to\; \mathrm{T}_P X \oplus \mathrm{T}_P Y \;\xrightarrow{\rho}\; \mathrm{T}_P M \;\to\; 0.$$

We pick a set of vectors $\mathcal{C}$ in $\mathrm{T}_P X \oplus \mathrm{T}_P Y$ such that $\rho(\mathcal{C})$ is a positive basis for $\mathrm{T}_P M$. We say that a basis $\mathcal{B}$ for $\mathrm{T}_P(X \cap Y)$ is positive *if the basis $\{\mathcal{B}, \mathcal{C}\}$ is a positive basis for $\mathrm{T}_P X \oplus \mathrm{T}_P Y$. This definition is independent of the choices and it turns $X \cap Y$ into an oriented submanifold of M.*

[1344]More examples for the failure of the Whitney trick in dimension 4, i.e. for fact that the conclusion of Theorem 94.16 does not hold in dimension 4, are given in [**Strl04**, Section 15.1] and [**Gil81**].

[1345]Recall that on page 1718 we defined the orientation of a 0-dimensional manifold M as a map $M \to \{\pm 1\}$.

PROOF. We leave the elementary proof to the reader. ■

Example. Let $n \in \mathbb{N}$ and let $0 \leq k \leq l \leq n$. We consider $\mathbb{C}\mathrm{P}^n$ with the complex submanifolds

$$\begin{aligned} A &= \{[z_0 : \dots : z_l : 0 : \dots : 0] \,|\, (z_0, \dots, z_l) \in \mathbb{C}^{l+1} \setminus \{0\}\} \\ B &= \{[0 : \dots : 0 : z_k : \dots : z_n] \,|\, (z_k, \dots, z_n) \in \mathbb{C}^{n+1-k} \setminus \{0\}\} \end{aligned}$$

and

$$C = \{[0 : \dots : 0 : z_k : \dots : z_l : 0 : \dots : 0] \,|\, (z_k, \dots, z_l) \in \mathbb{C}^{l+1-k} \setminus \{0\}\}.$$

As usual we equip $\mathbb{C}\mathrm{P}^n$ and A, B, C with the natural orientations coming from Proposition 12.7. Then $A \cap B = C$ and one can easily show that the orientation of $A \cap B$ equals the orientation of C.

The following theorem is a generalization of Theorem 94.18 and thus also of Theorem 94.5.

Theorem 94.20. *Let M be a compact oriented m-dimensional smooth manifold. Furthermore let $X \subset M$ be a closed oriented submanifold of codimension k and let Y be a compact oriented proper submanifold of codimension l. Then*[1346]

$$\underbrace{\mathrm{PD}_M([X])}_{\in \mathrm{H}^k(M,\partial M)} \cup \underbrace{\mathrm{PD}_M([Y])}_{\in \mathrm{H}^l(M)} = \mathrm{PD}_M([X \cap Y]) \in \mathrm{H}^{k+l}(M, \partial M; \mathbb{Z}).$$

PROOF. A proof is given in [**Bre93**, Theorem VI.11.9]. ■

Remark.

(1) Theorem 94.20 can be summarized as saying that cup products correspond to taking intersections of submanifolds. With this point of view, the symbol "$\cup$" for the cup product does not seem to be a great choice.
(2) It is conceivable that the conclusion of Theorem 94.20 also holds if we have a decomposition $\partial M = A \cup B$ of the boundary and we have proper submanifold X and Y with $\partial X \subset A$ and $\partial Y \subset B$. We leave the task of verifying such a generalization to other authors.

Examples.

(1) Let M be the smooth manifold that is the complement in S^3 of the three open solid tori and the open ball that are shown in Figure 1320. Furthermore let X and Y be the surfaces shown in that figure. Basically the same argument as on page 2278 shows that $[X]$ and $[Y]$ are non-trivial in $\mathrm{H}_2(M; \mathbb{Z})$. On the other hand the class $[X \cap Y] \in \mathrm{H}_1(M; \mathbb{Z})$ is clearly trivial. Therefore we deduce from Theorem 94.20 that $\mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]) = 0$.
(2) Using Theorem 94.20 and using the example preceding that theorem one can fairly easily give a completely geometric calculation of the cup product of $\mathbb{C}\mathrm{P}^n$.

Remark. Using Theorem 94.20 one can easily modify the proof of Proposition 94.14 to show that given any $n \geq 2$ every tubular neighborhood of $\mathbb{C}\mathrm{P}^{n-1}$ in $\mathbb{C}\mathrm{P}^n$ is non-trivial.

[1346] The sign should be taken with a slight grain of salt. It is conceivable that the sign depends on the various dimensions.

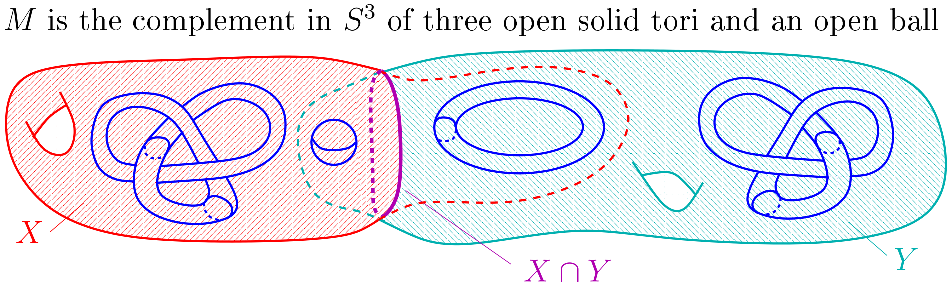


FIGURE 1320

Exercises for Chapter 94.

Exercise 94.1. Let M be a smooth manifold that admits two compact codimension-zero submanifolds A and B such that $A \cap B$ is a union of boundary components of A and a union of boundary components of B. By the Mayer–Vietoris Theorem 46.10 for Manifolds we obtain a long exact sequence

$$\cdots \to \mathrm{H}_n(A \cap B) \xrightarrow{i_{A\cap B} \oplus -i_{A\cap B}} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i_A + i_B} \mathrm{H}_n(M) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A \cap B) \to \dots$$

Let X be a closed oriented n-dimensional submanifold of M that intersects $A \cap B$ transversally.

(a) Show that $X \cap (A \cap B)$ is a closed $(n-1)$-dimensional submanifold of $A \cap B$.

We equip $X \cap (A \cap B)$ with the orientation coming from $X \cap (A \cap B) = \partial(X \cap A)$. We view the fundamental class $[X]$ as an element in $\mathrm{H}_n(M)$ and we view the fundamental class $[X \cap (A \cap B)]$ as an element in $\mathrm{H}_{n-1}(A \cap B)$.

(b) Show that $\partial_n([X]) = \pm[X \cap (A \cap B)] \in \mathrm{H}_{n-1}(A \cap B)$.

(c) Determine the correct sign in (b).

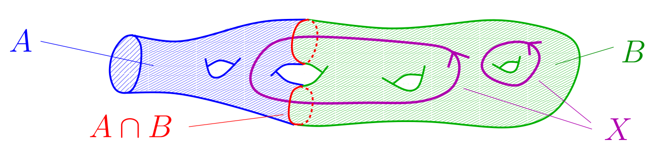


FIGURE 1321. Illustration of Exercise 94.1.

Exercise 94.2. Let A, B be two 1-dimensional oriented submanifolds of S^2 with $A \cdot B = 0$. Sketch a proof for the claim that A and B can be made disjoint via isotopies of A and B.

Exercise 94.3. Let M be a compact oriented m-dimensional smooth manifold. Furthermore let $k \in \{1, \dots, m-1\}$, let X be a proper oriented k-dimensional submanifold of M and let Y be a proper oriented $(m-k)$-dimensional submanifold of M. We denote by $i\colon X \to M$ and $j\colon Y \to M$ the inclusion maps. We could define the algebraic intersection number of X and Y via the formula in Theorem 94.5, i.e. we could define

$$X \cdot Y \;:=\; \langle \mathrm{PD}_M(i_*([X]) \cup \mathrm{PD}_M(j_*([Y])), [M] \rangle.$$

What, if any, conditions do you need to impose on the boundaries of X and Y for this definition to make any sense?
Remark. This exercise can be viewed as a continuation of the discussion on page 2273.

Exercise 94.4. Let M be a closed oriented connected m-dimensional smooth manifold. Furthermore let $p\colon \widetilde{M} \to M$ be a finite covering.

(a) Let X, Y be a complementary pair of submanifolds of M. We write $\widetilde{X} := p^{-1}(X)$ and $\widetilde{Y} := p^{-1}(Y)$. What is the relationship between $X \cdot Y$ and $\widetilde{X} \cdot \widetilde{Y}$? It suffices to give a short justification.
(b) Let X be a closed oriented submanifold of M. What is the relationship between $\mathrm{PD}_{\widetilde{M}}([\widetilde{X}])$ and $p^*(\mathrm{PD}_M([X]))$?
(c) Reprove (a) using (b) and Theorem 94.5.

Exercise 94.5. Let M be an oriented m-dimensional smooth manifold, let X be a k-dimensional oriented proper submanifold and let Y be an $(m-k)$-dimensional oriented proper submanifold of M. We suppose that $X \cap Y \cap \partial M = \emptyset$. What is the relationship between the algebraic intersection number $X \cdot Y$ in M and $DX \cdot DY$ in $\mathrm{D}\,M$?
Here, as always, given a smooth manifold N we denote by $\mathrm{D}\,M = N \cup_{\partial N = \partial N'} N'$ the double of N. We view $\mathrm{D}\,M$ as an oriented smooth manifold where the orientation is chosen such that the orientation of $\mathrm{D}\,M$ agrees with the orientation on N.

Exercise 94.6. Let $n \in \mathbb{Z}$. In Exercise 12.3 we showed that

$$W_n \ := \ \{[x : y : z] \in \mathbb{C}\mathrm{P}^2 \,|\, x^n + y^n + z^n = 0\}$$

is a closed 1-dimensional complex submanifold of the complex projective space $\mathbb{C}\mathrm{P}^2$. Show that $[W_n] = n \cdot [\mathbb{C}\mathrm{P}^1] \in \mathrm{H}_2(\mathbb{C}\mathrm{P}^2; \mathbb{Z})$.
Hint. Use Lemma 94.10.

Exercise 94.7. Let $\Sigma \subset S^2 \times (-1, 1)$ be a closed surface. Suppose that there exists a $P \in S^2$ such that Σ intersects $\{P\} \times [-1, 1]$ in a single point. Show that if this intersection point is a transversal intersection point, then Σ intersects every path $\gamma\colon [0, 1] \to S^2 \times [-1, 1]$ from a point in $S^2 \times \{-1\}$ to a point in $S^2 \times \{1\}$.

Exercise 94.8. We consider the 2-dimensional smooth manifold M that is shown together with the two proper submanifolds X and Y that are shown in Figure 1322. Show that X and Y cannot be made disjoint by a smooth isotopy that keeps the boundaries of X and Y fixed.

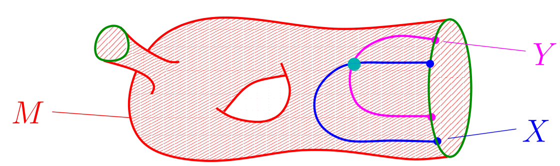


FIGURE 1322. Illustration of Exercise 94.8.

Exercise 94.9. By accident you ended up buying a clock where both hands have the same length, so you can not distinguish the hand which shows the hour from the hand that indicates the minute. Fortunately, very often you can still tell the precise time. For example, in the configuration in Figure 1323 it does not make sense for the hand that is indicating the hour to be just *before* 9 whereas the hand showing the minute is at 15. We refer to Figure 1323 for an illustration.

Of course there will be some configurations when it is indeed impossible to tell which hand is which. How many such confusions moments will there be in a 12 hour period?
Hint. Consider a suitable map $\gamma\colon S^1 \to S^1 \times S^1$.

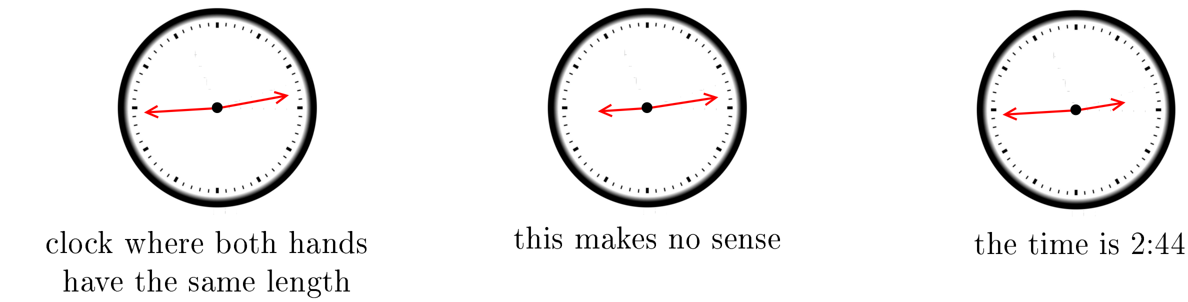


FIGURE 1323

Exercise 94.10. Let C and D be closed curves in $\mathbb{R}^2$ that intersect transversally. Give a rigorous proof that the number of intersection points between C and D is an even number.

Exercise 94.11. Let $\alpha\colon S^1 \to S^1 \times S^2$ be a smooth embedding. We assume that there exists a $z \in S^1$ such that $\gamma(S^1)$ intersects the submanifold $\{z\} \times S^2$ in a single point P and we assume that the intersection point is transversal intersection point of sign $+1$. Show that $\alpha\colon S^1 \to S^1 \times S^2$ is smoothly isotopic to the embedding $\beta\colon S^1 \to S^1 \times S^2$ that is given by $t \mapsto (t, P)$.
Remark. This statement is sometimes referred to as the *light bulb trick*.

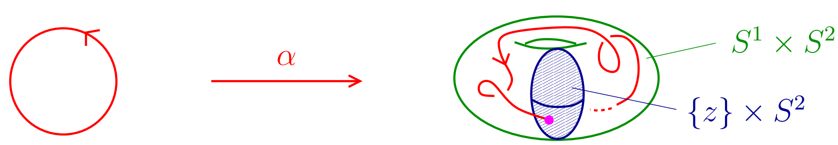


FIGURE 1324. Illustration for Exercise 94.11.

Exercise 94.12.

(a) Let $p\colon \widetilde{M} \to M$ be a finite covering and let X and Y be two submanifolds of M. Show that if X and Y can be made disjoint, then any *component* of $p^{-1}(X)$ can be made disjoint from any *component* of $p^{-1}(Y)$.

(b) We consider the surface M of genus two together with the two submanifolds X and Y that are shown in Figure 1325. Show that X and Y cannot be made disjoint.
Remark.

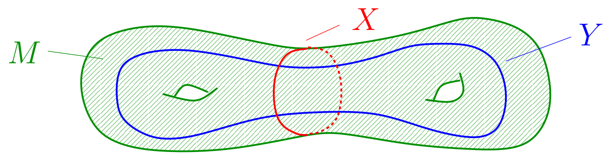


FIGURE 1325. Illustration for Exercise 94.12.

Exercise 94.13. Let M be a compact oriented 2-dimensional smooth manifold. We consider the following construction:

$$\begin{array}{rcl} \mathrm{H}_1(M, \partial M) \times \mathrm{H}_1(M, \partial M) & \to & \mathbb{Z} \\ (\sigma, \tau) & \mapsto & \begin{array}{c}\text{algebraic intersection number } S \cdot T \text{ of proper transverse} \\ \text{submanifolds } S \text{ and } T \text{ of } M \text{ with } [S] = \sigma \text{ and } [T] = \tau. \end{array} \end{array}$$

Does this give a well-defined map? We refer to Figure 1326 for an illustration.

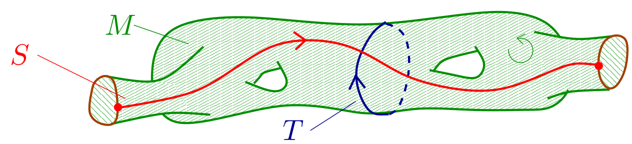


FIGURE 1326

Exercise 94.14. We consider the surface Σ of genus 2 together with the four oriented curves a_1, b_1, a_2, b_2 that are shown in Figure 1327.

(a) Let X be a closed oriented 1-dimensional submanifold of Σ that intersects a_1, b_1, a_2, b_2 transversally. Show that

$$[X] \; = (X \underset{\uparrow}{\cdot} b_1) \cdot [a_1] \; - \; (X \underset{\uparrow}{\cdot} a_1) \cdot [b_1] \; + \; (Y \underset{\uparrow}{\cdot} b_2) \cdot [a_2] \; - \; (X \underset{\uparrow}{\cdot} a_2) \cdot [b_2] \; \in \; \mathrm{H}_1(\Sigma).$$

these are the algebraic intersection numbers of oriented curves

(b) Let X and Y be two closed oriented 1-dimensional submanifolds of Σ that are both transverse to a_1, a_2, b_1, b_2. Show that if X and Y have the same algebraic intersection numbers with a_1, a_2, b_1, b_2, then $[X] = [Y] \in \mathrm{H}_1(\Sigma)$.

Remark. The above gives a practical way, for Σ and similarly for many other smooth manifolds, to determine precisely the homology class represented by a submanifold. In particular it can be used to deal with the type of problems as in Exercise 48.16.

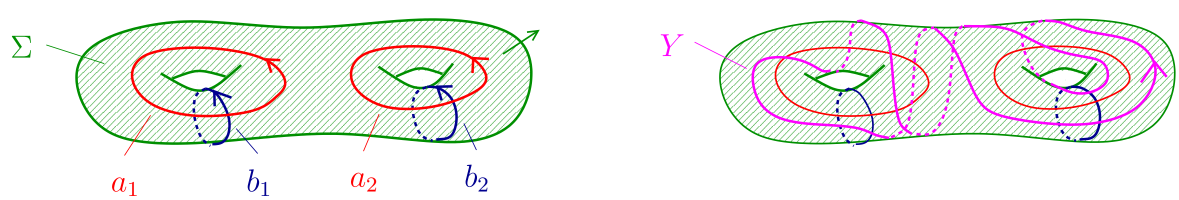


FIGURE 1327. Illustration for Exercise 94.14.

95. Intersections of cycles and the cup product

In this chapter we generalize some of the results from Chapter 94. In particular we will define the "algebraic intersection number" of two maps of topological manifolds into a given topological manifold, as long as the maps intersect "nicely". This notion generalizes the algebraic intersection number of two transverse submanifolds of a given smooth manifold that we introduced on page 2271.

Furthermore we introduce the algebraic intersection number of two singular cycles that intersect "nicely". Afterwards we will see in Theorems 95.7 and 95.9 how these algebraic intersection numbers relate to the cup product. These theorems can be viewed as a generalization of Theorem 94.5.

For many applications the results from Chapter 94 are good enough and for many readers it might make sense to move on to the next chapter. The main application of the results of this chapter will be the Lefschetz-Hopf Theorem 96.9 and the Poincaré-Hopf Theorem 97.5.

Since much of this chapter is quite technical we start out with the following motivating question.

Question 95.1. *As shown in Figure 1328 we consider the disk $\overline{B}^2$ together with the four points $P_\pm = (\pm 1, 0)$ and $Q_\pm = (0, \pm 1)$. Do there exist disjoint paths α and β such that α connects P_- to P_+ and β connects Q_- to Q_+?*

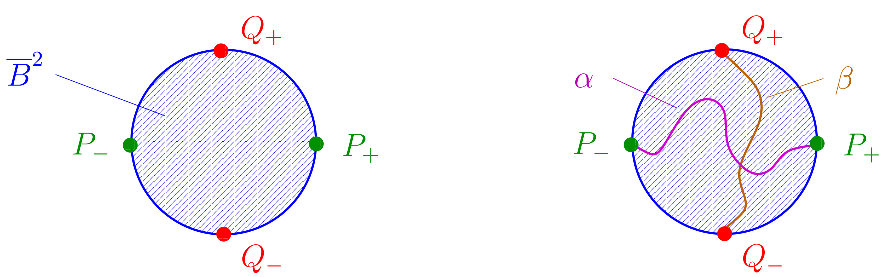


Figure 1328

It is pretty clear that the answer should be no. In fact once we have developed all the tools we will find it quite easy to answer Question 95.1 in the negative. We refer to Lemma 95.8 for the details.

95.1. The algebraic intersection number of maps of topological manifolds.

In this section we introduce the algebraic intersection number of two maps of topological manifolds into a given topological manifold. The precise setup and the definitions require some preparations. First we recall the following definition from page 414.

Definition. Let W be a compact topological manifold and let $A \subset W \setminus \partial W$ be a subset. A *regular neighborhood of* A is a compact codimension-zero submanifold N of W with the following properties:

(1) N is contained in $W \setminus \partial W$ and A is contained in the interior of N.
(2) A is a deformation retract of N.
(3) ∂N is a deformation retract of $N \setminus A$.

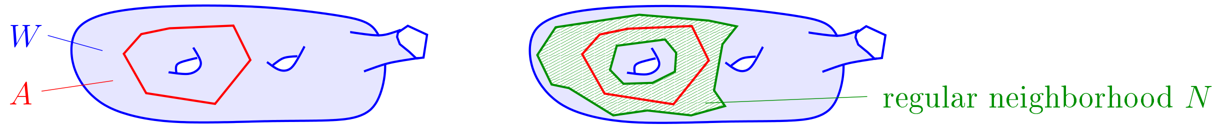


FIGURE 1329

The following proposition can be viewed as a variation on the Poincaré Duality Theorem 88.1.

Proposition 95.2. *Let Z be a compact oriented m-dimensional topological manifold and let A be a closed submanifold of ∂Z.*

(1) *Let $N \subset \partial Z$ be a regular neighborhood of A. We denote by $j\colon A \to N$ the inclusion map. Given any $l \in \mathbb{N}_0$ the following two maps are isomorphisms:*[1347]

$$\mathrm{H}^l(Z, \partial Z \setminus A) \xrightarrow[\cong]{\cap[Z]} \mathrm{H}_{m-l}(Z, N) \xleftarrow[\cong]{j_*} \mathrm{H}_{m-l}(Z, A).$$

(2) *We make the following technical hypothesis:*
(∗) *Given any neighborhood U of $A \subset \partial Z$ there exists a regular neighborhood N of $A \subset \partial Z$ with $N \subset U$.*

Under this hypothesis the isomorphism

$$\mathrm{PD}_Z\colon\ \mathrm{H}^l(Z, \partial Z \setminus A) \ \xrightarrow{\cong}\ \mathrm{H}_{m-l}(Z, A)$$

given in (1) does not depend on the choice of the regular neighborhood N. We denote the inverse of this map by PD_Z as well.

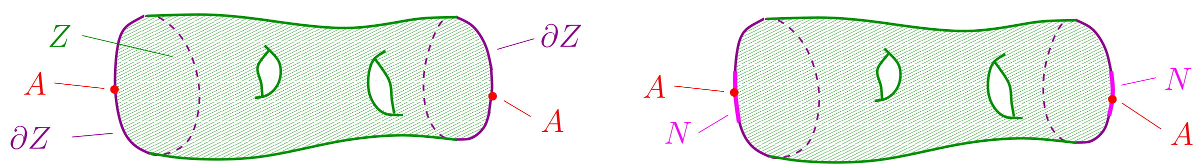


FIGURE 1330. Illustration for Proposition 95.2.

PROOF (∗).

(1) Let $N \subset \partial Z$ be a regular neighborhood of A. We denote by $i\colon \partial Z \setminus \mathring{N} \to \partial Z \setminus A$ and $j\colon A \to N$ the inclusion maps. We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}^l(Z, \partial Z \setminus A) \arrow[rr, "\cap[Z]", "\cong"'] \arrow[dr, "i_*"'] & & \mathrm{H}_{m-l}(Z, N) & \mathrm{H}_{m-l}(Z, A). \arrow[l, "j_*"', "\cong"] \\
& \mathrm{H}^l(Z, \partial Z \setminus \mathring{N}) \arrow[ur, "\cong", "\cap[Z]"'] & &
\end{tikzcd}$$

[1347]It follows from Proposition 74.12 (3) that the cap product $\cap[Z]\colon\ \mathrm{H}^l(Z, \partial Z \setminus A) \to \mathrm{H}_{m-l}(Z, N)$ is actually defined. It is also worth pointing out that it is not possible to go directly from $\mathrm{H}^l(Z, \partial Z \setminus A)$ to $\mathrm{H}_{m-l}(Z, A)$ via capping with $[Z]$, since $(\partial Z, \partial Z \setminus A, A)$ is not an excisive triad.

We make the following observations:

(a) The diagonal cap product exists by Proposition 44.3 (4) together with Proposition 74.12 (4).

(b) It follows from Lemma 83.8 (1) that the triangle commutes.

(c) It follows from the hypothesis that N is a regular neighborhood, together with Lemma 10.1 (1), that the maps i and j are both homotopy equivalences. Thus we obtain from Corollary 43.18 and Lemma 73.13 that the induced maps i^* and j_* are isomorphisms.

(d) It follows from the Poincaré Duality Theorem 88.1, which we can apply by Proposition 44.3 (4), that the right diagonal map is an isomorphism.

The above argument shows that the two horizontal maps are both isomorphisms.

(2) We need to show that the map in (1) does not depend on the choice of the regular neighborhood N. Thus let M and N be regular neighborhoods of $A \subset \partial Z$. It follows immediately from our hypothesis $(*)$ that it suffices to consider the case that $M \subset N$. We consider the following diagram:

$$\begin{tikzcd}
 & \mathrm{H}_{m-l}(Z,M) \arrow[dd] & \\
\mathrm{H}^l(Z,\partial Z\setminus A) \arrow[ur, "\cap[Z]", "\cong"'] \arrow[dr, "\cap[Z]", "\cong"'] & & \mathrm{H}_{m-l}(Z,A). \arrow[ul, "\cong"'] \arrow[dl, "\cong"] \\
 & \mathrm{H}_{m-l}(Z,N) &
\end{tikzcd}$$

Here all unmarked maps are induced by inclusions. The triangle to the left commutes by Lemma 83.8 and the triangle to the right commutes by the functoriality of cohomology groups. But this shows that the upper and the lower path from the left to the right give exactly the same map. ■

Definition. Let Z be a topological manifold. We say a proper submanifold X of Z is *well-behaved* if given any open neighborhood U of $\partial X \subset \partial Z$ the submanifold ∂X of ∂Z admits a regular neighborhood N with $N \subset U$.

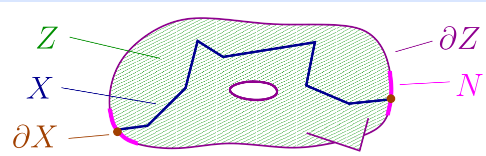


FIGURE 1331

Remark. Almost any proper submanifold X of some topological manifold Z that appears in nature is well-behaved. For example it follows from Proposition 64.12 and Theorem 10.3 that X is well-behaved if ∂X is a PL-submanifold of ∂Z or if ∂X is a smooth submanifold of ∂Z.

In the following we will study maps between topological manifolds. To simplify the discussion we use the following convention.

Convention. Let M be a topological manifold and let X be a submanifold of M. In our notation and our language often we do not distinguish between X as a submanifold and between the embedding $X \to M$.

Next we fix a convenient notation.

Notation. Given a map $f\colon Y \to X$ between topological spaces we write $|f| = f(Y)$.

Now we can give the first key definition of this chapter.

Definition. Let M be an m-dimensional topological manifold, let X be a k-dimensional topological manifold and let Y be an $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be maps.

(1) We say a point $P \in |f| \cap |g|$ is a *nice intersection point of f and g* if given any neighborhood U of P there exists a *nice neighborhood* Z, this means a submanifold $Z \subset M \setminus \partial M$ such that the following holds:
   (a) Z is homeomorphic to $\overline{B}^m$.
   (b) P is contained in[1348] $\mathring{Z} = Z \setminus \partial Z$ and Z is contained in U.
   (c) $|f| \cap |g| \cap Z = \{P\}$.
   Before we continue we set $\widetilde{X} := f^{-1}(|f| \cap Z)$ and we set $\widetilde{Y} := g^{-1}(|g| \cap Z)$. With this notation we also demand the following:
   (d) $\widetilde{X}$ is a compact codimension-zero submanifold of X and $\widetilde{Y}$ is a compact codimension-zero submanifold of Y.
   (e) The maps $f\colon \widetilde{X} \to Z$ and $g\colon \widetilde{Y} \to Z$ are both proper embeddings.
   (f) $f(\widetilde{X})$ and $g(\widetilde{Y})$ are both well-behaved submanifolds of Z.
   (g) $f(\widetilde{X})$ is homeomorphic to $\overline{B}^k$ and $g(\widetilde{Y})$ is homeomorphic to $\overline{B}^{m-k}$.
   In Figure 1332 we illustrate the definition for two different nice intersection points.
(2) We say that f and g *intersect nicely*, if all points in $|f| \cap |g|$ are nice intersection points.

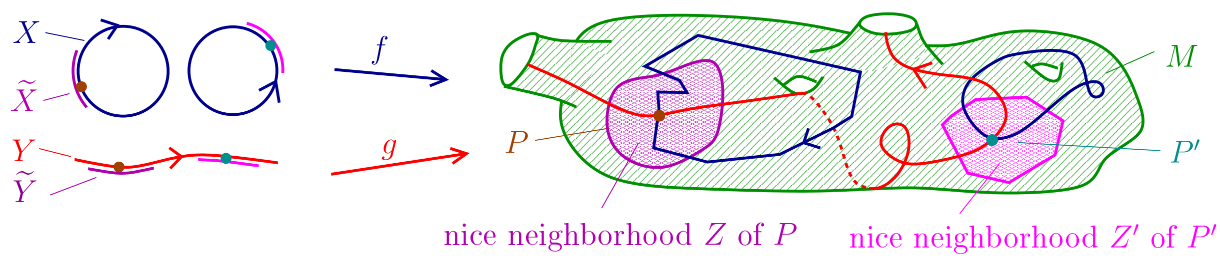


FIGURE 1332

Remark. By the above convention we now also defined the notion of a nice intersection point of two submanifolds and we defined the notion that two submanifolds intersect nicely.

Convention. Let W be an oriented topological manifold. We equip any codimension-zero submanifold of W with the orientation coming from Lemma 86.17.

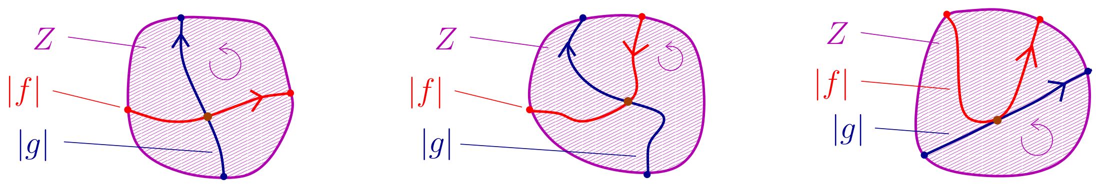
[1348] Note that by (a) we know that Z is a codimension-zero submanifold of M. Since $Z \subset M \setminus \partial M$ we know by Proposition 44.3 that the boundary of Z as a topological manifold agrees with the boundary of Z as a subset of M.

Now we are ready to give the second key definition of this chapter.

Definition. Let M be an *oriented* m-dimensional topological manifold, let X be an *oriented* k-dimensional topological manifold and let Y be an *oriented* $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X\to M$ and $g\colon Y\to M$ be maps and let P be a nice intersection point of f and g. We pick a nice neighborhood Z of P. We define the *index of f and g at P* as follows:[1349]

$$\operatorname{index}(f,g,P) \;=\; \Big\langle \overbrace{\underbrace{\mathrm{PD}_Z(\underbrace{f_*([\widetilde{X}])}_{\in \mathrm{H}_k(Z,\partial f(\widetilde{X}))})}_{\in \mathrm{H}^{m-k}(Z,\partial Z\setminus \partial f(\widetilde{X}))} \cup \underbrace{\mathrm{PD}_Z(\underbrace{g_*([\widetilde{Y}])}_{\in \mathrm{H}_{m-k}(Z,\partial(g(\widetilde{Y})))})}_{\in \mathrm{H}^{k}(Z,\partial Z\setminus \partial g(\widetilde{Y}))}}^{\in \mathrm{H}^m(Z,\partial Z)},\; \overbrace{[\,Z\,]}^{\in \mathrm{H}_m(Z,\partial Z)} \Big\rangle \;\in\; \mathbb{Z}.$$

Example. In Figure 1333 we show, in three cases, the indices of nice intersection points. The first two indices can be determined with some effort using Proposition 88.2 together with Proposition 80.11. Alternatively see Proposition 95.5 below. Finally we outsource the calculation of the third index to Exercise 95.2.



index$(f,g,P)=+1$ index$(f,g,P)=-1$ index$(f,g,P)=0$

FIGURE 1333

Next we need to show that the index of a nice intersection point is well-defined.

Lemma 95.3. *Let M, $f\colon X\to M$ and $g\colon Y\to M$ as in the previous definition and let P be a nice intersection point of f and g. The corresponding index* index(f,g,P) *is well-defined, i.e. it is independent of the choice of the nice neighborhood Z of P.*

SKETCH OF A PROOF $(*)$. By the "given any neighborhood U of P" aspect of the definition of a nice intersection point it suffices to deal with nice neighborhoods Z_1 and Z_2 with

[1349] Here are a few clarifications:

(1) By hypothesis we know that Z, $\widetilde{X}$ and $\widetilde{Y}$ are compact codimension-zero submanifolds of M, X and Y. As we just remarked, by Lemma 86.17 they inherit an orientation from the given orientations on M, X and Y. As always we denote by $[Z]$, $[\widetilde{X}]$ and $[\widetilde{Y}]$ the corresponding fundamental classes.
(2) By hypothesis $f(\widetilde{X})$ and $g(\widetilde{Y})$ are well-behaved submanifolds of Z. Thus we obtain from Proposition 95.2 the corresponding maps PD_Z.
(3) By hypothesis $f(\widetilde{X})$ and $g(\widetilde{Y})$ are compact. It follows from Proposition 44.2 (4b) that $\partial f(\widetilde{X})$ and $\partial g(\widetilde{Y})$ are compact. By Lemma 2.17 (2) this implies that $\partial Z\setminus\partial f(\widetilde{X})$ and $\partial Z\setminus \partial g(\widetilde{Y})$ are open subsets of ∂Z. Since P is the only intersection point of f and g in Z we see that these two open subsets cover all of ∂Z. It now follows from Proposition 74.12 (4) that the cup product is actually defined.

$Z_1 \subset \mathring{Z}_2$. We define $\widetilde{X}_1$, $\widetilde{X}_2$, $\widetilde{Y}_1$ and $\widetilde{Y}_2$ in the obvious way. We pick regular neighborhoods A_1 of $\partial X_1 \subset \partial Z_1$ and B_1 of $\partial Y_1 \subset \partial Z_1$. Similarly we pick A_2 and B_2 with the obvious meaning.

Let $i \in \{1,2\}$. We write $C_i := \partial Z_i \setminus \mathring{A}_i$ and $D_i := \partial Z_i \setminus \mathring{B}_i$. Similar to the proof of the Excision Theorem for Topological Manifolds 44.10 we use the Topological Collar Neighborhood Theorem 44.5 to find open subsets $\widetilde{A}_i$, $\widetilde{B}_i$, $\widetilde{C}_i$ and $\widetilde{D}_i$ of ∂Z_i that admit deformation retractions to A_i, B_i, C_i respectively D_i. Now we consider the following maps of triads of topological spaces:

$$\begin{array}{ccccccccc} Z_1 & & Z_1 & & Z_2 & & Z_2 & & Z_2 \\ A_1 & \to & \widetilde{A}_1 & \to & \widetilde{A}_1 \cup ((\mathring{Z}_2 \setminus Z_1) \cup \widetilde{A}_2) & \leftarrow & \widetilde{A}_2 & \leftarrow & A_2 \\ \partial Z_1 \setminus A_1 & & \partial Z_1 \setminus A_1 & & (\partial Z_1 \setminus A_1) \cup (\mathring{Z}_2 \setminus Z_1) \cup (\partial Z_2 \setminus A_2) & & \partial Z_2 \setminus A_2 & & \partial Z_1 \setminus A_2. \end{array}$$

It follows from Proposition 74.12 (4) that all these triads are excisive. Evidently the analogous statement also holds if we replace the A's by B's. Now the lemma can be deduced from Proposition 82.4 and Lemma 83.8 together with Lemma 87.24. We leave it to the reader to fill in the details. ■

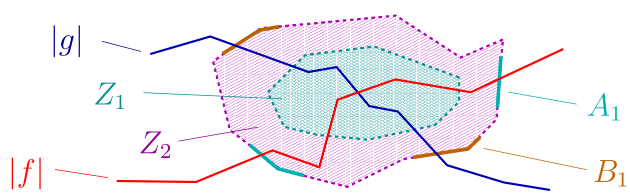


FIGURE 1334. Illustration for the proof of Lemma 95.3.

The following lemma is a generalization of Lemma 94.2.

Lemma 95.4. *Let M be an m-dimensional topological manifold, let X be a compact k-dimensional topological manifold and let Y be a compact $(m-k)$-dimensional topological manifold. Furthermore let $f: X \to M$ and $g: Y \to M$ be maps. If f and g intersect nicely, then $|f| \cap |g|$ is finite.*

PROOF (∗). Let P be a nice intersection point of f and g. We pick a nice neighborhood Z for P. Note that this means in particular that Z is a codimension-zero submanifold of M with $Z \subset M \setminus \partial M$, $P \in Z \setminus \partial Z$ and with $|f| \cap |g| \cap Z = \{P\}$. By Proposition 44.3 (3) we know that $Z \setminus \partial Z$ is an open subset of M. We have thus shown that P is an isolated point, in the sense of the definition on page 174, of the subspace $|f| \cap |g|$. Since by hypothesis every intersection point is nice we see that $|f| \cap |g|$ is a discrete subset of M. Since X and Y are compact we know by Lemma 2.40 together with Lemma 2.16 (2) that $|f| \cap |g|$ is also compact. Thus we see that $|f| \cap |g|$ is a discrete compact topological space. By Lemma 2.18 (1) this implies that $|f| \cap |g|$ is a finite set. ■

By Lemma 95.4 it now makes sense to introduce the following definition which can be viewed as an analogue of the definition on page 2271.

Definition. Let M be an oriented m-dimensional topological manifold. Let X be a compact oriented k-dimensional topological manifold and let Y be a compact oriented $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be maps. If f and g intersect nicely, then we define the *algebraic intersection number of f and g* to be

$$f \cdot g \;:=\; \underset{\substack{\uparrow \\ \text{finite sum by Lemma 95.4}}}{\sum_{P \in |f| \cap |g|}} \operatorname{index}(f,g,P).$$

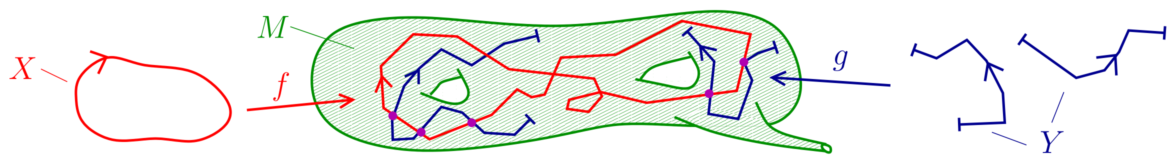


FIGURE 1335

Remark. Recall that by our convention on page 2304 we do not distinguish between a submanifold X of M and the actual embedding $X \to M$. In particular in the above definition we also introduced the algebraic intersection number $X \cdot Y$ of two suitable submanifolds of M.

95.2. **Transverse intersection points of maps.** In this section we will see that the notion of a nice intersection points incorporates the notion of a transverse intersection point that we introduced on pages 2270. We will also see that the concept of the algebraic intersection number that we defined in the previous section is a generalization of the algebraic intersection number that we introduced on page 2271.

The following definition can be viewed as a generalization of the concepts we introduced on pages 306 and 2270.

Definition. Let M be an m-dimensional smooth manifold and let $k \in \{1,\dots,m-1\}$. Furthermore let X be a k-dimensional smooth manifold and let Y be a $(m-k)$-dimensional smooth manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be smooth maps.

(1) We say a point $P \in |f| \cap |g|$ is a *transverse intersection point of f and g* if $P \in M \setminus \partial M$, if $f^{-1}(\{P\})$ contains a single point x_0, if $g^{-1}(\{P\})$ contains a single point y_0, if $x_0 \in X \setminus \partial X$ and $y_0 \in Y \setminus \partial Y$, and if $f_*(\mathrm{T}_{x_0} X) + g_*(\mathrm{T}_{y_0} Y) = \mathrm{T}_P M$.

(2) Now we assume that M, X and Y are oriented. Let P be a transverse intersection point of f and g. We set $\operatorname{sign}(f,g,P) := +1$ if

$$(f_*(\text{positive basis of } \mathrm{T}_{x_0} X), g_*(\text{positive basis of } \mathrm{T}_{y_0} Y)) \;=\; \text{positive basis of } \mathrm{T}_P M,$$

and otherwise we set $\operatorname{sign}(f,g,P) := -1$.

Remark. If X and Y are compact proper submanifolds of M and if f and g are just the inclusion maps, then the above definitions are precisely the definitions provided on pages 306 and 2270.

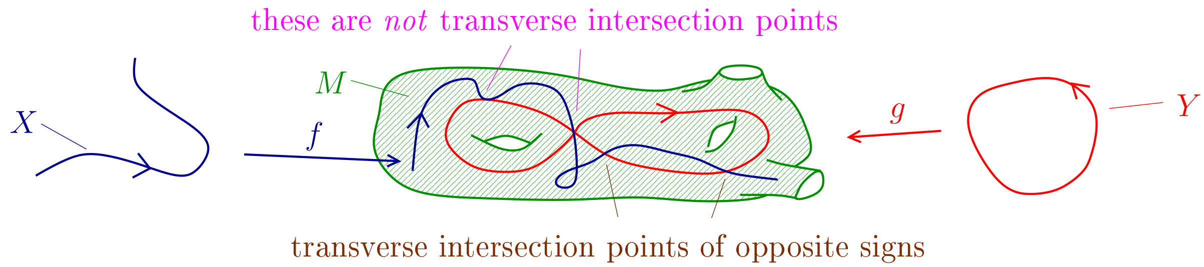


FIGURE 1336

Proposition 95.5. *Let M be an m-dimensional smooth manifold and let $k \in \{1, \ldots, m-1\}$. Furthermore let X be a k-dimensional smooth manifold and let Y be a $(m-k)$-dimensional smooth manifold. Finally let $f: X \to M$ and $g: Y \to M$ be smooth maps.*

(1) *Let $P \in |f| \cap |g|$. If P is a transverse intersection point of f and g, then P is a nice intersection point of f and g.*
(2) *If M, X and Y are oriented, then for any transverse intersection point P of f and g we have $\operatorname{index}(f, g, P) = \operatorname{sign}(f, g, P)$.*

PROOF. The alert reader will doubtlessly notice that the proof of the proposition mostly recycles ideas from the proof of Proposition 94.8 and Lemma 94.9.

Let U be an open neighborhood of P. It follows from Exercise 8.1, which can be solved using the Inverse Function Theorem 6.18, that there exists an open neighborhood A of x_0 such that the restriction of f to $f: A \to M$ is a smooth embedding and such that $f(A) \subset U$. Furthermore, using the hypothesis that $f^{-1}(\{P\}) = \{x_0\}$ and using Exercise 6.25 we see that, after possibly shrinking A, we can assume that $f^{-1}(f(A)) = A$.

The same way we can evidently find an open neighborhood B of y_0 such that $g: B \to M$ is a smooth embedding, such that $g(B) \subset U$ and such that $g^{-1}(g(B)) = B$.

By Proposition 8.1 we know that $f(A)$ and $g(B)$ are smooth submanifolds of M. By our hypothesis we know that P is a transverse intersection point of $f(A)$ and $g(B)$. It follows from Lemma 6.52 that there exists a chart $\Phi: V \to W$ for M around P with

$$\Phi(V \cap f(X)) \;=\; W \cap (\mathbb{R}^k \times \{0\}) \quad \text{and} \quad \Phi(V \cap g(Y)) \;=\; W \cap (\{0\} \times \mathbb{R}^{m-k}).$$

Next note that after possibly rescaling W we can assume that $\overline{B}^k \times \overline{B}^{m-k} \subset W$. We set $Z := \Phi^{-1}(\overline{B}^k \times \overline{B}^{m-k})$ and $\widetilde{X} := f^{-1}(Z)$ and $\widetilde{Y} := g^{-1}(Z)$. Note that by construction the maps $\Phi \circ f: \widetilde{X} \to \overline{B}^k \times \{0\}$ and $\Phi \circ g: \widetilde{Y} \to \{0\} \times \overline{B}^{m-k}$ are diffeomorphisms. It follows easily from the construction that Z has all the properties stated in the definition of a nice intersection point. This concludes the proof of Statement (1).

Now assume that M, X and Y are oriented. We set $\sigma := \operatorname{sign}(f, g, P)$. We make the following preparations:

(1) We equip each $\overline{B}^n$ with the usual orientation coming from $\mathbb{R}^n$.

(2) After possibly composing $\Phi\colon Z \to \overline{B}^k \times \overline{B}^{m-k}$ with reflections in some $(x_i = 0)$-hyperplanes we can assume that the diffeomorphisms $\Phi\circ f\colon f^{-1}(Z) = \widetilde{X} \to \overline{B}^k \times \{0\}$ and $\Phi \circ g\colon g^{-1}(Z) = \widetilde{Y} \to \{0\} \times \overline{B}^{m-k}$ are both orientation-preserving.

(3) We equip the topological manifold $\overline{B}^k \times \overline{B}^{m-k}$ with the product orientation as defined on page 1969. It follows from Proposition 80.11 that with this orientation the inclusion map $\overline{B}^k \times \overline{B}^{m-k} \to \mathbb{R}^m$ is orientation-preserving.

(4) Elementary linear algebra shows that (2) implies that $\Phi\colon Z \to \overline{B}^k \times \overline{B}^{m-k}$ is orientation-preserving if and only if $\sigma = +1$.

(5) It follows from Proposition 72.2 and Lemma 86.12 that $\Phi_*([Z]) = \sigma \cdot [\overline{B}^k \times \overline{B}^{m-k}]$.

Now we perform the following calculation:

$$\begin{aligned}
&\overbrace{\big\langle \mathrm{PD}_Z(f_*[\widetilde{X}])\cup \mathrm{PD}_Z(g_*[\widetilde{Y}]), [Z]\big\rangle}^{=\mathrm{index}(f,g,P),\text{ by definition}} \underset{\downarrow}{\overset{\text{by Lemma 74.6 (3) and Lemma 81.10}}{=}} \big\langle \Phi^*(\mathrm{PD}_Z(f_*[\widetilde{X}]))\cup\Phi^*(\mathrm{PD}_Z(g_*[\widetilde{Y}])), \Phi_*([Z])\big\rangle \\
&\underset{\uparrow}{=} \big\langle \sigma\cdot \mathrm{PD}_{\overline{B}^k\times\overline{B}^{m-k}}([\overline{B}^k\times 0]) \cup \sigma\cdot \mathrm{PD}_{\overline{B}^k\times\overline{B}^{m-k}}([0\times\overline{B}^{m-k}])), \sigma\cdot[\overline{B}^k\times\overline{B}^{m-k}]\big\rangle \\
&\text{by (4) and the pretty elementary Exercise 88.16} \\
&= \sigma^3\cdot \underbrace{\big\langle \mathrm{PD}_{\overline{B}^k\times\overline{B}^{m-k}}([\overline{B}^k\times 0]) \cup \mathrm{PD}_{\overline{B}^k\times\overline{B}^{m-k}}([0\times\overline{B}^{m-k}])), [\overline{B}^k\times\overline{B}^{m-k}]\big\rangle}_{=1 \text{ by Proposition 88.2}} = \underbrace{\mathrm{sign}(f,g,P)}_{=\sigma}.
\end{aligned}$$

■

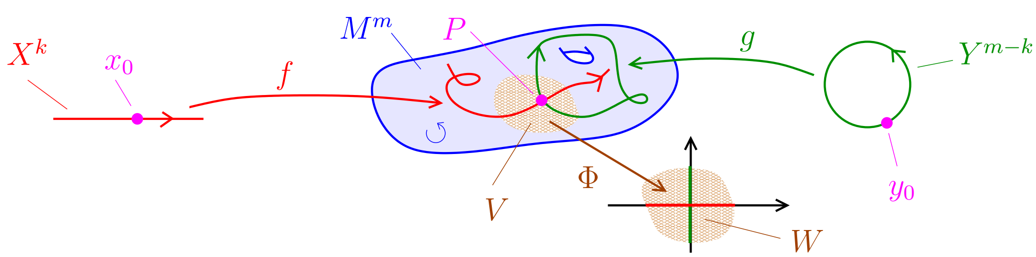


FIGURE 1337. Illustration for the proof of Proposition 95.5.

The following is an immediate corollary to Proposition 95.5.

Corollary 95.6. *Let M be an oriented m-dimensional smooth manifold, let X be an oriented k-dimensional submanifold and let Y be an oriented $(m-k)$-dimensional submanifold of M. We denote by $i\colon X \to M$ and $j\colon Y \to M$ the inclusion maps. If the submanifolds are transverse in the sense of the definition on page 306, then the maps $i\colon X \to M$ and $j\colon Y \to M$ are intersect nicely and the two definitions of the algebraic intersection given on pages 2271 and 2307 agree.*

By Corollary 95.6 the following theorem is a generalization of Theorem 94.5.

Theorem 95.7. *Let M be a compact oriented connected m-dimensional topological manifold. Furthermore let A and B be two compact $(m-1)$-dimensional submanifolds of ∂M with $\partial A = \partial B = A\cap B$ and with $\partial M = A\cup B$. Let X be a compact oriented k-dimensional* *topological manifold and let Y be a compact oriented $(m-k)$-dimensional topological manifold. Finally let $f\colon X \to M$ and $g\colon Y \to M$ be two proper maps with $f(\partial X) \subset A$ and $g(\partial Y) \subset B$. If f and g intersect nicely, then*

$$f \cdot g = \Big\langle \overbrace{\underbrace{\mathrm{PD}_M(\underbrace{f_*([X])}_{\in \mathrm{H}_k(M,A)})}_{\in \mathrm{H}^{m-k}(M,B)} \cup \underbrace{\mathrm{PD}_M(\underbrace{g_*([Y])}_{\in \mathrm{H}_{m-k}(M,B)})}_{\in \mathrm{H}^{k}(M,A)}}^{\in \mathrm{H}^m(M,\partial M)}, \overbrace{[M]}^{\in \mathrm{H}_m(M,\partial M)} \Big\rangle \in \mathbb{Z}.$$

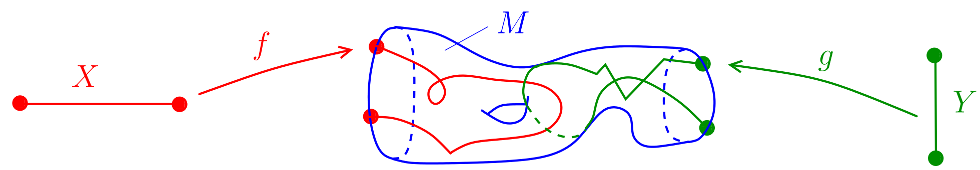


FIGURE 1338. Illustration of Theorem 95.7.

Remark. It follows immediately from Proposition 95.5 that Theorem 95.7 contains our earlier Theorem 94.5 as a special case.

In Theorem 95.14 below we will give a result which takes more time to formulate and which implies the above Theorem 95.7. To keep at least the flow of the definitions and statements coherent we decided to formulate Theorem 95.14 in this section. But rather awkwardly this means that we now provide a proof of Theorem 95.7 that relies on the future Theorem 95.14.

PROOF OF THEOREM 95.7 USING THEOREM 95.14. It follows easily from Lemma 95.12 that there exists a k-cycle $\sigma \in \mathrm{C}_k(M,A)$ and an $(m-k)$-cycle $\tau \in \mathrm{C}_{m-k}(M,B)$ with the following properties:

(1) we have $[\sigma] = f_*([X])$ and $[\tau] = g_*([Y])$,
(2) the cycles σ and τ are transverse,
(3) the equality $f \cdot g = \sigma \cdot \tau$ holds.

It follows that

$$f \cdot g \;=\; \sigma \cdot \tau \underset{\substack{\uparrow \\ \text{Theorem 95.14}}}{=} \langle \mathrm{PD}_M([\sigma]) \cup \mathrm{PD}_M([\tau]), [M] \rangle \underset{\substack{\uparrow \\ \text{since } [\sigma] = f_*([X]) \text{ and } [\tau] = g_*([Y])}}{=} \langle \mathrm{PD}_M(f_*[X]) \cup \mathrm{PD}_M(g_*[Y]), [M] \rangle.$$

■

As promised, we now find it quite easy to answer Question 95.1 in the negative.

Lemma 95.8. *We consider the disk $\overline{B}^2$ together with the four points $P_\pm = (\pm 1, 0)$ and $Q_\pm(0, \pm 1)$. If $\alpha\colon [-1,1] \to \overline{B}^2$ is a proper map with $\alpha(\pm 1) = P_\pm$ and if $\beta\colon [-1,1] \to \overline{B}^2$ is a proper map with $\beta(\pm 1) = Q_\pm$, then α and β intersect in at least one point.*

PROOF. We write $I = [-1,1]$. Suppose that there exist proper maps $\alpha\colon I \to \overline{B}^2$ and $\beta\colon I \to \overline{B}^2$ with $\alpha(\pm 1) = P_\pm$ and with $\beta(\pm 1) = Q_\pm$ such that $|\alpha| \cap |\beta| = \varnothing$. We introduce the following notation:

(1) We consider the "straight paths" $\alpha'\colon I \to \overline{B}^2$ given by $\alpha'(t) = (t,0)$ and $\beta'\colon I \to \overline{B}^2$ given by $\beta'(t) = (0,t)$.
(2) We write $A = \left\{e^{it} \,\middle|\, t \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]\right\}$ and $B = \left\{e^{it} \,\middle|\, t \in \left[-\frac{3\pi}{4}, -\frac{\pi}{4}\right] \cup \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]\right\}$.
(3) We denote by $\mathrm{PD}_{\overline{B}^2}$ the isomorphisms from the Poincaré Duality Theorem 88.1 corresponding to the decomposition $\partial \overline{B}^2 = A \cup B$.

We refer to Figure 1339 for an illustration. We have

$$\begin{aligned}
0 \overset{\substack{\text{since } |\alpha|\cap|\beta|=\varnothing\\ \downarrow}}{=} \alpha\cdot\beta &\overset{\substack{\text{Theorem 95.7}\\ \downarrow}}{=} \left\langle \mathrm{PD}_{\overline{B}^2}(\alpha_*([I])) \cup \mathrm{PD}_{\overline{B}^2}(\beta_*([I]), [\overline{B}^2]\right\rangle \\
&\underset{\substack{\uparrow\\ \text{by considering the long exact sequences in homology}\\ \text{of the pairs } (\overline{B}^2, A) \text{ and } (\overline{B}^2, B) \text{ we see that}\\ \alpha_*([I]) = \alpha'_*([I]) \text{ and } \beta_*([I]) = \beta'_*([I])}}{=} \left\langle \mathrm{PD}_{\overline{B}^2}(\alpha'_*([I])) \cup \mathrm{PD}_{\overline{B}^2}(\beta'_*([I]), [\overline{B}^2]\right\rangle \underset{\substack{\uparrow\\ \text{Theorem 95.7}}}{=} \alpha'\cdot\beta' \underset{\substack{\uparrow\\ \alpha',\beta' \text{ intersect}\\ \text{nicely in one}\\ \text{point}}}{=} \pm 1.
\end{aligned}$$

We have thus obtained a contradiction. ■

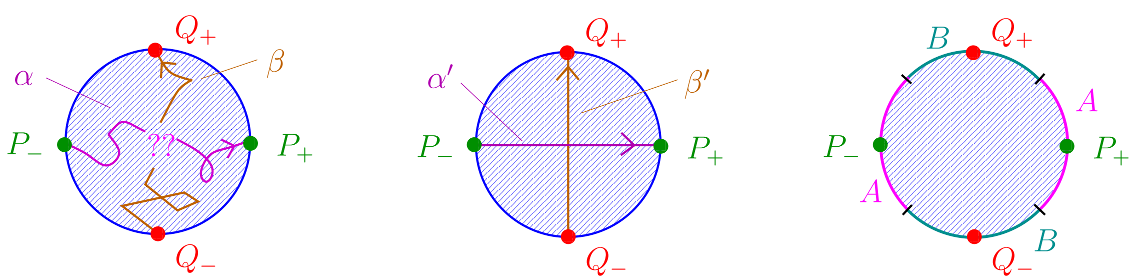


FIGURE 1339

95.3. **Intersections of cycles.** Now we turn to the study of singular chains in a topological manifold. To be on the safe side in our subsequent definitions we introduce the following convention.

Convention.

(1) Let X be a topological space. When we write "let $\sigma = a_1 \cdot \sigma_1 + \dots + a_r \cdot \sigma_r$ be a singular m-chain in X" it is is understood that the σ_i are pairwise different and that all the a_i are non-zero. We write $|\sigma| = |\sigma_1| \cup \dots \cup |\sigma_r|$.
(2) Let M be an oriented m-dimensional topological manifold M and let $\sigma\colon \Delta^k \to M$ and $\tau\colon \Delta^{m-k} \to M$ be singular simplices. Since each Δ^n is in particular a topological manifold it makes sense to say that the singular simplices intersect nicely. Furthermore, throughout this chapter we equip the topological manifold Δ^n with the orientation given by the discussion on page 1075 together with Proposition 86.11. In particular, if $\sigma\colon \Delta^k \to M$ and $\tau\colon \Delta^{m-k} \to M$ intersect nicely, then we obtain an algebraic intersection number $\sigma \cdot \tau$.

Definition. Let M be an m-dimensional topological manifold and let $\sigma = a_1\cdot\sigma_1+\cdots+a_r\cdot\sigma_r$ be a singular k-chain and let $\tau = b_1\cdot\tau_1+\cdots+b_s\cdot\tau_s$ be a singular $(m-k)$-chain in M. We say that σ and τ *intersect nicely* if the following two conditions are satisfied:

(1) For all $i \in \{1,\dots,r\}$ and $j\in\{1,\dots,s\}$ the maps $\sigma_i\colon \Delta^k \to M$ and $\tau_j\colon \Delta^{m-k}\to M$ intersect nicely.

(2) For all $(i,j)\neq(i',j')\in\{1,\dots,r\}\times\{1,\dots,s\}$ we have
$$(|\sigma_i|\cap|\tau_j|)\cap(|\sigma_{i'}|\cap|\tau_{j'}|) \;=\; \emptyset.$$
Put differently, any intersection point is the intersection point of a unique pair of singular simplices.

If this is the case and if M is oriented, then we define the *algebraic intersection number of σ and τ* to be
$$\sigma\cdot\tau \;\;:=\; \sum_{i,j}(a_i\cdot b_j)\cdot(\underset{\uparrow}{\sigma_i\cdot\tau_j}).$$
algebraic intersection number of $\sigma_i\colon\Delta^k\to M$ and $\tau_j\colon\Delta^{m-k}\to M$

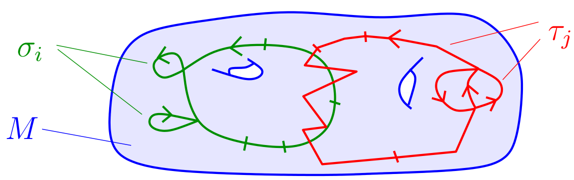


FIGURE 1340

Before we can state the main theorem of this chapter we recall the formulation of the Poincaré Duality theorem for (not necessarily compact) topological manifolds without boundary. Thus let W be an oriented m-dimensional topological manifold without boundary and let $K\subset W$ be a compact subset. By Theorem 87.10 there exists a unique lift of the orientation classes of W to a class in $\mathrm{H}_m(W,W\setminus K)$. Given any $k\in\mathbb{N}_0$ we denote by
$$-\cap W\colon\; \mathrm{H}^k(W,W\setminus K)\;\to\;\mathrm{H}_{m-k}(W)$$
the map that is given by capping with this unique lift. Lemma 88.18 says that these maps define a well-defined map
$$-\cap W\colon\; \mathrm{H}^k_{\mathrm{c}}(W)\;\to\;\mathrm{H}_{m-k}(W).$$
Furthermore the Poincaré Duality Theorem 88.19 says that this map is an isomorphism. We denote by
$$\mathrm{PD}_W\colon\; \mathrm{H}_{m-k}(W)\;\to\;\mathrm{H}^k_{\mathrm{c}}(W)$$
the inverse map. Finally we remind the reader that given a topological space X we denote by $\epsilon_X=\epsilon\colon\; \mathrm{H}_0(X)\to\mathbb{Z}$ the augmentation map.

Now we can formulate the main theorem of this chapter.

Theorem 95.9. *Let M be an oriented m-dimensional topological manifold without boundary. Let $\sigma \in \mathrm{C}_k(M)$ and $\tau \in \mathrm{C}_{m-k}(M)$ be two cycles. If σ and τ intersect nicely, then*[1350]

$$\sigma \cdot \tau \;=\; \epsilon_M\Big(\underbrace{\big(\overbrace{\mathrm{PD}_M([\sigma])}^{\in \mathrm{H}_c^{m-k}(M)} \cup \overbrace{\mathrm{PD}_M([\tau])}^{\in \mathrm{H}_c^{k}(M)}\big)}_{\in \mathrm{H}_c^m(M)} \cap M\Big).$$

95.4. **Proof of Theorem 95.9.** Before we can provide the proof of Theorem 95.9 we need to recall some definitions and we need to prove several lemmas. First let us recall the following definition from page 120.

Definition. Let X be a topological space.

(1) Let A and B be disjoint subsets of X. We say U and V are *separating open neighborhoods for A and B* if U and V are disjoint, if U is an open neighborhood of A and if V is an open neighborhood of B.
(2) A topological space X is called *normal* if any two disjoint closed subsets A and B admit separating open neighborhoods.

Proposition 95.10.

(1) *Every topological manifold is normal.*
(2) *Let M be a topological manifold and let A and B be two closed disjoint subsets of M. If $U' \subset \partial M$ and $V' \subset \partial M$ are separating open neighborhoods for $A \cap \partial M$ and $B \cap \partial M$, then there exist separating open neighborhoods U and V of A and B such that $U \cap \partial M = U'$ and $V \cap \partial M = V'$.*

FIGURE 1341. Illustration of Proposition 95.10.

Remark. For many situations it suffices to find separating open neighborhoods for *compact* subsets A and B of a Hausdorff space X. In this situation the existence of such open neighborhoods is proved in Lemma 2.46.

SKETCH OF PROOF (∗).

(1) First note that by Corollary 9.2 the topological space M is metrizable, i.e. M admits a metric such that the topology from the metric agrees with the given topology. It now follows from Proposition 2.47 that M is normal.
(2) Let M be a topological manifold and let A and B be two closed disjoint subsets of M. We pick a collar neighborhood $\partial M \times [0, 1]$ as given by the Topological Collar Neighborhood Theorem 44.5.

    Suppose that we are given separating neighborhoods $U' \subset \partial M$ and $V' \subset \partial M$ for $A \cap \partial M$ and $B \cap \partial M$. By Proposition 44.2 the complement $M \setminus \partial M$ is again a smooth

[1350] We refer to page 2008 for the definition of the cup product on cohomology with compact support.

manifold. By (1) it is normal. Thus there exist separating open neighborhoods $\widetilde{U}$ and $\widetilde{V}$ for the closed subsets $A \cap (M \setminus \partial M)$ and $B \cap (M \setminus \partial M)$. It follows from Lemma 6.7 (1) that $\widetilde{U}$ and $\widetilde{V}$ are open subsets of M. It is now straightforward to verify that $U := \widetilde{U} \cup ([0,1) \times U')$ and $V := \widetilde{V} \cup ([0,1) \times V')$ have all the desired properties. ■

Lemma 95.11. *Let M be an oriented m-dimensional topological manifold with $\partial M = \varnothing$ and let $U \subset M$ be an open subset. Let $i\colon U \to M$ be the inclusion map. For any compact subset $K \subset U$ and any $k \in \mathbb{N}_0$ the map*

$$i^*\colon\ \mathrm{H}^{m-k}(M, M \setminus K) \ \to\ \mathrm{H}^{m-k}(U, U \setminus K)$$

is an isomorphism and the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}^{m-k}_{\mathrm{c}}(U) \arrow[rr, "\cap U = \mathrm{PD}_U^{-1}", "\cong"'] \arrow[dd, "i_*"] & & \mathrm{H}_k(U) \arrow[dd, "i_*"] \\
& \mathrm{H}^{m-k}(U, U \setminus K) \arrow[ul] \arrow[ur, "\cap U"'] \arrow[dd, "(i^*)^{-1}"'] & \\
\mathrm{H}^{m-k}_{\mathrm{c}}(M) \arrow[rr, "\cap M = \mathrm{PD}_M^{-1}", "\cong"'] & & \mathrm{H}_k(M). \\
& \mathrm{H}^{m-k}(M, M \setminus K) \arrow[ul] \arrow[ur, "\cap M"'] &
\end{tikzcd}$$

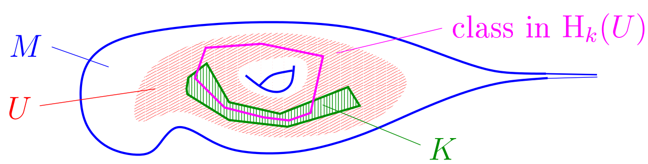


FIGURE 1342. Illustration for Lemma 95.11.

PROOF. We make the following comments and observations:

(i) By excision, see page 1840, the map $i^*\colon\ \mathrm{H}^{m-k}(M, M \setminus K) \to \mathrm{H}^{m-k}(U, U \setminus K)$ is an isomorphism.

(iii) The triangles on the top and the bottom commute by definition of PD_U and PD_M, see Lemma 88.18.

(iv) The square to the left commutes by definition of the map $i_*\colon\ \mathrm{H}^{m-k}_{\mathrm{c}}(U) \to \mathrm{H}^{m-k}_{\mathrm{c}}(M)$, see page 1908.

(v) It follows from Lemma 83.8 and Corollary 87.12 (2) that the square to the right commutes.

(vi) The rectangle in the back commutes by Lemma 88.25. ■

As in the case of Theorem 94.5 it is perhaps instructive to first prove Theorem 95.9 in the special case that the cycles do not intersect.

PROOF OF THEOREM 95.9 FOR DISJOINT CYCLES. Let M be an oriented m-dimensional topological manifold with $\partial M = \varnothing$. Let $\sigma \in \mathrm{C}_k(M)$ and $\tau \in \mathrm{C}_{m-k}(M)$ be two cycles such that $|\sigma| \cap |\tau| = \varnothing$. We have to show that $\mathrm{PD}_M([\sigma]) \cup \mathrm{PD}_M([\tau]) = 0$.

By Proposition 95.10 we can find separating open neighborhoods U and V for $|\sigma|$ and $|\tau|$. Given a compact subset J of M and given $i \in \mathbb{N}_0$ we denote by $\varphi_J\colon\ \mathrm{H}^i(M, M\setminus J) \to \mathrm{H}^i_c(M)$ the natural map.

Claim.

(1) There exists a compact subset $K \subset U$ and a cohomology class $\alpha \in \mathrm{H}^{m-k}(M, M \setminus K)$ such that $\varphi_K(\alpha) = \mathrm{PD}_M([\sigma])$.
(2) There exists a compact subset $L \subset V$ and a cohomology class $\beta \in \mathrm{H}^k(M, M \setminus L)$ such that $\varphi_L(\beta) = \mathrm{PD}_M([\tau])$.

We consider $\mathrm{PD}_U([\sigma]) \in \mathrm{H}^{m-k}_c(U)$. By definition of cohomology with compact support there exists a compact subset K of U such that $\mathrm{PD}_U([\sigma])$ lies in the image of the natural map $\mathrm{H}^{m-k}(U, U \setminus K) \to \mathrm{H}^{m-k}_c(U)$. The first statement of the claim now follows from Lemma 95.11. The proof of the second statement is evidently basically the same. ⊞

We consider the following diagram

$$\begin{tikzcd}
\mathrm{H}^{m-k}_c(M) \times \mathrm{H}^k_c(M) \arrow[rr, "\cup"] & & \mathrm{H}^m_c(M) \\
\mathrm{H}^{m-k}(M, M\setminus K) \times \mathrm{H}^k(M, M\setminus L) \arrow[u, "\varphi_K\times\varphi_L"] \arrow[r, "\cup"] & \mathrm{H}^m(M, \underbrace{(M\setminus K) \cup (M\setminus L)}_{=M}). \arrow[ur, "\varphi_M"'] &
\end{tikzcd}$$

$$(\mathrm{PD}_M(\alpha), \mathrm{PD}_M(\beta)) \mapsfrom (\alpha, \beta)$$

The diagram commutes by the definition of the cup product on cohomology with compact support, see page 2008. It follows from the observation that the lower right group is zero that the cup product of $\mathrm{PD}_M([\sigma])$ and $\mathrm{PD}_M([\tau])$ is zero. ■

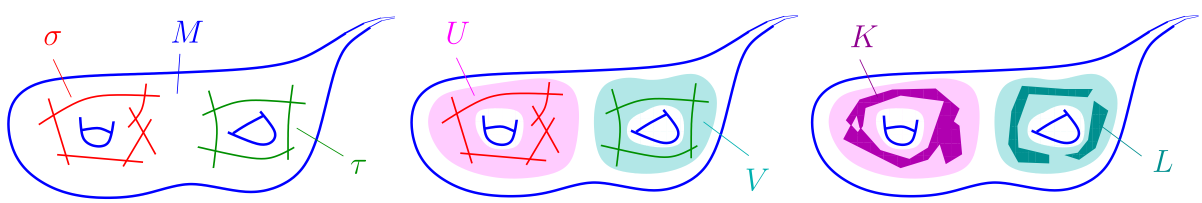


FIGURE 1343. Illustration for the proof of Theorem 95.9 for disjoint cycles.

Lemma 95.12. *Let X be a compact oriented, n-dimensional topological manifold. Furthermore let $\varphi_i\colon \Delta^n \to X \setminus \partial X$, $i = 1, \ldots, r$, be orientation-preserving embeddings such that the images are disjoint. We write $W = X \setminus (\varphi_1(\mathring{\Delta}^n) \cup \cdots \cup \varphi_r(\mathring{\Delta}^n))$. Given any representative $\mu \in \mathrm{C}_{n-1}(\partial X)$ of the fundamental class of ∂X there exists a singular n-chain $\sigma \in \mathrm{C}_n(W)$ such that*

$$\partial(\sigma + \varphi_1 + \cdots + \varphi_r) = \mu \quad \text{and with} \quad [\sigma + \varphi_1 + \cdots + \varphi_r] \;=\; [X].$$

Example. In the proof below of Theorem 95.9 we will apply Lemma 95.12 to the case that $X = \Delta^n$ itself is the standard n-simplex. This particular situation is illustrated in Figure 1344.

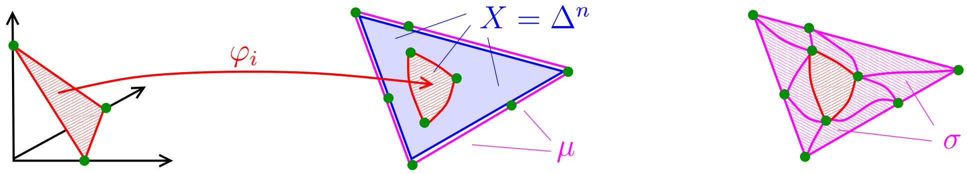


FIGURE 1344

PROOF. By Proposition 44.3 we know that W is a compact n-dimensional topological manifold with $\partial W = \varphi_1(\partial\Delta^n)\cup\cdots\cup\varphi_r(\partial\Delta^n)\cup\partial X$. Since the φ_i are orientation-preserving we obtain from Proposition 87.27 that $-\partial\varphi_1 - \cdots - \partial\varphi_r + \mu \in \mathrm{C}_{n-1}(\partial W)$ represents the fundamental class of ∂W. Again by Proposition 87.27 we know that there exists a singular n-chain $\sigma \in \mathrm{C}_n(W)$ with $\partial\sigma = -\partial\varphi_1 - \cdots - \partial\varphi_r + \mu$. It follows from Proposition 68.4 that $\sigma + \varphi_1 + \cdots + \varphi_r$ represents the fundamental class of X. ■

The formulation of the following, rather technical lemma, might initially be somewhat intimidating. We hope that Figure 1345 helps somewhat with keeping track of the various objects. Also note that the lemma can be viewed as a cousin of first degree to Lemma 89.5.

Lemma 95.13. *Let M be an oriented m-dimensional topological manifold and let Z be a compact codimension-zero submanifold of M. Furthermore let C and D be two compact codimension-zero submanifolds of ∂Z with $C \cup D = \partial Z$ and with $\partial C = \partial D = C \cap D$.*[1351] *Let $U \subset M$ be an open subset with $U \cap \partial Z \subset C$ and let $K \subset U$ be a compact subset. We consider the three inclusion maps $p\colon (U, \varnothing) \to (U, U\setminus \mathring{Z})$, $g\colon (Z\cap U, C\cap U) \to (Z, C)$ and $z\colon (Z\cap U, C\cap U) \to (U, U\setminus\mathring{Z})$. We note that the induced map*

$$z_*\colon\ \mathrm{H}_k(Z\cap U, C\cap U) \ \to\ \mathrm{H}_k(U, U\setminus\mathring{Z})$$

is an isomorphism by the Excision Theorem 44.10 for topological manifolds. With this preparation we can now formulate the conclusion of the lemma. Namely, the following diagram commutes:

$$\begin{tikzcd}
& \mathrm{H}_k(M) & & \mathrm{H}^{m-k}(M, M\setminus K) \arrow[ll, "\cap M"'] \arrow[dl] \arrow[dd] \\
\mathrm{H}_k(U) \arrow[ur] \arrow[d, "p_*"] & & \mathrm{H}^{m-k}(U, U\setminus K) \arrow[ll, "\cap U"'] & \\
\mathrm{H}_k(U, U\setminus\mathring{Z}) \arrow[r, "(z_*)^{-1}"] & \mathrm{H}_k(Z\cap U, C\cap U) \arrow[r, "g_*"] & \mathrm{H}_k(Z, C) & \mathrm{H}^{m-k}(Z, D). \arrow[l, "\cap[Z]"']
\end{tikzcd}$$

Here all the unmarked maps are induced by the obvious inclusion maps of (pairs) of topological spaces. Also note that the bottom cap product $\cap[Z]\colon \mathrm{H}^{m-k}(Z, D) \to \mathrm{H}_k(Z, C)$ is defined since our hypothesis on C and D together with Proposition 74.12 (5) imply that (Z, C, D) is an excisive triad.

[1351] In fact in the proof we only need that we have a decomposition $\partial Z = C \cup D$ such that (Z, C, D) is an excisive triad.

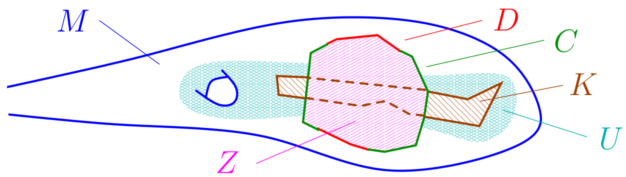


FIGURE 1345. Illustration of Lemma 95.13.

PROOF. We make the following preparations:

(a) We pick a representative $\nu_M \in \mathrm{H}_m(M, M \setminus K)$ for the lift of the orientation of M and similarly we pick $\nu_U \in \mathrm{H}_m(U, U \setminus K)$.

(b) We consider the two obvious inclusion maps $f\colon (Z, \partial Z) \to (Z, \overbrace{Z \setminus (\mathring{Z} \cap K)}^{=(Z\setminus K)\cup C})$ and $g\colon (Z \cap U, (Z \cap U) \setminus (\mathring{Z} \cap K)) \to (Z, Z \setminus (\mathring{Z} \cap K))$. It follows from the Excision Theorem 43.20 that the map $g_*\colon \mathrm{H}_m(Z\cap U, (Z\cap U)\setminus(\mathring{Z}\cap K)) \to \mathrm{H}_m(Z, Z\setminus(\mathring{Z}\cap K))$ is an isomorphism. We write $\nu_Z := g_*^{-1}(f_*([Z]))$.

Next we consider the following expanded version of the diagram shown in the lemma:

$$\begin{tikzcd}
 & \mathrm{H}_k(M) & & \mathrm{H}^{m-k}(M, M\setminus K) \arrow[ll, "\cap\nu_M"'] \arrow[dl] \arrow[ddd] \\
\mathrm{H}_k(U) \arrow[ur] \arrow[d] & & \mathrm{H}^{m-k}(U, U\setminus K) \arrow[ll, "\cap\nu_U"'] \arrow[d] & \\
\mathrm{H}_k(U, U\setminus \mathring{Z}) \arrow[r, "(z_*)^{-1}"] & \mathrm{H}_k(Z\cap U, C\cap U) \arrow[dr] & \mathrm{H}^{m-k}(Z\cap U, (Z\cap U)\setminus K). \arrow[l, "\cap\nu_Z"'] & \\
 & & \mathrm{H}_k(Z, C) \arrow[d, "="] & \mathrm{H}^{m-k}(Z, Z\setminus K) \arrow[l, "\cap f_*([Z])"] \arrow[ul] \arrow[d] \\
 & & \mathrm{H}_k(Z, C) & \mathrm{H}^{m-k}(Z, D). \arrow[l, "\cap[Z]"]
\end{tikzcd}$$

It suffices to show that all the five regions of the diagram commute:

(1) The parallelogram on the top of the diagram commutes by Lemma 83.8 applied to the map $h\colon (U, U \setminus K, \varnothing) \to (M, M \setminus K, \varnothing)$, together with the observation that Lemma 87.24 implies that $h_*(\nu_U) = \nu_M$.
(2) The 4-gon on the right commutes since all maps are induced by inclusions.
(3) For the lower parallelogram we obtain commutativity from applying Lemma 83.8 to the inclusion map[1352] $g\colon (Z \cap U, Z \cap U \setminus K, C \cap U) \to (Z, Z \setminus K, C)$.
(4) To show commutativity of the lower rectangle we apply Lemma 83.8 to the inclusion map $f\colon (Z, D, C) \to (Z, Z \setminus K, C)$.

[1352] Here we use the symbol for this map of triples of topological spaces. We feel that recycling the name g is justified since this map of triples "contains" our original map $g\colon (Z\cap U, C\cap U) \to (Z, C)$ of pairs of topological spaces.

Finally we turn to the upper rectangle. We need to show that the following rectangle commutes:

$$\begin{tikzcd}
\mathrm{H}_k(U) \arrow[d, "p_*"] & & \mathrm{H}^{m-k}(U,U\setminus K) \arrow[ll, "\cap\nu_U"'] \arrow[d, "z^*"] \\
\mathrm{H}_k(U,U\setminus\mathring{Z}) & \mathrm{H}_k(Z\cap U,C\cap U) \arrow[l, "z_*"'] & \mathrm{H}^{m-k}(Z\cap U,(Z\cap U)\setminus K). \arrow[l, "\cap\nu_Z"']
\end{tikzcd}$$

Now let $\beta \in \mathrm{H}^{m-k}(U, U \setminus K)$. We have the following equalities in $\mathrm{H}_k(U, U \setminus \mathring{Z})$:

$$\underset{\substack{\uparrow \\ \text{Lemma 83.8 applied to the map} \\ p\colon (U,U\setminus K,\varnothing)\to(U,U\setminus K,U\setminus\mathring{Z})}}{p_*(\beta\cap\nu_U) = \beta\cap p_*(\nu_U)} \overset{\substack{\text{it follows from Lemma 87.31 that} \\ \text{we have } p_*(\nu_U)=z_*(\nu_Z) \text{ in } \mathrm{H}_m(U,U\setminus K\cup U\setminus\mathring{Z}) \\ \downarrow}}{=} \underset{\substack{\uparrow \\ \text{Lemma 83.8 applied to the map} \\ z\colon (Z\cap U,Z\cap U\setminus K,C\cap U)\to(U,U\setminus K,U\setminus\mathring{Z})}}{\beta\cap z_*(\nu_Z) = z_*(z^*(\beta)\cap\nu_Z).}$$

We have thus shown that the diagram commutes. ■

After all these preparations we can finally attack Theorem 95.9.

Proof of Theorem 95.9. Let M be an oriented m-dimensional topological manifold with $\partial M = \varnothing$. Also let $\sigma = a_1 \cdot \sigma_1 + \cdots + a_r \cdot \sigma_r \in \mathrm{C}_k(M)$ and $\tau = b_1 \cdot \tau_1 + \cdots + b_s \cdot \tau_s \in \mathrm{C}_{m-k}(M)$ be two singular cycles that intersect nicely.

Claim 1. We can assume that we are in the following setting: There exists an $n \in \mathbb{N}_0$, there exist distinct points $P_1, \ldots, P_n \in M$ and there exist disjoint m-dimensional submanifolds $Z_1, \ldots, Z_n$ of M such that the following hold:

(1) For $i \neq j$ we have $|\sigma_i| \cap |\tau_j| = \varnothing$.
(2) For each $i \in \{1, \ldots, n\}$ we have $|\sigma_i| \cap |\tau_i| = \{P_i\}$.
(3) For $i = n+1, \ldots, \min\{r, s\}$ we have $|\sigma_i| \cap |\tau_i| = \varnothing$.
(4) For $i \in \{1, \ldots, n\}$ the following hold:[1353]
  (a) P_i is contained in $Z_i \setminus \partial Z_i$.
  (b) Z_i is homeomorphic to $\overline{B}^m$.
  (c) The map $\sigma_i\colon \Delta^k \to M$ is actually a proper embedding $\sigma_i\colon \Delta^k \to Z_i$ such that $X_i := \sigma_i(\Delta^k)$ is a well-behaved submanifold of Z_i. By the definition, see page 2303 this means that the submanifold $\partial X_i = \sigma_i(\partial\Delta^k)$ of ∂Z_i admits a regular neighborhood A_i.
  (d) Similarly to (c) the map $\tau_i\colon \Delta^{m-k} \to M$ is a proper embedding $\tau_i\colon \Delta^{m-k} \to Z_i$ such that $Y_i := \tau_i(\Delta^{m-k})$ is a well-behaved submanifold of Z_i. This means that the submanifold $\partial Y_i = \tau_i(\partial\Delta^k)$ of ∂Z_i admits a regular neighborhood B_i.
  (e) We have $A_i \cap B_i = \varnothing$.
  (f) For $j \neq i$ we have $|\sigma_j| \cap Z_i = \varnothing$ and $|\tau_j| \cap Z_i = \varnothing$.

[1353] These statements imply in particular that P_i is a nice intersection point of σ_i and τ_i.

The proof of the claim is built out of the following ingredients:

(i) The fact, established in Exercise 6.25, that given any map $f\colon \Delta^l \to M$ and given any $P \in M \setminus f(\Delta^l)$ there exists an open neighborhood U of P with $f(\Delta^l) \cap U = \varnothing$.
(ii) The "given any neighborhood U" feature of the definition of a nice intersection point.
(iii) By Lemma 41.1 we know that any Δ^l is homeomorphic to $\overline{B}^l$.
(iv) We use that it follows from (iii) and the definition of a nice intersection point that any $f(\widetilde{X})$ is homeomorphic to Δ^k and that any $g(\widetilde{Y})$ is homeomorphic to Δ^{m-k}. (Here we use the notation from the definition of a nice intersection point, i.e. $\widetilde{X}$ and $\widetilde{Y}$ are defined on page 2304.) We combine this piece of information with Lemma 95.12.

We leave it to the reader to work out the elementary albeit slightly messy details. ⊞

Next we introduce the following notation:

(5) We write $Z = Z_1 \cup \dots \cup Z_n$.
(6) We write $\widetilde{\sigma} = a_{n+1} \cdot \sigma_{n+1} + \dots + a_r \cdot \sigma_r$ and we denote by $|\widetilde{\sigma}|$ the union of the images of $\sigma_{n+1}, \dots, \sigma_r$. Similarly we define $\widetilde{\tau}$ and $|\widetilde{\tau}|$.

We refer to Figure 1346 on the left for an illustration.

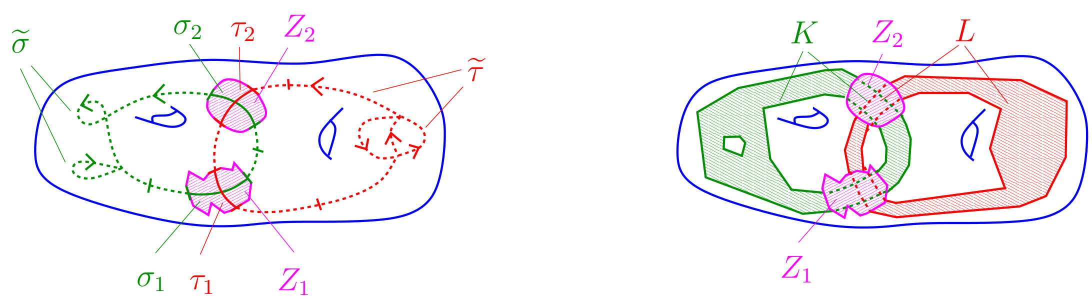


FIGURE 1346

Claim 1. There exist compact subsets K and L of M with the following properties:

(a) There exists a cohomology class $\varphi \in \mathrm{H}^{m-k}(M, M \setminus K)$ such that the image of φ in $\mathrm{H}_c^{m-k}(M)$ equals $\mathrm{PD}_M([\sigma])$.
(b) There exists a cohomology class $\psi \in \mathrm{H}^k(M, M \setminus L)$ such that the image of ψ in $\mathrm{H}_c^k(M)$ equals $\mathrm{PD}_M([\tau])$.
(c) We have $K \cap L \cap (M \setminus Z) = \varnothing$.
(d) For each $i \in \{1, \dots, n\}$ we have $K \cap \partial Z_i \subset A_i$ and $L \cap \partial Z_i \subset B_i$.

We refer to Figure 1346 on the right for an illustration.

By Proposition 44.3 we know that $W := M \setminus \mathring{Z}$ is an m-dimensional topological manifold. It follows from Proposition 95.10 (2) that there exist *disjoint* open neighborhoods $\widetilde{U} \subset W$ of $|\widetilde{\sigma}|$ and $\widetilde{V} \subset W$ of $|\widetilde{\tau}|$ such that for each $i \in \{1, \dots, n\}$ we have $\widetilde{U} \cap \partial Z_i = \mathring{A}_i$ and $\widetilde{V} \cap \partial Z_i = \mathring{B}_i$. Next we write $U = \widetilde{U} \cup \bigcup\limits_{i=1}^{n} \mathring{Z}_i$ and $V = \widetilde{V} \cup \bigcup\limits_{i=1}^{n} \mathring{Z}_i$. Note that it follows from the elementary Lemma 2.6 (2) that U and V are open subsets of M. We refer to Figure 1347 for an illustration.

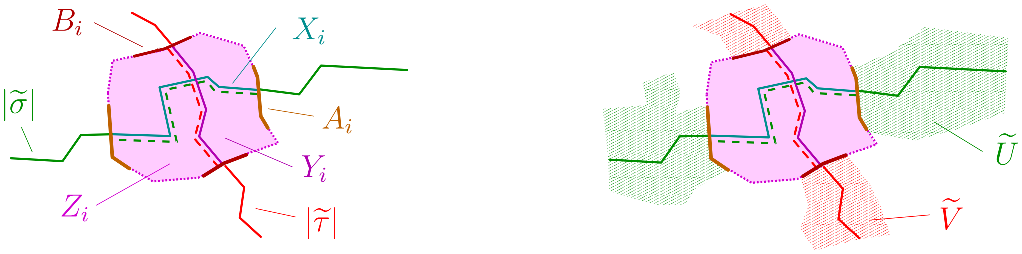


FIGURE 1347

We denote by $i\colon U \to M$ the inclusion map. Evidently we have $\sigma \in \mathrm{C}_k(U)$. Since U is an open subset of M we can thus consider $\mathrm{PD}_U([\sigma]) \in \mathrm{H}_{\mathrm{c}}^{m-k}(U)$. By Lemma 77.2 there exists a compact subset K of U and a cohomology class $\widetilde{\varphi} \in \mathrm{H}^{m-k}(U, U \setminus K)$ such that $\mathrm{PD}_U([\sigma])$ equals the image of $\widetilde{\varphi}$ under the natural map $\mathrm{H}^{m-k}(U, U \setminus K) \to \mathrm{H}_{\mathrm{c}}^{m-k}(U)$. By excision, see page 1840, the map $i^*\colon\ \mathrm{H}^{m-k}(M, M \setminus K) \to \mathrm{H}^{m-k}(U, U \setminus K)$ is an isomorphism. We set $\varphi := (i^*)^{-1}(\widetilde{\varphi})$. By Lemma 95.11 we know that the image of φ in $\mathrm{H}_{\mathrm{c}}^{m-k}(M)$ equals $\mathrm{PD}_M([\sigma])$.

By precisely the same argument we can also find a compact subset L of V and some cohomology class $\psi \in \mathrm{H}^k(M, M \setminus L)$ such that the image of ψ in $\mathrm{H}_{\mathrm{c}}^k(M)$ equals $\mathrm{PD}_M([\tau])$.

Finally note that by construction K and L, and also φ and ψ satisfy all the desired conditions set out in (a), (b), (c) and (d). ⊞

We continue with some more observations and notation.

(7) For $i = 1, \dots, n$ we write $K_i = K \cap Z_i$ and $L_i = L \cap Z_i$. Note that by construction we have $K_i \cap L_i \subset \mathring{Z}_i = Z_i \setminus \partial Z_i$.

(8) Given any compact subset T of M we denote by $o\colon (M, \varnothing) \to (M, M \setminus T)$ the obvious map. Also we denote by $o^*\colon\ \mathrm{H}^*(M, M \setminus T) \to \mathrm{H}_{\mathrm{c}}^*(M)$ the obvious map.

(9) We denote by $z\colon Z \to M$ the inclusion map and we denote by z also all the inclusion maps of pairs of topological spaces where the total spaces are Z and M.

(10) Given $i \in \{1, \dots, n\}$ we denote by $z_i\colon Z_i \to Z$ the inclusion map. We adopt the same convention as in (9) for pairs of topological spaces.

(11) We view Z as an m-dimensional submanifold of M and we equip Z with the orientation coming from Lemma 86.17. Given any subset $W \subset Z$ with $\partial Z \subset W$ we denote by $[Z]$ also the image of the fundamental class of Z under the map $\mathrm{H}_m(Z, \partial Z) \to \mathrm{H}_m(Z, W)$. We adopt the same convention for the $[Z_i]$.

(12) Recall that for each $i \in \{1, \dots, n\}$ we have $K \cap \partial Z_i \subset A_i$ and $L \cap \partial Z_i \subset B_i$. This implies that we obtain the maps of pairs $(Z_i, \partial Z_i \setminus A_i) \to (Z_i, Z_i \setminus K_i)$ and $(Z_i, \partial Z_i \setminus B_i) \to (Z_i, Z_i \setminus L_i)$. We refer to each of these maps as the canonical map c.

(13) For each $i \in \{1, \dots, n\}$ we have maps of pairs $d\colon (Z_i, \partial Z_i \setminus A_i) \to (Z_i, \partial Z_i \setminus \partial X_i)$ and $d\colon (Z_i, \partial Z_i \setminus B_i) \to (Z_i, \partial Z_i \setminus \partial Y_i)$. Since A_i is a regular neighborhood of ∂X_i and since B_i is a regular neighborhood of ∂Y_i we obtain from Lemma 73.13 (4) that these maps induce isomorphisms of relative cohomology groups.

(14) By Theorem 87.10 there exists a unique lift $\mu \in \mathrm{H}_m(M, M \setminus (K \cap L))$ of the orientation classes of M.

With the above definitions and observations it now makes sense to consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_{\mathrm{c}}^{m-k}(M) \times \mathrm{H}_{\mathrm{c}}^{k}(M) \arrow[r, "\cup"] & \mathrm{H}_{\mathrm{c}}^{m}(M) \arrow[r, "\cap M"] & \mathrm{H}_0(M) \arrow[r, "\epsilon_M"] \arrow[d, "\mathrm{id}"'] & \mathbb{Z} \arrow[d, "\mathrm{id}"'] \\
\mathrm{H}^{m-k}(M, M\setminus K) \times \mathrm{H}^{k}(M, M\setminus L) \arrow[u, "o^*\times o^*"'] \arrow[r, "\cup"] \arrow[d, "z^*\times z^*"] & \mathrm{H}^{m}(M, M\setminus(K\cap L)) \arrow[u, "o^*"'] \arrow[r, "\cap\mu"] \arrow[d, "z^*"] & \mathrm{H}_0(M) \arrow[r, "\epsilon_M"] & \mathbb{Z} \\
\mathrm{H}^{m-k}(Z, Z\setminus K) \times \mathrm{H}^{k}(Z, Z\setminus L) \arrow[r, "\cup"] \arrow[d, "\bigoplus_{i=1}^{n} z_i^*\times z_i^*"] & \mathrm{H}^{m}(Z, Z\setminus K\cap L) \arrow[r, "\cap[Z]"] \arrow[d, "\bigoplus_{i=1}^{n} z_i^*"] & \mathrm{H}_0(Z) \arrow[r, "\epsilon_Z"] \arrow[u, "z_*"'] & \mathbb{Z} \arrow[u, "\mathrm{id}"] \\
\bigoplus_{i=1}^{n} \mathrm{H}^{m-k}(Z_i, Z_i\setminus K_i)\times \mathrm{H}^{k}(Z_i, Z_i\setminus L_i) \arrow[r, "\oplus\cup"] \arrow[d, "\bigoplus_{i=1}^{n} c^*\times c^*"] & \bigoplus_{i=1}^{n} \mathrm{H}^{m}(Z_i, Z_i\setminus K_i\cap L_i) \arrow[r, "\oplus\cap[Z_i]"] \arrow[d, "\bigoplus_{i=1}^{n} c^*"] & \bigoplus_{i=1}^{n} \mathrm{H}_0(Z_i) \arrow[r, "\sum \epsilon_{Z_i}"'] \arrow[u, "\sum_{i=1}^{n} z_{i*}"'] & \mathbb{Z} \arrow[u, "\mathrm{id}"] \\
\bigoplus_{i=1}^{n} \mathrm{H}^{m-k}(Z_i, \partial Z_i\setminus \mathring{A}_i)\times \mathrm{H}^{k}(Z_i, \partial Z_i\setminus \mathring{B}_i) \arrow[r, "\oplus\cup"] & \bigoplus_{i=1}^{n} \mathrm{H}^{m}(Z_i, \partial Z_i) \arrow[r, "\oplus\cap[Z_i]"] & \bigoplus_{i=1}^{n} \mathrm{H}_0(Z_i) \arrow[r, "\sum \epsilon_{Z_i}"'] \arrow[u, "\mathrm{id}"'] & \mathbb{Z} \arrow[u, "\mathrm{id}"] \\
\bigoplus_{i=1}^{n} \mathrm{H}^{m-k}(Z_i, \partial Z_i\setminus \partial X_i)\times \mathrm{H}^{k}(Z_i, \partial Z_i\setminus \partial Y_i) \arrow[u, "\cong", "\bigoplus_{i=1}^{n} d^*\times d^*"'] \arrow[r, "\oplus\cup"] & \bigoplus_{i=1}^{n} \mathrm{H}^{m}(Z_i, \partial Z_i) \arrow[u, "\mathrm{id}"'] \arrow[r, "\oplus\cap[Z_i]"] & \bigoplus_{i=1}^{n} \mathrm{H}_0(Z_i) \arrow[r, "\sum \epsilon_{Z_i}"'] \arrow[u, "\mathrm{id}"'] & \mathbb{Z}. \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

Claim 2. The above diagram commutes.

In the following we deal with all of the fifteen squares of the diagram.

(a) The square on the top left commutes by definition of the cup product on cohomology with compact support, see page 2008.
(b) The left squares in the second row and also in the fourth row commute by Proposition 82.4 (1).
(c) The left square in the third row commutes by Lemma 83.9 (1) together with the aforementioned Proposition 82.4 (1).
(d) The middle square in the first row commutes by definition of the map $\cap M$.
(e) It follows from Corollary 87.12 that $z_*([Z]) = \mu \in \mathrm{H}_m(M, M \setminus (K \cap L))$. Therefore we obtain from Lemma 83.8 (1) that the middle square in the second row commutes.
(f) It follows from the remark on page 2121 that $\sum_{i=1}^{n} z_{i*}([Z_i]) = [Z]$. Now we obtain from Lemma 83.8 (1) and Lemma 83.9 (2) that the middle square in the third row commutes.
(g) The squares that involve the augmentation maps commute by the discussion on page 1088.
(h) It follows from Lemma 83.8 (1) and our conventions regarding the meaning of $[Z_i]$ that the middle square in the fourth row commutes.
(i) It follows from Proposition 82.4 that the bottom left square commutes.
(j) The remaining squares, that we did not discuss, commute for the trivial reason that the opposite maps in the square are actually on the nose the same map. ⊞

We introduce one last piece of notation:

(15) For each $i \in \{1,\dots,n\}$ we denote by $\mathrm{PD}_{Z_i}\colon \mathrm{H}_k(Z_i,\partial X_i) \to \mathrm{H}^{m-k}(Z_i,\partial Z_i\setminus\partial X_i)$ the isomorphism from Proposition 95.2.

Claim 3. Let $i \in \{1,\dots,n\}$.

(i) For the above $\varphi \in \mathrm{H}^{m-k}(M, M\setminus K)$ we have $(c^*\circ z_i^*\circ z^*)(\varphi) = a_i\cdot d^*(\mathrm{PD}_{Z_i}([X_i]))$.
(ii) For the above $\psi \in \mathrm{H}^{k}(M, M\setminus L)$ we have $(c^*\circ z_i^*\circ z^*)(\psi) = b_i\cdot d^*(\mathrm{PD}_{Z_i}([Y_i]))$.
(iii) We have

$$\epsilon_{Z_i}\Big(\big((d^*(\mathrm{PD}_{Z_i}([X_i]))\cup d^*(\mathrm{PD}_{Z_i}([Y_i]))\big)\cap[Z_i]\Big) \;=\; \mathrm{index}(\sigma_i,\tau_i,P_i).$$

We provide the proof of the three statements:

(i) We introduce the following notation:
(α) We set $U := (M\setminus Z)\cup\bigcup\limits_{i=1}^{n}\mathring{Z}_i\cup\bigcup\limits_{i=1}^{n}\mathring{A}_i = M\setminus\bigcup\limits_{i=1}^{n}(\partial Z_i\setminus\mathring{A}_i)$. It is elementary to see that U is in fact an open subset of M.
(β) We write $C := A_i$ and $D := \partial Z_i\setminus\mathring{A}_i$. Note that it follows from Proposition 44.3 that C and D are compact codimension-zero submanifolds of ∂Z_i with $C\cup D = \partial Z_i$ and with $C\cap D = \partial C = \partial D$.
(γ) We denote by $p\colon (U,\varnothing)\to(U,U\setminus\mathring{Z}_i)$, $\widetilde{z}\colon (Z_i\cap U, C\cap U)\to(U,U\setminus\mathring{Z}_i)$ and $g\colon (Z_i\cap U, C\cap U)\to(Z_i,C)$ the obvious inclusion maps.

We perform the following calculation in $\mathrm{H}_k(Z_i,A_i)$:

$$\underbrace{(c^*\circ z_i^*\circ z^*)(\varphi)}_{\in\,\mathrm{H}^{m-k}(Z_i,\partial Z_i\setminus\mathring{A}_i)}\cap[Z_i] \;\underset{\uparrow}{=}\; g_*(\widetilde{z}_*^{-1}(p_*([\sigma]))) \;\overset{\text{since } \sigma = a_1\cdot\sigma_1+\dots+a_r\cdot\sigma_r,\text{ since } p_*(\sigma_j)=0 \text{ for } j\neq i \text{ and since } [\sigma_i]=[X_i]}{\overset{\downarrow}{=}}\; g_*(\widetilde{z}_*^{-1}(p_*(a_i\cdot[X_i]))) \;\underset{\uparrow}{=}\; a_i\cdot[X_i].$$

this follows from Lemma 95.13, applied to the above choices of C and D, and the observation that $\varphi\cap M = [\sigma]$ with $|\sigma|\subset U$, note that here we use that we have $U\cap\partial Z_i\subset C$ and $K\subset U$ — by Lemma 87.24

Now recall that in the proof of Proposition 95.2 we saw that the isomorphism PD_{Z_i} is given as follows:

$$\begin{tikzcd}[column sep=small] \mathrm{H}^{m-k}(Z_i,\partial Z_i\setminus\partial X_i) \arrow[r, "d^*", "\cong"'] & \mathrm{H}^{m-k}(Z_i,\partial Z_i\setminus\mathring{A}_i) \arrow[r, "\cap[Z_i]", "\cong"'] & \mathrm{H}_k(Z_i,A_i) & \mathrm{H}_k(Z_i,\partial X_i). \arrow[l, "j_*"', "\cong"] \arrow[lll, bend left, "=:\mathrm{PD}_{Z_i}"] \end{tikzcd}$$

Since the inclusion induced map $j_*\colon \mathrm{H}_k(Z_i,\partial X_i)\to\mathrm{H}_k(Z_i,A_i)$ is an isomorphism we now see that $(c^*\circ z_i^*\circ z^*)(\varphi) = a_i\cdot d^*(\mathrm{PD}_{Z_i}([X_i]))$.

(ii) The proof of this statement is evidently the same as the proof of (2).
(iii) This statement follows immediately from the definitions together with the fact, obtained in Claim 2, that the bottom squares in the above diagram commute. ⊞

After all these preparations we can prove the desired equality:

$$\begin{aligned}
\epsilon_M\big((\mathrm{PD}_M([\sigma])\cup \mathrm{PD}_M([\tau]))\cap M\big) &\overset{\substack{\text{by construction of } \varphi \text{ and } \psi \text{ in Claim 1 and the first row of the commutative diagram from Claim 2}\\ \downarrow}}{=} \epsilon_M((\varphi\cup\psi)\cap\mu)\\
&\underset{\substack{\uparrow\\ \text{by the second, third and fourth row of the commutative diagram from Claim 2}}}{=} \sum_{i=1}^{n}\epsilon_{Z_i}\Big(\big(((c^*\circ z_i^*\circ z^*)(\varphi)\cup(c^*\circ z_i^*\circ z^*)(\psi)\big)\cap[Z_i]\Big)\\
&\underset{\substack{\uparrow\\ \text{by Claim 3}}}{=} \sum_{i=1}^{n} a_i\cdot b_i\cdot \mathrm{index}(\sigma_i,\tau_i,P_i) \underset{\substack{\uparrow\\ \text{by the setting from page 2318}}}{=} \sigma\cdot\tau.
\end{aligned}$$

This concludes the proof of the theorem. ■

We conclude this section with the following variation on Theorem 95.9.

Theorem 95.14. *Let M be a compact orientable connected m-dimensional topological manifold. Furthermore let A and B be compact $(m-1)$-dimensional submanifolds of ∂M with $\partial A=\partial B=A\cap B$ and with $\partial M=A\cup B$. Finally let $\sigma\in \mathrm{C}_k(M,A)$ and $\tau\in\mathrm{C}_{m-k}(M,B)$ be two cycles. If σ and τ intersect nicely, then*

$$\sigma\cdot\tau \;=\; \Big\langle \underbrace{\overbrace{\mathrm{PD}_M([\sigma])}^{\in \mathrm{H}^{m-k}(M,B)}\cup\overbrace{\mathrm{PD}_M([\tau])}^{\in\mathrm{H}^k(M,A)}}_{\in\mathrm{H}^m(M,\partial M)}\Big),[M]\Big\rangle.$$

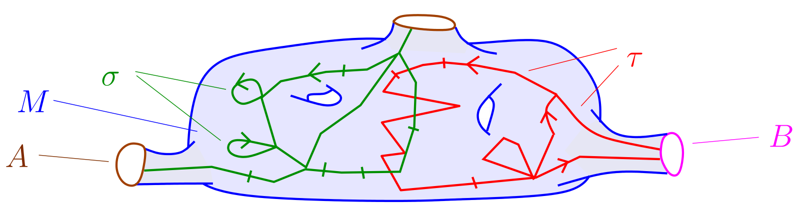


FIGURE 1348. Illustration of Theorem 95.14.

SKETCH OF PROOF. The proof of the theorem follows along the lines of the proof of Theorem 95.9. The key difference is that we need to use Poincaré Duality for non-compact topological manifolds with boundary. More precisely, let U be an oriented m-dimensional topological manifold with boundary. By Theorem 87.30, given any compact set K of U the orientation defines a natural homology class μ_K in $\mathrm{H}_m(U,(U\setminus K)\cup\partial U)$. The same construction as in Lemma 88.18, this time using Lemma 87.31 instead of Corollary 87.12, shows that capping with these homology classes defines a well-defined map

$$\mathrm{PD}_U\colon\ \mathrm{H}^k_{\mathrm{c}}(U)\ \to\ \mathrm{H}_{m-k}(U,\partial U).$$

A moderately heroic effort shows that the proof of the Poincaré Duality Theorem 88.1, as provided in Section 88.4 can be modified to show that the above map PD_U is an isomorphism. Once this is out of the way it is fairly straightforward to modify the proof of Theorem 95.9 to obtain a proof of Theorem 95.14. ■

95.5. **The algebraic $\mathbb{F}_2$-intersection number** $(*)$**.** In this section we quickly discuss the "$\mathbb{F}_2$-analogue" of the results of the previous sections.

Definition. Let M be an m-dimensional topological manifold, let X be a k-dimensional topological manifold and let Y be an $(m-k)$-dimensional topological manifold. Furthermore let $f\colon X \to M$ and $g\colon Y \to M$ be maps. If f and g intersect nicely, then we define

$$\text{the algebraic } \mathbb{F}_2\text{-intersection number of } f \text{ and } g \ := \ f \cdot_{\mathbb{F}_2} g \ := \ \#(|f| \cap |g|) \bmod 2.$$

If X and Y are in fact submanifolds and f and g are the inclusion map, then we write $X \cdot_{\mathbb{F}_2} Y := f \cdot_{\mathbb{F}_2} g$.

We recall the following notation.

Notation.

(1) In Corollary 87.5 we saw that any compact n-dimensional topological manifold W comes with its $\mathbb{F}_2$-fundamental class $[W]_{\mathbb{F}_2} \in \mathrm{H}_n(W, \partial W; \mathbb{F}_2)$. If W is in fact a proper submanifold of some other topological manifold M, then frequently we denote by $[W]_{\mathbb{F}_2} \in \mathrm{H}_n(M, \partial M; \mathbb{F}_2)$ the image of $[W]_{\mathbb{F}_2}$ under the inclusion induced map $\mathrm{H}_n(W, \partial W; \mathbb{F}_2) \to \mathrm{H}_n(M, \partial M; \mathbb{F}_2)$.

(2) In this section, given a compact topological manifold we denote by PD_M the Poincaré Duality isomorphisms with $\mathbb{F}_2$-coefficients provided by the Poincaré Duality Theorem 88.1.

Now we can formulate the following theorem, which is a variation on Theorem 95.7.

Theorem 95.15. *Let M be a compact connected m-dimensional topological manifold. Furthermore let A and B be compact $(m-1)$-dimensional submanifolds of ∂M such that $\partial A = \partial B = A \cap B$ and with $\partial M = A \cup B$. Let X be a compact k-dimensional topological manifold and let Y be a compact $(m-k)$-dimensional topological manifold. Finally let $f\colon X \to M$ and $g\colon Y \to M$ be two proper maps with $f(\partial X) \subset A$ and $g(\partial Y) \subset B$. If f and g intersect nicely, then*

$$f \cdot_{\mathbb{F}_2} g \ = \ \Big\langle \overbrace{\underbrace{\mathrm{PD}_M(\underbrace{f_*([X]_{\mathbb{F}_2})}_{\in \mathrm{H}_k(M,A;\mathbb{F}_2)})}_{\in \mathrm{H}^{m-k}(M,B;\mathbb{F}_2)} \cup \underbrace{\mathrm{PD}_M(\underbrace{g_*([Y]_{\mathbb{F}_2})}_{\in \mathrm{H}_{m-k}(M,B;\mathbb{F}_2)})}_{\in \mathrm{H}^{k}(M,A;\mathbb{F}_2)}}^{\in \mathrm{H}^m(M,\partial M;\mathbb{F}_2)}, \overbrace{[M]_{\mathbb{F}_2}}^{\in \mathrm{H}_m(M,\partial M;\mathbb{F}_2)} \Big\rangle \ \in \ \mathbb{F}_2.$$

In particular, if X and Y are already submanifolds of M, then

$$X \cdot_{\mathbb{F}_2} Y \ = \ \langle \mathrm{PD}_M([X]_{\mathbb{F}_2}) \cup \mathrm{PD}_M([Y]_{\mathbb{F}_2}), [M]_{\mathbb{F}_2} \rangle.$$

Proof. By Proposition 86.13 every topological manifold is $\mathbb{F}_2$-orientable. With this observation the proof of the theorem is a variation, and basically a simplification, of the proof of Theorem 94.5. We leave it to the skeptical reader to fill in the details. ■

Example. Similar to the discussion on page 2277 we can use Theorem 95.15 to show that some homology classes are in fact non-zero. For example in Figure 1349 we show the Klein bottle K together with two closed 1-dimensional submanifolds. These intersect in precisely one point, so their algebraic $\mathbb{F}_2$-intersection number is necessarily non-zero. Theorem 95.15 implies that $[X]_{\mathbb{F}_2}$ and $[Y]_{\mathbb{F}_2}$ are non-zero in $\mathrm{H}_1(K;\mathbb{F}_2)$.

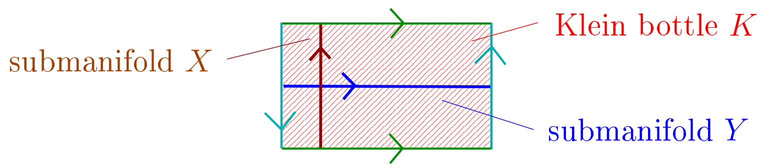


FIGURE 1349

Remark. A discussion of intersection numbers in topological manifolds is also given in [**Dol80**, Chapter VIII.13].

Exercises for Chapter 95.

Exercise 95.1. Is every topological space that is Hausdorff also normal?

Exercise 95.2. We consider the manifold $M = Z = [0,1]^2$ and the two embeddings $f, g\colon [0,1] \to Z$ that are shown in Figure 1333. Note that $P = (0,0)$ is the unique intersection point of $|f|$ and $|g|$. Show that $\operatorname{index}(f,g,P) = 0$.

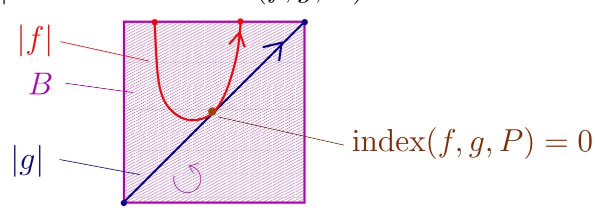


FIGURE 1350. Illustration of Exercise 95.2.

Exercise 95.3. Can one use the Generalized Alexander Duality Theorem 92.10 to answer Question 95.1? Or some other of the earlier theorems? If yes, how would you do it? You could also try to restrict yourselves to the case that the map α is injective.

Exercise 95.4. Let M be a compact oriented connected n-dimensional topological manifold and let $N \subset M$ be a compact oriented proper submanifold of codimension one. Finally let $\gamma\colon S^1 \to M$ be a map that intersects N nicely. Show that

$$\langle \mathrm{PD}_M([N]), \gamma_*([S^1])\rangle \;=\; \gamma \cdot N.$$

Exercise 95.5. In Figure 1333 we saw that $-1, 0, 1$ can be indices of nice intersection points of oriented submanifolds. Can any other numbers occur as indices of nice intersection points?

Exercise 95.6. Let $n \in \mathbb{N}$. By the discussion on page 1845 we know that for any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}^k(\mathbb{RP}^n;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0,\dots,n, \\ 0, & \text{otherwise.}\end{cases}$$

Use Theorem 95.15 to show that the obvious map

$$\mathbb{F}_2[x]/(x^{n+1}) \;\to\; (\mathrm{H}^*(\mathbb{RP}^n;\mathbb{F}_2), \cup)$$

is a ring isomorphism.

Remark. In Proposition 90.16 we used a rather different approach to calculating the cup product on $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2)$.

96. The Lefschetz-Hopf Theorem

First we recall the following definition from page 1623.

Definition. Let M be a compact topological manifold and let $f\colon M \to M$ be a map. We define the *Lefschetz number of f* to be

$$\Lambda(f) \;:=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \operatorname{tr}(f_*\colon\ \mathrm{H}_n(M;\mathbb{Z}) \to \mathrm{H}_n(M;\mathbb{Z})).$$

The following is one of our main results on the Lefschetz number.

Theorems 65.10 and 85.11 (Lefschetz Fixed Point Theorem) *Let M be a compact smooth or topological manifold and let $f\colon M \to M$ be a map. If the Lefschetz number $\Lambda(f)$ is non-zero, then f has a fixed point.*

In this chapter we will prove the Lefschetz-Hopf Theorems 96.5 and 96.9. These allow us to compute the Lefschetz number $\Lambda(f)$ as the sum of "indices" of the fixed points of a given map $f\colon M \to M$. In particular, if there are no fixed points, then this sum is zero. Thus the Lefschetz-Hopf Theorems are generalizations of the above version of the Lefschetz Fixed Point Theorem.

Recall that in Theorem 65.17 we used the Lefschetz Fixed Point Theorem 65.10 to show show that if a closed smooth manifold has non-zero Euler characteristic, then every vector field on M has at least one zero. In the following chapter we will use the Lefschetz-Hopf Theorems 96.5 and 96.9 to prove the Poincaré-Hopf Theorem 97.5 which is a significant generalization of Theorem 65.17.

96.1. **The Lefschetz number as an intersection number.** We start out with recalling and introducing some notation. We will make use of this notation throughout this chapter.

Notation. Let M be an n-dimensional topological respectively smooth manifold.

(1) It follows from Propositions 6.5 and 80.9 that we can view $M \times M$ as an $2n$-dimensional topological respectively smooth manifold.
(2) We denote by $p_1, p_2\colon M \times M \to M$ the projection onto the first respectively second factor.
(3) We write $\Delta_M := \{(x,x) \in M \times M \,|\, x \in M\}$.
(4) Let $f\colon M \to M$ be a map. We denote by $\operatorname{id} \times f\colon M \to M \times M$ the map that is given by $x \mapsto (x, f(x))$. We write $\operatorname{Gr}(f) = (\operatorname{id} \times f)(M)$. Note that $\operatorname{Gr}(\operatorname{id}) = \Delta_M$.
(5) The map $\operatorname{id} \times f\colon M \to M$ is easily seen to be a (smooth) proper embedding. Thus we can view $\operatorname{Gr}(f)$ as a proper n-dimensional submanifold of $M \times M$.
(6) Now assume that M is oriented.
    (a) We equip $M \times M$ with the product orientation that we defined on pages 304 and .
    (b) We equip $\operatorname{Gr}(f)$ with the unique orientation that turns $\operatorname{id} \times f$ into an orientation-preserving map.

(c) If M is furthermore closed, then we denote as usual by $[\mathrm{Gr}(f)] \in \mathrm{H}_n(\mathrm{Gr}(f);\mathbb{Z})$ the fundamental class of $\mathrm{Gr}(f)$. By the usual slight abuse of notation we denote by $[\mathrm{Gr}(f)] \in \mathrm{H}_n(M \times M;\mathbb{Z})$ the homology class represented by this submanifold. Note that $[\mathrm{Gr}(f)] = (\mathrm{id} \times f)_*([M]) \in \mathrm{H}_n(M \times M;\mathbb{Z})$.

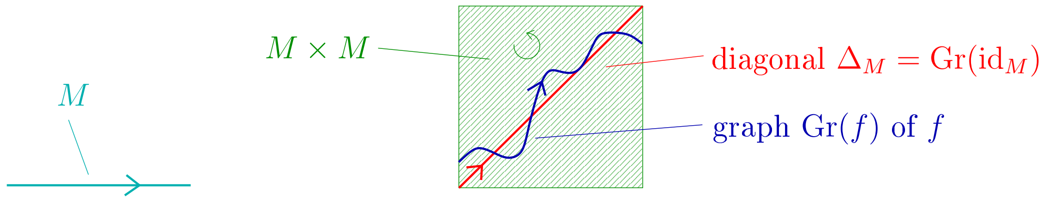


FIGURE 1351

To formulate the main result of this section we will also need the following definition.

Definition.

(1) Given a finitely generated abelian group we write $\mathrm{F}H := H/\operatorname{Tor}(H)$, where $\operatorname{Tor}(H)$ denotes the torsion subgroup of H.
(2) Let W be a closed oriented $2n$-dimensional topological manifold. We refer to the map

$$\begin{array}{rcl} Q_W\colon\ \mathrm{FH}_n(W;\mathbb{Z}) \times \mathrm{FH}_n(W;\mathbb{Z}) & \to & \mathbb{Z} \\ (a,b) & \mapsto & \langle \mathrm{PD}_W(a) \cup \mathrm{PD}_W(b), [W]\rangle \end{array}$$

as the *intersection form of W*.

Remark. We will discuss the intersection form in great detail in Chapter 100.

Now we can formulate the following, potentially quite counterintuitive proposition.

Proposition 96.1. *Let M be a closed oriented connected non-empty n-dimensional topological manifold.*

(1) *We have $Q_{M\times M}([\Delta_M],[\Delta_M]) = \chi(M)$.*
(2) *Given any map $f\colon M \to M$ we have $Q_{M\times M}([\mathrm{Gr}(f)],[\Delta_M]) = \Lambda(f)$.*

Remark. Sometimes in the literature, in line with the notation introduced on page 2307, the statement of Proposition 96.1 (1) gets shortened to the potentially ambiguous statement "$M \cdot M = \chi(M)$".

Examples.

(1) Suppose that in Proposition 96.1 we actually have a smooth manifold M and a smooth map $f\colon M \to M$. In this case we can view Δ_M and $\mathrm{Gr}(f)$ as oriented n-dimensional submanifolds of $M \times M$. If these two submanifold are transverse, then it follows from Theorem 94.5 that we can calculate $Q_{M\times M}$ geometrically, more precisely, we have the following equality:

$$Q_{M\times M}([\mathrm{Gr}(f)],[\Delta_M]) \;=\; \begin{array}{c}\text{algebraic intersection number } \mathrm{Gr}(f)\cdot\Delta_M \text{ of the oriented}\\ \text{submanifolds } \Delta_M \text{ and } \mathrm{Gr}(f)\text{, as defined on page 2271.}\end{array}$$

For example, let us consider the map $f\colon S^1 \to S^1$ that is illustrated in Figure 1352. We see that

$$\mathrm{Gr}(f)\cdot\Delta_M \underset{\substack{\uparrow \\ \text{there are three intersection points} \\ \text{with signs } -1,\, +1 \text{ and } -1}}{=} -1+1-1 \;=\; -1 \underset{\substack{\uparrow \\ \text{since } \mathrm{tr}(f \circlearrowleft \mathrm{H}_0(S^1;\mathbb{Z}))=1 \\ \text{and } \mathrm{tr}(f \circlearrowleft \mathrm{H}_1(S^1;\mathbb{Z}))=2}}{=} (-1)^0\cdot 1+(-1)^1\cdot 2 \;=\; \Lambda(f).$$

Fortunately this is consistent with Proposition 96.1 (2).

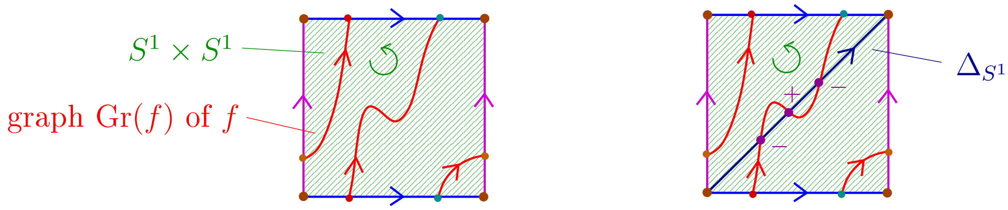


FIGURE 1352

(2) Let $n \in \mathbb{N}$. We have

$$Q_{S^n\times S^n}([\Delta_{S^n}],[\Delta_{S^n}]) \underset{\substack{\uparrow \\ \text{see Lemma 84.3}}}{=} (1\;\;1)\begin{pmatrix} 0 & 1 \\ (-1)^{n^2} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \;=\; 1+(-1)^{n^2} \;=\; \chi(S^n).$$

Again this is consistent with Proposition 96.1 (1).

The proof of Proposition 96.1 requires some preparations. In particular we will need the following lemma.

Lemma 96.2. *Let M be a closed oriented connected non-empty n-dimensional topological manifold, let $i \in \{0,\dots,n\}$ and let $v_1,\dots,v_m$ be a basis of $\mathrm{FH}^i(M;\mathbb{Z})$. There exists a basis $\widetilde{v}_1,\dots,\widetilde{v}_m$ of $\mathrm{FH}^{n-i}(M;\mathbb{Z})$ such that for any $k,l \in \{1,\dots,m\}$ we have $\langle v_k \cup \widetilde{v}_l, [M]\rangle = \delta_{kl}$.*

PROOF OF LEMMA 96.2. The statement of the lemma is precisely the content of Exercise 88.12. For the reader's convenience we nonetheless provide the proof. For $j = 1,\dots,m$ we set $\sigma_j := v_j \cap [M] \in \mathrm{FH}_{n-i}(M;\mathbb{Z})$. By the Poincaré Duality Theorem 88.1 we know that $\sigma_1,\dots,\sigma_m$ is a basis for $\mathrm{FH}_{n-i}(M;\mathbb{Z})$. Next note that it follows from the Universal Coefficient Theorem 75.13 for Cohomology Groups together with Lemma 75.5 (5) and Proposition 85.13 (4) that the map

$$\begin{array}{rcl} \mathrm{FH}^{n-i}(M;\mathbb{Z}) & \to & \mathrm{Hom}(\mathrm{FH}_{n-i}(M;\mathbb{Z}),\mathbb{Z}) \\ \varphi & \mapsto & (\sigma \mapsto \langle \varphi,\sigma\rangle) \end{array}$$

is an isomorphism. We denote by $\widetilde{v}_1,\dots,\widetilde{v}_m$ the cohomology classes with $\langle \widetilde{v}_k,\sigma_l\rangle = \delta_{kl}$. By the above $\widetilde{v}_1,\dots,\widetilde{v}_m$ are a basis for $\mathrm{FH}^{n-i}(M;\mathbb{Z})$. Now we see that for any $k,l \in \{1,\dots,m\}$ we have

$$\langle v_k \cup \widetilde{v}_l,[M]\rangle \overset{\substack{\text{by Lemma 83.4 (1), under the usual identification } \mathrm{H}_0(M;\mathbb{Z})=\mathbb{Z} \\ \downarrow}}{=} (v_k \cup \widetilde{v}_l)\cap[M] \underset{\substack{\uparrow \\ \text{Lemma 83.7}}}{=} \widetilde{v}_l \cap (v_k \cap [M]) \;=\; \widetilde{v}_l \cap \sigma_k \overset{\downarrow}{=} \langle \widetilde{v}_l,\sigma_k\rangle \;=\; \delta_{kl}.$$

■

Lemma 96.2 allows us to formulate our next lemma, which will be a key ingredient in the proof of Proposition 96.1.

Lemma 96.3. *Let M be a closed oriented connected n-dimensional topological manifold. For $i = 0, \dots, n$ we pick a basis $v_1^i, \dots, v_{m_i}^i$ of $\mathrm{FH}^i(M;\mathbb{Z})$. By Lemma 96.2 there exists a basis $\widetilde{v}_1^{n-i}, \dots, \widetilde{v}_{m_i}^{n-i}$ of $\mathrm{FH}^{n-i}(M;\mathbb{Z})$ such that for any $k, l \in \{1, \dots, m_i\}$ we have $\langle v_k^i \cup \widetilde{v}_l^{n-i}, [M]\rangle = \delta_{kl}$.*

(1) *We have the following equality in $\mathrm{H}^n(M \times M;\mathbb{Z})$:*

$$\mathrm{PD}_{M\times M}([\Delta_M]) \;=\; \sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)} \cdot \sum_{r=1}^{m_i} p_1^*(\widetilde{v}_r^{n-i}) \cup p_2^*(v_r^i).$$

(2) *Given any map $f\colon M \to M$ we have the following equality in $\mathrm{H}^n(M \times M;\mathbb{Z})$:*

$$\mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]) \;=\; \sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)} \cdot \sum_{k=1}^{m_i}\sum_{l=1}^{m_i} \underbrace{\langle v_k^i \cup f^*(\widetilde{v}_l^{n-i}), [M]\rangle}_{\in\mathbb{Z}} \cdot p_1^*(\widetilde{v}_k^{n-i}) \cup p_2^*(v_l^i).$$

Proof of Lemma 96.3. First we show that Statement (1) follows almost immediately from Statement (2). Indeed, we have

$$\mathrm{PD}_{M\times M}([\overbrace{\mathrm{Gr}(\mathrm{id}_M)}^{=\Delta_M}]) \underset{\substack{\uparrow\\ \text{Statement (2) applied to } f=\mathrm{id}_M}}{=} \sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_i} \underbrace{\langle v_r^i \cup \widetilde{v}_s^{n-i}, [M]\rangle}_{=\delta_{rs}} \cdot p_1^*(\widetilde{v}_r^{n-i}) \cup p_2^*(v_s^i).$$

Thus it remains to prove Statement (2) of Lemma 96.3. We start out the proof of Statement (2) with the following claim.

Claim. For $j \in \{0, \dots, n\}$ and $r, s \in \{1, \dots, m_j\}$ there exist a unique $a_{j,r,s} \in \mathbb{Z}$ such that

$$\mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]) \;=\; \sum_{j=0}^{n}\sum_{r=1}^{m_j}\sum_{s=1}^{m_j} a_{j,r,s} \cdot p_1^*(\widetilde{v}_r^{n-j}) \cup p_2^*(v_s^j) \;\in\; \mathrm{H}^n(M \times M;\mathbb{Z}).$$

Recall that by Proposition 85.13 (4) we know that the homology groups of M are all finitely generated. Thus it follows from the Künneth Theorem 84.16 for Cohomology Groups that there exists a short exact sequence of the following form:

$$0 \to \bigoplus_{j=0}^{n} \mathrm{H}^{n-j}(M;\mathbb{Z}) \otimes \mathrm{H}^j(M;\mathbb{Z}) \xrightarrow[\varphi\otimes\psi \,\mapsto\, p_1^*(\varphi)\cup p_2^*(\psi)]{} \mathrm{H}^n(M\times M;\mathbb{Z}) \to \bigoplus_{j=0}^{n+1} \overbrace{\mathrm{Tor}(\mathrm{H}^{n+1-j}(M;\mathbb{Z}), \mathrm{H}^j(M;\mathbb{Z}))}^{\text{torsion group by Lemma 57.17 (8)}} \to 0.$$

The Künneth Theorem 84.16 also says that the short exact sequence splits. Since the group on the right is torsion it follows easily, using some of the elementary properties of tensor product that we listed in Lemma 57.3, that the map

$$\bigoplus_{j=0}^{n} \underbrace{\mathrm{FH}^{n-j}(M;\mathbb{Z})}_{=\bigoplus_r \mathbb{Z}\cdot\widetilde{v}_r^{n-j}} \otimes \underbrace{\mathrm{FH}^j(M;\mathbb{Z})}_{=\bigoplus_s \mathbb{Z}\cdot v_s^j} \xrightarrow{\varphi\otimes\psi\mapsto p_1^*(\varphi)\cup p_2^*(\psi)} \mathrm{FH}^n(M\times M;\mathbb{Z})$$

is an isomorphism. The claim follows immediately from this fact. ⊞

First we remind the reader of a few facts on the cup, cap and cross product. To shorten the discussion we leave it to the reader to figure out what the various symbols stand for.

(a) Given $\varphi \in \mathrm{H}^i(M;\mathbb{Z})$ and $\sigma \in \mathrm{H}_i(M;\mathbb{Z})$ we have $\varphi \cap \sigma = \langle \varphi, \sigma\rangle \in \mathrm{H}_0(M;\mathbb{Z}) = \mathbb{Z}$.
(b) By Lemma 83.7 we have $(\psi \cup \varphi) \cap \sigma = \varphi \cap (\psi \cap \sigma)$.
(c) By Proposition 81.8 we have $\varphi \cup \psi = (-1)^{\deg(\varphi)\cdot\deg(\psi)} \cdot \psi \cup \varphi$.
(d) Given $g\colon X \to Y$ we know by Lemma 74.6 (3) we have $\langle g^*(\varphi), \sigma\rangle = \langle \varphi, g_*(\sigma)\rangle$.
(e) Given $g\colon X \to Y$ we know by Lemma 81.10 that $g^*(\varphi) \cup g^*(\psi) = g^*(\varphi \cup \psi)$.
(f) By Proposition 80.10 we have $[M\times M] = [M]\times[M]$. Furthermore, by Proposition 84.2 we have $p_1^*([M]^*) \cup p_2^*([M]^*) = [M \times M]^*$.
(g) By the Product Theorem 84.1 we have
$$\big(p_1^*(\alpha) \,\cup\, p_2^*(\beta)\big) \,\cap (\mu \times \nu) \;=\; (-1)^{\deg(\alpha)\cdot\deg(\beta)+\deg(\beta)\cdot\deg(\mu)} \cdot (\alpha \cap \mu) \times (\beta \cap \nu).$$
(h) By Exercise 80.2 the cross product on H_0 corresponds to multiplication under the augmentation maps.
(i) Given any $k \in \mathbb{Z}$ we have $(-1)^k = (-1)^{k^2}$.

Given $i \in \{0, \dots, n\}$ and $k, l \in \{1, \dots, m_i\}$ we calculate that

$$\begin{array}{rll}
\multicolumn{3}{l}{\langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}) \cup \mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]), [M\times M]\rangle} \\
& = (-1)^n\cdot\langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}), \mathrm{PD}_{M\times M}([\mathrm{Gr}(f)])\cap[M\times M]\rangle & \text{by (a), (b), (c) and (i)} \\
& = (-1)^n\cdot\langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}), [\mathrm{Gr}(f)]\rangle & \text{by definition of } \mathrm{PD}_{M\times M} \\
& = (-1)^n\cdot\langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}), (\mathrm{id}\times f)_*([M])\rangle & \text{by definition of } [\mathrm{Gr}(f)] \\
& = (-1)^n\cdot\langle (\mathrm{id}\times f)^*(p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i})), [M]\rangle & \text{by (d)} \\
& = (-1)^n\cdot\langle (p_1\circ(\mathrm{id}\times f))^*(v_k^i) \cup (p_2\circ(\mathrm{id}\times f))^*(\widetilde{v}_l^{n-i}), [M]\rangle & \text{by (e)} \\
& = (-1)^n\cdot\langle v_k^i \cup f^*(\widetilde{v}_l^{n-i}), [M]\rangle & \text{since } p_1\circ(\mathrm{id}\times f) = \mathrm{id} \\
& & \text{and } p_2\circ(\mathrm{id}\times f) = f.
\end{array}$$

Next we perform an alternative calculation:

$$\begin{array}{rll}
\multicolumn{3}{l}{\langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}) \cup \mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]), [M\times M]\rangle} \\
& = \Big\langle (p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i})) \cup \sum\limits_{j=0}^{n}\sum\limits_{r=1}^{m_j}\sum\limits_{s=1}^{m_j} a_{j,r,s}\cdot p_1^*(\widetilde{v}_r^{n-j}) \cup p_2^*(v_s^j), [M\times M]\Big\rangle & \text{by the claim} \\
& = (-1)^{(n-i)(n-j)}\cdot \sum\limits_{j=0}^{n}\sum\limits_{r=1}^{m_j}\sum\limits_{s=1}^{m_j} a_{j,r,s}\cdot \langle \underbrace{p_1^*(\overbrace{v_k^i\cup\widetilde{v}_r^{n-j}}^{\in \mathrm{H}^{i+n-j}(M)}}_{=0,\text{ for } i+n-j>n})\cup \underbrace{p_2^*(\overbrace{\widetilde{v}_l^{n-i}\cup v_s^j}^{\in\mathrm{H}^{n-i+j}(M)}}_{=0,\text{ for } n-i+j>n}), [M\times M]\rangle & \text{by (c) and (e)} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =0,\text{ for } i\neq j & \\
& = (-1)^{n-i}\cdot\sum\limits_{r=1}^{m_i}\sum\limits_{s=1}^{m_i} a_{i,r,s}\cdot\langle p_1^*(v_k^i \cup \widetilde{v}_r^{n-i}) \cup p_2^*(\widetilde{v}_l^{n-i} \cup v_s^i), [M]\times[M]\rangle & \text{by (f) and (i)} \\
& = (-1)^{n-i}\cdot\sum\limits_{r=1}^{m_i}\sum\limits_{s=1}^{m_i} a_{i,r,s}\cdot\langle v_k^i \cup \widetilde{v}_r^{n-i}, [M]\rangle\cdot\langle \widetilde{v}_l^{n-i} \cup v_s^i, [M]\rangle & \text{by (a), (g) and (h)} \\
& = (-1)^{n-i}\cdot\sum\limits_{r=1}^{m_i}\sum\limits_{s=1}^{m_i} a_{i,r,s}\cdot(-1)^{i\cdot(n-i)}\cdot\underbrace{\langle v_k^i \cup \widetilde{v}_r^{n-i}, [M]\rangle}_{=\delta_{kr}}\cdot\underbrace{\langle v_s^i \cup \widetilde{v}_l^{n-i}, [M]\rangle}_{=\delta_{ls}} & \text{by (c)} \\
& = (-1)^{n-i}\cdot\sum\limits_{r=1}^{m_i}\sum\limits_{s=1}^{m_i} a_{i,r,s}\cdot(-1)^{i\cdot(n-i)}\cdot\delta_{kr}\cdot\delta_{ls} \;=\; (-1)^{n-i+i\cdot(n-i)}\cdot a_{i,k,l}. &
\end{array}$$

Comparing these two calculations of $\langle p_1^*(v_k^i) \cup p_2^*(\widetilde{v}_l^{n-i}) \cup \mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]), [M \times M]\rangle$ we see that $(-1)^{i\cdot(n-i-1)} \cdot \langle v_k^i \cup f^*(\widetilde{v}_l^{n-i}), [M]\rangle = a_{i,k,l}$. But that is exactly what we needed to show. ■

Now we are finally ready to provide the proof of Proposition 96.1 (1). Note, that as we pointed out in Lemma 65.6, we have $\Lambda(\mathrm{id}) = \chi(M)$. Thus we see that Proposition 96.1 (1) is just a special case of Proposition 96.1 (2). Nonetheless, to reduce the notational mess we first prove Proposition 96.1 (1) separately.

Proof of Proposition 96.1 (1). We start out with the following claim which gives us a description of $\mathrm{PD}_{M\times M}([\Delta_M])$ that differs slightly from the one provided by Lemma 96.3 (1).

Claim. We have

$$\mathrm{PD}_{M\times M}([\Delta_M]) \;=\; \sum_{j=0}^{n} (-1)^j \cdot \sum_{s=1}^{m_j} p_1^*(v_s^j) \cup p_2^*(\widetilde{v}_s^{n-j}).$$

Let $j \in \{0, \dots, n\}$. Recall that we have $v_r^{n-j} \cup \widetilde{v}_s^j = \delta_{rs}$. From Proposition 81.8 we obtain that $\widetilde{v}_s^j \cup v_r^{n-j} = (-1)^{j\cdot(n-j)} \cdot \delta_{rs}$. In other words, we have $\widetilde{v}_s^j \cup (-1)^{j\cdot(n-j)} \cdot v_r^{n-j} = \delta_{rs}$. Now we see that

$$\mathrm{PD}_{M\times M}([\Delta_M]) \underset{\substack{\uparrow \\ \text{by Lemma 96.3 (1), applied to the bases } \{\widetilde{v}_r^j\}_{r=1,\dots,m_j} \text{ and the} \\ \text{corresponding dual bases } \{(-1)^{j\cdot(n-j)}\cdot v_s^j\}_{s=1,\dots,m_j}}}{=} \sum_{j=0}^{n} \overbrace{(-1)^{j\cdot(n-j-1)} \cdot (-1)^{j\cdot(n-j)}}^{=(-1)^j} \cdot \sum_{s=1}^{m_j} p_1^*(v_s^j) \cup p_2^*(\widetilde{v}_s^{n-j}).$$

⊞

Now we calculate that

$$\begin{aligned}
&\mathrm{PD}_{M\times M}([\Delta_M]) \cup \mathrm{PD}_{M\times M}([\Delta_M]) = \\
&\underset{\substack{\uparrow \\ \text{by Lemma 96.3 (1) and the claim}}}{=} \Big(\sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)} \cdot \sum_{r=1}^{m_i} p_1^*(\widetilde{v}_r^{n-i}) \cup p_2^*(v_r^i)\Big) \cup \Big(\sum_{j=0}^{n} (-1)^j \cdot \sum_{s=1}^{m_j} p_1^*(v_s^j) \cup p_2^*(\widetilde{v}_s^{n-j})\Big) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n} (-1)^{i\cdot(n-i-1)+j} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_j} p_1^*(\widetilde{v}_r^{n-i}) \cup p_2^*(v_r^i) \cup p_1^*(v_s^j) \cup p_2^*(\widetilde{v}_s^{n-j}) \\
&\underset{\substack{\uparrow \\ \text{by Proposition 81.8 and Lemma 81.10}}}{=} \sum_{i=0}^{n}\sum_{j=0}^{n} (-1)^{i\cdot(n-i-1)+j} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_j} (-1)^{i\cdot j+j\cdot(n-i)} \cdot \underbrace{p_1^*(\underbrace{\overbrace{v_s^j \cup \widetilde{v}_r^{n-i}}^{\in \mathrm{H}^{j+n-i}(M)}}_{=0 \text{ if } j+n\text{-}i>n}) \cup p_2^*(\underbrace{\overbrace{v_r^i \cup \widetilde{v}_s^{n-j}}^{\in \mathrm{H}^{i+n-j}(M)}}_{=0 \text{ if } i+n\text{-}j>n})}_{=0, \text{ for } i\neq j} \\
&\underset{\substack{\uparrow \\ \text{since } i+i^2 \text{ is even}}}{=} \sum_{i=0}^{n} (-1)^{i\cdot(n-i-1)} \cdot \sum_{r=1}^{m_i}\sum_{s=1}^{m_i} (-1)^{i\cdot(n-i)} \cdot p_1^*(\underbrace{v_s^i \cup \widetilde{v}_r^{n-i}}_{=\delta_{sr}\cdot[M]^*}) \cup p_2^*(\underbrace{v_r^i \cup \widetilde{v}_s^{n-i}}_{=\delta_{rs}\cdot[M]^*}) \\
&= \sum_{i=0}^{n} (-1)^i \cdot \sum_{r=1}^{m_i} \underbrace{p_1^*([M]^*) \cup p_2^*([M]^*)}_{=[M\times M]^* \text{ by Proposition 84.2}} \;=\; \chi(M) \cdot [M \times M]^*.
\end{aligned}$$

The desired statement now follows from the observation that by definition of $[M \times M]^*$ we have $\langle [M \times M]^*, [M \times M]\rangle = 1$. ■

Proof of Proposition 96.1 (2). The proof of Proposition 96.1 (2) is a generalization of the proof of Proposition 96.1 (1). It turns out that this argument is even less readable. In the following we quickly run through the argument and we only explain the steps that differ from the proof of Proposition 96.1 (2). We calculate that

$$\begin{aligned}
&\mathrm{PD}_{M\times M}([\mathrm{Gr}(f)]) \cup \mathrm{PD}_{M\times M}([\Delta_M]) = \\
&\underset{\uparrow}{=} \Big(\sum_{i=0}^{n}(-1)^{i\cdot(n-i-1)}\cdot\sum_{r=1}^{m_i}\sum_{s=1}^{m_i}\langle v_r^i\cup f^*(\widetilde{v}_s^{n-i}),[M]\rangle\cdot p_1^*(\widetilde{v}_r^{n-i})\cup p_2^*(v_s^i)\Big)\cup\Big(\sum_{j=0}^{n}(-1)^j\cdot\sum_{t=1}^{m_j}p_1^*(v_t^j)\cup p_2^*(\widetilde{v}_t^{n-j})\Big)
\end{aligned}$$

by Lemma 96.3 (2) and the rewriting procedure as in the claim in the proof of Proposition 96.1 (1)

$$\begin{aligned}
&= \sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i\cdot(n-i-1)+j}\cdot\sum_{r=1}^{m_i}\sum_{s=1}^{m_i}\sum_{t=1}^{m_j}\langle v_r^i\cup f^*(\widetilde{v}_s^{n-i}),[M]\rangle\cdot p_1^*(\widetilde{v}_r^{n-i})\cup p_2^*(v_s^i)\cup p_1^*(v_t^j)\cup p_2^*(\widetilde{v}_t^{n-j}) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i\cdot(n-i-1)+j}\cdot\sum_{r=1}^{m_i}\sum_{s=1}^{m_i}\sum_{t=1}^{m_j}(-1)^{i\cdot j+j\cdot(n-i)}\cdot\langle v_r^i\cup f^*(\widetilde{v}_s^{n-i}),[M]\rangle\cdot\underbrace{p_1^*(v_t^j\cup\widetilde{v}_r^{n-i})\cup p_2^*(v_s^i\cup\widetilde{v}_t^{n-j})}_{=0\text{ unless } i=j} \\
&= \sum_{i=0}^{n}(-1)^{i\cdot(n-i-1)+i}\cdot\sum_{r=1}^{m_i}\sum_{s=1}^{m_i}\sum_{t=1}^{m_i}(-1)^{i\cdot i+i\cdot(n-i)}\cdot\langle v_r^i\cup f^*(\widetilde{v}_s^{n-i}),[M]\rangle\cdot p_1^*(\underbrace{v_t^i\cup\widetilde{v}_r^{n-i}}_{=\delta_{tr}\cdot[M]^*})\cup p_2^*(\underbrace{v_s^i\cup\widetilde{v}_t^{n-i}}_{=\delta_{st}\cdot[M]^*}) \\
&= \sum_{i=0}^{n}(-1)^{i}\cdot\sum_{r=1}^{m_i}\langle v_r^i\cup f^*(\widetilde{v}_r^{n-i}),[M]\rangle\cdot[M\times M]^* \underset{\uparrow}{=} \sum_{i=0}^{n}(-1)^{i}\cdot\sum_{r=1}^{m_i}\Big\langle v_r^i\cup\sum_{s=1}^{m_i}a_{rs}^i\cdot\widetilde{v}_s^{n-i},[M]\Big\rangle\cdot[M\times M]^*
\end{aligned}$$

we define a_{rs}^i in the obvious way such that this equality holds

$$= \sum_{i=0}^{n}(-1)^{i}\cdot\sum_{r=1}^{m_i}\sum_{s=1}^{m_i}a_{rs}^i\cdot\underbrace{\langle v_r^i\cup\widetilde{v}_s^{n-i},[M]\rangle}_{=\delta_{rs}}\cdot[M\times M]^* = \sum_{i=0}^{n}(-1)^{i}\cdot\underbrace{\sum_{r=1}^{m_i}a_{rr}^i}_{=\mathrm{tr}(f^*\circlearrowright\mathrm{H}^i(M;\mathbb{Z}))}\cdot[M\times M]^* \underset{\uparrow}{=} \Lambda(f)\cdot[M\times M]^*.$$

by Exercise 75.12

As in the proof of Proposition 96.1 (1) we are now done. ■

96.2. **The Lefschetz-Hopf Theorem I.** In Proposition 96.1 we saw that we can express the Lefschetz number of a self-map of a manifold as an intersection number. Furthermore in Theorem 94.5 and in the more fancy Theorem 95.9 we learned how to determine intersection numbers using signs and indices of intersection points. The combination of the results, together with some fairly standard arguments, allows to prove two versions of the Lefschetz-Hopf Theorem. In this section we will use the more elementary Theorem 94.5 to prove the more basic version of the Lefschetz-Hopf Theorem.

Definition.

(1) Let M be a smooth manifold and let $f\colon M\to M$ be a smooth map.
   (a) Let $x\in M$ be a fixed point of f. Note that the differential $\mathrm{D}\,f_x$ is an endomorphism $\mathrm{D}\,f_x\colon\ \mathrm{T}_x\,M\to\mathrm{T}_x\,M$. We say that the fixed point x is *non-degenerate* if the endomorphism $\mathrm{id}_{\mathrm{T}_x\,M}-\mathrm{D}\,f_x$ of $\mathrm{T}_x\,M$ is invertible. Otherwise we say that the fixed point is *degenerate*.
   (b) Given a non-degenerate fixed point $x\in M$ we define

$$\mathrm{index}(f,x)\ :=\ \mathrm{sign}(\det(\mathrm{id}-\mathrm{D}\,f_x))\ =\ \frac{\det(\mathrm{id}-\mathrm{D}\,f_x)}{\|\det(\mathrm{id}-\mathrm{D}\,f_x)\|}.$$

(2) Let X be a topological space and let $f\colon X \to X$ be a map. We say a fixed point x of f is *isolated* if there exists a neighborhood U of x which does not contain another fixed point.

Example. Let $n \in \mathbb{Z}$. We consider the map $f\colon S^1 \to S^1$ that is given by $f(z) = z^n$. An elementary calculation shows that for each fixed point $x \in S^1$ the differential $\mathrm{D} f_x$ is given by multiplication by n.

(1) If $n = 1$, then every point on S^1 is a fixed point of f, and each fixed point is degenerate.
(2) If $n \neq 1$, then the fixed points of f are $x_j = e^{2\pi \mathrm{i} j/(n-1)}$, $j = 1, \ldots, |n-1|$. If $n \geq 1$, then the index of every fixed point is $\mathrm{sign}(1-n) = -1$, and if $n \leq 0$, then the index of every fixed point is $\mathrm{sign}(1-n) = +1$.

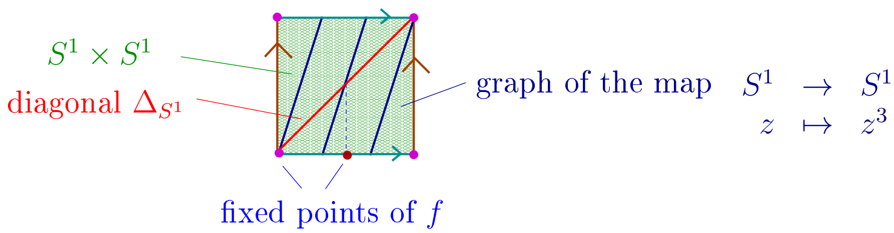


FIGURE 1353

Lemma 96.4.

(1) *Let M be a smooth manifold and let $f\colon M \to M$ be a smooth map. Every non-degenerate fixed point is also an isolated fixed point.*
(2) *Let X be a topological space and let $f\colon X \to X$ be a map. We assume that X is compact and Hausdorff. In this setting every fixed point of f is isolated if and only if f has only finitely many fixed points.*

PROOF ($*$).

(1) Using charts we see that we only need to consider the case that M is an open subset of some $\mathbb{R}^n$ and we can assume that the fixed point is $x = 0$. We consider the map $g\colon M \to \mathbb{R}^n$ that is given by $x \mapsto x - f(x)$. By our hypothesis we know that $\mathrm{D}\, g_0 = \mathrm{id}_{T_0 M} - \mathrm{D}\, f_0$ is invertible. It follows from the Inverse Function Theorem 6.18 that the map $x \mapsto g(x) = x - f(x)$ is a local diffeomorphism. In particular there exists an open neighborhood U of 0 such that the map $x \mapsto g(x) = x - f(x)$ is injective. But since $g(0) = 0$ this implies that U does not contain any other fixed point of f.
(2) This statement is the content of Exercise 3.17. For the reader's peace of mind we provide the proof. Thus let X be a topological space that is compact and Hausdorff. Furthermore let $f\colon X \to X$ be map. We denote by $F \subset X$ be the fixed point set of f.

    If F is finite, then it follows easily from the hypothesis that X is Hausdorff that F is a discrete subset of X. Conversely, suppose that F is a discrete subset of X. Note that it follows from Lemmas 2.40 and 2.17 together with Proposition 3.12 that

Δ_X and $\mathrm{Gr}(f)$ are closed subsets of $X \times X$. Thus $\Delta_X \cap \mathrm{Gr}(f)$ itself is a closed subset of $X \times X$, in particular it is a closed subset of Δ_X. We denote by $p\colon \Delta_X \to X$ the projection onto the first factor. Evidently this map is a homeomorphism and $F = p(\Delta_X \cap \mathrm{Gr}(f))$. Thus we have shown that F is a closed subset of X and by hypothesis we know that F is discrete. Since X is compact we obtain from Lemma 2.18 (1) that F is indeed finite. ■

Now we can formulate the first version of the Lefschetz-Hopf Theorem.

Theorem 96.5. (Lefschetz-Hopf Theorem I) *Let M be a compact orientable smooth manifold and let $f\colon M \to M$ be a smooth map. If every fixed point of f is non-degenerate and if there is no fixed point on ∂M, then*[1354]

$$\sum_{\substack{x \text{ a fixed}\\ \text{point of } f}} \mathrm{index}(f,x) \;=\; \Lambda(f).$$

Remark. If a smooth map has no fixed points, then the left hand side of the Lefschetz-Hopf Theorem 96.5 is evidently zero. Thus we see that the Lefschetz-Hopf Theorem 96.5 is a generalization of the Lefschetz Fixed Point Theorem 65.10, applied to the field $\mathbb{F} = \mathbb{Q}$.

Our first example hopefully puts any lingering worries about signs to rest.

Example. Let $n \in \mathbb{Z} \setminus \{1\}$. As on page 2334 we consider the map $f\colon S^1 \to S^1$ that is given by $f(z) = z^n$. We perform the following calculation:

$$\sum_{\substack{x \text{ a fixed}\\ \text{point of } f}} \mathrm{index}(f,x) \;\underset{\uparrow}{=}\; \sum_{j=1}^{|n-1|} \mathrm{index}(f, e^{2\pi i j/(n-1)}) \;\underset{\uparrow}{=}\; \sum_{j=1}^{|n-1|} -\operatorname{sign}(n-1) \;=\; -(n-1).$$

both equalities follow from the discussion on page 2334

Fortunately this is consistent with the Lefschetz-Hopf Theorem 96.5 and the following calculation:

$$\Lambda(f) \;=\; \underbrace{\mathrm{tr}(f_*\colon\ \mathrm{H}_0(S^1;\mathbb{Z}) \to \mathrm{H}_0(S^1;\mathbb{Z}))}_{=1,\text{ see page 1088}} - \underbrace{\mathrm{tr}(f_*\colon\ \mathrm{H}_1(S^1;\mathbb{Z}) \to \mathrm{H}_1(S^1;\mathbb{Z}))}_{=n,\text{ by Lemma 45.10}} \;=\; 1-n.$$

Just for fun let us consider a second example.

Example. Let $n \in \mathbb{N}$. We consider $M = \mathbb{R}^n \cup \{\infty\}$. We denote by 0 the origin of $\mathbb{R}^n$. In Exercise 6.13 we showed that M is an n-dimensional smooth manifold with the atlas given by the obvious chart for $\mathbb{R}^n$ together with the chart

$$\begin{array}{rcl} M \setminus \{0\} & \to & \mathbb{R}^n \\ x & \mapsto & \begin{cases} \frac{1}{\|x\|^2}\cdot x, & \text{if } x \neq \infty, \\ 0, & \text{if } x = \infty. \end{cases} \end{array}$$

Furthermore we showed that the map

$$\begin{array}{rcl} f\colon M & \to & M \\ x & \mapsto & \begin{cases} 3x, & \text{if } x \neq \infty, \\ \infty, & \text{if } x = \infty \end{cases} \end{array}$$

is smooth, that it has two fixed points, namely the origin 0 and ∞, and that $\mathrm{D}\,f_0 = 3\cdot\mathrm{id}$ and $\mathrm{D}\,f_\infty = \frac{1}{3}\cdot\mathrm{id}$. Thus we see

$$\sum_{\substack{x \text{ a fixed}\\ \text{point of } f}} \mathrm{index}(f,x) = \mathrm{index}(f,0) + \mathrm{index}(f,\infty) = \operatorname{sign}(\underbrace{(1-3)^n}_{\det(\mathrm{id}-\mathrm{D}\,f_0)}) + \operatorname{sign}(\underbrace{(1-\tfrac{1}{3})^n}_{\det(\mathrm{id}-\mathrm{D}\,f_\infty)}) = (-1)^n + 1.$$

[1354] Note that by Lemma 96.4 we know that the left-hand side is a finite sum.

Conveniently enough this is consistent with the Lefschetz-Hopf Theorem 96.5 and the following calculation:

$$\Lambda(f) \underset{\uparrow}{=} \Lambda(\mathrm{id}) = \chi(S^n) = 1 + (-1)^n.$$

follows from Lemma 65.7, since f is easily seen to be homotopic to id

In Lemma 2.44 we gave an explicit diffeomorphism between $\Phi\colon S^n \to M = \mathbb{R}^n \cup \{\infty\}$ that sends the South Pole to the origin and that sends the North Pole to ∞. For $n = 1$ and $n = 2$ we illustrate the map $\Phi^{-1} \circ f \circ \Phi\colon S^n \to S^n$ in Figure 1354

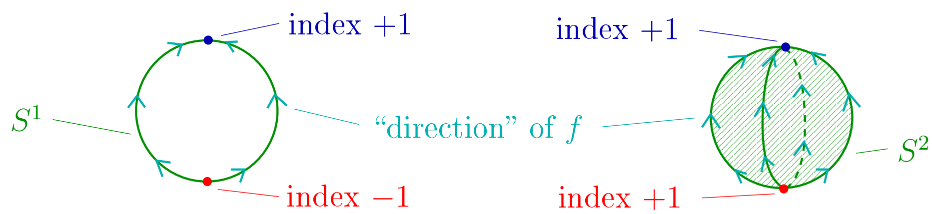


FIGURE 1354

PROOF OF THE LEFSCHETZ-HOPF THEOREM 96.5. Let M be a compact orientable n-dimensional smooth manifold and let $f\colon M \to M$ be a smooth map such that every fixed point is non-degenerate. We pick an orientation. First we assume that M is closed.

We pick an orientation for M. We equip $\mathrm{Gr}(f) \subset M \times M$ and $\Delta_M \subset M \times M$ with the orientation which turns the projection onto the first factor into an orientation-preserving diffeomorphism.

Claim. Let x be a fixed point of f.[1355]

(1) The point (x, x) is a transverse intersection point of the submanifolds $\mathrm{Gr}(f)$ and Δ_M if and only if x is non-degenerate.
(2) If (x, x) is a transverse intersection point of $\mathrm{Gr}(f)$ and Δ_M, then the sign of the intersection point is given by $\mathrm{index}(f, x) = \frac{\det(\mathrm{id} - \mathrm{D} f_x)}{\|\det(\mathrm{id} - \mathrm{D} f_x)\|}$.

Note that we have a natural identification $\mathrm{T}_{(x,x)}(M \times M) = \mathrm{T}_x M \times \mathrm{T}_x M$. We pick a positive basis $v_1, \dots, v_n$ for $\mathrm{T}_x M$. By the definition of the product orientation on $M \times M$, see page 304, we see that the basis $(v_1, 0), \dots, (v_n, 0), (0, v_1), \dots, (0, v_n)$ is a positive basis for $\mathrm{T}_{(x,x)}(M \times M)$. Next note that with respect to this basis we have

$$\underbrace{\begin{pmatrix} \mathrm{id}_n \\ \mathrm{D} f_x \end{pmatrix}}_{\substack{\text{positive basis} \\ \text{for } \mathrm{T}_{(x,x)} \mathrm{Gr}(f)}} \text{ followed by } \underbrace{\begin{pmatrix} \mathrm{id}_n \\ \mathrm{id}_n \end{pmatrix}}_{\substack{\text{positive basis} \\ \text{for } \mathrm{T}_{(x,x)} \Delta_M}} \begin{matrix}\text{which} \\ \text{gives us} \\ \text{the matrix}\end{matrix} \begin{pmatrix} \mathrm{id}_n & \mathrm{id}_n \\ \mathrm{D} f_x & \mathrm{id}_n \end{pmatrix} \begin{matrix}\text{which has} \\ \text{the same} \\ \text{determinant as}\end{matrix} \begin{pmatrix} \mathrm{id}_n & 0 \\ \mathrm{id}_n & \mathrm{id}_n - \mathrm{D} f_x \end{pmatrix}.$$

The desired statements follow immediately from this observation and the definitions. ⊞

[1355] We refer to pages 2270 and 2271 for the definition of a transverse intersection point and the sign of a transverse intersection point.

Now we perform the following calculation:

$$\Lambda(f) \underset{\substack{\uparrow\\ \text{Proposition 96.1}}}{=} \mathrm{Gr}(f)\cdot\Delta_M \underset{\substack{\uparrow\\ \text{by Theorem 94.5 and Claim (1)}}}{=} \sum_{\substack{(x,x)\text{ an intersection point}\\ \text{of }\mathrm{Gr}(f)\text{ and }\Delta_M}} \text{sign of }(x,x) \underset{\substack{\uparrow\\ \text{by Claim (2)}}}{=} \sum_{\substack{x\text{ a fixed}\\ \text{point of }f}} \mathrm{index}(f,x).$$

We have thus proved the theorem in the special case that M is closed and orientable.

Now we allow that the smooth manifold M has non-empty boundary. As on page 1163 we consider the double $\mathrm{D}\,M := (M\times\{1\})\sqcup_{\partial M\times\{1\}=\partial M\times\{2\}}(M\times\{2\})$. By Lemma 44.12 we know that $\mathrm{D}\,M$ is a closed orientable smooth manifold. We also consider the "double of f", i.e. we consider the map

$$\begin{array}{rcl}\psi\colon\ \mathrm{D}\,M & \to & \mathrm{D}\,M\\ [(P,\epsilon)] & \mapsto & [(f(P),\epsilon)].\end{array}$$

Let $x_1,\dots,x_k$ be the fixed points of $f\colon M\to M$. Recall that we demand that no fixed point of f lies on ∂M. It follows immediately that the fixed points of ψ are precisely $(x_1,1),\dots,(x_k,1)$, $(x_1,2),\dots,(x_k,2)$. Now we see that

$$\begin{aligned}2\cdot\sum_{i=1}^k \mathrm{index}(f,x_i) &\underset{\substack{\uparrow\\ \text{it is elementary to see that}\\ \mathrm{index}(f,x_i)=\mathrm{index}(\psi,(x_i,1))=\mathrm{index}(\psi,(x_i,2))}}{=} \sum_{i=1}^k \mathrm{index}(\psi,(x_i,1))+\sum_{i=1}^k \mathrm{index}(\psi,(x_i,2)) \underset{\substack{\uparrow\\ \text{by the above, since D}\,M\text{ is a closed}\\ \text{orientable smooth manifold}}}{=} \Lambda(\psi\colon\ \mathrm{D}\,M\to\mathrm{D}\,M)\\ &\underset{\substack{\uparrow\\ \text{follows from Exercise 65.15}}}{=} 2\cdot\Lambda(f\colon M\to M)+\Lambda(f\colon\partial M\to\partial M) \underset{\substack{\uparrow\\ \text{since there are no fixed points on }\partial M\text{ and since}\\ \partial M\text{ is a closed orientable smooth manifold}\\ \text{we know by the above that }\Lambda(f\colon\partial M\to\partial M)=0}}{=} 2\cdot\Lambda(f).\end{aligned}$$

Dividing both sides by 2 gives us the desired equality. ■

96.3. **The Lefschetz-Hopf Theorem II.** In this section we will formulate and prove an advanced version of the Lefschetz-Hopf Theorem 96.5 which puts fewer restrictions on the fixed points. The down-side is that in general we need to work with a more tricky definition of the index of a fixed point.

We start out with the following not overly exciting lemma.

Lemma 96.6. *Let M be an n-dimensional topological manifold and let $f\colon M\to M$ be a map. Let $z\in M\setminus\partial M$ be an isolated fixed point of $f\colon M\to M$. Given any neighborhood U of $(z,z)\in M\times M$ there exists a neighborhood V of $z\in M$ and a neighborhood W of $z\in M$ with the following properties:*

1. *We have $V\times W\subset U$.*
2. *z is the only fixed point of f contained in V.*
3. *V and W are both submanifolds of M and both are homeomorphic to $\overline{B}^n$.*
4. *We have $V\subset\mathring{W}$ and $f(V)\subset\mathring{W}$.*

Proof. This lemma follows fairly easily from Exercise 6.6. We leave it to the reader to fill in the details. ■

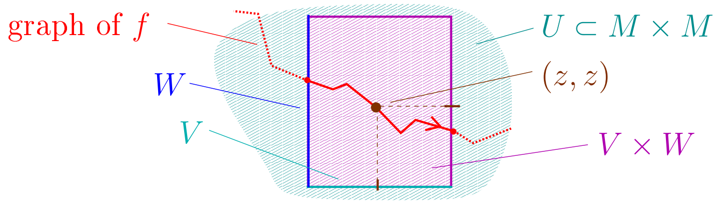


FIGURE 1355. Illustration of Lemma 96.6.

The lemma allows us to introduce the following definition.

Definition. Let M be an n-dimensional topological manifold and let $f\colon M \to M$ be a map. Furthermore let $z \in M \setminus \partial M$ be an isolated fixed point of f. We pick neighborhoods V and W of z that satisfy Properties (2) and (4) of Lemma 96.6. Furthermore we pick an embedding $\Phi\colon W \to \mathbb{R}^n$ with $\Phi(z) = 0$. We define the *index* $\operatorname{index}(f,z)$ *of the fixed point* z as the degree of the following homomorphism:

$$\overbrace{\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})}^{\cong \mathbb{Z} \text{ by Lemma 44.1 (2)}} \xleftarrow{\cong} \mathrm{H}_n(\Phi(V), \Phi(V) \setminus \{0\}) \xrightarrow{\mathrm{id} - (\Phi \circ f \circ \Phi^{-1})_*} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}).$$

this map is induced by the inclusion $\Phi(V) \subset \mathbb{R}^n$, it is an isomorphism by the Excision Theorem 43.20, here we also use that we obtain from the Topological Invariance of Domain Theorem 50.6 that $\Phi(V)$ is a neighborhood of 0

map defined since $f(V) \subset W$ and since x is the only fixed point of f contained in V

We illustrate this definition in Figure 1356.

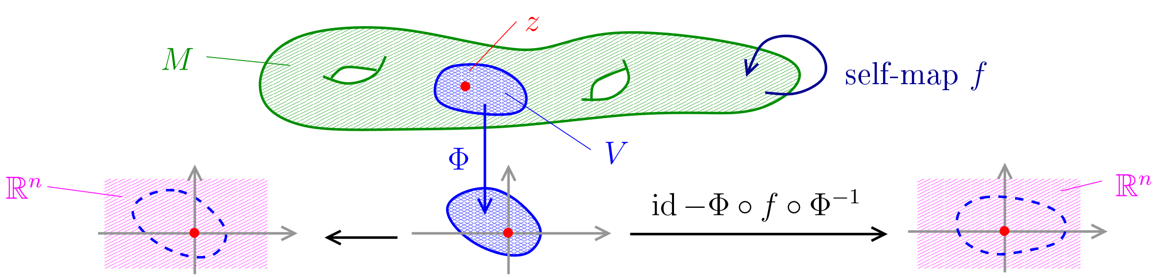


FIGURE 1356

Example. Let $n \in \mathbb{N}$. We consider the map

$$\begin{aligned} f\colon \mathbb{R}^2 = \mathbb{C} &\to \mathbb{C} = \mathbb{R}^2 \\ z &\mapsto z + z^n. \end{aligned}$$

The point $z = 0$ is evidently an isolated fixed point. Note that $f'(z) = 1 + nz^{n-1}$. It follows that for $n \geq 2$ we have $\mathrm{id} - \mathrm{D}\, f_0 = \mathrm{id} - \mathrm{id} = 0$. Thus we we see that for $n \geq 2$ the fixed point 0 is degenerate. Since 0 is an isolated fixed point we can nonetheless define the index using the approach above. We set $V = \overline{B}^2$ and $W = \overline{B}^2_3$ and $\Phi = \mathrm{id}$. Since

$$\mathrm{id} - (\Phi \circ f \circ \Phi^{-1}) \ = \ \mathrm{id} - f \ = \ \mathrm{id} - (\mathrm{id} + (z \mapsto z^n)) \ = \ -(z \mapsto z^n)$$

we obtain easily from Lemma 45.10 that $\operatorname{index}(f, 0) = -n$.

The following lemma says that the above index is well-defined.

Lemma 96.7. *Let M be an n-dimensional topological manifold and let $f\colon M \to M$ be a map. For an isolated fixed point $z \in M \setminus \partial M$ of f the above definition of* $\operatorname{index}(f,z)$ *does not depend on any choices.*

Proof (∗). Let V_1, W_1 and V_2, W_2 be neighborhoods of z that satisfy the properties stated in Lemma 96.6 (2) and (4). We pick embeddings $\Phi_1\colon W_1 \to \mathbb{R}^n$ and $\Phi_2\colon W_2 \to \mathbb{R}^n$ as in the definition of the index. We need to show that the choices lead to the same index.

First note that by the "given any neighborhood" feature of Lemma 96.6 we can assume that $V_1 \times W_1 \subset V_2 \times W_2$. Next we set $\epsilon := +1$ if $\Phi_2 \circ \Phi_1^{-1}$ is orientation-preserving, otherwise we set $\epsilon := -1$. Next we consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \arrow[ddd, "\mathrm{id}"] & \mathrm{H}_n(\Phi_1(V_1), \Phi_1(V_1)\setminus\{0\}) \arrow[l, "\iota"', "\cong"] \arrow[rr, "\mathrm{id}-(\Phi_1\circ f\circ\Phi_1^{-1})_*"] \arrow[dr, "\mathrm{id}-(\Phi_1\circ f\circ\Phi_1^{-1})_*"'] \arrow[ddd, "\epsilon\cdot(\Phi_2\circ\Phi_1^{-1})_*"'] & & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \arrow[ddd, "\mathrm{id}"] \\
 & & \mathrm{H}_n(\Phi_1(f(V_1)), \Phi_1(f(V_1))\setminus\{0\}) \arrow[ur, "\iota"'] \arrow[d, "\epsilon\cdot(\Phi_2\circ\Phi_1^{-1})_*"] & \\
 & & \mathrm{H}_n(\Phi_2(f(V_1)), \Phi_2(f(V_1))\setminus\{0\}) \arrow[dr, "\iota"] & \\
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \arrow[d, "\mathrm{id}"] & \mathrm{H}_n(\Phi_2(V_1), \Phi_2(V_1)\setminus\{0\}) \arrow[l, "\iota"', "\cong"] \arrow[ur, "\mathrm{id}-(\Phi_2\circ f\circ\Phi_2^{-1})_*"'] \arrow[rr, "\mathrm{id}-(\Phi_2\circ f\circ\Phi_2^{-1})_*"'] \arrow[d, "\iota"] & & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \arrow[d, "\mathrm{id}"] \\
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) & \mathrm{H}_n(\Phi_2(V_2), \Phi_2(V_2)\setminus\{0\}) \arrow[l, "\iota"', "\cong"] \arrow[rr, "\mathrm{id}-(\Phi_2\circ f\circ\Phi_2^{-1})_*"] & & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}).
\end{tikzcd}$$

We make the following clarifications and observations:

(1) The maps indicated by ι are the natural inclusion induced maps.
(2) Since $\Phi_1\colon W_1 \to \mathbb{R}^n$ is an embedding and since $V_1 \subset W_1$ we can consider the inverse $\Phi_1^{-1}\colon \Phi_1(V_1) \to V_1$. Furthermore, since $V_1 \subset V_2$ we can consider $\Phi_2(V_1)$ and we have $\Phi_2(V_1) \subset \Phi_2(V_2)$.
(3) It follows fairly immediately from the definition of ϵ that the diagram commutes.
(4) The left horizontal maps are isomorphisms by the Excision Theorem 43.20.

It follows from the above that the top and the bottom horizontal maps from left to right agree. But this means that we obtain the same degree. ■

The following lemma shows that the two notions of the index of a fixed point of a smooth map, introduced on pages 2333 and 2338 are consistent.

Lemma 96.8. *Let M be a smooth manifold, let $f\colon M \to M$ be a smooth map and let $z \in M \setminus \partial M$ be a non-degenerate fixed point of f. The fixed point z is isolated, and the index of z, as defined on page 2333, agrees with the index of z as defined on page 2338.*

Proof (∗). Let M be a smooth manifold, let $f\colon M \to M$ be a smooth map and let $z \in M \setminus \partial M$ be a non-degenerate fixed point of f. For the purpose of this proof we denote by $\operatorname{index}^{\mathrm{sm}}(z)$ the index of z as defined on page 2333 and we denote by $\operatorname{index}^{\mathrm{top}}(z)$ the index of z as defined on page 2338.

First note that we already showed in Lemma 96.4 that z is an isolated fixed point. Thus it remains to show that $\operatorname{index}^{\mathrm{top}}(z) = \operatorname{index}^{\mathrm{sm}}(z)$. From our smooth atlas for M we pick a chart $\Phi\colon W \to \mathbb{R}^n$ around z with $\Phi(z) = 0$. It follows fairly easily from the hypothesis that z is a non-degenerate fixed point that there exists an open neighborhood V around z such that $V \subset W$, such that $\Phi(V) \subset W$, and such that V contains no other fixed point of f. We set $\Theta := \mathrm{D}\,\Phi_z\colon\ \mathrm{T}_z\,M \to \mathrm{T}_0\,\mathbb{R}^n = \mathbb{R}^n$.

Now we consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \arrow[rr, bend left, "\cdot\,\operatorname{index}^{\mathrm{top}}(z)"] \arrow[dr, bend right, "(\mathrm{id}-\Theta\circ\mathrm{D}\,f_z\circ\Theta^{-1})_*"'] & \mathrm{H}_n(\Phi(V),\Phi(V)\setminus\{0\}) \arrow[l, "\cong"'] \arrow[r, "\mathrm{id}-(\Phi\circ f\circ\Phi^{-1})_*"'] & \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \arrow[d, "\mathrm{id}"] \\
 & \mathrm{H}_n(Y,Y\setminus\{0\}) \arrow[u, "\cong"] \arrow[r, "(\mathrm{id}-\Phi\circ f\circ\Phi^{-1})_*"] \arrow[d, "(\mathrm{id}-\Theta\circ\mathrm{D}\,f_z\circ\Theta^{-1})_*"'] & \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \arrow[dl, "\mathrm{id}"] \\
 & \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}). &
\end{tikzcd}$$

We make the following clarifications and observations:

(1) By construction V and W satisfy the properties of Lemma 96.6 (2) and (4). This implies that the composition of the top two horizontal maps is given by multiplication by $\operatorname{index}^{\mathrm{top}}(z)$.
(2) It follows immediately from Lemma 45.17 that there exists an open neighborhood Y of 0, contained in $\Phi(V)$, such that the map $\mathrm{id}-\Phi\circ f\circ\Phi^{-1}$ is homotopic, as a map of pairs $(\Phi(V),\Phi(V)\setminus\{0\}) \to (\mathbb{R}^n,\mathbb{R}^n\setminus\{0\})$, to the map $\mathrm{D}(\mathrm{id}-\Phi\circ f\circ\Phi^{-1})_0$.
(3) By the chain rule we have $\mathrm{D}(\mathrm{id}-\Phi\circ f\circ\Phi^{-1})_0 = \mathrm{id}-\Theta\circ\mathrm{D}\,f_z\circ\Theta^{-1}$.
(4) It follows from (2) together with Proposition 43.17 and some elementary considerations that the diagram commutes.
(5) It follows from the Excision Theorem 43.20 that the two maps decorated with "$\cong$" are indeed isomorphisms.

We see that

$$\begin{aligned}
\operatorname{index}^{\mathrm{top}}(z) &\overset{\substack{\text{by the above diagram and the above discussion}\\ \downarrow}}{=} \deg((\mathrm{id}_{\mathbb{R}^n}-\Theta\circ\mathrm{D}\,f_z\circ\Theta^{-1})_*) \overset{\substack{\text{by Lemma 45.18}\\ \downarrow}}{=} \operatorname{sign}(\det(\mathrm{id}_{\mathbb{R}^n}-\Theta\circ\mathrm{D}\,f_z\circ\Theta^{-1}))\\
&\underset{\substack{\uparrow\\ \text{basic linear algebra}}}{=} \operatorname{sign}(\det(\mathrm{id}_{\mathrm{T}_z\,M}-\mathrm{D}\,f_z)) \underset{\substack{\uparrow\\ \text{by the definition on page 2333}}}{=} \operatorname{index}^{\mathrm{sm}}(z).
\end{aligned}$$

■

By Lemma 96.8 the following theorem is a generalization of our previous Lefschetz-Hopf Theorem 96.9.

Theorem 96.9. (Lefschetz-Hopf Theorem II) *Let M be a compact orientable topological manifold and let $f\colon M \to M$ be a map which has no fixed points on ∂M. If f has* *finitely many fixed points, then*

$$\sum_{\substack{x \text{ a fixed} \\ \text{point of } f}} \operatorname{index}(f,x) \;=\; \Lambda(f).$$

Remark.

(1) The Lefschetz-Hopf Theorem has its origin in the work of Heinz Hopf [**Hopf29**, Section 8]. Various variations on the Lefschetz-Hopf Theorem 96.9 can be found in [**GD03**, Chapter V.16] and [**JM06**, Chapter 2.4].

(2) As on page 2335 we see that the Lefschetz-Hopf Theorem 96.9 is a generalization of the Lefschetz Fixed Point Theorem 85.11 for compact orientable topological manifolds.

Example. This time we consider S^2 with the avatar given by $\overline{B}^2/S^1$. We consider the map $f\colon \overline{B}^2/S^1 \to \overline{B}^2/S^1$ that is illustrated, to the best of the author's abilities, in Figure 1357.[1356] It has precisely one fixed point, namely $P := [S^1]$. We see that

$$\operatorname{index}(f,P) \underset{\substack{\uparrow \\ \text{Lefschetz-Hopf Theorem 96.9}}}{=} \Lambda(f) \underset{\substack{\uparrow \\ \text{since } f \text{ is homotopic to id}}}{=} \Lambda(\mathrm{id}) = \chi(S^2) = 2.$$

Note that with a little bit of care, say using Lemma 6.13, one can arrange that f is smooth. Also note that $\mathrm{D}\, f = \mathrm{id}$. Thus $P = [S^1]$ is a degenerate fixed point of f.[1357]

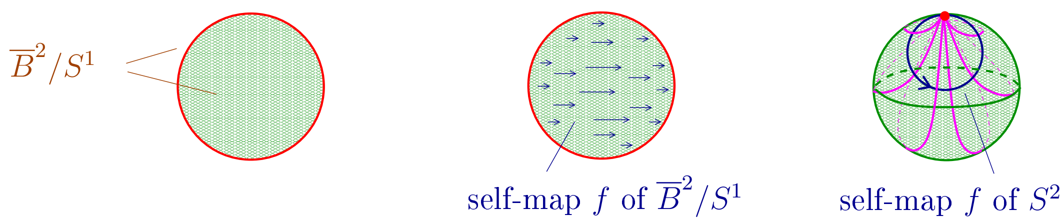


FIGURE 1357

The proof of the Lefschetz-Hopf Theorem 96.9 rests heavily on the shoulders of the following two lemmas.

Lemma 96.10. *Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^n$ be two compact connected n-dimensional topological submanifolds of $\mathbb{R}^n$ that contain 0 in the interior. We assume that $V \subset \mathring{W}$. Let $f\colon V \to \mathbb{R}^n$ be a map with $f(V) \subset \mathring{W}$ and which has a unique fixed point, namely 0. We set $G := \{(v, f(v)) \,|\, v \in V\}$ and we set $D := \{(v,v) \,|\, v \in V\}$.*

(1) *G and D are well-behaved proper submanifolds of $V \times W$, in the sense of the definition on page 2303.*

[1356] An alternative description of this map is given by considering $S^2 = \mathbb{R}^2 \cup \{\infty\}$. In this setting our map corresponds to the map $f\colon \mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}$ that is given by $f(x) = x + (1,0)$ for all $x \in \mathbb{R}^2$ and $f(\infty) = \infty$.

[1357] Incidentally this example gives an affirmative answer to the question asked in Exercise 2.12.

(2) *The following equality holds:*[1358][1359]

$$\Big\langle \underbrace{\mathrm{PD}_{V\times W}([G])}_{\in \mathrm{H}^n((V\times W)\setminus \partial G)} \cup \underbrace{\mathrm{PD}_{V\times W}([D])}_{\in \mathrm{H}^n((V\times W)\setminus \partial D)}, [V\times W]\Big\rangle \;=\; \deg\begin{pmatrix} \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \\ \uparrow \cong \\ \mathrm{H}_n(V, V\setminus\{0\}) \\ \downarrow (\mathrm{id}-f)_* \\ \mathrm{H}_n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \end{pmatrix}.$$

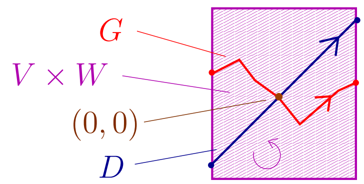


FIGURE 1358. Illustration for Lemma 96.10.

SKETCH OF PROOF. First we prove Statement (1). More precisely, we will show that G is a well-behaved proper submanifold of $V\times W$. The proof that D is also a well-behaved submanifold is basically identical. Note that by Exercise 6.10 we know that G is a proper submanifold of $V\times W$. Next note by definition of a well-behaved submanifold we need to show that given any open neighborhood U of $\partial G \subset \partial(V\times W) = (\partial V\times W)\cup(V\times\partial W)$ the submanifold ∂G of $\partial(V\times W)$ admits a regular neighborhood N with $N\subset U$. Note that using the fact that ∂V is compact and using the hypothesis that $f(V)\subset \mathring{W}$ it is fairly elementary to see that there exists an $\epsilon>0$ such that for every $x\in\partial G$ we have $\{x\}\times \overline{B}{}^n_\epsilon(f(x))\subset U$. It is straightforward to verify that given any $\nu\in(0,\epsilon)$ the set

$$N_\nu \;:=\; \{(x,y)\in\partial V\times W \,|\, x\in\partial V \text{ and } y\in \overline{B}{}^n_\nu(f(x))\} \;\subset\; \partial(V\times W)$$

is a regular neighborhood of ∂G with $N_\nu\subset U$. This concludes the proof of Statement (1).

Next we turn to the proof of Statement (2). The key idea is to "tilt" the situation to turn the diagonal D into a "horizontal submanifold". Afterwards we can use Proposition 88.2 to perform the desired calculation.

Now we carry out the details. To do so we introduce the following objects:

(a) We pick a compact submanifold $\widetilde{W}$ of $\mathbb{R}^n$ with $W\subset\widetilde{W}$ and $(\mathrm{id}-f)(W)\subset\widetilde{W}$.
(b) We consider the maps

$$\begin{array}{rcl} l\colon V & \to & G \\ v & \mapsto & (v,f(v)) \end{array} \qquad \begin{array}{rcl} \Theta\colon V\times W & \to & V\times\widetilde{W} \\ (v,w) & \mapsto & (v,w-f(v)) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} q\colon V\times\widetilde{W} & \to & \widetilde{W} \\ (v,w) & \mapsto & w. \end{array}$$

(c) Note that by (1) we can pick a regular neighborhood N_G for $\partial G\subset\partial(V\times W)$ with $N_G\subset\partial V\times W$ and with $N_G\cap\partial D=\varnothing$.

[1358]By (1) we know that G is a well-behaved proper submanifold of $V\times W$. Thus we can consider the isomorphism $\mathrm{PD}_{V\times W}\colon\ \mathrm{H}_n(V\times W,\partial G)\to \mathrm{H}^n(V\times W,(V\times W)\setminus\partial G)$ that we introduced in Proposition 95.2. Similarly we consider the isomorphism $\mathrm{PD}_{V\times W}$ with G replaced by D.

[1359]Since 0 is the only fixed point of f we see that $(\mathrm{id}-f)(V\setminus\{0\})\subset\mathbb{R}^n\setminus\{0\}$.

(d) We denote by $\mu_{\widetilde{W},0} \in \mathrm{H}_n(\widetilde{W}, \widetilde{W} \setminus \{0\}) \cong \mathbb{Z}$ the standard generator, as defined on page 2097. Furthermore we denote by $\mu^*_{\widetilde{W},0} \in \mathrm{H}^n(\widetilde{W}, \widetilde{W} \setminus \{0\}; \mathbb{Z}) \cong \mathbb{Z}$ the corresponding dual.

Many of the above objects are illustrated in Figure 1359. Now we can perform the desired calculation:[1360]

$$\begin{aligned}
\big\langle \mathrm{PD}_{V\times W}([G]) \cup \mathrm{PD}_{V\times W}([D]), [V\times W] \big\rangle
&\underset{\substack{\text{by Lemmas 83.4 and 83.7, under the natural}\\ \text{identification } \mathrm{H}_0(V\times W)=\mathbb{Z}}}{=} \mathrm{PD}_{V\times W}([D]) \cap \big(\overbrace{\mathrm{PD}_{V\times W}([G]) \cap [V\times W]}^{=[G]\in \mathrm{H}_n(V\times W, N_G) \text{ by definition of } \mathrm{PD}_{V\times W}}\big)\\
&\underset{\substack{\text{by Lemma 83.8 and since the map}\\ \Theta\colon V\times W\to\Theta(V\times W)\subset V\times\widetilde{W} \text{ is an}\\ \textit{orientation-preserving} \text{ homeomorphism}}}{=} \mathrm{PD}_{\Theta(V\times W)}(\Theta_*([D])) \cap \Theta_*([G])\\
&\underset{\substack{\text{follows from an argument similar}\\ \text{to the one provided in Lemma 89.5}}}{=} \mathrm{PD}_{V\times\widetilde{W}}(\Theta_*([D])) \cap \Theta_*([G])\\
&\underset{\substack{\text{since } \Theta(D)=V\times\{0\} \text{ and } l(V)=G \text{ and}\\ \text{since both maps are orientation-preserving}}}{=} \mathrm{PD}_{V\times\widetilde{W}}([V\times\{0\}]) \cap \Theta_*(l_*([V]))\\
&\underset{\text{this can be deduced fairly easily from Proposition 88.2}}{=} (-1)^n \cdot q^*(\mu^*_{\widetilde{W},0}) \cap \Theta_*(l_*([V]))\\
&\underset{\text{by Lemmas 83.4 and 74.6 (3)}}{=} (-1)^n \cdot \big\langle \mu^*_{\widetilde{W},0}, q_*(\Theta_*(l_*([V]))) \big\rangle\\
&\underset{\substack{\text{since } q\circ\Theta\circ l = f-\mathrm{id}\colon V\to\widetilde{W} \text{ and since we know}\\ \text{from Lemma 45.18 that } (f-\mathrm{id})_* = (-1)^n\cdot(\mathrm{id}-f)_*}}{=} \big\langle \mu^*_{\widetilde{W},0}, (\mathrm{id}-f)_*([V]) \big\rangle.\\
&\underset{\text{this follows from the following commutative diagram}}{=} \deg\Big(\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \xleftarrow{\cong} \mathrm{H}_n(V, V\setminus\{0\}) \xrightarrow{(\mathrm{id}-f)_*} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\})\Big).
\end{aligned}$$

$$\begin{tikzcd}
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) & \mathrm{H}_n(V, V\setminus\{0\}) \arrow[l, "\cong"'] \arrow[r, "(\mathrm{id}-f)_*"] & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \\
& \mathrm{H}_n(V, \partial V) \arrow[u] \arrow[r, "(\mathrm{id}-f)_*"] & \mathrm{H}_n(\widetilde{W}, \widetilde{W}\setminus\{0\}) \arrow[u]
\end{tikzcd}$$

together with the fact that $[V]$ and $\mu_{\widetilde{W},0}$ are both mapped to the standard generator of $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\})$

Thus we have proved the desired equality. ■

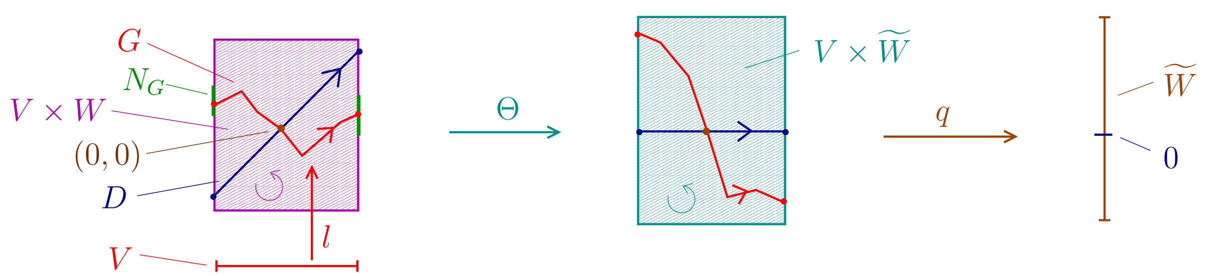


FIGURE 1359. Illustration for the proof of Lemma 96.10.

[1360]Careful reading of the equalities shows that it is in fact at times difficult to determine which relative (co-) homology groups the objects lie in. In particular on several occasions we implicitly use Lemma 83.8 and Lemma 74.6 (3) to go from one relative homology group to another. Writing down the calculation in full detail would make the proof even more unreadable that it already is.

Lemma 96.11. *Let M be a closed oriented n-dimensional topological manifold. Furthermore let $f\colon M \to M$ be a map. Finally let $z \in M$ be an isolated fixed point of f. We consider the two submanifolds $\operatorname{Gr}(f)$ and Δ_M of the $2n$-dimensional topological manifold $M \times M$. The following two statements hold:*

(1) *The point $(z,z) \in M \times M$ is a nice intersection point of the two submanifolds $\operatorname{Gr}(f)$ and Δ_M, in the sense of the definition on page 2304.*
(2) *We have the equality*

$$\operatorname{index}(f,z) \;=\; \underbrace{\operatorname{index}(\operatorname{Gr}(f),\Delta_M,(z,z)).}_{\substack{\text{index of the nice intersection point } (z,z),\\ \text{as defined on page 2305}}}$$

Remark. On page 2341 we gave an example of a map $f\colon S^2 \to S^2$ that has a unique fixed point P, which has index 2. It follows from Lemma 96.11 that (P,P) is a nice intersection point of the submanifolds $\operatorname{Gr}(f)$ and Δ_{S^2} of $S^2 \times S^2$ of index 2. This gives an affirmative answer to the question posed in Exercise 95.5.

Proof (∗).

(1) Let $z \in M$ be an isolated fixed point of $f\colon M \to M$. We need to show that the point $(z,z) \in M \times M$ is a nice intersection point of the two submanifolds $\operatorname{Gr}(f)$ and Δ_M. Thus let U be an open neighborhood of (z,z) in $M \times M$. By Lemma 96.6 there exists a neighborhood V of $z \in M$ and a neighborhood W of $z \in M$ with the following properties:
   (i) We have $V \times W \subset U$.
   (ii) z is the only fixed point of f contained in V.
   (iii) V and W are both submanifolds of M and both are homeomorphic to $\overline{B}^n$.
   (iv) We have $V \subset \mathring{W}$ and $f(V) \subset \mathring{W}$.
   By (i) it remains to show that $Z := V \times W$ is a nice neighborhood of (z,z) in the sense of the definition on page 2304. Most of the conditions are easily verified. For example by (iii) we know that $V \times W$ is homeomorphic to $\overline{B}^{2n}$. Furthermore, by (ii) we know that $\operatorname{Gr}(f) \cap \Delta_M \cap (V \times W) = \{(z,z)\}$. Finally note that it follows almost immediately from Lemma 96.10 (1) that $\operatorname{Gr}(f)$ and Δ_M are both well-behaved proper submanifolds of $V \times W$.

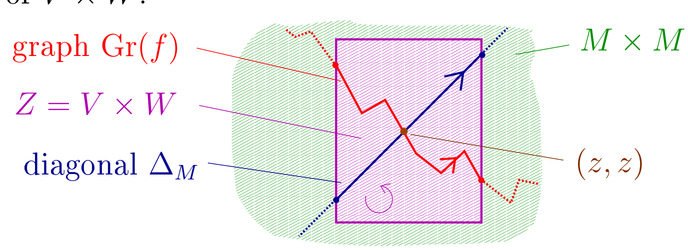


Figure 1360. Illustration for the proof of Lemma 96.11.

(2) To simplify the notation we can and will assume that $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^n$, $z = 0$ and $\Phi = \mathrm{id}$. But now in this setting the desired equality is an immediate consequence of Lemma 96.10 (2). ■

Now that we have Lemma 96.11 under our belt the proof of the Lefschetz-Hopf Theorem 96.9 is formally very similar to the proof of the Lefschetz-Hopf Theorem 96.5.

Proof of the Lefschetz-Hopf Theorem 96.9. Let M be a compact orientable topological manifold and let $f\colon M \to M$ be a map which has only finitely many fixed points $z_1, \dots, z_k$, and all of them lie in $M \setminus \partial M$. We pick an orientation for M.

First we assume that M is actually closed. Now we easily obtain the desired equality:

$$\Lambda(f) \underset{\substack{\uparrow\\ \text{by Proposition 96.1}}}{=} Q_{M\times M}([\mathrm{Gr}(f)],[\Delta_M]) \underset{\substack{\uparrow\\ \text{by Lemma 96.11 (1)}\\ \text{and Theorem 95.7}}}{=} \mathrm{Gr}(f)\cdot\Delta_M \underset{\substack{\uparrow\\ \text{by definition, see page 2307}}}{=} \sum_{i=1}^k \mathrm{index}(\mathrm{Gr}(f),\Delta_M,(z_i,z_i)) \underset{\substack{\uparrow\\ \text{by Lemma 96.11 (2)}}}{=} \sum_{i=1}^k \mathrm{index}(f,z_i).$$

Now we assume that M has non-empty boundary. Once again we consider the double $\mathrm{D}\,M := (M \times \{1\}) \sqcup_{\partial M\times\{1\}=\partial M\times\{2\}} (M \times \{2\})$. By Lemma 44.12 we know that $\mathrm{D}\,M$ is a closed orientable topological manifold. We consider the map

$$\begin{array}{rcl} \psi\colon \mathrm{D}\,M & \to & \mathrm{D}\,M \\ {[(P,\epsilon)]} & \mapsto & [(f(P),\epsilon)]. \end{array}$$

Since all of the fixed points of f lie in $M \setminus \partial M$ we see that the fixed points of ψ are $(z_1,1),\dots,(z_k,1),(z_1,2),\dots,(z_k,2)$. Now we see that

$$\begin{array}{rcl} 2\cdot\sum_{i=1}^k \mathrm{index}(f,z_i) & \underset{\substack{\uparrow\\ \text{it follows immediately from the definitions that}\\ \mathrm{index}(f,z_i)=\mathrm{index}(\psi,(z_i,1))=\mathrm{index}(\psi,(z_i,2))}}{=} & \sum_{i=1}^k \mathrm{index}(\psi,(z_i,1)) + \sum_{i=1}^k \mathrm{index}(\psi,(z_i,2)) \underset{\substack{\uparrow\\ \text{by the above, since D}\,M\text{ is a closed}\\ \text{orientable topological manifold}}}{=} \Lambda(\psi\colon \mathrm{D}\,M \to \mathrm{D}\,M) \\ & \underset{\substack{\uparrow\\ \text{follows from Exercise 65.15}}}{=} & 2\cdot\Lambda(f\colon M\to M) + \Lambda(f\colon \partial M \to \partial M) \underset{\substack{\uparrow\\ \text{since there are no fixed points on }\partial M\text{ and}\\ \text{since }\partial M\text{ is a closed orientable topological manifold}\\ \text{we know by the above that }\Lambda(f\colon\partial M\to\partial M)=0}}{=} 2\cdot\Lambda(f). \end{array}$$

Dividing both sides by 2 yields the desired result. ■

Exercises for Chapter 96.

Exercise 96.1. We consider the surface $M \subset \mathbb{R}^3$ as shown in Figure 1361. Furthermore we consider the map $\varphi\colon M \to M$ which is given by "rotating M around the skewer by an angle of π".

(a) For $i = 0,1,2$ determine $\mathrm{tr}(\varphi_*\colon \mathrm{H}_i(M;\mathbb{Z}) \to \mathrm{H}_i(M;\mathbb{Z}))$.
(b) For each fixed point x of φ determine $\mathrm{index}(\varphi,x)$.

Remark. Evidently you should use the Lefschetz-Hopf Theorem 96.9 to check your calculations in (a) and (b) for consistency.

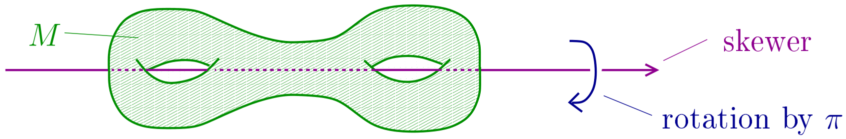


FIGURE 1361. Illustration of Exercise 96.1.

Exercise 96.2.

(a) Let $k, l \in \mathbb{Z}$. We consider the sphere $S^3 = \{(z,w) \in \mathbb{C}^2 \,|\, |z|^2 + |w^2| = 1\}$ and the self-map

$$\begin{array}{rcl} f\colon S^3 & \to & S^3 \\ (z,w) & \mapsto & (z^k, w^l). \end{array}$$

(b) We view S^2 as a subset of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. We consider the map

$$\begin{array}{rcl} g\colon S^2 & \to & S^2 \\ (z,t) & \mapsto & (z^2, t) \end{array}$$

(c) Let $n \in \mathbb{N}$. We consider the maps

$$\begin{array}{rcl} h\colon \mathbb{C} \cup \{\infty\} & \to & \mathbb{C} \cup \{\infty\} \\ z & \mapsto & \begin{cases} z^n, & \text{if } z \in \mathbb{C}, \\ \infty, & \text{if } z = \infty \end{cases} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} k\colon \mathbb{C} \cup \{\infty\} & \to & \mathbb{C} \cup \{\infty\} \\ z & \mapsto & \begin{cases} z + z^n, & \text{if } z \in \mathbb{C}, \\ \infty, & \text{if } z = \infty. \end{cases} \end{array}$$

In each case determine all fixed points, determine whether they are non-degenerate and determine the index of each fixed point.

97. The Poincaré-Hopf Theorem

Now we return to the study of vector fields on smooth manifolds.[1361] Recall that the Hairy Ball Theorem 45.14 says that there does not exist a nowhere vanishing vector field on any even-dimensional sphere. Also recall that later on we proved a significant generalization of this result. Namely, using the Lefschetz Fixed Point Theorem 65.9 we proved the following proposition.

Theorem 65.17. *Let M be a closed smooth manifold. If M admits a nowhere vanishing vector field, then $\chi(M) = 0$.*

In Chapter 95 we proved the Lefschetz-Hopf Theorem 96.9 which is a significant generalization of the Lefschetz Fixed Point Theorem 65.10 that we had just mentioned. In this chapter we will use this recently acquired theorems to prove the Poincaré-Hopf Theorem 97.5 which is a significant generalization of the above Theorem 65.17.

97.1. **The index of a zero of a vector field.** In the following discussion it will be helpful to study zeros of vector fields in more detail. As we will, not all zeros are created equal but they actually come in different flavors. More precisely, we have the following definition.

Definition. Let M be an n-dimensional smooth manifold, let v be a vector field on M and let $x \in M \setminus \partial M$ be an isolated zero of v. We pick a smooth embedding $\Psi\colon \overline{B}^n \to M \setminus \partial M$ with $\Psi(0) = x$ and such that x is the only zero of v on $\Psi(\overline{B}^n)$. We denote by $w = \Psi^*(v)$ the pullback vector field on $\overline{B}^n$ as defined on page 1632. In other words, given $y \in \overline{B}^n$ we write $w(y) := \mathrm{D}\,\Psi^{-1}_{\Psi(y)}(v(\Psi(y)))$. We define

$$\begin{array}{rclrcl} \operatorname{index}(v,x) & := & \text{degree of the map} \quad \Theta_{\Psi,v}\colon S^{n-1} & \to & S^{n-1} \\ & & y & \mapsto & \frac{w(y)}{\|w(y)\|}. \end{array}$$

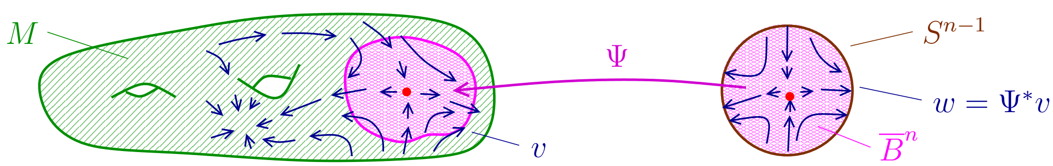


Figure 1362

Example. We consider the vector fields on $\mathbb{R}^2 = \mathbb{C}$ with isolated zero at the origin that are shown in Figure 1363. In the following we use the identity chart $\Psi = \mathrm{id}$ to determine the various indices. In fact, using Lemmas 45.10 and 45.11, we obtain the following table:

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
vector field v at $(x,y)=z$	(x,y)	$(-x,-y)$	$(-y,x)$	$(x,-y)$	$(\lvert x\rvert+\lvert y\rvert,0)$	z^2	z^{-2}
degree of $\Theta_{\Psi,v}\colon S^1 \to S^1$	$+1$	$+1$	$+1$	-1	0	$+2$	-2.

[1361] We refer to page 1631 for the definition of a vector field on a given smooth manifold M.

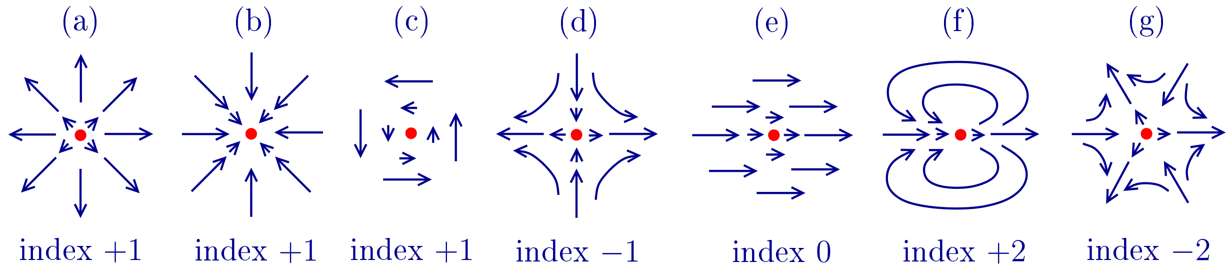


FIGURE 1363

Lemma 97.1. *Let M be an n-dimensional smooth manifold.*

(1) *Let v be a vector field on M. The definition of the index* $\operatorname{index}(v,x)$ *of an isolated zero $x \in M \setminus \partial M$ of a vector field v does not depend on the choice of the smooth embedding $\Psi\colon \overline{B}^n \to M$.*

(2) *If v and w are two vector fields on M that agree on an open subset $U \subset M$ and if z is an isolated zero of v in U, then* $\operatorname{index}(v,z) = \operatorname{index}(w,z)$.

PROOF (∗). The second statement is a straightforward consequence of the first statement. Thus it remains to prove the first statement. Let $\Psi\colon \overline{B}^n \to M$ be a smooth embedding as in the definition of $\operatorname{index}(v,x)$. We need to show that any other such smooth embedding leads to the same index. The proof is performed in two stages.

(a) Let $A \in \mathrm{O}(n)$ be an orthogonal matrix. We denote by $\rho_A\colon \mathbb{R}^n \to \mathbb{R}^n$ the map that is given by multiplication by the matrix A. Next we consider the smooth embedding $\Psi \circ \rho_A\colon \overline{B}^n \to M$. A straightforward calculation using the chain rule, see Proposition 6.15, shows that $\Theta_{\Psi\circ\rho_A,v} = \rho_A^{-1} \circ \Theta_{\Psi,v} \circ \rho_A\colon S^{n-1} \to S^{n-1}$. It follows from Lemma 45.11 that $\deg(\Theta_{\Psi\circ\rho_A,v}) = \deg(\Theta_{\Psi,v})$.

(b) Now assume we are given some other smooth embedding $\Xi\colon \overline{B}^n \to M$. We consider

$$N := \underbrace{\{y \in M \mid v(y) \neq 0\}}_{\text{open subset by Exercise 65.16}} \cup \{x\}.$$

It follows from Exercise 65.16 and the fact that x is an isolated zero that N is an open subset of M. In particular we see that N is an n-dimensional submanifold of M. Since x is the only zero of v in $\Psi(\overline{B}^n)$ and $\Xi(\overline{B}^n)$ we see that $\Psi\colon \overline{B}^n \to M$ and $\Xi\colon \overline{B}^n \to M$ are tubular maps, in the sense of the definition on page 367, for the 0-dimensional submanifold $\{x\}$ of N. By Proposition 8.26 there exists a diffeotopy $H\colon N \times [0,1] \to N$ rel $\{x\}$ and a matrix $A \in \mathrm{O}(n)$ such that $H_0 = \mathrm{id}$ and such that $H_1 \circ \Xi = \Psi \circ \rho_A\colon \overline{B}^n \to N$. Now we see that

$$\deg(\Theta_{\Xi,v}) = \deg(\Theta_{H_0\circ\Xi,v}) \underset{\substack{\uparrow \\ \text{the maps } \Theta_{H_t\circ\Xi,v} \text{ with } t\in[0,1] \text{ define a homotopy between } \Theta_{H_0\circ\Xi,v} \\ \text{and } \Theta_{H_1\circ\Xi,v}\text{, thus by Lemma 45.11 (3) the maps have the same degree}}}{=} \deg(\Theta_{H_1\circ\Xi,v}) = \deg(\Theta_{\Psi\circ\rho_A,v}) \underset{\substack{\uparrow \\ \text{by (1)}}}{=} \deg(\Theta_{\Psi,v}).$$

■

The next example shows that any integer can occur as the index of an isolated zero of a vector field.

Example. Let $n \in \mathbb{N}_{\geq 2}$ and let $d \in \mathbb{Z}$. It follows from Lemmas 45.10 and 45.12 that there exists a self-map $\varphi\colon S^{n-1} \to S^{n-1}$ of degree d. Let $\lambda \in \mathbb{R}_{>0}$. It is straightforward to verify that

$$\begin{aligned} v(\varphi)\colon \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto \begin{cases} 0, & \text{if } x = 0, \\ \lambda \cdot x \cdot \varphi(\frac{x}{\|x\|}), & \text{if } x \neq 0 \end{cases} \end{aligned}$$

is a continuous map, in particular it defines a vector field on $\mathbb{R}^n$. Evidently it has a single isolated zero at the origin and the corresponding index is precisely $d = \deg(\varphi)$.

The next lemma can be convenient for calculating the index of a zero of a vector field.

Lemma 97.2. *Let $X \subset \mathbb{R}^n$ be an open subset and let $\epsilon > 0$. Furthermore suppose we are given a vector field $w = (u, v)\colon X \times (-\epsilon, \epsilon) \to \mathbb{R}^n \times \mathbb{R}$ with an isolated zero at $(P, 0)$.*

(1) *If $v(x,t) > 0$ whenever $t > 0$ and if $v(x,t) < 0$ whenever $t < 0$, then*
$$\operatorname{index}(w,(P,0)) \;=\; \operatorname{index}(u,P).$$
(2) *If $v(x,t) < 0$ whenever $t > 0$ and if $v(x,t) > 0$ whenever $t < 0$, then*
$$\operatorname{index}(w,(P,0)) \;=\; -\operatorname{index}(u,P).$$

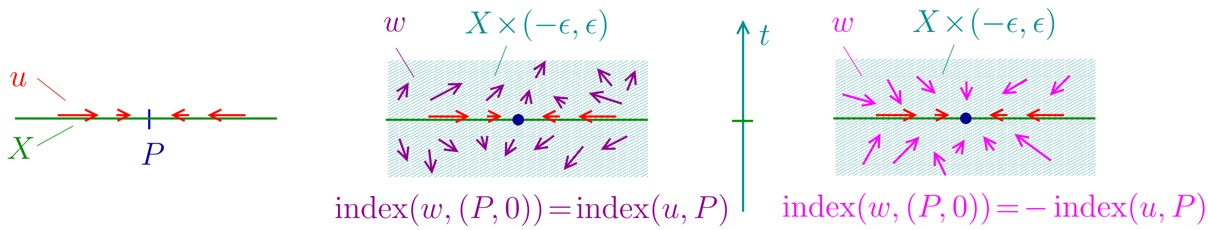


FIGURE 1364. Illustration of Lemma 97.2.

PROOF. We leave it to the reader to verify that the two statements follow easily from the definitions together with Exercise 45.13. ■

The following lemma says that isolated zeros of a vector field of index 0 can be eliminated by adjusting the vector field in the neighborhood of the zero.

Lemma 97.3. *Let M be an n-dimensional smooth manifold, let v be a vector field on M and let x be an isolated zero of v with* $\operatorname{index}(v,x) = 0$. *Let $\Psi\colon \overline{B}^n \to M \setminus \partial M$ be a smooth embedding with $\Psi(0) = x$ and such that x is the only zero of v in $\Psi(\overline{B}^n)$. There exists a vector field w on M that agrees with v on $M \setminus \Psi(B^n)$ and which has no zeros on $\Psi(\overline{B}^n)$.*

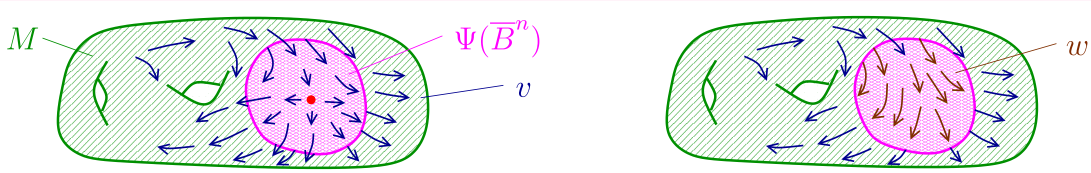


FIGURE 1365. Illustration of Lemma 97.3.

Sketch of proof. By Corollary 53.6 we know that the map $\deg\colon \pi_{n-1}(S^{n-1}) \to \mathbb{Z}$ is an isomorphism. In particular this implies, using our hypothesis that $\operatorname{index}(v,x) = 0$, that there exists a homotopy $H\colon S^{n-1} \times [0,1] \to S^{n-1}$ from the map $\Theta_{\Psi,v}\colon S^{n-1} \to S^{n-1}$ to a constant map. Using the homotopy H it is now straightforward to write down an explicit vector field w on M with the following properties:

(1) On $M \setminus \Psi(B^n)$ the vector field w equals v.
(2) On $\Psi(\overline{B}^n_{\frac{1}{2}})$ the vector field w is "constant" and non-zero.
(3) On $\Psi(\overline{B}^n \setminus B_{\frac{1}{2}})$ the vector field interpolates between (1) and (2).

We leave it to the reader to fill in the details. ■

The following proposition gives in particular a converse to Theorem 65.17.

Proposition 97.4. *Let M be a closed connected smooth manifold.*

(1) *There exists a smooth vector field u on M with finitely many zeros such that*

$$\sum_{x \text{ a zero of } u} \operatorname{index}(u,x) \;=\; \chi(M).$$

(2) *If $\chi(M) = 0$, then M admits a nowhere vanishing smooth vector field.*

Sketch of proof. Let M be a closed connected smooth manifold.

(1) First note that by Theorem 64.2 we know that M admits a smooth simplicial structure $(K = (V,S), \Theta\colon |K| \to M)$. Inductively we can define a smooth vector field u on M with the following properties:
   (a) The vector field u is defined to be the zero vector field on the vertices of the triangulation.
   (b) Suppose the vector field u is defined on the k-skeleton of M, then on every $(k+1)$-simplex s we extend the vector field u in the following way:
      (i) the vector field vanishes on the barycenter $\underline{s}$ of s,
      (ii) for all other points in the open simplex the vector field "points towards" the barycenter $\underline{s}$ of s.

      This can be done explicitly.[1362]

   We sketch the construction of the vector field u in Figure 1366. Note that the zeros of the vector field u are precisely the barycenters of the original simplices. Furthermore,

[1362]For the resilient reader we provide the gory details. Suppose we are given a vector field v on $\partial\Delta^{k+1}$. We denote by $b := (\frac{1}{k+1}, \dots, \frac{1}{k+1})$ the barycenter of Δ^{k+1}. We denote by $r\colon \Delta^{k+1} \setminus \{b\} \to \partial\Delta^{k+1}$ the map that is given by sending x to the unique intersection point of the ray $b + \mathbb{R}_{\geq 0} \cdot (x-b)$ with $\partial\Delta^{k+1}$. Next we consider the map

$$\begin{array}{rcl} \Delta^{k+1} & \to & \mathbb{R}^{k+2} \\ x & \mapsto & \begin{cases} \frac{\|x-b\|}{\|r(x)-b\|} \cdot (v(r(x)) + (b-x) \cdot \|x - r(x)\|), & \text{if } x \neq b, \\ 0, & \text{if } x = b. \end{cases} \end{array}$$

It is now fairly elementary to see that this vector field, transported back to M, has the desired properties. As always, with a little more care, using say Lemma 6.13, one can arrange that the vector field is actually smooth.

one can fairly easily verify that given a k-simplex s we have[1363] $\operatorname{index}(u, \underline{s}) = (-1)^k$. Thus we see that

$$\sum_{x \text{ a zero of } u} \operatorname{index}(u, x) \;=\; \sum_{s \text{ a } k\text{-simplex}} \operatorname{index}(u, \underline{s}) \;=\; \sum_{s \text{ a } k\text{-simplex}} (-1)^k \underset{\substack{\uparrow \\ \text{Corollary 61.25 (5)}}}{=} \chi(M).$$

This argument is also sketched in [**Stee51**, p. 202].

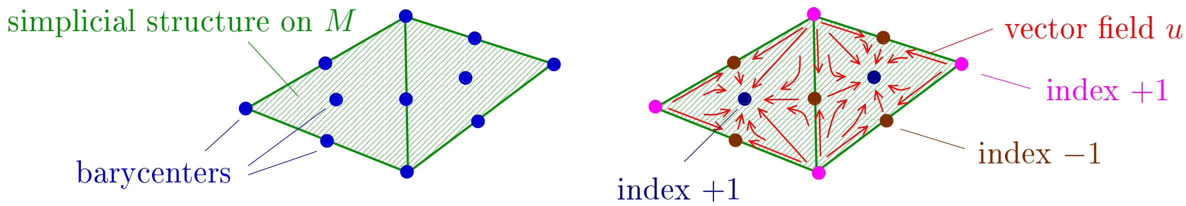


FIGURE 1366. First illustration for the proof of Proposition 97.4.

(2) The second statement is proved in [**Bre93**, Corollary VII.14.5], [**Stee51**, Corollary 39.8], [**Hirs76**, Theorem 5.2.10] and alternatively also in [**Tur90**, Lemma 6.3.4]. In the following we provide a terse sketch of the proof given in [**Tur90**, Lemma 6.3.4]. We continue with the discussion of (1) and we make the following observations and constructions.
  (a) Some thought shows that it follows from our hypothesis that M is connected and that $\chi(M) = 0$ that we can find a perfect match between adjacent barycenters. More precisely, we can find a decomposition $S = \bigsqcup_{i \in I} \{s_i, t_i\}$ of the simplex set, such that each s_i is a codimension one face of t_i.
  (b) Given each pair $\{s_i, t_i\}$ we consider a small suitable neighborhood of the 1-simplex $\{\underline{s_i}, \underline{t_i}\}$ of the barycentric subdivision. This neighborhood contains two critical points. As of right now, on this 1-simplex the vector field points from $\underline{s}_i$ to $\underline{t}_i$. We can replace the vector field u on this neighborhood by a different vector field v that agrees with u on the boundary, which on the 1-simplex now points from $\underline{t}_i$ to $\underline{s}_i$ and which has no critical points. This replacement is illustrated in Figure 1367.

Hopefully this outline is convincing. Evidently filling in all details is a nightmarish task. Thankfully the details are provided in [**Tur90**, Lemma 6.3.4]. ■

97.2. **The Poincaré-Hopf Theorem.** Now we can formulate the namesake of this chapter.

Theorem 97.5. (Poincaré-Hopf Theorem) *Let M be compact smooth manifold. Let v be a vector field*[1364] *on M with finitely many zeros. If M has boundary, then we insist that*

[1363] The fact that $\operatorname{index}(u, \underline{s}) = (-1)^k$ can be seen as follows. First note that the vector field u points in k directions towards $\underline{s}$ and in $(n-k)$ directions away from $\underline{s}$. Using a suitable embedding $\Psi \colon \overline{B}^m \to M$ with $\Psi(0) = \underline{s}$ we see that the corresponding map $\Theta_{\Psi,v} \colon S^{n-1} \to S^{n-1}$ is given by the diagonal matrix with k entries equal to -1 and $(n-k)$-entries equal to $+1$. Thus it follows from Lemma 45.11 that the degree of this map equals $(-1)^k$.

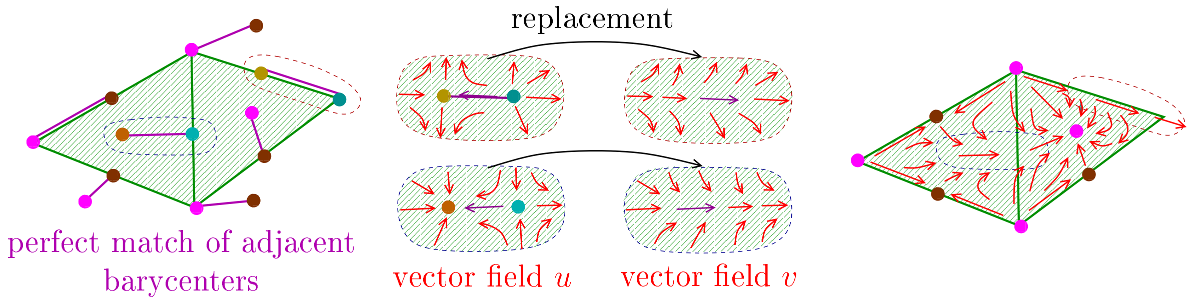


FIGURE 1367. Second illustration for the proof of Proposition 97.4.

along the boundary ∂M the vector field v is pointing outward. Then we have the formula

$$\sum_{x \text{ a zero of } v} \operatorname{index}(v,x) \;=\; \chi(M).$$

Remark.

(1) In the 2-dimensional setting the Poincaré-Hopf Theorem was first proved by Henri Poincaré [**Poi1885**] in the late 19th century. The generalization to the higher-dimensional case was first proved by Heinz Hopf [**Hopf27**] in 1927. There are many different proofs of the Poincaré-Hopf Theorem, sometimes with somewhat different hypotheses, in the literature, see e.g. [**BoT82**, Theorem 11.25], [**Vic94**, Theorem 7.27], [**Bre93**, Theorem VI.12.13], [**Miln65a**, p. 35], [**DFN85**, Theorem 15.3.2], [**BaT03**, Theorem 7.6.5] and [**MTo97**, Theorem 12.1]. See also the sketch in [**Bra07**, p. 69]. The proof we provide is partly inspired by the proofs in [**DFN85**] and [**Vic94**].

(2) The proof of the Poincaré-Hopf Theorem 97.5 relies very much on the Lefschetz-Hopf Theorem 96.9. For readers who prefer to work with the weaker Lefschetz-Hopf Theorem 96.5 we leave it as a little challenge to deduce from it the conclusion of the Poincaré-Hopf Theorem 97.5 under the stronger hypotheses that the vector field is smooth and that the indices of the zeros are ± 1.

Examples.

(1) In Figure 1000 we see two vector fields v and w on S^2. Each of these vector fields has precisely two zeros, namely the North Pole N and the South Pole S. Furthermore, considering the table on page 2347, it it pretty clear that for both vector fields both zeros have index $+1$. In particular we see that for both vector fields these two indices add up reassuringly to $\chi(S^2) = 2$.

(2) In Figure 1000 we see a vector field on the surface Σ of genus two with two zeros P and Q. The table on page 2347 strongly suggests that $\operatorname{index}(v,P) = -2$. By the discussion on page 1363 we also know that $\chi(\Sigma) = -2$. Thus it follows from the Poincaré-Hopf Theorem 97.5 that $\operatorname{index}(v,Q) = 0$. By Lemma 97.3 this means that we can modify the vector field v to obtain a vector field which has no zero at Q.

[1364] It is perhaps worth recalling that for us vector fields are assumed to be continuous. We do not make any assumptions on differentiability.

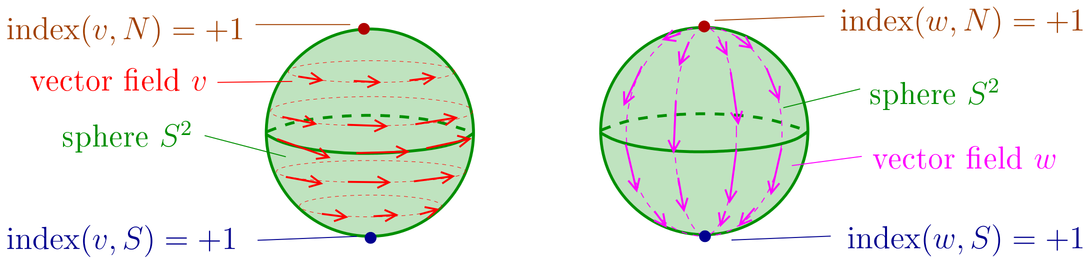


FIGURE 1368

Thus we have found a vector field on Σ with a single zero, namely at P. This gives a belated solution to Exercise 65.12.

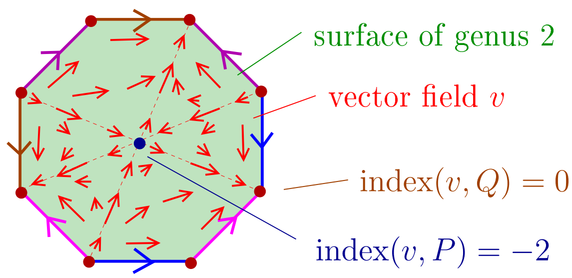


FIGURE 1369

It is perhaps worth stating the following immediate corollary to the Poincaré-Hopf Theorem 97.5. Note that this corollary generalized the statement of Theorem 65.17 to smooth manifolds with non-empty boundary.

Corollary 97.6. *Let M be a compact smooth manifold. If M admits a nowhere vanishing vector field v that either points outward along all of ∂M or that points inward along all of ∂M, then $\chi(M) = 0$.*

PROOF. First note that after possibly replacing the vector field v by $-v$ we can assume that v points outward along ∂M. If v has no zeros, then the sum over the indices of the zeros is also zero. Thus it follows from the Poincaré-Hopf Theorem 97.5 that $\chi(M) = 0$. ■

In preparation for the proof of the Poincaré-Hopf Theorem 97.5 let us first let us make a little remark and prove a useful lemma.

Remark. Let M be a topological manifold. Furthermore let $U \subset M$ be an open subset and let $f\colon U \to M$ be map. If x is an isolated fixed point of f, then we can define index$(f, x) \in \mathbb{Z}$ the same way as we did for the case $U = M$ on page 2338.

Lemma 97.7. *Let U be an open neighborhood of the origin $0 \in \mathbb{R}^n$ which contains the closed ball $\overline{B}^n$. Furthermore let $v\colon U \to \mathbb{R}^n$ be a vector field on U which has a single zero, namely at 0. We consider the map*

$$\begin{aligned} f\colon U &\to \mathbb{R}^n \\ x &\mapsto x - v(x). \end{aligned}$$

The point $x = 0$ is an isolated fixed point of f and we have the following equality:

$$\underbrace{\operatorname{index}(v,0)}_{\substack{\textit{index of a zero of a}\\ \textit{vector field, see page 2347}}} \quad = \quad \underbrace{\operatorname{index}(f,0)}_{\substack{\textit{index of a fixed}\\ \textit{point, see page 2338}}}.$$

Proof. Since v has a single zero at $x = 0$ we see that $x = 0$ is the only fixed point of f, in particular it is an isolated fixed point. We consider the embedding $\Psi = \operatorname{id}\colon \overline{B}^n \to \mathbb{R}^n$ to compute the indices $\operatorname{index}(v,0)$ and $\operatorname{index}(f,0)$. We consider the map $\Theta\colon S^{n-1} \to S^{n-1}$ that is given by $x \mapsto \frac{v(x)}{\|v(x)\|}$. Furthermore we consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_{n-1}(S^{n-1}) \arrow[rr, "\iota"] \arrow[dr, "\mathrm{id}"] \arrow[ddd, "\Theta_*"] & & \mathrm{H}_{n-1}(\mathbb{R}^n \setminus \{0\}) & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \arrow[l, "\partial_n", "\cong"'] \\
& \mathrm{H}_{n-1}(S^{n-1}) \arrow[r, "\iota"] \arrow[d, "\Theta_* = (\mathrm{id} - f)_*"] & \mathrm{H}_{n-1}(\overline{B}^n \setminus \{0\}) \arrow[u, "\iota", "\cong"'] \arrow[dd, "(\mathrm{id} - f)_* = \mathrm{id} - f_*"] & \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\}) \arrow[l, "\partial_n", "\cong"'] \arrow[u, "\iota", "\cong"'] \arrow[dd, "\mathrm{id} - f_*"] \\
& \mathrm{H}_{n-1}(\mathbb{R}^n \setminus \{0\}) \arrow[dr, "\mathrm{id}"] & & \\
\mathrm{H}_{n-1}(S^{n-1}) \arrow[rr, "\iota"] \arrow[ur, "\iota"] & & \mathrm{H}_{n-1}(\mathbb{R}^n \setminus \{0\}) & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}). \arrow[l, "\partial_n", "\cong"']
\end{tikzcd}$$

We make the following observations and clarifications:

(1) The maps ι are the various inclusion induced maps.

(2) The map

$$\begin{array}{rcl} H\colon S^{n-1} \times [0,1] & \to & \mathbb{R}^n \setminus \{0\} \\ (x,t) & \mapsto & \frac{v(x)}{(1-t)\cdot\|v(x)\|+t} \end{array}$$

is a homotopy between $H_0 = \Theta$ and $H_1 = v = \operatorname{id} - f$. It follows from the existence of such a homotopy together with Proposition 42.5 that we have the equality $\Theta_* = (\operatorname{id} - f)_*\colon \mathrm{H}_{n-1}(S^{n-1}) \to \mathrm{H}_{n-1}(\mathbb{R}^n \setminus \{0\})$.

(3) It follows from the naturality of the connecting homomorphisms, see Corollary 43.16 that the squares to the right commute. All the remaining parts of the diagram commute by the functoriality of homology groups.

It follows from the above discussion that

$$\underset{\substack{\uparrow \\ \text{by the definition on page 2347}}}{\operatorname{index}(v,0)} = \underset{\substack{\uparrow \\ \text{by the above discussion}}}{\deg\begin{pmatrix} \mathrm{H}_{n-1}(S^{n-1}) \\ \downarrow \Theta_* \\ \mathrm{H}_{n-1}(S^{n-1}) \end{pmatrix}} = \underset{\substack{\uparrow \\ \text{see definition on page 2338}}}{\deg\begin{pmatrix} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\ \iota \uparrow \cong \\ \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\}) \\ \downarrow \mathrm{id} - f_* \\ \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \end{pmatrix}} = \operatorname{index}(f,0). \qquad \blacksquare$$

Now we can provide the proof of the Poincaré-Hopf Theorem 97.5. To simplify the discussion we first deal with the special case that the smooth manifold is closed and oriented. We will deal with the general case right afterwards.

Proof of the Poincaré-Hopf Theorem for closed orientable M. Let M be a closed orientable n-dimensional smooth manifold. Let v be a vector field on M with finitely many zeros $P_1, \dots, P_k$. The idea now is similar to the idea behind the proof of Theorem 65.17. Namely we intend to use the vector field v to produce a self-map f of M. We will then roll out the Lefschetz-Hopf Theorem 96.5 to obtain the desired result.

As the reader surely remembers, in Proposition 65.18 we gave two different constructions of a self-map f from a given vector field v. The first approach used more machinery whereas the second proof was much more hands-on. It is technically slightly tricky to adapt the first approach to our current needs. Thus in the following we present a slight refinement of the second construction of the self-map f. We refer to the proof of Proposition 65.18 for details.

(1) By Exercise 9.4, which is a variation on Proposition 9.1, we can assume the following:
   (a) M is a submanifold of some $\mathbb{R}^m$.
   (b) For each zero P_i there exists an open neighborhood U_i of P_i that is "flat", in the sense that there exists a $y_i \in \mathbb{R}^{m-n}$ such that $U_i \subset \mathbb{R}^n \times \{y_i\}$.

(2) We consider the map

$$\begin{array}{rcl} F\colon M \times \mathbb{R}_{\geq 0} & \to & \mathbb{R}^m \\ (P,t) & \mapsto & P - t \cdot v(P). \end{array}$$

(3) There exists an $\epsilon > 0$ such that $Z := \{P + w \mid P \in M \text{ and } w \in (\mathrm{V}_P\, M)^\perp \text{ with } \|w\| \leq \epsilon\}$ is a neighborhood of M and such that the map

$$\begin{array}{rcll} q\colon Z & \to & M & \\ P+w & \mapsto & P & \quad \text{where } P \in M \text{ and } w \in (\mathrm{V}_P\, M)^\perp \text{ with } \|w\| \leq \epsilon \end{array}$$

is well-defined and continuous.

(4) There exists a $\mu > 0$ such that $F(M \times [0,\mu]) \subset Z$.

(5) We set $f := q \circ F_\mu\colon M \to M$.

(6) The map $q \circ F\colon M \times [0,\mu] \to M$ defines a homotopy between the identity and f.

Claim. Let $P \in M$.

(i) P is a zero of v if and only if P is a fixed point of f.

(ii) If P is a zero of v, then $\operatorname{index}(v,P) = \operatorname{index}(f,P)$.

Now we provide the proof of the two statements.

(i) Evidently if P is a zero of v, then it is a fixed point of f. Now suppose that P is not a zero of v. We can write $F_\mu(P) = P - \mu \cdot v(P) =: Q + w$ with $Q \in M$ and $w \in (\mathrm{V}_Q\, M)^\perp$ with $\|w\| \leq \epsilon$. Since $\mathrm{V}_P\, M \cap \mathrm{V}_P\, M^\perp = \{0\}$ and since $v(P)$ is a non-zero vector in $\mathrm{V}_P\, M$ we see that $P \neq Q$. By the definition in (5) we have $f(P) = Q$. Thus we have shown that $f(P) \neq P$.

(ii) It follows from (1b) and our explicit construction of f that we are essentially in the setting of Lemma 97.7, which then provides us with the desired equality. ⊞

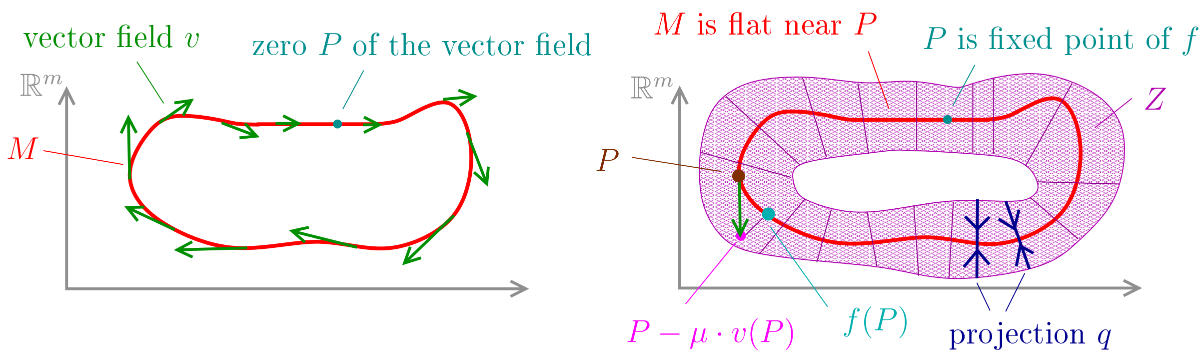


FIGURE 1370. Illustration for the proof of the Poincaré-Hopf Theorem 97.5.

Now we see that

$$\sum_{P \text{ a zero of } v} \operatorname{index}(v,P) \underset{\substack{\uparrow\\ \text{by the claim}}}{=} \sum_{P \text{ a fixed point of } f} \operatorname{index}(f,P) \underset{\substack{\uparrow\\ \text{by the Lefschetz-Hopf Theorem 96.9}\\ \text{note that here we use that } M \text{ is orientable}}}{=} \Lambda(f) \overset{\substack{\text{follows from Lemma 65.7 and the fact, mentioned in (6), that } f \text{ is homotopic to id}\\ \downarrow}}{=} \Lambda(\mathrm{id}) \underset{\substack{\uparrow\\ \text{see Lemma 65.6}}}{=} \chi(M).$$

■

Next we drop the hypothesis that M needs to be closed.

PROOF OF THE POINCARÉ-HOPF THEOREM 97.5 FOR COMPACT ORIENTABLE M. In the following let M be a compact orientable n-dimensional smooth manifold and let v be a vector field on M that points outward on ∂M and that has only finitely many zeros $P_1, \dots, P_k$.

We perform the following steps:

(1) By the Collar Neighborhood Theorem 8.12 we know that there exists a collar neighborhood $[-1,0] \times \partial M$ for $\partial M = \{0\} \times \partial M$. Possibly after rescaling the collar neighborhood we can arrange that none of the points $P_1, \dots, P_k$ lies in $[-1,0] \times \partial M$.

(2) Given $(t,x) \in [-1,0] \times \partial M$ we can use the isomorphism from Proposition 6.51 (3) to make the identification $\mathrm{T}_{(t,x)}([-1,0] \times \partial M) = (\mathrm{T}_t \mathbb{R}) \times \mathrm{T}_x \partial M = \mathbb{R} \times \mathrm{T}_x \partial M$. In particular on $[-1,0] \times \partial M$ we can write the vector field v as $v(t,x) = (v_1(t,x), v_2(t,x))$ with $v_1(t,x) \in \mathbb{R}$ and $v_2(t,x) \in \mathrm{T}_x \partial M$.

(3) It follows easily from the hypothesis that v points outward, see page 291 for the definition of an outward pointing tangent vector, and the fact that ∂M is compact that there exists an $\epsilon \in [-1,0)$ such that $v_1(t,x) > 0$ for every $x \in \partial M$ and every $t \in [\epsilon, 0]$.

(4) After further rescaling the collar neighborhood we might as well assume that $\epsilon = -1$.

(5) By Proposition 97.4 we know that there exists a vector field u on ∂M with finitely many zeros $Q_1, \dots, Q_l$ and such that the sum of the indices of the zeros of u equals precisely $\chi(\partial M)$.

(6) We consider the double $\mathrm{D}\, M := M \sqcup_{\partial M = \partial M'} M'$ where M' is a copy of M. By Lemma 44.12 we know that $\mathrm{D}\, M$ is a closed orientable n-dimensional smooth manifold.

(7) We denote by $\rho\colon \mathrm{D}\, M \to \mathrm{D}\, M$ the natural involution swapping the two copies of M.

(8) Our next goal is to construct a vector field w on M that equals v outside of the collar neighborhood, that equals u on ∂M and that interpolates between these vector fields on the collar neighborhood without creating new zeros. In fact the following vector field has precisely those properties:

$$w(x) := \begin{cases} v(x), & \text{if } x \in M \setminus ([-1,0] \times \partial M), \\ (\underbrace{|t| \cdot v_1(y,t)}_{\in \mathbb{R}}, \underbrace{|t| \cdot v_2(y,t) + (1+t) \cdot u(y)}_{\in \mathrm{T}_y \partial M}), & \text{if } x = (t,y) \text{ with } t \in [-1,0] \ \& \ y \in \partial M. \end{cases}$$

We use the natural involution $\rho\colon \mathrm{D}\,M \to \mathrm{D}\,M$ to extend this vector field to $\mathrm{D}\,M$. More precisely, given $x' \in M'$ we set $w(x') = \mathrm{D}\,\rho_{\rho(x')}(w(\rho(x))$. Note that w is indeed a vector field on $\mathrm{D}\,M$, i.e. there are no continuity issues along $\partial M = \partial M'$.

(9) The zeros of w are $P_1, \dots, P_k$, $\rho(P_1), \dots, \rho(P_k)$ and $Q_1, \dots, Q_l$. We make the following three observations:
   (a) We have $\mathrm{index}(w, P_i) = \mathrm{index}(v, P_i)$, since w equals v in a neighborhood of P_i.
   (b) We have $\mathrm{index}(w, \rho(P_i)) = \mathrm{index}(w, P_i)$, since ρ is a diffeomorphism.
   (c) It follows from our hypothesis that v points *outward* together with Lemma 97.2 that we have $\mathrm{index}(w, Q_j) = -\,\mathrm{index}(u, Q_j)$.

Now we have the following equality:

$$\sum_{i=1}^{k} \underbrace{\mathrm{index}(w, P_i)}_{\mathrm{index}(v,P_i) \text{ by (8a)}} + \sum_{i=1}^{k} \underbrace{\mathrm{index}(w, \rho(P_i))}_{\substack{= \mathrm{index}(v, P_i) \text{ by (8b)} \\ \text{together with (8a)}}} + \sum_{j=1}^{l} \underbrace{\mathrm{index}(w, Q_j)}_{= - \mathrm{index}(u, Q_j) \text{ by (8c)}} \overset{\substack{\text{since } \mathrm{D}\,M \text{ is a } \textit{closed} \text{ orientable smooth manifold we already know from the above that} \\ \text{the conclusion of the Poincaré-Hopf Theorem 97.5 holds for } \mathrm{D}\,M}}{\overset{\downarrow}{=}} \chi(\mathrm{D}\,M).$$

From this equality, together with (8a), (8b) and (8c) and Lemma 55.5 we obtain the following equality:

$$2 \cdot \sum_{i=1}^{k} \mathrm{index}(v, P_i) - \underbrace{\sum_{j=1}^{l} \mathrm{index}(u, Q_j)}_{= \chi(\partial M) \text{ by (5)}} = \underbrace{2 \cdot \chi(M) - \chi(\partial M).}_{\substack{= \chi(\mathrm{D}\,M) \text{ by Lemma 55.5} \\ \text{and Theorem 64.5}}}$$

Cleaning up the bottom equality we obtain the desired equality. ■

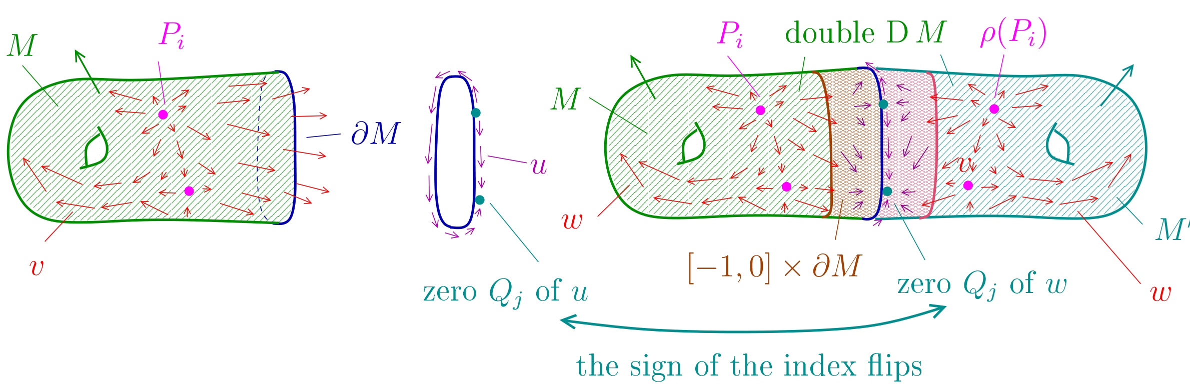


FIGURE 1371

Finally we drop the hypothesis that M needs to be orientable.

Proof of the Poincaré-Hopf Theorem. Let M be a compact n-dimensional smooth manifold and let v be a vector field on M that points outward on ∂M and that has only finitely many zeros $P_1, \ldots, P_k$. Evidently we only need to consider the case that M is connected. Furthermore, by the previous discussions it remains to consider the case that M is non-orientable.

By Proposition 17.3 there exists a 2-fold covering $p\colon \widetilde{M} \to M$ where $\widetilde{M}$ is a compact orientable smooth manifold and p is a local diffeomorphism. We equip $\widetilde{M}$ with the vector field p^*v that is the pullback of v under p, see Lemma 65.14. In other words, p^*v is the unique vector field on $\widetilde{M}$ that satisfies $\mathrm{D}\, p_x(p^*v(x)) = v(p(x))$ for every $x \in \widetilde{M}$. We see that

$$2\cdot\sum_{i=1}^{k}\operatorname{index}(v,P_i) \underset{\substack{\uparrow\\ \text{each } P_i \text{ has two preimages and the indices}\\ \text{of the preimages agree with the index of } P_i}}{=} \sum_{i=1}^{k}\sum_{Q\in p^{-1}(P_i)}\operatorname{index}(p^*v,Q) \underset{\substack{\uparrow\\ \text{by the above discussion,}\\ \text{since } \widetilde{M} \text{ is orientable}}}{=} \chi(\widetilde{M}) \underset{\substack{\uparrow\\ \text{by Proposition 37.4}\\ \text{and Theorem 64.5}}}{=} 2\cdot\chi(M).$$

■

97.3. **The Gauss map** $(*)$**.** In this section we will introduce the Gauss map and we will determine its degree. We will not make use of the results of this section. Hence on several occasions we will skip some details.

The following definition introduces the protagonist of this section.

Definition. Let $n \in \mathbb{N}_0$ and let $M \subset \mathbb{R}^{n+1}$ be a closed oriented submanifold of codimension one. We refer to

$$\begin{aligned} \mathfrak{G}\colon M &\to S^n \\ P &\mapsto \begin{array}{l} \text{the unique } \mathfrak{G}(P) \in \mathbb{R}^{n+1} \text{ that has the following properties:} \\ (1)\ \ \mathfrak{G}(P) \text{ is normal to the visual tangent space } \mathrm{V}_P\, M, \\ (2)\ \ \mathfrak{G}(P) \text{ has length one, i.e. } \mathfrak{G}(P) \in S^n, \\ (3)\ \ \mathfrak{G}(P) \text{ followed by a positive basis for } \mathrm{V}_P\, M \text{ is a positive basis for } \mathbb{R}^{n+1} \end{array} \end{aligned}$$

as the *Gauss map of* M. Using charts one can write down an explicit local description of the Gauss map which can be used to show that the Gauss map is smooth.

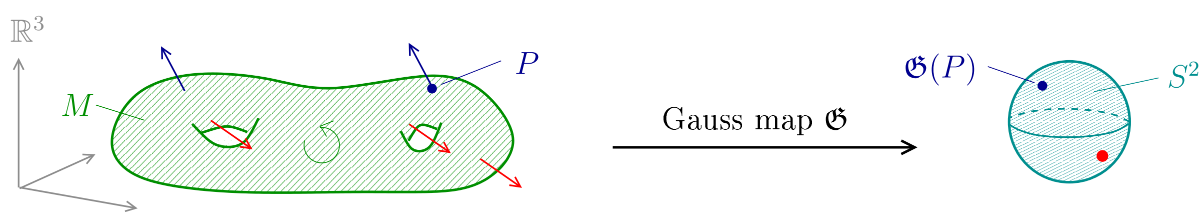


Figure 1372

Example. Let $n \in \mathbb{N}$. We consider the submanifold $S^n \subset \mathbb{R}^{n+1}$ with the usual orientation introduced on page 299. It follows almost immediately from the definitions and conventions that the corresponding Gauss map $\mathfrak{G}\colon S^n \to S^n$ is the identity.

The following theorem was first proved by Heinz Hopf [**Hopf25**] in 1925.

Theorem 97.8. *Let $M \subset \mathbb{R}^{n+1}$ be a closed oriented submanifold of codimension one. If n is even, then*

$$\deg(\textit{Gauss map } \mathfrak{G}\colon M \to S^n) \;=\; \tfrac{1}{2}\chi(M).$$

Remark.

(1) Alternative proofs of Theorem 97.8 are given in [**Bre93**, Theorem VI.12.11], [**Cib16**], [**Sakk90**], [**Tu11**, p. 174] and [**GP74**, p. 198].
(2) Note that in general the conclusion of Theorem 97.8 does not hold if n is odd. For example if we consider $M = S^1 \subset \mathbb{R}^2$, then we know by the above that the Gauss map $\mathfrak{G}$ is the identity, hence its degree equals one, whereas $\chi(S^1) = 0$.

prop:degree-by-local-info

Sketch of a proof of Theorem 97.8. Let $n \in \mathbb{N}_0$ and let $M \subset \mathbb{R}^{n+1}$ be a closed oriented submanifold of codimension one. Recall that above we observed that the Gauss map $\mathfrak{G}\colon M \to S^n$ is smooth. Thus we can appeal to Sard's Theorem 6.63 which implies in particular that there exists some $z \in S^n$ such that z and $-z$ are both regular values of the Gauss map $\mathfrak{G}$. Note that $\mathrm{O}(n+1)$ acts transitively on S^n. Thus, after a rotation, we can assume that $z = e_{n+1} = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. Next we consider the vector field

$$\begin{array}{rcl} v\colon M & \to & \mathbb{R}^{n+1} \\ P & \mapsto & \text{the orthogonal projection of } z = e_{n+1} \text{ to } \mathrm{V}_P\, M. \end{array}$$

Note that for $Q \in M$ we have $v(Q) = 0$ if and only if $\mathfrak{G}(Q) = z$ or $\mathfrak{G}(Q) = -z$.

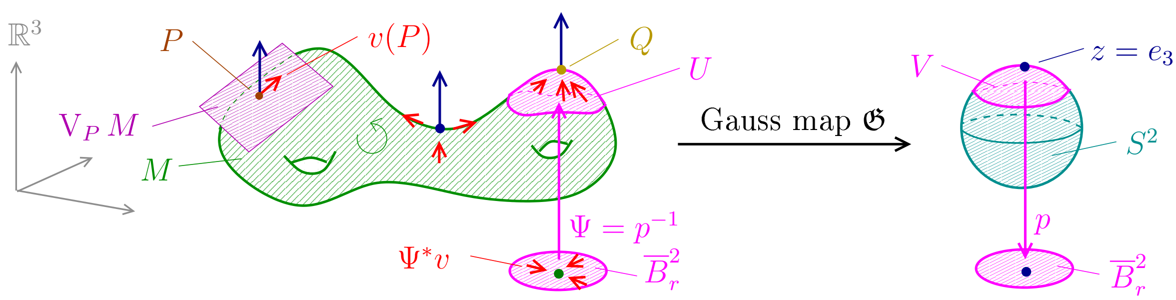


Figure 1373. First illustration for the proof of Theorem 97.8.

Let $Q \in M$ with $\mathfrak{G}(Q) = \pm z$. By design we know that Q is a regular value of $\mathfrak{G}$. By Lemma 6.54 (1) we know that $\mathfrak{G}$ is a local diffeomorphism at Q, thus it makes sense to introduce the following definition:

$$\deg(\mathfrak{G}, Q) \;=\; \begin{cases} +1, & \text{if } \mathfrak{G} \text{ is orientation-preserving at } Q \\ -1, & \text{if } \mathfrak{G} \text{ is orientation-reversing at } Q. \end{cases}$$

Claim. Let $Q \in M$.

(1) If $\mathfrak{G}(Q) = z$, then $\operatorname{index}(v, Q) = (-1)^n \cdot \deg(\mathfrak{G}, Q)$.
(2) If $\mathfrak{G}(Q) = -z$, then $\operatorname{index}(v, Q) = \deg(\mathfrak{G}, Q)$.

First we consider the case that we are given $Q \in M$ with $\mathfrak{G}(Q) = z$. We make the following observations and preparations.

(a) We denote by $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ the projection onto the first n coordinates.
(b) It follows fairly easily from Exercise 6.15 that there exists a neighborhood U of $Q \in M$, a neighborhood V of $z \in S^n$ and an $r \in \mathbb{R}_{>0}$ with the following properties:
  (i) U contains no other point x with $\mathfrak{G}(x) = z$.
  (ii) We have $\mathfrak{G}(U) \subset V$.
  (iii) The restriction of the projection $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ to U and V defines diffeomorphisms $p\colon U \to \overline{B}^n_r$ and $p\colon V \to \overline{B}^n_r$. It follows immediately from the orientation conventions and some elementary linear algebra that both maps are orientation-preserving if n is even and both maps are orientation-reversing if n is odd.
(c) We set $\Psi := p^{-1}\colon \overline{B}^n_r \to U$.
(d) Note that by (b) we know that M is locally a graph around Q. More precisely, there exists a map $f\colon \overline{B}^n_r \to \mathbb{R}$ such that $M \cap U = \{(x, f(x)) \,|\, x \in \overline{B}^n_r\}$. Using this observation one can write down an explicit expression of $\mathfrak{G}$. This explicit calculation then shows that for $x \in M$ we have

$$\mathfrak{G}(x) \;=\; \mathfrak{G}(Q) - v(x) + s(x) \qquad \text{with } s(x) = O(\underset{\substack{\uparrow \\ \text{usual euclidean norm on } \mathbb{R}^{n+1}}}{\|}x - Q\|^2).$$

Hopefully Figure 1374 makes this assertion plausible. We leave it to the reader to fill in the details.
(e) We denote by $\rho\colon \overline{B}^n_r \to \overline{B}^n_r$ the map given by $\rho(x) = -x$.
(f) It follows almost immediately from (c), the observation that $p(\mathfrak{G}(Q)) = 0$ and from Exercise 18.10 that, after possibly shrinking r, the maps $p \circ v$ and $\rho \circ p \circ \mathfrak{G}$ are homotopic as maps of pairs $(U, U \setminus \{Q\}) \to (\overline{B}^n_r, \overline{B}^n_r \setminus \{0\})$.

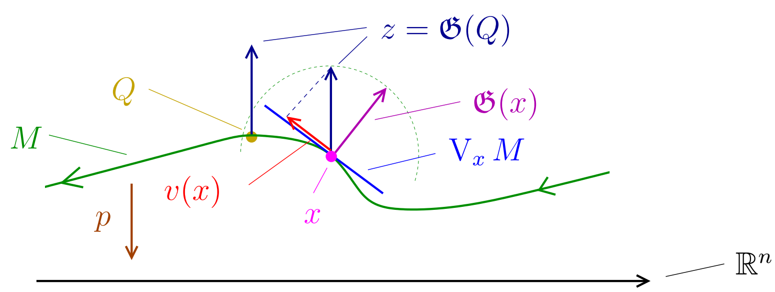


FIGURE 1374. Second illustration for the proof of Theorem 97.8.

We now see that

definition of index(v,Q), see page 2347, applied with our given Ψ — by Exercises 18.2 and 45.4 — by (f) together with Proposition 43.17 and together with Lemma 45.18

$$\begin{aligned}
\operatorname{index}(v,Q) &\overset{\downarrow}{=} \deg\left(\begin{array}{rl} S_r^{n-1} \!\!\!&\to S_r^{n-1} \\ x \!\!\!&\mapsto \frac{p(v(\Psi(x)))}{\|p(v(\Psi(x)))\|}\end{array}\right) \overset{\downarrow}{=} \deg\big((p\circ v\circ\Psi)_* \circlearrowleft \mathrm{H}_n(\overline{B}_r^n, \overline{B}_r^n\setminus\{0\})\big) \overset{\downarrow}{=} \\
&= \deg\big((\rho\circ p\circ \mathfrak{G}\circ\Psi)_* \circlearrowleft \mathrm{H}_n(\overline{B}_r^n, \overline{B}_r^n\setminus\{0\})\big) \\
\text{by Lemma 45.18} \ \rightarrow \ &= (-1)^n \cdot \deg\big((p\circ \mathfrak{G}\circ\Psi)_* \circlearrowleft \mathrm{H}_n(\overline{B}_r^n, \overline{B}_r^n\setminus\{0\})\big) \\
&\underset{\uparrow}{=} (-1)^n \cdot \deg(\mathfrak{G}, Q).
\end{aligned}$$

this follows from the definition of $\deg(\mathfrak{G},Q)$ together with Proposition 45.16 and the fact that we know that p and Ψ are either both orientation-preserving or they are both orientation-reversing

Now we consider the case that $\mathfrak{G}(Q) = -z$. The calculation is basically identical to the above. We only need to make the observation that in this setting the map p is orientation-preserving if and only if n is odd, whereas before p was orientation-preserving if and only if n is even. Thus we obtain the same result as above, except that by Lemma 45.18 we pick up a sign $(-1)^n$. ⊞

Now finally we assume that n is even. We perform the following calculation:

since z and $-z$ are regular values we can apply Proposition 69.7 to z and to $-z$

$$\begin{aligned}
2\cdot\deg(\text{Gauss map } \mathfrak{G}\colon M\to S^n) &\overset{\downarrow}{=} \sum_{\substack{Q\in M \text{ with} \\ \mathfrak{G}(Q)=z}} \operatorname{index}(\mathfrak{G},Q) + \sum_{\substack{Q\in M \text{ with} \\ \mathfrak{G}(Q)=-z}} \operatorname{index}(\mathfrak{G},Q) \\
&\underset{\uparrow}{=} \sum_{\substack{Q\in M \text{ with} \\ v(Q)=0}} \operatorname{index}(\mathfrak{G},Q) \underset{\uparrow}{=} \sum_{\substack{Q\in M \text{ with} \\ v(Q)=0}} \operatorname{index}(v,Q) \underset{\uparrow}{=} \chi(M).
\end{aligned}$$

since $v(Q)=0$ if and only if $\mathfrak{G}(Q)=z$ or $\mathfrak{G}(Q)=z$ — by the claim and since n is even — by the Poincaré-Hopf Theorem 97.5 ■

Exercises for Chapter 97.

Exercise 97.1. We consider the two vector fields v and w that are shown in Figure 1375 with a single zero at P. On page 2347 we already saw that $\operatorname{index}(v,P) = +1$. What is $\operatorname{index}(w,P)$?

FIGURE 1375. Illustration of Exercise 97.1.

Exercise 97.2.

(a) We consider the vector field v on $\mathbb{R}^2$ shown in Figure 1376. What is the index of v at the unique zero?

(b) Sketch a vector field v on $\mathbb{R}^2$ with a zero at some point P, such the corresponding index equals $+3$.

(c) Let $n \geq 2$. Show that for every $k \in \mathbb{Z}$ there exists a vector field v on $\mathbb{R}^n$ with a single zero at the origin 0 such that $\operatorname{index}(v, 0) = k$.

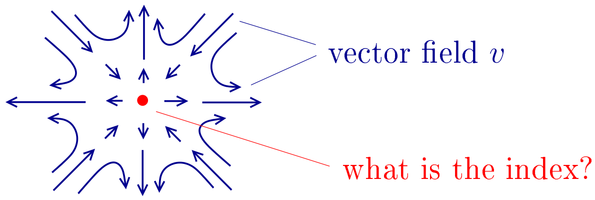


FIGURE 1376. Illustration of Exercise 97.2 (a).

Exercise 97.3. Let M be a closed oriented n-dimensional smooth manifold. We consider the tangent bundle

$$\mathrm{T}\,M \;:=\; \bigsqcup_{x\in M} (\mathrm{T}_x\,M \times \{x\}).$$

(a) Show that $\mathrm{T}\,M$ is naturally an oriented $2n$-dimensional smooth manifold such that the map $\varphi\colon M \to \mathrm{T}\,M$ given by $x \mapsto (0, x) \in \mathrm{T}_x\,M \times \{x\}$ is a smooth embedding.

(b) Show that there exists a smooth embedding $\psi\colon M \to \mathrm{T}\,M$, that is homotopic to φ, such that the submanifolds $\varphi(M)$ and $\psi(M)$ of $\mathrm{T}\,M$ intersect transversally and such that $\varphi(M) \cdot \psi(M) = \chi(M)$. Here the left-hand side is the algebraic intersection number as defined on page 2271.
Hint. Use Proposition 97.4.
Remark. In the literature the statement we just proved sometimes gets shortened to "$M \cdot M = \chi(M)$".

Exercise 97.4. We say that two vector fields v and w on a smooth manifold M are equivalent if there exists a diffeotopy $H\colon M \times [0, 1] \to M$ with $H_0 = \mathrm{id}$ and such that $(H_1)_*(v) = w$, i.e. such that for every $x \in M$ we have $(\mathrm{D}\,H_1)_x(v(x)) = w_{H_1(x)}$. Are all nowhere vanishing smooth vector fields on the torus $S^1 \times S^1$ equivalent?

98. Homology, cohomology and submanifolds

In this chapter, given an oriented, compact smooth manifold M we will relate the following objects:

(1) Codimension one submanifolds of M,
(2) cohomology classes in $\mathrm{H}^1(M;\mathbb{Z})$ of dimension one,
(3) homology classes in $\mathrm{H}_{n-1}(M,\partial M;\mathbb{Z})$ of codimension one,
(4) maps $M \to S^1$.

98.1. **Codimension one submanifolds.**

Let M be a compact oriented connected n-dimensional topological manifold. Every compact oriented proper submanifold $Y \subset M$ of *codimension* k gives rise to a homomorphism

$$\begin{array}{rcl} \mathrm{H}_k(M;\mathbb{Z}) & \to & \mathbb{Z} \\ \sigma & \mapsto & \langle \underbrace{\mathrm{PD}_M([Y])}_{\in \mathrm{H}^k(M;\mathbb{Z})}, \sigma\rangle. \end{array}$$

In many circumstances the next proposition gives an explicit description of this homomorphism $\mathrm{H}_k(M;\mathbb{Z}) \to \mathbb{Z}$:

Proposition 98.1. *Let M be a compact oriented connected n-dimensional topological manifold and let $Y \subset M$ be a compact oriented proper submanifold of* codimension k. *Let $f\colon X \to M$ be a map from a closed k-dimensional topological manifold X to M. If f is transverse to Y, then*

$$\langle \underbrace{\mathrm{PD}_M([Y])}_{\in \mathrm{H}^k(M;\mathbb{Z})}, \underbrace{f_*([X])}_{\in \mathrm{H}_k(M;\mathbb{Z})} \rangle \;=\; f \cdot Y.$$

Remark. Let M be a compact oriented connected n-dimensional topological manifold and let $Y \subset M$ be a compact oriented proper submanifold of codimension one. In Proposition 10.9 we saw that Y admits a tubular neighborhood of the form $[-1,1] \times Y$ such that the product orientation of $[-1,1] \times Y$ matches the orientation of M. If $f\colon S^1 \to M$ is a map, then loosely speaking, the algebraic intersection number $f \cdot Y$ counts (with signs) how often f goes from from $Y \times \{\mp 1\}$ to $Y \times \{\pm 1\}$.

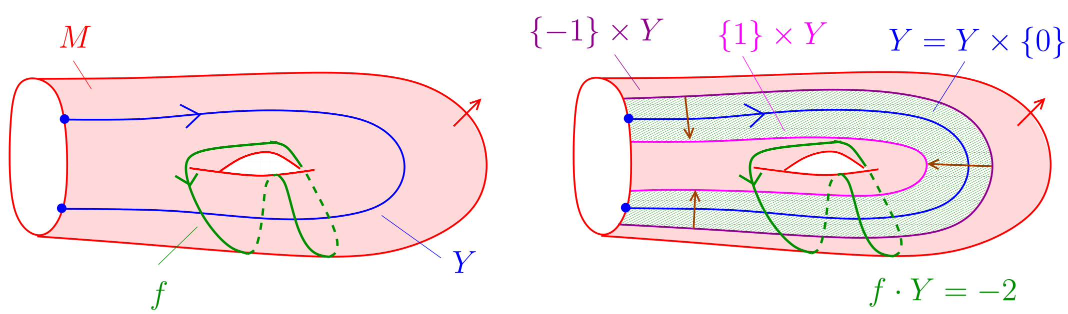


Figure 1377

Proof. We have

$$\begin{aligned} f\cdot Y &\underset{\substack{\uparrow\\ \text{by Theorem 95.7 since } f \text{ and } Y \text{ are transverse}}}{=} \epsilon_M\big(\big(\mathrm{PD}_M(f_*([X]))\cup\mathrm{PD}_M([Y])\big)\cap[M]\big) \underset{\substack{\uparrow\\ \text{by Lemma 83.7 and the discussion on page 2026}}}{=} \epsilon_M\big(\mathrm{PD}_M([Y])\cap\big(\mathrm{PD}_M(f_*([X])\cap[M])\big)\big)\\ &\underset{\substack{\uparrow\\ \text{definition of } \mathrm{PD}_M(f_*([X]))}}{=} \epsilon_M(\mathrm{PD}_M([Y])\cap f_*([X])) \underset{\substack{\uparrow\\ \text{Lemma 83.4}}}{=} \langle \mathrm{PD}_M([Y]), f_*([X])\rangle. \end{aligned}$$

■

Now we can prove the following proposition which gives an easy criterion for showing that a codimension-one submanifold represents a non-trivial class in homology.

Proposition 98.2. *Let M be a compact oriented n-dimensional smooth manifold and let N be a compact orientable non-empty proper submanifold of codimension one. If the complement $M\setminus N$ is connected, then $[N]\in \mathrm{H}_{n-1}(M,\partial M;\mathbb{Z})$ is a primitive*[1365] *element of infinite order.*

Example. Proposition 98.2 can in particular be seen as a generalization of the argument given on page 2277.

Remark. In any statement of a lemma, proposition or theorem it is always a good exercise to see whether the hypothesis stated are actually necessary. In the following we will discuss whether the conditions in Proposition 98.2 that M needs to be compact and oriented are necessary.

(1) We consider the real projective plane $\mathbb{RP}^2$ and the 1-dimensional submanifold N that is illustrated in Figure 1378 to the left. We see that $\mathbb{RP}^2\setminus N$ is connected, but since $\mathrm{H}_1(\mathbb{RP}^2;\mathbb{Z})\cong\mathbb{F}_2$ we see that N cannot possibly represent an element of infinite order. On the other hand, an "$\mathbb{F}_2$-version" of Proposition 98.2, which by Proposition 86.13 also applies to non-orientable smooth manifolds, shows that N represents a non-trivial element in $\mathrm{H}_1(\mathbb{RP}^2;\mathbb{F}_2)$.
(2) Let M be a non-compact oriented n-dimensional smooth manifold and let N be a compact orientable non-empty proper submanifold of codimension one. If ∂M is compact, then we will see in Exercise 98.1 using Proposition 11.4 and 98.2, that the homology class $[N]\in\mathrm{H}_{n-1}(M;\mathbb{Z})$ is still an element of infinite order.

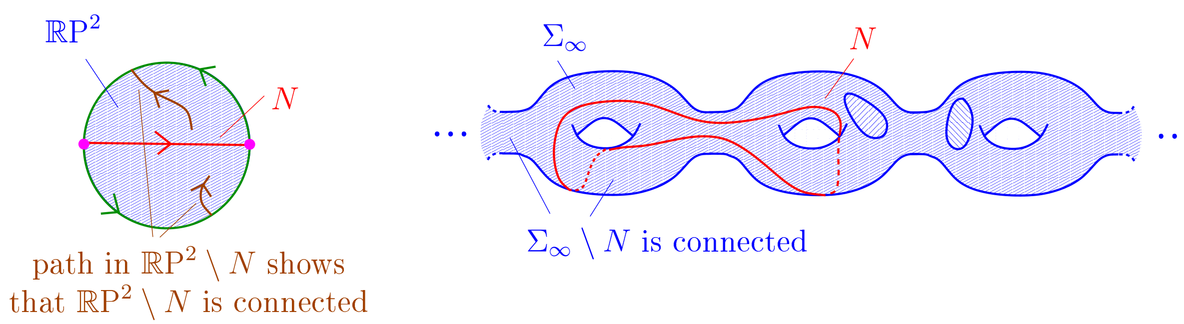


Figure 1378

[1365] Recall that an element g of an additive abelian group G is called *primitive* if there does not exist an $h\in G$ and an $n\geq 2$ with $n\cdot h=g$.

PROOF. Let M be a compact oriented n-dimensional smooth manifold and let N be a compact orientable proper submanifold of codimension one such that $M \setminus N$ is connected. By Proposition 10.9 we can find a trivial tubular neighborhood $[-1,1] \times N$ in M. By hypothesis $M \setminus N$ is connected. It follows from Proposition 10.11, Lemma 10.1 and Lemma 6.9 (3) that $M \setminus (-1,1) \times N$ is path-connected.

It follows from our hypothesis that N is non-empty, together with Proposition 44.2 (5), that we can pick a point $P \in N \setminus \partial N$. We write $P_\pm = P \times \{\pm 1\} \in [-1,1] \times N$. We denote by $\alpha\colon [0, \frac{1}{2}] \to [-1,1] \times N$ the "horizontal path" in $[-1,1] \times N$ from P_- to P_+. Since $M \setminus (-1,1) \times N$ is path-connected we can connect the two points P_+ to P_- via a path $\beta\colon [\frac{1}{2}, 1] \to M \setminus (-1,1) \times N$. We refer to Figure 1379 for an illustration.

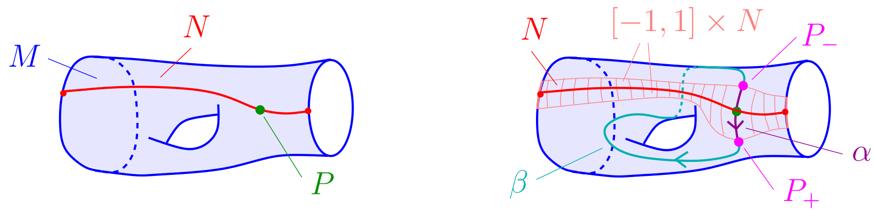


FIGURE 1379. Illustration for the proof of Proposition 98.2.

We consider the map $\gamma\colon S^1 = [0,1] \to M$ that is defined by α on $[0, \frac{1}{2}]$ and that is given by β on $[\frac{1}{2}, 1]$. We consider the following homomorphism

$$\begin{array}{rcl} \Psi\colon \mathrm{H}_{n-1}(M, \partial M; \mathbb{Z}) & \to & \mathbb{Z} \\ \sigma & \mapsto & \langle \underbrace{\mathrm{PD}_M(\sigma)}_{\in \mathrm{H}^1(M;\mathbb{Z})}, \underbrace{\gamma_*([S^1])}_{\in \mathrm{H}_1(M;\mathbb{Z})} \rangle. \end{array}$$

We want to show that $[N] \in \mathrm{H}_{n-1}(M, \partial M; \mathbb{Z})$ is a primitive element of infinite order. It suffices to show that $\Psi([N]) = \pm 1 \in \mathbb{Z}$. In fact we have

$$\Psi([N]) \underset{\substack{\uparrow \\ \text{by definition}}}{=} \langle \mathrm{PD}_M([N]), \gamma_*([S^1]) \rangle \underset{\substack{\uparrow \\ \text{by Proposition 98.1 since} \\ \gamma \text{ and } N \text{ are clearly transverse}}}{=} \gamma \cdot N \underset{\substack{\uparrow \\ \text{by construction of } \gamma}}{=} \pm 1,$$

as desired. ■

We conclude this section with the following lemma.

Lemma 98.3. (∗) *Let M be a compact orientable n-dimensional smooth manifold with at most one boundary component and which furthermore satisfies $\mathrm{H}_1(M; \mathbb{Z}) = 0$. (For example we could take $M = \overline{B}^n$ or $M = S^n$). Then for every connected submanifold N of codimension one with $N \cap \partial M = \emptyset$ there exists a compact n-dimensional submanifold W of M with $\partial W = N$.*

PROOF (∗). By Proposition 10.9 we can find a trivial tubular neighborhood $[-1,1] \times N$ in M. It follows from the argument of the proof of Proposition 98.2 and our assumption that M is orientable and the fact that M is compact that $X := M \setminus (-1,1) \times N$ is disconnected. Furthermore it follows from Proposition 6.30 that X is a submanifold of M with $\partial X = \partial M \cup (\{-1\} \times N) \cup (\{1\} \times N)$. Since ∂M consists of at most one component, since M is connected and since X is disconnected we see that there exists a component Y of X with $\partial X = \{\epsilon\} \times N$ for some $\epsilon \in \{-1, 1\}$. By adding "half" of the tubular neighborhood to Y we obtain a submanifold with $\partial Y = M$. ■

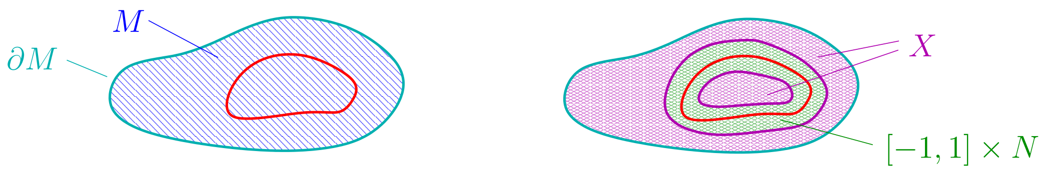


FIGURE 1380. Illustration for the proof of Lemma 98.3.

98.2. **Representing homology classes by submanifolds II.** First we recall the following definition from page 1760.

Definition. Let M be a smooth manifold and let A be a union of components of ∂M.

(1) If N is a compact oriented k-dimensional submanifold of M with $\partial N \subset A$, then we denote by $[N]$ the image of the fundamental class $[N]$ in $\mathrm{H}_k(M, A; \mathbb{Z})$ under the inclusion induced map $\mathrm{H}_k(N, \partial N; \mathbb{Z}) \to \mathrm{H}_k(M, A; \mathbb{Z})$.
(2) We say that a homology class $z \in \mathrm{H}_k(M, A; \mathbb{Z})$ *is represented by a submanifold* if there exists a compact oriented proper k-dimensional submanifold N with $\partial N \subset A$ such that $[N] = z$.

Let M be an orientable, compact n-dimensional smooth manifold. We saw in Theorem 94.5 that we can calculate the cup product of $\alpha \in \mathrm{H}^k(M, \partial M; \mathbb{Z})$ and $\beta \in \mathrm{H}^{n-k}(M; \mathbb{Z})$ if the Poincaré duals $\mathrm{PD}_M(\alpha) \in \mathrm{H}_{n-k}(M; \mathbb{Z})$ and $\mathrm{PD}_M(\beta) \in \mathrm{H}_k(M, \partial M; \mathbb{Z})$ are represented by transverse submanifolds. This rekindles our interest in the following question.

Question 70.11. *Let M be an n-dimensional smooth manifold and let A be a union of components of ∂M. For which $k \in \mathbb{N}_0$ can homology classes in $\mathrm{H}_k(M, A; \mathbb{Z})$ be represented by submanifolds?*

We already studied Question 70.11 in great detail in Section 70.4. For example in Corollary 70.14 and Corollary 70.17 we proved the following:

(1) If $n \geq 2$, then every class in $\mathrm{H}_1(M, A; \mathbb{Z})$ can be represented by a submanifold.
(2) If $n \geq 4$, then every class in $\mathrm{H}_2(M, A; \mathbb{Z})$ can be represented by a connected submanifold.

Instead of moving on to the next highest dimension, we now actually consider homology classes of *codimension one*. More precisely, we want to prove the following proposition.

Proposition 98.4. *Let M be a compact orientable n-dimensional smooth manifold and let $A \subset \partial M$ be a union of boundary components. Every class in $\mathrm{H}_{n-1}(M, A; \mathbb{Z})$ can be represented by a submanifold.*

Remark.

(1) In Proposition 98.6 we will prove a refinement of Proposition 98.4.
(2) In Proposition **??** we will prove the codimension-two analogue of Proposition 98.4. Furthermore in Proposition **??** we will prove an "$\mathbb{F}_2$-coefficient"-analogue of Proposition 98.4.

Proof. Let M be a compact orientable n-dimensional smooth manifold. In the following we deal with the case that $A = \partial M$. The general case is proved in a similar fashion, but now one needs to use Exercise 79.5 (4) instead of Proposition 79.10. We leave it to the reader to work out the details.

Now let $\sigma \in \mathrm{H}_{n-1}(M, \partial M; \mathbb{Z})$. We write $\phi = \mathrm{PD}_M(\sigma) \in \mathrm{H}^1(M;\mathbb{Z})$. By Proposition 79.10 or alternatively by Proposition 120.20 there exists a smooth map $f\colon M \to S^1$ such that $\phi = f^*([S^1]^*)$ where $[S^1]^* \in \mathrm{H}^1(S^1;\mathbb{Z})$ is the dual fundamental class.

By Sard's Theorem 6.63 and Proposition 6.62 (2) there exists a regular value $z \in S^1$. To simplify the notation we assume that in fact $z = 1$ is a regular value. We write $N := f^{-1}(\{1\})$. By the Regular Value Theorem 11.3 (1), (2), (3) we know that N is a compact orientable $(n-1)$-dimensional smooth manifold. Furthermore we know from Theorem 11.3 (4) that there exists an $\epsilon \in (0, \pi)$ and a tubular neighborhood $[-\epsilon, \epsilon] \times N$ of N such that the following diagram commutes

$$\begin{tikzcd}
{[-\epsilon,\epsilon] \times N} \arrow[r, hook] \arrow[d, "(s,x)\mapsto s"] & M \arrow[d, "f"] \\
{[-\epsilon,\epsilon]} \arrow[r, "s\mapsto e^{2\pi \mathrm{i} s}"] & S^1.
\end{tikzcd}$$

We pick the orientation for N such that the inclusion map $[-\epsilon, \epsilon] \times N \to M$ is orientation-preserving. It remains to prove that $[N] = \sigma \in \mathrm{H}_{n-1}(M, \partial M; \mathbb{Z})$. Recall that $\phi = \mathrm{PD}_M(\sigma)$, i.e. $\sigma = \phi \cap [M]$. Therefore it suffices to show that $[N] = \phi \cap [M] \in \mathrm{H}_{n-1}(M, \partial M; \mathbb{Z})$.

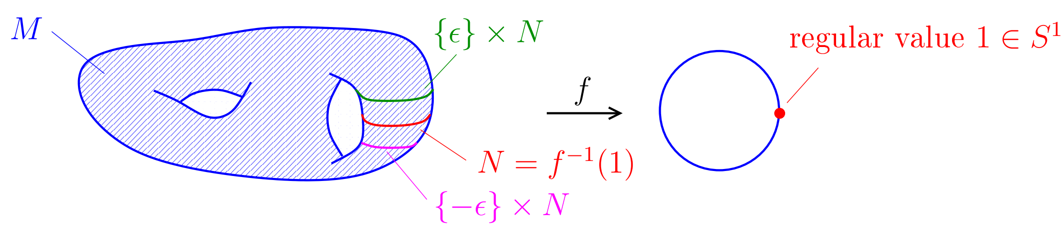


Figure 1381. Illustration for the proof of Proposition 98.4.

We introduce the following notation:

(1) We write $I = [-\epsilon, \epsilon]$ and $\mathring{I} = (-\epsilon, \epsilon)$,
(2) we denote by $i\colon I \times N \to M$ the inclusion map and we denote by i also all maps of pairs where the first map is given by the inclusion map,
(3) we denote by q all maps of pairs of topological spaces where the first map is just the identity on M,
(4) finally we denote by $p\colon I \times N \to I$ the projection map.

We consider the following diagram[1366]

$$\begin{array}{ccccc}
\mathrm{H}^1(M) & \times & \mathrm{H}_n(M,\partial M) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-1}(M,\partial M) \\
{\scriptstyle q^*}\uparrow & & \downarrow{\scriptstyle q_*} & & {\scriptstyle q_*}\downarrow{\scriptstyle =} \\
\mathrm{H}^1(M,M\setminus(\mathring{I}\times N)) & \times & \mathrm{H}_n(M,M\setminus(\mathring{I}\times N)\cup\partial M) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-1}(M,\partial M) \\
{\scriptstyle i^*}\downarrow{\scriptstyle\cong} & & \uparrow{\scriptstyle i_*} & & {\scriptstyle i_*}\uparrow \\
\mathrm{H}^1(I\times N,\partial I\times N) & \times & \mathrm{H}_n(I\times N,\partial(I\times N)) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{n-1}(I\times N,I\times\partial N) \\
{\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle -\times[N]} & & {\scriptstyle -\times[N]}\uparrow \\
\mathrm{H}^1(I,\partial I) & \times & \mathrm{H}_1(I,\partial I) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_0(I).
\end{array}$$

The proposition follows from the following claim.

Claim.

(1) The diagram commutes.[1367]
(2) The map $i^*\colon\ \mathrm{H}^1(M,M\setminus(\mathring{I}\times N))\to\mathrm{H}^1(I\times N,\partial I\times N)$ is an isomorphism.
(3) We have $(q^*\circ i_*^{-1}\circ p^*)([I]^*)=\phi$.
(4) We have $i_*([I]\times[N])=q_*([M])$.

We turn to the proof of these four statements:

(1) The top two squares commute by Lemma 83.8 applied to the two maps

$$\begin{array}{rcl} q\colon (M,\varnothing,\partial M) & \to & (M,M\setminus(\mathring{I}\times N),\partial M) \end{array}$$

and

$$\begin{array}{rcl} i\colon (I\times N,\partial I\times N,I\times\partial N) & \to & (M,M\setminus(\mathring{I}\times N),\partial M). \end{array}$$

Finally regarding the bottom square, note that

$$p^*([I]^*)\cap([I]\times[N]) \underset{\substack{\uparrow\\ \text{Proposition 80.10}}}{=} p^*([I]^*)\cap([I\times N]) \underset{\substack{\uparrow\\ \text{Proposition 84.2 (2)}}}{=} [\{0\}\times N] = (\overbrace{[I]^*\cap[I]}^{=[\{0\}]})\times[N].$$

Since $[I]^*\in\mathrm{H}^1(I,\partial I)$ and $[I]\in\mathrm{H}_1(I,\partial I)$ are generators it follows from this calculation that the bottom square commutes.

(2) It follows from excision for cohomology groups, see page 1840, that i^* is indeed an isomorphism. We leave it to the reader to verify the details.

(3) We identify the interval I with its image in S^1 under the orientation-preserving map defined by $s\mapsto e^{2\pi i s}$. We denote by $j\colon I\to S^1$ and $j\colon(I,\partial I)\to(S^1,S^1\setminus\mathring{I})$ the inclusion maps. Furthermore we denote by $r\colon(S^1,\varnothing)\to(S^1,S^1\setminus\mathring{I})$ the obvious map

[1366]For completeness' sake we recall that the relative cap product is only defined for excisive triads, see page 2005. So one should think for a second about why the second cap product is in fact defined.

[1367]Note that here for the first three rows we use the convention of page 2007 for what "commutativity" actually means.

of pairs of topological spaces. We consider the following diagram

$$\begin{tikzcd}
\mathrm{H}^1(S^1) \arrow[r, "f^*"] & \mathrm{H}^1(M) \\
\mathrm{H}^1(S^1, S^1 \setminus \mathring{I}) \arrow[u, "r^*"] \arrow[r, "f^*"] \arrow[d, "\cong"', "j^*"] & \mathrm{H}^1(M, M \setminus (\mathring{I} \times N)) \arrow[u, "q^*"'] \arrow[d, "i^*"', "\cong"] \\
\mathrm{H}^1(I, \partial I) \arrow[r, "f^*=p^*"] & \mathrm{H}^1(I \times N, \partial I \times N).
\end{tikzcd}$$

Here the map j^* is an isomorphism by excision for cohomology groups. Also note that at the bottom horizontal map we have $f^* = p^*$ by the discussion on page 2367. Furthermore we note that by the functoriality of cohomology groups both squares commute. Finally we point out that it follows immediately from Lemma 87.24 that $r^*((j^*)^{-1}([I]^*)) = [S^1]^*$. The desired statement is a consequence of the observation that $\phi = f^*([S^1]^*)$.

(4) This statement follows immediately from Lemma 87.24. ■

98.3. **The basic Thom-Pontryagin Theory.** In the proof of Proposition 98.4 we saw that a map from a smooth manifold N to S^1 gives rise to a codimension-one submanifold. In this section we will go in the opposite direction.

Definition. Let M be a compact oriented n-dimensional smooth manifold and let N be a proper oriented $(n-1)$-dimensional submanifold. By Proposition 10.9 there exists a trivial tubular neighborhood of the form $[-1,1]\times N$ such that the product orientation of $[-1,1]\times N$ matches the orientation of $[-1,1]\times N$, viewed as a codimension-zero submanifold of M. We refer to the map[1368]

$$\begin{array}{rcl} \rho_N\colon M & \to & S^1 \\ P & \mapsto & \begin{cases} e^{\pi i t}, & \text{if } P=(t,Q)\in[-1,1]\times N, \\ -1, & \text{otherwise} \end{cases} \end{array}$$

as the *Thom-Pontryagin map.*[1369] The definition of ρ_N is illustrated in Figure 1382.

Remark. The Thom-Pontryagin map ρ_N that we had just constructed depends on the choice of a tubular neighborhood. But it follows from the uniqueness of tubular neighborhoods, see Proposition 10.7, together with Lemma 10.4 (2) that any other choice of a tubular neighborhood $[-1,1]\times N$ leads to a map $M\to S^1$ that is homotopic to the original map.

[1368]It follows easily from Proposition 8.2 and Lemma 3.10 that this map is indeed continuous.

[1369]Lev Pontryagin (1908-1988) was a Soviet topologist, he made many major contributions to topology, despite having lost his eyesight in an accident when he was 14.

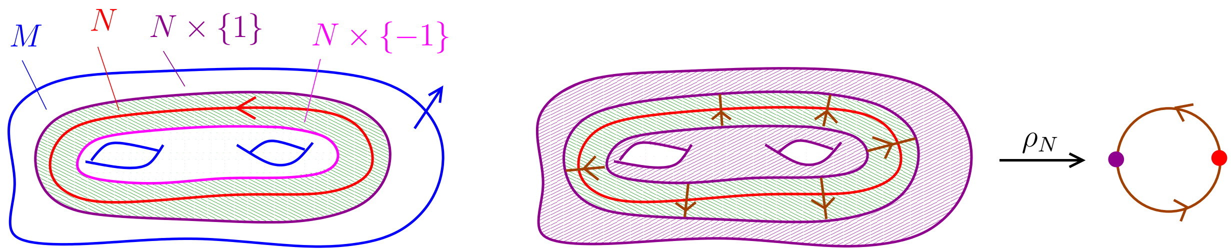


FIGURE 1382

Lemma 98.5. *Let M be a compact oriented n-dimensional smooth manifold and let N be a proper oriented proper $(n-1)$-dimensional submanifold. We have*

$$\rho_{N*} \;=\; \mathrm{PD}_M([N]) \;\in\; \mathrm{Hom}(\mathrm{H}_1(N;\mathbb{Z}),\mathbb{Z}) \underset{\substack{\uparrow \\ \text{identification from Proposition 75.18}}}{=} \mathrm{H}^1(N;\mathbb{Z}).$$

PROOF. We had implicitly verified this lemma in the above proof of Proposition 98.4. ∎

Example. We return to the example that we had discussed on page 1945. More precisely, let Σ be the surface of genus 2. The closed curves x_1, y_1, x_2, y_2 shown in Figure 1383 represent a basis for $\mathrm{H}_1(\Sigma;\mathbb{Z})$. We consider the epimorphism $\phi\colon \mathrm{H}_1(\Sigma) \to \mathbb{Z}$ that is given by $\phi(x_1) = 0$, $\phi(y_1) = 1$, $\phi(x_2) = -2$ and $\phi(y_2) = 1$. By Proposition 79.10 we know that ϕ can be realized by some map $g\colon \Sigma \to S^1$. Now we will construct such a map explicitly.

We consider the closed curve C that is illustrated in Figure 1383. We claim that the corresponding map $\rho_C\colon \Sigma \to S^1$ has the desired property. Put differently, we claim that $\rho_{C*} = \phi \in \mathrm{Hom}(\mathrm{H}_1(N;\mathbb{Z}),\mathbb{Z})$. It suffices to show that the two homomorphisms agree on the basis $[x_1], [y_1], [x_2], [y_2]$ of $\mathrm{H}_1(\Sigma;\mathbb{Z})$. For example we have

$$\rho_{C*}([x_2]) \underset{\substack{\uparrow \\ \text{Lemma 98.5}}}{=} \langle \mathrm{PD}_M([C]), [x_2]\rangle \underset{\substack{\uparrow \\ \text{Proposition 98.1}}}{=} x_2 \cdot C \underset{\substack{\uparrow \\ \text{see Figure 1383}}}{=} -2 \;=\; \phi(x_2).$$

It is straightforward to verify that the same equalities hold for all the other basis elements.

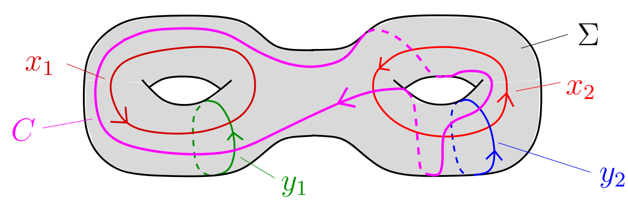


FIGURE 1383

Now we have the tools to prove the following refinement of Proposition 98.6.

Proposition 98.6. *Let M be a compact orientable n-dimensional smooth manifold and let $\phi \in \mathrm{H}_{n-1}(M,\partial M;\mathbb{Z})$. Let $\partial\phi \in \mathrm{H}_{n-2}(\partial M)$ be the image of ϕ under the connecting homomorphism in the long exact sequence of the pair $(M,\partial M)$. If $F \subset \partial M$ is a submanifold that represents $\partial\phi \in \mathrm{H}_{n-2}(A)$, then ϕ can be represented by a submanifold N of M with $\partial N = F$.*

PROOF (∗). Let M be a compact oriented n-dimensional smooth manifold. Suppose we are given a homology class $\phi \in \mathrm{H}_{n-1}(M, \partial M; \mathbb{Z})$ and a submanifold $F \subset \partial M$ that represents $\partial\phi \in \mathrm{H}_{n-2}(\partial M)$. As above we pick a trivial tubular neighborhood $[-1,1] \times F$ in ∂M such that the product orientation of $[-1,1] \times F$ matches the orientation of ∂M. Next we consider the corresponding Thom-Pontryagin map

$$\begin{array}{rcl} \rho_F \colon \partial N & \to & S^1 \\ P & \mapsto & \begin{cases} e^{\pi i t}, & \text{if } P = (t, Q) \in [-1,1] \times F, \\ -1, & \text{otherwise} \end{cases} \end{array}$$

Note that ρ_F is smooth on the closed subset $[-\frac{1}{2}, \frac{1}{2}] \times F$. It follows quite easily from the Whitney Approximation Theorem 9.3 (1d) that ρ_F is homotopic rel $[-\frac{1}{2}, \frac{1}{2}] \times F$ to a smooth map $f \colon \partial N \to S^1$ which is "close to ρ_F" in the sense that it has the property that for $P \notin [-\frac{1}{2}, \frac{1}{2}] \times F$ we have $f(P) \notin \{e^{\pi i t} \,|\, t \in [-\frac{1}{4}, \frac{1}{4}]\}$. Now note that it follows from Proposition 88.26 together with Proposition 79.10 (3) that there exists a smooth map $g \colon M \to S^1$ that represents $\mathrm{PD}_M(\phi) \in \mathrm{H}^1(M, \partial M; \mathbb{Z})$ and that agrees with f on ∂M.

By Sard's Theorem 6.63 and Proposition 6.62 (2) there exists an $s \in (-\frac{1}{4}, \frac{1}{4})$ such that $e^{\pi i s}$ is a regular value of g. We write $N := g^{-1}(\{z\})$. By the proof of Proposition 98.4 (1) we know that N is a smooth proper submanifold that represents ϕ. Note that by construction we have $\partial N = N \cap \partial M = \{s\} \times F$. Using a collar neighborhood of ∂M it is straightforward to find an isotopy that arranges that $\partial N = \{0\} \times F = F$. We leave it to the reader to fill in the few remaining details. ■

Later on we will make use of the following useful corollary.

Corollary 98.7. *Let M be a closed orientable n-dimensional smooth manifold and let C and D be two closed oriented $(n-1)$-dimensional submanifolds. If $[C] = [D] \in \mathrm{H}_n(M; \mathbb{Z})$, then there exists a compact oriented proper n-dimensional submanifold $W \subset [0,1] \times M$ such that $\partial W = (\{0\} \times C) \cup (\{1\} \times -D)$.*

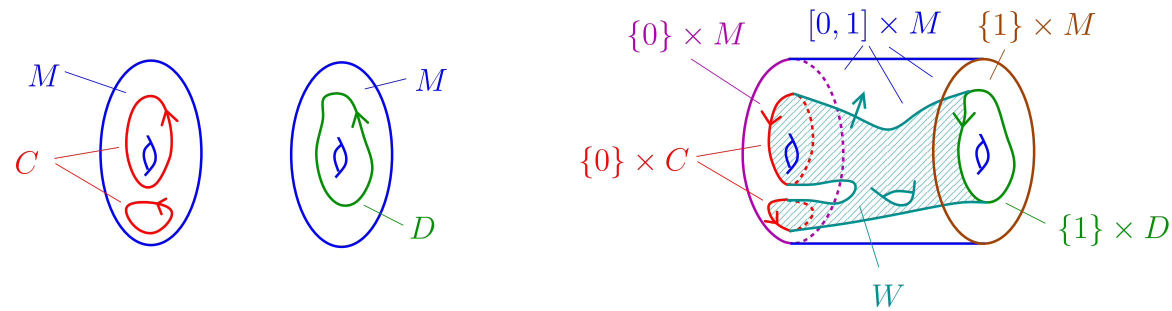


FIGURE 1384. Illustration of Corollary 98.7.

PROOF. We consider the following diagram:

$$\begin{array}{c} \ldots \longrightarrow \mathrm{H}_n([0,1]\times M) \xrightarrow{\partial} \mathrm{H}_{n-1}((\{0\}\times M) \cup (\{1\}\times M)) \xrightarrow{i_*} \mathrm{H}_{n-1}([0,1]\times M) \longrightarrow \ldots \\ \hspace{9em} {\scriptstyle p_*}\big\downarrow \cong \\ \hspace{9em} \mathrm{H}_n(M) \end{array}$$

Here the horizontal sequence is the long exact sequence in homology corresponding to the pair $([0,1]\times M, (\{0\}\times M)\cup(\{1\}\times M))$. Furthermore $p\colon [0,1]\times M \to M$ denotes the obvious projection. Our hypothesis that $[C]=[D]\in \mathrm{H}_n(M;\mathbb{Z})$ implies immediately that $p_*([\{0\}\times C]) = p_*([\{1\}\times D])$. Since p_* is an isomorphism and since the sequence is exact we see that there exists a homology class $\varphi\in \mathrm{H}_n([0,1]\times M)$ with $\partial\varphi = [\{0\}\times C]-[\{1\}\times D]$. Finally note that it follows immediately from Proposition 98.6 that there exists in particular a compact oriented proper n-dimensional submanifold W of $[0,1]\times M$ with $[W]=\varphi$ and with $\partial W = (\{0\}\times C)\cup(\{1\}\times -D)$. ■

In the remainder of this chapter we study an amusing generalization of the above Thom-Pontryagin map.

Definition. Let M be a compact oriented n-dimensional smooth manifold. Furthermore let $N_1,\dots,N_k$ be disjoint proper oriented $(n-1)$-dimensional submanifolds. As on page 2369 we pick a trivial tubular neighborhood of the form $[-1,1]\times(N_1\cup\dots\cup N_k)$. We consider the wedge $\bigvee_{j=1}^k S^1_j$ given by identifying the points $-1\in S^1_j$ to a single point $*$. We again refer to the map

$$\begin{array}{rcl}\rho_{N_1,\dots,N_k}\colon M & \to & \bigvee\limits_{j=1}^k S^1_j\\ P & \mapsto & \begin{cases} e^{\pi i t}\in S^1_j, & \text{if } P=(t,Q)\in[-1,1]\times N_j,\\ *, & \text{otherwise}\end{cases}\end{array}$$

as the *Thom-Pontryagin map*. The definition of ρ_N is illustrated in Figure 1385.

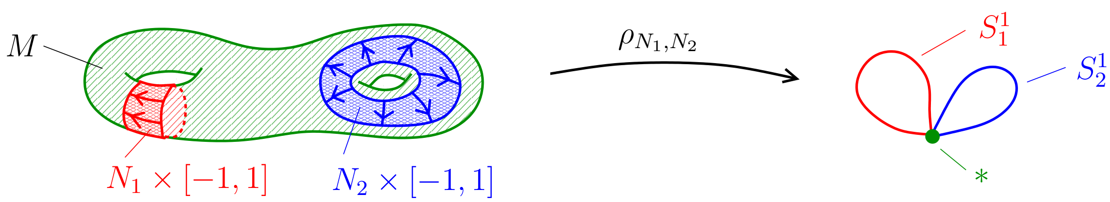


FIGURE 1385

Before we continue we recall that in Lemma 20.4 we gave an identification

$$\langle t_1,\dots,t_k\rangle \;=\; \pi_1\Big(\bigvee_{j=1}^k S^1_j, *\Big)$$

where each t_j is represented by a loop that goes once around the j-th circle S^1_j in a positive direction.

Proposition 98.8. *Let M be a compact oriented connected n-dimensional smooth manifold and let $N_1,\dots,N_k$ be disjoint proper oriented non-empty $(n-1)$-dimensional submanifolds. If $M\setminus(F_1\cup\dots\cup F_k)$ is connected, then for any $x_0\in M$ the map*

$$\rho_{N_1,\dots,N_k}\colon \pi_1(M,x_0)\;\to\;\pi_1\Big(\bigvee_{j=1}^k S^1_j,*\Big)\;=\;\langle t_1,\dots,t_k\rangle$$

is an epimorphism.

PROOF. We write $N = N_1 \cup \cdots \cup N_k$ and $\rho = \rho_{N_1,\dots,N_k}$. We pick a trivial tubular neighborhood of the form $[-1,1] \times N$. It follows from our hypothesis that $M \setminus N$ is connected, together with Proposition 10.11 and Lemma 10.1 that $M \setminus (-1,1) \times N$ is also connected. Since M is connected it follows from Proposition 15.11 that it suffices to consider a base point x_0 that lies in $M \setminus (-1,1) \times N$.

For $j = 1,\dots,k$ we do the following:

(1) we pick a point $P_i \in N_i$,
(2) since $M \setminus (-1,1) \times N$ is connected we can pick a path α_j from x_0 to $(-1,P_j)$ and we can pick a path γ_j from $(1,P_j)$ to x_0,
(3) we denote by β_j the "horizontal" path in $[-1,1] \times \mathbb{N}_j$ from $(-1,P_j)$ to $(1,P_j)$,
(4) we write $\delta_j := \alpha_j * \beta_j * \gamma_j$.

It follows easily from the definitions that for $j = 1,\dots,k$ we have $\rho_*([\delta_j]) = t_j$. We have thus shown that the generating set $\{t_1,\dots,t_k\}$ lies in the image of ρ_*. In other words, ρ_* is indeed an epimorphism. ■

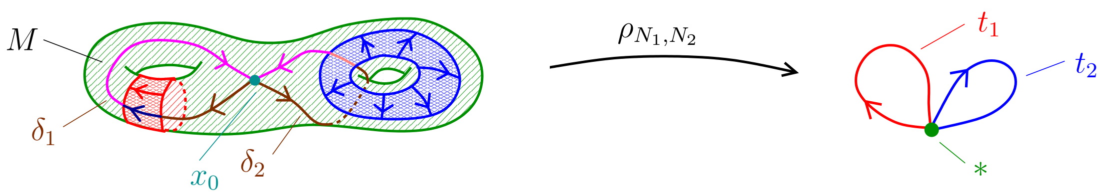


FIGURE 1386. Illustration for the proof of Proposition 98.8.

Example. Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. As should be pretty clear from considering Figure 1387, the surface Σ_g contains g disjoint curves $N_1,\dots,N_g$ such that the complement is connected. Thus it follows from Proposition 98.8 that there exists an epimorphism

$$\pi_1(\Sigma_g) \;\to\; \langle x_1,\dots,x_g\rangle.$$

In fact, as we will see in Exercise 98.2, it is not difficult to show the existence of such an epimorphism on a more algebraic level.

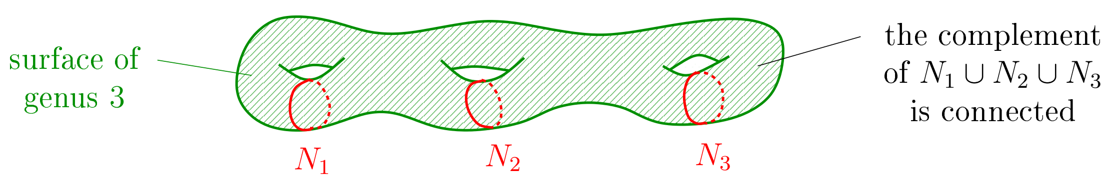


FIGURE 1387

The previous example raises the following thought-provoking question.

Question 98.9. *Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g.*

(1) *What is the maximal number $k = k(g) \in \mathbb{N}_0$ for which we can find k disjoint curves $F_1,\dots,F_k$ on Σ_g such that $\Sigma_g \setminus (F_1 \cup \cdots \cup F_k)$ is connected?*
(2) *What is the maximal $l = l(g) \in \mathbb{N}_0$ for which there exists an epimorphism from $\pi_1(\Sigma_g)$ onto a free group of k generators?*

In summary, as of right now we know the following:

$$\begin{array}{ccccccc} g & \leq & k(g) & \leq & l(g) & \leq & 2g. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by the above example} & & \text{Proposition 98.8} & & \text{Proposition 21.20} & \end{array}$$

98.4. **Cyclic coverings of manifolds** $(*)$**.** Our goal in this section is to give explicit constructions of cyclic coverings of manifolds. To simplify the language will use the following conventions throughout this section.

Convention.

(1) We will not distinguish between two coverings $p\colon X \to B$ and $q\colon T \to B$ that are equivalent in the sense of the definition on page 832.
(2) We will use Proposition 14.11 as an excuse to suppress all base points from our notation.[1370]

The following proposition combines several results on coverings that we proved a while ago.

Proposition 98.10. *Let M be a connected smooth manifold and let $\varphi\colon \pi_1(M) \to G$ be an epimorphism onto an abelian group. There exists a unique covering $p\colon \widetilde{M} \to M$ such that $p_*(\pi_1(\widetilde{M})) = \ker(\varphi)$. Furthermore this covering has the following properties:*

(1) *The covering is regular and the deck transformation group is isomorphic to G.*
(2) *The index of the covering is precisely $|G|$.*

Proof. The proposition follows immediately from Propositions 29.5 and 29.8 and 34.6 together with Corollary 34.7. ■

We sort-of introduced the following definition on page 911.

Definition. We say a covering $p\colon X \to Y$ is (infinite) cyclic if p is a regular covering and if the deck transformation group is an (infinite) cyclic group.

Let M be a smooth manifold M. Given any primitive $\phi \in \mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ and given $n \in \mathbb{N}$ we know by Proposition 98.10 that there exist corresponding (infinite) cyclic covers of M. Our goal is to a very explicit constructions of these (infinite) cyclic covers for many smooth manifolds.

In the remainder of this section we will use the following notation:

Notation. Let M be a connected smooth manifold.

(1) We use Proposition 75.18 to make the identification $\mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$. It follows almost immediately from the definitions that under this identification the primitive elements in $\mathrm{H}^1(M;\mathbb{Z})$ correspond to the epimorphisms $\pi_1(M) \to \mathbb{Z}$.

[1370] More precisely, using Proposition 14.11 it is not difficult to show that for any topological space X, any abelian group G and any two paths $p, q\colon [0,1] \to X$ from a point x_0 to a point x_1 we have $p_* = q_*\colon \mathrm{Hom}(\pi_1(X,x_0),G) \to \mathrm{Hom}(\pi_1(X,x_1),G)$. In particular, given a path-connected topological space X and $x_0, x_1 \in X$ the groups $\mathrm{Hom}(\pi_1(X,x_0),G)$ and $\mathrm{Hom}(\pi_1(X,x_0),G)$ are canonically isomorphic.

(2) Let $\phi \in \mathrm{H}^1(M;\mathbb{Z})$ again be a primitive element. Furthermore let $n \in \mathbb{N}$. In the following we denote by $\phi_n\colon \pi_1(M) \to \mathbb{Z}_n$ the epimorphism that is given by the composition $\pi_1(M) \xrightarrow{\phi} \mathbb{Z} \to \mathbb{Z}_n$.

At this stage it is worth recalling the following definitions from page 360.

Definition. Let M be an oriented smooth manifold and let $F \subset M$ be a compact oriented proper submanifold of codimension one. We pick an orientation-preserving tubular map $\Phi\colon [-1,1] \times F \to M$. We define

$$M \setminus\!\!\setminus F \;:=\; (M \setminus F) \sqcup (F \times \{\pm 0\})$$

with the topology that is generated by the following sets:

(1) the usual open subset of $M \setminus F$,
(2) the sets of the form $\Phi(U \times (0,\epsilon)) \cup (U \times \{+0\})$ where U is an open subset of F and where $\epsilon \in (0,1)$,
(3) the sets of the form $\Phi(U \times (-\epsilon,0)) \cup (U \times \{-0\})$ where U is an open subset of F and where $\epsilon \in (0,1)$.

We refer to $M \setminus\!\!\setminus F$ as *M cut along F*. This definition is illustrated in Figure 1388. We also introduce the following extra notation:

(4) We denote by $F_\pm$ the images of $F \times \{\pm 0\}$ in $M \setminus\!\!\setminus F$. Furthermore we denote by $i_\pm\colon F_\pm \to M \setminus\!\!\setminus F$ the obvious inclusion maps.
(5) We denote by $\sim_F$ the equivalence relation on the set $M \setminus\!\!\setminus F$ that is generated by $(P,+0) \sim (P,-0)$ for $P \in F$.
(6) We denote by χ_F the map

$$\begin{array}{rcl} \chi_F\colon M \setminus\!\!\setminus F & \to & M \\ P & \mapsto & \begin{cases} P, & \text{if } P \in M \setminus F, \\ Q, & \text{if } P = (Q, \pm 0). \end{cases} \end{array}$$

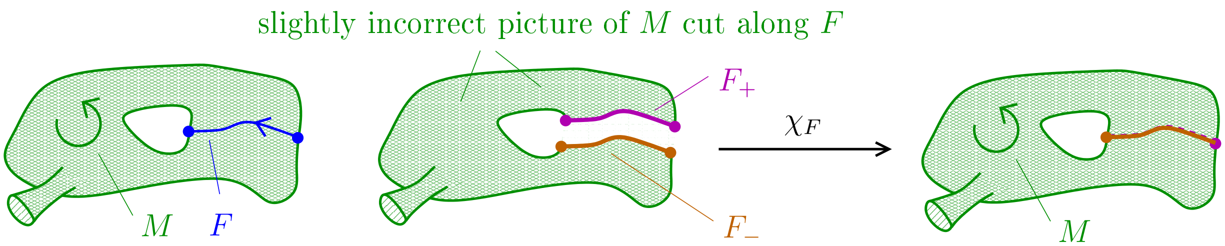


FIGURE 1388

Now we can give the explicit construction of the cyclic coverings.

Proposition 98.11. *Let M be compact oriented connected smooth manifold. Furthermore let $\phi \in \mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ be a primitive element. Finally let $F \subset M$ be a* *compact oriented proper codimension-one submanifold*[1371] *that is dual to* ϕ *in the sense that* $\mathrm{PD}_M([F]) = \phi$. *We denote by* $i_\pm\colon F \to F_\pm \subset M \setminus\!\setminus F$ *the two obvious inclusion maps.*

(1) *We write*

$$\widetilde{M}_\infty := \Big(\bigsqcup_{k\in\mathbb{Z}} (M \setminus\!\setminus F) \times \{k\}\Big)/\sim \qquad \begin{array}{c} \textit{where } (i_-(P),k) \sim (i_+(P),k+1) \\ \textit{for } P \in F \textit{ and } k \in \mathbb{Z}. \end{array}$$

The map

$$\begin{array}{rcl} p\colon \widetilde{M}_\infty & \to & M \\ [(P,k)] & \mapsto & [\chi_F(P)] \end{array}$$

is the infinite cyclic covering corresponding to ϕ.

(2) *Let* $n \in \mathbb{N}$. *We write*

$$\widetilde{M}_n := \Big(\bigsqcup_{k\in\mathbb{Z}_n} (M \setminus\!\setminus F) \times \{k\}\Big)/\sim \qquad \begin{array}{c} \textit{where } (i_-(P),k) \sim (i_+(P),k+1) \\ \textit{for } P \in F \textit{ and } k \in \mathbb{Z}_n. \end{array}$$

The map

$$\begin{array}{rcl} p\colon \widetilde{M}_n & \to & M \\ [(P,k)] & \mapsto & [\chi_F(P)] \end{array}$$

is the finite cyclic covering corresponding to ϕ_n.

Example. We consider the 2-dimensional smooth manifold M together with the submanifold F that are shown in Figure 1389. We illustrate the explicit construction of the infinite cyclic covering $\widetilde{M}_\infty$. The reader will not fail to spot the uncanny similarities with Lemma 25.12 and Figure 484.

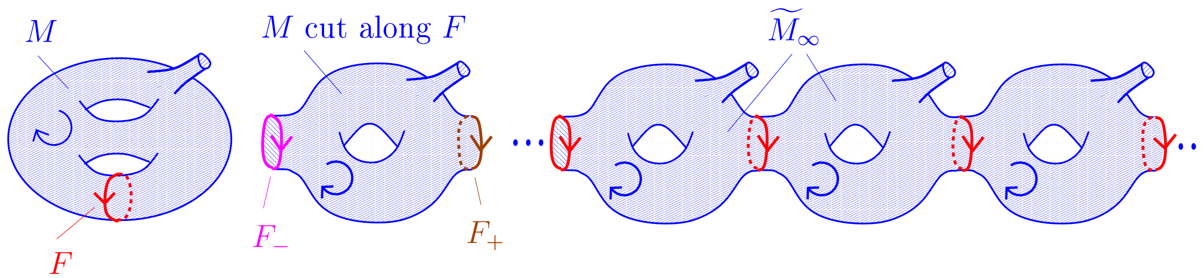


FIGURE 1389. Illustration of Proposition 98.11.

PROOF.

(1) We pick a tubular map $\Phi\colon [-1,1] \times F \to M$ for F. It follows almost immediately from the definitions that the open sets $M \setminus F$ and $\Phi((-1,1) \times F)$ are uniformly covered in the sense of the definition on page 493. Since these two open subsets cover all of M we see that $p\colon \widetilde{M}_\infty \to M$ is indeed a covering.

[1371]Note that we do not assume that F is connected.

In the following we will show that $p\colon \widetilde{M}_\infty \to M$ is indeed the covering corresponding to $\ker(\phi)$.[1372] We consider the following map:

$$\begin{array}{rcl} \alpha\colon M & \to & S^1 \\ P & \mapsto & \begin{cases} e^{\pi i t}, & \text{if } P = \Phi(t,Q) \text{ with } t \in [-1,1] \text{ and } Q \in F, \\ -1, & \text{otherwise} \end{cases} \end{array}$$

This is of course the Thom-Pontryagin map corresponding to the tubular map that we introduced on page 2369. Let $q\colon \mathbb{R} \to S^1$ be the universal covering that is given by $t \mapsto e^{\pi i t}$. Furthermore let

$$\begin{array}{rcl} r\colon \alpha^*\mathbb{R} = \{(P,t) \in M \times \mathbb{R} \,|\, \alpha(P) = p(t)\} & \to & M \\ (P,t) & \mapsto & P \end{array}$$

be the pullback of the covering q. Note that by Lemma 25.16 (3) we know that $r\colon \alpha^*\mathbb{R} \to M$ is also a covering.

Claim. The covering $p\colon \widetilde{M}_\infty \to M$ is equivalent to the covering $r\colon \alpha^*\mathbb{R} \to M$.

The key to proving the claim is to consider the map

$$\begin{array}{rcl} \beta\colon \widetilde{M}_\infty & \to & \mathbb{R} \\ [P] & \mapsto & \begin{cases} 2i-1, & \text{if } P \in (M \setminus \Phi([-1,1] \times F)) \times \{i\}, \\ 2i+t, & \text{if } P \in \Phi(\{t\} \times F) \times \{i\} \text{ for some } t \in [-1,0), \\ 2i+t-2, & \text{if } P \in \Phi(\{t\} \times F) \times \{i\} \text{ for some } t \in (0,1], \\ 2i, & \text{if } P \in F_- \times \{i\} \text{ or } P \in F_+ \times \{i+1\}. \end{cases} \end{array}$$

Using Lemma 2.35 (2) one easily verify that this map is continuous. We refer to Figure 1390 for an illustration. Next we consider the map

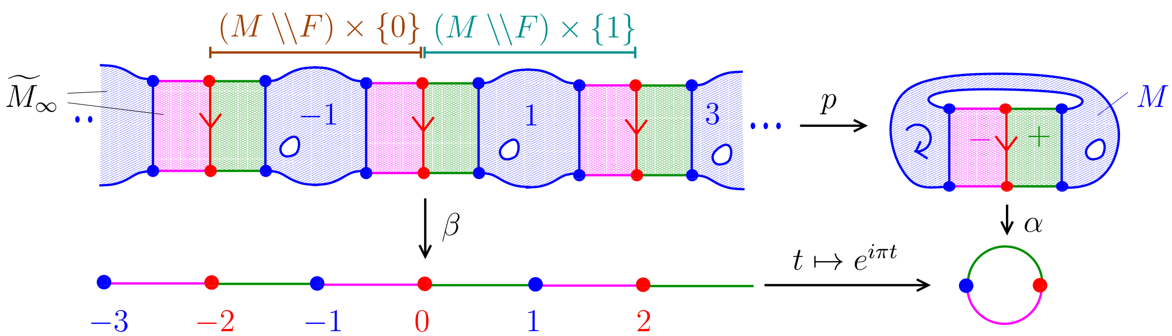


FIGURE 1390. Illustration for the proof of Proposition 98.11 (1).

$$\begin{array}{rcl} \Theta\colon \widetilde{M}_\infty & \to & \alpha^*\mathbb{R} \\ P & \mapsto & (p(P), \beta(P)). \end{array}$$

It is straightforward to see that Θ is a bijection. We leave it to the reader to show that Θ is actually a homeomorphism. One can easily verify that $\Theta \circ r = p$. In summary we conclude that the two coverings q and r are indeed equivalent. ⊞

[1372] An alternative proof of this statement is given in [**Herr19**, Lemma 1.31].

Now we see that

$$q_*(\pi_1(\widetilde{M}_\infty)) \underset{\substack{\uparrow \\ \text{since } q \text{ and } r \text{ are} \\ \text{equivalent coverings}}}{=} r_*(\pi_1(\alpha^*\mathbb{R})) \underset{\substack{\uparrow \\ \text{Lemma 25.16}}}{=} \alpha_*^{-1}(\pi_1(\mathbb{R})) \underset{\substack{\uparrow \\ \text{since } \pi_1(\mathbb{R})=0}}{=} \ker(\alpha_*) \underset{\substack{\uparrow \\ \text{Lemma 98.5}}}{=} \ker(\phi).$$

We have thus shown that $q_*\colon \widetilde{M}_\infty \to M$ is indeed the covering corresponding to $\ker(\phi)$.

(2) The proof of this statement is very similar to the proof of (1). We just need to replace the covering $q\colon \mathbb{R} \to S^1$ by the covering $q\colon S^1 \to S^1$ given by $z \mapsto z^n$. We leave it to the reader to once again lessen the burden on the author by filling in the details. ■

98.5. **Curves on 2-dimensional smooth manifolds** $(*)$**.** We conclude this chapter with a discussion of curves on 2-dimensional smooth manifolds. The results of this section are nice-to-have, but they will not be used in the remainder of these notes. Thus it is safe to skip this section in the unlikely event that the reader is in a hurry.

Let us now recall the following definition from page 544: A *closed oriented curve* C in a smooth manifold M is a closed oriented connected 1-dimensional submanifold of M or of ∂M. In this section we want to study curves on 2-dimensional smooth manifolds. Our main focus will be curves on the torus.

Before we state the results we recall that on page 1223 we saw that a closed oriented curve C in a smooth manifold M defines a homology class $[C] \in \mathrm{H}_1(M;\mathbb{Z})$. We start out with the following lemma.

Lemma 98.12. *Let $T = S^1 \times S^1$ be the torus and let C be a closed oriented curve on T.*
(1) *If $T \setminus C$ is connected, then $[C]$ is a primitive element of $\mathrm{H}_1(T;\mathbb{Z})$.*
(2) *If $T \setminus C$ is disconnected, then C is the boundary of an embedded disk.*

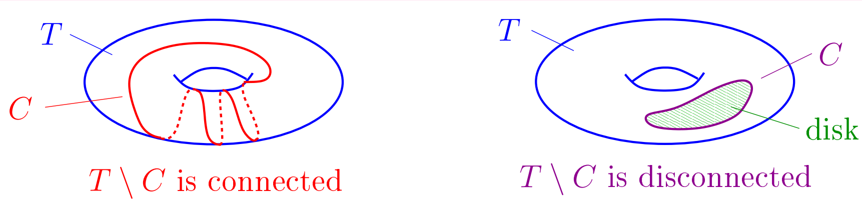


FIGURE 1391. Illustration of Lemma 98.12.

PROOF. If $T \setminus C$ is connected, then it follows from Proposition 98.2 that $[C]$ is a primitive element of $\mathrm{H}_1(T;\mathbb{Z})$. Now suppose that $T\setminus C$ is disconnected. By the Tubular Neighborhood Theorem 8.24 there exists a tubular neighborhood $[-1,1] \times C$. From Lemma 10.1 and Proposition 10.11 we obtain that $T \setminus (-1,1) \times C$ is also disconnected. Since C is connected this implies that $T \setminus (-1,1) \times C$ has two components A and B, one of which contains $\{-1\} \times C$ and one which contains $\{1\} \times C$.[1373] By Proposition 6.30 we know that A and B are compact 2-dimensional smooth manifolds, in particular they are finite CW-complexes.

[1373] Why is that?

We have

$$\chi(A)+\chi(B) \underset{\substack{\uparrow \\ \text{Lemma 55.5}}}{=} \underbrace{\chi(T)}_{=0}+\underbrace{\chi(\{-1\}\times C)}_{=0}+\underbrace{\chi(\{1\}\times C)}_{=0} = 0.$$

Since A and B have boundary it follows from the classification of compact 2-dimensional smooth manifolds and the calculation of Euler characteristics, see Lemma 55.6, that one of A or B is diffeomorphic to the disk $\overline{B}^2$. ■

Lemma 98.13. *Let $T = S^1 \times S^1$ be the torus. Let C and D be two closed oriented curves on T. If C and D are disjoint and if $[C]$ and $[D]$ are both non-zero*[1374] *in $H_1(T;\mathbb{Z})$, then $[C] = \pm[D]$.*

Proof. From Theorem 94.5 together with our hypothesis that $C \cap D = \emptyset$ we obtain that $\mathrm{PD}_T([C]) \cup \mathrm{PD}_T([D]) = 0$. It follows easily from the calculation of the cup product of the torus, see Lemma 81.14, that $\mathrm{PD}_T([C]) = k \cdot \mathrm{PD}_T([D])$ for some $k \in \mathbb{Z}$. We obtain from Lemma 98.12 that $k \in \{-1, 1\}$. ■

Lemma 98.14. *Let $T = S^1 \times S^1$ be the torus. Let C and D be two closed oriented curves on T. Suppose that $[C] = [D] \neq 0$, then the following hold:*

(1) *The two closed oriented curves are smoothly isotopic.*
(2) *If C and D are furthermore disjoint, then there exists an embedded annulus $X \subset T$ with $\partial X = C \cup -D$.*

Furthermore, the same statements also hold if we replace the torus $T = S^1 \times S^1$ by the annulus $S^1 \times [-1, 1]$.

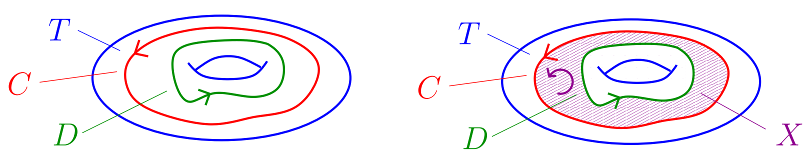


Figure 1392. Illustration of Lemma 98.14 (2)

Sketch of proof. We provide a sketch of a proof for the various statements:

(1) By the Transversality Theorem 9.10 (2) we can assume that C and D are already transverse. It follows from the calculation of the cup product of the torus and Theorem 94.5 that the algebraic intersection number is 0. The argument is now similar to the argument of the proof of Theorem 94.16, but this time we use Lemma 98.12 to find the desired embedded disks. We refer to Figure 1393 for an illustration.

    It is a good, and not entirely trivial, mental exercise to fill in the details. The argument is sketched in more detail in [**Rolf90**, Lemma 5 of Chapter 2.C].

(2) The proof of this statement is similar to the proof of Lemma 98.12 (2). We leave it to the reader to fill in the details.

The proof in the case that $T = S^1 \times [-1, 1]$ is easier. We will discuss it in Exercise 99.1. ■

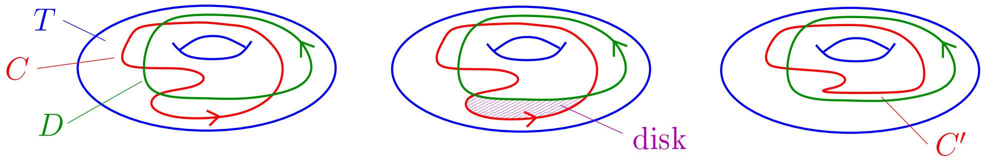


FIGURE 1393

For completeness sake' we state the following proposition, even though we will not make use of it.

Proposition 98.15. (∗) *Let F be a 2-dimensional smooth manifold and let C and D be two closed oriented curves on F.*

(1) *If C and D are homotopic, then C and D are smoothly isotopic.*
(2) *If C and D are furthermore disjoint and if they are not null-homotopic, then there exists an embedded annulus $X \subset F$ with $\partial X = C \cup -D$.*

PROOF.

(1) This statement is [**FaM11**, Proposition 1.10]. (Closely related statements are also proved in [**Baer28**], [**Eps66**, Theorem 2.1].) Unfortunately all the references are somewhat cavalier when it comes to distinguishing continuous and smooth maps, but with some effort one can see that all the references provide smooth isotopies.
(2) If F is orientable, then we will provide the proof in Exercise 98.7. The proof of the non-orientable case is not much harder. ■

Exercises for Chapter 98.

Exercise 98.1. Let M be a *non-compact* oriented n-dimensional smooth manifold with compact boundary. Furthermore let N be a compact orientable non-empty proper submanifold of codimension-one such that $M \setminus N$ is connected. Show that $[N] \in \mathrm{H}_{n-1}(M; \mathbb{Z})$ is an element of infinite order.
Hint. Use Propositions 11.4 and 98.2.

Exercise 98.2. Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. By Proposition 22.3 we know that there exists an isomorphism

$$\pi_1(\Sigma_g) \;\cong\; \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \cdot \dots \cdot [x_g, y_g] \rangle.$$

Use this isomorphism to give a purely algebraic proof that $\pi_1(\Sigma_g)$ admits an epimorphism onto a free group on g generators.

Exercise 98.3. Here is one possible approach to answering Question 98.9 (1). Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. Let $F_1, \dots, F_k$ be disjoint curves on Σ_g such that $\Sigma_g \setminus (F_1 \cup \dots \cup F_k)$ is connected. By the Tubular Neighborhood Theorem 8.24 we can pick a tubular neighborhood $\bigcup_{i=1}^{k} [-1, 1] \times F_i$ for the submanifold $F_1 \cup \dots \cup F_k \subset \Sigma_g$. We consider

[1374] Is this condition that $[C]$ and $[D]$ are both non-zero necessary?

the compact orientable 2-dimensional smooth manifold $X := \Sigma_g \setminus \bigcup_{i=1}^{k} (-1,1) \times F_i$. What can you say about the Euler characteristic of X? What does the classification of compact orientable 2-dimensional smooth manifolds tell you?

Exercise 98.4. Let Σ be a compact orientable connected 2-dimensional smooth manifold and let $A, B \subset \Sigma$ be two disjoint non-separating curves.

(a) Show that there exists a closed 1-dimensional smooth manifold $C \subset \Sigma$ that intersects each of A and B transversally in a single point.
Remark. Exercise 48.7 might come in handy.

(b) Can one also arrange that C is connected?

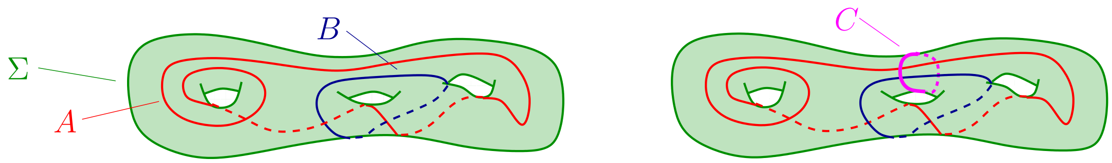


FIGURE 1394. Illustration for Exercise 98.4.

Exercise 98.5. Let M be a compact oriented connected smooth manifold and let F be a compact oriented proper submanifold. We use the notation from page 2375. In particular we denote by $i_\pm \colon F \to M \setminus\!\!\setminus F$ the two inclusion maps and we denote by $\chi_F \colon M \setminus\!\!\setminus F \to M$ the obvious projection map. Show that there exists a long exact sequence of the form

$$\dots \to \mathrm{H}_i(F) \xrightarrow{i_- - i_+} \mathrm{H}_i(M \setminus\!\!\setminus F) \xrightarrow{\chi_F} \mathrm{H}_i(M) \xrightarrow{\partial} \mathrm{H}_{i-1}(F) \to \dots$$

Exercise 98.6. Let (X, x_0) be a topological space that is path-connected, locally path-connected and semi-locally simply connected. Furthermore let $\phi \colon \pi_1(X, x_0) \to \langle t \rangle$ be an epimorphism. We denote by $p \colon \widetilde{X}_\infty \to X$ the covering corresponding to $\ker(\phi)$. By Proposition 34.6 we have an explicit identification of the group $\langle t \rangle$ with the deck transformation group of $p \colon \widetilde{X}_\infty \to X$. Given $\sigma \in \mathrm{C}_n(\widetilde{X}_\infty)$ and given $k \in \mathbb{Z}$ we write $t^k \cdot \sigma := (t^k)_*(\sigma)$.

(a) Let $n \in \mathbb{N}_0$ and let $\sigma, \mu \in \mathrm{C}_n(\widetilde{X}_\infty)$. Show that $p_*(\sigma) = p_*(\mu) \in \mathrm{C}_n(X)$ if and only if there exists a $k \in \mathbb{Z}$ with $\sigma = t^k \cdot \mu$.

(b) Let $n \in \mathbb{N}_0$. Show that the following sequence is exact:

$$0 \to \mathrm{C}_n(\widetilde{X}_\infty) \xrightarrow{t - \mathrm{id}} \mathrm{C}_n(\widetilde{X}_\infty) \xrightarrow{p_*} \mathrm{C}_n(X) \to 0.$$

Remark. By Proposition 43.11 we obtain a corresponding long exact sequence

$$\dots \to \mathrm{H}_{n+1}(X) \xrightarrow{\partial} \mathrm{H}_n(\widetilde{X}_\infty) \xrightarrow{t-\mathrm{id}} \mathrm{H}_n(\widetilde{X}_\infty) \xrightarrow{p_*} \mathrm{H}_n(X) \xrightarrow{\partial} \mathrm{H}_{n-1}(\widetilde{X}_\infty) \to \dots .$$

(c) As a reality check, what does the long exact sequence in (b) look like for the covering $p \colon \mathbb{R} \to S^1$?

(d) Let $r \in \mathbb{N}$. We denote by $\widetilde{X}_r$ the covering of X corresponding to the epimorphism $\ker(\phi \colon \pi_1(X, x_0) \to \langle t \rangle = \mathbb{Z} \to \mathbb{Z}_r)$. Show that there exists an exact sequence relating the homology groups of $\widetilde{X}_\infty$ and $\widetilde{X}_r$.

Exercise 98.7. Let F be an oriented 2-dimensional smooth manifold and let C and D be two closed oriented *disjoint* curves on F that are not null-homotopic. Assume that C and D are homotopic. Show that there exists an embedded annulus $X \subset F$ with $\partial X = C \cup -D$.
Hint. Use Exercise 99.2. You might also want to use some of the ideas of the proof of Lemma 98.12.

Exercise 98.8. Let $T = S^1 \times S^1$ be the torus and let C and D be two closed oriented 1-dimensional submanifolds of T with $[C] = [D] \in \mathrm{H}_1(T;\mathbb{Z})$. Use the results from Section 98.5 to give a down-to-earth proof of the statement that there exists a compact oriented proper 2-dimensional submanifold Ω of $T \times [0,1]$ such that

$$\partial\Omega \;=\; (\{0\}\times) \cup (\{1\} \times -D).$$

Remark. Evidently this is just a special case of Corollary 98.7. But it is fun to give a bare-hands proof in this special case.

Exercise 98.9. Let Σ be the surface of some genus $g \in \mathbb{N}_0$ and let $f\colon \Sigma \to \Sigma$ be some orientation-preserving self-diffeomorphism. We consider the corresponding mapping torus

$$\mathrm{Tor}(\Sigma, f) \;:=\; (\Sigma \times [0,1]) \,/\, \sim \qquad \text{where } (x,0) \sim (f(x),1) \text{ for all } x \in \Sigma.$$

Let F be a surface of some genus k. Suppose that there exists a map $\varphi\colon F \to \mathrm{Tor}(\Sigma, f)$ such that $\varphi_*([F]) = [\{0\} \times \Sigma] \in \mathrm{H}_2(\mathrm{Tor}(\Sigma, f))$. Show that $k \geq g$.
Hint. Make clever use of Proposition 98.11 and Exercise 70.3.

Exercise 98.10. Let M be a closed oriented n-dimensional smooth manifold. Furthermore let $\phi \in \mathrm{H}_{n-1}(M;\mathbb{Z})$, let $k \in \mathbb{N}$ and let F be a closed oriented $(n-1)$-dimensional submanifold that represents $k \cdot \phi$. Show that we can decompose $F = F_1 \sqcup \dots \sqcup F_k$ where each F_i represents ϕ.
Hint. Consider the Thom-Pontryagin map $\rho = \rho_F\colon M \to S^1$ that we introduced on page 2369. Note that $\rho^{-1}(\{1\}) = F$. Let $p\colon S^1 \to S^1$ be the covering map given by $z \mapsto z^k$. Show that there exists a map $\widetilde{\rho}\colon M \to S^1$ such that $\rho = p \circ \widetilde{\rho}$.

Exercise 98.11. Given a closed oriented 2-dimensional smooth manifold S with components $S_1, \dots, S_m$ we define

$$\chi_-(S) \;:=\; \sum_{i=1}^{m} \max\{-\chi(S_i), 0\}.$$

Let M be a closed oriented 3-dimensional smooth manifold. We consider the map

$$\begin{array}{rcl} x_M\colon \mathrm{H}_2(M;\mathbb{Z}) & \to & \mathbb{N}_0 \\ \sigma & \mapsto & \min\left\{\chi_-(S) \;\middle|\; \begin{array}{c} S \text{ is a closed oriented 2-dimensional} \\ \text{submanifold of } M \text{ with } [S] = \sigma \end{array}\right\}. \end{array}$$

Show that the map x_M is a seminorm, i.e. show that it has the following two properties:

(a) Given any $\sigma \in \mathrm{H}_2(M;\mathbb{Z})$ and $k \in \mathbb{Z}$ we have $x_M(k \cdot \sigma) = |k| \cdot x_M(\sigma)$.
Hint. Use Exercise 98.10.

(b) Given $\mu, \nu \in \mathrm{H}_2(M;\mathbb{Z})$ we have $x_M(\mu + \nu) \leq x_M(\mu) + x_N(\nu)$.
Hint. Use the Transversality Theorem 9.10 and Proposition 70.21.

(c) Show that there exists a unique seminorm $x_M\colon \mathrm{H}_2(M;\mathbb{R}) \to \mathbb{R}_{\geq 0}$ such that the map $\mathrm{H}_2(M;\mathbb{Z}) \xrightarrow{[\sigma]\mapsto[\sigma\otimes 1_{\mathbb{R}}]} \mathrm{H}_2(M;\mathbb{R}) \to \mathbb{R}_{\geq 0}$ agrees with the above map x_M.

Remark. This seminorm $x_M\colon \mathrm{H}_2(M;\mathbb{Z}) \to \mathbb{R}_{\geq 0}$ is usually called the *Thurston norm of M*. It was introduced by William Thurston [**Thu86**] in 1976 and, as is shown in [**Thu86**] and [**Gab83**], it has many remarkable properties.

Exercise 98.12. We continue with the definitions introduced in Exercise 98.11. Let Σ be a closed connected 2-dimensional smooth manifold with $\chi(\Sigma) \leq 0$.

(1) Let $f\colon \Sigma \to \Sigma$ be a diffeomorphism. We consider the mapping torus $\mathrm{Tor}(\Sigma, f)$. Show that $x([\Sigma]) = \chi_-(\Sigma)$.
Hint. Let F be any closed oriented 2-dimensional submanifold of $\mathrm{Tor}(\Sigma, f)$ with $[F] = [\Sigma]$. Using Proposition 98.11 we obtain a map $F \to \Sigma \times \mathbb{R}$. Consider the map $F \to \Sigma \times \mathbb{R} \to \Sigma$ and use Exercise 69.4.

(2) We can to consider the Thurston norm of $N \times S^1$.
 (a) Show that $x_{\Sigma^1\times S^1}([\Sigma \times \{*\}] = \chi_-(\Sigma)$.
 (b) Show that for any $\sigma \in \mathrm{H}_1(\Sigma)$ we have $x_{\Sigma^1\times S^1}([S^1] \times \sigma) = 0$.
 (c) Show that for any $\phi \in \mathrm{H}_2(\Sigma \times S^1;\mathbb{R})$ we have

$$x_{\Sigma\times S^1}(\phi) \;=\; \langle \underbrace{\mathrm{PD}_{\Sigma\times S^1}}_{\in \mathrm{H}^1(M;\mathbb{R})}, \underbrace{[\{*\}\times S^1]}_{\in \mathrm{H}_1(M;\mathbb{Z})} \rangle.$$

Hint. Either use Proposition 79.12 and (1). Alternatively use (2a) and (2b) together with Exercise 98.11 (b).

Exercise 98.13. Let M be a closed oriented n-dimensional smooth manifold. Given $\phi \in \mathrm{H}^1(M;\mathbb{Z})$ we define

$$x_M(\sigma) \;=\; \inf\{\underset{\substack{\uparrow\\ \text{simplicial volume as defined in Exercise 68.16}}}{\|N\|_{\mathbb{R}}} \,|\, N \text{ is a closed oriented submanifold of } M \text{ with } \mathrm{PD}_M([N]) = \phi\}.$$

Let N be a closed oriented connected $(n-1)$-dimensional smooth manifold. Given any cohomology class $\phi \in \mathrm{H}^1(N \times S^1)$ determine $x_{N\times S^1}(\phi)$ in terms of $\|N\|_{\mathbb{R}}$ and $\phi(\{*\} \times S^1)$.
Hint. Use Proposition 79.12 and (a).

99. Applications to knot theory

After so much theory we want to practice what we have learned by applying our recent results to knot theory.

99.1. **The longitude of a knot.** We start out our discussion of knots with the following definition.

Definition. Let $K \subset S^3$ be an oriented knot.

(1) A *thickening of* K is an orientation-preserving smooth embedding $F\colon \overline{B}^2 \times K \to S^3$ such that $F(0,P) = P$ for all $P \in K$.
(2) Given a thickening F of K we refer to $X_K := S^3 \setminus F(B^2 \times K)$ as the *knot exterior*.
(3) Given a thickening F of K and given $* \in K$ we refer to the oriented submanifold $\mu_K := F(S^1 \times \{*\})$ of $S^3 \setminus K$ as the *meridian* of K.
(4) By Lemma 68.16 (5) the inclusion induced map $\mathrm{H}_1(X_K;\mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_1(S^3 \setminus K;\mathbb{Z})$ is an isomorphism and we have $\mathrm{H}_1(X_K;\mathbb{Z}) = \mathbb{Z}\cdot[\mu_K]$. We denote by ϕ_K the isomorphism

$$\begin{array}{rcl} \phi_K\colon\ \mathrm{H}_1(S^3 \setminus K;\mathbb{Z}) \xleftarrow{\cong} \mathrm{H}_1(X_K;\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z} \\ {}[\mu_K] & \to & 1. \end{array}$$

(5) Suppose we are given a thickening F for K. We define

$$\text{self-linking number of } F \ := \ \phi_K(\underbrace{F_*([\{1\} \times K])}_{\in \mathrm{H}_1(X_K;\mathbb{Z})}).$$

Remark. It follows from the uniqueness of tubular neighborhoods, see Proposition 8.26, that the exterior X_K and the meridian μ_K are, in an appropriate sense, independent of the choice of the thickening. Therefore often we just say "the exterior" and "the meridian".

Lemma 99.1. *Let K be an oriented knot in S^3.*

(1) *Let $F\colon \overline{B}^2 \times K \to S^3$ be a thickening of K and let $s \in \mathbb{Z}$. Let $\gamma\colon S^1 \to K$ be an orientation-preserving diffeomorphism. Similar to Lemma 10.4 we consider the diffeomorphism*

$$\begin{array}{rcl} \Psi\colon \overline{B}^2 \times K & \to & \overline{B}^2 \times K \\ \underset{\substack{\uparrow \\ \text{where } z \in S^1}}{(w,\gamma(z))} & \mapsto & (w\cdot\gamma(z^s),\gamma(z)). \end{array}$$

The map $F \circ \Psi$ is a thickening with

$$\textit{self-linking number of } F\circ\Psi \ = \ \textit{self-linking number of } F + s.$$

(2) *Given any $n \in \mathbb{Z}$ there exists a thickening of K with self-linking number n.*
(3) *If $F, F'\colon \overline{B}^2 \times K \to S^3$ are two thickenings of K with self-linking number n, then there exists a diffeotopy $H\colon S^3 \times [0,1] \to S^3$ rel K with $H_0 = \mathrm{id}$ and $H_1 \circ F = F'$.*

Given $K \subset \mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ the analogous statements hold if we consider thickenings that are contained in $\mathbb{R}^3$. Furthermore, in this setting the diffeotopy in (3) of $\mathbb{R}^3$ can be assumed to have compact support.

Remark. Given an oriented knot K and $n \in \mathbb{Z}$ sometimes we use the uniqueness statement of Lemma 99.1 to talk about *the thickening of K with self-linking number n*. Sometimes we shorten this cumbersome expression to *the n-thickening of K*.

PROOF.

(1) We denote by m the self-linking number of F. We write $\lambda := F(\{1\} \times K)$ and $\mu := F(S^1 \times \{1\})$ and we view λ and μ as oriented 1-dimensional submanifolds with the orientations given by K and S^1. We calculate that

$$\text{self-linking number of } F\circ\Psi = \phi_K(F_*(\Psi_*([\{1\}\times K]))) \underset{\substack{\uparrow \\ \text{follows from Lemma 52.7 (2)}}}{=} \phi_K(s\cdot[\mu]+[\lambda]) \underset{\substack{\uparrow \\ \text{since } \phi_K(\mu)=1 \text{ and} \\ \text{by definition of } m}}{=} s+m.$$

(2) From the Tubular Neighborhood Theorem 8.24 we know that there exists a thickening $F\colon \overline{B}^2 \times K \to S^3$. We denote by m the self-linking number of F. The desired statement now follows from (1) applied to $n-m$.

(3) The uniqueness statement is an immediate consequence of (1) together with Proposition 10.7 and Lemma 10.4 (3).

The same arguments as above also apply if we replace S^3 by $\mathbb{R}^3$. ■

Definition. Let $K \subset S^3$ be an oriented knot. We pick a thickening $F\colon \overline{B}^2 \times K \to S^3$ of self-linking number zero. We refer to $\lambda_K = F(\{1\} \times K)$ as a *longitude* of K.[1375]

Examples.

(1) In Figure 1395 we show the longitude and the meridian for the trivial knot and the figure-8 knot.

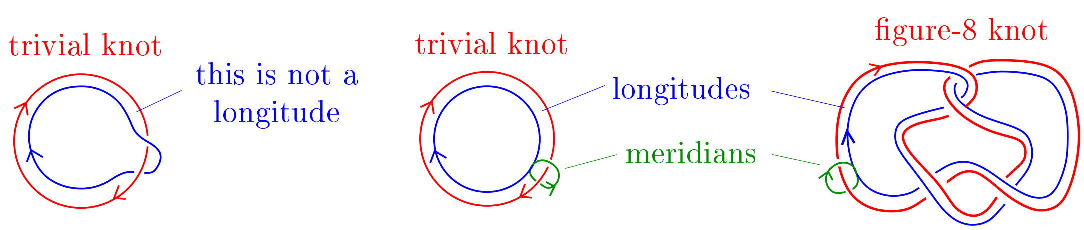


FIGURE 1395

(2) In Figure 1396 we show the trefoil knot with three wannabe longitudes. It is not entirely clear which one is the right longitude. We will soon develop a tool with which it will be straightforward to settle this issue.

On several occasions the following lemma comes in handy.

Lemma 99.2. *Let K be an oriented knot and let $F\colon \overline{B}^2 \times K \to S^3$ be a thickening with self-linking number zero. We denote by $X_K = S^3 \setminus F(B^2 \times K)$ the corresponding knot*

[1375] It follows from Lemma 99.1 (3) that the longitude is in an appropriate way uniquely determined.

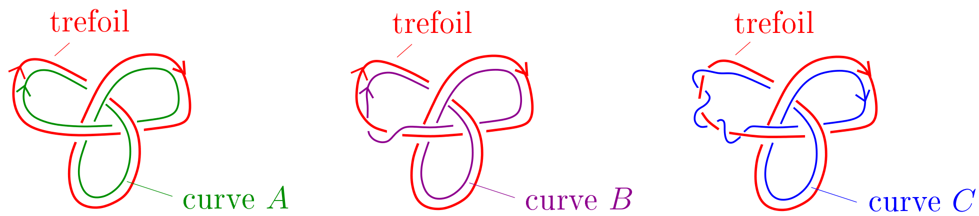


FIGURE 1396

exterior. Furthermore we denote as usual by $\mu_K = F(S^1 \times \{*\})$ and $\lambda_K = F(\{1\} \times K)$ the corresponding meridian and longitude.

(1) *The classes $[\mu_K]$ and $[\lambda_K]$ form a basis for $\mathrm{H}_1(\partial X_K;\mathbb{Z})$.*
(2) *There exists a short exact sequence of the following form*

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}_2(X_K,\partial X_K;\mathbb{Z}) & \to & \overbrace{\mathrm{H}_1(\partial X_K;\mathbb{Z})}^{=\mathbb{Z}\cdot[\mu_K]\oplus\mathbb{Z}\cdot[\lambda_K]} & \to & \overbrace{\mathrm{H}_1(X_K;\mathbb{Z})}^{=\mathbb{Z}\cdot[\mu_K]} & \to & 0 \\ & & & & [\mu_K] & \mapsto & [\mu_K] & & \\ & & & & [\lambda_K] & \mapsto & 0. & & \end{array}$$

(3) *The longitude λ_K is the unique, up to smooth isotopy, closed oriented curve on the torus $F(S^1 \times K)$ that is homologous to $K = \{0\} \times K$ in $F(\overline{B}^2 \times K)$ and that is null-homologous in X_K.*
(4) *If we view μ_K and λ_K as closed oriented curves on the oriented torus $F(S^1 \times K)$, then the algebraic intersection number equals $\mu_K \cdot \lambda_K = +1$.*
(5) *If denote by $i\colon \mu_K \to \partial X_K$ the inclusion map, then*

$$i^*([\mu_K]) \cap [\partial X_K] \;=\; -[\lambda_K] \;\in\; \mathrm{H}_1(\partial X_K;\mathbb{Z}).$$

(6) *Recall that given an oriented curve C we denote by $-C$ the curve with the opposite orientation. With this notation we have $\mu_{-K} = -\mu_K$ and $\lambda_{-K} = -\lambda_K$.*

PROOF.

(1) This statement follows from Lemma 46.12 (2).
(2) We consider the long exact sequence of homology groups of the pair $(X_K, \partial X_K)$:

$$\underbrace{\mathrm{H}_2(X_K;\mathbb{Z})}_{=\,0 \text{ by Lemma 68.16 (4)}} \;\to\; \mathrm{H}_2(X_K,\partial X_K;\mathbb{Z}) \;\to\; \mathrm{H}_1(\partial X_K;\mathbb{Z}) \;\xrightarrow{\phi}\; \mathrm{H}_1(X_K;\mathbb{Z}).$$

By definition we have $\phi([\lambda_K]) = 0$. Furthermore we know from Lemma 68.16 that the right-hand map is an epimorphism. The statement follows immediately from these observations.
(3) It follows from (1) that the kernel of $\mathrm{H}_1(\partial X_K;\mathbb{Z}) \to \mathrm{H}_1(X_K;\mathbb{Z})$ equals $\mathbb{Z}\cdot[\lambda_K]$. Since $\mathrm{H}_1(F(\overline{B}^2 \times K);\mathbb{Z}) = \mathbb{Z}\cdot[K] = \mathbb{Z}\cdot[\lambda_K]$ we see that there exists a unique homology class that has the properties we are interested in. The desired statement now follows from Lemma 98.14 (1).

(4) This statement follows from the definitions and the easy observation that the oriented submanifolds $S^1 \times \{1\}$ and $\{1\} \times S^1$ of the oriented smooth manifold $S^1 \times S^1$ have algebraic intersection number one.

(5) We calculate that

$$\begin{aligned} i^*([\mu_K]) \cap [\partial X_K] &\underset{\substack{\uparrow \\ \text{since } F_*([S^1\times S^1]) = [\partial F(\overline{B}^2 \times K)] = -[\partial X_K]}}{=} i^*([\mu_K]) \cap F_*(-[S^1 \times K]) \underset{\substack{\uparrow \\ \text{Lemma 83.8}}}{=} F_*\big([S^1 \times \{*\}]^* \cap (-[S^1 \times K])\big) \\ &\underset{\substack{\uparrow \\ \text{see page 2022}}}{=} -F_*([\{1\} \times K]) \underset{\substack{\uparrow \\ \text{by definition of } \lambda_K}}{=} -[\lambda_K]. \end{aligned}$$

(6) This statement follows easily from the definitions. We leave it to the reader to provide the details. ■

99.2. **Detecting non-invertible and chiral knots.** Now that we have the notion of the self-linking number of thickening a knot we can also outline how to answer several questions on symmetries of knots we had asked a long time ago.

We start out with the following definition.

Definition. Let K be an oriented knot.

(1) The *mirror* K^{mir} is defined as the reflection of K in any hyperplane of $\mathbb{R}^4$.[1376][1377] We give the mirror K^{mir} the orientation that turns the reflection into an orientation-preserving diffeomorphism $K \to K^{\mathrm{mir}}$.
(2) The *reverse* K^{rev} is defined as the knot K with the reverse orientation.
(3) The *inverse* K^{inv} is defined as the mirror of the reverse of K.

Remark. The names "reverse" and "inverse" at times also get used with different meanings. We follow [**Con70**, p. 336] for our naming convention.

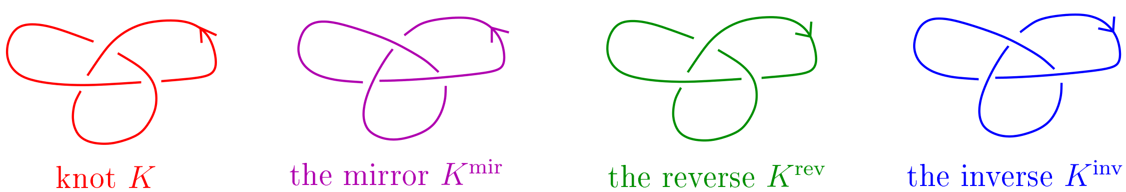


FIGURE 1397

Now the question arises whether an oriented knot is smoothly isotopic to any of its doppelganger. This leads us to the following definition.

Definition. Let K be an oriented knot.

(1) We say that K is *amphichiral* if it is smoothly isotopic to its mirror image. Otherwise we call the knot K *chiral*.[1378]

[1376]Note that it follows from Exercise 27.1 that, up to a smooth isotopy, the definition of K^{mir} does not depend on the choice of the hyperplane.

[1377]Note that on page 1807 we introduced the mirror of an unoriented knot. For the purpose of this section it is better to work with oriented knots.

(2) We say K is *reversible* if it is smoothly isotopic to its reverse.
(3) We say K is *invertible* if it is smoothly isotopic to its inverse.

Example. As we saw in Figure 225, it is basically clear that the trivial knot is amphichiral and reversible. Furthermore, in Figure 504 we showed that the figure-8 knot J is amphichiral. Also note that in Exercise 8.29 we showed that the trefoil and the figure-8 knot are reversible.

This lead us to the following two questions.

Question 8.41. *Is every knot reversible?*

Question 27.9. *Is the trefoil chiral?*

Before we show how to answer both questions let us introduce the following simple-minded definition.

Definition.

(1) A *group system* is a system (G, g_1, g_2) consisting of a group G together with two elements $g_1, g_2 \in G$.
(2) A *isomorphism between group systems* (G, g_1, g_2) *and* (H, h_1, h_2) is a group isomorphism $\varphi\colon G \to H$ with $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$.

The following proposition explains why we care about group systems.

Proposition 99.3.

(1) *Let $K \subset S^3$ be an oriented knot. We pick a thickening $F\colon \overline{B}^2 \times K \to S^3$ of K with self-linking number zero.*[1379] *Furthermore we pick an orientation-preserving diffeomorphism $\gamma\colon S^1 \to K$. We write*

$$x_{K,F} := F(1,\gamma(1)), \qquad \begin{array}{rcl} \mu_{K,F}\colon S^1 & \to & S^3 \setminus K \\ z & \mapsto & F(z,\gamma(1)), \end{array} \qquad \begin{array}{rcl} \lambda_{K,F,\gamma}\colon S^1 & \to & S^3 \setminus K \\ z & \mapsto & F(1,\gamma(z)). \end{array}$$

The isomorphism type of the group system

$$(\pi_1(S^3 \setminus K, x_{K,F}), [\mu_{K,F}], [\lambda_{K,F,\gamma}])$$

depends only on the choice of the oriented knot K. Any group system that is isomorphic to the above group system is called a group system for K.

(2) *Let K and J be two oriented knots. If K and J are smoothly isotopic, then the corresponding group systems are isomorphic.*
(3) *Given an oriented knot K with a group system $(\pi_K, \mu_K, \lambda_K)$ the following hold:*
   (a) *a group system for the mirror K^{mir} is given by* $(\pi_K, \mu_K^{-1}, \lambda_K)$
   (b) *a group system for the reverse K^{rev} is given by* $(\pi_K, \mu_K^{-1}, \lambda_K^{-1})$
   (c) *a group system for the inverse K^{inv} is given by* $(\pi_K, \mu_K, \lambda_K^{-1})$.

Remark. Friedhelm Waldhausen [**Wald68b**, Corollary 6.5] (see also [**BZH14**, Theorem 3.16]) showed in 1968 that the converse to Proposition 99.3 (2) holds. Thus we are certainly on the right track.

[1378] On page 781 we introduced these notions of unoriented knots.

[1379] We know by Lemma 99.1 (2) that such a thickening always exists.

PROOF.

(1) This statement follows easily from Lemma 99.1 (3).
(2) This statement follows equally easily from the Isotopy Extension Theorem 8.27.
(3) Let $F\colon \overline{B}^2 \times K \to S^3$ be a thickening of K with self-linking number zero and let $\gamma\colon S^1 \to K$ be an orientation-preserving diffeomorphism. Throughout the proof of (3) we work with the group system $(\pi_1(S^3 \setminus K, x_{K,F}), [\mu_{K,F}], [\lambda_{K,F,\gamma}])$ for K.
In the following we denote by ρ the reflection in a hyperplane of $\mathbb{R}^4$ and we set $K^{\mathrm{mir}} = \rho(K)$. Finally we denote by $\varphi\colon S^1 \to S^1$ and $\psi\colon \overline{B}^2 \to \overline{B}^2$ the maps given by complex conjugation.
(a) First note that the map[1380]

$$\begin{array}{rcl} F^{\mathrm{mir}}\colon \overline{B}^2 \times K^{\mathrm{mir}} & \to & S^3 \\ (x,y) & \mapsto & \rho(F(\psi(x),\rho(y))) \end{array}$$

is a thickening of K^{mir} with self-linking number zero. Furthermore note that $\gamma^{\mathrm{mir}} := \rho \circ \gamma\colon S^1 \to K^{\mathrm{mir}}$ is orientation-preserving. Thus we see that ρ_* induces an isomorphism between the group system $(\pi_1(S^3 \setminus K, x_{K,F}), [\mu_{K,F}]^{-1}, [\lambda_{K,F,\gamma}])$ and the group system $(\pi_1(S^3 \setminus K^{\mathrm{mir}}), x_{K^{\mathrm{mir}},F^{\mathrm{mir}}}, [\mu_{K^{\mathrm{mir}},F^{\mathrm{mir}}}], [\lambda_{K^{\mathrm{mir}},F^{\mathrm{mir}},\gamma^{\mathrm{mir}}}])$.
(b) First note that the map

$$\begin{array}{rcl} F^{\mathrm{rev}}\colon \overline{B}^2 \times K^{\mathrm{rev}} & \to & S^3 \\ (x,y) & \mapsto & F(\psi(x),y) \end{array}$$

is a thickening of K^{rev} with self-linking number zero. Furthermore note that $\gamma^{\mathrm{rev}} := \gamma \circ \varphi\colon S^1 \to K^{\mathrm{rev}}$ is orientation-preserving. Finally note that the group system $(\pi_1(S^3 \setminus K^{\mathrm{rev}}, x_{K^{\mathrm{rev}},F^{\mathrm{rev}}}), [\mu_{K^{\mathrm{rev}},F^{\mathrm{rev}}}], [\lambda_{K^{\mathrm{rev}},F^{\mathrm{rev}},\gamma^{\mathrm{rev}}}])$ equals the group system $(\pi_1(S^3 \setminus K, x_{K,F}), [\mu_{K,F}^{-1}], [\lambda_{K,F,\gamma}^{-1}])$.
(c) This statement follows immediately from (a) and (b). ■

On page 800 we saw that looking at finite quotients of groups can be a convenient tool for distinguishing groups. We now apply the same idea to group systems. There are various ways of making this idea precise. We will work with the following approach.

Definition. Let G be a *finite* group and let $x \in G$.

(1) Given a group system (π, g_1, g_2) where π is a finitely generated group we define

$$P_G^x(\pi, g_1, g_2) \;:=\; \underbrace{\sum_{\{\varphi\in \mathrm{Hom}(\pi,G)\,|\,\varphi(g_1)=x\}}}_{\text{finite sum by Exercise 21.3}} \varphi(g_2) \;\in\; \underbrace{\mathbb{Z}[G]}_{\text{group ring as defined on page 1355}}.$$

It is clear that isomorphic group systems give the same element in $\mathbb{Z}[G]$.
(2) Let K be a knot. Following [**Eis07**, Definition 1.2] we refer to

$$P_G^x(K) \;:=\; P_G^x(\text{group system of } K) \;\in\; \mathbb{Z}[G],$$

as the (G,x)-*coloring polynomial of* K.

[1380]In particular note that the "extra ψ" ensures this map F^{mir} is indeed orientation-preserving.

The art now is to find, given a knot, a finite group G and element $x \in G$ such that the (G,x)-coloring polynomial has content.

Example. We consider the Kinoshita-Terasaka knot and the Conway knot that are shown in Figure 1398. This pair of knots is notoriously hard to distinguish. Now let G be the Mathieu group M_{11}, this is the unique simple group of order $7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and the smallest of the sporadic simple groups, see [**CCNPW85**] for details. As is explained in [**Eis07**, p. 315], one way of describing $G = M_{11}$ is to note that it is the subgroup of the alternating group A_{11} that is generated by

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 5 & 7 & 10 & 1 & 8 & 6 & 11 & 9 & 4 \end{pmatrix}.$$

Note that x has order 11 in A_{11} and thus also in $G = M_{11}$. By [**Eis07**, p. 317] we have the following table:

$$\begin{array}{lcl} P_G^x(K) & = & 1 + 11x^3 + 11x^7, \\ P_G^x(K^{\text{mir}}) & = & 1 + 11x^4 + 11x^8, \\ P_G^x(K^{\text{rev}}) & = & 1 + 11x^4 + 22x^8, \\ P_G^x(K^{\text{inv}}) & = & 1 + 22x^3 + 11x^7, \end{array} \qquad \begin{array}{lcl} P_G^x(C) & = & 1 + 11x^3 + 11x^7, \\ P_G^x(C^{\text{mir}}) & = & 1 + 11x^4 + 11x^8, \\ P_G^x(C^{\text{rev}}) & = & 1 + 11x^4 + 11x^6 + 11x^8, \\ P_G^x(C^{\text{inv}}) & = & 1 + 11x^3 + 11x^5 + 11x^7. \end{array}$$

Since x has order 11 in G we see that all these elements of $\mathbb{Z}[G]$ are different. This shows that these eight oriented knots are pairwise non-smoothly isotopic. In particular this shows that the Kinoshita-Terasaka knot and the Conway knot are neither amphichiral, nor reversible nor invertible.

Kinoshita-Terasaka knot — Conway knot

FIGURE 1398

For the record let us formulate the following proposition.

Proposition 99.4. *There exists an oriented knot that is neither amphichiral, nor reversible nor invertible.*

We can now finally answer Question 27.9 in the affirmative.

Proposition 99.5. *The trefoil is chiral.*

PROOF. We equip the trefoil K with either orientation. In Proposition 27.6 we showed that there exists an epimorphism $\pi_1(S^3 \setminus K) \to S_3$. For a suitable choice of $x \in S_3$ one sees, using Proposition 99.3 and using the knot coloring polynomial $P^x_{S_3}$ that K is neither smoothly isotopic to it mirror nor to its inverse. Thus the unoriented trefoil is chiral. We will work out the details in Exercise 99.4. ■

99.3. **Seifert surfaces.** The following is the key object in this section.

Definition. Let $K \subset S^3$ be a knot. A *Seifert surface*[1381] *for* K is a compact orientable connected 2-dimensional submanifold F of S^3 such that $\partial F = K$. If K itself is oriented, then we equip a Seifert surface F with the orientation such that $\partial F = K$ as oriented smooth manifolds.

Examples.

(1) In Figure 1399 we show Seifert surfaces for the trivial knot, the trefoil and the figure-8 knot.

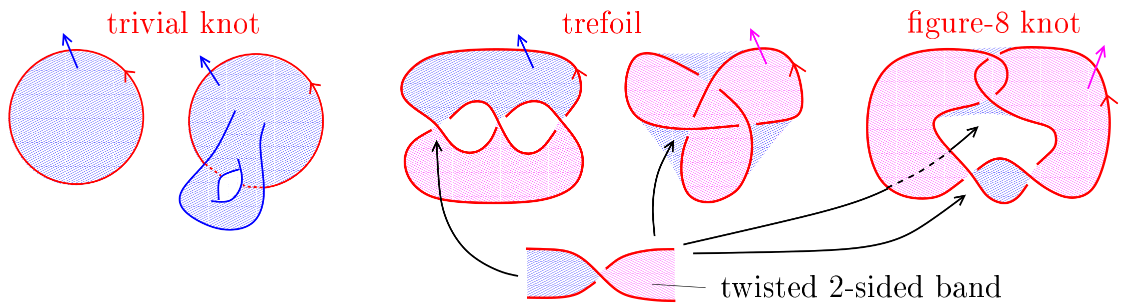


FIGURE 1399

(2) In Figure 1400 we see two surfaces of minimal surface area that bound the trefoil, both were manufactured by Jürgen Neukirch. The one on the left is orientable, thus it is a Seifert surface. The one on the right is diffeomorphic to the Möbius band, thus it is not a Seifert surface.

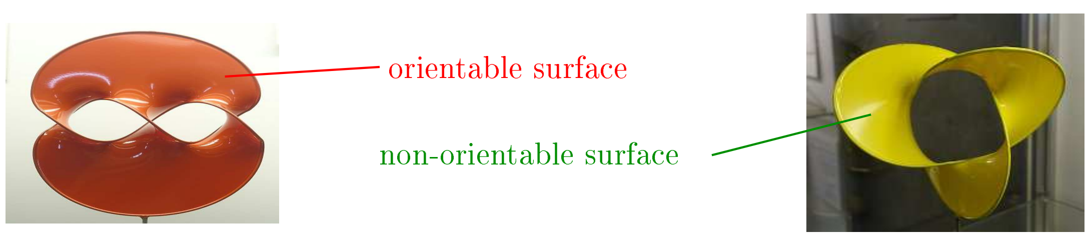


FIGURE 1400

The following proposition is arguably one of the most useful results in knot theory.

Proposition 99.6. *Every knot* $K \subset S^3$ *admits a Seifert surface.*

Remark.

(1) The proposition was first proved in 1930 by F. Frankl and Lev Pontryagin [**FPo30**] in 1930. Shortly afterwards Herbert Seifert [**Sei34**] gave a different proof. More precisely Seifert found an algorithm that given a diagram for a knot produces an explicit Seifert surface for the knot. Seifert's algorithm is also described in most books on knot theory, see e.g. [**Rolf90**, p. 120] or [**Lic97a**, Theorem 2.2].

(2) A slight generalization of the argument we give proves the following stronger statement: given any closed orientable n-dimensional smooth manifold $V \subset S^{n+2}$ there

[1381] Seifert surfaces are named after Herbert Seifert (1907-1996) whom we had already encountered at the Mayer–Vietoris Theorem 20.2.

exists a compact orientable $(n+1)$-dimensional submanifold $W \subset S^{n+2}$ such that $\partial W = V$. We refer to [**Bre93**, Theorem VII.14.7], [**Er69**, Lemma 2.2] and [**BDR17**] for details. The proof in these two references is close to the approach taken in the proof of Proposition 99.6, but it circumvents the use of Proposition 98.4 by a more formal argument.

PROOF. Let K be a knot. We need to show that there exists a compact orientable connected 2-dimensional submanifold F of S^3 such that $\partial F = K$.

First note that by Lemma 99.1 (2) we can pick a thickening $\overline{B}^2 \times K$ of the knot K with self-linking number zero. We denote by $X := S^3 \setminus (B^2 \times K)$ the knot exterior. Note that by Lemma 99.2 (2) we know that there exists a unique homology class $\sigma \in H_2(X, \partial X; \mathbb{Z})$ such that $\partial\sigma = [\{1\} \times K] \in H_1(\partial X; \mathbb{Z})$. It follows from Proposition 98.6 that there exists a compact oriented proper 2-dimensional submanifold Σ of X such that $[\Sigma] = \sigma$ and such that $\partial\Sigma = \{1\} \times K$. We apply Proposition 8.22 to

$$([0,1] \times K) \cup \Sigma \;\subset\; (\overline{B}^2 \times K) \cup X \;=\; S^3$$

and we obtain a compact orientable 2-dimensional submanifold Z whose boundary equals K. This submanifold might not be connected, but the component of Z that contains K is the desired Seifert surface. ■

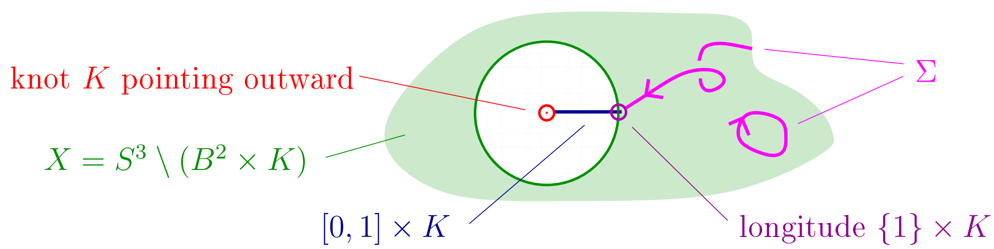


FIGURE 1401. Illustration for the proof of Proposition 99.6.

Definition. Let F be a compact orientable connected 2-dimensional smooth manifold with m boundary components. By the Surface Classification Theorem 23.4 there exists a unique $g \in \mathbb{N}_0$ such that F is homeomorphic to the surface of genus g minus m open disks. We refer to g as the *genus of F*.

Given a Seifert surface it is sometimes not entirely obvious how to determine the genus. The following lemma gives an easy criterion to determine the genus of a Seifert surface.

Lemma 99.7. *Let F be a compact orientable connected 2-dimensional submanifold with one boundary component. If $\Gamma \subset F$ is a topological graph that is a deformation retract of F, then*

$$\textit{genus of } F \;=\; \tfrac{1}{2}(1-\chi(F)) \;=\; \tfrac{1}{2}(1 - \#\textit{vertices of } \Gamma + \#\textit{edges of } \Gamma).$$

PROOF. We have

$$\text{genus of } F \underset{\substack{\uparrow \\ \text{by Proposition 48.9 we} \\ \text{have } \chi(F) = 1-2g}}{=} \tfrac{1}{2}(1-\chi(F)) \underset{\substack{\uparrow \\ \text{by Corollary 55.2 since } \Gamma \text{ is a} \\ \text{deformation retract of } F}}{=} \tfrac{1}{2}(1-\chi(\Gamma)).$$

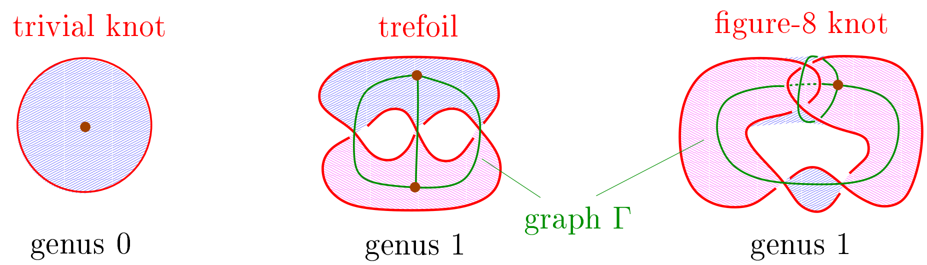


FIGURE 1402

■

If a knot K bounds a Seifert surface of genus g, then the construction sketched in Figure 1403 shows pretty convincingly that K also bounds Seifert surfaces of any genus $h \geq g$.

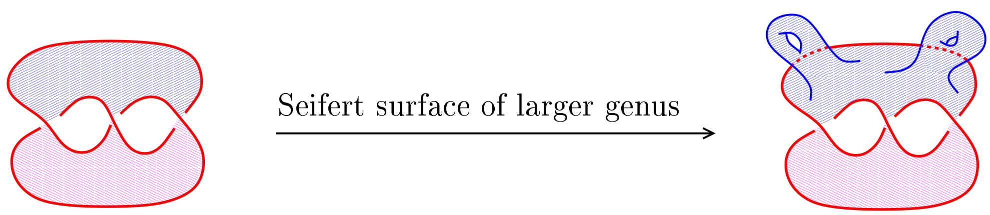


FIGURE 1403

So the interesting question is, what is the minimal genus of a Seifert surface for a given knot? This leads us to the following definition.

Definition. Given a knot $K \subset S^3$ we define

$$\operatorname{genus}(K) := \text{minimal genus of a Seifert surface of } K.$$

Examples.

(1) As we have seen in Figure 1402, the trivial knot has genus 0. More interestingly, we showed in Proposition 8.47 that if a knot has genus 0, then it is smoothly isotopic to the trivial knot.

(2) We had also just seen that the trefoil admits a Seifert surface of genus 1. Furthermore in Proposition 27.6 (2) we saw that the trefoil is not equivalent to the trivial knot. It follows from (1) that

$$\operatorname{genus}(\text{trefoil}) = 1.$$

We conclude this section with the following question.

Question 99.8. *We consider the knot K in Figure 1404. It admits a Seifert surface of genus two. Does it also admit a Seifert surface of genus one?*

99.4. **Linking numbers.** We recall the following definition from page 2233.

Definition. An *m-component link* is a closed 1-dimensional submanifold of S^3.

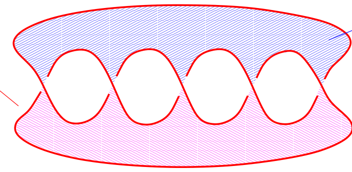


FIGURE 1404

Remark. It follows from Theorem 7.5 that a link is the same as a 1-dimensional submanifold of S^3 that is diffeomorphic to the disjoint union of finitely many copies of S^1.

Convention. When we write, "let $L = L_1 \sqcup \dots \sqcup L_m$ be an *m-component link*", then it is understood that each L_i is non-empty. In particular each L_i is a knot in its own right.

The following definition is the obvious generalization of the definition on page 386.

Definition. We say that *two m-component links $L = L_1 \sqcup \dots \sqcup L_m$ and $J = J_1 \sqcup \dots \sqcup J_m$ are smoothly isotopic* if there exists a smooth map

$$\begin{array}{rcl} F\colon (S^1_1 \sqcup \dots \sqcup S^1_m) \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & F(z,t) \end{array}$$

such that the following hold:

(1) for each $i \in \{1,\dots,m\}$ we have $F(S^1_i \times \{0\}) = K_i$ and $F(S^1_i \times \{1\}) = J_i$, and
(2) for each $t \in [0,1]$ the map $F_t\colon S^1 \sqcup \dots \sqcup S^1 \to S^3$ is a smooth embedding.

If L and J are oriented links, then we also demand that the isotopy is orientation-preserving, i.e. the map $F_1 \circ F_0^{-1}\colon K \to J$ is orientation-preserving

Examples.

(1) We say $L = L_1 \sqcup \dots \sqcup L_m \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ is a *trivial link* if it lies in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ and[1382] if it bounds m disjoint disks in $\mathbb{R}^2 \times \{0\}$. Using Theorem 8.36 one can easily verify that all trivial links are in fact smoothly isotopic.
(2) In Figure 1405 we show four examples of oriented links: the 2-component trivial link, two versions $\mathrm{H}^{\pm}$ of the Hopf link and one further link. The Hopf link is in fact related to the Hopf map

$$\begin{array}{rcl} H\colon S^3 = \{(z_1,z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 = \mathbb{C}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1 : z_2] \end{array}$$

from page 1071. More precisely, if P, Q are two distinct points in S^2, then the preimage $\mathrm{H}^{-1}(\{P,Q\})$ is a 2-component link that is smoothly isotopic to the Hopf links without orientation. This can be seen as follows: for $P = [1:0]$ and $Q = [0:1]$ one can verify "by hand" that the preimage is smoothly isotopic to a Hopf link. Given any two other distinct points P' and Q' in S^2 we can find disjoint smooth paths from P to P' and Q to Q', the preimages of these paths give rise to a smooth isotopy of links.

[1382] Is the second condition necessary?

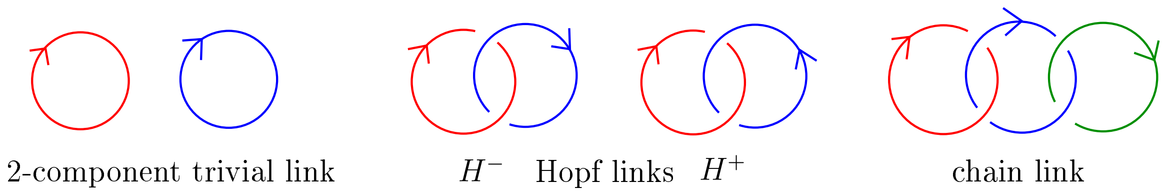


FIGURE 1405

Our goal now is to distinguish links that are not smoothly isotopic. It follows immediately from the definition that if two links $L = L_1 \sqcup \cdots \sqcup L_m$ and $J = J_1 \sqcup \cdots \sqcup J_m$ are smoothly isotopic, then the components are smoothly isotopic knots. Therefore we are mostly interested in distinguishing links for which the components are smoothly isotopic.

The following proposition generalizes Proposition 8.43. The proof is verbatim the same.

Proposition 99.9. *Let $L = L_1 \sqcup \cdots \sqcup L_m$ and $J = J_1 \sqcup \cdots \sqcup J_m$ be two m-component links. If the links are smoothly isotopic, then the following two statements hold:*

(1) *There exists an orientation-preserving diffeomorphism $\Phi\colon S^3 \to S^3$ with $\Phi(L_i) = J_i$ for $i = 1, \ldots, m$. If L and J are oriented, then there exists such a diffeomorphism Φ which for each $i \in \{1, \ldots, m\}$ restricts to an orientation-preserving diffeomorphism $L_i \to J_i$.*
(2) *The link complements $S^3 \setminus L$ and $S^3 \setminus J$ are diffeomorphic, in particular homeomorphic.*

Remark. By the Gordon-Luecke Theorem 8.46 we pointed out that for knots the converse to Proposition 8.43 (2) holds. More precisely, if K and J are two knots such that there exists an orientation-preserving diffeomorphism between the knot complements $S^3 \setminus K$ and $S^3 \setminus J$, then K and J are smoothly isotopic. The analogous statement is no longer true for links. In Figure 1406 we show two links $L = K \sqcup J$ and $L' = K' \sqcup J'$. In Exercise 99.7 we will see that $S^3 \setminus L$ and $S^3 \setminus L'$ are diffeomorphic. Even though we cannot prove it at the moment, it should be fairly clear that L and L' are not smoothly isotopic. This shows in particular that the problem of classifying links cannot be reduced to the problem of classifying the complements of links.

FIGURE 1406

To distinguish links with components that are smoothly isotopic we will now introduce an invariant of oriented links.

Definition. Let $K \subset S^3$ be an oriented knot. As before we denote by μ_K the meridian as defined on page 390 and as above we denote by ϕ_K the epimorphism $\phi_K\colon\ \mathrm{H}_1(S^3 \setminus K; \mathbb{Z}) \to \mathbb{Z}$

that is given by $\phi_K([\mu_K]) = 1$. If $J \subset S^3$ is an oriented knot that is disjoint from K, then we define the *linking number of K and J* to be

$$\operatorname{lk}(K, J) \ := \ \phi_K([J])$$

where $[J] \in \mathrm{H}_1(S^3 \setminus K; \mathbb{Z})$ denotes the image of the fundamental class of the oriented smooth manifold J under the inclusion $J \to S^3 \setminus K$.

Remark. We continue with the above notation. If we denote by $[\mu_K^*] \in \mathrm{H}^1(S^3 \setminus K; \mathbb{Z})$ the unique generator with $\langle [\mu_K^*], [\mu_K] \rangle = 1$, then it follows immediately from the definitions that for any oriented knot J disjoint from K we have

$$\operatorname{lk}(K, J) \ = \ \langle [\mu_K^*], [J] \rangle.$$

Example. If K is the trivial knot, then we saw in Proposition 27.4 that $S^3 \setminus K$ is diffeomorphic to the open solid torus $B^2 \times S^1$ and it is fairly straightforward to determine $\phi_K \colon \mathrm{H}_1(S^3 \setminus K; \mathbb{Z}) \to \mathbb{Z}$ applied to a given curve. This allows us to determine the linking numbers of the oriented links shown in Figure 1407.

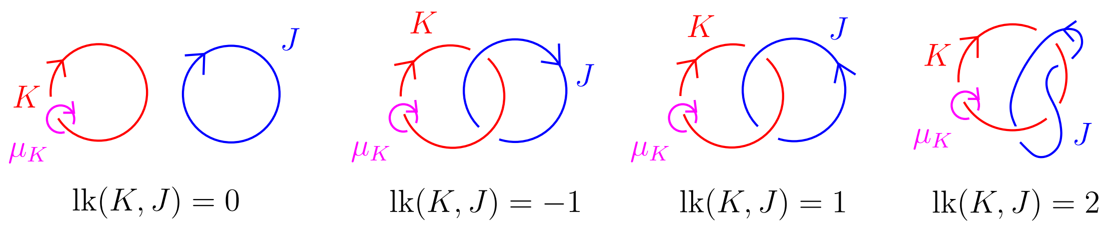


FIGURE 1407

Proposition 99.10.

(1) *Let $L = L_1 \sqcup \cdots \sqcup L_m$ and $J = J_1 \sqcup \cdots \sqcup J_m$ be two m-component oriented links. If the links L and J are smoothly isotopic, then for all $r, s \in \{1, \ldots, m\}$ we have*

$$\operatorname{lk}(L_r, L_s) \ = \ \operatorname{lk}(J_r, J_s).$$

(2) *Let $K \sqcup J$ be an oriented 2-component link. If we change the orientation of one of the components of the link, then the sign of the linking number flips.*

PROOF. The first statement is an immediate consequence of Proposition 99.9 (1). The second statement follows easily from the definitions. ■

Example. It follows from Proposition 99.10 together with the calculation of the linking numbers in Figure 1407 that the two Hopf links H^- and H^+ are not smoothly isotopic.

The following proposition says that the linking number $\operatorname{lk}(K, J)$ is symmetric in the oriented knots K and J.

Proposition 99.11. *For any oriented 2-component link $J \sqcup K$ we have*

$$\operatorname{lk}(J, K) \ = \ \operatorname{lk}(K, J).$$

Proof (∗). We pick a thickening for the link $L = J \sqcup K$ and use it to define the exteriors X_L, X_K and X_J. Throughout the proof we consider the following inclusion maps

$$\begin{tikzcd}
\partial X_K \arrow[dr, hook, "f"] & & X_J \\
& X_L \arrow[ur, hook, "j"] \arrow[dr, hook, "k"'] & \\
\partial X_J \arrow[ur, hook, "g"'] & & X_K.
\end{tikzcd}$$

Furthermore, throughout the proof we use the isomorphisms from Proposition 41.5 and page 1088 to make the identifications $\mathrm{H}_0(\partial X_J;\mathbb{Z}) = \mathrm{H}_0(X_L;\mathbb{Z}) = \mathrm{H}_0(X_J;\mathbb{Z}) = \mathbb{Z}$. We start out with the following claim.

Claim. We have

$$\mathrm{lk}(K,J) \;=\; \big(\underbrace{k^*([\mu_K^*])}_{\in \mathrm{H}^1(X_L;\mathbb{Z})} \cup \underbrace{j^*([\mu_J^*])}_{\in \mathrm{H}^1(X_L;\mathbb{Z})}\big) \cap \underbrace{g_*([\partial X_J])}_{\in \mathrm{H}_2(X_L;\mathbb{Z})}.$$

We denote by $[J] \in \mathrm{H}_1(J;\mathbb{Z})$ the fundamental class. We have the following equalities:

$$\begin{array}{l}
(k^*([\mu_K^*]) \cup j^*([\mu_J^*])) \cap g_*([\partial X_J]) \overset{\substack{\text{Proposition 82.3}\\ \downarrow}}{=} -(j^*([\mu_J^*]) \cup k^*([\mu_K^*])) \cap g_*([\partial X_J]) \\
\underset{\substack{\uparrow\\ \text{Lemma 83.7}}}{=} -k^*([\mu_K^*]) \cap (j^*([\mu_J^*]) \cap g_*([\partial X_J])) \underset{\substack{\uparrow\\ \text{Lemma 81.10 applied to } g}}{=} -k^*([\mu_K^*]) \cap g_*(g^* j^* [\mu_J^*] \cap [\partial X_J]) \\
\underset{\substack{\uparrow\\ \text{Lemma 99.2 (3)}}}{=} k^*([\mu_K^*]) \cap g_*([\lambda_J]) \underset{\substack{\uparrow\\ \text{Lemmas 74.6 (3) and 83.4}}}{=} \langle [\mu_K^*], k_* g_*([\lambda_J])\rangle \underset{\substack{\uparrow\\ \text{since } [\lambda_J] = [J] \in \mathrm{H}_1(X_K)}}{=} \langle [\mu_K^*], k_* g_*([J])\rangle \underset{\substack{\uparrow\\ \text{see page 2396}}}{=} \mathrm{lk}(K,J).
\end{array}$$

⊞

Now we conclude that

$$\begin{array}{l}
\mathrm{lk}(K,J) \underset{\substack{\uparrow\\ \text{by the claim}}}{=} (j^*([\mu_J^*]) \cup k^*([\mu_K^*])) \cap g_*([\partial X_J]) \underset{\substack{\uparrow\\ \text{by Proposition 82.3 and since by Corollary 87.28}\\ \text{we have } g_*([\partial X_J]) + f_*([\partial X_K]) = 0 \in \mathrm{H}_2(X_L;\mathbb{Z})}}{=} (-k^*([\mu_K^*]) \cup j^*([\mu_J^*])) \cap (-f_*([\partial X_K])) \\
\underset{\substack{\uparrow\\ \text{by the claim}}}{=} \mathrm{lk}(J,K).
\end{array}$$

■

To determine the linking number in more complicated cases it is helpful to have different ways to calculate the linking number.

Proposition 99.12. *Let $K \sqcup J$ be an oriented 2-component link. If F is a Seifert surface for K that is transverse to J, then*

$$\mathrm{lk}(K,J) \;=\; F \cdot J$$

where $F \cdot J$ denotes the algebraic intersection number of F and J.

Example. In Figure 1408 we show an oriented link $K \sqcup J$ and a Seifert surface F for K. Clearly we have $F \cdot J = 0$, thus it follows from Proposition 99.12 that $\mathrm{lk}(K,J) = 0$.

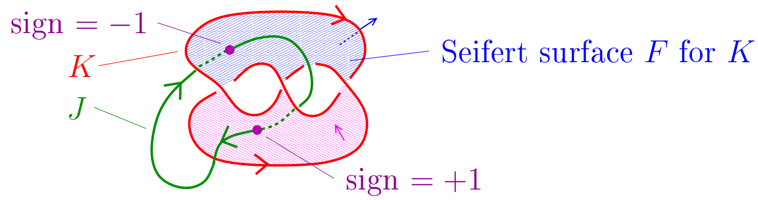


FIGURE 1408

Remark. The combination of Proposition **??** and Theorem 94.5, and some mild massaging, generalizes the statement of Proposition 99.12 to the higher-dimensional setting and it gives an alternative proof of Proposition 99.12.

PROOF. Let $K \sqcup J$ be an oriented 2-component link. We pick a tubular neighborhood $\overline{B}^2 \times K$ for K. After possibly shrinking the tubular neighborhood we can and will assume that $S^1 \times K$ intersects F transversally in a single curve[1383] and that $\overline{B}^2 \times K$ does not intersect J. We write $X := S^3 \setminus (B^2 \times K)$ and we write $F' := F \cap X$. It follows from our choice of tubular neighborhood and from Lemma 6.52 that F' is a compact proper submanifold of F.

We start out with the following claim.

Claim. We have $\mathrm{PD}_X([F']) = [\mu_K^*] \in \mathrm{H}^1(X;\mathbb{Z})$.

By definition of the Poincaré duality isomorphism PD_X the claim is equivalent to showing that $[\mu_K^*] \cap [X] = [F']$. From Lemma 99.2 (2) we know that the connecting homomorphism $\partial\colon \mathrm{H}_2(X, \partial X;\mathbb{Z}) \to \mathrm{H}_1(\partial X;\mathbb{Z})$ is a monomorphism. Therefore it suffices to show that $\partial([\mu_K^*] \cap [X]) = \partial[F']$. We denote by $i\colon \partial X \to X$ the inclusion map. We calculate that

$$\partial([\mu_K^*]\cap[X]) \underset{\substack{\uparrow\\ \text{Proposition 88.26}}}{=} -i^*([\mu_K^*])\cap[\partial X] \underset{\substack{\uparrow\\ \text{Lemma 99.2 (5)}}}{=} [\lambda_K] \underset{\substack{\uparrow\\ \text{Lemma 99.2 (4)}}}{=} [\partial F'] \underset{\substack{\uparrow\\ \text{Proposition 87.27}}}{=} \partial([F']). \qquad \boxplus$$

Now we calculate that

$$\begin{aligned} F\cdot J \overset{\substack{\text{Lemma 94.4}\\ \downarrow}}{=} J\cdot F \overset{\substack{\text{since } J\subset X\\ \downarrow}}{=} J\cdot F' &\overset{\substack{\text{Theorem 94.5}\\ \downarrow}}{=} (\mathrm{PD}_X([J])\cup \mathrm{PD}_X([F']))\cap[X] \overset{\substack{\text{Lemma 83.7}\\ \downarrow}}{=} \mathrm{PD}_X([F'])\cap(\mathrm{PD}_X([J])\cap[X]) \\ &\underset{\substack{\uparrow\\ \text{by the claim we have } \mathrm{PD}_X([F'])=[\mu_K^*] \text{ and}\\ \text{by definition we have } \mathrm{PD}_X([J])\cap[X]=[J]}}{=} [\mu_K^*]\cap[J] = \langle[\mu_K^*],[J]\rangle \underset{\substack{\uparrow\\ \text{see page 2396}}}{=} \mathrm{lk}(K,J). \end{aligned}$$

We have thus proved the desired equality. ■

Next we will formulate a proposition that in many "real-life" situations is often the most efficient way to compute linking numbers. Before we formulate the proposition, recall that on page 787 we introduced the notion of a knot diagram and the notion of an over- and undercrossing. These notion can be generalized in a pretty obvious way to the notion of a link diagram.

[1383]This last statement requires a little bit of thought. We leave the technical details to the reader.

To enhance readability we will formulate the next proposition in a slightly informal way.

Proposition 99.13. *Let $K \sqcup J \subset S^3$ be an oriented 2-component link that is given by a diagram in the xy-plane. Then*

$$\operatorname{lk}(K,J) \;=\; \begin{array}{rl} & \frac{1}{2}\cdot \textit{number of positive crossings between } K \textit{ and } J \\ - & \frac{1}{2}\cdot \textit{number of negative crossings between } K \textit{ and } J \end{array}$$

where a crossing P is called a positive (respectively negative) crossing if the direction of the upper strand together with the direction of the lower strand give a positive (respectively negative) basis for $\mathbb{R}^2$. We refer to Figure 1409 for an illustration.

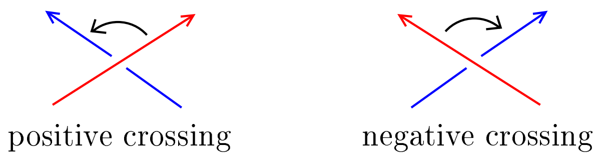


FIGURE 1409. Illustration for Proposition 99.13.

Examples.

(1) In Figure 1410 we show two oriented links and we compute the linking number using Proposition 99.13 by considering the signs of the crossings between K and J. As a reality check, on the left we consider the Hopf link that we had already considered in Figure 1407. In both cases we obtain that the linking number equals -1.

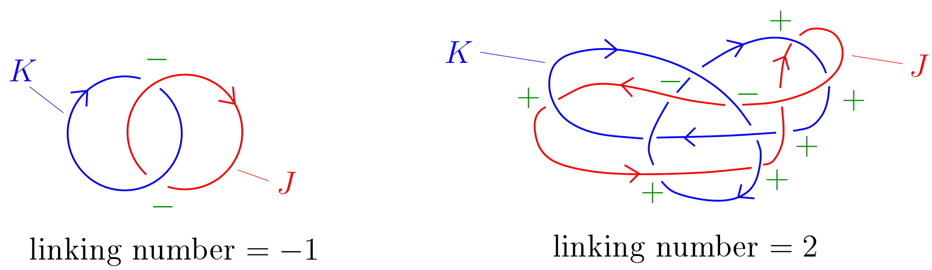


FIGURE 1410

(2) Now we can settle the question from page 2385, which of the three closed curves shown in Figure 1411 is actually the longitude of the trefoil. More precisely, by definition the longitude λ_K satisfies $\phi_K([\lambda_K]) = 0$, but again by definition this means that $\operatorname{lk}(K,\lambda_K) = 0$. Conversely, loosely speaking any curve that runs "parallel to K" and that has linking number zero with K, is a longitude. Using Proposition 99.13 we calculate the linking numbers of the oriented knot K with the oriented curves A, B and C. We obtain that C is the only curve with linking number zero. Thus C is the longitude.

Remark. Let $K \sqcup J \subset S^3$ be an oriented 2-component link that is given by a diagram in the xy-plane. An amusing exercise shows that

$$\operatorname{lk}(K,J) \;=\; \begin{array}{rl} & \text{number of positive crossings where } K \text{ crosses over } J \\ - & \text{number of negative crossings where } K \text{ crosses over } J. \end{array}$$

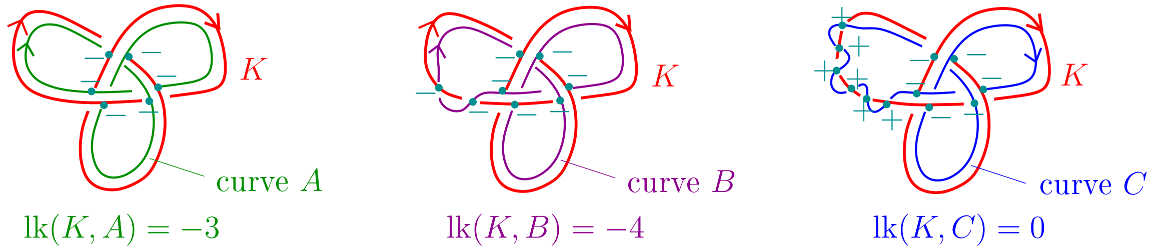


FIGURE 1411

SKETCH OF PROOF. [1384] We introduce the following notation:

(1) Given $t \in \mathbb{R}$ we consider the map
$$\begin{array}{rcl}\lambda_t\colon \mathbb{R}^3 & \to & \mathbb{R}^3 \\ (x,y,z) & \mapsto & (x,y,z+t).\end{array}$$
We extend this definition to a homeomorphism of S^3 by setting $\lambda_t(\infty)=\infty$.[1385]

(2) By Proposition 99.6 the knot J admits a Seifert surface F.

(3) We pick $r \in \mathbb{R}$ such that the minimal z-coordinate of any point on $\lambda_r(F)$ is greater than the maximal z-coordinate of any point in K.

(4) We write $F' = ((J \times [0,1]) \cup F)/\sim$ where we identify any point in $J \subset F$ with the corresponding point in $J \times \{1\}$. Note that F' is a compact 2-dimensional smooth manifold that inherits an orientation from F.

(5) We consider the map[1386]
$$\begin{array}{rcl}\Psi\colon F' = ((J\times[0,1])\cup F)/\sim & \to & \mathbb{R}^3 \\ Q & \mapsto & \begin{cases} \lambda_{t\cdot r}(P) & \text{if } Q=(P,t)\in J\times[0,1], \\ \lambda_r(Q), & \text{if } Q \in F.\end{cases}\end{array}$$

Then
$$\mathrm{lk}(K,J) \underset{\substack{\uparrow \\ \text{by the same argument} \\ \text{as in Proposition 99.12}}}{=} K\cdot\Psi \underset{\substack{\uparrow \\ \text{by choice of } r \text{ we have} \\ K\cap\Psi(F)=\varnothing}}{=} K\cdot \Psi|_{J\times[0,1]}.$$

The right-hand side can be computed easily, in fact the points in $K\cap\Psi(J\times[0,1])$ correspond precisely to the crossings of the diagrams where K crosses over J and one can easily convince oneself that the signs agree. The proposition now follows from the reformulation of the linking number in the remark preceding the proof. ■

In the following lemma we give a 4-dimensional interpretation of the linking number.

[1384] A proof can evidently only be as precise as the input. Since we did not bother to completely rigorously formulate the statement we can also not expect a completely rigorous proof. But with some effort one can formulate both the statement and the proof rigorously.

[1385] Why is this map continuous?

[1386] Note that the map $\Psi\colon F' \to S^3$ is not necessarily injective. In particular the image is not necessarily a Seifert surface for J.

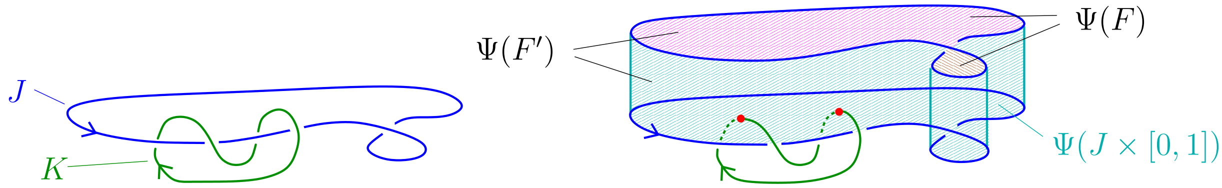


FIGURE 1412. Illustration of the proof of Proposition 99.13.

Lemma 99.14. *Let K, K' be two disjoint oriented knots.*

(1) *There exist oriented proper 2-dimensional submanifolds S and S' of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$ such that S and S' are transverse.*
(2) *Let S, S' be two oriented proper 2-dimensional submanifolds of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$. If S and S' are transverse, then*

$$\operatorname{lk}(K, K') \; = \; S \cdot_{\overline{B}^4} S'.$$

Remark. The combination of Proposition **??** and Theorem 94.5 generalizes the statement of Lemma 99.14 to the higher-dimensional setting and it gives an arguably mathematically purer proof of Lemma 99.14.

SKETCH OF PROOF. Let K, K' be two disjoint oriented knots.

(1) By Proposition 99.6 and the Transversality Theorem 9.10 (2) there exists a Seifert surface F for K that is transverse to K'. Furthermore let F' be a Seifert surface for K'. We pick a collar neighborhood $[0,2] \times S^3$ for $S^3 = \partial \overline{B}^4$. We use the collar neighborhood $[0,1] \times S^3$ to define the push-in[1387] $\widetilde{F}$ of F and we use the collar neighborhood $[0,2] \times S^3$ to define the push-in $\widetilde{F}'$ of F'. It follows easily from Lemma 8.23 that $\widetilde{F}$ and $\widetilde{F}'$ have the desired properties.
(2) First we note that

$$\operatorname{lk}(K, K') \underset{\substack{\uparrow \\ \text{Proposition 99.12}}}{=} F \cdot_{S^3} K' \underset{\substack{\uparrow \\ \text{by inspection of the} \\ \text{intersection points}}}{=} \widetilde{F} \cdot_{\overline{B}^4} \widetilde{F}' \underset{\substack{\uparrow \\ \text{follows easily from the definitions}}}{=} -(-\widetilde{F}) \cdot_{-\overline{B}^4} (-\widetilde{F}').$$

Now let S, S' be two oriented proper 2-dimensional submanifolds of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$. We think of $S^4 = \overline{B}^4 \cup (-\overline{B}^4)$ as the union of two copies of $\overline{B}^4$ with opposite orientations, the first copy contains S, S' and the second contains $-\widetilde{F}, -\widetilde{F}'$. (See Figure 1413 for an illustration.) It follows fairly easily from the Collar Neighborhood Theorem 8.20 that $S \cup -\widetilde{F}$ and $S' \cup -\widetilde{F}'$ are oriented submanifolds of

[1387] We refer to page 367 for the definition of push-ins.

S^4.[1388] We calculate that

$$S \cdot_{\overline{B}^4} S' - \operatorname{lk}(K,K') \underset{\substack{\uparrow \\ \text{by the above}}}{=} S \cdot_{\overline{B}^4} S' + (-\widetilde{F}) \cdot_{-\overline{B}^4} (-\widetilde{F}') \underset{\substack{\uparrow \\ \text{since } S\cap \widetilde{F}' = S'\cap \widetilde{F} = \varnothing}}{=} (S \cup -\widetilde{F}) \cdot_{S^4} (S' \cup -\widetilde{F}') \underset{\substack{\uparrow \\ \text{by Theorem 95.7 and} \\ \text{since } \mathrm{H}^2(S^4;\mathbb{Z})=0}}{=} 0.$$

We have thus shown the desired equality $S \cdot_{\overline{B}^4} S' = \operatorname{lk}(K,K')$. ■

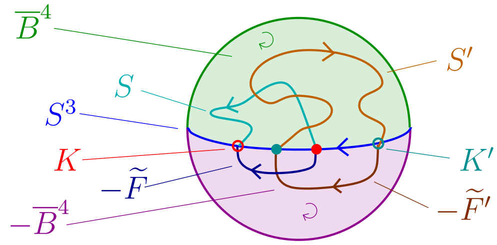


FIGURE 1413. Illustration for the proof of Lemma 99.14.

The following proposition gives two more alternative ways to define or calculate or interpret the linking number.

Proposition 99.15. *Let $K \sqcup J \subset \mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be an oriented 2-component link.*

(1) *If $\varphi\colon S^1 \to K$ and $\psi\colon S^1 \to J$ are two orientation-preserving diffeomorphisms, then*[1389]
$$\operatorname{lk}(K,J) \;=\; \deg \begin{pmatrix} S^1 \times S^1 & \to & S^2 \\ (z,w) & \mapsto & \frac{\varphi(z)-\psi(w)}{\|\varphi(z)-\psi(w)\|} \end{pmatrix}.$$

(2) *Let $f\colon [0,2\pi] \to K$ and $g\colon [0,2\pi] \to J$ be two maps such that f and g descend to orientation-preserving diffeomorphisms from $[0,2\pi]/0 \sim 2\pi$ to K and J. Then*
$$\operatorname{lk}(K,J) \;=\; \frac{1}{4\pi} \int_{s=0}^{s=2\pi} \int_{t=0}^{t=2\pi} \frac{\det \overbrace{\big((f(s)-g(t)) \;\; f'(s) \;\; g'(t)\big)}^{3\times 3\text{–matrix}}}{\|f(s)-g(t)\|^3}\, dt\, ds.$$

PROOF.

(1) We will prove this statement in Exercise 99.14. Alternatively see [**Rolf90**, p. 133].
(2) In Exercise 99.15 we will see that this statement can be deduced from Statement (1). Alternatively see [**RN11**, Section 5], [**Rolf90**, p. 133] and [**tD00**, p. 284]. ■

Remark.

(1) The linking number of an oriented 2-component link was initially introduced by Gauss in 1833 by the integral in Proposition 99.15. We refer to Figure 1414 for a scan of the relevant paragraphs in [**Gaus1877**, p. 605]. The term *geometria situs* is hereby

[1388]The word "submanifold of S^4" is used in a slightly ambiguous way since it depends on whether we view S^4 as a topological manifold or a smooth manifold. In this case we view S^4 only as a topological manifold.

[1389]Here we use the notion of "degree" that we introduced on page 1737.

the earlier name for topology. We refer to [**Epp99**, p. 73] for more information on the work of Gauss regarding the linking number.

(2) It is not immediately clear why the integral in Proposition 99.15 is an invariant of the equivalence class of the link $K \sqcup J$. How would you prove this statement, without using Proposition 99.15 (2)?
(3) It is clear from the second reformulation of the linking number given in Proposition 99.15 that the linking number is symmetric. How does the symmetry follow from the first reformulation?

Von der *Geometria Situs*, die LEIBNITZ ahnte und in die nur einem Paar Geometern (EULER und VANDERMONDE) einen schwachen Blick zu thun vergönnt war, wissen und haben wir nach anderthalbhundert Jahren noch nicht viel mehr wie nichts.

Eine Hauptaufgabe aus dem *Grenzgebiet* der *Geometria Situs* und der *Geometria Magnitudinis* wird die sein, die Umschlingungen zweier geschlossener oder unendlicher Linien zu zählen.

Es seien die Coordinaten eines unbestimmten Punkts der ersten Linie x, y, z; der zweiten x', y', z' und

$$\iint \frac{(x'-x)(\mathrm{d}y\,\mathrm{d}z'-\mathrm{d}z\,\mathrm{d}y')+(y'-y)(\mathrm{d}z\,\mathrm{d}x'-\mathrm{d}x\,\mathrm{d}z')+(z-z')(\mathrm{d}x\,\mathrm{d}y'-\mathrm{d}y\,\mathrm{d}x')}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}} = V$$

dann ist dies Integral durch beide Linien ausgedehnt

$$= 4m\pi$$

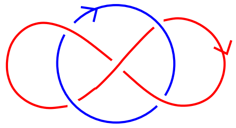
und m die Anzahl der Umschlingungen.

FIGURE 1414

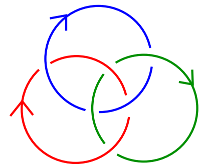
Example. Given any topological invariant the question that arises is, how good is it at distinguishing objects. It is quite easy to find examples of links which show that the linking number is far from being a complete invariant. In Figure 1415 we show the Whitehead link and the Borromean rings. Both are links with zero linking numbers and unknotted components but which are appear to be non-trivial. The Whitehead link is named after John Henry Constantine Whitehead who also introduced the concept of CW-complexes. The Borromean rings are named after the Borromeo family that carries the Borromean rings in its coat of arms since medieval days.

We record the following question for later on.

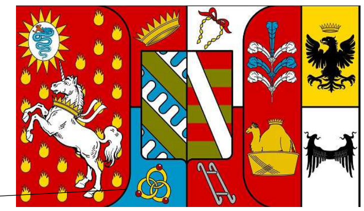
Question 99.16. *How can we show that neither the Whitehead link nor the Borromean rings are smoothly isotopic to the trivial link?*



Whitehead link



Borromean rings



coat of arms of the Borromeo family

FIGURE 1415

Remark. We can of course try to determine the fundamental groups of the complements of the trivial links and the Whitehead link and the Borromean rings. But then the question arises, how can we distinguish these groups?

99.5. **Alexander duality and linking numbers** ($*$)**.** Let Y be a compact non-empty subset of S^3 that admits a regular neighborhood. (For example, by Propositions 10.13, 10.9 and 10.11 the subset Y could be a link or a spatial graph.) We can compute the isomorphism types of the first homology group of the complement $S^3 \setminus Y$ as follows:

$$\mathrm{H}_1(S^3 \setminus Y;\mathbb{Z}) \underset{\substack{\uparrow \\ \text{Alexander Duality} \\ \text{Theorem 92.1}}}{\xrightarrow{\cong}} \mathrm{H}^1(Y;\mathbb{Z}) \underset{\substack{\uparrow \\ \text{Universal Coefficient} \\ \text{Theorem 75.13}}}{\xrightarrow{\cong}} \mathrm{Hom}(\mathrm{H}_1(Y;\mathbb{Z}),\mathbb{Z}) \underset{\substack{\uparrow \\ \text{non-natural isomorphism}}}{\cong} \mathrm{H}_1(Y;\mathbb{Z})/\text{torsion}.$$

The catch with the Alexander Duality Theorem 92.1 is that it is quite hard to "see" the isomorphism. For example, suppose we are given a basis for $\mathrm{H}_1(Y;\mathbb{Z})/\text{torsion}$, how do we get a basis for $\mathrm{H}_1(S^3 \setminus Y;\mathbb{Z})$? In many situations the following lemma resolves that issue.

Lemma 99.17. *Let Y be a compact non-empty subset of S^3 that admits a regular neighborhood $N(Y)$. We suppose that $\mathrm{H}_1(Y;\mathbb{Z})$ is a free abelian group*[1390] *of rank k. Let $v(x), C_n$ be oriented curves in $N(Y)$ and let $D_1,\dots,D_k$ be oriented curves in $S^3 \setminus N(Y)$. If*

$$\det\Big(\underbrace{(\mathrm{lk}(C_i,D_j)_{i,j=1,\dots,k})}_{k\times k\text{–matrix}}\Big) \;=\; \pm 1,$$

then $[D_1],\dots,[D_k]$ form a basis of $\mathrm{H}_1(S^3 \setminus Y;\mathbb{Z})$.

PROOF. Let Y be a compact non-empty subset of S^3 that admits a regular neighborhood $N = N(Y)$. We denote by $\mathring{N}$ the interior of N. We consider the homomorphisms

$$\begin{array}{rcl} \Omega\colon \mathbb{Z}^k & \to & \mathrm{H}_1(S^3\setminus\mathring{N};\mathbb{Z}) \\ (a_1,\dots,a_k) & \mapsto & \sum\limits_{i=1}^k a_i\cdot[D_i] \end{array} \quad\text{and}\quad \begin{array}{rcl} \Theta\colon \mathrm{H}_1(S^3\setminus\mathring{N};\mathbb{Z}) & \to & \mathbb{Z}^k \\ \sigma & \mapsto & (\phi_{C_1}(\sigma),\dots,\phi_{C_k}(\sigma)). \end{array}$$

It follows from our hypothesis and the definition of linking numbers that $\Theta\circ\Omega$ is an epimorphism. Furthermore we know from the Alexander Duality Theorem 92.1 and the Universal Coefficient Theorem 75.13 that $\mathrm{H}_1(S^3\setminus\mathring{N};\mathbb{Z})$ is also isomorphic to $\mathbb{Z}^k$. But by

[1390]Is it possible that $\mathrm{H}_1(Y;\mathbb{Z})$ is not a free abelian group?

Lemma 19.8 (5) an epimorphism between two free abelian groups of the same rank is an isomorphism. Thus we see that Θ and Ω are isomorphisms.

Since $\{[D_1], \dots, [D_k]\}$ is the image under Ω of the standard basis of $\mathbb{Z}^k$ we see that $[D_1], \dots, [D_k]$ form a basis for $\mathrm{H}_1(S^3 \setminus \mathring{N}; \mathbb{Z})$. But by Lemma 10.1 the inclusion induced map $\mathrm{H}_1(S^3 \setminus Y; \mathbb{Z}) \to \mathrm{H}_1(S^3 \setminus \mathring{N}; \mathbb{Z})$ is an isomorphism. Thus we see that $[D_1], \dots, [D_k]$ form a basis of $\mathrm{H}_1(S^3 \setminus Y; \mathbb{Z})$.[1391] ■

Example. In Figure 1416 we consider a spatial graph G together with a regular neighborhood N. By Lemma 55.12 we know that $\mathrm{H}_1(G; \mathbb{Z}) \cong \mathbb{Z}^2$. In Figure 1416 we also see two oriented curves C_1 and C_2 in N and two oriented curves D_1 and D_2 in $S^3 \setminus \mathring{N}$. Using Proposition 99.13 one can easily calculate that

$$\begin{pmatrix} \mathrm{lk}(C_1, D_1) & \mathrm{lk}(C_1, D_2) \\ \mathrm{lk}(C_2, D_1) & \mathrm{lk}(C_2, D_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

It follows from Lemma 99.17 that $[D_1], [D_2]$ form a basis for $\mathrm{H}_1(S^3 \setminus G; \mathbb{Z})$.

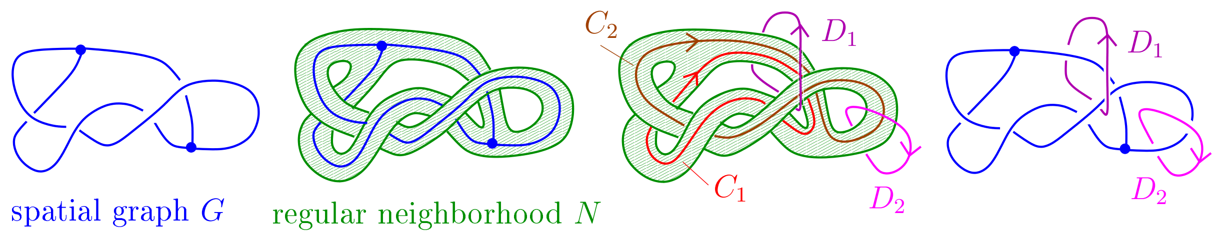


FIGURE 1416

In Lemma 92.3 we computed the homology groups of the complement of a link, but we could not determine explicit generators. The following lemma now solves that problem. In particular it gives an answer to Question 92.4.

Lemma 99.18. *Let $L = L_1 \sqcup \dots \sqcup L_m$ be an m-component link. We pick a thickening $\overline{B}^2 \times L$ for L.*

(1) *A basis for $\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}) \cong \mathbb{Z}^m$ is given by the homology classes that are represented by the meridians $\mu_1, \dots, \mu_m$.*
(2) *A basis for $\mathrm{H}_2(S^3 \setminus L; \mathbb{Z}) \cong \mathbb{Z}^{m-1}$ is given by considering the fundamental classes of all but one of the tori $S^1 \times L$, $i \in \{1, \dots, m\}$. More precisely, for any $j \in \{1, \dots, m\}$ a basis for $\mathrm{H}_2(S^3 \setminus L; \mathbb{Z}) \cong \mathbb{Z}^{m-1}$ is given by*

$$[S^1 \times L_1], \dots, [S^1 \times L_{i-1}], [S^1 \times L_{i+1}], \dots, [S^1 \times L_m].$$

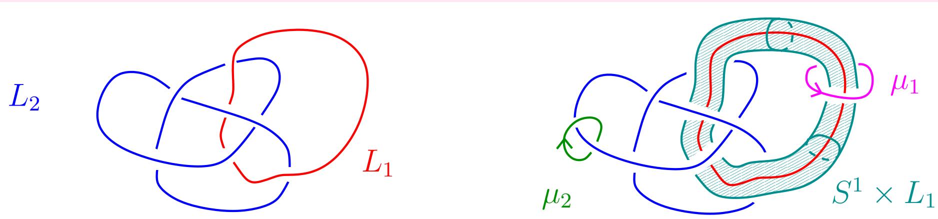


FIGURE 1417

[1391]The alert reader will have noticed that the proof resembles the argument given in Lemma 74.7.

Sketch of proof. Let $L = L_1 \sqcup \cdots \sqcup L_m$ be an m-component link. We know from Lemma 92.3 that $\mathrm{H}_1(S^3 \setminus L; \mathbb{Z}) \cong \mathbb{Z}^m$ and that $\mathrm{H}_2(S^3 \setminus L; \mathbb{Z}) \cong \mathbb{Z}^{m-1}$. We pick a thickening $\overline{B}^2 \times L$ for L. By the Collar Neighborhood Theorem 8.12 we can extend this thickening to a thickening $2\overline{B}^2 \times L$.

(1) The first statement follows from Lemma 99.17 applied to $\mathrm{C}_i = \{0\} \times L_i$, $i = 1, \ldots, m$ and $D_i := \mu_i$, $i = 1, \ldots, m$.
(2) To simplify the notation we consider the case $j = m$. We write $X' := S^3 \setminus \frac{1}{2}B^2 \times L$. For $i = 1, \ldots, m$ we write $T_i := S^1 \times L_i$. For $i = 1, \ldots, m$ we pick a point P_i on $\frac{1}{2}S^1 \times L_i \subset \partial X'$. Since $S^3 \setminus (2B^2 \times L)$ is path-connected we can pick for each $i \in \{1, \ldots, m-1\}$ a path γ_i from P_i to P_m that intersects T_i transversally in a single point and that does not intersect any of the T_j with $j \neq i, m$. Now we consider the map

$$\begin{array}{rcl} \Psi\colon\ \mathrm{H}_1(X', \partial X'; \mathbb{Z}) \times \mathrm{H}_2(X'; \mathbb{Z}) & \to & \mathbb{Z} \\ (\sigma, \tau) & \mapsto & \langle \mathrm{PD}_{X'}(\sigma) \cup \mathrm{PD}_{X'}(\tau), [X'] \rangle. \end{array}$$

It follows easily from Theorem 95.7 that for $i, j \in \{1, \ldots, m-1\}$ we have the equality $\Psi([\gamma_i], [T_j]) = \delta_{ij}$. By Lemma 74.8 this implies that $[T_1], \ldots, [T_{m-1}]$ do indeed form a basis of $\mathrm{H}_2(X'; \mathbb{Z})$. ■

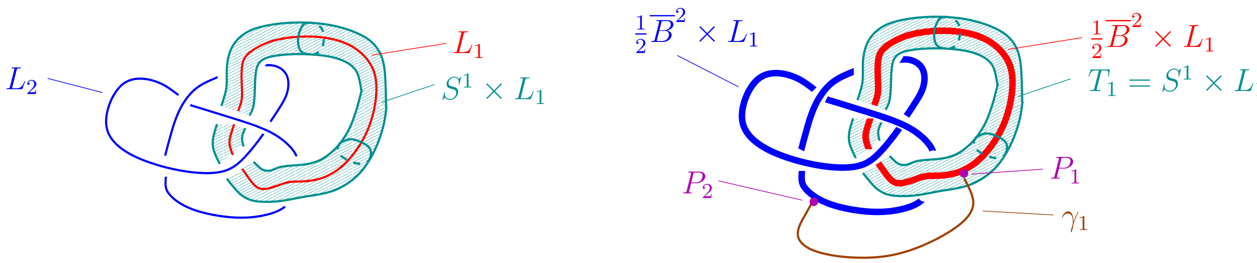


Figure 1418

Exercises for Chapter 99.

Exercise 99.1. Let $A = S^1 \times [-1, 1]$ be the annulus. Let C and D be two closed oriented curves on A. Suppose that $[C] = [D] \neq 0$. Show that the following two statements hold:

(a) the two closed oriented curves are smoothly isotopic,
(b) if C and D are furthermore disjoint, then there exists an embedded annulus $X \subset A$ with $\partial A = C \cup -D$.

Hint. Using the fact that A is an annulus you can find a smooth isotopy that makes the curves disjoint. At that point it is perhaps better to first prove (2).

Exercise 99.2. Let F be an oriented connected 2-dimensional smooth manifold and let C and D be distinct boundary curves. Suppose that C and D are homotopic. Show that F is diffeomorphic to the annulus $S^1 \times [-1, 1]$.

Exercise 99.3. Let K be a knot together with a diagram. In Section 27.6 we gave an explicit algorithm that determines the Wirtinger presentation for $\pi_1(S^3 \setminus K)$. Explain what element in this presentation corresponds to the longitude. Alternatively, for readers who like more explicit questions, determine the group system for the trefoil and the figure-8 knot.

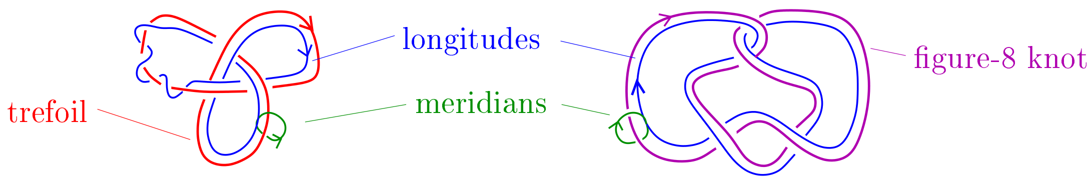


FIGURE 1419. Illustration for Exercise 99.3.

Exercise 99.4. Show that the trefoil K is chiral.
Hint. In Proposition 27.6 we showed that there exists an epimorphism $\pi_1(S^3 \setminus K) \to S_3$.

Exercise 99.5. Find explicit Seifert surfaces for the knots K_1 and K_2 shown in Figure 1420.
Remark. The answer to a question can only be as rigorous as the input. Since the input is a picture it suffices here of course to draw a convincing picture of a Seifert surface.

FIGURE 1420

Exercise 99.6. Let $\varphi\colon \overline{B}^3 \to S^3$ and $\psi\colon \overline{B}^3 \to S^3$ be two smooth embeddings with disjoint images. Let $K \subset \varphi(\overline{B}^3)$ and $J \subset \psi(\overline{B}^3)$ be oriented knots. Show that $\operatorname{lk}(K, J) = 0$.

Exercise 99.7. We consider the two links shown in Figure 1406. Show that the link complements are diffeomorphic.

FIGURE 1421

Exercise 99.8. Is every knot $K \subset S^3$ the boundary of a *non-orientable* compact surface that is smoothly embedded in S^3?

Exercise 99.9. The Borromean rings, shown in Figure 1415, have the property that if you remove one component, then the remaining two knots form a trivial link. Does there exist a four component link with the property that if you remove any of the four components, then you get a three component trivial link?

Exercise 99.10. Let $L = K \sqcup J$ and $L' = K' \sqcup J'$ be two oriented links such that the complements are diffeomorphic. Does it follow that $\operatorname{lk}(K,J) = \pm \operatorname{lk}(K',J')$?

Exercise 99.11. An m-component link $L = L_1 \sqcup \cdots \sqcup L_m \subset S^3$ is called *split link* if there exist disjoint closed balls $B_1, \dots, B_m$ such that each L_i is contained in B_i. Furthermore L is called a *boundary link* if there exist m disjoint Seifert surfaces $F_1, \dots, F_m$.

(a) A split link is easily seen to be a boundary link. Do there exist boundary links that are not split?

(b) Show that if $L = L_1 \sqcup \cdots \sqcup L_m$ is a boundary link, then there exists an epimorphism from $\pi_1(S^3 \setminus L)$ onto a free group of rank m.
Hint. You could try to find an interesting map from $S^3 \setminus L$ onto the wedge of m circles.

Exercise 99.12. Let L and M be two links in S^3. We pick an orientation-preserving embedding $\varphi\colon \overline{B}^3 \to S^3$ with $L \subset \varphi(B^3)$ and we pick an orientation-preserving embedding $\psi\colon \overline{B}^3 \to S^3$ with $M \subset \psi(B^3)$. Finally we pick two orientation-preserving embeddings $\alpha, \beta\colon \overline{B}^3 \to S^3$ with disjoint images. We refer to $\alpha(\varphi^{-1}(L)) \cup \beta(\psi^{-1}(M))$ as the *split union $L \sqcup M$ of the links L and M*. Show that the smooth isotopy class of the split union $L \sqcup M$ is independent of all the choices and that the smooth isotopy class of the split union only depends on the smooth isotopy type of L and M.
Remark. Evidently you want to use Theorem 8.36. But is that really enough?

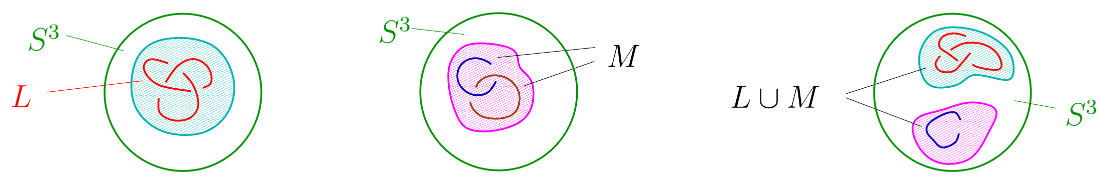


FIGURE 1422. Illustration of Exercise 99.12.

Exercise 99.13. Let $K \sqcup J$ be an oriented link in $\mathbb{R}^3$ and let $K^* \sqcup J^*$ be a mirror image as defined on page 780. As usual we view $\mathbb{R}^3$ as a subset of $\mathbb{R}^3 \cup \{\infty\} = S^3$. Determine the relationship between $\operatorname{lk}(K,J)$ and $\operatorname{lk}(K^*,J^*)$.

Exercise 99.14. Let $K \sqcup J \subset \mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be an oriented 2-component link. Show that if $\varphi\colon S^1 \to K$ and $\psi\colon S^1 \to J$ are two orientation-preserving diffeomorphisms, then[1392]

$$\operatorname{lk}(K,J) \;=\; \deg \left(\begin{array}{ccc} S^1 \times S^1 & \to & S^2 \\ (z,w) & \mapsto & \frac{\varphi(z)-\psi(w)}{\|\varphi(z)-\psi(w)\|} \end{array} \right).$$

Hint. Use Sard's Theorem 6.63, Proposition 99.13 and Proposition 69.7.

Exercise 99.15. Let $K \sqcup J \subset \mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be an oriented 2-component link. Furthermore let $f\colon [0,2\pi] \to K$ and $g\colon [0,2\pi] \to J$ be two maps such that f and g descend

[1392]Here we use the notion of "degree" that we introduced on page 1737.

to orientation-preserving diffeomorphisms from $[0,2\pi]/0 \sim 2\pi$ to K and J. Show that the following equality holds:

$$\operatorname{lk}(K,J) \;=\; \frac{1}{4\pi} \int\limits_{s=0}^{s=2\pi} \int\limits_{t=0}^{t=2\pi} \frac{\det \overbrace{\big((f(s)-g(t)) \;\; f'(s) \;\; g'(t)\big)}^{3\times 3\text{–matrix}}}{\|f(s)-g(t)\|^3}\, dt\, ds.$$

Hint. Use Exercise 99.14 together with Proposition 79.7, Exercises 78.4 and 78.5.

Exercise 99.16. Let $n \geq 4$ and let $K \subset S^n$ be a closed orientable smooth submanifold of codimension 2. Show that there exists a compact orientable submanifold W of S^n with $\partial W = K$.

Remark. Note that in general the analogous statement is not true if we do not impose any conditions on the codimension. For example by Proposition 9.1 we can view $\mathbb{C}\mathrm{P}^2$ as a smooth submanifold of some $S^n = \mathbb{R}^n \cup \{\infty\}$. But by the proof of Corollary **??** we know that $\mathbb{C}\mathrm{P}^2$ is not the boundary of any compact orientable smooth manifold, let alone one which is embedded in S^n.

Exercise 99.17. In Figure 99.17 we show two smooth embeddings of the genus two handlebody into $\mathbb{R}^3$. Are these two embeddings smoothly isotopic?

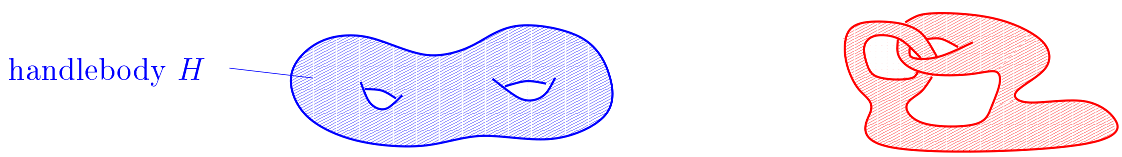


FIGURE 1423. Illustration of Exercise 99.17.

100. The intersection form I: Definition and calculations

In this chapter we will introduce and study the intersection form of compact oriented even-dimensional topological manifolds. Before we do so it is convenient to remind ourselves of a few basic notions and results in algebra.

100.1. **Pairings and forms.** On page 2192 we introduced the notion of a pairing over a commutative ring R. We recall the definition and we introduce some new notions.

Definition. Let R be a commutative ring.

(1) A *pairing over* R is an R-bilinear map $V \times W \to S$ where V, W and S are R-modules. indexpairing!symmetric
(2) We say that a pairing $\varphi\colon V \times V \to S$ is *symmetric* if $\varphi(v, w) = \varphi(w, v)$ for all $v, w \in V$. Furthermore we call the pairing *anti-symmetric* if $\varphi(v, w) = -\varphi(w, v)$ for all $v, w \in V$.
(3) We say that two pairings (V, φ) and (W, ψ) are *isometric*, written $(V, \varphi) \cong (W, \psi)$, if there exists an isometry, i.e. an isomorphism $f\colon V \to W$ of R-modules such that $\varphi(v_1, v_2) = \psi(f(v_1), f(v_2))$ for all $v_1, v_2 \in V$.
(4) Given a pairing $\varphi\colon V \times V \to S$ we denote by $-\varphi$ the pairing that is given by $(-\varphi)(v, w) := -\varphi(v, w)$ for all $v, w \in V$.
(5) Given two pairings $\varphi\colon V \times V \to S$ and $\psi\colon W \times W \to S$ we define the *direct sum* of φ and ψ to be the pairing

$$\begin{array}{rcl} \varphi \oplus \psi\colon (V \oplus W) \times (V \oplus W) & \to & S \\ (v \oplus w, v' \oplus w') & \mapsto & \varphi(v, v') + \psi(w, w'). \end{array}$$

(6) Given a pairing (V, φ) and $n \in \mathbb{Z}$ we define

$$n \cdot (V, \varphi) \;:=\; \begin{cases} \overbrace{(V, \varphi) \oplus \cdots \oplus (V, \varphi)}^{n\text{-summands}}, & \text{if } n \geq 0, \\ \underbrace{(V, -\varphi) \oplus \cdots \oplus (V, -\varphi)}_{|n|\text{-summands}}, & \text{if } n \leq 0. \end{cases}$$

(7) Given a pairing $(V, \varphi\colon V \times V \to S)$ and a submodule W of V we denote by $\varphi|_W$ the pairing that is given by restricting the map $\varphi\colon V \times V \to S$ to $W \times W \to S$.

In this chapter we will mostly be interested in the following special type of a pairing.

Definition. Let R be a commutative ring. A *form*[1393] is a finitely generated free R-module V together with a pairing $\varphi\colon V \times V \to R$ which is either symmetric or anti-symmetric. We refer to the rank of V also as the rank of the form.[1394]

For the reader's convenience we recall the following definition from page 2192.

[1392]The various definitions of "form" in the literature vary wildly. It is always a good idea, reading a paper or a book, to look up the precise definition of "form" that is used by the author.

[1393]The rank of a finitely generated free module over a commutative ring is well-defined, see e.g. [**Lam99**, p. 4]. Note though that the rank of a finitely generated free module over a non-commutative ring is not necessarily well-defined. More precisely, there exists a non-commutative ring R and $m \neq n \in \mathbb{N}$ such that R^m is isomorphic to R^n. We refer to [**Lam99**, p. 4] for such examples.

Definition. Let R be a commutative ring. A form $\varphi\colon V\times V\to R$ is called *non-singular* if the map

$$\begin{array}{rcl} V & \to & \operatorname{Hom}_R(V,R) \\ v & \mapsto & \left(\begin{array}{rcl} V & \to & R \\ w & \mapsto & \varphi(v,w) \end{array}\right) \end{array}$$

is an isomorphisms.[1394]

In many situations it is convenient to represent a form by a matrix. We recall the relevant definition from the linear algebra courses.

Definition. Let R be a commutative ring and let $\varphi\colon V\times V\to R$ be a form. Given any basis $\{v_1,\dots,v_k\}$ of the free R-module V we refer to the $(k\times k)$–matrix

$$(\varphi(v_i,v_j))_{i,j=1,\dots,k}\ \in\ \mathrm{M}(k\times k,R)$$

as a *matrix representing the form* φ. Note that by Lemma 90.1 the form is non-singular if and only if the matrix representing the form is invertible.

The following elementary lemma is usually proved in a standard course on linear algebra. It is also straightforward to verify it using only the relevant definitions.[1395]

Lemma 100.1. *Let R be a commutative ring and let $\varphi\colon V\times V\to R$ be a form. Let $\{v_1,\dots,v_k\}$ and $\{w_1,\dots,w_k\}$ be bases of the free R-module V. We write*

$$A\ :=\ (\varphi(v_i,v_j))_{i,j=1,\dots,k}\quad\text{and}\quad B\ :=\ (\varphi(w_i,w_j))_{i,j=1,\dots,k}.$$

If $P=(p_{ij})_{i,j=1,\dots,k}\in\mathrm{GL}(k,R)$ denotes the base change matrix that is uniquely defined by the property that $v_i=\sum\limits_{j=1}^{k}p_{ij}w_j$ for $i=1,\dots,k$, then

$$A\ =\ P^TBP.$$

The following lemma is even more elementary, it follows basically immediately from the definitions.

Lemma 100.2.

(1) *Let $\varphi\colon V\times V\to R$ be a form. If A is a matrix that represents φ, then the matrix $-A$ represents the form $-\varphi$.*
(2) *Let $\varphi\colon V\times V\to R$ and $\psi\colon W\times W\to R$ be two forms. If A is a matrix that represents φ and if B is a matrix that represents ψ, then the matrix $A\oplus B:=\begin{pmatrix}A&0\\0&B\end{pmatrix}$ represents the form $\varphi\oplus\psi$.*
(3) *Let $\varphi\colon V\times V\to R$ be a form and let $n\in\mathbb{N}_0$. If A is a matrix that represents φ, then the matrix $n\cdot A:=\underbrace{A\oplus\cdots\oplus A}_{n\text{-times}}$ represents the form $n\cdot\varphi$.*

Before we can turn to topology we need to recall the following definition from page 1621.

[1394]Note that a form is by definition (anti-) symmetric. It follows from this fact that the given condition is satisfied if and only if the condition with the two entries of φ swapped is satisfied. Thus the definition we give here is indeed consistent with the definition from page 2192.

[1395]It is easy though to get confused.

Definition. Given a finitely generated abelian group H we refer to

$$\mathrm{F}H \ := \ H/\operatorname{Tor}(H) \quad \text{ where } \operatorname{Tor}(H) \text{ denotes the torsion subgroup of } H$$

as the *maximal torsion-free quotient* of H.

The following lemma summarizes everything we need to know about maximal torsion-free quotients.

Lemma 100.3.

(1) *The maps*

$$H \ \mapsto \ \mathrm{F}H \qquad \textit{and} \qquad (\alpha\colon G \to H) \ \mapsto \ \left(\begin{array}{rcl} \alpha\colon \ \mathrm{F}G & \to & \mathrm{F}H \\ {[g]} & \mapsto & [\alpha(g)] \end{array} \right)$$

define a covariant functor from the category of finitely generated abelian groups to the category of finitely generated free abelian groups.

(2) *For every homomorphism $\alpha\colon A \to B$ between two finitely generated abelian groups the following diagram commutes:*[1396]

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "a\mapsto[a]"'] & B \arrow[d, "b\mapsto[b]"] \\ \mathrm{F}A \arrow[r, "\alpha"] & \mathrm{F}B. \end{tikzcd}$$

(3) *Any (anti-) symmetric pairing*

$$\varphi\colon H \times H \ \to \ \mathbb{Z}$$

over a finitely generated abelian group descends to a well-defined form

$$\begin{array}{rcl} \mathrm{F}H \times \mathrm{F}H & \to & \mathbb{Z} \\ ([g],[h]) & \mapsto & \varphi(g,h). \end{array}$$

Proof. In Lemma 65.4 we already showed that if H is a finitely generated abelian group, then $\mathrm{F}H$ is a finitely generated free abelian group. The remaining statements of the lemma are basically elementary. ■

Convention. On many occasions we do not distinguish in the notation between an element in an abelian group H and the element it represents in the maximal torsion-free quotient $\mathrm{F}H$. Furthermore, given a homomorphism $\alpha\colon A \to B$ between two finitely generated abelian groups we often denote the induced map $\mathrm{F}A \to \mathrm{F}B$ by α as well.

100.2. **The definition of the intersection form of topological manifolds.** Let M be a compact oriented m-dimensional topological manifold and let A and B be compact $(m-1)$-dimensional submanifolds of ∂M with $A \cup B = \partial M$ and with $A \cap B = \partial A = \partial B$.

[1396] In fancy speak this means that the projection maps $H \to \mathrm{F}H$ define a natural transformation from the identity functor to the functor $H \mapsto \mathrm{F}H$.

Recall that by the Poincaré Duality Theorem 88.1 we know that for any $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{H}^{m-k}(M,A;\mathbb{Z}) & \to & \mathrm{H}_k(M,B;\mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism. As on page 2153 we denote by

$$\mathrm{PD}_M\colon\ \mathrm{H}_k(M,A;\mathbb{Z}) \ \to\ \mathrm{H}^{m-k}(M,B;\mathbb{Z})$$

the inverse of this isomorphism. This leads us to the following definition.

Definition. Let M be a compact oriented topological manifold of the even dimension $2n$. We refer to the map[1397][1398]

$$\begin{array}{rcl} Q_M\colon\ \mathrm{FH}_n(M;\mathbb{Z}) \times \mathrm{FH}_n(M;\mathbb{Z}) & \to & \mathbb{Z} \\ (a,b) & \mapsto & \Big\langle \underbrace{\underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})}}_{\in \mathrm{FH}^{2n}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle \end{array}$$

as the *intersection form of M*.

Remark. Since $\mathrm{PD}_M\colon \mathrm{H}_n(M;\mathbb{Z}) \to \mathrm{H}^n(M,\partial M;\mathbb{Z})$ is an isomorphism it is clear that the intersection form is basically just a reformulation of the cup product pairing

$$\begin{array}{rcl} \mathrm{H}^n(M,\partial M;\mathbb{Z}) \times \mathrm{H}^n(M,\partial M;\mathbb{Z}) & \to & \mathrm{H}^{2n}(M,\partial M;\mathbb{Z}) \\ (\alpha,\beta) & \mapsto & \alpha \cup \beta \end{array}$$

on cohomology groups in the middle dimension that we introduced in Proposition 90.2. The big psychological advantage of homology is that very often one can "see homology classes". For example, in many cases homology classes are represented by submanifolds and we can then use Theorem 94.5 to compute the intersection form of a smooth manifold. Sometimes we can also represent homology classes by explicit cycles and we can use Theorem 95.9 to compute the intersection form.

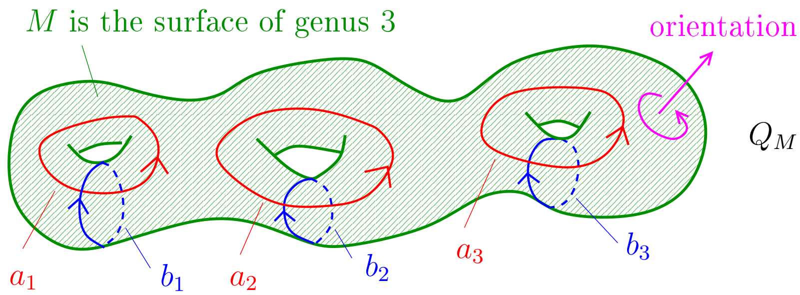
Examples.

(1) First we consider the surface M of genus 3. In principle we could obtain the intersection form from the calculation of the cup product that we had performed in Lemma 81.16. But it is much more fun and instructive to use Theorem 94.5 instead. More precisely, we consider the surface M that is illustrated in Figure 1424. We give it the usual orientation. We already saw in the discussion on page 1270 that the closed oriented curves $a_1, b_1, a_2, b_2, a_3, b_3$ illustrated in Figure 1424 represent a basis for $\mathrm{H}_1(M;\mathbb{Z})$. It follows easily from Theorem 94.5 that with respect to this basis the intersection form is represented by the matrix shown in Figure 1424. An analogous statement holds of course for any surface of genus $g \geq 1$.[1399]

[1397] Note that the homology group $\mathrm{H}_n(M;\mathbb{Z})$ in the middle dimension is not necessarily torsion-free. For example $M = S^1 \times \mathbb{RP}^3$ is orientable, but it follows easily from the Künneth Formula 58.8 and the calculation of the homology groups of $\mathbb{RP}^3$ in Proposition 48.10 that $\mathrm{H}_2(S^1 \times \mathbb{RP}^3;\mathbb{Z}) \cong \mathbb{Z}_2$.

[1398] As mentioned above, in the notation usually we do not distinguish between an element in $\mathrm{H}_n(M;\mathbb{Z})$ and the element it represents in $\mathrm{FH}_n(M;\mathbb{Z})$.

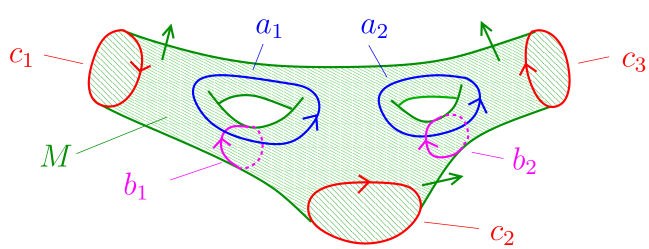
[1399] What do the curves a_i and b_i look like in the more precise description of the surface of genus 3 as a dodecagon with identified sides?



$$Q_M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

FIGURE 1424

(2) Next we let M be the surface of genus two with three disks removed that is shown in Figure 1425. By the discussion on page 1274 we know that a basis for $H_1(M;\mathbb{Z})$ is represented by the curves a_1, b_1, a_2, b_2 together with two out of the three boundary curves c_1, c_2, c_3. It follows from Theorem 94.5 that with respect to the basis $[a_1], [b_1], [a_2], [b_2], [c_1], [c_2]$ the intersection form is represented by the matrix given in Figure 1425.



$$Q_M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

FIGURE 1425

(3) Next let us consider the $2n$-dimensional torus $M = S^n \times S^n$. By the Künneth Formula 58.8 we know that a basis for $H_n(S^n \times S^n;\mathbb{Z})$ is given by $a = [S^n \times \{*\}]$ and $b = [\{*\} \times S^n]$. We have two approaches for determining the intersection with respect to this basis:
   (a) We can use the Product Theorem 84.1 and its corollaries. More precisely, we first determine the Poincaré duals of $[S^n \times \{0\}]$ and $[\{0\} \times S^n]$ as in Proposition 84.2 (2) and then we can apply Lemma 84.3.
   (b) As on page 2278 we can use algebraic intersection numbers and Theorem 94.5 to determine the intersection form.

   In both cases we see that with respect to this basis the intersection form of $S^n \times S^n$ is represented by the matrix

$$\begin{pmatrix} Q_M(a,a) & Q_M(a,b) \\ Q_M(b,a) & Q_M(b,b) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.$$

(4) We consider the complex manifold $\mathbb{C}\mathrm{P}^2$. As always we equip $\mathbb{C}\mathrm{P}^2$ with the natural orientation coming from Proposition 12.7. It follows from Proposition 94.11 that the intersection form

$$Q_{\mathbb{C}\mathrm{P}^2} \colon H_2(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \times H_2(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \to \mathbb{Z}$$

is represented by the 1×1–matrix (1).

(5) Let M be a closed oriented $2n$-dimensional topological manifold. As usual we denote by $-M$ the same topological manifold with the opposite orientation. It follows immediately from the definitions together with Proposition 72.2 that $Q_{-M} = -Q_M$.

(6) For the complex projective space $\mathbb{C}\mathrm{P}^2$ we use the common notation that we write $\overline{\mathbb{C}\mathrm{P}}^2$ instead of $-\mathbb{C}\mathrm{P}^2$. With this notation it follows from (4) and (5) that the intersection form

$$Q_{\overline{\mathbb{C}\mathrm{P}}^2}\colon\ \mathrm{H}_2(\overline{\mathbb{C}\mathrm{P}}^2;\mathbb{Z}) \times \mathrm{H}_2(\overline{\mathbb{C}\mathrm{P}}^2;\mathbb{Z}) \ \to\ \mathbb{Z}$$

is represented by the 1×1–matrix (-1).

100.3. **Properties of the intersection form.** In the following proposition we summarize some of the key properties of the intersection form. The proposition shows in particular that the intersection form is a form in the sense of the definition on page 2410.

Proposition 100.4. *Let M be a compact oriented $2n$-dimensional topological manifold. Then the following hold:*

(1) *The group $\mathrm{H}_n(M;\mathbb{Z})$ is finitely generated and $\mathrm{FH}_n(M;\mathbb{Z})$ is a finitely generated free abelian group.*

(2) *If n is even, then the intersection form Q_M is symmetric, otherwise it is anti-symmetric.*

(3) *If M is closed, then the intersection form is non-singular.*

The proof of Proposition 100.4 (3) relies on the following reformulation of the intersection form, which will also be useful later, on a separate occasion.

Lemma 100.5. (∗) *Let M be a compact oriented connected $2n$-dimensional topological manifold. We denote by $i\colon (M,\varnothing) \to (M,\partial M)$ the inclusion map. We write*

$$\begin{array}{rcl} \Phi\colon\ \mathrm{H}_n(M;\mathbb{Z}) \xrightarrow{i_*} \mathrm{H}_n(M,\partial M;\mathbb{Z}) \xrightarrow{\mathrm{PD}_M} \mathrm{H}^n(M;\mathbb{Z}) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(\mathrm{FH}_n(M;\mathbb{Z}),\mathbb{Z}). \\ \gamma & \mapsto & (a \mapsto \langle \gamma, a\rangle) \end{array}$$

For any $a, b \in \mathrm{FH}_n(M;\mathbb{Z})$ we have

$$Q_M(a,b) \ =\ \Phi(b)(a).$$

Proof of Lemma 100.5 (∗). First we consider the following diagram:

$$\begin{tikzcd} \mathrm{H}_n(M;\mathbb{Z}) \times \mathrm{H}_n(M;\mathbb{Z}) \arrow[r, "\mathrm{id}\times i_*"] \arrow[d, "\mathrm{PD}_M\times\mathrm{PD}_M"'] & \mathrm{H}_n(M;\mathbb{Z}) \times \mathrm{H}_n(M,\partial M;\mathbb{Z}) \arrow[d, "\mathrm{PD}_M\times\mathrm{PD}_M"] \\ \mathrm{H}^n(M,\partial M;\mathbb{Z}) \times \mathrm{H}^n(M,\partial M;\mathbb{Z}) \arrow[r, "\mathrm{id}\times i^*"] \arrow[d, "\cup"'] & \mathrm{H}^n(M,\partial M;\mathbb{Z}) \times \mathrm{H}^n(M;\mathbb{Z}) \arrow[d, "\cup"] \\ \mathrm{H}^{2n}(M,\partial M;\mathbb{Z}) \arrow[r, "="] & \mathrm{H}^{2n}(M,\partial M;\mathbb{Z}). \end{tikzcd}$$

By Proposition 88.26 we know that the upper square commutes. We obtain from Proposition 82.4, applied to the inclusion map $f\colon (M,\varnothing,\partial M) \to (M,\partial M,\partial M)$, that the lower square also commutes.

Now let $a, b \in \mathrm{FH}_n(M;\mathbb{Z})$. We calculate that

$$\begin{array}{rcll}
Q_M(b,a) & \overset{\text{definition of } Q_M}{\downarrow}\!= & \langle \mathrm{PD}_M(b) \cup \mathrm{PD}_M(a), [M]\rangle & \overset{\text{Lemma 83.4 (1)}}{\downarrow}\!= \ (\mathrm{PD}_M(b) \cup \mathrm{PD}_M(a)) \cap [M] \\
 & \underset{\text{by the above commutative diagram}}{=\!\uparrow} & (\mathrm{PD}_M(b) \cup \mathrm{PD}_M(i_*(a))) \cap [M] & \underset{\text{Lemma 83.7}}{=\!\uparrow} \ (\mathrm{PD}_M(i_*(a)) \cap (\mathrm{PD}_M(b) \cap [M]) \\
 & \underset{\text{definition of } \mathrm{PD}_M(b)}{=\!\uparrow} & \mathrm{PD}_M(i_*(a)) \cap b & \underset{\text{Lemma 83.4 (1)}}{=\!\uparrow} \ \langle \mathrm{PD}_M(i_*(a)), b\rangle \underset{\text{definition of } \Phi \text{ and ev}}{=\!\uparrow} \Phi(a)(b).
\end{array}$$

■

Proof of Proposition 100.4.

(1) We proved this statement in Proposition 85.13.
(2) Given $v, w \in \mathrm{FH}_n(M;\mathbb{Z})$ we have the following equality:

$$\overbrace{\Big\langle \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^n(M,\partial M;\mathbb{Z})}, [M] \Big\rangle}^{=Q_M(v,w)} \underset{\text{Proposition 82.3}}{=\!\uparrow} (-1)^{n \cdot n} \cdot \overbrace{\Big\langle \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^n(M,\partial M;\mathbb{Z})}, [M] \Big\rangle}^{=Q_M(w,v)}.$$

The statement follows from the observation that $(-1)^{n \cdot n} = (-1)^n$.
(3) Now suppose that M is closed. We consider the following diagram:

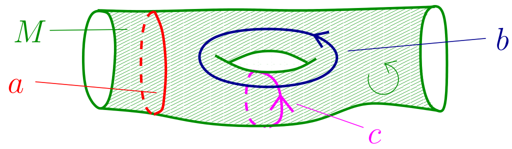
$$\begin{tikzcd}
\mathrm{FH}_n(M;\mathbb{Z}) \arrow[rr, "b \mapsto (a \mapsto Q_M(b,a))"] \arrow[dr, "\cong", "\mathrm{PD}_M"'] & & \mathrm{Hom}(\mathrm{FH}_n(M;\mathbb{Z}),\mathbb{Z}). \\
 & \mathrm{FH}^n(M;\mathbb{Z}) \arrow[ur, "\mathrm{ev}", "\mathrm{ev}"'] &
\end{tikzcd}$$

We make the following observations:
(a) Since M is closed the map PD_M does indeed take values in the cohomology of M. Furthermore, by the Poincaré Duality Theorem 88.1 the map PD_M is an isomorphism.
(b) It follows from the Universal Coefficient Theorem 75.13 for Cohomology Groups together with Lemma 75.5 (5) and Proposition 85.13 (4) that the evaluation map

$$\mathrm{ev}\colon\ \mathrm{FH}^n(M;\mathbb{Z}) \ \to \ \mathrm{Hom}(\mathrm{FH}_n(M;\mathbb{Z}),\mathbb{Z})$$

is an isomorphism.
(c) The diagram commutes by Lemma 100.5.
(d) We had just seen that the two diagonal maps are isomorphisms and that the diagram commutes. Thus we obtain that the top map is an isomorphism. By the definition on page 2411 this means precisely that the intersection form is non-singular. ■

Remark. Let M be a compact oriented $2n$-dimensional topological manifold. In Proposition 100.4 we just saw that if M is closed, then the intersection form is non-singular. If M is not closed, then this conclusion does not necessarily hold. For example in Figure 1426 we let M be the torus minus two open disks. We show a basis $\{a, b, c\}$ for $\mathrm{H}_1(M;\mathbb{Z})$. Using Theorem 94.5 one can easily see that the intersection form of M with respect to the given basis is represented by the matrix given in the figure. We see that the intersection form is anything but non-singular, in fact it is not even non-degenerate.[1400] Note though



$$Q_M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

FIGURE 1426

that in Proposition 100.15 and Proposition 100.21 we will see that even to a topological manifold that is not closed we can associate two pairings that are non-singular or at least non-degenerate.

In the next lemma we want to see to what degree the isometry type of the intersection form is an invariant of the homeomorphism type of the smooth manifold.

Lemma 100.6. *Let M and N be compact oriented connected $2n$-dimensional topological manifolds. Let $f\colon (M, \partial M) \to (N, \partial N)$ be a homotopy equivalence of pairs of topological spaces.*[1401]

(1) *If $f_*([M]) = [N]$, then f induces an isometry $Q_M \cong Q_N$.*
(2) *If $f_*([M]) = -[N]$, then f induces an isometry $Q_M \cong -Q_N$.*

Note that if f is a homeomorphism, then it follows from Proposition 72.2 that the first case occurs precisely when f is orientation-preserving and the second case occurs precisely when f is orientation-reversing.

PROOF (∗). [1402] Let $f\colon (M, \partial M) \to (N, \partial N)$ be a homotopy equivalence of pairs of topological spaces. It follows from Corollary 43.18 together with Corollary 87.4 (3) that there exists an $\varepsilon \in \{-1, 1\}$ which satisfies $f_*([M]) = \varepsilon \cdot [N]$.

Claim. For any $c \in \mathrm{H}_n(M; \mathbb{Z})$ we have $\mathrm{PD}_N(f_*(c)) = \varepsilon \cdot (f^{-1})^*(\mathrm{PD}_M(c)) \in \mathrm{H}^n(N, \partial N; \mathbb{Z})$.

So let $c \in \mathrm{H}_n(M; \mathbb{Z})$. We have

$$(f^{-1})^*(\mathrm{PD}_M(c)) \cap [N] \underset{\substack{\uparrow \\ \text{by the definition of } \varepsilon}}{=} (f^{-1})^*(\mathrm{PD}_M(c)) \cap \varepsilon \cdot f_*([M]) \underset{\substack{\uparrow \\ \text{Lemma 83.8}}}{=} \varepsilon \cdot f_*(\mathrm{PD}_M(c) \cap [M]) \underset{\substack{\uparrow \\ \text{definition of } \mathrm{PD}_M}}{=} \varepsilon \cdot f_*(c).$$

It follows from the definition of PD_N that we get the desired equality. ⊞

[1400] In fact the annulus $[0, 1] \times S^1$ also works as an example, but it is less fun.

[1401] Note that by Corollary 43.18 we know that the induced map $f_*\colon \mathrm{H}_{2n}(M, \partial M; \mathbb{Z}) \to \mathrm{H}_{2n}(N, \partial N; \mathbb{Z})$ is an isomorphism. By Corollary 87.4 the fundamental classes are generators of the homology groups. Thus one of the two cases has to occur.

[1402] The proof is perhaps slightly longer than one might expect. In a sense the statement of the lemma is basically trivial, but we have to pay a price for using the slightly awkward Poincaré Duality isomorphism $\mathrm{PD}_M\colon \mathrm{H}_n(M; \mathbb{Z}) \to \mathrm{H}^{2n-n}(M, \partial M; \mathbb{Z})$.

Now let $a, b \in \mathrm{FH}_n(M;\mathbb{Z})$. Then

$$\begin{array}{rcl} Q_N(f_*(a), f_*(b)) & \overset{\text{by definition of } Q_N}{\overset{\downarrow}{=}} & \langle \mathrm{PD}_N(f_*(a)) \cup \mathrm{PD}_N(f_*(b)), [N]\rangle \overset{\text{by the claim}}{\overset{\downarrow}{=}} \\ & = & \langle \varepsilon \cdot (f^{-1})^*(\mathrm{PD}_M(a)) \cup \varepsilon \cdot (f^{-1})^*(\mathrm{PD}_M(b)), \varepsilon \cdot f_*([M])\rangle \\ & \underset{\text{by Propositition 82.4 (1) and since } \varepsilon^3=\varepsilon}{\underset{\uparrow}{=}} & \varepsilon \cdot \langle (f^{-1})^*(\mathrm{PD}_M(a) \cup \mathrm{PD}_M(b)), f_*([M])\rangle \underset{\text{by Lemma 74.6 (3) and the definition of } Q_M}{\underset{\uparrow}{=}} \varepsilon \cdot Q_M(a,b). \end{array}$$

The lemma now follows from the observation that it follows from the hypothesis that f is a homotopy equivalence together with Corollary 42.8 that $f_*\colon \mathrm{FH}_n(M;\mathbb{Z}) \to \mathrm{FH}_n(N;\mathbb{Z})$ is an isomorphism. ■

100.4. **The intersection form and degree-one maps.** As we remarked on page 2413, the intersection form of a compact oriented $2n$-dimensional topological manifold contains basically the same information as the cup product pairing

$$\begin{array}{rcl} \mathrm{H}^n(M,\partial M;\mathbb{Z}) \times \mathrm{H}^n(M,\partial M;\mathbb{Z}) & \to & \mathrm{H}^{2n}(M,\partial M;\mathbb{Z}) \\ (\alpha,\beta) & \mapsto & \alpha \cup \beta. \end{array}$$

Given a map $f\colon X \to Y$ between topological spaces it follows from Lemma 81.10 that the cup product behaves well under the induced maps f^* on cohomology groups. It stands to reason that the intersection form should also behave well with respect to induced maps on homology.

So suppose we are given a degree-one map $f\colon X \to Y$ between closed oriented connected $2n$-dimensional topological manifolds. By the aforementioned Lemma 81.10 we know that for any $\alpha \in \mathrm{FH}^i(Y;\mathbb{Z})$ and $\beta \in \mathrm{FH}^j(Y;\mathbb{Z})$ we have $f^*\alpha \cup f^*\beta = f^*(\alpha \cup \beta)$. It seems reasonable to expect that given $\alpha \in \mathrm{H}_n(X;\mathbb{Z})$ and $\beta \in \mathrm{H}_n(X;\mathbb{Z})$ the following equality holds:

$$(*) \qquad Q_X(\alpha,\beta) \;=\; Q_Y(f_*(\alpha), f_*(\beta)).$$

In fact here is a diagram that looks awfully relevant:

$$\begin{tikzcd} \mathrm{H}_n(X;\mathbb{Z}) \times \mathrm{H}_n(X;\mathbb{Z}) \arrow[r, "\mathrm{PD}\times\mathrm{PD}", "\cong"'] \arrow[d, "f_*\times f_*"] & \mathrm{H}^n(X;\mathbb{Z}) \times \mathrm{H}^n(X;\mathbb{Z}) \arrow[r, "\cup"] & \mathrm{H}^{2n}(X;\mathbb{Z}) \arrow[r, "\langle -,[X]\rangle"] & \mathbb{Z} \arrow[d, "\mathrm{id}"'] \\ \mathrm{H}_n(Y;\mathbb{Z}) \times \mathrm{H}_n(Y;\mathbb{Z}) \arrow[r, "\mathrm{PD}\times\mathrm{PD}", "\cong"'] & \mathrm{H}^n(Y;\mathbb{Z}) \times \mathrm{H}^n(Y;\mathbb{Z}) \arrow[r, "\cup"] \arrow[u, "f^*\times f^*"] & \mathrm{H}^{2n}(Y;\mathbb{Z}) \arrow[r, "\langle -,[Y]\rangle"] \arrow[u, "f^*", "\cong"'] & \mathbb{Z}. \end{tikzcd}$$

If this diagram did commute, then $(*)$ would hold. One could now waste a lot of time trying to convince oneself that the diagram commutes. Or alternatively one might notice, as we do in the following example, that $(*)$ just does not hold in general.

Example. We consider the map $f\colon S^1 \times S^1 \to S^2$ that is illustrated in Figure 1427. It follows easily from Lemma 72.3 that this is a degree-one map. We consider the two homology classes α and β on the torus that are sketched in Figure 1427. Note that it follows easily from Theorem 94.5 that $Q_{S^1\times S^1}(\alpha,\beta) = +1$. On the other hand note that we have $f_*(\alpha) = f_*(\beta) = 0 \in \mathrm{H}_1(S^2;\mathbb{Z})$, which evidently implies that $Q_{S^2}(f_*(\alpha), f_*(\beta)) = 0$. Thus in this case the equality $(*)$ does not hold.[1403]

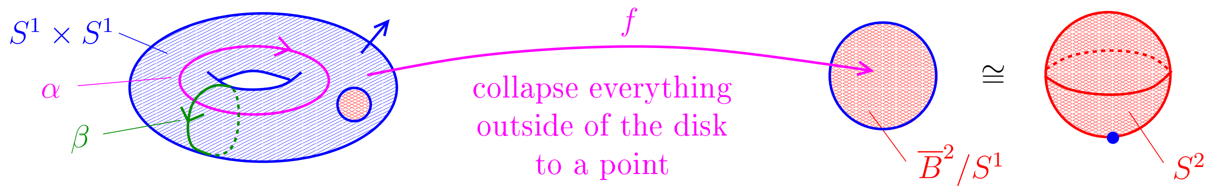


FIGURE 1427

Remark. With this discussion we also learned an interesting lesson:

Not every diagram commutes, not even up to sign.

100.5. **The connected sum of topological manifolds.** To get more examples of interesting intersection forms we want to study the intersection forms of the connected sum of smooth and topological manifolds. Recall that on page 377 we introduced the connected sum of two *smooth* manifolds. In this section we will now also define the connected sum of topological manifolds. The reader who is not that interested in topological manifolds can safely move on to the next section.

For those who are staying, let us recall the definition of the connected sum of two *oriented* smooth manifolds.

Definition. Let $n \in \mathbb{N}$ and let M and N be two connected oriented non-empty n-dimensional smooth manifolds. Furthermore let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\psi\colon \overline{B}^n \to N \setminus \partial N$ be smooth embeddings such that φ is orientation-preserving and such that ψ is orientation-reversing. We define the *connected sum of the oriented smooth manifolds M and N* as

$$M\#N := \big(M \setminus \varphi(B^n)\big) \sqcup \big(N \setminus \psi(B^n)\big)/\sim \ \text{ where } \varphi(P)\sim\psi(P) \text{ for all } P\in S^{n-1}.$$

In Proposition 8.35 we showed that the connected sum of two oriented smooth manifolds is again a smooth manifold.

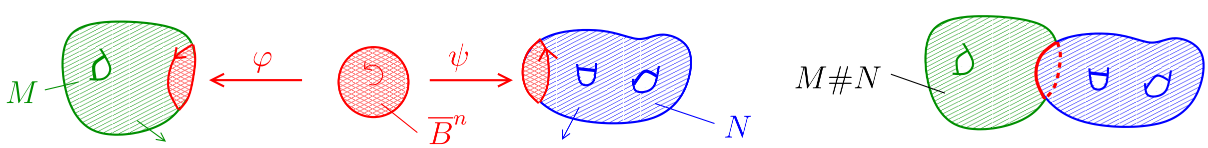


FIGURE 1428

Now we intend to generalize this notion to the setting of topological manifolds. The idea is to suitably adapt the concepts used in the above definition. Browsing through earlier chapters we see the following:

(1) On page 116 we defined an embedding to be a map $f\colon X \to Y$ between two topological spaces that is a homeomorphism onto its image.
(2) On page 2105 we introduced the notion of an orientation of a topological manifold and of an orientation-preserving map.

[1403]This still begs the question, in the above diagram, which of the squares does not commute?

So are we done? Can we just copy-paste the above definition?

As we had just mentioned, in Proposition 8.35 we showed that the connected sum of two oriented smooth manifolds is again a smooth manifold. Recall that the proof of Proposition 8.35 relied heavily on Proposition 8.1 which says that the image of a smooth embedding $f\colon X \to Y \setminus \partial Y$ is a smooth submanifold of the smooth manifold Y.

The question arises, whether the analogous statement holds for topological manifolds. In other words, we have the following question.

Question 100.7. *Let X be an n-dimensional topological manifold. Is the image of an embedding $f\colon \overline{B}^n \to X \setminus \partial X$ necessarily a topological submanifold of X?*

For better or worse the answer to Question 100.7 is no.[1404] Namely consider the embedding $f\colon \overline{B}^3 \to \mathbb{R}^3$ that is illustrated in Figure 1429. We refer to the image $f(\overline{B}^3)$ as the *Alexander horned ball*.[1405] It is clear that this is a weird embedding. We leave it to the reader to make the embedding precise and to show that $f(\overline{B}^3)$ is not a topological submanifold of $\mathbb{R}^3$. But even without filling in all details this map should shatter the reader's confidence that Question 100.7 has a positive answer.

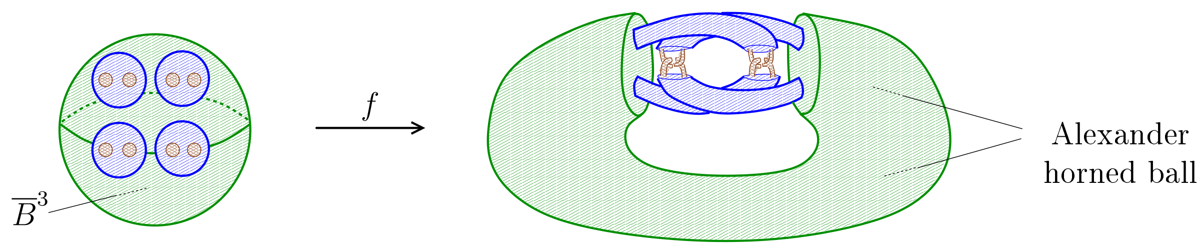


FIGURE 1429

This stupefying example leads us to the following definition.

Definition. Let X and Y be topological manifolds. We say an embedding $f\colon X \to Y$ is *locally flat* if $f(X)$ is a submanifold of Y.

With this definition we can now finally give a reasonable definition of the connected sum of two *topological* manifolds. For completeness' sake we now also define the connected sum if one or both topological manifolds involved are non-orientable.

Definition. Let $n \in \mathbb{N}$. Let M and N be two connected non-empty n-dimensional topological manifolds.

(1) Suppose that at least one of M or N is non-orientable. We pick two locally flat embeddings $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\psi\colon \overline{B}^n \to N \setminus \partial N$ and we define the *connected sum of M and N* as

$$M\#N := \big(M\setminus\varphi(B^n)\big) \sqcup \big(N\setminus\psi(B^n)\big)/\sim \text{ where } \varphi(P)\sim\psi(P) \text{ for all } P\in S^{n-1}.$$

[1404] That is not really a surprise, on page 2065 we already saw wild knots in $\mathbb{R}^3$ and on page 1297 we saw the Alexander horned sphere. Both give examples of embeddings of topological manifolds such that the image is not a submanifold.

[1405] On page 1297 we referred to $f(S^2)$ as the Alexander horned sphere.

(2) If both M and N are orientable, then we pick orientations for M and N. Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\psi\colon \overline{B}^n \to N \setminus \partial N$ be locally flat embeddings such that φ is orientation-preserving and such that ψ is orientation-reversing. As before we define the *connected sum of the oriented topological manifolds M and N* as

$$M\#N := \big(M \setminus \varphi(B^n)\big) \sqcup \big(N \setminus \psi(B^n)\big)/\sim \text{ where } \varphi(P)\sim\psi(P) \text{ for all } P\in S^{n-1}.$$

We refer to Figure 1430 for an illustration. For completeness we define the connected sum of an empty manifold with another connected topological manifold M to be M, i.e. we set $\varnothing\#M := M\#\varnothing := M$.

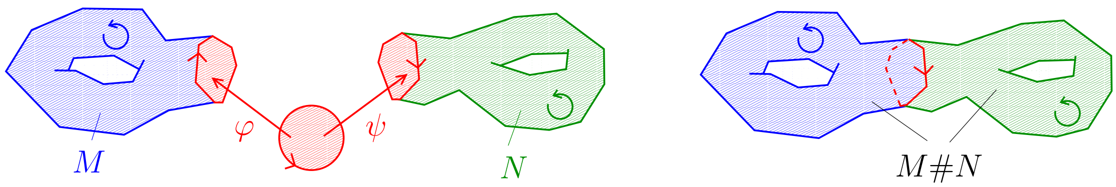


FIGURE 1430

The following proposition is the analogue of Proposition 8.35.

Proposition 100.8. *Let $n \in \mathbb{N}$. Let M and N be two connected n-dimensional topological manifolds. (If both M and N are orientable, then we demand that M and N are oriented.) Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\psi\colon \overline{B}^n \to N \setminus \partial N$ be two locally flat embeddings. (If M and N are orientable, then we demand that φ is orientation-preserving and that ψ is orientation-reversing.) The following five statements hold:*

(1) *The resulting connected sum $M\#N$ is an n-dimensional topological manifold such that $M \setminus \varphi(B^n)$ and $N \setminus \psi(B^n)$ are submanifolds.*
(2) *If M and N are closed, then $M\#N$ is also a closed topological manifold.*
(3) *If $n \geq 2$ or if at least one of M or N is closed, then $M\#N$ is also connected.*
(4) *The homeomorphism type of $M\#N$ does not depend on the choice of φ and ψ.*
(5) *If M and N are both oriented, then the topological manifold $M\#N$ admits a unique orientation that coincides with the orientations of the submanifolds $M \setminus \varphi(B^n)$ and $N \setminus \psi(B^n)$.*

Convention. We always equip the connected sum of two oriented connected n-dimensional topological manifolds with the orientation from Proposition 100.8 (5).

Most of the statements of Proposition 100.8 are easy to prove.

PROOF OF PROPOSITION 100.8 (1), (2), (3) AND (5). The proof is basically identical to the proofs of the corresponding statements in the smooth setting that we proved in Proposition 8.35. There are just a few minor modifications that we need to make:

(1) We used Proposition 8.2 to show that the image of a smooth embedding is a submanifold. This feature is now built into the definition of a locally flat embedding.
(2) We need to replace Propositions 8.2 and 8.15 by Propositions 44.3 and 44.8. ■

The fact that the connected sum operation does not really depend on the choice of the locally flat topological embeddings is much more difficult to prove. The key to doing so is the following difficult theorem.

Theorem 100.9. (Annulus Theorem) *Let $n \in \mathbb{N}$ and let $\varphi, \psi \colon \overline{B}^n \to \mathbb{R}^n$ be two locally flat embeddings. If $\varphi(\overline{B}^n) \subset \psi(B^n)$, then $\psi(\overline{B}^n) \setminus \varphi(B^n)$ is homeomorphic to $[0,1] \times S^{n-1}$.*

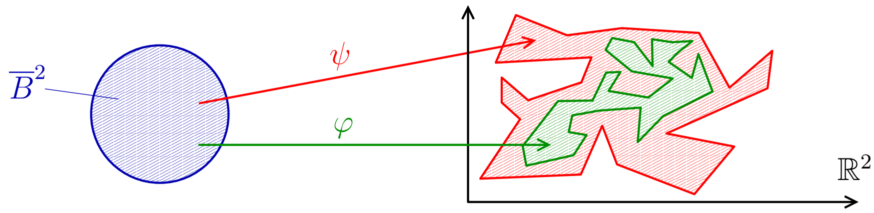


FIGURE 1431. Illustration of the Annulus Theorem 100.9.

PROOF. The case $n = 1$ is trivial. For $n = 2$ and $n = 3$ the Annulus Theorem follows from the work of Tibor Radó [**Rad26**] and Edwin Moise [**Moi52, Moi77**]. The Annulus Theorem was proved for dimensions $n \geq 5$ by Rob Kirby [**Kir69**] and in dimension 4 by Frank Quinn [**Qu82**, p. 506]. We also refer to [**Edw84**, p. 247] for more information. ■

Using the Annulus Theorem 100.9 one can now prove the following theorem which can be viewed as the topological analogue of Theorem 8.36.

Theorem 100.10. *Let M be an n-dimensional topological manifold. (If M is orientable, then we pick an orientation for M.) In the following let $\varphi_1, \ldots, \varphi_m \colon \overline{B}^n \to M \setminus \partial M$ and $\psi_1, \ldots, \psi_m \colon \overline{B}^n \to M \setminus \partial M$ be two sets of m locally flat embeddings of $\overline{B}^n$ with disjoint images. (If M is orientable, then we demand that all φ_i and ψ_i are orientation-preserving or that all are orientation-reversing.) If $n \geq 2$ or if $m = 1$, then there exists a homeotopy*

$$F \colon M \times [0,1] \;\to\; M$$

rel some neighborhood of ∂M from the identity $F_0 = \mathrm{id}_M$ to a homeomorphism $F_1 \colon M \to M$ such that for each $i \in \{1, \ldots, m\}$ we have $F_1 \circ \varphi_i = \psi_i$. (Note that if M is orientable, then the homeomorphism F_1 is in fact orientation-preserving by Exercise 86.3).

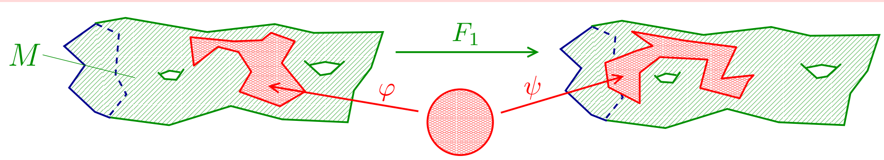


FIGURE 1432. Illustration of Theorem 100.10.

Remark. We consider $\overline{B}^3$ as a submanifold of $\mathbb{R}^3$. Since the image of the embedding $\varphi \colon \overline{B}^3 \to \mathbb{R}^3$ that is given by the Alexander horned ball is not a submanifold of the topological manifold we see that there is no homeotopy $F \colon \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ with $F_1 \circ \varphi = \mathrm{id}$. In particular we see that in the formulation of Theorem 100.10 we cannot drop the hypothesis that the embeddings are locally flat.

Proof of Theorem 100.10. This theorem follows from the Annulus Theorem 100.9 together with reasonably elementary arguments. The details are worked out in [**FNOP19**, Proof of Theorem 5.11]. Alternatively the reader is encouraged to assemble the argument by using the obvious analogues Exercise 17.6 and Exercise 8.7, which were both formulated in the setting of smooth manifolds, in the setting of topological manifolds. ■

Proof of Proposition 100.8 (4). Thanks to our preparations the proof is basically identical to the proof of the corresponding statement in the smooth setting that we proved in Proposition 8.35. All we need to is to replace Theorem 8.36 by Theorem 100.10. ■

100.6. **The intersection form and the connected sum operation.** First let use see how the homology groups and fundamental groups behave under the connected sum operation.

Proposition 100.11. *Let $n \in \mathbb{N}$ and let M and N be two closed oriented connected n-dimensional topological manifolds. Furthermore let $\varphi\colon \overline{B}^n \to M$ be an orientation-preserving locally flat embedding and let $\psi\colon \overline{B}^n \to N$ be an orientation-reversing locally flat embedding. Then $\mathrm{H}_0(M\#N;R) \cong \mathrm{H}_n(M\#N;R) \cong R$. Furthermore, for any $i \neq 0, n$ and any commutative ring R the following inclusion induced maps are isomorphisms*

$$\mathrm{H}_i(M\#N;R) \xleftarrow{\cong} \mathrm{H}_i(M\setminus\varphi(B^n);R) \oplus \mathrm{H}_i(N\setminus\psi(B^n);R) \xrightarrow{\cong} \mathrm{H}_i(M;R) \oplus \mathrm{H}_i(N;R).$$

Furthermore if $n \geq 3$, then the following inclusion induced maps are isomorphisms

$$\pi_1(M\#N) \xleftarrow{\cong} \pi_1(M \setminus \varphi(B^n)) * \pi_1(N \setminus \psi(B^n)) \xrightarrow{\cong} \pi_1(M) * \pi_1(N).$$

Proof ($*$). We proved the statements in Proposition 20.12 and Proposition 68.13 for smooth manifolds. The proof for topological manifolds is very similar, we just need to note that it follows from by Proposition 74.12 (5) and the Mayer–Vietoris Theorem 74.14 that the Mayer–Vietoris argument in the proof of Lemma 68.14 works for topological manifolds. ■

The next proposition says that the isomorphism of homology groups from Proposition 100.11 induces an isometry of intersection forms.

Proposition 100.12. *Let $n \geq 1$ and let M and N be closed oriented $2n$-dimensional topological manifolds. Furthermore let $\varphi\colon \overline{B}^n \to M$ be an orientation-preserving locally flat embedding and let $\psi\colon \overline{B}^n \to N$ be an orientation-reversing locally flat embedding. The isomorphism $\mathrm{H}_n(M\#N;\mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_n(M;\mathbb{Z}) \oplus \mathrm{H}_n(N;\mathbb{Z})$ from Proposition 100.11 induces an isometry*

$$Q_{M\#N} \xrightarrow{\cong} Q_M \oplus Q_N$$

of forms.

Example. We consider the surface Σ of genus two. In Figure 1433 we recall that we can view Σ as the connected sum of two tori. It thus follows from Proposition 100.12, the discussion on page 2414 and Lemma 100.2 (2) that the intersection form on $\mathrm{H}_1(\Sigma;\mathbb{Z}) \cong \mathbb{Z}^4$

is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Not surprisingly this recovers the result from page 2413.

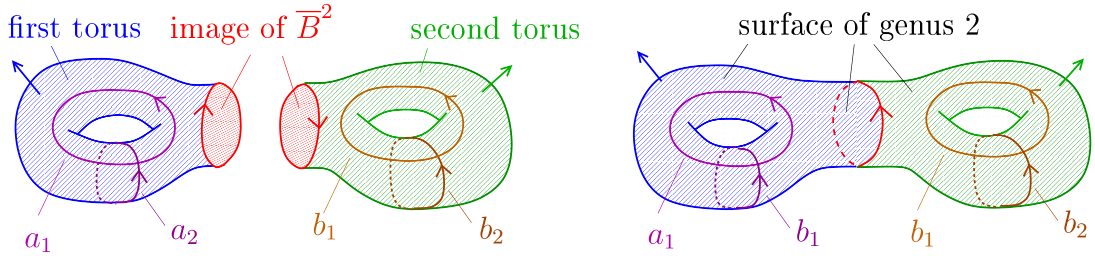


FIGURE 1433

Remark. Let $n \geq 1$ and let M and N be closed oriented $2n$-dimensional topological manifolds. Furthermore let $\varphi\colon \overline{B}^n \to M$ be an orientation-preserving locally flat embedding and let $\psi\colon \overline{B}^n \to N$ be an orientation-reversing locally flat embedding. Figure 1433 suggests the following approach to proving Proposition 100.12:

(1) First we pick n-dimensional cycles $a_1, \dots, a_k$ in $M \setminus \varphi(\overline{B}^{2n})$ such that $[a_1], \dots, [a_k]$ form a basis of $\mathrm{FH}_n(M;\mathbb{Z})$ and which intersect nicely, in the sense of the definition on page 2312.

(2) Similarly we pick transverse n-dimensional cycles $b_1, \dots, b_l$ in $N \setminus \psi(\overline{B}^{2n})$ such that $[b_1], \dots, [b_l]$ form a basis of $\mathrm{FH}_n(N;\mathbb{Z})$ and which intersect nicely.

(3) It follows immediately from Proposition 100.11 that $[a_1], \dots, [a_k], [b_1], \dots, [b_l]$ form a basis for $\mathrm{FH}_n(M\#N;\mathbb{Z})$.

(4) In Theorem 95.9 we saw that the intersection form of cycles that intersect nicely in terms of the algebraic intersection number that we introduced on page 2312. We use this fact together with the basis from (3) to determine the intersection form of $M\#N$. Here we make the following observations:
   (a) By the orientation conventions from Proposition 100.8 (3) the algebraic intersection numbers of the a_i in M are the same as in $M\#N$.
   (b) Similarly the algebraic intersection numbers of the b_i in N are the same as in $M\#N$,
   (c) Since the a_i do not intersect the b_j we see that the algebraic intersection number $a_i \cdot b_j = 0$ for any i, j.

   Summarizing we obtain immediately from Theorem 95.9 that $Q_M \oplus Q_N \cong Q_{M\#N}$.

The above argument sounds very convincing. But there is a major catch: how can we find such $a_1, \dots, a_k$ and $b_1, \dots, b_l$? The problem is not that we need to avoid $\varphi(\overline{B}^{2n})$ and $\psi(\overline{B}^{2n})$.[1406] The problem is to find such cycles that are transverse. Even though we have

[1406]Why is this not a problem?

some tools, e.g. we have Theorem 9.15 and the Transversality Theorem 9.10, it is very hard to turn the above approach into a rigorous proof. Furthermore, any attempt to make this approach work would inevitably end up with an unreadable mess.

After the discussion in the previous remark we recall the advice from Rob Kirby [**Kir89**, p. 21], see page 2276:

> "Think with [algebraic] intersection [numbers], prove with cup products."

Thus we will use cup products to prove Proposition 100.12.

Proof of Proposition 100.12 (∗). Throughout the proof let $n \geq 1$. We start out with a lengthy preparation. Thus let X and Y be two oriented connected $2n$-dimensional topological manifolds such that ∂X and ∂Y are both homeomorphic to S^{2n-1}. We equip ∂X and ∂Y with the orientation coming from Proposition 86.19. Let $\theta\colon \partial X \to \partial Y$ be an an orientation-reversing homeomorphism. We write $Z = X \cup_\theta Y$. In the following we view X and Y as subsets of Z. By Proposition 44.8 we know that X and Y are in fact codimension-zero submanifolds of Z and that we can equip Z with an orientation such that the induced orientations on X and Y are the given orientations.

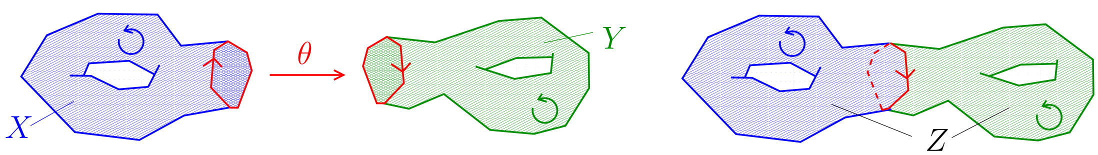


Figure 1434. Illustration for the proof of Proposition 100.12.

We start out with the following claim.

Claim. The inclusions $i\colon X \to Z$ and $j\colon Y \to Z$ induce an isometry

$$Q_X \oplus Q_Y \xrightarrow{\cong} Q_Z.$$

As in Proposition 100.12 we see that the two inclusion maps i and j induce an isomorphism $\mathrm{H}_n(X;\mathbb{Z}) \oplus \mathrm{H}_n(Y;\mathbb{Z}) \to \mathrm{H}_n(Z;\mathbb{Z})$. Evidently this isomorphism descends to an isomorphism $\mathrm{FH}_n(X;\mathbb{Z}) \oplus \mathrm{FH}_n(Y;\mathbb{Z}) \to \mathrm{FH}_n(Z;\mathbb{Z})$. Therefore it suffices to prove the following three statements.

(1) for $a, b \in \mathrm{FH}_n(X;\mathbb{Z})$ we have $Q_X(a,b) = Q_Z(i_*(a), i_*(b))$,
(2) for $a, b \in \mathrm{FH}_n(Y;\mathbb{Z})$ we have $Q_Y(a,b) = Q_Z(j_*(a), j_*(b))$,
(3) for $a \in \mathrm{FH}_n(X;\mathbb{Z})$ and $b \in \mathrm{FH}_n(Y;\mathbb{Z})$ we have $Q_Z(i_*(a), j_*(b)) = 0$.

We start out with the proof of the first statement. Thus let $a, b \in \mathrm{FH}_n(X;\mathbb{Z})$. We write $\mathring{X} = X \setminus \partial X$. We denote by $p\colon (Z, \varnothing) \to (Z, Z \setminus \mathring{X})$ and $i\colon (X, \partial X) \to (Z, Z \setminus \mathring{X})$ the two obvious inclusion maps of pairs of topological spaces. It is a consequence of the Excision Theorem 43.20 for Topological Manifolds that $i_*\colon \mathrm{FH}_k(X, \partial X;\mathbb{Z}) \to \mathrm{FH}_k(Z, Z \setminus \mathring{X};\mathbb{Z})$ is an isomorphism for every $k \in \mathbb{N}_0$. Then

$$\begin{aligned} Q_X(a,b) &\underset{\substack{\uparrow \\ \text{by definition of } Q_X}}{=} \langle \mathrm{PD}_X(a) \cup \mathrm{PD}_X(b), [X] \rangle \underset{\substack{\uparrow \\ \text{Lemma 74.6 (3) applied to } i\colon (X,\partial X) \to (Z, Z \setminus \mathring{X}), \\ \text{together with Proposition 82.4 (1)}}}{=} \big\langle (i^*)^{-1}(\mathrm{PD}_X(a)) \cup (i^*)^{-1}(\mathrm{PD}_X(b)), i_*([X]) \big\rangle \\ &\underset{\substack{\uparrow \\ \text{first note that by Lemma 87.24 we have, } i_*([X]) = p_*([Z]), \text{ next apply} \\ \text{Lemma 74.6 (3) to } p, \text{ finally apply Proposition 82.4 (1) to } p}}{=} \big\langle p^*((i^*)^{-1}(\mathrm{PD}_X(a))) \cup p^*((i^*)^{-1}(\mathrm{PD}_X(b))), [Z] \big\rangle \\ &\underset{\substack{\uparrow \\ \text{by Lemma 89.5 we know that} \\ p^*(i^*)^{-1}(\mathrm{PD}_X(c)) = \mathrm{PD}_Z(i_*(c))}}{=} \big\langle \mathrm{PD}_Z(i_*(a)) \cup \mathrm{PD}_Z(i_*(b)), [Z] \big\rangle \underset{\substack{\uparrow \\ \text{definition of } Q_Z}}{=} Q_Z(i_*(a), i_*(b)). \end{aligned}$$

The second statement is of course proved the same way. We turn to the proof of the third statement. So let $a \in \mathrm{FH}_n(X;\mathbb{Z})$ and $b \in \mathrm{FH}_n(Y;\mathbb{Z})$. The above argument shows that we can compute $Q_Z(i_*(a), j_*(b))$ in terms of the cup product in Z of an element in the image $\mathrm{H}^n(Z, Z \setminus \mathring{X}) \to \mathrm{H}^n(Z)$ and an element in the image of $\mathrm{H}^n(Z, Z \setminus \mathring{Y};\mathbb{Z}) \to \mathrm{H}^n(Z;\mathbb{Z})$. But $(Z \setminus \mathring{X}) \cup (Z \setminus \mathring{Y}) = Z$. So it follows from Proposition 82.4 (1) that this cup product is zero. ⊞

After these preparations it is very easy to prove the actual statement of the proposition. Thus let M and N be closed oriented $2n$-dimensional topological manifolds and let $\varphi\colon \overline{B}^{2n} \to M$ be an orientation-preserving locally flat embedding and let $\psi\colon \overline{B}^{2n} \to N$ be an orientation-reversing locally flat embedding. We write $X = M \setminus \varphi(B^{2n})$ and $Y = N \setminus \psi(B^{2n})$. It follows from the definition of a locally flat embedding together with Proposition 44.3 that X and Y are compact topological manifolds. With this notation we can write $M = X \cup \overline{B}^{2n}$ and $N = Y \cup \overline{B}^{2n}$. It follows from the claim and the fact that $\mathrm{H}_n(\overline{B}^{2n};\mathbb{Z}) = 0$ that the inclusions induce isometries

$$Q_{M\#N} \xleftarrow{\cong} Q_X \oplus Q_Y \xrightarrow{\cong} Q_M \oplus Q_N.$$

We have thus proved the desired statement. ■

We recall the following question.

Question 68.15. *Are the topological spaces $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ and $S^2 \times S^2$ homeomorphic?*

These topological spaces are hard to distinguish since we saw that they are simply connected and that their (co-) homology groups agree. Now we want to look at the intersection forms to see whether they can help us in distinguishing these three smooth manifolds.

Example. On page 2415 we saw that the intersection form of $\mathbb{C}\mathrm{P}^2$ is represented by the 1×1–matrix (1) and we saw that the intersection form of $\overline{\mathbb{C}\mathrm{P}}^2$ is represented by the 1×1–matrix (-1). From Proposition 100.12 we thus obtain:

(1) the intersection form of $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, whereas

(2) the intersection form of $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Finally we recall that on page 2414 we showed that

(3) the intersection form of $S^2 \times S^2$ is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Motivated by Question 68.15 and Lemma 100.6 the following question arises.

Question 100.13. *Are the forms over $\mathbb{Z}^2$ represented by the matrices*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

isometric (possibly up to a sign reversal)?

In Chapterr̃efch:intersection-forms-ii we will develop the theory of non-singular forms to answer Question 100.13.

100.7. **The asymmetric intersection pairings** (∗)**.** On page 2416 we saw that the intersection form of a topological manifold with non-empty boundary can be degenerate. In this and the following section we will associate to a compact oriented topological manifold the asymmetric intersection pairings which are non-singular and the reduced intersection form which is at least non-degenerate.

Before we provide the definitions let us recall and introduce a new definition.

Definition. Let R be a commutative ring and let V, W and S be R-modules. Recall that according to the definition on page 2192 we say that a pairing $\langle\ ,\ \rangle \colon V \times W \to R$ is *non-singular* if the induced maps

$$\begin{array}{rcl} V & \to & \operatorname{Hom}_R(W,S) \\ v & \mapsto & \left(\begin{array}{rcl} W & \to & S \\ w & \mapsto & \langle v,w\rangle \end{array} \right) \end{array} \quad \text{and} \quad \begin{array}{rcl} W & \to & \operatorname{Hom}_R(V,S) \\ w & \mapsto & \left(\begin{array}{rcl} V & \to & S \\ v & \mapsto & \langle v,w\rangle \end{array} \right) \end{array}$$

are isomorphisms. We now say that the pairing is *non-degenerate* if the above two maps are both monomorphisms. Otherwise we say that the pairing is *degenerate*.

Example. Let R be a commutative domain and let $A \in \mathrm{M}(n \times n; R)$. Elementary algebra, similar to the proof of Lemma 90.1, shows that the form

$$\begin{array}{rcl} R^n \times R^n & \to & R \\ (v,w) & \mapsto & v^T A w \end{array}$$

is non-singular if and only if $\det(A)$ is a unit in R and it is non-singular if and only if $\det(A)$ is non-zero.

For the record let us state the following lemma that follows almost immediately from the definitions.

Lemma 100.14. *Let R be a commutative domain. Let V, W and S be R-modules and let*

$$\langle\ ,\ \rangle \colon V \times W \to S$$

be a pairing. The following two statements are equivalent:

(1) *The pairing is non-degenerate.*

(2) *For every $v \in V$ there exists a $w \in W$ with $\langle v, w\rangle \neq 0$ and for every $w \in W$ there exists a $v \in V$ with $\langle v, w\rangle \neq 0$.*

In Proposition 100.4 we saw that if we are given a *closed* oriented $2n$-dimensional topological manifold M, then the intersection form

$$\begin{array}{rrcl} Q_M\colon & \mathrm{FH}_n(M;\mathbb{Z}) \times \mathrm{FH}_n(M;\mathbb{Z}) & \to & \mathbb{Z} \\ & (a,b) & \mapsto & \Big\langle \underbrace{\underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})}}_{\in \mathrm{FH}^{2n}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle. \end{array}$$

is non-singular. On page 2416 we saw that if M is *not closed*, then this conclusion does not necessarily hold. More precisely we gave an explicit example of a compact orientable smooth manifold, namely the torus minus two open disks, such that the intersection form is degenerate, in particular it is singular.

We move on to the definition of asymmetric intersection pairings on topological manifolds.

Definition. Let M be a compact oriented n-dimensional topological manifold and suppose that we are given two compact $(n-1)$-dimensional submanifolds A and B of ∂M. We suppose that one of the following two conditions is satisfied:[1407]

(i) We have $\partial M = A \cup B$ and $A \cap B = \partial A = \partial B$.

(ii) We have $\mathring{A} \cap \mathring{B} = \emptyset$.

We write $X := \partial M \setminus \mathring{A}$ and $Y := \partial M \setminus \mathring{B}$.[1408] For each $k \in \mathbb{N}_0$ we refer to the pairing[1409]

$$\begin{array}{rrcl} Q^{\mathrm{as}}_{M,A,B}\colon & \mathrm{FH}_k(M,A;\mathbb{Z}) \times \mathrm{FH}_{n-k}(M,B;\mathbb{Z}) & \to & \mathbb{Z} \\ & (v,w) & \mapsto & \Big\langle \underbrace{\underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^{n-k}(M,X;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^{k}(M,Y;\mathbb{Z})}}_{\in \mathrm{H}^{n}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle \end{array}$$

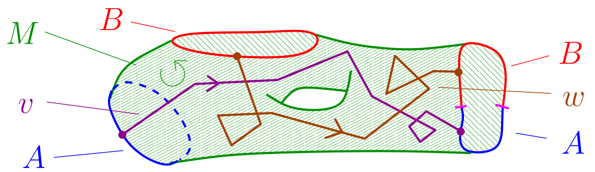
as an *asymmetric intersection pairing of M.* When A and B are clear from the context we write Q^{as}_M instead of the cumbersome $Q^{\mathrm{as}}_{M,A,B}$.

Remark. In the previous definition, if we take $A = B = \emptyset$, then we obtain precisely the intersection form as introduced on page 2413, in other words, $Q^{\mathrm{as}}_{M,\emptyset,\emptyset} = Q_M$.

[1407] For the most part we will be interested in Case (1), only in one setting we will find it more pleasing to work with the setting of Case (2).

[1408] Note that by Proposition 44.3 we know that X and Y are also compact submanifolds of ∂M with $\partial X = \partial A = A \cap X$ and $\partial Y = \partial B = B \cap Y$.

[1409] Note that in the settings (1) and (2), it follows from Proposition 74.12 (5) and (3) that (M, A, B) is an excisive triad. In particular it follows, see the definition on page 2005, that the cup product is actually defined.



$$Q^{\mathrm{as}}_{M,A,B}(v,w) = +1 - 1 = 0$$

FIGURE 1435

Example. Let M be the torus minus two open balls. We are interested in the (asymmetric) intersection pairings

$$Q_M \colon\ \mathrm{H}_1(M;\mathbb{Z}) \times \mathrm{H}_1(M;\mathbb{Z}) \ \to\ \mathbb{Z} \qquad \text{and} \qquad Q^{\mathrm{as}}_{M,\varnothing,\partial M} \colon\ \mathrm{H}_1(M;\mathbb{Z}) \times \mathrm{H}_1(M,\partial M;\mathbb{Z}).$$

We consider the four curves a, b, c, d on M that are shown in Figure 1436. Note that the curves a, b, c represent a basis for $\mathrm{H}_1(M;\mathbb{Z})$, whereas the curves a, b, d represent a basis for $\mathrm{H}_1(M,\partial M;\mathbb{Z})$.[1410] Using these bases and using Theorem 94.5 we see that

$$Q_M \text{ is represented by } \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \text{ and } Q^{\mathrm{as}}_{M,\varnothing,\partial M} \text{ is represented by } \begin{array}{c c} & \begin{array}{ccc} a & b & d \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Note that the former matrix has determinant zero whereas the latter matrix has determinant 1. This calculation, together with Lemma 90.1 shows that the former pairing is degenerate whereas the latter pairing is non-singular.[1411] As we will see in the next proposition, it is not a coincidence that in this particular example the asymmetric intersection pairing is non-singular.

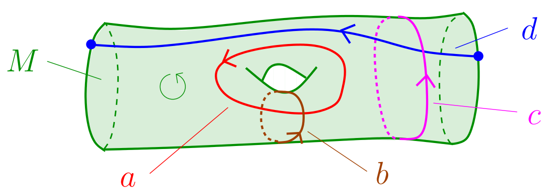


FIGURE 1436

Proposition 100.15. *Let M be a compact oriented n-dimensional topological manifold. Furthermore suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$.*

(1) *Given any $k \in \mathbb{N}_0$ the following pairing is non-singular:*[1412]

$$Q^{\mathrm{as}}_{M,A,B} \colon\ \mathrm{FH}_k(M,A;\mathbb{Z}) \times \mathrm{FH}_{n-k}(M,B;\mathbb{Z}) \ \to\ \mathbb{Z}.$$

[1410] Why is that?

[1411] In fact, using this observation and Exercise 90.1 (c) one can easily answer the previous footnote.

For the remaining two statements of the proposition we assume that $n = 2k$ is even. We denote by $p_*\colon \mathrm{H}_k(M;\mathbb{Z}) \to \mathrm{H}_k(M, \partial M;\mathbb{Z})$ the obvious map.

(2) *The following diagram commutes:*

$$\begin{tikzcd}
\mathrm{FH}_k(M;\mathbb{Z}) \times \mathrm{FH}_k(M;\mathbb{Z}) \arrow[r, "Q_M"] \arrow[d, "\mathrm{id} \times p_*"] & \mathbb{Z} \arrow[d, "\mathrm{id}"] \\
\mathrm{FH}_k(M;\mathbb{Z}) \times \mathrm{FH}_k(M, \partial M;\mathbb{Z}) \arrow[r, "Q^{\mathrm{as}}_{M,\varnothing,\partial M}"] & \mathbb{Z}.
\end{tikzcd}$$

(3) *Let $v_1, \dots, v_m$ be a basis of the free abelian group $\mathrm{FH}_k(M;\mathbb{Z})$. By* (1) *together with Exercise 90.1* (a) *there exists a unique "dual" basis $v_1^*, \dots, v_m^*$ of the free abelian group $\mathrm{FH}_k(M, \partial M;\mathbb{Z})$ with $Q^{\mathrm{as}}_{M,\varnothing,\partial M}(v_i, v_j^*) = \delta_{ij}$ for all $i, j \in \{1, \dots, m\}$. With this notation the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{FH}_k(M;\mathbb{Z}) \arrow[r, "p_*"] & \mathrm{FH}_k(M, \partial M;\mathbb{Z}) \\
\mathbb{Z}^m \arrow[u, "\cong", "e_i \mapsto v_i"'] \arrow[r, "\text{multiplication by}", "\text{matrix } (Q_M(v_i, v_j))_{i,j=1,\dots,m}"'] & \mathbb{Z}^m. \arrow[u, "e_i \mapsto v_i^*", "\cong"']
\end{tikzcd}$$

Put differently, let $(Q_M(v_i, v_j))_{i,j=1,\dots,m}$ be the $(m \times m)$–matrix that represents the intersection form Q_M with respect to the basis $v_1, \dots, v_m$. This matrix represents the map $p_*\colon \mathrm{FH}_k(M;\mathbb{Z}) \to \mathrm{FH}_k(M, \partial M;\mathbb{Z})$ with respect to the bases $v_1, \dots, v_m$ and $v_1^*, \dots, v_m^*$.

Proof.

(1) We consider the map

$$\Psi\colon\ \mathrm{FH}_k(M, A;\mathbb{Z}) \xrightarrow{\mathrm{PD}_M} \mathrm{FH}^{n-k}(M, B;\mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(\mathrm{FH}_{n-k}(M, B;\mathbb{Z}), \mathbb{Z}).$$

First note that basically the same argument as in the proof of Lemma 100.5 shows that for every $a \in \mathrm{FH}_k(M, A;\mathbb{Z})$ and every $b \in \mathrm{FH}_{n-k}(M, B;\mathbb{Z})$ we have $Q^{\mathrm{as}}_{M,A,B}(a, b) = \Psi(b)(a)$. Furthermore note that as in the proof of Proposition 100.4 (3) we see that the two maps PD_M and $\mathrm{ev}\colon \mathrm{FH}^{n-k}(M, B;\mathbb{Z}) \to \mathrm{Hom}(\mathrm{FH}_{n-k}(M, B;\mathbb{Z}), \mathbb{Z})$ are isomorphisms. In particular $\Psi = \mathrm{ev} \circ \mathrm{PD}$ is an isomorphism. Together with the remark on page 2193 this discussion implies that $Q^{\mathrm{as}}_{M,A,B}$ is non-singular.

(2) This statement follows easily from the definitions together with Lemma 83.8 and Proposition 82.4. In fact we will work out the details in the proof of Lemma 100.16 (2).

(3) Let $i \in \{1, \dots, m\}$. We write $p_*(v_i) = \sum\limits_{j=1}^{m} b_{ji} \cdot v_j^*$. Let $j \in \{1, \dots, m\}$. We see that

$$b_{ji} \underset{\substack{\uparrow \\ \text{since } Q^{\mathrm{as}}_{M,\varnothing,\partial M}(v_j, v_k^*) = \delta_{jk}}}{=} Q^{\mathrm{as}}_{M,\varnothing,\partial M}\Big(v_j, \sum_{k=1}^{m} b_{ki} \cdot v_k^*\Big) = Q^{\mathrm{as}}_{M,\varnothing,\partial M}(v_j, p_*(v_i)) \underset{\substack{\uparrow \\ \text{follows from (2)}}}{=} Q_M(v_j, v_i).$$

[1412]This statement is a generalization of Proposition 100.4 (3).

Since by definition the matrix (b_{ij}) represents the map p_* with respect to the given bases we obtain the desired result. ■

In the remainder of this section we prove several properties of the asymmetric intersection pairings.

Lemma 100.16. *Let M be a compact oriented connected n-dimensional topological manifold and suppose that we are given two compact $(n-1)$-dimensional submanifolds A and B of ∂M. We suppose that one of the following two conditions is satisfied:*

(i) *We have $\partial M = A \cup B$ and $A \cap B = \partial A = \partial B$.*

(ii) *We have $\mathring{A} \cap \mathring{B} = \varnothing$.*

The following two statements hold.

(1) *If we are given $v \in \mathrm{H}_k(M, A; \mathbb{Z})$ and $w \in \mathrm{H}_{n-k}(M; \mathbb{Z})$, then*
$$Q^{\mathrm{as}}_{M,A,B}(v,w) \;=\; (-1)^{k\cdot(n-k)} \cdot Q^{\mathrm{as}}_{M,B,A}(w,v).$$

(2) *Suppose we are given two compact $(n-1)$-dimensional submanifolds A' and B' of ∂M that also satisfy (i) or (ii) and that furthermore satisfy $A \subset A'$ and $B \subset B'$. We denote by $p\colon (M,A) \to (M,A')$ and $q\colon (M,B) \to (M,B')$ the obvious maps. For any $v \in \mathrm{H}_k(M, A'; \mathbb{Z})$ and $w \in \mathrm{H}_{n-k}(M, B'; \mathbb{Z})$ we have the following equality:*
$$Q^{\mathrm{as}}_{M,A',B'}(p_*(v), q_*(w)) \;=\; Q^{\mathrm{as}}_{M,A,B}(v,w).$$

Proof (∗). We write $X = \partial M \setminus \mathring{A}$ and $Y = \partial M \setminus \mathring{B}$.

(1) Given $v \in \mathrm{H}_k(M, A; \mathbb{Z})$ and $w \in \mathrm{H}_{n-k}(M, B; \mathbb{Z})$ we have the following equality:
$$\overbrace{\Big\langle \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^{n-k}(M,X;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^{k}(M,Y;\mathbb{Z})}, [M] \Big\rangle}^{=Q^{\mathrm{as}}_{M,A,B}(v,w)} \underset{\substack{\uparrow \\ \text{by Proposition 82.3}}}{=} (-1)^{(n-k)\cdot k} \cdot \overbrace{\Big\langle \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^{k}(M,Y;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^{n-k}(M,X;\mathbb{Z})}, [M] \Big\rangle}^{=Q^{\mathrm{as}}_{M,B,A}(w,v)}.$$

(2) Not surprisingly we write $X' = \partial M \setminus \mathring{A}'$ and $Y' = \partial M \setminus \mathring{B}'$. Note that by hypothesis we have $X' \subset X$ and $Y' \subset Y$. Let $f\colon (M,X') \to (M,X)$ and $g\colon (M,Y') \to (M,Y)$ be the obvious maps. Given $v \in \mathrm{H}_k(M, A'; \mathbb{Z})$ and $w \in \mathrm{H}_{n-k}(M, B'; \mathbb{Z})$ we have the following equality:
$$\begin{aligned}
\overbrace{\Big\langle \underbrace{\mathrm{PD}_M(p_*(v))}_{\in \mathrm{H}^{n-k}(M,X';\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(q_*(w))}_{\in \mathrm{H}^{k}(M,Y';\mathbb{Z})}, [M] \Big\rangle}^{=Q^{\mathrm{as}}_{M,A',B'}(p_*(v),q_*(w))} &\underset{\substack{\text{follows immediately from Lemma 83.8} \\ \downarrow}}{=} \Big\langle \underbrace{f^*(\mathrm{PD}_M(v))}_{\in \mathrm{H}^{n-k}(M,X';\mathbb{Z})} \cup \underbrace{g^*(\mathrm{PD}_M(w))}_{\in \mathrm{H}^{k}(M,Y';\mathbb{Z})}, [M] \Big\rangle \\
&\underset{\substack{\text{Proposition 82.4} \\ \downarrow}}{=} \Big\langle \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^{n-k}(M,X;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^{k}(M,Y;\mathbb{Z})}, [M] \Big\rangle \;=\; Q^{\mathrm{as}}_{M,A,B}(v,w).
\end{aligned}$$
■

Our next result on asymmetric intersection pairing deals with the interplay between the asymmetric intersection pairing on a topological manifold and the asymmetric intersection pairings on its boundary.

Proposition 100.17. *Let M be a compact oriented n-dimensional topological manifold. Suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. Let $a \in \mathrm{H}_k(A, \partial A; \mathbb{Z})$ and let $c \in \mathrm{H}_{n-k+1}(M, A; \mathbb{Z})$. We denote by $w\colon (A, \partial A) \to (M, B)$ the obvious map and we denote by $\partial\colon \mathrm{H}_{n-k+1}(M, A; \mathbb{Z}) \to \mathrm{H}_{n-k}(A; \mathbb{Z})$ the connecting homomorphism of the long exact sequence in homology of the pair (M, A). In this setting the following equality holds:*

$$Q^{\mathrm{as}}_{A,\varnothing,\partial A}(\partial c, a) \;=\; Q^{\mathrm{as}}_{M,A,B}(c, w_*(a)).$$

In other words, the following diagram commutes in a suitable sense:

$$\begin{tikzcd}
\mathrm{H}_{n-k+1}(M, A; \mathbb{Z}) \arrow[d, "\partial"] & \times & \mathrm{H}_k(M, B; \mathbb{Z}) \arrow[drr, "\mathrm{Q}^{\mathrm{as}}_{M,A,B}"] & & \\
 & & & & \mathbb{Z}. \\
\mathrm{H}_{n-k}(A; \mathbb{Z}) & \times & \mathrm{H}_k(A, \partial A; \mathbb{Z}) \arrow[uu, "w_*"'] \arrow[urr, "Q^{\mathrm{as}}_{A,\varnothing,\partial A}"'] & &
\end{tikzcd}$$

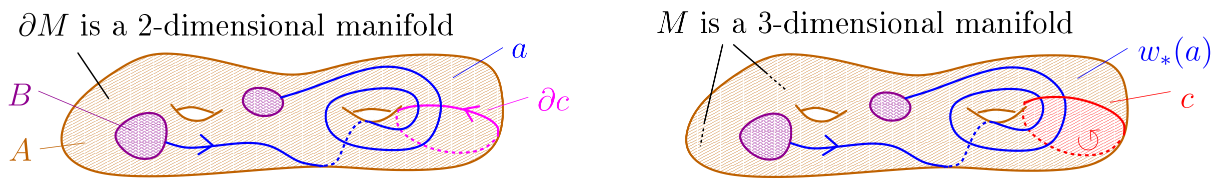


FIGURE 1437. Illustration for Proposition 100.17

In the proof of Proposition 100.17 we will need the following generalization of Lemma 82.6.

Lemma 100.18. *Let (X, A, B) be an excisive triad of topological spaces and let R be a commutative ring. We denote by $i\colon (A, A \cap B) \to (X, B)$ and $j\colon (A, A \cap B) \to (A \cup B, B)$ the inclusion maps and we denote by δ the connecting homomorphisms in the long exact sequence in cohomology corresponding to the pair (X, A) and the triple $(X, A \cup B, B)$. Note that it follows from Lemma 74.13* (b) *that $j^*\colon \mathrm{H}^{p+q}(A \cup B, B; R) \to \mathrm{H}^{p+q}(A, A \cap B; R)$ is an isomorphism. The following diagram commutes*

$$\begin{tikzcd}
\mathrm{H}^p(A, A \cap B; R) & \times & \mathrm{H}^q(A; R) \arrow[d, "\delta"] \arrow[r, "\cup"] & \mathrm{H}^{p+q}(A, A \cap B; R) \arrow[d, "(j^*)^{-1}", "\cong"'] \\
 & & & \mathrm{H}^{p+q}(A \cup B, B; R) \arrow[d, "\delta"] \\
\mathrm{H}^p(X, B; R) \arrow[uu, "i^*"] & \times & \mathrm{H}^{q+1}(X, A; R) \arrow[r, "\cup"] & \mathrm{H}^{p+q+1}(X, A \cup B; R)
\end{tikzcd}$$

up to the sign $(-1)^p$. Recall that according to the convention on page 2007 "commutes" means that if we start with an element in $\mathrm{H}^p(X, B; R)$ and an element in $\mathrm{H}^q(A; R)$, then the two routes to $\mathrm{H}^{p+q+1}(X, A \cup B; \mathbb{Z})$ lead to the same element up to multiplication by $(-1)^p$.

PROOF OF LEMMA 100.18 $(*)$. We leave it to the long-suffering reader to go through the slightly painful task of modifying the proof of Lemma 82.6 to get the desired result. ■

Proof of Proposition 100.17 (∗). We consider the following diagram of maps

$$\begin{tikzcd}
\mathrm{H}^k(A,\partial A;\mathbb{Z}) & \times & \mathrm{H}^{n-k}(A;\mathbb{Z}) \arrow[r, "\cup"] \arrow[dd, "\delta"] & \mathrm{H}^n(A,\partial A;\mathbb{Z}) \arrow[d, "(u^*)^{-1}", "\cong"'] \arrow[dr, bend left, "\langle -,[A]\rangle"] & \\
 & & & \mathrm{H}^n(\partial M,B;\mathbb{Z}) \arrow[r, "\langle -,p_*([\partial M])\rangle"] \arrow[d, "\delta"] & \mathbb{Z}. \\
\mathrm{H}^k(M,B;\mathbb{Z}) \arrow[uu, "w^*"] & \times & \mathrm{H}^{n-k+1}(M,A;\mathbb{Z}) \arrow[r, "\cup"] & \mathrm{H}^{n+1}(M,\partial M) \arrow[ur, bend right, "\langle -,[M]\rangle"'] &
\end{tikzcd}$$

We make the following clarifications and observations:

(1) By Proposition 74.12 we know that (M,A,B) is an excisive triad.
(2) We denote by $u\colon (A,\partial A)\to(\partial M,B)$ the inclusion map. It follows from (1) and Lemma 74.13 (b) that the map u induces an isomorphism on cohomology groups.
(3) We denote by $\delta\colon \mathrm{H}^n(A;\mathbb{Z})\to\mathrm{H}^{n+1}(M,A;\mathbb{Z})$ and $\delta\colon \mathrm{H}^n(\partial M,B;\mathbb{Z})\to\mathrm{H}^{n+1}(M,\partial M;\mathbb{Z})$ the connecting homomorphisms in cohomology of the pair (M,A) and of the triple $(M,\partial M,B)$.
(4) It follows from (1) and Lemma 100.18 that the left part of the diagram commutes up to the sign $(-1)^k$.
(5) We denote by $p\colon (\partial M,\varnothing)\to(\partial M,B)$ the obvious map. By Lemma 87.24 we have $p_*([\partial M])=u_*([A])\in\mathrm{H}_n(\partial M,B;\mathbb{Z})$. Together with Lemma 74.6 (3) it follows that the top right triangle commutes.
(6) Note that by Proposition 87.27 we know that $\partial([M])=[\partial M]\in\mathrm{H}_n(\partial M;\mathbb{Z})$. Together with Lemma 83.10 this implies that the bottom right triangle commutes.

Next we consider the following two diagrams:

$$\begin{tikzcd}
\mathrm{H}_k(M,A;\mathbb{Z}) \arrow[r, "\partial"] \arrow[d, "\mathrm{PD}_M"'] & \mathrm{H}_k(A;\mathbb{Z}) \arrow[d, "\mathrm{PD}_A"] \\
\mathrm{H}^{n-k}(M,B;\mathbb{Z}) \arrow[r, "w^*"] & \mathrm{H}^{n+1-k}(A,\partial A;\mathbb{Z})
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\mathrm{H}_k(A,\partial A;\mathbb{Z}) \arrow[r, "w_*"] \arrow[d, "\mathrm{PD}_A"'] & \mathrm{H}_k(M,B;\mathbb{Z}) \arrow[d, "\mathrm{PD}_M"] \\
\mathrm{H}^{n-k}(A;\mathbb{Z}) \arrow[r, "\delta"] & \mathrm{H}^{n-k+1}(M,A;\mathbb{Z}).
\end{tikzcd}$$

By Proposition 88.27 we know that the diagram to the left commutes up to the factor $(-1)^k$ whereas the diagram to the right commutes as it is. With all these preparations we can now easily carry out the desired calculation:

$$\begin{aligned}
Q_{A,\varnothing,\partial A}(\partial c,a) &\overset{\substack{\text{by definition of } Q_A\\ \downarrow}}{=} \langle \mathrm{PD}_A(\partial c)\cup\mathrm{PD}_A(a),[A]\rangle \overset{\substack{\text{since the second diagram commutes up to multiplication by } (-1)^k\\ \downarrow}}{=} (-1)^k\cdot\langle w^*(\mathrm{PD}_M(c))\cup\mathrm{PD}_A(a),[A]\rangle \\
&\underset{\substack{\uparrow\\ \text{since the first diagram commutes}\\ \text{up to multiplication by } (-1)^k}}{=} \langle \mathrm{PD}_M(c)\cup\delta(\mathrm{PD}_A(a)),[M]\rangle \underset{\substack{\uparrow\\ \text{since the third diagram commutes}}}{=} \langle \mathrm{PD}_M(c)\cup\mathrm{PD}_M(w_*(a)),[M]\rangle \\
&= Q^{\mathrm{as}}_{M,A,B}(c,w_*(a)).
\end{aligned}$$

■

The following proposition relates the asymmetric intersection pairing on a topological manifold to the asymmetric intersection pairing on a codimension-zero submanifold.

Proposition 100.19. *Let M be a compact oriented m-dimensional topological manifold and let $W \subset M$ be a compact codimension-zero submanifold with corner.*[1413] *Let A and B be disjoint unions of components of ∂M with $\partial_1 W \subset A$. We denote by $i\colon (W, \partial_1 W) \to (M, A)$, $w\colon (W, \partial_0 W) \to (M, M \setminus \mathring{W})$ and $p\colon (M, B) \to (M, M \setminus \mathring{W})$ the obvious maps. For every $\alpha \in \mathrm{H}_k(W, \partial_1 W)$ and $\beta \in \mathrm{H}_{n-k}(M, B)$ we have the following equality:*[1414]

$$Q^{\mathrm{as}}_{W,\partial_1 W,\partial_0 W}(\alpha, (w_*)^{-1}(p_*(\beta))) \;=\; Q^{\mathrm{as}}_{M,A,B}(i_*(\alpha), \beta).$$

In other words, the following diagram commutes in a suitable sense:

$$\begin{tikzcd}
\mathrm{H}_k(W, \partial_1 W; \mathbb{Z}) \times \mathrm{H}_{n-k}(W, \partial_0 W; \mathbb{Z}) \arrow[d, "w_*"', "\cong"] \arrow[dd, "i_*"] \arrow[dr, "Q^{\mathrm{as}}_{W,\partial_1 W,\partial_0 W}"] & \\
\mathrm{H}_{n+k}(M, M \setminus \mathring{W}; \mathbb{Z}) & \mathbb{Z}. \\
\mathrm{H}_k(M, A; \mathbb{Z}) \times \mathrm{H}_{n-k}(M, B; \mathbb{Z}) \arrow[u, "p_*"] \arrow[ur, "Q^{\mathrm{as}}_{M,A,B}"'] &
\end{tikzcd}$$

The analogous statement also holds with the roles of the first and the second entry swapped.

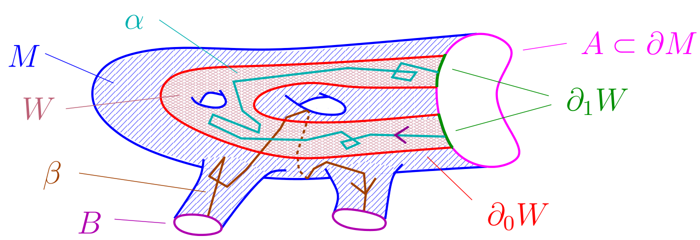


FIGURE 1438. Illustration for Proposition 100.19.

PROOF (∗). We perform the following calculation:

$$\begin{aligned}
Q^{\mathrm{as}}_{W,\partial_1 W,\partial_0 W}(\alpha, (w_*)^{-1}(p_*(\beta)) &= \langle \mathrm{PD}_W(\alpha) \cup \mathrm{PD}_W((w_*)^{-1}(p_*(\beta))), [W]\rangle \\
&\underset{\substack{\uparrow \\ \text{by Lemma 89.6}}}{=} \langle w^*((w^*)^{-1}(\mathrm{PD}_W(\alpha)) \cup w^*(\mathrm{PD}_M(\beta)), [W]\rangle \underset{\substack{\uparrow \\ \text{by Lemma 74.6 (3)}}}{=} \langle (w^*)^{-1}(\mathrm{PD}_W(\alpha)) \cup \mathrm{PD}_M(\beta), w_*([W])\rangle \\
&\underset{\substack{\uparrow \\ \text{by Lemma 87.24 we have } w_*([W]) = p_*([M])}}{=} \langle (w^*)^{-1}(\mathrm{PD}_W(\alpha)) \cup \mathrm{PD}_M(\beta), p_*([M])\rangle \underset{\substack{\uparrow \\ \text{Lemma 74.6 (3)}}}{=} \langle p^*((w^*)^{-1}(\mathrm{PD}_W(\alpha))) \cup p^*(\mathrm{PD}_M(\beta)), [M]\rangle \\
&\underset{\substack{\uparrow \\ \text{by Lemma 89.5}}}{=} \langle \mathrm{PD}_M(i_*(\alpha)), \mathrm{PD}_M(\beta), [M]\rangle \;=\; Q^{\mathrm{as}}_{M,A,B}(i_*(\alpha), \beta).
\end{aligned}$$

■

The case we will use most often is the special case that $\partial_0 W = \partial W$, $\partial_1 W = \varnothing$, $A = \varnothing$ and $B = \partial M$. In this case Proposition 100.20 specializes to the following slightly more readable proposition.

[1413] We refer to Lemma 89.1 for the definition of $\partial_0 W$ and $\partial_1 W$.

[1414] Here we use that we know by the Excision Theorem 89.2 that the map $w\colon (W, \partial W) \to (M, M \setminus \mathring{W})$ induces an isomorphism on homology groups.

Proposition 100.20. *Let M be a compact oriented m-dimensional topological manifold and let $W \subset M \setminus \partial M$ be a compact m-dimensional submanifold. We denote by $i\colon W \to M$, $w\colon (W, \partial W) \to (M, M \setminus \mathring{W})$ and $p\colon (M, \partial M) \to (M, M \setminus \mathring{W})$ the obvious maps. For every $\alpha \in \mathrm{H}_k(W)$ and $\beta \in \mathrm{H}_{n-k}(M, \partial M)$ we have the following equality:*

$$Q^{\mathrm{as}}_{W,\varnothing,\partial W}(\alpha, (w_*)^{-1}(p_*(\beta))) \;=\; Q^{\mathrm{as}}_{M,\varnothing,\partial M}(i_*(\alpha), \beta).$$

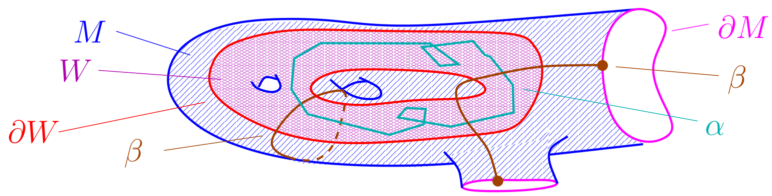


FIGURE 1439. Illustration for Proposition 100.20.

100.8. **The reduced intersection form** $(*)$**.** After the exhaustive and exhausting discussion of the asymmetric intersection pairings in the previous section, let us now move on to our second variation on the intersection form.

Proposition 100.21. *Let M be a compact oriented $2n$-dimensional topological manifold. We denote by $i\colon \partial M \to M$ the inclusion map.*

(1) *Let $\alpha \in \mathrm{FH}_n(M;\mathbb{Z})$. The following two statements are equivalent:*
   (a) $\alpha \in \operatorname{im}(i_*\colon \mathrm{FH}_n(\partial M;\mathbb{Z}) \to \mathrm{FH}_n(M;\mathbb{Z}))$.
   (b) *For every $\beta \in \mathrm{FH}_n(M;\mathbb{Z})$ we have $Q_M(\alpha, \beta) = 0$.*

(2) *We write $\widehat{\mathrm{H}}_n(M;\mathbb{Z}) := \operatorname{coker}(i_*\colon \mathrm{H}_n(\partial M;\mathbb{Z}) \to \mathrm{H}_n(M;\mathbb{Z}))$. The form*

$$\begin{array}{rcl} Q^{\mathrm{red}}_M\colon \mathrm{F}\widehat{\mathrm{H}}_n(M;\mathbb{Z}) \times \mathrm{F}\widehat{\mathrm{H}}_n(M;\mathbb{Z}) & \to & \mathbb{Z} \\ ([\varphi], [\psi]) & \mapsto & Q_M(\varphi, \psi) \end{array}$$

   is well-defined and non-degenerate. We refer to this form as the reduced intersection form of M.

(3) *If the intersection form Q_M is non-degenerate, then the obvious map $\mathrm{FH}_n(M;\mathbb{Z}) \to \mathrm{FH}_n(M;\mathbb{Z})$ is an isomorphism and it induces an isometry $Q_M \to Q^{\mathrm{red}}_M$.*

PROOF. We denote by $i\colon \partial M \to M$ and $j\colon (M, \varnothing) \to (M, \partial M)$ the obvious maps. Before we do anything else we recall that we have a long exact sequence

$$\ldots \to \mathrm{H}_n(\partial M;\mathbb{Z}) \xrightarrow{i_*} \mathrm{H}_n(M;\mathbb{Z}) \xrightarrow{j_*} \mathrm{H}_n(M, \partial M;\mathbb{Z}) \xrightarrow{\partial} \mathrm{H}_{n-1}(\partial M) \to \ldots$$

Now we turn to the actual proofs of the statements.

(1) Let $\alpha \in \mathrm{H}_n(M;\mathbb{Z})$. First we prove "(a)$\Rightarrow$(b)". Thus we assume that $\alpha = i_*(\gamma)$ for some $\gamma \in \mathrm{H}_n(\partial M;\mathbb{Z})$. Let $\beta \in \mathrm{H}_n(M;\mathbb{Z})$. We perform the following calculation:

$$\begin{aligned} Q_M(\alpha,\beta) &\underset{\substack{\uparrow \\ \text{by definition of } Q_M}}{=} \langle \mathrm{PD}(i_*(\gamma)) \cup \mathrm{PD}(\beta), [M]\rangle \underset{\substack{\uparrow \\ \text{Lemma 83.7 (3)}}}{=} \langle \mathrm{PD}(\beta), j_*(\underbrace{\mathrm{PD}(i_*(\gamma)) \cap [M]}_{= i_*(\gamma) \text{ by definition of PD}})\rangle \\ &= \langle \mathrm{PD}(\beta), j_*(i_*(\gamma))\rangle \underset{\substack{\uparrow \\ \text{since } j_* \circ i_* = 0}}{=} 0. \end{aligned}$$

Now we prove "(b)⇒(a)". Thus we assume that $Q_M(\alpha,\beta)=0$ for all $\beta\in \mathrm{FH}_n(M;\mathbb{Z})$. This implies for any $\beta\in\mathrm{FH}_n(M;\mathbb{Z})$ we have the following equalities:

$$Q^{\mathrm{as}}_{M,\partial M,\varnothing}(j_*(\alpha),\beta) \underset{\substack{\uparrow \\ \text{by Proposition 100.15 (2)}}}{=} Q_M(\alpha,\beta\rangle \underset{\substack{\uparrow \\ \text{by assumption}}}{=} 0.$$

By Proposition 100.15 (1) we know that the pairing $Q^{\mathrm{as}}_{M,\partial M,\varnothing}$ is non-singular, in particular it is non-degenerate. Thus it follows from the above, together with Lemma 100.14, that $j_*(\alpha)=0\in \mathrm{FH}_n(M;\mathbb{Z})$. It follows from the above long exact sequence, together with Exercise 57.2, that $\mathrm{im}(i_*\colon \mathrm{FH}_n(\partial M;\mathbb{Z})\to \mathrm{FH}_n(M;\mathbb{Z}))$.

(2) It follows immediately from (1) "(a)⇒(b)" that the form Q^{red}_M is well-defined. Furthermore it follows from (1) "(b)⇒(a)" and Lemma 100.14 that Q^{red}_M is non-degenerate.

(3) This statement follows immediately from (1) and Lemma 100.14. ■

Examples.

(1) One last time we consider again our favorite example M, namely the torus minus two open disks. We use the notation from Figure 1440. A basis for $\mathrm{H}_1(M;\mathbb{Z})$ is given by a,b,c and the image of $\mathrm{H}_1(\partial M;\mathbb{Z})\to \mathrm{H}_1(M;\mathbb{Z})$ is given by $\mathbb{Z}\cdot c$. With respect to these bases we see that

$$Q_M \text{ is represented by } \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \text{ and } Q^{\mathrm{red}}_M \text{ is represented by } \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array}.$$

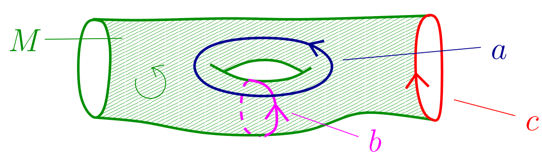


FIGURE 1440

(2) In Exercise 100.6 we will provide an example of a compact orientable 4-dimensional smooth manifold M for which the reduced intersection form is not non-singular. We will obtain many more such examples later on in Proposition **??**.

Exercises for Chapter 100.

Exercise 100.1. We equip $\mathrm{H}_1(S^1\times S^1;\mathbb{Z})$ with the standard basis $\mu=[S^1\times\{*\}]$ and $\lambda=[\{*\}\times S^1]$. Show that

$$Q_{S^1\times S^1}(a\cdot\mu+b\cdot\lambda, c\cdot\mu+d\cdot\lambda) \;=\; \det\begin{pmatrix} a & b \\ c & d\end{pmatrix}.$$

Remark. The exercise is really quite easy, but it does produce a neat formula.

Exercise 100.2. Let M be a compact oriented topological manifold. Recall that we say that M is *amphichiral* if M admits an orientation-reversing self-homeomorphism, otherwise we say that M is *chiral*. For which $k, l \in \mathbb{N}_0$ is the connected sum $k \cdot \mathbb{C}\mathrm{P}^2 \# l \cdot \overline{\mathbb{C}\mathrm{P}}^2$ chiral? More precisely:

(a) for the values that you exclude, show that $k \cdot \mathbb{C}\mathrm{P}^2 \# l \cdot \overline{\mathbb{C}\mathrm{P}}^2$ is not chiral,
(b) for all the other values show explicitly that $k \cdot \mathbb{C}\mathrm{P}^2 \# l \cdot \overline{\mathbb{C}\mathrm{P}}^2$ is chiral.

Exercise 100.3. Let $n \in \mathbb{N}$ and let M and M' be two closed oriented connected non-empty n-dimensional topological manifolds. Let $X \subset M$ and $X' \subset M'$ be submanifolds that are homeomorphic to $\overline{B}^n$ and let $\varphi\colon \partial X \to \partial X'$ be an orientation-reserving homeomorphism. Show that $(M \setminus \mathring{X}) \cup_{\varphi\colon \partial X \to \partial X'} (M \setminus \mathring{X}')$ is homeomorphic to the connected sum $M \# N$.
Hint. Use the Alexander trick from Exercise 3.29.
Remark. Initially it might sound like this statement holds "by definition". But in fact in Exercise **??** we will see that the obvious analogue in the setting of smooth manifolds does *not* hold.

Exercise 100.4. Let R be a commutative ring and let $\varphi\colon V \times V \to R$ be a form. The submodule $R := \{v \in V \,|\, \varphi(v, w) = 0 \text{ for all } w \in V\}$ is called the *radical* of the form φ. Show that if R is a field, then the map

$$\begin{array}{rcl} V/R \times V/R & \to & R \\ ([v],[w]) & \mapsto & \varphi(v,w) \end{array}$$

is well-defined and non-singular.
Remark. Recall that a form is by our definition symmetric or anti-symmetric and V is a finite-dimensional vector space over the field R.

Exercise 100.5. Let M be a compact oriented 4-dimensional topological manifold and let $i\colon \partial M \to M$ be the inclusion map.

(a) Show that if $i_*\colon \mathrm{H}_1(\partial M; \mathbb{Z}) \to \mathrm{H}_1(M; \mathbb{Z})$ is a monomorphism, then the intersection form Q_M is non-singular.
(b) Show that if $i_*\colon \mathrm{H}_1(\partial M; \mathbb{Q}) \to \mathrm{H}_1(M; \mathbb{Q})$ is a monomorphism, then the intersection form Q_M is non-degenerate.

Exercise 100.6. Let $n \in \mathbb{N}$. We consider the map

$$\begin{array}{rcl} \varphi\colon \overline{B}^2 \times S^1 & \to & \overline{B}^2 \times S^1 \subset \partial(\overline{B}^2 \times \overline{B}^2) \\ (z,w) & \mapsto & (z, w \cdot z^n). \end{array}$$

Furthermore we consider [1415]

$$M \;:=\; (\overline{B}^2 \times \overline{B}^2) \underset{\substack{\overline{B}^2 \times S^1 \to \partial(\overline{B}^2 \times \overline{B}^2) \\ (z,w) \mapsto \varphi(z,w)}}{} (\overline{B}^2 \times \overline{B}^2).$$

One can show by hand that M is a compact orientable 4-dimensional smooth manifold.[1416]

(a) Show that $\pi_1(M) = 0$, $\mathrm{H}_2(M; \mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{H}_2(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$.

[1415] Later on page 2475 we will say that M is obtained from the 4-ball $\overline{B}^2 \times \overline{B}^2$ by attaching a 2-handle.
[1416] We will carry out the details in Proposition 103.1, but this is irrelevant for the present exercise.

(b) Show that $H_1(\partial M;\mathbb{Z}) \cong \mathbb{Z}_n$ and $H_2(\partial M;\mathbb{Z}) = 0$. (In fact very well-trained eyes will spot that ∂M is diffeomorphic to the lens space $L(n,1)$.)

(c) Show that if $n \geq 2$, then the intersection form Q_M is represented by one of the two 1×1-matrices $(-n)$ or (n).
Hint. Use Proposition 100.15 (3). *Remark.* Note that it follows from (b) that the intersection form equals the reduced intersection form. Thus we have shown, as promised on page 2436, that there exists a compact orientable 4-dimensional smooth manifold M for which the reduced intersection form is not non-singular.

Exercise 100.7. Let M be a compact oriented n-dimensional topological manifold and let A, B be disjoint unions of components of ∂M. Furthermore let V and W be compact codimension-zero submanifolds of M as in Figure 1441 and let $\alpha \in H_k(M, A)$ and $\beta \in H_{n-k}(X, B)$.

(a) Inspired by Figure 1441 state a formula which computes $Q_M(\alpha, \beta)$ in terms of suitable intersection numbers of V and W.
Hint. What is the role of C and D in Figure 1441?

(b) Prove (a).

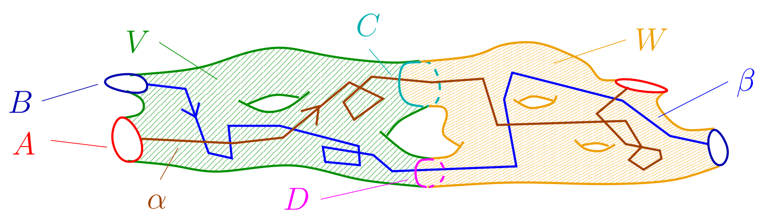


FIGURE 1441. Illustration of Exercise 100.7.

101. The intersection form II: Algebra and some topology

In Chapter 100 we introduced the intersection form of compact oriented even-dimensional topological manifolds. On page 2426 we considered the intersection forms of $S^2 \times S^2$, $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$. At that point it became clear that we have to invest some time to study (anti-) symmetric forms as purely algebraic objects.

We start out with the following lemma that summarizes all the basic facts about the interplay between forms and matrices that we will need in this chapter.

Lemma 101.1. *Let R be a commutative ring.*

(1) *Let A be a (anti-) symmetric*[1417] *$n \times n$–matrix over R. We consider the map*

$$\begin{array}{rcl} \varphi(A)\colon R^n \times R^n & \to & R \\ (v,w) & \mapsto & v^T A w. \end{array}$$

The following statements hold:

(a) *$\varphi(A)$ is an (anti-) symmetric form over R.*

(b) *The form $\varphi(A)$ is non-singular, in the sense of the definition on page 2411, if and only if A is invertible, i.e. if $\det(A)$ is a unit in R*

(c) *If $P \in \mathrm{GL}(n,R)$ is an invertible matrix, then the forms $\varphi(P^T A P)$ and $\varphi(A)$ are isometric.*

(d) *If B is a (anti-) symmetric $n \times n$-matrix over R such that $\varphi(A)$ and $\varphi(B)$ are isometric, then there exists a $P \in \mathrm{GL}(n,R)$ with $B = P^T A P$.*

(2) *Let (V,ψ) be an (anti-) symmetric form over R of rank n. If A is a matrix that represents ψ, then $(V,\psi) \cong (R^n, \varphi(A))$. In particular every (anti-) symmetric form over R is isometric to $\varphi(A)$ for some (anti-) symmetric $n \times n$–matrix A over R.*

Convention. Lemma 101.1 allows us to go back and forth between (anti-) symmetric forms and (anti-) symmetric matrices. Thus in the following discussion we will blur the distinction between these two concepts. For example, given a (anti-) symmetric matrix A over a commutative ring we will often just refer to the (anti-) symmetric form $\varphi(A)$ as the *form A*.

Proof (∗).

(1) (a) This statement follows almost immediately from the definitions and the observation that for any $v, w \in R^n$ we have

$$\varphi(A)(w,v) \;=\; w^T A v \;\underset{\substack{\uparrow \\ \text{for a } (1\times 1)\text{–matrix } (x) \text{ we have } (x)^T = (x)}}{=}\; (w^T A^T v)^T \;=\; v^T A^T w \;=\; \varphi(A^T)(v,w).$$

(b) The proof of this statement is an instructive exercise in linear algebra. We will not rob the reader of the joy of doing this exercise.

(c) We claim that the isomorphism $v \mapsto Pv$ defines an isometry from the form $(R^n, \varphi(P^T A P))$ to the form $(R^n, \varphi(A))$. Indeed, for any $v, w \in R^n$ we have

$$\varphi(A)(Pv, Pw) \;=\; (Pv)^T A P w \;=\; v^T (P^T A P) w \;=\; \varphi(P^T A P)(v,w).$$

[1417]Recall that a matrix A is called *anti-symmetric* if $A^T = -A$.

(d) Let B be a (anti-) symmetric $n \times n$-matrix over R and let $f\colon R^n \to R^n$ be an isometry between $\varphi(A)$ and $\varphi(B)$, i.e. let f be an isomorphism such that for all $v, w \in R^n$ we have $v^T A w = \varphi(A)(v, w) = \varphi(B)(f(v), f(w)) = f(v)^T A f(w)$. Let P be the matrix representing f with respect to the standard basis of R^n. Since f is an isomorphism we know that $P \in \mathrm{GL}(n, R)$. Finally a straightforward calculation shows that $B = P^T A P$.

(2) Let $\psi\colon V \times V \to R$ be an (anti-) symmetric form over R. Let $v_1, \ldots, v_n$ be a basis of V. We denote by $A = (\psi(v_i, v_j))_{i,j=1,\ldots,n}$ the corresponding matrix representing the form ψ. We consider the isomorphism

$$\begin{array}{rcl} \Theta\colon R^k & \to & V \\ (\lambda_1, \ldots, \lambda_n) & \mapsto & \lambda_1 v_1 + \cdots + \lambda_n v_n. \end{array}$$

We claim that Θ is an isometry from the form $(R^n, \varphi(A))$ to the form (V, ψ). Indeed, if we denote by $e_1, \ldots, e_n$ the standard basis for R^n, then for any $i, j \in \{1, \ldots, n\}$ we have

$$\psi(\Theta(e_i), \Theta(e_j)) \;=\; \psi(v_i, v_j) \;=\; ij\text{-entry of } A \;=\; e_i^T A e_j \;=\; \varphi(A)(e_i, e_j).$$

By bilinearity of forms we obtain the corresponding statement for all pairs of elements in R^n. ■

Lemma 101.1 (1b), (1d) and (2) shows that the following definition makes sense.

Definition. Let R be a commutative ring. We denote by R^* the multiplicative units of R. Let (V, ψ) be a non-singular (anti-) symmetric form over R of rank n. We pick a basis $v_1, \ldots, v_n$ of V. We refer to

$$\det(\psi) \;:=\; \det(\psi(v_i, v_j)_{i,j=1,\ldots,n}) \;\in\; R^*/\{a^2 \,|\, a \in R^*\}$$

as the *determinant of the form* (V, ψ). Note that for $R = \mathbb{Z}$ the determinant is a well-defined element of $\{\pm 1\}$.

Example.

(1) For a matrix $A \in \mathrm{GL}(n, R)$ we have of course $\det(\varphi(A)) = \det(A) \in R^*/\{a^2 \,|\, a \in R^*\}$.

(2) We work with $R = \mathbb{Z}$. We set $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $\det(A) = 1$ and $\det(A) = -1$ we see that $\varphi(A)$ and $\varphi(B)$ are non-isometric forms over $\mathbb{Z}^2$.

101.1. **Non-singular anti-symmetric forms.** In this section we will completely classify non-singular anti-symmetric forms over the real numbers and over the integers.

First we consider the matrix

$$S \;:=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is anti-symmetric and it has determinant 1. From Lemma 101.1 we obtain an anti-symmetric form $\varphi(S)$. The following proposition shows that $\varphi(S)$ is in some sense the only example of a non-singular anti-symmetric form over the real numbers or over the integers.

Proposition 101.2. *If (V,φ) is a non-singular anti-symmetric form over $\mathbb{R}$ or over $\mathbb{Z}$, then there exists a $k \in \mathbb{N}_0$ such that*[1418] $(V,\varphi) \cong k \cdot \varphi(S)$, *i.e. φ is represented by the $2k \times 2k$–matrix*

$$\underbrace{\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix}}_{k \text{ diagonal blocks of } 2\times 2\text{-matrices}}.$$

In particular the rank of V is even.

Remark. Propositions 100.4 and 101.2 imply in particular that the isometry type of the intersection of a surface of genus g is determined by the formal properties of an intersection form. In particular we did not need to go through the discussion on page 2413 if we just need to know the *isometry type* of the intersection form.

Sketch of proof. First we remark that throughout the proof on many occasions we use Lemma 101.1 without saying so explicitly.

After this preamble, let (V,φ) be a non-singular anti-symmetric form over $\mathbb{R}$ of rank n. We pick an $(n \times n)$–matrix A that represents φ. Since φ is anti-symmetric we see that all terms of A on the diagonal are zero. In the following we consider the usual three types of elementary matrices:

$$D_\lambda := \begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}, \lambda \neq 0, \quad P := \begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix} \quad \text{and} \quad Q_\lambda := \begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \lambda & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}.$$

We make the following elementary observations:

(i) Replacing a matrix B by $D_\lambda^T B D_\lambda$ means that first we multiply a column by λ, and then we multiply the corresponding row by λ.
(ii) Replacing a matrix B by $P^T B P$ means that first we exchange two columns, and then we exchange the corresponding two rows.
(iii) Replacing a matrix B by $Q_\lambda^T B Q_\lambda$ means that first we add λ-times one column to another column, and then we do the same operation for the corresponding rows.

Now we can perform the following sequence of operations to A:

$$A \overset{(1)}{=} \begin{pmatrix} 0 & * & * & \dots & * \\ * & 0 & * & \dots & * \\ * & * & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ * & * & * & * & 0 \end{pmatrix} \xrightarrow{(2)} \begin{pmatrix} 0 & 1 & * & \dots & * \\ -1 & 0 & * & \dots & * \\ * & * & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ * & * & * & * & 0 \end{pmatrix} \xrightarrow{(3)} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & * & \dots & * \\ 0 & * & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ 0 & * & * & * & 0 \end{pmatrix} \xrightarrow{(4)} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ 0 & 0 & * & * & 0 \end{pmatrix}$$

where we perform the following steps:

(1) We write out A, here and throughout the steps we use the convention that $*$ can mean any real number, the only condition being that all matrices are anti-symmetric.

[1418] Here, when we write $k \cdot \varphi(S)$ we use the definition from page 2410.

(2) Since $\det(A) \neq 0$ there exists at least one entry on the first row which is non-zero. After operations of type (ii) and (i) we can arrange that the $(1, 2)$-entry equals 1. By the anti-symmetry we see that the $(2, 1)$-entry now equals -1.
(3) Using operations of type (iii) we can "clean up" the remaining terms on the first row and column.
(4) Similarly, we can use operations of type (iii) to "clean up" the remaining terms on the second row and column.

At this stage we can continue with the same process, applied to the $(n-2)\times(n-2)$–matrix at the right bottom. Eventually we will have turned A into a matrix of the promised type.

Now suppose that (V, φ) is a non-singular anti-symmetric form over $\mathbb{Z}$. In Exercise 101.6 we will modify the above proof to deal with this new situation. ■

In a sense Proposition 101.2 is rather disappointing. We had hoped that intersection forms will give us new meaningful invariants. But Proposition 101.2 together with Proposition 100.4 shows that intersection forms of closed even-dimensional topological manifolds do not give us any extra information that would allow us to distinguish. manifolds.

Nonetheless, Proposition 101.2 has just enough content to answer Question 88.8 where we had asked which integers appear as Euler characteristics of closed orientable connected even-dimensional topological manifolds.[1419] Now we can provide a complete answer to this question.

Proposition 101.3.
(1) *Let $k \in \mathbb{N}$. Given any $n \in \mathbb{Z}$ there exists a closed orientable connected $4k$-dimensional smooth manifold with Euler characteristic n.*
(2) (a) *The Euler characteristic of any closed orientable connected topological manifold of dimension 2 is even and ≤ 2. Conversely, any even number ≤ 2 is the Euler characteristic of such a smooth manifold.*
(b) *Let $k \in \mathbb{N}$. The Euler characteristic of any closed orientable connected topological manifold of dimension $4k+2$ is even. Conversely, any even number is the Euler characteristic of such a smooth manifold.*

Remark. In Exercise 106.2 we will deal with the possible Euler characteristics of smooth manifolds with possibly non-empty boundary.

Proof. We start out with two claims that give us the stated restrictions on Euler characteristics of closed orientable topological manifolds.

Claim. Let $k \in \mathbb{N}_0$. The Euler characteristic of every closed orientable topological manifold M of dimension $4k + 2$ is even.

[1419] This question arose from the fact, shown in Proposition 88.7, that the Euler characteristic of every closed odd-dimensional topological manifold is necessarily zero. So it was natural to wonder what is happening for closed even-dimensional topological manifolds.

Indeed, we have

$$\chi(M) \underset{\substack{\uparrow\\ \text{Proposition 85.34 (4)}}}{=} \sum_{i=0}^{4k+2}(-1)^i\cdot\dim_{\mathbb{R}}(\mathrm{H}_i(M;\mathbb{R})) \underset{\substack{\uparrow\\ \text{by Theorem 88.6 we have } \dim_{\mathbb{R}}(\mathrm{H}_i(M;\mathbb{R}))=\dim_{\mathbb{R}}(\mathrm{H}_{4k+2-i}(M;\mathbb{R}))}}{=} \dim_{\mathbb{R}}(\mathrm{H}_{2k+1}(M;\mathbb{R})) + \sum_{i=0}^{2k}(-1)^i\cdot\overbrace{2\cdot\dim_{\mathbb{R}}(\mathrm{H}_i(M;\mathbb{R}))}^{\text{even}}$$

$$\underset{\substack{\uparrow\\ \text{by Proposition 100.4 and the facts that } M \text{ is closed and that } 2k+1 \text{ is odd we know}\\ \text{that the intersection form on } \mathrm{H}_{2k+1}(M;\mathbb{R}) \text{ is anti-symmetric and non-singular,}\\ \text{by Proposition 101.2 we thus know that } \dim_{\mathbb{R}}(\mathrm{H}_{2k+1}(M,\mathbb{R})) \text{ is even}}}{\equiv} 0 \mod 2.$$

⊞

Claim. The Euler characteristic of every every closed orientable connected 2-dimensional topological manifold M is ≤ 2.

Of course this statement follows from the Surface Classification Theorem 23.4 together with the calculation of the Euler characteristics on page 1363. But one can prove the claim directly. First the case that M is empty is trivial. Now assume that M is non-empty. In this case we have the following (in-) equalities:

$$\chi(M) \underset{\substack{\uparrow\\ \text{by Proposition 55.1}}}{=} \sum_{i\in\mathbb{N}_0}(-1)^i\cdot\mathrm{rank}(\mathrm{H}_i(M)) \underset{\substack{\uparrow\\ \text{by Theorem 87.3}}}{=} \overbrace{\mathrm{rank}(\mathrm{H}_0(M))}^{=1 \text{ by Proposition 41.5}} - \mathrm{rank}(\mathrm{H}_1(M)) + \overbrace{\mathrm{rank}(\mathrm{H}_2(M))}^{=1 \text{ by Theorem 87.1}} \leq 2.$$

⊞

It remains to show that besides these two restrictions on the possible values of Euler characteristics, all other values can be realized. Our arguments in all of the three cases rely on the following observation.

Observation.

(i) On page 1364 we saw that for every $m \in \mathbb{N}_0$ we have $\chi(\mathbb{C}\mathrm{P}^m) = m+1$.
(ii) Lemma 55.7 says that if M and N are two compact oriented connected *even-dimensional* smooth manifolds, then $\chi(M\#N) = \chi(M)+\chi(N)-2$.
(iii) There is any number of reasons, e.g. we could use Lemma 55.8, why given any $m \in \mathbb{N}$ the Euler characteristic of the m-dimensional torus $(S^1)^m$ is zero.

Now we turn to the various cases:

(1) Let $k \in \mathbb{N}$ and let $n \in \mathbb{Z}$. Note that $\chi(\mathbb{C}\mathrm{P}^k) = 2k+1$ is odd and ≥ 3. From the above observations it follows that
$$\chi(n\cdot\mathbb{C}\mathrm{P}^{2k}\#nk\cdot(S^1)^{4k}) \;=\; n\cdot(2k+1)-nk\cdot 2 \;=\; n.$$
(2) (a) Evidently we have $\chi(S^2)=2$. Note let $n=2m$ be an even integer with $n \leq 0$. It follows from the above observations that
$$\chi((m+1)\cdot(S^1\times S^1)) \;=\; (m+1-1)\cdot 2 \;=\; n.$$
(b) Let $k \in \mathbb{N}$ and let n be an even integer. From the above observations it follows that
$$\chi\big(n\cdot\mathbb{C}\mathrm{P}^{2k+1}\#\tfrac{n}{2}(2k+1)\cdot(S^1)^{4k+2}\big) \;=\; n\cdot(2k+2)-\tfrac{n}{2}(2k+1)\cdot 2 \;=\; n. \qquad \blacksquare$$

101.2. **Interlude: Sphere-packing.** In this short section we want to consider sphere packings. The relationship with our actual goal of dealing with symmetric forms is arguably tenuous but nonetheless fascinating.

Definition. Let $v_1, \ldots, v_n$ be linearly independent vectors in $\mathbb{R}^n$. We refer to the group

$$\mathbb{Z} \cdot v_1 + \cdots + \mathbb{Z} \cdot v_n \;=\; \left\{ \sum_{i=1}^{n} a_i \cdot v_i \,\middle|\, a_1, \ldots, a_n \in \mathbb{Z} \right\}$$

as the *lattice spanned by* $v_1, \ldots, v_n$.[1420]

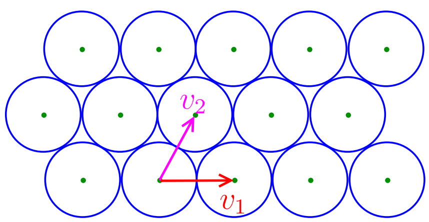
We start out with the following theorem. The fact that strictly speaking we did not define all the terms will not bother us.

Theorem 101.4.

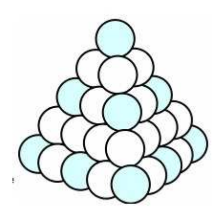
(1) *A densest packing of disks of radius* $\frac{1}{2}$ *in* $\mathbb{R}^2$ *is given by setting the centers on the "hexagonal" lattice spanned by* $v_1 = (1, 0)$ *and* $v_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ *shown in Figure 1442 on the left. The density equals* $\frac{\pi}{\sqrt{12}}$.

(2) *No packing of balls of the same radius in* $\mathbb{R}^3$ *has density greater than that of the "face-centered cubic packing" shown in Figure 1442 in the middle and on the right. The density equals* $\frac{\pi}{\sqrt{18}}$.

Proof. The first statement was proved by L. Fejes Tóth [**FT43, FT72**] in 1943. It is much harder to prove the second statement. This statement had been conjectured by Kepler in 1613 and it was finally proved by Thomas Hales [**Hale00, Hale05, Hale17**] in 2005. ■



densest disk packing



ball packings of highest density

Figure 1442

For a mathematician it is natural to wonder what are the highest densities of packings in higher dimensions. Not surprisingly this is a very hard problem. Astonishingly the next case that was dealt with is the 8-dimensional setting. More precisely, the following theorem was proved by Maryna Viazovska [**Via17**] in 2016.

[1420] This is perhaps not the standard definition of a lattice, but who cares?

Theorem 101.5. *We consider the following matrix*

$$E_8 := \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

No packing of unit balls in $\mathbb{R}^8$ has density greater than that given by the lattice spanned by the eight column vectors of the E_8–matrix. Furthermore, this packing has density $\frac{\pi^4}{384}$.

Remark. We also refer to [**Bae17**] for other interesting appearances of the E_8–matrix, for example its relationship to the regular icosahedron

Shortly afterwards the 24-dimensional case was settled in [**CKMRV17**], in this case a densest packing is given by the "Leech lattice". OK, that was fun, but now let us return to our actual problem, and let us see whether this section was of any use.

101.3. **Non-singular symmetric forms over the real numbers.** In an earlier section we saw that non-singular *anti-symmetric* forms are almost depressingly boring mathematical objects. In this and the next section we will consider non-singular *symmetric* forms. These will, fortunately, turn out to be much more interesting.

Examples. We list some symmetric integral matrices with determinant equal to ± 1. By Lemma 101.1 these define in particular non-singular forms over $\mathbb{R}$ and also over $\mathbb{Z}$.

(1) The 1×1-matrices $(+1)$ and (-1).
(2) The slightly mysterious 8×8–matrix E_8 from the previous section.[1421]
(3) The matrix

$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is symmetric and it has determinant -1.

It is natural to ask whether there is some type of analogue of Proposition 101.2 for symmetric forms. In fact using the methods of the proof of Proposition 101.2 one can prove the following statement.

Proposition 101.6. *If (V, φ) is a non-singular symmetric form over $\mathbb{R}$ of rank n, then there exists a $k \in \mathbb{N}_0$ such that $(V, \varphi) \cong k \cdot (+1) \oplus (n-k) \cdot (-1)$.*

Remark. Recall that Proposition 101.2 dealt with anti-symmetric forms over $\mathbb{R}$ and also with anti-symmetric forms over $\mathbb{Z}$. In contrast Proposition 101.6 only deals with symmetric forms over $\mathbb{R}$. It is a very instructive exercise to run through the proof of Proposition 101.6 and to determine where it breaks down if we work over $\mathbb{Z}$.

[1421] A miracle! The matrix E_8 actually has determinant one.

Sketch of proof. Let (V, φ) be a non-singular symmetric form over $\mathbb{R}$ of rank n. In the subsequent arguments below we use the conventions and notions we introduced in Proposition 101.2. In particular we will also use the operations of type (i), (ii) and (iii) that we introduced in the proof of Proposition 101.2.

We pick an $(n \times n)$–matrix A that represents φ. Now we can perform the following sequence of operations to A:

$$A \stackrel{(1)}{=} \begin{pmatrix} * & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{pmatrix} \xrightarrow{(2)} \begin{pmatrix} \epsilon_1 & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{pmatrix} \xrightarrow{(3)} \begin{pmatrix} \epsilon_1 & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{pmatrix}$$

where we perform the following steps:

(1) We write out A. Here and throughout the steps we use the convention that $*$ can mean anything, the only condition being that all matrices are symmetric.
(2) Since $\det(A) \neq 0$ there exists at least one entry on the first row which is non-zero. After operations of type (iii) and (i) we can arrange that the $(1,1)$-entry equals some $\epsilon_1 \in \{-1, 1\}$.
(3) Using operations of type (iii) we can "clean up" the remaining terms on the first row and column.

At this stage we can continue with the same process, applied to the $(n-1) \times (n-1)$–matrix at the right bottom. Eventually we will have turned A into a diagonal matrix where all entries lie in $\{-1, 1\}$. After operations of type (ii) we can arrange that the first diagonal entries are equal to $+1$ and the final ones are equal to -1. ■

Let $n \in \mathbb{N}_0$. Proposition 101.6 raises the question whether or not the symmetric forms defined by $k \cdot (+1) \oplus (n-k) \cdot (-1)$ are isometric for different values of $k \in \{0, \dots, n\}$. To answer this question it is helpful to introduce the following definition.

Definition. Let (V, φ) be a symmetric form over the real numbers.

(1) We say φ is *positive-definite* if $\varphi(v, v) > 0$ for all $v \neq 0$.
(2) We say φ is *negative-definite* if $\varphi(v, v) < 0$ for all $v \neq 0$.
(3) We say φ is *indefinite* if φ is neither positive-definite nor negative-definite.

Examples.

(1) The form $(+1)$ is positive-definite and the form (-1) is negative-definite.
(2) The form E_8 is positive-definite. Evidently one can show this by a brute force calculation. A more conceptual proof is given in [**HGK10**, Theorem 1.16.14].
(3) The form H is indefinite, indeed, we have

$$\varphi(H)\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = (1 \ \ 1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2$$

and

$$\varphi(H)\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = (1 \ \ -1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2.$$

To distinguish non-isometric forms we need to introduce some invariants of forms.

Definition. Let (V,φ) be a symmetric form over the real numbers. We define

$b^+(\varphi) :=$ maximal dimension of a vector subspace U of V such that $\varphi|_U$ is positive-definite,
$b^-(\varphi) :=$ maximal dimension of a vector subspace U of V such that $\varphi|_U$ is negative-definite.

Furthermore we define the *signature* $\operatorname{sign}(\varphi)$ as

$$\operatorname{sign}(\varphi) \;:=\; b^+(\varphi) - b^-(\varphi).$$

Example. We consider the symmetric form

$$\begin{aligned} \varphi\colon \mathbb{R}^2\times\mathbb{R}^2 &\;\to\; \mathbb{R} \\ \left(\begin{pmatrix} v_1 \\ v_2\end{pmatrix}, \begin{pmatrix} v_1 \\ v_2\end{pmatrix}\right) &\;\mapsto\; (v_1\ v_2)\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} v_1 \\ v_2\end{pmatrix}. \end{aligned}$$

In Figure 1443 we show the subsets

$$X^+ \;:=\; \{v\in V\,|\,\varphi(v,v)>0\}\cup\{0\} \quad\text{ and }\quad X^- \;:=\; \{v\in V\,|\,\varphi(v,v)<0\}\cup\{0\}.$$

These subsets are evidently *not* vector subspaces of $V=\mathbb{R}^2$. The maximal dimension of a vector subspace contained in X^+ and X^- is in both cases equal to one. Thus we see that $b^+(\varphi)=b^-(\varphi)=1$. In particular we see that $\operatorname{sign}(\varphi)=0$.

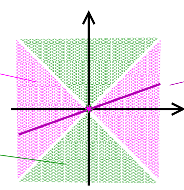


FIGURE 1443

The following lemma is an immediate consequence of the definitions.

Lemma 101.7. *Let (V,φ) and (W,ψ) be two non-singular symmetric forms over $\mathbb{R}$. If (V,φ) and (W,ψ) are isometric, then*

$$\dim_{\mathbb{R}}(V)=\dim_{\mathbb{R}}(W), \quad b^\pm(\varphi)=b^\pm(\psi) \quad\textit{and}\quad \operatorname{sign}(\varphi)=\operatorname{sign}(\psi).$$

In the following proposition we determine $b^\pm$ and the signature for a non-singular form that is given by a diagonal matrix.

Proposition 101.8. *If D is a diagonal $(n\times n)$–matrix where the diagonal entries $d_1,\dots,d_n$ are non-zero real numbers, then*

$$b^+(\varphi(D)) \;=\; \textit{number of positive } d_i\textit{'s} \quad\textit{and}\quad b^-(\varphi(D)) \;=\; \textit{number of negative } d_i\textit{'s}.$$

In particular

$$\operatorname{sign}(\varphi(D)) \;=\; \textit{number of positive } d_i\textit{'s} \;-\; \textit{number of negative } d_i\textit{'s}.$$

PROOF. After reordering the diagonal entries we can assume that the first k entries of D are positive and that the remaining $(n-k)$ entries are negative. We write

$$\begin{aligned} V^+ &\;:=\; \{(x_1,\dots,x_k,0,\dots,0)\in\mathbb{R}^n\,|\,x_1,\dots,x_k\in\mathbb{R}\} \qquad\text{and} \\ V^- &\;:=\; \{(0,\dots,0,y_1,\dots,y_{n-k})\in\mathbb{R}^n\,|\,y_1,\dots,y_{n-k}\in\mathbb{R}\} \end{aligned}$$

It is clear that $\varphi(D)|_{V^+}$ is positive-definite and that $\varphi(D)|_{V^-}$ is negative-definite. Thus $b^+(\varphi(D)) \geq k$ and $b^-(\varphi(D)) \geq n-k$. If remains to prove the following claim.

Claim. For any symmetric form (W,ψ) over $\mathbb{R}$ we have $b^+(\psi)+b^-(\psi) \leq \dim_{\mathbb{R}}(W)$.

Thus let W^+ be a positive-definite subspace of W and let W^- be a negative-definite subspace of W. The intersection $W^+ \cap W^-$ is a subspace that is positive-definite and negative-definite. So for any non-zero vector $v \in W^+ \cap W^-$ we have $\psi(v,v) > 0$ and $\psi(v,v) < 0$. This is only possible if no such v exists, i.e. we have shown that the intersection $W^+ \cap W^- = \{0\}$. Now we see that

$$\dim(W^+)+\dim(W^-) \underset{\substack{\uparrow \\ \text{linear algebra}}}{=} \dim(W^+ + W^-)+\dim(W^+\cap W^-) \underset{\substack{\uparrow \\ W^+\cap W^-=\{0\}}}{=} \dim(W^+ + W^-) \underset{\substack{\uparrow \\ \text{since } W^+ + W^- \subset W}}{\leq} \dim(W).$$ ■

Corollary 101.9.

(1) *Let (V,φ) be a non-singular symmetric form over $\mathbb{R}$. We write $n=\dim_{\mathbb{R}}(V)$. The following statements hold:*
   (a) *We have $b^{\pm}(-\varphi)=b^{\mp}(\varphi)$, in particular we have $\operatorname{sign}(-\varphi)=-\operatorname{sign}(\varphi)$.*
   (b) *We have $b^+(\varphi)+b^-(\varphi)=\dim_{\mathbb{R}}(V)$.*
   (c) *We have $b^+(\varphi)=\frac{1}{2}(n+\operatorname{sign}(\varphi))$.*
   (d) *If we write $k=b^+(\varphi)$, then the form (V,φ) is isometric to $k\cdot(+1)\oplus(n-k)\cdot(-1)$.*
   (e) *We have $\operatorname{sign}(\varphi)\equiv n \bmod 2$.*

(2) *If (V,φ) and (W,ψ) are two non-singular symmetric forms over $\mathbb{R}$, then*
$$\operatorname{sign}(\varphi\oplus\psi) \;=\; \operatorname{sign}(\varphi)+\operatorname{sign}(\psi).$$

PROOF.

(1) (a) This statement follows immediately from the definitions.
   (b) By Proposition 101.6 and Lemma 101.7 it suffices to prove the statement for non-singular symmetric forms that are represented by diagonal matrices. But for those forms the statement follows immediately from Proposition 101.8.
   (c) This statement is an immediate consequence of (b) and the definition of the signature.
   (d) This statement follows from Proposition 101.6, Lemma 101.7 and Proposition 101.8.
   (e) We have $\operatorname{sign}(\varphi)=b^+(\varphi)-b^-(\varphi)\equiv b^+(\varphi)+b^-(\varphi)=n \bmod 2$.

(2) This statement follows immediately from Statement (1d), Proposition 101.6 applied to φ and ψ and Proposition 101.8. ■

Example. Given a non-singular symmetric form φ on $\mathbb{R}^3$ one can visualize it by considering the "unit-sphere"
$$S(\varphi) \;:=\; \{v\in\mathbb{R}^3 \,|\, \varphi(v,v)=1\}.$$
Corollary 101.9 (3) implies in particular that every non-singular symmetric form on $\mathbb{R}^3$ is isometric to one of the following four forms

(a) $3\cdot(1)$,	(b) $2\cdot(1)\oplus(-1)$
(c) $2\cdot(-1)\oplus(1)$	(d) $3\cdot(-1)$.

In the last case the unit-sphere is the empty set. In the other cases the unit-spheres are sketched in Figure 1444.

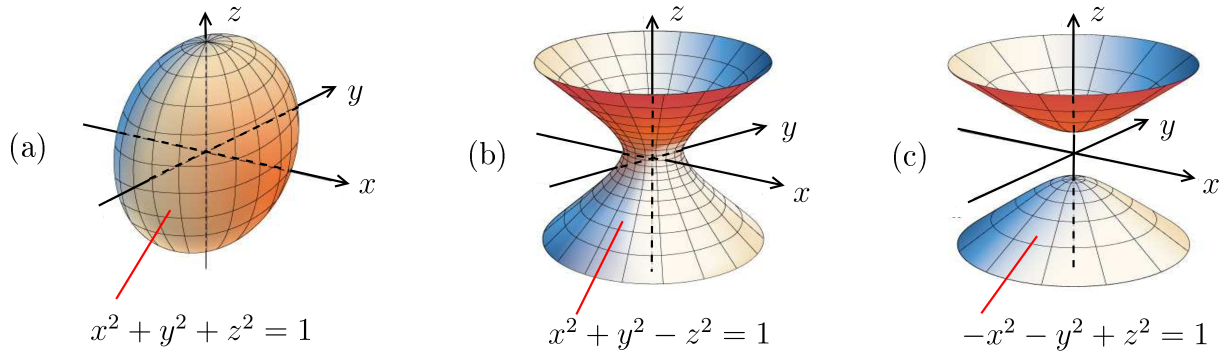


FIGURE 1444

The following theorem says that the dimension and the signature are a complete invariant for isometry types of non-singular symmetric forms over $\mathbb{R}$.

Theorem 101.10. (Sylvester's Theorem a. k. a. Sylvester's law of inertia) *Let (V,φ) and (W,ψ) be two non-singular symmetric forms over $\mathbb{R}$. Then*

$$(V,\varphi) \text{ and } (W,\psi) \text{ are isometric} \quad\Longleftrightarrow\quad \dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(W) \text{ and } \operatorname{sign}(\varphi) = \operatorname{sign}(\psi).$$

PROOF. The "$\Rightarrow$"-direction is precisely Lemma 101.7. It remains to prove the "$\Leftarrow$"-direction. So suppose we are given two non-singular symmetric forms (V,φ) and (W,ψ) over $\mathbb{R}$ with same dimension and same signature. It follows from our hypothesis together with Corollary 101.9 (c) and (d) that there exist $n,k \in \mathbb{N}_0$ such that (V,φ) and (W,ψ) are both isometric to $k\cdot(+1)\oplus(n-k)\cdot(-1)$. But this implies that (V,φ) and (W,ψ) are themselves isometric. ∎

Remark. It is also very easy to classify symmetric forms over $\mathbb{R}$ that are not necessarily non-singular. More precisely, given a symmetric form (V,φ) over $\mathbb{R}$ we define the *nullity* $\sigma(\varphi)$ as

$$\nu(\varphi) \;:=\; \text{maximal dimension of a subspace } W \text{ of } V \text{ such that } \varphi|_W \text{ is the zero form.}$$

It is an amusing little exercise to modify the proof of Sylvester's Theorem 101.10 to show that two symmetric forms over $\mathbb{R}$ are isometric if and only if their dimensions, signatures and nullities agree.

Now it is time to apply our recently acquired knowledge to topology.

Definition.

(1) If (V,φ) is a symmetric form over the integers, then we define

$$\begin{aligned} g\otimes\mathbb{R}\colon V\otimes\mathbb{R}\times V\otimes\mathbb{R} \;&\to\; \mathbb{R} \\ (v\otimes r, w\otimes s) \;&\mapsto\; r\cdot\varphi(v,w)\cdot s. \end{aligned}$$

Furthermore we define

$$\operatorname{sign}(\varphi) \ := \ \operatorname{sign}(\varphi \otimes \mathbb{R}).$$

(2) Let $k \geq 1$ and let M be a compact oriented $4k$-dimensional topological manifold. We define the *signature of* M to be[1422]

$$\operatorname{sign}(M) \ := \ \operatorname{sign}(Q_M).$$

For convenience we define the signature of the empty topological manifold to be zero.

Remark. The signature of a manifold was first introduced in 1923 by Hermann Weyl [**Wey23**, p. 432] in a little known paper written in Spanish. As is explained in [**Eckm06**], Hermann Weyl deliberately chose this obscurity since at this point, according to Hermann Weyl, algebraic topology was not considered "a respectable field of mathematics".

Lemma 101.11. *Let $k \in \mathbb{N}$.*

(1) *Let M be a compact oriented $4k$-dimensional topological manifold. The form $Q_M \otimes \mathbb{R}$ is isometric to the form*

$$\begin{array}{rcl} \mathrm{H}^{2k}(M, \partial M; \mathbb{R}) \times \mathrm{H}^{2k}(M, \partial M; \mathbb{R}) & \to & \mathbb{R} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [M] \rangle. \end{array}$$

(2) *The signature is additive under the connected sum operation and under disjoint union.*

(3) *For every compact oriented $4k$-dimensional topological manifold we have*

$$\operatorname{sign}(-M) \ = \ -\operatorname{sign}(M).$$

(4) *Let M and N be two compact oriented $4k$-dimensional topological manifolds.*
  (a) *If M and N are homeomorphic, then* $\operatorname{sign}(M) = \pm \operatorname{sign}(N)$.
  (b) *If the pairs $(M, \partial M)$ and $(N, \partial N)$ are homotopy equivalent, then we have the equality* $\operatorname{sign}(M) = \pm \operatorname{sign}(N)$.

Proof.

(1) This statement follows easily from the various definitions and Corollary 57.21.
(2) It follows easily from Lemma 83.9 that the signature is additive under disjoint union. If we are given closed topological manifolds, then it follows immediately from Proposition 100.12 and Corollary 101.9 that the signature is additive under the connected sum operation.
(3) The third statement is an immediate consequence of Lemma 100.6 and Corollary 101.9.
(4) The last statement is also an immediate consequence of Lemma 100.6 and Corollary 101.9. ■

[1422] Recall that on page 2413 we introduced the intersection form $Q_M\colon \mathrm{FH}_{2k}(M;\mathbb{Z}) \times \mathrm{FH}_{2k}(M;\mathbb{Z}) \to \mathbb{Z}$. Furthermore, recall that by Proposition 100.4 the intersection form Q_M is indeed symmetric, so it makes sense to consider its signature.

Examples. We have

$$\begin{array}{ccccccccc}
\operatorname{sign}(\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2) & = & \operatorname{sign}(\mathbb{C}\mathrm{P}^2)+\operatorname{sign}(\mathbb{C}\mathrm{P}^2) & = & \operatorname{sign}(1)+\operatorname{sign}(1) & = & 1+1 & = & 2 \\
& \uparrow & & \uparrow & & & & & \\
& \text{Lemma 101.11 (2) and (3)} & & \text{see page 2415} & & & & & \\
& \downarrow & & \downarrow & & & & & \\
\operatorname{sign}(\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2) & = & \operatorname{sign}(\mathbb{C}\mathrm{P}^2)-\operatorname{sign}(\mathbb{C}\mathrm{P}^2) & = & \operatorname{sign}(1)-\operatorname{sign}(1) & = & 1-1 & = & 0.
\end{array}$$

Furthermore we have

$$\operatorname{sign}(S^2\times S^2) \underset{\substack{\uparrow \\ \text{see page 2414}}}{=} \operatorname{sign}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underset{\substack{\uparrow \\ \text{Lemma 101.1}}}{=} \operatorname{sign}\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right) = \operatorname{sign}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \underset{\substack{\uparrow \\ \text{Proposition 101.8}}}{=} 0.$$

The above examples, together with Lemmas 100.6 and 101.11 (4) now allow us to give a partial answer to Question 68.15.

Corollary 101.12. *The topological space $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ is neither homotopy equivalent (in particular it is not homeomorphic) to $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ nor to $S^2\times S^2$.*

Remark. Corollary 101.12 shows in particular that in the definition of the connected sum of two orientable (topological) smooth manifolds we needed to work with *oriented* (topological) smooth manifolds.

Unfortunately we are not yet able to distinguish the two smooth manifolds $S^2\times S^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$. In fact Sylvester's Theorem 101.10 might suggest that we are again at a dead end. But this is not the case since Sylvester's Theorem is a statement about *real* symmetric forms, but we are really dealing with symmetric forms defined over the integers. We will exploit this extra piece of information in the next section.

101.4. **Non-singular symmetric forms over the integers.** We consider the matrices

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and } D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It follows from Sylvester's Theorem 101.10 that the matrices give rise to *isometric forms over the real numbers*. The question arises, whether the forms on $\mathbb{Z}^2$ defined by these matrices are isometric. To get an idea let us consider what we can say about the "squares $\varphi(v,v)$" defined by these forms. Note that given any $v=\begin{pmatrix} a \\ b \end{pmatrix}\in\mathbb{R}^2$ we have

$$\varphi(H)(v,v) = {}_{(a\ b)}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2ab \quad \text{and}\varphi(H)(v,v) = {}_{(a\ b)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = a^2-b^2.$$

We see that for the form to the left we only get even integers whereas to the right any integer can occur. This already shows that the two forms are not isometric over the integers. It follows immediately from this discussion and the earlier calculations of the intersection forms that the intersection forms of $S^2\times S^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ are not isometric, not even up to sign. This allows us to settle the remaining case of Question 68.15. More precisely, by Lemma 100.6 we have the following proposition.

Proposition 101.13. *The topological space $S^2 \times S^2$ is not homotopy equivalent (in particular it is not homeomorphic) to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.*

This encouraging discussion leads us to the following definition.

Definition. A symmetric form (V, φ) over the integers is called *even* if

$$\varphi(v, v) \equiv 0 \mod 2 \quad \text{ for all } v \in V,$$

otherwise the intersection form is called *odd.* We refer to the property of φ being even or odd as the *parity* of the form.

The following lemma summarizes some basic facts about the parity of symmetric forms.

Lemma 101.14.

(1) *Isometric forms have the same parity.*
(2) *For a symmetric form φ the parities of φ and $-\varphi$ agree.*
(3) *The direct sum of an odd form with any other form is also odd.*
(4) *If A is a symmetric square matrix over the integers, then*

$$\varphi(A) \text{ is even} \quad \Longleftrightarrow \quad \text{all diagonal entries of } A \text{ are even.}$$

Examples. It follows from Lemma 101.14 that the forms $(+1)$, (-1) and $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ are odd and that the forms E_8 and $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are even.

PROOF. The only statement which requires a little thought is the fact that if $A = (a_{ij})$ is a symmetric $(n \times n)$–matrix with even diagonal entries, then $\varphi(A)$ is even. Thus let $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$. We calculate that

$$\varphi(A)(v, v) \; = \; v^T A v \; = \; \sum_{i=1}^{n} \sum_{j=1}^{n} v_i v_j \cdot a_{ij} \underset{\substack{\uparrow \\ \text{since } A \text{ is symmetric}}}{=} 2 \sum_{i=1}^{n} \sum_{j=1}^{i-1} v_i v_j \cdot a_{ij} + \sum_{i=1}^{n} v_i^2 a_{ii} \underset{\substack{\uparrow \\ \text{since the } a_{ii} \text{ are even}}}{\equiv} 0 \bmod 2. \qquad \blacksquare$$

We just saw that the two matrices

$$H \; = \; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{ and } D \; = \; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

give rise to *isometric forms over the real numbers* and that the two matrices give rise to *non-isometric forms over the integers.* The question arises, how much more complicated is the classification of non-singular symmetric forms over the integers than the classification of non-singular forms over the real numbers. In particular the following question arises.

Question 101.15. *Can we classify non-singular symmetric forms over the integers?*

The following is the key theorem in the classification of non-singular *indefinite* symmetric forms over $\mathbb{Z}$.

Theorem 101.16. *If φ_1 and φ_2 are two non-singular* indefinite *symmetric forms over the integers, then the following holds:*

$$\varphi_1 \text{ and } \varphi_2 \text{ are isometric} \quad \Longleftrightarrow \quad \varphi_1 \text{ and } \varphi_2 \text{ have the same rank, signature and parity.}$$

Example. Using the parity we had just seen that the intersection forms of the closed oriented 4-dimensional smooth manifolds $M = S^2 \times S^2$ and $N = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ are not isometric, even up to sign. Now we consider the closed oriented 4-dimensional smooth manifolds that we obtain from M and N by performing the connected sum with $\mathbb{C}\mathrm{P}^2$. In other words, we are considering $M\#\mathbb{C}\mathrm{P}^2 = (S^2 \times S^2)\#\mathbb{C}\mathrm{P}^2$ and $N\#\mathbb{C}\mathrm{P}^2 = \mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2\#\mathbb{C}\mathrm{P}^2$. It follows from Proposition 100.12 that the intersection forms of these two smooth manifolds are represented by the matrices

$$A \;=\; \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Both forms are non-singular indefinite and odd. Thus it follows from Theorem 101.16, or alternatively from Exercise 101.11, that the intersection forms are isometric. Thus we cannot distinguish $M\#\mathbb{C}\mathrm{P}^2 = (S^2 \times S^2)\#\mathbb{C}\mathrm{P}^2$ and $N\#\mathbb{C}\mathrm{P}^2 = \mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2\#\mathbb{C}\mathrm{P}^2$ using their intersection forms. In fact, it turns out that there is a good reason why we have a hard time distinguishing them: as is shown in [**GoS99**, p. 151] the two smooth manifolds $(S^2 \times S^2)\#\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2\#\mathbb{C}\mathrm{P}^2$ are in fact diffeomorphic, in fact with good eyes one can write down an explicit diffeomorphism.

Proof. The proof of the theorem is provided in [**MH73**, Theorem II.5.3] (see also [**GoS99**, Theorem 1.2.14] for an outline of the proof). The proof relies crucially on a significant input from algebraic number theory, namely the Hasse-Minkowski theorem. This theorem says that given $a_1, \ldots, a_n \in \mathbb{Q} \setminus \{0\}$ the equation

$$x_1^2 \cdot a_1 + \cdots + x_n^2 \cdot a_n \;=\; 0$$

has a non-trivial integral solution if and only if it has a non-trivial real solution and if for each prime p the equation has also has a non-trivial solution over the p-adic numbers $\mathbb{Q}_p$. A proof of the Hasse-Minkowski theorem is given in [**Ser73**, Chapter IV Theorem 8], [**BSh66**, Section 7.1] or [**Neuk99**, p. 385]. ∎

As an immediate corollary to Theorem 101.16 we obtain the classification of non-singular *indefinite odd* forms. More precisely, we have the following corollary.

Corollary 101.17. *If φ is a non-singular* indefinite *odd symmetric form over the integers, then*

$$\varphi \;\cong\; b^+(\varphi) \cdot (1) \,\oplus\, b^-(\varphi) \cdot (-1).$$

Proof. Let φ be a non-singular indefinite odd symmetric form. It follows from Proposition 101.8 that the rank, signature and parity of the two non-singular indefinite odd forms in the statement of the corollary agree. So these two forms are isometric by Theorem 101.16. ∎

Now we turn to the study of even forms. We have the following, perhaps somewhat surprising proposition.

Proposition 101.18. *Let φ be a non-singular symmetric form over the integers. If φ is even, then* $\operatorname{sign}(\varphi) \equiv 0 \bmod 8$.

Example. As a reality check, the two even non-singular symmetric forms that we are acquainted with are $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E_8. These forms a non-singular even forms and the signatures are 0 and 8, so that matches Proposition 101.18.

PROOF. Given a non-singular symmetric form (W,ψ) over $\mathbb{Z}$ we say that an element $v \in W$ is *characteristic* if for all $w \in W$ we have

$$\varphi(v,w) \ \equiv \ \varphi(w,w) \ \bmod \ 2.$$

We make the following observations:

(1) It follows immediately from the definitions that (W,ψ) is even if and only if 0 is a characteristic element.

(2) If $W = k \cdot (1) \oplus l \cdot (-1)$, then it follows from a straightforward calculation as in the proof of Lemma 101.14 that an element $u \in \mathbb{Z}^{k+l}$ is characteristic if and only if all its entries are odd.

It follows from Observation (1) that the following claim implies in particular the desired statement of Proposition 101.18.

Claim. Let (V,φ) be a non-singular symmetric form over $\mathbb{Z}$. If $v \in V$ is a characteristic element, then

$$\varphi(v,v) \ \equiv \ \operatorname{sign}(\varphi) \ \bmod \ 8.$$

Let $v \in V$ be a characteristic element of (V,φ). It is straightforward to verify that $v + e_- + e_+$ is a characteristic element in $(V \oplus \mathbb{Z} \oplus \mathbb{Z}, \varphi \oplus (1) \oplus (-1))$, where e_- and e_+ are generators of the first respectively the second $\mathbb{Z}$-summand. Note that, regardless of φ, the form $\varphi \oplus (1) \oplus (-1))$ is indefinite and odd. Thus by Corollary 101.17 there exists an isometry[1423]

$$\Phi\colon \varphi \oplus (1) \oplus (-1) \ \xrightarrow{\cong} \ (b^+(\varphi)+1)\cdot(1) \oplus (b^-(\varphi)+1)\cdot(-1).$$

We write $u := \Phi(v + e_- + e_+)$. It is clear that u is a characteristic element for the form on the right-hand side. The above Observation (2) tells us that each entry of u is an odd integer. Now we calculate that

$$\begin{aligned}
\varphi(v,v) & \overset{\substack{\text{straightforward calculation} \\ \downarrow}}{=} (\varphi \oplus (1) \oplus (-1))(v+e_-+e_+, v+e_-+e_+) \overset{\substack{\text{since } \Phi \text{ is an isometry} \\ \downarrow}}{=} \\
& = ((b^+(\varphi)+1)\cdot(1) \oplus (b^-(\varphi)+1)\cdot(-1))(u,u) \\
& = \sum_{i=1}^{b^+(\varphi)+1} (\underbrace{i\text{-th entry of } u}_{\text{odd integer}})^2 + \sum_{i=b^+(\varphi)+2}^{b^+(\varphi)+b^-(\varphi)+2} (-1)\cdot(\underbrace{i\text{-th entry of } u}_{\text{odd integer}})^2 \\
& \underset{\substack{\uparrow \\ \text{since for each odd number } k \text{ we have } k^2 \equiv 1 \bmod 8}}{\equiv} \sum_{i=1}^{b^+(\varphi)+1} 1 + \sum_{i=b^+(\varphi)+2}^{b^+(\varphi)+b^-(\varphi)+2} (-1) \ = \ (b^+(\varphi)+1) - (b^-(\varphi)+1) \ = \ \operatorname{sign}(\varphi) \bmod 8.
\end{aligned}$$

[1424]

[1423] Note that both sides are non-singular odd indefinite forms, regardless of what non-singular symmetric form φ we actually consider.

■

Now we can provide the classification of non-singular indefinite *even* symmetric forms.

Corollary 101.19. *If φ is a non-singular* indefinite *even symmetric form over the integers, then*

$$\varphi \;\cong\; \frac{\operatorname{sign}(\varphi)}{8}\cdot E_8 \oplus \frac{\operatorname{rank}(\varphi)-|\operatorname{sign}(\varphi)|}{2}\cdot H.$$

Proof. First let φ be a non-singular even symmetric form over the integers. Note that it follows from Proposition 101.18 that $\operatorname{sign}(\varphi)$ is indeed divisible by 8. Furthermore it follows from $\operatorname{rank}(\varphi) = b^+(\varphi) + b^-(\varphi)$ and $\operatorname{sign}(\varphi) = b^+(\varphi) - b^-(\varphi)$ that $\operatorname{rank}(\varphi) - |\operatorname{sign}(\varphi)|$ is divisible by 2. So it makes sense to write down the right-hand side.

Next we recall that the two non-singular symmetric forms E_8 and H are even, that $\operatorname{rank}(E_8) = \operatorname{sign}(E_8) = 8$ and that $\operatorname{rank}(H) = 2$ and $\operatorname{sign}(H) = 0$. Now assume that φ is furthermore indefinite. It follows from the above observation regarding E_8 and H that the rank, signature and parity of the two non-singular indefinite even forms in the statement of the corollary agree. So these two forms are isometric by Theorem 101.16. ■

Now we turn to the study of non-singular *definite* symmetric forms over the integers. We start out with some more examples of such forms. The following lemma is proved in [**MH73**, Lemmas 6.1 and 6.2] and [**Ser73**, p. 51].

Lemma 101.20. *We denote by $e_1, \dots, e_{4m}$ the standard basis for $\mathbb{Z}^{4m}$. We denote by Γ_{4m} the submodule of $\mathbb{Z}^{4m}$ spanned by the vectors $\{e_i + e_j\}_{i,j=1,\dots,4m}$ and $\frac{1}{2}(e_1 + \cdots + e_{4m})$.*

1. *The restriction of the usual form $\langle\, , \rangle$ on $\mathbb{Z}^{4m}$ to Γ_{4m} is a non-singular positive-definite symmetric form.*
2. *The form Γ_{4m} is odd if and only if m is odd.*
3. *The form Γ_8 is isometric to E_8.*
4. *The form Γ_{16} is not isometric to $E_8 \oplus E_8$.*

It turns out that there are many more *definite* symmetric forms than indefinite symmetric forms and that there is no known classification of such forms. For example, by [**MH73**, p. 28] and [**Ser73**, p. 55] we have the following table

dimension	8	16	24	32	40
number of isometry types of non-singular positive-definite *even* symmetric forms	1	2	24	$\geq 10^7$	$\geq 10^{51}$

Here in dimension 8, up to isometry, the unique example is $\Gamma_8 = E_8$ whereas in dimension 16 the unique examples are $\Gamma_8 \oplus \Gamma_8 = E_8 \oplus E_8$ and Γ_{16}.

Similar results are also known for the number of isometry classes of non-singular positive-definite *odd* symmetric forms of a given rank. We refer to [**CSl99**, p. 416] and [**Kin03**, Table 2] for details.

[1424] Why is it true that for each odd number k we have $k^2 \equiv 1 \bmod 8$?

101.5. **Forms over $\mathbb{F}_2$.** Considering the power of the intersection form of a closed *oriented* even-dimensional topological manifold it is natural to ask, whether one can obtain similar results for closed (possibly) *non-orientable* even-dimensional topological manifolds. In fact we have the following definition.

Definition. Let M be a closed $2n$-dimensional topological manifold M (not necessarily orientable). We denote by $[M]_{\mathbb{F}_2} \in \mathrm{H}_{2n}(M;\mathbb{F}_2)$ the $\mathbb{F}_2$-fundamental class given by Corollary 87.5. Given $k \in \mathbb{N}_0$ we denote by

$$\mathrm{PD}_M^{\mathbb{F}_2}\colon\ \mathrm{H}_k(M;\mathbb{F}_2) \ \to\ \mathrm{H}^{2n-k}(M;\mathbb{F}_2)$$

the inverse of the isomorphism given by the Poincaré Duality Theorem 88.1. We can now define the $\mathbb{F}_2$-*valued intersection form* as follows:

$$\begin{array}{rcl} Q_M^{\mathbb{F}_2}\colon\ \mathrm{H}_n(M;\mathbb{F}_2)\times \mathrm{H}_n(M;\mathbb{F}_2) & \to & \mathbb{F}_2 \\ (a,b) & \mapsto & \big\langle \mathrm{PD}_M^{\mathbb{F}_2}(a)\cup \mathrm{PD}_M^{\mathbb{F}_2}(b), [M]_{\mathbb{F}_2}\big\rangle. \end{array}$$

The above definition is completely analogous to the definition of the $\mathbb{Z}$-valued intersection form on page 2413. Not surprisingly the $\mathbb{F}_2$-valued intersection forms shares many properties of the $\mathbb{Z}$-valued intersection form.

Proposition 101.21.

(1) *The $\mathbb{F}_2$-valued intersection form $Q_M^{\mathbb{F}_2}$ on a closed $2n$-dimensional topological manifold M is symmetric and non-singular.*

(2) *The $\mathbb{F}_2$-valued intersection forms of homeomorphic closed topological manifolds are isometric.*

Proof. The proof is a very mild variation on the proofs of Proposition 100.4 and Lemma 100.6. ■

Examples.

(1) Similar to the discussion on page 2414 one can show that the $\mathbb{F}_2$-valued intersection form of the torus $S^1\times S^1$ is represented by the matrx $\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$.

(2) Proposition 90.16, together with the obvious analogue of Proposition 100.12, implies that the $\mathbb{F}_2$-valued intersection form of $\mathbb{RP}^2\#\mathbb{RP}^2$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$.

The question arises, whether we can classify non-singular symmetric forms over $\mathbb{F}_2$. For better or worse it turns out that there is a straightforward classification. To state the classification we need the following definition which is inspired by our definition on page 2452.

Definition. Let V be a finite-dimensional $\mathbb{F}_2$-vector space. We say a form $\lambda\colon V\times V\to \mathbb{F}_2$ is *even* if $\lambda(v,v)=0\in\mathbb{F}_2$ for all $v\in V$. Otherwise we say that the symmetric form is *odd*. We call the property of a form to be even or odd to be the *parity* of the form.

Example. As we had just seen, the $\mathbb{F}_2$-valued intersection form of the torus $S^1\times S^1$ is even whereas the $\mathbb{F}_2$-valued intersection form of the torus $\mathbb{RP}^2\#\mathbb{RP}^2$ is odd. Thus we have found a new, and particularly complicated, proof that these two 2-dimensional smooth manifolds are not homeomorphic.

The following proposition gives a complete classification of non-singular symmetric forms over $\mathbb{F}_2$ up to isometry.

Proposition 101.22. *Two non-singular symmetric forms (V,κ) and (W,λ) over $\mathbb{F}_2$ are isometric if and only if $\dim(V)=\dim(W)$ and if κ and λ have the same parity.*

Proof. We will prove Proposition 101.22 in Exercise 101.13. For peace of mind we point out that the proposition is proved in [**Hopk15**, Theorem 2.7]. It can also be deduced from [**Szy97**, Corollary 6.3.1]. ■

The proposition shows that, unfortunately, we extract only very little information from the $\mathbb{F}_2$-valued intersection form.

Remark. Let V be a finite-dimensional vector space over $\mathbb{F}_2$ and let $\langle\,,\,\rangle\colon V\times V\to\mathbb{F}_2$ be an even form. It follows from Proposition 101.22 that V admits a *symplectic basis*, i.e. there exists a basis $a_1,\dots,a_k,b_1,\dots,b_k$ for V such that the following conditions are satisfied:

(1) For every $i,j\in\{1,\dots,k\}$ we have $\langle a_i,a_j\rangle=\langle b_i,b_j\rangle=0$.
(2) For every $i,j\in\{1,\dots,k\}$ we have $\langle a_i,a_j\rangle=\delta_{ij}$.

This observation will come in handy later on.

Exercises for Chapter 101.

Exercise 101.1. Let φ be a symmetric form on a finite-dimensional real vector space V. Prove the following statement: If φ is positive-definite, then φ is non-singular.

Exercise 101.2. Let R be a commutative ring. In this exercise we will see that non-singular symmetric forms over R give rise to a group. (We refer to page 2410 for the definition of a non-singular symmetric form.) Note that the set of symmetric non-singular forms $\varphi\colon R^k\times R^k\to R$ with the direct sum operation is a monoid.

We say a symmetric form φ is *hyperbolic* if it is isometric to a form of the form

$$\begin{array}{rcl}\psi\colon (V\oplus V)\times(V\oplus V) & \to & \mathbb{R}\\ (v_1\oplus v_2),(w_1\oplus w_2) & \mapsto & (v_1^T\,v_2^T)\begin{pmatrix}0 & \mathrm{id}\\ \mathrm{id} & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}.\end{array}$$

(a) Show that the monoid of symmetric non-singular forms $\varphi\colon R^k\times R^k\to R$ modulo the submonoid generated by hyperbolic forms is an abelian group. We refer to this group as the *Witt group* $W(R)$.
(b) Show that the maps

$$\begin{array}{rcl}W(\mathbb{R}) & \to & \mathbb{Z}\\ \varphi & \mapsto & \mathrm{sign}(\varphi)\end{array}\qquad\text{and}\qquad\begin{array}{rcl}W(\mathbb{Z}) & \to & \mathbb{Z}\\ \varphi & \mapsto & \mathrm{sign}(\varphi\otimes\mathbb{R})\end{array}$$

are isomorphisms.

(c) Show that $W(\mathbb{Q})$ admits an epimorphism onto $\mathbb{Z} \oplus (\mathbb{Z}_2)^\infty$.
Hint. Consider determinants of matrices representing a given form.
Remark. The group $W(\mathbb{Q})$ is computed in [**MH73**, Chapter IV.2] or alternatively in [**Lam05**, Chapter VI.4].

Exercise 101.3. Let M be a closed oriented 3-dimensional smooth manifold. Show that $\operatorname{sign}(M \times [0,1]) = 0$.

Exercise 101.4.

(a) Let M be a closed oriented connected $4k$-dimensional topological manifold and let N be a closed oriented $4l$-dimensional topological manifold. Show that

$$\operatorname{sign}(M \times N) \;=\; \operatorname{sign}(M) \cdot \operatorname{sign}(N).$$

(b) We extend the definition of the signature from page 2450 by defining $\operatorname{sign}(M) = 0$ for any closed oriented connected topological manifold whenever $\dim(M) \not\equiv 0 \bmod 4$. Show that the conclusion of (a) holds without any restrictions on the dimensions of M and N.

Remark. This exercise is carried out in [**Brun05**, Lemma 2.6].

Exercise 101.5. Let M be a closed non-orientable $4k$-dimensional smooth manifold. Let $p\colon \widetilde{M} \to M$ be the orientation covering that we introduced on page 531. Show that $\operatorname{sign}(\widetilde{M}) = 0$.

Exercise 101.6.

(a) Explain how to modify the proof of Proposition 101.2 so that it also holds for the integers.
(b) Explain why the proof of Proposition 101.6 cannot be generalized to the integers.

Exercise 101.7. Let A and C be invertible real $k \times k$-matrices. We assume that C is symmetric.

(a) Show that the signature of the matrix $\begin{pmatrix} 0 & A \\ A^T & C \end{pmatrix}$ is zero.
(b) Does the conclusion of (a) still hold if A and C are not invertible?

Exercise 101.8. For which $k \in \mathbb{Z}$ does there exist a closed oriented 3-dimensional smooth manifold N with $\operatorname{sign}(S^1 \times N) = k$?

Exercise 101.9. Let $A \in \mathrm{GL}(n, \mathbb{R})$. Show that

$$b^+(\varphi(A)) \;=\; \max\left\{ k \in \mathbb{N}_0 \;\middle|\; \begin{array}{c} \text{there exists a } k\text{-dimensional subspace } V \\ \text{of } \mathbb{R}^n \text{ with } v^T A v > 0 \text{ for all } v \in V \cap S^{n-1} \end{array} \right\}.$$

Exercise 101.10. Show that the map

$$\begin{array}{rcl} \{A \in \mathrm{GL}(n,\mathbb{R}) \mid A = A^T\} & \to & \mathbb{Z} \\ A & \mapsto & \operatorname{sign}(A) \end{array}$$

is continuous.
Hint. You can work either with the definition of the signature (or better, with the reformulation given in Exercise 101.9) or you could use Proposition 3.37.

Exercise 101.11. Show "by hand" that the forms on $\mathbb{Z}^3$ defined by the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are isometric.

Exercise 101.12. Let $\varphi\colon V \times V \to \mathbb{Z}$ be a form over $\mathbb{Z}$. Show that there exists a non-degenerate form $\psi\colon W \times W \to \mathbb{Z}$ such that φ is isometric to the orthogonal sum of ψ with a form that is identically zero.

Exercise 101.13. Show that two non-singular symmetric forms (V, κ) and (W, λ) over $\mathbb{F}_2$ are isometric if and only if $\dim(V) = \dim(W)$ and if κ and λ have the same parity.

102. The intersection form III: 4-dimensional manifolds

We have now introduced lots of invariants of topological spaces, namely the fundamental group, higher homotopy groups, homology groups and the cohomology groups together with the cup product. The question arises, whether we are done at some point, or whether this list of ever new invariants continues forever.

In an attempt to respond to this question we want to consider closed oriented simply connected 4-dimensional (topological) smooth manifolds and their intersection forms. What follows is a long survey on some amazing results. Unfortunately the proofs of any of these statements goes well beyond what we can cover in these introductory notes.

The following question will guide us through the chapter.

Question 102.1.

(1) *Is the intersection form a complete invariant of closed oriented simply connected 4-dimensional (topological) smooth manifolds?*[1425]
(2) *Which isometry types of non-singular symmetric forms are realized by closed oriented simply connected 4-dimensional (topological) smooth manifolds?*

102.1. The K3-surface and Rokhlin's Theorem.

In the following we will give a mostly chronological account of the developments regarding the two (or possibly four?) questions posed in Question 102.1.

In Theorem 101.5 we introduced the initially rather unexpected form E_8. One might wonder whether a form as strange as E_8 can ever appear "in nature". As we will now see, somewhat surprisingly this is indeed the case.

Definition. We refer to

$$\mathrm{K3} \;:=\; \big\{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}\mathrm{P}^3 \,\big|\, z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\big\}$$

as the K3-*surface*. Sometimes the K3-surface is also called the *Kummer surface*.

Remark. The odd name "K3-surface" comes about as follows.

(1) First of all, as we will see in Proposition 102.2, K3 is a *2-dimensional* complex manifold, i.e. it is a complex *surface*.
(2) According to [**OTL16**] the definition of the K3-surface is already implicit in the work of Srinivasa Ramanujan in the 1910s. The K3-surface was (re-) discovered by André Weil [**Weil58**] who named it in honor of three algebraic geometers, Kummer, Kähler and Kodaira, and also the mountain K2 in Pakistan which is the second highest mountain on earth. More precisely, André Weil wrote: "Dans la seconde partie de mon rapport, il s'agit des varietés kähleriennes dites K3, ainsi nommées en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire".

The following proposition summarizes some of the key aspects of the K3-surface.

[1425] Note that for a simply connected 4-dimensional topological manifold M we have $\mathrm{H}_1(M;\mathbb{Z}) = 0$ by the Hurewicz Theorem 52.5. If M is closed and orientable, then we have $\mathrm{H}_3(M;\mathbb{Z}) = 0$ by the Poincaré Duality Theorem 88.1. Thus we see that for closed oriented simply connected 4-dimensional topological manifolds the intersection form contains all the (co-) homological data we could ask for.

FIGURE 1445

Proposition 102.2. *The* K3*-surface is a closed simply connected complex manifold of complex dimension 2 or equivalently of real dimension 4. Furthermore we have*

$$\textit{intersection form of the K3-surface} \;\cong\; 2\cdot E_8 \oplus 3\cdot H.$$

PROOF. All the statements are proved in [**GoS99**, Theorem 1.3.8] or alternatively [**McS16**, p. 176]. The intersection form is also calculated in [**Huy16**, Proposition 3.5]. The calculation of the intersection form, perhaps somewhat disappointingly, makes use of Corollary 101.19. ■

Remark. The K3-surface, albeit with a somewhat different definition as ours, plays a big role in (algebraic) geometry, as is indicated by the fact that a whole book [**Huy16**] is dedicated to the study of the K3-surface(s). Note that it follows from [**Huy16**, Theorem 2.4.2 and 7.1.1] that the K3-surface as defined above, viewed as a smooth manifold, admits many non-biholomorphic complex structures.[1426]

In the following we want to discuss one particular aspect of the K3-surface. Namely, the calculation of the intersection form of the K3-surface, of the intersection form of $S^2 \times S^2$ and Proposition 100.12 raises the question whether one can split off copies of $S^2 \times S^2$ form the K3-surface. More precisely the following question arose almost immediately after the introduction of the K3-surface.

Question 102.3.

(1) *Does there exist a closed oriented 4-dimensional topological manifold M such that the* K3*-surface is homeomorphic to $M \# S^2 \times S^2$?*
(2) *Does there exist a closed oriented 4-dimensional smooth manifold M such that the* K3*-surface is diffeomorphic to $M \# S^2 \times S^2$?*

We will return to the question shortly. But before we do so, we continue with our chronological discussion. Next in line is a result proved by Vladimir Rokhlin in 1952. To formulate Rokhlin's Theorem we need the notion of a "spin structure". The definition of a "spin structure" is for example given in [**GoS99**, Chapter 1.4.2], but the precise definition is of no concern to us and we will use this notion as a black box. We just note that in the following we say that a smooth manifold is *spin*, if it admits a spin structure. We now state the key result on the existence of spin structures in our context.

[1426]We had seen already on page 449 that there exist complex manifolds that are diffeomorphic but not biholomorphic.

Proposition 102.4. *Let X be a closed oriented 4-dimensional smooth manifold.*

(1) *If X is spin, then the intersection form Q_X is even.*
(2) *If the intersection form Q_X is even and if $\mathrm{H}_1(X;\mathbb{Z})$ has no 2-torsion, then the smooth manifold X is spin.*

Proof. The proposition is the content of [**GoS99**, Corollary 5.7.6], which in turn is proved using the "Wu formula". The proposition is also discussed in [**Scor05**, p. 163]. ■

Now let us state Rokhlin's celebrated theorem.

Theorem 102.5. (Rokhlin 1952) *Let X be a closed oriented 4-dimensional smooth manifold. If X is spin, then $\operatorname{sign}(X)$ is divisible by 16.*

Proof. This theorem was proved by Vladimir Rokhlin [**Rok52**][**Rok86a**, p. 21] in 1952. More modern proofs of Rokhlin's Theorem are given in [**FK78**], [**LaM89**, Theorem II.2.13 and Corollary IV.1.2] and [**Sav02**, Theorem 2.1]. ■

Remark. We will now see that the converse to neither of the two statements in Proposition 102.4 holds.

(1) We consider the *Enriques surfaces*

$$X := \underbrace{\big\{[z_0:z_1:z_2:z_3] \in \mathbb{C}\mathrm{P}^3 \,\big|\, z_0^4+z_1^4+z_2^4+z_3^4=0\big\}}_{=\text{K3-surface}}\big/(z_0,z_1,z_2,z_3) \sim (\overline{z}_1,-\overline{z}_0,\overline{z}_3,-\overline{z}_2).$$

We make the following observations:
(a) It is elementary to show that the equivalence relation $\sim$ on K3 defines a smooth free orientation-preserving action by the group $\mathbb{Z}_2$.
(b) It follows from (a) and Propositions 6.32 and 6.47 that the Enriques surface is a closed oriented connected 4-dimensional smooth manifold.
(c) Since the K3-surface is simply connected it follows from (a), together with Lemma 16.5, Proposition 16.9 and Theorem 16.16, that $\pi_1(X) \cong \mathbb{Z}_2$. In particular we obtain from the Hurewicz Theorem 52.5 that $\mathrm{H}_1(X;\mathbb{Z})$ has 2-torsion.
(d) As is shown in [**Hab82**], the intersection form Q_X is isometric to $E_8 \oplus H$, in particular Q_X is even and $\operatorname{sign}(X) = 8$.

Note that it follows from (d) together with Rokhlin's Theorem 102.5 that X cannot be spin. In particular we see that the converse to Proposition 102.4 (1) does not hold. We refer to [**Hab82**], [**FS84**], [**GoS99**, p. 93] and [**LaM89**, p. 90] for more details on the Enriques surface.

(2) We consider the smooth manifold $S^1 \times \mathbb{R}\mathrm{P}^3$. It follows from the discussion on page **??** that $S^1 \times \mathbb{R}\mathrm{P}^3$ is parallelizable, i.e. its tangent bundle is a trivial bundle. This implies that $S^1 \times \mathbb{R}\mathrm{P}^3$ admits a spin structure. Since $\mathrm{H}_2(S^1 \times \mathbb{R}\mathrm{P}^3;\mathbb{Z}) = 0$ we see that its intersection form is even. But $\mathrm{H}_1(S^1 \times \mathbb{R}\mathrm{P}^3) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ has 2-torsion. Thus we see that the converse to Proposition 102.4 (2) does not hold.

The combination of Proposition 102.4 and Rokhlin's Theorem 102.5 gives us the following variation on Rokhlin's Theorem. This was historically the first result to address Question 102.1 (2).

Theorem 102.6. (Rokhlin 1952) *Let X be a closed oriented 4-dimensional smooth manifold. If the intersection form Q_X is even and if $\mathrm{H}_1(X;\mathbb{Z})$ has no 2-torsion (e.g. if X is simply connected), then* $\mathrm{sign}(X)$ *is divisible by 16.*[1427]

In a different direction the following theorem was the first noteworthy result towards an affirmative answer to Question 102.1 (1).

Theorem 102.7. (Milnor 1958) *Let X and Y be two closed oriented simply connected 4-dimensional topological manifolds. Then*

$$X \text{ and } Y \text{ are homotopy equivalent} \quad \Longleftrightarrow \quad Q_X \text{ is isometric to } \pm Q_Y.$$

Proof. This theorem was proved for smooth manifolds by John Milnor [**Miln58b**, Theorem 3] in 1958, building on work of John Whitehead [**WhdJ49a**]. Furthermore [**GoS99**, Theorem 1.2.25] and [**Kir89**, Theorem 2.1] sketch a proof that also works for topological manifolds. ■

102.2. **The work of Michael Freedman and Simon Donaldson.** After the somewhat preliminary results by Vladimir Rokhlin and John Milnor the early 1980s saw two major breakthroughs by Michael Freedman and Simon Donaldson.

First, in 1981 Michael Freedman [**Fre82**] gave a complete answer to Question 102.1 in the category of topological manifolds.[1428] More precisely, Michael Freedman [**Fre82**] proved the following theorem.

Theorem 102.8. (Freedman 1981) *Let Q be a nonsingular symmetric form over the integers.*

(1) *If Q is even, then up to an orientation-preserving homeomorphism, there exists a unique closed oriented simply connected topological manifold whose intersection form is isometric to Q.*
(2) *If Q is odd, then up to orientation-preserving homeomorphisms, there exist precisely two closed oriented simply connected topological manifolds whose intersection forms are isometric to Q. These two topological manifolds are distinguished by the Kirby-Siebenmann invariant which is discussed in the subsequent remark.*

Remark.

(1) Let X be a simply connected closed topological manifold. By the Hurewicz Theorem 53.5 we know that the Hurewicz homomorphism $\pi_2(X) \to \mathrm{H}_2(X;\mathbb{Z})$ is an isomorphism. In particular, if $\pi_2(X)$ vanishes, then $\mathrm{H}_2(X;\mathbb{Z}) = 0$. This fact shows that Theorem 102.8, applied to the form on the trivial group $\{0\}$, implies that every homotopy 4-sphere is homeomorphic to S^4. We will discuss this in greater detail in Chapter **??**.

[1427] Recall that from Proposition 101.18 we know, for purely algebraic reasons, that the signature of a non-singular even symmetric form is divisible by 8.

[1428] Michael Freedman (*1951) is an American mathematician, in 1986 he was awarded the Fields medal for the proof of Theorem 102.8.

(2) The two homeomorphism types of Theorem 102.8 are distinguished by the Kirby-Siebenmann invariant[1429] $\mathrm{ks}(M) \in \mathrm{H}^4(M;\mathbb{Z}_2) = \mathbb{Z}_2$ of a closed connected non-empty 4-dimensional topological manifold M. This invariant is defined in [**KSi77**, p. 318] or alternatively in [**FQ90**, Section 10.2B] and [**Rudy16**, Chapter 3.4]. The Kirby-Siebenmann invariant has the following properties:
   (a) By [**Rudy16**, Theorem 3.4.3] this invariant vanishes if X admits a PL-structure.[1430] Recall that in Theorem 64.14 we saw that every smooth manifold admits a PL-structure. In particular we see that the Kirby-Siebenmann invariant vanishes for smooth manifolds. We refer to [**Cer68**, p. IX] and [**Scor05**, p. 220] for more details.
   (b) By [**FNOP19**, Theorem 7.3] the Kirby-Siebenmann invariant is additive under the connected sum operation.

(3) Freedman's Theorem also answers Question 102.3 (1) in the affirmative. More precisely, by the existence statement of Theorem 102.8 (1) there exists a closed oriented simply connected 4-dimensional topological manifold X with $Q_X \cong E_8$. It follows from Propositions 100.11 and 100.12, together with the uniqueness statement of Theorem 102.8 (1), there exists an orientation-preserving homeomorphism of the K3-surface to $X\#X\#3\cdot(S^2\times S^2)$.

(4) We conclude this remark with a quick foray into the high-dimensional world. For $n \geq 3$ Wall [**Wall62**] (see also [**Ran02**, p. 269]) classified diffeomorphism classes of pairs $(M, \partial M)$ where M is an orientable $(n-1)$-connected $2n$-dimensional smooth manifold and where the boundary ∂M is a homotopy sphere.[1431] The classification goes via the intersection form and one extra invariant which lies in the homotopy group $\pi_{n-1}(\mathrm{SO}(n))$.

Example. On page 2415 we saw that the intersection form of $\mathbb{C}\mathrm{P}^2$ is represented by the (1×1)–matrix (1). By (2b) we know that the Kirby-Siebenmann invariant of $\mathbb{C}\mathrm{P}^2$ vanishes. Theorem 102.8 (2) together with Remark (2) says that there exists a closed oriented simply connected 4-dimensional topological manifold, which usually is denoted by $*\mathbb{C}\mathrm{P}^2$, with the same intersection form as $\mathbb{C}\mathrm{P}^2$ but with non-zero Kirby-Siebenmann invariant. Theorem 102.8 (2) also says that $*\mathbb{C}\mathrm{P}^2$ is unique up to an orientation-preserving homeomorphism. The topological manifolds $\mathbb{C}\mathrm{P}^2$ and $*\mathbb{C}\mathrm{P}^2$ are not homeomorphic since the Kirby-Siebenmann invariants differ. But note that by Theorem 102.7 the topological manifolds $\mathbb{C}\mathrm{P}^2$ and $*\mathbb{C}\mathrm{P}^2$ are homotopy equivalent.

Among many other consequences we obtain the following rather surprising result.

Theorem 102.9. *There exist closed orientable 4-dimensional topological manifolds that do not admit a smooth structure.*

[1429] Here it is worth recalling that Proposition 75.15 (3) says that for a closed connected non-empty 4-dimensional topological manifold we have $\mathrm{H}^4(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

[1430] We refer to page 1599 for the definition of a PL-structure.

[1431] See page 2161 for the definition of a homotopy sphere

PROOF. At this point we can combine the above results to give two types of examples of such smooth manifolds:

(1) It follows from Theorem 102.8 (2) and the above Remark (3) that given any non-singular odd symmetric form Q there exists a closed oriented 4-dimensional topological manifold X with $Q_X \cong Q$ and non-trivial Kirby-Siebenmann invariant. As we pointed out above in Remark (2a), this topological manifold does not admit a smooth structure.
(2) If we apply Freedman's Theorem 102.8 (1) to the non-singular positive-definite even form E_8 we obtain a certain closed oriented simply connected 4-dimensional topological manifold which, by Rokhlin's Theorem 102.6, cannot admit a smooth structure. ■

In a remarkable twist, shortly after Freedman proved his results on the intersection forms of 4-dimensional topological manifolds, Simon Donaldson[1432], see [**Don83**, Theorem A] and [**Don87**, Theorem 1], proved the following result regarding intersection forms of 4-dimensional smooth manifolds.[1433]

Theorem 102.10. (Donaldson 1982) *Let X be a closed oriented connected 4-dimensional smooth manifold. If Q_X is positive-definite, then Q_X is represented by the identity matrix.*

Example. We consider the positive definite symmetric form $E_8 \oplus E_8$. It has signature 16, hence Rokhlin's Theorem 102.6 has no objections. But Donaldson's Theorem 102.10 implies that this form cannot be the intersection form of a closed oriented connected 4-dimensional smooth manifold.

To understand the significance of Donaldson's Theorem, often referred to as *Donaldson's Theorem A*, recall that on page 2455 we saw that there are more than 10^{51} isometry classes of non-singular positive-definite symmetric forms of rank 40. By Freedman's Theorem 102.8 all of them arise as intersection forms of closed oriented connected 4-dimensional topological manifolds. By Donaldson's Theorem 102.10 the only one which appears as the intersection form of a closed oriented connected 4-dimensional smooth manifold is the one represented by the identity matrix.[1434]

Remark. It is difficult to exaggerate the importance of Donaldson's Theorem in mathematics.[1435] For example it is at the heart of the proof of Theorem **??** which says that $\mathbb{R}^4$

[1432]Simon Donaldson (*1957) is an English mathematician. He proved Theorem 102.10 when he was a second year PhD student. The Fields medalist Michael Atiyah described in [**At87**] the work of Simon Donaldson as follows: "In 1982, when he was a second-year graduate student, Simon Donaldson proved a result that stunned the mathematical world." Simon Donaldson was awarded the Fields medal in 1986.

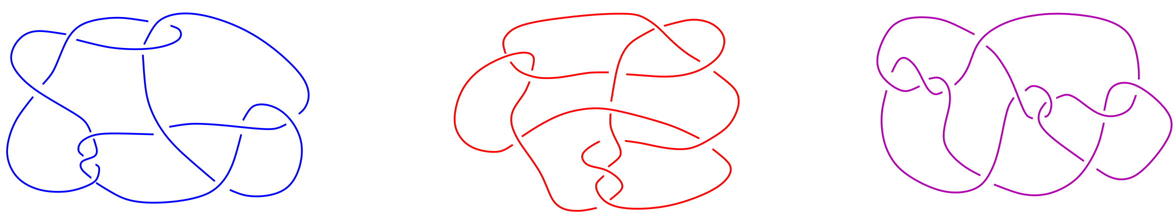
[1433]An exposition of the proof is also given in [**Law85**] and a different proof, using Seiberg-Witten invariants is given in [**Nic00**, Theorem 2.4.18] and a proof using Heegaard Floer homology is given in [**OS03**, Theorem 9.1].

[1434]The identity matrix represents, by Proposition 100.12 and the discussion on page 2415, the intersection form of the connected sum of 40 copies of $\mathbb{CP}^2$.

[1435]Admittedly it is also hard to underestimate the role of Donaldson's Theorem outside of mathematics.

FIGURE 1446. The Fields medalists Michael Freedman, Gerhard Faltings and Simon Donaldson at the ICM 1986 in Berkeley.

admits uncountably many smooth structures. But Donaldson's Theorem can also appear in totally unexpected situations. For example in 2017 Duncan McCoy **[McCoy17]** used Donaldson's Theorem, and many other ingredients, to show that if K is an alternating knot with unknotting number one[1436], then such a crossing change can already be performed in an alternating diagram. For example, according to **[LiM]** the three alternating knots shown in Figure 1447 have unknotting number one. By the aforementioned result we now know that we can turn these three knots into the trivial knot by changing a crossing in the alternating diagrams shown. Which crossings do you need to change?



the knot $11a_{12}$ the knot $11a_{41}$ the knot $11a_{56}$

FIGURE 1447

Now that we have digested Donaldson's Theorem 102.10, the question arises, what can we say about *indefinite* intersection forms of closed oriented 4-dimensional smooth manifolds? First recall that in Corollary 101.17 we saw that every non-singular indefinite odd symmetric form is isometric to $m \cdot (1) \oplus n \cdot (-1)$ for some $m, n \in \mathbb{N}$. Next note that according to Proposition 100.12 and the discussion on page 2415 every such form is realized by the smooth manifold $m \cdot \mathbb{CP}^2 \# n \cdot \overline{\mathbb{CP}}^2$.

Therefore, in the following we only need to discuss the realizability of non-singular indefinite *even* symmetric forms. Recall that in Corollary 101.19 we showed that every non-singular even indefinite symmetric form is isometric to $m \cdot H \oplus n \cdot E_8$ for some $m \in \mathbb{N}_0$ and $n \in \mathbb{Z} \setminus \{0\}$. The following theorem, proved by Mikio Furuta **[Furu01]** in 2001 gives some restrictions on the possible values of m and n.

[1436] We refer to page 809 for the definition of the unknotting number of a knot. A knot is called *alternating* if it admits a diagram that is alternating, i.e. which has the property that each undercrossing is followed by an overcrossing and vice versa.

Theorem 102.11. (Furuta's 10/8-Theorem) *If M is a closed oriented connected 4-dimensional smooth manifold with* indefinite *even intersection form, then*

$$b_2(M) \;\geq\; \frac{10}{8}\cdot \operatorname{sign}(M)+2.$$

In particular[1437] $Q_M \cong m\cdot H\oplus n\cdot E_8$ *with* $m\geq |n|+1$.

Remark. Furuta's 10/8-Theorem allows us to answer Question 102.3 (2) in the negative. More precisely, suppose that we have a decomposition of the K3-surface as a connected sum $M\#N$. We make the following four observations:

(1) Without loss of generality we have $\operatorname{sign}(M)\geq \operatorname{sign}(N)$.
(2) It follows from Proposition 102.2, Lemma 101.14 (3) and Proposition 100.12 that Q_M and Q_N are even.
(3) From Lemma 101.11 (2) and Proposition 102.2 we obtain that $\operatorname{sign}(M)+\operatorname{sign}(N)=16$.
(4) By Proposition 102.2 and Proposition 100.11 we know that $b_2(M)+b_2(N)=22$.

If we combine these facts with Rokhlin's Theorem 102.6 we obtain easily that we have $\operatorname{sign}(M)=16$. Furthermore, it follows from Donaldson's Theorem 102.10 that Q_M is indefinite. Now Furuta's 10/8-Theorem implies that $b_2(M)\geq \frac{10}{8}\cdot 16+2=22$, but by Proposition 102.2 this equals $b_2(\mathrm{K3})$. Thus we obtain from Proposition 100.11 that $b_2(N)=0$.[1438] We can conclude that we cannot split off a copy of $S^2\times S^2$ from K3. This answers Question 102.3 (2) in the negative.

Furuta's 10/8 Theorem is not yet optimal since it does not quite close the gap between the forms which we can realize by smooth manifolds and the forms which we can exclude. More precisely it follows from Proposition 100.12, the calculation of the intersection forms of the K3-surface, see Proposition 102.2, and of the intersection form of $S^2\times S^2$ that for any $m\in\mathbb{Z}\setminus\{0\}$ and any $n\geq 3|m|$ there exists a closed oriented simply connected 4-dimensional smooth manifold with intersection form isometric to $m\cdot E_8\oplus n\cdot H$. More precisely, we have

$$\text{intersection form of } m\cdot \mathrm{K3}\,\#\,(n-3|m|)\cdot(S^2\times S^2) \;\cong\; m\cdot E_8\oplus n\cdot H.$$

The following conjecture predicts that this result is optimal.

Conjecture 102.12. (11/8-Conjecture) *If M is a closed oriented 4-dimensional smooth manifold with indefinite even intersection form, then*

$$b_2(M)\;\geq\;\frac{11}{8}\cdot\operatorname{sign}(M).$$

Equivalently, if $Q_M\cong m\cdot H\oplus n\cdot E_8$ with $n\neq 0$, then $m\geq \frac{3}{2}|n|$.[1439]

[1437]This second statement follows from the first statement together with Corollary 101.19 and the calculation of the rank and signature of H and E_8.

[1438]By Proposition 100.11 and Proposition 102.2 we also know that $\pi_1(N)=0$. Recall that by the Hurewicz Theorem 53.5 we know that if $\pi_1(N)=0$ and $\mathrm{H}_2(N;\mathbb{Z})=0$, then $\pi_2(N)=0$. Thus it follows from Theorem **??** that N is homeomorphic to S^4. Since Question **??** is still open, we do not know whether N is diffeomorphic to S^4.

[1439]As before, the fact that these two statements are equivalent follows from Corollary 101.19 and the calculation of the rank and signature of H and E_8.

Remark.

(1) A proof of the 11/8-Conjecture would imply, by Freedman's Theorem 102.8, that any closed oriented simply connected 4-dimensional smooth manifold is *homeomorphic* to either a connected sum of the form $m \cdot \mathbb{C}\mathrm{P}^2 \# n \cdot \overline{\mathbb{C}\mathrm{P}}^2$ or alternatively to a connected sum of the form $m \cdot \mathrm{K3} \# n \cdot (S^2 \times S^2)$.

(2) For simply connected smooth manifolds currently the best result is [**HLSX18**, Corollary 1.13] which says that if M is a closed oriented simply connected 4-dimensional smooth manifold that is not homeomorphic to S^4, $S^2 \times S^2$ or the K3-surface and whose intersection form is indefinite and even, then $b_2(M) \geq \frac{10}{8} \cdot |\operatorname{sign}(M)| + 4$.

The other question which arises in this context is whether closed orientable 4-dimensional smooth manifolds that are homeomorphic are also necessarily diffeomorphic. It turns out that in many cases the answer is very much no. For example the following theorem was proved in 1998 by Ron Fintushel and Ron Stern [**FS98**].

Theorem 102.13. *There exist closed orientable 4-dimensional smooth manifolds X_i, $i \in \mathbb{N}$ such that each X_i is homeomorphic to the* K3*-surface, but such that the X_i are pairwise non-diffeomorphic.*

Similar results have been obtained for many other 4-dimensional smooth manifolds, e.g. for $\mathbb{C}\mathrm{P}^2 \# k \overline{\mathbb{C}\mathrm{P}}^2$ with $k = 9$ by Simon Donaldson [**Don87**] in 1987, $k = 8$ by Dieter Kotschick [**Kot89**] in 1989 and $k = 7$ by Jongil Park [**Park05**] in 2005, and since then also for $k = 6, 5, 4, 3, 2$, see e.g. [**AP08, AP10**] for more details on the history of this problem.

The following question is surely one of the most interesting problem in the theory of 4-dimensional smooth manifolds.

Question 102.14. *Does every closed oriented connected 4-dimensional smooth manifold M admit infinitely many pairwise non-diffeomorphic smooth structures? Put differently, given any such M, do there always exist infinitely many smooth manifolds M_i, $i \in \mathbb{N}$ which are homeomorphic to M but which are pairwise non-diffeomorphic?*

Note that an affirmative answer to Question 102.14 would also give a positive answer to Question **??**.

The following theorem, proved by C. T. C. Wall [**Wall64**, Theorem 3] in 1964 (see also [**Scor05**, p. 149] and [**Qu83**, Theorem 1.1]) shows that in some sense homeomorphic simply connected smooth manifolds are not that far from being diffeomorphic.

Theorem 102.15. *Let M and N be two closed oriented simply connected 4-dimensional topological manifolds. If M and N are homeomorphic, then there exists a $k \in \mathbb{N}$ such that $M \# k \cdot (S^2 \times S^2)$ and $N \# k \cdot (S^2 \times S^2)$ are diffeomorphic.*

This raises the question, whether in the above theorem $k = 1$ suffices. More precisely the following question is still open.

Question 102.16. *Let M and N be two closed oriented simply connected 4-dimensional topological manifolds that homeomorphic. Is $M \# (S^2 \times S^2)$ diffeomorphic to $N \# (S^2 \times S^2)$?*

There are some indications, see [**JZ18**], that the answer to the above question should be no.

Exercises for Chapter 102.

Exercise 102.1. As we discussed on 2464, it follows from Theorem 102.8 that there exists a closed orientable simply connected 4-dimensional topological manifold $*\mathbb{C}\mathrm{P}^2$ whose intersection form is represented by the (1×1)-matrix (1) and with $\mathrm{ks}(*\mathbb{C}\mathrm{P}^2) = 1$.

Let M and N be two closed oriented simply connected 4-dimensional topological manifolds. Show that there exist $m, m', n, n' \in \mathbb{N}_0$ and $\epsilon \in \{0, 1\}$ such that the topological manifolds $M \# m \cdot \mathbb{C}\mathrm{P}^2 \# m' \cdot \overline{\mathbb{C}\mathrm{P}^2} \# \epsilon \cdot *\mathbb{C}\mathrm{P}^2$ and $N \# n \cdot \mathbb{C}\mathrm{P}^2 \# n' \cdot \overline{\mathbb{C}\mathrm{P}^2}$ are homeomorphic.
Hint. Make use of the properties of the Kirby-Siebenmann invariant discussed on page 2464. Furthermore it is strongly advisable to use Theorem 101.16.

Exercise 102.2. Let M and N be two closed oriented simply connected 4-dimensional smooth manifolds. Show that there exist $m, m', n, n' \in \mathbb{N}_0$ such that $M \# m \cdot \mathbb{C}\mathrm{P}^2 \# m' \cdot \overline{\mathbb{C}\mathrm{P}^2}$ is diffeomorphic to $N \# n \cdot \mathbb{C}\mathrm{P}^2 \# n' \cdot \overline{\mathbb{C}\mathrm{P}^2}$.
Remark. Make use of Exercise 102.1 and use the fact from page 2453 that the two smooth manifolds $(S^2 \times S^2) \# \mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2 \# \mathbb{C}\mathrm{P}^2$ are diffeomorphic.

Part XI

Handle Structures and Morse Theory

As we have seen, CW-complexes and simplicial complexes have turned out to be useful tools in topology since often they make it possible to break a given topological space into smaller, more manageable pieces. In Theorems 64.2 and 64.5 we saw that every smooth manifold admits a simplicial structure, in particular it admits a CW-structure. Thus in principle we can study a smooth manifold by starting with a 0-dimensional CW-complex and attaching cells of dimensions $1, 2, \dots$. The catch is that the inbetween objects are usually not smooth manifolds.

In this section we will introduce handle decompositions of smooth manifolds. These are in many ways quite similar to CW-structures, except that now all intermediate objects are smooth manifolds. In Chapter 104 we will use Morse theory to show that every compact smooth manifold admits a handle decomposition.

103.1. **Handle attachments.** The following definition is inspired by the definition of a thickened knot that we gave on page 2384.

Definition. Let W be an m-dimensional smooth manifold. A *thickened l-sphere* is a smooth embedding $\overline{B}^{m-l} \times S^l \to W$.

Our first proposition introduces the notion of a handle attachment and it summarizes some of the key properies.

Proposition 103.1. *Let M be a compact n-dimensional smooth manifold. Furthermore suppose that we are given a thickened $(k-1)$-sphere in the boundary ∂M, i.e. suppose we are given a smooth embedding $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$. We refer to*[1440][1441]

$$M \cup_\varphi h^k \;:=\; M \cup_{\partial M \xleftarrow{\varphi} \overline{B}^{n-k} \times S^{k-1}} \big(\overline{B}^{n-k} \times \overline{B}^k\big)$$

*as the result of a k-*handle attachment *to M. The following statements hold:*

(1) (a) *The topological space $M \cup_\varphi h^k$ is an n-dimensional topological manifold.*
 (b) *Every choice of a collar*[1442]$\Theta\colon [0,1] \times \partial M \to M$ *equips $M \cup_\varphi h^k$ canonically with the structure of an n-dimensional smooth manifold. This smooth structure has the property that M and $\overline{B}^{n-k} \times \overline{B}^{k-1}$ are both submanifolds with corner, where the corner set in both cases is given by $\varphi(S^{n-k-1} \times S^{k-1})$.*
 (c) *Let Θ and Ξ be two collars. By* (1b) *these two collars give rise to two smooth structures $\mathcal{A}_\Theta$ and $\mathcal{A}_\Xi$ on the topological manifold $M \cup_\varphi h^k$. Given any neighborhood U of the image of $\varphi(\overline{B}^{n-k} \times S^{k-1})$ in $M \cup_\varphi h^k$ there exists a homeotopy $H\colon (M \cup_\varphi h^k) \times [0,1] \to M \cup_\varphi h^k$ with the following four properties:*
  (i) $H_0 = \mathrm{id}$.
  (ii) $H_1\colon (M \cup_\varphi h^k, \mathcal{A}_\Theta) \to (M \cup_\varphi h^k, \mathcal{A}_\Xi)$ *is a diffeomorphism.*
  (iii) *For each $t \in [0,1]$ the map h_t restricts to a diffeomorphism of M and it restricts to a diffeomorphism of $\overline{B}^{n-k} \times \overline{B}^k$.*
  (iv) *$H(x,t) = x$ for all $x \notin U$ and all $t \in [0,1]$.*

(d) *Any two choices of collar neighborhoods give rise to diffeomorphic smooth structures on M.*

(2) *$M \cup_\varphi h^k$ is compact.*

(3) *The boundary of the smooth manifold $M \cup_\varphi h^k$ is given by*

$$\partial(M \cup_\varphi h^k) \;=\; (\partial M \setminus \varphi(B^{n-k} \times S^{k-1})) \cup_{\varphi|_{S^{n-k-1} \times S^{k-1}}} (S^{n-k-1} \times \overline{B}^k).$$

(4) *If $\varphi, \psi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ are two smooth embeddings that are smoothly isotopic, then $M \cup_\varphi h^k$ and $M \cup_\psi h^k$ are diffeomorphic. Furthermore, given any neighborhood U of ∂M one can find a diffeomorphism that is the identity on the common subset $M \setminus U$.*

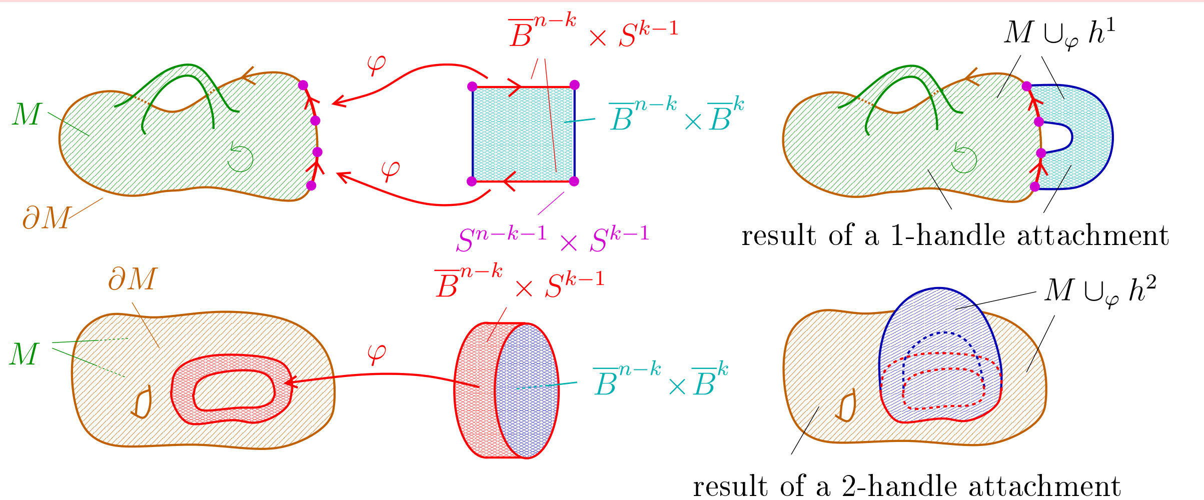


FIGURE 1448. Illustration of Proposition 103.1.

Remark.

(1) The fact that the precise smooth structure on $M \cup_\varphi h^k$ depends on a choice of a collar is a nuisance. Usually we use (1c) as an excuse to ignore this issue, albeit with a slightly bad conscience.

(2) The modification of the boundary that we saw in Proposition 103.1 (5) is called a *surgery along* $\varphi(\{0\} \times S^{k-1}) \subset \partial M$. We will study surgeries in greater detail in Chapter **??**.

(3) We deal with the question of whether $M \cup_\varphi h^k$ is orientable respectively connected in Lemma 103.8 and Proposition 103.9.

SKETCH OF PROOF.

[1440]Recall that by the definition on page 200 this means that

$$M \cup_\varphi h^k \;:=\; \big(M \sqcup (\overline{B}^{n-k} \times \overline{B}^k)\big)/\sim \quad \text{where } \varphi(P) \sim P \text{ for } P \in \overline{B}^{n-k} \times S^{k-1}.$$

[1441]The fact that in the notation the map φ now goes from right to left is a little awkward, but all other conventions lead sooner or later also to slightly unnatural results.

[1442]Recall that by the Collar Neighborhood Theorem 8.12 a collar $[0,1] \times \partial M \to M$ always exists.

(1) The proof of Statement (1) is quite similar to the proof of Proposition 8.15 (1).

(a) Let $\Theta\colon [0,1]\times\partial M \to M$ be a collar. We use this collar and the standard product neighborhood $[0,1]\times\overline{B}^{n-k}\times S^{k-1}\subset \overline{B}^{n-k}\times\overline{B}^{k}$ together with the argument from the proof of Proposition 8.15 (1) to find suitable canonical charts for all points that do not lie on $\varphi(S^{n-k-1}\times S^{k-1})$. Furthermore, for points on $\varphi(S^{n-k-1}\times S^{k-1})$ one can easily find "charts" such that the image equals $\{(x_1,\dots,x_n)\in\mathbb{R}^n \,|\, x_{n-1}\leq 0 \text{ or } x_n\geq 0\}$. Similar to the proof of Proposition 80.9 we compose those charts with the map

$$\begin{array}{rcl}\Psi\colon \mathbb{R}^{n-2}\times\{(x,y)\in\mathbb{R}^2\,|\,x\leq 0 \text{ or } y\geq 0\} & \to & \mathbb{R}^{n-2}\times\mathbb{R}\times\mathbb{R}_{\geq 0}\\ (x_1,\dots,x_{n-2},\underbrace{r\cos(\varphi),r\sin(\varphi)}_{r\in\mathbb{R}_{\geq 0}\text{ and }\varphi\in[0,\frac{3}{2}\pi]}) & \mapsto & (x_1,\dots,x_{n-2},r\cos(\tfrac{2}{3}\varphi),r\sin(\tfrac{2}{3}\varphi)).\end{array}$$

If one interprets the above discussion with good will, then one can see that this defines a smooth atlas on $M\cup_\varphi h^k$ which has all the desired properties.

(b) We proved this statement already in (a).

(c) As in the proof of Proposition 8.15 (1c) this statement follows without too much effort from the uniqueness statement for collars, see Proposition 8.13.

(d) This statement is an immediate consequence of (c).

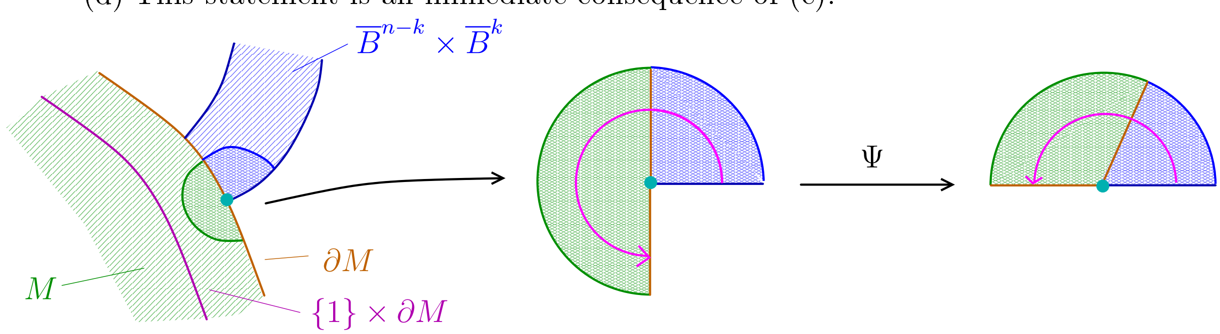


FIGURE 1449. Illustration for the proof of Proposition 103.1 (1).

(2) It follows from the compactness of M and $\overline{B}^{n-k}\times\overline{B}^{k}$ together with Lemma 3.21 (4) that $M\cup_\varphi h^k$ is compact.

(3) Similar to the proof of Proposition 8.15 (4) one can prove the statement about the description of $\partial(M\cup_\varphi h^k)$ quite easily using Proposition 6.27.

(4) It follows from (1c) that throughout the proof of (4) we can work with the same collar neighborhood $[0,1]\times\partial M$. Now let $\varphi,\psi\colon \overline{B}^{n-k}\times S^{k-1}\to\partial M$ be two smooth embeddings. We suppose that they are smoothly isotopic. Let $F\colon \overline{B}^{n-k}\times S^{k-1}\times[0,1]\to\partial M$ be a smooth isotopy with $F_0=\varphi$ and $F_1=\psi$. By the Isotopy Extension Theorem 8.27 together with Lemma 8.4 there exists a diffeotopy $G\colon\partial M\times[0,1]\to\partial M$ with $G_t=\mathrm{id}$ for $t\in[0,\frac14]$ and with $G_t\circ\varphi=\psi$ for all $t\in[\frac34,1]$. One can now easily

verify that the map

$$\begin{array}{rcl} \Xi\colon M\cup_\varphi h^k & \to & M\cup_\psi h^k \\ [x] & \mapsto & \begin{cases} [x], & \text{if } x\in M\setminus([0,1]\times\partial M), \\ [(t,G(1-t,y))], & \text{if } x=(t,y)\in[0,1]\times\partial M, \\ [x], & \text{if } x\in \overline{B}^{n-k}\times\overline{B}^k \end{cases} \end{array}$$

is a diffeomorphism. Finally suppose that we are given a neighborhood U of ∂M. Evidently we can pick a collar neighborhood $[0,1]\times\partial M$ that is contained in $M\setminus U$. The diffeomorphism that we just constructed is now the identity on the common subset $M\setminus U$.

As on so many other occasions we leave it to the reader to fill in the annoying details. ■

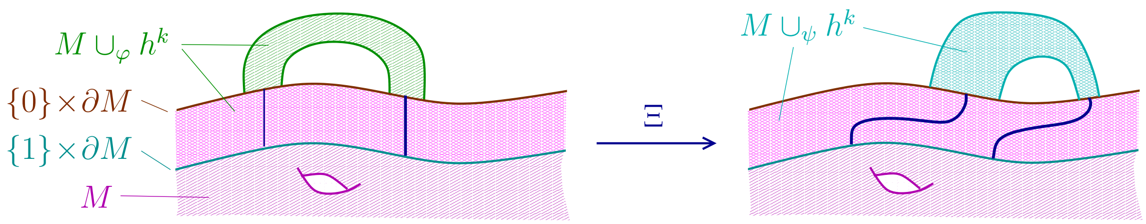


FIGURE 1450. Illustration for the proof of Proposition 103.1 (2).

Remark. The approach in the proof of Proposition 103.1 for defining charts for $M\cup_\varphi h^k$ at the points in $\varphi(S^{n-k-1}\times S^{k-1})$ is sometimes called *rounding corners* or *smoothing corners.*

Proposition 103.1 motivates the following definition.

Definition. Let M be a compact n-dimensional smooth manifold. Suppose we are given a thickened $(k-1)$-sphere $\varphi\colon \overline{B}^{n-k}\times S^{k-1}\to\partial M$.

(1) We say

$$M\cup_\varphi h^k \;:=\; M\cup_{\partial M\xleftarrow{\varphi}\overline{B}^{n-k}\times S^{k-1}}\left(\overline{B}^{n-k}\times\overline{B}^k\right)$$

is *obtained from M by attaching the k-handle $\overline{B}^{n-k}\times\overline{B}^k$ along φ.* In particular we refer to the image of $\overline{B}^{n-k}\times\overline{B}^k$ in $M\cup_\varphi h^k$ as a *k-handle* or as a *handle of index k.*

(2) (a) We refer to the image of $\{0\}\times S^{k-1}$ in ∂M as the *attaching sphere of the handle.*
(b) We refer to the image of $\{0\}\times\overline{B}^k$ in $M\cup_\varphi h^k$ as the *core of the handle.*
(c) We refer to the image of $S^{n-k}\times\{0\}$ in $\partial(M\cup_\varphi h^k)$ as the *belt sphere of the handle.*

We refer to Figure 1451 for an illustration of these definitions.

Example. Let M be a compact n-dimensional smooth manifold.

(1) Let us consider the extreme cases of attaching a 0-handle and attaching an n-handle.
   (a) It follows basically from the definitions that attaching a 0-handle is the same as taking the disjoint union with $\overline{B}^n$.

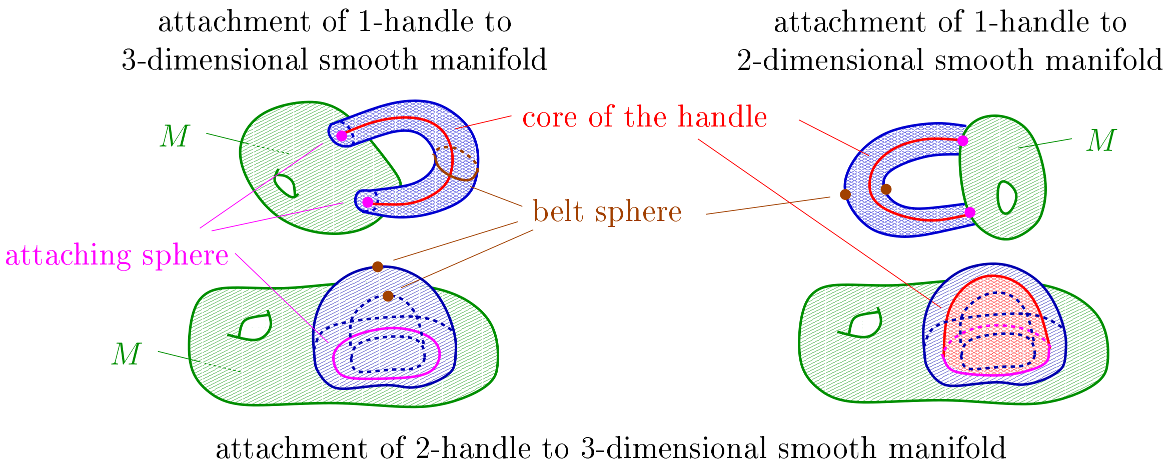


FIGURE 1451

(b) It follows immediately from Proposition 8.2 (3) that any smooth embedding $\varphi\colon \overline{B}^0 \times S^{n-1} \to \partial M$ is a diffeomorphism onto a component of ∂M. Thus an n-handle attachment is the same as gluing an n-ball to a boundary component that is diffeomorphic to S^{n-1} via the diffeomorphism φ. Note that in Proposition **??** we will see that in general the result depends on the choice of φ.

We illustrate these two examples in Figure 1452.

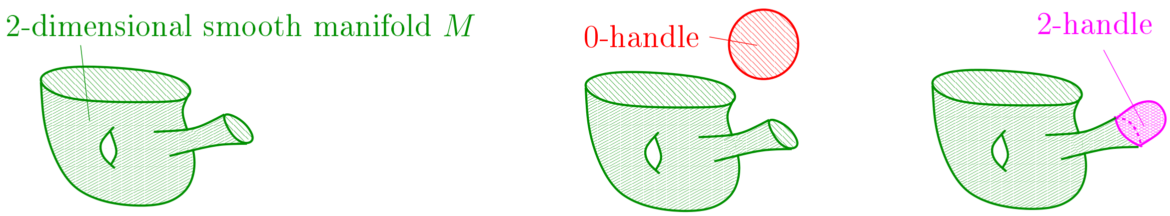


FIGURE 1452

(2) Next we turn to attachments of 1-handles. As is shown in Figure 1453, the effect of attaching a 1-handle depends on the choice of the attaching map $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$. In Lemma 103.10 we will see that the four examples of 1-handle attachments that we see in Figure 1453 exhaust in a suitable sense all possibilities for 1-handle attachments.

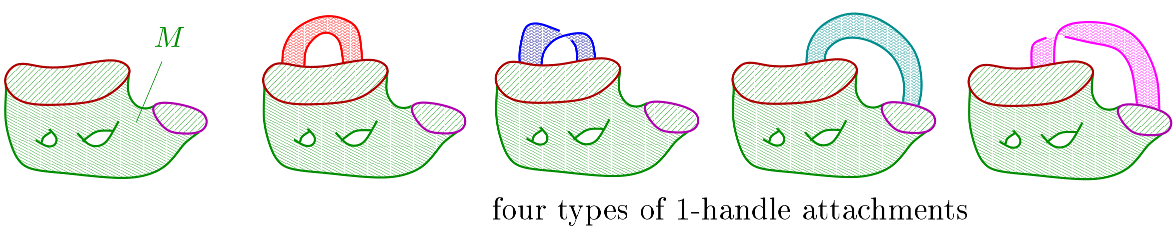


FIGURE 1453

(3) For handle attachments that are not of index 0, 1 and $\dim(M)$ there is no straightforward classification of all possibilities. For example we consider the smooth manifold

$M = \overline{B}^4$. Let $\gamma\colon S^1 \to S^3$ be a smooth embedding. Note that $K = \gamma(S^1)$ is a knot. In Lemma 99.1 we saw that for each $d \in \mathbb{Z}$ there exists an essentially unique tubular map $\overline{B}^2 \times K \to S^3$ of self-linking number d. Any such tubular map describes a 2-handle attachment to $\overline{B}^4$. There is no reason to believe that all these 2-handle attachments can be classified easily.

103.2. **Handle decompositions.** The following is a good example of a self-explanatory definition.

Definition. Let M be a compact n-dimensional smooth manifold. A *handle decomposition for M* is a diffeomorphism

$$\Theta\colon M \xrightarrow{\cong} \varnothing \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_m} h^{k_m}$$

where for every $i = 1, \ldots, m$ the map φ_i is a thickened $(k_i - 1)$-sphere in the boundary of the smooth manifold $\varnothing \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_{i-1}} h^{k_{i-1}}$. Usually we suppress the name of the diffeomorphism, in this case Θ, from the notation.

Examples.

(1) In Figure 1454 we show a handle decomposition for the surface of genus $g = 2$. This handle decomposition consists of one 0-handle, $2g$ 1-handles and one 2-handle.

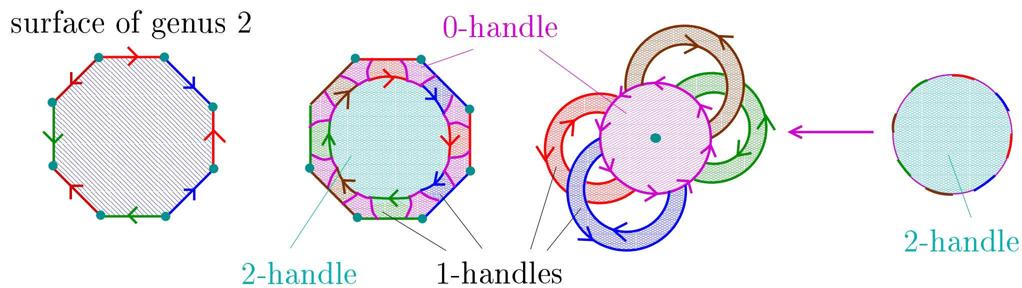


FIGURE 1454

(2) In Figure 1455 we show a handle decomposition for the non-orientable surface of genus $k = 3$. This handle decomposition consists of one 0-handle, k 1-handles and one 2-handle.

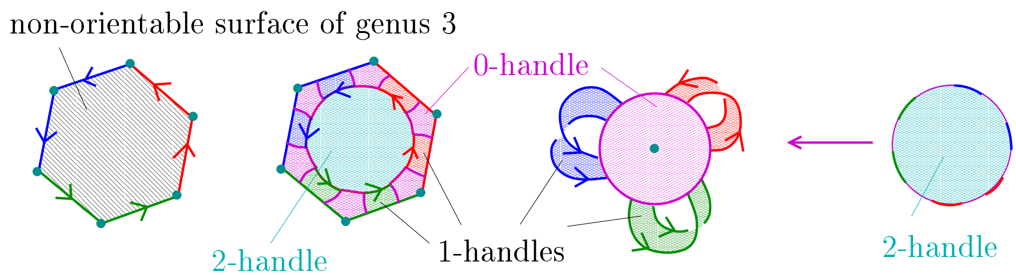


FIGURE 1455

We formulate the next two types of examples as a lemma.

Lemma 103.2.

(1) *Let $n \in \mathbb{N}$. We consider the thickened $(n-1)$-sphere that is given by the identity*
$$\mathrm{id}\colon \overline{B}^0 \to S^{n-1} \ \to \ \partial(\overline{B}^n).$$
If we attach an n-handle to $\overline{B}^n$ along id *we obtain a smooth manifold that is diffeomorphic to S^n. In particular we see that S^n admits a handle decomposition with a single 0-handle and a single n-handle.*

(2) *Let $m, k \in \mathbb{N}$. We consider the thickened $(k-1)$-sphere*
$$\begin{array}{rcl} \varphi\colon \overline{B}^m \times S^{k-1} & \to & S^{m+k-1} = \partial(\overline{B}^{m+k}) \\ (x,y) & \mapsto & \dfrac{(\frac{1}{2}\cdot x, y)}{\|(\frac{1}{2}\cdot x, y)\|} \\ \uparrow & & \\ \text{with } (x,y) \in \overline{B}^m \times S^{k-1} \subset \mathbb{R}^m \times \mathbb{R}^k = \mathbb{R}^{m+k} & & \end{array}$$
If we attach a k-handle to the 0-handle $\overline{B}^{m+k}$ along the thickened $(k-1)$-sphere φ, then we obtain a smooth manifold that is diffeomorphic to $\overline{B}^m \times S^k$. In particular we see that $\overline{B}^m \times S^k$ admits a handle decomposition with a single 0-handle and a single k-handle.

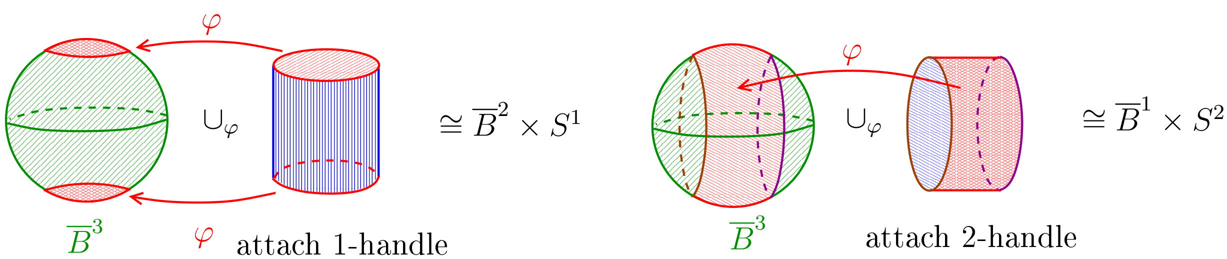


FIGURE 1456. Illustration of Lemma 103.2 (2).

PROOF.

(1) Let $n \in \mathbb{N}$. As on page 199 we consider the maps
$$\begin{array}{rcl} g_\pm\colon \overline{B}^n = \overline{B}^n_\pm & \to & S^n \\ (x_1,\dots,x_n) & \mapsto & \big((-1)^n\cdot x_1,\dots,x_n, \pm\sqrt{1-x_1^2-\dots-x_n^2}\big). \end{array}$$
We leave it to the reader to verify that the map[1443]
$$\begin{array}{rcl} \overline{B}^n_+ \cup_{S^{n-1} \xleftarrow{\mathrm{id}} S^{n-1}} \overline{B}^n_- & \to & S^n \\ {[P]} & \mapsto & \begin{cases} g_+(P), & \text{if } P \in \overline{B}^n_+, \\ g_-(P), & \text{if } P \in \overline{B}^n_- \end{cases} \end{array}$$
is an orientation-preserving diffeomorphism.

[1443]As we mentioned in the formulation of Proposition 103.1, the precise smooth structure on $\overline{B}^n_+ \cup_{S^{n-1} \xleftarrow{\mathrm{id}} S^{n-1}} \overline{B}^n_-$ depends on the choice of a collar. We follow the convention from page 352, namely we consider $\overline{B}^n_\pm$ with the standard collar that we introduced on page 351. With this convention the given map is indeed an honest-to-God diffeomorphism.

(2) We will prove this statement in Exercise 103.2. ■

In the above we have now seen that some of our favorite smooth manifolds admit a handle decomposition. In Chapter 104 we will prove the Handle Decomposition Theorem 104.11 which states that in fact every compact smooth manifold admits a handle decomposition. In the remainder of this chapter we perform the less exciting task of proving several basic facts about handle attachments and handle decompositions.

103.3. **Handle rearrangements.** The following proposition shows that when we attach several handles, then we can rearrange the handle attachments so that they are done with increasing indices.

Proposition 103.3. *Let M be a compact n-dimensional smooth manifold. Suppose that for $i = 1, \dots, m$ we are given smooth embeddings*

$$\varphi_i \colon \overline{B}^{n-k_i} \times S^{k_i-1} \;\to\; \partial(M \cup_{\varphi_1} h^{k_1} \cdots \cup_{\varphi_{i-1}} h^{k_i}).$$

Furthermore let

$$\psi \colon \overline{B}^{n-l} \times S^{l-1} \;\to\; \partial(M \cup_{\varphi_1} h_1^{k_1} \cdots \cup_{\varphi_m} h_m^{k_m})$$

be a smooth embedding. If for $i = 1, \dots, m$ we have $l \leq k_i$, then there exists a smooth embedding

$$\widetilde{\psi} \colon \overline{B}^{n-l} \times S^{l-1} \;\to\; \textit{interior of } (\partial M \cap \partial(M \cup_{\varphi_1} h_1^{k_1} \cdots \cup_{\varphi_m} h_m^{k_m})) \textit{ in } \partial M$$

such that $M \cup_{\varphi_1} h_1^{k_1} \cdots \cup_{\varphi_m} h_m^{k_m} \cup_\psi h^l$ is diffeomorphic to[1444] *$M \cup_{\widetilde{\psi}} h^l \cup_{\varphi_1} h_1^{k_1} \cdots \cup_{\varphi_m} h_m^{k_m}$. Furthermore, given any neighborhood U of ∂M one can find a diffeomorphism that is the identity on the common subset $M \setminus U$.*

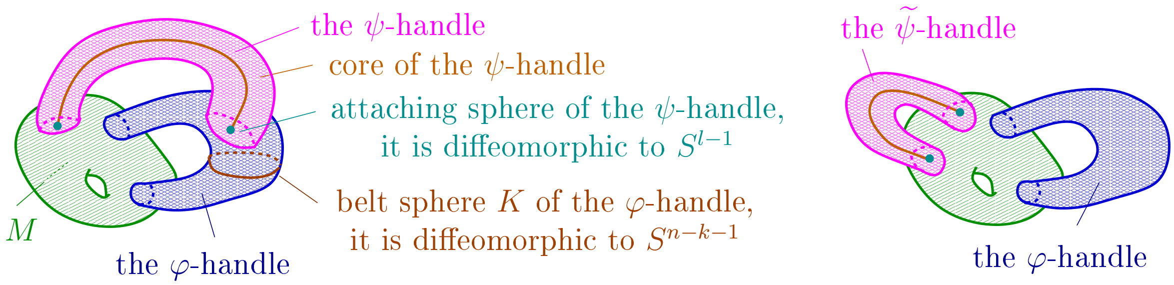


FIGURE 1457. Illustration of Proposition 103.3.

Remark. Colloquially replacing the ψ-handle by the $\widetilde{\psi}$-handle in (the proof of) Proposition 103.3 is called a *handle slide*

PROOF. To simplify the notation we prove the proposition for the special case $m = 1$. The proof of the general case is not much harder and we leave it to the reader to fill in the details.

Thus suppose that we are given a smooth embedding $\varphi \colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ and suppose that we are given a smooth embedding $\psi \colon \overline{B}^{n-l} \times S^{l-1} \to \partial(M \cup_\varphi h^k)$ with $l \leq k$.

[1444]Note that it follows from the fact that ψ takes values in the interior of $\partial M \cap \partial(M \cup_{\varphi_1} h_1^{k_1} \cdots \cup_{\varphi_m} h_m^{k_m}))$ in ∂M that the maps $\varphi_1, \dots, \varphi_m$ can also be used to attach successively handles to $M \cup_{\widetilde{\psi}} h^l$.

To simplify the notation we make the obvious identification $S^{l-1} = \{0\} \times S^{l-1} \subset \overline{B}^{n-l} \times S^{l-1}$. We write $f := \psi|_{S^{l-1}}$. Note that $f(S^{l-1})$ is precisely the attaching sphere of the ψ-handle. We denote by $K = S^{n-k-1} \times \{0\} \subset \partial(M \cup_\varphi h^k)$ the belt sphere of the φ-handle.

Claim. There exists a smooth isotopy $F\colon S^{l-1} \times [0,1] \to \partial(M \cup_\varphi h^k)$ such that $F_0 = f$ and $F_1(S^{l-1}) \subset \partial(M \cup_\varphi h^k) \setminus K$.

Note that we know by Proposition 8.1 and Proposition 103.1 (1) that $f(S^{l-1})$ and K are both submanifolds of $\partial(M \cup_\varphi h^k)$. By the Transversality Theorem 9.10 there exists a smooth isotopy $F\colon S^{l-1} \times [0,1] \to \partial(M \cup_\varphi h^k)$ with $F_0 = f$ such that $F_1(S^{l-1})$ is transverse to the belt sphere K. Note that

$$\dim(F_1(S^{l-1})) + \dim(K) \;=\; l-1+n-k-1 \;\underset{\substack{\uparrow \\ \text{by hypothesis we have } l \le k}}{\le}\; n-2 \;<\; n-1 \;=\; \dim(\partial(M \cup_\varphi h^k)).$$

It follows immediately from the definition of transverse intersection and this inequality of dimensions that $F_1(S^{l-1})$ is actually disjoint from the belt sphere K. ⊞

We continue with the following technical enhancement of the previous claim.

Claim. There exists a smooth isotopy $\Psi\colon (\overline{B}^{n-l} \times S^{l-1}) \times [0,1] \to \partial(M \cup_\varphi h^k)$ with $\Psi_0 = \psi$ and such that $\Psi_1(\overline{B}^{n-l} \times S^{l-1}) \subset \partial M \setminus \varphi(\overline{B}^{n-k} \times S^{k-1})$.

It follows from Proposition 103.1 (1) that $\varphi(S^{n-k-1} \times \overline{B}^k)$ is a tubular neighborhood for the belt sphere $K = S^{n-k-1} \times \{0\} \subset \partial(M \cup_\varphi h^k)$. We obtain from the previous claim together with Exercise 10.3 and Lemma 8.7 a smooth isotopy $G\colon f(S^{l-1}) \times [0,1] \to \partial(M \cup_\varphi h^k)$ such that $G_0 = \mathrm{id}$ and such that $G_1(f(S^{l-1})) \subset \partial(M \cup_\varphi h^k) \setminus \varphi(\overline{B}^{n-k} \times S^{k-1})$. Next note that using the Isotopy Extension Theorem 8.27 we can extend G to a smooth isotopy $G\colon (\psi(\overline{B}^{n-l} \times S^{l-1})) \times [0,1] \to \partial(M \cup_\varphi h^k)$. Unfortunately a priori we do not know that $G_1(\psi(\overline{B}^{n-l} \times S^{l-1})) \subset \partial M \setminus \varphi(\overline{B}^{n-k} \times S^{k-1})$. But it follows immediately from the "Tube Lemma", see Exercise 3.6, that there exists an $\epsilon > 0$ such that the map

$$\begin{array}{rcl} \Psi\colon (\overline{B}^{n-l} \times S^{l-1}) \times [0,1] & \to & \partial(M \cup_\varphi h^k) \\ ((v,P),t) & \mapsto & G(\psi(\underbrace{((1-t)+\epsilon \cdot t) \cdot v, P}_{\in \overline{B}^{n-l} \times S^{l-1}}), t) \end{array}$$

has all the desired properties. ⊞

It follows easily from Proposition 103.1 (5) that $\widetilde{\psi} := \Psi_1$ has all the desired properties. ■

Proposition 103.3 leads us to the following definition.

Definition. Let M be a compact n-dimensional smooth manifold. Suppose we are given a handle decomposition where the handles are attached with non-decreasing indices, i.e. we

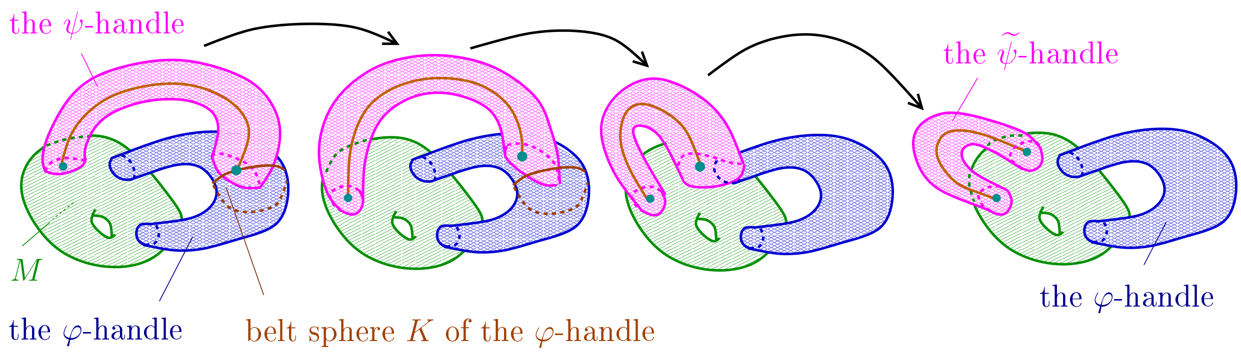


FIGURE 1458. Illustration for the proof of Proposition 103.3.

have a handle decomposition of the following form:

$$M = \varnothing \underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{\text{0-handles}} \cup \ldots \underbrace{\cup_{\varphi_{n,1}} h^n \cdots \cup_{\varphi_{n,r_n}} h^n}_{n\text{-handles}}.$$

We introduce the following two concepts:

(1) Given $i \in \{0, \ldots, n\}$ we refer to

$$M_i := \varnothing \underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{\text{0-handles}} \cup \ldots \underbrace{\cup_{\varphi_{i,1}} h^i \cdots \cup_{\varphi_{i,r_i}} h^i}_{i\text{-handles}}.$$

as the *i-skeleton of the handle decomposition*.

(2) We say that a handle decomposition is *standard* if it is of the above form and if for each $i \in \{0, \ldots, n\}$ the attaching maps $\varphi_{i,1}, \ldots, \varphi_{i,r_i}$ take values in ∂M_{i-1}.

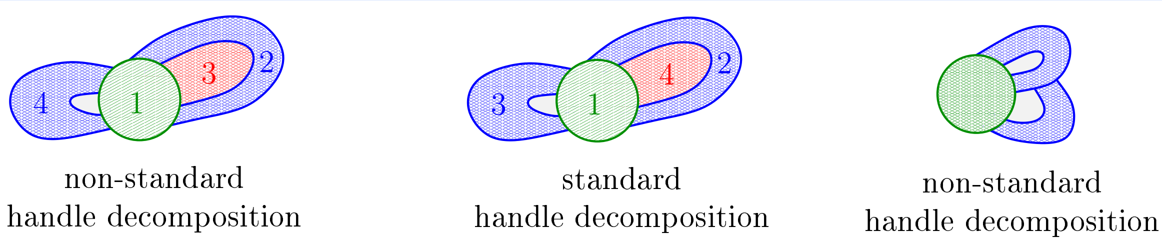


FIGURE 1459

Corollary 103.4. *Let M be a compact n-dimensional smooth manifold. If M admits a handle decomposition, then it also admits a handle decomposition with the same number of handles of each index, but which is now standard.*

PROOF. This follows immediately from iteratively applying Proposition 103.3 to reorder the handles. ■

Remark. In Chapter **??** we will see several other ways for modifying a given handle decomposition. But for the time being the above rearranging of handle attachments is good enough.

103.4. **Dual handle decomposition.** In this section we will see that we can associate to each handle decomposition a "dual" handle decomposition. More precisely, we have the following proposition.

Proposition 103.5. *Let M be a closed n-dimensional smooth manifold together with a handle decomposition*
$$M \;=\; \varnothing \cup_{\varphi_0} \underbrace{h^{k_0}}_{\text{0-th handle}} \cup \cdots \cup_{\varphi_m} \underbrace{h^{k_m}}_{m\text{-th handle}}.$$
There exists canonical handle decomposition
$$M \;=\; \varnothing \cup_{\psi_m} \underbrace{\widetilde{h}^{n-k_m}}_{\text{0-th dual handle}} \cup \cdots \cup_{\psi_0} \underbrace{\widetilde{h}^{n-k_0}}_{m\text{-th dual handle}},$$
called dual handle decomposition, *with the following properties:*

1. *For each $i \in \{0, \dots, m\}$ the following statements hold:*
   (a) *The i-th handle of the dual handle decomposition equals, as a subset of M, the $(m-i)$-th handle of the original handle decomposition.*
   (b) *The indices of the handles are reversed, in the sense that*
   $$\begin{array}{c}\textit{index of the } i\textit{-th handle of}\\ \textit{the dual handle decomposition}\end{array} \;=\; n - \begin{array}{c}\textit{index of the } (m-i)\textit{-th handle of}\\ \textit{the original handle decomposition.}\end{array}$$
   In particular the number i-handles in the dual handle decomposition equals the number of $(n-i)$-handles in the original decomposition.
   (c) *The roles of attaching spheres and belt spheres is reversed in the sense that*
   $$\begin{array}{c}\textit{belt sphere of the } i\textit{-th handle of}\\ \textit{the dual handle decomposition}\end{array} \;=\; \begin{array}{c}\textit{attaching sphere of the } (m-i)\textit{-th handle}\\ \textit{of the original handle decomposition.}\end{array}$$
   Furthermore the same statement, with the roles of attaching and belt spheres reversed, also holds.
2. *If the original handle decomposition is standard, then so is the dual handle decomposition.*
3. *The dual of the dual handle decomposition is the original handle decomposition.*

Example. In Figure 1460 we show to the left the a handle decomposition of S^2. But as we see, the decomposition of the sphere into subsets can also be read the other way and we obtain the dual handle decomposition.

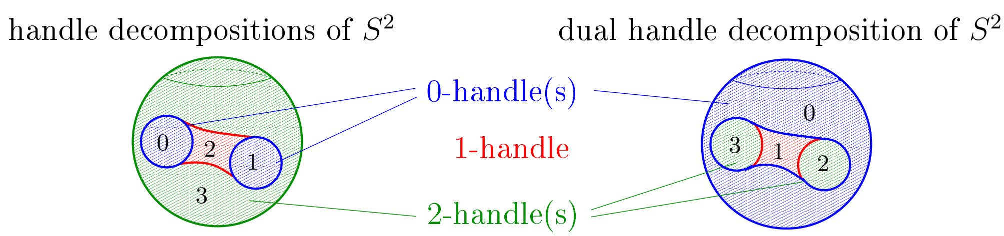


FIGURE 1460

Remark.

(1) It follows immediately from the Proposition 103.5 that every handle decomposition of a closed connected non-empty n-dimensional smooth manifold admits at least one n-handle. This gives in particular a solution to Exercise 103.3 (a).
(2) The handle decomposition that we construct in the proof of Proposition 103.5 is sometimes called the *dual handle decomposition*.

The following lemma is the key ingredient in the proof of Proposition 103.5.

Lemma 103.6. *Let X be an n-dimensional smooth manifold and let $\varphi\colon \overline{B}^{n-k}\times S^{k-1}\to\partial X$ be a smooth embedding. Let $X\cup_\varphi h^k := X\cup_{\partial X\xleftarrow{\varphi}\overline{B}^{n-k}\times S^{k-1}}(\overline{B}^{n-k}\times\overline{B}^k)$ be the result of attaching the corresponding k-handle to X. Furthermore let Y be an n-dimensional smooth manifold and let $\gamma\colon\partial Y\to\partial(X\cup_\varphi h^k)$ be a diffeomorphism. We consider the maps*

$$\begin{aligned}\psi\colon \overline{B}^k\times S^{n-k-1} &\to \partial Y\\ (x,y)&\mapsto \gamma^{-1}(y,x)\end{aligned}\quad\textit{and}\quad\begin{aligned}\delta\colon\partial X &\to \partial(Y\cup_\psi h^k)\\ P &\mapsto \begin{cases}\gamma^{-1}(P), & \textit{if } P\in\partial X\setminus\varphi(B^{n-k}\times S^{k-1}),\\ \varphi^{-1}(P), & \textit{if } P\in\varphi(B^{n-k}\times S^{k-1}).\end{cases}\end{aligned}$$

The obvious map from

$$\underbrace{\big(X\cup_{\partial X\xleftarrow{\varphi}\overline{B}^{n-k}\times S^{k-1}}(\overline{B}^{n-k}\times\overline{B}^k)\big)}_{=X\cup_\varphi h^k}\cup_{\partial(X\cup_\varphi h^k)\xleftarrow{\gamma}\partial Y} Y$$

to

$$\underbrace{\big(Y\cup_{\partial Y\xleftarrow{\psi}\overline{B}^k\times S^{n-k-1}}(\overline{B}^{n-k}\times\overline{B}^k)\big)}_{=Y\cup_\psi h^k}\cup_{\partial(Y\cup_\psi h^k)\xleftarrow{\delta}\partial X} X$$

is a diffeomorphism.

Lemma 103.6 can be summarized as follows: If a smooth manifold W is obtained from X by attaching a k-handle and then gluing on Y, then we can also obtain W from Y by attaching an $(n-k)$-handle and gluing on X.

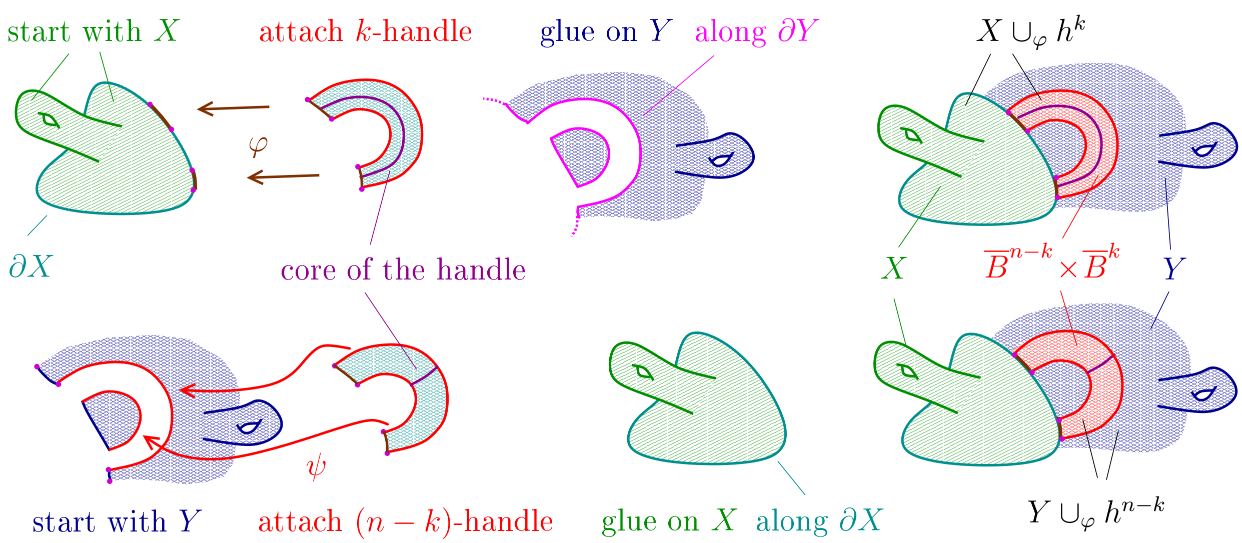


FIGURE 1461. Illustration of Lemma 103.6.

Sketch of proof of Lemma 103.6. Once one has digested the statement of the lemma, one realizes, that the lemma comes close to being a tautology. One can easily verify that the "obvious" maps in both directions are well-defined and inverses of one another. Furthermore using Lemma 3.44 (4) it is straightforward to verify that these maps are continuous, thus both maps are homeomorphisms. One still needs to show that both maps are in fact smooth. This joyless task is best left to the reader's officemate. ■

Proof of Proposition 103.5. Let $M = \varnothing \cup_{\varphi_0} h^{k_0} \cup \dots \cup_{\varphi_m} h^{k_m}$ be a handle decomposition. Given $i = 0, \dots, m$ we write $X_i := \varnothing \cup_{\varphi_0} h^{k_0} \cup \dots \cup_{\varphi_{i-1}} h^{k_{i-1}}$, $M_i := X_i \cup_{\varphi_i} h^{k_i}$ and $Y_i := M \setminus \mathring{M}_i$. Note that Y_i is an n-dimensional smooth manifold by Proposition 6.30. Also note that $Y_m = M \setminus M = \varnothing$. Since M is closed we can apply Lemma 103.6 iteratively altogether $(m+1)$ times. We see that $M = \varnothing \cup_{\psi_m} h^{n-k_m} \cup \dots \cup_{\psi_0} h^{n-k_0}$ where for $i = 0, \dots, m$ the map ψ_i is given by

$$\begin{array}{rcl} \psi_i \colon \overline{B}^{k_i} \times S^{n-k_i-1} & \to & \partial(\varnothing \cup_{\psi_m} h^{n-k_m} \cup \dots \cup_{\psi_{i+1}} h^{n-k_{i+1}}) \\ (x, y) & \mapsto & [(y, x)]. \end{array}$$

It is straightforward to verify that this dual handle decomposition has all the properties stated in the proposition. ■

We conclude this section with the following perhaps slightly surprising corollary.

Corollary 103.7. *Let M be a* closed *connected non-empty n-dimensional smooth manifold. For every handle decomposition of M we have the inequality:*

$$1 + \#(n-1)\text{-handles} - \#n\text{-handles} \;\geq\; \text{minimal size of a generating set of } \pi_1(M).$$

Proof. Let M be a closed connected non-empty n-dimensional smooth manifold that is equipped with a handle decomposition of M. We have the following (in-) equalities:

$$\overbrace{\begin{array}{c} 1 + \#(n-1)\text{-handles} \\ - \#n\text{-handles} \end{array}}^{\text{original handle decomposition}} \underset{\substack{\uparrow \\ \text{Proposition 103.5}}}{=} \overbrace{\begin{array}{c} 1 + \#1\text{-handles} \\ - \#0\text{-handles} \end{array}}^{\text{dual handle decomposition}} \underset{\substack{\uparrow \\ \text{Corollary 103.12}}}{\geq} \begin{array}{c} \text{minimal size of a} \\ \text{generating set of } \pi_1(M). \end{array}$$

■

103.5. **Handle attachments: Connectedness and orientability.** In this section we will see how handle attachments affect connectedness and orientability. In the subsequent section we will see how handle attachments affect various algebraic invariants. First let us deal with the basic concept of connectedness.

Lemma 103.8. *Let M be a compact n-dimensional smooth manifold. Suppose that we are given a thickened $(k-1)$-sphere in the boundary ∂M, i.e. suppose we are given a smooth embedding $\varphi \colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$. The following statements hold:*

$(k>1)$ *The inclusion induced map $\pi_0(M) \to \pi_0(M \cup_\varphi h^k)$ is a bijection.*

$(k=1)$ (a) *If the attaching map $\varphi \colon \overline{B}^{n-1} \times S^0 \to \partial M$ takes values in a single component of ∂M, then the inclusion induced map $\pi_0(M) \to \pi_0(M \cup_\varphi h^k)$ is a bijection.*

(b) *If the attaching map* $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$ *takes values in two components of* ∂M*, then the number of components goes down by one.*

$(k=0)$ $M \cup_\varphi h^0$ *has one extra component, namely the 0-handle.*

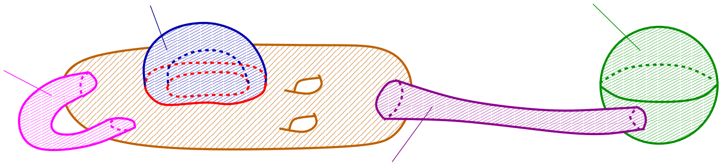


FIGURE 1462. Illustration of Lemma 103.8.

PROOF. The statements for $k > 1$ and $k = 1$ follow easily from Lemma 2.62. The statement for $k = 0$ follows from the fact that adding a 0-handle is just taking the disjoint union with the 0-handle. ■

Next we study the slightly subtle question how attaching a handle affects orientability.

Proposition 103.9. *Let* M *be a compact oriented* n*-dimensional smooth manifold. Suppose that we are given a thickened* $(k-1)$*-sphere in the boundary* ∂M*, i.e. suppose we are given a smooth embedding* $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$*. The following statements hold:*

(i) *Let* $\tau\colon S^{k-1} \to S^{k-1}$ *be a reflection in some hyperplane of* $\mathbb{R}^k$*. The corresponding map* $\varphi \circ (\mathrm{id}_{\overline{B}^{n-k}} \times \tau)\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ *is also a thickened* $(k-1)$*-sphere and the smooth manifold* $M \cup_{\varphi\circ(\mathrm{id}\times\tau)} h^k$ *is diffeomorphic to* $M \cup_\varphi h^k$*.*

(ii) *If* $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ *is orientation-reversing, then there exists an orientation-reversing smooth embedding* $\psi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ *such that* $M \cup_\psi h^k$ *is diffeomorphic to* $M \cup_\varphi h^k$*.*

Now we consider the various dimensions separately:

$(k\geq 2)$ $M \cup_\varphi h^k$ *admits an orientation which agrees with the given orientation on* $M \setminus \partial M$*.*

$(k=1)$ (a) *If*[1445] $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$ *is orientation-preserving or orientation-reversing, then* $M \cup_\varphi h^k$ *admits an orientation that agrees with the given orientation on* $M \setminus \partial M$*.*

(b) *We suppose that the image of* φ *lies in a single component of* M*. If* (a) *is not satisfied, then* $M \cup_\varphi h^k$ *is non-orientable.*

$(k=0)$ $M \cup_\varphi h^0 = M \sqcup \overline{B}^n$ *is again orientable.*

PROOF.

(i) The reflection $\tau\colon S^{k-1} \to S^{k-1}$ extends in an obvious way to a diffeomorphism $\tau\colon \overline{B}^k \to \overline{B}^k$. The diffeomorphism from $M \cup_{\varphi\circ(\mathrm{id}\times\tau)} h^k$ to $M \cup_\varphi h^k$ is given by the identity on M and by $\mathrm{id}_{\overline{B}^{n-k}} \times \tau$ on $\overline{B}^{n-k} \times \overline{B}^k$.

[1445] Recall that according to the orientation convention on page 1719 we have an orientation-preserving diffeomorphism $\overline{B}^{n-1} \times S^0 \to -\overline{B}^{n-1} \sqcup \overline{B}^{n-1}$.

(ii) This statement follows immediately from (1).

After these preparations we turn to the proofs of the main statements of the proposition.

($k\geq 2$) This statement follows immediately from (ii) together with Proposition 8.15 (6).
($k=1$) (a) This statement follows immediately from (ii) together with Proposition 8.15 (6).
(b) This statement is basically a special case of Exercise 8.25.
($k=0$) This case is trivial. ■

Proposition 103.9 (3) motivates the following definition.

Definition. Let M be a compact oriented n-dimensional smooth manifold. We say a thickened 0-sphere $\varphi\colon \overline{B}^n \times S^0 \to \partial M$ is *orientable* if φ is orientation-preserving or if it is orientation-reversing. Otherwise we say that the thickened 0-sphere is *non-orientable*.

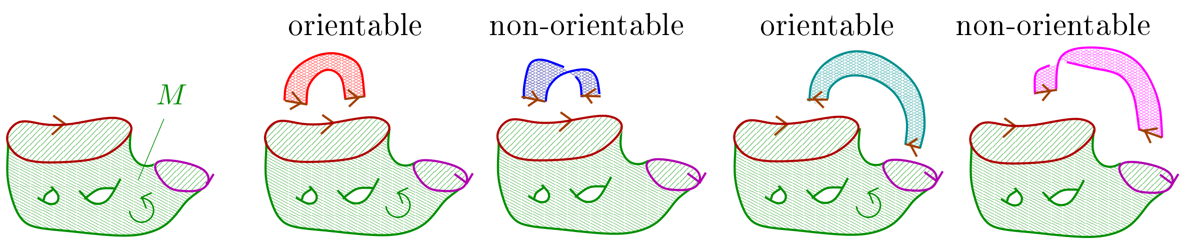


FIGURE 1463

The following lemma basically shows that there are not that many different possibilities for attaching 1-handles.

Lemma 103.10. *Let M be a compact oriented n-dimensional smooth manifold, let F be a component of ∂M and let $\varphi, \psi\colon \overline{B}^{n-1} \times S^0 \to F$ be two thickened 0-spheres. If φ and ψ are both orientable or if they are both non-orientable, then $M \cup_\varphi h^1$ and $M \cup_\psi h^1$ are diffeomorphic.*

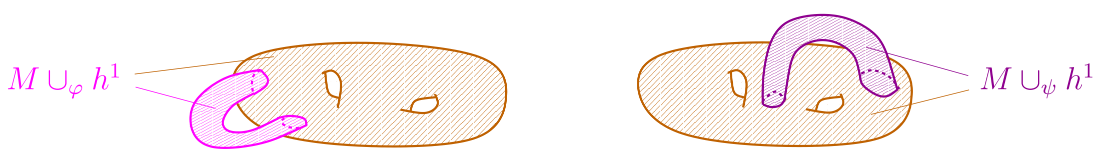


FIGURE 1464. Illustration of Lemma 103.10.

PROOF. Let $\varphi, \psi\colon \overline{B}^{n-1} \times S^0 \to F$ be two thickened 0-spheres. First assume that $n \geq 3$. As in Proposition 103.9 (1) let $\tau\colon S^0 \to S^0$ be the map that swaps the two points ± 1. By Proposition 103.9 we can, if convenient, replace φ by $\varphi \circ (\mathrm{id} \times \tau)$ and we can replace ψ by $\psi \circ (\mathrm{id} \times \tau)$. It follows from this observation that we can arrange that the maps $\varphi\colon \overline{B}^{n-1} \times \{1\} \to \partial M$ and $\psi\colon \overline{B}^{n-1} \times \{1\} \to \partial M$ are both orientation-preserving. It follows from our orientation hypothesis on φ and ψ that $\varphi\colon \overline{B}^{n-1} \times \{-1\} \to \partial M$ and $\psi\colon \overline{B}^{n-1} \times \{-1\} \to \partial M$ are both orientation-preserving or they are both orientation-reversing.

Since $n = \dim(M) \geq 3$ we have $\dim(\partial M) \geq 2$. Thus we can apply Theorem 8.36 which implies that the two maps φ and ψ are smoothly isotopic. Thus it follows from Proposition 103.1 that $M \cup_\varphi h^1$ and $M \cup_\psi h^1$ are diffeomorphic.

It remains to deal with the cases $n = 1$ and $n = 2$. The case $n = 1$ is basically trivial. Finally we will deal with the case $n = 2$ in Exercise 103.1. ■

103.6. **Handle attachments: Algebraic invariants.** In this section we will discuss how a handle attachment alters some of our favorite algebraic invariants of topological spaces, namely the fundamental group and the homology groups. The statements below are basically the same as for cell attachments that we gave in Propositions 37.11 and Lemma 46.19. First we have the following analogue of Proposition 37.11.

Proposition 103.11. *Let M be a compact n-dimensional smooth manifold, let $k \in \mathbb{N}$ and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. We pick $(x_0, y_0) \in S^{n-k-1} \times S^{k-1}$ and we set $P := \varphi(x_0, y_0) \in \partial M$. The following statements hold:*

($k>2$) *The inclusion induced map $\pi_1(M, P) \to \pi_1(M \cup_\varphi h^k, P)$ is an isomorphism.*

($k=2$) *Let $\beta\colon S^1 \to M$ be the loop that is given by $z \mapsto \varphi(x_0, z)$. The inclusion induced map $\pi_1(M, P) \to \pi_1(M \cup_\varphi h^2, P)$ descends to an isomorphism*

$$\pi_1(M, P)/\langle\langle[\beta]\rangle\rangle \xrightarrow{\cong} \pi_1(M \cup_\varphi h^2, P).$$

($k=1$) *We set $P' := \varphi(x_0, -y_0) \in \partial M$.*

(a) *Suppose that P and P' lie in the same component of M. Let γ be a loop in $M \cup_\varphi h^1$ that is given by the concatenation of the obvious path in the 1-handle from P to P' and some path in M from P' to P. With this notation the map*

$$\pi_1(M, P) * \langle t\rangle \to \pi_1(M \cup_\varphi h^1, P)$$

that is given by the inclusion induced map $\pi_1(M, P) \to \pi_1(M \cup_\varphi h^1, P)$ and given by $t \mapsto [\gamma]$ is an isomorphism.

(b) *Suppose that P and P' lie in two different components of M. Then we get an isomorphism*

$$\pi_1(M, P) * \pi_1(M, P') \to \pi_1(M \cup_\varphi h^1, P).$$

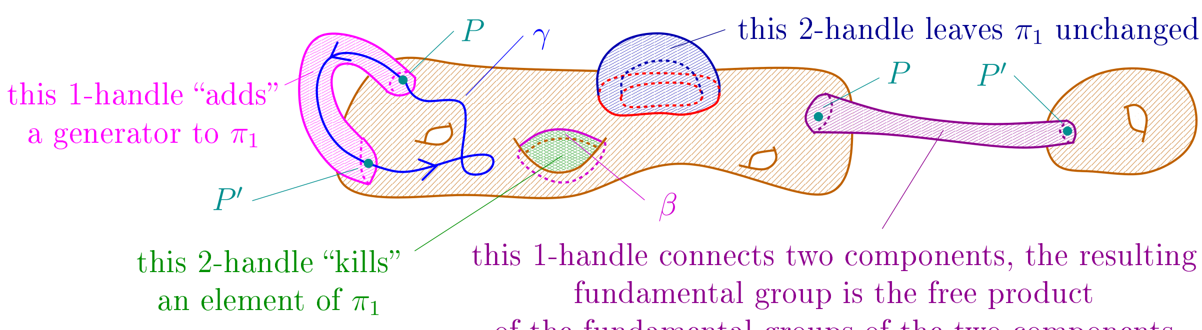


FIGURE 1465. Illustration of Proposition 103.11.

PROOF. The statements follow fairly easily from a slight generalization of the Seifert–van Kampen Theorem 22.2 and the HNN-Seifert–van Kampen Theorem 26.3 (b). Overall the proof is very similar to the proof of Proposition 37.11 and we leave it to the reader to make the obvious alterations to obtain a proof for the present proposition. ■

Corollary 103.12. *Let M be a compact connected non-empty smooth manifold. For every handle decomposition of M we have the inequality:*

$$1 + \#1\text{-handles} - \#0\text{-handles} \;\geq\; \text{minimal size of a generating set of } \pi_1(M).$$

Proof. Let M be a compact connected non-empty smooth manifold that is equipped with a handle decomposition. We denote by r the number of 0-handles and we denote by s the number of 1-handles. First note, that by Corollary 103.4 we can assume that the handle decomposition is standard. We pick a base point P in one of the 0-handles. By the definition of a generating set, see page 592, it suffices to prove the following claim.

Claim. There exists an epimorphism from a group of rank $1+s-r$ onto $\pi_1(M,P)$.

Given $k \in \mathbb{N}_0$ we denote by M^k the k-skeleton of M. It follows fairly easily from our hypothesis that M is connected, together with Lemma 103.8 and Proposition 103.11, that $\pi_1(M^1,P)$ is a free group on $1+s-r$ generators. Next note that it follows from Proposition 103.11 that the inclusion induced map $\pi_1(M^1,P) \to \pi_1(M,P)$ is an epimorphism. Thus we have shown, as desired, that there exists an epimorphism from a group of rank $1+s-r$ onto $\pi_1(M,P)$. ■

Next we study the effect of a handle attachment to homology groups. The following lemma is an analogue of Lemma 46.19.

Lemma 103.13. *Let M be a compact n-dimensional smooth manifold, let $k \in \mathbb{N}_0$, let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere and let $x \in \overline{B}^{n-k}$. We denote by $i\colon M \to M \cup_\varphi h^k$ the inclusion map. There exists a natural exact sequence of the form*[1446]

$$0 \to \mathrm{H}_k(M) \xrightarrow{i_*} \mathrm{H}_k(M \cup_\varphi h^k) \to \underbrace{\widetilde{\mathrm{H}}_{k-1}(S^{k-1})}_{\cong \mathbb{Z}} \xrightarrow{\substack{\text{induced by the map } y \mapsto \varphi(x,y)}} \mathrm{H}_{k-1}(M) \xrightarrow{i_*} \mathrm{H}_{k-1}(M \cup_\varphi h^k) \to 0.$$

Furthermore, for any $l \neq k-1, n$ the inclusion induced map

$$i_*\colon \mathrm{H}_l(M) \xrightarrow{\cong} \mathrm{H}_l(M \cup_\varphi h^k)$$

is an isomorphism.

Remark. By considering the 0-th homology groups one can deduced Lemma 103.8 from Lemma 103.13.

[1446] Note this statement also holds for $k=0$, we just need to remember that by Lemma 43.1 we have $\widetilde{\mathrm{H}}_{-1}(\emptyset) = \mathbb{Z}$.

Proof. We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_j(M\cup_\varphi h^k) \arrow[r, "\partial"] & \widetilde{\mathrm{H}}_{j-1}(\varphi(\overline{B}^{n-k}\times S^{k-1})) \arrow[r] & \mathrm{H}_{j-1}(M)\oplus\overbrace{\widetilde{\mathrm{H}}_{j-1}(\varphi(\overline{B}^{n-k}\times\overline{B}^{k}))}^{=0} \arrow[r] & \mathrm{H}_{j-1}(M\cup_\varphi h^k) \arrow[r] & {} \\
& \widetilde{\mathrm{H}}_{j-1}(S^{k-1}) \arrow[u, "\cong"] \arrow[ur] & & &
\end{tikzcd}$$

Here the upper sequence is the long exact sequence coming from a slight generalization of the Mayer-Vietoris Theorem 46.10 for Smooth Manifolds. The maps emanating from below are the obvious maps. The statements of the lemma follow from staring at this diagram for a few short minutes. ■

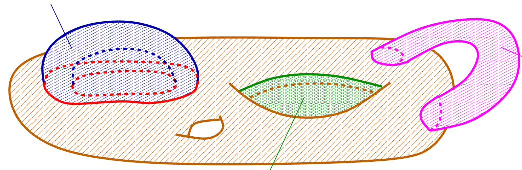


Figure 1466. Illustration of Lemma 103.13.

103.7. **Handle attachments: Invariants of the boundary** (∗)**.** In this section, given a handle attachment $M\cup_\varphi h^k$ we will study the interaction between algebraic invariants of $M\cup_\varphi h^k$ and its boundary $\partial(M\cup_\varphi h^k)$. Again we first consider fundamental groups.

Lemma 103.14. *Let M be a connected n-dimensional smooth manifold such that ∂M is also connected. Let $\varphi\colon \overline{B}^{n-k}\times S^{k-1}\to\partial M$ be a thickened $(k-1)$-sphere. The following statements hold:*

(1) *If $1\le k<n-1$, then the boundary $\partial(M\cup_\varphi h^k)$ is connected and non-empty.*
(2) (a) *If $1\le k\le n-3$ and if $\pi_1(\partial M)\to\pi_1(M)$ is an isomorphism, then the induced map $\pi_1(\partial(M\cup_\varphi h^k))\to\pi_1(M\cup_\varphi h^k)$ is also an isomorphism.*
   (b) *If $1\le k\le n-2$ and if $\pi_1(\partial M)\to\pi_1(M)$ is an epimorphism, then the induced map $\pi_1(\partial(M\cup_\varphi h^k))\to\pi_1(M\cup_\varphi h^k)$ is also an epimorphism.*

Proof. First we prove Statement (1). By Proposition 103.1 we have

$$\partial(M\cup_\varphi h^k) \;=\; \underbrace{(\partial M\setminus\varphi(B^{n-k}\times S^{k-1}))}_{\substack{\text{connected by Lemma 2.63,}\\ \text{since } \partial M \text{ is connected}}}\cup_{\varphi|\underbrace{S^{n-k-1}\times S^{k-1}}_{\neq\varnothing\text{ since }1\le k<n\text{-}1}}\underbrace{(S^{n-k-1}\times\overline{B}^k)}_{\substack{\text{connected since}\\ k<n-1}}.$$

It follows from Lemma 2.63 that $\partial(M\cup_\varphi h^k)$ is indeed connected. Since $n-k-1\ge 0$ we see that the boundary is furthermore non-empty.

We turn to the proof of Statement (2). We pick and immediately ignore a base point in $\varphi(S^{n-k-1}\times S^{k-1}$. We consider the following diagram:

$$\begin{array}{ccc}
\pi_1\big(\overbrace{(\partial M\setminus\varphi(B^{n-k}\times S^{k-1}))\underset{S^{n-k-1}\times S^{k-1}}{\cup}(S^{n-k-1}\times\overline{B}^k)}^{=\partial(M\cup_\varphi h^k)}\big) & \longrightarrow & \pi_1\big(\overbrace{M\underset{\overline{B}^{n-k}\times S^{k-1}}{\cup}(\overline{B}^{n-k}\times\overline{B}^k)}^{=M\cup_\varphi h^k}\big) \\
& \searrow & \uparrow \\
& & \pi_1\big(\partial M\underset{S^{n-k-1}\times S^{k-1}}{\cup}(S^{n-k-1}\times\overline{B}^k)\big).
\end{array}$$

All the maps are induced by inclusions, hence the diagram commutes. First we consider the case that $1\leq k\leq n-3$. In this setting we need to show that the top map is an isomorphism. This statement is a consequence of the following two observations:

(i) Note that it follows from Lemma 22.11 (1) and our hypothesis that $k\leq n-3$ that the map $\pi_1(\partial M\setminus\varphi(B^{n-k}\times S^{k-1}))\to\pi_1(\partial M)$ is an isomorphism. Next note that it follows from this observation together with the Seifert–van Kampen Theorem 22.2 and Lemma 21.22 (1) that the diagonal map is an isomorphism.

(ii) By hypothesis we know that $\pi_1(\partial M)\to\pi_1(M)$ is an isomorphism. It follows from this observation together with the Seifert–van Kampen Theorem 22.2 and Lemma 21.22 (1) that the vertical map is an isomorphism.

Finally we consider the case that $k\leq n-2$. In this setting we perform the same argument as above, except that need to replace Lemma 22.11 (1) by Lemma 22.11 (2) and that we need to replace Lemma 21.22 (1) by Lemma 21.22 (2). ■

Corollary 103.15. *Let $n\in\mathbb{N}$ with $n\geq 4$. Let M be a compact connected non-empty n-dimensional smooth manifold that is equipped with a handle decomposition.*

(1) *If all handles have index $\leq n-3$, then the boundary ∂M is connected and the inclusion induced map $\pi_1(\partial M)\to\pi_1(M)$ is an isomorphism.*

(2) *If all handles have index $\leq n-2$, then the boundary ∂M is connected and the inclusion induced map $\pi_1(\partial M)\to\pi_1(M)$ is an epimorphism.*

Proof. Let M be a compact connected non-empty n-dimensional smooth manifold that is equipped with a handle decomposition. If M has a single 0-handle, then the statement follows immediately from Lemma 103.14.

Now suppose that M has in fact more than one 0-handle. In Corollary **??** we will see that we can arrange that the handle decomposition has a single 0-handle. Alternatively we can argue directly as follows. Let $A_1,\dots,A_k$ be the 0-handles. It follows easily from the hypothesis that M is connected that there exist 1-handles $B_1,\dots,B_{k-1}$ such that $W:=(A_1\cup\dots\cup A_k)\cup(B_1\cup\dots\cup B_{k-1})$ is connected. It is straightforward to verify that $\pi_1(W)$ is trivial and that ∂W is connected. We obtain M from W by attaching handles of index ≥ 1. Thus the desired statement again follows from Lemma 103.14. ■

We continue with the "homological analogue" of Lemma 103.14.

Lemma 103.16. *Let M be an n-dimensional smooth manifold. Suppose we are given a thickened $(k-1)$-sphere $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$. Let $j \in \mathbb{N}_0$. The following statements hold:*

(1) *If $j \leq n-k-2$ and if $\mathrm{H}_i(\partial M) \to \mathrm{H}_i(M)$ is an isomorphism for $i = 0, \dots, j$, then $\mathrm{H}_j(\partial(M \cup_\varphi h^k)) \to \mathrm{H}_j(M \cup_\varphi h^k)$ is also an isomorphism for $i = 0, \dots, j$.*

(2) *If $j \leq n-k-1$ and if $\mathrm{H}_i(\partial M) \to \mathrm{H}_i(M)$ is an isomorphism for $i = 0, \dots, j-1$ and if it is an an epimorphism for $i = j$, then the analogous statements hold for the maps $\mathrm{H}_j(\partial(M \cup_\varphi h^k)) \to \mathrm{H}_j(M \cup_\varphi h^k)$.*

PROOF. The lemma follows from applying the Mayer-Vietoris Theorem 46.10 to the decomposition $\partial M = (\partial M \setminus \varphi(B^{n-k} \times S^{k-1})) \cup (\overline{B}^{n-k} \times S^{k-1})$, to $M \cup_\varphi h^k$ and finally to $\partial(M \cup_\varphi h^k) = (\partial M \setminus \varphi(B^{n-k} \times S^{k-1})) \cup (S^{n-k-1} \times \overline{B}^k)$. Since we will not make use of the lemma we leave the pleasure of filling in the details to the reader. ■

The following is an immediate corollary to Lemma 103.16.

Corollary 103.17. *Let M be a compact non-empty n-dimensional smooth manifold that is equipped with a handle decomposition and let $j \in \mathbb{N}_0$.*

(1) *If all handles have index $\leq n-j-2$, then for $i = 0, \dots, j$ the inclusion induced map $\mathrm{H}_j(\partial M) \to \mathrm{H}_j(M)$ is an isomorphism.*

(2) *If all handles have index $\leq n-j-1$, then for $i = 0, \dots, j-1$ the inclusion induced map $\mathrm{H}_i(\partial M) \to \mathrm{H}_i(M)$ is an isomorphism and the inclusion induced map $\mathrm{H}_j(\partial M) \to \mathrm{H}_j(M)$ is an epimorphism.*

103.8. **Handle decompositions and CW-structures** (∗)**.** We conclude this chapter with a proposition that shows that a handle decomposition of a smooth manifold M gives rise to a CW-complex that is *homotopy equivalent* to the smooth manifold M. We will not really make use of this result, but it is good to know it exists, and in principle it can be used to deduce some results on handle decompositions from our long experience with CW-structures.

Proposition 103.18. *Every compact smooth manifold M that is equipped with a handle decomposition is homotopy equivalent to a* CW*-complex X such that for each $k \in \mathbb{N}_0$ the number of k-handles of M equals the number of k-cells of X.*

FIGURE 1467. Illustration of Proposition 103.18.

Remark. We will prove Proposition 103.18 using the Homotopy Pushout Theorem 39.18, which comes with a rather intimidating proof. A slightly more direct proof for Proposition 103.18 is given in [**Miln63a**, Theorem 3.4].

As we mentioned in the remark, the key to proving Proposition 103.18 is the following theorem whose easy formulation belies the difficulty of the proof.

Theorem 39.18. (Homotopy Pushout Theorem) *Suppose we are given the following commutative diagram of maps between topological spaces:*

$$\begin{tikzcd} Y \arrow[d, "\varphi_Y"'] & A \arrow[l, "f"'] \arrow[r, hook, "i"] \arrow[d, "\varphi_A"] & X \arrow[d, "\varphi_X"] \\ Y' & A' \arrow[l, "f'"'] \arrow[r, hook, "i'"] & X'. \end{tikzcd}$$

If the vertical maps are homotopy equivalences and if the maps i and i' to the right are closed cofibrations[1447]*, then the induced map*

$$Y \cup_A X \;\to\; Y' \cup_{A'} X'$$

between the pushouts is a homotopy equivalence.

Proof of Proposition 103.18. Let M be a compact smooth manifold with a given handle decomposition. By Corollary 103.4 we can arrange, without changing the number of handles of any index, that the handle decomposition is standard.

Given $k \in \mathbb{N}_0$ we denote, as on page 2481, by M^k the k-skeleton of the handle decomposition. In other words, M^k is is the union of all handles of dimension $\leq k$. Furthermore, given any CW-complex X we denote by X^k its k-skeleton, i.e. X^k is the union of cells of dimension $\leq k$. We will prove the following slight refinement of the original statement of Proposition 103.18.

Statement. There exists a CW-complex X and a homotopy equivalence $g\colon M \to X$ with the following two properties:

(1) for each $k \in \mathbb{N}_0$ we have $g(M^k) \subset X^k$,
(2) for each $k \in \mathbb{N}_0$ the number of k-cells of X equals the number of k-handles of M.

By an elementary induction argument it suffices to prove the following claim.

Claim. Let N be a compact n-dimensional smooth manifold that is equipped with a standard handle decomposition. We suppose that there exists a homotopy equivalence $f\colon N \to X$ to some CW-complex X such that for every $k \in \mathbb{N}_0$ we have $f(N^k) \subset X^k$. Let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial N$ be a smooth embedding. There exists a cellular map[1448] $\psi\colon S^{k-1} \to X$ and a homotopy equivalence[1449]

$$g\colon \underbrace{\big(N \sqcup \big(\overline{B}^{n-k} \times \overline{B}^{k}\big)\big)/\sim}_{=N\cup_\varphi h^k} \;\to\; \underbrace{\big(X \sqcup \overline{B}^{k}\big)/\sim}_{\psi(x)\sim x \text{ for all } x\in S^{k-1}}.$$

[1447] We refer to page 1023 for the definition of a closed cofibration. To get the hang of the argument it is not that important though to recall the definition of a closed cofibration.

[1448] We equip S^{k-1} with the canonical CW-structure that we introduced on page 935.

[1449] Note that it follows from Lemma 36.32 and the fact that ψ is cellular that $(X \sqcup \overline{B}^k)/\sim$ is a CW-complex with one extra k-cell.

We consider the map $\eta\colon S^{k-1} \to X$ that is given by $\eta(x) := (g \circ \varphi)(0,x)$. By the Cellular Approximation Theorem 38.13 there exists a homotopy $F\colon S^{k-1} \times [0,1] \to X$ from $F_0 = \eta\colon S^{k-1} \to X$ to a cellular map $F_1 = \psi\colon S^{k-1} \to X$. We obtain the following commutative diagram:

$$\begin{tikzcd}
N & \partial N \arrow[l, hook'] & \overline{B}^{n-k} \times S^{k-1} \arrow[l, "\varphi"'] \arrow[r, hook] & \overline{B}^{n-k} \times \overline{B}^{k} \\
N \arrow[u, "\mathrm{id}"] \arrow[d, "f"'] & & S^{k-1} \arrow[ll, "x \mapsto \varphi(0,x)"'] \arrow[r, hook] \arrow[u, "x\mapsto(0,x)"'] \arrow[d] & \overline{B}^{k} \arrow[u, "x\mapsto(0,x)"'] \arrow[d] \\
X \arrow[d, "\mathrm{id}"'] & & S^{k-1} \arrow[ll, "\eta"'] \arrow[r, hook] \arrow[d, "x\mapsto(x,0)"] & \overline{B}^{k} \arrow[d, "x\mapsto(x,0)"] \\
X & & S^{k-1} \times [0,1] \arrow[ll, "F"'] \arrow[r, hook] & \overline{B}^{k} \times [0,1] \\
X \arrow[u, "\mathrm{id}"] & & S^{k-1} \arrow[ll, "\psi"'] \arrow[r, hook] \arrow[u, "x\mapsto(x,1)"'] & \overline{B}^{k}. \arrow[u, "x\mapsto(x,1)"']
\end{tikzcd}$$

It is clear that all of the vertical maps are homotopy equivalences. Furthermore note that it follows immediately from Proposition 39.4 and 39.12 that all of the right horizontal maps are closed cofibrations. Thus it follows from the Homotopy Pushout Theorem 39.18, applied altogether four times, that the induced maps between the various pushouts have homotopy inverses. Combining these four homotopy equivalences, respectively their homotopy inverses, we obtain the promised homotopy equivalence

$$\underbrace{N \cup_{\overline{B}^{n-k} \times S^{k-1}} (\overline{B}^{n-k} \times \overline{B}^{k})}_{=(N \sqcup (\overline{B}^{n-k} \times \overline{B}^{k}))/\sim} \;\to\; \underbrace{X \cup_{S^{k-1}} \overline{B}^{k}}_{=(X \sqcup \overline{B}^{k})/\sim}.$$

■

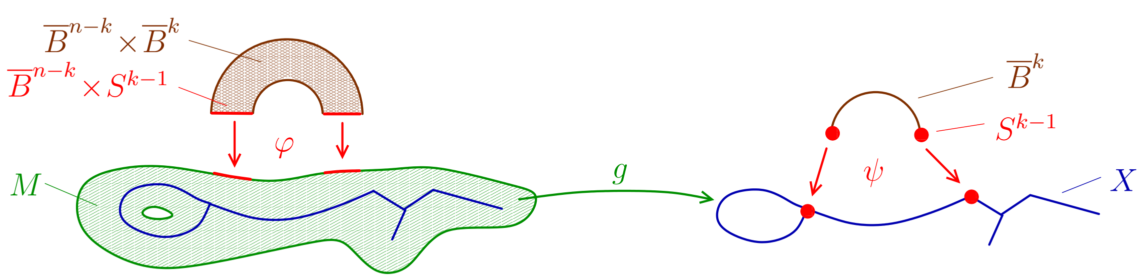


FIGURE 1468. Illustration for the proof of Proposition 103.18.

Exercises for Chapter 103.

Exercise 103.1. Let M be a compact oriented 2-dimensional smooth manifold and let F be a component of ∂M. Show that any two orientation-preserving 1-handle attachments to F lead to diffeomorphic smooth manifolds.

Exercise 103.2. Let $m, k \in \mathbb{N}$. We consider the thickened $(k-1)$-sphere

$$\begin{array}{rcl} \varphi\colon \overline{B}^m \times S^{k-1} & \to & S^{m+k-1} = \partial(\overline{B}^{m+k}) \\ (x,y) & \mapsto & \dfrac{(\frac{1}{2}\cdot x, y)}{\|(\frac{1}{2}\cdot x, y)\|}. \end{array}$$

(a) Show that $\overline{B}^{m+k} \cup_\varphi h^k$ is homeomorphic to $\overline{B}^m \times S^k$.
(b) Show that $\overline{B}^{m+k} \cup_\varphi h^k$ is diffeomorphic to $\overline{B}^m \times S^k$.[1450]

Exercise 103.3. Let M be a closed connected non-empty n-dimensional smooth manifold.
(a) Show that every handle decomposition of M admits at least one n-handle.
(b) Suppose we are given a handle decomposition of M with a unique n-handle. Show that this n-handle has non-empty intersection with every other handle.

Exercise 103.4. On page 2477 we saw that the surface of genus 2 admits a handle decomposition with four 1-handles. Does it also admit a handle decomposition with less than four 1-handles?

Exercise 103.5. Let M be a compact oriented n-dimensional smooth manifold and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. Show that if $k \geq 2$ and $n \geq k+3$, then there exists an epimorphism $\pi_1(\partial M) \to \pi_1(\partial(M \cup_\varphi h^k))$.

Exercise 103.6. Let $K \subset S^3$ be a knot. We pick a tubular neighborhood $\overline{B}^2 \times K$ and as on page 1728 we consider the knot exterior $X_K := S^3 \setminus (B^2 \times K)$. Can we add handles to X_K to obtain a smooth manifold that is diffeomorphic to S^3?

Exercise 103.7. We consider the 3-dimensional smooth manifold M shown in Figure 1469. Furthermore we consider the thickened 1-sphere $\varphi\colon \overline{B}^1 \times S^1 \to \partial M$ shown in Figure 1469. Can you recognize the diffeomorphism type of $M \cup_\varphi h^2$? Can you prove that it is what you think it is?

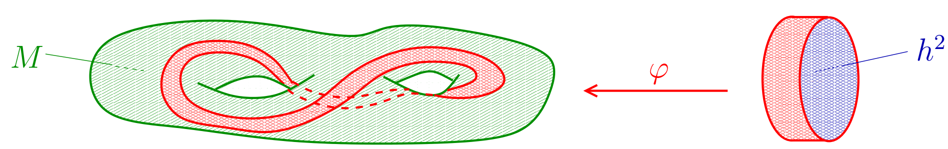


FIGURE 1469. Illustration for Exercise 103.7.

Exercise 103.8. Let M be a connected n-dimensional smooth manifold. Furthermore let $\varphi\colon \overline{B}^{n-1} \times S^0 \to \partial M$ be a thickened 0-sphere in ∂M. Let $\tau\colon \overline{B}^{n-1} \to \overline{B}^{n-1}$ be the reflection in a hyperplane. We consider the thickened 0-sphere

$$\begin{array}{rcl} \overline{B}^n \times S^0 & \to & \partial M \\ (P,\epsilon) & \mapsto & \begin{cases} \varphi(\tau(P),-1), & \text{if } \epsilon = -1, \\ \varphi(P,1), & \text{if } \epsilon = 1. \end{cases} \end{array}$$

Is it possible that $M \cup_\varphi h^1$ and $M \cup_\psi h^1$ are diffeomorphic?

[1450]This exercise is meant only for the heroically inclined reader.

Exercise 103.9. Let M be a compact n-dimensional smooth manifold such that ∂M is connected. Let $k \in \mathbb{N}_0$ and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. Show that if $\in \{1, \ldots, n-1\}$, then $\partial(M \cup_\varphi h^k)$ is also connected.

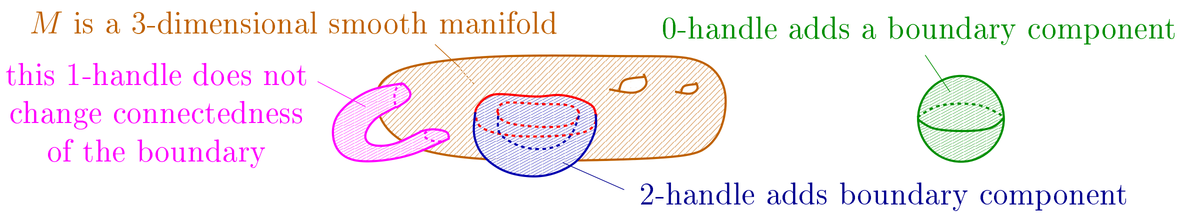


FIGURE 1470. Illustration of Exercise 103.9.

Exercise 103.10. Let M be a compact n-dimensional smooth manifold.

(a) Let $k \in \mathbb{N}_0$ and let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to \partial M$ be a thickened $(k-1)$-sphere. Prove that
$$\chi(M \cup_\varphi h^k) \;=\; \chi(M) + (-1)^k.$$

(b) Show that given any handle decomposition for M the following equality holds:
$$\sum_{k \in \mathbb{N}_0} (-1)^k \cdot \text{number of } k\text{-handles} \;=\; \chi(M).$$

Exercise 103.11. Let $k \in \mathbb{N}$ and let M be a compact orientable $2k$-dimensional smooth manifold which admits a handle decomposition such that all handles have index $\leq k$. Is the inclusion induced map $\mathrm{H}_k(\partial M) \to \mathrm{H}_k(M)$ necessarily an epimorphism?

Exercise 103.12. Let M be a a compact smooth that is equipped with a standard handle decomposition. Given $k \in \mathbb{N}_0$ we denote by M^k the k-skeleton of the handle decomposition, i.e. M^k is the union of all handles up to dimension $\leq k$. Let P be a point in the 0-skeleton and let $i \in \mathbb{N}$.

(a) Show that for any $k > i$ the inclusion induced map $\pi_i(M^k, P) \to \pi_i(M, P)$ is an isomorphism.

(b) Show that the inclusion induced map $\pi_k(M^k, P) \to \pi_k(M, P)$ is an epimorphism.

Hint. Use the Whitney Approximation Theorem 9.3 and Transversality Theorem 9.10 applied to a suitable choice of proper submanifolds.

Remark. This exercise can be viewed as a handle analogue of the corresponding statement for CW-complexes that we proved in Proposition 40.9.

104. Morse theory I: Closed smooth manifolds

In the last chapter we introduced handle decompositions and showed explicitly that some of our favorite compact smooth manifolds admit handle decompositions. In this chapter we will show, using Morse theory, that every compact smooth manifold admits a handle decomposition.

104.1. Non-degenerate critical points.

The following definition is just a special case of the definition on page 308.

Definition. Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. We say $P \in M$ is a *regular point* if $\mathrm{D}\, f_P\colon\, \mathrm{T}_P\, M \to \mathbb{R}$ is non-zero, otherwise we say P is a *critical point*.

In our previous discussions we were trying hard to avoid critical points. In the following we will see that the study of critical points can actually be quite fruitful. In particular, as we will see shortly, critical points come in different flavors which have a great influence on the topology of the smooth manifold.

Lemma 104.1. *Let M be an n-dimensional smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Let $P \in M \setminus \partial M$ be a critical point. We pick a chart $\Phi\colon U \to V$ for P. We write $Q := \Phi(P)$. The congruence*[1451] *class of the Hessian matrix*[1452]

$$\mathrm{Hess}(f \circ \Phi^{-1})_Q \;:=\; \left(\frac{\partial^2}{\partial x_i \partial x_j}\Big|_Q f \circ \Phi^{-1}\right)_{i,j=1,\dots,n} \;\in\; \mathrm{M}(n \times n, \mathbb{R})$$

is independent of the choice of the chart.

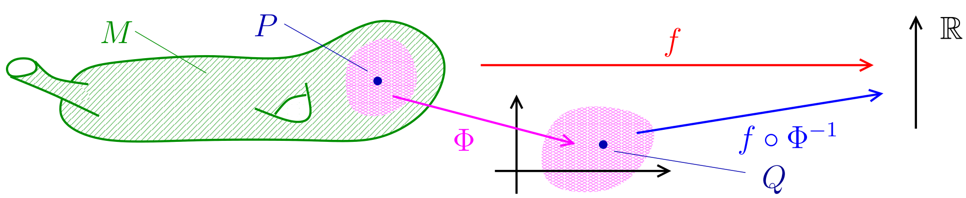


Figure 1471. Illustration for Lemma 104.1.

Proof. There are two approaches to proving the lemma:

(1) Let $\Psi\colon X \to Y$ be some other chart around P. We set $R := \Psi(Q)$ and we set $Z := \mathrm{D}(\Psi \circ \Phi^{-1})_P$. A slightly heroic calculation, see [**Gaul82**, p. 51], shows that $Z^T \cdot \mathrm{Hess}(f \circ \Psi^{-1})_R \cdot Z = \mathrm{Hess}(f \circ \Phi^{-1})_P$.
(2) The statement can also be proved more elegantly by defining intrinsically a symmetric form $\mathrm{H}_P\, f\colon\, \mathrm{T}_P\, M \times \mathrm{T}_P\, M \to \mathbb{R}$ which now happens to be represented by the matrix $\mathrm{Hess}(f \circ \Phi^{-1})_P$. At this point the lemma is a consequence of the purely algebraic Lemma 101.1. This approach is taken in [**Miln63a**, p. 4] and [**Nic11**, p. 6]. ■

[1451] Here recall that two real $n \times n$-matrices G and H are called *congruent* if there exists a matrix $Z \in \mathrm{GL}(n, \mathbb{R})$ such that $Z^T G Z = H$.

[1452] Implicitly we use throughout that the Hessian matrix is symmetric. The fact that given a smooth function the partial derivatives commute is variously known (depending on the geographic location of the author) as the Schwarz Theorem, the Young Theorem or the Clairaut Theorem.

It follows from Lemma 104.1, common sense and the combination of Lemma 101.1 (1c) and Proposition 101.8 that the following definition is independent of any choices.

Definition. Let M be an n-dimensional smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Let $P \in M \setminus \partial M$ be a critical point. We pick a chart $\Phi\colon U \to V$ for P and we write $Q := \Phi(P)$.

(1) We say the critical point P is *nondegenerate* if $\det(\mathrm{Hess}(f \circ \Phi^{-1})_Q) \neq 0$. Otherwise we call the critical point *degenerate*.
(2) We refer to[1453]

$$\mathrm{index}(f, P) \;:=\; \text{number of negative eigenvalues of } \mathrm{Hess}(f \circ \Phi^{-1})_Q$$

as the *index of the critical point P*.

Examples.

(1) The "mother of all examples" are the functions

$$\begin{array}{rcl} \mathbb{R}^n & \to & \mathbb{R} \\ (x_1, \ldots, x_n) & \mapsto & c + \sum\limits_{i=1}^{n} a_i x_i^2 \end{array}$$

with $c, a_1, \ldots, a_n \in \mathbb{R}$. Evidently the origin 0 is a critical point. With respect to the identity chart the Hessian $\mathrm{Hess}(f)_0$ is the diagonal matrix with diagonal entries $a_1, \ldots, a_n$. Thus the origin is a nondegenerate critical point if and only if all a_i are non-zero. Finally, we have the following equality:

$$\mathrm{index}(f, 0) \;=\; \text{number of negative } a_i.$$

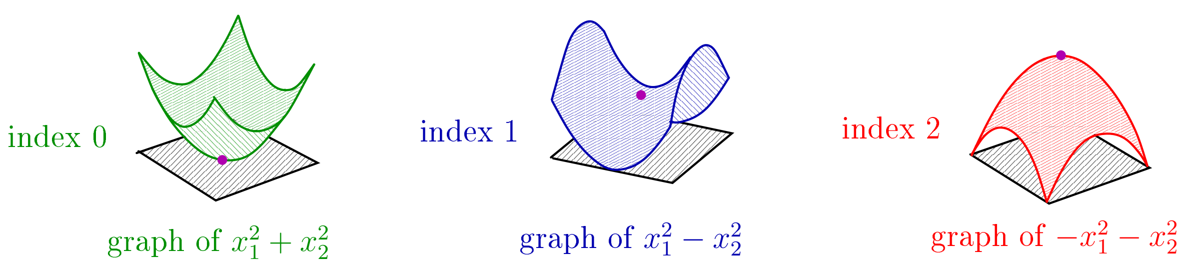


Figure 1472

(2) We consider the smooth manifold $M \subset \mathbb{R}^3$ that is shown in Figure 1473 together with the "height function" $f\colon M \to \mathbb{R}$ that is defined by $f(x, y, z) = z$. In Figure 1473 we indicate all critical points of f and we show their indices.

[1453] Alternatively one could use Proposition 101.8 to define

$$\mathrm{index}(f, P) \;:=\; b^-(\mathrm{H}_P\, f\colon\ \mathrm{T}_P\, M \times \mathrm{T}_P\, M \to \mathbb{R})$$

where $\mathrm{H}_P\, f$ is the intrinsic Hessian mentioned above and where $b^-(\varphi)$ of a symmetric form $\varphi\colon V \times V \to \mathbb{R}$ is defined, see page 2447, as the maximal dimension of a vector subspace W of V for which the symmetric form $\varphi|_W$ is negative definite.

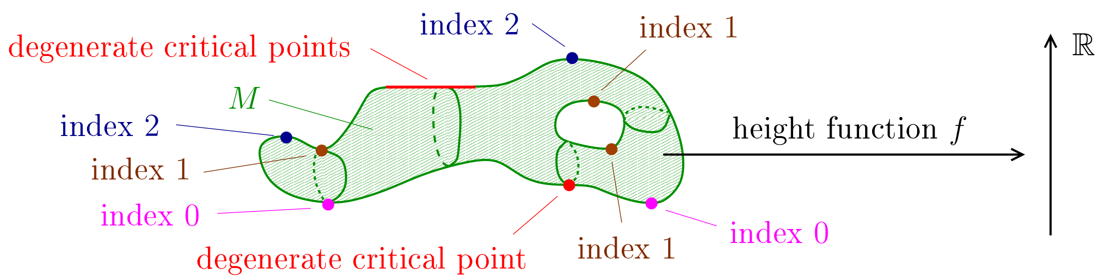


FIGURE 1473

The following result says that, up to taking an appropriate chart, the above mother of all examples is the only example of a non-degenerate critical point. The result was first proved by Marston Morse [**MorsM34**] in 1934.

Theorem 104.2. (Morse Lemma) *Let M be an n-dimensional smooth manifold. Furthermore let $f\colon M \to \mathbb{R}$ be a smooth function and let $P \in M \setminus \partial M$ be a critical point. If P is non-degenerate, then there exists a chart $\Phi\colon U \to V$ around P with $U \subset M \setminus \partial M$ and with $\Phi(P) = 0$ such that for every $(x_1, \dots, x_n) \in V$ we have*

$$(f \circ \Phi^{-1})(x_1, \dots, x_n) \;=\; f(P) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$$

where $k = \operatorname{index}(f, P)$.

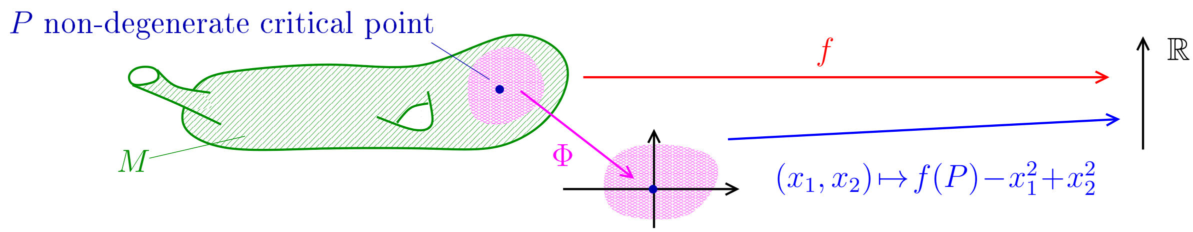


FIGURE 1474. Illustration of the Morse Lemma 104.2.

Remark.

(1) Recall that the Inverse Mapping Theorem 6.40 says that if the differential $\mathrm{D}\, f_P$ of a given map $f\colon M \to N$ between two smooth manifolds at a point $P \in M \setminus \partial M$ is invertible, then the map is locally a diffeomorphism. The Morse Lemma is a similar statement: at a non-degenerate critical point the local behavior of a smooth map $f\colon M \to \mathbb{R}$ is determined by the second partial derivatives.

(2) Our proof of the Morse Lemma 104.2 is basically the same proof as in [**Miln63a**, Lemma 2.2]. Somewhat different proofs are given in [**Nic11**, Theorem 1.12] and [**Lau11**, Section 3.3.2].

(3) Let M be an n-dimensional smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Furthermore let $P \in M \setminus \partial M$ be a non-degenerate critical point. It follows easily from the Morse Lemma 104.2 that
 (a) P is a local maximum if and only if $\operatorname{index}(f, P) = n$.
 (b) P is a local minimum if and only if $\operatorname{index}(f, P) = 0$.

One of the key ingredients to the proof of the Morse Lemma 104.2 is the following proposition.

Proposition 6.16. *Let $U \subset \mathbb{R}^n$ be an open convex subset that contains the origin and let $f\colon U \to \mathbb{R}$ be a smooth function with $f(0) = 0$. There exist smooth maps $a_1, \dots, a_n\colon U \to \mathbb{R}$ such that for $i = 1, \dots, n$ we have $a_i(0) = \frac{\partial f}{\partial x_i}(0)$ and such that*

$$f(x_1, \dots, x_n) \;=\; \sum_{i=1}^{n} a_i(x_1, \dots, x_n) \cdot x_i \quad \text{for all } x = (x_1, \dots, x_n) \in U.$$

This proposition gives us the following corollary, which is actually the content of Exercise 6.14.

Corollary 104.3. *Let $U \subset \mathbb{R}^n$ be an open convex subset that contains the origin and let $f\colon U \to \mathbb{R}$ be a smooth function with $f(0) = 0$. If 0 is a critical point of f, then there exist smooth functions $h_{ij}\colon U \to \mathbb{R}$, $i, j = 1, \dots, n$ such that $h_{ij} = h_{ji}$ and such that*

$$f(x) \;=\; \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij}(x) \cdot x_i \cdot x_j \quad \text{for all } x \in U.$$

Proof of Corollary 104.3. First note that by Proposition 6.16 there exist smooth maps $a_1, \dots, a_n\colon U \to \mathbb{R}$ such that for $i = 1, \dots, n$ we have $a_i(0) = \frac{\partial f}{\partial x_i}(0) = 0$ and such that

$$f(x) \;=\; \sum_{i=1}^{n} a_i(x) \cdot x_i \quad \text{for all } x = (x_1, \dots, x_n) \in U.$$

Since 0 is a critical point we know that $a_1(0) = \dots = a_n(0) = 0$. Thus, given $i \in \{1, \dots, n\}$ we can once again apply the ever-useful Proposition 6.16 to the smooth map a_i and we obtain smooth maps $b_{i1}, \dots, b_{in}\colon U \to \mathbb{R}$ with $b_{ij}(0) = (\frac{\partial}{\partial x_j} a_i)(0) = (\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} f)(0)$ and such that

$$a_i(x) \;=\; \sum_{j=1}^{n} b_{ij}(x) \cdot x_j \quad \text{for all } x \in U.$$

It now clear that the maps $h_{ij} := \frac{1}{2}(b_{ij} + b_{ji})$ have the desired properties. ∎

Now we can provide the proof of the Morse Lemma 104.2.

Proof of the Morse Lemma 104.2. We make a few preparations.

(1) We can assume that M is an open convex neighborhood U of $0 \in \mathbb{R}^n$, that $P = 0$ is the non-degenerate critical point of $f\colon U = M \to \mathbb{R}$ and that $f(0) = 0$.
(2) Throughout the proof we use the usual notation that given $x \in \mathbb{R}^n$ we denote its coordinates by $x_1, \dots, x_n$.
(3) Let V and W be open neighborhoods of $0 \in \mathbb{R}^n$. We say two smooth maps $g\colon V \to \mathbb{R}$ and $h\colon W \to \mathbb{R}$ are *equivalent*, if there exist open sets $0 \in V' \subset V$ and $0 \in W' \subset W$ and a diffeomorphism $\Psi\colon V' \to W'$ with $\Psi(0) = 0$ such that $(g \circ \Psi^{-1})(y) = h(y)$ for all $y \in W'$. Note that this is indeed an equivalence relation.

With these preparations we can reformulate our goal as follows.

Goal. We want to show that $f\colon U \to \mathbb{R}$ is equivalent to a map that is of the form

$$(x_1, \dots, x_n) \;\mapsto\; \epsilon_1 \cdot x_1^2 + \dots + \epsilon_n \cdot x_n^2$$

for some $\epsilon_1,\ldots,\epsilon_n \in \{-1,1\}$.

Note that once we have proved this goal we are done, since it follows from Lemma 104.1 that that the number of negative ϵ_i's in a map as above equals precisely index(f,P).

Now let us turn to the task of achieving this goal. First note that by Corollary 104.3 we know that there exist smooth functions $h_{ij}\colon U\to\mathbb{R}$, $i,j=1,\ldots,n$ such that $h_{ij}=h_{ji}$ and such that f is given by

$$(x_1,\ldots,x_n) \;\mapsto\; \sum_{i=1}^{n}\sum_{j=1}^{n} h_{ij}(x)\cdot x_i\cdot x_j.$$

The idea now is to "diagonalize" the matrices $(h_{ij}(x))_{i,j=1,\ldots,n}$. At this point it is worth recalling that in Proposition 101.6 we gave a constructive proof of the fact that given any real symmetric $(n\times n)$-matrix $A=(a_{ij})$ with $\det(A)\neq 0$ there exists a matrix $P\in \mathrm{GL}(n,\mathbb{R})$ and $\epsilon_1,\ldots,\epsilon_n\in\{-1,1\}$ such that P^TAP is the diagonal matrix with entries $\epsilon_1,\ldots,\epsilon_n$. The idea now is to imitate the proof of Proposition 101.6. As in the proof of Proposition 101.6 we proceed iteratively. This leads us to the following claim.

Claim. Given any $s\in\{0,\ldots,n\}$ the map $f\colon U\to\mathbb{R}$ is equivalent to a map of the form

$$(x_1,\ldots,x_n) \;\mapsto\; \sum_{i=1}^{s}\sum_{j=1}^{s} \mu_{ij}(x)\cdot x_i\cdot x_j \;+\; \sum_{i=s+1}^{n}\epsilon_i\cdot x_i^2,$$

where the μ_{ij} are smooth functions with $\mu_{ij}=\mu_{ji}$ and with $\epsilon_i\in\{\pm 1\}$.

Recall that our goal was precisely to prove the claim for $s=0$. As we remarked above, for $s=n$ there is nothing left to show, since we already saw that it follows from Corollary 104.3 that f is of the required form. Next suppose that the statement holds for some $s\in\{1,\ldots,n\}$. We need to show that the statement also holds for $s-1$. We might as well assume that f itself is already of the form

$$(x_1,\ldots,x_n) \;\mapsto\; \sum_{i=1}^{s}\sum_{j=1}^{s} \mu_{ij}(x)\cdot x_i\cdot x_j \;+\; \sum_{i=s+1}^{n}\epsilon_i\cdot x_i^2 \qquad \text{with } \mu_{ij}=\mu_{ji}.$$

Since 0 is a non-degenerate critical point we know that the matrix $(\mu_{ij}(0))_{i,j=1,\ldots,s}$ is invertible. As we saw in the proof of Proposition 101.6, it is an elementary linear algebra fact that there exists a matrix $A\in\mathrm{GL}(s,\mathbb{R})$ such that the ss-entry of $A^T\cdot(\mu_{ij}(0))_{i,j=1,\ldots,s}\cdot A$ is non-zero. By applying the diffeomorphism Φ that is given by the invertible matrix $\begin{pmatrix} A & 0\\ 0 & \mathrm{id}_{n-s}\end{pmatrix}$ we see that we might as well assume that $\mu_{ss}(0)\neq 0$. After possibly shrinking U we can assume that $\mu_{ss}(x)\neq 0$ for all $x\in U$. Now we consider the map

$$\begin{array}{rcl} \Phi\colon U & \to & \mathbb{R}^n \\ x=(x_1,\ldots,x_n) & \mapsto & \Big(x_1,\ldots,x_{s-1},\sqrt{|\mu_{ss}(x)|}\cdot\Big(x_s+\sum\limits_{i=1}^{s-1}\frac{\mu_{is}(x)}{|\mu_{ss}(x)|}\cdot x_i\Big),x_{s+1},\ldots,x_n\Big). \end{array}$$

Evidently $\Phi(0)=0$. A straightforward calculation shows that $\det(\mathrm{D}\,\Phi_0)=\sqrt{|\mu_{ss}(0)|}\neq 0$. It follows from the Inverse Function Theorem 6.18 that, possibly after shrinking U even further, the map $\Phi\colon U\to\mathbb{R}^n$ is a diffeomorphism onto its image. It remains to show that

$f \circ \Phi^{-1}$ is of the form

$$y = (y_1, \dots, y_n) \;\mapsto\; \sum_{i=1}^{s-1}\sum_{j=1}^{s-1} \nu_{ij}(y) \cdot y_i \cdot y_j \;+\; \sum_{i=s}^{n} \epsilon_i \cdot y_i^2 \qquad \text{with } \nu_{ij} = \nu_{ji}.$$

Thus let $y = (y_1, \dots, y_n) \in \Phi(U)$. We write $(x_1, \dots, x_n) = x := \Phi^{-1}(y)$. Note that by definition of Φ we have

$$(y_1, \dots, y_n) \;=\; \Big(x_1, \dots, x_{s-1}, \sqrt{|\mu_{ss}(x)|} \cdot \big(x_s + \sum_{i=1}^{s-1} \frac{\mu_{is}(x)}{|\mu_{ss}(x)|} \cdot x_i\big), x_{s+1}, \dots, x_n\Big).$$

We perform the following little calculation:

$$\begin{aligned} y_s^2 \;&=\; |\mu_{ss}(x)| \cdot \big(x_s + \sum_{i=1}^{s-1} \frac{\mu_{is}(x)}{|\mu_{ss}(x)|} \cdot x_i\big) \cdot \big(x_s + \sum_{j=1}^{s-1} \frac{\mu_{js}(x)}{|\mu_{ss}(x)|} \cdot x_j\big) \\ &=\; |\mu_{ss}(x)| \cdot \big(x_{ss}^2 + \sum_{i=1}^{s-1}\sum_{j=1}^{s-1} \frac{\mu_{is}(x) \cdot \mu_{js}(x)}{|\mu_{ss}(x)|^2} \cdot x_i \cdot x_j + 2 \cdot \sum_{i=1}^{s-1} \frac{\mu_{is}(x)}{|\mu_{ss}(x)|} \cdot x_i \cdot x_s\big). \end{aligned}$$

Finally note that

$$\begin{aligned} (f \circ \Phi^{-1})(y) \;=\; f(x) \;&\overset{\substack{\text{follows from the above description of } f \\ \downarrow}}{=}\; \sum_{i=1}^{s-1}\sum_{j=1}^{s-1} \mu_{ij}(x) \cdot x_i \cdot x_j \;+\; 2 \cdot \sum_{i=1}^{s} \mu_{is} \cdot x_i \cdot s_s \;+\; \sum_{i=s+1}^{n} \epsilon_i \cdot x_i^2 \;\overset{\substack{\text{follows immediately from the above calculation of } {\color{red}y_s^2} \\ \downarrow}}{=} \\ &=\; \sum_{i=1}^{s-1}\sum_{j=1}^{s-1} \underbrace{\Big(\mu_{ij}(x) - \frac{\mu_{is}(x) \cdot \mu_{js}(x)}{\mu_{ss}(x)}\Big)}_{=:\widetilde{\mu}_{ij}(x)} \cdot x_i \cdot x_j + \frac{\mu_{ss}(x)}{|\mu_{ss}(x)|} \cdot {\color{red}y_s^2} + \sum_{i=s+1}^{n} \epsilon_i \cdot x_i^2 \\ &\underset{\substack{\uparrow \\ \text{follows from } x_i = y_i \text{ for } i \neq s, \text{ from } x = \Phi^{-1}(y) \text{ and} \\ \text{from the fact that the sign of } \mu_{ss}(x) \text{ is constant on } U}}{=}\; -\sum_{i=1}^{s-1}\sum_{j=1}^{s-1} \widetilde{\mu}_{ij}(\Phi^{-1}(y)) \cdot y_i \cdot y_j \;+\; \underbrace{\frac{\mu_{ss}(0)}{|\mu_{ss}(0)|}}_{\in\{-1,1\}} \cdot y_s^2 + \sum_{i=s+1}^{n} \epsilon_i \cdot y_i^2. \end{aligned}$$

This concludes the induction step. ■

Corollary 104.4. *Let M be a smooth manifold and let $f \colon M \to \mathbb{R}$ be a smooth function such that no critical point is contained in ∂M.*

1. *The set of non-degenerate critical points of f is a discrete subset of M.*
2. *If M is compact and if all critical points are non-degenerate, then there exist only finitely many non-degenerate critical points of f.*

Remark. As shown in Figure 1475, the set of non-degenerate critical points of a smooth function is not necessarily a closed subset. This shows in particular that in Corollary 104.4 (2) we really need to assume that all critical points are non-degenerate.

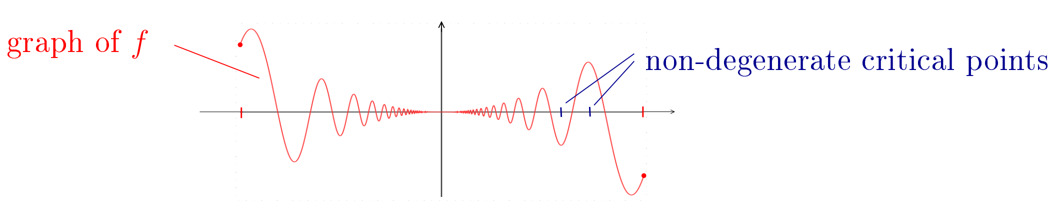


FIGURE 1475

PROOF.

(1) It follows immediately from the Morse Lemma 104.2 that every non-degenerate critical point admits an open neighborhood which contains no other critical point. But this means, by definition, that the non-degenerate critical points of f form a discrete subset of M.
(2) In Exercise 6.29 we showed that the set of critical points of f is a closed subset of M. Since we assume that all critical points are non-degenerate we see that the set of non-degenerate critical points is closed. By (1) we also know that this set is discrete. It follows from the hypothesis that M is compact, together with Lemma 2.18 (2), that the set of non-degenerate critical points is indeed finite. ■

104.2. **Existence of Morse functions.** The following definition introduces the key character of the reminder of this chapter.

Definition. Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. We say f is a *Morse function* if all critical points are non-degenerate and if no critical point lies on ∂M.

Before we discuss the general existence of Morse functions on smooth manifolds let us consider some explicit examples of Morse functions.

Examples.

(1) In Figure 1476 we show three surfaces S^2, F and T in $\mathbb{R}^3$. In each case the "height function"
$$\begin{array}{rcl} h\colon \mathbb{R}^3 & \to & \mathbb{R} \\ (x,y,z) & \mapsto & z \end{array}$$
is a Morse function. Evidently S^2 and F are diffeomorphic, but the numbers of critical points differ. Just for fun we point out that August Möbius [**Möb1863**] in

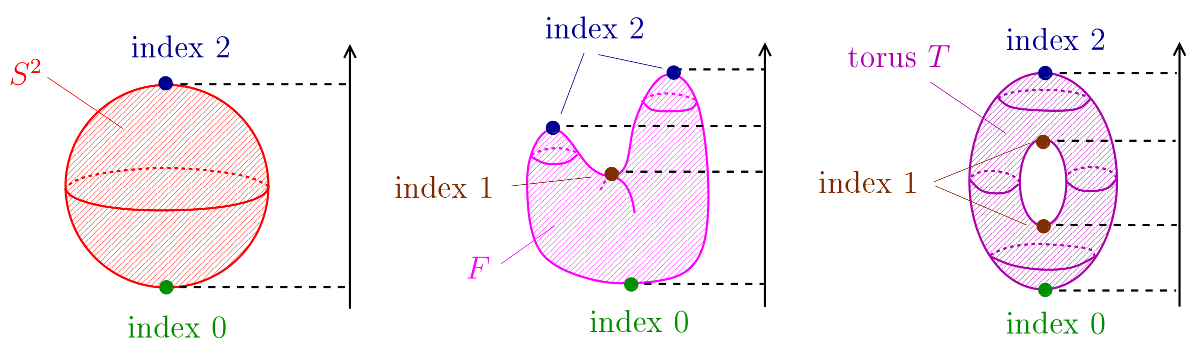


FIGURE 1476

1863 drew basically the same pictures, see Figure 1477.

(2) Let $n \in \mathbb{N}_0$. We pick real numbers $c_0 < c_1 < \cdots < c_n$. We consider the following functions on $\mathbb{RP}^n$ and $\mathbb{CP}^n$:
$$\begin{array}{rcl} f\colon \mathbb{RP}^n & \to & \mathbb{R} \\ [x_0 : \cdots : x_n] & \mapsto & \dfrac{\sum\limits_{j=0}^{n} c_j \cdot x_j^2}{x_0^2 + \cdots + x_n^2} \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{CP}^n & \to & \mathbb{R} \\ [z_0 : \cdots : z_n] & \mapsto & \dfrac{\sum\limits_{j=0}^{n} c_j \cdot |z_j|^2}{|z_0|^2 + \cdots + |z_n|^2}. \end{array}$$

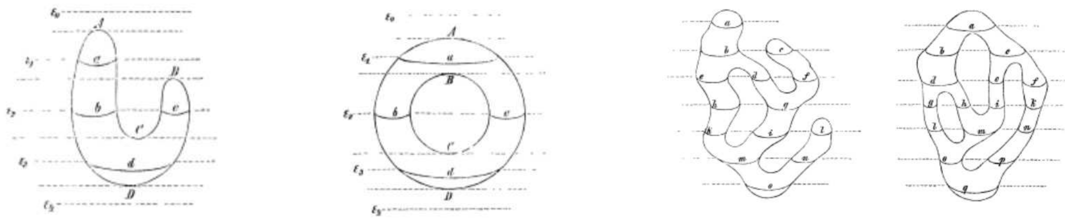


FIGURE 1477. Figures taken from [**Möb1863**].

Using the explicit charts given in Exercise 6.22 and Lemma 12.5 we will verify in Exercise 104.3 that both maps are Morse functions, that both maps have precisely $n+1$ critical points and that these are of index $0, 1, \ldots, n$ respectively $0, 2, \ldots, 2n$.

(3) Let $n \in \mathbb{N}$. In Lemma 6.55 we showed that $\mathrm{SO}(n)$ is a closed smooth manifold of dimension $\frac{1}{2}n(n-1)$. Let $1 < c_1 < \cdots < c_n$ be real numbers. In [**Mata02**, Lemma 3.12] it is shown that the map

$$\begin{array}{rcl} \mathrm{SO}(n) & \to & \mathbb{R} \\ A = (a_{ij}) & \mapsto & \sum\limits_{j=1}^{n} c_j \cdot a_{jj} \end{array}$$

is a Morse function. Furthermore it is shown that the critical points are precisely the diagonal matrices in $\mathrm{SO}(n)$. With very similar ideas one can also obtain explicit Morse functions on the smooth manifolds $\mathrm{U}(n)$ and $\mathrm{SU}(n)$. We refer to [**Mata02**, Examples 3.13 and 3.14] for details.

The following proposition provides a rich source of Morse functions.

Proposition 104.5. *Let M be a closed smooth submanifold of some $\mathbb{R}^n$.*

(1) *The set of $Q \in \mathbb{R}^n$ for which the map $M \to \mathbb{R}$ given by $x \mapsto \|x-Q\|^2$ is a Morse function is a set of full measure.*[1454]

(2) *The set of $v \in \mathbb{R}^n$ for which the map $M \to \mathbb{R}$ given by $x \mapsto \langle x, v\rangle$ is a Morse function is a set of full measure.*

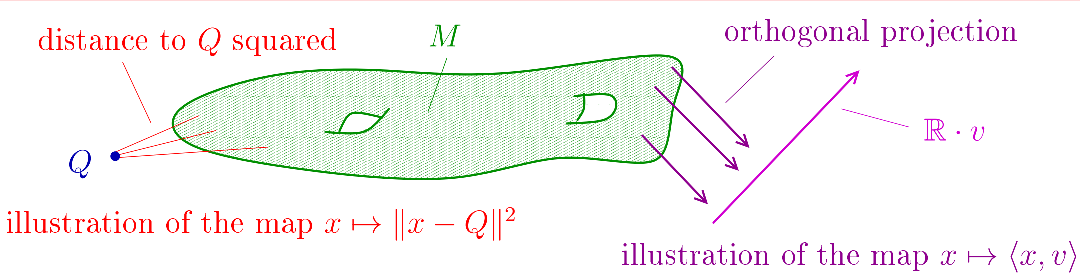


FIGURE 1478. Illustration of Proposition 104.6.

[1454] We refer to page 316 for the definition of a subset of full measure.

Sketch of proof. Let M be a closed smooth submanifold of some $\mathbb{R}^n$.

(1) Given $Q \in \mathbb{R}^n$ we consider the map

$$\begin{array}{rcl} f_Q\colon M & \to & \mathbb{R} \\ x & \mapsto & \|x-Q\|^2. \end{array}$$

Furthermore we consider

$$N \;:=\; \{(p,w) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, p \in M \text{ and } w \in (\mathrm{V}_P\, M)^\perp\}.$$

It is not particularly difficult to show that N is an n-dimensional submanifold of $\mathbb{R}^{2n}$. Next we consider the "endpoint map"

$$\begin{array}{rcl} E\colon N & \to & \mathbb{R}^n \\ (p,w) & \mapsto & p+w \end{array}$$

which is evidently smooth. We say $Q \in \mathbb{R}^n$ is a *focal point* if Q is a critical value of E. A slightly heroic calculation, see [**Miln63a**, Lemma 6.5], shows that Q is a focal point if and only if the map $f_Q\colon M \to \mathbb{R}$ has nondegenerate critical points. The desired statement now follows from the above discussion together with Sard's Theorem 6.63 which tells us that the set of critical values of E is of measure zero in $\mathbb{R}^n$.

(2) This statement is proved in [**Nic11**, Corollary 1.25]. ■

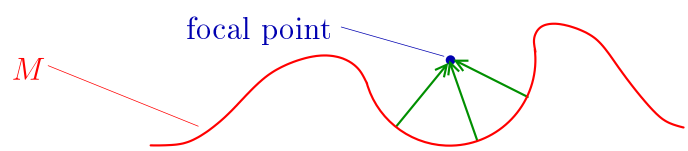


Figure 1479. Illustration for the proof of Proposition 104.6 (1).

Proposition 104.6. *Every closed smooth manifold admits a Morse function.*

Remark.

(1) There are many refinements of Proposition 104.6 in the literature. For example, by [**Mata02**, Theorem 2.20] every smooth function on a closed smooth manifold can be approximated by Morse functions.

(2) Using a doubling argument and collar neighborhoods it is a routine exercise to show, using Proposition 104.6, that also every compact smooth manifold admits a Morse function. But as we will see in Chapter 105, for compact smooth manifolds it is much more convenient to work with Morse functions that "respect the boundary". In Proposition 105.4 we will provide such a refinement of Proposition 104.6 for smooth manifolds with boundary.

(3) In Proposition 105.10 we will see that one can always find "self-indexing" Morse functions.

Proof. Let M be a closed smooth manifold. It follows from Proposition 9.1 together with Proposition 8.1 that we can assume that M is a submanifold of some $\mathbb{R}^n$. It now follows from Proposition 104.5 that M admits a Morse function. ■

104.3. **The Smooth Manifold Product Theorem.** In this section and the following section we will see that there is a close relationship between the topology of a smooth manifold and the non-degenerate critical points of a Morse function.

Notation. Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Given $a \in \mathbb{R}$ we write

$$M_a \ := \ f^{-1}((-\infty, a]) \ = \ \{x \in M \mid f(x) \le a\}.$$

Note that if a is a regular value and if $\partial M \cap f^{-1}(\{a\}) = \varnothing$, then it follows from the Regular Value Theorem 6.53 that $f^{-1}(\{a\})$ is a closed submanifold of M and it follows from Exercise 6.31 that M_a is a submanifold of M with boundary $f^{-1}(\{a\}) \cup (\partial M \cap f^{-1}((-\infty, a]))$.

Example. In Figure 1480 we show a smooth manifold M together with the "length" function $f\colon M \to \mathbb{R}$ given by $(x, y, z) \mapsto x$. For various $a_i \in \mathbb{R}$ we show the corresponding preimage $f^{-1}(\{a_i\})$. The submanifold M_{a_i} is of course everything "to the $\le$-left" of $f^{-1}(\{a_i\})$.

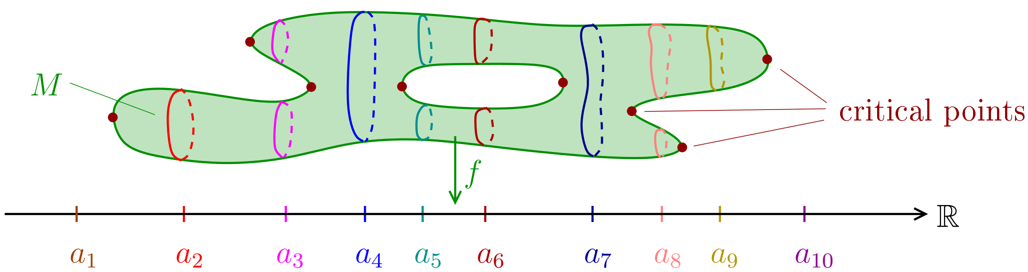


FIGURE 1480

Theorem 104.7. *Let M be a compact smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Let $a < b \in \mathbb{R}$ such that $f^{-1}([a, b])$ does not contain a critical point of f. If $f^{-1}([a, b]) \cap \partial M = \varnothing$, then there exists a diffeomorphism $M_a \to M_b$. Furthermore, given any $\epsilon > 0$ one can find diffeomorphism $M_a \to M_b$ that is the identity on the common subset $M_{a-\epsilon}$.*

Example. In Figure 1480 there is no critical point in $f^{-1}([a_5, a_6])$. Thus we obtain from Theorem 104.7 that the manifolds M_{a_5} and M_{a_6} are diffeomorphic. Furthermore, we can find a diffeomorphism that is the identity on "most of M_{a_5}".

Our proof of Theorem 104.7 rests on the following result, that will turn out to be of independent interest later on.

Theorem 104.8. (Smooth Manifold Product Theorem) *Let M be a compact smooth manifold, let $f\colon M \to \mathbb{R}$ be a smooth function and let $a < b \in \mathbb{R}$. We suppose that for every $P \in f^{-1}([a, b])$ the differential $\mathrm{D} f_P\colon \mathrm{T}_P M \to \mathbb{R}$ is non-zero.*[1455] *Furthermore we suppose that the following conditions are satisfied:*

(1) *$f^{-1}(\{a\})$ is a closed submanifold of M or it is a union of components of ∂M.*
(2) *$f^{-1}(\{b\})$ is a closed submanifold of M or it is a union of components of ∂M.*

Then $f^{-1}([a, b])$ is diffeomorphic to $[a, b] \times f^{-1}(\{a\})$.[1456]

Sketch of a proof of Theorem 104.8. Note that using Proposition 9.1 and Proposition 8.1 we can assume that M is a submanifold of some $\mathbb{R}^n$. Given $P \in M$ we denote by[1457] $\operatorname{grad} f_P \in \mathrm{V}_P\, M$ the gradient of f at P.[1458] Note that our hypothesis on f just says that given any $P \in f^{-1}([a,b])$ the gradient is non-zero. Thus given any $P \in f^{-1}([a,b])$ we can define

$$v(P) \;:=\; \frac{\operatorname{grad} f_P}{\|\operatorname{grad} f_P\|^2} \;\in\; \mathrm{V}_P\, M.$$

Claim. If $\gamma\colon [0,\epsilon] \to f^{-1}([a,b])$ is a solution to the differential equation $\gamma'(t) = v(\gamma(t))$, then $(f \circ \gamma)'(t) \equiv 1$.

We perform the following calculation:

$$\begin{array}{rcll}
(f\circ\gamma)'(t) &=& \mathrm{D}(f\circ\gamma)_t \overset{\substack{\text{chain rule}\\ \downarrow}}{=} \mathrm{D}\, f_{\gamma(t)}(\gamma'(t)) & \overset{\substack{\text{definition of gradient, here we set } P:=\gamma(t)\\ \downarrow}}{=} \langle \gamma'(t), \operatorname{grad} f_P\rangle \\
&& \underset{\substack{\uparrow\\ \text{by hypothesis we have } \gamma'(t)=v(\gamma(t))}}{=} \langle v(\gamma(t)), \operatorname{grad} f_P\rangle & \underset{\substack{\uparrow\\ \text{since } P=\gamma(t) \text{ and by definition of } v}}{=} \left\langle \tfrac{1}{\|\operatorname{grad} f_P\|^2}\cdot \operatorname{grad} f_P, \operatorname{grad} f_P\right\rangle \;=\; 1.
\end{array}$$

⊞

We set $W := f^{-1}(\{a\})$. It remains to prove the following claim.

Claim.

(1) Given any $P \in W$ there exists a unique smooth map $\gamma_P\colon [a,b] \to f^{-1}([a,b])$ such that $\gamma_P(a) = P$ and such that for each $t \in [a,b]$ we have $\gamma_P'(t) = v(\gamma_P(t))$.
(2) The map

$$\begin{array}{rcl} \Theta\colon [a,b]\times W &\to& M \\ (t,P) &\mapsto& \gamma_P(t) \end{array}$$

is a smooth embedding with image $f^{-1}([a,b])$.

[1455]Let $P \in M$. We recall the following definition from page 308:

(1) If $P \notin M \setminus \partial M$, then P is a regular point if $\mathrm{D}\, f_P\colon\ \mathrm{T}_P\, M \to \mathbb{R}$ is non-zero.
(2) If $P \in \partial M$, then P is a regular point if the map $\mathrm{D}\, f_P\colon\ \mathrm{T}_P\, \partial M \to \mathbb{R}$ is non-zero.

Thus we see that for points in $M \setminus \partial M$ our condition just says that we demand that P is a regular point. For points on the boundary our condition is more relaxed.

[1456]To make sense of the proposition we need to make the following two observations:

(i) Our hypotheses (1) and (2), together with Exercise 6.31, imply that Lemma 6.25 that $f^{-1}([a,b])$ is naturally a smooth manifold.
(ii) Our hypothesis (1), together with the combination of Lemma 6.25 and Proposition 6.27 shows that $f^{-1}(\{a\})$ is a closed smooth manifold in it own right. It follows from Proposition 6.51 that the product $[a,b] \times f^{-1}(\{a\})$ is naturally a smooth manifold.

Thus it actually does make sense to say that $f^{-1}([a,b])$ is diffeomorphic to $[a,b] \times f^{-1}(\{a\})$.

[1457]Recall that on page 292 we introduced the visual tangent space $\mathrm{V}_P\, M$ which, by the discussion on page 491, is naturally isomorphic to $\mathrm{T}_P\, M$.

[1458]The gradient $\operatorname{grad} f_P \in \mathrm{V}_P\, M$ of f at $P \in M$ is defined as the unique vector in $\mathrm{V}_P\, M$ which satisfies $\langle w, \operatorname{grad} f_P\rangle = \mathrm{D}\, f_P(w) \in \mathbb{R}$ for all $w \in \mathrm{V}_P\, M$. It is fairly elementary to show that $P \mapsto \operatorname{grad} f_P$ is a smooth map $M \to \mathbb{R}^n$.

We now provide the proofs of the two statements of the claim.

(1) This statement follows from the previous claim and standard theory of differential equations.
(2) It follows from (1) that the map $\Theta\colon [a,b] \times W \to M$ is well-defined. Furthermore it follows again from standard theory of differential equations that the map is smooth and that for each $t \in [a,b]$ the map $W \to M$ given by $P \mapsto \gamma_P(t)$ is a smooth embedding with image $f^{-1}(\{t\})$. It follows easily from this observation and the previous claim that the map $\Theta\colon [a,b] \times W \to M$ is actually a smooth embedding with image $f^{-1}([a,b])$. ⊞

We leave it to the reader to dig out their favorite books on differential equations to fill in the details. Very similar statements are also proved in [**Miln63a**, p. 10] and [**Lee02**, Theorem 9.12, Corollary 9.17]. ■

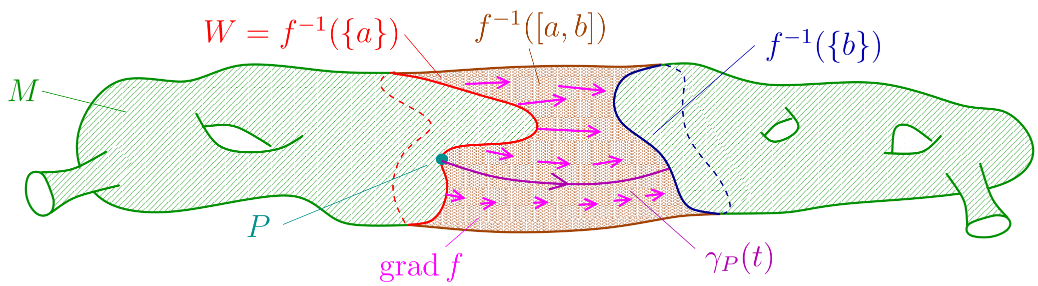


FIGURE 1481. Illustration for the proof of Theorem 104.7.

Now we have no troubles with providing the desired proof of Theorem 104.7.

PROOF OF THEOREM 104.7. Let M be a compact smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Let $a < b \in \mathbb{R}$ such that $f^{-1}([a,b])$ does not contain a critical point of f. We suppose that $f^{-1}([a,b]) \cap \partial M = \varnothing$. By the Regular Value Theorem 6.53 we know that $W := f^{-1}(\{a\})$ is a closed smooth submanifold of M. By Theorem 104.8 we know that $f^{-1}([a,b])$ is diffeomorphic to $[a,b] \times W = [a,b] \times f^{-1}(\{a\})$. It now follows from Exercise 8.13 (or alternatively from Lemma 105.2) that $M_b = M_a \cup f^{-1}([a,b])$ is diffeomorphic to M_a. Furthermore, if we are given some $\epsilon > 0$, then it follows from Exercise 8.13 that we can find a diffeomorphism that is the identity on $M_{a-\epsilon}$. ■

104.4. **The Handle Addition Theorem.** Let M be a compact smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Furthermore let $a < b \in \mathbb{R}$ such that a and b are regular values and such that $f^{-1}([a,b]) \cap \partial M = \varnothing$. Note that the Regular Value Theorem 6.53 implies that $f^{-1}(\{a\})$ and $f^{-1}(\{b\})$ are closed submanifolds of M. Note that if the stronger condition holds that $f^{-1}([a,b])$ does not contain a critical point of f, then the Smooth Manifold Product Theorem 104.8 says that M_b is diffeomorphic to M_a. But what happens if $f^{-1}((a,b))$ contains a critical point?

On the left of Figure 1482 we show a 2-dimensional smooth manifold M together with a smooth function $f\colon M \to \mathbb{R}$ such that there exists a unique critical point of index $k = 1$ in $f^{-1}((a,b))$. On the left of Figure 1482 we show the result of attaching a handle of index $k = 1$ to M_a. Some goodwill shows that M_b is diffeomorphic to the result of attaching this handle of index $k = 1$ to M_a. In fact given any $\epsilon > 0$ one can find a diffeomorphism that is the identity on the common subset $M_{a-\epsilon}$.

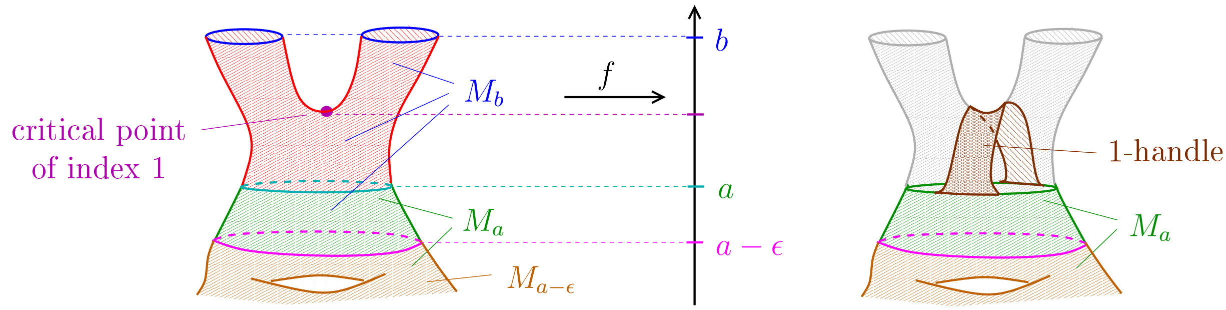


FIGURE 1482

The following all-important theorem shows that a suitable generalization of the above observation holds in all reasonable settings.

Theorem 104.9. (Handle Addition Theorem) *Let M be a compact smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Furthermore let $a < b \in \mathbb{R}$ such that a and b are regular values and such that $f^{-1}([a,b]) \cap \partial M = \varnothing$.*

(1) *Suppose there exists a single critical point $P \in f^{-1}([a,b])$. We set $k := \operatorname{index}(f, P)$. If P is non-degenerate, then M_b is diffeomorphic to the result of attaching a k-handle to M_a.*
(2) *Suppose there exists a single critical value $c \in [a,b]$. Let $P_1, \dots, P_m$ be the preimages of c. We suppose that $P_1, \dots, P_m$ are non-degenerate critical points. Let $k_1, \dots, k_m$ be the indices of the critical points. Then M_b is diffeomorphic to the result of attaching m handles of index k_i, $i = 1, \dots, m$ to M_a along thickened spheres with disjoint images.*

Furthermore, in both settings, given any $\epsilon > 0$ there exists an diffeomorphism as above which is the identity on the common subset $M_{a-\epsilon}$.

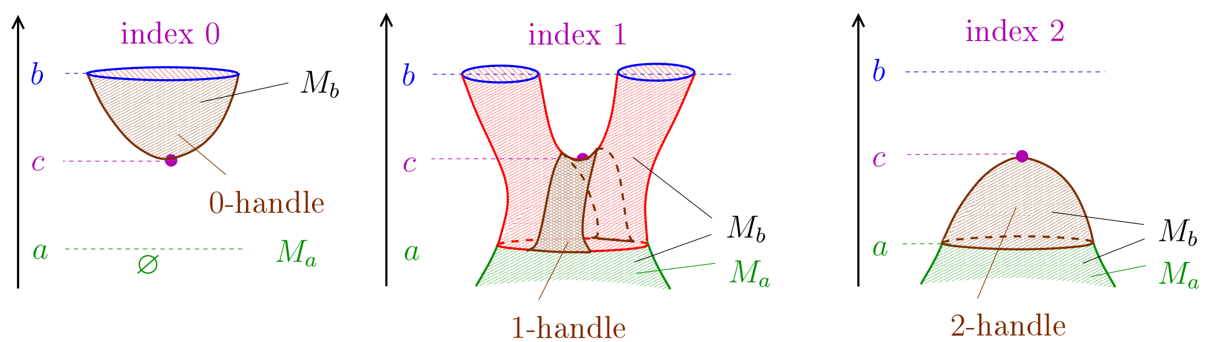


FIGURE 1483. Illustration for Theorem 104.9 (1).

Example. In Figure 1484 we return to the setting of Figure 1480. We show all the critical points and we indicate the indices. Furthermore, reading from left to right we show how M gets built out of the empty manifold by various handle attachments. To simplify our pictures we do not distinguish between a smooth manifold that is the result of attaching a

k-handle and a smooth manifold that is diffeomorphic to the result of attaching a k-handle.

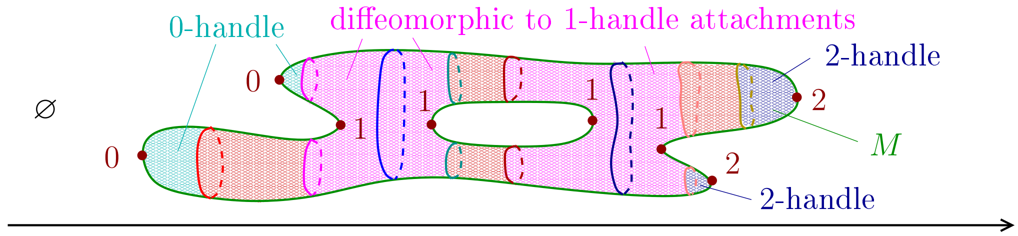


FIGURE 1484

Remark. We provide an almost complete proof of Theorem 104.9. Only at the very end of the proof, on page 2512, when we need to prove that a certain map is a diffeomorphism, we not feel like dealing with the technical challenge of verifying that a given homeomorphism is in fact a diffeomorphism. We refer to our usual sources, namely [**Nic11**, Theorem 2.7 and Remark 2.9], [**Miln63a**, Theorem 3.2, Remark on p. 17, Remark 3.3] and [**Kos93**, Chapter VII.2] for more details.

PROOF. Let M be a compact smooth manifold, let $f\colon M \to \mathbb{R}$ be a smooth function and let $a < b \in \mathbb{R}$ be regular values such that $f^{-1}([a,b]) \cap \partial M = \varnothing$. In the following we will only prove Statement (1). Statement (2) can be proved either by generalizing the proof of Statement (1) or by using Exercise 104.7. We leave it to the reader to fill in the details for the proof of Statement (2).

Thus we suppose there exists a single critical point $P \in f^{-1}([a,b])$ and we suppose that P is non-degenerate. We set $c := f(P)$ and we set $k := \operatorname{index}(f, P)$. We need to show that M_b is diffeomorphic to the result of attaching a k-handle to M_a.

We make the following preparations:

(1) We set $l := n - k$.
(2) We make the identification $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$. Given $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ we write $x^2 = x_1^2 + \dots + x_k^2$ and similarly given $y = (y_1, \dots, y_l) \in \mathbb{R}^l$ we write $y^2 = y_1^2 + \dots + y_l^2$.
(3) By the Morse Lemma 104.2 we can identify a neighborhood of P in $M \setminus \partial M$ with a neighborhood V of $(0,0) \in \mathbb{R}^k \times \mathbb{R}^l$ such that $P = (0,0)$ and such that for every $(x,y) \in V$ we have $f(x,y) = c - x^2 + y^2$.
(4) Since a and b are regular values we know that $c \in (a,b)$. Thus we can pick an $\eta > 0$ such that $a \le c - \eta < c + \eta \le b$ and such that $\overline{B}^n_{\sqrt{2}\eta}(0) \subset V$. Note that by our hypothesis there are no critical points in $f^{-1}([a, c-\eta])$ and there are no critical points in $f^{-1}([c+\eta, b])$. Thus it follows from Theorem 104.7 that it suffices to show that $M_{c+\eta}$ is diffeomorphic to the result of attaching a k-handle to $M_{c-\eta}$.

Note that only in the neighborhood V do we have completely control over f. Unfortunately a priori the difference $M_b \setminus \mathring{M}_a$ is not contained in the neighborhood V. To deal with this issue we introduce an auxiliary function. First note that by Lemma 6.13 there exists a smooth map $\phi\colon [0,\infty) \to \mathbb{R}_{\ge 0}$ with $\phi(0) > \eta$, $\phi(t) = 0$ for $t \ge 2\eta$ and such that $\phi'(t) \in (-1, 0]$ for all $t \in [0,\infty)$. We refer to Figure 1485 for an illustration.

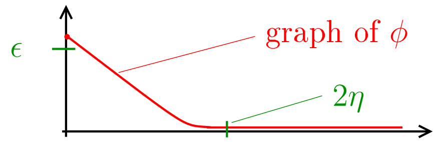


FIGURE 1485. First illustration for the proof of Theorem 104.9.

Next we consider the map

$$\begin{aligned} F\colon M &\to \mathbb{R} \\ P &\mapsto \begin{cases} f(x,y) - \phi(x^2+2y^2) = c - x^2 + y^2 - \phi(x^2+2y^2), & \text{if } P = (x,y) \in \overline{B}^n_{\sqrt{2\eta}}(0), \\ f(P), & \text{otherwise.} \end{cases} \end{aligned}$$

The following claim summarizes some of the key properties of F.

Claim 1.

(1) The map F is smooth.
(2) The critical points of F are precisely the critical points of f.
(3) Given any $r \in \mathbb{R}$ we have $f^{-1}((-\infty, r]) \subset F^{-1}((-\infty, r])\}$.
(4) We consider the ellipsoid $E := \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + 2y^2 \leq 2\eta\}$. Given any $r \in \mathbb{R}$ we have $F^{-1}((-\infty, r]) \setminus f^{-1}((-\infty, r]) \subset E \subset \overline{B}^n_{\sqrt{2\eta}}(0)$.
(5) The smooth manifold $F^{-1}((-\infty, c-\eta])$ is diffeomorphic to $F^{-1}((-\infty, c+\eta])$.
(6) We have $F^{-1}((-\infty, c+\eta]) = f^{-1}((-\infty, c+\eta])$.

In particular we have

$$M_{c-\eta} = f^{-1}((-\infty, c-\eta]) \underset{\substack{\uparrow \\ \text{by (4) the difference is contained in } E \subset \overline{B}^n_{\sqrt{2\eta}}(0)}}{\overset{(3)}{\subset}} F^{-1}((-\infty, c-\eta]) \underset{\substack{\uparrow \\ \text{diffeomorphic by (5)}}}{\cong} F^{-1}((-\infty, c+\eta]) \overset{(6)}{=} f^{-1}((-\infty, c+\eta]) = M_{c+\eta}.$$

The various level sets of f and F are illustrated in Figure 1486.

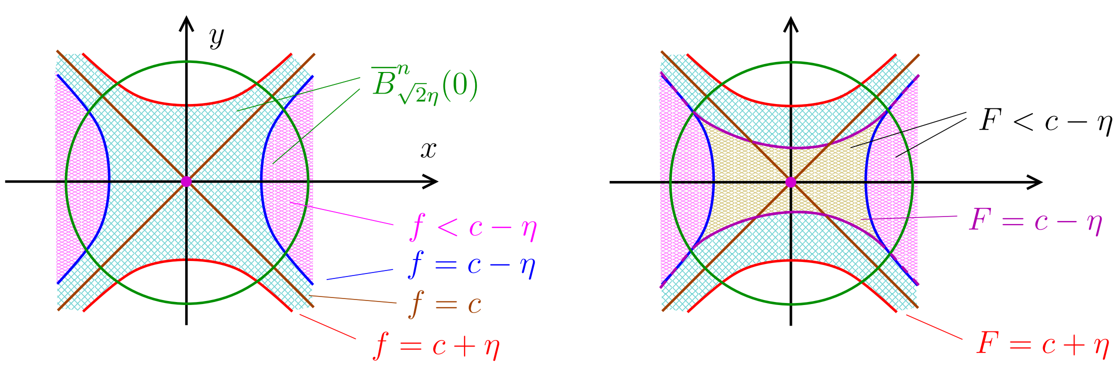


FIGURE 1486. Second illustration for the proof of Theorem 104.9.

We turn to the proof of the claim. We prove the statements in a somewhat different order.

(1) Note that ϕ is zero outside of $\overline{B}^n_{\sqrt{2}\eta}(0)$. It follows from this observation together with the fact $\overline{B}^n_{\sqrt{2}\eta}(0)$ is contained in V that F is smooth.

(2) It is clear that we only need to consider the points (x, y) with $x^2 + 2y^2 \le 2\eta$. For those points we compute, using the chain rule from Proposition 6.15 that

$$\mathrm{D}\, f_{(x,y)} = (2x, 2y) \ \text{ and } \ \mathrm{D}\, F_{(x,y)} = (2x\cdot\overbrace{(-1-\underbrace{\phi'(x^2+2y^2)}_{>-1 \text{ since } \phi'(t)\in(-1,0]})}^{<0}, 2y\cdot\overbrace{(1-2\cdot\underbrace{\phi'(x^2+2y^2)}_{\le 0 \text{ since } \phi'(t)\in(-1,0]}))}^{\ge 1}).$$

We see that $\mathrm{D}\, f_{(x,y)} = (0, 0)$ if and only if $\mathrm{D}\, F_{(x,y)} = (0, 0)$.

(3) Since $\phi(x, y) \ge 0$ for all $(x, y) \in \overline{B}^n_{\sqrt{2}\eta}(0)$ we see that $F(P) \le f(P)$ for all $P \in M$. But this implies that for any $r \in \mathbb{R}$ we have $f^{-1}((-\infty, r]) \subset F^{-1}((-\infty, r])$.

(4) Note that the two maps $f\colon M \to \mathbb{R}$ and $F\colon M \to \mathbb{R}$ agree outside of $E \subset \overline{B}^n_{\sqrt{2}\eta}(0)$. The desired statement is an immediate consequence of this observation.

(6) Note that given any $(x, y) \in E$ we have

$$\underset{}{F(x,y)} \underset{\substack{\uparrow \\ \text{since } \phi(x,y)\ge 0}}{\le} f(x,y) \underset{\substack{\uparrow \\ \text{since } E\subset\overline{B}^n_{\sqrt{2}\eta}(0)\subset V}}{=} c - x^2 + y^2 \le c + \tfrac{1}{2}x^2 + y^2 \underset{\substack{\uparrow \\ \text{since } (x,y)\in E=\{(x,y)\,|\,x^2+2y^2\le 2\eta\}}}{\le} c + \eta.$$

The desired equality follows from this observation together with (4).

(5) We need to show that the smooth manifold $F^{-1}((-\infty, c-\eta])$ is diffeomorphic to $F^{-1}((-\infty, c-\eta])$. By Theorem 104.7 together with (2) it suffices to prove that the preimage $F^{-1}([c-\eta, c+\eta])$ does not contain a critical point of f. This statement follows from the following inclusions and statements:

$$\begin{aligned}\overbrace{F^{-1}([c-\eta,c+\eta])}^{\substack{\text{by construction we have } \phi(0)>\eta \\ \text{thus } F(0,0)=c-\phi(0)<c-\eta \\ \text{thus this set does not contain } (0,0)}} &= F^{-1}((-\infty,c+\eta])\setminus F^{-1}((-\infty,c-\eta)) \\ &\underset{\substack{\uparrow \\ \text{follows from (3) and (6)}}}{\subset} f^{-1}((-\infty,c+\eta])\setminus f^{-1}((-\infty,c-\eta)) = \underbrace{f^{-1}([c-\eta,c+\eta])}_{\substack{\text{by hypothesis this contains} \\ \text{a single critical point of } f, \\ \text{namely } (0,0)}}.\end{aligned}$$

⊞

By Claim 1 it suffices to prove that the smooth manifold $F^{-1}((-\infty, c-\eta])$ is diffeomorphic to the result of attaching a k-handle to $M_{c-\eta} = f^{-1}((-\infty, c-\eta])$. This step is performed in the following claim.

Claim 2. We consider the thickened $(k-1)$-sphere

$$\begin{aligned}\varphi\colon \overline{B}^{n-k} \times S^{k-1} &\to f^{-1}(\{c-\eta\}) \subset \partial M_{c-\eta} \\ (x, y) &\mapsto \Big(\sqrt{2\cdot\eta\cdot x^2+\eta}\cdot y, \sqrt{2\eta}\cdot x\Big).\end{aligned}$$

The smooth manifold $M_{c-\varphi} \cup_\varphi h^k = f^{-1}((-\infty, c-\eta]) \cup_\varphi h^k$ is diffeomorphic to the smooth manifold $F^{-1}((-\infty, c-\eta])$.

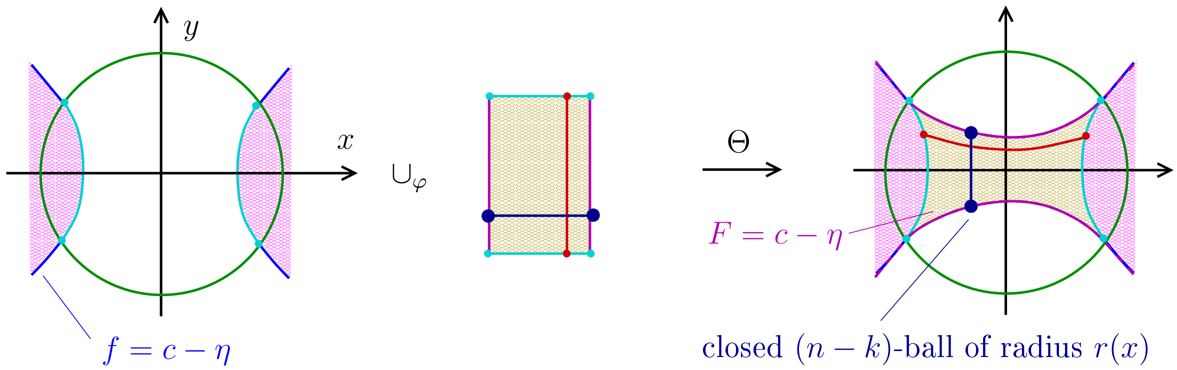


FIGURE 1487. Third illustration for the proof of Theorem 104.9.

We only provide a sketch of the proof of Claim 2. As so often trying to show that a given map is really a diffeomorphism quickly leads to unreadable arguments. The full details for the proof are provided in [**Kos93**, p. 129]. First note that a fairly straightforward argument shows that for any $x \in \overline{B}^k_{\sqrt{\eta}}(0)$ there exists an $r(x) \in \mathbb{R}_{>0}$ such that

$$\{y \in \overline{B}^l \,|\, F(x,y) \leq c+\eta\} \;=\; \overline{B}^{n-k}_{r(x)}(0).$$

Now the desired diffeomorphism is given by

$$\begin{aligned} f^{-1}((-\infty, c-\eta]) \cup_\varphi (\overline{B}^l \times \overline{B}^k) &\to F^{-1}((-\infty, c-\eta]) \\ P &\mapsto \begin{cases} P, & \text{if } P \in f^{-1}((-\infty, c-\eta]), \\ \left(\sqrt{2\eta x^2+\eta}\cdot y,\, x \cdot r(\sqrt{2\eta x^2+\eta})\right), & \text{if } P = (x,y) \in \overline{B}^l \times \overline{B}^k. \end{cases} \end{aligned}$$

This completes the sketch of the proof of the claim. ⊞

Finally suppose we are given an $\epsilon > 0$. We need to show that there exists an diffeomorphism as above which is the identity on the common subset $M_{a-\epsilon}$. As the reader will surely have noticed, this is an immediate consequence of our construction together with the "furthermore" statement of Theorem 104.7. ■

104.5. **The Handle Decomposition Theorem.** With the results from the previous section it is now pretty straightforward to prove the following result.

Proposition 104.10. *Let M be a closed smooth manifold. If $f\colon M \to \mathbb{R}$ is a Morse function, then there exists a standard handle decomposition of M such that for each $k \in \mathbb{N}_0$ the number of k-handles equals the number of critical points of f of index k.*

PROOF. Let M be a closed smooth manifold and furthermore let $f\colon M \to \mathbb{R}$ be a Morse function. Recall that by Corollary 104.4 (2) we know that f has only finitely many critical points. We order the corresponding critical values by $\lambda_1 < \cdots < \lambda_s$. Next we pick real numbers $\mu_0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{s-1} < \lambda_s$ and furthermore we pick some $\mu_s \geq \max(f\colon M \to \mathbb{R})$. Note that $M_{\mu_0} = \varnothing$ and $M_{\mu_s} = M$. It follows from Theorem 104.9 that each M_{μ_i} is diffeomorphic to the result of attaching finitely many handles to $M_{\mu_{i-1}}$.[1459] In fact Theorem 104.9 says that we can arrange that the handles are in one-to-one correspondence to the critical points with critical value λ_i and that the indices of the handles correspond precisely to the indices of the critical points. This shows that $M = M_{\mu_s}$ admits a handle decomposition with the predicted number of handles. Finally we apply Corollary 103.4 to conclude that M actually admits a standard handle decomposition with the same number of handles with the same indices. ■

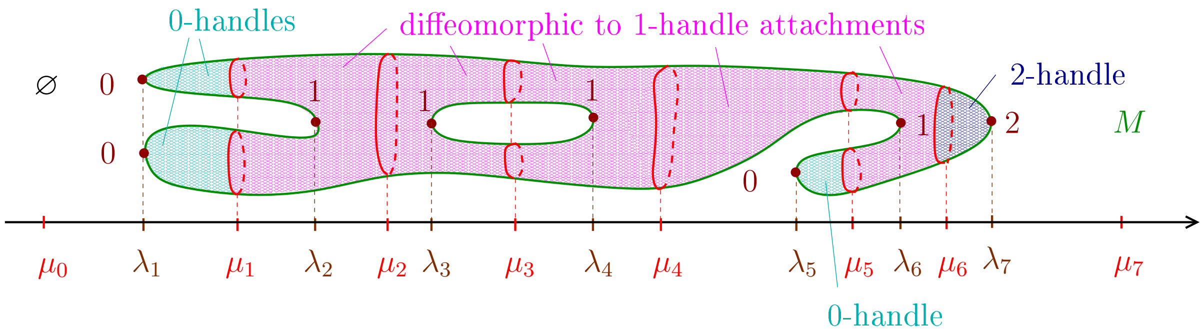


FIGURE 1488. Illustration for the proof of Proposition 104.10.

Remark.

(1) In Proposition 105.9 we will prove a converse to Proposition 104.10, namely we will see that handle decompositions give rise to Morse functions.
(2) Let M be a closed smooth manifold and let $f\colon M \to \mathbb{R}$ is a Morse function. In the proof of Proposition 104.10 we just saw that f gives rise to a handle decomposition of M. Evidently $-f\colon M \to \mathbb{R}$ is also a Morse function. The corresponding handle decomposition is for all intents and purposes the same as the dual handle decomposition that we introduced in Proposition 104.10. Since we will not make use of this statement we will also not bother with providing a proof.

Now we can easily prove the main theorem of this chapter.

Theorem 104.11. (Handle Decomposition Theorem) *Every closed smooth manifold admits a standard handle decomposition.*

Remark.

(1) In the literature the Handle Decomposition Theorem is often called the Handle Presentation Theorem.
(2) We will deal with compact smooth manifolds with non-empty boundary in the following chapter. In particular we will prove the Relative Handle Decomposition Theorem 105.6 that says that every compact smooth manifold admits a handle structure rel a given union of boundary components. As we point out in Theorem 105.7, this implies in particular that every compact smooth manifold admits a standard handle decomposition.

[1459] Note that implicitly we use our hypothesis that M is closed, since to apply Theorem 104.9 we need to ensure that $f^{-1}([\mu_{i-1}, \mu_i]) \cap \partial M = \varnothing$.

Proof. Let M be a closed smooth manifold. By Proposition 104.6 we know that there exists a Morse function $f\colon M \to \mathbb{R}$. It follows from Proposition 104.10 that M admits a standard handle decomposition. ■

We can now give a new proof for some of the statements of Proposition 64.6.

Proposition 64.6. *Let M be a closed n-dimensional smooth manifold.*

(1) *For every $k > n$ we have $\mathrm{H}_k(M;\mathbb{Z}) = 0$.*
(3) *Given any $P \in M$ the fundamental group $\pi_1(M,P)$ is finitely presented.*
(4) *All homology groups of M are finitely generated abelian groups.*

Remark.

(1) Morse theory can also be used to prove the above formulation of Proposition 64.6 for compact smooth manifolds with non-empty boundary. In fact there are at least two approaches to achieving this goal:
   (a) Either one applies the above result to the double $\mathrm{D}\,M$ of a compact smooth manifold with boundary and one applies a little bit of algebra, e.g. Lemma 21.16.
   (b) Alternatively one can wait for Theorem 105.7 where we prove the generalization of the Handle Decomposition Theorem 104.11 to compact smooth manifolds with non-empty boundary.
(2) In Theorem 106.11 and Proposition 106.10 we will give explicit upper bounds on the "size" of the homology groups of a closed smooth manifold in terms of the number of critical points of a Morse function respectively the number of handles in a handle decomposition.

Sketch of proof. Let M be a closed n-dimensional smooth manifold. By the Handle Decomposition Theorem 104.11 we know that M admits a handle decomposition. It follows easily from a straightforward induction argument using Proposition 103.11 that for any $P \in M$ the fundamental group $\pi_1(M,P)$ is finitely presented. Similarly it follows from a straightforward induction argument using Lemma 103.13 that for every $k > n$ we have $\mathrm{H}_k(M;\mathbb{Z}) = 0$ and, using the elementary group-theoretic Lemma 19.6 (1), we see that all homology groups of M are finitely generated abelian groups. ■

The next proposition is also a straightforward consequence of our results.

Proposition 104.12. *Let M be a closed smooth manifold.*

(1) *If f is a Morse function for M, then M is homotopy equivalent to a* CW*-complex where the number of k-cells equals precisely the number of critical points of index k.*
(2) *The smooth manifold is homotopy equivalent to a finite* CW*-complex.*

Example. Let $n \in \mathbb{N}_0$. Recall that on page 2502 we showed that there exists a Morse function $f\colon \mathbb{RP}^n \to \mathbb{R}$ with precisely one critical point for each index $0,\dots,n$. It follows from Proposition 104.12 that $\mathbb{RP}^n$ is homotopy equivalent to a CW-complex with precisely one cell of dimension $0,\dots,n$ and no other cells. In the same vein the above results show that $\mathbb{CP}^n$ is homotopy equivalent to a CW-structure with precisely one cell of degree $0,2,\dots,2n$ and no other cells. This discussion almost recovers the result of Lemma 36.1.

Proof.

(1) This statement follows immediately from Corollary 104.4 (2), Theorem 104.9 and Proposition 103.18.
(2) This statement is a consequence of (1) and Proposition 104.6. Put differently, it is a consequence of the Handle Decomposition Theorem 104.11 and Proposition 103.18. ■

Remark.
(1) There are various other proofs that Morse functions give rise to CW-complexes that are homotopy equivalent to M, see e.g. [**BaH04**, Theorem 3.28], [**Miln65b**, Chapter 3] and [**Mata02**, Theorem 4.18]. Terse argument are also given in [**Nic11**, Theorem 2.10] and [**Wall16**, Proposition 5.3.1]. Furthermore an even terser argument is given in [**Hirs76**, Theorem 6.4.1].
(2) Given a closed smooth manifold there is a notion of a more refined version of a Morse function, namely of a Morse-Smale function. We will not provide the definition, instead we refer to [**BaH04**, p. 7] for the definition. But we try to pique the reader's interest with the following two statements:
   (a) The Kupka-Smale Theorem, proved independently by Ivan Kupka [**Kupk63**] and Steven Smale [**Sma63**], says that every closed smooth manifold admits a Morse-Smale function. Alternatively see [**BaH04**, Theorem 6.6] for a proof.
   (b) In [**Lau92**, p. 222] (see also [**BaH04**, p. 179]) it is shown that a Morse-Smale function gives rise to a CW-complex that is actually *homeomorphic* to M and such that for each $k \in \mathbb{N}_0$ the critical points of index k are in one-to-one correspondence to the k-cells.
(3) Proposition 104.12 (2) is related to several earlier results:
   (a) In Theorems 64.2 we showed that every compact smooth manifold M admits a finite simplicial structure. By Theorem 64.5 this implies in particular that M admits a finite CW-structure.
   (b) In Corollary 63.37 we used the Nerve Theorem 63.32 to show that every compact smooth manifold is weakly homotopy equivalent to a finite CW-complex.
   (c) Finally in Corollary 85.18 we used Borsuk's Theorem 85.3 to show that every topological manifold is homotopy equivalent to a CW-complex.

104.6. **Reeb's Theorem.** Theorem 104.9 also allows us to prove the following theorem which was first proved by Georges Reeb [**Reeb46**] in 1946.

Theorem 104.13. (Reeb's Theorem) *Let M be a closed n-dimensional smooth manifold. If M admits a smooth function $f\colon M \to \mathbb{R}$ with precisely two critical points which are both non-degenerate, then M is* homeomorphic *to S^n.*

Remark. For the record we mention that by [**Miln64b**, Theorem 1'] the conclusion of Theorem 104.13 also holds if the critical points are degenerate.

Reeb's Theorem 104.13 is an immediate consequence of Theorem 104.9 together with the following proposition.

Proposition 104.14. *Let $n \in \mathbb{N}$.*

(1) *If $\varphi\colon S^{n-1} \to S^{n-1}$ is a homeomorphism, then $\overline{B}^n \cup_\varphi \overline{B}^n$ is homeomorphic to S^n.*

(2) *Let M be a closed n-dimensional smooth manifold. If M admits a handle decomposition consisting of a single 0-handle and a single n-handle, then M is* homeomorphic *to S^n.*

Proof.

(1) Let $\varphi\colon S^{n-1} \to S^{n-1}$ be a homeomorphism. We apply the Alexander trick to get an extension of $\varphi\colon S^{n-1} \to S^{n-1}$ to the ball $\overline{B}^n$. More precisely we consider the following map:

$$\begin{aligned} \Phi\colon \overline{B}^n &\to \overline{B}^n \\ x &\mapsto \begin{cases} \|x\| \cdot \varphi(\frac{x}{\|x\|}), & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

By Exercise 3.29 we know that this map Φ is a homeomorphism. Finally we consider the following sequence of homeomorphisms:

$$\begin{aligned} \overline{B}^n_+ \cup_{\varphi\colon S^{n-1} \to S^{n-1}} \overline{B}^n_- &\xrightarrow{\cong} \overline{B}^n_+ \cup_{\mathrm{id}\colon S^{n-1} \to S^{n-1}} \overline{B}^n_- \xrightarrow[\text{from page 199}]{\text{homeomorphism}} S^n. \\ [P] &\mapsto \begin{cases} [\Phi(P)], & \text{if } P \in \overline{B}^n_+, \\ [P], & \text{if } P \in \overline{B}^n_-. \end{cases} \end{aligned}$$

(2) Let M be a closed n-dimensional smooth manifold that M admits a handle decomposition consisting of a single 0-handle and a single n-handle. This means that there exists a diffeomorphism $\varphi\colon S^{n-1} \to S^{n-1}$ such that M is diffeomorphic to[1460] $\overline{B}^n \cup_\varphi \overline{B}^n$. It now follows from (1) that M is homeomorphic to S^n. ■

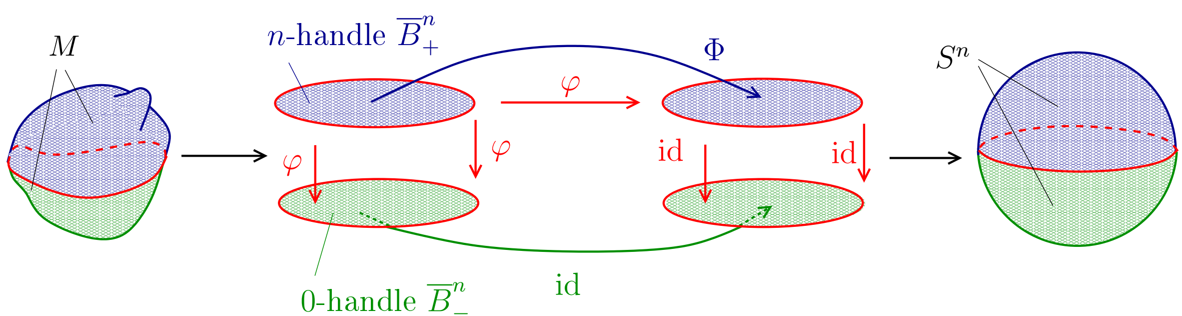


Figure 1489. Illustration for the proof of Theorem 104.13.

Reeb's Theorem 104.13 and Proposition 104.14 raise several fantastic questions. First of all, it is natural to ask whether the analogue of Proposition 104.14 (1) holds if we replace "homeomorphism" by "diffeomorphism". Put differently, we have the following question.

Question 104.15. *Let $\varphi\colon S^{n-1} \to S^{n-1}$ be a diffeomorphism. Is the smooth manifold $\overline{B}^n \cup_\varphi \overline{B}^n$* diffeomorphic *to S^n?*

[1460]Note that by Proposition 8.15 we can equip $\overline{B}^n \cup_\varphi \overline{B}^n$ with a canonical smooth structure.

Next let us reflect whether we can upgrade the composition of maps towards the end of the proof Proposition 104.14 into a diffeomorphism. The good news is that, by Exercise 8.16, we can replace the map on the right-hand side by a diffeomorphism. The bad news is that, as we saw in Exercise 3.29, the map $\Phi\colon B^n \to \overline{B}^n$ given by the Alexander trick is not smooth at the origin, in particular Φ is not diffeomorphism. But perhaps we can pick Φ more cleverly? This leads us to our second question.

Question 104.16. *Let $n \in \mathbb{N}$. Does every diffeomorphism $\varphi\colon S^n \to S^n$ extend to a diffeomorphism of $\overline{B}^{n+1}$?*

In Propositions **??** and **??** we will provide answers to Questions 104.16 and 104.15.

Exercises for Chapter 104.

Exercise 104.1. Show that every Morse function is a height function. More precisely, let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a Morse function. Show that there exists a smooth embedding $\varphi\colon M \to \mathbb{R}^n$ such that $f = h \circ \varphi$ where $h\colon \mathbb{R}^n \to \mathbb{R}$ is the map given by $(x_1, \ldots, x_n) \mapsto x_n$.

Exercise 104.2. Sketch Morse functions on the torus $T = [0,1]^2/\sim$, the real projection plane $\mathbb{RP}^2 = \overline{B}^2/x \sim -x$ and the surface of genus two $E_8/\sim$ such that each Morse function has a single local maximum and a single local minimum.

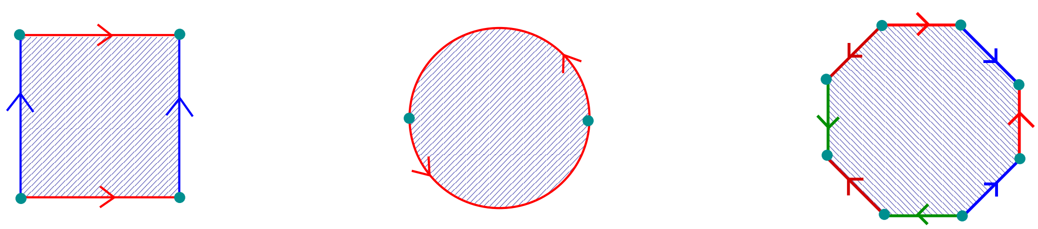


FIGURE 1490. Illustration for Exercise 104.2.

Exercise 104.3. Let $n \in \mathbb{N}_0$. We pick real numbers $c_0 < c_1 < \cdots < c_n$. We consider the following functions on $\mathbb{RP}^n = S^n/\{\pm\,\mathrm{id}\}$ and $\mathbb{CP}^n = S^{2n+1}/S^1$:

$$\begin{array}{rcl} f\colon \mathbb{RP}^n & \to & \mathbb{R} \\ [x_0 : \cdots : x_n] & \mapsto & \sum\limits_{j=0}^{n} c_j \cdot x_j^2 \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{CP}^n & \to & \mathbb{R} \\ [z_0 : \cdots : z_n] & \mapsto & \sum\limits_{j=0}^{n} c_j \cdot |z_j|^2. \end{array}$$

(a) Show that f is a Morse function, that f has precisely $n+1$ critical points and that these are of index $0, 1, \ldots, n$.
(b) Show that g is a Morse function, that g has precisely $n+1$ critical points and that these are of index $0, 2, \ldots, 2n$.

Remark. You might want to use the explicit charts given in Exercise 6.22 and Lemma 12.5.

Exercise 104.4. Let M be a smooth manifold and let $f\colon M \to \mathbb{R}$ be a Morse function. Furthermore let $\lambda\colon M \to \mathbb{R}$ be a smooth function such that $\mathrm{D}\,\lambda_P \neq 0$ for every $P \in M$. Show that $\lambda \cdot f\colon M \to \mathbb{R}$ is also a Morse function and that the critical points, and the corresponding indices, of λf agree with the critical points, and the corresponding indices, of λ.

Exercise 104.5. Let M and N be two closed smooth manifolds and let $f\colon M \to \mathbb{R}$ and $g\colon N \to \mathbb{R}$ be Morse functions.

(a) Let $a, b \in \mathbb{R}$ with $a > \max(|f|)$ and $b > \max(|g|)$. Show that the map

$$\begin{array}{rcl} f \cdot g\colon M \times N & \to & \mathbb{R} \\ (x, y) & \mapsto & (a + f(x)) \cdot (b + g(y)) \end{array}$$

is a Morse function.

(b) Determine the critical points, and their indices, of the Morse function of $f \cdot g$ in terms of the critical points, and their indices, of f and g.

Exercise 104.6. Let M and N be two closed n-dimensional smooth manifolds and let $f\colon M \to \mathbb{R}$ and $g\colon N \to \mathbb{R}$ be Morse functions with unique critical points of index 0 and n. Show that the connected sum $M\#N$ admits a Morse function φ with unique critical points of index 0 and n and such that for $k \in \{1, \ldots, n-1\}$ we have

$$\begin{array}{rl} \text{number of critical points of } \varphi\colon M\#N \to \mathbb{R} = & \text{number of critical points of } f\colon M \to \mathbb{R} \\ & + \text{ number of critical points of } g\colon N \to \mathbb{R}. \end{array}$$

Exercise 104.7. Let M be a smooth manifold, let $f\colon M \to \mathbb{R}$ be a smooth function and let $P \in M \setminus \partial M$ be a non-degenerate critical point of f. Let U be a compact neighborhood of $P \in M \setminus \partial M$ which contains no other critical point. Show that there exists an $\epsilon > 0$ such that for every $\mu \in [0, \epsilon]$ there exists a smooth function $g\colon M \to \mathbb{R}$ with the following four properties:

(1) The function g agrees with f outside of U.
(2) P is a critical point of g with $\operatorname{index}(g, P) = \operatorname{index}(f, P)$.
(3) $g(P) = f(P) + \mu$.
(4) P is the only critical point of g in U.

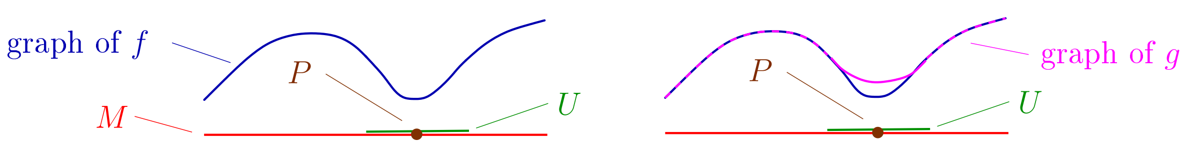


FIGURE 1491. Illustration of Exercise 104.7.

105. Morse theory II: The Relative Handle Decomposition Theorem

In Proposition 104.6 we showed that every *closed* smooth manifold admits a Morse function. Using Theorem 104.9 this allowed us to prove the Handle Decomposition Theorem 104.11 which says that every *closed* smooth manifold admits a handle decomposition.

As we remarked on page 2504, it is not difficult to prove, using Proposition 104.6, that in fact every *compact* smooth manifold M admits a Morse function $f\colon M \to \mathbb{R}$. But this is not enough to prove the Handle Decomposition Theorem 104.11 for compact smooth manifolds with *non-empty boundary* since Theorem 104.9 has the hypothesis that $f^{-1}([a,b]) \cap \partial M = \varnothing$.

In this chapter we will see that compact smooth manifolds have in fact a handle decomposition. In fact we will do more, namely we will develop the theory of relative handle decompositions and we will study more refined versions of Morse functions. As we will see on several occasions, this extra effort for smooth manifolds with boundary will also come in handy in the study of *closed* smooth manifolds.

105.1. **Relative Handle Decompositions.** For smooth manifolds with non-empty boundary it will be convenient to study relative handle decompositions. This concept generalizes the notion of a handle decomposition that we introduced on page 2477.

Definition. Let M be a compact n-dimensional smooth manifold and let A be a (possibly empty) union of components of ∂M.

(1) A *handle decomposition for M rel A* is a diffeomorphism
$$\Theta\colon M \xrightarrow{\cong} ([0,1]\times A) \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_m} h^{k_m}$$
where for $i = 0,\dots,m$ the map φ_i is a thickened (k_i-1)-sphere in the boundary of the smooth manifold $([0,1]\times A)\cup_{\varphi_0} h^{k_0} \cup\cdots\cup_{\varphi_{i-1}} h^{k_{i-1}}$ and where $\Theta^{-1}\colon [0,1]\times A \to M$ is a collar of A.[1461]

(2) The notion of a *standard handle decomposition rel A* is defined the same way as we did for the case $A = \varnothing$ on page 2481.

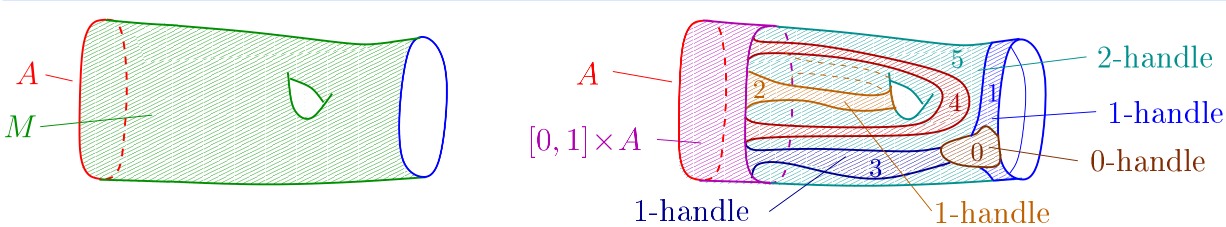


Figure 1492

The following proposition is basically a generalization of Proposition 103.5.

Proposition 105.1. *Let M be a compact n-dimensional smooth manifold. Furthermore let $\partial M = A \sqcup B$ be a decomposition of ∂M into two disjoint unions of components A*

[1461]We refer to page 351 for the definition of a collar.

and B. Finally let

$$M \;=\; ([0,1]\times A)\cup_{\varphi_0} \underbrace{h^{k_0}}_{\text{0-th handle}} \cup\cdots\cup_{\varphi_m} \underbrace{h^{k_m}}_{m\text{-th handle}}$$

be a handle decomposition of M rel A. We consider

$$M' \;:=\; (M\setminus([0,1)\times A))\cup_{B=B\times\{0\}}([0,1]\times B)$$

and we set $A' := \{1\}\times A\subset M'$ and $B' := \{1\}\times B\subset M'$.

(0) *There exists a diffeomorphism from M to M' that restricts to the obvious diffeomorphism $A\to A'$ and $B\to B'$.*

There exists canonical handle decomposition

$$M' \;=\; ([0,1]\times B')\cup_{\psi_m} \underbrace{\widetilde{h}^{n-k_m}}_{\text{0-th dual handle}} \cup\cdots\cup_{\psi_0} \underbrace{\widetilde{h}^{n-k_0}}_{m\text{-th dual handle}},$$

of M' rel B' called dual handle decomposition*, with the following properties:*

(1) *For each $i\in\{0,\dots,m\}$ the following statements hold:*
  (a) *The i-th handle of the dual handle decomposition equals, as a subset of $M\cap M'$, the $(m-i)$-th handle of the original handle decomposition.*
  (b) *The indices of the handles are reversed, in the sense that*

$$\begin{matrix}\textit{index of the } i\textit{-th handle of}\\ \textit{the dual handle decomposition}\end{matrix} \;=\; n - \begin{matrix}\textit{index of the } (m-i)\textit{-th handle of}\\ \textit{the original handle decomposition.}\end{matrix}$$

  In particular the number of i-handles in the dual handle decomposition equals the number of $(n-i)$-handles in the original decomposition.
  (c) *The roles of attaching spheres and belt spheres is reversed in the sense that*

$$\begin{matrix}\textit{belt sphere of the } i\textit{-th handle of}\\ \textit{the dual handle decomposition}\end{matrix} \;=\; \begin{matrix}\textit{attaching sphere of the } (m-i)\textit{-th handle}\\ \textit{of the original handle decomposition.}\end{matrix}$$

  Furthermore the same statement, with the roles of attaching and belt spheres reversed, also holds.

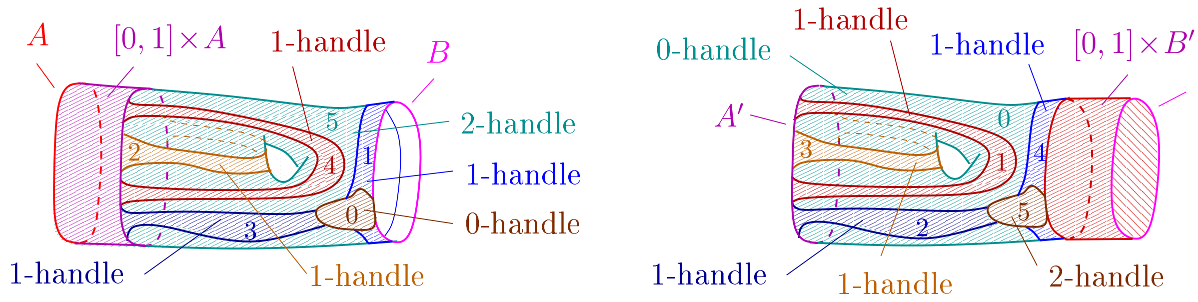
(2) *If the original handle decomposition is standard, then so is the dual handle decomposition.*

(3) *The dual of the dual handle decomposition is the original handle decomposition.*

Example. In Figure 1493 we show again M, the torus minus two open disks, with two boundary components A and B. To the left we show the above handle decomposition of M rel A. To the right we see the corresponding dual handle decomposition of M' rel B'.

In the proof of Proposition 105.1 we will make use of the following lemma, which also doubles as Exercise 8.13.

Lemma 105.2. *Let M be a compact smooth manifold and let C be a union of components of ∂M. Let $\varphi\colon C\to C$ be a diffeomorphism. Given any neighborhood U of $C\subset M$ there*



standard handle decomposition of M rel A dual handle decomposition of M' rel B'

FIGURE 1493. Illustration of Proposition 105.1.

exists a diffeomorphism

$$M \quad \to \quad \underbrace{(M \sqcup ([0,1] \times C))/\sim}_{\text{smooth manifold by Proposition 8.15}} \qquad \text{where } x \sim \varphi(x,0) \text{ for } x \in C$$

that is the identity on the common subset $M \setminus U$ and that restricts to the obvious diffeomorphism from C to $\{1\} \times C$.

Remark. In the setting of Lemma 105.2, we sometimes refer to $[0,1] \times C$ as an *external collar*.

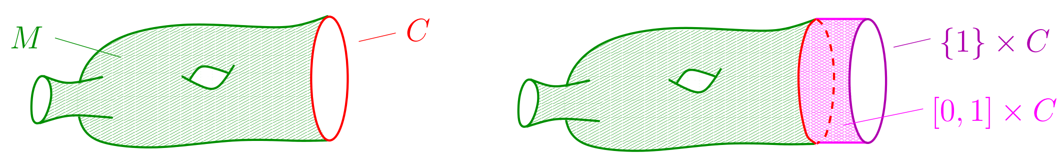


FIGURE 1494. Illustration for Lemma 105.2.

PROOF OF LEMMA 105.2. As is explained in Proposition 8.15 (1), to say that the topological space $(M \sqcup ([0,1] \times C))/\sim$ is a smooth manifold we actually need to pick a collar $[-1,0] \times C$ with $C = \{0\} \times C$ to define the actual smooth atlas. Since U is a neighborhood of C and since C is compact there exists an $\epsilon > 0$ such that $[-\epsilon, 0] \times C \subset U$. Using Lemma 6.13 one can easily show that there exists a smooth function $\eta\colon [-\epsilon, 0] \to [-\epsilon, 1]$ with $\eta(t) = t$ for $t \in [-\epsilon, -\frac{\epsilon}{2}]$, $\eta(0) = 1$ and such that $\eta'(t) > 0$ for all t. The desired diffeomorphism is now given by the identity on $M \setminus ([-\epsilon, 0] \times C)$ and by stretching $[-\epsilon, 0] \times C$ to $[-\epsilon, 1] \times C$ via the map $(t, P) \mapsto (\eta(t), P)$. ■

PROOF OF PROPOSITION 105.1.

(0) It follows from Proposition 8.2 that $W := M \setminus ([0,1) \times A)$ is a smooth manifold. It follows from Lemma 105.2 that both $M = ([0,1] \times A) \cup W$ and $M' = W \cup ([0,1] \times B)$ are diffeomorphic to W.

(1)-(3) The proof of the remaining statements is basically identical to the proof of Proposition 103.5. We only need a minuscule generalization of Lemma 103.6, namely instead of considering diffeomorphisms $\gamma\colon \partial Y \to \partial(X \cup_\varphi h^k)$ we also need to allow diffeomorphisms $\gamma\colon C \to D$ where C is a component of ∂Y and D is a component of $\partial(X \cup_\varphi h^k)$ that contains $h^k \cap \partial(X \cup_\varphi h^k)$. We leave it to the reader to fill in the details. ■

The following proposition says that we can "stack" relative handle decompositions.

Proposition 105.3. *Let M be a compact n-dimensional smooth manifold and let N be a compact n-dimensional smooth manifold. Furthermore let A be a union of boundary components of M and let B be a union of boundary components of N. Next let*

$$\begin{array}{rrcl} & M & = & ([0,1] \times A) \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_r} h^{k_r} \\ \text{and} & N & = & ([0,1] \times B) \cup_{\psi_0} h^{l_0} \cup \cdots \cup_{\psi_s} h^{l_s} \end{array}$$

be handle decompositions rel A respectively rel B. Finally let $\Theta \colon B \to \partial M \setminus A$ be a diffeomorphism. We consider the smooth manifold $M \cup_\Theta N$. There exists a handle decomposition of the form

$$M \cup_\Theta N \xrightarrow{\cong} ([0,1] \times A) \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_r} h^{k_r} \cup_{\widetilde{\psi}_0} h^{l_0} \cup \cdots \cup_{\widetilde{\psi}_s} h^{l_s}$$

rel A, where the attaching maps $\widetilde{\psi}_0, \ldots, \widetilde{\psi}_s$ are obtained in a hopefully obvious way from the attaching maps $\psi_0, \ldots, \psi_s$.

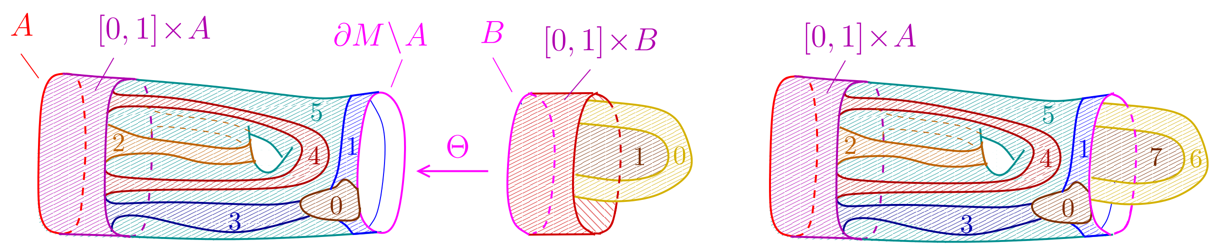


FIGURE 1495. Illustration of Proposition 105.3.

PROOF. We have the following diffeomorphisms:

$$\begin{aligned} M \cup_\Theta N &= M \cup_\Theta (([0,1]\times B) \cup_{\psi_0} h^{l_0} \cup \cdots \cup_{\psi_s} h^{l_s}) = (M \cup_\Theta ([0,1]\times B)) \cup_{\psi_0} h^{l_0} \cup \cdots \cup_{\psi_s} h^{l_s} \\ &\underset{\substack{\uparrow \\ \text{Lemma 105.2}}}{\cong} M \cup_{\widetilde{\psi}_0} h^{l_0} \cup \cdots \cup_{\widetilde{\psi}_s} h^{l_s} = (([0,1]\times A) \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_r} h^{k_r}) \cup_{\widetilde{\psi}_0} h^{l_0} \cup \cdots \cup_{\widetilde{\psi}_s} h^{l_s}. \end{aligned}$$

■

Examples.

(1) Let M be a compact smooth manifold with a handle decomposition

$$M = \varnothing \cup_{\varphi_0} h^{k_0} \cup \cdots \cup_{\varphi_r} h^{k_r}.$$

We can view this as a handle decomposition rel $\varnothing$. By Proposition 105.1 we can equip M with the dual handle decomposition, which is de facto a handle decomposition rel ∂M. If we glue two copies of M along the common boundary, i.e. if we consider the double $\mathrm{D}\,M$, then it follows from Proposition 105.3 that we can stack the original handle decomposition and the dual decompositions. In summary we obtain a handle decomposition for the double $\operatorname{D} M$ of the form

$$\operatorname{D} M \;=\; \varnothing \cup_{\varphi_0} h^{k_0} \cup \dots \cup_{\varphi_r} h^{k_r} \;\cup_{\widetilde{\psi}_r} h^{n-k_r} \cup \dots \cup_{\widetilde{\psi}_0} h^{n-k_0}.$$

(2) First let $k, l \in \mathbb{N}$.

(a) In Exercise 103.2 we showed that $\overline{B}^k \times S^l$ admits a handle decomposition rel $\varnothing$ consisting of one 0-handle and one l-handle.

(b) By (a) together with Proposition 105.1 we see that $\overline{B}^k \times S^l$ admits a handle decomposition rel $\partial(\overline{B}^k \times S^l)$ consisting of one $(k+l)$-handle and one k-handle.

Now let $m, n \in \mathbb{N}$. In Lemma 27.1 we showed that S^{m+n-1} is diffeomorphic to $(S^{m-1} \times \overline{B}^n) \cup_{S^{m-1} \times S^{n-1}} (\overline{B}^m \times S^{n-1})$. Note that by Proposition 105.3 we can stack the handle decomposition of $S^{m-1} \times \overline{B}^n$ rel $\varnothing$ with the handle decomposition of $\overline{B}^m \times S^{n-1}$ rel $\partial(\overline{B}^m \times S^{n-1})$. Thus we see that S^{m+n-1} admits a handle decomposition consisting of four handles of index 0, $m-1$, m and $m+n-1$.

105.2. **Morse functions for smooth manifolds with boundary.** Our goal will be to show the existence of suitable relative handle decompositions. The key tool to doing so is the following generalization of Proposition 104.6.

Proposition 105.4. *Let M be a compact smooth manifold and let A and B be two disjoint unions of components of ∂M. There exists a Morse function $f\colon M \to [0,1]$ such that $A = f^{-1}(\{0\})$ and such that $B = f^{-1}(\{1\})$.*

Example. In Figure 1496 we show a surface M with A and B as in Proposition 105.4. We also show a different embedding of M into $\mathbb{R}^3$ such that the height function $(x, y, z) \mapsto z$ has the desired properties.

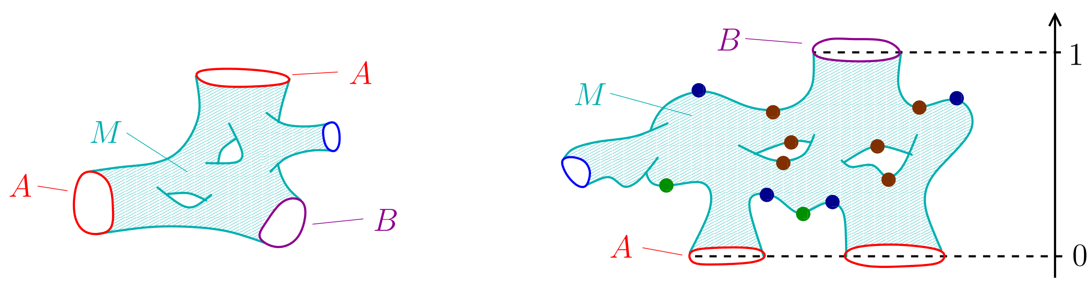


FIGURE 1496

The proof of Proposition 105.4 requires some technical preparations. First let us introduce the following definition.

Definition. Let M be a smooth manifold, let $f\colon M \to \mathbb{R}$ be a smooth function and let A be a union of components of ∂M. We say f *is pointing inwards on* A if for every $P \in A$ and every inward pointing tangent vector[1462] $P \in \operatorname{T}_P M$ we have $\operatorname{D} f_P(v) > 0$. Similarly we are defining that f is *pointing outwards* on A.

[1462] We refer to page 291 for the definition of an inward pointing tangent vector.

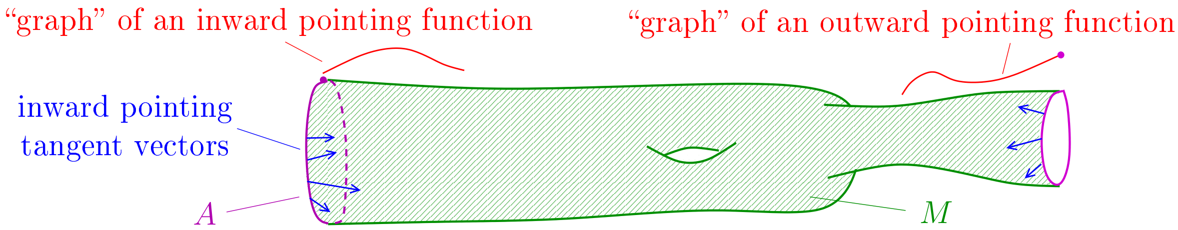


FIGURE 1497

Lemma 105.5. *Let N be a closed smooth manifold and let $f\colon [0,1]\times N\to\mathbb{R}$ be a smooth function that is pointing inwards on $\{0\}\times N$ and that has no critical points. There exists a smooth function $g\colon [0,1]\times N\to\mathbb{R}$ with the following properties:*

1. *The map g agrees with f on $[\frac{1}{2},1]\times N$.*
2. *The map g has no critical points and it is pointing inwards on $\{0\}\times N$.*
3. *The map g is constant on $\{0\}\times N$ and the value on $\{0\}\times N$ equals the minimum of g on $[0,1]\times N$.*

SKETCH OF A PROOF OF LEMMA 105.5. We pick some $s\in\mathbb{R}$ that is less than the minimum of f on $[0,1]\times N$. Using Lemma 6.13 it is pretty straightforward to see that there exist smooth maps $\alpha,\beta\colon[0,1]\to\mathbb{R}$ with $\alpha+\beta\equiv 1$, $\alpha(t)=1$ for sufficiently small t and $\beta(t)=1$ for $t\in[\frac{1}{2},1]$ such that the map

$$\begin{array}{rcl} g\colon [0,1]\times N & \to & \mathbb{R} \\ (t,P) & \mapsto & \alpha(t)\cdot(s+t)+\beta(t)\cdot f(P,t) \end{array}$$

has all the desired properties. We leave it to the reader to fill in the dull technical details. ■

SKETCH OF A PROOF OF PROPOSITION 105.4. Let M be a compact smooth manifold and let A and B be two disjoint unions of components of ∂M. A slight refinement of Proposition 11.8 shows that we can view M as a submanifold of $\{x\in\mathbb{R}^n\,|\,\|x\|\in[1,2]\}$ such that $\varphi(A)\subset S^{n-1}_1$ and such that $\varphi(B)\subset S^{n-1}_2$. Using the Collar Neighborhood Theorem 8.12 we obtain collar neighborhoods $[0,1]\times A$ and $[0,1]\times B$.

As in the proof of Proposition 104.5 one can show that given any $\epsilon>0$ there exists a $Q\in\overline{B}^n_\epsilon$ such that

$$\begin{array}{rcl} f_Q\colon M & \to & \mathbb{R} \\ x & \mapsto & \|x-Q\|^2 \end{array}$$

is a Morse function. It is straightforward to see that by picking a sufficiently small $\epsilon>0$ we can arrange that $f_Q\colon M\to\mathbb{R}$ is pointing inwards on A and that it is pointing outwards on B. Thus, Using Lemma 105.5, applied to $[0,1]\times A$ and $[0,1]\times B$ we can modify f_Q to obtain a Morse function on M that is constant on A and that is constant on B and such that all other values lie between the values on A and B. After rescaling we have obtained the desired Morse function.

A more detailed and authoritative proof of Proposition 105.4 is given in [**Miln65b**, Theorem 2.5]. ■

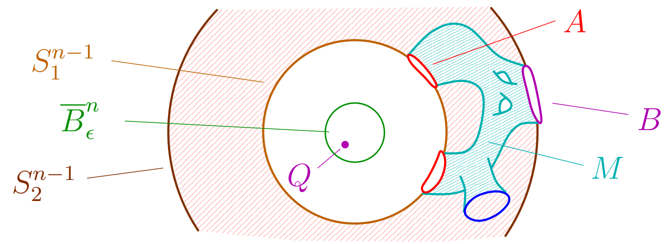


FIGURE 1498. Illustration of the proof of Proposition 105.4.

105.3. **The Relative Handle Decomposition Theorem.** Our goal in this section is to prove the following refinement of the Handle Decomposition Theorem 104.11.

Theorem 105.6. (Relative Handle Decomposition Theorem) *Let M be a compact n-dimensional smooth manifold. Given any union A of components of ∂M there exists a handle decomposition of M rel A.*

If we apply the Relative Handle Decomposition Theorem 105.6 to $A = \varnothing$ we obtain for free the following generalization of the Handle Decomposition Theorem 104.11.

Theorem 105.7. (Handle Decomposition Theorem) *Every compact smooth manifold admits a standard handle decomposition.*

PROOF OF THE RELATIVE HANDLE DECOMPOSITION THEOREM 105.6. Let M be a compact smooth manifold and let A be a union of components of ∂M. We denote by B the union of the remaining components of ∂M. By Proposition 105.4 there exists a Morse function $f\colon M \to [0,1]$ such that $A = f^{-1}(\{0\})$ and such that $B = f^{-1}(\{1\})$. We make the following two observations:

(a) Since $\partial M = A \cup B$ we have $f^{-1}((0,1)) \cap \partial M = \varnothing$.
(b) Note that $A \cup B = \partial M$. Thus, by definition of a Morse function, A and B do not contain critical points, in other words, 0 is not a critical value and 1 is not a critical value.

By Corollary 104.4 (2) the Morse function has only finitely many critical points. We order the critical values by $0 < \lambda_1 < \cdots < \lambda_s < 1$. (Note that by (b) we know that $0 < \lambda_1$ and $\lambda_s < 1$.) We pick real numbers $0 < \epsilon < \mu_0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{s-1} < \lambda_s < 1 = \mu_s$.

Note that it follows from (a) and the Regular Value Theorem 6.53 that $f^{-1}(\{\epsilon\})$ is a closed submanifold of M. Thus it follows from the Smooth Manifold Product Theorem 104.8 that $M_\epsilon = f^{-1}([0,\epsilon])$ is diffeomorphic to $[0,1] \times A$.

Next note that it follows from Theorem 104.9 and (a) that for each $i \in \{1,\dots,s\}$ there exists a diffeomorphism from M_{μ_i} to the result of attaching finitely many handles to $M_{\mu_{i-1}}$. Furthermore, Theorem 104.9 says that we can find such a diffeomorphism that is the identity on the common subset M_ϵ. This shows that $M = M_{\mu_s}$ admits a handle decomposition rel A.

Finally note that it follows immediately from iteratively applying Proposition 103.3 that we can rearrange our handle attachments to obtain a standard handle decomposition of M rel A. ■

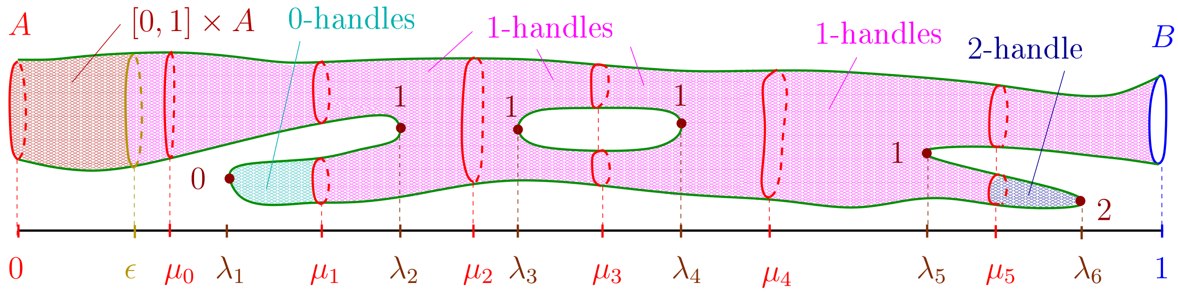


FIGURE 1499. Illustration for the proof of Theorem 105.6.

We conclude this section with the following generalization of Proposition 104.12.

Proposition 105.8. *Every compact smooth manifold is homotopy equivalent to a finite* CW*-complex.*

PROOF. This statement follows immediately from the Handle Decomposition Theorem 105.7 together with Proposition 103.18. ■

105.4. **Constructing Morse functions from handle decompositions.** By the Handle Addition Theorem 104.9 we know that Morse functions give rise to handle decompositions. In this section we will see that the we can also go backwards. For example we have the following converse to Theorem 104.9.

Proposition 105.9. *Let M be a compact smooth manifold and suppose we are given a decomposition $\partial M = A \sqcup B$ where A and B are unions of components of ∂M.*

(1) *Let $\varphi\colon \overline{B}^{n-k} \times S^{k-1} \to B$ be a thickened $(k-1)$-sphere and let $f\colon M \to \mathbb{R}$ be a map that points inwards on A and that points outwards on B. Then there exists a smooth map $g\colon M \cup_\varphi h^k \to \mathbb{R}$ with the following properties:*
   (a) *The restriction of g to M equals f.*
   (b) *g points outwards on $\partial(M \cup_\varphi h^k) \setminus A$.*
   (c) *In the handle $\overline{B}^{n-k} \times \overline{B}^k$ the map g has precisely one critical point, and this critical point has index k.*
(2) *If M is equipped with a handle decomposition $M \cong ([0,1] \times A) \cup_{\varphi_1} h^{k_1} \cup \cdots \cup_{\varphi_m} h^{m_k}$ rel A, then there exists a smooth map $g\colon M \to \mathbb{R}$ with the following properties:*
   (a) *On $[0,1] \times A$ the map g equals the projection onto $[0,1]$,*
   (b) *The map g has precisely m critical points $P_1, \ldots, P_m$, each of which is non-degenerate and such that for $i = 1, \ldots, m$ the index of the critical point P_i equals the index of the handle φ_i.*

PROOF.

(1) This statement is proved by running the proof of Theorem 104.9 "backwards". In particular the idea is, to define g on the k-handle to be a suitable modification of the

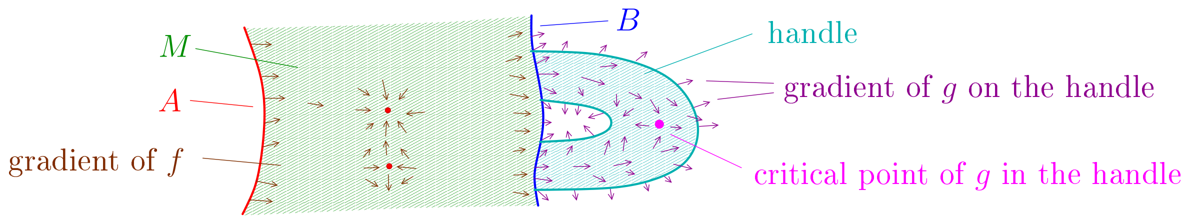


FIGURE 1500. Illustration of Proposition 105.9.

function

$$\begin{array}{rcl} \overline{B}^k \times \overline{B}^{n-k} & \to & \mathbb{R} \\ (x_1,\dots,x_k,y_1,\dots,y_{n-k}) & \mapsto & -x_1^2 - \dots - x_k^2 + y_1^2 + \dots + y_{n-k}^2. \end{array}$$

We refer to [**Kos93**, p. 130] for the details.

(2) This statement follows immediately from applying (1) altogether m times. ■

We conclude this section with a neat refinement of Proposition 104.6. To formulate the result we introduce the following definition.

Definition. Let M be a compact n-dimensional smooth manifold and suppose we are given a decomposition $\partial M = A \sqcup B$ where A and B are unions of components of ∂M. A Morse function $f\colon M \to \mathbb{R}$ is called *self-indexing with respect to A and B*, if the following conditions are satisfied.

(1) We have $f|_A \equiv -\frac{1}{2}$ and $f|_B \equiv n + \frac{1}{2}$.
(2) If P is a critical point of f, then $f(P) = \operatorname{index}(f, P)$.

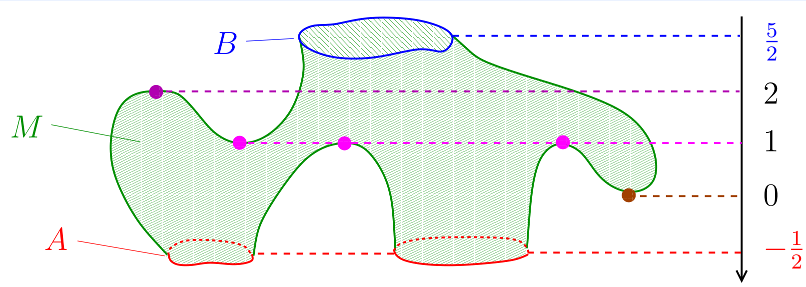


FIGURE 1501. Illustration of Proposition 105.10.

Proposition 105.10. *Let M be a compact n-dimensional smooth manifold and suppose we are given a decomposition $\partial M = A \sqcup B$ where A and B are unions of components of ∂M. There exists a self-indexing Morse function $f\colon M \to \mathbb{R}$ with respect to A and B.*

SKETCH OF PROOF. Let M be a compact n-dimensional smooth manifold and suppose we are given a decomposition $\partial M = A \sqcup B$ where A and B are unions of components of ∂M. By the Handle Decomposition Theorem 104.11 there exists a standard handle decomposition

$$M = ([0,1]\times A) \underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{0\text{-handles}} \cup \dots \underbrace{\cup_{\varphi_{n,1}} h^n \cdots \cup_{\varphi_{n,r_n}} h^n}_{n\text{-handles}}$$

rel A. Given $i \in \{-1,\dots,n\}$ we denote by M_i the i-skeleton of the relative handle decomposition and we set $B_i := \partial M_i \setminus A$. It suffices to prove the following claim.

Claim. For $i = -1,\dots,n$ there exists a Morse function f_i on M_i with the following properties:

(1) We have $f|_A \equiv -\frac{1}{2}$ and $f|_{B_i} \equiv i + \frac{1}{2}$.
(2) If P is a critical point of f_i, then $f_i(P) = \operatorname{index}(f_i, P)$.

For $i = -1$ we have $M_i = [0,1] \times A$ and we choose the map that is given by $(t,a) \mapsto t - \frac{1}{2}$. Now suppose the claim holds for some $i \in \{-1,\dots,n-1\}$. By Proposition 105.9 we can extend the map $f_i \colon M_i \to \mathbb{R}$ to a Morse function on $g \colon M_{i+1} \to \mathbb{R}$ such that g has precisely r_{i+1} extra critical points, and each of these critical points has index $i+1$. Note that the value of g on these critical points is necessarily greater than $i + \frac{1}{2}$ and that the value of g on each $S^{n-k-1} \times B^k$ is greater than $i + \frac{1}{2}$. By rescaling, see Exercise 104.4, we can arrange that the value of g at each of these critical points is precisely $i+1$ and we can arrange that the value of g on each $S^{n-k-1} \times B^k$ is greater than $i + \frac{1}{2}$. The values of g on B_i now lie in $[i+\frac{1}{2}, i+\frac{3}{2})$ and g points outwards on B_i. Using a collar neighborhood for B_i together with Lemma 105.5 it is now fairly straightforward to arrange that the map g actually equals $i + \frac{3}{2}$ on B_i. As always, the adventurous reader is warmly invited to fill in the details. ⊞

For full details we refer to the original proof by Stephen Smale [**Sma61b**, Theorem C]. Alternatively the statement is also proved in [**Miln65b**, p. 45] and [**Nic11**, Chapter 2.4]. ■

105.5. **Handle decompositions of topological manifolds.** The notion of a handle decomposition also makes sense, with a few suitable modifications in the definition, for compact topological manifolds. So the question arises, does every compact topological manifold admit a handle decomposition? It turns out that this is a subtle question.

(1) By Theorem **??** every compact topological manifold of dimension ≤ 3 admits a smooth structure. Thus it follows from Theorem 105.7 that every such topological manifold admits a handle decomposition.
(2) Rob Kirby and Larry Siebenmann [**KSi77**, Essay III.2, Theorem 2.1] (see also [**Kupe17b**, Theorem 1.22]) showed that every compact topological manifold of dimension ≥ 6 admits a topological handle decomposition. Later Frank Quinn [**Qu82**, Theorem 2.3.1] extended this result to the 5-dimensional setting.
(3) It follows from Proposition **??** that every closed 4-dimensional topological manifold that admits a topological handle decomposition also admits a smooth structure. But in Theorem 102.9 we saw that there exist closed 4-dimensional topological manifolds that do not admit a smooth structure. In particular there exist closed 4-dimensional topological manifolds that do not admit a handle decomposition.

Exercises for Chapter 105.

Exercise 105.1. In Figure 1502 we show once again M, the torus with two disks removed. Let A be one of the two boundary components. In Figure 1492 we showed that M admits a handle decomposition rel A with one 0-handle, three 1-handles and one 2-handle. Does there exist a handle decomposition of M rel A with fewer handles?

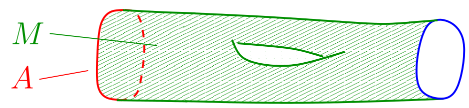


FIGURE 1502. Illustration of Exercise 105.1.

Exercise 105.2. Let M be a compact connected non-empty smooth manifold that is equipped with a handle decomposition. On page 2522 we saw that the double $\mathrm{D}\,M$ comes with a canonical handle decomposition. For which dimensions of M can you use this fact to argue that the inclusion induced map $\pi_1(M) \to \pi_1(\mathrm{D}\,M)$ is an isomorphism?

Exercise 105.3. Give an explicit self-indexing Morse function for the torus $S^1 \times S^1$ and the Klein bottle.

FIGURE 1503. Illustration for Exercise 105.3.

106. Handle homology and the Morse inequalities

In this chapter we will see that a handle decomposition of a compact smooth manifold gives rise to a chain complex, namely the handle chain complex. This chain complex can be viewed as an analogue of the cellular chain complex of a CW-complex. We will see that the corresponding homology groups, called handle homology groups, are naturally isomorphic to the usual singular homology groups. We will use this isomorphism to prove the Morse inequalities, which show that the "size" of singular homology groups gives a lower bound on the number of singularities of a Morse function on a given smooth manifold.

106.1. **Handle homology.** In Chapter 48 we introduced the cellular chain complex of a CW-complex X. Recall that this chain complex has the following two features:

(1) The chain groups are defined to be $\mathrm{C}_k^{\mathrm{CW}}(X) := \mathrm{H}_k(X^k, X^{k-1})$. By Lemma 48.1 this chain group can be viewed as the free abelian groups on the set of k-cells.
(2) The corresponding cellular homology groups are naturally isomorphic to the singular homology groups.

In this section we will outline how one can perform something very similar in the setting of smooth manifolds with a given handle decomposition.

The following lemma is an analogue of Lemma 48.1.

Lemma 106.1. *Let M be a compact n-dimensional smooth manifold together with a standard handle decomposition*

$$M \cong \varnothing \underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{\text{0-handles}} \cup \ldots \underbrace{\cup_{\varphi_{n,1}} h^n \cdots \cup_{\varphi_{n,r_n}} h^n}_{n\text{-handles}}.$$

Let $k \in \mathbb{N}_0$. As on page 2481 we denote by M^k the k-skeleton of the handle decomposition, i.e. M^k is union of all handles of index $\leq k$.[1463]

(1) *Let $k \in \{0, \ldots, n\}$. Given $i \in \{1, \ldots, r_i\}$ we denote by $\Phi_{k,i} \colon \overline{B}_i^k \to \{0\} \times \overline{B}_i^k \to M^k$ the map that is given by the inclusion map of the core of the i-th k-handle. For every $k, j \in \mathbb{N}_0$ the following map is an isomorphism:*[1464]

$$\bigoplus_{i=1}^{r_i} \Phi_{k,i*} \colon \bigoplus_{i=1}^{r_i} \mathrm{H}_j(\overline{B}_i^k, \partial \overline{B}_i^k) \;\longrightarrow\; \mathrm{H}_j(M^k, M^{k-1}).$$

In particular the following holds:
   (a) *For $i \neq k$ we have $\mathrm{H}_i(M^k, M^{k-1}) = 0$.*
   (b) *The group $\mathrm{H}_k(M^k, M^{k-1})$ is a free abelian group where the rank equals the number of k-handles.*

(2) *For $i > k$ we have $\mathrm{H}_i(M^k) = 0$.*
(3) *For $i < k$ the inclusion $j \colon M^k \to M$ induces an isomorphism $j_* \colon \mathrm{H}_i(M^k) \xrightarrow{\cong} \mathrm{H}_i(M)$.*
(4) *The inclusion $j \colon M^k \to M$ induces an epimorphism $j_* \colon \mathrm{H}_k(M^k) \twoheadrightarrow \mathrm{H}_k(M)$.*

[1463] Note that this definition makes sense since the handle decomposition is standard.

[1464] Note that by definition of a standard handle decomposition, see page 2481, the restriction of $\Phi_{k,i}$ to $\partial \overline{B}_i^k$ does indeed take values in M^{k-1}.

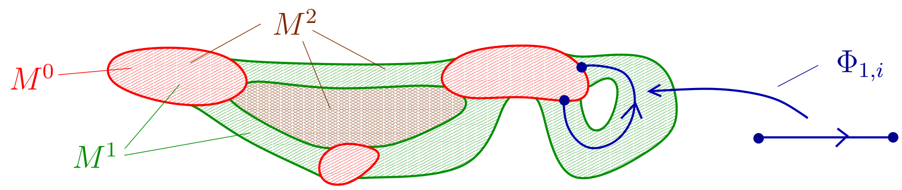


FIGURE 1504. Illustration for Lemma 106.1.

Remark. The statement of Lemma 106.1 (3) and (4) is very similar in spirit to the content of Exercise 103.12.

PROOF. Not surprisingly the proof of the lemma is very similar to the proof of Lemma 48.1. But completeness sake let us go through the motions.

(1) We consider the following diagram

$$\begin{array}{ccc} \bigoplus\limits_{i=1}^{r_i} \mathrm{H}_k(\overline{B}_i^k, \partial \overline{B}_i^k) & \xrightarrow{\quad\bigoplus \Phi_{k,i*}\quad} & \mathrm{H}_k(M^k, M^{k-1}) \\ \downarrow & & \uparrow \\ \bigoplus\limits_{i=1}^{r_i} \mathrm{H}_k(\overline{B}^{n-k} \times \overline{B}_i^k, \overline{B}^{n-k} \times \partial \overline{B}_i^k) & \xrightarrow{\bigoplus \Phi_{k,i*}} & \bigoplus\limits_{i=1}^{r_i} \mathrm{H}_k(\Phi_{k,i}(\overline{B}^{n-k} \times \overline{B}_i^k), \Phi_{k,i}(\overline{B}^{n-k} \times \partial \overline{B}_i^k)). \end{array}$$

The left vertical map is induced by the inclusions $x \mapsto (0, x)$. It is an isomorphism by Corollary 43.18. The bottom horizontal map is induced by homeomorphisms, thus it is also an isomorphism. Next note that the right vertical map is induced by the obvious inclusions. It follows almost immediately from the Excision Theorem 89.2 that the right vertical map is an isomorphism. Clearly he diagram clearly commutes. Hence we get that the top horizontal map is indeed an isomorphism. Finally the "in particular" statements now follow immediately from Lemma 45.1.

(2) We fix $i \in \mathbb{N}$. We want to show that $\mathrm{H}_i(M^k) = 0$ for $k = 0, \ldots, i-1$. First of all we have $\mathrm{H}_i(M^0) = 0$ since M^0 is diffeomorphic to a disjoint union of n-balls. Now assume that we already know that $\mathrm{H}_i(M^k) = 0$ for some $k < i-1$. We want to show that $\mathrm{H}_i(M^{k+1}) = 0$. We consider the long exact sequence of the pair (M^{k+1}, M^k):

$$\ldots \to \mathrm{H}_{i+1}(M^{k+1}, M^k) \to \underbrace{\mathrm{H}_i(M^k)}_{=0} \to \mathrm{H}_i(M^{k+1}) \to \underbrace{\mathrm{H}_i(M^{k+1}, M^k)}_{=0 \text{ by (1) since } i>k+1} \to \ldots$$

Thus we obtain that indeed $\mathrm{H}_i(M^{k+1}) = 0$.

(3) We fix $i \in \mathbb{N}_0$. Let $m > i$. This time we consider the following excerpt from the long exact sequence of the pair (M^{m+1}, M^m):

$$\ldots \to \underbrace{\mathrm{H}_{i+1}(M^{m+1}, M^m)}_{=0 \text{ by (1) since } i<m} \to \mathrm{H}_i(M^m) \to \mathrm{H}_i(M^{m+1}) \to \underbrace{\mathrm{H}_i(M^{m+1}, M^m)}_{=0 \text{ by (1) since } i<m} \to \ldots$$

This discussion implies in particular that we have inclusion induced isomorphisms $\mathrm{H}_i(M^k) \xrightarrow{\cong} \mathrm{H}_i(M^{k+1}) \xrightarrow{\cong} \ldots \xrightarrow{\cong} \mathrm{H}_i(M^n) = \mathrm{H}_i(M)$.

(4) In (3) we showed that the inclusion induced map $\mathrm{H}_k(M^{k+1}) \to \mathrm{H}_k(M)$ is an isomorphism. Thus it suffices to show that the inclusion induced map $\mathrm{H}_k(M^k) \to \mathrm{H}_k(M^{k+1})$ is an epimorphism. To show this we consider the following excerpt from long exact sequence of the pair (M^{k+1}, M^k):

$$\ldots \to \mathrm{H}_{k+1}(M^{k+1}, M^k) \to \mathrm{H}_k(M^k) \to \mathrm{H}_k(M^{k+1}) \to \underbrace{\mathrm{H}_k(M^{k+1}, M^k)}_{=0 \text{ by } (1)} \to \ldots$$

We see that the inclusion induced map $\mathrm{H}_k(M^k) \to \mathrm{H}_k(M^{k+1})$ is indeed an epimorphism. ■

Lemma 106.2. *Let M be compact smooth manifold together with a standard handle decomposition. Given $k \in \mathbb{N}_0$ we denote by $d = d_k$ the map*

$$\begin{tikzcd} \mathrm{H}_k(M^k, M^{k-1}) \arrow[r, "\partial_k"] \arrow[rr, bend right, "=:d=d_k"'] & \mathrm{H}_{k-1}(M^{k-1}) \arrow[r, "j_{k-1}"] & \mathrm{H}_{k-1}(M^{k-1}, M^{k-2}) \end{tikzcd}$$

where ∂_k is the connecting map in the long exact sequence of the pair (M^k, M^{k-1}) and where j_{k-1} is the map induced by the projection $\mathrm{C}_{k-1}(M^{k-1}) \to \mathrm{C}_{k-1}(M^{k-1}, M^{k-2})$. These maps have the property that for every $k \in \mathbb{N}_0$ we have $d_k \circ d_{k+1} = 0$.

Proof. This proof is easy. We just need to copy-paste the proof of Lemma 48.2. Thus we consider the following commutative diagram of maps

$$\begin{tikzcd} & \mathrm{H}_k(M^k) \arrow[dr, "j_k"] & & \\ \mathrm{H}_{k+1}(M^{k+1}, M^k) \arrow[ur, "\partial_{k+1}"] \arrow[rr, "d_{k+1}"] & & \mathrm{H}_k(M^k, M^{k-1}) \arrow[rr, "d_k"] \arrow[dr, "\partial_k"'] & & \mathrm{H}_{k-1}(M^{k-1}, M^{k-2}) \\ & & & \mathrm{H}_{k-1}(M^{k-1}) \arrow[ur, "j_{k-1}"'] & \end{tikzcd}$$

The map $\partial_k \circ j_k \colon \mathrm{H}_k(M^k) \to \mathrm{H}_{k-1}(M^{k-1})$ is the composition of two successive maps in the long exact sequence of the pair (M^k, M^{k-1}), i.e. the map $\partial_k \circ j_k$ is the zero map. Since the diagram commutes we obtain that $d_k \circ d_{k+1} = 0$. ■

Definition. Let M be compact smooth manifold together with a standard handle decomposition.

(1) We refer to the map d_k from Lemma 106.2 as the *handle boundary map*.
(2) We write $\mathrm{C}_k^{\mathrm{han}}(M) := \mathrm{H}_k(M^k, M^{k-1})$ and we refer to $(\mathrm{C}_*^{\mathrm{han}}(M), d_*)$ as the *handle chain complex.*[1465]
(3) We denote the homology groups of the handle chain complex by $\mathrm{H}_k^{\mathrm{han}}(M)$ and we refer to these groups as the *handle homology groups of the smooth manifold M with the given handle decomposition.*

[1465]It follows from Lemma 106.2 that $(\mathrm{H}_k(M^k, M^{k-1}), d_k)_{k\in\mathbb{N}_0}$ is indeed a chain complex.

In Proposition 48.4 we showed that for a CW-complex the singular and cellular homology groups are naturally isomorphic. The following proposition proves the analogous statement in our present setting.

Proposition 106.3. *Let M be compact smooth manifold together with a standard handle decomposition. Given any $k \in \mathbb{N}_0$ there exists a uniquely determined natural*[1466] *isomorphism*

$$\Phi_M \colon \mathrm{H}_k(M) \xrightarrow{\cong} \mathrm{H}_k^{\mathrm{han}}(M)$$

with the property that the following diagram commutes:[1467]

$$\begin{tikzcd}
 & \mathrm{H}_k(M) \arrow[dr, "\Phi_M"] & \\
\mathrm{H}_k(M^k) \arrow[ur] \arrow[dr, "j_k"'] & & \mathrm{H}_k^{\mathrm{han}}(M). \\
 & \ker(d_k) \arrow[ur, two heads] &
\end{tikzcd}$$

In particular, in light of the above and Corollary 42.8, the isomorphism type of the handle homology groups depends only on the homotopy type of M.

Proof of Proposition 106.3. The proof of Proposition 106.3 is almost embarrassingly similar to the proof of Proposition 48.4 (1). For the reader's convenience we nonetheless provide an abbreviated version of the argument. As in the proof of Proposition 48.4 (1) we add a couple of maps to the commutative diagram that we had already encountered in the proof of Lemma 106.2:

$$\begin{tikzcd}
 & & & & & 0 & \\
0 \arrow[drr] & & & \mathrm{H}_k(M^{k+1}) \arrow[urr] & & & \\
 & & \mathrm{H}_k(M^k) \arrow[ur, two heads] \arrow[dr, hook, "j_k"] & & & & \\
\mathrm{H}_{k+1}(M^{k+1}, M^k) \arrow[urr, "\partial_{k+1}"] \arrow[rrr, "d_{k+1}"] & & & \mathrm{H}_k(M^k, M^{k-1}) \arrow[rrr, "d_k"] \arrow[dr, "\partial_k"'] & & & \mathrm{H}_{k-1}(M^{k-1}, M^{k-2}). \\
 & & & & \mathrm{H}_{k-1}(M^{k-1}) \arrow[urr, hook, "j_{k-1}"'] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

Recall that by Lemma 106.1 (1) and (2) we know that $\mathrm{H}_k(M^{k+1}, M^k) = 0$, $\mathrm{H}_k(M^{k-1}) = 0$ and $\mathrm{H}_{k-1}(M^{k-2}) = 0$. It follows almost immediately from these facts that the diagonal

[1466] We leave it to the reader to figure out what "natural" should mean in this context.

[1467] Here we again denote by $j_k \colon \mathrm{H}_k(M^k) \to \mathrm{H}_k(M^k, M^{k-1})$ the obvious map. In the long exact sequence of the pair (M^k, M^{k-1}) we see that the image of $\mathrm{H}_k(M^n)$ in $\mathrm{H}_k(M^k, M^{k-1})$ lies in the kernel of the connecting homomorphism ∂_k. This implies that the map $j_k \colon \mathrm{H}_k(M^k) \to \mathrm{H}_k(M^k, M^{k-1})$ takes values in $\ker(d_k)$.

sequences in the above diagram are exact. Next we consider the diagram:

$$\begin{tikzcd}[column sep=huge]
\mathrm{H}_k(M^k)/\operatorname{im}(\partial_{k+1}) \arrow[r, "\text{isomorphism since the left}\atop\text{diagonal sequence is exact}"] \arrow[d, "\cong"', "\text{a diagram chase shows that}\atop j_k\text{ induces an isomorphism}"] & \mathrm{H}_k(M^{k+1}) \arrow[r, "\text{isomorphism by}\atop\text{Lemma 106.1 (3)}"] & \mathrm{H}_k(M). \arrow[d, "=:\Phi_M"] \\
\ker(d_k)/\operatorname{im}(d_{k+1}) \arrow[rr, "\mathrm{id}"'] & & \mathrm{H}^{\mathrm{han}}_k(M)
\end{tikzcd}$$

Next note that by construction the isomorphism Φ_M has the property that the following diagram commutes:

$$\begin{tikzcd}
& \mathrm{H}_k(M) \arrow[dr, "\Phi_M"] & \\
\mathrm{H}_k(M^k) \arrow[ur, two heads] \arrow[dr, "j_k"'] & & \mathrm{H}^{\mathrm{han}}_k(M). \\
& \ker(d_k) \arrow[ur, two heads] &
\end{tikzcd}$$

Note that the top-left diagonal map is an epimorphism by Lemma 106.1 (4). This implies immediately that Φ_M is uniquely determined.

Finally we leave it to the reader who figured out what it means for the isomorphism to be "natural" to actually show that it is natural. ■

In the next section we will discuss a practical approach to calculating the boundary maps in the handle chain complex. But even without this knowledge we can already obtain interesting consequences of Proposition 106.3. For example we can prove the following corollary.[1468]

Corollary 106.4. *Let M be a compact smooth manifold. Given any handle decomposition for M the following equality holds:*

$$\sum_{k\in\mathbb{N}_0}(-1)^k\cdot\textit{number of }k\textit{-handles} \;=\; \chi(M).$$

Proof. Let M be a compact smooth manifold together with a handle decomposition. Given $k\in\mathbb{N}_0$ we denote by c_k the number of k-handles. By Corollary 103.4 we can find a *standard* handle structure on M such that for every $k\in\mathbb{N}_0$ the number of k-handles also equals c_k. Using this new handle decomposition we perform the following calculation:

$$\begin{aligned}
\chi(M) \;&\overset{\text{see definition on page 1367}}{=}\; \sum_{k\in\mathbb{N}_0}(-1)^k\cdot\operatorname{rk}(\mathrm{H}_k(M)) \;\overset{\text{Proposition 106.3}}{=}\; \sum_{k\in\mathbb{N}_0}(-1)^k\cdot\operatorname{rk}(\mathrm{H}^{\mathrm{han}}_k(M))\\
&\underset{\text{Lemma 55.3}}{=}\; \sum_{k\in\mathbb{N}_0}(-1)^k\cdot\operatorname{rk}(\mathrm{C}^{\mathrm{han}}_k(M)) \;\underset{\text{by Lemma 106.1}}{=}\; \sum_{k\in\mathbb{N}_0}(-1)^k\cdot c_k.
\end{aligned}$$

■

[1468] Admittedly we could already have proved that particular corollary earlier on, namely using Lemma 103.13. This was precisely the content of Exercise 103.10.

Next we obtain the following corollary.

Corollary 106.5. *The Euler characteristic of every closed odd-dimensional smooth manifold is zero.*

Remark. In Proposition 88.7 we used Poincaré Duality to prove a generalization of Corollary 106.5 to compact odd-dimensional topological manifolds.

PROOF. Let M be a closed $(2n+1)$-dimensional smooth manifold. By the Handle Decomposition Theorem 104.11 we can equip M with a handle decomposition. By Proposition 103.5 we can also consider the corresponding dual handle decomposition. Now we see that

$$\begin{aligned}\chi(M) &\underset{\text{Corollary 106.4}}{\overset{\downarrow}{=}} \sum_{k=0}^{2n+1}(-1)^k\cdot \begin{array}{c}\text{number of } k\text{-handles in}\\ \text{the handle decomposition}\end{array} \underset{\text{by definition of the dual handle decomposition}}{\overset{\downarrow}{=}} \sum_{k=0}^{2n+1}(-1)^k\cdot \begin{array}{c}\text{number of } (2n+1-k)\text{-handles in}\\ \text{the dual handle decomposition}\end{array}\\ &\underset{\text{substitution } m=2n+1-k}{\overset{\uparrow}{=}} -\sum_{m=0}^{2n+1}(-1)^m\cdot \begin{array}{c}\text{number of } m\text{-handles in the}\\ \text{dual handle decomposition}\end{array} \underset{\text{Corollary 106.4}}{\overset{\uparrow}{=}} -\chi(M).\end{aligned}$$

Thus we see that $\chi(M)=0$. ■

We conclude this section with an even more pleasing corollary.

Corollary 106.6. *Let M be a closed smooth manifold. For every Morse function $f\colon M\to\mathbb{R}$ we have*

$$\sum_{k\in\mathbb{N}_0}(-1)^k\cdot \#\textit{critical points of } f \textit{ of index } k \;=\; \chi(M).$$

Remark.

(1) For $M=S^2$ the statement of Corollary 106.6 implicitly appears already in a paper written by Clerk Maxwell [**Max1870**] in 1870 with the beautiful title "on hills and dales".

(2) In Exercise 106.13 we will see that Corollary 106.6 can also be deduced from the Poincaré-Hopf Theorem 97.5.

PROOF. Let $f\colon M\to\mathbb{R}$ be a Morse function on a closed smooth manifold. It follows from Proposition 104.10 that M admits a handle decomposition such that for each $k\in\mathbb{N}_0$ the number of k-handles equals the number of critical points of index k. The desired equality now follows immediately from Corollary 106.4. ■

106.2. **The boundary maps in the handle chain complex.** The following proposition is a useful tool for calculating the boundary maps in a given handle chain complex.

Proposition 106.7. *Let M be an oriented n-dimensional smooth manifold together with a standard handle decomposition*

$$M\cong\varnothing\,\underbrace{\cup_{\varphi_{0,1}}h^0\cdots\cup_{\varphi_{0,r_0}}h^0}_{\text{0-handles}}\cup\ldots\underbrace{\cup_{\varphi_{n,1}}h^n\cdots\cup_{\varphi_{n,r_n}}h^n}_{n\text{-handles}}.$$

We assume that the inclusion maps of all the handles are orientation-preserving. Given $k \in \mathbb{N}_0$ and given $i \in \{1, \ldots, r_i\}$ we denote by $\Phi_{k,i}\colon \{0\} \times \overline{B}_i^k \to M^k$ the map that is given by the inclusion map of the core of the handle. By Lemma 106.1 we know that

$$\mathcal{B}_k \ := \ \{\Phi_{k,1*}([\{0\} \times \overline{B}^k]), \ldots, \Phi_{k,r_k*}([\{0\} \times \overline{B}^k])\}$$

is a basis of $\mathrm{H}_k^{\mathrm{han}}(M) := \mathrm{H}_k(M^k, M^{k-1})$. Given $j \in \{1, \ldots, r_{k+1}\}$ and $i \in \{1, \ldots, r_k\}$ we have the following equality:[1469]

(i,j)-entry of the matrix representing $d_{k+1}\colon \mathrm{H}_{k+1}^{\mathrm{han}}(M) \to \mathrm{H}_k^{\mathrm{han}}(M)$ with respect to the bases $\mathcal{B}_{k+1}$ and $\mathcal{B}_k$	$=$	*algebraic intersection number in ∂M_k of the attaching sphere of the $(k+1,j)$-handle with the belt sphere of the (k,i)-handle.*

Example. We consider the 3-dimensional smooth manifold M which is shown in Figure 1505. It has three 0-handles, three 1-handles and two 2-handles. Using Proposition 106.7 one sees that the corresponding handle chain complex is isomorphic to the following chain complex:

$$0 \to \begin{matrix}\mathbb{Z}\\ \mathbb{Z}\end{matrix} \xrightarrow{\begin{pmatrix} 0 & -1 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}} \begin{matrix}\mathbb{Z}\\ \mathbb{Z}\\ \mathbb{Z}\end{matrix} \xrightarrow{\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}} \begin{matrix}\mathbb{Z}\\ \mathbb{Z}\\ \mathbb{Z}\end{matrix} \to 0. \qquad \text{Thus we obtain that} \qquad \mathrm{H}_k^{\mathrm{han}}(M) \cong \begin{cases} 0, & \text{if } k \geq 3, \\ \mathbb{Z}, & \text{if } k = 2, \\ \mathbb{Z}, & \text{if } k = 1, \\ \mathbb{Z}^2, & \text{if } k = 0. \end{cases}$$

By Lemma 103.2 (2) we know that the component of M that is given by gluing the first 2-handle to the first 0-handle is diffeomorphic to $\overline{B}^1 \times S^2$. But what we can say about the other component of M? We will answer this question on page **??**.

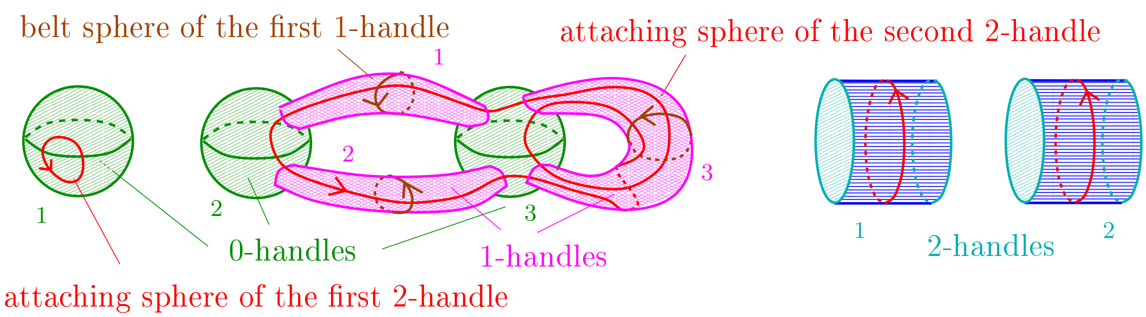


FIGURE 1505

Remark. Let M be an oriented smooth manifold that is equipped with a standard handle decomposition.

(1) If the various attaching spheres and the belt spheres intersect transversally, then we know by Theorem 94.5 that the algebraic intersection number of the attaching spheres and the belt spheres can be determined in a pleasingly geometric way. If

[1469] For the right-hand side note that ∂M_k is a compact oriented $(n-1)$-dimensional smooth manifold, the attaching sphere of the $(k+1,j)$-handle is an oriented k-dimensional sphere in ∂M_k and the belt sphere of the (k,i)-handle is an oriented $(n-k-1)$-dimensional sphere in ∂M_k. So, according to the definition on page 2280, it makes sense to consider in ∂M_k the algebraic intersection number of the two spheres.

they do not intersect transversally, do no despair, since we can iteratively apply the Transversality Theorem 9.10 to find an isotopy of the attaching spheres so that the resulting attaching spheres intersect the belt spheres transversally. By Proposition 106.7 (4) these isotopies do not change the diffeomorphism type of the smooth manifold. Put differently, after possibly fiddling a little bit with the handle decomposition we can compute the handle homology groups using intersection numbers of transverse spheres.

(2) In (1) we just argued that the boundary maps in the handle chain complex can be determined geometrically. In [**DGK19**] this approach is used to define handle homology without referring to singular homology. Furthermore in [**DGK19**] it is shown using the Cerf Theorem **??**, but again without making use of singular homology groups, that the isomorphism types of these geometrically defined handle homology groups only depend on the diffeomorphism type of the underlying smooth manifold. An outline of the argument is also given in [**GoS99**, Chapter 4.2].

In fact this approach, together with Lemma 103.6, can be used to give an alternative proof of some of the statements of the Poincaré Duality Theorem 88.1. We refer to [**Kos93**, Theorem VII.5.1] and [**DGK19**] for details.

Proof. Let M be an oriented n-dimensional smooth manifold together with a standard handle decomposition

$$M \cong \varnothing \underbrace{\cup_{\varphi_{0,1}} h^0 \cdots \cup_{\varphi_{0,r_0}} h^0}_{0\text{-handles}} \cup \ldots \underbrace{\cup_{\varphi_{n,1}} h^n \cdots \cup_{\varphi_{n,r_n}} h^n}_{n\text{-handles}}$$

which has the property that the inclusion maps of all the handles are orientation-preserving. Let $k \in \mathbb{N}_0$, let $j \in \{1, \ldots, r_{k+1}\}$ and let $i \in \{1, \ldots, r_k\}$. We introduce the following notation:

(1) We denote by W the result of attaching to M^{k-1} all k-handles *except* for the i-th k-handle.
(2) We denote by $\alpha\colon \overline{B}^{n-k} \times \overline{B}^k \to M^k$ the embedding corresponding to the (k, i)-handle. Since the handle decomposition is standard we see that $\alpha(\overline{B}^{n-k} \times S^{k-1}) \subset M^{k-1} \subset W$. Furthermore note that $\alpha(S^{n-k-1} \times \{0\})$ is the belt sphere of the (k, i)-handle.
(3) Let $\beta\colon \overline{B}^{n-k-1} \times \overline{B}^{k+1} \to M^{k+1}$ be the embedding corresponding to the $(k+1, j)$-handle. Note that $\beta(\{0\} \times S^k)$ is the attaching sphere of the $(k+1, j)$-handle. Since we are dealing with a standard handle decomposition we see that $\beta(\{0\} \times S^k) \subset \partial M_k$.

We refer to Figure 1506 for an illustration.

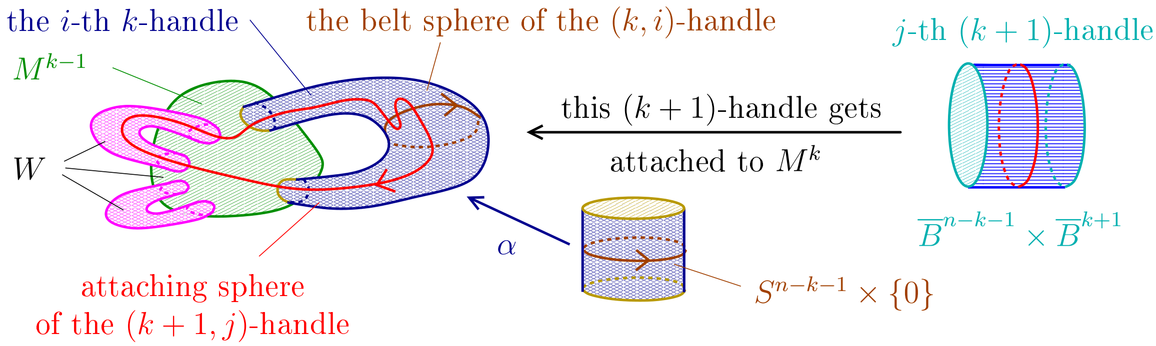


FIGURE 1506. Illustration for the proof of Proposition 106.7.

Next we consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_{k+1}(\{0\}\times\overline{B}^{k+1}, \{0\}\times S^k) \arrow[r, "\beta_*"] \arrow[d, "\partial_{k+1}"'] & \mathrm{H}_{k+1}(M^{k+1}, M^k) \arrow[r, "\partial_{k+1}"] & \mathrm{H}_k(M^k) \arrow[d] \\
\mathrm{H}_k(\{0\}\times S^k) \arrow[r, "\beta_*"] \arrow[d, "\beta_*"'] & \mathrm{H}_k(M^k, W) \arrow[dd, bend left, "h_*^{-1}"] & \mathrm{H}_k(M^k, M^{k-1}) \arrow[l] \\
\mathrm{H}_k(\partial M_k, \partial M_k \setminus \alpha(S^{k-1}\times B^{n-k})) \arrow[ur] & & \\
\mathrm{H}_k(\alpha(S^{n-k-1}\times\overline{B}^k), \alpha(S^{n-k-1}\times S^{k-1})) \arrow[r] \arrow[u, "h_*"', "\cong"] & \mathrm{H}_k(\alpha(\overline{B}^{n-k}\times\overline{B}^k), \alpha(\overline{B}^{n-k}\times S^{k-1})) \arrow[uu, "h_*", "\cong"'] \arrow[d, "\alpha_*^{-1}"] & \\
\mathrm{H}_k(S^{n-k-1}\times\overline{B}^k, S^{n-k-1}\times S^{k-1}) \arrow[r] \arrow[u, "\alpha_*"', "\cong"] \arrow[d, "\sigma\mapsto Q({[S^{n-k-1}\times\{0\}]},\sigma)"] & \mathrm{H}_k(\overline{B}^{n-k}\times\overline{B}^k, \overline{B}^{n-k}\times S^{k-1}) \arrow[d, "\sigma\mapsto\widetilde{Q}({[\overline{B}^{n-k}\times\{0\}]},\sigma)"'] \arrow[dr, "n\cdot{[\{0\}\times\overline{B}^k]}\mapsto n"] & \\
\mathbb{Z} \arrow[r, "\mathrm{id}"'] & \mathbb{Z} \arrow[r, "\mathrm{id}"'] & \mathbb{Z}.
\end{tikzcd}$$

We make the following clarifications and observations:

(1) By a slight abuse of notation we denote all of the following inclusion maps by h:

$$\begin{aligned}
h\colon \alpha(\overline{B}^{n-k}\times\overline{B}^k) &\to M^k \\
h\colon (\alpha(\overline{B}^{n-k}\times\overline{B}^k), \alpha(\overline{B}^{n-k}\times S^{k-1})) &\to (M^k, W) \\
h\colon (\alpha(S^{n-k-1}\times\overline{B}^k), \alpha(S^{n-k-1}\times S^{k-1})) &\to (\partial M_k, \partial M_k \setminus \alpha(S^{k-1}\times B^{n-k})).
\end{aligned}$$

(2) It follows from the Excision Theorem 89.2 that the maps

$$\begin{aligned}
h_*\colon \mathrm{H}_k(\alpha(\overline{B}^{n-k}\times\overline{B}^k), \alpha(\overline{B}^{n-k}\times S^{k-1})) &\to \mathrm{H}_k(M^k, W) \quad \text{and} \\
h_*\colon \mathrm{H}_k(\alpha(S^{n-k-1}\times\overline{B}^k), \alpha(S^{n-k-1}\times S^{k-1})) &\to \mathrm{H}_k(\partial M_k, \partial M_k\setminus\alpha(S^{k-1}\times B^{n-k}))
\end{aligned}$$

are isomorphisms.

(3) We consider the two asymmetric intersection pairings

$$\begin{aligned} Q\colon\ \mathrm{FH}_{n-k-1}(S^{n-k-1}\times\overline{B}^k)\times\mathrm{FH}_k(S^{n-k-1}\times\overline{B}^k,S^{n-k-1}\times S^{k-1}) &\to \mathbb{Z}\\ \widetilde{Q}\colon\ \mathrm{FH}_{n-k}(\overline{B}^{n-k}\times\overline{B}^k,S^{n-k-1}\times\overline{B}^k)\times\mathrm{FH}_k(\overline{B}^{n-k}\times\overline{B}^k,\overline{B}^{n-k}\times S^{k-1}) &\to \mathbb{Z}\end{aligned}$$

as defined on page 2428.

(4) The undecorated maps are the obvious maps induced by inclusions.

(5) It follows from the naturality of the connecting homomorphism that the top rectangle in the diagram commutes.

(6) The regions between the second and the fifth row commute by the functoriality of relative homology groups.

(7) It follows from Proposition 100.17 that the bottom left square commutes.

(8) Finally note that it follows from Proposition 88.2 that $Q([\overline{B}^{n-k}\times\{0\}],[\{0\}\times\overline{B}^k])=1$. It follows from this observation that the bottom right triangle commutes.

We denote by $\Xi\colon\ \mathrm{H}_{k+1}(\{0\}\times\overline{B}^{k+1},\{0\}\times S^k)\to\mathbb{Z}$ the composition of the blue maps, i.e. the composition of the maps on right-hand path. We can now prove the desired equality:

$$\begin{aligned}
&(i,j)\text{-entry of the matrix representing } d_{k+1} \underset{\text{this equality holds basically by definition}}{=} \Xi([\{0\}\times\overline{B}^{k+1}])\\
&\underset{\text{intersection form of } S^{n-k-1}\times\overline{B}^k}{=} Q([S^{n-k-1}\times\{0\}],\alpha_*^{-1}(h_*^{-1}([\beta(\{0\}\times S^k)]))) \underset{\text{intersection form of } \alpha(S^{n-k-1}\times\overline{B}^k)}{=} Q([\alpha(S^{n-k-1}\times\{0\})],h_*^{-1}([\beta(\{0\}\times S^k)]))\\
&\uparrow \text{by the commutative diagram and since by Proposition 87.27 we have } \partial([\{0\}\times\overline{B}^{k+1}])=[\{0\}\times S^k] \qquad \uparrow \text{since } \alpha \text{ is an orientation-preserving embedding}\\
&= Q_{\partial M_k}([\underbrace{\alpha(S^{n-k-1}\times\{0\})}_{\text{belt sphere}}],[\underbrace{\beta(\{0\}\times S^k)}_{\text{attaching sphere}}]).\\
&\uparrow \text{by Proposition 100.19}
\end{aligned}$$

This is precisely the equality that we had intended to show. ■

106.3. **The Morse inequalities.** Given a topological space X or a smooth manifold M we show in this section that the "size" of homology groups gives rise to lower bounds on the number of cells of a CW-structure for X or number of handles in a handle decomposition of M. Furthermore we will see that the homology groups give lower bounds on the number of non-degenerate critical points of a Morse function on M.

The key to proving these results is some charming algebra. To formulate the main algebraic statement we need to introduce the following notation.

Notation. Given a chain complex C_* and given $j\in\mathbb{N}_0$ we denote by $b_j(C)\in\mathbb{N}_0\cup\{\infty\}$ the rank of $\mathrm{H}_j(C)$ and we denote by $q_j(C)\in\mathbb{N}_0\cup\{\infty\}$ the minimal number of generators of the torsion subgroup $\operatorname{Tor}\mathrm{H}_j(C)$.

Proposition 106.8. *Let C_* be a chain complex*[1470] *of finitely generated free abelian groups.*

(a) *Given any $k \in \mathbb{N}_0$ we have the inequality*
$$\sum_{j=0}^{k} (-1)^j \cdot \operatorname{rank}(\mathrm{C}_{k-j}) \;\geq\; q_k(C) + \sum_{j=0}^{k} (-1)^j \cdot b_{k-j}(C).$$
(b) *Given any $k \in \mathbb{N}_0$ we have the inequality*
$$\operatorname{rank}(\mathrm{C}_k) \;\geq\; q_k(C) + q_{k-1}(C) + b_k(C).$$

Remark.

(α) (a) Let $k \in \mathbb{N}_0$. If we add Inequality (a) for k and $k-1$, then we obtain precisely Inequality (b). In other words, Proposition 106.8 (b) is a consequence of Proposition 106.8 (a). In Exercise 106.3 we will see that the converse is not true. Put differently, in general the inequalities one obtains from (a) are stronger than the inequalities from (b).

(b) By adding and subtracting equations one can also show easily that the following two statements are equivalent:

(i) For every $k \in \mathbb{N}_0$ the inequality in (a) is an equality.

(ii) For every $k \in \mathbb{N}_0$ the inequality in (b) is an equality.

(β) We refer to [**Far04**, Chapter 1] for an alternative proof of Proposition 106.8 and for a generalization to chain complexes over more general rings.

The proof of Proposition 106.8 relies on the following elementary lemma.

Lemma 106.9. *If $F \xrightarrow{\varphi} G \to H \to 0$ is an exact sequence where F and G are finitely generated free abelian groups, then*
$$\operatorname{rank}(F) \;\geq\; \text{minimal number of generators of the torsion subgroup of } H.$$

Proof of Lemma 106.9. We write $m = \operatorname{rank}(F)$ and $n = \operatorname{rank}(G)$. Furthermore we set $k := \min\{m, n\}$. Since $\mathbb{Z}$ is a principal ideal domain it follows from the Smith Normal Form Theorem, see Exercise 19.18, that we can find bases for F and G such that the only entries of the corresponding $(n \times m)$-matrix A representing φ are on the diagonal. We denote by $a_1, \ldots, a_k$ these diagonal entries. Then we see that
$$\text{torsion subgroup of } H \;\cong\; \text{torsion subgroup of } \mathbb{Z}^n/A\mathbb{Z}^m \;\cong\; \bigoplus_{a_i \neq 0} \mathbb{Z}_{a_i}.$$
Thus we see that the torsion subgroup of H admits a generating set with at most k elements. Since $k \leq m$ we see that the minimal number of generators of the torsion subgroup of H is $\leq \operatorname{rank}(F)$. ■

Proof of Proposition 106.8.

(a) Let (C_*, ∂_*) be a chain complex of finitely generated free abelian groups. For each $k \in \mathbb{N}_0$ we write $Z_k := \ker(\partial_k \colon C_k \to C_{k-1})$, $B_k := \operatorname{im}(\partial_{k+1} \colon C_{k+1} \to C_k)$ and

[1470] Note that according to our convention on page 1086 a chain complex "starts" at degree 0.

$H_k = Z_k/B_k = \mathrm{H}_k(C_*)$. Basically by definition we have for each $k \in \mathbb{N}_0$ the short exact sequences

$$\text{(a)}\quad 0 \to B_k \to Z_k \to H_k \to 0 \quad\text{and}\quad \text{(b)}\quad 0 \to Z_k \to \mathrm{C}_k \xrightarrow{\partial} B_{k-1} \to 0.$$

Given $k \in \mathbb{N}_0$ we perform the following calculation:

$$\begin{aligned}\sum_{j=0}^{k}(-1)^j\cdot\mathrm{rank}(\mathrm{C}_{k-j}) &\underset{\substack{\uparrow\\ \text{follows from Lemma 19.6 (2) applied to the above two short exact sequences}}}{=} \sum_{j=0}^{k}(-1)^j\cdot(\mathrm{rank}(H_{k-j})+\mathrm{rank}(B_{k-j})+\mathrm{rank}(B_{k-j-1}))\\ &\underset{\substack{\uparrow\\ \text{all the other terms cancel}}}{=} \mathrm{rank}(B_k)+\sum_{j=0}^{k}(-1)^j\cdot\mathrm{rank}(H_{k-j}) \underset{\substack{\uparrow\\ \text{by Lemma 106.9 applied to the exact sequence (a)}}}{\geq} q_k(C)+\sum_{j=0}^{k}(-1)^j\cdot\overbrace{\mathrm{rank}(H_{k-j})}^{=b_{k-j}(C)}.\end{aligned}$$

(b) As mentioned above, this inequality follows immediately from adding the inequality (a) for k and $k-1$. Indeed, one sees that almost all of the terms cancel and thus one obtains the promised inequality. ■

Evidently the idea now is to apply Proposition 106.8 to topology. To do so it is convenient to introduce the following definition.

Notation. Given a topological space X and given $j \in \mathbb{N}_0$ we write[1471]

$$b_j(X) := b_j(\mathrm{C}_*(X)) = \mathrm{rank}(\mathrm{H}_j(X)) \qquad\text{and}\qquad q_j(X) := q_j(\mathrm{C}_*(X)).$$

In other words, $q_j(X)$ is the minimal number of generators of the torsion subgroup $\mathrm{Tor}\,\mathrm{H}_j(X)$.

Proposition 106.10.

(1) *Let M be a compact smooth manifold that is equipped with a handle decomposition. Given $k \in \mathbb{N}_0$ we denote by d_k the number of k-handles in the handle decomposition.*
  (a) *Given any $k \in \mathbb{N}_0$ we have the inequality*
$$\sum_{j=0}^{k}(-1)^j\cdot d_{k-j} \;\geq\; q_k(M)+\sum_{j=0}^{k}(-1)^j\cdot b_{k-j}(M).$$
  (b) *Given any $k \in \mathbb{N}_0$ we have the inequality*
$$d_k \;\geq\; q_k(M)+q_{k-1}(M)+b_k(M).$$

(2) *Let X be a* CW*-complex with finitely many cells in each dimension. Given $k \in \mathbb{N}_0$ we denote by d_k the number of k-cells. The same inequalities as in* (1) *hold.*

Example. Let $n \in \mathbb{N}_0$. By Proposition 48.10 we know that given any $j \in \mathbb{N}_0$ we have

$$\mathrm{H}_j(\mathbb{RP}^n) \;\cong\; \begin{cases}\mathbb{Z}, & \text{if } j=0 \text{ or if } j=n \text{ and } n \text{ is odd},\\ \mathbb{Z}_2, & \text{if } j \text{ is odd and } j<n,\\ 0, & \text{if } j \text{ is even and } 0<j\neq 0 \text{ or if } j>n.\end{cases}$$

[1471] On page 1366 we called $b_j(X)$ the j-th Betti number of X.

It follows almost immediately from this calculation that for any $j \in \{0,\ldots,n\}$ we have $q_j(\mathbb{RP}^n) + q_{j-1}(\mathbb{RP}^n) + q_{j-1}(\mathbb{RP}^n) = 1$. By Proposition 106.10 (1b) this implies that any handle decomposition of $\mathbb{RP}^n$ has at least one handle of each index $0,\ldots,n$. Furthermore by Proposition 106.10 (2b) this calculation implies that any CW-decomposition of $\mathbb{RP}^n$ has at least one cell of each index $0,\ldots,n$. In particular this shows that the CW-structure of $\mathbb{RP}^n$ that we introduced in Lemma 36.1 is "minimal". This observation also gives a solution to Exercise 48.9.

Proof.

(1) Let M be a compact smooth manifold that is equipped with a handle decomposition. Given $k \in \mathbb{N}_0$ we denote by d_k the number of k-handles in the handle decomposition. It follows from Corollary 103.4 that without loss of generality we can assume that the handle decomposition is standard. This allows us to consider the handle chain complex $\mathrm{C}_*^{\mathrm{han}}(M)$. Now we can prove Inequality (a):

$$\sum_{j=0}^{k}(-1)^j\cdot d_{k-j} \underset{\text{by Lemma 106.1 we have } d_i=\operatorname{rank}(\mathrm{C}_i^{\mathrm{han}}(M))}{\overset{\downarrow}{=}} \sum_{j=0}^{k}(-1)^j\cdot\operatorname{rank}(\mathrm{C}_{k-j}^{\mathrm{han}}(M)) \underset{\text{Proposition 106.10}}{\overset{\downarrow}{\geq}} q_k(\mathrm{C}_*^{\mathrm{han}}(M)) + \sum_{j=0}^{k}(-1)^j\cdot b_{k-j}(\mathrm{C}_{k-j}^{\mathrm{han}}(M))$$
$$\underset{\text{by Proposition 106.3 we know that } \mathrm{H}_j(\mathrm{C}_*^{\mathrm{han}}(M))\cong \mathrm{H}_j(M)}{\overset{\uparrow}{=}} q_k(M) + \sum_{j=0}^{k}(-1)^j\cdot b_{k-j}(M).$$

Finally, as we pointed out above, Inequality (b) is an immediate consequence of Inequality (a).

(2) Let X be a CW-complex with finitely many cells in each dimension. The proof in this setting is almost identical to the proof in (1), we just need to work with the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X)$ and we make use of the fact that by Lemma 48.1 we know that $d_i = \operatorname{rank}(\mathrm{C}_i^{\mathrm{CW}}(X))$ and we use that by Proposition 48.4 we know that $\mathrm{H}_j(\mathrm{C}_*^{\mathrm{CW}}(X)) \cong \mathrm{H}_j(X)$. ■

Now we can formulate what is arguably the most interesting result of this section, since it connects two initially rather disparate objects.

Theorem 106.11. (Morse Inequalities) *Let M be a closed smooth manifold together with a Morse function $f\colon M\to\mathbb{R}$. Given $k\in\mathbb{N}_0$ we write*

$$d_k \;:=\; \text{number of critical points of index } k.$$

The following two inequalities hold:

(a) *Given any $k \in \mathbb{N}_0$ we have the inequality*
$$\sum_{j=0}^{k}(-1)^j\cdot d_{k-j} \;\geq\; q_k(M) + \sum_{j=0}^{k}(-1)^j\cdot b_{k-j}(M).$$

(b) *Given any $k \in \mathbb{N}_0$ we have the inequality*
$$d_k \;\geq\; q_k(M) + q_{k-1}(M) + b_k(M).$$

Examples. Let $n \in \mathbb{N}_0$.

(1) Recall that on page 2541 we just showed that for every $j \in \{0, \dots, n\}$ we have $q_j(\mathbb{R}\mathrm{P}^n) + q_{j-1}(\mathbb{R}\mathrm{P}^n) + q_{j-1}(\mathbb{R}\mathrm{P}^n) = 1$. By Theorem 106.11 (b) this implies that any Morse function on $\mathbb{R}\mathrm{P}^n$ has at least one critical point of each index $0, \dots, n$. In particular the Morse function on $\mathbb{R}\mathrm{P}^n$ that we gave on page 2502 has the minimal number of critical points.

(2) On page 1262 we showed that given any $j \in \mathbb{N}_0$ we have

$$\mathrm{H}_j(\mathbb{C}\mathrm{P}^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } j = 0, 2, 4, \dots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

As in (1) it follows that the Morse function on $\mathbb{C}\mathrm{P}^n$ that we gave on page 2502 has the minimal number of critical points.

(3) Let M be a closed connected non-empty n-dimensional smooth manifold. By Theorem 66.6, or alternatively by Theorem 87.1 and Proposition 87.22 we know the following:
(a) If M is orientable, then $\mathrm{H}_n(M) \cong \mathbb{Z}$ and $\mathrm{Tor}(\mathrm{H}_{n-1}(M)) = 0$.
(b) If M is non-orientable, then $\mathrm{H}_n(M) \cong \mathbb{Z}$ and $\mathrm{Tor}(\mathrm{H}_{n-1}(M)) \cong \mathbb{Z}_2$.
Rather charmingly, in either case we see that $q_n(M) + q_{n-1}(M) + b_n(M) = 1$, which matches the fact that any Morse function has to have at least one global maximum, i.e. one critical point of index n.

Proof of Theorem 106.11. Let M be a closed smooth manifold together with a Morse function $f\colon M \to \mathbb{R}$. By Proposition 104.10 we know that M admits a standard handle decomposition of M such that for each $k \in \mathbb{N}_0$ the number of k-handles equals the number of critical points of f of index k. The inequalities now follow from the inequalities of Proposition 106.10 (1). ■

Remark. Let M be a closed smooth manifold and let $f\colon M \to \mathbb{R}$ be a Morse function. Given $j \in \mathbb{N}_0$ we denote by d_j the number of critical points of f of index j. It follows from the Morse inequalities proved in Theorem 106.11, together with the de Rham Theorem 79.6 and Corollary 75.20 (2), that for every $k \in \mathbb{N}_0$ we have the inequality

$$\sum_{j=0}^{k} (-1)^j \cdot d_{k-j} \;\;\geq\;\; \sum_{j=0}^{k} (-1)^j \cdot \dim_{\mathbb{R}}(\underbrace{\mathrm{H}^{k-j}_{\mathrm{dR}}(M)}_{\substack{\uparrow \\ \text{the de Rham cohomology groups of } M}}).$$

This inequality can also proved directly using purely analytic methods, we refer to [**Cha93**, Appendix] and [**Wit82**] for details.

The question arises how "sharp" the Morse inequalities are. This leads us to the following definition.

Definition. Let M be a closed smooth manifold.

(1) A Morse function $f\colon M \to \mathbb{R}$ is called *perfect* if for every $k \in \mathbb{N}_0$ we have

$$\text{number of critical points of index } k \;=\; q_k(M) + q_{k-1}(M) + b_k(M).$$

(2) A handle decomposition is called *minimal* if for every $k \in \mathbb{N}_0$ we have
$$\text{number of handles of index } k \;=\; q_k(M) + q_{k-1}(M) + b_k(M).$$

Remark. Let M be a closed smooth manifold.

(1) Recall that by Propositions 104.10 and 105.9 we can go back and forth between Morse functions and handle decompositions. In particular we see that M admits a perfect Morse function if and only if M admits a minimal handle decomposition.
(2) Note that by the remark on page 2540 we know that if a Morse function is perfect, then the inequalities stated in Theorem 106.11 (a) are also equalities. The same logic applies to the inequalities on the number of handles stated in Proposition 106.10 (1a).

It is now natural to ask which closed smooth manifolds admit perfect Morse functions. It follows from Corollaries 103.12 and 103.7 that the fundamental group gives additional lower bounds on the number of critical points of a Morse function. Thus in the following discussion we restrict ourselves to the case of simply connected smooth manifolds. So now we have a reasonable question:

Question 106.12. *Does every closed simply connected smooth manifold admit a perfect Morse function?*

In fact in 1962 Steven Smale [**Sma62**, Theorem 6.1] proved the following theorem which gives an affirmative answer to Question 106.12 for high-dimensional smooth manifolds.[1472]

Theorem 106.13. *Let M be a closed smooth manifold. If M is simply connected and if* $\dim(M) \geq 6$*, then f admits a perfect Morse function.*

Remark. In [**Shark93**, Chapter VII] Theorem 106.13 gets generalized, in a suitable sense, to certain closed non-simply connected smooth manifolds.

Thus let us turn our gaze towards the low-dimensional case. It follows from the classification of 2-dimensional smooth manifolds, see Theorem 23.4, and the resolution of the Poincaré Conjecture in dimension 3, see Theorem **??**, that Question 106.12 has an affirmative answer in dimensions 2 and 3 for the silly reason that the spheres are the only closed simply connected smooth manifolds in those dimensions. Furthermore for 5-dimensional smooth manifolds it follows from the classification of closed simply connected 5-dimensional smooth manifolds given by Denis Barden in 1965, see [**Bard65**, Corollary 2.2.2], that Question 106.12 has an affirmative answer.

Finally we turn to the 4-dimensional setting. Since low-dimensional topologists usually prefer to think about handles we switch from Morse functions to handles.

Lemma 106.14. *Let M be a closed orientable simply connected 4-dimensional smooth manifold. A handle decomposition is minimal if and only if it has no 1-handles and no 3-handles.*

Proof. Let M be a closed orientable simply connected 4-dimensional smooth manifold. Note that in Exercise 88.19 we saw that it follows easily from the Hurewicz Theorem 52.5,

[1472] We will sketch a proof in Exercise **??**.

Corollary 17.4, the Poincaré Duality Theorem 88.1 and the Universal Coefficient Theorem 75.13 for Cohomology Groups that $\mathrm{H}_1(M) = 0$, $\mathrm{H}_3(M) = 0$ and that $\mathrm{H}_2(M)$ is torsion-free. In particular we see that $q_k(M) + q_{k-1}(M) + b_k(M) = 0$ for $k = 1$ and $k = 3$.

Now we turn to the actual proof of the lemma. First note that the above calculation shows that a minimal handle decomposition has no 1-handles and no 3-handles. Conversely, suppose that we have a handle decomposition with no 1-handles and no 3-handles. It follows easily from Lemma 103.8 that the handle decomposition contains precisely one 0-handle and, say by considering the dual handle decomposition, we see that the handle decomposition contains precisely one 4-handle. Finally note that

$$\#\text{2-handles} \underset{\substack{\uparrow \\ \text{by Corollary 106.4}}}{=} \chi(M) - 2 \underset{\substack{\uparrow \\ \text{since } \mathrm{H}_1(M)=\mathrm{H}_3(M)=0 \\ \text{and since } \mathrm{H}_0(M)=\mathrm{H}_4(M)\cong\mathbb{Z}}}{=} b_2(M) \underset{\substack{\uparrow \\ \text{since } q_1(M)=q_2(M)=0}}{=} q_2(M) + q_1(M) + b_2(M).$$

We have thus shown that the handle decomposition is minimal. ■

This leads us to following open problem, see [**Kir97**, Propblem 4.18].

Question 106.15.

(1) *Does every M closed simply connected 4-dimensional smooth manifold admit a handle decomposition without 1-handles?*
(2) *Does every M closed simply connected 4-dimensional smooth manifold admit a handle decomposition without 1- and 3-handles?*

Note that a positive answer to Question 106.15 (2) would imply, by Reeb's Theorem 104.13 that every closed simply connected 4-dimensional smooth manifold M with $\mathrm{H}_2(M) = 0$ is homeomorphic to S^4. By Freedman's Theorem **??** this statement is correct. But this indicates that in all likelihood it will be very difficult to show that Question 106.15 (2) has a positive answer.

Remark. Chapters 104, 105 and 106 contain almost everything that we want to say about Morse theory. More detailed accounts of Morse theory are given in [**Miln63a, Nic11, Mata02, Lau11, Kos93, Knu15**]. There are also many variations on standard Morse theory. We want to mention two of them:

(1) Instead of smooth maps $M \to \mathbb{R}$ one can also consider smooth maps $M \to S^1$. The notion of a Morse function also makes sense, with basically the same definition, in this setting. The corresponding theory is developed in [**Nov81, Nov82, Paj06, Far04**]. In particular one can use "twisted homology" groups to give lower bounds on the number of critical points of a Morse function $f\colon M \to S^1$. We refer to [**Far04**, Corollary 3.3] and [**Nov81, Nov82**] for details.
(2) Given a CW-complex or an (abstract) simplicial complex Robin Forman [**Form95, Form98**] developed the, initially surely counterintuitive, notion of "discrete Morse theory". We refer to [**Form02, Knu15, Scov19**] and [**Koz08**, Chapter 11] for details.

Exercises for Chapter 106.

Exercise 106.1. Consider the 3-dimensional smooth manifold that is sketched in Figure 1507.

(a) Determine $\pi_1(M)$.
(b) Make an educated guess what well-known smooth manifold M is diffeomorphic to.

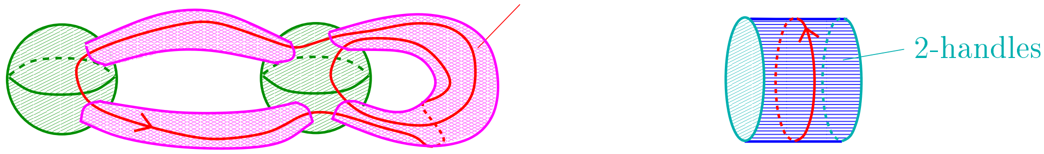


FIGURE 1507. Illustration of Exercise 106.1.

Exercise 106.2.

(a) Let M be a compact orientable connected 2-dimensional topological manifold. Show that $\chi(M) \leq 2$.
Remark. Evidently this follows from the Surface Classification Theorem 23.4. But it would be more elegant to have a proof that does not rely on this result.
(b) Show that given any $k \in \mathbb{Z}_{\leq -2}$ there exists a compact orientable connected 2-dimensional smooth manifold M with $\chi(M) = k$.
(c) Let $n \in \mathbb{N}_{\geq 3}$. Show that given any $k \in \mathbb{Z}$ there exists a compact orientable connected n-dimensional smooth manifold M with $\chi(M) = k$.

Exercise 106.3. Give an example of a chain complex of finitely generated free abelian groups where the inequality of Proposition 106.8 (b) holds for all $k \in \mathbb{N}_0$, but such that the inequality of Proposition 106.8 (a) does not hold for all $k \in \mathbb{N}_0$.

Exercise 106.4. Given a closed connected non-empty smooth manifold M and given a Morse function $f\colon M \to \mathbb{R}$ we denote by $d_k(M, f)$ the number of critical points of f of index k. From the Morse Inequalities of Theorem 106.11 and from Corollaries 103.12 and 103.7 we obtain lower bounds on the number of critical points $d_k(M, f)$ of a Morse function f in terms of the homology groups and the fundamental group of M. Show that these lower bounds are not optimal, i.e. show that there exists a closed connected non-empty smooth manifold M that does not admit a Morse function $f\colon M \to \mathbb{R}$ such that these lower bounds are equalities.
Hint. Apply the Morse Inequalities of Theorem 106.11 to finite covers of a suitable M. For example you could take M to be the mapping torus $\operatorname{Tor}((S^1)^m, \varphi)$ for a suitable diffeomorphism of a high-dimensional torus $(S^1)^m$.

Exercise 106.5. Let M be a smooth homology 3-sphere, i.e. let M be a 3-dimensional smooth manifold such that for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(M; \mathbb{Z}) \cong \mathrm{H}_k(S^3; \mathbb{Z})$. Show that if $\pi_1(M)$ is non-trivial, then every Morse function on M has at least six critical points.
Remark. In Proposition 66.12 we saw that the Poincaré Homology Sphere is an example of such M.

Exercise 106.6. Let M be a closed smooth homology 5-sphere, i.e. let M be a 5-dimensional smooth manifold such that for ever $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(M;\mathbb{Z}) \cong \mathrm{H}_k(S^5;\mathbb{Z})$. Show that if $\pi_1(M)$ is non-trivial, then every Morse function on M has at least ten critical points.
Remark. In Exercise 106.5 we dealt with the case of smooth homology 3-spheres. Furthermore note that in Exercise 66.13 and in Proposition 107.5 we give examples of such M.

Exercise 106.7. Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. We pick $r, s \in \mathbb{Z}$ such that $qr - ps = -1$. We write

$$A = \begin{pmatrix} q & p \\ s & r \end{pmatrix} \quad \begin{array}{c}\text{and we consider the orientation-}\\ \text{reversing diffeomorphism}\end{array} \quad \begin{array}{rcl} \varphi\colon \mathbb{R}^2/\mathbb{Z}^2 & \to & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [Av]. \end{array}$$

Let $\theta\colon \mathbb{R}^2/\mathbb{Z}^2 \to (S^1)^2$ be the usual diffeomorphism that is given by $[(x,y)] \mapsto (e^{2\pi \mathrm{i} x}, e^{2\pi \mathrm{i} y})$. We set

$$M := ((\overline{B}^2 \times S^1) \sqcup (\overline{B}^2 \times S^1))/\sim \qquad \text{where } (x,y) \sim (\theta \circ \varphi \circ \theta^{-1})(x,y) \text{ for } x, y \in S^1 \times S^1.$$

By Proposition 8.15 we know that M is a closed 3-dimensional smooth manifold.

(a) Show that M admits a handle decomposition with a unique handle in dimensions 0,1,2 and 3.
(b) Show that for every $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_n^{\mathrm{hand}}(M) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3, \\ 0, & \text{if } n = 2 \text{ or } n > 3, \\ \mathbb{Z}_p, & \text{if } n = 1. \end{cases}$$

Remark. In Lemma **??** we will see that M is diffeomorphic to the lens space $L(p,q)$. Thus this exercise together with Proposition 106.3 gives us a new calculation of the singular homology groups of lens spaces.

Exercise 106.8. Let $g, n \in \mathbb{N}_0$. As usual we denote by $\Sigma_{g,n}$ the surface of genus g minus n open disks. What is the minimal number of handles in a handle decomposition of $\Sigma_{g,n}$?

Exercise 106.9. Let $n \in \mathbb{N}$. What is the minimal number k of critical points of a Morse function on the n-dimensional torus $(S^1)^n$? You need to show that there exists a Morse function such that the number of critical points equals k and you need to show that every Morse function has at least k critical points.

Exercise 106.10. Let M be a compact smooth manifold and let A be a union of components of ∂M. Suppose we are given a handle decomposition of M rel A. Define a relative handle chain complex $\mathrm{C}_*^{\mathrm{han}}(M,A)$ such that the corresponding homology groups $\mathrm{H}_*^{\mathrm{han}}(M,A)$ are isomorphic to the relative singular homology groups of (M,A).

Exercise 106.11. Let M be a compact smooth manifold and let A be two a union of components of ∂M. Furthermore let $f\colon M \to [0,1]$ be a Morse function such that $A = f^{-1}(\{0\})$ and such that $\partial M = f^{-1}(\{1\})$. Formulate a result that gives a lower bound on the number of critical points of such f in terms of the relative homology groups $\mathrm{H}_*(M,A)$.
Remark. In Proposition 105.4 we saw that such a Morse function always exists.
Hint. Make use of Exercise 106.10.

Exercise 106.12. Let M be a closed smooth manifold and let $f\colon M \to \mathbb{R}$ be a Morse function. Given $k \in \mathbb{N}_0$ we denote by d_k the number of critical points of index k. The Morse function is called *lacunary* if for every $k \in \mathbb{N}_0$ at least one of d_k and d_{k+1} is zero. Show that every lacunary Morse function is perfect.
Remark. This statement is sometimes called the *Morse lacunary principle.*

Exercise 106.13. Let M be a closed smooth manifold and let $f\colon M \to \mathbb{R}$ be a smooth map. Furthermore let g be a Riemannian structure on M, as defined on page 872. Given $P \in M$ we define the *gradient of f at P* to be the unique vector $\operatorname{grad} f_P \in \mathrm{T}_P\, M$ that satisfies

$$g_P(\operatorname{grad} f_P, w) \;=\; \mathrm{D}\, f_P(w) \;\in\; \mathrm{T}_{f(P)}\, \mathbb{R} = \mathbb{R} \qquad \text{for all } w \in \mathrm{T}_P\, M.$$

One can easily show that $P \mapsto \operatorname{grad} f_P$ defines a vector field on M in the sense of the definition on page 1185.

(a) We suppose that f is a Morse function. Show that every zero of the vector field $\operatorname{grad} f$ is isolated.
(b) Let P be a critical point of f. Show that

$$\underbrace{\operatorname{index}(\operatorname{grad} f, P)}_{\text{see page 2347 for the definition}} \;=\; (-1)^{\operatorname{index}(f,P)}.$$

Remark. This exercise shows that Corollary 106.6 can also be deduced from the Poincaré-Hopf Theorem 97.5.

107. Constructing smooth manifolds using handles

In this chapter we will do the following:

(1) We will use handles to construct interesting compact and closed smooth manifolds out of finite presentations of groups.
(2) We will show that every finite CW-complex is homotopy equivalent to a compact smooth manifold.

107.1. **Finite presentations of groups and handle decompositions.**

Proposition 107.1. *Let $\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$ be a finite presentation. Given any $n \geq 4$ there exists a compact orientable connected non-empty n-dimensional smooth manifold with the following properties:*

(1) $\pi_1(M) \cong \pi$.
(2) M admits a standard handle decomposition with precisely one 0-handle, k 1-handles, l 2-handles and no handles of index > 2.

Proof. Let $\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$ be a finite presentation and let $n \geq 4$. We start with the 0-handle $\overline{B}^n$. Since $n \geq 2$ we can can pick k *orientation-preserving* thickened 0-spheres $\varphi_1, \dots, \varphi_k \colon \overline{B}^{n-1} \times S^0 \to \partial(\overline{B}^n)$ with disjoint images and we denote by W the result of attaching the corresponding k 1-handles to the 0-handle $\overline{B}^n$. Note that W, and thus also ∂W, is orientable by Proposition 103.9.

We pick a base point $P \in \partial W$. It follows from Proposition 103.11 that we can make an identification $\pi_1(W, P) = \langle g_1, \dots, g_k \rangle$. Since $n \geq 3$ we know by Lemma 103.14 that the inclusion induced map $\pi_1(\partial W, P) \to \pi_f 1(W, P)$ is an epimorphism. Therefore we can pick loops $\alpha_1, \dots, \alpha_l \colon (S^1, *) \to (\partial W, P)$ that represent $r_1, \dots, r_l \in \pi_1(W, P) = \langle g_1, \dots, g_k \rangle$.

By hypothesis we have $\dim(\partial W) = n - 1 \geq 3$. Thus we can apply Theorem 9.15 and we obtain that the map $\alpha_1 \sqcup \dots \sqcup \alpha_l \colon S^1 \sqcup \dots \sqcup S^1 \to \partial W$ is in fact homotopic to a smooth embedding $\beta_1 \sqcup \dots \sqcup \beta_l \colon S^1 \sqcup \dots \sqcup S^1 \to \partial W$. It follows from Proposition 8.1 and the Tubular Neighborhood Theorem 8.24[1473] that we can find smooth embeddings $\psi_1, \dots, \psi_l \colon \overline{B}^{n-2} \times S^1 \to \partial W$ with disjoint images such that for each $i \in \{1, \dots, l\}$ and each $x \in S^1$ we have $\psi_i((0, x)) = \beta_i(x)$. Finally we consider

$$M \;:=\; W \cup_{\psi_1} h^2 \cdots \cup_{\psi_l} h^2 \;=\; \varnothing \cup \overline{B}^n \cup_{\varphi_1} h^1 \cup \dots \cup_{\varphi_k} h^1 \cup_{\psi_1} h^2 \cdots \cup_{\psi_l} h^2.$$

The Proposition follows from the following calculation:

$$\begin{aligned} \pi_1(M, P) &\overset{\substack{\text{by Proposition 103.11, note that the } x_i \text{ are necessary since we lost track of the base points} \\ \downarrow}}{\cong} \langle g_1, \dots, g_k \rangle / \langle\!\langle x_1 r_1 x_1^{-1}, \dots, x_l r_l x_l^{-1} \rangle\!\rangle \\ &\underset{\substack{\uparrow \\ \text{by Lemma 21.2 (5)}}}{\cong} \langle g_1, \dots, g_k \rangle / \langle\!\langle r_1, \dots, r_l \rangle\!\rangle \underset{\substack{\uparrow \\ \text{by definition}}}{=} \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle. \end{aligned}$$

■

Now we can provide a new proof of the following proposition.

[1473] Note that we can apply the Tubular Neighborhood Theorem 8.24 since ∂W is orientable.

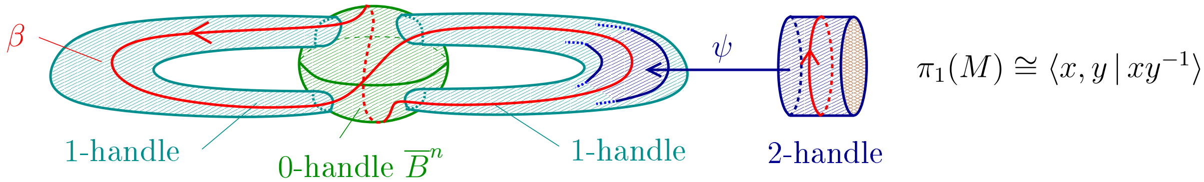


FIGURE 1508. Illustration for the proof of Proposition 107.1.

Proposition 22.10. *Let $\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l\rangle$ be a finite presentation. Given any $n \geq 4$ there exists a closed orientable connected non-empty n-dimensional smooth manifold W with $\pi_1(W) \cong \pi$.*

Remark. Proposition 22.10 is a key ingredient of the proof of the fact, stated in Theorem 28.4, that closed smooth manifolds of a fixed dimension ≥ 4 cannot be classified.

PROOF. Let $\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l\rangle$ be a finite presentation and let $n \geq 4$. By Proposition 107.1 there exists a compact orientable connected non-empty n-dimensional smooth manifold with the following properties:

(1) $\pi_1(M) \cong \pi$.
(2) M admits a handle decomposition $M = \varnothing \cup h^0 \cup_{\varphi_1} h^1 \cdots \cup_{\varphi_k} h^1 \cup_{\psi_1} h^2 \cdots \cup_{\psi_l} h^2$.

By Lemma 44.12 the double $W := \mathrm{D}\, M$ is a closed orientable connected non-empty n-dimensional smooth manifold. Thus it suffices to prove the following claim.

Claim. There exists an isomorphism $\pi_1(\mathrm{D}\, M) \cong \pi$.

For fun we provide two different proofs.

(1) Note that by Corollary 103.15 we know that the boundary ∂M is connected. Therefore it follows from the Seifert-van Kampen Theorem 22.2 we have an isomorphism $\pi_1(\mathrm{D}\, M) \cong \pi_1(M) *_{\pi_1(\partial M)} \pi_1(M)$. Since $n \geq 4$ and since M has only handles of index ≤ 2 we obtain from Corollary 103.15 that the inclusion induced map $\pi_1(\partial M) \to \pi_1(M)$ is an epimorphism. It follows from the group-theoretic Lemma 44.15 that the map $\pi_1(M) \to \pi_1(\mathrm{D}\, M)$ is actually an isomorphism.
(2) On page 2523 we pointed out that it is a consequence of Propositions 105.1 and 105.3 that the double admits a handlebody presentation of the following form:

$$\mathrm{D}\, M = \underbrace{\varnothing \cup h^0 \cup_{\varphi_1} h^1 \cdots \cup_{\varphi_k} h^1 \cup_{\psi_1} h^2 \cdots \cup_{\psi_l} h^2}_{=M} \cup_{\alpha_l} h^{n-2} \cdots \cup_{\alpha_1} h^{n-2} \cup_{\beta_k} h^{n-1} \cdots \cup_{\beta_1} h^{n-1} \cup h^n,$$

where the $(n-2)$-handles corresponding to $\alpha_l, \dots, \alpha_1$ are dual to the 2-handles given by $\psi_l, \dots, \psi_1$ and the $(n-1)$-handles corresponding to $\beta_k, \dots, \beta_1$ are dual to the 1-handles given by $\varphi_k, \dots, \varphi_1$.

(a) First let us deal with the case that $n \geq 5$. In this case $\mathrm{D}\, M$ is obtained from M by attaching handles of index ≥ 3. But by Proposition 103.11 we know that handle attachments of index ≥ 3 do not affect the fundamental group. Thus we see that $\pi_1(\mathrm{D}\, M) \cong \pi_1(M) \cong \pi$.

(b) The case $n = 4$ is a little bit more interesting. We have

$$\begin{aligned}\pi_1(\mathrm{D}\,M) &\overset{\text{by Proposition 103.11}}{=} \pi_1(M)/\langle\langle \text{attaching circles of the dual 2-handles } \alpha_l,\dots,\alpha_1\rangle\rangle \\ &\underset{\text{Proposition 105.1 (1c)}}{=} \pi_1(M)/\langle\langle \text{belt circles of the 2-handles } \psi_l,\dots,\psi_1\rangle\rangle \underset{\text{since belt circles are null-homotopic}}{\cong} \pi_1(M) \cong \pi.\end{aligned}$$

■

This is a good moment to reminder the reader of the following question.

Question 22.14. *Is every finitely presented group the fundamental group of a closed (or at least compact) orientable connected 3-dimensional smooth manifold?*

We will answer Question 22.14 in Theorem **??**.

107.2. **Integral homology spheres and homotopy spheres.** We remind the reader of the following definitions from page 1670 and page 2161.

Definition. Let M be an n-dimensional smooth manifold M.

(1) If for every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(M) \cong \mathrm{H}_k(S^n)$, then we refer to M as a *smooth homology n-sphere.*
(2) If M is a homology sphere and if M is simply connected, then we refer to M as a *smooth homotopy n-sphere.*

Our goal in this section is to apply the construction used in Proposition 22.10 to construct smooth homology and potentially interesting smooth homotopy spheres.

We start out with the following proposition, which the reader might recognize as Exercise 88.21.

Proposition 107.2. *Let M be a compact orientable n-dimensional topological manifold. For every $j \in \mathbb{N}_0$ there exists an isomorphism*

$$\mathrm{H}_j(\mathrm{D}\,M) \;\cong\; \mathrm{H}_j(M) \oplus \mathrm{Hom}(\mathrm{H}_{n-j}(M), \mathbb{Z}) \oplus \mathrm{Tor}(\mathrm{H}_{n-j-1}(M)).$$

Proof. Let $i\colon M \to \mathrm{D}\,M$ be the natural inclusion and let $r\colon \mathrm{D}\,M \to M$ be the folding map that we introduced on page 1164. Note that by definition we have $r \circ i = \mathrm{id}_M$. We consider the long exact sequence in homology of the pair $(\mathrm{D}\,M, M)$:

$$\dots \;\to\; \mathrm{H}_j(M) \xrightarrow{i_*} \mathrm{H}_j(\mathrm{D}\,M) \;\to\; \mathrm{H}_j(\mathrm{D}\,M, M) \;\to\; \dots$$

Now we see that we have the following isomorphisms:

$$\begin{aligned}\mathrm{H}_j(\mathrm{D}\,M) &\underset{\substack{\text{since } r\circ i=\mathrm{id}_M \text{ we see that the above long exact sequence breaks up into short exact sequences,}\\ \text{furthermore it follows from Lemma 46.2 that each short exact sequence splits and}\\ \text{that the middle term is isomorphic to the direct sum of the two other terms}}}{\cong} \mathrm{H}_j(M) \oplus \mathrm{H}_j(\mathrm{D}\,M, M) \overset{\text{Excision Theorem 44.10, here the blue } M \text{ stands for the second copy of } M \text{ in the double D}\,M}{\cong} \mathrm{H}_j(M) \oplus \mathrm{H}_j(M, \partial M) \\ &\underset{\text{Poincaré Duality Theorem 88.1}}{\cong} \mathrm{H}_j(M) \oplus \mathrm{H}^{n-j}(M) \underset{\text{Universal Coefficient Theorem 75.13}}{\cong} \mathrm{H}_j(M) \oplus \mathrm{Hom}(\mathrm{H}_{n-j}(M), \mathbb{Z}) \oplus \mathrm{Tor}(\mathrm{H}_{n-j-1}(M)).\end{aligned}$$

■

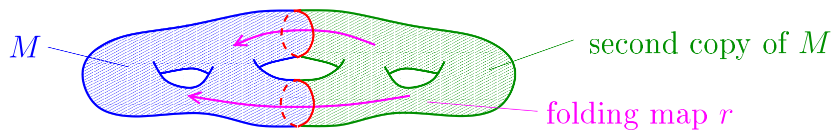


FIGURE 1509. Illustration for the proof of Proposition 107.2.

Now we can prove the following refinement of Proposition 22.10.

Corollary 107.3. *Let $\pi = \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_l \rangle$ be a finite presentation.*

(1) *Given any $n \geq 5$ there exists a closed orientable connected non-empty n-dimensional smooth manifold W with $\pi_1(W) \cong \pi$ and such that*[1474]

$$\mathrm{H}_i(W) \;\cong\; \begin{cases} \pi_{\mathrm{ab}}, & \text{if } i = 1, \\ \mathbb{Z}^{l-k+\mathrm{rk}(\pi_{\mathrm{ab}})}, & \text{if } i = 2, \\ \mathbb{Z}^{l-k+\mathrm{rk}(\pi_{\mathrm{ab}})} \oplus \mathrm{Tor}(\pi_{\mathrm{ab}}), & \text{if } i = n-2, \\ \mathbb{Z}^{\mathrm{rk}(\pi_{\mathrm{ab}})}, & \text{if } i = n-1, \\ \mathbb{Z}, & \text{if } i = 0 \text{ or if } i = n, \\ 0, & \text{else.} \end{cases}$$

(2) *There exists a closed orientable connected non-empty smooth manifold W of dimension $n = 4$ with $\pi_1(W) \cong \pi$ and such that the homology groups are as in (1), except that for $i = 2$ we take the direct sum of the terms that we wrote down for $i = 2$ and $i = n - 2$.*

PROOF. As in the proof of Proposition 22.10 that we just gave on page 2550 we use Proposition 107.1 to obtain a compact orientable connected non-empty n-dimensional smooth manifold with the following properties:

(1) $\pi_1(M) \cong \pi$.
(2) M admits a handle decomposition with precisely one 0-handle, k 1-handles, l 2-handles and no handles of index > 2.

In the above proof of Proposition 22.10 we saw that the fundamental group of $W := \mathrm{D}\,M$ is isomorphic to π and that W is a closed orientable connected non-empty n-dimensional smooth manifold. It remains to show that the homology groups of $W = \mathrm{D}\,M$ are as stated. In fact it follows from Proposition 107.2 that it suffices to prove the following claim.

Claim.

$$\mathrm{H}_0(M) \cong \mathbb{Z}, \quad \mathrm{H}_1(M) \cong \pi_{\mathrm{ab}}, \quad \mathrm{H}_2(M) \cong \mathbb{Z}^{l-k+\mathrm{rk}(\pi_{\mathrm{ab}})} \quad \text{and} \quad \mathrm{H}_i(M) = 0 \text{ for } i \geq 3.$$

The statements for $\mathrm{H}_0(M)$ and $\mathrm{H}_1(M)$ follow immediately from Proposition 41.5 and the Hurewicz Theorem 52.5. Next note that that

$$\mathrm{H}_i(M) \underset{\substack{\uparrow \\ \text{Proposition 106.3}}}{\cong} \mathrm{H}_i^{\mathrm{han}}(M) \underset{\substack{\uparrow \\ \text{since } M \text{ has no handles of index} \geq 3}}{=} \begin{cases} 0, & \text{if } i > 2, \\ \ker(\mathrm{C}_2^{han}(M) \to \mathrm{C}_1^{\mathrm{han}}(M)), & \text{if } i = 2. \end{cases}$$

[1474] We denote by π_{ab} the abelianization of π, as defined on page 639.

Note that this observation for $i = 2$, together with Lemma 19.8, shows that $\mathrm{H}_2(M)$ is a free abelian group. Finally note that

$$\mathrm{rk}(\mathrm{H}_2(M)) \underset{\substack{\uparrow \\ \text{by definition of } \chi(M)}}{=} \chi(M) + \sum_{i\neq 2} (-1)^{i+1}\cdot \mathrm{rk}(\mathrm{H}_i(M)) \underset{\substack{\uparrow \\ \text{by Corollary 106.4 and the above calculations}}}{=} (1-k+l) - 1 + \mathrm{rk}(\pi_{\mathrm{ab}}) = l - k + \mathrm{rk}(\pi_{\mathrm{ab}}).$$

■

As we mentioned, our goal is to construct smooth homology spheres and smooth homotopy spheres. Corollary 107.3 leads us to the following group theoretic definitions.

Definition.

(1) The *deficiency* of a finite presentation $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l\rangle$ is defined to be $k - l$.
(2) A finite presentation is called *balanced*, if its deficiency equals zero.
(3) A group π is called *perfect* if its abelianization π_{ab} is trivial.

The following proposition gives us some interesting examples of (presentations of) groups.

Proposition 107.4.

(1) *For every closed connected non-empty 3-dimensional smooth manifold M the fundamental group $\pi_1(M)$ admits a balanced presentation.*
(2) *The binary icosahedral group, that we introduced on page 1671, is a perfect group of order 120 that admits a balanced presentation.*
(3) *For every odd prime p the group $\mathrm{SL}(2, \mathbb{F}_p)$ is perfect and it admits a balanced presentation.*

Proof.

(1) We will prove this statement in Exercise 107.2. Alternatively the statement is proved in Theorem **??**.
(2) Let π be the binary icosahedral group. In Proposition 66.12 we showed that π is a perfect group of order 120. It remains to show that π admits a balanced presentation. There are two possible approaches:
   (a) In [**KSc79**, Chapter II] it is shown that the binary icosahedral group is isomorphic to $\langle x, y \,|\, x^3 = y^5, x^3 = (xy)^2\rangle$.
   (b) Recall that by definition $\pi \subset \mathrm{SU}(2)$. Thus it makes sense to consider the quotient $M := S^3/\pi$, which we called the Poincaré Homology Sphere. In Proposition 66.12 we showed that M is a closed orientable 3-dimensional smooth manifold. Thus it follows from (1) that $\pi \cong \pi_1(M)$ admits a balanced presentation.
(3) In Exercise 107.3 we will see that $\mathrm{SL}(2, \mathbb{F}_p)$ is perfect. The much more subtle fact that $\mathrm{SL}(2, \mathbb{F}_p)$ admits a balanced presentation is shown in [**CR80**]. ■

In Proposition 66.11 we showed that there exist smooth homology 3-spheres. The combination of Corollary 107.3 and Proposition 107.4 gives us the high-dimensional analogue.

Proposition 107.5. *Given any $n \geq 4$ there exists a smooth homology n-sphere with non-trivial fundamental group.*

Remark.

(1) In Exercise 66.13 we used a different construction to obtain high-dimensional smooth homology spheres.
(2) Michel Kervaire [**Kerv69**, Theorem 1] completely determined which finitely presented groups can occur as fundamental groups of smooth homology n-spheres for $n \geq 4$.

The construction that we used in the proof of Proposition 107.1 and that we used in the above proof of Propositions 22.10 is a never-ending source of joy, since even for presentations of the trivial group one obtains interesting examples of smooth manifolds:

Example. Let $k \in \mathbb{Z}$. We consider the presentation

$$\langle x, y \,|\, \underbrace{xyx = yxy}_{=(1)}, \underbrace{x^{k+1} = y^k}_{=(2)} \rangle.$$

We will now see that this is a presentation of the trivial group. Indeed, we have

(a) $y \;=\; x^{-1}y^{-1}yxy \;=\; (yx)^{-1}yxy \underset{\substack{\uparrow \\ \text{by (1)}}}{=} (yx)^{-1}x(yx)$

(b) $y^{k+1} \underset{\substack{\uparrow \\ \text{follows from taking} \\ \text{the } (k+1)\text{-st power of (a)}}}{=} (yx)^{-1}x^{k+1}(yx) \underset{\substack{\uparrow \\ \text{by (2)}}}{=} (yx)^{-1}y^k(yx) = x^{-1}y^kx \underset{\substack{\uparrow \\ \text{by (2)}}}{=} x^{-1}x^{k+1}x = x^{k+1} \underset{\substack{\uparrow \\ \text{by (2)}}}{=} y^k.$

Thus $y^{k+1} = y^k$ which implies that $y = 1$. From (1) it follows that also $x = 1$. We have thus shown that this is indeed a presentation of the trivial group.

Let $n \geq 4$. We apply the construction of the proof of Propositions 22.10 and 107.1 to the above balanced presentation of the trivial group. By Corollary 107.3 we obtain a smooth homotopy n-sphere.

The following question arises:

Question 107.6. *Are the smooth homotopy n-spheres that we just constructed*[1475] *diffeomorphic to S^n?*

For $n = 4$ and some special cases Question 107.6 was answered in the affirmative [**Gom91**]. In general though, at least for $n = 4$ Question 107.6 seems to be wide open. We refer to [**GoS99**, p. 149] and [**Kir89**, p. 18] for more information.

We have now seen that finite presentations of groups give rise to interesting smooth manifolds. The question arises, to what degree different presentations give rise to different smooth manifolds. It is worth recalling the following theorem.

Theorem 21.7. (Tietze Theorem) *Any two finite presentations for some given group are related by a finite sequence of Tietze transformations, i.e. by a sequence of the following*

[1475] Evidently there are various choices involved in these constructions.

two transformations:

(a) $$\langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle \quad\leftrightsquigarrow\quad \langle x_1,\dots,x_k \mid r_1,\dots,r_l,s\rangle$$

where s is a word in in $r_1,\dots,r_l$, and

(b) $$\langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle \quad\leftrightsquigarrow\quad \langle x_1,\dots,x_k,x \mid r_1,\dots,r_l,s\cdot x\cdot t\rangle$$

where $x\neq x_1,\dots,x_k$ and where s,t are words in $x_1,\dots,x_k$.

In our context, where we study balanced presentations of the trivial group, we could like to go from one presentation to another while keeping the deficiency fixed. This leads us to the following definition.

Definition. Let $\langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle$ be a presentation. An *Andrews-Curtis transformation* is one of the following ways to obtain a new presentation:

(a) $$\langle x_1,\dots,x_k \mid r_1,\dots,r_i,\dots,r_l\rangle \quad\leftrightsquigarrow\quad \langle x_1,\dots,x_k \mid r_1,\dots,x_j^{-\epsilon}\cdot r_i\cdot x_j^{\epsilon},\dots r_l\rangle$$

where $j\in\{1,\dots,k\}$ and $\epsilon\in\{\pm 1\}$

(b) $$\langle x_1,\dots,x_k \mid r_1,\dots,r_i,\dots,r_l\rangle \quad\leftrightsquigarrow\quad \langle x_1,\dots,x_k, \mid r_1,\dots,r_i\cdot r_j,\dots,r_l\rangle$$

where $i\neq j\in\{1,\dots,l\}$

(c) $$\langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle \quad\leftrightsquigarrow\quad \langle x_1,\dots,x_k,x \mid r_1,\dots,r_l,x\rangle$$

where $x\neq x_1,\dots,x_k$.

The following is now a slight reformulation of the Andrews-Curtis Conjecture which we first encountered in Section 21.4.

Conjecture 21.18. (Andrews-Curtis Conjecture) *Any balanced presentation of the trivial group can be turned into the empty presentation by a finite sequence of Andrews-Curtis transformations.*

In the Handle Cancellation Theorem **??** and the Handle Addition Theorem **??** we will see that the Andrews-Curtis transformations have analogues for handle decompositions. In particular, as is argued on [**GoS99**, p. 149], an affirmative answer to the Andrews-Curtis Conjecture 21.18 would imply that Question 107.6 has a positive answer.

For better or worse the evidence towards the Andrews-Curtis Conjecture 21.18 is rather shaky. For example, let $k\in\mathbb{Z}$. As above we consider the following balanced presentation of the trivial group:

$$\langle x,y \mid xyx=yxy, x^{k+1}=y^k\rangle.$$

For most k it is unknown whether the presentation can be turned into the empty presentation by a finite sequence of Andrews-Curtis transformations. In fact the consensus seems to be that the Andrews-Curtis Conjecture 21.18 is most likely false.

Remark.

(1) More information and some partial results regarding the Andrews-Curtis Conjecture 21.18 can be found in [**Barm11**, Chapter 11].
(2) Sergey Matveev [**Matv87**, p. 917] (see also [**Kupe17a**, Proposition 4.2] and [**Barm11**, Chapter 11]) showed that the Andrews-Curtis Conjecture 21.18 is equivalent to a special case of the Zeeman Conjecture 119.20.

107.3. **Handle decompositions and CW-structures.** In Proposition 103.18 we saw that every compact smooth manifold M that is equipped with a handle decomposition is homotopy equivalent to a CW-complex X such that for each $k \in \mathbb{N}_0$ the number of k-handles of M equals the number of k-cells of X. In this section we will prove the following converse to this statement.

Proposition 107.7. *Given any finite* CW*-complex X there exists a compact orientable smooth manifold M that is homotopy equivalent to X and which admits a handle decomposition such that for each $k \in \mathbb{N}_0$ the number of k-cells of X equals the number of k-handles of M.*

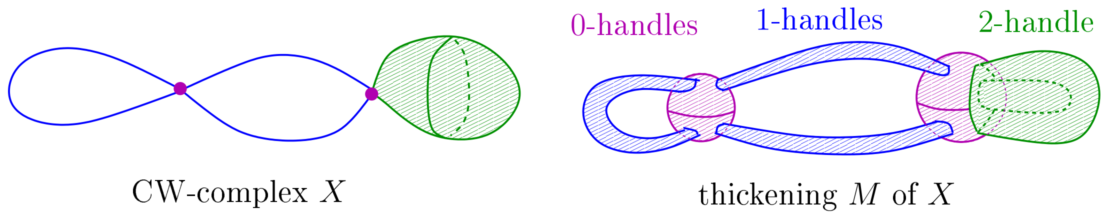


FIGURE 1510. Illustration of Proposition 107.7.

Remark.

(1) In Proposition 107.7 we do not make any claims about the dimension of M. We leave it to the reader to figure out what is the lowest dimension for M that one can achieve with our proof.
(2) Let X be a finite CW-complex. Any compact smooth manifold M that is homotopy equivalent to X is called a *thickening of* X. Proposition 107.7 thus says in particular that every finite CW-complex admits a thickening.
(3) The theory of thickenings of finite CW-complexes is developed in detail in [**Maz63**] and [**Wall66b**]. In particular given a finite k-dimensional CW-complex it follows from [**Wall66b**, p. 76] (see also [**BCFHLN19**, Theorem 4.1] that the the following statements hold:
   (a) Given any $n \geq 2k$ there exists an n-dimensional thickening of X.
   (b) If $n \geq 2k+1$ and if $n \geq 6$, then any two n-dimensional thickenings of X are actually diffeomorphic.

   We refer to [**BCFHLN19**, Chapter 4] for more information on thickenings.
(4) The topic of finding smooth manifolds that "look" like a given topological space is also discussed in [**BuS10**, Chapter 2].

The proof of Proposition 107.7 rests on the following lemma.

Lemma 107.8. *If $N \geq 2r+2$, then for every smooth embedding $\varphi\colon S^r \to \mathbb{R}^N$ there exists an orientation-preserving smooth embedding $\Phi\colon \overline{B}^{N-r} \times S^r \to \mathbb{R}^N$ such that $\Phi(0,P) = \varphi(P)$ for all $P \in S^r$.*

PROOF. The statement of the lemma is basically the content of the Tubular Neighborhood Theorem 8.24 (5). For the reader's convenience we recall the proof. First we consider the standard smooth embedding $\theta\colon S^r \to \mathbb{R}^{r+1} \times \mathbb{R}^{N-r-1} = \mathbb{R}^N$ which is given by $x \mapsto (x,0)$. This smooth embedding can be "thickened" up. For example we could consider the smooth embedding

$$\begin{array}{rcl}\Theta\colon \overline{B}^{N-r}\times S^r & \to & \mathbb{R}^{r+1}\times\mathbb{R}^{N-r-1}=\mathbb{R}^N\\ (\underbrace{(x,y)}_{\substack{\uparrow\\ \text{with } x\in\mathbb{R} \text{ and } y\in\mathbb{R}^{N-r-1}}},P) & \mapsto & ((1+\frac{x}{2})\cdot P,y).\end{array}$$

After possibly composing Θ with a reflection in the last coordinate we see that Θ is orientation-preserving. By hypothesis we have $N\geq 2r+2$. Thus we obtain from Proposition 11.11 that there exists a smooth isotopy $F\colon S^r\times[0,1]\to\mathbb{R}^N$ with $F_0=\theta$ and $F_1=\varphi$. By the Isotopy Extension Theorem 8.27 there exists a diffeotopy $G\colon\mathbb{R}^N\times[0,1]\to\mathbb{R}^N$ with $G_1\circ\theta=\varphi$. The map $\Phi:=G_1\circ\Theta\colon\overline{B}^{N-r}\times S^r\to\mathbb{R}^N$ has all the desired properties. ■

The heart of the proof of Proposition 107.7 is contained in the following lemma.

Lemma 107.9. *Let X be a finite* CW*-complex and let $\alpha\colon S^{k-1}\to X$ be a map. If X is homotopy equivalent to a compact orientable smooth manifold M, then there exists a compact orientable smooth manifold N and an orientation-preserving thickened $(k-1)$-sphere $\varphi\colon\overline{B}^{\dim(N)-k}\times S^{k-1}\to\partial N$ such that the following are satisfied:*

1. *N is also homotopy equivalent to X.*
2. *$N\cup_\varphi h^k$ is homotopy equivalent to $X\cup_\alpha\overline{B}^k$.*
3. *If M is equipped with a handle decomposition, then we can equip N with a handle decomposition such that for each $k\in\mathbb{N}_0$ the number of k-handles of M equals the number of k-handles of N.*

Proof of Proposition 107.7. Note that in the setting of Lemma 107.9 we know by Propositions 103.1 and 103.9 that $N\cup_\varphi h^k$ is again a compact orientable smooth manifold with one extra k-handle. Thus we see that Proposition 107.7 follows immediately from Lemma 107.9 and an elementary induction argument. ■

Proof of Lemma 107.9. Let $f\colon X\to M$ be a homotopy equivalence between a finite CW-complex X and a compact orientable m-dimensional smooth manifold M. Furthermore let $\alpha\colon S^{k-1}\to X$ be a map.

We pick some $r\geq\max\{k,2k-\dim(M)\}$. We write $\widetilde{M}:=M\times\overline{B}^r$ and we equip the product $\widetilde{M}$ with the smooth atlas provided by Proposition 80.9. Note that the map $i\colon M\to\widetilde{M}$ given by $x\mapsto(x,0)$ is a homotopy equivalence.

Claim. We write $\widetilde{f}:=i\circ f\colon X\to\widetilde{M}$. There exists a homotopy $F\colon S^{k-1}\times[0,1]\to\widetilde{M}$ from the map $\widetilde{f}\circ\alpha\colon S^{k-1}\to\widetilde{M}=M\times\overline{B}^r$ to a smooth embedding $\theta\colon S^{k-1}\to\partial\widetilde{M}$.

By the combination of the Whitney Approximation Theorem 9.3 and the Transversality Theorem 9.10 we know that the map $\widetilde{f}\circ\alpha\colon S^{k-1}\to\widetilde{M}$ is homotopic to a smooth map that is transverse to the proper submanifold $M\times\{0\}$ of $\widetilde{M}=M\times\overline{B}^r$. Note that

$$\dim(S^{k-1})+\dim(M\times\{0\}) \;=\; k-1+\dim(M) \underset{\substack{\uparrow\\ \text{since } r\geq k}}{<} r+\dim(M) \;=\; \dim(M\times\overline{B}^r).$$

Thus the image of a map $S^{k-1} \to \widetilde{M}$, which is transverse to $M \times \{0\}$, actually misses $M \times \{0\}$. Therefore we now know that $\widetilde{f} \circ \alpha \colon S^{k-1} \to \widetilde{M}$ is homotopic to a map that misses $M \times \{0\}$. Next, using the deformation retraction from $\overline{B}^k \setminus \{0\}$ to S^{k-1} we see that $\widetilde{f} \circ \alpha \colon S^{k-1} \to \widetilde{M}$ is actually homotopic to a map $\varphi \colon S^{k-1} \to M \times S^{k-1} \subset \partial \widetilde{M}$. Finally, since

$$2(k-1) \underset{\substack{\uparrow \\ \text{since } r \geq 2k - \dim(M)}}{<} \dim(M) + r - 1 \;=\; \dim(\partial \widetilde{M})$$

we obtain from Theorem 9.15 that $\varphi \colon S^{k-1} \to \partial \widetilde{M}$ is homotopic to a smooth embedding $\theta \colon S^{k-1} \to \partial \widetilde{M}$. ⊞

Next we pick an $n \in \mathbb{N}$ with $n \geq \max\{2 \cdot \dim(\widetilde{M}) + 1, 2k\}$. Since $n \geq 2 \cdot \dim(\widetilde{M}) + 1$ we obtain immediately from Proposition 11.8 that there exists a proper smooth embedding $\widetilde{M} \to H_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\}$. In other words, we can assume that $\widetilde{M}$ is a proper smooth submanifold of H_n.

By the General Tubular Neighborhood Theorem 10.5 the proper submanifold $\widetilde{M}$ of H_n admits a tubular neighborhood $N \subset H_n$. By definition of a tubular neighborhood, see page 418, we know that N is a compact codimension-zero submanifold of H_n with corner. Furthermore, by Proposition 10.11 we know that the inclusion $j \colon \widetilde{M} \to N$ is a homotopy equivalence. As we mentioned in Lemma 89.1, by rounding corners as in the proof of Proposition 80.9 we can view N as an n-dimensional smooth manifold.

The map $j \circ \theta \colon S^{k-1} \to \partial \widetilde{M} \subset \partial N$ takes values in ∂H_n. Since $N \cap \partial H_n$ is a codimension-zero submanifold of ∂H_n we see that there exists an open neighborhood U of $\theta(S^{k-1})$ in ∂H_n that is contained in N. Since ∂H_n is diffeomorphic to $\mathbb{R}^{n-1}$ and since, by our choice of n, we have $n - 1 \geq 2(k-1) + 2$, we obtain from Lemma 107.8 that there exists an orientation-preserving smooth embedding $\Theta \colon \overline{B}^{n-k} \times S^{k-1} \to \partial H_n$ such that $\Theta(0, P) = \theta(P)$ for all $P \in S^{k-1}$. After possibly "shrinking" the balls we can arrange that the image of Θ actually lies in $U \subset N \cap \partial H_n$.

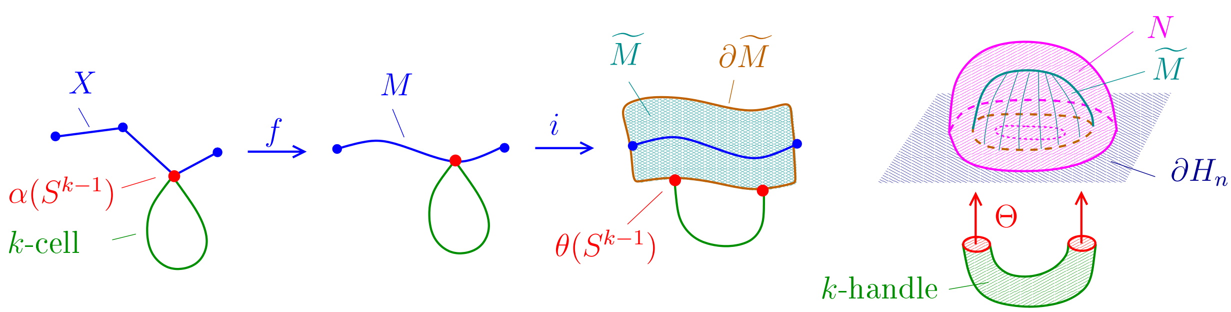


Figure 1511. Illustration for the proof of Proposition 107.7.

Statement (2) of the lemma is a consequence of the following claim.

Claim. The topological space $X \cup_\alpha \overline{B}^k$ is homotopy equivalent to the smooth manifold $N \cup_\Theta h^k$.

The experienced reader will doubtlessly spot that this is the right moment to roll out the Homotopy Pushout Theorem 39.18. To apply this great theorem let us consider the following commutative diagram:

$$\begin{tikzcd}
X \arrow[d, "f"] \arrow[dd, bend right, "\widetilde{f}:="'] & S^{k-1} \arrow[l, "\alpha"'] \arrow[r, hook] \arrow[d, "\mathrm{id}"] & \overline{B}^k \arrow[d, "\mathrm{id}"] \\
M \arrow[d, "i"] & S^{k-1} \arrow[l, "f\circ\alpha"'] \arrow[r, hook] \arrow[d, "\mathrm{id}"] & \overline{B}^k \arrow[d, "\mathrm{id}"] \\
\widetilde{M} \arrow[d, "\mathrm{id}"] & S^{k-1} \arrow[l, "\widetilde{f}\circ\alpha"'] \arrow[r, hook] \arrow[d, "x\mapsto(x,0)"] & \overline{B}^k \arrow[d, "x\mapsto(x,0)"] \\
\widetilde{M} & S^{k-1}\times[0,1] \arrow[l, "F"'] \arrow[r, hook] & \overline{B}^k\times[0,1] \\
\widetilde{M} \arrow[u, "\mathrm{id}"] \arrow[d, "j"] & S^{k-1} \arrow[l, "\theta"'] \arrow[r, hook] \arrow[u, "x\mapsto(x,1)"] \arrow[d, "\mathrm{id}"] & \overline{B}^k \arrow[u, "x\mapsto(x,1)"] \arrow[d, "\mathrm{id}"] \\
N \arrow[d, "\mathrm{id}"] & S^{k-1} \arrow[l, "j\circ\theta"'] \arrow[r, hook] \arrow[d, "x\mapsto(0,x)"] & \overline{B}^k \arrow[d, "x\mapsto(0,x)"] \\
N & \overline{B}^{n-k}\times S^{k-1} \arrow[l, "\Theta"'] \arrow[r, hook] & \overline{B}^{n-k}\times\overline{B}^k.
\end{tikzcd}$$

All of the vertical maps are clearly homotopy equivalences. Furthermore note that it follows immediately from Proposition 39.4 and 39.12 that all of the right horizontal maps are closed cofibrations. Thus it follows from the Homotopy Pushout Theorem 39.18, applied altogether six times, that the induced maps between the various pushouts have homotopy inverses. Combining these six homotopy equivalences, respectively their homotopy inverses, we obtain the promised homotopy equivalence from $X \cup_\alpha \overline{B}^k$ to $N \cup_\Theta h^k$. ⊞

Finally assume that M is equipped with a handle decomposition. We sketch the proof how we can find a handle decomposition for N with the same number of handles. First, except for a slight sense of uneasy about rounding corners, it is pretty clear that we can equip $\widetilde{M} = M \times \overline{B}^r$ with a handle decomposition that is given by taking the product of each handle of M with $\overline{B}^r$. Now we consider the actual smooth manifold N which we are interested in. Note that N is by definition of a tubular neighborhood the total space of a linear $\overline{B}^s$-bundle $p\colon N \to \widetilde{M}$. This bundle might well be non-trivial, thus there is no reason to assume that N is actually a product $\widetilde{M} \times \overline{B}^s$. But, it follows from the same argument as in Proposition **??** and the fact that each handle is contractible that the restriction of p to each handle is actually a trivial linear $\overline{B}^s$-bundle. Thus we see that the handle structure on $\widetilde{M}$ does give rise to a handle structure on N with exactly the same number of handles of each index. ■

The next proposition explains what type of closed smooth manifolds one can obtain by the ever-popular doubling procedure to the smooth manifolds constructed in the proof of Proposition 107.7.

Proposition 107.10. *Let X be a finite k-dimensional* CW*-complex. There exists an $N \geq 2k+1$ such that for every $n \geq N$ there exists a closed orientable n-dimensional smooth manifold M which admits a standard handle decomposition and which admits a homotopy equivalence*

$$\varphi\colon X \;\to\; M^{n-k-1}$$

from X to the $(n-k-1)$-skeleton of M.

Remark. Note that in the setting of Proposition 107.10 we obtain from Exercise 103.12 and Lemma 106.1 (3) and (4), together with Proposition 40.7 and Corollary 42.8, that the map $X \to M^{n-k-1} \to M$ induces an isomorphism of homotopy groups and homology groups up to dimension $n-k-2$.

Proof. Let X be a finite k-dimensional CW-complex. By Proposition 107.7 there exists a homotopy equivalence $\psi\colon X \to V$ from X to a compact smooth manifold V which has the nice property that it admits a handle decomposition such that all handles are of index $\leq k$. By Corollary 103.4 we can arrange that this handle decomposition is standard. After possibly replacing V by some $\overline{B}^s \times V$ we can arrange that $\dim(V) \geq 2k+1$. We set $N := \dim(V)$. Now let $n \geq N$. We set $W := \overline{B}^{n-N} \times V$ and we set $M := \mathrm{D}\,W$. We equip the double $M = \mathrm{D}\,W$ with the handle decomposition that we described on page 2522, namely which is given by the handle decomposition on M and the dual handle decomposition on the second copy of M. In particular we see that we can equip M with a handle decomposition such that the $(n-k-1)$-skeleton equals W. The map $\varphi\colon X \to M^{n-k-1} = V \times \overline{B}^{n-N}$ given by $P \mapsto (\psi(P), 0)$ is now the promised homotopy equivalence from X to the $(n-k-1)$-skeleton of M. ■

In Proposition 107.7 we just showed that every finite CW-complex X is homotopy equivalent to a compact smooth manifold X. The following question, which implicitly we had already asked on page 2178, arises.

Question 107.11. *Given a finite* CW*-complex X, what extra conditions do we need to impose on X to ensure that X is homotopy equivalent to a* closed *smooth or topological manifold?*

We make a couple of initial observations. Let X be a CW-complex.

(1) If X is homotopy equivalent to a closed orientable n-dimensional topological manifold, then it follows from the Poincaré Duality Theorem 88.1, together with Proposition 42.5, Lemmas 73.13 and 83.8, that there exists a class $\sigma \in \mathrm{H}_n(X;\mathbb{Z})$ such that for each $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \cap\sigma\colon\ \mathrm{H}^{n-k}(X;\mathbb{Z}) & \to & \mathrm{H}_k(X;\mathbb{Z}) \\ \psi & \mapsto & \psi \cap \sigma \end{array}$$

is an isomorphism.

(2) If X is homotopy equivalent to a closed *non-orientable* topological manifold, then it follows from Proposition 86.15 that X admits a 2-fold covering $\widetilde{X}$ which is homotopy equivalent to a closed orientable topological manifold.

(3) If X is homotopy equivalent to a closed (orientable) n-dimensional topological manifold, then every finite covering of X is also homotopy equivalent to a closed (orientable) n-dimensional topological manifold. In particular given any finite covering $\widetilde{X}$ the properties stated in (1) and (2) need to hold.

(4) These observations in (1), (2) and (3) already give lots of restrictions. We will generalize these conditions on page **??**. There we will show that any CW-complex X which is homotopy equivalent to a closed orientable n-dimensional topological manifold, is in fact a "Poincaré duality space".

The extra conditions stated in (2), (3) and (4) vanish, if X is a simply connected CW-complex. Thus we have arrived at the following subtle variation on Question 107.11

Question 107.12. *Let X be a finite simply connected* CW*-complex X which admits a class $\sigma \in \mathrm{H}_n(X;\mathbb{Z})$ as in* (1)*. Is X homotopy equivalent to a closed n-dimensional smooth or topological manifold?*

It turns out that the answer to Question 107.12 is no. Namely, there are other, more subtle bundle-theoretic obstructions to being homotopy equivalent to a closed smooth or topological manifold. Slightly more precisely, the obstruction is given by showing that the "Spivak Normal Fibration" of a given finite CW-complex has the structure of a vector bundle. In fact in [**DaP15**, p. 21] (see also [**MM79**, p. 32]) it is shown that there exists a finite simply connected 5-dimensional CW-complex X such that the cohomology ring is isomorphic to the cohomology ring of $S^2 \times S^3$ but such that X is not homotopy equivalent to a closed topological manifold. Similar examples of such simply connected CW-complexes are given in [**GiS65**, Theorem 5.2]. We also refer to [**CLM21**, Chapter 5.5] and to [**CW21**, Chapter 1.1.2] for details.

Exercises for Chapter 107.

Exercise 107.1. In the proof of Proposition 22.10 that we gave on page 2550 we used the following statement. Let M be a compact connected non-empty smooth manifold that admits a handle decomposition such that all handles have index ≤ 2. If $\dim(M) \geq 4$, then the inclusion induced map $\pi_1(M) \to \pi_1(\mathrm{D}\,M)$ is an isomorphism. Does this statement also hold if $\dim(M) = 3$?

Exercise 107.2. Let M be a closed connected 3-dimensional smooth manifold. Show that $\pi_1(M)$ admits a balanced presentation.
Hint. By the Handle Decomposition Theorem 104.11 we know that M admits a handle decomposition. By Corollaries 106.4 and 106.5 we know that

$$\sum_{k=0}^{3} (-1)^k \cdot \text{number of } k\text{-handles} \;=\; 0.$$

You could first consider the case that the handle decomposition contains precisely one 0-handle and one 3-handle. Afterwards try to tackle the general case.

Exercise 107.3. Let k be a field with at least three elements. Show that $\mathrm{SL}(2,k)$ is perfect, i.e. show that the abelianization is the trivial group.
Hint. Use that for any non-zero $d \in k$ and $a \in k$ we have

$$\begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & (d^2-1)a \\ 0 & 1 \end{pmatrix}.$$

Exercise 107.4. Give an example of a finite CW-complex X which is not homotopy equivalent to a closed topological manifold but which admits a class $\sigma \in \mathrm{H}_n(X;\mathbb{Z})$ such that for each $k \in \mathbb{N}_0$ the map

$$\begin{array}{rccc} \cap\sigma\colon & \mathrm{H}^{n-k}(X;\mathbb{Z}) & \to & \mathrm{H}_k(X;\mathbb{Z}) \\ & \psi & \mapsto & \psi \cap \sigma \end{array}$$

is an isomorphism.
Hint. You could consider the wedge of a sphere with a suitable CW-complex with non-trivial fundamental group.

108. The classification of compact 2-dimensional smooth manifolds II

In this fairly short chapter we will finally complete the proof of the classification of compact 2-dimensional smooth manifolds. But first, as a warm-up, let us go one dimension lower, namely let us give a proof for half of the content of Theorem 7.5.

Proposition 108.1. *Every compact connected non-empty 1-dimensional smooth manifold is diffeomorphic either to S^1 or to $[0,1]$.*

Remark. As we explained on page 330, using the fact that topological manifolds are by definition second-countable it is not difficult to deduce from Proposition 108.1 that every non-compact connected 1-dimensional smooth manifold is diffeomorphic to either the open interval $(0,1)$ or the half-open interval $[0,1)$.

Proof. Let M be a compact connected non-empty 1-dimensional smooth manifold. By the Handle Decomposition Theorem 104.11 we can equip M with a standard handle decomposition.

Claim. There exists a standard handle decomposition for M with a single 0-handle.

Since M is non-empty we know that M has at least one 0-handle. Now suppose that our handle decomposition has more than one 0-handle. Since M is connected and since the handle decomposition is standard it follows from Lemma 103.8 that there exists a 1-handle H that gets glued to two different 0-handles D and E. It is elementary to see that $D \cup E \cup H$ is diffeomorphic to $[-1,1]$. Thus we can combine $D \cup E \cup H$ to a single 0-handle. Note that the resulting handle decomposition is still standard. We now proceed iteratively. ⊞

We now fix a handle decomposition for M with a single 0-handle D. If the handle decomposition does not contain a 1-handle, then $M = D$, in particular M is diffeomorphic to $[0,1]$.

Now suppose that the handle decomposition contains at least one 1-handle. Since the handle decomposition is standard we know that all attaching maps of the 1-handles take values in $\partial M^1 = \partial D$. Now let $\varphi\colon \{0\} \times S^0 \to \partial D$ be an attaching map of a 1-handle. Note that the map is injective. Since both sets consist of precisely two elements we see that the attaching map is a bijection. In particular we see that $D \cup_\varphi (\{0\} \times \overline{B}^1)$ is a closed 1-dimensional smooth manifold. It is elementary to see that this smooth manifold is diffeomorphic to S^1. Since this smooth manifold is closed we see that there are no other 1-handles. Thus we have shown that M is diffeomorphic to S^1. ■

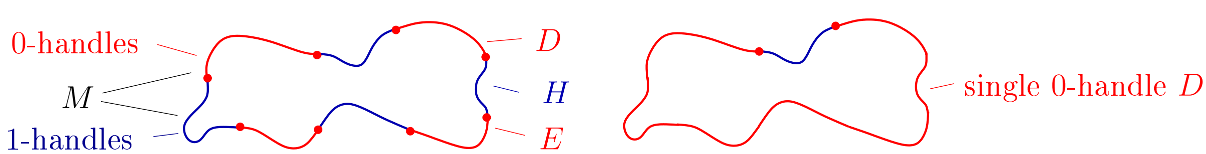


Figure 1512. Illustration for the proof of Proposition 108.1.

108.1. **Handle decompositions of 2-dimensional smooth manifolds.** In this section we will study handle decompositions of some of our favorite 2-dimensional smooth manifolds. We will also learn how to modify handle decompositions of 2-dimensional smooth manifolds. In the next section we will use the results from this section to give a surprisingly short proof of the statement that any compact connected non-empty 2-dimensional smooth manifold is diffeomorphic to one of the standard examples.

We start out with the following lemma which is very specific to the 2-dimensional setting.

Lemma 108.2. *Let M be a 2-dimensional smooth manifold, let C be a boundary component and let $\varphi\colon S^1 \to C$ be a diffeomorphism. The diffeomorphism type of $M \cup_\varphi h^2$ does not depend on the choice of φ.*

Remark. The proof of Lemma 108.2 rests on the fact, proved in Proposition 30.1 (2), that for $n = 2$ every diffeomorphism of S^{n-1} extends to a diffeomorphism of $\overline{B}^n$. Note that by Theorem **??** and the Cerf Theorem 8.44 the analogous statement also holds for $n = 3$ and $n = 4$. In particular the obvious analogue of Lemma 108.2 also holds in those dimensions. But we will see in Proposition **??** that the analogue of Lemma 108.2 does *not* hold for $n = 7$.

Proof. Let M be a 2-dimensional smooth manifold, let C be a boundary component of M and let $\varphi, \psi\colon S^1 \to C$ be two diffeomorphisms. We consider the self-diffeomorphism $f := \varphi \circ \psi^{-1}\colon S^1 \to S^1$. By Proposition 30.1 (2) the self-diffeomorphism f of S^1 extends to a self-diffeomorphism F of the disk $\overline{B}^2$. Now we obtain from Proposition 8.15 (7) that $M \cup_\varphi h^2 = M \cup_{C \xleftarrow{\varphi} S^1} \overline{B}^2$ is diffeomorphic to $M \cup_\psi h^2 = M \cup_{C \xleftarrow{\psi} S^1} \overline{B}^2$. ■

In the following we will state and prove a few results which are not germane to the 2-dimensional setting, but which also have analogues in the higher-dimensional setting, which we will eventually discuss in Chapter **??**. In this and the following section we will use slightly informal language and we will partially work with figures. Nonetheless the mathematically mature reader should have no troubles with making the whole discussion completely and utterly rigorous.

First we formulate the following special case of Proposition 103.1 (4).

Lemma 108.3. *Let M be a 2-dimensional smooth manifold. Given any smooth isotopy $F\colon (\overline{B}^1 \times S^0) \times [0,1] \to \partial M$ there exists a diffeomorphism $M \cup_{F_0} h^1 \to M \cup_{F_1} h^1$. Any such replacement of the 1-handle given by F_0 by the 1-handle given by F_1 is called a* handle slide.

Example. For what follows it will be very helpful to get a little experience with what handle slides can do.

Lemma 108.4. *Every closed connected non-empty 2-dimensional smooth manifold admits a handle decomposition with the following properties:*

1. *There exists a single 0-handle.*
2. *The attaching maps of all of the 1-handles take values in the boundary of the 0-handle.*[1476]
3. *There exists a single 2-handle.*

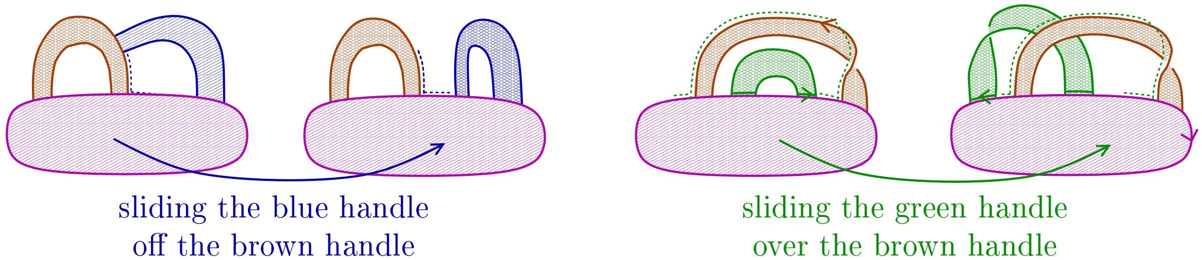


FIGURE 1513

Remark. In Corollary **??** we will see that the conclusion of Lemma 108.4 in fact holds in all dimensions.

SKETCH OF PROOF. By the Handle Decomposition Theorem 104.11 we can equip M with a standard handle decomposition. Since M is non-empty we know that there exists at least one 0-handle. Suppose that M has more than one 0-handle. Since M is connected it follows from the fact that the handle decomposition is standard together with Lemma 103.8 that there exists a 1-handle H that gets glued to two different 0-handles D and E. In Exercise 108.2 we will see that $D \cup E \cup H$ is again diffeomorphic to a disk. Thus we can combine $D \cup E \cup H$ to a single 0-handle. It is straightforward to verify that the new handle decomposition is again standard and that the number of 2-handles is unaffected. We now proceed iteratively till we get down to a single 0-handle.

Next we apply the above procedure to the dual handle decomposition and we can arrange that the dual handle decomposition has a single 0-handle. Furthermore, as mentioned above, we can do this without changing the fact that the dual handle decomposition is standard and without changing the fact that the dual handle decomposition has a single 2-handle.

In summary, we have found a standard handle decomposition with a single 0-handle and a single 2-handle. ■

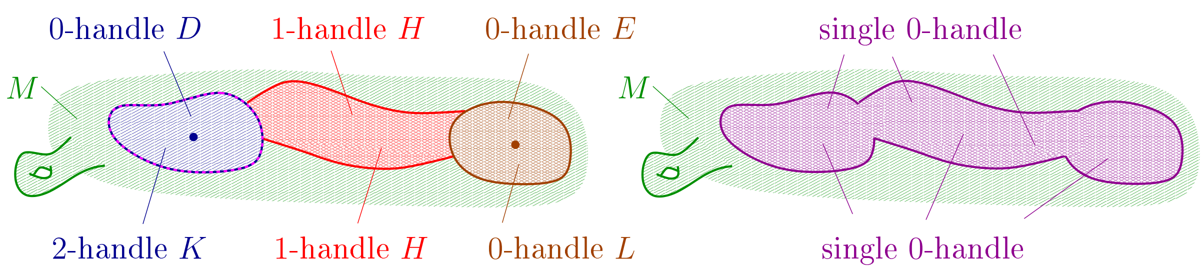


FIGURE 1514. Illustration of the proof of Lemma 108.4 (1).

Definition.

(1) We say a handle decomposition of 2-dimensional smooth manifold is *completely standard*, if it is of the form described in Lemma 108.4.

[1476]In fact this statement is in this context equivalent to saying that the handle decomposition is standard in the sense of the definition on page 2481.

(2) Let M be a 2-dimensional smooth manifold that is equipped with a completely standard handle decomposition. We equip $\overline{B}^1 \times \overline{B}^1$ with the product orientation and we equip $\overline{B}^1 \times S^0 = \overline{B}^1 \times \partial\overline{B}^1 \subset \overline{B}^1 \times \overline{B}^1$ with the boundary orientation. As on page 2486 we say that a 1-handle with attaching map $\varphi\colon \overline{B}^1 \times S^0 \to \partial\overline{B}^2$ is *orientable* if the attaching map is either orientation-preserving on both components of $\overline{B}^1 \times S^0$ or if it is orientation-reversing on both components of $\overline{B}^1 \times S^0$. Otherwise we say that the 1-handle is *non-orientable*.

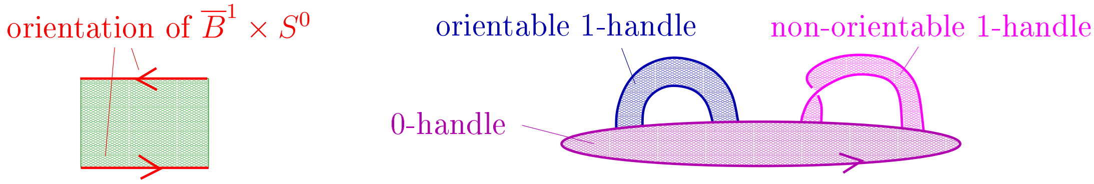


FIGURE 1515

Lemma 108.5. *Let M be a 2-dimensional smooth manifold M with a completely standard handle decomposition. The following two statements are equivalent:*

(1) *M is orientable.*
(2) *All 1-handles are orientable.*

PROOF. In Proposition 8.15 (6) we dealt with the orientability of smooth manifolds that are given by gluing oriented smooth manifolds along unions of boundary components. In our setting we only glue along codimension-zero submanifolds of the boundary. But the same statements hold and we see that our lemma is basically a special case of Proposition 8.15 (6). ■

Convention.

(1) Let M be a 2-dimensional smooth manifold M with a single 0-handle $\overline{B}^2$. If M is orientable, then we equip M with the orientation such that the natural map $\overline{B}^2 \to M$ given by the 0-handle is orientation-preserving.
(2) Let M be a closed 2-dimensional smooth manifold. If we are given a handle decomposition for M with a single 2-handle, then in the figures we show the 0-handles and the 1-handle attachments and we indicate with a discrete $\cup h^2$ the 2-handle. Note that by Lemma 108.2 the 2-handle attachment is basically unambiguous.

Examples.

(1) In Figure 1516 we show several completely standard handle decompositions for several of our favorite surfaces.
(2) (a) In Figure 1517 we show a completely standard handle decomposition for the surface of genus wo. This handle decomposition has an obvious generalizations to higher genera.
    (b) In Figure 1518 we show a completely standard handle decomposition for the non-orientable surface of genus 3, as defined on page 206. Again htis example has obvious generalizations to higher genera.

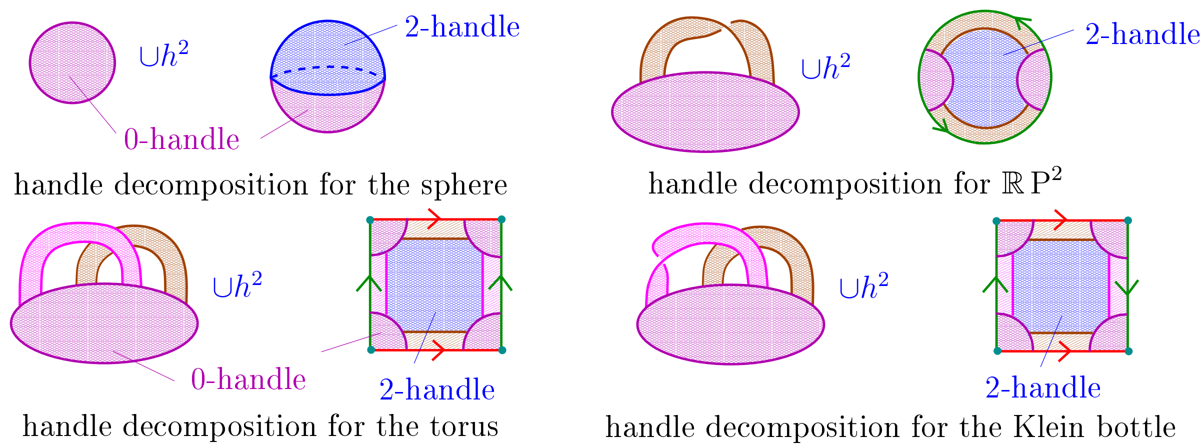


FIGURE 1516

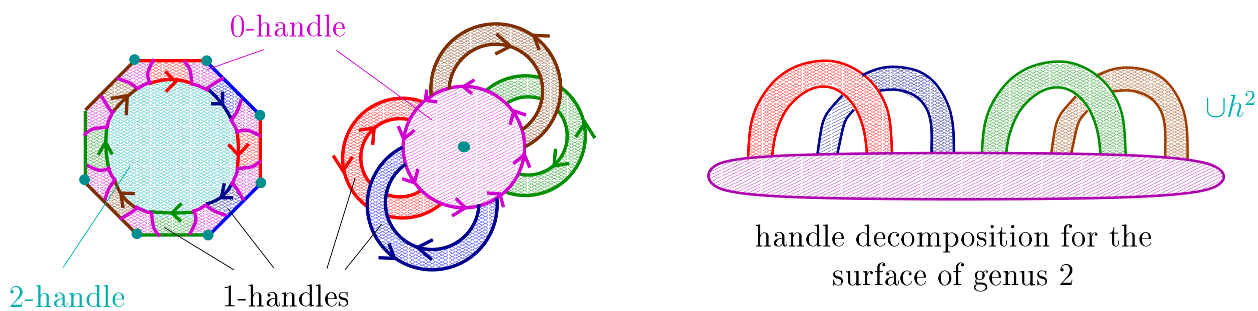


FIGURE 1517

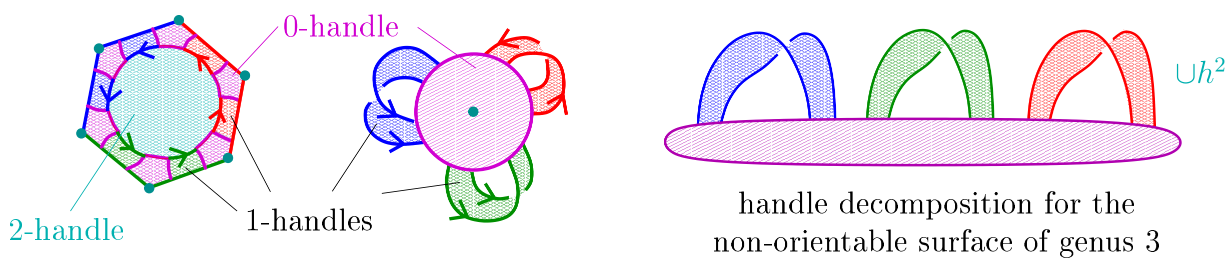


FIGURE 1518

Lemma 108.6. *Let M and N be two closed 2-dimensional smooth manifolds. We assume that both are equipped with a completely standard handle decomposition. Then $M\#N$ has a completely standard handle decomposition such that the 1-handles are given by "juxtaposing" the 1-handle attachments for M and the 1-handle attachments for N. We will not give a formal definition of "juxtaposing", instead we refer to Figure 1519 for a hopefully unambiguous "definition by illustration".*

PROOF. We denote by $D = \overline{B}^2$ the unique 0-handle of M and we denote by K the unique 2-handle of M. Furthermore we denote by $E = \overline{B}^2$ the unique 0-handle of N and we denote by L the unique 2-handle of N. It follows almost immediately from Exercise 103.3 that we

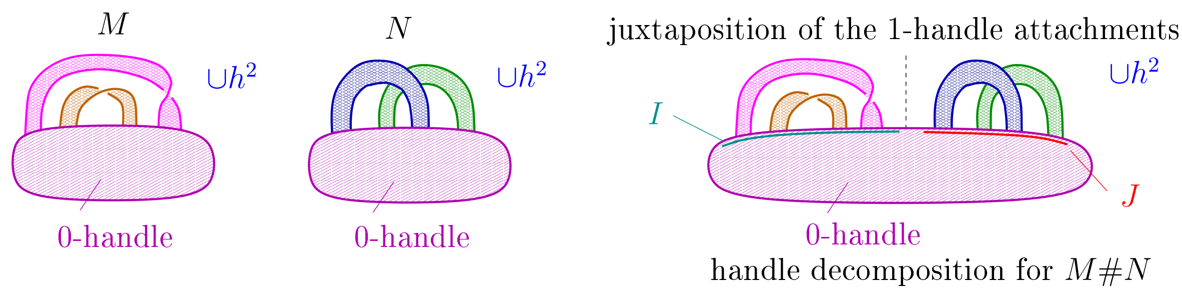


FIGURE 1519. Illustration for Lemma 108.6.

can pick a smooth embedding $\varphi\colon \overline{B}^2 \to M$ such that $\varphi(\overline{B}^2_{\leq 0})$ is contained in the 0-handle D, such that $\varphi\colon \overline{B}^2_{\leq 0} \to D = \overline{B}^2$ is orientation-preserving and such that $\varphi(\overline{B}^2_{\geq 0})$ is contained in the 2-handle K. Similarly we pick a smooth embedding $\varphi\colon \overline{B}^2 \to N$, with the only real difference being, that we demand that the map $\psi\colon \overline{B}^2_{\leq 0} \to E$ is orientation-reversing. Now we have

$$\begin{aligned} M\#N &\overset{\substack{\text{by definition of the connected sum, note that if } M \text{ and } N \text{ are oriented,}\\ \text{then the orientation conventions work out}\\ \downarrow}}{=} (M \setminus \varphi(B^2)) \sqcup (N \setminus \psi(B^2))/\varphi(x) \sim \psi(x) \text{ for } x \in S^1 \\ &= \overbrace{(D \setminus \varphi(B^2_{\leq 0})) \cup (E \setminus \psi(B^2_{\leq 0}))}^{\text{diffeomorphic to } \overline{B}^2} \ \cup\ \text{1-handles of } M \ \cup\ \text{1-handles of } N \\ &\qquad \cup\ \underbrace{(K \setminus \varphi(B^2_{\geq 0})) \cup (L \setminus \psi(B^2_{\geq 0}))}_{\text{diffeomorphic to } \overline{B}^2}. \end{aligned}$$

Thus we have obtained a handle decomposition for $M\#N$ with a single 0-handle, a single 2-handle, and such that the handles of M and N get attached to disjoint semicircles in the boundary of the 0-handle. In other words, the 1-handle attachments are juxtaposed. ■

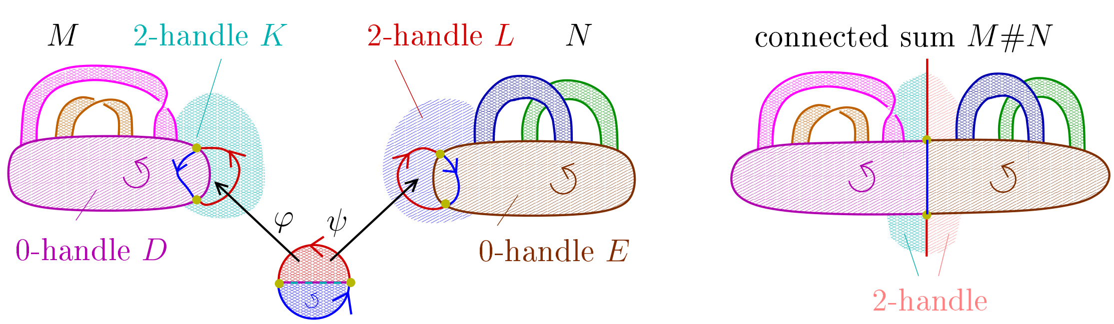


FIGURE 1520. Illustration for the proof of Lemma 108.6.

Examples.

(1) In Figure 1521 we use Lemma 108.6 to give a handle decomposition of the connected sum $\mathbb{RP}^2\#\mathbb{RP}^2$ of two copies of the real projective plane $\mathbb{RP}^2$. Furthermore we show a handle slide that results in the handle decomposition of the Klein bottle that we

already saw in Figure 1516. This argument shows in particular that $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ is diffeomorphic to Klein bottle. Thus we have obtained a new proof of Lemma 8.33.

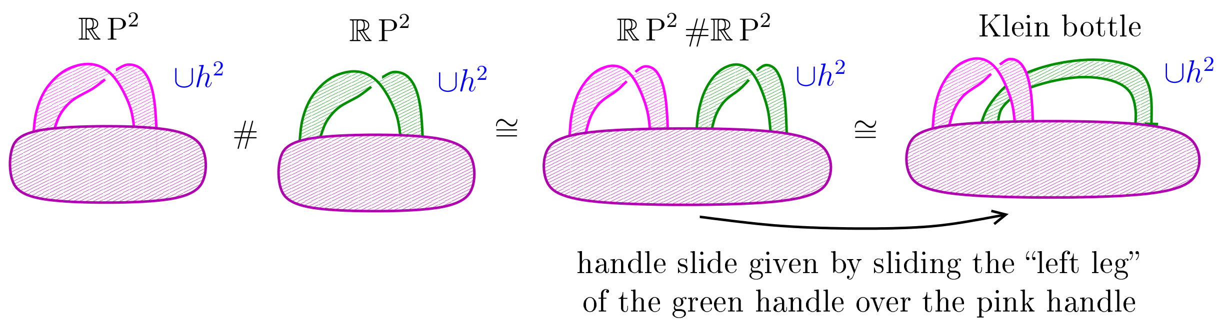


FIGURE 1521

(2) (a) In Figure 1522 we show, using altogether three handle slides, that the connected sum $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ is diffeomorphic to the connected sum of $\mathbb{R}\mathrm{P}^2$ with the torus.

(b) Now let $g \in \mathbb{N}_0$. We consider the connected sum of $2g+1$ copies of $\mathbb{R}\mathrm{P}^2$ as given by attaching $2g+1$ non-orientable 1-handles to a single 0-handle. We iteratively apply the procedure in Figure 1522 to the three right-most non-orientable 1-handles. We turn the two right-most non-orientable 1-handles into a pair of interlocking orientable 1-handles. Eventually we see that the connected sum of $2g+1$ copies of $\mathbb{R}\mathrm{P}^2$ is diffeomorphic to the connected sum of $\mathbb{R}\mathrm{P}^2$ with the surface of genus g.

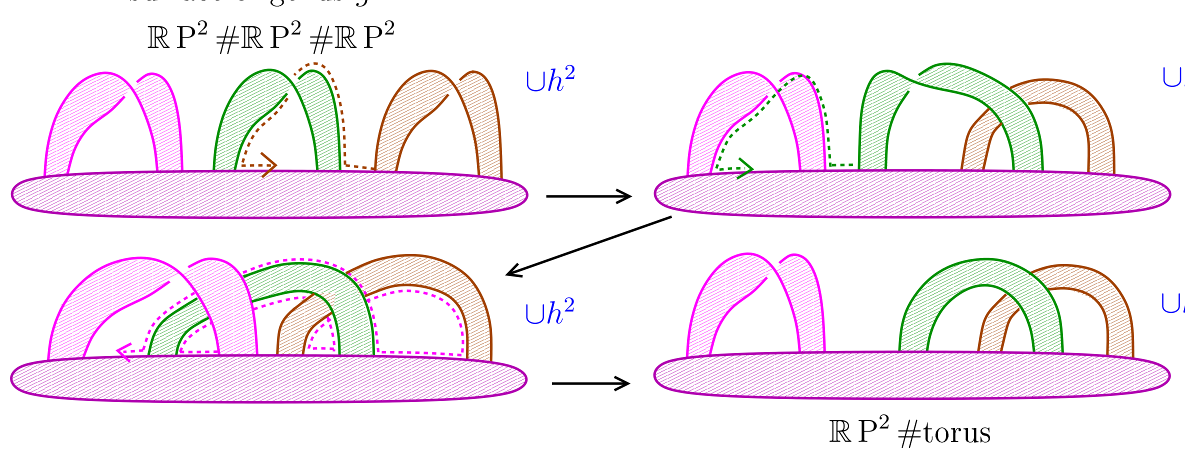


FIGURE 1522

108.2. **The classification of compact 2-dimensional smooth manifolds.** We recall the following notation that we introduced on page 669.

Notation. Given $g, n \in \mathbb{N}_0$ we write[1477]

$$\Sigma_{g,n} \ := \ \text{the surface of genus } g \text{ minus } n \text{ open disks.}$$

Furthermore for $g \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we write

$$N_{g,n} \ := \ \text{the non-orientable surface of genug } g \text{ minus } n \text{ open disks.}$$

In Theorem 23.6 we showed that the above surfaces are pairwise non-homeomorphic, in particular we showed they are pairwise non-diffeomorphic. In this section we prove that any compact connected 2-dimensional smooth manifold is diffeomorphic to one of the above surfaces. This finally completes the classification of compact surfaces that we started in Chapter 23.

Here is the precise theorem we intend to prove.

Theorem 108.7. *Let Σ be a compact connected non-empty 2-dimensional smooth manifold.*

(1) *If Σ is orientable, then Σ is diffeomorphic to the surface $\Sigma_{g,n}$ for some $g, n \in \mathbb{N}_0$.*
(2) *If Σ is non-orientable, then Σ is diffeomorphic to the surface $N_{k,n}$ for some $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.*

Remark.

(1) Our proof of Theorem 108.7 is inspired by the ambitious [**GoS99**, Exercise 4.2.4]. The key idea was communicated to us by Robert Gompf. The proof we provide is also outlined in [**Mart16**, Theorem 6.1.6].
(2) In Exercise **??** we will also sketch an alternative approach to proving Theorem 108.7.
(3) One can prove Theorem 108.7 through Morse theory, without using handles. This approach is taken in [**Mata02**, Chapter 5.2] and [**Hirs76**, Chapter 3].
(4) On page 675 we discuss several other approaches to proving the analogue of Theorem 108.7 in the setting of topological manifolds.

To increase readability we split the proof of Theorem 108.7 into two parts. First we treat closed manifolds, afterwards we will prove the general case.

Proof of Theorem 108.7 for closed manifolds. By the Handle Decomposition Theorem 104.11 together with Lemma 108.4 it suffices to prove the proposition for a closed 2-dimensional smooth manifold that is equipped with a completely standard handle decomposition, i.e. that is equipped with a handle decomposition with the following properties:

(1) There exists a single 0-handle D.
(2) The attaching maps of all of the 1-handles take values in the boundary of the 0-handle.
(3) There exists a single 2-handle.

We prove this statement by induction on the number of 1-handles. If we have zero 1-handles, then M is given by gluing a 2-handle to a 0-handle via some attaching map $\varphi\colon S^1 \to S^1$. Now we see that we have the following diffeomorphisms:

$$M \;=\; \overline{B}^2 \cup_\varphi h^2 \;=\; \overline{B}^2 \cup_{S^1 \xleftarrow{\varphi} S^1} \overline{B}^2 \;\underset{\substack{\uparrow \\ \text{by Lemma 108.2 all attaching maps give} \\ \text{the same diffeomorphism type}}}{\cong}\; \overline{B}^2 \cup_{S^1 \xleftarrow{\mathrm{id}} S^1} \overline{B}^2 \;\underset{\substack{\uparrow \\ \text{see Lemma 103.2}}}{\cong}\; S^2.$$

Next suppose that the claim holds whenever the number of 1-handles is $\leq k$. Thus let M be a smooth manifold as above with $k+1$ 1-handles.

[1477] We refer to page 668 for the definition of “minus n open disks”.

Perhaps slightly surprisingly the case that the smooth manifold is non-orientable is slightly easier. So let us deal with this case initially. Since M is now assumed to be non-orientable we obtain from Lemma 108.5 that there exists a non-orientable 1-handle. We denote the corresponding attaching map by α. We denote by $\psi_1, \dots, \psi_k$ the attaching maps of the remaining 1-handles. Let $I \subset \partial D$ be a compact interval that contains the image of the attaching map of the chosen 1-handle and let $J \subset \partial D$ be the compact interval with $\partial D = I \cup J$ and $I \cap J = \partial I = \partial J$. Here comes the key observation: *Since the 1-handle is non-orientable we see that $\partial(\overline{B}^2 \cup_\alpha h^1)$ is connected. Thus we can iteratively slide all handle attachments $\psi_1, \dots, \psi_k$ that end up on I across the chosen 1-handle so that they take values in the interval J.* (We refer to Figure 1523 for an illustration.) We denote the new handle attachment maps by $\widetilde{\psi}_1, \dots, \widetilde{\psi}_k$.

We now see that

$$
\begin{aligned}
M &\cong \overline{B}^2 \cup_\alpha h^1 \cup_{\psi_1} h^1 \cdots \cup_{\psi_k} h^1 \cup h^2 \overset{\substack{\text{follows from Lemmas 108.2 and 108.3}\\ \downarrow}}{\cong} \overline{B}^2 \cup_\alpha h^1 \cup_{\widetilde{\psi}_1} h^1 \cdots \cup_{\widetilde{\psi}_k} h^1 \cup h^2 \\
&\underset{\substack{\uparrow\\ \text{by Lemma 108.6}}}{\cong} \underbrace{(\overline{B}^2 \cup_\alpha h^1 \cup h^2)}_{\cong \mathbb{RP}^2\text{, see Figure 1516}} \# \underbrace{(\overline{B}^2 \cup_{\widetilde{\psi}_1} h^1 \dots \cup_{\widetilde{\psi}_k} h^1 \cup h^2)}_{\text{by induction } \cong k \cdot \mathbb{RP}^2 \text{ or } \cong \Sigma_{\frac{k}{2}}} \underset{\substack{\uparrow\\ \text{see Figure 1522 where we see}\\ \text{that } \mathbb{RP}^2 \# \Sigma_l \cong (2l+1)\cdot\mathbb{RP}^2}}{\cong} (k+1)\cdot\mathbb{RP}^2 \underset{\substack{\uparrow\\ \text{by Lemma 8.33 (2)}}}{\cong} \text{non-orientable surface of genus } k+1.
\end{aligned}
$$

We have thus proved the induction step if M is non-orientable.

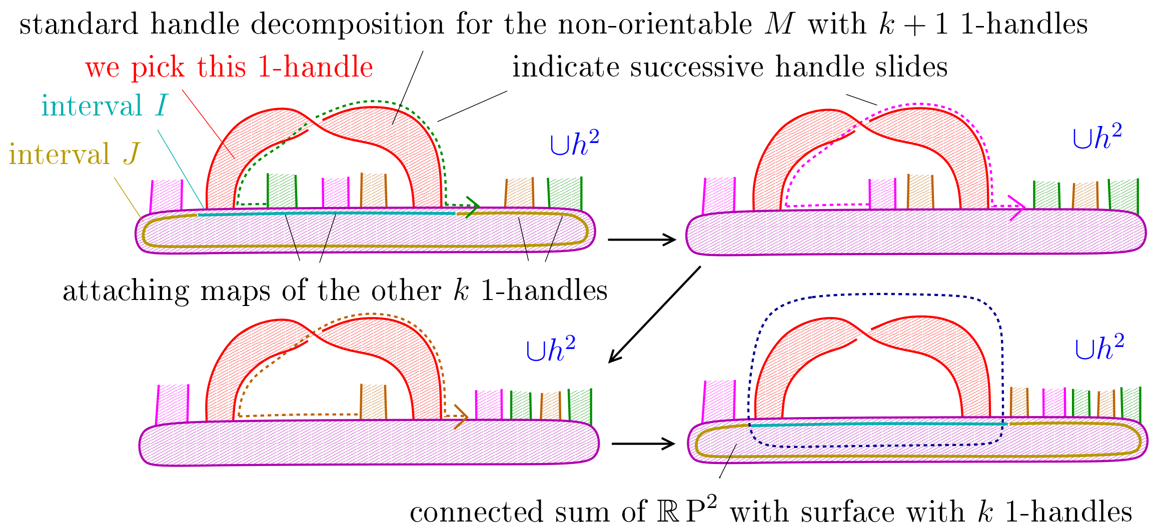


FIGURE 1523. First illustration for the proof of Theorem 108.7.

Now we turn to the case that M is orientable. We pick any 1-handle. Since M is orientable the chosen 1-handle is orientable. Note that this implies that the boundary of $\overline{B}^2 \cup_\alpha h^1$ is disconnected. But since we work with a completely standard handle decomposition we know that, once we attached all 1-handles, the boundary is connected. This implies that there exists at least one other 1-handle that connects the two components of

the boundary of $\overline{B}^2 \cup_\alpha h^1$. We pick such a 1-handle and we call it the *dual handle*. We denote its attaching map by β. We denote by $\psi_1, \dots, \psi_{k-1}$ the attaching maps of the remaining 1-handles. Let $I \subset \partial D$ be a compact interval that contains the image of the attaching maps α and β and let $J \subset \partial D$ be the compact interval with $\partial D = I \cup J$ and $I \cap J = \partial I = \partial J$. Here comes the key observation: *It is straightforward to see that $\partial(\overline{B}^2 \cup_\alpha h^1 \cup_\beta h^1)$ is connected. Thus we can iteratively slide all handle attachments $\psi_1, \dots, \psi_k$ that end up on I across the two 1-handles so that they take values in the interval J.* (We refer to Figure 1524 for an illustration.) We denote the new handle attachment maps by $\widetilde{\psi}_1, \dots, \widetilde{\psi}_{k-1}$.
We now see that

$$M \cong \overline{B}^2 \cup_\alpha h^1 \cup_\beta h^1 \cup_{\psi_1} h^1 \cdots \cup_{\psi_{k-1}} h^1 \cup h^2 \overset{\text{follows from Lemmas 108.2 and 108.3}}{\cong} \overline{B}^2 \cup_\alpha h^1 \cup_\beta h^1 \cup_{\widetilde{\psi}_1} h^1 \cdots \cup_{\widetilde{\psi}_{k-1}} h^1 \cup h^2$$

$$\underset{\text{by Lemma 108.6}}{\cong} \underbrace{(\overline{B}^2 \cup_\alpha h^1 \cup_\beta h^1 \cup h^2)}_{\cong \text{ torus, see Figure 1516}} \; \# \; \underbrace{(\overline{B}^2 \cup_{\widetilde{\psi}_1} h^1 \dots \cup_{\widetilde{\psi}_{k-1}} h^1 \cup h^2)}_{\substack{\text{since all bands are orientable we know that} \\ \text{this is an orientable surface, thus by induction} \\ \text{we know that it is diffeomorphic to } \Sigma_{\frac{k-1}{2}}}} \overset{\text{follows from Lemma 8.33 (1) or alternatively the discussion on page 2569}}{\cong} \text{surface of genus } \tfrac{k+1}{2}.$$

We have thus now proved the induction step also in the case that M is orientable. ■

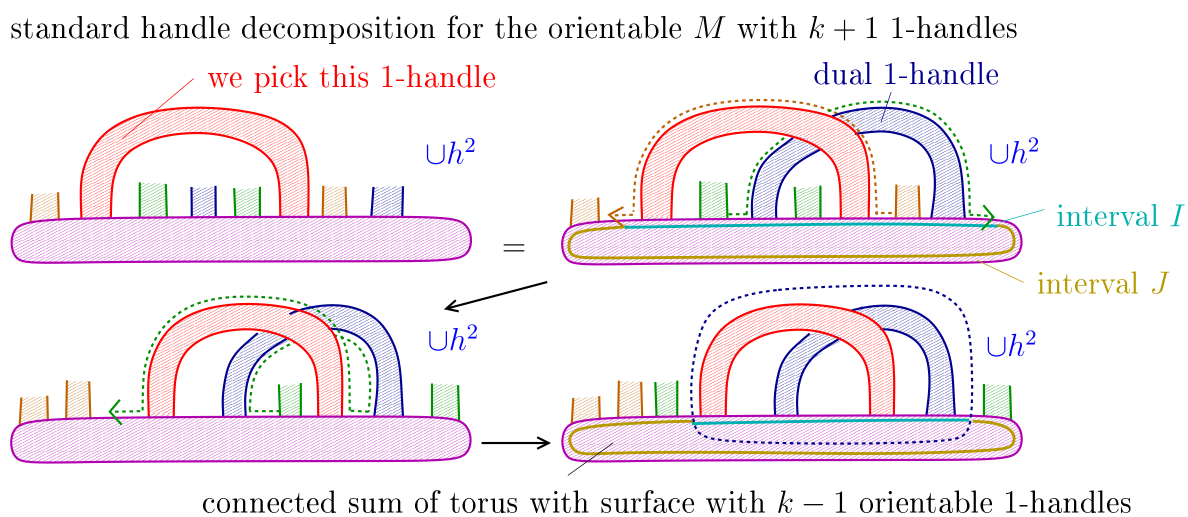


FIGURE 1524. Second illustration for the proof of Theorem 108.7.

In the following we will finally deal with 2-dimensional smooth manifolds with non-empty boundary.

PROOF OF THEOREM 108.7 FOR ALL MANIFOLDS. Let M be a compact connected non-empty 2-dimensional smooth manifold. By Proposition 6.27 the boundary ∂M is a closed 1-dimensional smooth manifold. Since ∂M is compact it has finitely many components that we denote by $C_1, \dots, C_n$. (Note that we allow the case that $n = 0$.) Each component

C_i is compact and has no boundary. Thus it follows from Proposition 108.1 that for each $i = 1, \ldots, n$ there exists a diffeomorphism $f_i \colon S^1 \to C_i$.

We denote by $\widehat{M}$ the result of gluing a disk to each boundary component. More precisely, we consider

$$\widehat{M} \;:=\; \Big(M \sqcup \bigsqcup_{i=1}^{n} (\overline{B}^2 \times \{i\})\Big) \,/ \sim \;\text{ where } f_i(P) \sim (P, i) \text{ for } P \in S^1 \text{ and } i \in \{1, \ldots, n\}.$$

By Proposition 8.15 we know that $\widehat{M}$ is a closed connected non-empty 2-dimensional smooth manifold that contains M as a smooth submanifold and which has the property that the obvious maps $\overline{B}^2 \times \{i\} \to \widehat{M}$ are all smooth embeddings. Furthermore it follows from Proposition 8.15 (6) and the observation that $\overline{B}^2$ admits an orientation-reversing diffeomorphism, that M is orientable if and only if $\widehat{M}$ is orientable.

Now suppose that M is orientable. By the above $\widehat{M}$ is also orientable. Since $\widehat{M}$ is closed orientable connected and non-empty we know by the above that there exists a diffeomorphism $\Phi \colon \Sigma_g \to \widehat{M}$ from the surface of some genus g to $\widehat{M}$. Then Φ restricts to a diffeomorphism

$$\Phi \colon \underbrace{\Sigma_g \setminus \bigcup_{i=1}^{n} \Phi^{-1}\big(B^2 \times \{i\}\big)}_{\substack{= \Sigma_{g,n} \text{ since the maps } \overline{B} \times \{i\} \to \Sigma_g \\ \text{are all smooth embeddings}}} \;\to\; M.$$

If M is non-orientable, then $\widehat{M}$ is closed connected non-orientable and non-empty, thus we know that it is diffeomorphic to the non-orientable surface N_g for some g. As above we see that M is diffeomorphic to $N_{g,n}$. ■

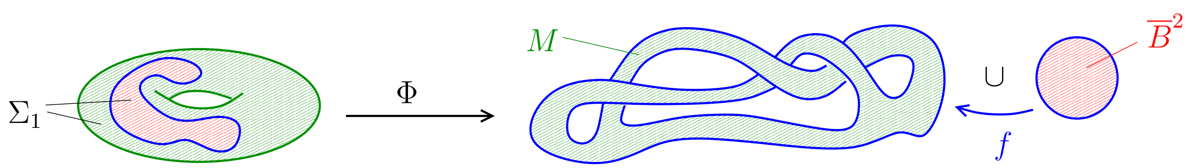


FIGURE 1525. Third illustration for the proof of Theorem 108.7.

Exercises for Chapter 108.

Exercise 108.1. We denote by $p \colon \mathbb{R}^3 \to \mathbb{R}$ the projection onto the third coordinate. Let $\gamma \colon S^1 \to \mathbb{R}^3$ be a smooth embedding such that the map $p \circ \gamma \colon S^1 \to \mathbb{R}$ has precisely two critical points. Show that $\gamma(S^1)$, viewed as a submanifold of $\mathbb{R}^3 \cup \{\infty\} = S^3$ is smoothly isotopic to the trivial knot.

Exercise 108.2. We consider the attaching map

$$\begin{array}{rcl} \varphi \colon \overline{B}^1 \times S^0 & \to & \overline{B}^2 \times \{\pm 1\} \\ (y, \epsilon) & \mapsto & \Big(\big(\epsilon \cdot \sqrt{1 - \tfrac{y^2}{4}}, \tfrac{y}{2}\big), \epsilon\Big). \end{array}$$

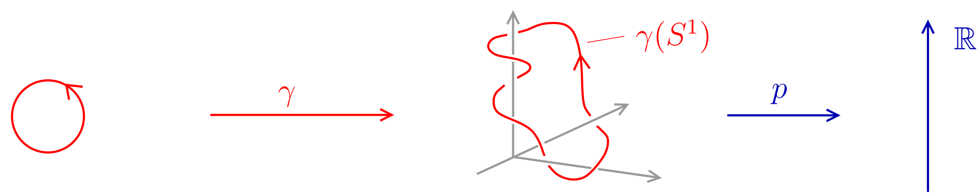


FIGURE 1526. Illustration of Exercise 108.1.

Show that $(\overline{B}^1 \times \overline{B}^1) \cup_{\varphi\colon \overline{B}^1 \times S^0 \to S^1 \times \{\pm 1\}} (\overline{B}^2 \times \{\pm 1\})$ is diffeomorphic to $\overline{B}^2$.

Remark.

(1) It follows easily from this exercise together with Theorem 8.36 that any attaching of a 1-handle to two disks is diffeomorphic to $\overline{B}^2$.

(2) By "rotating around the x-axis" one sees that a similar statement also holds in the higher-dimensional setting.

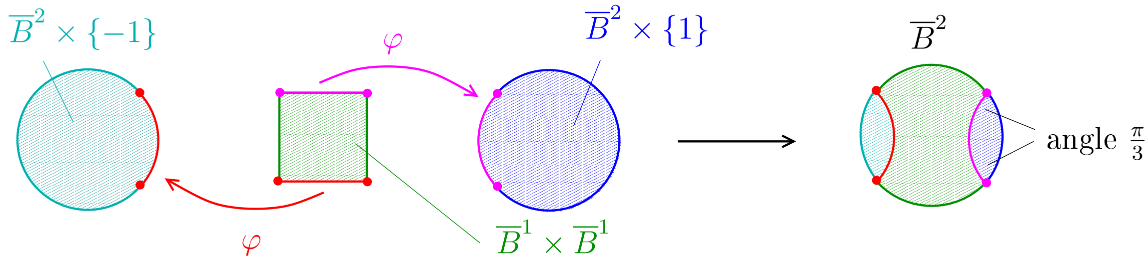


FIGURE 1527. Illustration for Exercise 108.2.

Part XII

Steenrod operations

109. Steenrod Operations

In this chapter we introduce the Steenrod operations. These are cohomology operations which satisfy several axioms. In the first three sections we will state the axioms and draw several interesting conclusions. Afterwards we will sketch a proof for the existence of the Steenrod operations.

109.1. **The axioms of Steenrod operations.**

Definition.

(1) Let A and B be abelian groups and let $m, n \in \mathbb{N}_0$. Any natural transformation from the contravariant functor

$$\mathrm{H}^m(-;A)\colon \text{category of topological spaces} \quad \to \quad \text{category of sets}$$

to the contravariant functor

$$\mathrm{H}^n(-;B)\colon \text{category of topological spaces} \quad \to \quad \text{category of sets}$$

is called a *cohomology operation of degree $n-m$*. In other words, a cohomology operation consists for any topological space X of a map $\Phi_X\colon\ \mathrm{H}^m(X;A) \to \mathrm{H}^n(X;B)$ of sets such that for any map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}^m(Y;A) \arrow[r, "f^*"] \arrow[d, "\Phi_Y"'] & \mathrm{H}^m(X;A) \arrow[d, "\Phi_X"] \\ \mathrm{H}^n(Y;B) \arrow[r, "f^*"] & \mathrm{H}^n(X;B). \end{tikzcd}$$

(2) A cohomology operation is called *additive* if each map $\mathrm{H}^m(X;A) \to \mathrm{H}^n(X;B)$ is a homomorphism.

(3) A cohomology operation is called *non-trivial* if there exists a topological space X such that the map $\mathrm{H}^m(X;A) \to \mathrm{H}^n(X;B)$ is non-trivial in the sense that the image is not just the trivial element.

With the obvious modifications the same definitions also apply to relative cohomology viewed as functors from the category of pairs of topological spaces to the category of sets.

Perhaps somewhat surprisingly we have already assembled a substantial list of non-trivial cohomology operations.

Examples.

(1) Let $n \in \mathbb{N}_0$ and let G be an abelian group. Given a pair of topological spaces (X, A) we consider the connecting homomorphism $\mathrm{H}^n(X,A;G) \to \mathrm{H}^{n+1}(X;G)$. It follows from Lemma 73.15 (3) that these homomorphisms define an additive cohomology operation of degree one.

(2) Let $\varphi\colon A \to B$ be a homomorphism between two abelian groups and let $n \in \mathbb{N}_0$. On page 1823 we saw that given any topological space X we obtain an induced homomorphism $\varphi_*\colon\ \mathrm{H}^n(X;A) \to \mathrm{H}^n(X;B)$. By Lemma 73.9 we know that these maps define an additive cohomology operation of degree zero.

(3) Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups. On page **??** we introduced the n-th Bockstein homomorphism $\beta_n\colon\ \mathrm{H}^n(X;C) \to \mathrm{H}^{n+1}(X;A)$. In

Lemma **??** we saw that the n-th Bockstein homomorphisms define an additive cohomology operation of degree one.

(4) Let $k \in \mathbb{N}$. It follows from Proposition 82.4 (1) that for any $m \in \mathbb{N}_0$ and any commutative ring R the maps[1478]

$$\begin{array}{rcl} \mathrm{H}^m(X,A;R) & \to & \mathrm{H}^{km}(X,A;R) \\ \varphi & \mapsto & \varphi^k = \underbrace{\varphi \cup \dots \cup \varphi}_{k \text{ times}} \end{array}$$

define a cohomology operation of degree $km-m$. Note that this type of a cohomology operation is in general not additive. For example, if $k \geq 2$, then it follows easily from the calculation of the cup product of $\mathbb{C}\mathrm{P}^k$, see Proposition 90.7, that the map $\mathrm{H}^2(\mathbb{C}\mathrm{P}^k;\mathbb{Z}) \to \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^k;\mathbb{Z})$ given by $\varphi \mapsto \varphi^k$ is not a homomorphism.

(5) If R is a commutative ring with $1+1=0$, then the cohomology operation $\varphi \mapsto \varphi^2$ is in fact additive. Indeed for any $\varphi, \psi \in \mathrm{H}^m(X,A;R)$ we have[1479]

$$\begin{array}{rcl} (\varphi+\psi)^2 & = & \varphi\cup\varphi+\varphi\cup\psi+\psi\cup\varphi+\psi\cup\psi \\ & \underset{\substack{\uparrow \\ \text{by Proposition 81.8 we have } \psi\cup\varphi = (-1)^{m^2}\cdot(\varphi\cup\psi)}}{=} & \varphi\cup\varphi+(1+(-1)^{m^2})\cdot\varphi\cup\psi+\psi\cup\psi \underset{\substack{\uparrow \\ \text{since } 1-1=0=1+1 \text{ in } R}}{=} \varphi\cup\varphi+\psi\cup\psi = \varphi^2+\psi^2. \end{array}$$

Note that all of the cohomology operations that we have introduced above are of non-negative degree. This raises the following amusing question.

Question 109.1. *Does there exist a non-trivial cohomology operation of negative degree?*

We will come back to this question in Proposition **??**.

The following theorem gives us new and interesting cohomology operations. The theorem was first proved by Norman Steenrod [**Stee47**] in 1947.

Theorem 109.2. *There exists a family*

$$\mathrm{Sq}^i\colon\ \mathrm{H}^n(X,A;\mathbb{Z}_2) \;\to\; \mathrm{H}^{n+i}(X,A;\mathbb{Z}_2) \qquad \textit{with } i \in \mathbb{N}_0 \textit{ and } n \in \mathbb{N}_0$$

of additive cohomology operations which satisfies the following axioms:

(1) $\mathrm{Sq}^0 = \mathrm{id}$.
(2) *For any* $\varphi \in \mathrm{H}^i(X,A;\mathbb{Z}_2)$ *we have* $\mathrm{Sq}^i(\varphi) = \varphi^2 = \varphi\cup\varphi \in \mathrm{H}^{2i}(X,A;\mathbb{Z}_2)$.
(3) *For any* $\varphi \in \mathrm{H}^j(X,A;\mathbb{Z}_2)$ *with* $i > j$ *we have* $\mathrm{Sq}^i(\varphi) = 0$.
(4) *Let* (X,A,B) *be an excisive triad of topological spaces. For any* $\varphi \in \mathrm{H}^r(X,A;\mathbb{Z}_2)$ *any* $\psi \in \mathrm{H}^s(X,B;\mathbb{Z}_2)$ *and any* $k \in \mathbb{N}_0$ *we have*[1480]

$$\mathrm{Sq}^k(\varphi\cup\psi) \;=\; \sum_{i=0}^{k} \underbrace{\mathrm{Sq}^i(\varphi)}_{\in \mathrm{H}^{r+i}(X,A;\mathbb{Z}_2)} \;\cup\; \underbrace{\mathrm{Sq}^{k-i}(\psi)}_{\in \mathrm{H}^{s+k+i}(X,B;\mathbb{Z}_2)} \;\in\; \mathrm{H}^{r+s+k}(X,A\cup B;\mathbb{Z}_2).$$

This equality is often called the Cartan formula.

[1478] We refer to page 2005 for the definition of the relative cup product.

[1479] This can be viewed as the cohomological "freshman's dream". Perhaps it should called the incoming PhD student's dream.

In the next two sections we will discuss some properties and applications of these cohomology operations. Only after these two sections, once we are convinced that Steenrod operations are indeed useful and interesting, will we provide a proof for Theorem 109.2.

Remark.

(1) The first three axioms of Theorem 109.2 can be summarized as follows: Given any $\varphi \in \mathrm{H}^k(X, A; \mathbb{Z}_2)$ we obtain the following table

$\mathrm{Sq}^0(\varphi)$	$\mathrm{Sq}^1(\varphi)$	$\dots$	$\mathrm{Sq}^{k-1}(\varphi)$	$\mathrm{Sq}^k(\varphi)$	$\mathrm{Sq}^{k+1}(\varphi)$	$\dots$
φ	?	$\dots$	?	φ^2	0	$\dots$

(2) Throughout the remainder of this chapter we will almost invariably consider cohomology with $\mathbb{Z}_2$-coefficients. We point out that by Proposition 81.8 the cup product on the $\mathbb{Z}_2$-cohomology $\mathrm{H}^*(X, A; \mathbb{Z}_2)$ of a given pair of topological spaces (X, A) is actually *commutative*.

The following theorem is [**SE62**, Theorem VIII.3.10].

Theorem 109.3. *Any two families of additive cohomology operations that satisfy Axioms* (1) *to* (4) *are identical.*

This theorem allows us to give a name to the above cohomology operations.

Definition. We refer to the cohomology operations from Theorem 109.2 as the *Steenrod operations.*[1481]

The following theorem shows that the first Steenrod operation Sq^1 is actually an old acquaintance of ours.

Theorem 109.4. *The first Steenrod operation* Sq^1 *is the Bockstein homomorphism corresponding to the short exact sequence* $0 \to \mathbb{Z}_2 \xrightarrow{[a]\mapsto[2a]} \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$.

Proof. We will not make use of this theorem, thus we decline to provide a proof. The theorem is proved in [**Bre93**, p. 418] or alternatively in [**MTa68**, p. 23] or [**Hat02**, Theorem 4.L.12]. ■

Evidently our goal now is to compute Steenrod operations in specific examples. We start out with the following slightly disappointing lemma.

Lemma 109.5. *For any wedge of finitely many spheres all Steenrod operations in degrees* ≥ 1 *are zero.*

Proof. We will prove the lemma in Exercise 109.2. ■

In remainder of this section we intend to compute the Steenrod operations for the real projective spaces $\mathbb{R}\mathrm{P}^n$ with $n \in \mathbb{N}_0$. Before we consider the real projective spaces in detail it is convenient to reformulate the Cartan formula.

[1480]We refer to page 1851 for the definition of an excisive triad. By Proposition 74.12 most "reasonable" triads are in fact excisive. In fact we will only use the special cases that $A = B$, or that at least one of A or B is the empty set. We refer again to page 2005 for the definition of the relative cup product.

[1481]Sometimes they are also called Steenrod squares, and the term Steenrod operations is used for a more general concept.

For the reformulation we need the following notation:

Notation. Let (X,A) be a pair of topological spaces. Given $\varphi \in \mathrm{H}^k(X,A;\mathbb{Z}_2)$ we write

$$\mathrm{Sq}(\varphi) \;:=\; \mathrm{Sq}^0(\varphi)+\mathrm{Sq}^1(\varphi)+\dots+\mathrm{Sq}^k(\varphi) \;\in\; \mathrm{H}^*(X,A;\mathbb{Z}_2)=\bigoplus_{n\in\mathbb{N}_0}\mathrm{H}^n(X,A;\mathbb{Z}_2).$$

Furthermore we extend the homomorphism Sq in the obvious way to a homomorphism $\mathrm{Sq}\colon\, \mathrm{H}^*(X;\mathbb{Z}_2)\to \mathrm{H}^*(X;\mathbb{Z}_2)$.

Remark. Let (X,A) be a pair of topological spaces and let $\varphi\in \mathrm{H}^k(X,A;\mathbb{Z}_2)$. Recall that by Axiom (3) given $\varphi\in\mathrm{H}^k(X,A;\mathbb{Z}_2)$ we have $\mathrm{Sq}^i(\varphi)=0$ for $i>k$. Thus we can write

$$\mathrm{Sq}(\varphi) \;=\; \mathrm{Sq}^0(\varphi)+\mathrm{Sq}^1(\varphi)+\dots+\mathrm{Sq}^k(\varphi) \;=\; \sum_{i\in\mathbb{N}_0}\mathrm{Sq}^i(\varphi).$$

With this notation we can formulate the following lemma.

Lemma 109.6. *Let (X,A) be a pair of topological spaces.*

(1) *Given any $\varphi,\psi\in\mathrm{H}^*(X,A;\mathbb{Z}_2)$ we have*
$$\mathrm{Sq}(\varphi\cup\psi) \;=\; \mathrm{Sq}(\varphi)\cup\mathrm{Sq}(\psi).$$
(2) *Given any $\varphi\in\mathrm{H}^*(X,A;\mathbb{Z}_2)$ and given any $k\in\mathbb{N}_0$ we have $\mathrm{Sq}(\varphi^k)=\mathrm{Sq}(\varphi)^k$.*

Proof. The first statement follows easily from the Cartan formula. Indeed, we see that

$$\begin{aligned}\mathrm{Sq}(\varphi\cup\psi) &= \sum_{k\in\mathbb{N}_0}\mathrm{Sq}^k(\varphi\cup\psi) \;\overset{\substack{\text{Cartan formula, i.e. Axiom (4)}\\ \downarrow}}{=} \sum_{k\in\mathbb{N}_0}\sum_{i=0}^{k}\mathrm{Sq}^i(\varphi)\cup\mathrm{Sq}^{k-i}(\psi)\\ &= \sum_{r\in\mathbb{N}_0}\sum_{s\in\mathbb{N}_0}\mathrm{Sq}^r(\varphi)\cup\mathrm{Sq}^s(\psi) = \Big(\sum_{r\in\mathbb{N}_0}\mathrm{Sq}^r(\varphi)\Big)\cup\Big(\sum_{s\in\mathbb{N}_0}\mathrm{Sq}^s(\psi)\Big)=\mathrm{Sq}(\varphi)\cup\mathrm{Sq}(\psi).\end{aligned}$$

The second statement is an immediate consequence of the first statement. ■

Before we state a corollary we introduce the following semi-obvious definition.

Definition. Let $r,s\in\mathbb{N}_0$. If $0\le r\le s$, then we set $\binom{s}{r}=\frac{s!}{r!(s-r)!}$, otherwise we set $\binom{s}{r}=0$.

Now we can state the promised corollary.

Corollary 109.7. *Let X be a topological space. For any cohomology class $\varphi\in\mathrm{H}^1(X;\mathbb{Z}_2)$ of degree* one*, any $k\in\mathbb{N}$ and any $i\in\mathbb{N}$ we have*

$$\mathrm{Sq}^i(\varphi^k) \;=\; \binom{k}{i}\cdot\varphi^{k+i} \;\in\; \mathrm{H}^{k+i}(X;\mathbb{Z}_2).$$

Examples.

(1) By Proposition 90.16 we know that given any $n\in\mathbb{N}_0$ we have a ring isomorphism $(\mathrm{H}^*(\mathbb{RP}^n;\mathbb{Z}_2),\cup)\cong\mathbb{Z}_2[x]/(x^{n+1})$, where $x\in\mathrm{H}^1(\mathbb{RP}^n;\mathbb{Z}_2)$, and we have a ring isomorphism $(\mathrm{H}^*(\mathbb{RP}^\infty;\mathbb{Z}_2),\cup)\cong\mathbb{Z}_2[x]$, where $x\in\mathrm{H}^1(\mathbb{RP}^\infty;\mathbb{Z}_2)$. Since Steenrod operations are additive we see that the result of Corollary 109.7 gives us the promised calculations of all Steenrod operations for the real projective spaces.

(2) In Exercise 109.3 we will use a similar argument to calculate the Steenrod operations on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}_2)$.

Proof. We calculate that

$$\mathrm{Sq}(\varphi^k) \underset{\substack{\uparrow\\ \text{Lemma 109.6 (2)}}}{=} \mathrm{Sq}(\varphi)^k \underset{\substack{\uparrow\\ \text{by Axiom (3)}\\ \text{since } \varphi\in \mathrm{H}^1(X;\mathbb{Z}_2)}}{=} (\mathrm{Sq}^0(\varphi)+\mathrm{Sq}^1(\varphi))^k \underset{\substack{\uparrow\\ \text{Axioms (1) and (2)}}}{=} (\varphi+\varphi^2)^k \underset{\substack{\uparrow\\ \text{since the ring } (\mathrm{H}^*(X;\mathbb{Z}_2),\cup)\\ \text{is commutative}}}{=} \varphi^k\cup(1+\varphi)^k = \sum_{j=0}^k \binom{k}{j}\varphi^{k+j}.$$

The desired conclusion follows from comparing the terms in degree $k+i$ on both sides. ■

109.2. **Steenrod operations and connecting homomorphisms.** This section is completely dedicated to the proof of the following proposition. In the subsequent section we will see that the proposition has powerful applications.

Proposition 109.8. *If (X,A) is a pair of topological spaces, then the Steenrod operations commute with the connecting homomorphisms in cohomology. More precisely, given any $i,n\in\mathbb{N}_0$ the following diagram commutes:*

$$\begin{tikzcd} \mathrm{H}^n(A;\mathbb{Z}_2) \arrow[r, "\delta"] \arrow[d, "\mathrm{Sq}^i"'] & \mathrm{H}^{n+1}(X,A;\mathbb{Z}_2) \arrow[d, "\mathrm{Sq}^i"] \\ \mathrm{H}^{n+i}(A;\mathbb{Z}_2) \arrow[r, "\delta"] & \mathrm{H}^{n+i+1}(X,A;\mathbb{Z}_2). \end{tikzcd}$$

In the proof of Proposition 109.8 we will make use of the cross product on cohomology which we now introduce.

Definition. Let R be a commutative ring, let X be a topological space and let (Y,B) be a pair of topological spaces. Furthermore let $\varphi\in\mathrm{H}^m(X;R)$ and $\psi\in\mathrm{H}^n(Y,B;R)$. We denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the projections. We define the *cross product*[1482]

$$\varphi\times\psi \;:=\; \underbrace{p^*(\varphi)}_{\substack{\uparrow\\ \text{lies in } \mathrm{H}^m(X\times Y;R)}} \cup \underbrace{q^*(\psi)}_{\substack{\uparrow\\ \text{lies in } \mathrm{H}^n(X\times Y,X\times B;R)}} \;\in\; \mathrm{H}^{m+n}(X\times Y,X\times B;R).$$

The following proposition basically says that in Axiom (4), i.e. in the Cartan formula, we can replace the cup product by the cross product.

Proposition 109.9. *Let X be a topological space and let (Y,B) be a pair of topological spaces. For any $\varphi\in\mathrm{H}^r(X;\mathbb{Z}_2)$, $\psi\in\mathrm{H}^s(Y,B;\mathbb{Z}_2)$ and $n\in\mathbb{N}_0$ the following equality holds:*

$$\mathrm{Sq}^n(\varphi\times\psi) \;=\; \sum_{i=0}^n \mathrm{Sq}^i(\varphi)\times\mathrm{Sq}^{n-i}(\psi) \;\in\; \mathrm{H}^{r+s+n}(X\times Y,X\times B;\mathbb{Z}_2).$$

[1482] Recall that on page 1964, given two *homology* classes $\mu\in\mathrm{H}_m(X;\mathbb{Z})$ and $\nu\in\mathrm{H}_n(Y;\mathbb{Z})$ we defined the *cross product* $\mu\times\nu\in\mathrm{H}_{m+n}(X\times Y;\mathbb{Z})$. It should usually be clear from the context whether in a given situation we mean the cross product on homology or on cohomology.

PROOF (∗). Let X be a topological space and let (Y,B) be a pair of topological spaces. Furthermore let $\varphi \in \mathrm{H}^r(X;\mathbb{Z}_2)$ and $\psi \in \mathrm{H}^s(Y,B;\mathbb{Z}_2)$. We denote by $p\colon X\times Y \to X$ and $q\colon X \times Y \to Y$ the projections. We start out with the following claim.

Claim. We have the following equalities:[1483]

(a) $p^*(\varphi) = \varphi \times 1_Y \in \mathrm{H}^r(X\times Y;\mathbb{Z}_2)$ and $q^*(\psi) = 1_X \times \psi \in \mathrm{H}^s(X\times Y;\mathbb{Z}_2)$.
(b) $\varphi\times\psi = (\varphi\times 1_Y)\cup(1_X\times\psi) \in \mathrm{H}^{r+s}(X\times Y;\mathbb{Z}_2)$.

Let us provide the proof of the claim.

(a) We perform the following easy calculation:

$$\varphi \times 1_Y = p^*(\varphi)\cup q^*(1_Y) \underset{\substack{\uparrow \\ \text{by definition we have} \\ p^*(1_X)=q^*(1_Y)=1_{X\times Y}}}{=} p^*(\varphi)\cup 1_{X\times Y} \underset{\substack{\uparrow \\ 1_{X\times Y}\text{ is multiplicatively neutral,} \\ \text{see Proposition 81.7}}}{=} p^*(\varphi).$$

Evidently the same argument also shows that $q^*(\psi) = 1_X\times\psi$.

(b) This statement follows from (a) and the definition of the cross product.

⊞

Now we turn to the proof of the actual statement of the proposition. We have the following equalities in $\mathrm{H}^{r+s+n}(X\times Y;\mathbb{Z}_2)$:

$$\begin{aligned}
\mathrm{Sq}^n(\varphi\times\psi) &\overset{\substack{\text{part (b) of the claim}\\ \downarrow}}{=} \mathrm{Sq}^n\big((\varphi\times 1_Y)\cup(1_X\times\psi)\big) \overset{\substack{\text{by the Cartan formula, i.e. by Axiom (4)}\\ \downarrow}}{=} \sum_{i=0}^n \mathrm{Sq}^i(\varphi\times 1_Y)\cup \mathrm{Sq}^{n-i}(1_X\times\psi) \\
&\underset{\substack{\uparrow\\ \text{part (a) of the claim}}}{=} \sum_{i=0}^n \mathrm{Sq}^i(p^*(\varphi))\cup\mathrm{Sq}^{n-i}(q^*(\psi)) \underset{\substack{\uparrow\\ \text{since }\mathrm{Sq}^i\text{ and }\mathrm{Sq}^{n-i}\text{ are natural transformations}}}{=} \sum_{i=0}^n p^*(\mathrm{Sq}^i(\varphi))\cup q^*(\mathrm{Sq}^{n-i}(\psi)) \\
&\underset{\substack{\uparrow\\ \text{by definition of the cross product}}}{=} \sum_{i=0}^n \mathrm{Sq}^i(\varphi)\times\mathrm{Sq}^{n-i}(\psi).
\end{aligned}$$

We have thus obtained the desired equality. ■

The proof of Proposition 109.8 rests on the following lemma.

Lemma 109.10. *Let X be a topological space, let (Y,B) be a pair of topological spaces and let R be a commutative ring. If $\varphi\in\mathrm{H}^m(X;R)$ and $\psi\in\mathrm{H}^n(B;R)$, then*

$$\underset{\substack{\uparrow\\ \textit{connecting homomorphism}\\ \textit{in cohomology for the}\\ \textit{pair } (X\times Y,X\times B)}}{\delta}(\overbrace{\varphi\times\psi}^{\in \mathrm{H}^{m+n}(X\times B;R)}) = (-1)^m\cdot\varphi\times\overbrace{\underset{\substack{\uparrow\\ \textit{connecting homomorphism}\\ \textit{in cohomology for the pair } (Y,B)}}{\delta}\psi}^{\in\mathrm{H}^{n+1}(Y,B;R)} \in \mathrm{H}^{m+n+1}(X\times Y, X\times B;R).$$

[1483] Recall that given a topological space W we denote by $1_Z\in\mathrm{H}^0(W;\mathbb{Z}_2)$ the cohomology class that is represented by the cocycle that assigns to each singular 0-simplex the value $1\in\mathbb{Z}_2$.

Proof of Lemma 109.10. This equality follows fairly easily from Lemma 81.2, the definition of the cross product via the cup product and the explicit description of the connecting homomorphism on page 1820. We leave it to the reader to fill in the details. ■

Proof of Proposition 109.8. We start out with a warm-up case.

Claim. Let Z be a topological space and let $a < b$ be two real numbers. The conclusion of the proposition holds for the pair $(Z \times [a,b], Z \times \partial([a,b]))$ of topological spaces.

Let $\theta \in \mathrm{H}^n(Z \times \partial([a,b]);\mathbb{Z}_2) = \mathrm{H}^n(Z \times \{a,b\};\mathbb{Z}_2)$. Recall that we need to show that $\mathrm{Sq}^i(\delta\theta) = \delta(\mathrm{Sq}^i(\theta))$. First note that θ can be written as $\theta = \varphi \times \psi$ for some $\varphi \in \mathrm{H}^n(Z;\mathbb{Z}_2)$ and some $\psi \in \mathrm{H}^0(\partial([a,b]);\mathbb{Z}_2) = \mathrm{H}^0(\{a,b\};\mathbb{Z}_2)$.[1484] Note that we have the following equalities in $\mathrm{H}^{n+i+1}(Z \times [a,b], Z \times \partial([a,b]);\mathbb{Z}_2)$:

$$
\begin{aligned}
\mathrm{Sq}^i(\delta\theta) &\overset{\substack{\text{since } \theta=\varphi\times\psi \\ \downarrow}}{=} \mathrm{Sq}^i(\delta(\varphi\times\psi)) \overset{\substack{\text{by Lemma 109.10, we also} \\ \text{use that } (-1)^n = 1 \in \mathbb{Z}_2 \\ \downarrow}}{=} \mathrm{Sq}^i(\varphi\times\delta\psi) \overset{\substack{\text{Proposition 109.9} \\ \downarrow}}{=} \sum_{j=0}^{i} \mathrm{Sq}^{i-j}(\varphi)\times\mathrm{Sq}^j(\delta\psi) \overset{\substack{\text{since } \delta\psi \in \mathrm{H}^1([a,b],\partial([a,b]);\mathbb{Z}_2) \\ \text{and since } \mathrm{H}^i([a,b],\partial([a,b]);\mathbb{Z}_2)=0 \\ \text{for every } i\geq 2 \text{ we know that} \\ \mathrm{Sq}^j(\delta\psi)=0 \text{ for } j\geq 1 \\ \downarrow}}{=} \mathrm{Sq}^i(\varphi)\times\mathrm{Sq}^0(\delta\psi) \\
&\underset{\substack{\uparrow \\ \text{Axiom (1) says} \\ \text{that } \mathrm{Sq}^0=\mathrm{id}}}{=} \mathrm{Sq}^i(\varphi)\times\delta\psi \underset{\substack{\uparrow \\ \text{Lemma 109.10}}}{=} \delta(\mathrm{Sq}^i(\varphi)\times\psi) \underset{\substack{\uparrow \\ \text{by Proposition 109.9 together with Axioms (1) and (3)} \\ \text{and the fact that } \psi\in\mathrm{H}^0(\partial([a,b]);\mathbb{Z}_2)}}{=} \delta(\mathrm{Sq}^i(\varphi\times\psi)) = \delta(\mathrm{Sq}^i(\theta)).
\end{aligned}
$$

⊞

The remainder of the proof of the proposition consists in cleverly reducing the general case to the above special case. Thus let (X,A) be a pair of topological spaces. We denote by $i\colon A \to X$ the inclusion map. We set

$$\mathrm{Cyl}(i) = \begin{array}{c}\text{mapping cylinder of the}\\ \text{inclusion } i\colon A\to X\end{array} := ((A\times[0,1])\sqcup X)/\sim \quad \text{where } (a,1)\sim a \text{ for all } a\in A.$$

We point out that by Exercise 24.8 and Lemma 24.6 we know that the obvious inclusions $A\times[0,1] \to \mathrm{Cyl}(i)$ and $X \to \mathrm{Cyl}(i)$ are embeddings. Thus we can view $A\times[0,1]$ and X as subsets of $\mathrm{Cyl}(i)$. Now we consider the following subsets of the mapping cylinder $\mathrm{Cyl}(i)$:

$$B := (A\times[\tfrac{1}{2},1])\cup X \quad \text{and} \quad C := A\times\{0\}.$$

We refer to Figure 1528 for an illustration. The five statements in the following claim break

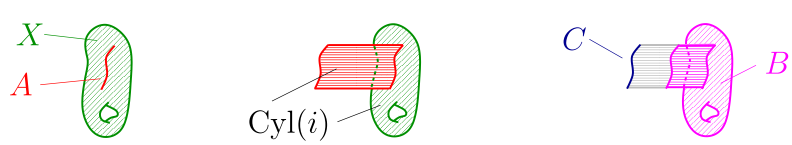


Figure 1528

the proof of the proposition into five manageable steps.

[1484] This statement follows quite easily from the fact that $Z\times\partial([a,b]) = (Z\times\{a\})\sqcup(Z\times\{b\})$, the observation that $\mathrm{H}^0(\{a,b\}) = \mathbb{Z}\cdot 1_{\{a\}}\oplus\mathbb{Z}\cdot 1_{\{b\}}$, Proposition 81.7 and Lemma 83.9. We leave it to the assiduous reader to assemble these steps in the right logical order.

Claim.

(1) The conclusion of the proposition holds for $(A \times [0, \frac{1}{2}], A \times \partial([0, \frac{1}{2}]))$.
(2) The conclusion of the proposition holds for $(\mathrm{Cyl}(i), B \cup C)$.
(3) The conclusion of the proposition holds for $(\mathrm{Cyl}(i), C)$.
(4) The conclusion of the proposition holds for $(\mathrm{Cyl}(i), A \times [0, 1])$.
(5) The conclusion of the proposition holds for (X, A).

The five statements of the claim are illustrated in Figure 1529.

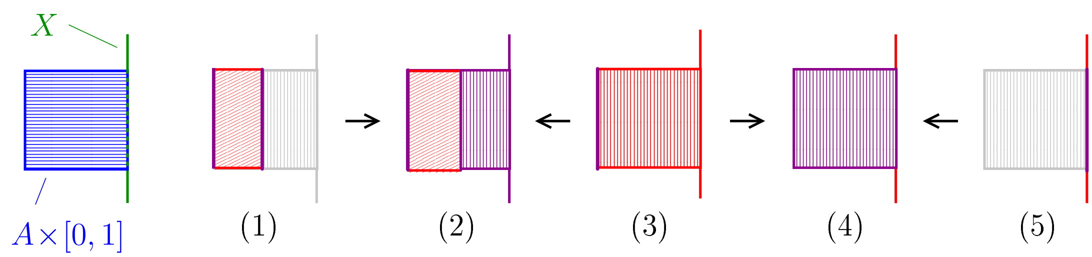


FIGURE 1529

Before we work our way through the five steps we need to make two related observations. Let $f\colon (X, A) \to (Y, B)$ be a map of pairs of topological spaces. We obtain the following cubical diagram:

$$\begin{tikzcd}
\mathrm{H}^n(Y;\mathbb{Z}_2) \arrow[rr, "\delta"] \arrow[dr, "f^*"] \arrow[dd, "\mathrm{Sq}^i"'] & & \mathrm{H}^{n+1}(Y,B;\mathbb{Z}_2) \arrow[dr, "f^*"] \arrow[dd, "\mathrm{Sq}^i"] & \\
& \mathrm{H}^n(X;\mathbb{Z}_2) \arrow[rr, "\delta"] \arrow[dd, "\mathrm{Sq}^i"'] & & \mathrm{H}^{n+1}(X,A;\mathbb{Z}_2) \arrow[dd, "\mathrm{Sq}^i"] \\
\mathrm{H}^{n+i}(B;\mathbb{Z}_2) \arrow[rr, "\delta"] \arrow[dr, "f^*"] & & \mathrm{H}^{n+i+1}(Y,B;\mathbb{Z}_2) \arrow[dr, "f^*"] & \\
& \mathrm{H}^{n+i}(A;\mathbb{Z}_2) \arrow[rr, "\delta"] & & \mathrm{H}^{n+i+1}(X,A;\mathbb{Z}_2).
\end{tikzcd}$$

It follows from the naturality of the Steenrod operations that the parallelograms to the left and right commute. Furthermore, it follows from the naturality of the connecting homomorphism, see Lemma 73.15, that the top and bottom parallelograms commute. Using these facts we can make the following two observations:

(i) If the square in the front commutes and if the map f^* to the bottom right is an isomorphism, then it follows from a stroll through the cube, as indicated in Figure 1530, that the square in the back also commutes.
(ii) If the square in the back commutes and if the map f^* to the top left is an isomorphism, then it follows as in (i) that the square in the front also commutes.

Now we turn to the actual proof of the five steps in the claim.

(1) This statement follows immediately from the previous claim.
(2) We consider the inclusion map $f\colon (A \times [0, \frac{1}{2}], A \times \partial([0, \frac{1}{2}])) \to (\mathrm{Cyl}(i), B \cup C)$ and we consider the corresponding cube. By (1) we know that the square in the front commutes. We leave it to the reader to cleverly apply the Excision Theorem for

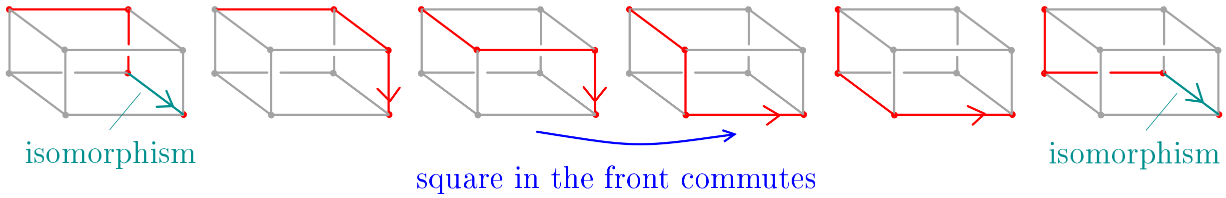


FIGURE 1530

Cohomology Groups 74.1 to show that the map f^* to the bottom right of the cube is an isomorphism. Thus we obtain from (i) that the square in the back commutes.

(3) We consider the inclusion map $f\colon (\mathrm{Cyl}(i), C) \to (\mathrm{Cyl}(i), B \cup C)$ and we consider the corresponding cube. By (2) we know that the square in the back commutes. Since on the first entry f is the identity we see that the top left map f^* is an isomorphism. Thus we obtain from (ii) that the square in the front also commutes.

(4) We consider the inclusion map $f\colon (\mathrm{Cyl}(i), A \times \{0\}) \to (\mathrm{Cyl}(i), \overbrace{A \times [0,1]}^{=C})$. By (3) we know that the square in the front commutes. Since $A \times \{0\}$ is a deformation retract of $A \times [0,1]$ we obtain from Lemma 73.13 (4) that the map f^* to the bottom right of the cube is an isomorphism. Thus we obtain from (i) that the square in the back also commutes.

(5) We consider the obvious inclusion

$$\begin{aligned} f\colon X &\to \mathrm{Cyl}(i) \\ x &\mapsto [(x,1)]. \end{aligned}$$

It also defines a map $f\colon (X, A) \to (\mathrm{Cyl}(i), A \times [0,1])$ of pairs of topological spaces. We consider the corresponding cube. By (4) we know that the square in the back commutes. By Lemma 24.8 (1) the map $f\colon X \to \mathrm{Cyl}(i)$ is in fact a homotopy equivalence. Thus it follows from Lemma 73.13 that the top left f^* is an isomorphism. Therefore we obtain from (ii) that the square in the front commutes.

■

109.3. **Steenrod operations and suspensions.** In Proposition 109.8 we saw that Steenrod operations commute with connecting homomorphisms in cohomology. In this section we will use this fact to show that Steenrod operations behave well under taking suspensions. This fact will help us to obtain new information on homology groups of spheres.

First recall that in Proposition 45.7 we saw that taking the suspension $\Sigma(X)$ of a topological space shifts the reduced homology groups "upwards by one". The following proposition says that the same phenomenon occurs when we consider cohomology groups.

Proposition 109.11. *Let X be a non-empty topological space, let $n \in \mathbb{N}$[1485] and let G be an abelian group. As on page 1178 we denote by C_-X and C_+X the "lower and upper half" of the suspension $\Sigma(X)$. We refer to Figure 1531 for an illustration.*

(1) *The following maps are isomorphisms:*[1486]

$$\mathrm{H}^n(X;G) \xrightarrow{\delta} \mathrm{H}^{n+1}(C_+X,X;G) \leftarrow \mathrm{H}^{n+1}(\Sigma(X),C_-X;G) \rightarrow \mathrm{H}^{n+1}(\Sigma(X);G).$$

Here the map to the left is the connecting homomorphism of the pair (C_+X,X) from Lemma 73.15 The remaining two horizontal maps are induced by the obvious inclusion maps.

(2) *The isomorphism*

$$\Sigma_X \colon \mathrm{H}^n(X;G) \xrightarrow{\cong} \mathrm{H}^{n+1}(\Sigma(X);G)$$

that is given by the maps in (1) is in fact a natural isomorphism.

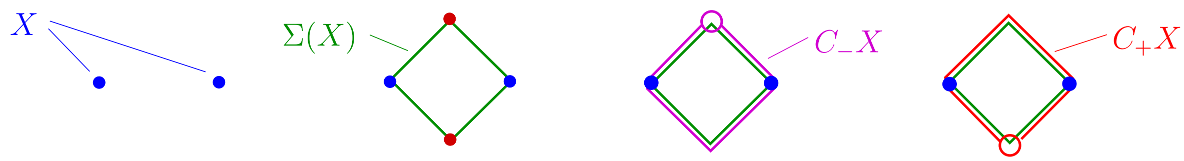


FIGURE 1531. Illustration for Proposition 109.11.

PROOF. The proof is a slight variation on the proof of Proposition 45.7. We leave it to the reader to fill in the details. ■

The following proposition shows that Steenrod operations behave well under taking suspensions.

Proposition 109.12. *Given any topological space X, any $n \in \mathbb{N}$ and any $i \in \mathbb{N}_0$ the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}^n(X;\mathbb{Z}_2) & \xrightarrow[\cong]{\Sigma_X} & \mathrm{H}^{n+1}(\Sigma(X);\mathbb{Z}_2) \\ \downarrow \mathrm{Sq}^i & & \downarrow \mathrm{Sq}^i \\ \mathrm{H}^{n+i}(X;\mathbb{Z}_2) & \xrightarrow[\cong]{\Sigma_X} & \mathrm{H}^{n+i+1}(\Sigma(X);\mathbb{Z}_2). \end{array}$$

Remark. The proposition says in particular that if there exists a non-trivial Steenrod operation on a topological space X, then its suspension inherits a non-trivial Steenrod operation. This is in so far interesting as we know by Lemma 82.7 that the cup product vanishes on suspensions. The fact that suspensions can have non-trivial Steenrod operations will allow us to prove several hitherto inaccessible results.

[1485] Basically the same statement works for $n = 0$, the only nuisance is that in that case we need to work with reduced cohomology groups.

[1486] Here we use Lemma 24.2 (3) to identify X with the subset $C_-X \cap C_+X$ of $\Sigma(X)$.

Proof. We consider the following diagram

$$\begin{tikzcd}
\mathrm{H}^n(X;\mathbb{Z}_2) \arrow[r, "\delta"'] \arrow[rrr, bend left, "\Sigma_X"] \arrow[d, "\mathrm{Sq}^i"] & \mathrm{H}^{n+1}(C_+X,X;\mathbb{Z}_2) \arrow[d, "\mathrm{Sq}^i"] & \mathrm{H}^{n+1}(\Sigma(X),C_-X;\mathbb{Z}_2) \arrow[l, "\cong"'] \arrow[r, "\cong"'] \arrow[d, "\mathrm{Sq}^i"] & \mathrm{H}^{n+1}(\Sigma(X);\mathbb{Z}_2) \arrow[d, "\mathrm{Sq}^i"] \\
\mathrm{H}^{n+i}(X;\mathbb{Z}_2) \arrow[r, "\delta"] \arrow[rrr, bend right, "\Sigma_X"'] & \mathrm{H}^{n+1+i}(C_+X,X;\mathbb{Z}_2) & \mathrm{H}^{n+1+i}(\Sigma(X),C_-X;\mathbb{Z}_2) \arrow[l, "\cong"] \arrow[r, "\cong"] & \mathrm{H}^{n+1+i}(\Sigma(X);\mathbb{Z}_2).
\end{tikzcd}$$

The first square commutes by Proposition 109.8. The two other squares commute by the naturality of the Steenrod operation Sq^i. Thus we see that, as desired, the diagram commutes. ■

Next we recall that by the definitions on pages 1071 and 2212 we have three different types of Hopf maps:

Definition. We consider the following maps

$$\underbrace{\begin{array}{rcl} H\colon S^3 & \to & S^2=\mathbb{CP}^1 \\ \underbrace{(z_1,z_2)}_{\in\mathbb{C}^2} & \mapsto & [z_1:z_2] \end{array}}_{\text{the Hopf map}} \quad \underbrace{\begin{array}{rcl} \mathrm{H}_{\mathbb{H}}\colon S^7 & \to & S^4=\mathbb{HP}^1 \\ \underbrace{(z_1,z_2)}_{\in\mathbb{H}^2} & \mapsto & [z_1:z_2] \end{array}}_{\text{quaternionic Hopf map}} \quad \text{and} \quad \underbrace{\begin{array}{rcl} \mathrm{H}_{\mathbb{O}}\colon S^{15} & \to & S^8=\mathbb{OP}^1 \\ \underbrace{(z_1,z_2)}_{\in\mathbb{O}^2} & \mapsto & [z_1:z_2]. \end{array}}_{\text{octonionic Hopf map}}$$

In Theorem 91.3 we showed that the Hopf map $H\colon S^3\to S^2$ is not null-homotopic. The key to doing so were the following two observations:

(1) The mapping cone $\mathrm{Cone}(H\colon S^3\to S^2)$ is homeomorphic to $\mathbb{CP}^2$.
(2) The cup product on $\mathbb{CP}^2$ has the property that for a generator $\varphi\in \mathrm{H}^2(\mathbb{CP}^2;\mathbb{Z})$ the square φ^2 is non-zero.

Since the Steenrod operations are a generalization of the mod 2 "cup product square" and since Steenrod operations behave well with respect to suspensions it seems reasonable that we can use Steenrod operations to show that the suspensions of the Hopf maps are not null-homotopic.

Indeed, we can prove the following theorem.

Theorem 109.13.

(1) *For any $n\geq 2$ the $(n-2)$-fold suspension*

$$\Sigma^{n-2}(H)\colon \underbrace{\Sigma^{n-2}(S^3)}_{=S^{n+1}} \;\to\; \underbrace{\Sigma^{n-2}(S^2)}_{=S^n}$$

of the Hopf map defines a non-trivial element in $\pi_{n+1}(S^n)$. In particular for any $n\geq 2$ we have $\pi_{n+1}(S^n)\neq 0$.

(2) *For any $n\geq 4$ the $(n-4)$-fold suspension*

$$\Sigma^{n-4}(\mathrm{H}_{\mathbb{H}})\colon \underbrace{\Sigma^{n-4}(S^7)}_{=S^{n+3}} \;\to\; \underbrace{\Sigma^{n-4}(S^4)}_{=S^n}$$

of the quaternionic Hopf map defines a non-trivial element in $\pi_{n+3}(S^n)$*. In particular for any* $n \geq 4$ *we have* $\pi_{n+3}(S^n) \neq 0$.

(3) *For any* $n \geq 8$ *the* $(n-8)$*-fold suspension*

$$\Sigma^{n-8}(\mathrm{H}_{\mathbb{O}})\colon \underbrace{\Sigma^{n-8}(S^{15})}_{=S^{n+7}} \to \underbrace{\Sigma^{n-8}(S^{8})}_{=S^{n}}$$

of the octonionic Hopf map defines a non-trivial element in $\pi_{n+7}(S^n)$*. In particular for any* $n \geq 8$ *we have* $\pi_{n+7}(S^n) \neq 0$.

Before we turn to the proof of Theorem 109.13 we recall, for the reader's convenience, the following, frequently overlooked, proposition which says that taking suspensions and taking mapping cones commutes.

Proposition 24.16. *If* $f\colon X \to Y$ *is a map between topological spaces, then there exists a homeomorphism*

$$g\colon\ \mathrm{Cone}(\Sigma(f)\colon \Sigma(X) \to \Sigma(Y)) \ \xrightarrow{\cong}\ \Sigma(\mathrm{Cone}(f\colon X \to Y)).$$

We also state the following lemma which we are already dimly aware of.

Lemma 109.14. *If a map* $\varphi\colon S^m \to S^n$ *is homotopic to a constant map, then the corresponding mapping cone* $\mathrm{Cone}(\varphi\colon S^m \to S^n)$ *is homotopy equivalent to* $S^{m+1} \vee S^n$.

Proof. We have

$$\mathrm{Cone}(\varphi\colon S^m \to S^n) \underset{\substack{\uparrow \\ \text{by Lemma 24.12, since } \varphi \text{ is homotopic to a constant map}}}{\simeq} \mathrm{Cone}(\text{constant map}\colon S^m \to S^n) \underset{\substack{\uparrow \\ \text{Lemma 91.1}}}{\cong} S^{m+1} \vee S^n.$$

■

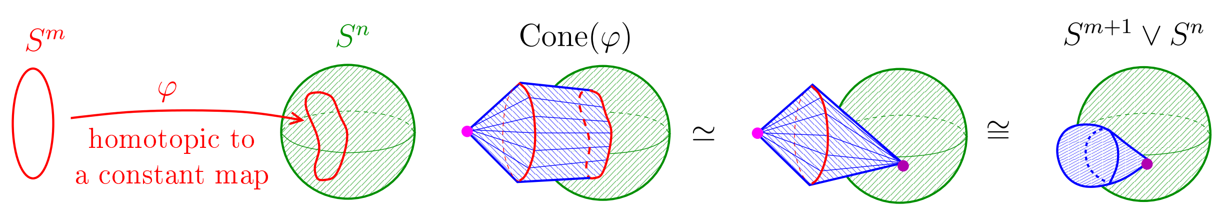


Figure 1532. Illustration of Lemma 109.14.

Now we turn to the proof of Theorem 109.13. For clarity's sake we first provide a detailed proof of Theorem 109.13 (1). Afterwards we explain how to modify that proof to obtain statements (2) and (3) of Theorem 109.13.

Proof of Theorem 109.13 (1). In the following we will show that $\Sigma(H)\colon S^4 \to S^3$ is not homotopic to a constant map. The general statement regarding the $(n-2)$-fold suspension $\Sigma^{n-2}(H)$ follows easily from iterating the argument below.

For once we do a proof by contradiction. So let us assume that the map $\Sigma(H)\colon S^4 \to S^3$ is in fact homotopic to a constant map. We make the following preparations:

(1) To simplify the notation we denote the mapping cone of a map $f\colon X \to Y$ by $\mathrm{C}(f)$ instead of $\mathrm{Cone}(f)$. Furthermore we write $\mathbb{F} = \mathbb{Z}_2$.

(2) By Lemma 91.2 there exists a homeomorphism $f\colon\ \mathrm{C}(H\colon S^3 \to S^2) \xrightarrow{\cong} \mathbb{CP}^2$.

(3) We denote by $g\colon \Sigma(\mathrm{C}(H)) \xrightarrow{\cong} \mathrm{C}(\Sigma(H))$ the homeomorphism from Proposition 24.16.

(4) Since we assume that the map H is homotopic to a constant map we obtain from Lemma 109.14 that there exists a homotopy equivalence $h\colon \mathrm{C}(\Sigma(H)) \to S^3 \vee S^5$.

Now we consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{F}) \arrow[r, "f^*", "\cong"'] \arrow[d, shift right, "x\mapsto x\cup x"'] \arrow[d, shift left, "\mathrm{Sq}^2"] & \mathrm{H}^2(\mathrm{C}(H);\mathbb{F}) \arrow[r, "\Sigma", "\cong"'] \arrow[d, "\mathrm{Sq}^2"] & \mathrm{H}^3(\Sigma(\mathrm{C}(H));\mathbb{F}) \arrow[d, "\mathrm{Sq}^2"] & \mathrm{H}^3(\mathrm{C}(\overbrace{\Sigma(H)}^{S^4\to S^3});\mathbb{F}) \arrow[l, "\cong"', "g^*"] \arrow[d, "\mathrm{Sq}^2"] & \mathrm{H}^3(S^5\vee S^3;\mathbb{F}) \arrow[l, "h^*"', "\cong"] \arrow[d, "\mathrm{Sq}^2"] \\
\mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{F}) \arrow[r, "f^*", "\cong"'] & \mathrm{H}^4(C(H);\mathbb{F}) \arrow[r, "\Sigma", "\cong"'] & \mathrm{H}^5(\Sigma(\mathrm{C}(H));\mathbb{F}) & \mathrm{H}^5(\mathrm{C}(\Sigma(H));\mathbb{F}) \arrow[l, "\cong"', "g^*"] & \mathrm{H}^5(S^5\vee S^3;\mathbb{F}). \arrow[l, "h^*"', "\cong"]
\end{tikzcd}$$

We make the following observations:

(1) The maps Σ are the suspension isomorphisms given by Proposition 109.11.
(2) It follows from Axiom (2) that the two vertical maps on the left are actually the same.
(3) It follows from the naturality of Sq^2 that the first, third and the fourth square commute.
(4) It follows from Proposition 109.12 that the second square commutes.
(5) Since f and g are homeomorphisms we see that the induced maps f^* and g^* are isomorphisms. Furthermore, since h is a homotopy equivalence we obtain from Lemma 73.13 (3) that the horizontal maps h^* are isomorphisms.
(6) By Proposition 90.13 we know that there exists a unique non-trivial $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{F})$ and that this element satisfies $x \cup x \neq 0 \in \mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{F}) \cong \mathbb{F}$. In particular we see that the left vertical homomorphism is an isomorphism and that it is non-trivial.
(7) By Lemma 109.5 we know that the vertical homomorphism to the right is trivial.

In summary we have shown that the above diagram commutes, we have shown that all horizontal maps are isomorphisms, we have seen that the left vertical map is non-trivial and have pointed out that the right vertical map is trivial. But this is absurd. In other words, we have obtained a contradiction to our hypothesis that the map $\Sigma(H)\colon S^4 \to S^3$ is null-homotopic. ■

Proof of Theorem 109.13 (2) and (3). As we will see shortly, the proof for the quaternionic and octonionic Hopf maps is basically the same as the proof for the "classical" Hopf map.

Let us first consider the quaternionic Hopf. By Lemma 91.2 (2) there exists a homeomorphism $f\colon \mathrm{C}(H_{\mathbb{H}}) \xrightarrow{\cong} \mathbb{H}\mathrm{P}^2$. If $\Sigma(H_{\mathbb{H}})\colon S^8 \to S^5$ was null-homotopic we would, as in the proof of (1), obtain a commutative diagram of the following form:

$$\begin{tikzcd}
\mathrm{H}^4(\mathbb{H}\mathrm{P}^2;\mathbb{F}) \arrow[r, "f^*", "\cong"'] \arrow[d, shift right, "x\mapsto x\cup x"'] \arrow[d, shift left, "\mathrm{Sq}^4"] & \mathrm{H}^4(\mathrm{C}(H_{\mathbb{H}});\mathbb{F}) \arrow[r, "\Sigma"] \arrow[d, "\mathrm{Sq}^4"] & \mathrm{H}^5(\Sigma(\mathrm{C}(H_{\mathbb{H}}));\mathbb{F}) \arrow[d, "\mathrm{Sq}^4"] & \mathrm{H}^5(\mathrm{C}(\overbrace{\Sigma(H_{\mathbb{H}})}^{S^8\to S^5});\mathbb{F}) \arrow[l, "\cong"', "g^*"] \arrow[d, "\mathrm{Sq}^4"] & \mathrm{H}^5(S^9\vee S^5;\mathbb{F}) \arrow[l, "h^*"', "\cong"] \arrow[d, "\mathrm{Sq}^4"] \\
\mathrm{H}^8(\mathbb{H}\mathrm{P}^2;\mathbb{F}) \arrow[r, "f^*", "\cong"'] & \mathrm{H}^8(C(H_{\mathbb{H}});\mathbb{F}) \arrow[r, "\Sigma"] & \mathrm{H}^9(\Sigma(\mathrm{C}(H_{\mathbb{H}}));\mathbb{F}) & \mathrm{H}^9(\mathrm{C}(\Sigma(H_{\mathbb{H}}));\mathbb{F}) \arrow[l, "\cong"', "g^*"] & \mathrm{H}^9(S^9\vee S^5;\mathbb{F}). \arrow[l, "h^*"', "\cong"]
\end{tikzcd}$$

By Proposition 90.20 (1) the cup product on the left is non-trivial. Thus the same logic as in the proof of (1) leads to a contradiction.

The argument for the octonionic Hopf map is almost identical to the argument for the quaternionic Hopf map. We just need to redo the above argument with Lemma 91.2 (3) and with Proposition 90.20 (2). ■

For the record we jot down the following proposition that we proved in the proof of Theorem 109.13. We will make use of it later on.

Proposition 109.15. $(*)$ *Given any $n \in \mathbb{N}_0$ the maps*

$$\begin{array}{lrcl} (1) & \mathrm{Sq}^2\colon \mathrm{H}^{2+n}(\mathrm{Cone}(\Sigma^n(H));\mathbb{Z}_2) & \to & \mathrm{H}^{4+n}(\mathrm{Cone}(\Sigma^n(H));\mathbb{Z}_2) \\ (2) & \mathrm{Sq}^4\colon \mathrm{H}^{4+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{H}}));\mathbb{Z}_2) & \to & \mathrm{H}^{8+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{H}}));\mathbb{Z}_2) \\ (3) & \mathrm{Sq}^8\colon \mathrm{H}^{8+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{O}}));\mathbb{Z}_2) & \to & \mathrm{H}^{16+n}(\mathrm{Cone}(\Sigma^n(H_{\mathbb{O}}));\mathbb{Z}_2) \end{array}$$

are isomorphisms.

109.4. **Proof of Theorem 109.2 I.** In the final two sections we will give a partial proof of Theorem 109.2. More precisely, we will construct the Steenrod operations and we will provide the proof of some of the properties. The proof is split into two parts:

(1) The goal of this section is to construct a certain interesting sequence of natural maps $\Delta_n\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$, $n \in \mathbb{N}_0$, where the zeroth map Δ_0 is a diagonal approximation.
(2) In the subsequent section we will show how this sequence of maps can be used to define the Steenrod operations.

In preparation for the construction of the sequence Δ_n we recall some definitions and results from earlier on. First recall that on page 1433 we introduced the tensor product $A_* \otimes B_*$ of two chain complexes A_* and B_*. We will make use of the following harmless lemma.

Lemma 84.7. *Let A_* and B_* be chain complexes. The unique map*

$$T\colon A_* \otimes B_* \;\to\; B_* \otimes A_*$$

that for $a \in A_k$ and $b \in B_l$ is given by

$$a \otimes b \;\mapsto\; (-1)^{kl} \cdot b \otimes a$$

is a chain map.

Since we will make use of this later on, we point out that T is natural in A_* and B_* and we point out that $T^2 = \mathrm{id}$. Next we recall the following definition from page 1984.

Definition. A *diagonal approximation* consists of a natural chain map

$$\Phi\colon\ \mathrm{C}_*(X) \;\to\; \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

for each topological space X, such that for each $x \in X$, viewed as a singular 0-simplex, the following equality holds in $(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_0 = \mathrm{C}_0(X) \otimes \mathrm{C}_0(X)$:

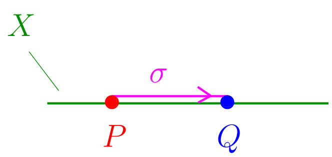
$$\Phi(x) \;=\; x \otimes x.$$

Example. We will try to rise to the challenge of finding a suitable illustration for a diagonal approximation Φ. We start out with the following observation: if $\sigma\colon \Delta^1 \to X$ is a singular

1-simplex from a point P to a point Q, then $\Phi(\sigma) \in (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_1$ needs to satisfy

$$\partial\Phi(\sigma) \underset{\substack{\uparrow \\ \text{since } \Phi \text{ is a chain map}}}{=} \Phi(\partial\sigma) = \Phi(Q-P) = \Phi(Q) - \Phi(P) \underset{\substack{\uparrow \\ \text{since } \Phi \text{ is determined on singular 0-simplices}}}{=} Q \otimes Q - P \otimes P.$$

For example $\Phi(\sigma) = (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_1 = \mathrm{C}_1(X) \otimes \mathrm{C}_0(X) \oplus \mathrm{C}_0(X) \otimes \mathrm{C}_1(X)$ could be of the form $\alpha \otimes P + Q \otimes \beta$ where $\alpha\colon \Delta^1 \to X$ and $\beta\colon \Delta^1 \to X$ are singular simplices from P to Q. This situation is illustrated, with artistic liberties, in Figure 1533.



diagonal approximation
$\Delta_0(\sigma) = \alpha \otimes P + Q \otimes \beta$

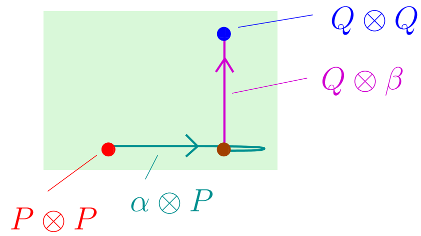


FIGURE 1533

The following proposition contains the two key results on diagonal approximations.

Proposition 81.6.
(1) *There exist diagonal approximations.*
(2) *Given any two diagonal approximations Φ and Ψ there exists a natural chain homotopy equivalence between Φ and Ψ.*

PROOF. We gave an explicit example of a diagonal approximation in Lemma 81.5. In Proposition 80.17 we gave a proof of (2) using the Acyclic Model Theorem 80.16 (2). ■

The following theorem shows that any diagonal approximation can be extended to a sequence of maps.

Theorem 109.16.
(1) *Given any topological space X there exists a sequence of natural maps*

$$\Delta_n\colon\ \mathrm{C}_*(X) \ \to\ \mathrm{C}_*(X) \otimes \mathrm{C}_*(X), \quad n \in \mathbb{N}_0$$

of degree n[1487] *such that the following two conditions are satisfied:*
 (a) *The map Δ_0 is a diagonal approximation.*
 (b) *For every $n \in \mathbb{N}_0$ the following equality holds:*[1488]

$$\partial \circ \Delta_{n+1} + (-1)^n \cdot \Delta_{n+1} \circ \partial \ = \ (T + (-1)^{n+1} \cdot \mathrm{id}) \circ \Delta_n.$$

(2) *Given two sequences Δ_n, Δ'_n, $n \in \mathbb{N}_0$, as in (1) there exists a sequence of natural maps*

$$H_n\colon\ \mathrm{C}_*(X) \ \to\ \mathrm{C}_*(X) \otimes \mathrm{C}_*(X), \quad n \in \mathbb{N}_0$$

of degree n such that the following two conditions are satisfied:
(a) *H_0 is the zero map.*
(b) *For every $n \in \mathbb{N}_0$ the following equality holds:*[1489]

$$\partial \circ H_{n+1} + (-1)^n \cdot H_{n+1} \circ \partial \ = \ \Delta_n - \Delta'_n - (T + (-1)^n \cdot \mathrm{id}) \circ H_n.$$

In the proof of Theorem 109.16 we will need the following innocuous variation on the definition of a chain map that we had given on page 1086.

Definition.

(1) A *chain map $f\colon C_* \to D_*$ of degree d* between two chain complexes consists of a family $\{f_n\colon \mathrm{C}_n \to D_{n+d}\}_{n\in\mathbb{N}_0}$ of maps such that for each $n \in \mathbb{N}_0$ the following equality holds:
$$f_{n-1}\circ\partial_n \;=\; \partial_{n+d}\circ f_n.$$
(2) An *anti-chain map $f\colon C_* \to D_*$ of degree d* is basically the same data as a chain map of degree d, except that for each $n \in \mathbb{N}_0$ the following equality holds:
$$f_{n-1}\circ\partial_n \;=\; -\partial_{n+d}\circ f_n.$$

Remark. Given an anti-chain map $\{f_n\}_{n\in\mathbb{N}_0}$ of degree d one can easily see that the maps $\{(-1)^n\cdot f_n\}_{n\in\mathbb{N}_0}$ form a chain map of degree d. This little trick makes it possible to translate all kinds of statements about chain maps to anti-chain maps. For example, as in Lemma 41.7 we see that an anti-chain map induces a map on homology groups.

In the proof of Theorem 109.16 we will need the following consequence of the Acyclic Model Theorem 80.13.

Theorem 109.17. (Acyclic Model Theorem) *Let $\mathcal{Top}$ be the category of topological spaces and let $\mathcal{C}$ be the category of generalized[1490] chain complexes.*

(1) *Let $S\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ be a natural chain map of degree d. If we have $d \geq 1$, then for any topological space X and any $k\in\mathbb{N}_0$ there exists a natural map $P_X\colon \mathrm{C}_k(X) \to (\mathrm{C}_*(X)\otimes\mathrm{C}_*(X))_{k+d+1}$ such that for any $k\in\mathbb{N}_0$ we have*
$$\partial\circ P_X + P_X\circ\partial \;=\; S_X\colon\, \mathrm{C}_k(X) \;\to\; (\mathrm{C}_*(X)\otimes\mathrm{C}_*(X))_{k+d}.$$
(2) *For a natural anti-chain map $S\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ of degree d the analogous data exists except that now we have*
$$\partial\circ P_X - P_X\circ\partial \;=\; S_X\colon\, \mathrm{C}_k(X) \;\to\; (\mathrm{C}_*(X)\otimes\mathrm{C}_*(X))_{k+d}.$$

Proof of the Acyclic Model Theorem 109.17.

(1) Given any topological space X we consider $G(X) := (\mathrm{C}_*(X)\otimes\mathrm{C}_*(X))[-d]$. Here we use the notation introduced on page 2039, namely given a chain complex D_* we denote by $D_*[-d]$ the chain complex D_* "shifted by d to the right". This shift turns $G(X)$ into a generalized chain complex. Furthermore, this shift turns $S\colon \mathrm{C}_*(X) \to G(X)$ into a natural chain map of degree 0.

[1487] By a map of degree n between chain complexes C_* and D_* we mean a collection of maps $\mathrm{C}_k \to D_{k+n}$, $k\in\mathbb{N}_0$.

[1488] Note that both sides are maps $\mathrm{C}_*(X) \to \mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ of degree n.

[1489] In other words, the map H_{n+1} is a chain homotopy between Δ_n and Δ'_n up to the "error term" $-(T+(-1)^n\cdot\mathrm{id})\circ H_n$.

[1490] Recall that a generalized chain complex is like a chain complex except that we allow chain groups in negative degrees.

Next note that for any $k \in \mathbb{N}_0$ and any $i \geq 0$ we have

$$\begin{array}{rcl} \mathrm{H}_i(G(\Delta^k)) & = & \mathrm{H}_i((\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))[-d]) \;=\; \mathrm{H}_{i+d}(\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)) \\ & \underset{\substack{\uparrow \\ \text{by the Künneth Theorem 58.7} \\ \text{for chain complexes}}}{=} & \bigoplus_{p+q=i+d} \mathrm{H}_p(\Delta^k) \otimes \mathrm{H}_q(\Delta^k) \underset{\substack{\uparrow \\ \text{by the calculation of } \mathrm{H}_*(\Delta^k) \\ \text{and since } d \geq 1}}{=} 0. \end{array}$$

Thus we see that all the hypotheses of the Acyclic Model Theorem 80.13 are satisfied. Therefore we see that for any topological space X and any $k \in \mathbb{N}_0$ there exists a natural map $P_X \colon \mathrm{C}_k(X) \to G_{k+1}(X)$ such that for any $k \in \mathbb{N}_0$ we have

$$\partial \circ P_X + P_X \circ \partial \;=\; S_X \colon \mathrm{C}_k(X) \;\to\; G_k(X).$$

(2) Suppose we are given a natural anti-chain map S of degree d. As in the remark preceding the theorem we can turn S into a natural chain map T by setting $T_k = (-1)^k \cdot S_k$, $k \in \mathbb{N}_0$. We apply (1) and we obtain natural maps Q_X. Now we define $P_k := (-1)^k \cdot (Q_X)_k$, $k \in \mathbb{N}_0$. One easily verifies that the desired equality holds. ■

Now, finally, we can turn to the actual proof of Theorem 109.16.

Proof of Theorem 109.16.

(1) We construct the maps $\Delta_n \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$, $n \in \mathbb{N}_0$ iteratively.
   (i) First of all, we know by the Proposition 81.6 (1) above that there exists a diagonal approximation $\Delta_0 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$.
   (ii) Note that $T \circ \Delta_0 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ is another diagonal approximation. It follows from Proposition 81.6 (2) that there exists a natural chain homotopy $\Delta_1 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ between Δ_0 and $T \circ \Delta_0$. In other words, there exists a natural map $\Delta_1 \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ of degree one such that

$$\partial \circ \Delta_1 + \Delta_1 \circ \partial \;=\; \underbrace{T \circ \Delta_0 - \Delta_0}_{=(T-\mathrm{id})\circ \Delta_0}.$$

Thus we are done with the construction of the map Δ_1. This step is illustrated in Figure 1534.

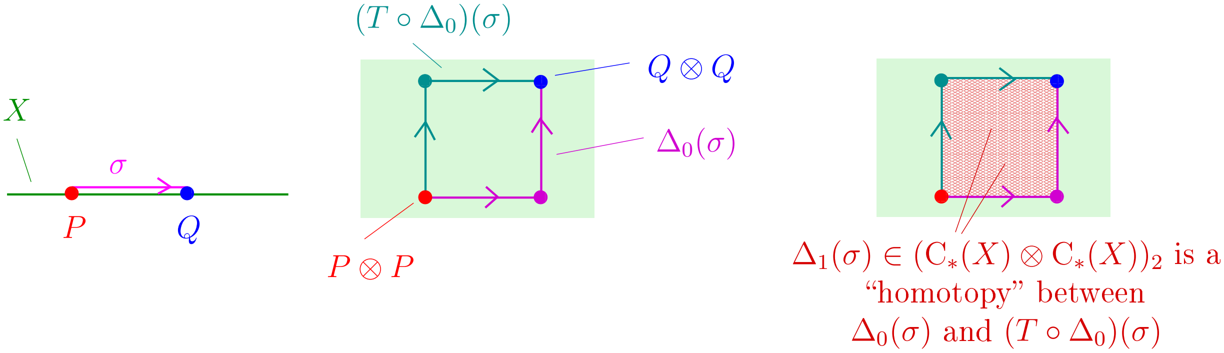


Figure 1534

(iii) The starting point for defining the next map Δ_2 is the following amusing little calculation:[1491]

$$\begin{aligned} 0 \underset{\text{since } T^2=\mathrm{id}}{=} (T^2-\mathrm{id})\circ\Delta_0 &= (T+\mathrm{id})\circ(T-\mathrm{id})\circ\Delta_0 \underset{\text{by (ii)}}{=} (T+\mathrm{id})\circ(\partial\circ\Delta_1+\Delta_1\circ\partial) \\ &\underset{\text{since } T \text{ and id are chain maps they commute with the boundary maps}}{=} \partial\circ(T+\mathrm{id})\circ\Delta_1+(T+\mathrm{id})\circ\Delta_1\circ\partial. \end{aligned}$$

This shows that the map $(T+\mathrm{id})\circ\Delta_1\colon\ \mathrm{C}_*(X)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ is a natural anti-chain map of degree one. By the Acyclic Model Theorem 109.17 (2) there exists a natural map $\Delta_2\colon\ \mathrm{C}_*(X)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ of degree two such that

$$\partial\circ\Delta_2-\Delta_2\circ\partial \;=\; (T+\mathrm{id})\circ\Delta_1.$$

This conclude the construction of the map Δ_2.

(iv) Just for kicks we do the calculation for the next stage as well. We perform the following calculation:

$$\begin{aligned} 0 \underset{\text{since } T^2=\mathrm{id}}{=} (T^2-\mathrm{id})\circ\Delta_1 &= (T-\mathrm{id})\circ(T+\mathrm{id})\circ\Delta_1 \underset{\text{by (iii)}}{=} (T-\mathrm{id})\circ(\partial\circ\Delta_2-\Delta_2\circ\partial) \\ &\underset{\text{since } T \text{ and id are chain maps they commute with the boundary maps}}{=} \partial\circ(T-\mathrm{id})\circ\Delta_2-(T-\mathrm{id})\circ\Delta_2\circ\partial. \end{aligned}$$

This shows that the map $(T-\mathrm{id})\circ\Delta_2\colon\ \mathrm{C}_*(X)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ is a natural chain map of degree two. By the Acyclic Model Theorem 109.17 (1) there exists a natural map $\Delta_3\colon\ \mathrm{C}_*(X)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ of degree three such that

$$\partial\circ\Delta_3+\Delta_3\circ\partial \;=\; (T-\mathrm{id})\circ\Delta_2.$$

This concludes the construction of the map Δ_3.

(iv) Now we iterate this procedure and we obtain the desired sequence of natural maps Δ_n.

(2) Given two sequences Δ_n,Δ'_n as in (1) we define the desired maps H_n, $n\in\mathbb{N}_0$ in a fashion that is quite similar to the approach taken in (1).

(i) We set $H_0=0$.

(ii) Next we point out that by definition Δ_0 and Δ'_0 are diagonal approximations. Thus it follows from Proposition 81.6 (2) that there exist natural maps H_1 such that $\partial\circ H_1+H_1\circ\partial=\Delta_0-\Delta'_0$.

(iii) We calculate that

$$\begin{aligned} &\partial\circ(\Delta_1-\Delta'_1-(T-\mathrm{id})\circ H_1)+(\Delta_1-\Delta'_1-(T-\mathrm{id})\circ H_1)\circ\partial \\ &\quad= (\partial\circ\Delta_1+\Delta_1\circ\partial)-(\partial\circ\Delta'_1+\Delta'_1\circ\partial)-\partial\circ(T-\mathrm{id})\circ H_1-(T-\mathrm{id})\circ H_1\circ\partial \\ &\quad\underset{\substack{\text{since } \partial\circ\Delta_1+\Delta_1\circ\partial=(T-\mathrm{id})\circ\Delta_0 \text{ (same for } \Delta'_0,\Delta'_1) \\ \text{and since } T \text{ and id are chain maps}}}{=} (T-\mathrm{id})\circ(\Delta_0-\Delta'_0)-(T-\mathrm{id})\circ(\partial\circ H_1+H_1\circ\partial) \underset{\text{since } \partial\circ H_1+H_1\circ\partial=\Delta_0-\Delta'_0}{=} 0. \end{aligned}$$

[1491] These are equalities of maps $\mathrm{C}_*(X)\to\mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$.

This shows that $\Delta_1 - \Delta_1' - (T - \mathrm{id}) \circ H_1$ is an anti-chain map of degree one. By the Acyclic Model Theorem 109.17 (2) there exists a natural map H_2 with

$$H_2 \circ \partial - \partial \circ H_2 \;=\; -(T - \mathrm{id}) \circ H_1 + \Delta_1 - \Delta_1'.$$

(iv) Now we iterate this procedure to obtain the desired sequence of natural maps. One easily sees that we get the correct pattern for the signs involved. ■

Remark. If there did exist a diagonal approximation $\Delta_0\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ that is symmetric in the sense that $T \circ \Delta_0 = \Delta_0$, then we could take $\Delta_1 = 0$ and we could also take all higher Δ_n to be zero. Later, in Chapter 109, we will see that this is in general not the case. So even though diagonal approximations are symmetric on the degree 0 level, they cannot be chosen to be symmetric in higher degrees.

109.5. **Proof of Theorem 109.2 II.** In this section we will use the sequence of natural maps $\Delta_n\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$, $n \in \mathbb{N}_0$, from Theorem 109.16 to define the Steenrod operations.

To get an inspiration for how to define the Steenrod operations we first recall the definition of the cup product. To do so we need the following lemma which is inspired by the definitions on from page 1979.

Lemma 109.18. *Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes and let R be a commutative ring. Given $p, q \in \mathbb{N}_0$ we consider the map*[1492]

$$\begin{aligned} \Xi_{p,q}\colon \operatorname{Hom}(\mathcal{C}_p, R) \otimes \operatorname{Hom}(\mathcal{D}_q, R) &\to \operatorname{Hom}((\mathcal{C} \otimes \mathcal{D})_{p+q}, R) \\ \varphi \otimes \psi &\mapsto \left(\begin{array}{r} (\mathcal{C}_* \otimes \mathcal{D}_*)_{p+q} \xrightarrow{\text{projection}} \mathcal{C}_p \otimes \mathcal{D}_q \xrightarrow{\varphi \otimes \psi} R \otimes R \to R \\ a \otimes b \mapsto a \cdot b \end{array} \right). \end{aligned}$$

The corresponding map

$$\Xi := \bigoplus_{p,q \in \mathbb{N}_0} \Xi_{p,q}\colon \operatorname{Hom}(\mathcal{C}_*, R) \otimes \operatorname{Hom}(\mathcal{D}_*, R) \;\to\; \operatorname{Hom}(\mathcal{C}_* \otimes \mathcal{D}_*, R)$$

is a cochain map.

Proof. The statement follows basically immediately from the definitions. ■

The following lemma gives us a slight reformulation of the original definition of the cup product.

Lemma 109.19. *Let X be a topological space and let R be a commutative ring. Furthermore let $\Delta_0\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ be a diagonal approximation. We consider the cochain map*

$$\begin{tikzcd} \operatorname{Hom}(\mathrm{C}_*(X), R) \otimes \operatorname{Hom}(\mathrm{C}_*(X), R) \arrow[r, "\Xi"] \arrow[rr, bend right, "=: h_0"'] & \operatorname{Hom}(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X), R) \arrow[r, "\Delta_0^*"] & \operatorname{Hom}(\mathrm{C}_*(X), R). \end{tikzcd}$$

[1492] The tensor product of cochain complexes is defined basically the same way as we defined the tensor product of chain complexes, see page 1433.

The corresponding map

$$\begin{array}{rcl} \mathrm{H}^p(X;R)\times \mathrm{H}^q(X;R) & \to & \mathrm{H}^{p+q}(X;R) \\ ([\varphi],[\psi]) & \mapsto & [h_0(\varphi\otimes\psi)] \end{array}$$

equals the cup product as defined on page 1980.

Proof (∗). We denote by $d\colon X\to X\times X$ the diagonal map that is given by $d(x)=(x,x)$. Let $\Theta\colon\ \mathrm{C}_*(X\times X)\to \mathrm{C}_*(X)\otimes \mathrm{C}_*(X)$ denote a Eilenberg-Zilber map from Theorem 80.1. For the purpose of this proof the only thing we need to know about the Eilenberg-Zilber map Θ is that, basically by definition, the map $\Theta\circ d\colon\ \mathrm{C}_*(X)\to \mathrm{C}_*(X\times X)\to \mathrm{C}_*(X)\otimes \mathrm{C}_*(X)$ is a diagonal approximation. We consider the following diagram:

$$\begin{tikzcd} \mathrm{Hom}(\mathrm{C}_*(X),R)\otimes\mathrm{Hom}(\mathrm{C}_*(X),R) \arrow[r, "\Xi"] \arrow[rr, bend left, "=:h_0"] & \mathrm{Hom}(\mathrm{C}_*(X)\otimes\mathrm{C}_*(X),R) \arrow[r, "\Delta_0^*"] \arrow[d, "\Theta^*"] & \mathrm{Hom}(\mathrm{C}_*(X),R) \\ & \mathrm{Hom}(\mathrm{C}_*(X\times X),R). \arrow[ur, "d^*"'] & \end{tikzcd}$$

The lower sequence of maps is at the heart of the definition of the cup product on page 1980. Since $\Theta\circ d$ is a diagonal approximation we obtain from Proposition 81.6 that Δ_0 and $\Theta\circ d$ are chain homotopic. But by Lemma 73.8 (4) this means that the two maps induce the same map on cohomology. The lemma follows almost immediately from the above observations. ∎

Now we turn to the actual construction of the Steenrod operations. To do so we pick a sequence of natural maps

$$\Delta_n\colon\ \mathrm{C}_*(X)\ \to\ \mathrm{C}_*(X)\otimes \mathrm{C}_*(X),\quad n\in\mathbb{N}_0$$

which satisfies the conditions set out in Theorem 109.16 (1). In light of Lemma 109.19 it is clear what we want to do next. Namely we want to replace Δ_0 by the "higher maps" Δ_n. This leads us straight to the following definition.

Definition. Let X be a topological space and let R be a commutative ring. Given $i\in\mathbb{N}_0$ we define

$$\begin{tikzcd} \mathrm{Hom}(\mathrm{C}_*(X),R)\otimes\mathrm{Hom}(\mathrm{C}_*(X),R) \arrow[r, "\Xi"] \arrow[rr, bend right, "=:h_i"'] & \mathrm{Hom}(\mathrm{C}_*(X)\otimes\mathrm{C}_*(X),R) \arrow[r, "\Delta_i^*"] & \mathrm{Hom}(\mathrm{C}_*(X),R). \end{tikzcd}$$

Recall that Δ_i is a map of degree i. Since we dualize we see that h_i is a map of degree $-i$. For convenience we define $h_i:=0$ for $i\in\mathbb{Z}_{<0}$.

Now we can finally define the Steenrod operations.

Theorem 109.20. *Let X be a topological space, let $A \subset X$ be a subset and let R be a commutative ring. If $1+1=0$ in R, then for any $q \in \mathbb{N}_0$ and any $i \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl} \mathrm{Sq}^i\colon\ \mathrm{H}^q(X,A;R) & \to & \mathrm{H}^{q+i}(X,A;R) \\ [\underbrace{\varphi}_{\in \mathrm{C}^q(X;R)}] & \mapsto & [h_{q-i}(\underbrace{\varphi \otimes \varphi}_{\in \mathrm{C}^q \otimes \mathrm{C}^q})] \end{array}$$

is a well-defined natural homomorphism.

Remark. The proof of Theorem 109.20 that we give below is a long and winding calculation, with few interesting view points. Perhaps more interesting than the question, why the definition works, is the question, why does the map

$$\begin{array}{rcl} \mathrm{C}^q(X,A;R) \otimes \mathrm{C}^r(X,A;R) & \to & \mathrm{C}^{q+r+j}(X,A;R) \\ \varphi \otimes \psi & \mapsto & h_j(\underbrace{\varphi \otimes \psi}_{\in \mathrm{C}^q \otimes \mathrm{C}^r}) \end{array}$$

not descend to a map on cohomology?

The proof of Theorem 109.20 requires some preparations. First of all, note that the equalities in Theorem 109.16 translate into the following lemma.

Lemma 109.21. *Let X be a topological space. Furthermore let R be a commutative ring with $1_R + 1_R = 0_R$. For any $\alpha, \beta \in \mathrm{C}^*(X;R)$ we have the following equality:*

$$h_{n+1}(\delta(\alpha \otimes \beta)) + \delta(h_{n+1}(\alpha \otimes \beta)) \;=\; h_n(\alpha \otimes \beta + \beta \otimes \alpha) \;\in\; \mathrm{C}^*(X;R).$$

Proof. Recall, see Theorem 109.16, that one of the defining properties of the sequence Δ_n is that for every $n \in \mathbb{N}_0$ the following equality holds:

$$\partial \circ \Delta_{n+1} + (-1)^n \cdot \Delta_{n+1} \circ \partial \;=\; (T + (-1)^{n+1} \cdot \mathrm{id}) \circ \Delta_n.$$

We dualize this equality, we use the equality $1+1=0$, we use the fact, shown in Lemma 109.18 and that Ξ is a cochain map. After performing all these steps one obtains the desired equality. We leave it to the reader to fill in the few remaining details. ■

For elements of the form $\alpha \otimes \alpha$ the following lemma shows how the maps h_n relate to coboundaries.

Lemma 109.22. *Let X be a topological space and let R be a commutative ring. If $1+1=0$ in R, then for any $\alpha \in \mathrm{C}^q(X;R)$ we have*

$$h_{n+1}(\delta\alpha \otimes \delta\alpha) \;=\; \delta h_{n+1}(\alpha \otimes \delta\alpha) + \delta h_n(\alpha \otimes \alpha) \;\in\; \mathrm{C}^{n+1}(X;R).$$

Proof. Since we assume that $1+1=0$ in R we can ignore conveniently enough ignore signs throughout the proof. Thus we compute

$$\begin{array}{rcl} h_{n+1}(\delta\alpha \otimes \delta\alpha) & \overset{\substack{\text{since } \delta(c\otimes d) = \delta c \otimes d + c \otimes \delta d \\ \downarrow}}{=} & h_{n+1}(\delta(\alpha \otimes \delta\alpha)) \quad \overset{\substack{\text{Lemma 109.21} \\ \downarrow}}{=} \; \delta h_{n+1}(\alpha \otimes \delta\alpha) + h_n(\alpha \otimes \delta\alpha + \delta a \otimes \alpha) \\ & \underset{\substack{\uparrow \\ \text{definition of } \delta \text{ on tensor product}}}{=} & \delta h_{n+1}(\alpha \otimes \delta\alpha) + h_n(\delta(\alpha \otimes \alpha)) \underset{\substack{\uparrow \\ \text{by Lemma 109.21 and since } h_{n-1}(\underbrace{\alpha\otimes\alpha+\alpha\otimes\alpha}_{=0 \text{ since } 1+1=0}) = 0}}{=} \delta h_{n+1}(\alpha \otimes \delta\alpha) + \delta h_n(\alpha \otimes \alpha). \end{array}$$

■

In the proof of Theorem 109.20 we will need the following little lemma.

Lemma 109.23. *Let X be a topological space and let R be a commutative ring with $1+1=0$. For any $\alpha,\beta\in \mathrm{C}^q(X;R)$ the following equality holds:*

$$h_n((\alpha+\beta)\otimes(\alpha+\beta)) = h_n(\alpha\otimes\alpha)+h_n(\beta\otimes\beta)+h_{n+1}(\delta(\alpha\otimes\beta))+\delta(h_{n+1}(\alpha\otimes\beta)).$$

Proof of Lemma 109.23. This follows from "multiplying out", the linearity of h_n and the equality

$$h_n(\alpha\otimes\beta+\beta\otimes\alpha) \underset{\substack{\uparrow\\ \text{by Lemma 109.21}}}{=} h_{n+1}(\delta(\alpha\otimes\beta))+\delta(h_{n+1}(\alpha\otimes\beta)).$$

■

Now, after these lengthy preparations we can provide the proof of Theorem 109.20.

Proof of Theorem 109.20. Let X be a topological space, let $A\subset X$ be a subset and let R be a commutative ring. First we consider the case that $A=\varnothing$. Let $\varphi\in \mathrm{C}^q(X;R)$ be a representative of some element in $\mathrm{H}^q(X;R)$.

(1) Note that $\varphi\otimes\varphi$ is of degree $2q$. Thus $h_{q-i}(\varphi\otimes\varphi)$ is of degree $2q-(q-i)=q+i$.
(2) Let $\varphi\in \mathrm{C}^q(X;R)$ be a cocycle. We calculate that

$$\delta h_{q-i}(\varphi\otimes\varphi) \underset{\substack{\uparrow\\ \text{Lemma 109.21}}}{=} h_{q-i}(\delta(\varphi\otimes\varphi))+h_{q-i-1}(\overbrace{\varphi\otimes\varphi+\varphi\otimes\varphi}^{=0\text{ since }1+1=0}) \underset{\substack{\uparrow\\ \text{definition of }\delta\text{ on tensor product}}}{=} h_{q-i}(\delta\varphi\otimes\varphi+\varphi\otimes\delta\varphi) \underset{\substack{\uparrow\\ \text{since }\varphi\text{ is a cocycle}}}{=} 0.$$

Thus we have shown that $h_{q-i}(\varphi\otimes\varphi)$ is a cocycle. In other words, we have shown that it defines an element in cohomology.
(3) Let $\psi\in \mathrm{C}^{q-1}(X;R)$. We have the following equality in $\mathrm{H}^{q+i}(X;R)$:

$$[h_{q-i}((\varphi+\delta\psi)\otimes(\varphi+\delta\psi))] \underset{\substack{\uparrow\\ \text{by Lemma 109.23, here we use that }\delta(\varphi\otimes\delta\psi)=\delta\varphi\otimes\delta\psi+\varphi\otimes\delta\delta\psi=0}}{=} [h_{q-i}(\varphi\otimes\varphi)+h_{q-i}(\delta\psi\otimes\delta\psi)] \underset{\substack{\uparrow\\ \text{Lemma 109.22}}}{=} [h_{q-i}(\varphi\otimes\varphi)].$$

This shows that the map Sq^i is well-defined, i.e. it does not depend on the choice of the representative of the cohomology class.
(4) It follows quite easily from Lemma 109.23 that Sq^i is in fact a homomorphism.

Finally we consider the case that $A\neq\varnothing$. Using the observation on page 1829 we make the identification

$$\mathrm{C}^q(X,A;R) \;=\; \{\varphi\in \mathrm{C}^q(X;R)=\mathrm{Hom}(\mathrm{C}_q(X),R)\,|\,\varphi\text{ vanishes on }\mathrm{C}_q(A)\subset \mathrm{C}_q(X)\}.$$

It follows from the naturality of h_n that if $\varphi\in \mathrm{C}^q(X)$ vanishes on $\mathrm{C}_q(A)$, then $h_{q-i}(\varphi\otimes\varphi)$ vanishes on the subgroup $\mathrm{C}_*(A)\otimes \mathrm{C}_*(X)+\mathrm{C}_*(X)\otimes \mathrm{C}_*(A)$ of $\mathrm{C}_*(X)\otimes \mathrm{C}_*(X)$. Thus we see that the map

$$\begin{array}{rcl}\mathrm{Sq}^i\colon\ \mathrm{H}^q(X,A;R) & \to & \mathrm{H}^{q+i}(X,A;R)\\ [\underbrace{\varphi}_{\in \mathrm{C}^q(X;R)}] & \mapsto & [h_{q-i}(\varphi\otimes\varphi)]\end{array}$$

is well-defined. ■

We continue with the following proposition.

Proposition 109.24. *The definition of the maps* Sq^i *does not depend on the choice of the natural maps* Δ_n.

PROOF. Suppose we are given another sequence of maps $\widetilde{\Delta}_n$ that have the properties described in Theorem 109.16 (1). We define the maps $\widetilde{h}_i$ in the obvious way. By Theorem 109.16 (2) there exists a sequence of natural maps

$$\mathrm{H}_n\colon\ \mathrm{C}_*(X)\ \to\ \mathrm{C}_*(X)\otimes \mathrm{C}_*(X),\quad n\in\mathbb{N}_0$$

of degree n such that $H_0=0$ and such that for every $n\in\mathbb{N}_0$ the following equality holds:

$$(*)\qquad \partial\circ \mathrm{H}_{n+1}+(-1)^n\cdot \mathrm{H}_{n+1}\circ\partial\ =\ \Delta_n-\widetilde{\Delta}_n-(T+(-1)^n\cdot\mathrm{id})\circ \mathrm{H}_n\,.$$

Given a cocycle $\varphi\in \mathrm{C}^q(X,A;\mathbb{Z}_2)$ and $i\in\mathbb{N}_0$ we see that we have the following equality in $\mathrm{H}^{2q-i}(X,A;R)$:

$$\begin{array}{rcl}
[h_i(\varphi\otimes\varphi)]-[\widetilde{h}_i(\varphi\otimes\varphi)] & \overset{\substack{\text{definition of } h_i,\widetilde{h}_i\\ \downarrow}}{=} & [\Delta_i^*(\Xi(\varphi\otimes\varphi))-\widetilde{\Delta}_i^*(\Xi(\varphi\otimes\varphi))]\\
& \underset{\substack{\uparrow\\ \text{by } (*) \text{ and since } 1+1=0}}{=} & [\mathrm{H}_{i+1}^*(\delta(\Xi(\varphi\otimes\varphi))+\underbrace{\delta(\mathrm{H}_{i+1}^*(\varphi\otimes\varphi))}_{\text{coboundary}}+\mathrm{H}_i^*((T+\mathrm{id})^*(\Xi(\varphi\otimes\varphi)))]\\
& \underset{\substack{\uparrow\\ \text{since } \Xi \text{ is a cochain map and}\\ \text{since } \Xi \text{ commutes with } T}}{=} & [\mathrm{H}_{i+1}^*(\Xi(\underbrace{\delta(\varphi\otimes\varphi)}_{=0\text{ since }\varphi\text{ cocycle}}))+\mathrm{H}_i^*(\Xi(\underbrace{\varphi\otimes\varphi+\varphi\otimes\varphi}_{=0\text{ since }1+1=0}))]\ =\ 0.
\end{array}$$

We have thus shown that the two definitions using Δ_n and $\widetilde{\Delta}_n$ agree. ■

By Proposition 109.24 the maps Sq^i are unambiguously defined. Thus we can make the following definition.

Definition. Given a pair of topological spaces (X,A) and $i\in\mathbb{N}_0$ we refer to each map

$$\mathrm{Sq}^i\colon\ \mathrm{H}^q(X,A;\mathbb{Z}_2)\ \to\ \mathrm{H}^{q+i}(X,A;\mathbb{Z}_2)$$

as the *i-th Steenrod operation.*

By Theorem 109.20 the Steenrod operations are additive cohomology operations. To complete the proof of Theorem 109.2 we need to verify that the maps Sq^i have all the other desired properties. In other words we need to prove the following theorem.

Theorem 109.25. *The Steenrod operations* Sq^i *have the following properties:*

(1) $\mathrm{Sq}^0=\mathrm{id}$.
(2) *For any* $\varphi\in \mathrm{H}^i(X,A;\mathbb{Z}_2)$ *we have* $\mathrm{Sq}^i(\varphi)=\varphi^2$.
(3) *For any* $\varphi\in \mathrm{H}^j(X,A;\mathbb{Z}_2)$ *with* $j<i$ *we have* $\mathrm{Sq}^i(\varphi)=0$.
(4) *For any* $\varphi\in \mathrm{H}^r(X,A;\mathbb{Z}_2)$ *and* $\psi\in \mathrm{H}^s(X,A;\mathbb{Z}_2)$ *we have*

$$\mathrm{Sq}^k(\varphi\cup\psi)\ =\ \sum_{i=0}^k \mathrm{Sq}^i(\varphi)\cup \mathrm{Sq}^{k-i}(\psi).$$

Proof.

(1) The notation might suggest that this statement is surely trivial, but that is not the case. For time, space and mental energy reasons we will not give the proof, which at first glance does not seem terribly enlightening anyway. Instead we refer to [**Spa95**, p. 274] for details.

(2) First we deal with the absolute case. Thus let $\varphi \in \mathrm{C}^i(X;R)$ be a cocycle. We calculate

$$\mathrm{Sq}^i(\varphi) \underset{\substack{\uparrow \\ \text{by definition}}}{=} [h_{i-i}(\varphi \otimes \varphi)] = [h_0(\varphi \otimes \varphi)] \underset{\substack{\uparrow \\ \text{Lemma 109.19}}}{=} [\varphi] \cup [\varphi].$$

The case that we are dealing with a cocycle $\varphi \in \mathrm{C}^i(X, A; R)$ is proved basically the same way.

(3) Let $\varphi \in \mathrm{C}^j(X, A; \mathbb{Z}_2)$ be a cocycle and suppose that $i > j$. We have

$$\mathrm{Sq}^i([\varphi]) \underset{\substack{\uparrow \\ \text{by definition}}}{=} [h_{j-i}(\varphi)] \underset{\substack{\uparrow \\ \text{since } j-i<0 \text{ we have } h_{j-i}=0}}{=} [0].$$

(4) This statement is proved in [**Spa95**, p. 274]. ■

Exercises for Chapter 109.

Exercise 109.1.

(a) Show that for any $k \in \mathbb{N}_{\geq 3}$ and $m \in \mathbb{N}$ the map $\mathrm{H}^m(X; \mathbb{Z}_2) \to \mathrm{H}^{mk}(X; \mathbb{Z}_2)$ given by $\varphi \mapsto \varphi^k$ is not an additive cohomology operation.

(b) Is the map $\mathrm{H}^1(X; \mathbb{Z}_3) \to \mathrm{H}^3(X; \mathbb{Z}_3)$ given by $\varphi \mapsto \varphi^3$ an additive cohomology operation?

Exercise 109.2. Let $X = S^m \vee S^n$ be the wedge of two spheres. Show that for $i \geq 1$ all Steenrod operations $\mathrm{Sq}^i\colon \mathrm{H}^k(X; \mathbb{Z}_2) \to \mathrm{H}^k(X; \mathbb{Z}_2)$ are zero.
Hint. Reread the proof of Lemma 81.11.

Exercise 109.3. Find a closed formula for the i-th Steenrod operation on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}_2)$.
Hint. Use Proposition 90.13 (2).

110. The Adem relations

The following theorem was first proved by José Ádem [**Ade52, Ade57**] in 1952.

Theorem 110.1. (Adem Relations) *For any $0 < a < 2b$ we have the following equality of cohomology operations:*[1493]

$$\mathrm{Sq}^a \circ \mathrm{Sq}^b \;=\; \sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \cdot \mathrm{Sq}^{a+b-j} \circ \mathrm{Sq}^j \,.$$

These equalities are called the Adem relations.

Remark. As is remarked on [**SE62**, p. 2], the Adem relations are a consequence of the properties of Steenrod operations that were stated in Theorem 109.2. This observation also follows from the uniqueness result stated in Theorem 109.3.

Proof. Since the proof would consume too much of our valuable time we will not provide it. Instead we refer to the original paper by Adem [**Ade52, Ade57**] or alternatively to [**BuM82**], [**SE62**, Theorem VIII.1.5], [**MTa68**, p. 23], or [**Hat02**, Theorem 4.L.13] for a proof. ■

Even though we will not prove the Adem relations, we will shamelessly exploit them in this chapter. Afterwards the conscientious reader will surely feel compelled to read the proof of the Adem relations.

Just to get a feeling for the Adem relations we determine for small values of a what the relations really look like.

Example. For $a = 1$ and for any $b \in \mathbb{N}$ we have

$$\mathrm{Sq}^1 \circ \mathrm{Sq}^b \;=\; \underbrace{\sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \cdot \mathrm{Sq}^{a+b-j} \circ \mathrm{Sq}^j}_{\text{with } a=1} \;=\; \underbrace{\binom{b-1}{1} \cdot \mathrm{Sq}^{1+b} \circ \mathrm{Sq}^0}_{j=0} \underset{\substack{\uparrow \\ \text{since } \mathrm{Sq}^0 = \mathrm{id}}}{=} (b-1) \cdot \mathrm{Sq}^{1+b} \,.$$

Furthermore, for $a = 2$ and any $b \in \mathbb{N}_{\geq 2}$ we have

$$\mathrm{Sq}^2 \circ \mathrm{Sq}^b \;=\; \underbrace{\binom{b-1}{2} \cdot \mathrm{Sq}^{2+b} \circ \mathrm{Sq}^0}_{j=0} + \underbrace{\binom{b-2}{0} \cdot \mathrm{Sq}^{1+b} \circ \mathrm{Sq}^1}_{j=1} \underset{\substack{\uparrow \\ \text{since } \mathrm{Sq}^0 = \mathrm{id}}}{=} \binom{b-1}{2} \cdot \mathrm{Sq}^{2+b} + \mathrm{Sq}^{1+b} \circ \mathrm{Sq}^1 \,.$$

In particular setting $b = 2$ we see that

$$\mathrm{Sq}^2 \circ \mathrm{Sq}^2 \;=\; \mathrm{Sq}^3 \circ \mathrm{Sq}^1 \,.$$

[1493]These are equalities of cohomology operations $\mathrm{H}^n(X, A; \mathbb{Z}_2) \to \mathrm{H}^{n+a+b}(X, A; \mathbb{Z}_2)$. In particular, since we work over $\mathbb{Z}_2$-coefficients, we only care about the binomial coefficient modulo 2. We will discuss the mod 2 reductions of binomial coefficients in Lemma 110.3.

110.1. **Application of the Adem relations to homotopy groups of spheres.** Recall that in Theorem 109.13 we used Steenrod operations to show that the three Hopf maps $H\colon S^3 \to S^2$, $H_{\mathbb{H}}\colon S^7 \to S^4$ and $\mathrm{H}_{\mathbb{O}}\colon S^{15} \to S^8$ and their suspensions define non-trivial elements in the corresponding homotopy groups of spheres.

In this section we will use the Adem relations to show that the 2-fold iteration of these Hopf maps also define non-trivial elements in the homotopy groups of spheres. More precisely, we have the following theorem.

Theorem 110.2.

(1) *For any $n \geq 2$ the map*

$$\Sigma^{n-2}\big(\underbrace{\overbrace{H}^{S^3\to S^2} \circ \overbrace{\Sigma(H)}^{S^4\to S^3}}_{\text{map } S^4\to S^2}\big)\colon S^{n+2} \to S^n$$

defines a non-trivial element in $\pi_{n+2}(S^n)$. In particular for any $n \geq 2$ we have $\pi_{n+2}(S^n) \neq 0$.

(2) *For any $n \geq 4$ the map*

$$\Sigma^{n-4}\big(\underbrace{\overbrace{H_{\mathbb{H}}}^{S^7\to S^4} \circ \overbrace{\Sigma^3(H_{\mathbb{H}})}^{S^{10}\to S^7}}_{\text{map } S^{10}\to S^4}\big)\colon S^{n+6} \to S^n$$

defines a non-trivial element in $\pi_{n+6}(S^n)$. In particular for any $n \geq 4$ we have $\pi_{n+6}(S^n) \neq 0$.

(3) *For any $n \geq 8$ the map*

$$\Sigma^{n-8}\big(\underbrace{\overbrace{\mathrm{H}_{\mathbb{O}}}^{S^{15}\to S^8} \circ \overbrace{\Sigma^7(\mathrm{H}_{\mathbb{O}})}^{S^{22}\to S^{15}}}_{\text{map } S^{22}\to S^8}\big)\colon S^{n+14} \to S^n$$

defines a non-trivial element in $\pi_{n+14}(S^n)$. In particular for any $n \geq 8$ we have $\pi_{n+14}(S^n) \neq 0$.

For the reader's convenience we recall the two lemmas which will play a key role in the proof of Theorem 110.2.

Lemma 91.4. *Let $f\colon A \to X$ be a map between two topological spaces. We denote by $j\colon X \to \mathrm{Cone}(f\colon A \to X)$ the obvious inclusion. Then there exists a natural long exact sequence of the form*

$$\ldots \to \mathrm{H}^{k-1}(A;\mathbb{Z}_2) \xrightarrow{\delta} \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f\colon A \to X);\mathbb{Z}_2) \xrightarrow{j^*} \mathrm{H}^k(X;\mathbb{Z}_2) \xrightarrow{f^*} \mathrm{H}^k(A;\mathbb{Z}_2) \xrightarrow{\delta} \ldots$$

As an example for Lemma 91.4 let us consider the mapping cone of the iterated suspension of the Hopf map.

Example. Let $k \in \mathbb{N}_0$. We consider $\Sigma^k(H)\colon S^{k+3} \to S^{k+2}$, i.e. we consider the k-fold suspension of the Hopf map $H\colon S^3 \to S^2$. As we saw in Lemma 91.5, it follows almost immediately from the long exact sequence of Lemma 91.4 that the maps

$$\widetilde{\mathrm{H}}^{k+2}(\mathrm{Cone}(\Sigma^k(H)\colon S^{k+3} \to S^{k+2});\mathbb{Z}_2) \underset{\cong}{\xrightarrow{j^*}} \mathrm{H}^{k+2}(S^{k+2};\mathbb{Z}_2)$$

and

$$\mathrm{H}^{k+3}(S^{k+3};\mathbb{Z}_2) \xrightarrow[\cong]{\delta} \widetilde{\mathrm{H}}^{k+4}(\mathrm{Cone}(\Sigma^k(H)\colon S^{k+3} \to S^{k+2});\mathbb{Z}_2)$$

are isomorphisms, and that all the other reduced cohomology groups of the mapping cone vanish.

In the following we recall three statements of Lemma 24.9 which are at the moment relevant to us. It might be helpful to have a peek at Figure 1535.

Lemma 24.9. *Let $f\colon A \to X$ be a map between topological spaces.*

(1) *The inclusion $X \to \mathrm{Cone}(f\colon A \to X)$ is an embedding.*
(2) *The map*
$$\begin{array}{rcl} h\colon \mathrm{Cone}(f\colon A \to X)/X & \to & \Sigma(A) \\ [(a,t)] & \mapsto & [(a,1-2t)] \end{array}$$
is a homeomorphism.
(4) *The subset X is a deformation retract of the open neighborhood that is given by the complement of the cone point.*

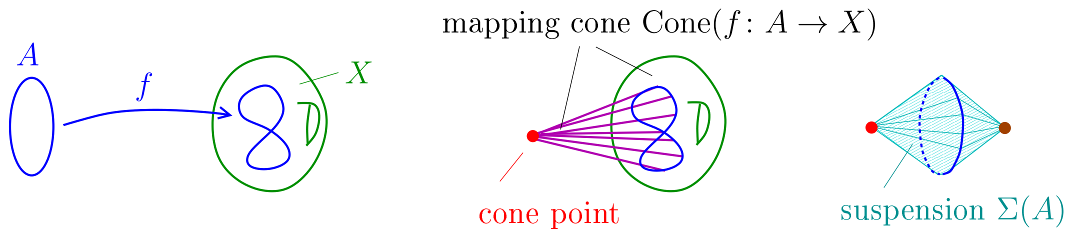


FIGURE 1535. Illustration of Lemma 24.9.

Now we turn to the proof of Theorem 110.2. As in the proof of Theorem 109.13 we first consider the case of the classical Hopf map, i.e. we first prove Statement (1).

PROOF OF THEOREM 110.2 (1). Let $n \in \mathbb{N}_{\geq 2}$. We write

$$\Sigma^{n-2}(H \circ \Sigma(H)) \underset{\substack{\uparrow \\ \text{Lemma 24.4 (2)}}}{=} \underbrace{\overbrace{\Sigma^{n-2}(H)}^{S^{n+1}\to S^n}}_{=:g} \circ \underbrace{\overbrace{\Sigma^{n-1}(H)}^{S^{n+2}\to S^{n+1}}}_{=:f} \colon S^{n+2} \to S^n.$$

We need to show that $g \circ f\colon S^{n+2} \to S^n$ defines a non-trivial element in $\pi_{n+2}(S^n)$. As in the proof of Theorem 109.13 we do a proof by contradiction. Thus suppose that there exists a homotopy from $g \circ f\colon S^{n+2} \to S^n$ to a constant map. By Lemma 40.4 we can use this homotopy to construct a map[1494] $\varphi\colon \mathrm{Cone}(S^{n+2}) = \overline{B}^{n+3} \to S^n$ such that $\varphi|_{S^{n+2}} = g \circ f$.

[1494]As we discussed on page 692, given any $m \in \mathbb{N}_0$ the map $\mathrm{Cone}(S^m) \to \overline{B}^{m+1}$ given by $[(v,t)] \mapsto v \cdot t$ is a homeomorphism. We will use this map to make the identification $\overline{B}^{m+1} = \mathrm{Cone}(S^m)$.

We consider the map[1495]

$$\begin{array}{rcl}\phi\colon \overbrace{(\mathrm{Cone}(S^{n+2})\sqcup S^{n+1})/\sim}^{=\mathrm{Cone}(f\colon S^{n+2}\to S^{n+1})} & \to & S^n \\ [P] & \mapsto & \begin{cases} g(P), & \text{if } P\in S^{n+1}, \\ \varphi(P), & \text{if } P\in \mathrm{Cone}(S^{n+2})=\overline{B}^{n+3}. \end{cases}\end{array}$$

Our main object of interest is now, perhaps rather confusingly, the mapping cone of ϕ. This mapping cone comes with two interesting maps:

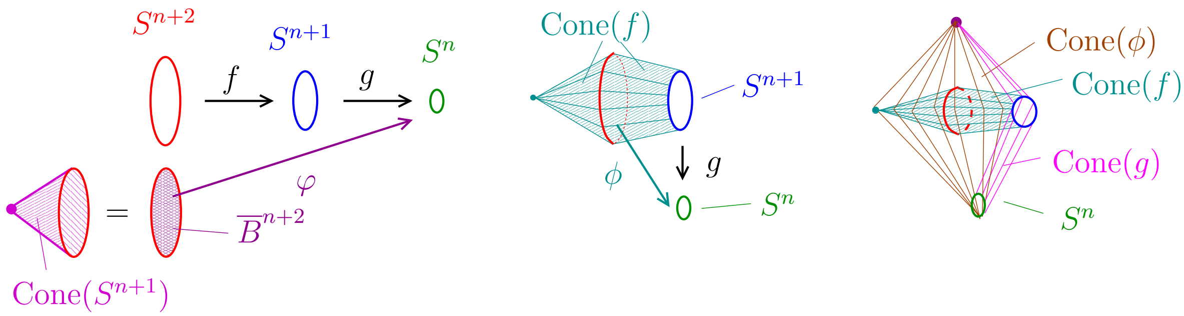


FIGURE 1536. Illustration for the proof of Theorem 110.2.

(1) Since we have an inclusion $S^{n+1}\to \mathrm{Cone}(f)$ and since $\phi|_{S^{n+1}}=g$ we obtain, see Lemma 24.10 (3), also an inclusion map

$$i\colon\ \mathrm{Cone}(g\colon S^{n+1}\to S^n)\ \to\ \mathrm{Cone}(\phi\colon\ \mathrm{Cone}(f)\to S^n).$$

(2) We also consider the projection map

$$p\colon\ \mathrm{Cone}(\phi\colon\ \mathrm{Cone}(f)\to S^n)\ \to\ \mathrm{Cone}(\phi\colon\ \mathrm{Cone}(f)\to S^n)/S^n.$$

In the following two claims we will study the maps on cohomology induced by the inclusion i and the projection p.

Claim 1.

(a) The induced map

$$i^*\colon\ \mathrm{H}^k(\mathrm{Cone}(\phi);\mathbb{Z}_2)\ \to\ \mathrm{H}^k(\mathrm{Cone}(g);\mathbb{Z}_2)$$

is an isomorphism for $k\neq n+1$.

(b) We have $\mathrm{H}^{n+1}(\mathrm{Cone}(\phi);\mathbb{Z}_2)=0$.

[1495]It follows from $\varphi|_{S^{n+2}}=g\circ f$ that this map is well-defined. Furthermore we obtain immediately from Lemma 3.44 (5) that this map is continuous.

We consider the following diagram:

$$\begin{tikzcd}[column sep=small]
 & \mathrm{H}^{k-2}(S^{n+2};\mathbb{Z}_2) \arrow[d] & & & \mathrm{H}^{k-1}(S^{n+2};\mathbb{Z}_2) \arrow[d] & \\
{} \arrow[r] & \mathrm{H}^{k-1}(\mathrm{Cone}(f);\mathbb{Z}_2) \arrow[r] \arrow[d] & \mathrm{H}^k(\mathrm{Cone}(\phi\colon \mathrm{Cone}(f)\to S^n);\mathbb{Z}_2) \arrow[r] \arrow[d, "i^*"] & \mathrm{H}^k(S^n;\mathbb{Z}_2) \arrow[r, "\phi^*"] \arrow[d, "\mathrm{id}"] & \mathrm{H}^k(\mathrm{Cone}(f);\mathbb{Z}_2) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \mathrm{H}^{k-1}(S^{n+1};\mathbb{Z}_2) \arrow[r] \arrow[d, "f^*"] & \mathrm{H}^k(\mathrm{Cone}(g\colon S^{n+1}\to S^n);\mathbb{Z}_2) \arrow[r] & \mathrm{H}^k(S^n;\mathbb{Z}_2) \arrow[r, "f^*"] & \mathrm{H}^k(S^{n+1};\mathbb{Z}_2) \arrow[r] \arrow[d, "f^*"] & {} \\
 & \mathrm{H}^{k-1}(S^{n+2};\mathbb{Z}_2) & & & \mathrm{H}^k(S^{n+2};\mathbb{Z}_2) &
\end{tikzcd}$$

Here the two horizontal and the two vertical sequences are obtained from Lemma 91.4. The diagram commutes by the naturality of the long exact sequence. The two statements of the claim follow easily from the above diagram (see also the discussion on page 2601) and the calculation of the cohomology groups of spheres with $\mathbb{Z}_2$-coefficients. ⊞

Claim 2. The map

$$p^*\colon\ \mathrm{H}^k(\mathrm{Cone}(\phi)/S^n;\mathbb{Z}_2)\ \to\ \mathrm{H}^k(\mathrm{Cone}(\phi);\mathbb{Z}_2)$$

is an isomorphism for $k \neq n, n+1$.

We consider the following diagram:

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{H}^{k-1}(S^n;\mathbb{Z}_2) \arrow[r, "\delta"] & \mathrm{H}^k(\mathrm{Cone}(\phi), S^n;\mathbb{Z}_2) \arrow[r] \arrow[d] & \mathrm{H}^k(\mathrm{Cone}(\phi);\mathbb{Z}_2) \arrow[r] & \mathrm{H}^k(S^n;\mathbb{Z}_2) \arrow[r] & \ldots \\
 & & \mathrm{H}^k(\mathrm{Cone}(\phi)/S^n;\mathbb{Z}_2) \arrow[ur, "p^*"'] & & &
\end{tikzcd}$$

Here on top we see the long exact sequence in cohomology of the pair $(\mathrm{Cone}(\phi), S^n)$. Furthermore the vertical map is the natural map from Proposition 74.2 which has in particular the property that it makes the diagram commute. Furthermore, it follows from Proposition 74.2 (2) and Lemma 24.9 (6) that the vertical map is an isomorphism. The desired statement follows immediately from this discussion and the calculation of the cohomology groups of S^n. ⊞

Next we consider the following diagram.

$$\begin{tikzcd}
\mathrm{H}^n(\mathrm{Cone}(\Sigma^{n-2}(H));\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2", "\cong"'] \arrow[d, "="] & \mathrm{H}^{n+2}(\mathrm{Cone}(\Sigma^{n-2}(H));\mathbb{Z}_2) \arrow[d, "="'] & \\
\mathrm{H}^n(\mathrm{Cone}(g);\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2", "\cong"'] & \mathrm{H}^{n+2}(\mathrm{Cone}(g);\mathbb{Z}_2) & \\
\mathrm{H}^n(\mathrm{Cone}(\phi);\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2"] \arrow[u, "i^*", "\cong"'] & \mathrm{H}^{n+2}(\mathrm{Cone}(\phi);\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2"] \arrow[u, "\cong", "i^*"'] & \mathrm{H}^{n+4}(\mathrm{Cone}(\phi);\mathbb{Z}_2) \\
& \mathrm{H}^{n+2}(\mathrm{Cone}(\phi)/S^n;\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2"] \arrow[u, "p^*", "\cong"'] & \mathrm{H}^{n+4}(\mathrm{Cone}(\phi)/S^n;\mathbb{Z}_2) \arrow[u, "\cong", "p^*"'] \\
& \mathrm{H}^{n+2}(\Sigma(\mathrm{Cone}(f));\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2", "\cong"'] \arrow[u, "h^*", "\cong"'] \arrow[d, "\theta^*", "\cong"'] & \mathrm{H}^{n+4}(\Sigma(\mathrm{Cone}(f));\mathbb{Z}_2) \arrow[u, "\cong", "h^*"'] \arrow[d, "\cong", "\theta^*"'] \\
& \mathrm{H}^{n+2}(\mathrm{Cone}(\Sigma(f));\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2", "\cong"'] \arrow[d, "="'] & \mathrm{H}^{n+4}(\mathrm{Cone}(\Sigma(f));\mathbb{Z}_2) \arrow[d, "="] \\
& \mathrm{H}^{n+2}(\mathrm{Cone}(\Sigma^n(H));\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2", "\cong"'] & \mathrm{H}^{n+4}(\mathrm{Cone}(\Sigma^n(H));\mathbb{Z}_2).
\end{tikzcd}$$

We make the following clarifications and observations:

(1) At the top and bottom we use that by definition $f = \Sigma^{n-1}(H)\colon S^{n+2} \to S^{n+1}$ and $g = \Sigma^{n-2}(H)\colon S^{n+1} \to S^n$.
(2) We obtain from Proposition 109.15 (1) that the top and bottom horizontal maps Sq^2 are isomorphisms.
(3) It follows from the above two claims that the vertical maps i^* and p^* are isomorphisms.
(4) The map $h\colon \mathrm{Cone}(\phi\colon \mathrm{Cone}(f) \to S^n)/S^n \to \Sigma(\mathrm{Cone}(f))$ is the homeomorphism from Lemma 24.9 (4).
(5) The map $\theta\colon \mathrm{Cone}(\Sigma(f)\colon \Sigma(S^{n+2}) \to \Sigma(S^{n+1})) \to \Sigma(\mathrm{Cone}(f\colon S^{n+2} \to S^{n+1}))$ is the homeomorphism from Proposition 24.16.
(6) It follows from the naturality of the Steenrod operations that all the squares commute.
(7) The combination of the above arguments implies that both of the maps Sq^2 in the third row are isomorphisms.

Finally we consider the following diagram.

$$\begin{tikzcd}
\overbrace{\mathrm{H}^n(\mathrm{Cone}(\phi);\mathbb{Z}_2)}^{\neq 0} \arrow[r, "\mathrm{Sq}^2", "\cong"'] \arrow[dr, "\mathrm{Sq}^1"'] & \mathrm{H}^{n+2}(\mathrm{Cone}(\phi);\mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^2", "\cong"'] & \mathrm{H}^{n+2}(\mathrm{Cone}(\phi);\mathbb{Z}_2). \\
& \underbrace{\mathrm{H}^{n+1}(\mathrm{Cone}(\phi);\mathbb{Z}_2)}_{=0} \arrow[ur, "\mathrm{Sq}^3"'] &
\end{tikzcd}$$

We make the following observations:

(1) By the discussion preceding the diagram we know that the two maps on the top are isomorphisms.
(2) As we saw on page 2600, we have the Adem relation $\mathrm{Sq}^2 \circ \mathrm{Sq}^2 = \mathrm{Sq}^3 \circ \mathrm{Sq}^1$. This implies that the diagram commutes.
(3) We have

$$\mathrm{H}^n(\mathrm{Cone}(\phi);\mathbb{Z}_2) \underset{\substack{\uparrow \\ \text{Claim 1 (a)}}}{\cong} \mathrm{H}^n(\mathrm{Cone}(\overbrace{\Sigma^{n-2}(H)}^{=g});\mathbb{Z}_2) \underset{\substack{\uparrow \\ \text{see the example on page 2601}}}{\cong} \mathrm{H}^n(S^n;\mathbb{Z}_2) \cong \mathbb{Z}_2.$$

(4) In Claim 1 (b) we had shown that the group $\mathrm{H}^{n+1}(\mathrm{Cone}(\phi);\mathbb{Z}_2)$ is trivial.

By (1) and (3) the two horizontal maps in the diagram give us an isomorphism of a non-trivial group. On the other hand, by (2) and (4) this isomorphism factors through a trivial group. This is absurd. We have thus obtained a contradiction. ■

Now we turn to the proof of the remaining two statements of Theorem 110.2.

Proof of Theorem 110.2 (2) and (3). The proof for the quaternionic Hopf map $H_{\mathbb{H}}$ and the octonion Hopf map $\mathrm{H}_{\mathbb{O}}$ is almost the same as the above proof that deals with the classical Hopf map H. We just need to replace Proposition 109.15 (1) by Proposition 109.15 (2) and (3) and we need to use the following Adem relations:

$$\begin{aligned} \mathrm{Sq}^4 \circ \mathrm{Sq}^4 &= \mathrm{Sq}^6 \circ \mathrm{Sq}^2 + \mathrm{Sq}^7 \circ \mathrm{Sq}^1 \\ \mathrm{Sq}^8 \circ \mathrm{Sq}^8 &= \mathrm{Sq}^{12} \circ \mathrm{Sq}^4 + \mathrm{Sq}^{14} \circ \mathrm{Sq}^2 + \mathrm{Sq}^{15} \circ \mathrm{Sq}^1 . \end{aligned}$$

These relations can be obtained easily by hand from Theorem 110.1. Alternatively one can use the Adem relations calculator:

`https://math.berkeley.edu/~kruckman/adem/`

As in the proof for the Hopf maps one sees that the initial Steenrod operations on the right hand side all end up in a cohomology group which is zero. Thus one obtains again a contradiction to the hypothesis that the maps $\Sigma^{n-4}(H_{\mathbb{H}} \circ \Sigma^3(H_{\mathbb{H}}))\colon S^{n+6} \to S^n$ and $\Sigma^{n-8}(H_{\mathbb{O}} \circ \Sigma^7(H_{\mathbb{O}}))\colon S^{n+14} \to S^n$ are null-homotopic. ■

110.2. **Decomposing Steenrod operations and the Hopf invariant revisited.** The key to extracting extra mileage out of Steenrod operations is the following innocuous lemma from elementary number theory.

Lemma 110.3. *Let $n,k \in \mathbb{N}_0$. If we write $n = \sum\limits_{i\in\mathbb{N}_0} n_i \cdot 2^i$ and $k = \sum\limits_{i\in\mathbb{N}_0} k_i \cdot 2^i$ with $n_i, k_i \in \{0,1\}$, then*

$$\binom{n}{k} = \prod_{i\in\mathbb{N}_0} \binom{n_i}{k_i} \mod 2.$$

Remark.

(1) Note that for $r, s \in \{0,1\}$ it follows easily from the definition on page 2579 that

$$\binom{s}{r} = \begin{cases} 1, & \text{if } r \leq s, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let $n \in \mathbb{N}$ and let $k \in \{0, \ldots, d\}$. We define n_i, k_i as above. It follows from Lemma 110.3 and (1) that $\binom{n}{k}$ is odd if and only if for each i we have $k_i \leq n_i$. If we draw the usual Pascal triangle, then by shading odd and even terms in different colors we obtain a Sierpinski triangle. We refer to Figure 1537 for an illustration.[1496]

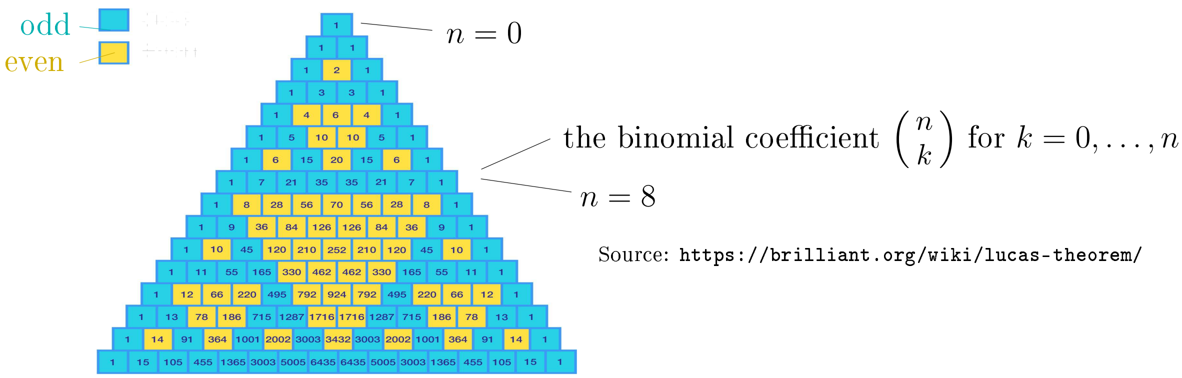


FIGURE 1537

(3) In fact the obvious analogue of Lemma 110.3 holds with the prime 2 replaced by any prime p. This more general statement is known as Lucas' Theorem, a proof is for example provided in [**Fi47**].

PROOF. Let $n, k \in \mathbb{N}_0$. We write $n = \sum\limits_{i \in \mathbb{N}_0} n_i \cdot 2^i$ and $k = \sum\limits_{i \in \mathbb{N}_0} k_i \cdot 2^i$ with $n_i, k_i \in \{0,1\}$ In the polynomial ring $\mathbb{Z}_2[x]$ we have the following equalities[1497]

$$\sum_{i=0}^{n} \binom{n}{i} x^i \underset{\substack{\uparrow \\ \text{binomial formula}}}{=} (1+x)^n = (1+x)^{\sum n_i 2^i} = \prod_{i \in \mathbb{N}_0} (1+x)^{2^i n_i} \underset{\substack{\uparrow \\ \text{since in } \mathbb{Z}_2[x] \text{ we have } (a+b)^2 = a^2+b^2}}{=} \prod_{i \in \mathbb{N}_0} (1+x^{2^i})^{n_i} \underset{\substack{\uparrow \\ \text{binomial formula}}}{=} \prod_{i \in \mathbb{N}_0} \sum_{j_i=0}^{n_i} \binom{n_i}{j_i} x^{j_i \cdot 2^i}.$$

[1496] A more elaborate version of this figure can be found here:

`https://upload.wikimedia.org/wikipedia/commons/8/87/Sierpinski_Pascal_triangle.svg`

[1497] The argument has some similarities with the proof of the claim on page 2209.

Now we compare the x^k-coefficients, we see that[1498]

$$\binom{n}{k} \;=\; x^k\text{-coefficient of } \sum_{i=0}^{n}\binom{n}{i}x^i \underset{\substack{\uparrow \\ \text{by the above equality}}}{=} x^k\text{-coefficient of } \prod_{i\in\mathbb{N}_0}\sum_{j_i=0}^{n_i}\binom{n_i}{j_i}x^{j_i\cdot 2^i} \underset{\substack{\uparrow \\ \text{by the uniqueness of the 2-adic description} \\ \text{of a natural number we only get } x^k \\ \text{if each variable } j_i \text{ equals } k_i}}{=} \prod_{i\in\mathbb{N}_0}\binom{n_i}{k_i}.$$

We have thus proved the desired equality. ■

This lemma allows us to prove the following theorem.

Theorem 110.4. *Let $i \in \mathbb{N}_0$. If i is not a power of 2, then Sq^i is decomposable, in the sense that it can be written as the sum of compositions of Steenrod operations of smaller degree.*

Example. By the discussion on page 2600 we know that for any $k \in \mathbb{N}_0$ we have

$$\mathrm{Sq}^{2k+1} \;=\; \mathrm{Sq}^1 \circ \mathrm{Sq}^{2k} \quad\text{ and }\quad \mathrm{Sq}^{4k+2} \;=\; \mathrm{Sq}^2 \circ \mathrm{Sq}^{4k} + \mathrm{Sq}^{4k+1} \circ \mathrm{Sq}^1\,.$$

In particular we see that

$$\mathrm{Sq}^3 = \mathrm{Sq}^1 \circ \mathrm{Sq}^2, \quad \mathrm{Sq}^5 = \mathrm{Sq}^1 \circ \mathrm{Sq}^4, \quad \mathrm{Sq}^6 = \mathrm{Sq}^2 \circ \mathrm{Sq}^4 + \mathrm{Sq}^5 \circ \mathrm{Sq}^1 \quad\text{ and }\; \mathrm{Sq}^7 = \mathrm{Sq}^1 \circ \mathrm{Sq}^6\,.$$

Proof. Suppose that we are given $i \in \mathbb{N}_0$ that is not a power of two. This means that we can write $i = a + b$ where $b = 2^k$ and $a \in \{1, \ldots, 2^k - 1\}$. Since $0 < a < 2b$ we obtain an Adem relation for $\mathrm{Sq}^a \circ \mathrm{Sq}^b$ from Theorem 110.1. This relation can be rewritten as follows:

$$\binom{b-1}{a} \cdot \mathrm{Sq}^{a+b} \;=\; \underbrace{\mathrm{Sq}^a \circ \mathrm{Sq}^b + \sum_{j=1}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \cdot \mathrm{Sq}^{a+b-j} \circ \mathrm{Sq}^j}_{\text{compositions of Steenrod operations of degree } < a+b}\,.$$

It remains to show that $\binom{b-1}{a} = 1 \bmod 2$. We calculate

$$\binom{b-1}{a} \overset{\substack{\text{by Lemma 110.3 where } b-1 = \sum_{i\in\mathbb{N}_0} d_i\cdot 2^i \text{ and } a = \sum_{i\in\mathbb{N}_0} c_i \cdot 2^i \text{ with } c_i, d_i \in \{0,1\} \\ \downarrow}}{=} \prod_{i\in\mathbb{N}_0}\binom{d_i}{c_i} \underset{\substack{\uparrow \\ \text{since } b-1 = 2^k - 1 = 1 + 2 + \cdots + 2^{k-1} \text{ we have } d_0 = \cdots = d_{k-1} = 1 \text{ and } d_k = d_{k+1} = \cdots = 0 \\ \text{furthermore, since } a < 2^k - 1 \text{ we have } c_k = c_{k+1} = \cdots = 0}}{=} \prod_{i=0}^{k-1}\binom{1}{c_i} \cdot \prod_{i=k}^{\infty}\binom{0}{0} \;=\; 1 \mod 2.$$

■

Corollary 110.5. *Let X be a topological space and let $n \in \mathbb{N}_0$. We consider the Steenrod operation*

$$\mathrm{Sq}^i \colon\; \mathrm{H}^n(X;\mathbb{Z}_2) \;\to\; \mathrm{H}^{n+i}(X;\mathbb{Z}_2).$$

If the cohomology groups inbetween vanish, i.e. if $\mathrm{H}^{n+1}(X;\mathbb{Z}_2) = \ldots = \mathrm{H}^{n+i-1}(X;\mathbb{Z}_2) = 0$, and if i is not a power of 2, then the Steenrod operation is the zero map.

[1498] Note that the equalities also make sense for $k > n$. In this setting one sees easily that all terms are in fact zero.

Proof. Since i is not a power of 2 we know by Theorem 110.4 that Sq^i can be written as the sum of compositions of Steenrod operations of smaller degree. But by our hypothesis for any $j \in \{1, \ldots, i-1\}$ we have $\mathrm{H}^{n+j}(X; \mathbb{Z}_2) = 0$. But this implies in particular that for any such j we have $\mathrm{Sq}^j \colon \mathrm{H}^n(X; \mathbb{Z}_2) \to \mathrm{H}^{n+j}(X; \mathbb{Z}_2) = 0$. ∎

$$
\begin{tikzcd}
\mathrm{H}^n(X;\mathbb{Z}_2) \arrow[rrrrrrr, bend left, "\mathrm{Sq}^i"] \arrow[rrrr, bend right, "\mathrm{Sq}^j"'] & 0 & 0 & 0 & 0 \arrow[rrr, bend right] & 0 & 0 & \mathrm{H}^{n+i}(X;\mathbb{Z}_2)
\end{tikzcd}
$$

Figure 1538. Illustration for the proof of Corollary 110.5.

In Question 84.19 we asked whether all (reasonable) superalgebras can be realized as cohomology rings. We will now see that this is not the case. Before we can state the corresponding proposition we need to introduce the following definition.

Definition. Let $m \in \mathbb{N}$ be an even number and let $s \in \mathbb{Z}$. We define $R_m(s)$ to be the superalgebra which is given by the groups[1499]

$$
R_m(s) \;=\; \begin{cases} \mathbb{Z}\cdot 1, & \text{in degree } 0, \\ \mathbb{Z}\cdot x, & \text{in degree } m, \\ \mathbb{Z}\cdot y, & \text{in degree } 2m, \\ 0, & \text{in all other degrees} \end{cases}
$$

and where the multiplication is given by $x \cdot x = s \cdot y$. In other words, the multiplication table is given as follows

	1	x	y
1	1	x	y
x	x	$s\cdot y$	0
y	y	0	$0.$

Example.

(1) For any m the superalgebra $R_m(1)$ is evidently isomorphic to $\mathbb{Z}[x]/(x^3)$ where x is of degree m.
(2) By (1) together with Proposition 90.7 and Proposition 90.20 we have isomorphisms

$$
(\mathrm{H}^*(\mathbb{C}\mathrm{P}^2;\mathbb{Z}),\cup) \cong R_2(1), \;\; (\mathrm{H}^*(\mathbb{H}\mathrm{P}^2;\mathbb{Z}),\cup) \cong R_4(1) \text{ and } (\mathrm{H}^*(\mathbb{O}\mathrm{P}^2;\mathbb{Z}),\cup) \cong R_8(1).
$$

Next we prove the following proposition which gives a negative answer to Questions 84.19 and 90.21.

Proposition 110.6. *Let m be an even number and let s be an odd number. If m is not a power of 2, then there is no topological space X such that $(\mathrm{H}^*(X;\mathbb{Z}),\cup)$ is isomorphic to $R_m(s)$.*

In the proof of Proposition 110.6 we will need the following lemma.

[1499]It follows from the fact that m is even that this is indeed a superalgebra.

Lemma 110.7. *Let $m \in \mathbb{N}$. We denote by $\varphi\colon \mathbb{Z} \to \mathbb{Z}_m$ the canonical map. Given any topological space X there exists a long exact sequence*

$$\ldots \to \mathrm{H}^k(X;\mathbb{Z}) \xrightarrow{\cdot m} \mathrm{H}^k(X;\mathbb{Z}) \xrightarrow{\varphi_*} \mathrm{H}^k(X;\mathbb{Z}_m) \xrightarrow{\delta} \mathrm{H}^{k+1}(X;\mathbb{Z}) \to \ldots$$

Proof of Lemma 110.7. We denote by $\psi\colon \mathbb{Z} \to \mathbb{Z}$ the map that is given by multiplication by m. We have the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}_m \to 0.$$

By the discussion on page **??** we have the corresponding Bockstein sequence in cohomology which looks as follows:

$$\ldots \xrightarrow{\delta} \mathrm{H}^k(X;\mathbb{Z}) \xrightarrow{\psi_*} \mathrm{H}^k(X;\mathbb{Z}) \xrightarrow{\varphi_*} \mathrm{H}^k(X;\mathbb{Z}_2) \xrightarrow{\delta} \mathrm{H}^{k+1}(X;\mathbb{Z}) \to \ldots$$

By the almost, but not entirely trivial Lemma **??** we know that ψ_* is indeed multiplication by m. ■

Proof of Proposition 110.6. Let m be an even number. Furthermore let X be a topological space such that $(\mathrm{H}^*(X;\mathbb{Z}),\cup) = R_m(s)$ for some odd number s. We need to show that m is a power of 2. We denote by $\varphi\colon \mathbb{Z} \to \mathbb{Z}_2$ the canonical map. We consider the following diagram

$$\begin{tikzcd}
\mathrm{H}^m(X;\mathbb{Z}) \arrow[r, "x\mapsto x^2=x\cup x=s\cdot y"] \arrow[d, "\varphi_*"] & \mathrm{H}^{2m}(X;\mathbb{Z}) \arrow[d, "\varphi_*"] \\
\mathrm{H}^m(X;\mathbb{Z}_2) \arrow[r, "z\mapsto z^2=z\cup z"] \arrow[d] & \mathrm{H}^{2m}(X;\mathbb{Z}_2) \arrow[d] \\
\mathrm{H}^m(X;\mathbb{Z}_2) \arrow[r, "z\mapsto \mathrm{Sq}^m(z)"] & \mathrm{H}^{2m}(X;\mathbb{Z}_2).
\end{tikzcd}$$

We make the following clarifications and observations:

(1) It follows from Proposition 82.4 (2) that the upper square commutes.
(2) It follows from Lemma 110.7 and our hypothesis on the topological space X that the map $\varphi_*\colon \mathrm{H}^{2m}(X;\mathbb{Z}) \to \mathrm{H}^{2m}(X;\mathbb{Z}_2)$ is an epimorphism and that $\mathrm{H}^{2m}(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$. In particular we see that $\varphi_*(y) \in \mathrm{H}^{2m}(X;\mathbb{Z}_2)$ is a non-trivial element of order 2.
(3) Since s is odd we obtain from (2) that $\varphi_*(s\cdot y) = s\cdot\varphi_*(y)$ is a non-trivial element of $\mathrm{H}^{2m}(X;\mathbb{Z}_2)$.
(4) The lower square commutes by Axiom (2) of Steenrod operations, see Theorem 109.2 (2).
(5) By our hypothesis on X we know that the intermediate cohomology groups vanish, i.e. we have $\mathrm{H}^{m+1}(X;\mathbb{Z}_2) = \cdots = \mathrm{H}^{2m-1}(X;\mathbb{Z}_2) = 0$.

It follows from (3) and (4) that the bottom horizontal map has non-trivial image. But by Corollary 110.5 together with (5) this is only possible if m is indeed a power of 2. ■

As promised in the title of the section, we now return to the Hopf invariant. Let $f\colon S^{2n-1} \to S^n$ be a map where $n \geq 2$. Recall that on page 2217 we introduced the Hopf invariant $\mathrm{Hopf}(f) \in \mathbb{Z}$. The following lemma translates the Hopf invariant into the language of the above discussion. The lemma can also be viewed as an alternative definition of the Hopf invariant, at least up to sign.

Lemma 110.8. *Let $n \geq 2$. If $f\colon S^{2n-1} \to S^n$ is a map, then there exists an isomorphism*

$$(\mathrm{H}^*(\mathrm{Cone}(f);\mathbb{Z}),\cup) \;\cong\; R_n(\mathrm{Hopf}(f))$$

of superalgebras.

Proof. This statement follows immediately from Lemma 91.5 and the definition of the Hopf invariant, see page 2217. ■

The following proposition is the summary of our previous discussion of the Hopf invariant.

Proposition 110.9. *Let $n \geq 2$.*
1. *The Hopf invariant* $\mathrm{Hopf}\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$ *is a homomorphism.*
2. *The Hopf invariant* $\mathrm{Hopf}\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$ *is zero for odd n.*
3. *If $\mathbb{R}^n$ can be equipped with the structure of a division algebra, then there exists a map $f\colon S^{2n-1} \to S^n$ with* $\mathrm{Hopf}(f) = 1$.
4. *For $k = 1, 2, 4$ the Hopf maps $H, H_{\mathbb{H}}$ and $\mathrm{H}_{\mathbb{O}}$ give elements in $\pi_{4k-1}(S^{2k})$ with Hopf invariant* 1.
5. *Given any $k \in \mathbb{N}$ there exists a map $f\colon S^{4k-1} \to S^{2k}$ with* $\mathrm{Hopf}(f) = 2$.

Proof. The proof follows from Proposition 91.7, Corollary 91.8, Lemma 91.11 and Propositions 91.13 and 91.12. ■

We recall the following question that we asked a while ago.

Question 91.17. *For which $n \in \mathbb{N}_{\geq 2}$ does there exist a map $f\colon S^{2n-1} \to S^n$ with* $\mathrm{Hopf}(f) = 1$*?*

The following theorem is an immediate consequence of Proposition 110.6 and Lemma 110.8.

Theorem 110.10. *Let $n \geq 2$. If there exists a map $f\colon S^{2n-1} \to S^n$ with* $\mathrm{Hopf}(f) = 1$, *then n is a power of* 2.

The combination of Proposition 110.9 (3) and Theorem 110.10 gives us an alternative proof of the following theorem.

Theorem 90.23. (Hopf 1940) *If $\mathbb{R}^n$ admits the structure of a real division algebra, then n is a power of 2.*

It is an amusing mental exercise, left to the reader, to compare the two proofs of Theorem 90.23. The first one relies on the calculation of the cup product of $\mathbb{RP}^n \times \mathbb{RP}^n$ with $\mathbb{Z}_2$-coefficients and a fact in elementary number theory that is proved on page 2209. The second proof uses Steenrod operations and a different input from elementary number theory, namely Lemma 110.3. Note that in some sense the former proof is more satisfactory since we could provide full details, whereas the latter proof relies on the Adem relations that we did not prove.

For completeness we remark that in 1960 Frank Adams [**Adam60**] (see also [**AdA66**], [**May99a**, Chapter 24.6] and [**Hus93**, Theorem 15.4.3]) proved the following strengthening of Theorem 110.10. Note that this theorem finally gives a complete answer to Question 91.17.

Theorem 110.11. *Let $n \geq 2$. If there exists a map $f\colon S^{2n-1} \to S^n$ with* $\operatorname{Hopf}(f) = 1$, *then $n = 2, 4$ or $n = 8$.*

The combination of Proposition 110.9 (3) and Theorem 110.11 gives us an alternative proof of the following beautiful theorem.

Theorem 90.25. *Every finite-dimensional division algebra over $\mathbb{R}$ is of dimension 1,2, 4 or 8.*

110.3. **The Steenrod cyclic reduced powers** ($*$)**.** The Steenrod operations Sq^i are cohomology operations on cohomology groups with $\mathbb{Z}_2$-coefficients. It turns out that there exist also analogous cohomology operations for cohomology with $\mathbb{Z}_p$-coefficients. More precisely, the following theorem holds:

Theorem 110.12. *Let p be an odd prime. There exists a family*[1500]

$$P^i\colon\ \mathrm{H}^n(X,A;\mathbb{Z}_p) \ \to\ \mathrm{H}^{n+2i(p-1)}(X,A;\mathbb{Z}_p) \quad \textit{with } i \in \mathbb{N}_0 \textit{ and } n \in \mathbb{N}_0$$

of additive cohomology operations which satisfies the following axioms:

(1) $P^0 = \mathrm{id}$.
(2) *For any $\varphi \in \mathrm{H}^{2i}(X,A;\mathbb{Z}_p)$ we have $P^i(\varphi) = \varphi^p \in \mathrm{H}^{2pi}(X,A;\mathbb{Z}_p)$.*
(3) *For any $\varphi \in \mathrm{H}^j(X,A;\mathbb{Z}_p)$ with $i > j$ we have $P^i(\varphi) = 0$.*
(4) *Let (X,A,B) be an excisive triad of topological spaces. For any $\varphi \in \mathrm{H}^r(X,A;\mathbb{Z}_p)$ any $\psi \in \mathrm{H}^s(X,B;\mathbb{Z}_p)$ and any $k \in \mathbb{N}_0$ we have*[1501]

$$P^k(\varphi \cup \psi) \ = \sum_{i=0}^{k} \underbrace{P^i(\varphi)}_{\in \mathrm{H}^{r+2i(p-1)}(X,A;\mathbb{Z}_p)} \ \cup \ \underbrace{P^{k-i}(\psi)}_{\in \mathrm{H}^{s+2(k-i)(p-1)}(X,B;\mathbb{Z}_p)} \ \in\ \mathrm{H}^{r+s+2k(p-1)}(X,A\cup B;\mathbb{Z}_p).$$

This equality is often called the Cartan formula.

Proof. This theorem is proved in [**SE62**, Chapter VI]. A very different proof is given in [**Hat02**, Theorem 4.L.16]. ■

The following theorem plays the role of Theorem 110.1.

[1500]Note that if for argument's sake one plugs $p = 2$ into the statements of Theorem 110.12, then one does *not* get the statements of Theorem 109.2 since the degree formulas are different. In other words there is no easy way to combine Theorems 109.2 and 110.12 into a single statement.

[1501]We refer to page 1851 for the definition of an excisive triad. By Proposition 74.12 most "reasonable" triads are in fact excisive. In fact we will only use the special cases that $A = B$, or that at least one of A or B is the empty set. We refer again to page 2005 for the definition of the relative cup product.

Theorem 110.13. (Adem Relations) *Let p be an odd prime.*

(1) *For any $0 < a < pb$ we have the following equality of cohomology operations:*

$$P^a \circ P^b \;=\; \sum_{j=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+j} \cdot \binom{(p-1)(b-j)-1}{a-pj} \cdot P^{a+b-j} \circ P^j.$$

(2) *Given $i \in \mathbb{N}_0$ we denote by $\beta\colon \mathrm{H}^i(X;\mathbb{Z}_p) \to \mathrm{H}^{i+1}(X;\mathbb{Z}_p)$ the Bockstein homomorphism that corresponds to the short exact sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$. For any $0 < a \le pb$ the following mystifying equality holds:*

$$\begin{aligned} P^a \circ \beta \circ P^b \;=\; & \sum_{j=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+j} \cdot \binom{(p-1)(b-j)}{a-pj} \cdot \beta \circ P^{a+b-j} \circ P^j \\ & - \sum_{j=0}^{\lfloor \frac{a-1}{p} \rfloor} (-1)^{a+j} \cdot \binom{(p-1)(b-j)-1}{a-pj-1} \cdot P^{a+b-j} \circ \beta \circ P^j. \end{aligned}$$

These equalities are called the Adem relations.

Example. Let us consider the case of the prime $p = 3$. In this case the corresponding reduced Steenrod operations are additive cohomology operations of the form

$$P^i\colon\ \mathrm{H}^n(X, A; \mathbb{Z}_3) \;\to\; \mathrm{H}^{n+4i}(X, A; \mathbb{Z}_3).$$

For $a = 1$ and $b = 3$ we obtain the Adem relation

$$P^1 \circ P^3 \underset{\substack{\uparrow \\ \text{Theorem 110.13 (1)}}}{=} \sum_{j=0}^{\lfloor \frac{1}{3} \rfloor} (-1)^{1+j} \binom{2\cdot(3-j)-1}{1-3\cdot j} \cdot P^{4-j} \circ P^j \underset{\substack{\uparrow \\ \text{only one summand, namely for } j=0}}{=} (-1)\cdot\binom{5}{1}\cdot P^4 \;=\; -5\cdot P^4 \underset{\substack{\uparrow \\ \text{we work in } \mathbb{Z}_3}}{=} P^4.$$

The Steenrod cyclic reduced powers can, not surprisingly, be used to give more information on homotopy groups of spheres. We refer to [**SE62**, Chapter VI.5] and [**Hat02**, Example 4.L.6] for details.

Instead of providing more results on homotopy groups of spheres we want to give a different application of the Steenrod cyclic reduced powers, namely we want to prove the following proposition which gives a negative answer to Question 90.22.

Proposition 110.14. *There is no topological space X such that $\mathrm{H}^*(X;\mathbb{Z}_3)$ is isomorphic to $\mathbb{Z}_3[\varphi]/(\varphi^4)$ with $\varphi \in \mathrm{H}^8(X;\mathbb{Z}_3)$.*

PROOF. We provide a proof by contradiction. So suppose there exists a topological space X such that $\mathrm{H}^*(X;\mathbb{Z}_3)$ is isomorphic to $\mathbb{Z}_3[\varphi]/(\varphi^4)$ with $\varphi \in \mathrm{H}^8(X;\mathbb{Z}_3)$. We see that

$$0 \underset{\substack{\uparrow \\ \text{since we assume that} \\ \mathrm{H}^*(X;\mathbb{Z}_3) \cong \mathbb{Z}_3[\varphi]/(\varphi^4)}}{\neq} \varphi^3 \overset{\substack{\text{Theorem 110.12 (2)} \\ \downarrow}}{=} P^4(\varphi) \underset{\substack{\uparrow \\ \text{since } P^4 = P^1 \circ P^3}}{=} (P^1 \circ P^3)(\varphi) \;=\; P^1(\overbrace{P^3(\varphi)}^{\in \mathrm{H}^{8+3\cdot 4}(X;\mathbb{Z}_3)}) \underset{\substack{\uparrow \\ \text{since } \mathrm{H}^{20}(X;\mathbb{Z}_3) = 0}}{=} 0.$$

We have thus obtained a contradiction. ■

$$
\begin{tikzcd}[column sep=tiny]
& & & & & & & & H^8(X;\mathbb{Z}_3) \arrow[d, dash] & & & & & & & & & & & & & & & & H^{24}(X;\mathbb{Z}_2) \arrow[d, dash] \\
\mathbb{Z}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z}_3 \arrow[rrrrrrrrrrrrrrrr, bend left, "P^4 \text{ which equals } \varphi \mapsto \varphi^3"] \arrow[rrrrrrrrrrrr, bend right, "P^3"'] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z}_3 & 0 & 0 & 0 & 0 \arrow[rrrr, bend right, "P^1"'] & 0 & 0 & 0 & \mathbb{Z}_3
\end{tikzcd}
$$

FIGURE 1539. Illustration for the proof of Proposition 110.14.

Remark. In Exercise 110.2 we will use Proposition 110.14 to give a negative answer to Question 60.11. So with most reasonable interpretations it is fair to say that there is no smooth manifold that deserves the name $\mathbb{O}\mathrm{P}^n$ with $n \geq 3$.

Exercises for Chapter 110.

Exercise 110.1. Let X be a topological space and let $x \in \mathrm{H}^1(X;\mathbb{Z}_2)$ be a cohomology class of degree *one*. Show that for any $k \in \mathbb{N}$ and any $i \in \mathbb{N}$ we have

$$\mathrm{Sq}^i(x^{2^k}) \;=\; \begin{cases} x^{2^k}, & \text{if } i = 0, \\ x^{2^{k+1}}, & \text{if } i = 2^k, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 110.2. Let $n \geq 3$. Show that there is no closed orientable $8n$-dimensional smooth manifold that admits a CW-structure with precisely one cell in dimensions $0, 8, 16, \ldots, 8n$.
Remark. This exercise gives a negative answer to Question 60.11.

Part XIII

Homotopy Theory and Eilenberg-Maclane Spaces

111.1. **The definition of relative homotopy groups.** While studying homology groups we saw at some point that it was convenient to introduce the relative homology groups $\mathrm{H}_k(X, A)$ corresponding to a pair of topological spaces (X, A). In our latest attempt to understand homotopy groups we will now, perhaps not surprisingly, introduce relative homotopy groups. We start out with introducing the following notation.

Notation. Throughout this chapter we write $I = [0, 1]$ for the unit interval. Furthermore, given $n \in \mathbb{N}$ we use the following notation:

(1) we denote by $I^n \subset \mathbb{R}^n$ the cube in $\mathbb{R}^n$,
(2) we denote by ∂I^n the boundary of I^n viewed as a subset of $\mathbb{R}^n$,
(3) by a slight abuse of notation we make the identification $I^{n-1} = I^{n-1} \times \{0\}$,
(4) we write $J^{n-1} = \overline{\partial I^n \setminus I^{n-1}}$.

We refer to Figure 1540 for an illustration.

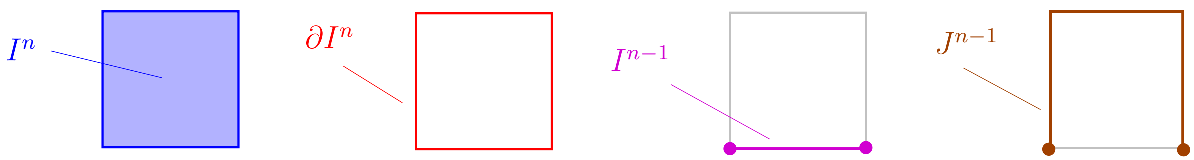


Figure 1540

The following definition is a variation on the notion of a pointed topological space that we introduced on page 476.

Definition. A *pointed pair of topological spaces* is a triple (X, A, x_0), where X is a topological space, A is a subset of X and x_0 is a point in A. A *morphism* between pointed pairs of topological spaces (X, A, x_0) and (Y, B, y_0) is a map $f : X \to Y$ with $f(A) \subset B$ and $f(x_0) = y_0$. We refer to the corresponding category as the *category of pointed pairs of topological spaces.*

Definition. Let (X, A, x_0) be a pointed pair of topological spaces. Given $n \geq 1$ we define the *n-th relative homotopy group of* (X, A, x_0) to be[1502]

$$\pi_n(X, A, x_0) \ := \ \text{homotopy classes of maps } (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\}).$$

The definition is illustrated in Figure 1541.

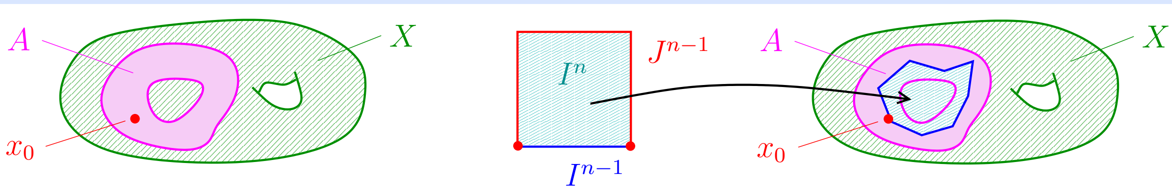


Figure 1541

[1502] On page 571 we defined the notion of maps between pairs of topological spaces and we defined the notion of a homotopy between such maps. These notions extend in an obvious way to triples of topological spaces.

Examples.

(1) Let (X, A, x_0) be a pointed pair of topological spaces. We want to get a better understanding of elements in $\pi_1(X, A, x_0)$. For $n = 1$ we have $I^1 = [0, 1]$, $\partial I^1 = \{0, 1\}$ and $J^0 = \{1\}$. A map $(I^1, \partial I^1, J^0) \to (X, A, \{x_0\})$ is thus the same as a path in X that starts at a point in A and that ends in x_0. We refer to Figure 1542 and 1543 for an illustration.

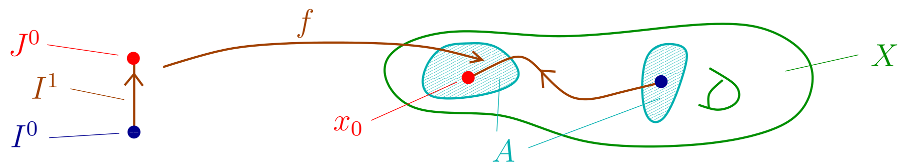


FIGURE 1542

the maps define the same element in $\pi_1(X, A, x_0)$

the maps define (presumably) three different elements in $\pi_1(X, A, x_0)$

FIGURE 1543

(2) Let (X, x_0) be a pointed topological space. For any $n \geq 1$ the map

$$\begin{array}{rcl} \pi_n(X, x_0) & \to & \pi_n(X, \{x_0\}, x_0) \\ [f\colon (I^n, \partial I^n) \to (X, \{x_0\})] & \mapsto & [f\colon (I^n, \partial I^n, J^{n-1}) \to (X, \{x_0\}, \{x_0\})] \end{array}$$

defines, basically by definition, a natural bijection. In the following we will use this natural bijection to make the identification $\pi_n(X, x_0) = \pi_n(X, \{x_0\}, x_0)$.

(3) Let X be a topological space, let A be a subset of X and let $x_0 \in A$ be a point. For any $n \geq 1$ the map

$$\begin{array}{rcl} \pi_n(X, x_0) & \to & \pi_n(X, A, x_0) \\ [f\colon (I^n, \partial I^n) \to (X, \{x_0\})] & \mapsto & [f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] \end{array}$$

is a natural map from absolute homotopy groups to relative homotopy groups.

In the previous discussion we already referred to $\pi_n(X, A, x_0)$ as a "relative homotopy group". This was slightly premature, but now we will actually deliver the goods and provide a group structure.

Definition. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \geq 2$.[1503] Given two maps $f, g\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ we define[1504]

$$\begin{array}{rcl} f * g\colon I^n & \to & X \\ (t_1, \dots, t_n) & \mapsto & \begin{cases} f(2t_1, t_2, \dots, t_n), & \text{if } t_1 \in [0, \frac{1}{2}], \\ g(2t_1 - 1, t_2, \dots, t_n) & \text{if } t_1 \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

Note that $f * g$ is again a map $(I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ of triples. The definition of $f * g$ is illustrated in Figure 1544.

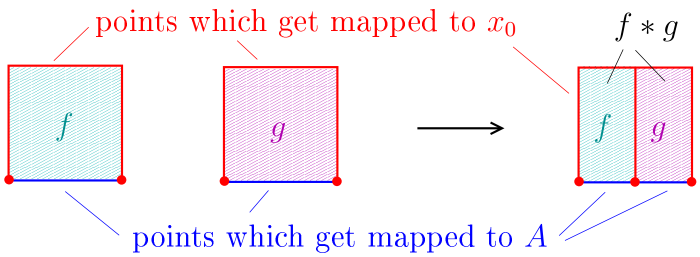


FIGURE 1544

The following proposition finally gives the justification for the name *relative homotopy group*.

Proposition 111.1. *Let (X, A, x_0) be a pointed pair of topological spaces.*

(1) *Let $n \geq 2$. The map*

$$\begin{array}{rcl}\pi_n(X,A,x_0) \times \pi_n(X,A,x_0) & \to & \pi_n(X,A,x_0) \\ ([f],[g]) & \mapsto & [f * g]\end{array}$$

is well-defined and it defines a group structure on $\pi_n(X, A, x_0)$. Furthermore the following two statements hold:

(a) *the trivial element in $\pi_n(X, A, x_0)$ is represented by the constant map $I^n \to X$ that sends every point in I^n to x_0,*

(b) *the inverse of the relative homotopy class $[f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})]$ is represented by the map*

$$\begin{array}{rcl}(I^n, \partial I^n, J^{n-1}) & \to & (X, A, \{x_0\}) \\ (t_1, t_2, \dots, t_n) & \mapsto & (1 - t_1, t_2, \dots, t_n).\end{array}$$

(2) *For $n \geq 2$ the above natural bijection $\pi_n(X, x_0) \xrightarrow{=} \pi_n(X, \{x_0\}, x_0)$ is in fact a group isomorphism.*

(3) *For $n \geq 2$ the above natural map $\pi_n(X, x_0) \to \pi_n(X, A, x_0)$ is a group homomorphism.*

(4) *For $n \geq 3$ the group $\pi_n(X, A, x_0)$ is abelian.*

(5) *Let $n \in \mathbb{N}$. Given a morphism $f\colon (X, A, x_0) \to (Y, B, y_0)$ of pointed pairs of topological spaces the map*

$$\begin{array}{rcl}f_*\colon \pi_n(X,A,x_0) & \to & \pi_n(Y,B,y_0) \\ [\varphi] & \mapsto & [f \circ \varphi]\end{array}$$

is well-defined. Furthermore the following four statements hold:

(a) *For $n \geq 2$ the induced map f_* is a group homomorphism.*

(b) *The maps f_* define a functor from the category of pointed pairs of topological spaces to the category of sets (respectively groups for $n \geq 2$).*

[1503]What, if anything, goes wrong for $n = 1$?

[1504]This map is continuous by Lemma 2.35 (2).

(c) *Let $n \geq 1$ and let $f, g\colon (X, A) \to (Y, B)$ be two maps between pairs of topological spaces. If $f_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, f(x_0))$ is a bijection for all $x_0 \in A$ and if g is homotopic*[1505] *to f, then $g_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, g(x_0))$ is also a bijection for all $x_0 \in A$.*

(d) *Let $f\colon (X, A) \to (Y, B)$ be a map between pairs of topological spaces and let $x_0 \in A$. If f is a homotopy equivalence, then the map*

$$f_*\colon \pi_n(X, A, x_0) \;\to\; \pi_n(Y, B, f(x_0))$$

is a bijection (in particular it is an isomorphism if $n \geq 2$).

Sketch of proof.

(1) The proof of the first statement is quite similar to the proof of Proposition 40.1. We leave it to the reader to fill in the details.

(3) This statement requires just slightly more thought than one might think. More precisely, given two maps f and g from $(I^n, \partial I^n, J^{n-1})$ to $(X, \{x_0\}, \{x_0\})$, the "product maps" $f * g$ considered on page 1055 for $\pi_n(X, \{x_0\})$ (and illustrated in Figure 679) and in the above proposition for $\pi_n(X, A, x_0)$ are not identical. Nonetheless, using Lemma 40.2 or Proposition 53.11 it is straightforward to show that they are homotopic. This observation shows that the above identification of sets is in fact an identification of groups.

(2) We already know that the map is a bijection. Thus it suffices to show that it is a group homomorphism. But this follows from (3) applied to $A = \{x_0\}$.

(4) The proof of this statement is an entertaining variation on the proof of Proposition 40.3. The proof is the content of Exercise 111.2.

(5) Statements (a) and (b) follow basically immediately from the definitions. Statement (c) and (d) are proved in a similar fashion to Proposition 40.7. ■

Remark.

(1) We will see shortly that the relative homotopy group $\pi_2(X, A, x_0)$ is in general not abelian.

(2) For want of a better idea, given a pointed pair of topological spaces (X, A, x_0) we refer to $\pi_1(X, A, x_0)$ also as a *relative homotopy group* even though strictly speaking we did not define a group structure on $\pi_1(X, A, x_0)$.

In light of Proposition 111.1 we adopt the following convention:

Convention. For $n \geq 3$ we usually use additive notation for the group structure on the relative homotopy groups $\pi_n(X, A, x_0)$.

Let X be a topological space and let $A \subset B \subset X$ be two subsets. In Proposition 43.15 we saw that this data gives rise to a long exact sequence of homology groups. Shortly we will see that a similar statement holds for relative homotopy groups. We did not define a group structure on $\pi_1(X, A, x_0)$ and we did not define a group structure on $\pi_0(X, x_0)$. (In fact we will see in Exercise 111.3 that there is no natural way to turn these sets into groups.) Nonetheless we would like to include these objects at the lower end of the long

[1505] Here we mean of course, "homotopic as a map between pairs of topological spaces".

exact sequence of (relative) homotopy groups. In fact, as it turns out, $\pi_1(X, A, x_0)$ and $\pi_0(X, x_0)$ have ever so slightly more structure than just the structure of a set.

Definition.

(1) A *pointed set* is a pair (S, s_0) where S is a set and $s_0 \in S$ is a point. We call s_0 the *trivial element* in S.

(2) A *morphism* from a pointed set (S, s_0) to a pointed set (T, t_0) is a map $f\colon S \to T$ with $f(s_0) = t_0$. Furthermore we adopt the following language:

   (a) Given a morphism $f\colon (S, s_0) \to (T, t_0)$ we refer to $\ker(f) := f^{-1}(t_0)$ as the *kernel of* f.

   (b) We say a morphism $f\colon (S, s_0) \to (T, t_0)$ is *trivial* if $\ker(f) = S$, put differently, if all elements in S get sent to t_0.

   (c) We say a morphism $f\colon (S, s_0) \to (T, t_0)$ is a *monomorphism* if $\ker(f) = \{s_0\}$.

   (d) We say a morphism $f\colon (S, s_0) \to (T, t_0)$ is an *epimorphism* if f is surjective.

(3) We say that a sequence

$$\dots \xrightarrow{f_{n+2}} (S_{n+1}, s_{n+1}) \xrightarrow{f_{n+1}} (S_n, s_n) \xrightarrow{f_n} (S_{n-1}, s_{n-1}) \xrightarrow{f_{n-1}} \dots$$

of morphisms of pointed sets is *exact* if for every $n \in \mathbb{N}_0$ we have the usual equality $\ker(f_n) = \operatorname{im}(f_{n+1})$.

Examples.

(1) Any group G gives rise to the pointed set $(G, \{e\})$ where $e \in G$ evidently denotes the trivial element. We will thus always view a group as a pointed set. Any group homomorphism is also a morphism of the corresponding pointed sets. Furthermore an exact sequence of homomorphisms of groups is also an exact sequence of pointed sets.

(2) If G is a group and H is a subgroup, then the set of right cosets G/H is a pointed set with trivial element given by $H \in G/H$. The obvious projection map $G \to G/H$ is a morphism of pointed sets.

The following little lemma will be proved in Exercise 111.4.

Lemma 111.2. *Let $\varphi\colon G \to H$ be a group homomorphism. Furthermore let (S, s_0) be a pointed set which is equipped with an H-action. Suppose that we have an exact sequence*

$$\{e\} \to G \xrightarrow{\varphi} H \xrightarrow{\psi} (S, s_0) \to \{e\}$$

where ψ is a morphism of pointed sets that preserves the H-action. Then $\varphi(G)$ is a subgroup of H whose index equals the cardinality of the set S.

Example. The previous lemma might sound rather artificial. But we have seen it before in group theory: suppose that H is a group and G is a subgroup. As above we view the set of right cosets G/H as a pointed set. Left-multiplication by G defines an action on G/H. Evidently the projection $G \to G/H$ preserves the right-action and we get a short exact sequence $\{e\} \to G \to H \to G/H \to \{e\}$ as in Lemma 111.2.

We continue with pointed sets that arise in topology.

Examples.

(1) Let (X, x_0) be a pointed topological space. Recall that on page 1054 we defined $\pi_0(X, x_0)$ to be the set of homotopy classes of maps from the point I^0 to X. In other words, $\pi_0(X, x_0)$ is the set of path-components of X. We view $\pi_0(X, x_0)$ as a pointed set where the trivial element is given by the map that sends the unique point in I^0 to x_0. A map $f\colon X \to Y$ between topological spaces defines a morphism $f_*\colon \pi_0(X, x_0) \to \pi_0(Y, f(x_0))$ of pointed sets in the obvious way. Sometimes, by an unfortunate abuse of language, we refer to $\pi_0(X, x_0)$ as a homotopy group of X.

(2) Similarly to the above, given a pointed pair of topological spaces (X, A, x_0) we view $\pi_1(X, A, x_0)$ as pointed set where the trivial element is once again represented by the constant path $I \to X$ given by sending all points to x_0.

We conclude this section with the following elementary lemma that will be proved in Exercise 111.6.

Lemma 111.3. *Let (X, A, x_0) be a pointed pair of topological spaces and let $n \geq 1$. Let $f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ be a map. If $f(I^n) \subset A$, then f represents the trivial element in $\pi_n(X, A, x_0)$.*[1506][1507]

Since it is much easier to convince oneself of a statement if one sees an explicit example, we show such a map with $f(I^n) \subset A$ in Figure 1545.

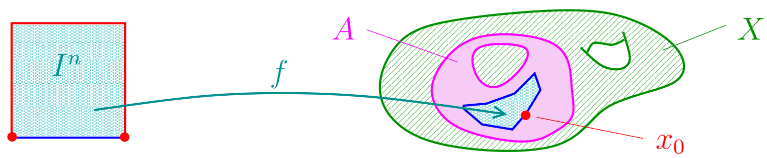


FIGURE 1545

111.2. **The long exact homotopy sequence of a pair.** In Proposition 43.15 we saw that we can associate to a pair (X, A) of topological spaces a long exact sequence of (relative) homology groups. The goal of this section is to do the same for homotopy groups. More precisely we will associate to a pointed pair (X, A, x_0) of topological spaces a long exact sequence of (relative) homotopy groups.

Proposition 111.4. *Let (X, A, x_0) be a pointed pair of topological spaces.*

(1) *For any $n \geq 1$ the map*

$$\begin{array}{rcl} \partial_n\colon \pi_n(X, A, x_0) & \to & \pi_{n-1}(A, x_0) \\ {[f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})]} & \mapsto & \left[\begin{array}{rcl} (I^{n-1}, \partial I^{n-1}) & \to & (A, \{x_0\}) \\ (x_1, \ldots, x_{n-1}) & \mapsto & f(x_1, \ldots, x_{n-1}, 0) \end{array} \right] \end{array}$$

[1506] Put differently, as for relative homology groups, all objects that "land in A" are in fact trivial in the relative homotopy group.

[1507] Note that by the above discussion of pointed sets the statement of Lemma 111.3 is also meaningful for $n = 1$.

is well-defined. (*See Figure 1546 for an illustration.*) *We refer to it as the* connecting morphism. *It has the following two properties:*

(a) *The connecting morphism is a morphism of pointed sets and it is a group homomorphism for $n \geq 2$.*

(b) *The connecting morphism is a natural transformation from the covariant functor $\pi_n(X, A, x_0)$ to the covariant functor $\pi_{n-1}(A, x_0)$. In everyday language this just simply means that for a map $f\colon (X, A, x_0) \to (Y, B, y_0)$ between pointed pairs of topological spaces the following diagram commutes:*

$$\begin{tikzcd} \pi_n(X, A, x_0) \arrow[r, "\partial"] \arrow[d, "f_*"] & \pi_{n-1}(A, x_0) \arrow[d, "f_*"] \\ \pi_n(Y, B, y_0) \arrow[r, "\partial"] & \pi_{n-1}(B, y_0). \end{tikzcd}$$

(2) *The following sequence is exact*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \to \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0).$$

Here $i\colon A \to X$ is the inclusion map and the map $\pi_n(X, x_0) \to \pi_n(X, A, x_0)$ is the natural map introduced on page 2617.

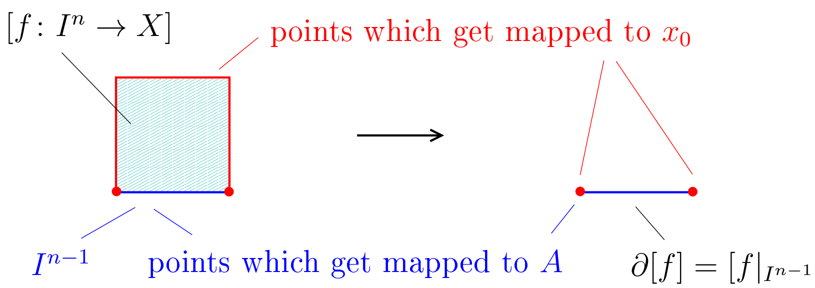


FIGURE 1546

Examples.

(1) We consider $X = \mathbb{R}^2$ and we consider $A = A_0 \sqcup A_1$ which is the union of disjoint closed disks A_0 and A_1 in $X = \mathbb{R}^2$. We denote by $x_0 \in A_0$ and $x_1 \in A_1$ the centers of the two disks. Furthermore we denote by $f\colon I = [0, 1] \to X$ a map with $f(0) = x_1$ and $f(1) = x_0$. (We refer to Figure 1547 for an illustration.) Finally, for any $n \in \mathbb{N}_0$ and any $i \in \{0, 1\}$ we denote by $c_{x_i}\colon I^n \to A_i$ the constant map that sends all points to x_i. It follows from Lemma 18.14 together with Lemma 18.13 and Propositions 18.16 (2) and 40.7 (2) that all the homotopy groups of A and X are generated by $[c_{x_0}]$ except for $\pi_0(A, x_0)$ that equals the set $\{[c_{x_0}], [c_{x_1}]\}$.

As discussed on page 2617, f defines an element in $\pi_1(X, A, x_0)$. Using the definitions one sees almost immediately that $\partial[f] = [c_{x_1}] \in \pi_0(A, x_0)$. It follows easily from this data and Proposition 111.4 (2) that $\pi_1(X, A, x_0)$ consists of precisely two elements, namely $[c_{x_0}]$ and $[f]$.

(2) Let $X = \mathbb{R}^2$. We denote by A the wedge of two circles, viewed as a subset of $X = \mathbb{R}^2$. Furthermore we denote by x_0 the wedge point. It follows from Proposition 111.4 (2)

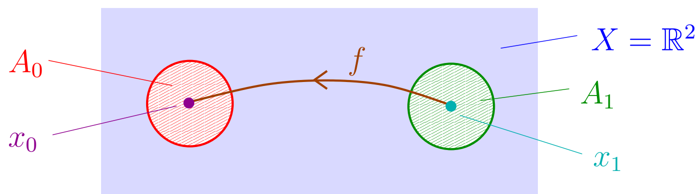


FIGURE 1547

that $\pi_2(X,A,x_0) \xrightarrow{\cong} \pi_1(A,x_0)$. In Lemma 22.4 we saw that $\pi_1(A,x_0)$ is non-abelian, thus $\pi_2(X,A,x_0)$ is also non-abelian.

In fact we can identify two elements in $\pi_2(X,A,x_0)$ that do not commute. We consider the two maps $f,g\colon (I^2,\partial I^2,J^1) \to (X,A,\{x_0\})$ that are illustrated in Figure 1548. It follows easily from the definitions that $\partial[f],\partial[g]$ are the standard generators of the free group[1508] $\pi_1(S^1\vee S^1,x_0)$. Since these do not commute we see that $[f]$ and $[g]$ do not commute either.

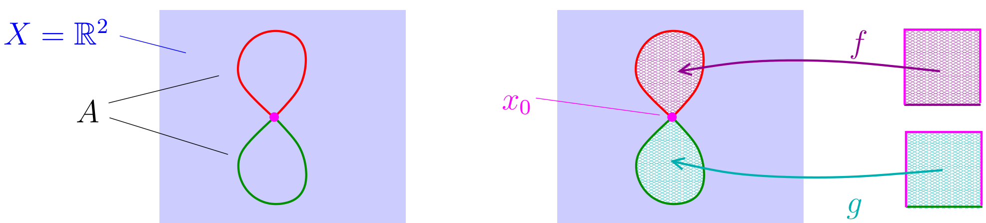


FIGURE 1548

PROOF. We start out with the proof of statement (1). In fact this statement follows mostly easily from the definitions. One only needs to think for a minute about the question why the connecting homomorphism is in fact a homomorphism.

We turn to the proof of statement (2). In fact we prove a slight variation on the second statement. More precisely, let X be a topological space, let A be a subset of X and let B be a subset of A. Finally let $x_0 \in B$. For any $n \geq 2$ the map

$$\begin{array}{rcl} \partial_n\colon \pi_n(X,A,x_0) & \to & \pi_{n-1}(A,B,x_0) \\ {[f\colon (I^n,\partial I^n,J^{n-1}) \to (X,A,\{x_0\})]} & \to & \left[\begin{array}{rcl} (I^{n-1},\partial I^{n-1},J^{n-2}) & \to & (A,B,\{x_0\}) \\ (x_1,\dots,x_{n-1}) & \mapsto & f(x_1,\dots,x_{n-1},0) \end{array}\right] \end{array}$$

is easily seen to be a well-defined morphism of pointed sets. Our goal is to prove the following claim.

Claim. The following sequence is exact[1509]

$$\dots \to \pi_n(A,B,x_0) \xrightarrow{i_*} \pi_n(X,B,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \xrightarrow{\partial_n} \pi_{n-1}(A,B,x_0) \to \dots \to \pi_1(X,A,x_0).$$

where $i\colon (A,B) \to (X,B)$ and $j\colon (X,B) \to (X,A)$ are the obvious morphisms of pointed pairs of topological spaces.

[1508]We refer to page 606 for the calculation of the fundamental group of $S^1 \vee S^1$.

[1509]This sequence plays of course the role of the long exact sequence in homology groups of a triple of topological spaces, see Proposition 43.15.

Note that this claim, applied to $B = \{x_0\}$ together with Proposition 111.1 (2) almost implies that the sequence of the proposition is exact. The only problem is that the sequence of the claim stops slightly earlier, more precisely it stops at the subindex one instead of subindex zero. Thus the only piece missing is the exactness of the sequence of the proposition at the last three pointed sets. We leave the verification of exactness at these three spots as an elementary exercise to the conscientious reader.

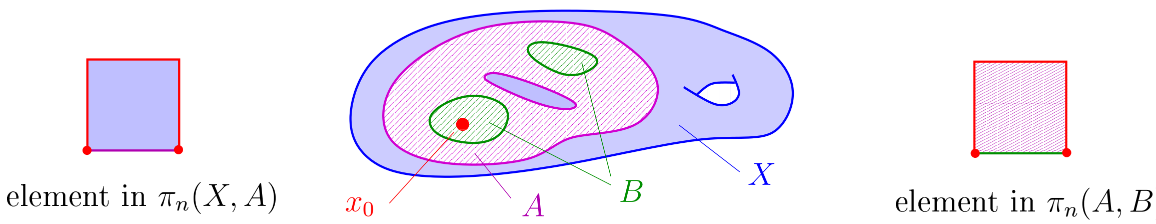


FIGURE 1549. First illustration for the proof of Proposition 111.4.

Now we turn to the proof of the claim. Not surprisingly the proof of the claim breaks up into three separate proofs, namely we need to verify exactness at the three different groups.

Exactness at $\pi_n(X, B, x_0)$. It follows easily from Lemma 111.3 that $j_* \circ i_* = (j \circ i)_*$ is the trivial morphism. We leave it to the reader to verify that $\ker(j_*) \subset \operatorname{im}(i_*)$.

Exactness at $\pi_n(X, A, x_0)$. It follows again easily from Lemma 111.3 and the definition of the connecting homomorphism ∂ that $\partial \circ j_*$ is the trivial morphism. Now suppose we are given $f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})$ such that the restriction of f to $I^{n-1} \times \{0\}$ represents the trivial element in $\pi_{n-1}(A, B, x_0)$. This means that there exists a homotopy $H\colon I^{n-1} \times I \to A$ (with all the required restrictions) from $f|_{I^n \times \{0\}}$ to the constant map. The map

$$\begin{array}{rcl} I^n \times [0,1] & \to & X \\ (x_1, \ldots, x_n, t) & \mapsto & \begin{cases} f\big(x_1, \ldots, x_{n-1}, \frac{t}{2} \cdot x_n + (1 - x_n)\big), & \text{if } x_n \in \big[\frac{t}{2}, 1\big] \\ H\big(x_1, \ldots, x_{n-1}, 2x_n - t\big), & \text{if } x_n \in \big[0, \frac{t}{2}\big] \end{cases} \end{array}$$

is continuous by Lemma 14.3 and it defines a homotopy (again with all the desired extra properties) from f to a map that represents an element in $\pi_n(X, B, x_0)$. The definition of the homotopy is sketched, to the best of my abilities, in Figure 1550.

Exactness at $\pi_n(A, B, x_0)$. First note that $i_* \circ \partial$ is the trivial morphism. Indeed, if $f\colon (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, \{x_0\})$ is a map, then $f\colon (I^n \times [0,1], \partial I^n \times [0,1]) \to (X, x_0)$ defines a homotopy from $\partial f = f|_{I^n \times \{0\}}$ to the constant map c_{x_0}. Thus $(i_* \circ \partial)[f] = [i_* \circ \partial f]$ is the trivial element in $\pi_n(X, B, x_0)$.

Now suppose that we are given a map $f\colon (I^n, \partial I^n, J^{n-1}) \to (A, B)$ such that $j \circ f$ represents the trivial element in $\pi_n(X, B, x_0)$. This means that there exists a homotopy H (with all the obvious conditions) from $j \circ f$ to the constant map c_{x_0}. We define

$$\begin{array}{rcl} g\colon I^n & \to & B \\ (x_1, \ldots, x_n) & \mapsto & H(x_1, \ldots, x_{n-1}, 0, x_n). \end{array}$$

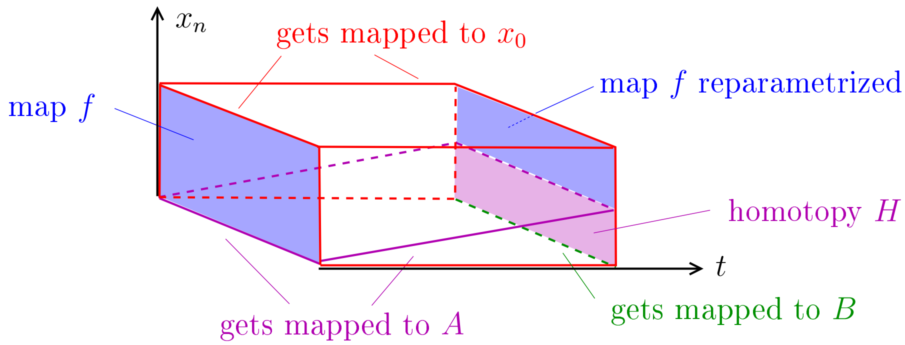


FIGURE 1550

We pick a homeomorphism $\Psi\colon I^2 \to I^2$ with the properties as illustrated in Figure 1551.[1510] (Here it is psychologically convenient to think of I^2 as the square in the plane with x_n and x_{n+1}-coordinates.) It follows immediately from the definitions that the map

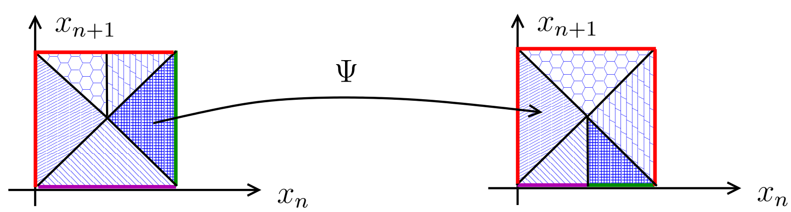


FIGURE 1551

$$\begin{array}{rcl} I^{n+1} & \to & X \\ (x_1,\ldots,x_n,x_{n+1}) & \to & H(x_1,\ldots,x_{n-1},\Psi(x_n,x_{n+1})) \end{array}$$

defines an element in $\pi_{n+1}(X,A,x_0)$ such that the connecting morphism sends it to the element in $\pi_n(A,B,x_0)$ represented by the map[1511]

$$\begin{array}{rcl} f*g\colon I^n & \to & X \\ (t_1,\ldots,t_n) & \mapsto & \begin{cases} f(t_1,\ldots,t_{n-1},2t_n), & \text{if } t_n\in[0,\frac{1}{2}], \\ g(t_1,\ldots,t_{n-1},2t_n-1), & \text{if } t_n\in[\frac{1}{2},1] \end{cases} \end{array}$$

It remains to show that $f*g$ and f represent the same element in $\pi_n(A,B,x_0)$. This can be done in a fashion similar to the above proof of exactness. More precisely, a homotopy between f and $f*g$, again with all the desired properties, is given by

$$\begin{array}{rcl} I^n\times[0,1] & \to & X \\ (x_1,\ldots,x_n,t) & \mapsto & \begin{cases} f\big(x_1,\ldots,x_{n-1},\frac{t}{2}\cdot x_n+(1-x_n)\big), & \text{if } x_n\in\big[\frac{t}{2},1\big] \\ H\big(x_1,\ldots,x_{n-1},2x_n-t\big), & \text{if } x_n\in\big[0,\frac{t}{2}\big] \end{cases} \end{array}$$ ■

[1510]The triangles in Figure 1551 indicate how one could write down an explicit homeomorphism.

[1511]Strictly speaking this definition of $f*g$ differs from the definition on page 2617 since we use a different coordinate to perform the product. But it is convenient to use the same notation.

111.3. **Paths acting on relative homotopy groups.** Let (X, x_0) be a pointed topological space. In Section 54.3 we saw that the fundamental group $\pi_1(X, x_0)$ of X acts on any homotopy group $\pi_n(X, x_0)$. In this section we will discuss an analogous construction for relative homotopy groups. In particular, given a pointed pair (X, A, x_0) of topological spaces we will see that the fundamental group $\pi_1(A, x_0)$ of A (let us stress that here we write A instead of X) acts on any relative homotopy group $\pi_n(X, A, x_0)$.

The following definition is quite similar to the definition on page 1352.

Definition. Let X be a topological space, let A be a subset of X and let $n \in \mathbb{N}_{\geq 2}$.[1512] Furthermore let $\gamma\colon [0,1] \to A$ be a path in the subset A[1513] from a point x_0 to a point x_1. Finally let

$$f\colon (I^n, \partial I^n, J^{n-1}) \;\to\; (X, A, \{x_1\})$$

be a map. We define

$$f^\gamma\colon (I^n, \partial I^n, J^{n-1}) \;\to\; (X, A, \{x_0\})$$

to be the map that is illustrated in Figure 1552. For our particularly pedantic readers we now also give a super-rigorous definition of f^γ: First of all, given $t \in [0,1]$ we define $\mathrm{C}_t = \left[\frac{t}{4}, \frac{4-t}{4}\right]^{n-1} \times [0, 1 - \frac{t}{2}]$. Now we define

$$\begin{aligned} f^\gamma\colon (I^n, \partial I^n, J^{n-1}) &\to (X, A, x_0) \\ (x_1, \dots, x_n) &\to \begin{cases} f\big(2x_1 - \frac{1}{2}, \dots, 2x_n\big), & \text{if } (x_1, \dots, x_n) \in \mathrm{C}_1 = \left[\frac{1}{4}, \frac{3}{4}\right]^{n-1} \times [0, \frac{1}{2}], \\ \gamma(t), & \text{if } (x_1, \dots, x_n) \in \partial\, \mathrm{C}_t \text{ for } t \in [0,1]. \end{cases} \end{aligned}$$

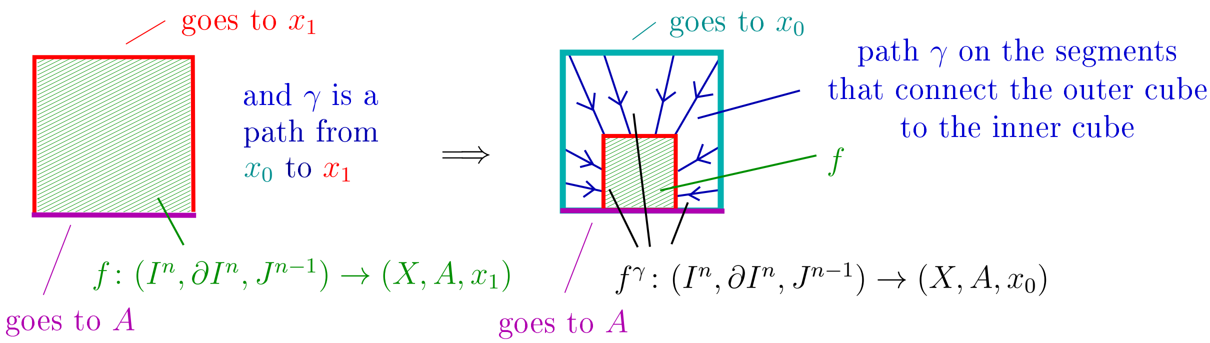


FIGURE 1552

The following proposition can be viewed as the "relative analogue" of Proposition 40.5.

Proposition 111.5. *Let X be a topological space, let A be a subset of X and let $n \in \mathbb{N}_{\geq 2}$.*

(1) *If $\gamma\colon [0,1] \to A$ is a path from x_0 to x_1, then the map*

$$\begin{aligned} \gamma_*\colon \pi_n(X, A, x_1) &\to \pi_n(X, A, x_0) \\ [f] &\mapsto [f^\gamma] \end{aligned}$$

is a well-defined map that is a group homomorphism.

[1512] What is wrong with $n = 1$?

[1513] Note that in contrast to the definition on page 1352 we only consider paths in A. At this point we do not allow paths in X.

(2) *If x_0 and x_1 are two points in A which lie in the same path-component of A, then $\pi_n(X,A,x_0)$ is isomorphic to $\pi_n(X,A,x_1)$.*
(3) *Let $\varphi\colon (X,A)\to(Y,B)$ be a map between pairs of topological spaces.*
 (a) *If $\gamma\colon[0,1]\to A$ is a path from x_0 to x_1, then the following diagram commutes*

$$\begin{tikzcd}
\pi_n(X,A,x_1) \arrow[r, "\gamma_*"] \arrow[d, "\varphi_*"] & \pi_n(X,A,x_0) \arrow[d, "\varphi_*"] \\
\pi_n(Y,B,\varphi(x_1)) \arrow[r, "(\varphi\circ\gamma)_*"] & \pi_n(Y,B,\varphi(x_0)).
\end{tikzcd}$$

 (b) *If $\varphi_*\colon \pi_n(X,A,x_0)\to\pi_n(Y,B,\varphi(x_0))$ is an isomorphism (epimorphism, monomorphism) for some base point $x_0\in A$, then for any other point x_1 in the same path component of A the map $\varphi_*\colon\pi_n(X,A,x_1)\to\pi_n(Y,B,\varphi(x_1))$ is also an isomorphism (epimorphism, monomorphism).*
(4) *Let γ and δ be two paths in A from x_0 to x_1. If γ and δ are path-homotopic in A, then the corresponding maps γ_* and δ_* from $\pi_n(X,A,x_1)$ to $\pi_n(X,A,x_0)$ agree.*
(5) *If γ is a path in A from x_0 to x_1 and if δ is a path in A from x_1 to x_2, then*[1514]

$$(\underbrace{\gamma*\delta}_{\text{path in } A \text{ from } x_0 \text{ to } x_2})_* = \gamma_*\circ\delta_*\colon \pi_n(X,A,x_2)\ \to\ \pi_n(X,A,x_0).$$

(6) *The map*

$$\begin{array}{rcl}\pi_1(A,x_0)\times\pi_n(X,A,x_0) & \to & \pi_n(X,A,x_0)\\ ([\gamma],[f]) & \mapsto & [\gamma]\cdot[f]:=[f^\gamma]\end{array}$$

defines an action (in the sense of the definition on page 185) of the group $\pi_1(A,x_0)$ on the group $\pi_n(X,A,x_0)$.
(7) *We denote by $i\colon A\to X$ the inclusion map and by $\varphi\colon\pi_n(X,x_0)\to\pi_n(X,A,x_0)$ the natural map. Then, with the convention from page 2007, the following diagram commutes*

$$\begin{tikzcd}
\pi_1(X,x_0)\times\pi_n(X,x_0) \arrow[d, shift left=6em, "\varphi_*"] \arrow[rr] & & \pi_n(X,x_0) \arrow[d, "\varphi_*"'] \\
\pi_1(A,x_0)\times\pi_n(X,A,x_0) \arrow[u, shift left=-6em, "i_*"] \arrow[rr] & & \pi_n(X,A,x_0).
\end{tikzcd}$$

(8) *If $\gamma\colon[0,1]\to A$ is a path from x_0 to x_1, then the following diagram commutes:*

$$\begin{tikzcd}
\pi_n(X,A,x_1) \arrow[r, "{[f]\mapsto[f^\gamma]}"] \arrow[d, "\partial"] & \pi_n(X,A,x_0) \arrow[d, "\partial"'] \\
\pi_{n-1}(A,x_1) \arrow[r, "{[f]\mapsto[f^\gamma]}"] & \pi_{n-1}(A,x_0).
\end{tikzcd}$$

PROOF. The proof of the first six statements of the proposition is basically identical to the proof of Proposition 54.9. The last two statements follow easily from the definitions. To protect the trees we will not carry out the details. ■

[1514]Here it is worth remembering that by definition, see page 465, the product path $\gamma*\delta$ is given by first traversing γ and then traversing δ.

Remark. Let X be a topological space, let A be a non-empty subset and let $n \in \mathbb{N}_{\geq 2}$. If A is path-connected, then it follows from Proposition 111.5 (2) that the isomorphism type of the relative higher homotopy groups $\pi_n(X, A, x_0)$ does not depend on the choice of a base point in A. Thus if we are only interested in the isomorphism type of the n-th relative homotopy group of (X, A), then sometimes we just write $\pi_n(X, A)$, i.e. we suppress the base point from the notation.

Recall that for the relative homotopy groups $\pi_n(X, A, x_0)$ with $n \geq 3$ we use the additive notation for the group structure. Thus Lemma 54.13, Propositions 40.3, 111.1 (4) and 111.5 lead us to the following observation:

Observation. Let (X, A, x_0) be a pointed pair of topological spaces. If $n \geq 3$, then the abelian group $\pi_n(X, A, x_0)$ admits a natural $\mathbb{Z}[\pi_1(A, x_0)]$-left module structure.

Example. We consider the wedge $X = S^1 \vee S^2$ where we identify $i \in S^1$ with $(0, 0, -1) \in S^2$. We denote by x_0 the wedge point. Furthermore let $A = S^1 \subset X$. We denote by $t \in \pi_1(S^1, i)$ the standard generator. (We refer to Figure 1553 for an illustration.) By Proposition 20.3 we know that the inclusion $A \to X$ induces an isomorphism $\pi_1(A, x_0) \to \pi_1(X, x_0)$. We use this isomorphism to make the identification $\pi_1(A, x_0) = \pi_1(X, x_0) = \langle t \rangle$. It follows from Proposition 111.4 and Proposition 111.5 (7) that the natural map $\pi_2(X, x_0) \to \pi_2(X, A, x_0)$ is an isomorphism of $\mathbb{Z}[t^{\pm 1}]$-modules. In particular, by Proposition 54.15, we have

$$\pi_2(X, A, x_0) \xleftarrow{\cong} \pi_2(S^1 \vee S^2) \xrightarrow{\cong} \mathbb{Z}[t^{\pm 1}].$$

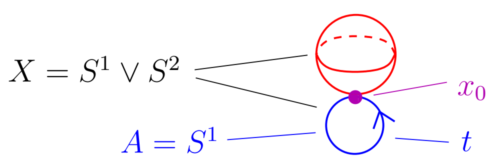


FIGURE 1553

111.4. **Alternative description of relative homotopy groups.** Let (X, x_0) be a pointed topological space. Recall that on page 1059 we had in particular introduced the following identifications:

$$\pi_n(X, x_0) \;=\; \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of maps } (I^n, \partial I^n) \to (X, x_0)\end{array} \;=\; \begin{array}{c}\text{set (of homotopy classes)}\\ \text{of maps } (S^n, *) \to (X, x_0).\end{array}$$

These identifications allowed us to go back and forth between different view points regarding homotopy groups.

Before we continue with our discussion of relative homotopy groups we also want to introduce a second view point regarding relative homotopy groups. The key to doing so is the following elementary lemma.

Lemma 111.6. *Let $n \in \mathbb{N}$. There exists an explicit*[1515] *map $\varphi \colon I^n \to \overline{B}^n$ with the following three properties:*

(1) *we have $\varphi(J^{n-1}) = * := (0, \ldots, 0, 1)$ and $\varphi(I^{n-1}) = S^{n-1}$,*

(2) *the map* φ *descends to a homeomorphism* $(I^n/J^{n-1}, I^{n-1}/\partial I^{n-1}) \xrightarrow{\cong} (\overline{B}^n, S^{n-1})$,
(3) *the restriction of* φ *to the map* $I^n \setminus \partial I^n \to B^n$ *is an orientation-preserving diffeomorphism.*

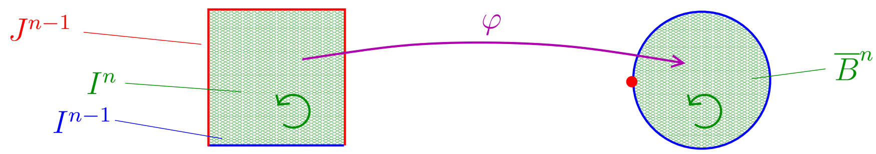


FIGURE 1554. Illustration for Lemma 111.6.

SKETCH OF A PROOF FOR LEMMA 111.6 (∗). We write $Q = (\frac{1}{2}, \dots, \frac{1}{2}, 0) \in I^n$. Given any point $x \in I^n \setminus \{Q\}$ we define

$$\rho(x) \ := \ \sup\left\{ \|rx\| \,\middle|\, r \in \mathbb{R}_{>0} \text{ and } r(x-Q)+Q \in I^n \right\}.$$

We consider the map

$$\begin{array}{rcl} \alpha\colon I^n/J^{n-1} & \to & \overbrace{\{(x_1,\dots,x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\}}^{=H_n} \cup \{\infty\} \\ [x] & \mapsto & \begin{cases} \frac{x-Q}{\|x-Q\|} \cdot \tan\left(\frac{\pi}{2} \cdot \frac{\|x-Q\|}{\rho(x)}\right), & \text{if } x \neq Q, \\ 0, & \text{if } x = Q. \end{cases} \end{array}$$

This map stretches each of the segments from Q to a point on J^{n-1} to an infinite ray emanating from the origin. In particular all points in J^{n-1} get sent to infinity, i.e. the map is well-defined. We refer to Figure 1555 for an illustration.

Similar to the proof of Proposition 2.53 we see that the map α is a homeomorphism that restricts to a homeomorphism $I^{n-1}/\partial I^{n-1} \to (\mathbb{R}^{n-1} \times \{0\}) \cup \{\infty\}$. Let $\beta\colon \mathbb{R}^n \cup \{\infty\} \to S^n$ be the inverse of the stereographic projection that we introduced in Lemma 2.44. Note that $\beta(\infty) = (0, \dots, 0, 1)$. Furthermore note that the homeomorphism β restricts to a homeomorphism[1516]

$$\beta\colon H^n \cup \{\infty\} \ \to \ S^n_{\geq 0} \ := \ \{(x_1, \dots, x_{n+1}) \in S^n \,|\, x_n \geq 0\}.$$

Finally we consider the following map:

$$\begin{array}{ccccccccc} I^n & \to & I^n/J^{n-1} & \xrightarrow{\alpha} & H_n \cup \{\infty\} & \xrightarrow{\beta} & S^n_{\geq 0} & \to & \overline{B}^n \\ & & & & & & (x_1, \dots, x_{n-1}, x_n, x_{n+1}) & \mapsto & (x_1, \dots, x_{n-1}, x_{n+1}). \end{array}$$

Using Exercise 6.17 one can easily verify that this map has all the desired properties.[1517] ∎

[1515]"Explicit" means that in the proof we explicitly write down such a map that does not depend on any choices. No reader will ever want to see the precise definition, but the fact that the map is explicitly given implies that there is no ambiguity involved when we write "let φ be the map from Lemma 111.6."

[1516]Note that the condition $x_n \geq 0$ is not a typo.

[1517]More precisely, the author is somewhat optimistic that the signs are right and that the map is indeed orientation-preserving.

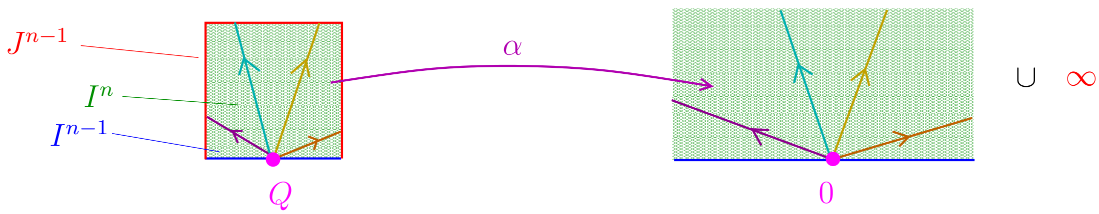


FIGURE 1555

Convention. Using the map

$$\varphi\colon (I^n, \partial I^n, J^{n-1}) \ \to \ (\overline{B}^n, S^{n-1}, *)$$

from Lemma 111.6 we make the following identifications

$$\pi_n(X,A,x_0) \ = \ \begin{array}{c}\text{set (of homotopy classes) of}\\ \text{maps } (I^n,\partial I^n,J^{n-1}) \to (X,A,x_0)\end{array} \ = \ \begin{array}{c}\text{set (of homotopy classes) of}\\ \text{maps } (\overline{B}^n,S^{n-1},*) \to (X,A,x_0).\end{array}$$

We refer to Figure 1556 for an illustration.

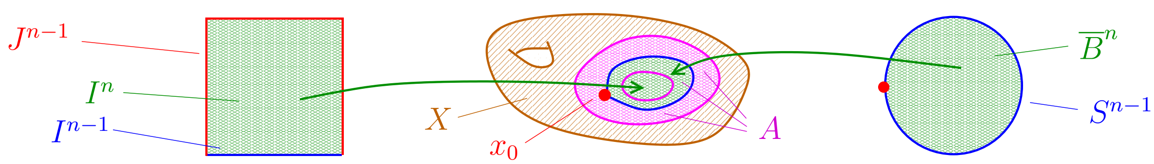


FIGURE 1556

Remark. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}$. As above we make the identifications

$$\pi_n(X,A,x_0) = \left\{\begin{array}{c}\text{homotopy classes of maps}\\ (\overline{B}^n,S^{n-1},*) \to (X,A,x_0)\end{array}\right\} \text{ and } \pi_{n-1}(A,x_0) = \left\{\begin{array}{c}\text{homotopy classes of maps}\\ (S^{n-1},*) \to (A,x_0)\end{array}\right\}.$$

We leave it to the interested reader to verify that the connecting homomorphism $\pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ from Proposition 111.4 is given by the map

$$\begin{array}{rcl}\partial_n\colon \pi_n(X,A,x_0) & \to & \pi_{n-1}(A,x_0)\\ [f\colon (\overline{B}^n,S^{n-1},*) \to (X,A,x_0)] & \mapsto & [f|_{S^{n-1}}\colon (S^{n-1},*) \to (A,x_0)].\end{array}$$

111.5. **Connected (pairs of) topological spaces.** For convenience we recall the following definition from page 1327.

Definition. Let $k \in \mathbb{N}_0$. We say a topological space X is *k-connected*, if $\pi_i(X) = 0$ for $i = 0, \ldots, k$.

This definition has the following obvious "relative" analogue.

Definition. Let X be a topological space, let $A \subset X$ and let $k \in \mathbb{N}$. We say (X, A) *is k-connected* if every path-component of X contains a point of A and if for every $i \in \{1, \ldots, k\}$ and every $x_0 \in A$ we have $\pi_i(X, A, x_0) = 0$.

The following lemma provides a convenient reformulation of the previous definition.

Lemma 111.7. *Let X be a topological space, let $A \subset X$ and let $k \in \mathbb{N}$. The following two statements are equivalent:*

1. *The pair (X, A) is k-connected.*
2. *Every path-component of X contains a point of A, furthermore for every $a \in A$ and every $j \in \{1, \ldots, k-1\}$ the inclusion induced map $\pi_j(A, a) \to \pi_j(X, a)$ is an isomorphism and the inclusion induced map $\pi_k(A, a) \to \pi_k(X, a)$ is an epimorphism.*

Example. Let X be a topological space and let $A \subset X$ be a deformation retract. It follows from Proposition 40.7 (2) that for any $x_0 \in A$ and any $i \in \mathbb{N}_0$ the inclusion induced map $\pi_i(A, x_0) \to \pi_i(X, x_0)$ is an isomorphism. It follows from Lemma 111.7 that (X, A) is k-connected for every $k \in \mathbb{N}$.

PROOF. We denote by $i \colon A \to X$ the inclusion map. Let $a \in A$. By Proposition 111.4 we have the following long exact sequence of homotopy groups:

$$\ldots \to \pi_k(A, a) \xrightarrow{i_*} \pi_k(X, a) \to \pi_k(X, A, a) \xrightarrow{\partial} \pi_{k-1}(A, a) \xrightarrow{i_*} \pi_{k-1}(X, a) \to \ldots$$

The lemma is an immediate consequence of the existence of this long exact sequence. ∎

The following proposition gives a convenient characterization of the vanishing relative homotopy groups.

Proposition 111.8. *Let X be a topological space and let $A \subset X$. For any $n \in \mathbb{N}$ the following two statements are equivalent:*

1. *Every map $(\overline{B}^n, S^{n-1}) \to (X, A)$ is homotopic rel S^{n-1} to a map $\overline{B}^n \to A$,*
2. *for every $x_0 \in A$ we have $\pi_n(X, A, x_0) = 0$.*

Furthermore, for $n = 0$ the above statement (1) is equivalent to

(2') *every path-component of X contains a point of A.*

Example. We consider the infinite cylinder $X = S^1 \times \mathbb{R}$ and the annulus $A = S^1 \times [-1, 1]$. Evidently A is a deformation retract of X. Thus, by the previous example the pair (X, A) is k-connected for every $k \in \mathbb{N}_0$. In Figure 1557 we show a map $f \colon (\overline{B}^2, S^1) \to (X, A)$. Furthermore we show a map $g \colon \overline{B}^2 \to A$ that is homotopic, rel S^1, to f.

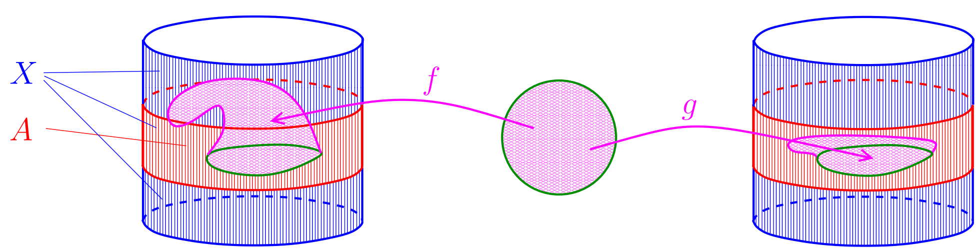


FIGURE 1557

PROOF OF PROPOSITION 111.8. First note that it is elementary to see that for $n = 0$ the statements (1) and (2') are equivalent. Now let $n \geq 1$. The implication (1) $\Rightarrow$ (2) is an immediate consequence of Lemma 111.3 and the by now well-known fact that there exists a homeomorphism $(I^n, \partial I^n) \xrightarrow{\cong} (\overline{B}^n, S^{n-1})$.

The most interesting statement of the proposition is the implication (2) $\Rightarrow$ (1). Thus we now assume that $\pi_n(X,A,x_0)=0$ for every $x_0\in A$. Let $f\colon(\overline{B}^n,S^{n-1})\to(X,A)$ be a map. We need to show that f is homotopic rel S^{n-1} to a map that takes values only in A. Let $\varphi\colon I^n\to\overline{B}^n$ be the map from Lemma 111.6 with $\varphi(J^{n-1})=(0,\dots,0,1)$. We set $x_0:=f(0,\dots,0,1)$. Note that the map $f\circ\varphi\colon I^n\to X$ is actually a map

$$f\circ\varphi\colon(I^n,\partial I^n,J^{n-1})\ \to\ (X,A,\{x_0\})$$

of triples. Since $\pi_n(X,A,x_0)=0$ we see that there exists a homotopy

$$F\colon(I^n,\partial I^n,J^{n-1})\times[0,1]\ \to\ (X,A,\{x_0\})$$

which satisfies $F_0=f\circ\varphi$ and $F_1=c_{x_0}$. Now we consider the map

$$\begin{array}{rcl} I^n\times[0,1] & \to & X \\ (x_1,\dots,x_n,t) & \mapsto & \begin{cases} F\big(x_1,\dots,x_{n-1},\frac{1}{1-\frac{t}{2}}\cdot(x_n-\frac{t}{2}),t\big), & \text{if } x_n\in\big[\frac{t}{2},1\big] \\ F\big(x_1,\dots,x_{n-1},0,2\cdot x_n\big), & \text{if } x_n\in\big[0,\frac{t}{2}\big]. \end{cases} \end{array}$$

The definition of the homotopy is sketched, with moderate success, in Figure 1558. This

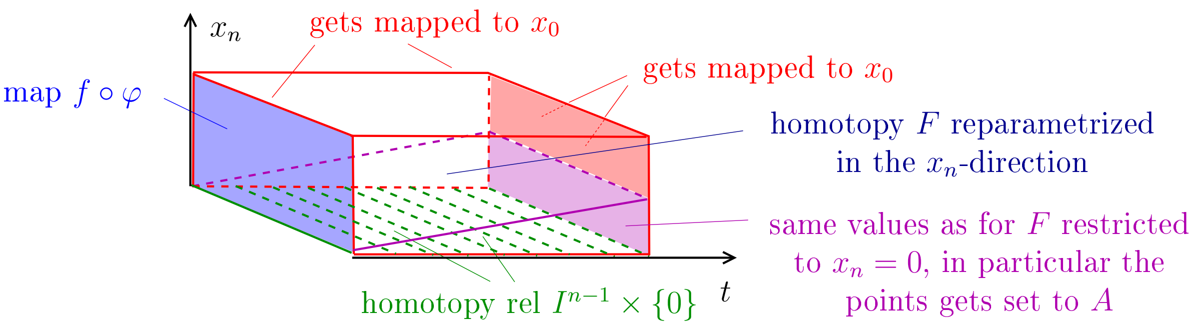


FIGURE 1558

map has the following properties:

(0) the map is continuous by Lemma 14.3,
(1) the map is a homotopy rel $I^{n-1}\times\{0\}$,
(2) the map is a homotopy from $f\circ\varphi$ to a map that takes values in A,
(3) the map sends $J^{n-1}\times[0,1]$ to x_0.

It follows from Property (3), and using the above map $\varphi\colon(I^n,J^{n-1})\to(\overline{B}^n,(0,\dots,0,1))$, that F defines a homotopy $G\colon\overline{B}^n\times[0,1]\to X$. Property (1) implies that G is a homotopy rel S^{n-1}. Furthermore Property (2) says that G is a homotopy from f to a map that takes values in A. Thus G is the desired a homotopy. ■

The following proposition gives us an important example of a k-connected pair of topological spaces.

Proposition 111.9. *Let X be a* CW*-complex and let $k\in\mathbb{N}_0$.*

(1) *If A is a subcomplex such that all cells in $X\setminus A$ have dimension greater than k, then the pair (X,A) is k-connected.*
(2) *The pair (X,X^k) is k-connected.*[1518]

PROOF.

(1) Let X be a CW-complex, let A be a subcomplex and let $k \in \mathbb{N}_0$. Suppose that all cells in $X \setminus A$ have dimension greater than k. This implies in particular that every path-component of X contains at least one point in A. The remainder of the proof is almost the same as the proof of Proposition 40.9. The key difference is that we need to replace the Cellular Approximation Theorem 38.13 by the Cellular Approximation Theorem 38.14 for Pairs of CW-complexes and that now we use the (1) $\Rightarrow$ (2) implication of Proposition 111.8. We leave it to the reader to carry out the modest modifications of the proof of Proposition 40.9.

(2) This statement is of course an immediate consequence of the first statement. ■

The following corollary is an extremely useful consequence of Lemma 111.7 and Proposition 111.9 (2). We had proved it earlier in the guise of Proposition 40.9.

Corollary 111.10. *Let X be a* CW*-complex, let $k \in \mathbb{N}_0$ and let $x \in X^k$. The following two statements hold:*

(1) *For $j = 1, \ldots, k-1$ the inclusion induced map $\pi_j(X^k, x) \to \pi_j(X, x)$ is an isomorphism.*

(2) *The inclusion induced map $\pi_k(X^k, x) \to \pi_k(X, x)$ is an epimorphism.*

111.6. **The relative Hurewicz Theorem.** In this section we state the "Relative Hurewicz Theorem" that generalizes the original Hurewicz Theorem 53.5 to the relative setting.

Definition. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}$. We denote by $[I^n] \in \mathrm{H}_n(I^n, \partial I^n; \mathbb{Z})$ the fundamental class of I^n.[1519] We refer to the map

$$\begin{array}{rcl} \Phi_{(X,A,x_0)} \colon \pi_n(X, A, x_0) & \to & \mathrm{H}_n(X, A; \mathbb{Z}) \\ [f \colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] & \mapsto & f_*([I^n]) \end{array}$$

as the *relative Hurewicz homomorphism.*

The following lemma summarizes some elementary properties of the relative Hurewicz homomorphism.

Lemma 111.11.

(1) *Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}_{\geq 2}$. The map*

$$\begin{array}{rcl} \Phi_{(X,A,x_0)} \colon \pi_n(X, A, x_0) & \to & \mathrm{H}_n(X, A; \mathbb{Z}) \\ [f \colon (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\})] & \mapsto & f_*([I^n]) \end{array}$$

is well-defined and it is a homomorphism.

[1518] Recall that X^k denotes the k-skeleton of X, i.e. the union of all cells of dimension $\leq k$.

[1519] Here we endow $I^n \subset \mathbb{R}^n$ with the usual orientation coming from $\mathbb{R}^n$.

(2) *Let (X, x_0) be a pointed topological space and let $n \geq 2$. The following diagram commutes*

$$\begin{tikzcd}
\pi_n(X,x_0) \arrow[r, "\Phi_{(X,x_0)}"] \arrow[d, "\cong"'] & \mathrm{H}_n(X;\mathbb{Z}) \arrow[d, "\cong"] \\
\pi_n(X,\{x_0\},x_0) \arrow[r, "\Phi_{(X,\{x_0\},x_0)}"] & \mathrm{H}_n(X,\{x_0\};\mathbb{Z}).
\end{tikzcd}$$

Here the left-hand vertical map is given by the identification from page 2617 and the right-hand vertical map is the obvious map from absolute to relative homology groups.

(3) *Let (X, A, x_0) be a pointed pair of topological spaces. The following diagram commutes*

$$\begin{tikzcd}
\ldots \arrow[r] & \pi_n(A,x_0) \arrow[r] \arrow[d] & \pi_n(X,x_0) \arrow[r] \arrow[d] & \pi_n(X,A,x_0) \arrow[r, "\partial"] \arrow[d] & \pi_{n-1}(A,x_0) \arrow[r] \arrow[d] & \ldots \arrow[r] & \pi_1(X,x_0) \arrow[d] \\
\ldots \arrow[r] & \mathrm{H}_n(A;\mathbb{Z}) \arrow[r] & \mathrm{H}_n(X;\mathbb{Z}) \arrow[r] & \mathrm{H}_n(X,A;\mathbb{Z}) \arrow[r, "\partial"] & \mathrm{H}_{n-1}(A;\mathbb{Z}) \arrow[r] & \ldots \arrow[r] & \mathrm{H}_1(X;\mathbb{Z}).
\end{tikzcd}$$

Here the horizontal sequences are the long exact sequences given by Corollary 43.16 and by Proposition 111.4 and the vertical maps are the Hurewicz homomorphisms.

Sketch of proof.

(1) The proof of the first statement is evidently just a straightforward variation on the proof of Lemma 53.1. We refer to [**Bre93**, Lemma VII.10.2] or [**Hat02**, Proposition 4.36] for details.
(3) We have to show that each of the three squares commutes.
   (a) The first square commutes basically by definition, see also Lemma 53.1 (3).
   (b) The fact that the second square commutes is an immediate consequence of the definitions of the "absolute" and the "relative" Hurewicz homomorphisms, see above and page 1324.
   (c) Finally it follows fairly easily from Lemma 87.24 and Proposition 87.27 that the third square commutes.
(2) This is just a special case of the second square considered in (3). ■

Again the following question arises: to what degree is the relative Hurewicz homomorphism in fact an isomorphism? One might hope that if a pair (X, A) is $(n-1)$-connected, then the Hurewicz homomorphism $\Phi_{(X,A,x_0)}\colon \pi_n(X, A, x_0) \to \mathrm{H}_n(X, A; \mathbb{Z})$ is an isomorphism. But the following example shows that this is *not* the case.

Example. We consider again $X = S^1 \vee S^2$ and $A = S^1 \subset X$. We denote by x_0 the wedge point. It follows easily from the long exact sequence of homotopy groups provided by Proposition 111.4 that (X, A) is 1-connected.[1520] We see that in this case the map

$$\Phi_{(X,A,x_0)}\colon \underbrace{\pi_2(X,A,x_0)}_{\cong \mathbb{Z}^\infty,\text{ see page 2628}} \to \underbrace{\mathrm{H}_2(X,A;\mathbb{Z})}_{\cong\mathbb{Z}}$$

is *not* an isomorphism.

The following lemma is the key to understanding why the relative Hurewicz homomorphism fails to be a monomorphism.

Lemma 111.12. *Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}_{\geq 2}$. Furthermore let $\sigma \in \pi_n(X, A, x_0)$. For any $g \in \pi_1(A, x_0)$ we have*

$$\Phi_{(X,A,x_0)}(g\cdot\sigma) \;=\; \Phi_{(X,A,x_0)}(\sigma).$$

Proof. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}_{\geq 2}$. Furthermore let $[f] \in \pi_n(X, A, x_0)$ and let $[\gamma] \in \pi_1(A, x_0)$. As in Proposition 54.9 (7) we see that the maps $f\colon (I^n, \partial I^{n-1}) \to (X, A)$ and $f^\gamma\colon (I^n, \partial I^{n-1}) \to (X, A)$ are homotopic. We obtain that

$$\Phi_{(X,A,x_0)}([\gamma]\cdot[f]) \;=\; \Phi_{(X,A,x_0)}([f^\gamma]) \;=\; (f^\gamma)_*([I^n]) \underset{\substack{\uparrow\\ \text{by Proposition 43.17, since } f \text{ and } f^\gamma \text{ are homotopic as maps } (I^n,\partial I^n)\to(X,A)}}{=} f_*([I^n]) \;=\; \Phi_{(X,A,x_0)}([f]).$$

■

The above lemma leads us to the following definition.

Definition. Let (X, A, x_0) be a pointed pair of topological spaces.[1521]

(1) For $n \in \mathbb{N}_{\geq 3}$ we define $\pi'_n(X, A, x_0)$ to be the quotient group[1522]

$$\pi'_n(X,A,x_0) \;:=\; \pi_n(X,A,x_0)/\{g\cdot\sigma-\sigma \,|\, g\in\pi_1(A,x_0) \text{ and } \sigma\in\pi_n(X,A,x_0)\}.$$

(2) We define[1523]

$$\pi'_2(X,A,x_0) \;:=\; \pi_2(X,A,x_0)/\langle\langle\{(g\cdot\sigma)\cdot\sigma^{-1} \,|\, g\in\pi_1(A,x_0), \sigma\in\pi_2(X,A,x_0)\}\rangle\rangle.$$

(3) By Lemma 111.12 the Hurewicz homomorphism

$$\Phi_{(X,A,x_0)}\colon \pi_n(X,A,x_0) \;\to\; \mathrm{H}_n(X,A;\mathbb{Z})$$

descends to a homomorphism

$$\Phi_{(X,A,x_0)}\colon \pi'_n(X,A,x_0) \;\to\; \mathrm{H}_n(X,A;\mathbb{Z}).$$

We refer to this homomorphism again as the *Hurewicz homomorphism*.

[1520] Alternatively we could view X as a CW-complex in the usual way. Then we obtain from Proposition 111.9 that the pair (X, A) is 1-connected.

[1521] Here we use the common, but sometimes confusing, convention that we write the group structure on the abelian group $\pi_n(X, A, x_0)$, $n \geq 3$, additively whereas we write the group structure on $\pi_2(X, A, x_0)$ multiplicatively. Evidently there is no rule without an exception, hence, if $\pi_2(X, A, x_0)$ happens to be abelian after all, then we also like to use additive notation for $\pi_2(X, A, x_0)$.

[1522] Let $n \in \mathbb{N}_{\geq 3}$. By Proposition 111.1 (4) we know that $\pi_n(X, A, x_0)$ is abelian. Therefore it follows from Exercise 54.8 that $\{g\cdot\sigma-\sigma \,|\, g\in\pi_1(A,x_0) \text{ and } \sigma\in\pi_n(X,A,x_0)\}$ is in fact a subgroup of $\pi_n(X, A, x_0)$.

[1523] Recall that given a group π and a subset $X \subset \pi$ we denote by $\langle\langle X\rangle\rangle$ the subgroup of π normally generated by X. We refer to page 627 for details.

Remark. Let (X, A, x_0) be a pointed pair of topological spaces. It is a fun exercise to show "by hand" that $\pi_2'(X, A, x_0)$ is an abelian group. We refer to [**Hat02**, Lemma 4.39] for details.

Example. Once again we consider $X = S^1 \vee S^2$ and $A = S^1 \subset X$. We denote by x_0 the wedge point. As usual we write $\pi_1(A, x_0) = \langle t \rangle$. On page 2628 we already saw that $\pi_2(X, A, x_0) \cong \mathbb{Z}[t^{\pm 1}]$ as $\mathbb{Z}[\pi_1(A, x_0)] = \mathbb{Z}[t^{\pm 1}]$-modules. Thus we have an isomorphism

$$\pi_2'(X, A, x_0) \;\cong\; \mathbb{Z}[t^{\pm 1}]/\{(t^k - 1) \cdot f \mid f \in \mathbb{Z}[t^{\pm 1}] \text{ and } k \in \mathbb{Z}\}.$$

It is an amusing algebra exercise to show that the group on the right-hand side is isomorphic to $\mathbb{Z}$.

The following theorem is a generalization of the Hurewicz Theorem 53.5 to the relative case.

Theorem 111.13. (Relative Hurewicz Theorem) *Let (X, A, x_0) be a pointed pair of topological spaces such that X and A are path-connected. Let $n \geq 2$. If the pair (X, A) is $(n-1)$-connected, then the following two statements hold:*

(1) *We have $\mathrm{H}_k(X, A; \mathbb{Z}) = 0$ for $k = 1, \ldots, n-1$.*
(2) *The Hurewicz homomorphism*

$$\Phi_{(X,A,x_0)} \colon \pi_n'(X, A, x_0) \;\to\; \mathrm{H}_n(X, A; \mathbb{Z})$$

is an isomorphism. In particular, if $\pi_1(A, x_0) = 0$, then the Hurewicz homomorphism

$$\Phi_{(X,A,x_0)} \colon \pi_n(X, A, x_0) \;\to\; \mathrm{H}_n(X, A; \mathbb{Z})$$

is an isomorphism.

Proof. We will not provide a proof of the Relative Hurewicz Theorem. A full proof is given in [**Bre93**, Theorem VII.10.7] or alternatively in [**Hat02**, Theorem 4.37] and [**Spa95**, Proposition 7.5.1]. If A is simply connected, then it is a great exercise to figure out to what degree the proof of the Hurewicz Theorem 53.5 can be modified to deal with the relative case. ■

The following theorem, which can be viewed as a corollary to the Relative Hurewicz Theorem 111.13, is a nice illustration of the usefulness of relative homotopy groups.

Theorem 111.14. *Let X and Y be two topological spaces and furthermore let $f \colon X \to Y$ be a map. We pick a base point $x_0 \in X$ and we write $y_0 = f(x_0)$.*

(1) *If for every $i \in \mathbb{N}_{\geq 2}$ the induced map*

$$f_* \colon \pi_i(X, x_0) \;\xrightarrow{\cong}\; \pi_i(Y, y_0)$$

is an isomorphism, then for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_* \colon \mathrm{H}_i(X; \mathbb{Z}) \;\xrightarrow{\cong}\; \mathrm{H}_i(Y; \mathbb{Z})$$

is also an isomorphism.

(2) *If X and Y are simply connected, then the converse to* (1) *holds. More precisely, in this case, if for every $i \in \mathbb{N}_{\geq 2}$ the induced map*

$$f_*\colon \mathrm{H}_i(X;\mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_i(Y;\mathbb{Z})$$

is an isomorphism, then for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_*\colon \pi_i(X,x_0) \xrightarrow{\cong} \pi_i(Y,y_0)$$

is also an isomorphism.

Remark. In some textbooks, see e.g. [**Bre93**, p. 481], [**DaK01**, p. 162] and [**Spa95**, p. 399], this theorem is referred to as the "Whitehead Theorem". We will reserve the name "Whitehead Theorem" for a much more interesting later result, namely Theorem 119.9.

Proof. The idea is to "replace" the map $f\colon X \to Y$ by an inclusion map so that we can apply the commutative diagram from Lemma 111.11. The key to doing so is the notion of the mapping cylinder that we introduced on page 697. Recall that the mapping cylinder corresponding to f is defined as the topological space

$$\mathrm{Cyl}(f) \;:=\; ((X \times [0,1]) \sqcup Y)/\sim \;\text{ where } (x,1) \sim f(x) \text{ for all } x \in X.$$

We consider the maps

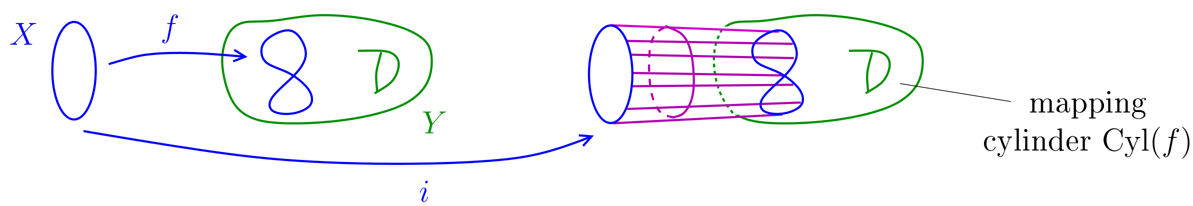


Figure 1559

$$\begin{array}{rl} i\colon X &\to \mathrm{Cyl}(f) \\ x &\mapsto (x,0) \end{array} \quad \text{respectively} \quad \begin{array}{rl} r\colon \mathrm{Cyl}(f) &\to Y \\ P &\mapsto \begin{cases} P, & \text{if } P \in Y, \\ f(Q), & \text{if } P = [(Q,t)] \text{ with } Q \in X, t \in [0,1]. \end{cases} \end{array}$$

Note that we have $r \circ i = f\colon X \to Y$. Also note that in Lemma 24.8 (2a) we showed that the retraction $r\colon \mathrm{Cyl}(f) \to Y$ is a homotopy equivalence. We use the inclusion $i\colon X \to \mathrm{Cyl}(f)$ to identify X with $i(X)$, i.e. we view X as a subset of $\mathrm{Cyl}(f)$.

In the following we prove Statement (2) of the theorem. Thus we assume that X and Y are simply connected and we assume that for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_*\colon \mathrm{H}_i(X;\mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_i(Y;\mathbb{Z})$$

is an isomorphism. We need to show that for every $i \in \mathbb{N}_{\geq 2}$ the induced map

$$f_*\colon \pi_i(X,x_0) \xrightarrow{\cong} \pi_i(Y,y_0)$$

is also an isomorphism. We consider the following maps

$$\begin{tikzcd}
 & & & \pi_2(Y,y_0) & & \\
\ldots \arrow[r] & \pi_3(\mathrm{Cyl}(f),X,x_0) \arrow[r, "\partial"] \arrow[d] & \pi_2(X,x_0) \arrow[r, "i_*"] \arrow[ur, "f_*"] \arrow[d] & \pi_2(\mathrm{Cyl}(f),x_0) \arrow[r] \arrow[u, "r_*"'] \arrow[d] & \pi_2(\mathrm{Cyl}(f),X,x_0) \arrow[r] \arrow[d] & 0 \\
\ldots \arrow[r] & \mathrm{H}_3(\mathrm{Cyl}(f),X;\mathbb{Z}) \arrow[r, "\partial"] & \mathrm{H}_2(X;\mathbb{Z}) \arrow[r, "i_*"] \arrow[dr, "f_*"'] & \mathrm{H}_2(\mathrm{Cyl}(f);\mathbb{Z}) \arrow[r] \arrow[d, "r_*"] & \mathrm{H}_2(\mathrm{Cyl}(f),X;\mathbb{Z}) \arrow[r] & 0 \\
 & & & \mathrm{H}_2(Y;\mathbb{Z}) & &
\end{tikzcd}$$

where the vertical maps between the second and the third row are the Hurewicz homomorphisms. The desired statement follows from the following observations:

(a) The triangles commute since $f = r \circ i$. Furthermore the squares commute by Lemma 111.11.
(b) By our hypothesis the maps $f_*\colon \mathrm{H}_*(X;\mathbb{Z}) \to \mathrm{H}_*(Y;\mathbb{Z})$ are isomorphisms. Furthermore, since r is a deformation retraction we obtain from Corollary 42.8 that the maps $r_*\colon \mathrm{H}_*(\mathrm{Cyl}(f);\mathbb{Z}) \to \mathrm{H}_*(Y;\mathbb{Z})$ are isomorphisms.
(c) Since $f_* = (r \circ i)_* = r_* \circ i_*$ we obtain immediately from our result in (b) that the maps $i_*\colon \mathrm{H}_*(X;\mathbb{Z}) \to \mathrm{H}_*(\mathrm{Cyl}(f);\mathbb{Z})$ are also isomorphisms.
(d) It follows from (c) and the bottom long exact sequence that all the relative homology groups $\mathrm{H}_*(\mathrm{Cyl}(f),X;\mathbb{Z})$ vanishes.
(e) It follows easily from (d), the hypothesis that X is simply connected and the Relative Hurewicz Theorem 111.13 that all the relative homotopy groups $\pi_*(\mathrm{Cyl}(f),X,x_0)$ vanish.
(f) Using (e) we obtain immediately from the top long exact sequence that the maps $i_*\colon \pi_*(X,x_0) \to \pi_*(\mathrm{Cyl}(f),x_0)$ are isomorphisms.
(g) By Proposition 40.7 (2) we know that the maps $r_*\colon \pi_*(\mathrm{Cyl}(f),x_0) \to \pi_*(Y,y_0)$ are isomorphisms.
(h) Finally note that it follows from $f_* = r_* \circ i_*$ together with (f) and (g) that the maps $f_*\colon \pi_*(X,x_0) \to \pi_*(Y,y_0)$ are indeed isomorphisms.

The proof of Statement (1) is very similar to the above argument. We leave it to the reader to fill in the details. ■

We conclude this section with a discussion of homotopy spheres. Recall that by the definition on page 2161 a homotopy n-sphere Σ is a closed n-dimensional topological manifold that is $(n-1)$-connected, i.e. the homotopy groups of Σ agree with the homotopy groups of S^n up to degree $n-1$.

The following pleasant corollary shows that in fact all homotopy groups of Σ are isomorphic to the corresponding homotopy groups of S^n.

Corollary 111.15. *Let $n \in \mathbb{N}_{\geq 2}$. If Σ is a homotopy n-sphere, then for every $i \in \mathbb{N}_0$ we have $\pi_i(\Sigma) \cong \pi_i(S^n)$.*

PROOF. Let Σ be a homotopy n-sphere. Note that by Theorem 111.14 (2) it suffices to find a map $f\colon S^n \to \Sigma$ such that for every $i \in \mathbb{N}_{i\geq 2}$ the induced map $f_*\colon \mathrm{H}_i(S^n;\mathbb{Z}) \to \mathrm{H}_i(\Sigma;\mathbb{Z})$ is an isomorphism.

So let us try to find such a map f. We pick a base point $x_0 \in \Sigma$. By the definition and Proposition **??** we know that Σ is a closed connected non-empty n-dimensional topological manifold with $\mathrm{H}_i(\Sigma;\mathbb{Z}) \cong \pi_i(\Sigma, x_0) = 0$ for $i = 1,\dots,n-1$. It follows from the Hurewicz Theorem 53.5 that the Hurewicz homomorphism

$$\begin{array}{rcl} \Phi\colon \pi_n(\Sigma, x_0) & \to & \mathrm{H}_n(\Sigma;\mathbb{Z}) \\ {[f\colon (S^n,*) \to (\Sigma, x_0)]} & \mapsto & f_*([S^n]) \end{array}$$

is an isomorphism. Note that the fact that $\pi_1(\Sigma, x_0) = 0$ together with Corollary 86.16 implies that Σ is orientable. Therefore by Theorem 87.1 we know that $\mathrm{H}_n(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$. Since the Hurewicz homomorphism is an isomorphism we can pick a map $f\colon S^n \to \Sigma$ such that $f_*([S^n]) \in \mathrm{H}_n(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$ is a generator.

We claim that f has the desired properties. By choice of f we know that the map $f_*\colon \mathrm{H}_n(S^n;\mathbb{Z}) \to \mathrm{H}_n(\Sigma;\mathbb{Z})$ is an epimorphism. But since both groups are isomorphic to $\mathbb{Z}$ we see that the map is in fact an isomorphism. Since S^n and Σ are path-connected we obtain from the discussion on page 1088 that $f_*\colon \mathrm{H}_0(S^n;\mathbb{Z}) \to \mathrm{H}_0(\Sigma;\mathbb{Z})$ is also an isomorphism. Furthermore for all other dimensions f_* induces an isomorphism of homology groups for the rather banal reasons that all other homology groups of S^n and Σ are zero. (Indeed, we saw above that the homology groups in dimension less than n vanish and it follows from Theorem 87.3 that the homology groups vanish in dimension greater than n.) ■

111.7. **Excision for relative homotopy groups?** We have now seen that the (relative) homotopy groups and (relative) homology groups have some properties in common:

(1) both invariants are functorial,
(2) in both cases homotopic maps induce the same maps on the invariants,
(3) a (pointed) pair of topological spaces gives rise to a long exact sequence.

These formal similarities between (relative) homotopy and (relative) homology groups makes one wonder whether an analogue of the Excision Theorem 43.19 holds for relative homotopy groups.

For better or worse this is not the case. More precisely, the next lemma shows that a naive analogue of the Excision Theorem 43.19 fails even in very simple situations.

Lemma 111.16. *We consider the 2-dimensional sphere $X = S^2$ together with the subset $A := S^2_{\leq 0} = \{(x,y,z) \in S^2 \,|\, z \leq 0\}$ given by the lower hemisphere and together with the subset $Z = \{(0,0,-1)\}$ that is given by the South Pole. Then*[1524]

$$\pi_3(X, A) \not\cong \pi_3(X \setminus Z, A \setminus Z).$$

PROOF. We start out with the following observations:

(1) On page 116 we saw that $A = S^2_{\leq 0}$ is homeomorphic to $\overline{B}^2$. In particular A is contractible which implies by Proposition 40.7 (2) that $\pi_2(A) = \pi_3(A) = 0$.

[1524] Here evidently the closure of Z is contained in the interior of A, i.e. the hypotheses for the Excision Theorem 43.19 for homology groups are satisfied.

FIGURE 1560

(2) The above homeomorphism also shows that $A \setminus Z$ is homeomorphic to $\overline{B}^2 \setminus \{(0)\}$ which is homotopy equivalent to S^1. This implies that $\pi_2(A \setminus Z) = \pi_3(A \setminus Z) = 0$.

(3) Finally in Lemma 2.44 we saw that $X \setminus Z = S^2 \setminus \{(0, 0, -1)\}$ is homeomorphic to $\mathbb{R}^2$ which implies that $\pi_2(X \setminus Z) = \pi_3(X \setminus Z) = 0$.

It follows from these observations and the long exact sequence of homotopy groups, provided by Proposition 111.4, that $\pi_3(X \setminus Z, A \setminus Z) \cong \pi_2(A \setminus Z) \cong \pi_2(S^1) = 0$ and furthermore that $\pi_3(X, A) \cong \pi_3(S^2)$. But in Theorem 91.3 we saw that $\pi_3(S^2)$ is non-trivial. We have thus shown that $\pi_3(X, A)$ is not isomorphic to $\pi_3(X \setminus Z, A \setminus Z)$. ■

Remark. We just used the fact that $\pi_3(S^2) \neq 0$ to show that relative homotopy groups do not satisfy excision. It is a good moment to remind ourselves of Theorem 57.27. The theorem suggests that if relative homotopy groups did satisfy excision, then it would follow from Proposition 111.4 that, at least morally, (abelianized) homotopy groups would agree with homology groups.[1525] But the calculations $\pi_3(S^2) \neq 0$ and $\mathrm{H}_3(S^2) = 0$ show that this is not the case.

In much more restricted circumstances there is in fact an excision theorem. For the record we state the following theorem. We will not make use of it and therefore we will not provide the proof (which by all accounts seems to be somewhat messy.)

Theorem 111.17. (Blakers-Massey Theorem)

(1) *Let X be a topological space and let A and B be subsets such that $X = \mathring{A} \cup \mathring{B}$. If $(A, A \cap B)$ is n-connected and if $(B, A \cap B)$ is m-connected, then the inclusion induced map*
$$\pi_k(A, A \cap B) \;\to\; \pi_k(X, B)$$
is an isomorphism for $k < m + n$ and it is an epimorphism for $k = m + n$.

(2) *The conclusion of (1) also holds if X is a CW-complex and A and B are two subcomplexes of X with $X = A \cup B$ such that $A \cap B$ is non-empty and connected.*

PROOF. The first statement was originally proved in 1952 by Albert Blakers and William Massey [**BlM52**]. Proofs can also be found in [**Gra75**, Corollary 16.27], [**tD08**, Theorem 6.4.1], [**tDKP70**, p. 211] or [**WhdG78**, Chapter VII.7]. The second statement can presumably be deduced from the first statement. Alternatively see [**Hat02**, Theorem 4.23]. ■

[1525] To turn this logic into a proper argument might be a little tricky because of base point issues.

Exercises for Chapter 111.

Exercise 111.1. Show that the three paths to the right of Figure 1543 define three different elements in $\pi_1(X, A, x_0)$.

Exercise 111.2. Let (X, A, x_0) be a pointed pair of topological spaces. Show that if $n \geq 3$, then the relative homotopy group $\pi_n(X, A, x_0)$ is abelian.

Exercise 111.3. Let X be a topological space and $x_0 \in X$. We consider the set $\pi_0(X, x_0)$. We all know that $\pi_0(X, x_0)$ is "not a group". But why not? After all, every non-empty set can be equipped with a group structure (why is that?). So why is not possible to equip $\pi_0(X, x_0)$ with a group structure in a "meaningful way"? More precisely, if we could equip it with a group structure in a reasonable sense, what properties would we demand? Why can these not be satisfied?

Exercise 111.4. Let $\varphi\colon G \to H$ be a group homomorphism. Furthermore let (S, s_0) be a pointed set which is equipped with an H-action. Suppose that we have an exact sequence

$$\{e\} \;\to\; G \;\xrightarrow{\varphi}\; H \;\xrightarrow{\psi}\; (S, s_0) \;\to\; \{e\}$$

where ψ is a morphism of pointed sets that preserves the H-action. Show that $\varphi(G)$ is a subgroup of H whose index equals the cardinality of the set S.

Exercise 111.5. Is a monomorphism of pointed sets necessarily injective?

Exercise 111.6. Provide a proof for Lemma 111.3.

Exercise 111.7. Let (X, A, x_0) be a pointed pair of topological spaces and let $n \in \mathbb{N}$. By the convention of page 2630 we can view elements in $\pi_n(X, A, x_0)$ as homotopy classes of maps $(\overline{B}^n, S^{n-1}, *) \to (X, A, \{x_0\})$. Using this point of view, how can we visualize the action of $\pi_1(A, x_0)$ on $\pi_n(X, A, x_0)$?

Exercise 111.8. Give an example of a connected CW-complex X, a connected subcomplex $A \subset X$ and an $n \in \mathbb{N}$ such that the homotopy group $\pi_n(X/A)$ is not isomorphic to the relative homotopy group $\pi_n(X, A)$.

112. Fibrations

In this chapter we introduce fibrations and we show that fibrations give rise to a long exact sequence of homotopy groups. In the subsequent two chapters we will give many more examples of fibrations and we will see that fibrations allow us to compute various interesting higher homotopy groups.

112.1. **Definition and examples of fibrations.**

The following is the key definition of this chapter.

Definition. Let $p\colon Y \to B$ be a map between topological spaces.

(1) Let X be a topological space. We say that the map $p\colon Y \to B$ has the *homotopy lifting property with respect to X* if given any map $F\colon X \times [0,1] \to B$ and given any map $\widetilde{f}\colon X \times \{0\} \to Y$ with $p \circ \widetilde{f} = F \circ i$ there exists a map $\widetilde{F}\colon X \times [0,1] \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\widetilde{f}"] \arrow[d, "i"] & Y \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "F"] \arrow[ur, dashed, "\widetilde{F}", "??"'] & B. \end{tikzcd}$$

Sometimes we refer to $\widetilde{F}\colon X \times [0,1] \to Y$ as a *lift* of $F\colon X \times [0,1] \to B$. (Note that we do *not* demand that the lift $\widetilde{F}$ is any way unique.) We refer to Figure 1561 for an illustration of the definition.

(2) We say that the map $p\colon Y \to B$ is a *fibration* if p has the homotopy lifting property with respect to all topological spaces.

(3) We say that the map $p\colon Y \to B$ is a *Serre fibration* if p has the homotopy lifting property with respect to all closed balls $\overline{B}^n$, $n \in \mathbb{N}_0$. So perhaps somewhat confusingly a Serre fibration is not necessarily a fibration.

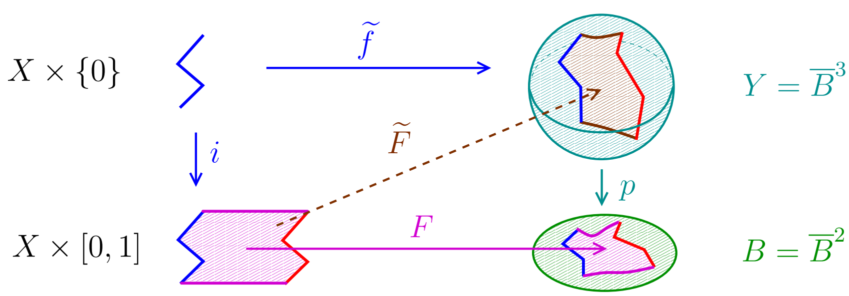


Figure 1561

Remark.

(1) Different textbooks mean different objects when they talk about "fibrations". Out definition coincides with the language used in [**Hat02, Rot88, DaK01**]. In [**Bre93**], what we call a "fibration" is called a "Hurewicz fibration".

(2) If $p\colon Y \to B$ is Serre fibration and if Y and B are CW-complexes, then it follows from **[StW84, Cau92]** that p is in fact a fibration. We will not make use of this result.

Examples.

(1) In Proposition 16.12 we already showed that every covering $p\colon Y \to B$ has the homotopy lifting property (in fact with a unique lift) with respect to every topological manifold, in particular with respect to any closed ball $\overline{B}^n$. This implies in particular that every covering is a Serre fibration.

(2) Let B and G be topological spaces. We denote by $p\colon B\times G \to B$ and $q\colon B\times G\to G$ the two obvious projections. In the following we show that the projection map $p\colon B\times G\to B$ is a fibration. Indeed, in the setting above the map

$$\begin{array}{rcl}\widetilde{F}\colon X\times[0,1] & \to & B\times G \\ (x,t) & \mapsto & \big(F(x,t), q\big(\underbrace{\widetilde{f}(x,0)}_{\in B\times G}\big)\big)\end{array}$$

makes the diagram commute. As is shown in Figure 1562, in contrast to the first example, the lift $\widetilde{F}$ is in general not unique.

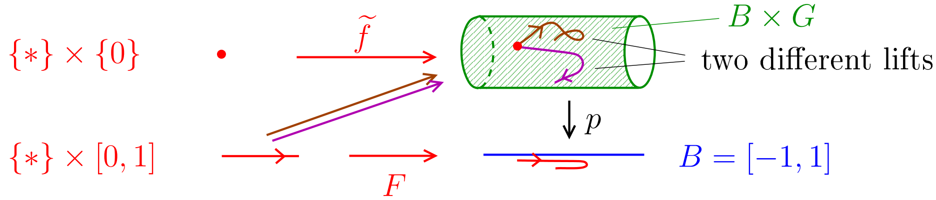


FIGURE 1562

(3) For any $n\in\mathbb{N}$ the map

$$\begin{array}{rcl}\overline{B}^{n+1} & \to & \overline{B}^n \\ (x_1,\dots,x_n,x_{n+1}) & \mapsto & (x_1,\dots,x_n)\end{array}$$

is a Serre fibration. With some effort this statement can be proved "by hand". The case $n=1$ is precisely the content of Exercise 112.1.

We conclude the list of examples with a "non-example". To make this example more readable we package it is as a lemma.

Lemma 112.1. *The map $p\colon S^1\to[-1,1]$ that is given by the projection onto the x-coordinate is not a Serre fibration.*

PROOF. We consider the topological space $X = \overline{B}^1 = [-1,1]$ together with the maps $\widetilde{f}\colon X\times\{0\}\to S^1$ and $F\colon[-1,1]\times[0,1]\to[-1,1]$ that are illustrated in Figure 1563. We claim that there is no lifting $\widetilde{F}\colon X\times[0,1]\to S^1$. More precisely, we prove the following claim.

Claim. If $\widetilde{F}\colon X\times[0,1]\to S^1$ is a map with $p\circ\widetilde{F}=F$, then $\widetilde{F}|_{X\times\{0\}}$ does not agree with $\widetilde{f}$.

Thus let $\widetilde{F}\colon X\times[0,1]\to S^1$ be a map with $p\circ\widetilde{F}=F$. We make the following observations:

(1) The preimage $F^{-1}(0)$ is connected by the definition of F.
(2) We obtain from (1) and Lemma 2.57 that the image $\widetilde{F}(F^{-1}(0))$ is also connected.
(3) The fact that $p\circ\widetilde{F}=F$ implies that $\widetilde{F}(F^{-1}(0))\subset p^{-1}(0)$.[1526]
(4) By definition of the map p we have $p^{-1}(0)=\{(0,\pm1)\}$.
(5) It follows from (2), (3) and (4) that $\widetilde{F}$ restricted to $F^{-1}(0)$ is constant.
(6) The map $\widetilde{f}$ is not constant on $F^{-1}(0)$.

The combination of (5) and (6) proves our claim. ■

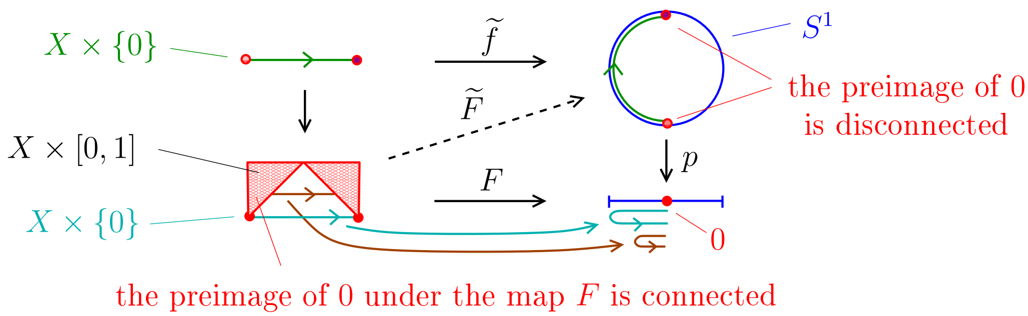


FIGURE 1563

We continue with introducing some language and notation.

Notation.

(1) Given a fibration or a Serre fibration $p\colon Y\to B$ we sometimes refer to B as the *base space* and to Y as the *total space*.[1527] Furthermore, given a point $*\in B$ we refer to $F:=p^{-1}(*)$ as a *fiber*. The last definition is somewhat dangerous since, as we see in the last of the above examples, the homeomorphism type $p^{-1}(*)$ depends on the choice of $*$. We will shortly discuss "uniqueness" of the fiber. Finally throughout this chapter we denote by $i\colon F\to Y$ the inclusion map.
(2) Given $n\in\mathbb{N}_0$ we define I, I^n and J^n as on page 2616.

Before we can formulate the main theorem of this chapter we need to make some preparations.

Lemma 112.2.

(1) *For each $n\in\mathbb{N}$ there exists an orientation-preserving homeomorphism*

$$\big(\overline{B}^n\times[0,1],\overline{B}^n\times\{0\}\cup S^{n-1}\times[0,1]\big)\xrightarrow{\cong}\big(\overline{B}^n\times[0,1],\overline{B}^n\times\{0\}\big)$$

[1526]Since I always find these arguments confusing let me give the argument: Let $y\in\widetilde{F}(F^{-1}(0))$. This means that there exists a $P\in F^{-1}(0)$ with $y=\widetilde{F}(P)$. We have $p(y)=p(\widetilde{F}(P))=F(P)$, but $P\in F^{-1}(0)$, thus $p(y)=0$. In other words, we have $y\in p^{-1}(0)$.

[1527]This mirrors the language that we introduced on page **??** for vector bundles.

of pairs of topological spaces. (This statement is illustrated in Figure 1564 on the left.)

(2) *For each $n \in \mathbb{N}$ there exists a homeomorphism*

$$(I^n, J^{n-1}) \xrightarrow{\cong} (I^n, I^{n-1} \times \{0\}).$$

of pairs of topological spaces. (This statement is illustrated in Figure 1564 on the right.)

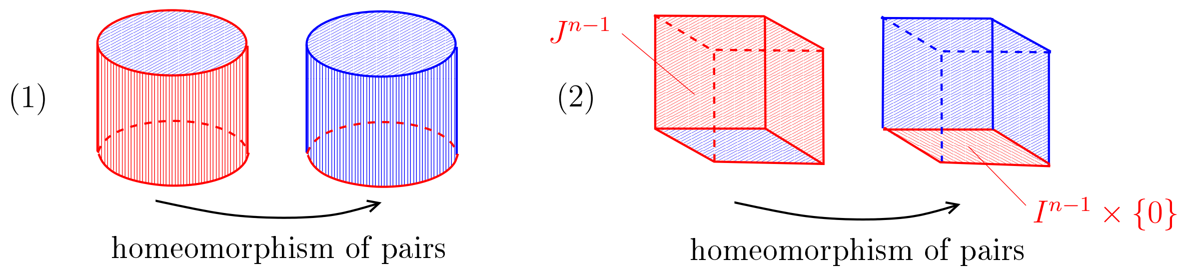


FIGURE 1564. Illustration of Lemma 112.2.

SKETCH OF PROOF (∗).

(1) For $n = 1$ such a homeomorphism is illustrated in Figure 1565. The general case is

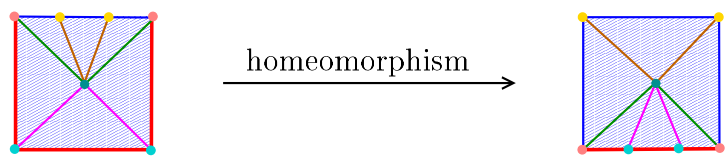


FIGURE 1565. Illustration for the proof of Lemma 112.2.

given by "rotating this homeomorphism around the $x_1 = 0$-axis in $\mathbb{R}^n$".

(2) Using the homeomorphism $\overline{B}^{n-1} \cong I^{n-1}$ from Proposition 2.53 (2) one can easily reduce the proof of (2) to the statement that we had just proved in (1). ∎

We will also soon need the following elementary lemma.

Lemma 112.3. *Let $p\colon Y \to B$ be a Serre fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$. Furthermore let $n \in \mathbb{N}$. Given any map $\varphi\colon (I^n, \partial I^n) \to (B, \{b_0\})$ there exists a map $\widetilde{\varphi}\colon (I^n, J^{n-1}) \to (Y, \{y_0\})$ such that the following diagram commutes:*

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ I^n \arrow[ur, dashed, "\widetilde{\varphi} \text{ with } J^{n-1} \mapsto y_0"] \arrow[r, "\varphi \text{ with } \partial I^n \mapsto b_0"'] & B. \end{tikzcd}$$

We refer to Figure 1566 for an illustration.

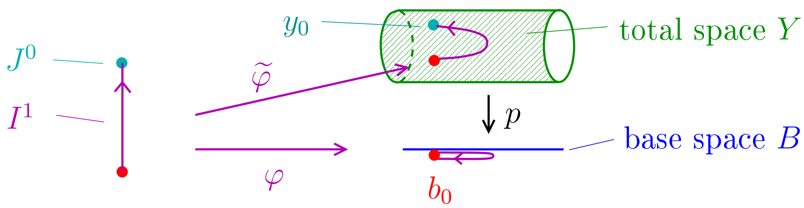


FIGURE 1566. Illustration of Lemma 112.3.

PROOF. Let $\Psi\colon (I^n, J^{n-1}) \xrightarrow{\cong} (I^n, I^{n-1} \times \{0\})$ be a homeomorphism as in Lemma 112.2 (2). We consider the diagram

$$\begin{tikzcd} I^{n-1} \times \{0\} \arrow[r, "\Psi^{-1}"] \arrow[d, hook] & J^{n-1} \arrow[r, "\text{constant map } y_0"] \arrow[d, hook] & Y \arrow[d, "p"] \\ I^{n-1} \times [0,1] \arrow[r, "\Psi^{-1}"'] \arrow[urr, dashed, "G"] & I^n \arrow[r, "\varphi"'] & B. \end{tikzcd}$$

Note that the diagram without the red dashed arrow commutes. Since p is a Serre fibration and since I^{n-1} is homeomorphic to the $(n-1)$-ball, see Proposition 2.52 (2), it follows that a map $G\colon I^{n-1} \times I \to Y$ exists that makes the diagram commute. It is now straightforward to verify that $G \circ \Psi\colon I^n \to B$ has the desired properties. ■

112.2. **The homotopy fiber of a fibration.** First we introduce the following, arguably long overdue notation.

Notation. Given two topological spaces X and Y we define $[X, Y]$ to be the set of homotopy equivalence classes of maps from X to Y.[1528]

The following lemma is a reformulation of Lemma 18.6.

Lemma 112.4. *Let X, Y and Z be topological spaces. The map*

$$\begin{array}{rcl} [Y,Z] \times [X,Y] & \mapsto & [X,Z] \\ ([g],[f]) & \mapsto & [g \circ f] \end{array}$$

is well-defined.

Now we can formulate the key technical proposition of this section.

Proposition 112.5. *Let $p\colon Y \to B$ be a fibration. Given $b \in B$ we write $F_b := p^{-1}(b)$. Let $\alpha\colon [0,1] \to B$ be a path from a point b_0 to a point b_1. We consider the following diagram*

$$\begin{tikzcd} F_{b_0} \times \{0\} \arrow[r, "(x,0)\mapsto x"] \arrow[d, "i"] & Y \arrow[d, "p"] \\ F_{b_0} \times [0,1] \arrow[r, "(x,t)\mapsto \alpha(t)"'] \arrow[ur, "H(\alpha)"] & B. \end{tikzcd}$$

[1528]Recall that in Lemma 18.4 we saw that "being homotopic" is an equivalence relation.

Since $p\colon Y \to B$ is a fibration there exists a diagonal map $H(\alpha)\colon F_{b_0} \times [0,1] \to Y$.[1529] *(We refer to Figure 1567 for an illustration.) We define $\Theta(\alpha) := H(\alpha)_1$. The following statements hold:*

(1) *The map $\Theta(\alpha)$ goes from F_{b_0} to F_{b_1}.*
(2) *The map $\Theta(\alpha)$, viewed as an element in $[F_{b_0}, F_{b_1}]$, is well-defined.*
(3) *The map $\Theta(\alpha)\colon F_{b_0} \to F_{b_1}$ is a homotopy equivalence.*
(4) *If two paths $\alpha, \beta\colon [0,1] \to B$ from b_0 to b_1 are path-homotopic, then $[\Theta(\alpha)] = [\Theta(\beta)]$.*
(5) *If c_b is the constant path at $b \in B$, then $[\Theta(c_b)] = [\mathrm{id}_{F_b}] \in [F_b, F_b]$.*
(6) *If α is a path in B from b_0 to b_1 and β is a bath in B from b_1 to b_2, then*

$$\underbrace{[\Theta(\alpha * \beta)]}_{\in[F_{b_0},F_{b_2}]} \;=\; \underbrace{[\Theta(\beta)]}_{\in[F_{b_1},F_{b_2}]} \circ \underbrace{[\Theta(\alpha)]}_{\in[F_{b_0},F_{b_1}]}.$$

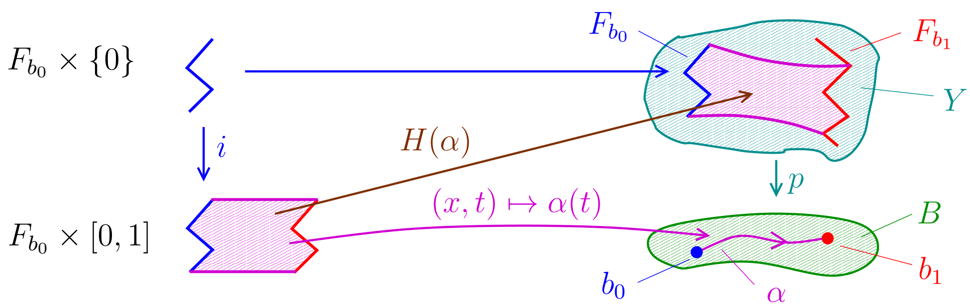


FIGURE 1567. Illustration of Proposition 112.5.

Remark. The picture behind the statement of Proposition 112.5 is that while walking along B we can drag the fiber along. Thus any path defines a map between the corresponding fibers.

PROOF (∗). Let $p\colon Y \to B$ be a fibration over a topological space B. Given $b \in B$ we write $F_b := p^{-1}(b)$. Let $\alpha\colon [0,1] \to B$ be a path from a point b_0 to a point b_1. We consider the following diagram

$$\begin{tikzcd}
F_{b_0} \times \{0\} \arrow[r, "(p,0)\mapsto p"] \arrow[d, "i"] & Y \arrow[d, "p"] \\
F_{b_0} \times [0,1] \arrow[r, "(x,t)\mapsto\alpha(t)"'] \arrow[ur, "H(\alpha)"] & B.
\end{tikzcd}$$

Since $p\colon Y \to B$ is by hypothesis a fibration there exists in fact such a diagonal map $H(\alpha)\colon F_{b_0} \times [0,1] \to Y$. Note that the map $H(\alpha)_0\colon F_{b_0} \to B$ is just the obvious inclusion. We turn to the proofs of the statements, albeit perhaps in an unexpected order:

(1) Since the above diagram commutes we see immediately that the map $H(\alpha)_1$ is a map from $F_{b_0} \to F_{b_1}$.

[1529] Recall that in general $H(\alpha)$ is not uniquely determined by the data, i.e. there might be many such maps.

(4) Let $\alpha, \beta\colon [0,1] \to B$ be two paths from b_0 to b_1. We pick maps $H(\alpha)$ and $H(\beta)$ as above, keeping in mind that $H(\alpha)$ and $H(\beta)$ are not uniquely determined. Now we assume that α and β are path-homotopic. We need to show the following claim.

Claim. The maps $H(\alpha)_1, H(\beta)_1\colon F_{b_0} \to F_{b_1}$ are homotopic.

We start out the proof of the claim with the following observation. It follows immediately from Lemma 112.2 (2), applied to $n = 2$, that there exists a self-homeomorphism

$$\Psi\colon [0,1]\times[0,1] \;\to\; [0,1]\times[0,1]$$

that restricts to a homeomorphism

$$\underbrace{([0,1]\times\{0,1\}) \cup (\{0\}\times[0,1])}_{=:U} \;\to\; [0,1]\times\{0\}.$$

For reassurance the construction of such a map is also sketched in Figure 1568. Now

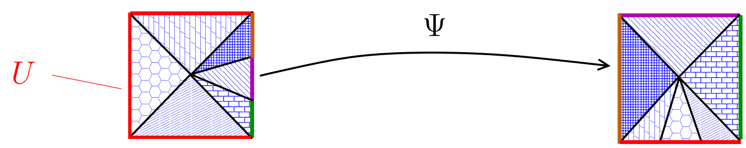


FIGURE 1568

we pick a homotopy $G\colon [0,1]\times[0,1] \to B$ rel $\{0,1\}$ from α to β. We consider the following diagram

$$\begin{tikzcd}
F_{b_0}\times[0,1]\times\{0\} \arrow[r, "\mathrm{id}_{F_{b_0}}\times\Psi^{-1}"] \arrow[d, hook] & \overbrace{(F_{b_0}\times[0,1]\times\{0,1\}) \cup (F_{b_0}\times\{0\})\times[0,1]}^{=F_{b_0}\times U} \arrow[r, "\Omega"] \arrow[d, hook] & Y \arrow[d, "p"] \\
F_{b_0}\times[0,1]\times[0,1] \arrow[r, "\mathrm{id}_{F_{b_0}}\times\Psi^{-1}"'] \arrow[urr, dashed, "\Xi"] & F_{b_0}\times[0,1]\times[0,1] \arrow[r, "(P,s,t)\mapsto G(s,t)"'] \arrow[ur, dashed, "\Xi\circ(\mathrm{id}_{F_{b_0}}\times\Psi)"'] \arrow[l, bend left, "\mathrm{id}_{F_{b_0}}\times\Psi"] & B.
\end{tikzcd}$$

We make the following clarifications and observations:

(a) The map Ω is defined as follows:[1530]
   (i) on $F_{b_0}\times[0,1]\times\{0\} = F_{b_0}\times[0,1]$ the map Ω is given by $H(\alpha)$,
   (ii) on $F_{b_0}\times[0,1]\times\{1\} = F_{b_0}\times[0,1]$ the map Ω is given by $H(\beta)$,
   (iii) on $F_{b_0}\times\{0\}\times[0,1]$ the map Ω is given by the projection onto F_{b_0} followed by the inclusion into Y.

(b) It follows immediately from the definitions that the diagram commutes.

(c) Since $p\colon Y \to B$ is a fibration we see that there exists a map Ξ that makes the outer diagram commute.

[1530]As on so many other occasions we can use Lemma 2.35 (2) to show that the map Ω is indeed continuous.

(d) The map $\Xi \circ (\mathrm{id}_{F_{b_0}} \times \Psi)\colon F_{b_0} \times [0,1] \times [0,1] \to Y$ is a homotopy between $H(\alpha)$ and $H(\beta)$. In particular the map $(\Xi \circ (\mathrm{id}_{F_{b_0}} \times \Psi))_1\colon F_{b_0} \times [0,1] \to Y$ is the desired homotopy between $H(\alpha)_1$ and $H(\beta)_1$. ⊞

(5) This statement follows from the observation that for the constant path c_b at a point $b \in B$ we can choose $H(c_b)\colon F_b \times [0,1] \to F_b$ to be the projection. Evidently we obtain that $\Theta(c_b) = H(c_b)_1 = \mathrm{id}_{F_b}$.

(6) Let α be a path in B from b_0 to b_1 and let β be a bath in B from b_1 to b_2. Almost the same way that we define the product $\alpha * \beta$ we combine $H(\alpha)$ and $H(\beta)$. More precisely, we consider the map[1531]

$$\begin{array}{rcl} H(\alpha * \beta)\colon F_{b_0} \times [0,1] & \to & X \\ (P,t) & \mapsto & \begin{cases} H(\alpha)(P,2t), & \text{if } t \in [0,\frac{1}{2}], \\ H(\beta)\big(\underbrace{H(\alpha(P),1)}_{\in F_{b_1}}, 2t-1\big), & \text{if } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

One can easily verify that this map has all the properties that we expect from a map that is called $H(\alpha * \beta)$. The desired statement is now an immediate consequence of the observation that $H(\alpha * \beta)_1 = H(\beta)_1 \circ H(\alpha)_1$.

(3) Let $\alpha\colon [0,1] \to B$ be a path from b_0 to b_1. We pick a map $H(\alpha)\colon F_{b_0} \times [0,1] \to X$ as above. We want to show that the corresponding map $H(\alpha)_1\colon F_{b_0} \to F_{b_1}$ is a homotopy equivalence. We denote by $\overline{\alpha}\colon [0,1] \to B$ the inverse path of α, i.e. the path from b_1 to b_0 that is given by $\overline{\alpha}(t) = \alpha(1-t)$. We pick a map $H(\overline{\alpha})\colon F_{b_1} \times [0,1] \to X$ as above. It remains to prove the following claim.

Claim.

(a) The map $H(\overline{\alpha})_1 \circ H(\alpha)_1\colon F_{b_0} \to F_{b_0}$ is homotopic to the identity of F_{b_0}.
(b) The map $H(\alpha)_1 \circ H(\overline{\alpha})_1\colon F_{b_1} \to F_{b_1}$ is homotopic to the identity of F_{b_1}.

We prove the first statement. The proof of the second statement is verbatim the same. We have

$$[\,H(\overline{\alpha})_1 \circ H(\alpha)_1\,] \underset{\substack{\uparrow \\ \text{by (6)}}}{=} [\,H(\alpha * \overline{\alpha})\,] \underset{\substack{\uparrow \\ \text{by (4) and Proposition 14.6 (3)}}}{=} [\,H(c_{b_0})\,] \underset{\substack{\uparrow \\ \text{by (5).}}}{=} [\,\mathrm{id}_{F_{b_0}}\,].$$

⊞

(2) We apply the argument from (4) to $\beta = \alpha$ together with two maps $H(\alpha)$ and $H(\beta)$ that make the diagram commute. We obtain by (3) that $H(\alpha)$ and $H(\beta)$ are homotopic, i.e. we see that $[H(\alpha)] \in [F_{b_0}, F_{b_1}]$ is well-defined. ■

Using Proposition 112.5, and the proof thereof, we almost immediately obtain the following proposition.

Proposition 112.6. *Let $p\colon Y \to B$ be a map between topological spaces. We assume that B is path-connected.*

(1) *If $p\colon Y \to B$ is a fibration, then for any two points $b_0, b_1 \in B$ the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are homotopy equivalent.*

[1531] It follows again from Lemma 14.3 that this map is continuous.

(2) *If $p\colon Y \to B$ is a Serre fibration, then for any two points $b_0, b_1 \in B$ such that the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are discrete subsets, the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are homotopy equivalent.*

Proof.

(1) This statement follows immediately from Proposition 112.5 (3) and the hypothesis that B is path-connected.
(2) Let $p\colon Y \to B$ be a Serre fibration with B a path-connected topological space. Let $b_0, b_1 \in B$ such that the fibers $F_{b_0} := p^{-1}(b_0)$ and $F_{b_1} := p^{-1}(b_1)$ are discrete subsets. In the proof of Proposition 112.5 (3) we used the homotopy lifting property for the topological spaces $X = F_{b_i} \times [0,1]^n$ with $i \in \{0,1\}$ and $n \in \{1,2\}$. Since F_{b_0} and F_{b_1} are discrete we see that each component of $X = F_{b_i} \times [0,1]^n$ is homeomorphic to $[0,1]^n$, i.e. each component is homeomorphic to $\overline{B}^n$. Since we can do the lifting component-wise it suffices in our context that we are given a Serre fibration. We leave it to the reader to fill in the details. ■

Proposition 112.6 (1) allows us to make the following definition.

Definition. Let $p\colon Y \to B$ be a fibration over a path-connected topological space B. We refer to the homotopy type of the fibers as the *homotopy fiber* of the fibration.

Remark. A more precise statement of Proposition 112.6 (1) holds: any path $\gamma\colon [0,1] \to B$ from b_0 to b_1 defines naturally a homotopy equivalence between the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$. We refer to [**DaK01**, Theorem 6.12] for the precise formulation and proof of this statement.

Examples.

(1) Let $p\colon \widetilde{X} \to X$ be a covering map over a path-connected topological space. As discussed on page 2643 such a map is a Serre fibration. It follows immediately from the definition of a covering map that for each $x \in X$ the fiber $p^{-1}(x)$ is a discrete subset of $\widetilde{X}$. Therefore it follows from Lemma 18.13 and Proposition 112.6 (2) that given any two points $x_0, x_1 \in X$ there exists a bijection $p^{-1}(x_0) \to p^{-1}(x_1)$. In other words, we have found a new proof for Lemma 16.1.
(2) Let us consider again the "non-example" from Lemma 112.1, i.e. we consider the map

$$\begin{array}{rcl} p\colon Y = S^1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\} & \to & B = [-1,1] \\ (x,y) & \mapsto & x. \end{array}$$

In this case $p^{-1}(-1) = \{(-1,0)\}$ and $p^{-1}(0) = \{(0,\pm 1)\}$ are discrete subsets, but the number of components is different. From Lemma 18.13 we deduce that $p^{-1}(-1)$ is not homotopy equivalent to $p^{-1}(0)$. Thus Proposition 112.6 (2) gives a new proof (after the more hands-on proof provided by Lemma 112.1) that p is not a Serre fibration.
(3) We consider the map $p\colon Y \to B$ given by the total space

$$Y \;:=\; \{\underbrace{(x,y)}_{\in \mathbb{R}^2} \,|\, y = 1 - x \text{ and } x \in [0,1]\} \;\cup\; \bigcup_{n \in \mathbb{N}} \{\underbrace{(x,y)}_{\in \mathbb{R}^2} \,|\, x \in [0,1] \text{ and } y = 1 + \tfrac{1}{n}\}$$

the base space $B = [0,1]$ and the obvious projection $p\colon Y \to X$. (This example is illustrated in Figure 1569.) It is an amusing exercise to show that $p\colon Y \to B$ is a Serre fibration. We saw in Exercise 18.16 that the fibers $p^{-1}(0)$ and $p^{-1}(1)$ are not homotopy equivalent. It follows from Proposition 112.6 (1) that $p\colon Y \to B$ is not a fibration.[1532]

We can also show more directly that p is not a fibration: We consider $X = p^{-1}(1)$, the usual projection map $f\colon X \times [0,1] \to B = [0,1]$ together with the obvious inclusion $\widetilde{f}\colon X \times \{0\} \to Y$. It is once again an amusing elementary exercise to show that there is no map $\widetilde{F}\colon X \times [0,1] \to B$ with all the desired properties. This shows that $p\colon Y \to B$ is not a fibration.

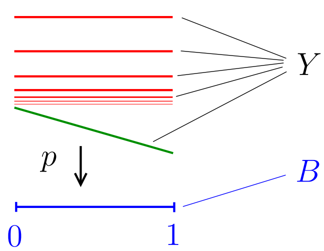


FIGURE 1569

112.3. **The long exact homotopy sequence of a Serre fibration.** For us the most interesting aspect of Serre fibrations is that they give rise to long exact sequences of homotopy groups. More precisely, we have the following theorem.

Theorem 112.7. *Let $p\colon Y \to B$ be a Serre fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$ and $F = p^{-1}(b_0)$.*

(1) *Given any $n \in \mathbb{N}$ the following map is well-defined:*

$$\begin{array}{rcl} \partial_n\colon \pi_n(B,b_0) & \to & \pi_{n-1}(F,y_0) \\ {[\varphi\colon (I^n,\partial I^n) \to (B,b_0)]} & \mapsto & \left[\begin{array}{c} \textit{pick a map } \widetilde{\varphi}\colon (I^n,J^{n-1}) \to (Y,\{y_0\}) \textit{ with } p\circ\widetilde{\varphi} = \varphi \\ (\textit{we know from Lemma 112.3 that such a map exists}) \\ \textit{and then restrict the map } \widetilde{\varphi} \textit{ to } I^{n-1} = I^{n-1}\times\{0\} \end{array}\right]. \end{array}$$

(The definition of ∂_n is moderately successfully illustrated in Figure 1570.) Furthermore, these maps ∂_n have the following properties:

(a) *The maps are morphisms of groups (if $n \geq 2$) and of pointed sets (if $n = 1$).*

[1532] The discussion of this example would be more intelligible if we followed the convention of [**Bre93**, p. 450] and if we referred to "fibrations" as "Hurewicz fibrations" to better distinguish them from "Serre fibrations".

(b) *The maps are natural in the following two senses: Suppose we are given another Serre fibration $p'\colon Y' \to B'$ and suppose we are given a commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[d, "\Phi"'] & B \arrow[d, "\phi"] \\ Y' \arrow[r, "p'"] & B'. \end{tikzcd}$$

If we write $y_0' = \Phi(y_0), b_0' = \phi(b_0)$ and $F' = (p')^{-1}(b_0')$, then for any $n \in \mathbb{N}$ the following diagram commutes

$$\begin{tikzcd} \pi_n(B,b_0) \arrow[r, "\partial_n"] \arrow[d, "\phi_*"'] & \pi_{n-1}(F,y_0) \arrow[d, "(\Phi|_F)_*"] \\ \pi_n(B',b_0') \arrow[r, "\partial_n"] & \pi_{n-1}(F',y_0'). \end{tikzcd}$$

(c) *Let $\alpha\colon [0,1] \to B$ be a path from b_0 to some point b_1.*[1533] *We write $F_1 = p^{-1}(b_1)$. If $\Theta(\alpha)\colon F \to F_1$ is a map as in Proposition 112.5, then the following diagram commutes*

$$\begin{tikzcd} \pi_n(B,b_0) \arrow[r, "\partial_n"] \arrow[d, "\alpha_*"'] & \pi_{n-1}(F,y_0) \arrow[d, "\Theta(\alpha)_*"] \\ \pi_n(B,b_1) \arrow[r, "\partial_n"] & \pi_{n-1}(F_1,y_1) \end{tikzcd}$$

where $y_1 = \Theta(\alpha)(y_0)$.

(2) *The following sequence is exact:*[1534]

$$\begin{aligned} \dots &\xrightarrow{\partial_{n+1}} \pi_n(F,y_0) \xrightarrow{i_*} \pi_n(Y,y_0) \xrightarrow{p_*} \pi_n(B,b_0) \xrightarrow{\partial_n} \\ &\xrightarrow{\partial_n} \pi_{n-1}(F,y_0) \xrightarrow{i_*} \dots\dots\dots \xrightarrow{p_*} \pi_1(B,b_0) \xrightarrow{\partial_1} \\ &\xrightarrow{\partial_1} \pi_0(F,y_0) \xrightarrow{i_*} \pi_0(Y,y_0) \xrightarrow{p_*} \pi_0(B,b_0). \end{aligned}$$

(3) *If the inclusion map $i\colon F \to Y$ is actually homotopic rel y_0 to the constant map $c_{y_0}\colon F \to \{y_0\}$, then for any $n \in \mathbb{N}_{\geq 2}$ the above long exact sequence breaks into a short exact sequence*

$$0 \;\to\; \pi_n(Y,y_0) \;\xrightarrow{p_*}\; \pi_n(B,b_0) \;\xrightarrow{\partial_n}\; \pi_{n-1}(F,y_0) \;\to\; 0.$$

Furthermore the short exact sequence splits, which implies that for any $n \in \mathbb{N}_{\geq 2}$ we have an isomorphism

$$\pi_n(B,b_0) \;\cong\; \pi_n(Y,y_0) \oplus \pi_{n-1}(F,y_0).$$

Before we head towards the proof of Theorem 112.7 we want to see what the theorem tells us in two types of fibrations that we had considered above, namely for coverings and products.

[1533] Here we allow of course that $b_1 = b_0$.

[1534] Here, at the lower end, we mean "exact" in the sense of the definition on page 2620.

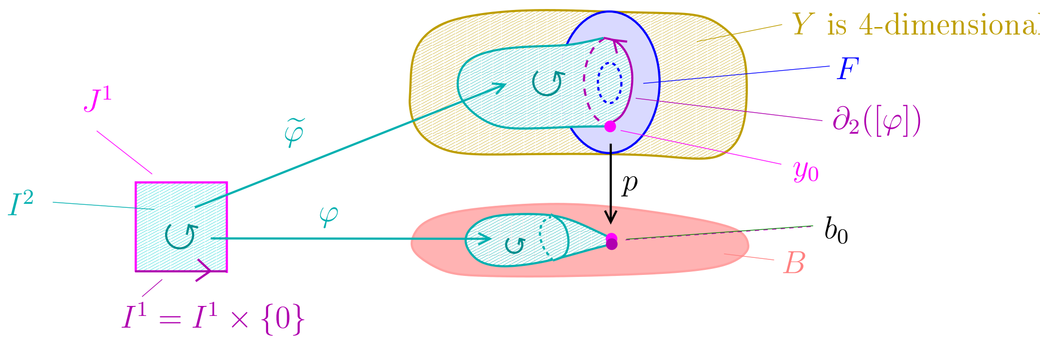


FIGURE 1570

Example. Let $p\colon Y \to B$ be a covering of path-connected topological spaces. As mentioned above, by Proposition 16.12 we know that p is a Serre fibration. We pick $y_0 \in Y$ and we write $b_0 = p(y_0)$ and $F = p^{-1}(b_0)$. It follows immediately from the definition of a covering map that F, equipped with the subspace topology coming from Y, is a discrete topological space, in particular $\#\pi_0(F, y_0) = \#F$ and $\pi_i(F, y_0) = 0$ for $i \geq 1$. From the long exact sequence provided by Theorem 112.7 we obtain the following two pieces of information:

(1) At the lower end the long exact sequence looks as follows:

$$\underbrace{\pi_1(F, y_0)}_{=\{e\}} \to \pi_1(Y, y_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial} \underbrace{\pi_0(F, y_0)}_{\#\pi_0(F,y_0)=\#F} \to \underbrace{\pi_0(Y, y_0)}_{=\{e\}} \to \underbrace{\pi_0(B, b_0)}_{=\{e\}}.$$

In other words we obtain the short exact sequence

$$\{e\} \to \pi_1(Y, y_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial} \pi_0(F, y_0) \to \{e\}.$$

We draw the following two conclusions:

(a) The map $p_*\colon \pi_1(Y, y_0) \to \pi_1(B, b_0)$ is a monomorphism.

(b) With a little bit of extra thought[1535] we obtain from the short exact sequence, Proposition 112.5, Theorem 112.7 (1c) and Lemma 111.2 that $p_*(\pi_1(Y, y_0))$ is a subgroup of $\pi_1(B, b_0)$ of index $\#\pi_0(F, y_0) = \#F = [Y : B]$.

These two statements are precisely the content of Lemma 15.9 and Lemma 16.15 (3).

(2) Since all other homotopy groups of F vanish we see that for any $n \geq 2$ the map $p_*\colon \pi_n(Y, x_0) \to \pi_n(B, b_0)$ is an isomorphism. This result recovers Proposition 40.13.

We move on to the next application. More precisely we want to apply Theorem 112.7 to the case of a Serre fibration of the form $X \times Y \to X$. This leads to a new proof of Propositions 16.20 and 40.8.

[1535] We leave this extra thinking as a mental exercise to the reader.

Proposition 40.8. *Let X and Y be two topological spaces and let $x_0 \in X$ and $y_0 \in Y$. We consider the maps*

$$\begin{array}{rcl} i\colon X & \to & X \times Y \\ x & \mapsto & (x, y_0) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon Y & \to & X \times Y \\ y & \mapsto & (x_0, y). \end{array}$$

For any $n \in \mathbb{N}$ the map[a]

$$\begin{array}{rcl} \Phi\colon \pi_n(X, x_0) \times \pi_n(Y, y_0) & \to & \pi_n(X \times Y, (x_0, y_0)) \\ (\varphi, \psi) & \to & i_*(\varphi) \cdot j_*(\psi) \end{array}$$

is an isomorphism.

[a] Note that for $n \geq 2$ the map is a group homomorphism since in this case the group on the right-hand side is abelian. It is a priori not completely clear that the map is also a homomorphism for $n = 1$, but we will see in the proof that the map is indeed a homomorphism.

Proof. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the two obvious projections. On page 2643 we saw that both projections are fibrations. We concentrate on the first projection map $p\colon X \times Y \to X$. It is a fibration with fiber $p^{-1}(x_0) = \{x_0\} \times Y$. Using the obvious homeomorphism $Y \to \{x_0\} \times Y$ we obtain from Theorem 112.7 the long exact sequence

$$\ldots \to \pi_n(Y, y_0) \xrightarrow{j_*} \pi_n(X \times Y, (x_0, y_0)) \xrightarrow{p_*} \pi_n(X, x_0) \to \pi_{n-1}(Y, y_0)) \xrightarrow{j_*} \ldots .$$

Evidently we have $q \circ j = \mathrm{id}_Y$ which implies that $q_* \circ j_* = \mathrm{id}_{\pi_1(Y, y_0)}$. It follows that for every $n \in \mathbb{N}_0$ the map $j_*\colon \pi_n(F, (x_0, y_0)) \to \pi_n(X \times Y, (x_0, y_0))$ is a monomorphism. Thus for each $n \in \mathbb{N}$ we see that

$$0 \to \pi_n(Y, y_0) \xrightarrow{j_*} \pi_n(X \times Y, (x_0, y_0)) \xrightarrow{p_*} \pi_n(X, x_0) \to 0$$

is exact. Once again using that $q_* \circ j_* = \mathrm{id}$ we obtain from Splitting Lemma 46.2 that the given map

$$\begin{array}{rcl} \Phi\colon \pi_n(X, x_0) \times \pi_n(Y, y_0) & \to & \pi_n(X \times Y, (x_0, y_0)) \\ (\varphi, \psi) & \to & i_*(\varphi) \cdot j_*(\psi) \end{array}$$

is indeed an isomorphism. ■

Even these modest applications should already convince the reader that it is worth proving Theorem 112.7. After the proof we will give many more examples of fibrations and we will find many more interesting applications.

112.4. **Proof of Theorem 112.7.** The proof of Theorem 112.7 requires some preparations, i.e. we first need to prove some propositions.

Proposition 112.8. *Let $p\colon Y \to B$ be a Serre fibration and let (K, L) be a pair of* CW-*complexes. We denote by $i\colon K \times \{0\} \cup L \times [0,1] \to K \times [0,1]$ the inclusion map. Given any map $F\colon K \times [0,1] \to B$ and given any map $\widetilde{f}\colon K \times \{0\} \cup L \times [0,1] \to Y$ with $p \circ \widetilde{f} = F \circ i$*

there exists a map $\widetilde{F}\colon K \times [0,1] \to Y$ *such that the following diagram commutes:*

$$\begin{tikzcd}
K \times \{0\} \cup L \times [0,1] \arrow[rr, "\widetilde{f}"] \arrow[d, hook, "i"] & & Y \arrow[d, "p"] \\
K \times [0,1] \arrow[rr, "F"'] \arrow[urr, dashed, "\widetilde{F}" near start, "??"'] & & B.
\end{tikzcd}$$

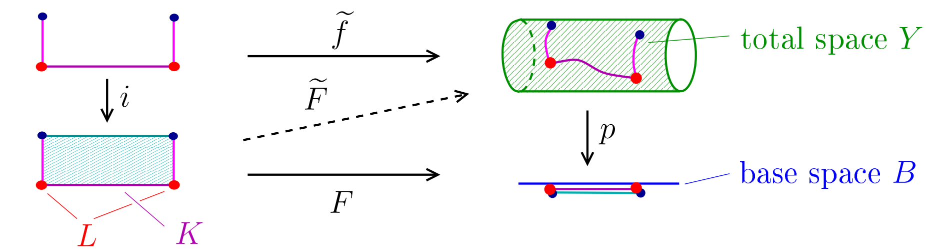


FIGURE 1571. Illustration of Proposition 112.8.

PROOF. We start out with a reformulation of our task: Given $n = 0, 1, 2, 3, \ldots$ we need to define maps

$$\widetilde{F}^n \colon (K \times \{0\}) \cup (K^n \cup L) \times [0,1] \;\to\; Y$$

that make the above diagram commute and which have the property that for $n < m$ the maps $\widetilde{F}^n$ and $\widetilde{F}^m$ agree on the common domain.

We set $\widetilde{F}^{-1} = \widetilde{f}$. Suppose we have already defined maps $\widetilde{F}^{-1}, \ldots, \widetilde{F}^{n-1}$ as above. Let $\{\Phi_i \colon (\overline{B}^n, S^{n-1}) \to (K^n, L^{n-1})\}_{i \in I}$ be the characteristic maps of the n-cells of K that do not lie in L. Recall that by Lemma 112.2 (1) there exists a homeomorphism

$$\Psi\colon \big(\overline{B}^n \times [0,1], (\overline{B}^n \times \{0\}) \cup (S^{n-1} \times [0,1])\big) \xrightarrow{\cong} \big(\overline{B}^n \times [0,1], \overline{B}^n \times \{0\}\big).$$

For each $i \in I$ we consider the following diagram

$$\begin{tikzcd}
\overline{B}^n \times \{0\} \arrow[r, "\Psi^{-1}"] \arrow[d, hook] & \overline{B}^n \times \{0\} \cup S^{n-1} \times \{1\} \arrow[r, "\widetilde{F}^{n-1} \circ \Phi_i"] \arrow[d, hook] & Y \arrow[d, "p"] \\
\overline{B}^n \times [0,1] \arrow[r, "\Psi^{-1}"'] \arrow[urr, dashed, "\Xi_i"'] & \overline{B}^n \times [0,1] \arrow[r, "F \circ \Phi_i"'] & B.
\end{tikzcd}$$

One can easily verify that the two rectangles commute. Since $p\colon Y \to B$ is a Serre fibration there exists a map $\Xi_i \colon \overline{B}^n \times [0,1] \to Y$ that makes the diagram commute. These maps $\{\Xi_i\}_{i \in I}$ can now be used to extend $\widetilde{F}^{n-1}$ to the desired map $\widetilde{F}^n$. ■

Proposition 112.9. *Let* $p\colon Y \to B$ *be a Serre fibration and let* (K, L) *be a pair of* CW-*complexes such that* L *is a deformation retract of* K. *We denote by* $i\colon L \to K$ *the inclusion map. Given any map* $f\colon K \to B$ *and given any map* $g\colon L \to Y$ *with* $f \circ i = p \circ g\colon L \to B$

there exists a map $\widetilde{f}\colon K \to Y$ *such that the following diagram commutes:*

$$\begin{tikzcd} L \arrow[r, "g"] \arrow[d, hook, "i"'] & Y \arrow[d, "p"] \\ K \arrow[r, "f"'] \arrow[ur, dashed, "\widetilde{f}"] & B. \end{tikzcd}$$

Example. Later on we will apply Proposition 112.9 to the special case that $K = \overline{B}^n$ and that $L = \{(0, \ldots, 0, 1)\}$. We refer to Figure 1572 for an illustration.

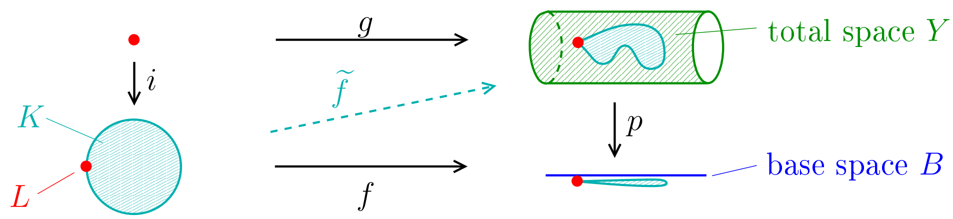


FIGURE 1572

PROOF. We make the following preparations:

(1) Since L is a deformation retract of K there exists by definition of a deformation retraction, see page 548, a homotopy $H\colon K \times [0,1] \to K$ rel L such that $H_0 = \mathrm{id}_K$ and $H_1(K) \subset L$. We denote by $r = H_1\colon K \to L$ the corresponding retraction.

(2) We denote by $\mathrm{proj}\colon L \times [0,1] \to L$ the obvious projection.

Now we consider the following commutative diagram

$$\begin{tikzcd} L \times \{0\} \arrow[r, hook] \arrow[d, hook] & K \times \{1\} \cup L \times [0,1] \arrow[r, "r \cup \mathrm{proj}"] \arrow[d, hook] & L \arrow[r, "g"] \arrow[d, hook] & Y \arrow[d, "p"] \\ K \times \{0\} \arrow[r, hook] & K \times [0,1] \arrow[r, "H"'] \arrow[urr, dashed, "\Phi"] & K \arrow[r, "f"'] & B. \end{tikzcd}$$

To be somewhat more precise, the map $r \cup \mathrm{proj}\colon K \times \{1\} \cup L \times [0,1]$ is defined via the retraction r on $K = K \times \{1\}$ and the projection $L \times [0,1] \to L$. The right-hand square commutes by our hypothesis. One easily verifies that the middle square commutes. It follows from Proposition 112.8 that there exists a map $\Phi\colon K \times [0,1] \to Y$ that makes the diagram commute. For any $k \in K$ we have

$$(p \circ \Phi)(k, 0) \;=\; f(H(k,0)) \;=\; f(\mathrm{H}_0(k)) \;=\; f(k)$$

and for any $l \in L$ we have

$$\Phi(l, 0) \;=\; (g \circ \mathrm{proj})(l, 0) \;=\; g(\mathrm{proj}(l, 0)) \;=\; g(l).$$

This shows that the restriction of Φ to $K = K \times \{0\}$ has the desired properties. ∎

Proposition 112.10. *Let $p\colon Y \to B$ be a Serre fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$. Let $B' \subset B$ be any subset with $b_0 \in B'$. We set $Y' := p^{-1}(B')$. Then*
$$p_*\colon \pi_n(Y, Y', y_0) \;\to\; \pi_n(B, B', b_0)$$
is an isomorphism for all $n \in \mathbb{N}$. (We refer to Figure 1573 for an illustration.)

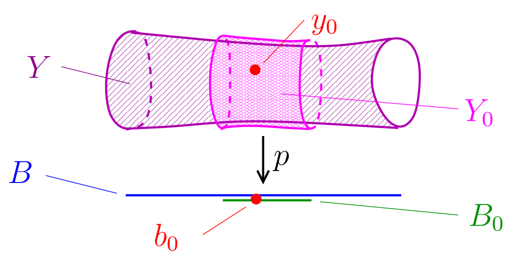


FIGURE 1573. Illustration of Proposition 112.10.

In the proof of Proposition 112.10 we will need the following elementary lemma.

Lemma 112.11. *Let $n \in \mathbb{N}$. If $x \in S^{n-1}$, then $(\overline{B}^n \times \partial[0,1]) \cup (\{x\} \times [0,1])$ is a deformation retract of $\overline{B}^n \times [0,1]$.*

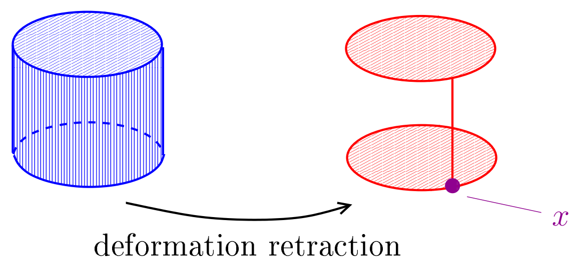


FIGURE 1574. Illustration of Lemma 112.11.

PROOF OF LEMMA 112.11 (∗). As usual we write $I = [0,1]$. Given $k \in \{1, \dots, n\}$ we consider
$$X_k \;:=\; (I^{n-1} \times \partial I) \;\cup\; (\{\underbrace{(1, 1, \dots, 1)}_{\in \mathbb{R}^{n-k}}\} \times I^k).$$
(We refer to Figure 1575 for an illustration.) Using Proposition 2.52 (2) and Lemma 3.32 (3) one can show easily that there exists a homeomorphism
$$(\overline{B}^n \times [0,1], (\overline{B}^n \times \{0,1\}) \cup (\{x\} \times [0,1])) \;\xrightarrow{\cong}\; (X_n, X_1).$$
Thus it suffices to show that each X_i is a deformation retract of X_{i+1}. But such deformation retractions one can actually write down by hand. ■

Now we can provide the proof of Proposition 112.10.

PROOF OF PROPOSITION 112.10. Let $p\colon Y \to B$ be a Serre fibration and let $n \in \mathbb{N}$. Furthermore let $y_0 \in Y$. We write $b_0 = p(y_0)$. Finally let $B' \subset B$ be any subset with $b_0 \in B'$. We set $Y' := p^{-1}(B')$. Throughout this proof we use the identification from page 2630, i.e. we view elements in a relative homotopy group $\pi_n(X, A, x_0)$ as homotopy classes of maps $(\overline{B}^n, S^{n-1}, *) \to (X, A, x_0)$.

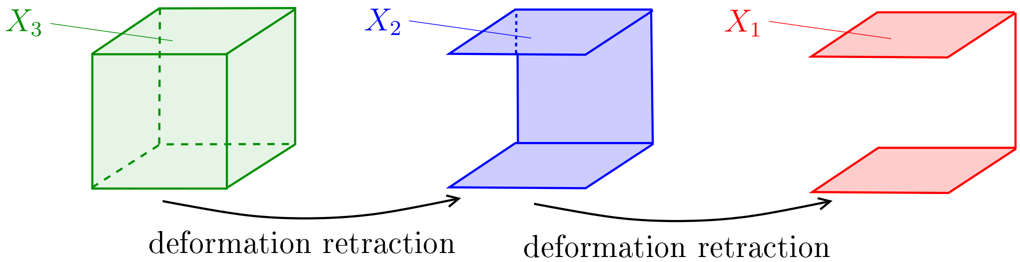


FIGURE 1575. Illustration for the proof of Lemma 112.11.

First we show that $p_*\colon \pi_n(Y, Y', y_0) \to \pi_n(B, B', b_0)$ is an epimorphism. Thus suppose we are given a map $f\colon (\overline{B}^n, S^{n-1}, *) \to (B, B', b_0)$. It follows from Proposition 112.9, applied to $(K, L) = (\overline{B}^n, *)$, that there exists a map $g\colon \overline{B}^n \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} \{*\} \arrow[r, "* \mapsto y_0"] \arrow[d, hook] & Y \arrow[d, "p"] \\ \overline{B}^n \arrow[r, "f"'] \arrow[ur, dashed, "g"] & B. \end{tikzcd}$$

Basically by definition g is a map $(\overline{B}^n, S^{n-1}, *) \to (Y, Y', y_0)$. Since $f = p \circ g$ we have $[f] = p_*([g])$.

Next we show that $p_*\colon \pi_n(Y, Y', y_0) \to \pi_n(B, B', b_0)$ is also a monomorphism. Therefore suppose that we are given two maps $g_0, g_1\colon (\overline{B}^n, S^{n-1}, *) \to (Y, Y', y_0)$ such that the equality $p_*([g_0]) = p_*([g_1]) \in \pi_n(B, B', b_0)$ holds. This equality means that $p \circ g_0$ is homotopic to $p \circ g_1$ via a homotopy $F\colon (\overline{B}^n, S^{n-1}, *) \times [0,1] \to (B, B', b_0)$. We consider the following commutative diagram (first without the dashed arrow):

$$\begin{tikzcd} (\overline{B}^n \times \{0,1\}) \cup (\{*\} \times [0,1]) \arrow[r, "\text{via } g_0, g_1"] \arrow[d, hook] & Y \arrow[d, "p"] \\ \overline{B}^n \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & B. \end{tikzcd}$$

By Proposition 112.9 and Lemma 112.11 there exists a map $\widetilde{F}\colon \overline{B}^n \times [0,1] \to Y$ such that the above diagram commutes. This map $\widetilde{F}\colon \overline{B}^n \times [0,1] \to Y$ is easily seen to be a homotopy between g_0 and g_1 which shows that $[g_0] = [g_1] \in \pi_n(Y, Y', y_0)$. ■

With these preparations it is surprisingly easy to provide a proof of Theorem 112.7 (1) and (2).

PROOF OF THEOREM 112.7 (1) AND (2). Let $p\colon Y \to B$ be a Serre fibration. Furthermore let $y_0 \in Y$. We write $b_0 = p(y_0)$ and $F = p^{-1}(b_0)$. We denote by $i\colon F \to Y$ the inclusion map. We consider the following diagram of maps

$$\begin{tikzcd}
\dots \arrow[r] & \pi_n(F,y_0) \arrow[r, "i_*"] & \pi_n(Y,y_0) \arrow[r] \arrow[ddr, bend right, "p_*"'] & \pi_n(Y,F,y_0) \arrow[r, "\partial_n"] \arrow[d, "p_*", "\cong"'] & \pi_{n-1}(F,y_0) \arrow[r, "i_*"] & \dots \\
& & & \pi_n(B,\{b_0\},b_0) \arrow[d, "="] & & \\
& & & \pi_n(B,b_0). \arrow[uur, bend right, "\partial_n\circ p_*^{-1}"'] & &
\end{tikzcd}$$

We make the following comments:

(a) The top sequence of maps is the long exact sequence of homotopy groups of the pointed pair of topological spaces (Y,F,y_0), see Proposition 111.4 for details.
(b) The top vertical map is an isomorphism by Proposition 112.10.
(c) The bottom vertical map is the identification given on page 2617.
(d) It follows immediately from the definitions that the left-hand triangle commutes.
(e) The right-hand triangle commutes by definition of the diagonal map.

It follows immediately from the definition of p_* that given any $n\in\mathbb{N}$ the map

$$\partial_n\circ p_*^{-1}\colon \pi_n(B,b_0) \;\to\; \pi_{n-1}(F,y_0)$$

is given by

$$[\varphi\colon (I^n,\partial I^n)\to(B,b_0)] \;\mapsto\; \left[\begin{array}{c}\text{the restriction of } \widetilde{\varphi} \text{ to } I^{n-1}=I^{n-1}\times\{0\}\\ \text{where } \widetilde{\varphi} \text{ is a map } (I^n,J^{n-1})\to(Y,\{y_0\})\\ \text{with } p\circ\widetilde{\varphi}=\varphi\end{array}\right].$$

This observation shows that the map given in statement (1) of the theorem is well-defined and that it is a homomorphism (respectively a morphism of pointed sets). We leave it to the few readers who made it that far to verify that the maps ∂_n have the two properties stated in (1b) and (1c).

Furthermore, since the top sequence is exact and since the vertical maps are isomorphisms we see that the sequence given in statement (2) of the theorem is indeed exact. ■

Proof of Theorem 112.7 (3). Let $p\colon Y\to B$ be a Serre fibration. Furthermore let $y_0\in Y$. We write $b_0=p(y_0)$ and $F=p^{-1}(b_0)$. We denote by $i\colon F\to Y$ the inclusion map. Now we suppose that the inclusion map $i\colon F\to Y$ is homotopic rel y_0 to the constant map $c_{y_0}\colon F\to\{y_0\}$. Put differently, we suppose that there exists a homotopy $G\colon F\times[0,1]\to Y$ rel y_0 such that $G_0=i$ and $G_1(f)=y_0$ for every $f\in F$.

It follows almost immediately from Lemmas 46.2 and 46.4 that it suffices to show that for any $n\in\mathbb{N}_{\geq 2}$ there exists a homomorphism $\lambda\colon \pi_{n-1}(F,y_0)\to\pi_n(B,b_0)$ such that $\partial_n\circ\lambda=\mathrm{id}_{\pi_{n-1}(F,y_0)}$. We consider the map[1536]

$$\begin{array}{rcl}\lambda\colon \pi_{n-1}(F,y_0) & \to & \pi_n(Y,F,y_0) \xrightarrow{p_*} \pi_n(B,\{b_0\},b_0)=\pi_n(B,b_0)\\ [\varphi\colon (I^{n-1},\partial I^{n-1})\to(F,y_0)] & \mapsto & \left[\begin{array}{c}(I^n,\partial I^n,J^{n-1})\to(Y,F,\{y_0\})\\ (x,t)\mapsto G(\varphi(x),t)\end{array}\right]\end{array}$$

[1536]It takes a second to actually verify that the map $(x,t)\mapsto G(\varphi(x),t)$ defines in fact a map $(I^n,\partial I^n,J^{n-1})\to(Y,F,\{y_0\})$.

It is elementary to verify that λ is in fact a homomorphism. It follows almost immediately from the definitions that $\partial_n \circ \lambda = \mathrm{id}_{\pi_{n-1}(F,y_0)}$. ■

112.5. **The path space fibration.** We recall the following, long dormant, definition from page 234.

Definition. Let X and Y be topological spaces.

(1) We denote by $X^Y = C(Y, X)$ the set of all maps[1537] from Y to X.
(2) The *compact-open topology* on X^Y is the topology generated, in the sense of the definition on page 105, by all the sets of the form
$$M(K, U) \ := \ \{f \in X^Y \,|\, f(K) \subset U\}$$
where $K \subset Y$ is compact and where $U \subset X$ is open.

In the following we state slightly simplified versions of two technical results which we proved a long time ago.

Proposition 5.4. *Let X be a topological space.*

(1) *The evaluation map*
$$\begin{array}{rcl} e\colon X^{[0,1]} \times [0,1] & \to & X \\ ((f\colon [0,1] \to X), y) & \mapsto & f(y) \end{array}$$
is continuous.
(2) *For each $y \in [0,1]$ the evaluation map*
$$\begin{array}{rcl} e\colon X^{[0,1]} & \to & X \\ (f\colon [0,1] \to X) & \mapsto & f(y) \end{array}$$
is continuous.

Proposition 5.6. *Let X and T be two topological spaces. Let $H\colon T \times [0,1] \to X$ be a map. Given $t \in T$ we denote by $H_t\colon [0,1] \to X$ the map defined by $H_t(y) = H(t, y)$. Then*
$$H\colon T \times [0,1] \to X \text{ is continuous} \ \Leftrightarrow \ \begin{array}{l} (1)\ \text{each } H_t\colon [0,1] \to X \text{ is continuous, and} \\ (2)\ \text{the map } T \to X^{[0,1]} \text{ given by } t \mapsto H_t \text{ is continuous.} \end{array}$$

This reminder allows us to introduce our latest best friends.

Definition. Let X be a topological space.

(1) We define
$$\textit{free path space } X^{[0,1]} \ := \ \text{set of all maps } [0,1] \to X$$
where $X^{[0,1]}$ is as usual equipped with the compact-open topology. Furthermore we introduce the
$$\textit{free loop space } \Omega(X) \ := \ \{f \in X^{[0,1]} \,|\, f(0) = f(1)\}.$$
We equip the free loop space with the compact-open topology coming from $X^{[0,1]}$.

[1537]Of course here we mean “all continuous maps”.

(2) Suppose we are also given a base point $x_0 \in X$. We introduce the

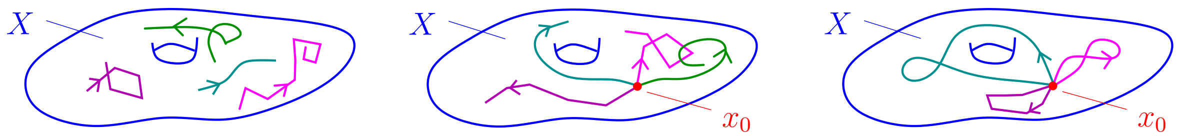
$$\textit{path space } P(X,x_0) \;:=\; \{f \in X^{[0,1]} \,|\, f(0)=x_0\}$$

and the

$$\textit{loop space } \Omega(X,x_0) \;:=\; \{f \in X^{[0,1]} \,|\, f(0)=f(1)=x_0\}.$$

We equip both sets with the compact-open topology coming from $X^{[0,1]}$.

(3) We denote by $c_{x_0}\colon [0,1] \to X$ the constant path given by sending all t's to x_0.



elements in the free path space $X^{[0,1]}$ — elements in the path space $P(X,x_0)$ — elements in the loop space $\Omega(X,x_0)$

FIGURE 1576

Examples.

(1) Let (X,x_0) be a pointed topological space and let $\alpha,\beta \in \Omega(X,x_0)$. It follows from Proposition 5.6 that a path in $\Omega(X,x_0)$ from α to β is basically the same as a homotopy between the maps α and β viewed as loops in (X,x_0). Thus we see that we have a natural identification

$$\pi_0(\Omega(X,x_0)) \;=\; \pi_1(X,x_0).$$

(2) Let (X,x_0) be a pointed topological space. It follows from Proposition 5.4 that the map

$$\begin{array}{rcl} e\colon P(X,x_0) & \to & X \\ f & \mapsto & f(1) \end{array}$$

is continuous. We can ask whether this map has a left-inverse. More precisely, the question is whether there exists a continuous map $s\colon X \to P(X,x_0)$ such that $e(s(x)) = x$ for all $x \in X$. This sounds like a rather random question, so let us reformulate it. The question is, can we *continuously* assign to each $x \in X$ a path from x_0 to x? Clearly that is possible say for $(X,x_0) = (\mathbb{R}^2,0)$, we just assign to each $x \in \mathbb{R}^2$ the direct path $e(x)$ that is given by $t \mapsto t\cdot x$. But can we do so say for $X = S^1$ or $X = S^2$?[1538] We will answer this question in Exercise 112.2.

For the record we state the following lemma.

Lemma 112.12. *The maps*

$$\begin{array}{rcl} (X,x_0) & \mapsto & (\Omega(X,x_0), c_{x_0}) \\ (f\colon (X,x_0)\to(Y,y_0)) & \mapsto & \left(\begin{array}{rcl} \Omega(X,x_0) & \to & \Omega(Y,y_0) \\ (\gamma\colon [0,1]\to X) & \mapsto & (f\circ\gamma\colon [0,1]\to Y) \end{array}\right) \end{array}$$

define a covariant functor from the category $\mathcal{PTop}$ of pointed topological spaces to the category $\mathcal{PTop}$ of pointed topological spaces. We refer to this functor as the loop space functor.

[1538] This question is somewhat similar to Question 51.7 and it also falls into the vaguely defined realm of "topological robotics".

PROOF. Let $f\colon (X,x_0)\to (Y,y_0)$ be a map between pointed topological spaces. It follows from Lemma 5.5 (1) that the map

$$\begin{array}{rcl} \Omega(X,x_0) & \to & \Omega(Y,y_0) \\ (\gamma\colon [0,1]\to X) & \mapsto & (f\circ\gamma\colon [0,1]\to Y) \end{array}$$

is actually continuous. With this observation out of the way it is straightforward to prove the lemma. ■

The following lemma says that in some sense (free) path spaces, in isolation, are not particularly interesting examples of topological spaces.

Lemma 112.13.

(1) *Let X be a topological space. We view X as a subset of the free path space $X^{[0,1]}$ via the inclusion $x\mapsto c_x$.*[1539] *Then X is a deformation retract of the free path space $X^{[0,1]}$.*

(2) *If (X,x_0) is a pointed topological space, then $\{c_{x_0}\}$ is a deformation retract of the path space $P(X,x_0)$, in particular the path space $P(X,x_0)$ is contractible.*

FIGURE 1577. Illustration of Lemma 112.13 (2).

PROOF. We prove statement (2). The proof of statement (1) is basically identical. Thus let (X,x_0) be a pointed topological space. We consider the "obvious" deformation retraction given by

$$\begin{array}{rcl} F\colon P(X,x_0)\times[0,1] & \to & P(X,x_0) \\ ((f\colon [0,1]\to X),t) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ s & \mapsto & f(s\cdot(1-t)) \end{array}\right) \end{array}$$

The only reason why we are not already done is that we still need to verify that F is in fact continuous. As we will see this is slightly delicate. We consider the map

$$G\colon \underbrace{X^{[0,1]}\times[0,1]}_{=:T} \to X^{[0,1]}$$

that is defined in exactly the same way as F. Since $F=G|_{P(X,x_0)\times[0,1]}$ it suffices to show that G is continuous. By the "⇒"-direction of Proposition 5.6 applied to $T=X^{[0,1]}\times[0,1]$ it suffices to show that the map

$$\begin{array}{rcl} H\colon [0,1]\times\overbrace{(X^{[0,1]}\times[0,1])}^{=T} & \to & X \\ (s,((g\colon [0,1]\to X),x)) & \mapsto & g(s\cdot x) \end{array}$$

[1539] Note that this inclusion $X\to X^{[0,1]}$ is by Lemma 5.3 an embedding.

is continuous. But the map H can be written as the composition

$$\begin{array}{rcccl} H\colon [0,1]\times X^{[0,1]}\times[0,1] & \to & X^{[0,1]}\times[0,1] & \to & X \\ (s,g,x) & \mapsto & (g,s\cdot(1-x)) & & \\ & & (g,t) & \mapsto & g(t). \end{array}$$

The first map is continuous by Lemma 3.8 and the second map is continuous by Proposition 5.4 (1). Thus we see that H itself is continuous. ■

In contrast to the previous proposition the following theorem shows that (free) path spaces give rise to interesting *maps*.

Theorem 112.14.

(1) *Let X be a topological space. The evaluation map*

$$\begin{array}{rcl} p\colon X^{[0,1]} & \to & X \\ (f\colon [0,1]\to X) & \mapsto & f(1) \end{array}$$

is a fibration. Furthermore, given any $x_0\in X$ the fiber $p^{-1}(x_0)$ is the topological space of all paths $[0,1]\to X$ that end *in x_0.*

(2) *Let (X,x_0) be a pointed topological space. The evaluation map*

$$\begin{array}{rcl} p\colon P(X,x_0) & \to & X \\ (f\colon [0,1]\to X) & \mapsto & f(1) \end{array}$$

is a fibration with fiber $p^{-1}(x_0)=\Omega(X,x_0)$.

Theorem 112.14 leads us to the following definition.

Definition.

(1) Let X be a topological space. We refer to the above fibration $p\colon X^{[0,1]}\to X$ as the *free path space fibration of X*.

(2) Let (X,x_0) be a pointed topological space. In the following we refer to the above fibration $p\colon P(X,x_0)\to X$ as the *path space fibration of* (X,x_0).

PROOF. We prove statement (2). The proof of statement (1) is basically identical and is thus left to the reader. Thus let (X,x_0) be a pointed topological space. First note that the map $p\colon P(X,x_0)\to X$ is continuous by Proposition 5.4 (2). Next recall that we need to solve the following type of extension problem for some topological space W and $c>0$: [1540]

$$\begin{tikzcd} W\times\{0\} \arrow[r, "\widetilde{f}"] \arrow[d, hook] & P(X,x_0)\subset X^{[0,1]} \arrow[d, "p"'] & \gamma \arrow[d, mapsto] \\ W\times[0,c] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & X. & \gamma(1) \end{tikzcd}$$

We refer to Figure 1578 for an illustration.

[1540] In the diagram we write $W\times[0,c]$ instead of the more customary $W\times[0,1]$ for purely psychological reasons, it makes it easier to distinguish the two types of intervals which arise throughout the proof and which play very different roles.

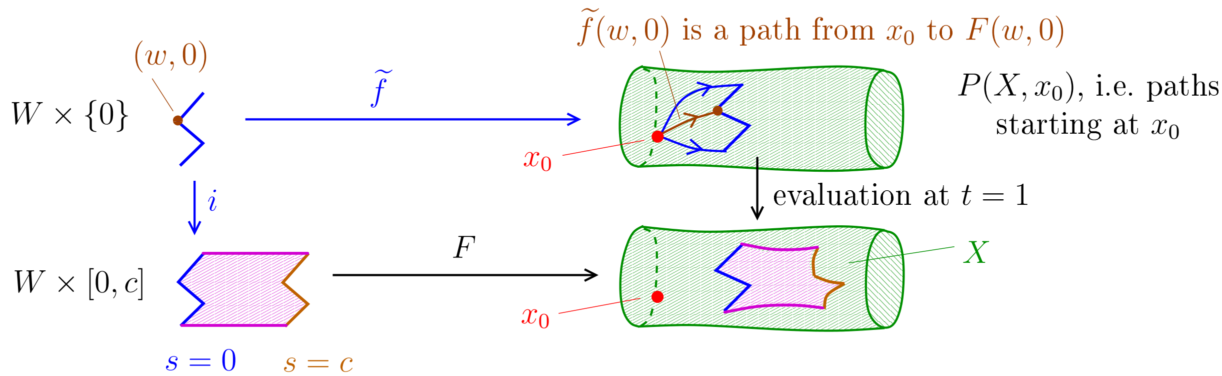


FIGURE 1578. Illustration for the proof of Theorem 112.14.

The idea for the proof is quite simple: Given $(w,s)\in W\times[0,c]$ we have to "systematically" find an element $\gamma\in P(X,x_0)$ with $p(\gamma)=F(w,s)$, i.e. we have to find a path $\gamma\colon[0,1]\to X$ from x_0 to $F(w,s)$. But there is an obvious path, namely we first travel along $\widetilde{f}(w,0)$ from x_0 to $F(w,0)$ and then we use the path $t\mapsto F(w,t)$ to get from $F(w,0)$ to $F(w,s)$. That is exactly what we will do, except that we need to be a little careful with our parametrizations.

Now we have to turn this idea into a proper proof. We consider the map

$$\begin{array}{rcl}\widetilde{F}\colon W\times[0,c] & \to & P(X,x_0)\\ (w,s) & \mapsto & \left(\begin{array}{rcl}[0,1] & \to & X\\ t & \mapsto & \begin{cases}\widetilde{f}(w,0)\big(\frac{t}{1-\frac{s}{2c}}\big), & \text{if } t\in[0,1-\frac{s}{2c}],\\ F(w,2\cdot(t-1+\frac{s}{2})), & \text{if } t\in[1-\frac{s}{2c},1]\end{cases}\end{array}\right).\end{array}$$

We claim that this map has all the desired properties:

(1) Let $(w,s)\in W\times[0,c]$. We make the following observations:
 (a) The given map $[0,1]\to X$ starts at x_0.
 (b) It follows easily from Lemma 2.35 (2) that the given map $[0,1]\to X$ is indeed continuous.

 In other words, the given map $[0,1]\to X$ does indeed define an element in $P(X,x_0)$.

(2) For each $(w,s)\in W\times[0,c]$ we have by construction that $\widetilde{F}(w,s)(t=1)=F(w,s)$. In other words, we have $p\circ\widetilde{F}=F$.

Thus it remains to prove the following claim:

Claim. The map $\widetilde{F}\colon W\times[0,c]\to P(X,x_0)$ is continuous.

We consider the map

$$\begin{array}{rcl}\widetilde{G}\colon W\times[0,c]\times[0,1] & \to & X\\ (w,s,t) & \mapsto & \widetilde{F}(w,s)(t)=\begin{cases}\widetilde{f}(w,0)\big(\frac{t}{1-\frac{s}{2c}}\big), & \text{if } t\in[0,1-\frac{s}{2c}],\\ F(w,2\cdot(t-1+\frac{s}{2})), & \text{if } t\in[1-\frac{s}{2c},1].\end{cases}\end{array}$$

It follows easily from the fact that F and $\widetilde{f}$ are continuous that the restrictions of $\widetilde{G}$ to the closed subsets

$$\begin{aligned} A_1 &:= W \times \{(s,t) \in [0,c] \times [0,1] \,|\, t \in [0, 1 - \tfrac{s}{2c}]\}, \\ A_2 &:= W \times \{(s,t) \in [0,c] \times [0,1] \,|\, t \in [1 - \tfrac{s}{2c}, 1]\} \end{aligned}$$

and

are continuous. Therefore it follows from Lemma 2.35 (2) that $\widetilde{G}$ is continuous. Finally we deduce from the "$\Rightarrow$"-direction of Proposition 5.6 applied to $T = W \times [0,c]$, $Y = [0,1]$ and $H = \widetilde{G}$ that $\widetilde{F}$ is also continuous. ■

We obtain the following amusing corollary which later on will turn out to play an important role.

Corollary 112.15. *Given any pointed topological space (X, x_0) and any $n \geq 1$ the map*[1541]

$$\begin{array}{rcl} \partial_n \colon \pi_n(X, x_0) & \xrightarrow{\cong} & \pi_{n-1}(\Omega(X, x_0), c_{x_0}) \\ {[f \colon ([0,1]^n, \partial([0,1]^n)) \to (X, x_0)]} & \mapsto & \left[\begin{array}{rcl} ([0,1]^{n-1}, \partial([0,1]^{n-1})) & \to & (\Omega(X,x_0), c_{x_0}) \\ x & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ t & \mapsto & f(x,t) \end{array} \right) \end{array} \right] \end{array}$$

is a natural isomorphism.

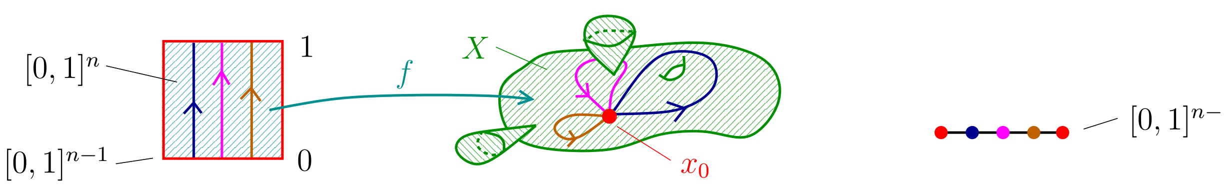


FIGURE 1579. Illustration of Corollary 112.15.

Remark. For $n = 1$ the statement of Corollary 112.15 recovers the result from page 2661 where we saw that there exists a natural identification

$$\pi_0(\Omega(X, x_0), c_{x_0}) = \pi_1(X, x_0).$$

PROOF. There are two approaches to proving the corollary, namely one can prove it by "the machine" and one can prove it "by hand".

(1) Let $f \colon (X, x_0) \to (Y, y_0)$ be a map between pointed topological spaces. It is clear that we obtain a commutative diagram

$$\begin{tikzcd} P(X, x_0) \arrow[r, "\gamma \mapsto f \circ \gamma"] \arrow[d, "p"'] & P(Y, y_0) \arrow[d, "p"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

[1541] Using Proposition 5.6 one can show directly, with some slight effort, that the map ∂_n is well-defined.

It follows from Theorems 112.14 (1) and (2) and 112.7 that we have commutative diagram of long exact sequences

$$\begin{array}{ccccccccc} \dots \to & \pi_n(P(X,x_0),c_{x_0}) & \to & \pi_n(X,x_0) & \xrightarrow{\partial} & \pi_{n-1}(\Omega(X,x_0),c_{x_0}) & \to & \pi_{n-1}(P(X,x_0),c_{x_0}) & \to \dots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \dots \to & \pi_n(P(Y,y_0),c_{y_0}) & \to & \pi_n(Y,y_0) & \xrightarrow{\partial} & \pi_{n-1}(\Omega(Y,y_0),c_{y_0}) & \to & \pi_{n-1}(P(Y,y_0),c_{y_0}) & \to \dots \end{array}$$

We obtain from Lemma 112.13 and Proposition 40.7 (2) that the homotopy groups of the path spaces $P(X,x_0)$ and $P(Y,y_0)$ are trivial. Thus the connecting homomorphisms are *natural* isomorphisms. We leave it to the reader to verify that in our situation the connecting homomorphisms from Theorem 112.7 are precisely the maps that we wrote down. Incidentally this then also shows that the maps, as written down, actually do make sense.

(2) In Exercise 112.4 we will prove the corollary "by hand". ■

Example. We consider the loop space of the 2-dimensional sphere with the base point given by the North Pole $N = (0,0,1)$. We calculate that

$$\pi_1(\Omega(S^2,N),c_N) \underset{\substack{\uparrow \\ \text{Corollary 112.15}}}{\xleftarrow{\cong}} \pi_2(S^2,N) \underset{\substack{\uparrow \\ \text{Corollary 53.6 (1)}}}{\xrightarrow{\cong}} \mathbb{Z}.$$

In Figure 1580 we see a loop $\{\gamma_z\}_{z\in S^1}$ of loops in the pointed topological space (S^2,N). Put differently, we see a loop in the pointed topological space $(\Omega(S^2,N),c_N)$. It is an amusing and instructive exercise to run through the above two isomorphisms to verify that our loop in the pointed topological space $(\Omega(S^2,N),c_N)$ is actually a generator of $\pi_1(\Omega(S^2,N),c_N)$.

So what does this example actually really mean? It means that even though each individual loop γ_z is null-homotopic in S^2, we cannot continuously deform all the loops γ_z altogether into the constant loop. It is a good idea to think about what it would mean for $[z \mapsto \gamma_z] \in \pi_1(\Omega(S^2,N),c_N)$ to be null-homotopic and why, pictorially, this is not the case in our situation.

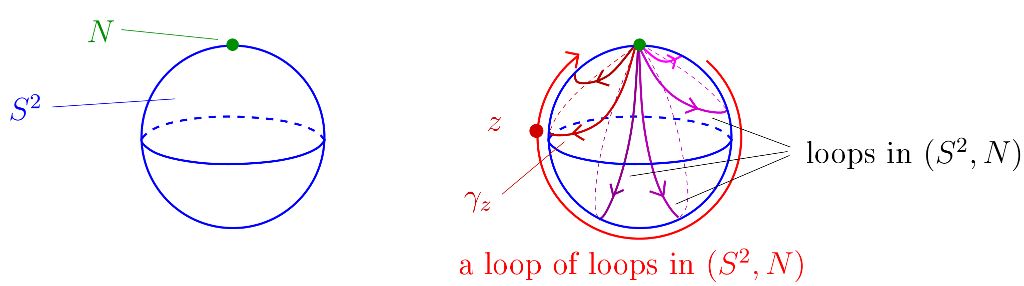


Figure 1580

Note that the statement of Corollary 112.15 can be summarized as follows: The loop space functor

$$\begin{array}{cccc} \Omega\colon & \begin{array}{c}\text{category } \mathcal{PTop} \text{ of pointed}\\ \text{topological spaces}\end{array} & \to & \begin{array}{c}\text{category } \mathcal{PTop} \text{ of pointed}\\ \text{topological spaces}\end{array} \\ & (X,x_0) & \mapsto & (\Omega(X,x_0),c_{x_0}) \end{array}$$

shifts homotopy groups down by one degree, i.e. for every pointed topological space (X, x_0) and every $n \in \mathbb{N}$ we have a natural isomorphism

$$\pi_n(X, x_0) \xrightarrow{\cong} \pi_{n-1}(\Omega(X, x_0), c_{x_0}).$$

This should be compared to the fact that in Proposition 45.7 we saw that the suspension functor

$$\begin{array}{rcl} \Sigma\colon \text{category } \mathcal{Top} \text{ of topological spaces} & \to & \text{category } \mathcal{Top} \text{ of topological spaces} \\ X & \mapsto & \Sigma(X) \end{array}$$

shifts reduced homology groups up by one degree, i.e. for every topological space X and any $n \in \mathbb{N}$ we have a natural isomorphism

$$\widetilde{\mathrm{H}}_n(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)).$$

In Exercise 112.8 we will discuss whether we can also shift homotopy groups *upwards* and whether we can shift reduced homology groups *downwards* in a functorial way.

112.6. **Every map can be replaced by a fibration.** In this section we will see that one can "replace" any map $f\colon X \to Y$ between topological spaces by a fibration $P_f \to Y$ that has very similar properties to the map f.

We start out with the notion of the pullback of a fibration. Perhaps not surprisingly the definition is almost identical to the definition of the pullback of a vector bundle that we gave on page **??**.

Lemma 112.16. *Let $f\colon A \to B$ be a map between topological spaces and let $q\colon Z \to B$ be a fibration. As in Lemma 25.16 (1) we consider the pullback*

$$f^*Z \;:=\; \{(a, z) \,|\, a \in A \text{ and } \overbrace{z \in q^{-1}(f(a))}^{\text{i.e. } q(z) = f(a)}\} \;\subset\; A \times Z.$$

The following statements hold:

(1) *The pullback map*

$$\begin{array}{rcl} p\colon f^*Z \;:=\; \{(a, z) \,|\, a \in A \text{ and } z \in q^{-1}(f(a))\} & \to & A \\ (a, z) & \mapsto & a \end{array}$$

is also a fibration. Furthermore for any $a \in A$ the fiber $p^{-1}(a)$ is homeomorphic to the fiber $q^{-1}(f(a))$.

(2) *The following diagram commutes*

$$\begin{tikzcd} f^*Z \arrow[r, "(a,z)\mapsto z"] \arrow[d, "p"] & Z \arrow[d, "q"] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

Proof.

(2) It is straightforward to verify that the diagram commutes.

(1) Let X be any topological space. We consider the following diagram where initially we ignore the diagonal arrows:

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\widetilde{g}"] \arrow[d, "i"] & f^*Z \arrow[r, "(a,z)\mapsto z"] \arrow[d, "p"] & Z \arrow[d, "q"] \\
X \times [0,1] \arrow[r, "G"'] \arrow[ur, dashed, "?\widetilde{G}?"] \arrow[urr, "\widetilde{f\circ G}"'] & A \arrow[r, "f"'] & B.
\end{tikzcd}$$

Here the maps $G\colon X \times [0,1] \to A$ and $\widetilde{g}\colon X \times \{0\} \to f^*Z$ are "the challenge" we have to deal with in the definition of a fibration. Since $q\colon Z \to B$ is a fibration there exists a map $\widetilde{f \circ G}\colon X \times [0,1] \to Z$ that makes the diagram commute. It is now straightforward to verify that the map

$$\begin{array}{rcl} X \times [0,1] & \to & f^*Z = \{(a,z) \,|\, a \in A \text{ and } z \in q^{-1}(f(a))\} \\ (x,t) & \mapsto & \big(G(x,t), \big(\widetilde{f \circ G}\big)(x,t)\big) \end{array}$$

makes the above diagram commute. We have thus shown that $p\colon f^*Z \to A$ is also a fibration.

Finally we have to prove the statement regarding the fibers. Thus let $a \in A$. It follows immediately from the definitions that $p^{-1}(a) = \{a\} \times q^{-1}(f(a)) \subset A \times Z$. It follows from Lemma 3.8 (2a) that the fiber $p^{-1}(a)$ is homeomorphic to the fiber $q^{-1}(f(a))$. ■

Definition. Using the notation from Lemma 112.16 we refer to $f^*Z \to A$ as the *pullback* of the fibration $q\colon Z \to B$ under the map $f\colon A \to B$.

Now we turn to the actual problem that we had stated in the beginning of this section.

Definition. Let $f\colon X \to Y$ be a map between topological spaces. Similar to the free path space fibration we now consider the fibration[1542]

$$\begin{array}{rcl} q\colon Y^{[0,1]} & \to & Y \\ \gamma & \mapsto & \gamma(0). \end{array}$$

We refer to the map

$$\begin{array}{rcl} p\colon P_f := f^*(Y^{[0,1]}) = \{(x,\gamma\colon [0,1] \to Y) \in X \times Y^{[0,1]} \,|\, \gamma(0) = f(x)\} & \to & Y \\ (x,\gamma) & \mapsto & \gamma(1) \end{array}$$

as the *mapping path fibration.*[1543]

In Proposition 39.7 we saw that any map $f\colon X \to Y$ between topological spaces can be "replaced" by a cofibration $X \to \mathrm{Cyl}(f)$. The following proposition is the analogue of Proposition 39.7 for fibrations.

[1542]This map is basically the same as the free path space fibration except that now we consider the evaluation $\gamma(0)$ instead of the evaluation $\gamma(1)$.

[1543]Note that it follows from Proposition 5.4 (2) that this map is in fact continuous.

Proposition 112.17. *Let $f\colon X \to Y$ be a map between topological spaces. We denote by $p\colon P_f \to Y$ the corresponding mapping path fibration.*

(1) *The map*
$$\begin{array}{rcl} h\colon X & \to & P_f \\ x & \mapsto & (x, \text{constant path } c_{f(x)}) \end{array}$$
is a homotopy equivalence and it has the property that it makes the following diagram commute:
$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & P_f \arrow[dl, "p"] \\ & Y. & \end{tikzcd}$$

(2) *The map $p\colon P_f \to Y$ is a fibration.*[1544] *Given any $y \in Y$ the fiber is given by*
$$\{(x,\gamma) \,|\, x \in X \text{ and } \gamma\colon [0,1] \to Y \text{ is a path from } f(x) \text{ to } y\}.$$

Definition. Let $f\colon X \to Y$ be a map between topological spaces. We suppose that Y is path-connected. We refer to the homotopy type of the fiber of the corresponding mapping path fibration $P_f \to Y$[1545] as the *homotopy fiber of f*.

Remark. If $f\colon X \to Y$ is a fibration, then it is shown in [**DaK01**, Theorem 6.18] that the map $h\colon X \to P_f$ in Proposition 112.17 (1) is a "fiber homotopy equivalence", thus, loosely speaking, the original fibration $f\colon X \to Y$ and the new fibration $p\colon P_f \to Y$ are essentially the same. We will not make use of this fact.

Example. Proposition 112.17 thus says that any map $f\colon X \to Y$ between topological spaces can be "replaced" by a fibration. As a reality check let us consider again the "non-example" from Lemma 112.1, i.e. we consider the map $f\colon S^1 \to [-1,1]$ that is given by the projection onto the x-coordinate. Lemma 112.1 says that f is not a fibration. Let us see what the above construction does in this case.

As pointed out in Proposition 112.17 (2), given any $y \in [-1,1]$ the preimage of y under the map $p\colon P_f \to Y = [-1,1]$ is given by
$$p^{-1}(y) \;=\; \{(x,\gamma) \,|\, x \in S^1 \text{ and } \gamma\colon [0,1] \to [-1,1] \text{ is a path from } f(x) \text{ to } y\}.$$
In Figure 1581 we strive to illustrate three points in the preimage of a point $y \in [-1,1]$. In Exercise 112.3 we will see that for each $y \in [-1,1]$ the preimage $p^{-1}(y)$ is homotopy equivalent to S^1. In particular this shows that the homotopy fiber of $f\colon S^1 \to [-1,1]$ is S^1. This exercise does not quite show yet that the map $p\colon P_f \to [-1,1]$ is a fibration, but at the very least it is consistent with Proposition 112.6 (1).

Proof of Proposition 112.17. Let $f\colon X \to Y$ be a map between topological spaces. We consider the map
$$\begin{array}{rcl} p\colon P_f = f^*(Y^{[0,1]}) = \{(x,\gamma\colon [0,1] \to Y) \in X \times Y^{[0,1]} \,|\, \gamma(0) = f(x)\} & \to & Y \\ (x,\gamma) & \mapsto & \gamma(1). \end{array}$$

[1544] This statement justifies the name "mapping path *fibration*".

[1545] Recall that Proposition 112.6 (1) says that the homotopy type of the fiber of a fibration over a path-connected topological space is well-defined.

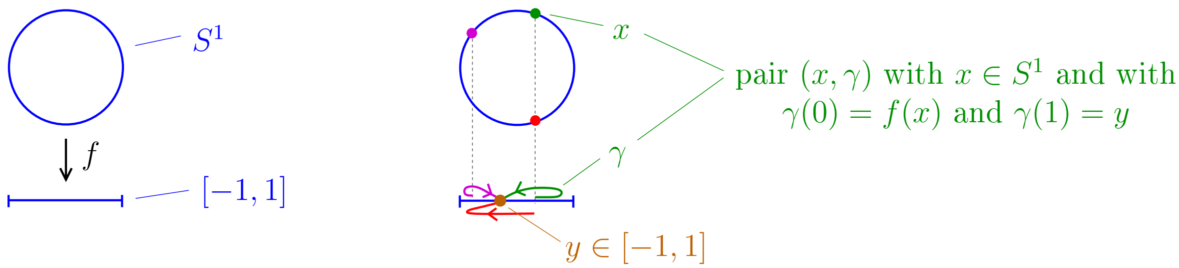


FIGURE 1581

We turn to the proof of the two statements:

(1) Given $y \in Y$ we denote, as usual, by $c_y\colon [0,1] \to Y$ the constant path given by $c_y(t) = y$ for all $t \in [0,1]$. We consider the map $h\colon X \to P_f$ that is given by $x \mapsto (x, c_{f(x)})$.[1546] We claim it has the desired properties. It is obvious that $p \circ h = f$, i.e. the diagram given in the proposition commutes. So it remains to show that h is a homotopy equivalence. We consider the map $g\colon P_f \to X$ given by $g(x,\gamma) = x$. Evidently we have $g \circ h = \mathrm{id}_X$. We are done once we have shown that $h \circ g$ is homotopic to id_{P_f}. In fact such a homotopy is given by[1547]

$$\begin{array}{rcl} H\colon P_f \times [0,1] & \to & P_f \\ \underset{\substack{\uparrow \\ \gamma\colon [0,1] \to Y \text{ is a path with } \gamma(0) = f(x)}}{((x,\gamma),t)} & \mapsto & \left(x, \left(\begin{array}{rcl} [0,1] & \to & Y \\ s & \mapsto & \gamma(s \cdot t) \end{array}\right)\right). \end{array}$$

One easily verifies that $H_0 = h \circ g$ and that $H_1 = \mathrm{id}_{P_f}$.

(2) We want to show that the map $p\colon P_f \to Y$ is a fibration. Thus let A be any topological space. Let $c > 0$. We consider the following "challenge diagram":[1548]

$$\begin{tikzcd} A \times \{0\} \arrow[r, "\widetilde{g}"] \arrow[d, "i"'] & P_f = \{(x, \gamma\colon [0,1] \to Y) \in X \times Y^{[0,1]} \,|\, \gamma(0) = f(x)\} \arrow[d, "p"', "\substack{(x,\gamma) \\ \downarrow \\ \gamma(1)}"] \\ A \times [0,c] \arrow[r, "G"'] \arrow[ur, dashed, "\widetilde{G}", "??"'] & Y. \end{tikzcd}$$

To define $\widetilde{G}$ we first split up the map $\widetilde{g}$ into its components. More precisely, we define maps $\widetilde{g}_1\colon A \to X$ and $\widetilde{g}_2\colon A \to Y^{[0,1]}$ via the equality

$$\widetilde{g}(a,0) \;=\; \big(\widetilde{g}_1(a), \widetilde{g}_2(a)\big) \;\in\; P_f \subset X \times Y^{[0,1]} \qquad \text{for all } a \in A.$$

[1546] It follows from Lemma 3.8 (2b) and Lemma 5.3 that this map is continuous.

[1547] An argument as in the proof of Lemma 112.13 (2) shows that the map H is indeed continuous.

[1548] As in the proof of Theorem 112.14 we prefer to work, for purely psychological reasons, with $A \times [0,c]$ instead of $A \times [0,1]$.

Note that by definition of P_f the path $\widetilde{g}_2(a) \in Y^{[0,1]}$ is a path from $f(\widetilde{g}_1(a))$ to $G(a,0)$. We need to define a suitable map $\widetilde{G}\colon A \times [0,c] \to P_f \subset X \times Y^{[0,1]}$. We consider the two factors separately:

(a) We already have the map $A \times [0,c] \to A \xrightarrow{\widetilde{g}_1} X$.
(b) To each $(a,s) \in A \times [0,c]$ we need to assign "systematically" a path $[0,1] \to Y$ from $f(x)$ to $G(a,s)$. The idea is quite simple: Given $(a,s) \in A \times [0,c]$ we consider the path $[0,1] \to Y$ that is given by first traveling along the path $\widetilde{g}_2(a)$ from $f(\widetilde{g}_1(a))$ to $G(a,0)$ and then we travel along the path $t \mapsto G(a,t)$ from $G(a,0)$ to $G(a,s)$. More precisely, similar to the proof of Theorem 112.14 we consider the map[1549]

$$\begin{array}{rcl} \widetilde{G}_2\colon A \times [0,c] & \to & Y^{[0,1]} \\ (a,s) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & Y \\ t & \mapsto & \begin{cases} \widetilde{g}_2(a)\big(\frac{t}{1-\frac{s}{2c}}\big), & \text{if } t \in [0, 1-\frac{s}{2c}], \\ G(a, 2\cdot(t-1+\frac{s}{2})), & \text{if } t \in [1-\frac{s}{2c}, 1] \end{cases} \end{array} \right). \end{array}$$

We refer to Figure 1582 for an illustration of the definition of $\widetilde{G}_2$.

Finally we consider the following map:

$$\begin{array}{rcl} \widetilde{G}\colon A \times [0,c] & \to & P_f \\ (a,s) & \mapsto & \big(\widetilde{g}_1(a), \widetilde{G}_2(a,s)\big). \end{array}$$

We make the following observations:

(a) It follows easily from the definitions that $\widetilde{G}$ takes values in P_f.
(b) It follows from the above and Lemma 3.8 (2b) that $\widetilde{G}$ is continuous.
(c) It is straightforward to verify that $\widetilde{G}$ makes the diagram commute.

We have thus verified that $\widetilde{G}$ has all the desired properties. ■

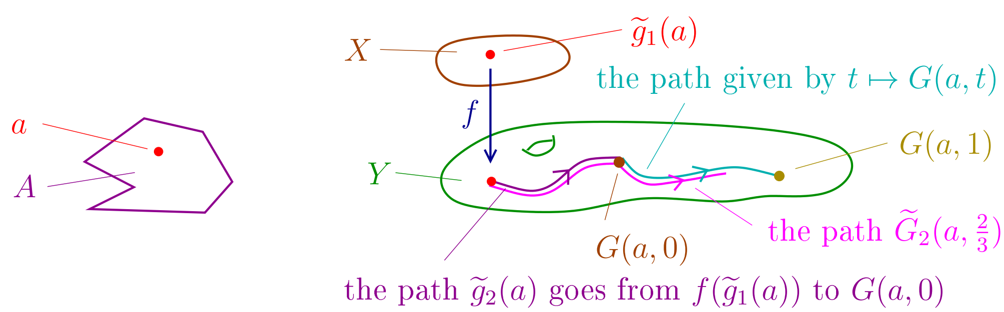


FIGURE 1582. Illustration for the proof of Proposition 112.17.

The statement of Proposition 112.17 that any map $f\colon X \to Y$ between topological spaces can be "replaced" by a fibration is quite surprising and perhaps also somewhat puzzling. Therefore we want to consider one more example.

[1549]The argument in the proof of Theorem 112.14 also shows that the map $\widetilde{G}_2$ does indeed take values in $Y^{[0,1]}$ and that the map $\widetilde{G}_2\colon A \times [0,c] \to Y^{[0,1]}$ is continuous.

Example. We denote by Σ the surface of genus 2 and we denote by T the torus. We consider the map $f\colon \Sigma \to T$ that is illustrated in Figure 1583. We denote by $p\colon P_f \to T$ the mapping path fibration and we denote by F the corresponding homotopy fiber. So what can we say about the mysterious homotopy fiber F?

We have the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & \pi_2(F) \arrow[r] & \pi_2(P_f) \arrow[r, "p_*"] & \pi_2(T) \arrow[r, "\partial"] & \pi_1(F) \arrow[r] & \pi_1(P_f) \arrow[r, "p_*"] & \pi_1(T) \arrow[r] & \dots \\
& & \pi_2(\Sigma) \arrow[u, "h_*", "\cong"'] \arrow[ur, "f_*"'] & & & \pi_1(\Sigma). \arrow[u, "h_*", "\cong"'] \arrow[ur, "f_*"'] & &
\end{tikzcd}$$

We make the following explanations and observations:

(1) The top sequence is the long exact sequence coming from Theorem 112.7. The vertical maps are induced by the homotopy equivalence $h\colon \Sigma \to P_f$ from Proposition 112.17 (1). In particular the vertical maps are isomorphisms by Proposition 40.7 (2).
(2) Since $p \circ h = f$ we see that the diagram commutes.
(3) By the discussion on page 1070 we know that $\pi_n(T) = \pi_n(\Sigma) = 0$ for $n \geq 2$.
(4) It is pretty clear that the map $f_*\colon \pi_1(\Sigma) \to \pi_1(T)$ is an epimorphism.
(5) We obtain from the above discussion and from Proposition 31.8 that F is path-connected, that $\pi_n(F) = 0$ for $n \geq 2$ and that $\pi_1(F)$ is an *infinitely generated* free group.

Thus we see that the homotopy groups of F are isomorphic to the homotopy groups of an infinite wedge of circles.

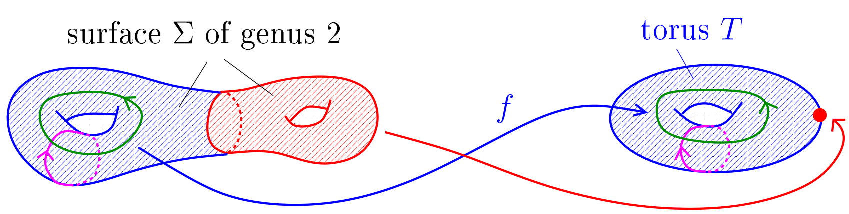


FIGURE 1583. A degree-one map from the surface of genus 2 to the torus.

112.7. **Adjoint functors** (∗)**.** In Chapter 39 we introduced the notion of a "cofibration" and we just got acquainted with the notion of a "fibration". In this section we will discuss the relationship between these two notions.

First we point out a formal similarity between fibrations and cofibrations, namely they both give rise to long exact sequences:

(1) Let $p\colon Y \to B$ be a fibration. Let $y_0 \in Y$. We write $b_0 = p(y_0)$ and we denote by $F = p^{-1}(b_0)$ the homotopy fiber of p. Let $i\colon F \to Y$ be the inclusion map. By Theorem 112.7 we obtain the following long exact sequence of homotopy groups:
$$\dots \xrightarrow{\partial} \pi_n(F, y_0) \xrightarrow{i_*} \pi_n(Y, y_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, y_0) \xrightarrow{i_*} \dots.$$
(2) Let $i\colon A \to X$ be a cofibration. We refer to the mapping cone $C := \mathrm{Cone}(i)$ as the *cofiber*. If $i(A)$ is a closed non-empty subset of X, then it follows easily from

Corollary 43.16 and Corollary 46.17 that we obtain a long exact sequence of reduced homology groups of the following form:

$$\ldots \xrightarrow{\partial} \widetilde{\mathrm{H}}_n(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{p_*} \widetilde{\mathrm{H}}_n(C) \xrightarrow{\partial} \widetilde{\mathrm{H}}_{n-1}(A) \xrightarrow{i_*} \ldots.$$

This basically recovers the statement of Lemma 46.18.

Here is another similarity between fibrations and cofibrations:

(1) Let $p\colon Y \to B$ be a fibration and let $f\colon A \to B$ be a map. As in Lemma 25.16 (1) we consider the pullback

$$\begin{tikzcd} f^*Y \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

where $f^*Y = \{(a,y) \in A \times Y \,|\, f(a) = p(y)\}$. In Lemma 112.16 we saw that the pullback $f^*Y \to A$ is also a fibration.

(2) Now let $i\colon A \to X$ be a cofibration and let $f\colon A \to B$ be a map to another topological space. As on page 732 we consider the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d] \\ X \arrow[r] & f_*X \end{tikzcd}$$

where $f_*X = (X \sqcup B)/i(a) \sim f(a)$ for all $a \in A$. By Proposition 39.17 (1) we will know that the pushout $B \to f_*X$ is again a cofibration.

So, to summarize: *the pullback of a fibration is again a fibration* and *the pushout of a cofibration is again a cofibration.*

The main goal of this section though is to formulate an arguably more direct relationship between the notions of a fibration and a cofibration. The key to doing so is the following definition from category theory.

Definition. We say that two covariant functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ between categories $\mathcal{C}$ and $\mathcal{D}$ form an *adjoint pair* if given any $c \in \mathrm{Ob}(\mathcal{C})$ and $d \in \mathrm{Ob}(\mathcal{D})$ there exists a bijection

$$\mathrm{Mor}_{\mathcal{D}}(F(c), d) \;\to\; \mathrm{Mor}_{\mathcal{C}}(c, G(d))$$

that is natural in c and d.[1550] We also say that the functor F is the *left adjoint of* G and that G is the *right adjoint of* F.

Examples.

(1) Let $\mathcal{S}$ be the category of sets and let $\mathcal{G}$ be the category of groups. We consider the two functors

$$\begin{array}{rcl} F\colon \mathcal{S} & \to & \mathcal{G} \\ S & \mapsto & \langle S\rangle \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G\colon \mathcal{G} & \to & \mathcal{S} \\ (\pi, \,\cdot\,) & \mapsto & \pi. \end{array}$$

[1550] We leave it to our reader to figure out what "natural in c and d" means.

In other words, F associates to a set S the free group $\langle S\rangle$ generated by S that we defined on page 591.[1551] Furthermore, G assigns to a group $(\pi,\,\cdot\,)$ the underlying set π.[1552] Using Lemma 19.14 one sees that for a set S and a group π the map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{S}}(S,\overbrace{\pi}^{=G(\pi,\,\cdot\,)}) & \to & \mathrm{Mor}_{\mathcal{G}}(\overbrace{\langle S\rangle}^{=F(S)},(\pi,\,\cdot\,)) \\ (\text{map } h\colon S\to\pi \text{ of sets}) & \mapsto & \begin{array}{l}\text{the unique homomorphism } \langle S\rangle\to(\pi,\,\cdot\,)\\ \text{that restricts to } h \text{ on the subset } S\subset\langle S\rangle\end{array} \end{array}$$

is a natural bijection. Thus we see that the functors F and G are adjoint.

(2) Let $\mathcal{AbGr}$ be the category of abelian groups and let A be an abelian group. We consider the two functors

$$\begin{array}{rcl} F\colon \mathcal{AbGr} & \to & \mathcal{AbGr} \\ B & \mapsto & A\otimes B \end{array} \quad\text{and}\quad \begin{array}{rcl} G\colon \mathcal{AbGr} & \to & \mathcal{AbGr} \\ B & \mapsto & \mathrm{Hom}(A,B). \end{array}$$

Now let $B,C\in\mathcal{AbGr}$. We consider the map[1553]

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{AbGr}}(\overbrace{A\otimes B}^{=F(B)},C) & \to & \mathrm{Mor}_{\mathcal{A}}(B,\overbrace{\mathrm{Hom}(A,C)}^{=G(C)}) \\ (f\colon A\otimes B\to C) & \mapsto & \left(\begin{array}{rcl} B & \to & \mathrm{Hom}(A,C) \\ b & \mapsto & \left(\begin{array}{rcl} A & \to & Y \\ a & \mapsto & f(a\otimes b)\end{array}\right)\end{array}\right). \end{array}$$

Using Lemma 57.2 (3) one can easily show that this map is a bijection. This shows that F and G form an adjoint pair.

(3) Let $\mathcal{Top}$ be the category of topological spaces. Rather similarly to the previous example we consider the two functors[1554]

$$\begin{array}{rcl} F\colon \mathcal{Top} & \to & \mathcal{Top} \\ X & \mapsto & X\times[0,1] \end{array} \quad\text{and}\quad \begin{array}{rcl} G\colon \mathcal{Top} & \to & \mathcal{Top} \\ X & \mapsto & X^{[0,1]}. \end{array}$$

Now let $X,Y\in\mathrm{Ob}(\mathcal{Top})$, i.e. let X and Y be topological spaces. We consider the map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Top}}(\overbrace{X\times[0,1]}^{=F(X)},Y) & \to & \mathrm{Mor}_{\mathcal{Top}}(X,\overbrace{Y^{[0,1]}}^{=G(Y)}) \\ (f\colon X\times[0,1]\to Y) & \mapsto & \left(\begin{array}{rcl} X & \to & Y^{[0,1]} \\ x & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & Y \\ t & \mapsto & f(x,t)\end{array}\right)\end{array}\right). \end{array}$$

[1551] In what sense is this a functor?

[1552] Such a functor, where we "forget" some of the given structure is often called a *forgetful functor*.

[1553] For abelian groups X and Y we have by definition that $\mathrm{Mor}_{\mathcal{AbGr}}(X,Y)=\mathrm{Hom}(X,Y)$.

[1554] Here we equip $X\times[0,1]$ with the product topology and we equip $X^{[0,1]}$ with the compact-open topology.

It follows from Proposition 5.6 that this map is well-defined and that it is a bijection.[1555] We have thus shown that F and G form an adjoint pair.

Now we can explain the formal relationship between fibrations and cofibrations. Let E and B be topological spaces and let $p\colon E \to B$ be a map. Recall that by definition $p\colon E \to B$ is a fibration if we can always complete the following diagram:

$$\begin{tikzcd}
Y \arrow[r, "\widetilde{f}"] \arrow[d, hook, "y\mapsto(y,0)"'] & E \arrow[d, "p"] \\
Y \times [0,1] \arrow[r, "F"] \arrow[ur, dashed, "\widetilde{F}"] & B.
\end{tikzcd}$$

Now we "dualize" this diagram as follows:

(1) we reverse the orientation of each arrow,
(2) we replace monomorphisms by epimorphisms,
(3) we replace $Y \times [0,1]$ by $Y^{[0,1]}$,
(4) for purely psychological reasons we replace $p\colon E \to B$ by $i\colon A \to X$.

We obtain the following diagram:

$$\begin{tikzcd}
Y & X \arrow[l, "\widetilde{f}"'] \arrow[dl, dashed, "\widetilde{F}"] \\
Y^{[0,1]} \arrow[u, two heads, "f\mapsto f(0)"] & A. \arrow[l, "f"] \arrow[u, "i"']
\end{tikzcd}$$

Finally recall that we had just shown in Example (3) that given two topological spaces A and B we have a natural identification

$$\mathrm{Mor}(A \times [0,1], B) \;=\; \mathrm{Mor}(A, B^{[0,1]}).$$

Thus (with a few other more or less mysterious changes) the previous challenge diagram translates into the following challenge diagram:

$$\begin{tikzcd}
X \arrow[r, "x\mapsto(x,0)"] \arrow[d, "\widetilde{f}"'] & X \times [0,1] \arrow[dl, dashed, "\widetilde{F}"] \\
Y & A \times [0,1]. \arrow[l, "f"] \arrow[u, "i"']
\end{tikzcd}$$

But this challenge diagram looks familiar: it is precisely the diagram that appears in the definition of a cofibration.

[1555] More precisely, if we consider this map in the category of sets, then it is obviously a bijection. Proposition 5.6 shows that the map f on the left is continuous if and only if the corresponding map on the right is continuous. This shows that we also have a bijection between the sets of continuous maps, i.e. between the sets of morphisms in the category of topological spaces.

We conclude this chapter with three quotes:

(1) "In the mathematical disciplines of algebraic topology and homotopy theory, Eckmann–Hilton duality in its most basic form, consists of taking a given diagram for a particular concept and reversing the direction of all arrows [...]."[1556] [1557]
(2) "Eckmann–Hilton duality can be extremely helpful as an organizational principle, reducing significantly what one has to remember, and providing valuable hints on how to proceed in various situations." [**Hat02**, p. 464]
(3) "In mathematics [...] currying[1558] is the technique of translating the evaluation of a function that takes multiple arguments into evaluating a sequence of functions, each with a single argument. For example, a function that takes two arguments, one from X and one from Y, and produces outputs in Z, by currying is translated into a function that takes a single argument from X and produces as outputs functions from Y to Z."[1559]

So we see that in the above discussion we performed Eckmann-Hilton duality combined with some currying.

Exercises for Chapter 112.

Exercise 112.1. Show that the projection map

$$\begin{array}{rcl} \overline{B}^2 & \to & \overline{B}^1 = [-1,1] \\ (x,y) & \mapsto & x \end{array}$$

is a Serre fibration.

Exercise 112.2. . Let $n \in \mathbb{N}_0$ and let x_0 be a point on S^n. It follows from Proposition 5.4 that the map

$$\begin{array}{rcl} e\colon P(S^n, x_0) & \to & S^n \\ \{f\colon [0,1] \to S^n \,|\, f(0) = x_0\} & \mapsto & f(1) \end{array}$$

is continuous. For which $n \in \mathbb{N}_0$ does e admit a left-inverse?

Exercise 112.3. We consider the map $f\colon S^1 \to [-1,1]$ that is given by the projection onto the x-coordinate. We define the mapping path fibration $q\colon P_f \to [-1,1]$ as in Proposition 112.17. Show that for each $y \in [-1,1]$ the preimage $q^{-1}(y)$ admits a deformation retraction to a circle.
Remark. See Figure 1581 for an illustration of the mapping path fibration.

[1556] `https://en.wikipedia.org/wiki/Eckmann-Hilton_duality`

[1557] Note that the notion of pullback and pushout are related by Eckmann-Hilton duality.

[1558] This notion is named after the American mathematician Haskell Brooks Curry (1900-1982) who was an American mathematician and logician. There are three programming languages named after him, namely Haskell, Brook and Curry.

[1559] `https://en.wikipedia.org/wiki/Currying#Algebraic_topology`

Exercise 112.4. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. Show "with your bare hands" that the map

$$\begin{array}{rcl} \partial_n \colon \pi_n(X, x_0) & \xrightarrow{\cong} & \pi_{n-1}(\Omega(X, x_0), c_{x_0}) \\ {[f \colon ([0,1]^n, \partial([0,1]^n)) \to (X, x_0)]} & \mapsto & \left[\begin{array}{rcl} ([0,1]^{n-1}, \partial([0,1]^{n-1})) & \to & (\Omega(X, x_0), c_{x_0}) \\ x & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ t & \mapsto & f(x,t) \end{array} \right) \end{array} \right] \end{array}$$

is a natural isomorphism.
Remark. This is of course the statement of Corollary 112.15.

Exercise 112.5. Let (X, x_0) be a pointed topological space. In Corollary 112.15 we give for any $n \geq 1$ a natural isomorphism

$$\pi_n(X, x_0) \xrightarrow{\cong} \pi_{n-1}(\Omega(X, x_0), c_{x_0}).$$

What does this statement mean for $n = 1$? It seems to say that $\pi_0(\Omega(X, x_0), c_{x_0})$ admits a natural group structure. Can you define such a natural group structure without using the above isomorphism?

Exercise 112.6. Convince yourself, without using Corollary 112.15, that the loop in the pointed topological space $(\Omega(S^2, N), c_N)$ that is shown in Figure 1580 is not null-homotopic.

Exercise 112.7. Let X be a topological space and let $x_0 \in X$ be a base point. Recall that the loop space $\Omega(X, x_0)$ is defined as the set $\Omega(X, x_0) = \{f \in X^{[0,1]} \,|\, f(0) = f(1) = x_0\}$ together with the compact-open topology. We denote by $\sim$ the equivalence relation on $\Omega(X, x_0)$ given by homotopy of paths. By definition we have $\pi_1(X, x_0) = \Omega(X, x_0)/\sim$. We equip $\pi_1(X, x_0)$ with the quotient topology. We can thus view $\pi_1(X, x_0)$ as a topological space. Show that $\pi_1(S^1, 1)$ is a discrete topological space.
Hint. It is possibly easier to show that for any pointed topological space (X, x_0) that admits a universal covering the fundamental group $\pi_1(X, x_0)$ is a discrete topological space.
Remark. There are examples of pointed topological spaces for which $\pi_1(X, x_0)$ is not a discrete topological space. For example the fundamental group of the Hawaiian Earrings, introduced in Exercise 25.1, is not a discrete topological space. We refer to **[Fab05, Fab11]** for details.

Exercise 112.8.

(a) Show that there is no functor

$$\Phi \colon \begin{array}{c}\text{category } \mathcal{PTop} \text{ of pointed} \\ \text{topological spaces}\end{array} \to \begin{array}{c}\text{category of } \mathcal{PTop} \text{ pointed} \\ \text{topological spaces}\end{array}$$

which admits for every $n \in \mathbb{N}$ a natural isomorphism

$$\pi_n(X, x_0) \xrightarrow{\cong} \pi_{n+1}(\Phi(X, x_0)).$$

(b) Show that there is no functor

$$\Psi \colon \text{category of topological spaces} \to \text{category of topological spaces}$$

which admits for every $n \in \mathbb{N}$ a natural isomorphism

$$\widetilde{\mathrm{H}}_n(X) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{n-1}(\Psi(X)).$$

Hint. In Exercise 43.4 (b) we proved the following statement. If X is a topological space with $\widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}$, then given any map $f\colon X \to X$ the induced map f_* on $\widetilde{\mathrm{H}}_0(X)$ is given by multiplication by some $\epsilon \in \{-1, 0, 1\}$.

113. Fiber bundles

In this chapter we will introduce fiber bundles. For us they will be the most important source of fibrations. In this and the following chapter one of our favorite games will be to find more and more fiber bundles to obtain lots of non-trivial information on homotopy groups of various topological spaces.

113.1. **The definition of fiber bundles.**

The following proposition says that being a Serre fibration is a "local property".

Proposition 113.1. *Let $p\colon Y \to B$ be a map between topological spaces. If for every $b \in B$ there exists an open neighborhood U of b such that $p\colon p^{-1}(U) \to U$ is a Serre fibration, then $p\colon Y \to B$ itself is a Serre fibration.*

Proof. Let $p\colon Y \to B$ be a map between topological spaces. In this proof we say $U \subset B$ is *small* if $p\colon p^{-1}(U) \to U$ is a Serre fibration.

Now assume that $p\colon Y \to B$ has the property that B can be covered by small sets. We want to show that this hypothesis implies that the map $p\colon Y \to B$ is a Serre fibration. Therefore let $n \in \mathbb{N}_0$, let $F\colon \overline{B}^n \times [0,1] \to B$ and let $\widetilde{f}\colon \overline{B}^n \times \{0\} \to Y$ be maps such that the rectangle in the following diagram commutes:

$$\begin{tikzcd}
\overline{B}^n \times \{0\} \arrow[r, "\widetilde{f}"] \arrow[d, hook] & Y \arrow[d, "p"] \\
\overline{B}^n \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & B.
\end{tikzcd}$$

We need to show that there exists a diagonal map $\widetilde{F}\colon \overline{B}^n \times [0,1] \to Y$ such that the above diagram commutes.

Claim. There exists a CW-structure X on $\overline{B}^n$ and real numbers $0 = s_0 < s_1 < \cdots < s_k = 1$ such that for each cell e of X and each $i \in \{0, \ldots, k-1\}$ the image $F(e \times [s_i, s_{i+1}])$ is small.

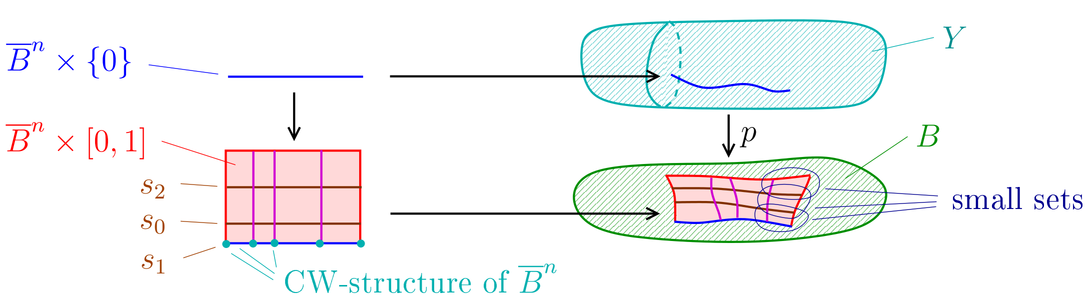


Figure 1584

By Lemma 41.1 we can identify $\overline{B}^n$ with $[0,1]^n$. Since B is by hypothesis covered by small open subsets and since any subset of a small set is again small we obtain from

Corollary 2.76 that there exists a $k \in \mathbb{N}$ such that each

$$F\Big(\underbrace{\left[\tfrac{a_1}{k}, \tfrac{a_1+1}{k}\right] \times \dots \left[\tfrac{a_n}{k}, \tfrac{a_n+1}{k}\right]}_{\subset [0,1]^n = \overline{B}^n} \times \underbrace{\left[\tfrac{a_{n+1}}{k}, \tfrac{a_{n+1}+1}{k}\right]}_{\subset [0,1]}\Big)$$

is small for all $a_1, \dots, a_{n+1} \in \{0, \dots, k-1\}$. We equip $[0,1]^n = \overline{B}^n$ with the CW-structure defined on page 962 given by the "subcubes" of side length $\frac{1}{k}$. Furthermore we set $s_i = \frac{i}{k}$, $i = 0, \dots, k$. By design this CW-structure and this choice of s_i has the desired properties. ⊞

We continue with the following claim.

Claim. Let $i \in \{0, \dots, k-1\}$. Given a map $\widetilde{F}_i \colon \overline{B}^n \times \{s_i\} \to Y$ with $p \circ \widetilde{F}_i = F|_{\overline{B}^n \times \{s_i\}}$ there exists a map $\widetilde{F}_{i+1} \colon \overline{B}^n \times [s_i, s_{i+1}] \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
\overline{B}^n \times \{s_i\} \arrow[r, "\widetilde{F}_i"] \arrow[d, hook] & Y \arrow[d, "p"] \\
\overline{B}^n \times [s_i, s_{i+1}] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}_{i+1}"] & B.
\end{tikzcd}$$

In fact the claim follows immediately from the argument of the proof of Proposition 112.8 applied to the CW-complex $K := X = \overline{B}^n$ and the subcomplex $L = \varnothing$. Indeed, in the proof of Proposition 112.8 we did not need to assume that $p \colon Y \to B$ is a Serre fibration, all we ever used is that for every cell e of B the image $F(e)$ is small in the above sense. But in our present setting this is guaranteed by our choice of the CW-structure on $\overline{B}^n$. ⊞

We set $\widetilde{F}_0 = \widetilde{f}_0 \colon \overline{B}^n \times \{0\} \to Y$. Iteratively applying the previous claim we obtain maps $\widetilde{F}_i \colon \overline{B}^n \times [s_i, s_{i+1}] \to Y$ that agree on the overlaps of the domains. By Lemma 14.3 all these maps together define a continuous map $\widetilde{F} \colon \overline{B}^n \times [0,1] \to Y$ which by construction has the desired properties. ■

As we saw on page 2643, every projection $p \colon B \times F \to B$ is a (Serre) fibration. This observation, together with Proposition 113.1 leads us naturally to the following definition.

Definition. Let F be a topological space.

(1) A map $p \colon Y \to B$ between two topological spaces is called an *F-bundle over B* if given any $b \in B$ there exists an open neighborhood U of b and a homeomorphism $\Phi \colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "(x,f)\mapsto x"] \\
& U. &
\end{tikzcd}$$

Alternatively we call such a map $p \colon Y \to B$ a *bundle projection* with *fiber F*. As for fibrations we sometimes refer to B as the *base space* and to Y as the *total space*. Furthermore we refer to the map $p \colon Y \to B$ as the *projection*.

(2) We say that two F-bundles $p\colon Y \to B$ and $q\colon Z \to B$ are *equivalent* if there exists a homeomorphism $f\colon Y \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & B. & \end{tikzcd}$$

The following proposition will be the incarnation of Theorem 112.7 that we will use most often.

Proposition 113.2. *Let $p\colon Y \to B$ be a fiber bundle with fiber F.*

(1) *The map $p\colon Y \to B$ is a Serre fibration.*

(2) *We pick $y_0 \in Y$ and we write $b_0 = p(y_0)$. We pick an identification $F \xrightarrow{\cong} p^{-1}(b_0)$. We denote by $i\colon F = p^{-1}(b_0) \to Y$ the inclusion. We use the fact from (1) that $p\colon Y \to B$ is a Serre fibration to define maps $\partial_n\colon \pi_n(B, b_0) \to \pi_{n-1}(F, y_0)$, $n \in \mathbb{N}$, as in Theorem 112.7. The following sequence is exact:*

$$\begin{array}{l} \dots \xrightarrow{\partial_{n+1}} \pi_n(F,y_0) \xrightarrow{i_*} \pi_n(Y,y_0) \xrightarrow{p_*} \pi_n(B,b_0) \xrightarrow{\partial_n} \\ \xrightarrow{\partial_n} \pi_{n-1}(F,y_0) \xrightarrow{i_*} \dots\dots\dots \xrightarrow{p_*} \pi_1(B,b_0) \xrightarrow{\partial_1} \\ \xrightarrow{\partial_1} \pi_0(F,y_0) \xrightarrow{i_*} \pi_0(Y,y_0) \xrightarrow{p_*} \pi_0(B,b_0). \end{array}$$

(3) *If we are given a commutative diagram*

$$\begin{tikzcd} Y' \arrow[r, "g"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

where $p'\colon Y' \to B'$ is a fiber bundle with fiber F', then given any base point $y_0' \in Y'$ the obvious diagram of the long exact sequences of homotopy groups corresponding to the fiber bundles $p\colon Y \to B$ and $p'\colon Y' \to B'$ is commutative.

(4) *If the fiber F and the base space B are path-connected, then the total space Y is also path-connected.*

Proof.

(1) This part of the proposition is an immediate consequence of the definitions, Proposition 113.1 and the discussion on page 2643.
(2) The second statement follows immediately from (1) and Theorem 112.7 (2).
(3) This statement follows easily from Theorem 112.7 (1b).
(4) This statement follows immediately from (2). ■

Proposition 113.2 sparks our interest in fiber bundles. Fortunately, in the following discussion we will see that fiber bundles are ubiquitous and can be found throughout the notes.

Examples.

(1) For any two topological spaces B and F the projection map $p\colon B\times F\to B$ onto the first factor is evidently a fiber bundle with fiber F. Any F-bundle that is equivalent to such a bundle is called *trivial*, otherwise it is called non-trivial.
(2) The "obvious projection map" from the Möbius band to the circle S^1, i.e. the map

$$\begin{array}{rcl} Y=([0,1]\times[-1,1])/(0,y)\sim(1,-y) & \to & S^1 \\ (x,y) & \mapsto & e^{2\pi \mathrm{i} x} \end{array}$$

is easily seen to be a fiber bundle with fiber $[-1,1]$. Since the Möbius band is not homeomorphic to the trivial bundle $S^1\times[-1,1]$ we see that the bundle is non-trivial.
(3) The same argument as in (2) shows that the Klein bottle is a non-trivial S^1-bundle over S^1. In Lemma 8.33 (1) we saw that the connected sum $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ is homeomorphic to the Klein bottle. We have thus shown that $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ is a non-trivial S^1-bundle over S^1. Amusingly an analogous statement holds in dimension four, namely the connected sum $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}^2}$ is a non-trivial S^2-bundle over S^2. It is a challenging exercise to prove this fact by hand, alternatively a proof can be found in [**GoS99**, p. 107] or [**Scor05**, p. 124]. Similarly, one can show that the connected sum $\mathbb{H}\mathrm{P}^2\#\overline{\mathbb{H}\mathrm{P}^2}$ is a non-trivial S^4-bundle over S^4.
(4) An n-dimensional vector bundle $p\colon W\to B$, in the sense of the definition on page **??**, is by definition an $\mathbb{R}^n$-bundle over B.

We consider the next example in greater detail.

Lemma 113.3.

(1) *Let X be a topological space and let $f\colon X\to X$ be a homeomorphism. The natural projection*

$$\begin{array}{rcl} \overbrace{\mathrm{Tor}(X,f)=([0,1]\times X)/(0,x)\sim(1,f(x))}^{\text{basically the mapping torus of }(X,f)} & \to & [0,1]/0\sim 1=S^1 \\ [(t,x)] & \mapsto & [t]=e^{2\pi \mathrm{i} t} \end{array}$$

is an X-bundle over S^1.
(2) *Let X be a topological space and let $f_0,f_1\colon X\to X$ be two homeomorphisms. If f_0 and f_1 are isotopic, then the two X-bundles $\mathrm{Tor}(X,f_0)\to S^1$ and $\mathrm{Tor}(X,f_1)\to S^1$ are equivalent.*
(3) *Let V be an n-dimensional real vector space and let $\varphi\colon V\to V$ be an isomorphism of vector spaces. We equip V with the topology given on page 116. For each $t\in[0,1)$ we equip $\{t\}\times V\subset M(V,\varphi)$ with the obvious vector space structure. This turns the projection map $M(V,\varphi)\to S^1$ into an n-dimensional vector bundle.*

Proof.

(1) We denote by $\pi\colon [0,1]\to[0,1]/0\sim 1$ the obvious projection map. We consider the two open sets $U_1:=\pi((\frac14,\frac34))$ and $U_2:=\pi([0,\frac13)\cup(\frac23,1])$ in $[0,1]/0\sim 1$. Furthermore

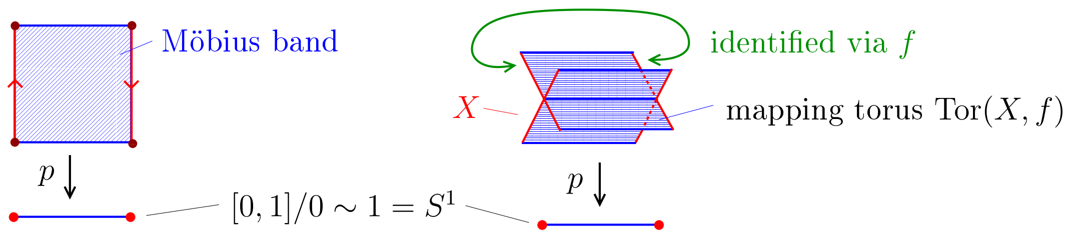


FIGURE 1585

we consider the maps

$$\begin{array}{rcl} p^{-1}(U_1) & \xrightarrow{\Phi_1} & U_1 \times X \\ [(t,x)] & \mapsto & (t,x) \end{array} \quad \text{and} \quad \begin{array}{rcl} p^{-1}(U_2) & \xrightarrow{\Phi_2} & U_2 \times X \\ [(t,x)] & \mapsto & \begin{cases} (t, f(x)), & \text{if } t \in [0, \frac{1}{3}), \\ (t,x) & \text{if } t \in (\frac{2}{3}, 1]. \end{cases} \end{array}$$

Note that the map on the right is indeed well-defined. Using Lemmas 3.22 and 14.3 one can show quite easily that both maps are homeomorphisms. Furthermore it is basically clear that Φ_1 and Φ_2 respect the projections onto U_1 and U_2. Therefore, since the two open subsets U_1 and U_2 cover all of $[0,1]/0 \sim 1$, we have shown that $p\colon \operatorname{Tor}(X,f) \to [0,1]/0 \sim 1$ is indeed an X-bundle.

(2) This statement will be proved in Exercise 113.3.

(3) The proof of this statement is basically the same as the proof of (1). We only have to make the extra observation that for each $i \in \{1,2\}$ and each $t \in U_i$ the map

$$\Phi_i \colon V = [\{t\} \times V] = p^{-1}(t) \ \to \ \{t\} \times V = V$$

is either given by id or ϕ. Thus in either case the map is given by an *isomorphism* of vector spaces. ■

Remark. The previous lemma shows that any self-homeomorphism of $\mathbb{R}^n$ gives rise to an $\mathbb{R}^n$-bundle over S^1, whereas any isomorphism of $\mathbb{R}^n$ gives rise to an n-dimensional vector bundle over S^1. The question arises whether every $\mathbb{R}^n$-bundle is secretly a vector bundle. More precisely, the question arises whether every $\mathbb{R}^n$-bundle is equivalent to an n-dimensional vector bundle. In Exercise 113.4 we will answer this question in the affirmative if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism.[1560]

It follows from the discussion on page 714 and Lemma 113.3 that the complement of the trefoil and the complement of the figure-8 knot are fiber bundles over S^1. Of course there are many other knots, for example consider the stevedore knot shown in Figure 1586. The following question arises:

Question 113.4. *Is the complement of the stevedore knot a fiber bundle over S^1?*

We continue with more examples of fiber bundles.

Examples.

[1560] Frankly, I am not sure whether the same statement holds if f is a homeomorphism.

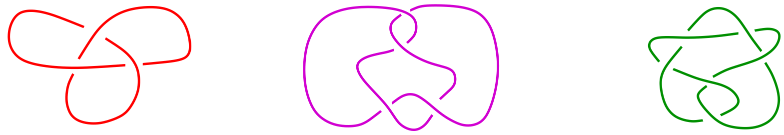


FIGURE 1586

(1) As we pointed out above, any n-dimensional vector bundle $p\colon W \to B$ is by definition an $\mathbb{R}^n$-bundle over B. In fact p gives rise to two other bundles which are much more interesting topological objects:
   (a) We denote by $s\colon B \to W$ the zero-section, i.e. the map given by sending each $b \in B$ to the zero-element of the vector space $p^{-1}(y)$. We write $W^* := W \setminus s(B)$. It follows easily from the definitions that $p\colon W^* \to B$ is a fiber bundle with fiber $\mathbb{R}^n \setminus \{0\}$.
   (b) Let $\lambda \in \mathbb{R}$. By definition of a vector bundle we have a well-defined multiplication by λ on each fiber. We define

$$SW \;:=\; W^*/\sim \qquad \text{where } v \sim w \text{ if } p(v) = p(w) \text{ and}$$
$$\qquad\qquad\qquad\qquad \text{if there exists a } \lambda > 0 \text{ with } \lambda v = w.$$

   The map $p\colon W^* \to B$ descends to a map $p\colon SW \to B$. We leave it as an entertaining exercise to verify that $p\colon SW \to B$ is an S^{n-1}-bundle.

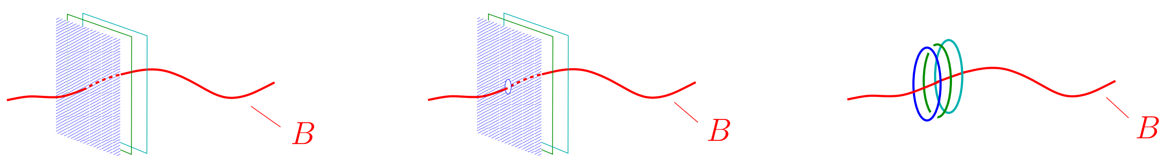


FIGURE 1587

(2) A smooth $\overline{B}^n$-bundle $p\colon W \to M$ over a smooth manifold M, in the sense of the definition on page 416, is by definition an $\overline{B}^n$-bundle in the above sense. Slightly more interesting, if M has no boundary and if we restrict p to a map $\partial W \to M$, then it follows easily from Proposition 80.9 (1) that the restriction of p to ∂W defines an S^{n-1}-bundle $p\colon \partial W \to M$.

The following lemma is almost verbatim the same as Lemma **??**.

Lemma 113.5. *Let $g\colon C \to B$ be a map between topological spaces and let $p\colon Y \to B$ be a fiber bundle with fiber F. As in Lemma 25.16 (1) we consider the pullback*

$$g^*Y \;:=\; \{(c,f) \,|\, c \in C \text{ and } f \in p^{-1}(g(c))\} \;\subset\; C \times Y,$$

which is equipped with the subspace topology. Then the map

$$\begin{array}{rcl} q\colon g^*Y \;:=\; \{(c,f) \,|\, c \in C \text{ and } f \in p^{-1}(g(c))\} & \to & C \\ (c,f) & \mapsto & c \end{array}$$

is also an F-bundle.

Proof. The proof is, not surprisingly, also almost identical to the proof of Lemma **??**. ■

Remark. The pullback of bundles satisfies basically the same statements as the pullback of vector bundles, see Lemma **??**. There is surely no need to spell out these properties again.

The following lemma classifies bundles over compact intervals $[a, b]$ and also over the circle S^1.

Lemma 113.6.

(1) *Every bundle over a compact interval* $[a, b]$ *is trivial.*
(2) *If* $p\colon Y \to S^1$ *is a fiber bundle with fiber* X*, then there exists a homeomorphism* $f\colon X \to X$ *such that* p *is equivalent to the above fiber bundle* $\mathrm{Tor}(X, f) \to S^1$.

Sketch of proof.

(1) This statement will be proved in Exercise 113.2.
(2) Suppose we are given a fiber bundle $p\colon Y \to S^1$ with fiber X. Let $\varphi\colon [0, 2\pi] \to S^1$ be the map that is given by $\varphi(t) = e^{\mathrm{i}t}$. It follows from (1) that the pullback bundle

$$q\colon \varphi^*Y = \{(t, v) \in [0, 2\pi] \times Y \,|\, v \in p^{-1}(\varphi(t))\} \;\to\; [0, 2\pi]$$

is trivial, i.e. there exists a homeomorphism $\Phi\colon [0, 2\pi] \times X \to \varphi^*Y$ such that the following diagram commutes

$$\begin{tikzcd}
{[0,2\pi]\times X} \arrow[rr, "\Phi"] \arrow[dr, "(t,x)\mapsto t"'] & & \varphi^*Y \arrow[dl, "q"] \\
& {[0,2\pi].} &
\end{tikzcd}$$

We define $f\colon X \to X$ to be the homeomorphism that is given by

$$\begin{array}{ccccccccc}
 & & & & \varphi(0)=1=\varphi(2\pi) & & & & \\
 & & & & \downarrow & & & & \\
X & \xrightarrow{x\mapsto(0,x)} & \{0\}\times X & \xrightarrow{\Phi} q^{-1}(0) \;\to & p^{-1}(1) & \leftarrow\; q^{-1}(2\pi) & \xleftarrow{\Phi} \{2\pi\}\times X & \xrightarrow{(2\pi,x)\mapsto x} & X. \\
 & & & (0,v) \;\mapsto & v & & & & \\
 & & & & v & \hookleftarrow\; (2\pi,v). & & &
\end{array}$$

We leave it as a mildly interesting exercise to the reader to verify that $p\colon Y \to S^1$ is equivalent to the fiber bundle $\mathrm{Tor}(X, f) \to S^1$. ■

For the record we mention the following generalization of Lemma 113.6 (1).

Proposition 113.7. *Let* X *be a topological space. If* X *is paracompact and contractible, then every bundle over* X *is trivial.*

Proof. The statement of the proposition is almost the same as the statement of Proposition **??** (3) for vector bundles. The reference we gave for vector bundles, namely **[Hat2**, Theorem 1.6], also applies to general bundles. ■

We conclude this section with an example that shows how the above results can be used in practice. Thus let $n \in \mathbb{N}$ and K be a closed k-dimensional submanifold of the smooth manifold S^n. We calculate that for $i = 1, \ldots, n-k-2$ we have

$$\mathrm{H}_i(S^n \setminus K; \mathbb{Z}) \underset{\substack{\uparrow \\ \text{Alexander Duality} \\ \text{Theorem 92.1}}}{\cong} \mathrm{H}^{n-i-1}(K; \mathbb{Z}) \underset{\substack{\uparrow \\ \text{by Proposition 75.15 (5) and the} \\ \text{fact that } n-i-1 > k = \dim(K)}}{=} 0.$$

The following proposition shows that not only do the first $n-k-2$ homology vanish, but the first $n-k-2$ homotopy groups also vanish.

Proposition 113.8. $(*)$ *Let $n \in \mathbb{N}$ and let K be a closed k-dimensional submanifold of the smooth manifold S^n. For $i = 1, \ldots, n-k-2$ we have $\pi_i(S^n \setminus K) = 0$.*

Example. Let K be the trefoil in $\mathbb{R}^3$ and let $\mu \subset \mathbb{R}^3$ be a meridian as defined on page 390. (See also Figure 1588 to the left.) In Lemma 46.14 we saw that μ represents a non-zero element in $\mathrm{H}_1(S^3 \setminus K)$. It follows from the naturality of the Hurewicz homomorphism, see Proposition 52.2 (4), that μ also represents a non-trivial element in $\pi_1(S^3 \setminus K)$. Via the usual smooth embedding $\mathbb{R}^3 \to \mathbb{R}^3 \times \{0\} \to \mathbb{R}^4$ we can view K and μ as submanifolds of $S^4 = \mathbb{R}^4 \cup \{\infty\}$. By Proposition 113.8 we have $\pi_1(S^4 \setminus K) = 0$. In particular we obtain that μ is trivial in $\pi_1(S^4 \setminus K)$. In fact in Figure 1588 we show an embedded disk D in $\mathbb{R}^4 \subset S^4$ whose boundary is given by μ and that does not intersect K. Here we use the convention from page 437, i.e. "pinkness" of a point measures how large is the fourth coordinate.

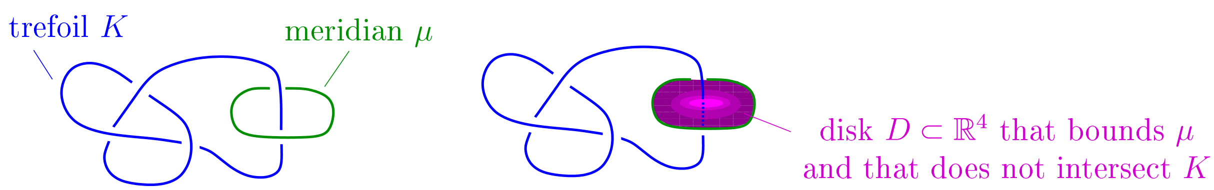


FIGURE 1588

PROOF $(*)$. Let $n \in \mathbb{N}$ and K be a closed k-dimensional submanifold of the smooth manifold S^n. If $n-k-2 \leq 0$, then there is nothing to prove. Thus we assume that $n-k-2 \geq 1$, i.e. we assume that $n \geq k+3$.

We want to show that $\pi_i(S^n \setminus K) = 0$ for $i = 1, \ldots, n-k-2$. We already know that the corresponding homology groups vanish. Thus by Corollary 53.7 it suffices to show that $\pi_1(S^n \setminus K) = 0$. That is precisely what we will do in the remainder of this proof.

In the following we consider the case that K is connected. We leave it to the reader to modify the proof to deal with the case that K is disconnected. By the General Tubular Neighborhood Theorem 10.5 the closed submanifold K admits a tubular neighborhood $p\colon W \to K$. Recall that W is a compact n-dimensional submanifold of S^n and that p is in particular a $\overline{B}^{n-k}$-bundle over K. As we pointed out on page 2684, the projection $p\colon \partial W \to K$ is an S^{n-k-1}-bundle.

We write $X = S^n \setminus \mathring{W}$. By Proposition 10.11 and Lemma 10.1 we know that $\pi_1(S^n \setminus K)$ is isomorphic to $\pi_1(X)$. Thus it suffices to show that X is simply connected.

By Proposition 6.30 we know that X is a compact n-dimensional submanifold of S^n with $W \cap X = \partial W = \partial X$. From Proposition 113.2 (4) we know that ∂W is connected. We pick a base point $x_0 \in \partial W$. By the Seifert–van Kampen Theorem 22.2 for smooth manifolds we know that the inclusion induced map

$$\pi_1(X, x_0) *_{\pi_1(\partial W, x_0)} \pi_1(W, x_0) \;\to\; \underbrace{\pi_1(S^n, x_0)}_{=0}$$

is an isomorphism. We recall that Lemma 21.22 says that $A *_G G \cong A$. Thus, if we want to show that $\pi_1(X, x_0) = 0$ it follows from Lemma 21.22 that it suffices to prove the following claim.

Claim. The inclusion induced map $\pi_1(\partial W, x_0) \to \pi_1(W, x_0)$ is an isomorphism.

We denote both of the inclusions $\partial W \to W$ and $S^{n-k-1} \to \overline{B}^{n-k}$ by i. We have a commutative diagram

$$\begin{tikzcd} \partial W \arrow[rr, "i"] \arrow[dr] & & W \arrow[dl] \\ & K & \end{tikzcd}$$

of fiber bundles with fibers S^{n-k-1} and $\overline{B}^{n-k}$. From Proposition 113.2 (3) we obtain the following commutative diagram of long exact sequences of morphisms between pointed sets:

$$\begin{tikzcd} \dots \arrow[r] & \pi_2(K) \arrow[r, "\partial"] \arrow[d, "="] & \pi_1(S^{n-k-1}) \arrow[r] \arrow[d, "i_*"] & \pi_1(\partial W) \arrow[r] \arrow[d, "i_*"] & \pi_1(K) \arrow[r, "\partial"] \arrow[d, "="] & \pi_0(S^{n-k-1}) \arrow[r] \arrow[d, "i_*"] & \dots \\ \dots \arrow[r] & \pi_2(K) \arrow[r, "\partial"] & \pi_1(\overline{B}^{n-k}) \arrow[r] & \pi_1(W) \arrow[r] & \pi_1(K) \arrow[r, "\partial"] & \pi_0(\overline{B}^{n-k}) \arrow[r] & \dots \end{tikzcd}$$

Since $n - k \geq 3$ we have $n - k - 1 \geq 2$, thus $\pi_1(S^{n-k-1}) = 0$ by Proposition 14.14. It follows from a slight generalization[1561] of the Five Lemma 43.12 that the middle vertical map $i_*\colon \pi_1(\partial W) \to \pi_1(W)$ is an isomorphism. ■

Remark. We want to sketch an alternative approach to proving Proposition 113.8. Thus let $n \in \mathbb{N}$, let K be a closed k-dimensional submanifold of the smooth manifold S^n and finally let $i \in \{1, \dots, n-k-2\}$. We want to show that $\pi_i(S^n \setminus K, x_0) = 0$ for some base point $x_0 \in S^n \setminus K$. Let $[\varphi\colon (S^i, *) \to (S^n \setminus K, x_0)] \in \pi_i(S^n \setminus K, x_0)$ be a random element. We make the following observations:

(1) By the Whitney Approximation Theorem 9.3 we can assume that φ is smooth.
(2) Since $i < n$ we know that $\pi_i(S^n, x_0) = 0$. It follows from Lemma 40.4 that there exists a map $\psi\colon \overline{B}^{i+1} \to S^n$ with $\psi|_{S^i} = \varphi$.
(3) Once again, by the Whitney Approximation Theorem 9.3 we can assume that ψ is also smooth.

[1561]More precisely, if one goes through the proof of the Five Lemma 43.12, then one realizes that it suffices that the right most vertical map is an injective morphism of pointed sets, i.e. the right most vertical map does not need to be a homomorphism of groups.

(4) Since $k+i+1 < n$ one would like to argue, similar to the Transversality Theorem 9.10, that we can homotop ψ rel S^i to a map that does not intersect K, i.e. we would like to homotop ψ rel S^i to a map $\overline{B}^{i+1} \to S^n \setminus K$. By Lemma 40.4 this would then imply that $[\varphi] = 0 \in \pi_i(S^n \setminus K, x_0)$.

This approach can be turned into a proper proof, but we will not do so.

113.2. **Hopf fibrations arising from complex and quaternionic projective spaces.** In this section we will see that the complex projective spaces $\mathbb{C}\mathrm{P}^n$ and the quaternionic projective spaces[1562] $\mathbb{H}\mathrm{P}^n$ give rise to interesting fiber bundles. In particular we will see that the Hopf map $S^3 \to S^2$ that we introduced on page 1071 is in fact an S^1-bundle.

In the following we introduce two straightforward higher-dimensional analogues of the original Hopf map.

Definition. Let $n \in \mathbb{N}$. We refer to the maps

$$\begin{array}{rcl} p\colon S^{2n+1} = \{(z_0,\dots,z_n) \in \mathbb{C}^{n+1} \,|\, |z_0|^2 + \dots + |z_n|^2 = 1\} & \to & \mathbb{C}\mathrm{P}^n \\ (z_0,\dots,z_n) & \mapsto & [z_0 : \dots : z_n] \end{array}$$

and[1563]

$$\begin{array}{rcl} q\colon S^{4n+3} = \{(h_0,\dots,h_n) \in \mathbb{H}^{n+1} \,|\, |h_0|^2 + \dots + |h_n|^2 = 1\} & \to & \mathbb{H}\mathrm{P}^n \\ (h_0,\dots,h_n) & \mapsto & [h_0 : \dots : h_n] \end{array}$$

as *Hopf maps.*

The following proposition, together with Proposition 113.2, explains our interest in the Hopf maps.[1564]

Proposition 113.9. *Let $n \in \mathbb{N}$.*
(1) *The Hopf map $p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is an S^1-bundle.*
(2) *The Hopf map $q\colon S^{4n+3} \to \mathbb{H}\mathrm{P}^n$ is an S^3-bundle.*

Proof. We only provide the proof of (2). The proof of (1) is a straightforward modification of the proof of (2). Given $j \in \{0,\dots,n\}$ we define

$$V_j \;:=\; \{[h_0 : \dots : h_n] \in \mathbb{H}\mathrm{P}^n \,|\, h_j \neq 0\}.$$

The same argument as in the proof of Lemma 12.5 shows that V_j is an open subset of $\mathbb{H}\mathrm{P}^n$.

Claim. The map

$$\begin{array}{rcl} \Phi_j\colon q^{-1}(V_j) = \Big\{(h_0,\dots,h_n) \in \mathbb{H}^{n+1} \,\Big|\, \sum\limits_{i=0}^{n} |h_i|^2 = 1 \text{ and } h_j \neq 0\Big\} & \to & V_j \times S^3 \\ (h_0,\dots,h_n) & \mapsto & \big([h_0 : \dots : h_n], \frac{1}{|h_j|}\cdot h_j\big) \end{array}$$

is a homeomorphism.

[1562] We refer to page 1472 for the slightly subtle definition of the quaternionic projective spaces.

[1563] Recall that on page 1462 we defined the norm of a quaternion $h = a\cdot 1 + b\cdot i + c\cdot j + d\cdot k$ as $|h| = \sqrt{a^2+b^2+c^2+d^2}$.

[1564] For mysterious reasons the Hopf maps are almost invariably called "Hopf fibrations", even though the name "Hopf fiber bundle" would be more precise.

We consider the map

$$\begin{array}{rcl}\Psi_j\colon V_j \times S^3 & \to & q^{-1}(V_j) \\ ([h_0 : \dots : \underbrace{h_j}_{\neq 0} : \dots : h_n], u) & \mapsto & \begin{array}{l}\text{the normalization to length one of the non-zero vector} \\ \big(u \cdot h_j^{-1} \cdot h_0, \dots, \underbrace{u \cdot h_j^{-1} \cdot h_j}_{=u}, \dots, u \cdot h_j^{-1} \cdot h_n\big) \in \mathbb{H}^{n+1}.\end{array}\end{array}$$

This map Ψ_j is well-defined. Indeed, a short calculation shows that if we pick a different representative for $[h_0 : \dots : h_n]$, i.e. if we plug in $[\lambda \cdot h_0 : \dots : \lambda \cdot h_n]$ for some $\lambda \in \mathbb{H} \setminus \{0\}$ then we obtain the same result.[1565] Using Lemma 3.22 one can show that the map Ψ_j is continuous. One easily verifies that Ψ_j and Φ_j are inverses of one another. ⊞

It is clear that the open sets V_j cover all of $\mathbb{H}\mathrm{P}^n$ and that the triangles

$$\begin{tikzcd} q^{-1}(V_j) \arrow[rr, "\Phi_j"] \arrow[dr, "q"'] & & V_j \times S^3 \arrow[dl, "(v,h)\mapsto h"] \\ & S^3 & \end{tikzcd}$$

commute. This shows, according to the definition on page 2680, that $q\colon S^{4n+3} \to \mathbb{H}\mathrm{P}^n$ is an S^3-bundle. ■

Proposition 113.9, combined with Proposition 113.2 gives us two interesting theorems.

Theorem 113.10.
(1) *For all $n \geq 3$ we have $\pi_n(S^3) \cong \pi_n(S^2)$.*
(2) *We have $\pi_3(S^2) \cong \mathbb{Z}$ and $\pi_3(S^2)$ is generated by the Hopf map $S^3 \to S^2$.*

The second statement gives in particular an answer to Question 91.15 (1).

PROOF.

(1) On page 197 we gave an identification of $\mathbb{C}\mathrm{P}^1$ with S^2. Thus Proposition 113.9 (1), applied to $n = 1$, tells us that the Hopf map $p\colon S^3 \to S^2 = \mathbb{C}\mathrm{P}^1$ defines an S^1-bundle. Proposition 113.2 gives us the long exact sequence[1566]

$$\dots \to \pi_n(S^1) \to \pi_n(S^3) \xrightarrow{p_*} \pi_n(S^2) \xrightarrow{\partial} \pi_{n-1}(S^1) \to \pi_{n-1}(S^3) \to \dots$$

The desired statement now follows from the example on page 1068 where we saw that $\pi_k(S^1) = 0$ for $k \geq 2$.

(2) In Corollary 53.6 (1) we saw that $\pi_3(S^3) = \mathbb{Z} \cdot [\mathrm{id}_{S^3}]$. By (1) the Hopf map $p\colon S^3 \to S^2 = \mathbb{C}\mathrm{P}^1$ induces an isomorphism $p_*\colon \pi_3(S^3) \xrightarrow{\cong} \pi_3(S^2)$. The image of $[\mathrm{id}_{S^3}]$ under

[1565] Note though that the order of the factors in the definition of Ψ_j is important since $\mathbb{H}$ is not commutative.

[1566] It is actually worth having a glance at the lower end of this sequence. We have the exact sequence

$$\pi_2(S^3) \xrightarrow{p_*} \pi_2(S^2) \xrightarrow{\partial} \pi_1(S^1) \to \pi_1(S^3).$$

From Proposition 40.10 we know that $\pi_1(S^3) = \pi_2(S^3) = 0$ and Proposition 16.17 we know that $\pi_1(S^1) \cong \mathbb{Z}$. Thus we obtain that $\pi_2(S^2) \cong \mathbb{Z}$. In Exercise 113.12 we will see that using this exact sequence we can obtain a new proof that id_{S^2} is a generator of $\pi_2(S^2)$.

p_* is by definition precisely the class in $\pi_3(S^2)$ represented by the Hopf map. Thus we see that $\pi_3(S^2) \cong \mathbb{Z}$ and that $\pi_3(S^2)$ is generated by the Hopf map. ■

Theorem 113.11.

(1) *For all $n \geq 2$ we have $\pi_n(S^4) \cong \pi_n(S^7) \oplus \pi_{n-1}(S^3)$.*

(2) *For $n \in \{1, \ldots, 6\}$ we have $\pi_n(S^4) \cong \pi_{n-1}(S^3)$.*

Remark. From Corollary 53.6 (1) we know that $\pi_7(S^7) \cong \mathbb{Z}$. Thus it follows from Theorem 113.11 (1) that $\pi_7(S^4) \cong \pi_7(S^7) \oplus \pi_6(S^3)$ admits an epimorphism onto $\mathbb{Z}$. In fact in Corollary 91.8 we already saw that the Hopf invariant is an epimorphism $\pi_7(S^4) \to \mathbb{Z}$.

Proof.

(1) By Lemma 60.9 (2) we can identify the quaternionic projective space $\mathbb{H}\mathrm{P}^1$ with S^4. Therefore if we apply Proposition 113.9 (2) with $n = 1$ we obtain an S^3-bundle $q\colon S^7 \to \mathbb{H}\mathrm{P}^1 = S^4$. We pick $y_0 \in S^7$ and we write $b_0 = q(y_0)$. We pick an identification $S^3 = q^{-1}(b_0)$ and we denote by $i\colon S^3 \to S^7$ the map that is given by the inclusion of the fiber $F = S^3$ into S^7. From Proposition 113.2 we obtain the following long exact sequence
$$\cdots \to \pi_n(S^3, y_0) \xrightarrow{i_*} \pi_n(S^7, y_0) \xrightarrow{q_*} \pi_n(S^4, b_0) \xrightarrow{\partial} \pi_{n-1}(S^3, y_0) \xrightarrow{i_*} \pi_{n-1}(S^7, y_0) \to \ldots .$$
By Proposition 40.10 we know that $\pi_3(S^7, y_0) = 0$. By definition this implies that the map $i\colon S^3 \to S^7$ is homotopic rel y_0 to the constant map $c_{y_0}\colon S^3 \to \{y_0\}$. By Theorem 112.7 (3) this implies that the above long exact sequence splits into short exact sequences
$$0 \to \pi_n(S^7, y_0) \xrightarrow{q_*} \pi_n(S^4, b_0) \xrightarrow{\partial} \pi_{n-1}(S^3, y_0) \to 0.$$
Furthermore Theorem 112.7 (3) says that these short exact sequences split, therefore we obtain the promised isomorphisms $\pi_n(S^4, b_0) \cong \pi_n(S^7, y_0) \oplus \pi_{n-1}(S^3, y_0)$.

(2) The statement holds trivially Proposition 40.10 for $n = 1$. For $n \geq 2$ this statement follows from (1) and Proposition 40.10 which says in this context that $\pi_n(S^7) = 0$ for $n = 1, \ldots, 6$. ■

As we will see, Proposition 113.2 is a goose that just cannot stop laying golden eggs. For example we have the following theorem.

Theorem 113.12. *We use the homeomorphism from page 197 to make the identification $\mathbb{C}\mathrm{P}^1 = S^2$ and as on page 194 we view $\mathbb{C}\mathrm{P}^1$ as a subset of each $\mathbb{C}\mathrm{P}^n$ and of $\mathbb{C}\mathrm{P}^\infty$.*

(1) *Let $n \in \mathbb{N}$. We denote by $p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ the Hopf map from page 2688. We have*
$$\pi_k(\mathbb{C}\mathrm{P}^n) = \begin{cases} \mathbb{Z} \cdot [\mathrm{id}_{\mathbb{C}\mathrm{P}^1}], & \text{if } k = 2, \\ 0, & \text{if } k = 1, 3, 4, 5, \ldots, 2n, \\ \mathbb{Z} \cdot [p], & \text{if } k = 2n+1. \end{cases}$$

(2) *We have*
$$\pi_k(\mathbb{C}\mathrm{P}^\infty) = \begin{cases} \mathbb{Z} \cdot [\mathrm{id}_{\mathbb{C}\mathrm{P}^1}], & \text{if } k = 2, \\ 0, & \text{if } k = 1, 3, 4, 5, 6, \ldots . \end{cases}$$

Remark. Let $k < l$. In Question 53.4 we had asked whether the complex projective space $\mathbb{C}\mathrm{P}^k$ is a retract of $\mathbb{C}\mathrm{P}^l$. In Proposition 90.14 we used the calculation of the cup product to give a negative answer. Now we can also use homotopy groups for the same purpose. More precisely, from Theorem 113.12, we obtain that $\pi_{2k+1}(\mathbb{C}\mathrm{P}^k) \cong \mathbb{Z}$ but that $\pi_{2k+1}(\mathbb{C}\mathrm{P}^l) = 0$. The usual argument, see Lemma 15.7 implies that once again we obtain a negative answer to Question 53.4, i.e. $\mathbb{C}\mathrm{P}^k$ is not a retract of $\mathbb{C}\mathrm{P}^l$.

Proof.

(1) By Proposition 113.9 (1) we know that for every $n \in \mathbb{N}$ the Hopf map $p\colon S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is an S^1-bundle. Thus we obtain from Proposition 113.2 the following long exact sequence of homotopy groups:

$$\ldots \to \pi_k(S^1) \to \pi_k(S^{2n+1}) \xrightarrow{p_*} \pi_k(\mathbb{C}\mathrm{P}^n) \xrightarrow{\partial} \pi_{k-1}(S^1) \to \ldots$$

We recall that $\pi_k(S^{2n+1}) = 0$ for $k = 1, \ldots, 2n$, that $\pi_{2n+1}(S^{2n+1}) \cong \mathbb{Z} \cdot [\mathrm{id}_{S^{2n+1}}]$ and that $\pi_k(S^1) = 0$ for $k \geq 2$. Putting everything together implies immediately that $\pi_k(\mathbb{C}\mathrm{P}^n) = 0$ for $k = 3, \ldots, 2n$ and that $\pi_{2n+1}(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z} \cdot [p]$.[1567] Only the lower end of the long exact sequence requires slightly more thought. More precisely, we have the exact sequence

$$\underbrace{\pi_2(S^{2n+1})}_{=0} \to \pi_2(\mathbb{C}\mathrm{P}^n) \xrightarrow{\partial} \pi_1(S^1) \to \underbrace{\pi_1(S^{2n+1})}_{=0} \to \pi_1(\mathbb{C}\mathrm{P}^n) \to \pi_0(S^1) \to \pi_0(S^{2n+1}).$$

Since S^1 and S^{2n+1} are path-connected we see that the map on the right is an isomorphism of pointed sets. It follows that $\pi_2(\mathbb{C}\mathrm{P}^n) \xrightarrow{\cong} \pi_1(S^1) \cong \mathbb{Z}$ and that $\pi_1(\mathbb{C}\mathrm{P}^n) = 0$. (We had of course known the latter statement for a long time, see e.g. page 997.) The only thing which is not entirely clearly from this argument is why $\mathrm{id}_{\mathbb{C}\mathrm{P}^1}$ represents a generator of $\pi_2(\mathbb{C}\mathrm{P}^n)$. One could either heroically fight one's way through the above isomorphism, in the same vein as in Exercise 113.12, and use that we know explicitly the generator of $\pi_1(S^1)$.

But fortunately we do not have to go down that route since in Corollary 53.6 (2) we already saw that $\mathrm{id}_{\mathbb{C}\mathrm{P}^1}$ represents a generator of $\pi_2(\mathbb{C}\mathrm{P}^n)$.

(2) We calculate that

$$\pi_k(\mathbb{C}\mathrm{P}^\infty) = \pi_k\Big(\bigcup_{k\in\mathbb{N}} \mathbb{C}\mathrm{P}^n\Big) \underset{\substack{\uparrow \\ \text{by Proposition 40.11 since, see Lemma 36.6,} \\ \text{we can view } \mathbb{C}\mathrm{P}^\infty \text{ as a CW-complex where} \\ \text{each } k\text{-skeleton is given by } \mathbb{C}\mathrm{P}^n \subset \mathbb{C}\mathrm{P}^\infty}}{=} \varinjlim \pi_k(\mathbb{C}\mathrm{P}^n) \underset{\substack{\uparrow \\ \text{by (1)}}}{=} \begin{cases} \mathbb{Z}\cdot[\mathrm{id}_{\mathbb{C}\mathrm{P}^1}], & \text{if } k = 2, \\ 0, & \text{if } k = 1, 3, 4, 5, \ldots. \end{cases}$$

■

113.3. **The Hopf fibration arising from the octonions** (∗)**.** In Proposition 113.9 we had in particular seen that the Hopf map $p\colon S^3 \to \mathbb{C}\mathrm{P}^1 = S^2$ is an S^1-bundle and that the Hopf map $q\colon S^7 \to \mathbb{H}\mathrm{P}^1 = S^4$ is an S^3-bundle. We exploited these fiber bundles in Theorems 113.10 and 113.11 to obtain non-trivial information on higher homotopy groups of spheres.

[1567] Here we use the trivial, but useful observation that $p_*([\mathrm{id}_{S^n}]) = [p]$.

In our discussion of division algebras in Section 60.1 we gave the quaternions as a first example of a non-commutative division algebra. The algebra structure on the quaternions is still nonetheless quite reasonable, in particular the algebra structure is associative which was enough to define the quaternionic projective spaces on page 1472.

Next on the list of ever weirder division algebras were the octonions $\mathbb{O}$ that we introduced on page 1465. Recall that the underlying vector space of the octonions is $\mathbb{R}^8$ but that this time the algebra structure is neither commutative nor associative. The only redeeming feature is that $\mathbb{O}$ is still a division algebra.

Since the octonions are no longer associative one cannot just copy-paste the definition of the quaternionic projective spaces on page 1472 to define octonionic projective spaces in any obvious way. But we can make a prediction: if there was a first octonionic projective space, then it would presumably give rise to a Hopf fibration $r\colon S^{15} \to \mathbb{O}P^1 = S^8$ with fiber S^7. Alas, octonionic projective spaces do not exist. But as we will see, we can still use octonions to produce such a fiber bundle. More precisely, our goal in this section is to prove the following proposition.

Proposition 113.13. *There exists a fiber bundle $p\colon S^{15} \to S^8$ with fiber S^7.*

The proof of Proposition 113.13 is rather longish, thus we will first point out that it allows us to prove the following theorem.

Theorem 113.14.

(1) *For all $n \geq 2$ we have $\pi_n(S^8) \cong \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)$.*
(2) *For $n \in \{1, \ldots, 14\}$ we have $\pi_n(S^8) \cong \pi_{n-1}(S^7)$.*

Proof. The proof is basically the same as the proof of Theorem 113.11. We just need to replace Proposition 113.9 (2) by Proposition 113.13. ■

We can copy-paste the remark after Theorem 113.11:

Remark. From Corollary 53.6 (1) we know that $\pi_{15}(S^{15}) \cong \mathbb{Z}$. Thus it follows from Theorem 113.11 (1) that $\pi_{15}(S^8)$ admits an epimorphism onto $\mathbb{Z}$. In fact in Corollary 91.8 we already saw that the Hopf invariant is an epimorphism $\pi_{15}(S^8) \to \mathbb{Z}$.

The remainder of this chapter is occupied with providing a proof of Proposition 113.13. We will need the notion of the join of two topological spaces that we introduced on page 207.

Definition. Let X and Y be two non-empty topological spaces. We define the *join* $X * Y$ to be the topological space obtained from $X \times [0,1] \times Y$, equipped with the product topology, by performing the following two types of identifications:

(1) for every $x \in X$ we identify all points in $\{x\} \times \{0\} \times Y$ to a single point and
(2) for every $y \in Y$ we identify all points in $X \times \{1\} \times \{y\}$ to a single point.

Furthermore, if one of X or Y is the empty topological space, then we define $X * Y$ to be the other topological space.

We remind the reader of the following lemma.

Lemma 3.50. *Given any $m, n \in \mathbb{N}_0$ the map*

$$\begin{array}{rcl} S^m * S^n & \to & S^{m+n+1} \\ [(x,t,y)] & \mapsto & \underbrace{\left(x \cdot \cos\left(\frac{\pi t}{2}\right), y \cdot \sin\left(\frac{\pi t}{2}\right)\right)}_{\in \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} = \mathbb{R}^{m+n+2}} \end{array}$$

is a homeomorphism.

The following proposition explains our sudden interest in the join of two topological spaces.

Proposition 113.15. *Let X be a topological space that is compact and Hausdorff. Furthermore let $\varphi\colon X \times X \to X$ be a map. Given $x \in X$ we consider the maps*[1568]

$$\begin{array}{rcl} L_x\colon X & \to & X \\ a & \mapsto & \varphi(x,a) \end{array} \quad \text{and} \quad \begin{array}{rcl} R_x\colon X & \to & X \\ a & \mapsto & \varphi(a,x). \end{array}$$

If each map $L_x\colon X \to X$ and each map $R_x\colon X \to X$ is a homeomorphism, then the map

$$\begin{array}{rcl} h_\varphi\colon X * X & \to & \Sigma(X) \\ [(x,t,y)] & \mapsto & [(\varphi(x,y), 2t-1)] \end{array}$$

is a fiber bundle with fiber X.[1569]

Example. We write $S^3 = \{z \in \mathbb{H} \,|\, |z| = 1\}$. By Lemma 60.1 (7) we know that for any $w, z \in \mathbb{H}$ we have $|w \cdot z| = |w| \cdot |z|$. Thus multiplication of quaternions defines a map $\varphi\colon S^3 \times S^3 \to S^3$. Given any $z \in S^3$ the map L_z is a homeomorphism. In fact an inverse is given by $L_{z^{-1}}$ since for any $x \in S^3$ we have

$$(L_{z^{-1}} \circ L_z)(x) \;=\; z^{-1} \cdot (z \cdot x) \underset{\substack{\uparrow \\ \text{since } \mathbb{H} \text{ is associative}}}{=} (z^{-1} \cdot z) \cdot x \;=\; x,$$

and similarly we have $L_z \circ L_{z^{-1}} = \mathrm{id}_{S^3}$. The same argument shows that R_z is also a homeomorphism. Thus we obtain from Proposition 113.15 and Lemma 3.50 a fiber bundle $p\colon S^7 \to S^4$ with fiber S^3. We leave it to the heroically inclined reader to verify that this S^3-bundle is equivalent to the S^3-bundle $q\colon S^7 \to \mathbb{H}\mathrm{P}^1 = S^4$ that we had constructed in the proof of Proposition 113.9.

Proof. We consider the following two subsets of $\Sigma(X)$:

$$C \;:=\; ([-1, \tfrac{1}{2}] \times X) \big/ \{-1\} \times X \quad \text{and} \quad D \;:=\; ([-\tfrac{1}{2}, 1] \times X) \big/ \{1\} \times X.$$

We refer to Figure 1589 for an illustration. Since the interiors of these two sets cover $\Sigma(X)$ it suffices to find trivializations of the map $h_\varphi\colon X * X \to \Sigma(X)$ with fiber X over C and over D.

We start out with finding a trivialization over C.

Claim. The map

$$\begin{array}{rcl} \Psi\colon X \times C & \to & X * X \\ (x, [(y,t)]) & \mapsto & [(x, \frac{2}{3}(t+1), L_x^{-1}(y))] \end{array}$$

[1568]Here the mnemonic device is that "L" should suggest "left-multiplication" and "R" should suggest "right-multiplication".

[1569]It is a good exercise to first verify that the map is actually well-defined.

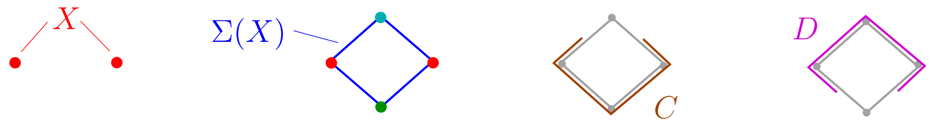


FIGURE 1589

is continuous and it restricts to a homeomorphism onto $h_\varphi^{-1}(C)$.

By now the reader is surely aware of the fact that whenever we mix quotients with products we need to tread carefully. Thus let us first consider the maps

$$\begin{array}{rcl} \alpha\colon X\times X & \to & X\times X \\ (x,y) & \mapsto & (x,\varphi(x,y))=(x,L_x(y)) \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon X\times X & \to & X\times X \\ (x,y) & \mapsto & (x,L_x^{-1}(y)). \end{array}$$

Note that these two maps are inverses of one another. Also note that α is continuous by Lemma 3.8 (2b). Furthermore note that α is a bijection, in fact the inverse is given by β. Since X is compact and Hausdorff we obtain from Proposition 3.12 and Proposition 2.43 (3) that α is a homeomorphism. In particular $\beta=\alpha^{-1}$ is also continuous.

Next we consider the map

$$\begin{array}{rcl} \psi\colon X\times([-1,\tfrac12]\times X) & \to & X\times[0,1]\times X \\ (x,t,y) & \mapsto & (x,\tfrac23(t+1),L_x^{-1}(y)). \end{array}$$

Since $\beta(x,y)=(x,L_x^{-1}(y))$ and since β is continuous we see that ψ is also continuous. Finally we consider the following commutative diagram

$$\begin{tikzcd} X\times[-1,\tfrac12]\times X \arrow[rr, "{(x,t,y)\mapsto(x,[(t,y)])}"] \arrow[d, "\psi"'] \arrow[drr, "f:=p\circ\psi"] & & \overbrace{X\times\big(([-1,\tfrac12]\times X)\,\big/\,(\{-1\}\times X)\big)}^{=X\times C} \arrow[d, "\Psi"] \\ X\times[0,1]\times X \arrow[rr, two heads, "{p(x,t,y):=[(x,t,y)]}"'] & & X*X. \end{tikzcd}$$

Claim. The map Ψ is continuous.

We had just seen that $f=p\circ\psi$ is continuous. Note that by Lemma 5.15 (1) the projection map $q\colon[-1,\tfrac12]\times X\to([-1,\tfrac12]\times X)\,\big/\,(\{-1\}\times X)$ is a quotient map. Since X is compact and Hausdorff we know by Lemma 2.73 that X is regionally compact.[1570] Thus it follows from Theorem 5.16 that the top horizontal map $\mathrm{id}_X\times q$ is a quotient map. Thus it follows from Lemma 5.17 (2) that ψ is continuous. ⊞

One easily verifies that Ψ defines a bijection

$$\Psi\colon \underbrace{X\times C}_{\subset X*X} \to \underbrace{h_\varphi^{-1}(C)}_{\subset\Sigma(X)}.$$

[1570]If the reader is unhappy about using Lemma 2.73 we point out that in our application of the proposition the topological space will be $X=S^n$ and thus obviously regionally compact.

Since X is compact we know by Lemma 3.48 (1) that the left-hand side is compact. By Lemma 24.2 (2) and our hypothesis that X is Hausdorff we know that the right-hand side is Hausdorff. Therefore it follows from Proposition 2.43 (3) that the continuous bijection $\Psi\colon X \times C \to h_\varphi^{-1}(C)$ is indeed a homeomorphism. ⊞

We consider the following diagram

$$\begin{tikzcd} X \times C \arrow[rr, "\Psi"] \arrow[dr, "(x,c)\mapsto c"'] & & h_\varphi^{-1}(C) \arrow[dl, "h_\varphi"] \\ & C. & \end{tikzcd}$$

It is straightforward to verify that this diagram commutes. Since the top map is homeomorphism by the previous claim we have found a trivialization of h_φ over C. We leave it to the reader to make the minor modifications needed to find a trivialization of h_φ over D.[1571] ■

Evidently the idea now is to replace the quaternions in the example on page 2693 by the octonions. In the following we recall the most salient bits from the discussion of octonions on page 1465. The octonions $\mathbb{O}$ are $\mathbb{R}^8$ together with an algebra structure over $\mathbb{R}$ that has the following properties:

(1) The octonions $\mathbb{O}$ are a division algebra, i.e. for all $a \neq 0$ and $b \in \mathbb{O}$ there exists an $x \in \mathbb{O}$ with $a \cdot x = b$ and a $y \in \mathbb{O}$ with $y \cdot a = b$.
(2) The algebra structure is neither commutative nor associative.
(3) Multiplication on $\mathbb{O}$ respects the euclidean norm, more precisely, given $a, b \in \mathbb{O}$ we have $\|a \cdot b\| = \|a\| \cdot \|b\|$.

It follows from (3) that the algebra multiplication induces a map $\varphi\colon S^7 \times S^7 \to S^7$.

Lemma 113.16. *Given any $z \in S^7$ the maps $L_z\colon S^7 \to S^7$ and $R_z\colon S^7 \to S^7$, as defined in Proposition 113.15, are homeomorphisms.*

PROOF. First note that we can *not* recycle the argument from page 2693 since there we used that the quaternions are associative.

By Corollary 50.8 (2) it suffices to show that the maps L_z and R_z are injective. First we show that given any $z \in S^7$ the map $L_z\colon S^7 \to S^7$ is injective. So suppose that we have $a, b \in S^7$ with $L_z(a) = L_z(b)$. We obtain that

$$\|a-b\| \underset{\substack{\uparrow \\ \text{since } z \in S^7}}{=} \|z\| \cdot \|a-b\| \underset{\substack{\uparrow \\ \text{by (3)}}}{=} \|z \cdot (a-b)\| \underset{\substack{\uparrow \\ \text{since } \mathbb{O} \text{ is an algebra}}}{=} \|z \cdot a - z \cdot b\| \underset{\substack{\uparrow \\ \text{by definition of } L_z}}{=} \|L_z(a) - L_z(b)\| \underset{\substack{\uparrow \\ \text{since } L_z(a) = L_z(b)}}{=} 0.$$

Thus we see that $a = b$. Almost the same argument shows that R_z is injective.[1572] ■

Now we are basically done with the proof of Proposition 113.13.

[1571] Note that in this second part we will need to use that the maps $R_x\colon X \to X$ are homeomorphisms.

[1572] At first glance it might appear odd that we never used that the octonions are a division algebra, we only used the innocuous looking third property. But it is clear that (3) already implies that $\mathbb{O}$ has no zero-divisors. By Lemma 60.5 this forces $\mathbb{O}$ to be a division algebra.

Proof of Proposition 113.13. It follows immediately from Lemma 113.16, Proposition 113.15 and Lemma 3.50 that there exists a fiber bundle $p\colon S^{15} \to S^8$ with fiber S^7. ■

Exercises for Chapter 113.

Exercise 113.1. Let F be a topological space and let $p\colon Y \to B$ be an F-bundle. Suppose that the fiber F and the base space B are path-connected. Show directly, without making use of Proposition 113.2 (2), that the total space Y is also path-connected.

Exercise 113.2. Let $p\colon Y \to [0,1]$ be a bundle with fiber F. Show with your bare hands (i.e. without using Proposition 113.7) that p is trivial, i.e. show that p is equivalent to the trivial bundle $F \times [0,1] \to [0,1]$.

Exercise 113.3. Let X be a topological space and let $f_0, f_1\colon X \to X$ be homeomorphisms. Suppose that f_0 and f_1 are isotopic. Show that the two X-bundles $\operatorname{Tor}(X, f_0) \to S^1$ and $\operatorname{Tor}(X, f_1) \to S^1$ are equivalent.

Exercise 113.4. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Show that the corresponding $\mathbb{R}^n$-bundle $\operatorname{Tor}(\mathbb{R}^n, f) \to S^1$ is equivalent to an n-dimensional vector bundle.
Hint. Use Lemma 45.17 and Lemma 113.3 (2), the latter is a. k. a. Exercise 113.3.

Exercise 113.5. The Ehresmann Fibration Theorem says the following:
Let $f\colon M \to N$ be a smooth map between two closed smooth manifolds. We suppose that N is connected. If f is a submersion, i.e. if given any $P \in M$ the map $\operatorname{D} f_P\colon \operatorname{T}_P M \to \operatorname{T}_{f(P)} N$ is an epimorphism, then f is a fiber bundle.
Provide a proof of the Ehresmann Fibration Theorem using the Submersion Theorem 6.56.

Exercise 113.6. Use the Ehresmann Fibration Theorem, see Exercise 113.5, to give an alternative proof of the fact that the Hopf map $S^3 \to \mathbb{C}\mathrm{P}^1$ defines an S^1-bundle over $\mathbb{C}\mathrm{P}^1 = S^2$.

Exercise 113.7. As on page 1071 we consider the Hopf map:

$$\begin{array}{rcl} H\colon S^3 = \{(z_1,z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 = \mathbb{C}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1 : z_2]. \end{array}$$

(a) Let $P, Q \in S^2$ be two distinct points. Show that $\mathrm{H}^{-1}(\{P\}) \cup \mathrm{H}^{-1}(\{Q\})$ is isotopic to a Hopf link, that we defined in Figure 1405.
(b) Let $P, Q, R \in S^2$ be two distinct points. Sketch the link $\mathrm{H}^{-1}(\{P\}) \cup \mathrm{H}^{-1}(\{Q\}) \cup \mathrm{H}^{-1}(\{R\})$.

Exercise 113.8. Let $n \in \mathbb{N}$. We consider the map

$$\begin{array}{rcl} \varphi\colon\ \mathrm{U}(n) = \{A \in \mathrm{M}(n \times n, \mathbb{C}) \,|\, A \cdot \overline{A}^T = \mathrm{id}\} & \to & \mathbb{C} \setminus \{0\} \\ A & \mapsto & \det(A). \end{array}$$

(a) Show that φ takes values in S^1.
(b) Show that the map $\varphi\colon\ \mathrm{U}(n) \to S^1$ is a fiber bundle.
Hint. You can use the Ehresmann Fibration Theorem from Exercise 113.5.

Exercise 113.9. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. By the discussion on page 2682 and Proposition 113.2 we know that the map

$$\begin{array}{rcl}\mathrm{Tor}(X,f) = (X \times [0,1])/(x,0) \sim (f(x),1) & \to & S^1 \\ [(x,t)] & \mapsto & e^{2\pi i t}\end{array}$$

is a fibration. We make the obvious identification $X = X \times \{0\} \subset \mathrm{Tor}(X,f)$. Let $k \in \mathbb{N}_0$ and let $\alpha\colon [0,1] \to S^1$ be the usual loop given by $\alpha(t) = e^{2\pi i t}$. By Proposition 112.5 and Lemma 42.2 we get a well-defined map

$$\Theta(\alpha)_*\colon \ \mathrm{H}_k(X;\mathbb{Z}) \ \to \ \mathrm{H}_k(X;\mathbb{Z}).$$

Show that $\Theta(\alpha)_* = (f_*)^{-1}$.

Exercise 113.10. Let W be an n-dimensional smooth manifold and let K be a compact proper connected k-dimensional submanifold with $k \leq n-3$. Show that the inclusion induced map $\pi_1(W \setminus K) \to \pi_1(W)$ is an isomorphism.
Hint. Use Lemma 21.22 (3).

Exercise 113.11. The same way that we defined the infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$, see page 942, we define the infinite quaternionic projective space $\mathbb{H}\mathrm{P}^\infty$ as the direct limit of the finite-dimensional quaternionic projective spaces $\mathbb{H}\mathrm{P}^k$. Show that for any $i \in \mathbb{N}_{\geq 2}$ we have an isomorphism $\pi_i(\mathbb{H}\mathrm{P}^\infty) \cong \pi_{i-1}(S^3)$.

Exercise 113.12. Let $p\colon S^3 \to S^2$ the Hopf fibration. In Proposition 113.9 we saw that it is an S^1-fiber bundle. We consider the connecting homomorphism $\partial\colon \pi_2(S^2) \to \pi_1(S^1)$ of the long exact sequence of homotopy groups from Proposition 113.2 (2). Show, just using the definition of the connecting homomorphism, that $\partial([\mathrm{id}_{S^2}]) = [\mathrm{id}_{S^1}]$.

Exercise 113.13. Show that the homotopy groups of S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$ are isomorphic.

Exercise 113.14. Let $m, k \in \mathbb{N}$ and let $p\colon M \to S^k$ be a fiber bundle with fiber S^m. We denote by $i\colon S^m \to M$ the homeomorphism to any of the fibers.

(a) Show that M is a closed $(m+k)$-dimensional topological manifold.
(b) We suppose that $m < k$ and we suppose that M is orientable. Show that the maps $p_*\colon \ \mathrm{H}_k(M) \to \mathrm{H}_k(S^k)$ and $i_*\colon \ \mathrm{H}_m(S^m) \to \mathrm{H}_m(M)$ are isomorphisms and show that $\mathrm{H}_i(M) = 0$ for $i \neq 0, k, m, m+n$.
Hint. Use Theorems 112.7 and 111.14 and use Poincaré Duality.
(c) We suppose again that M is orientable. Show that the conclusion in (b) does not necessarily hold if we drop the hypothesis that $m < k$.

114. Homotopy groups of Lie groups

In this chapter our focus will lie on the study of the homotopy groups of the matrix groups $\mathrm{O}(n)$ and $\mathrm{U}(n)$. Later on in the section we will see that we should also define a third type of groups, namely the groups $\mathrm{Sp}(n)$ which can be viewed as the quaternionic analogue of $\mathrm{O}(n)$ and $\mathrm{U}(n)$.

Since these groups are Lie groups and since this structure is helpful in our study of $\mathrm{O}(n)$ and $\mathrm{U}(n)$ we first give a short introduction to Lie groups.

114.1. **Topological groups and Lie groups.** First we recall the following definition from page 210.

Definition. A *topological group* is a topological space X together with a group structure such that the two maps

$$\begin{array}{rcl} X\times X & \to & X \\ (x,y) & \mapsto & x\cdot y \end{array} \quad \text{and} \quad \begin{array}{rcl} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are continuous. A *morphism between topological groups* X and Y is a homomorphism $f\colon X\to Y$ that is furthermore continuous.

Examples.

(1) On page 211 we already saw that for any $n\in\mathbb{N}$ the matrix groups $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are topological groups.

(2) Let X be a topological space and let $x_0\in X$ be a base point. We consider the loop space $\Omega(X,x_0)$, i.e. the set

$$\Omega(X,x_0) \;=\; \{\gamma\colon [0,1]\to X \,|\, \gamma(0)=\gamma(1)=x_0\}/\text{homotopy}$$

equipped with the compact-open topology. As in Exercise 112.7 we equip the set $\pi_1(X,x_0)=\Omega(X,x_0)/\sim$ with the quotient topology, i.e. we view $\pi_1(X,x_0)$ as a topological space. Evidently $\pi_1(X,x_0)$ admits a group structure, but somewhat surprisingly, the multiplication map $\pi_1(X,x_0)\times\pi_1(X,x_0)\to\pi_1(X,x_0)$ is in general *not* continuous. In fact such an example is given by the Hawaiian Earrings, introduced in Exercise 25.1. We refer to [**Fab11**] for details.[1573]

We leave the elementary proof of the following lemma to the reader.

Lemma 114.1. *Let G be a topological group and let H be a subgroup of G. The map*

$$\begin{array}{rcl} G\times H & \to & G \\ (g,h) & \mapsto & g\cdot h \end{array}$$

defines a right-action of H on G that is free and continuous.

To avoid any misunderstanding we introduce the following convention.

[1573] In fact, as is pointed out in [**Fab11**], the failure of $\pi_1(X,x_0)$ to be a topological group is another instance of the phenomenon, discussed in Section 5.2, that in general product and quotient topologies do not mix well.

Convention. Let G be a topological group and let H be a subgroup. We denote by G/H the topological space that is given by the equivalence relation[1574]

$$g_1 \sim g_2 \quad \Longleftrightarrow \quad \text{there exists an } h \in H \text{ with } g_1 \cdot h = g_2.$$

The following definition of a Lie group is quite similar to the above definition of a topological group.

Definition. A *Lie group* is a smooth manifold X with empty boundary together with a group structure such that the two maps[1575]

$$\begin{array}{rcl} X \times X & \to & X \\ (x,y) & \mapsto & x \cdot y \end{array} \quad \text{and} \quad \begin{array}{rcl} X & \to & X \\ x & \mapsto & x^{-1} \end{array}$$

are smooth.

Remark.

(1) It follows immediately from Lemma 6.23 (2) that a Lie group is also a topological group.
(2) As we will see in Exercise 114.2, strictly speaking one does not need to demand that the map $x \mapsto x^{-1}$ is smooth, since it follows already from the other properties.

Examples.

(1) It is straightforward to see that the smooth manifold $\mathbb{R}$ with the group structure given by addition and the smooth manifold S^1 with the group structure given by multiplication are Lie groups.
(2) The product $G_1 \times G_2$ of two Lie groups G_1 and G_2 admits a canonical smooth manifold structure by Proposition 6.51 and the group structures on G_1 and G_2 define a group structure on $G_1 \times G_2$. One can easily verify that $G_1 \times G_2$ is again a Lie group. In particular the torus $S^1 \times S^1$ is a Lie group.

As mentioned in the beginning of this chapter, we are now interested in the homotopy groups of the groups

$$\begin{array}{rcl} \mathrm{O}(n) & = & \{A \in \mathrm{M}(n \times n, \mathbb{R}) \,|\, A^T A = \mathrm{id}\} \\ \mathrm{U}(n) & = & \{A \in \mathrm{M}(n \times n, \mathbb{C}) \,|\, \overline{A}^T A = \mathrm{id}\}. \end{array}$$

As we will see later on, even if we are only interested in these groups it makes sense to extend our gaze and to consider the quaternionic analogue of $\mathrm{O}(n)$ and $\mathrm{U}(n)$:

Definition. Let $n \in \mathbb{N}$. We define.[1576]

$$\mathrm{Sp}(n) \;=\; \{A \in \mathrm{M}(n \times n, \mathbb{H}) \,|\, \overline{A}^T A = \mathrm{id}\}.$$

We will first discuss $\mathrm{Sp}(n)$ a little bit before we settle on a name for it:

[1574]The point of this convention is that in principle G/H could mean the quotient of the topological space G by the subset H.

[1575]Here we equip $X \times X$ with the product smooth manifold structure introduced in Proposition 6.51.

[1576]Recall that the conjugate of a quaternion $h = a\cdot 1 + b\cdot \mathrm{i} + c\cdot \mathrm{j} + d\cdot \mathrm{k}$ is defined as $\overline{h} = a\cdot 1 - b\cdot \mathrm{i} - c\cdot \mathrm{j} - d\cdot \mathrm{k}$. Given a matrix $A = (a_{ij})$ over $\mathbb{H}$ we define $\overline{A} = (\overline{a}_{ij})$.

Lemma 114.2.

(1) *We have* $\mathrm{Sp}(1) = S^3$.

(2) *Let* $n \in \mathbb{N}$. *Matrix multiplication turns* $\mathrm{Sp}(n)$ *into a group.*

(3) *Let* $n \in \mathbb{N}$. *We write* $J = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix}$. *The map*

$$\begin{array}{rcll} \Phi\colon \mathrm{Sp}(n) & \to & \{X \in \mathrm{U}(2n) \,|\, JXJ^{-1} = \overline{X}\} & \\ P & \mapsto & \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} & \text{where } A, B \in \mathrm{M}(n \times n, \mathbb{C}) \text{ with}^{1577}\ P = A + B\mathrm{j} \end{array}$$

is a group isomorphism.

Proof.

(1) In Lemma 60.1 (5) we saw that for any quaternion $h \in \mathbb{H}$ we have $h \cdot \overline{h} = \|h\|^2$. It follows immediately that $\mathrm{Sp}(1) = S^3$.

(2) We start out with the following observation: A straightforward calculation[1578] shows that for any two matrices $A, B \in \mathrm{M}(n \times n, \mathbb{H})$ we have $\overline{(AB)}^T = \overline{B}^T \overline{A}^T$. Now we turn to the proof that $\mathrm{Sp}(n)$ is a group:
   (a) The fact that $\mathbb{H}$ is associative implies immediately that the multiplication of matrices in $\mathrm{M}(n \times n, \mathbb{H})$ is associative.
   (b) The identity matrix is as usual a neutral element.
   (c) Given $A \in \mathrm{Sp}(n)$ we have by definition that $\overline{A}^T A = \mathrm{id}$, i.e. $\overline{A}^T$ is a left-inverse. The above observation implies that $\overline{A}^T$ is also a right-inverse.
   (d) It follows from the above observation that $\mathrm{Sp}(n)$ is closed under matrix multiplication and taking inverses.

   We have thus shown that $\mathrm{Sp}(n)$ is indeed a group.

(3) We will only use this isomorphism to justify the name for $\mathrm{Sp}(n)$, otherwise it will not play a role. Therefore we will not provide the fairly elementary proof of this statement, instead we refer to [**Bre93**, p. 103] for the proof. ■

Not surprisingly we will now see that $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ are Lie groups.

Lemma 114.3.

(1) *Let* $n \in \mathbb{N}$.
   (a) *The set* $\mathrm{O}(n)$ *is a submanifold of* $\mathrm{M}(n \times n, \mathbb{R}) = \mathbb{R}^{n^2}$ *of dimension* $\frac{1}{2}n(n-1)$.
   (b) *The set* $\mathrm{U}(n)$ *is a submanifold of* $\mathrm{M}(n \times n, \mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$ *of dimension* n^2.
   (c) *The set* $\mathrm{Sp}(n)$ *is a submanifold of* $\mathrm{M}(n \times n, \mathbb{H}) = \mathbb{H}^{n^2} = \mathbb{R}^{4n^2}$ *of dimension* $2n^2 + n$.

[1577] It follows from $\mathrm{i} \cdot \mathrm{j} = \mathrm{k}$ that any quaternion $a \cdot 1 + b \cdot \mathrm{i} + c \cdot \mathrm{j} + d \cdot \mathrm{k}$ can be uniquely written as $(a \cdot 1 + b \cdot \mathrm{i}) + (c \cdot 1 + d \cdot \mathrm{i}) \cdot \mathrm{j}$ where $a \cdot 1 + b \cdot \mathrm{i}$ and $c \cdot 1 + d \cdot \mathrm{i}$ are complex numbers. The same statement applies to matrices.

[1578] In the calculation we need to use the fact, mentioned in Lemma 60.1 (3), that for quaternions $w, z \in \mathbb{H}$ we have $\overline{z \cdot w} = \overline{w} \cdot \overline{z}$.

Furthermore, if we view $\mathrm{O}(n)$ *as a subspace of* $\mathrm{O}(n+1)$ *in the obvious way, then* $\mathrm{O}(n)$ *is a submanifold of* $\mathrm{O}(n+1)$. *The same statements apply to* $\mathrm{U}(n)$ *and* $\mathrm{Sp}(n)$.

(2) $\mathrm{O}(n)$, $\mathrm{U}(n)$ *and* $\mathrm{Sp}(n)$, *viewed as smooth manifolds in their own right, are closed, in particular they are compact.*

(3) *The usual matrix multiplication turns* $\mathrm{O}(n)$, $\mathrm{U}(n)$ *and* $\mathrm{Sp}(n)$ *into Lie groups.*

Remark.

(1) Recall that on page **??** we defined the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$ as follows:

$$\mathrm{Sp}(2g,\mathbb{Z}) \;=\; \{A\in \mathrm{GL}(2g,\mathbb{Z})\,|\,A^TJA=A\} \quad \text{where} \quad J=\begin{pmatrix} 0 & \mathrm{id}_g \\ -\mathrm{id}_g & 0\end{pmatrix}.$$

This definition, together with Lemmas 114.2 and 114.3, leads us to calling $\mathrm{Sp}(n)$ the *n-th compact symplectic group*.

(2) The statements regarding the groups $\mathrm{O}(n)$ and $\mathrm{U}(n)$ are mostly already contained in Lemma 6.55.

Proof. In Lemma 6.55 we had basically proved statements (1) and (2) for the groups $\mathrm{O}(n)$ and $\mathrm{U}(n)$. The only aspect that we did not mention explicitly so far is that $\mathrm{O}(n)$ is a submanifold of $\mathrm{O}(n+1)$. We leave it to the reader to verify that this statement holds. The proof of (1) and (2) for $\mathrm{Sp}(n)$ is basically the same.

In Lemma 3.53 we showed that $\mathrm{O}(n)$ and $\mathrm{U}(n)$ are topological groups. Almost the same argument given in that proof shows that matrix multiplication and taking inverses are smooth maps. This implies that $\mathrm{O}(n)$ and $\mathrm{U}(n)$ are Lie groups. Furthermore it is clear that multiplication in $\mathrm{Sp}(n)$ is smooth. It remains to show that the map

$$\begin{array}{rcl}\mathrm{Sp}(n) & \to & \mathrm{Sp}(n)\\ A & \mapsto & A^{-1}\end{array}$$

is smooth. We will show this fact in Exercise 114.4. ■

Basically by definition we have $\mathrm{O}(1)=S^0$ and $\mathrm{U}(1)=S^1$, furthermore by Lemma 114.2 (1) we have $\mathrm{Sp}(1)=S^3$. Thus it follows from Lemma 114.3 that these three spheres can be viewed as Lie groups. The question arises, whether perhaps other (all?) spheres can be equipped with the structure of a Lie group. The following lemma gives an easy way to rule out the existence of Lie group structures on some smooth manifolds.

Lemma 114.4. *Every Lie group is parallelizable.*[1579]

Examples. Recall that on page **??** we saw that $2n$-dimensional spheres with $n\geq 1$ are not parallelizable. Furthermore t on page **??** we showed that for $g\neq 1$ the surface of genus g is not parallelizable. Thus it follows from Lemma 114.4 that these spheres and surfaces cannot be equipped with a Lie group structure.

Sketch of proof. Let G be an n-dimensional Lie group with trivial element e. We denote by $R_g\colon G\to G$ the map that is given by right-multiplication by g. Since G is a Lie

[1579] Recall that according to the definition on page **??** a smooth manifold is called parallelizable if its tangent bundle is trivial.

group this map is in fact a diffeomorphism. The map

$$\begin{array}{rcl} G \times T_e G & \to & TG = \{(g,h) \,|\, g \in G \text{ and } w \in T_h G\} \\ (g,v) & \mapsto & (g,(R_g)_*(v)) \end{array}$$

is an isomorphism of vector bundles. (This statement is basically obvious, except that perhaps it is not completely clear why the map is continuous, we refer to [**Lee02**, Corollary 8.39] for more details.) But this shows that G is parallelizable. ■

The following lemma gives us more examples of Lie groups.

Lemma 114.5. *Let G be a Lie group and let K be a finite normal subgroup. The quotient G/K admits a unique smooth manifold structure such that the projection $G \to G/K$ is a local diffeomorphism. Furthermore the group structure on the quotient group G/K defines a Lie group structure on G/K.*

Example. It follows from Lemma 114.5 that for any $n \in \mathbb{N}$ the quotients $\mathrm{SU}(n)/\{\pm\,\mathrm{id}\}$ and $(\mathrm{SU}(n) \times \mathrm{SU}(n))/\{\pm(\mathrm{id},\mathrm{id})\}$ are Lie groups.

Proof. In Lemma 114.1 we already saw that the action of K on G is free and continuous. Since G is a Lie group one also sees that the action is smooth. Finally, since G is finite we get for free that the action is proper. Thus we obtain from Proposition 6.32 that G/K admits a unique smooth manifold structure (without boundary) such that the projection $G \to G/K$ is a local diffeomorphism. We leave it to the reader to verify that the induced multiplication map $G/K \times G/K \to G/K$ is again smooth. ■

Above we have seen that subgroups of matrix groups often give rise to Lie groups. Perhaps somewhat surprisingly, in a sense, all compact Lie groups arise that way. More precisely, the following theorem holds:[1580]

Theorem 114.6. *Any compact Lie group is isomorphic (in the obvious sense) to a Lie subgroup of* $\mathrm{GL}(n,\mathbb{R})$ *for some* $n \in \mathbb{N}$.

The theorem is a consequence of the so-called Peter-Weyl Theorem. We only stated the theorem for "general culture". We will not make use of it and thus we refer to [**BtD95**, Theorem III.4.1] or alternatively to [**Sep07**, Theorem 3.28] for a proof.

114.2. **The homotopy groups of** $\mathrm{O}(n)$**,** $\mathrm{U}(n)$ **and** $\mathrm{Sp}(n)$**.** In this chapter we want to see what we can say about the homotopy groups of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.

Convention.

(1) Given $n \in \mathbb{N}$ we have the obvious monomorphism

$$\begin{array}{rcl} \mathrm{O}(n) & \to & \mathrm{O}(n+1) \\ A & \mapsto & \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

Often we will use it to identify $\mathrm{O}(n)$ with its image in $\mathrm{O}(n+1)$. Similarly we view $\mathrm{U}(n)$ as a subgroup of $\mathrm{U}(n+1)$ and we view $\mathrm{Sp}(n)$ as a subgroup of $\mathrm{Sp}(n+1)$.

[1580]One could view Theorem 114.6 as an analogue of Proposition 9.1 or of the Whitney Embedding Theorem 11.14.

(2) We will view any subgroup of $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$ or $\mathrm{GL}(n,\mathbb{H})$ as a topological space that is equipped with the base point given by the identity matrix. Thus in the remainder of this chapter we will suppress the base point in the discussion of fundamental groups.

The following lemma evidently uses the convention from page 2699.

Lemma 114.7. *Let $n \in \mathbb{N}$. The maps*

$$\begin{array}{rcl} \mathrm{O}(n+1)/\mathrm{O}(n) & \xrightarrow{p} & S^n \\ [(v_1\,\dots\,v_{n+1})] & \mapsto & v_{n+1} \end{array} \qquad \begin{array}{rcl} \mathrm{U}(n+1)/\mathrm{U}(n) & \xrightarrow{q} & S^{2n+1} \\ [(v_1\,\dots\,v_{n+1})] & \mapsto & v_{n+1} \end{array} \quad \textit{and} \quad \begin{array}{rcl} \mathrm{Sp}(n+1)/\mathrm{Sp}(n) & \xrightarrow{r} & S^{4n+3} \\ [(v_1\,\dots\,v_{n+1})] & \mapsto & v_{n+1} \end{array}$$

are well-defined and they are homeomorphisms.

PROOF.

(1) First we show that the given map $p\colon \mathrm{O}(n+1)/\mathrm{O}(n) \to S^n$ is well-defined. To do so we consider the map[1581]

$$\begin{array}{rcl} \mathrm{O}(n+1) & \to & S^n \\ V := (v_1\,\dots\,v_{n+1}) & \mapsto & v_{n+1} = V\cdot e_{n+1}. \end{array}$$

Note that if we multiply V on the right by a matrix of the form $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, then the multiplication with e_{n+1} is not affected. Therefore the above map $\mathrm{O}(n+1) \to S^n$ descends to a map $\mathrm{O}(n+1)/\mathrm{O}(n) \to S^n$ which is exactly the map p we consider in the lemma. This shows that p is well-defined.

Also note that basic linear algebra tells us that given any point $v_{n+1} \in S^n$ there exist vectors $v_1,\dots,v_n \in \mathbb{R}^n$ such that $v_1,\dots,v_n,v_{n+1}$ form an orthonormal basis $\mathbb{R}^{n+1}$. This shows that the given map p is a surjection. Similarly, elementary linear algebra shows that p is injective. Finally note that S^n is Hausdorff and note that Lemmas 114.3 and 2.40 imply that $\mathrm{O}(n+1)/\mathrm{O}(n)$ is compact. Therefore it follows from Proposition 2.43 (3) that the map p is indeed a homeomorphism.

(2),(3) The proof for (2) and (3) is basically identical to the proof in (1), except that the linear algebra aspects require slightly more thought. ■

The following proposition gives us new interesting examples of fiber bundles.

Proposition 114.8. *Let $n \in \mathbb{N}$. The maps*

$$\begin{array}{rcl} \mathrm{O}(n+1) & \to & S^n \\ (v_1\,\dots\,v_{n+1}) & \mapsto & v_{n+1} \end{array} \qquad \begin{array}{rcl} \mathrm{U}(n+1) & \to & S^{2n+1} \\ (v_1\,\dots\,v_{n+1}) & \mapsto & v_{n+1} \end{array} \quad \textit{and} \quad \begin{array}{rcl} \mathrm{Sp}(n+1) & \to & S^{4n+3} \\ (v_1\,\dots\,v_{n+1}) & \mapsto & v_{n+1} \end{array}$$

are bundle projections with fibers given by $\mathrm{O}(n)$, $\mathrm{U}(n)$ *and* $\mathrm{Sp}(n)$ *respectively.*

Remark. Without too much effort one can show that the bundle projections $\mathrm{O}(n+1) \to S^n$ with fiber $\mathrm{O}(n)$ and $\mathrm{U}(n+1) \to S^{2n+1}$ with fiber $\mathrm{U}(n)$ from the proposition restrict to bundle projections $\mathrm{SO}(n+1) \to S^n$ with fiber $\mathrm{SO}(n)$ and $\mathrm{SU}(n+1) \to S^{2n+1}$ with fiber $\mathrm{SU}(n)$.

[1581] It follows easily from $A^TA = \mathrm{id}$ that every column v of A satisfies $\|v\| = 1$, i.e. every column lies in S^n.

We will outline two approaches to proving Proposition 114.8. The first approach relies on the following proposition that shows more generally that Lie groups (and sometimes topological groups) give rise to fiber bundles.

Proposition 114.9. $(*)$

(1) *Let G be a Lie group. If H is a closed subgroup of G, then the projection $G \to G/H$ is a bundle projection with fiber H.*
(2) *Let G be a topological group and H be a closed subgroup of G. If H admits the structure of a Lie group, then the projection $G \to G/H$ is a bundle projection with fiber H.*

Remark. One might hope that Proposition 114.9 (1) generalizes to all topological groups and subgroups. But as shown in [**Karu58**, p. 347], there exists a topological group G and a closed subgroup H such that the projection $G \to G/H$ is not a bundle projection.

Proof of Proposition 114.9 $(*)$.

(1) This statement is proved in [**Kiri08**, Theorem 2.11]. Alternatively, if G is a closed subgroup of some $\mathrm{GL}(n,\mathbb{C})$, then the proposition is proved in [**Bre93**, p. 110].
(2) This statement is proved in [**Pal61**, Corollary 4.1]. ■

First proof of Proposition 114.8 $(*)$. These three statements are an immediate consequence of Lemma 114.7 and Proposition 114.9 (1). ■

Second proof of Proposition 114.8. In the following we give an outline of a "bare hands only" proof that the first projection map $p\colon \mathrm{O}(n+1) \to S^n$ is indeed a bundle projection with fiber $\mathrm{O}(n)$. We leave it to the reader to modify the proof to deal with the two other cases.[1582]

Thus let $n \in \mathbb{N}$. As usual we equip $\mathbb{R}^{n+1}$ with the euclidean inner product $\langle\, ,\, \rangle$. In this proof we identify $\mathrm{O}(n+1)$ with the group of isometries of $(\mathbb{R}^{n+1}, \langle\, ,\, \rangle)$. More precisely, we do not distinguish in the notation between an isometry of $(\mathbb{R}^{n+1}, \langle\, ,\, \rangle)$ and the matrix it represents with respect to the standard basis of $\mathbb{R}^{n+1}$.

We consider the two compact subsets

$$V \;=\; \{(x_1,\dots,x_{n+1}) \in S^n \,|\, x_{n+1} \le \tfrac{1}{2}\} \quad \text{and } W \;=\; \{(x_1,\dots,x_{n+1}) \in S^n \,|\, x_{n+1} \ge -\tfrac{1}{2}\}.$$

We continue with the following claim.

Claim. Let $P \in V$. We write $W(P) := \operatorname{span}\{e_{n+1}, P\}$. There exists a unique orientation-preserving isometry $\alpha(P)$ of $\mathbb{R}^{n+1} = W(P) \oplus W(P)^\perp$ that satisfies the following two conditions:

(1) the restriction of $\alpha(P)$ to $W(P)$ is an orientation-preserving isometry of $W(P)$ that sends e_{n+1} to P,
(2) it equals the identity on $W(P)^\perp$.

We illustrate the definition of $\alpha(P)$ in Figure 1590. In the 3-dimensional setting the isometry $\alpha(P)$ is the unique rotation around "the $W(P)^\perp$-axis" that sends e_{n+1} to P.

[1582] Alternatively the reader can look up the proof of Proposition 114.9 (1).

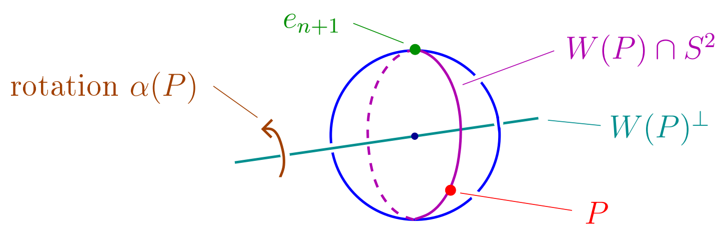


Figure 1590

Note that $(W(P), \langle\,,\,\rangle)$ and $(\mathbb{R}^2, \langle\,,\,\rangle)$ are both positive-definite and symmetric forms. Thus it follows from Exercise 101.1 that both forms are also non-singular. Therefore we obtain from Sylvester's Theorem 101.10 that the two forms $(W(P), \langle\,,\,\rangle)$ and $(\mathbb{R}^2, \langle\,,\,\rangle)$ are isometric. In the latter case the orientation-preserving isometries are given by $\mathrm{SO}(2)$. Since $\mathrm{SO}(2) = S^1$ acts freely and transitively on S^1 we see that given any two points $X, Y \in \mathbb{R}^2$ with $\|X\| = \|Y\|$ there exists a unique orientation-preserving isometry β of $(\mathbb{R}^2, \langle\,,\,\rangle)$ with $\beta(X) = Y$. Since $(W(P), \langle\,,\,\rangle)$ is isometric to $(\mathbb{R}^2, \langle\,,\,\rangle)$ the same uniqueness statement applies to $W(P)$ and the two points $e_{n+1}, P \in W(P)$. The claim immediately follows from this observation.

We leave it to the reader to verify that the map $\alpha\colon V \to \mathrm{O}(n+1)$ that we had just defined is indeed continuous. Next we consider the map

$$\begin{array}{rcl} f\colon V \times \mathrm{O}(n) & \to & \mathrm{O}(n+1) \\ (P, A) & \mapsto & \alpha(P) \circ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{array}$$

An entertaining exercise in linear algebra shows that the map f restricts to a bijection $V \times \mathrm{O}(n) \to p^{-1}(V)$. Since V and $\mathrm{O}(n)$ are compact we obtain from Proposition 2.43 (3) that f defines a homeomorphism $V \times \mathrm{O}(n) \to p^{-1}(V)$.

Now recall that for any $P \in V$ we have, by design, that $\alpha(P)(e_{n+1}) = P$. Using this observation we see almost immediately that the following diagram commutes:

$$\begin{tikzcd} V \times \mathrm{O}(n) \arrow[rr, "f", "\cong"'] \arrow[dr, "(P,A)\mapsto P"'] & & p^{-1}(V) \arrow[dl, "p"', "B\mapsto B\cdot e_{n+1}"] \\ & V. & \end{tikzcd}$$

This shows that we have found a trivialization of p over V. With very minor modifications one can also find a trivialization of p over W. Since the interiors of V and W cover all of S^n we see that these two trivializations show that $p\colon \mathrm{O}(n+1) \to S^n$ is indeed a fiber bundle with fiber $\mathrm{O}(n)$. ■

Now we can continue with the actual study of the homotopy groups of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$.

Proposition 114.10. *Let $n \in \mathbb{N}$.*

(1) *The inclusion induced map*[1583]

$$\pi_k(\mathrm{O}(n)) \;\to\; \pi_k(\mathrm{O}(n+1)) \quad \text{is an} \quad \begin{cases} \text{isomorphism for } k < n-1, \\ \text{epimorphism for } k = n-1. \end{cases}$$

(2) *The inclusion induced map*

$$\pi_k(\mathrm{U}(n)) \;\to\; \pi_k(\mathrm{U}(n+1)) \quad \text{is an} \quad \begin{cases} \text{isomorphism for } k < 2n, \\ \text{epimorphism for } k = 2n. \end{cases}$$

(3) *The inclusion induced map*

$$\pi_k(\mathrm{Sp}(n)) \;\to\; \pi_k(\mathrm{Sp}(n+1)) \quad \text{is an} \quad \begin{cases} \text{isomorphism for } k < 4n+2, \\ \text{epimorphism for } k = 4n+2. \end{cases}$$

Example.

(1) We know that $\mathrm{U}(1) = S^1$. Thus we obtain from Proposition 114.10 (2) that all unitary groups $\mathrm{U}(n)$ are path-connected.

(2) Recall that in Lemma 114.2 (1) we pointed out that $\mathrm{Sp}(1) = S^3$. Thus we obtain from Proposition 114.10 (3) that all compact symplectic groups $\mathrm{Sp}(n)$ are simply connected.

Proof. We prove the first statement. The proofs of the remaining two statements are almost identical.[1584] By Proposition 114.8 we have a bundle projection $p\colon \mathrm{O}(n+1) \to S^n$ with fiber $\mathrm{O}(n)$. From Proposition 113.2 we obtain a long exact sequence of the form

$$\ldots \to \pi_{k+1}(S^n) \xrightarrow{\partial} \pi_k(\mathrm{O}(n)) \to \pi_k(\mathrm{O}(n+1)) \to \pi_k(S^n) \xrightarrow{\partial} \ldots$$

The desired statement now follows from the familiar fact that by Proposition 40.10 we have $\pi_i(S^k) = 0$ for $i < k$. ■

As a corollary to Proposition 114.10 (1) we can now finally answer Question 51.5. More precisely, we have the following proposition.

Proposition 114.11.

(1) *For any $n \in \mathbb{N}$ the inclusions $i\colon \mathrm{SO}(n) \to \mathrm{SL}(n,\mathbb{R})$ and $j\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{GL}_+(n,\mathbb{R})$*[1585] *are homotopy equivalences.*

(2) *Let $n \in \mathbb{N}_{\geq 2}$. We have isomorphisms*

$$\pi_1(\mathrm{SO}(n)) \xrightarrow[i_*]{\cong} \pi_1(\mathrm{SL}(n,\mathbb{R})) \xrightarrow[j_*]{\cong} \pi_1(\mathrm{GL}_+(n,\mathbb{R})) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}, & \text{if } n \geq 3. \end{cases}$$

[1583] Recall that in Lemma 2.65 (1) we saw that $\mathrm{O}(n)$ is disconnected. Thus in principle it is not completely clear what we mean by $\pi_k(\mathrm{O}(n))$. But, as we pointed out on page 2703, we always consider $\mathrm{O}(n)$ as equipped with the base point that is given by the identity. Also note that it follows easily from Lemma 3.55 that for a topological group the isomorphism type of the homotopy groups does not depend on the choice of a base point.

[1584] The only catch is that one needs to be careful about the degrees involved.

(3) *In each of the three fundamental groups of* (2) *a generator of the group is represented by the loop*

$$\begin{array}{rcl} S^1 & \to & \mathrm{SO}(n) \\ e^{\mathrm{i}\varphi} & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-2} \end{pmatrix}. \end{array}$$

Proof.

(1) We proved this statement in Lemma 8.8.
(2) First we consider the case $n = 2$. We have isomorphisms

$$\underset{\substack{\uparrow \\ \text{Proposition 16.17}}}{\mathbb{Z} \cong} \pi_1(S^1, \mathrm{id}) = \pi_1(\mathrm{SO}(2), \mathrm{id}) \underset{\substack{\uparrow \\ \text{by (1) and Proposition 18.16 (1)}}}{\xrightarrow{\cong} \pi_1(\mathrm{SL}(2,\mathbb{R}), \mathrm{id}) \xrightarrow{\cong}} \pi_1(\mathrm{GL}_+(2,\mathbb{R}), \mathrm{id}).$$

Now we consider the case $n \geq 3$. We have isomorphisms

$$\mathbb{Z}_2 \underset{\substack{\uparrow \\ \text{Corollary 51.4}}}{\cong} \pi_1(\mathrm{SO}(3), \mathrm{id}) \underset{\substack{\uparrow \\ \text{by Proposition 114.10 and the} \\ \text{fact that by Lemma 2.65 (1)} \\ \mathrm{SO}(k) \text{ is the path-component} \\ \text{of } \mathrm{O}(k) \text{ that contains id}}}{\cong} \pi_1(\mathrm{SO}(n), \mathrm{id}) \underset{\substack{\uparrow \\ \text{by (1) and Proposition 18.16 (1)}}}{\xrightarrow{\cong} \pi_1(\mathrm{SL}(n,\mathbb{R}), \mathrm{id}) \xrightarrow{\cong}} \pi_1(\mathrm{GL}_+(n,\mathbb{R}), \mathrm{id}).$$

(3) We consider the loop

$$\begin{array}{rcl} \gamma\colon S^1 & \to & \mathrm{SO}(2) \\ e^{\mathrm{i}\varphi} & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}. \end{array}$$

Since $\mathrm{SO}(2) = S^1$ we know by Proposition 16.17 that γ represents a generator of $\pi_1(\mathrm{SO}(2))$. By Proposition 114.10 we know that the inclusion $\mathrm{SO}(2) \to \mathrm{SO}(n)$ induces an epimorphisms of fundamental groups. It follows immediately that the loop given in (3) is indeed a generator of $\pi_1(\mathrm{SO}(n))$. Since the inclusions i and j induce isomorphisms of fundamental groups we see that the given loop also represents a generator of $\pi_1(\mathrm{SL}(n,\mathbb{R}))$ and $\pi_1(\mathrm{GL}(n,\mathbb{R}))$. ■

114.3. **Spin groups.** Let $n \geq 3$. Recall that in Lemma 2.65 (1) we showed that $\mathrm{SO}(n)$ is path-connected and in Corollary 114.11 (2) we had just seen that $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$. By Lemma 16.15 this implies that the universal covering, as defined on page 835, of $\mathrm{SO}(n)$ is a 2-fold covering of $\mathrm{SO}(n)$.[1586] For example in Theorem 51.2 we showed that the 2-fold covering of $\mathrm{SO}(3)$ is diffeomorphic to $\mathrm{SU}(2)$.

Thus the following question arises:

[1585]Recall that we write $\mathrm{GL}_+(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det(A) > 0\}$.

[1586]Note that here we use that $\mathrm{SO}(n)$ is a smooth manifold, which implies by the discussion on page 835 that the universal covering does indeed exist.

Question 114.12. *Let $n \geq 3$.*

(1) *Is the universal covering of* $\mathrm{SO}(n)$ *a group?*
(2) *If yes, is it a group "we already know"?*

It turns out that the answer to Question 114.12 (1) is yes. In fact, the following, initially perhaps somewhat surprising proposition holds. The proof turns out to be almost disappointingly easy.

Proposition 114.13. *Let G be a connected Lie group and let $p\colon \widetilde{G} \to G$ be some covering (e.g. it could be the universal covering). We denote by 1_G the neutral element in G. We pick a point $1_{\widetilde{G}} \in p^{-1}(1_G)$. Then $\widetilde{G}$ admits a unique Lie group structure such that $p\colon \widetilde{G} \to G$ is a smooth map and a homomorphism and such that $1_{\widetilde{G}}$ is the neutral element in $\widetilde{G}$.*

Remark.

(1) Let G be a topological group that is connected and locally path-connected. The proof of Proposition 114.13 also shows that given any covering $p\colon \widetilde{G} \to G$ of G together with a choice of $\widetilde{e} \in \widetilde{G}$ with $p(\widetilde{e}) = e$ there exists a unique topological group structure on $\widetilde{G}$ with trivial element $\widetilde{e}$ which turns p into a group homomorphism.
(2) Let $k \in \mathbb{N}_{\geq 2}$. By Proposition 114.13 the universal covering $\widetilde{\mathrm{SL}(k,\mathbb{R})}$ of the Lie group $\mathrm{SL}(k,\mathbb{R})$ is again Lie group. In [**HallB15**, Theorem C.12] it is shown that $\widetilde{\mathrm{SL}(k,\mathbb{R})}$ is *not* isomorphic to any subgroup of $\mathrm{GL}(n,\mathbb{R})$ for any $n \in \mathbb{N}$. This shows that Theorem 114.6 does not extend to non-compact Lie groups.

Sketch of the proof. First note that we saw in Proposition 17.1 that $\widetilde{G}$ is a topological manifold that admits a unique smooth structure such that $p\colon \widetilde{G} \to G$ is a smooth map. Now we turn towards defining the Lie group structure on the smooth manifold $\widetilde{G}$. In fact we will give two equivalent definitions of the group structure on $\widetilde{G}$:

(1) First note that by Proposition 16.20 we know that $\widetilde{G} \times \widetilde{G}$ is also simply connected. We consider the maps

$$\begin{tikzcd} & \widetilde{G} \arrow[d, "p"] \\ \widetilde{G}\times\widetilde{G} \arrow[r, "(x,y)\mapsto p(x)\cdot p(y)"'] & G. \end{tikzcd}$$

It follows from Proposition 29.2 that there exists a unique lift

$$\begin{tikzcd} & \widetilde{G} \arrow[d, "p"] \\ \widetilde{G}\times\widetilde{G} \arrow[ur, dashed, "(1_{\widetilde{G}},1_{\widetilde{G}})\mapsto 1_{\widetilde{G}}"] \arrow[r, "(x,y)\mapsto p(x)\cdot p(y)"'] & G. \end{tikzcd}$$

(2) By Proposition 29.5 and Corollary 29.9 we can assume that $\widetilde{G}$ equals the canonical universal covering, i.e. we can assume that

$$\widetilde{G} \;=\; \{\text{paths in } G \text{ starting at } 1_G\}/\text{ homotopies that fix the starting point and the endpoint}$$

With that explicit description we define

$$\begin{array}{rcl} \widetilde{G} \times \widetilde{G} & \to & \widetilde{G} \\ ([a(t)], [b(t)]) & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & G \\ t & \mapsto & a(t) \cdot b(t) \end{array} \right]. \end{array}$$

Using the fact that the multiplication map $\widetilde{G} \times \widetilde{G} \to \widetilde{G}$ is continuous one can show fairly easily that this map is well-defined.

We leave it as a charming exercise to show that the two given definitions of the group structure agree and that in both cases we defined a Lie group structure on $\widetilde{G}$.[1587]

It remains to show that the group structure on $\widetilde{G}$ is the unique Lie group structure on the smooth manifold $\widetilde{G}$ for which $1_{\widetilde{G}}$ is the neutral element and that turns p into a homomorphism. Suppose we are given a smooth map $\varphi\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ that has the same properties. We obtain the following commutative diagram:

$$\begin{tikzcd} \widetilde{G} \times \widetilde{G} \arrow[r, "\varphi"] \arrow[d, "p\times p"'] \arrow[dr, "(x,y)\mapsto p(x)\cdot p(y)"] & \widetilde{G} \arrow[d, "p"] \\ G \times G \arrow[r, "(x,y)\mapsto x\cdot y"'] & G. \end{tikzcd}$$

Since the diagram commutes we see that $\varphi\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ is precisely a lift of the blue map $\widetilde{G} \times \widetilde{G} \to G$ that sends $(1_{\widetilde{G}}, 1_{\widetilde{G}})$ to $1_{\widetilde{G}}$. Thus by the uniqueness of the lift we see that the Lie group structure on $\widetilde{G}$ is uniquely determined by our wish list. ■

This proposition leads us to the following definition.

Definition. Let $n \geq 3$. We refer to the canonical universal covering[1588] of the pointed topological space $(\mathrm{SO}(n), \mathrm{id})$ as the *spin group* $\mathrm{Spin}(n)$. By Proposition 114.13 there exists a unique Lie group structure on $\mathrm{Spin}(n)$ for which the canonical base point is the neutral element and which has the property that the projection $p\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a smooth map and a group homomorphism.

The combination of Propositions 114.11 (2) and 114.13 and Corollary 29.9 shows that $\mathrm{Spin}(n)$ is indeed a Lie group and that $p\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a 2-fold covering that is a group homomorphism.

Remark. By Corollary 29.9 the universal covering of a smooth manifold, even if it is equipped with a base point, is only well-defined up to “equivalence”. This slightly awkward issue can be circumvented in two different ways:

(1) either we use the explicit model of the universal covering of a based smooth manifold in terms of equivalence classes of paths, or

[1587] The smoothness of the group multiplication is perhaps easier to prove with the first point of view, the fact that it is a group structure is possibly less confusing to verify with the second point of view.

[1588] We refer to page 835 for the definition of the canonical universal covering of a suitable pointed topological space and to the definition of the corresponding canonical base point.

(2) in this particular situation there is a more elegant solution, one can give a purely algebraic explicit description of $\mathrm{Spin}(n)$. We refer to [**Mein13**, Chapter 3.1] for details.

It is harder to give an answer to Question 114.12 (2). For reasonably low dimensions the following proposition gives a positive answer.

Proposition 114.14.

(1) *There exist Lie group isomorphisms*[1589]

$$\begin{array}{llcl} \text{(a)} & \mathrm{SO}(3) & \cong & \mathrm{SU}(2)/\{\pm\,\mathrm{id}\} \\ \text{(b)} & \mathrm{SO}(4) & \cong & (\mathrm{SU}(2)\times\mathrm{SU}(2))/\{\pm(\mathrm{id},\mathrm{id})\} \\ \text{(c)} & \mathrm{SO}(5) & \cong & \mathrm{Sp}(2)/\{\pm\,\mathrm{id}\}. \end{array}$$

(2) *There exist Lie group isomorphisms*

$$\begin{array}{llcl} \text{(a)} & \mathrm{Spin}(3) & \cong & \mathrm{SU}(2) \\ \text{(b)} & \mathrm{Spin}(4) & \cong & \mathrm{SU}(2)\times\mathrm{SU}(2) \\ \text{(c)} & \mathrm{Spin}(5) & \cong & \mathrm{Sp}(2) \\ \text{(d)} & \mathrm{Spin}(6) & \cong & \mathrm{SU}(4). \end{array}$$

Proposition 114.14 (b) thus says that for $n = 3, 4, 5, 6$ the spin group $\mathrm{Spin}(n)$ is isomorphic to Lie groups that we are already familiar with. But according to [**Mein13**, p. 53] "for $n \geq 7$, there are no further accidental isomorphisms of this type." In other words, in general the answer to Question 114.12 (2) is negative.

Proof.

(1) (a) This statement follows immediately from Lemma 51.1 and Theorem 51.2.

(b) In Proposition 60.2 we saw that the quaternion multiplication defines a group structure on $S^3 = \{h \in \mathbb{H} \,|\, |h| = 1\}$ where the trivial element is given by $1 \in \mathbb{H}$. Furthermore in Proposition 60.2 we had also seen that the Lie groups $\mathrm{SU}(2)$ and S^3 are isomorphic. Thus it suffices to show that the Lie groups $(S^3\times S^3)/\{\pm(1,1)\}$ and $\mathrm{SO}(4)$ are diffeomorphic. We consider the following map:

$$\begin{array}{rcl} \Phi\colon S^3\times S^3 & \to & \mathrm{SO}(4) \\ (q_1,q_2) & \mapsto & \left(\begin{array}{rcl} \mathbb{R}^4=\mathbb{H} & \to & \mathbb{R}^4=\mathbb{H} \\ z & \mapsto & q_1 z q_2^{-1} \end{array}\right). \end{array}$$

We make the following observations:

(i) In Exercise 60.1 we showed that the map $z \mapsto q_1 z q_2^{-1}$ does indeed define an element in $\mathrm{SO}(4)$.

(ii) One can easily verify that the map Φ is smooth and that it is a homomorphism.

(iii) We leave it to the energetic reader to verify that the differential of Φ at each point is invertible.

[1589] Note that it follows from Lemma 114.5 that the groups on the right-hand side are indeed Lie groups.

(iv) We note that the kernel of this homomorphism is given by $\{\pm(1,1)\}$. Indeed, the kernel of this homomorphism is given by the set of all $(q_1,q_2)\in S^3\times S^3$ such that $q_1zq_2^{-1}=z$ for all $z\in\mathbb{H}$. By setting $z=1$ we obtain that $q_1=q_2$. Furthermore, since $q_1zq_1^{-1}=z$ for all $z\in S^3$ we see that $q_1=q_2$ lies in the center of the group S^3. An elementary calculation shows that the center of S^3 is given by ±1. Thus the kernel of Φ contains precisely the two elements $(1,1)$ and $(-1,-1)$.

It follows from the above, Lemmas 3.22 and 114.5 and elementary group theory[1590] that the above map Φ descends to a smooth and injective map

$$\overline{\Phi}\colon (S^3\times S^3)/\{\pm(1,1)\} \;\to\; \mathrm{SO}(4)$$

such that the differential at each point is invertible. By Lemmas 114.3 and 114.5 both sides are closed connected 6-dimensional smooth manifolds. Thus it follows from Corollary 50.8 (2) that the map $\overline{\Phi}$ is also surjective. It follows from the above results and the Inverse Mapping Theorem 6.40 (2) that $\overline{\Phi}$ is in fact a diffeomorphism.

(c) Let V be the real vector space of quaternionic Hermitian 2×2-matrices with trace 0. In other words we consider

$$V \;=\; \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{M}(2\times2,\mathbb{H})\,\middle|\, a+d=0, a,d\in\mathbb{R} \text{ and } b=\overline{c}\right\}.$$

Note that V is a 5-dimensional real vector space. The group $\mathrm{Sp}(2)$ acts by conjugation on V.[1591] One easily verifies that the kernel is given by $\{\pm\mathrm{id}\}$. Furthermore by [**Pro07**, Chapter 4.5.3] there exists a positive-definite symmetric form on V that is preserved by this action. Thus, by Sylvester's Theorem 101.10 this action defines a homomorphism $\mathrm{Sp}(2)\to\mathrm{SO}(5)$. Similar to (b) we obtain a diffeomorphism $\mathrm{Sp}(2)/\{\pm\mathrm{id}\}\xrightarrow{\cong}\mathrm{SO}(5)$.

(2) (a) By (2a) we can make the identification $\mathrm{SO}(3)=\mathrm{SU}(2)/\{\pm\mathrm{id}\}$ of Lie groups. By Theorem 51.2 we know that $\mathrm{SU}(2)$ is homeomorphic to S^3, in particular we see that $\mathrm{SU}(2)$ is simply connected. By Lemma 114.5 the quotient map $\mathrm{SU}(2)\to\mathrm{SU}(2)/\{\pm1\}$ is a covering map, it is smooth and it is a homomorphism. Thus we see that $\mathrm{SU}(2)$ is in fact the universal covering of $\mathrm{SU}(2)/\{\pm\mathrm{id}\}=\mathrm{SO}(3)$ and we deduce from Proposition 114.13 that $\mathrm{Spin}(3)\cong\mathrm{SU}(2)$.

(b) By the same argument as in (2a) this statement can be deduced from (1b).

(c) By the same argument as in (2a) this statement can be deduced from (1c).

(d) We will not make use of this results, hence we will not provide a proof, instead we refer to [**Mein13**, p. 84] for details. ■

[1590] More precisely, we use the elementary fact that if $\alpha\colon G\to H$ is a group homomorphism, then the induced map $G/\ker(\alpha)\to H$ is injective.

[1591] The group $\mathrm{Sp}(2)$ acts evidently by conjugation on all quaternionic 2×2-matrices, in principle one now needs to verify that this action restricts to an action on V.

Remark. Initially the spin groups might appear as amusing curiosities. In fact they arise naturally in many parts of mathematics and physics. We refer to [**LaM89, Morg96, Mein13**] for more details.

114.4. **The Bott Periodicity Theorem.** We continue with the following definition.

Definition. We define[1592]

$$\mathrm{O} \;=\; \varinjlim \mathrm{O}(n) \;=\; \left\{ A=(a_{ij})_{i,j\in\mathbb{N}} \;\middle|\; \begin{array}{l}\text{there exists an } n\in\mathbb{N}\\ \text{and a } B\in\mathrm{O}(n) \text{ with}\end{array}\; A=\begin{pmatrix} B & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \ddots\end{pmatrix}\right\}$$

where O has the obvious group structure and where O has the topology that is defined by the condition that subset $U\subset \mathrm{O}$ is open if and only if for each n the intersection $U\cap \mathrm{O}(n)$ is open. Similarly we define

$$\mathrm{U} \;=\; \varinjlim \mathrm{U}(n), \quad \mathrm{Sp} \;=\; \varinjlim \mathrm{Sp}(n)$$

In Proposition 25.1 we saw that every direct system $\{G_i\}_{i\in I}$ of groups admits a direct limit $\varinjlim G_i$. In fact for the above direct limit $\mathrm{O}=\varinjlim \mathrm{O}(n)$ the group structure is the "obvious" one given above. The following shows that these group structure are continuous, i.e. O, U and Sp are topological groups.

Lemma 114.15. *The above group structures on* O, U *and* Sp *are continuous.*

Remark. Initially one might be tempted to dismiss Lemma 114.15 as obvious. But in general there is a subtle issue. If $\{G_i\}_{i\in I}$ is a direct system of topological groups, then the above multiplication on the direct limit $\varinjlim G_i$ might not be continuous. Such examples are for example provided in [**TSH98**]. If one thinks about it for a while one realizes that this failure for the multiplication map to be continuous is another instance of the observation made in Section 5.2, that mixing product and quotient topologies is fraught with dangers.

Proof. This lemma can surely be proved, with a little effort, "by hand". Alternatively one can use [**TSH98**, Theorem 2.7] which shows that for a direct system $\{X_i\}_{i\in\mathbb{N}}$ of compact topological groups the above multiplication on $\varinjlim X_i$ is indeed continuous. ■

For later usages of O, U and Sp we have to get the following technical lemma out of the way.

Lemma 114.16. $(*)$ *The topological spaces* O, U *and* Sp *admit* CW*-structures such that each* $\mathrm{O}(n)\subset\mathrm{O}$, $\mathrm{U}(n)\subset\mathrm{U}$ *and* $\mathrm{Sp}(n)\subset\mathrm{Sp}$ *is a subcomplex.*

Proof. In [**WhdJ44**] it is shown that we can equip the $\mathrm{O}(n)$ with CW-structures such that for each n the subset $\mathrm{O}(n)\subset\mathrm{O}(n+1)$ is actually a subcomplex. Analogous statements for $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ are proved in [**Yok55**, Theorem 5.3 and 5.5] or alternatively in [**Stro11**, Chapter 9.3.3]. It follows from Lemma 36.34 that O is a CW-complex. ■

[1592]When we talk about the direct limit we have to say what category we consider. Here we want to consider the category of groups and of topological spaces at the same time. Since we have obvious inclusions maps $\mathrm{O}(n)\to\mathrm{O}(n+1)$ we obtain from Lemma 25.3 that in both settings we obtain the same underlying set, which we can then equip with a group structure and a topology.

Now we turn to the study of the homotopy groups of O, U and Sp.

Lemma 114.17. *Given any $k \in \mathbb{N}$ we have*

$$\begin{array}{llcllll} (1) & \pi_k(\mathrm{O}) & = & \varinjlim \pi_k(\mathrm{O}(n)) & \xleftarrow{\cong} & \pi_k(\mathrm{O}(N)) & \text{for any } N \in \mathbb{N} \text{ with } N > k+1, \\ (2) & \pi_k(\mathrm{U}) & = & \varinjlim \pi_k(\mathrm{U}(n)) & \xleftarrow{\cong} & \pi_k(\mathrm{U}(N)) & \text{for any } N \in \mathbb{N} \text{ with } 2N > k, \\ (3) & \pi_k(\mathrm{Sp}) & = & \varinjlim \pi_k(\mathrm{Sp}(n)) & \xleftarrow{\cong} & \pi_k(\mathrm{Sp}(N)) & \text{for any } N \in \mathbb{N} \text{ with } 4N > k-2. \end{array}$$

PROOF (∗). We prove (1). The proofs of (2) and (3) are almost identical. We start out with the following claim.

Claim. Given any compact subset K of O there exists an $n \in \mathbb{N}$ with $K \subset \mathrm{O}(n)$.

Let K be a compact subset of O. Given $n \in \mathbb{N}$ we write $X_n := \{1, \dots, n\} \times \{1, \dots, n\}$. First recall that $\mathrm{O}(n)$ is equipped with the subspace topology coming from the inclusion map $g_n\colon \mathrm{O}(n) \subset \mathrm{M}(n \times n, \mathbb{R}) = \mathbb{R}^{X_n} = \mathbb{R}^{n^2}$. Given $n \in \mathbb{N}$ we denote by $f_n\colon \mathbb{R}^{X_n} \to \mathbb{R}^{X_{n+1}}$ the obvious inclusion map. These have the property that for any $n \in \mathbb{N}$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{O}(n) \arrow[r, "g_n"] \arrow[d, hook] & \mathbb{R}^{X_n} \arrow[d, hook, "f_n"] \\ \mathrm{O}(n+1) \arrow[r, "g_{n+1}"] & \mathbb{R}^{X_{n+1}}. \end{tikzcd}$$

The above inclusion maps $g_n\colon \mathrm{O}(n) \to \mathbb{R}^{X_n}$ induce a map $g\colon \mathrm{O} = \varinjlim \mathrm{O}(n) \hookrightarrow \varinjlim \mathbb{R}^{X_n}$. By our hypothesis that K is compact and Lemma 2.40 we know that $g(K) \subset \varinjlim \mathbb{R}^{X_n}$ is also compact. Basically the same argument as in the proof of Lemma 36.3 shows that there exists an $n \in \mathbb{N}$ with $g(K) \subset \mathbb{R}^{X_n}$. But this implies that $K \subset \mathrm{O}(n)$. ⊞

Now let $N \in \mathbb{N}$ with $N > k+1$. We have

$$\pi_k(\mathrm{O}) \underset{\substack{\uparrow \\ \text{by Proposition 40.11} \\ \text{and the claim}}}{=} \varinjlim \pi_k(\mathrm{O}(n)) \underset{\substack{\uparrow \\ \text{Proposition 114.10 (1)} \\ \text{and Lemma 25.2}}}{=} \pi_k(\mathrm{O}(N)).$$

■

We will now state two versions of the Bott Periodicity Theorem which implies in particular that the homotopy groups of O, U and Sp are periodic.

The first theorem requires just a little bit of preparation.

(1) As above we equip U, Sp and O with the base point given by the infinite identity matrix. We use these base points to form the loop spaces ΩU, $\Omega\,\mathrm{Sp}$ and ΩO.
(2) Given a point x in a topological space X we denote by c_x the corresponding constant path.
(3) We use notation (2) to equip each loop space $\Omega(X, x_0)$ with the base c_{x_0} and we can thus iterate the loop space construction.
(4) The obvious isomorphism $\mathbb{H} \xrightarrow{\cong} \mathbb{R}^4$ of real vector spaces defines inclusion maps $\mathrm{Sp}(n) \to \mathrm{O}(4n)$ and $\mathrm{Sp} \to \mathrm{O}$.
(5) The inclusion $\mathbb{R} \to \mathbb{H}$ defines inclusions $\mathrm{O}(n) \to \mathrm{Sp}(n)$ and thus $\mathrm{O} \to \mathrm{Sp}$.

Theorem 114.18. (Bott Periodicity Theorem) *There exist homotopy equivalences*

$$(1)\quad \mathrm{U} \to \Omega^2\,\mathrm{U} \qquad (2)\quad \mathrm{Sp} \to \Omega^4\,\mathrm{O} \qquad \textit{and} \qquad (3)\quad \mathrm{O} \to \Omega^4\,\mathrm{Sp}.$$

Before we talk about the proof of the Bott Periodicity Theorem 114.18 let us move on to the following direct consequence.

Theorem 114.19. (Bott Periodicity Theorem) *For any $i \in \mathbb{N}_0$ we have*[1593]

$$\begin{array}{lrcl}
(1) & \pi_{i+2}(\mathrm{U}) & \cong & \pi_i(\mathrm{U}), \\
(2) & \pi_{i+4}(\mathrm{Sp}) & \cong & \pi_i(\mathrm{O}), \\
(3) & \pi_{i+4}(\mathrm{O}) & \cong & \pi_i(\mathrm{Sp}).
\end{array}$$

In particular, combining (2) and (3) we have for any $i \in \mathbb{N}_0$ that

$$\begin{array}{lrcl}
(4) & \pi_{i+8}(\mathrm{O}) & \cong & \pi_i(\mathrm{O}), \\
(5) & \pi_{i+8}(\mathrm{Sp}) & \cong & \pi_i(\mathrm{Sp}).
\end{array}$$

Proof of the Bott Periodicity Theorems 114.18 and 114.19. First note that Theorem 114.19 follows immediately from Theorem 114.18 together with Corollary 112.15. All the known proofs are non-trivial and go well-beyond what we can do in these modest notes.

Theorem 114.18 was first proved by Raoul Bott [**Bot57, Bot59, Bot70**] in 1957. Another proof, using Morse theory, is given in [**Miln63a**, p. 129, 142]. There explicit maps that give these homotopy equivalences, they are described well in [**DL61**]. Nonetheless the description is too long for us to write down at this point.

There are many proofs of Theorem 114.19 which use a slightly different language, in particular it is at first glance not entirely clear whether they also imply Theorem 114.18. We refer to [**Stro11**, Chapter 32.6], [**AGP02**, Chapter 9.5], [**APr99**] and [**Hat2**, Theorem 2.16] for a discussions and proofs of Theorem 114.19. ■

Using the Bott Periodicity Theorem 114.19 we can completely determine the homotopy groups of O, U and Sp.

Proposition 114.20. *The homotopy groups of* O, U *and* Sp *are as follows:*

$i \bmod 8$	0	1	2	3	4	5	6	7
$\pi_i(\mathrm{U})$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\pi_i(\mathrm{O})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\pi_i(\mathrm{Sp})$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$.

Proof. The homotopy groups of U are straightforward to determine. More precisely, let $i \in \mathbb{N}_0$. We have

$$\pi_i(\mathrm{U}) \underset{\substack{\uparrow \\ \text{Bott Periodicity Theorem 114.19}}}{=} \pi_{i \bmod 2}(\mathrm{U}) \underset{\substack{\uparrow \\ \text{Lemma 114.17 (2)}}}{\overset{\cong}{\leftarrow}} \pi_{i \bmod 2}(\mathrm{U}(1)) \underset{\substack{\uparrow \\ \text{since } \mathrm{U}(1) = S^1}}{=} \pi_{i \bmod 2}(S^1) \underset{\substack{\uparrow \\ \text{see Lemma 2.65 and} \\ \text{Proposition 16.17}}}{\cong} \begin{cases} 0, & \text{if } i = 0 \bmod 2, \\ \mathbb{Z}, & \text{if } i = 1 \bmod 2. \end{cases}$$

[1593] For $i = 0$ we actually have an isomorphism of pointed sets.

Now we consider the homotopy groups of O and Sp. By the Bott Periodicity Theorem 114.19 it suffices to compute the homotopy groups $\pi_i(\mathrm{O})$ and $\pi_i(\mathrm{Sp})$ for $i = 0, 1, 2, 3$.[1594] So let $i \in \{0, 1, 2, 3\}$. We have

$$\pi_i(\mathrm{Sp}) \underset{\substack{\uparrow \\ \text{Lemma 114.17 (3)}}}{\xleftarrow{\cong}} \pi_i(\mathrm{Sp}(1)) \underset{\substack{\uparrow \\ \text{since } \mathrm{Sp}(1) = S^3}}{=} \pi_i(S^3) \underset{\substack{\uparrow \\ \text{Proposition 40.10 and Corollary 53.6 (1)}}}{\cong} \begin{cases} 0, & \text{if } i = 0, 1, 2, \\ \mathbb{Z}, & \text{if } i = 3. \end{cases}$$

We turn to the calculation of the homotopy groups of O. First we note that for any $i \geq 1$ we have

$$\pi_i(\mathrm{O}) \underset{\substack{\uparrow \\ \text{Lemma 114.17 (3)}}}{\xleftarrow{\cong}} \pi_i(\mathrm{O}(i+2)) \underset{\substack{\uparrow \\ \text{Lemmas 2.65 and 3.55}}}{\cong} \pi_i(\mathrm{SO}(i+2)).$$

Thus it remains to prove the following claim.

Claim.

(a)	$\pi_0(\mathrm{O}(2)) \cong \mathbb{Z}_2$	(c)	$\pi_2(\mathrm{SO}(4)) \cong 0$
(b)	$\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$	(d)	$\pi_3(\mathrm{SO}(5)) \cong \mathbb{Z}.$

We turn to the proof of these four separate claims.

(a) By Lemma 2.65 (1) we know that $\mathrm{O}(2)$ consists of two path components, thus $\pi_0(\mathrm{O}(2))$ is by definition a pointed set with two elements.
(b) By Proposition 114.11 we know that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$.
(c) We have the following isomorphisms:

$$\pi_2(\mathrm{SO}(4)) \underset{\substack{\uparrow \\ \text{by Proposition 114.14 (1b)} \\ \text{and Proposition 60.2}}}{\cong} \pi_2((S^3 \times S^3)/\mathbb{Z}_2) \underset{\substack{\uparrow \\ \text{by Lemma 114.5} \\ \text{and Proposition 40.13}}}{\cong} \pi_2(S^3 \times S^3) \underset{\substack{\uparrow \\ \text{Proposition 40.8}}}{\cong} \pi_2(S^3) \oplus \pi_2(S^3) \underset{\substack{\uparrow \\ \text{Proposition 40.10}}}{=} 0.$$

(d) We have the following isomorphisms:

$$\pi_3(\mathrm{SO}(5)) \underset{\substack{\uparrow \\ \text{Proposition 114.14 (1c)}}}{\cong} \pi_3(\mathrm{Sp}(2)/\{\pm \mathrm{id}\}) \underset{\substack{\uparrow \\ \text{by Lemma 114.5 and} \\ \text{Proposition 40.13}}}{\cong} \pi_3(\mathrm{Sp}(2)) \underset{\substack{\uparrow \\ \text{Proposition 114.10}}}{\cong} \pi_3(\mathrm{Sp}) \underset{\substack{\uparrow \\ \text{see above}}}{\cong} \mathbb{Z}.$$

■

114.5. **H-spaces** (∗)**.** In this final section of this chapter we want to state two general statements about homotopy groups of Lie groups. In fact these statements hold for a much wider class of topological spaces that we introduce now:

Definition. We say a topological space X is an *H-space*[1595] if there exists a map[1596]

$$\mu \colon X \times X \to X$$

[1594]We will see that $\pi_0(\mathrm{O})$ is the pointed set with two elements. It follows from the Bott Periodicity Theorem 114.19 that $\pi_{4+8k}(\mathrm{Sp})$ and $\pi_{8k}(\mathrm{O})$, $k \geq 1$, are also pointed sets with two elements. But these pointed sets are also groups. Since there is only one group structure on a pointed set with two elements we see that all these groups are isomorphic to $\mathbb{Z}_2$.

and an element $e \in X$ such that $\mu(e,e) = e$ and such that the maps

$$\begin{array}{rcl} (X,e) & \mapsto & (X,e) \\ x & \mapsto & \mu(x,e) \end{array} \quad \text{and} \quad \begin{array}{rcl} (X,e) & \mapsto & (X,e) \\ x & \mapsto & \mu(e,x) \end{array}$$

of pairs of topological spaces are homotopic to the identity.

Examples.

(1) Evidently Lie groups and more generally, topological groups are H-spaces, where μ is given by the group structure and e is the trivial element of the group. In particular we see that the Lie groups S^1 and S^3 are H-spaces.
(2) The topological space S^7 together with the multiplication given by the octonion structure on $\mathbb{O} = \mathbb{R}^8$ and with $e = 1 = (1,0,\dots,0) \in S^7$ is an H-space.[1597] Since octonion multiplication is not associative we see that, at least with this multiplication, S^7 is not a topological group.

We have just seen that the spheres S^1, S^3 and S^7 are H-space. Before we continue with more examples we point out that we have all the tools to show that no other spheres have H-space structures.

Theorem 114.21. *Let $n \in \mathbb{N}$. If S^n admits the structure of an H-space, then $n \in \{1,3,7\}$.*

Proof. Let $\mu\colon S^n \times S^n \to S^n$ be the multiplication map of an H-space structure and let $e \in X$ be the "neutral" element. It follows immediately from the definition of the bidegree, see page 2223, together with Proposition 42.5 and the existence of e, that the map μ has bidegree $(1,1)$. From Proposition 91.12 we obtain a map $S^{2n+1} \to S^{n+1}$ with Hopf invariant ± 1. By the Adams Theorem 110.11[1598] we know that $n+1 \in \{2,4,8\}$. ■

We continue with one more amusing example of an H-space.

Example. Given a non-zero polynomial $p(t) = \sum\limits_{i=0}^{n} a_i t^i \in \mathbb{C}[t]$ we define

$$[p(t)] \;:=\; [a_0 : a_1 : \cdots : a_n : 0 : \dots] \;\in\; \mathbb{C}\mathrm{P}^\infty.$$

Evidently every element in $\mathbb{C}\mathrm{P}^\infty$ is of that form. We consider the "multiplication" map

$$\begin{array}{rcl} \mu\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty & \to & \mathbb{C}\mathrm{P}^\infty \\ ([p(t)],[q(t)]) & \mapsto & [\underbrace{p(t)\cdot q(t)}_{\in \mathbb{C}[t]}]. \end{array}$$

We leave it to the reader to verify that μ is continuous. It is clear that left and right "multiplication" by $e = [1] = [1:0:\dots]$ is the identity. This shows that $\mathbb{C}\mathrm{P}^\infty$ together

---

[1595] According to [**Hub99**, p. 747] the term H-space was coined by Jean-Pierre Serre to honor the work of Heinz Hopf.

[1596] As usual, whenever we deal with topological spaces, all maps are understood to be continuous.

[1597] Recall that by the discussion on page 1465 we know that for octonions $w, z \in \mathbb{O} = \mathbb{R}^8$ we have $\|w \cdot z\| = \|w\| \cdot \|z\|$, in particular the multiplication restricts to a map on S^7. Also note that multiplication by $e = 1$ is the identity.

[1598] We partially proved the somewhat weaker Theorem 110.10.

with μ and e is an H-space. Note that this multiplication μ is commutative and associative, but it does not have inverses[1599], thus μ does not define a group structure on $\mathbb{CP}^\infty$.

We formulate the last example as a lemma.

Lemma 114.22. *Let (X, x_0) be a pointed topological space. We denote by $\Omega(X, x_0)$ the loop space as defined on page 2661.*

(1) *The map*

$$\begin{array}{rcl} \mu\colon \Omega(X,x_0)\times\Omega(X,x_0) & \to & \Omega(X,x_0) \\ (f\colon [0,1]\to X, g\colon [0,1]\to X) & \mapsto & (\underbrace{f * g}_{\substack{\text{multiplication of loops}\\ \text{as defined on page 465}}} : [0,1]\to X) \end{array}$$

is continuous.

(2) *The map $\mu\colon \Omega(X,x_0)\times\Omega(X,x_0)\to\Omega(X,x_0)$ from (1) together with the constant loop given by $e(t)\equiv x_0$, turns $\Omega(X,x_0)$ into an H-space.*

Remark.

(1) Note that for the H-space $\Omega(X, x_0)$ "multiplication" by e is not the identity, it is only homotopy equivalent to the identity.
(2) We had hinted at the existence of a multiplication map on $\Omega(X, x_0)$ in Exercise 112.5.

Proof (∗). Let (X, x_0) be a pointed topological space. To simplify the notation we write ΩX instead of $\Omega(X, x_0)$.

(1) We need to show that the map $\mu\colon \Omega X\times\Omega X\to\Omega X$ is in fact continuous. By the "⇒"-direction of Proposition 5.6, applied to $T = \Omega X\times\Omega X$ and the regionally compact topological space $Y = [0,1]$, it suffices to show that the following map is continuous:

$$\Phi\colon (\Omega X\times\Omega X)\times[0,1] \xrightarrow{\mu\times\mathrm{id}} \Omega X\times[0,1] \xrightarrow{(f,t)\mapsto f(t)} X.$$

On the subset $(\Omega X\times\Omega X)\times[0,\frac{1}{2}]$ this map is equal to

$$(\Omega X\times\Omega X)\times[0,\tfrac{1}{2}] \xrightarrow{((f,g)\mapsto f)\times(t\mapsto 2t))} \Omega X\times[0,1] \xrightarrow{(f,t)\mapsto f(t)} X.$$

The first map is evidently continuous. Furthermore, the second map is continuous by Proposition 5.4. Thus we have shown that the restriction of Φ to the subset $(\Omega X\times\Omega X)\times[0,\frac{1}{2}]$ is continuous. Basically the same argument shows that the restriction of Φ to $(\Omega X\times\Omega X)\times[\frac{1}{2},1]$ is continuous. it follows from Lemma 14.3 that Φ itself is continuous.

(2) Let $e\in\Omega(X,x_0)$ be the constant loop given by $e(t)\equiv x_0$. It is clear that $\mu(e,e)=e$. Thus it remains to show that the maps

$$\begin{array}{rcl} (\Omega(X,x_0),e) & \mapsto & (\Omega(X,x_0),e) \\ f & \mapsto & \mu(f,e) \end{array} \quad\text{and}\quad \begin{array}{rcl} (\Omega(X,x_0),e) & \mapsto & (\Omega(X,x_0),e) \\ f & \mapsto & \mu(e,f) \end{array}$$

[1599] Why are there elements in $\mathbb{CP}^\infty$ which do not have multiplicative inverses?

of pairs of topological spaces are homotopic to the identity. We consider the map

$$\begin{array}{rcl} H\colon \Omega(X,x_0)\times[0,1] & \to & \Omega(X,x_0) \\ ((f\colon[0,1]\to X),t) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ s & \mapsto & \begin{cases} f(\frac{2s}{t+1}), & \text{if } s\in[0,\frac{t+1}{2}], \\ x_0, & \text{otherwise.}\end{cases}\end{array}\right)\end{array}$$

We leave it to the underchallenged reader to show, using the "⇒"-direction of Proposition 5.6, that this map H is in fact continuous. Note that we have $H(e,t)=e$ for all $t\in[0,1]$. With these observations it is basically clear that it is the desired homotopy between the map $f\mapsto\mu(f,e)$ and the identity.

Almost exactly the same way we deal with the second map $f\mapsto\mu(e,f)$. ■

The following lemma is the only statement on H-spaces that we can prove with our limited space resources.

Lemma 114.23. *If X is a path-connected H-space, then $\pi_1(X)$ is abelian.*

Remark. Let (Y,y_0) be a pointed topological space. We recall the following two facts:

(a) Recall that in Corollary 112.15 we saw that given any $k\in\mathbb{N}$ there exists an isomorphism $\pi_{k+1}(Y,y_0)\xrightarrow{\cong}\pi_k(\Omega(Y,y_0))$.
(b) We pointed out above that the loop space $\Omega(Y,y_0)$ is an H-space.

The combination of (a), (b) and Lemma 114.23 implies that given any $k\in\mathbb{N}_{\geq 2}$ we have

$$\pi_k(Y,y_0)\underset{\substack{\uparrow\\ \text{by (a)}}}{\cong}\pi_{k-1}(\Omega(Y,y_0))\cong\ \dots\underset{\substack{\uparrow\\ \text{by (a)}}}{\cong}\pi_1(\overbrace{\Omega(\Omega\dots\Omega}^{k \text{ times}}(Y,y_0)))\underset{\substack{\uparrow\\ \text{by (b) and since } k\geq 2}}{=}\pi_1(\text{H-space})\underset{\substack{\uparrow\\ \text{Lemma 114.23}}}{=}\text{abelian.}$$

We have thus shown that $\pi_k(Y,y_0)$ is abelian for every $k\geq 2$. In other words, we have obtained a new proof for Proposition 40.3.

Just for kicks we provide two different proofs for Lemma 114.23. The first proof, which we provide here is "down to earth". The second proof, which we will provide in the next section, is more structural.

First proof of Lemma 114.23 (∗). We denote by $\mu\colon X\times X\to X$ and $e\in X$ the objects in the definition of an H-space. Let $f,g\colon[0,1]\to X$ be loops in e. We need to show that the path $f*g$ is path-homotopic to the path $g*f$.

We denote by $\mu(f,e)$ the path $t\mapsto\mu(f(t),e)$ and similarly we define $\mu(e,g)$. We consider the map

$$\begin{array}{rcl} F\colon[0,1]\times[0,1] & \to & X \\ (s,t) & \mapsto & \mu(f(s),g(t)).\end{array}$$

Note that this map is continuous by Lemma 3.8 (2b) and the fact that μ is continuous. Furthermore we consider the following path-homotopy:[1600]

$$\begin{array}{rcl} H\colon [0,1]\times[0,1] & \to & X \\ (s,t) & \mapsto & \begin{cases} F(2s\cdot(1-t,t)), & \text{if } s\le\frac12 \\ F((2-2s)\cdot(1-t,t)+(2s-1)\cdot(1,1)), & \text{if } s>\frac12. \end{cases} \end{array}$$

By carefully plugging in $t=0$ and $t=1$ we see that H is a path-homotopy from the loop $\mu(f,e)*\mu(e,g)$ to the loop $\mu(e,g)*\mu(f,e)$.

By definition of an H-space there exists a homotopy $A\colon X\times[0,1]\to X$ rel $\{e\}$ with $A_0=\mathrm{id}$ and $A_1=\mu(-,e)$ and there exists a homotopy $B\colon X\times[0,1]\to X$ rel $\{e\}$ with $B_0=\mathrm{id}$ and $B_1=\mu(e,-)$. We consider the maps

$$\begin{array}{rcl} A_f\colon [0,1]\times[0,1] & \to & X \\ (s,t) & \mapsto & A(\underbrace{f(s)}_{\in X},t) \end{array} \quad\text{and}\quad \begin{array}{rcl} B_g\colon [0,1]\times[0,1] & \to & X \\ (s,t) & \mapsto & B(\underbrace{g(s)}_{\in X},t). \end{array}$$

Note that A_f is a path-homotopy from the loop f to the loop $\mu(f,e)$ and B_g is a path-homotopy from the loop g to the loop $\mu(e,g)$. The path-homotopy between the loops $f*g$ and $g*f$ is given by combining H with A_f and B_g as indicated in Figure 1591. We leave it to the reader to conscientiously write down the precise homotopy. ■

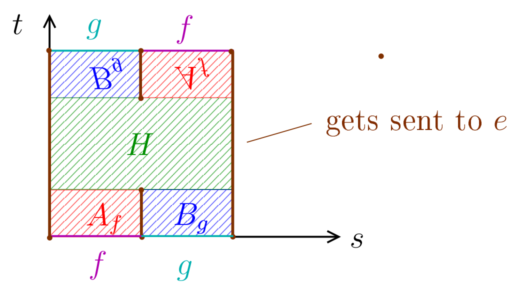


FIGURE 1591. Illustration for the proof of Lemma 114.23.

The following theorem was proved by William Browder [**Browd60**, Theorem 2] [**Browd61**, Theorem 6.11] in 1961.

Theorem 114.24. *Let X be a path-connected H-space. If the group $\bigoplus_{i=0}^{\infty}\mathrm{H}_i(X;\mathbb{Z})$ is finitely generated, then*

$$\min\{m\in\mathbb{N}_{\ge 2}\,|\,\pi_m(X)\ne 0\}$$

is odd. In particular we have $\pi_2(X)=0$.

Examples.

(1) On page 2716 we saw that $\mathbb{C}\mathrm{P}^\infty$ is an H-space. Furthermore in Corollary 53.6 (2) we saw that $\pi_2(\mathbb{C}\mathrm{P}^\infty)\ne 0$. Thus we see that in the statement of Theorem 114.24 we cannot drop the hypothesis that $\bigoplus_{i=0}^{\infty}\mathrm{H}_i(X;\mathbb{Z})$ is a finitely generated group.

[1600] It follows easily from Lemma 2.35 (2') that this map is continuous.

(2) Let $k \in \mathbb{N}$. It follows from Theorem 114.24, together with Theorem 113.12 and the discussion on page 1262, that the finite complex projective space $\mathbb{CP}^k$ does *not* admit an H-space structure.

We conclude this chapter with a corresponding theorem for Lie groups.

Theorem 114.25. *If G is a path-connected Lie group, then*

$$\min\{m \in \mathbb{N}_{\geq 2} \,|\, \pi_m(G) \neq 0\}$$

is odd. In particular $\pi_2(G) = 0$.

Remark. There are many direct proofs in the literature for the statement that given a path-connected Lie group G the second homotopy group $\pi_2(G)$ vanishes, see e.g. [**Carta36**] or [**Miln63a**, Theorem 21.7] or [**BtD95**, Proposition V.7.5].

Note that Theorem 114.25 has no conditions on the Lie group except that it needs to be path-connected. The reason why we can get away with so few conditions is given by the following proposition.

Proposition 114.26. *Let G be a Lie group. There exists a compact Lie subgroup K[1601] such that G is homeomorphic to $K \times \mathbb{R}^n$ for some $n \in \mathbb{N}_0$.*

Proof of Proposition 114.26. Like many other foundational results on Lie groups this proposition has its origins in the work of Henri Cartan [**Carta36**]. A full proof is given in [**Mal45**, Theorem 11] or [**Mos49**, Theorem 2]. A sketch of the proof is provided in [**Sam52**, Chapter 7]. ■

Example. We consider the group $\operatorname{Isom}(\mathbb{R}^n)$ of isometries of the vector space $\mathbb{R}^n$ equipped with the usual inner product. It is an elementary, but not totally trivial fact, see [**Rees83**, p. 11], that

$$\operatorname{Isom}(\mathbb{R}^n) \;=\; \{\text{all maps of the form } x \mapsto Ax + v \text{ where } A \in \mathrm{O}(n) \text{ and } v \in \mathbb{R}^n\}.$$

The maps

$$\begin{array}{rcl} \mathrm{O}(n) \times \mathbb{R}^n & \to & \operatorname{Isom}(\mathbb{R}^n) \\ (A, v) & \mapsto & (x \mapsto Ax + v) \end{array} \quad \text{and} \quad \begin{array}{rcl} \operatorname{Isom}(\mathbb{R}^n) & \to & \mathrm{O}(n) \times \mathbb{R}^n \\ f & \mapsto & (\underbrace{(x \mapsto (f(x) - f(0))}_{\text{represented by matrix in } \mathrm{O}(n)}, f(0)) \end{array}$$

are inverses of one another. Thus we can use the smooth manifold structure on $\mathrm{O}(n) \times \mathbb{R}^n$ to turn $\operatorname{Isom}(\mathbb{R}^n)$ into a smooth manifold and it is quasi-obvious to see that this turns $\operatorname{Isom}(\mathbb{R}^n)$ into a Lie group. We can view $\mathbb{R}^n$ and $\mathrm{O}(n)$ as subgroups of $\operatorname{Isom}(\mathbb{R}^n)$ in an obvious way. But one easily verifies that the map $\mathrm{O}(n) \times \mathbb{R}^n \to \operatorname{Isom}(\mathbb{R}^n)$ is not an isomorphism. In other words, as predicted by Proposition 114.26 the Lie group $\operatorname{Isom}(\mathbb{R}^n)$ splits topologically into a product. But the Lie group does not split as a Lie group.

Proof of Theorem 114.25. Let G be a path-connected Lie group. By Proposition 114.26 we know that G is homotopy equivalent to a compact path-connected Lie group K. By

[1601] In fact a Lie group contains a maximal compact Lie subgroup which is furthermore unique up to conjugation. We will not make use of this fact.

Proposition 40.7 (2) it suffices to prove the desired statement for K. Since K is in particular a compact smooth manifold we obtain from Proposition 64.6 that $\bigoplus_{i=0}^{\infty} \mathrm{H}_i(X;\mathbb{Z})$ is a finitely generated abelian group. Thus we can conclude the proof by appealing to Theorem 114.24. ■

114.6. **The Eckmann-Hilton Theorem** ($*$). In this section we will state and prove a theorem that was provided by Bruno Eckmann and Peter Hilton [**EcH62**] in 1962. The theorem is totally elementary, but at the same time it is surprisingly useful. In particular it will be the key to giving the second proof of Lemma 114.23.

The formulation of the Eckmann-Hilton Theorem requires the following harmless definition.

Definition. Let X be a set.

(1) A *binary operation* is nothing but a map $X \times X \to X$.
(2) A binary operation $\mu\colon X \times X \to X$ is called *unital* if there exists an element $1 \in X$ such that $\mu(1,x) = \mu(x,1) = x$ for all $x \in X$.

Example. The multiplication map on a group is evidently a unital binary operation.

The following theorem is in some sense quite elementary, but the cleverness lies in its formulation.

Theorem 114.27. (Eckmann-Hilton Theorem) *Let X be a set equipped with two binary operations "$\circ$" and "$\otimes$". Suppose the following two conditions are satisfied:*

(a) *"$\circ$" and "$\otimes$" are both unital.*
(b) *For every $a, b, c, d \in X$ we have the equality*
$$(a \otimes b) \circ (c \otimes d) \;=\; (a \circ c) \otimes (b \circ d).$$

Then "$\circ$" and "$\otimes$" are the same and they are commutative and associative.

Proof. First we show that the units of the two operations "$\circ$" and "$\otimes$" coincide:
$$1_\circ \;=\; 1_\circ \circ 1_\circ \;=\; (1_\otimes \otimes 1_\circ) \circ (1_\circ \otimes 1_\otimes) \underset{\substack{\uparrow \\ \text{by (b)}}}{=} (1_\otimes \circ 1_\circ) \otimes (1_\circ \circ 1_\otimes) \;=\; 1_\otimes \otimes 1_\otimes \;=\; 1_\otimes.$$

Next we show that the two operations "$\circ$" and "$\otimes$" are the same and that they are commutative. Thus let $x, y \in X$. Then
$$x \circ y = (1 \otimes x) \circ (y \otimes 1) \underset{\substack{\uparrow \\ \text{by (b)}}}{=} (1 \circ y) \otimes (x \circ 1) = y \otimes x = (y \circ 1) \otimes (1 \circ x) \underset{\substack{\uparrow \\ \text{again by (b)}}}{=} (y \otimes 1) \circ (1 \otimes x) = y \circ x.$$

Finally we show that the operations are associative. Thus let $x, y, z \in X$. We have the following equality:
$$(x \otimes y) \otimes z \;=\; (x \otimes y) \otimes (1 \otimes z) \underset{\substack{\uparrow \\ \text{one last time by (b)}}}{=} (x \otimes 1) \otimes (y \otimes z) \;=\; x \otimes (y \otimes z).$$
■

Now we can prove the following lemma, which in particular gives the promised second, more structural proof of Lemma 114.23.

Lemma 114.28. *Let X be a path-connected H-space and let $x_0 \in X$. In the following we denote by $\mu\colon X \times X \to X$ the multiplication map of the H-space. We consider the following two binary operations on $\pi_1(X, x_0)$:*

$$\begin{array}{rcl} \pi_1(X,x_0)\times\pi_1(X,x_0) & \xrightarrow{\circ} & \pi_1(X,x_0) \\ ([f],[g]) & \mapsto & [f * g] \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi_1(X,x_0)\times\pi_1(X,x_0) & \xrightarrow{\otimes} & \pi_1(X,x_0) \\ ([f],[g]) & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & X \\ t & \mapsto & \mu(f(t),g(t)) \end{array}\right]. \end{array}$$

The following two statements hold:

1. *The two binary operations agree.*
2. *The group $\pi_1(X, x_0)$ is abelian.*

Proof. Let X be a path-connected H-space and let $x_0 \in X$. We denote by $\mu\colon X \times X \to X$ the multiplication map of the H-space. Since μ is an H-space structure one sees easily that "$\otimes$" is a unital binary operation. Furthermore we have known ever since Proposition 14.7 that "$\circ$" is a unital binary operation. In Exercise 114.9 we will show that Condition (b) of the Eckmann-Hilton Theorem 114.27 is satisfied. Thus we obtain from the Eckmann-Hilton Theorem 114.27 the amusing fact that the two operations agree and we obtain that both operations are commutative. In particular this means that $\pi_1(X, x_0)$ is abelian. ■

Exercises for Chapter 114.

Exercise 114.1. Let X be the torus minus one open disk. Does X admit the structure of a topological group?

Exercise 114.2. Let M be smooth manifold without boundary and let $g\colon M \times M \to M$ be a map that is smooth and that defines a group structure on M. Show that (M, g) is a Lie group. In other words, show that the map $M \to M$ given by $x \mapsto x^{-1}$ is smooth.
Hint. Use the Implicit Function Theorem, see e.g. [**Lee02**, p. 661].

Exercise 114.3. Let $n \in \mathbb{N}$. We define

$$\mathrm{GL}(n,\mathbb{H}) \;=\; \left\{ A \in \mathrm{M}(n\times n,\mathbb{H}) \;\middle|\; \begin{array}{c} \text{there exist matrices } X, Y \in \mathrm{M}(n\times n,\mathbb{H}) \\ \text{such that } AX = YA = \mathrm{id} \end{array} \right\}.$$

Is $\mathrm{GL}(n, \mathbb{H})$ an open subset of $\mathrm{M}(n \times n, \mathbb{H})$?

Exercise 114.4. Let $n \in \mathbb{N}$. In the proof of Lemma 114.3 we already saw that $\mathrm{Sp}(n)$ is a smooth manifold and that matrix multiplication turns $\mathrm{Sp}(n)$ into a group. Show that the map

$$\begin{array}{rcl} \mathrm{Sp}(n) & \to & \mathrm{Sp}(n) \\ A & \mapsto & A^{-1} \end{array}$$

is a smooth map.
Note: The argument in the proof of Lemma 3.53 can be used to show that taking inverses in $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ is smooth. Does this help us in any way?
Remark. In principle one could use Exercise 114.2. But what do readers do who are still traumatized by the Implicit Function Theorem?

Exercise 114.5. Use the Ehresmann Fibration Theorem that we had stated in Exercise 113.5 to give a third proof that the map $p\colon \mathrm{O}(n+1) \to S^n$ from Proposition 114.8 is a fiber bundle.

Exercise 114.6. Determine all closed non-orientable 2-dimensional smooth manifolds that admit a Lie group structure.

Exercise 114.7. We define $\mathrm{SU} = \varinjlim \mathrm{SU}(n)$ the same way as we defined U. Determine the homotopy groups of SU.
Remark. You might want to use Exercise 113.8.

Exercise 114.8.

(a) Let X be a CW-complex and let $e \in X^0$ be a point in the 0-skeleton. Suppose we are given a map $\mu\colon X \times X \to X$ such that the two maps
$$\begin{array}{rcl} X & \to & X \\ x & \mapsto & \mu(x,e) \end{array} \quad \text{and} \quad \begin{array}{rcl} X & \to & X \\ x & \mapsto & \mu(e,x) \end{array}$$
are homotopic to the identity. Show that μ is homotopic to a map $\nu\colon X \times X \to X$ such that $\nu(x,e) = \nu(e,x) = x$ for all $x \in X$.

(b) Show that ν in (a) defines an H-space structure on X.

(c) Let X be a CW-complex. Suppose there exists a homotopy equivalence $f\colon X \to Y$ to an H-space Y. Show that X itself admits an H-space structure.

Exercise 114.9. Let (X, x_0) be a pointed path-connected H-space. Let $\mu\colon X \times X \to X$ be the multiplication of the H-space structure. We consider the following two binary operations on $\pi_1(X, x_0)$:
$$\begin{array}{rcl} \pi_1(X,x_0) \times \pi_1(X,x_0) & \xrightarrow{\circ} & \pi_1(X,x_0) \\ ([f],[g]) & \mapsto & [f * g] \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi_1(X,x_0) \times \pi_1(X,x_0) & \xrightarrow{\otimes} & \pi_1(X,x_0) \\ ([f],[g]) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ t & \mapsto & \mu(f(t),g(t)) \end{array} \right). \end{array}$$
Show that "$\circ$" and "$\otimes$" satisfy Condition (b) of the Eckmann-Hilton Theorem 114.27.

Exercise 114.10. Let (X, x_0) be a pointed topological space and let $k \in \mathbb{N}_{\geq 2}$. Define the multiplication $*\colon \pi_k(X,x_0) \times \pi_k(X,x_0) \to \pi_k(X,x_0)$ as on page 1055. Provide a different multiplication $\otimes\colon \pi_k(X,x_0) \times \pi_k(X,x_0) \to \pi_k(X,x_0)$ such that "$*$" and "$\otimes$" satisfy the conditions of the Eckmann-Hilton Theorem 114.27. Conclude, once again, that $\pi_k(X,x_0)$ is abelian.

115. Thom-Pontryagin Theory I: The basics

In this chapter we will introduce the Thom-Pontryagin approach to calculating homotopy groups of spheres. Furthermore we will see that it allows us to reprove several earlier results. In Chapters 116 and 117 we will use the Thom-Pontryagin approach to obtain new results.

Before we start with discussing the Thom-Pontryagin theory we fix some notations and conventions.

Notation and Convention. Let $n \in \mathbb{N}_0$.

(1) We denote by $*$ the point in $\overline{B}^n/S^{n-1}$ that is given by $[S^{n-1}]$.
(2) We denote by $*$ the "North Pole" $(0,\dots,0,1) \in S^n$.

We make the following identifications:

(a) We use the homeomorphism $S^n \to \mathbb{R}^n \cup \{\infty\}$ from Lemma 2.44, which is given by stereographic projection, that sends the North Pole $*$ to ∞ to make the identification $S^n = \mathbb{R}^n \cup \{\infty\}$.
(b) On page 182 we wrote down an explicit homeomorphism $\overline{B}^n/S^{n-1} \to S^n$. This homeomorphism sends $*$ to $(0,\dots,0,1)$. $(x_1,\dots,x_{n+1}) \mapsto (x_{n+1},x_2,\dots,x_n,-x_1)$. This way we obtain a homeomorphism $\overline{B}^n/S^{n-1} \to S^n$ that sends $*$ to $(0,\dots,0,1)$.
(c) Putting the two homeomorphisms from (a) and (b) together we obtain a homeomorphism

$$\varphi\colon \overline{B}^n/S^{n-1} \;\to\; \mathbb{R}^n \cup \{\infty\}.$$

It follows fairly easily from the definitions of the two maps in (a) and (b) that φ restricts to a diffeomorphism $B^n \to \mathbb{R}^n$ and that φ is of the form

$$x \;\mapsto\; \begin{cases} \eta(\|x\|)\cdot x, & \text{if } x \in B^n, \\ \infty, & \text{if } x = *, \end{cases}$$

where $\eta\colon [0,1) \to [0,\infty)$ is a strictly monotonously increasing function with $\eta(0) = 0$ and $\lim\limits_{t\to 1} \eta(t) = \infty$.

Summarizing we have the identifications $S^n = \mathbb{R}^n \cup \{\infty\} = \overline{B}^n/S^{n-1}$ and under these identifications the points $* \in S^n$, ∞ and $* \in \overline{B}^n/S^{n-1}$ get identified. We refer to Figure 1592 for an illustration.

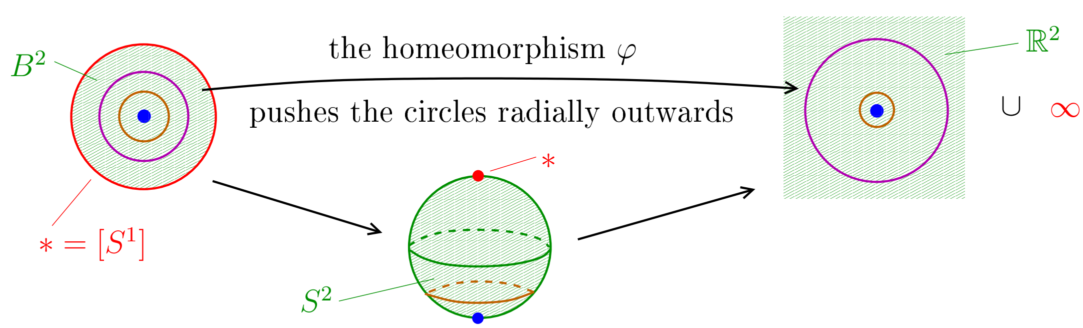


Figure 1592

115.1. **The generalized Thom-Pontryagin construction.** On page 2369 we saw that a proper oriented codimension-one submanifold of a compact oriented smooth manifold M gives rise, via the Thom-Pontryagin construction, to a map $M \to S^1$. In this section we will generalize this construction.

The key objects in the more general construction of Thom-Pontryagin maps are "thickened" submanifolds:

Definition. Let M be an orientable n-dimensional smooth manifold.

(1) Let N be a compact orientable proper k-dimensional submanifold of M. A *thickening of* N is an injective map $g\colon \overline{B}^{n-k} \times N \to M$ with the following properties:[1602][1603][1604]
   (a) for all $P \in N$ we have $g(0, P) = P$,
   (b) the image $g(\overline{B}^{n-k} \times N)$ is a submanifold with corner of M,
   (c) we have $g(\overline{B}^{n-k} \times N) \cap \partial M = g(\overline{B}^{n-k} \times \partial N)$.

   We refer to Figure 1593 for an illustration.

(2) A *thickened submanifold* of M is a pair (N, g) where N is a compact orientable proper[1605] submanifold of M and g is a thickening for N.

(3) Let $\varphi\colon M \to W$ be a smooth embedding of M into a smooth manifold W of the same dimension. Given a thickened k-dimensional submanifold $(N, g\colon \overline{B}^{n-k} \times N)$ of $M \setminus \partial M$ we equip $\varphi(N)$ with the "obvious thickening", namely with the thickening that is given by

$$\begin{array}{rcl} \varphi_*(g)\colon \overline{B}^{n-k} \times \varphi(N) & \to & W \\ (v, P) & \mapsto & g(v, \varphi^{-1}(P)). \end{array}$$

   Evidently $(\varphi(N), \varphi_*(g))$ is a thickened submanifold of W.

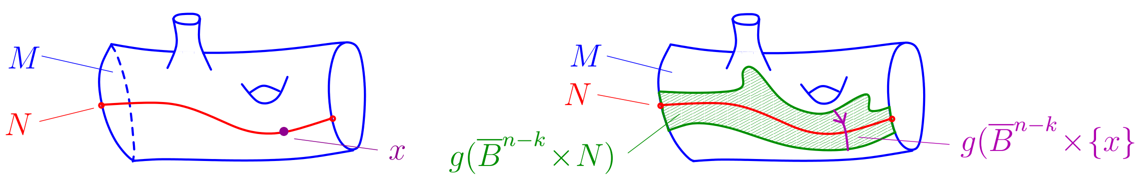


FIGURE 1593

The following definition can be viewed as a generalization of the Thom-Pontryagin map introduced on page 2369.

[1602]Here one really would like to say that $g\colon \overline{B}^{n-k} \times N \to M$ is an "smooth embedding", but if N has boundary, then it is not entirely clear in what sense $\overline{B}^{n-k} \times N$ is a smooth manifold and what "smooth embedding" is supposed to mean.

[1603]The definition looks very similar to the definition of a tubular neighborhood. In fact a thickening is the same data as a trivial tubular neighborhood, it is just phrased in slightly different words.

[1604]On page 2384 we introduced the thickening of an *oriented* knot. For knots the present definition is the same as before, except that we no longer demand that the knot is oriented.

[1605]Note that here, in the definition of a thickened submanifold, we sneaked in the adjectives "compact", "orientable" and "proper", so that we will not have to carry them around all the time.

Definition. Let M be an orientable n-dimensional smooth manifold. Given a thickened k-dimensional submanifold $(N, g\colon \overline{B}^{n-k} \times N \to M)$ we refer to the map[1606]

$$\begin{array}{rcl} \rho_{(N,g)}\colon M & \to & S^{n-k} = \overline{B}^{n-k}/S^{n-k-1} \\ P & \mapsto & \begin{cases} [x], & \text{if } P = g(x,y) \text{ with } (x,y) \in \overline{B}^{n-k} \times N, \\ *, & \text{otherwise} \end{cases} \end{array}$$

as the *Thom-Pontryagin map*. (We refer to Figure 1594 for an illustration.) Sometimes we refer to this approach of assigning to a thickened k-dimensional submanifold a map $M \to S^{n-k}$ as the *Thom-Pontryagin construction.*

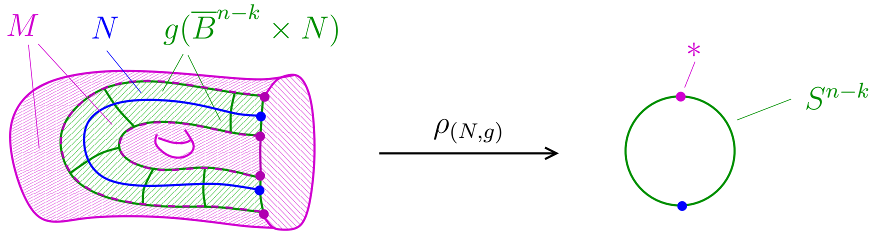


FIGURE 1594

Remark.

(1) Let N be a compact *oriented* proper codimension-one submanifold of an *oriented* smooth manifold M. It follows immediately from Proposition 10.9 that N admits a thickening $[-1, 1] \times N$ such that the inclusion $(-1, 1) \times N \to M$ is orientation-preserving.[1607] The map $\rho_N\colon M \to S^1$ that we defined on page 2369 is basically the same as the above map $\rho_{(N,g)}\colon M \to S^1$. The only difference is that the two maps differ by multiplication by $i \in S^1$.

(2) We saw in Proposition 94.14 that the submanifold $\mathbb{CP}^1$ of $\mathbb{CP}^2$ does *not* admit a trivial tubular neighborhood, in particular it does not admit a thickening.

Example. Let $n \in \mathbb{N}$ and let M be a closed orientable n-dimensional smooth manifold together with a 0-dimensional submanifold $N = \{P_1, \ldots, P_m\}$. Suppose N is equipped with a thickening

$$g = g_1 \sqcup \cdots \sqcup g_m\colon (\overline{B}^n \times \{P_1\}) \sqcup \cdots \sqcup (\overline{B}^n \times \{P_m\}) \ \to \ M.$$

From the Thom-Pontryagin construction we obtain a map $\rho_{(N,g)}\colon M \to S^n$. It might not be clear what this is good for, but it is kind of fun. This example is illustrated in Figure 1595.

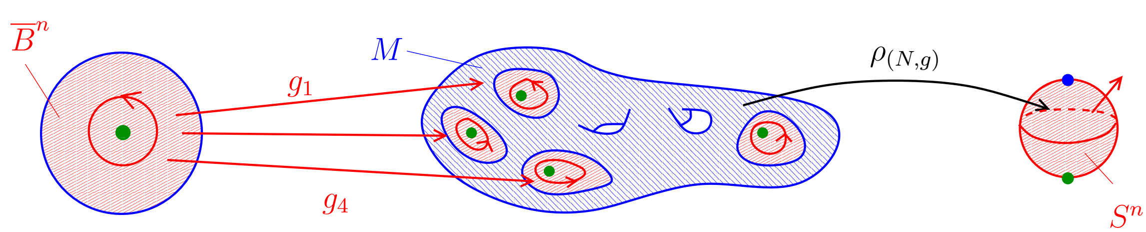


FIGURE 1595

[1606] It follows easily from Lemma 89.1 and Lemma 2.35 (2) that this map is indeed continuous.

[1607] Furthermore we saw in Proposition 10.7 that this thickening is essentially unique.

The following lemma gives us a first interesting example of a Thom-Pontryagin map.

Lemma 115.1. *Let $n \in \mathbb{N}$ and let $P \in B^n$. We equip the 0-dimensional smooth manifold $\{P\}$ with any thickening $g\colon \overline{B}^n \times \{P\} \to B^n$ that is orientation-preserving*[1608]*. The corresponding map*

$$\rho_{(\{P\},g)}\colon \overline{B}^n/S^{n-1} \;\to\; \overline{B}^n/S^{n-1}$$

is homotopic rel $*$ to the identity map of $\overline{B}^n/S^{n-1}$.

The proof of Lemma 115.1 relies on the following lemma which we will use on several occasions.

Lemma 115.2. *Let $m \in \mathbb{N}$. Given any real number $r \in (0,1)$ there exists a homotopy $F\colon \overline{B}^m/S^{m-1} \times [0,1] \to \overline{B}^m/S^{m-1}$ rel $*$ with the following three properties:*

1. $F_0 = \mathrm{id}$,
2. *the restriction of F_1 to B^m_r is given by multiplication by $\frac{1}{r}$,*
3. *for any point $x \notin B^m_r$ we have $F_1(x) = *$.*

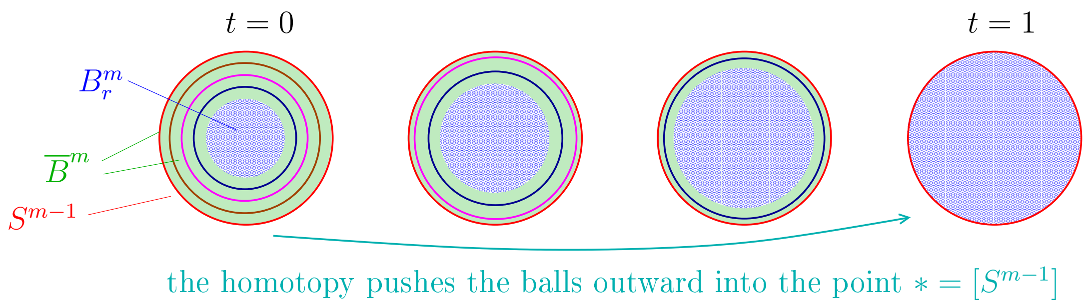


FIGURE 1596. Illustration of Lemma 115.2 (1).

PROOF. Let $r \in (0,1)$. We consider the map

$$\begin{array}{rcl} (\overline{B}^m/S^{m-1}) \times [0,1] & \to & \overline{B}^m/S^{m-1} \\ ([x],t) & \mapsto & \begin{cases} [x\cdot(1-t+t\cdot\frac{1}{r})], & \text{if } x \in B^m \text{ and } t < \frac{1-\|x\|}{(\frac{1}{r}-1)\cdot\|x\|} \\ *, & \text{otherwise.} \end{cases} \end{array}$$

We leave as to the engaged reader to verify that this map is indeed continuous. It is clear that F has the desired properties. ■

Now we can easily prove Lemma 115.1.

SKETCH OF PROOF OF LEMMA 115.1. It follows quite easily[1609] from Theorem 8.36 that we can assume without loss of generality that $P = 0$ and that we can assume that g is the thickening $g\colon \overline{B}^n \times \{0\} \to B^n$ given by $g(v,0) = \frac{1}{2}v$. By Lemma 115.2 there exists a homotopy $F\colon (\overline{B}^n/S^{n-1}) \times [0,1] \to \overline{B}^n/S^{n-1}$ rel $*$ with the following three properties:

(1) $F_0 = \mathrm{id}$,

[1608] We equip $\overline{B}^n$ and B^n with the usual orientation.

[1609] The slightly hesitant "quite easily" refers to the fact that, if one thinks about it, we need that the diffeotopy of $\mathbb{R}^{m+k}$ given by Theorem 8.36 has compact support.

(2) the restriction of F_1 to $B^n_{\frac{1}{2}}$ is given by the map $x \mapsto 2 \cdot x$,
(3) for any point $x \notin B^n_{\frac{1}{2}}$ we have $F_1(x) = *$.

If one thinks about it for a second, then one realizes that F_1 equals $\rho_{(0,g)}$ "on the nose". Thus we have found the desired homotopy. ■

115.2. **The Thom-Pontryagin construction and homotopy groups.** The following is the key observation which justifies our renewed interest in the Thom-Pontryagin construction.

Observation. Let $n, k \in \mathbb{N}_0$ and let $(N, g\colon \overline{B}^n \times N \to \mathbb{R}^{n+k})$ be a thickened k-dimensional submanifold of $\mathbb{R}^{n+k}$. We can view $(N, g\colon \overline{B}^n \times N \to \mathbb{R}^{n+k})$ also as a thickened submanifold of $\mathbb{R}^{n+k} \cup \{\infty\}$.[1610] Note that the corresponding Thom-Pontryagin map

$$\begin{array}{rcl} \rho_{(N,g)}\colon S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} & \to & \overline{B}^n / S^{n-1} = S^n \\ P & \mapsto & \begin{cases} [x], & \text{if } P = g(x,y) \text{ with } (x,y) \in \overline{B}^n \times N, \\ *, & \text{otherwise,} \end{cases} \end{array}$$

sends ∞ to $*$. In particular $\rho_{(N,g)}$ defines an element in $\pi_{n+k}(S^n, *)$. It is worth repeating this observation:

thickened k-dimensional submanifolds of $\mathbb{R}^{n+k}$ give rise to elements in $\pi_{n+k}(S^n, *)$.

The following questions immediately arise:

Question 115.3. *Let $n, k \in \mathbb{N}_0$.*
(1) *Which elements in $\pi_{n+k}(S^n, *)$ can be realized by Thom-Pontryagin maps?*
(2) *When do two thickened k-dimensional submanifolds of $\mathbb{R}^{n+k}$ give rise to the same element in $\pi_{n+K}(S^n, *)$.*

Before we address this question we intend to discuss two examples in greater detail:
(1) the case of thickened 0-dimensional submanifolds of some $\mathbb{R}^n$, and
(2) the case of thickened 1-dimensional submanifolds of $\mathbb{R}^3$.

We start out with the first case. Dealing with this case requires one definition.

Definition. Let $P \in \mathbb{R}^n$ be a point and let $g\colon \{P\} \times \overline{B}^n \to \mathbb{R}^n$ be a thickening of P. We equip $\{P\} \times \overline{B}^n = \overline{B}^n$ and $\mathbb{R}^n$ with the standard orientations. We define

$$\operatorname{sign}(g) := \begin{cases} +1, & \text{if } g \text{ is orientation-preserving,} \\ -1, & \text{if } g \text{ is orientation-reversing.} \end{cases}$$

Now we can discuss the case of thickened 0-dimensional submanifolds.

Example. Let $N = \{P_1, \dots, P_m\}$ be a 0-dimensional submanifold of some $\mathbb{R}^n$ and suppose N is equipped with a thickening

$$g = g_1 \sqcup \cdots \sqcup g_m\colon (\overline{B}^n \times P_1) \sqcup \cdots \sqcup (\overline{B}^n \times P_m) \to \mathbb{R}^n.$$

[1610] Why can we do so?

It follows almost immediately from Proposition 53.11 that

$$[\rho_{(N,g)}] \;=\; \sum_{i=1}^{m} \operatorname{sign}(g_i)\cdot[\mathrm{id}_{S^n}] \;\in\; \pi_n(S^n,*).$$

We continue our discussion of examples of the Thom-Pontryagin construction with knots and links. By the above discussion every thickened knot or link in $\mathbb{R}^3$, i.e. every thickened 1-dimensional submanifold of $\mathbb{R}^3$, gives rise to an element in $\pi_3(S^2,*)$. For the trefoil together with some thickening we illustrate the corresponding map $S^3 \to S^2$ in Figure 1597.

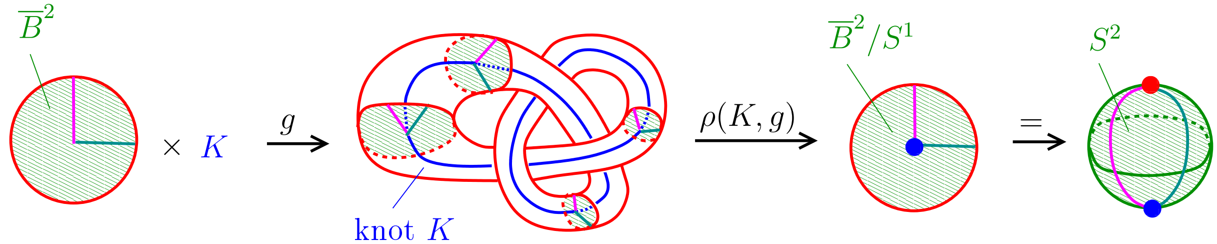


FIGURE 1597

Before we continue we need to introduce the following slight variation on a definition from page 2384.

Definition. Let K be a knot and let $g\colon \overline{B}^2 \times K \to S^3$ be a thickening. We pick an orientation for K and we equip $g(\{1\}\times K)$ with the corresponding definition. We define[1611]

$$\text{self-linking number of } g \;:=\; \operatorname{lk}(K, g(\{1\}\times K)).$$

It follows from Proposition 99.10 (2) that this definition does not depend on the choice of the orientation of K.

The following lemma deals with two basic, but important examples. Later on both examples will follow (at least up to sign) from a more general result, but for the geometrically inclined person it can be fun to consider the more direct arguments provided in the proof of the lemma.

Lemma 115.4. *Let K be the trivial knot and let $g\colon \overline{B}^2 \times K \to \mathbb{R}^3$ be a thickening.*

(1) *If g is a thickening with self-linking number zero, then $[\rho_{(K,g)}] = 0 \in \pi_3(S^2,*)$.*
(2) *If g is a thickening with self-linking number one, then[1612] $[\rho_{(K,g)}] = [H] \in \pi_3(S^2,*)$ where $H\colon (S^3,*) \to (S^2,*)$ denotes the Hopf map, that we defined on page 1071.*

SKETCH OF PROOF $(*)$. It follows easily from Lemma 99.1[1613] that any two thickenings with the same self-linking number give rise to the same element in $\pi_3(S^2,*)$. Furthermore it follows almost immediately from Lemma 27.1 that we can make the identification

$$S^3 \;=\; (S^1\times\overline{B}^2)\cup_{S^1\times S^1}(\overline{B}^2\times S^1)$$

[1611] We refer to page 2396 for the definition of the linking number $\operatorname{lk}(K,J)$ of two disjoint oriented knots K and J.

[1612] The sign should be correct. Hopefully. Perhaps.

[1613] Note that here we need that in the statement of Lemma 99.1 for $\mathbb{R}^3$ we stated that there exists a diffeotopy of $\mathbb{R}^3$ with compact support.

such that $K=\{0\}\times S^1$. Furthermore it follows from Lemma 27.2 that the obvious inclusion of $\overline{B}^2\times S^1$ into the above decomposition of S^3 is a thickening of the trivial knot $K=S^1$ with self-linking number zero.

(1) Let g be the thickening of the trivial knot K with self-linking number zero. We want to show that $[\rho_{(K,g)}]=0\in\pi_3(S^2,*)$. By Proposition 54.9 it suffices to show that the map $\rho_{(K,g)}\colon S^3\to S^2$ is homotopic to a constant map. (More precisely, Proposition 54.9 (7) says that in this context we do not need to worry about base points since S^2 is path-connected.) Using the above preparations we will prove in Exercise 115.1 that the map $\rho_{(K,g)}\colon S^3\to S^2$ is indeed homotopic to a constant map.

(2) First we note that under the above identification of S^3 the Hopf map is given as follows:[1614]

$$\begin{array}{rcl} H\colon \overbrace{(S^1\times\overline{B}^2)\cup_{S^1\times S^1}(\overline{B}^2\times S^1)}^{=S^3} & \to & \mathbb{C}\mathrm{P}^1 \\ [(z,w)] & \mapsto & \begin{cases} [\frac{w}{z}:1], & \text{if } z\in S^1 \text{ and } w\in\overline{B}^2, \\ [1:\frac{z}{w}], & \text{if } z\in\overline{B}^2 \text{ and } w\in S^1. \end{cases} \end{array}$$

Now note that it follows from Lemma 99.1 (1) that the map

$$\begin{array}{rcl} g\colon \overline{B}^2\times S^1 & \to & \overline{B}^2\times S^1 \subset S^3=(S^1\times\overline{B}^2)\cup_{S^1\times S^1}(\overline{B}^2\times S^1) \\ (z,w) & \mapsto & (z\cdot w,w) \end{array}$$

is a thickening of the trivial knot of self-linking number one. Furthermore by definition the corresponding map $\rho_{(K,g)}$ is easily seen to be the map[1615]

$$\begin{array}{rcl} \rho_{(K,g)}\colon (S^1\times\overline{B}^2)\cup_{S^1\times S^1}(\overline{B}^2\times S^1) & \to & \overline{B}^2/S^1 \\ [(z,w)] & \mapsto & \begin{cases} \infty=[S^1], & \text{if } z\in S^1 \text{ and } w\in\overline{B}^2, \\ [\frac{z}{w}], & \text{if } z\in\overline{B}^2 \text{ and } w\in S^1.. \end{cases} \end{array}$$

Comparing the definitions of the Hopf map H and the map $\rho_{(K,g)}$ on $\overline{B}^2\times S^1$ we see that we are perhaps not that far off.

Next we need to relate the targets $\mathbb{C}\mathrm{P}^1$ and $\overline{B}^2/S^1$. To do so we consider the map $\Theta\colon\mathbb{C}\mathrm{P}^1\to\overline{B}^2/S^1$ that is given by $\Theta([1:z])=\frac{z}{1+|z|}$ for $z\in\mathbb{C}$ and that is given by $\Theta([0:1])=[S^1]=*\in\overline{B}^2/S^1$. It follows easily from Proposition 2.43 (3) that Θ is a homeomorphism. The particularly audacious reader can verify that this map is homotopic to the "official" identification $\mathbb{C}\mathrm{P}^1=\overline{B}^2/S^1$ which is given by the identification $\mathbb{C}\mathrm{P}^1=S^2$ from page 197 and the above identification $S^2=\overline{B}^2/S^1$. Note that the image of the first solid torus $\overline{B}^2\times S^1$ under the map $\Theta\circ H$ equals the disk of radius $\frac{1}{2}$, i.e. it equals $\overline{B}^2_{\frac{1}{2}}\subset\overline{B}^2/S^1$.

[1614]As a reality check one can verify that the map is well-defined, i.e. for $[(z,w)]$ with $(z,w)\in S^1\times S^1$ we get consistent answers.

[1615]Why does it say $[\frac{z}{w}]$ instead of $[z\cdot w]$, which might look more natural?

We apply Lemma 115.2 to obtain a homotopy F of $\overline{B}^2/S^1$ with the following properties:
(a) $F_0 = \mathrm{id}$,
(b) the restriction of F_1 to $B^2_{\frac{1}{2}}$ is given by multiplication by 2,
(c) for any point $x \notin B^2_{\frac{1}{2}}$ we have $F_1(x) = *$.
The map $F_0 \circ \Theta \circ H = \Theta \circ H$ is homotopic rel $*$ to $F_1 \circ \Theta \circ H$. It is now straightforward to show that $F_1 \circ \Theta \circ H$ is homotopic rel $*$ to $\rho_{(K,g)}$. We leave it to the reader to fill in the details. ■

We conclude this list of basic examples of the Thom-Pontryagin construction with the following lemma.

Lemma 115.5. *Let $n, k \in \mathbb{N}_0$ with $n + k \geq 2$. Let (N_i, g_i), $i = 1, \dots, m$ be thickened k-dimensional submanifolds of $\mathbb{R}^{n+k}$. If the images $g_i(\overline{B}^n \times N_i)$ are contained in m disjoint closed smooth $(n+k)$-dimensional balls[1616], then*

$$\rho_{(N_1 \sqcup \cdots \sqcup N_m, g_1 \sqcup \cdots \sqcup g_m)} \;=\; \rho_{(N_1,g_1)} + \cdots + \rho_{(N_m,g_m)} \;\in\; \pi_{n+k}(S^n, *).$$

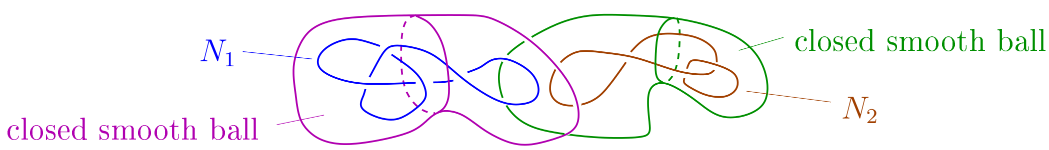


FIGURE 1598

PROOF. This lemma follows easily from Proposition 53.11. We leave it to the reader to fill in the details. ■

The following proposition gives an affirmative answer to Question 115.3.

Proposition 115.6. *Let $k, n \in \mathbb{N}_0$. Given any $\varphi \in \pi_{n+k}(S^n, *)$ there exists a thickened k-dimensional submanifold (N, g) of $\mathbb{R}^{n+k}$ with $[\rho_{(N,g)}] = \varphi$.*

PROOF. Throughout this proof we find it convenient to view each m-dimensional sphere as the quotient $\overline{B}^m/S^{m-1}$. So suppose we are given a map $\varphi\colon \overline{B}^{n+k}/S^{n+k-1} \to \overline{B}^n/S^{n-1}$ with $\varphi(*) = *$. As we pointed out in Corollary 9.4, it is a consequence of the Whitney Approximation Theorem 9.3 that we can arrange that φ is in fact a smooth map. It follows immediately from Sard's Theorem 6.63 and Proposition 6.62 (2) that there exists a regular value $z \in \overline{B}^n/S^{n-1}$ such that $\varphi^{-1}(z)$ does not contain the point $*$. By Proposition 8.29 (2)[1617] we can without loss of generality assume that the origin 0 is a regular value. By the Regular Value Theorem 11.1 we know that the preimage $N := \varphi^{-1}(0)$ is a closed orientable k-dimensional submanifold of B^{n+k}. Furthermore by statement (4) of the Regular Value

[1616]Here a "closed smooth $(n+k)$-dimensional ball" is shorthand for the image of a smooth embedding of $\overline{B}^{n+k}$ in $\mathbb{R}^{n+k}$.

[1617]To be more precise, we apply Proposition 8.29 to the smooth manifold $\overline{B}^n/S^{n-1} = \mathbb{R}^n \cup \{\infty\}$ with $P_1 = z$, $P_2 = \infty$ and $Q_1 = 0$ and $Q_2 = \infty$.

Theorem 11.1 we know that there exists an $\epsilon > 0$ and a thickening $g\colon \overline{B}^n \times N \to B^{n+k}$ such that $\varphi^{-1}(\overline{B}^n_\epsilon) = g(\overline{B}^n \times N)$ and such that the following diagram commutes[1618][1619]

$$\begin{tikzcd}
\overline{B}^n \times N \arrow[r, "g"] \arrow[d, "(p,x)\mapsto \epsilon\cdot p"] & B^{n+k} \arrow[d, "\varphi"] \\
\overline{B}^n_\epsilon \arrow[r, hook] & \overline{B}^n/S^{n-1}.
\end{tikzcd}$$

This thickening g allows us to consider the map $\rho_{(N,g)}\colon \overline{B}^{n+k}/S^{n+k-1} \to \overline{B}^n/S^{n-1}$. Note that in general φ and $\rho_{(N,g)}$ are quite different maps. (We refer to Figure 1599 for an illustration of the two maps φ and $\rho_{(N,g)}$.) But that does not bother us too much, since it suffices to prove the following claim.

Claim. The maps φ and $\rho_{(N,g)}$ from $\overline{B}^{n+k}/S^{n+k-1}$ to $\overline{B}^n/S^{n-1}$ are homotopic rel $*$.

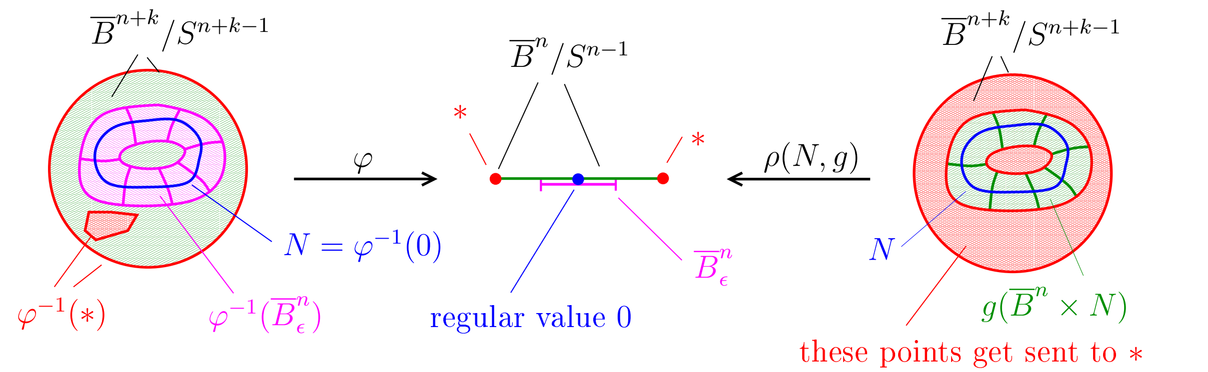


FIGURE 1599

First we remind the reader that in Lemma 115.2 (1) we showed that there exists a homotopy $F\colon (\overline{B}^n/S^{n-1}) \times [0,1] \to \overline{B}^n/S^{n-1}$ rel $*$ with the following properties:

(1) $F_0 = \mathrm{id}$,
(2) the restriction of F_1 to B^n_ϵ is given by multiplication by $\frac{1}{\epsilon}$,
(3) for any point $x \notin B^n_\epsilon$ we have $F_1(x) = *$.

We define $\psi = F_1 \circ \varphi$. It follows from (1) that ψ is homotopic rel $*$ to φ. Therefore we are done once we have shown that ψ in fact equals $\rho_{(N,g)}$. As a reminder, see the definition on page 2726, the map $\rho_{(N,g)}$ is given as follows:[1620]

$$\begin{array}{rcl}
\rho_{(N,g)}\colon \overline{B}^{n+k}/S^{n+k-1} & \to & S^n = \overline{B}^n/S^{n-1} \\
P & \mapsto & \begin{cases} x, & \text{if } P = g(x,y) \text{ with } (x,y) \in B^n \times N, \\ *, & \text{otherwise.} \end{cases}
\end{array}$$

[1618] Note the discretely hidden multiplication by ϵ in the left-vertical map.

[1619] Note that the above information implies in particular that $\varphi^{-1}(B^n_\epsilon) = g(B^n \times N)$.

[1620] Eyes which profited from a carrot-rich diet might notice that the description is not verbatim the same as on page 2726. But with either description of the map the points $P = g(x,y)$ with $x \in S^{n-1}$ and $y \in N$ get sent to $*$. So we really do consider the same map as before.

Now we verify that ψ does indeed agree with $\rho_{(N,g)}$. Thus first suppose we are given a point $P = g(x,y)$ with $(x,y) \in B^n \times N$. We have

$$\psi(P) \underset{\substack{\uparrow \\ \text{definition of } \psi}}{=} F_1(\varphi(g(x,y))) \underset{\substack{\uparrow \\ \text{by the above} \\ \text{commutative diagram}}}{=} F_1(\epsilon\cdot x) \underset{\substack{\uparrow \\ \text{by Property (2) of } F_1 \\ \text{and since } \epsilon\cdot x \in B^n_\epsilon}}{=} \tfrac{1}{\epsilon}\cdot\epsilon\cdot x = x = \rho_{(N,g)}(g(x,y)) = \rho_{(N,g)}(P).$$

Finally suppose that we are given a point $P \in \overline{B}^{n+k}/S^{n+k-1}$ that does not lie in the subset $g(B^n \times N)$. In this case we calculate that

$$\psi(P) \underset{\substack{\uparrow \\ \text{definition of } \psi}}{=} F_1(\varphi(P)) \underset{\substack{\uparrow \\ \text{since } \varphi^{-1}(B^n_\epsilon) = g(B^n\times N) \\ \text{and by choice of } P}}{=} F_1(\text{point that does not lie in } B^n_\epsilon) \underset{\substack{\uparrow \\ \text{by Property (3) of } F_1}}{=} * = \rho_{(N,g)}(P).$$

■

115.3. **The thickened cobordism groups and the Thom-Pontryagin Theorem.** Recall that the Thom-Pontryagin construction from page 2726 makes the assignment

$$\text{thickened } k\text{-dimensional submanifold of } \mathbb{R}^{n+k} \;\rightsquigarrow\; \text{map } (S^{n+k},*) \to (S^n,*).$$

Now we would like to answer Question 115.3 (2), i.e. we would like to determine when do two thickened k-dimensional submanifolds give rise to the same homotopy class of maps $(S^{n+k},*) \to (S^n,*)$. Loosely speaking we would like to fill in the left-hand side of

$$?? \;\rightsquigarrow\; \text{map } (S^{n+k}\times[0,1], *\times[0,1]) \to (S^n,*).$$

A quick look at the Thom-Pontryagin construction suggests that the left-hand side should be filled in with a suitable thickened $(k+1)$-dimensional submanifold of $\mathbb{R}^{n+k}\times[0,1]$.

This makes sense, except that in Proposition 6.51 we saw that for products of two smooth manifolds it is advisable to have the smooth manifold with boundary as a first factor. Thus we will consider submanifolds of $[0,1]\times\mathbb{R}^{n+k}$.[1621]

Before we continue we introduce the following straightforward convention.

Convention. Given a topological space X and an interval I and some $t \in I$ we make at times the identification $X = \{t\}\times X$ via the obvious bijection.

The above observation leads us to the following definition which looks awfully similar to the definition of the notion of a cobordism of smooth manifolds that we introduced on page **??**.

Definition. Let $k, m \in \mathbb{N}_0$.

(1) Let N_0 and N_1 be two closed orientable k-dimensional submanifolds of $\mathbb{R}^m$. A *cobordism between* N_0 *and* N_1 is a compact orientable proper $(k+1)$-dimensional submanifold W of $[0,1]\times\mathbb{R}^m$ with $\partial W = (\{0\}\times N_0)\cup(\{1\}\times N_1)$.[1622]

(2) Let (N_0,g_0) and (N_1,g_1) be two thickened k-dimensional submanifolds of $\mathbb{R}^m$. A *thickened cobordism between* (N_0,g_0) *and* (N_1,g_1) is a thickened $(k+1)$-dimensional

[1621] Perhaps in hindsight we should have defined homotopies as maps $[0,1]\times X \to Y$ instead of defining them as maps $X\times[0,1]\to Y$?

submanifold (W,G) of $[0,1]\times\mathbb{R}^m$ where W is a cobordism between N_0 and N_1 and G is a thickening of W with the following two properties:[1623]
(a) $G|_{\{0\}\times\overline{B}^m} = g_0$ under the identification $\{0\}\times\overline{B}^m = \overline{B}^m$,
(b) $G|_{\{1\}\times\overline{B}^m} = g_1$ under the identification $\{1\}\times\overline{B}^m = \overline{B}^m$.
If such a thickened cobordism exists, then we say that (N_0,g_0) and (N_1,g_1) are *cobordant*.[1624] If (N,g) is cobordant to the empty submanifold, then we say N is a *thickened null-cobordism* and we say that (N,g) is *null-cobordant*. The definition of a thickened cobordism is illustrated in Figure 1600.

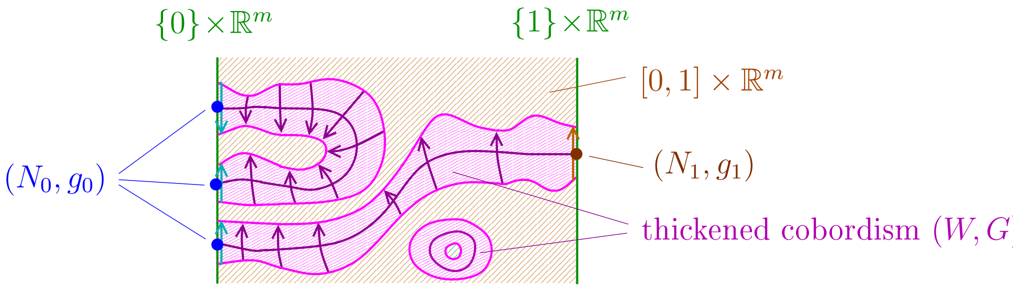


FIGURE 1600

The following lemma shows that diffeotopies give rise to thickened cobordisms.

Lemma 115.7. *Let (M,g) be a thickened submanifold of $\mathbb{R}^m$ and let φ be a diffeomorphism of $\mathbb{R}^m$. If φ is diffeotopic to the identity, then $(\varphi(M),\varphi_*(g))$ is cobordant to (M,g).*

Example. Let $m\in\mathbb{N}$. Any map $\mathbb{R}^m\to\mathbb{R}^m$ of the form $x\mapsto r\cdot x+P$ for some $r>0$ and $P\in\mathbb{R}^m$ is clearly diffeotopic to the identity. Thus Lemma 115.7 says that given a thickened submanifold (M,g) we can translate and shrink it without changing the thickened cobordism class.

PROOF. Let $(M,g\colon \overline{B}^{m-k}\times M\to\mathbb{R}^m)$ be a thickened k-dimensional submanifold of $\mathbb{R}^m$ and furthermore let $F\colon\mathbb{R}^m\times[0,1]\to\mathbb{R}^m$ be a diffeotopy from the identity to a diffeomorphism φ. It is straightforward to verify that the map

$$\begin{array}{rcl}[0,1]\times M & \to & [0,1]\times\mathbb{R}^m\\ (t,P) & \mapsto & (t,F(P,t))\end{array}$$

is a smooth embedding. We write $W:=F([0,1]\times M)$. It is straightforward to verify that W together with the thickening given by

$$\begin{array}{rcl}\overline{B}^{m-k}\times W & \to & [0,1]\times\mathbb{R}^m\\ (v,(t,F(P,t))) & \mapsto & (t,F(g(v,P),t))\end{array}$$

is a thickened cobordism between (M,g) and $(\varphi(M),\varphi_*(g))$. ■

[1622]Note that we work with submanifolds that are orientable, but we do not assume that they are equipped with an orientation.

[1623]Note that it follows from property (d) of a thickened submanifold, see page 2725, that we know that in fact $G|_{\{0\}\times\overline{B}^m}\subset\{0\}\times\mathbb{R}^m$ and $G|_{\{1\}\times\overline{B}^m}\subset\{1\}\times\mathbb{R}^m$.

[1624]We spare the reader the even more logical name "thickened cobordant".

Lemma 115.8. *"Being cobordant" is an equivalence relation on the set of thickened k-dimensional submanifolds of $\mathbb{R}^m$.*

SKETCH OF PROOF. To simplify the discussion we ignore throughout the proof the thickenings. The reader should have no major troubles modifying the proof to also include the thickenings.

It is straightforward to see that "being cobordant" is reflexive and symmetric: Indeed, given a closed k-dimensional submanifold N a cobordism from N to itself is given by $[0,1]\times N \subset [0,1]\times \mathbb{R}^m$. Furthermore, suppose there exists a cobordism W between N_0 and N_1. "Flipping" W along $\mathbb{R}^m \times \{\frac{1}{2}\}$ we obtain a cobordism between N_1 and N_0.

The proof of transitivity takes more efforts. Initially the idea is quite simple. Given a cobordism V from N_0 to N_1 and another cobordism W from N_1 to N_2 one would like to stick them together to obtain a cobordism from N_0 to N_2. Unfortunately, as we had discussed on page 363, the resulting object is in general not a (smooth) submanifold of the smooth manifold $\mathbb{R}^m$.

Fortunately we had already developed all the tools to deal with the situation. More precisely, in Proposition 8.21 we saw that we can modify the cobordisms V and W to obtain cobordisms V' and W' that are products on $[0,\epsilon]\times\mathbb{R}^m$ and $[1-\epsilon,1]\times\mathbb{R}^m$ for a suitable $\epsilon\in(0,\frac{1}{2})$. Now we can stick V' and W' together to obtain the desired cobordism X. We refer to Figure 1601 for an illustration. ■

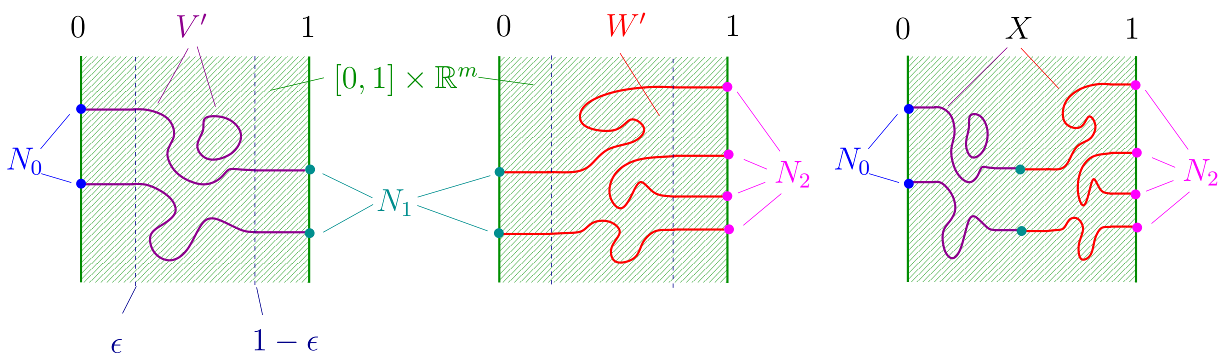


FIGURE 1601. Illustration for the proof of Lemma 1601.

The previous lemma allows us to make the following definition.

Definition. We denote by $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ the set of thickened cobordism classes of thickened k-dimensional submanifolds of $\mathbb{R}^m$. In light of the subsequent Proposition 115.9 we refer to $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ as a *thickened cobordism group.*

The definition of $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ is reminiscent of the definition of the cobordism group Ω_k that we gave on page **??**. The group structure on Ω_k that we had introduced in Lemma **??** is given by "disjoint union". We will now see that $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ also admits a group structure that is defined quite similarly.

Proposition 115.9. *Let $k\in\mathbb{N}_0$ and let $m\in\mathbb{N}_{\geq 2}$. Let (M,g) and (N,h) be two thickened k-dimensional submanifolds of $\mathbb{R}^m$. Since M and N are compact we can pick $r,s\in\mathbb{R}_{>0}$*

with $g(\overline{B}^{m-k} \times M) \subset B_r^m(0)$ *and* $h(\overline{B}^{m-k} \times N) \subset B_s^m(0)$. *Furthermore we pick two orientation-preserving smooth embeddings* $\varphi\colon \overline{B}_r^m(0) \to \mathbb{R}^m$ *and* $\psi\colon \overline{B}_s^m(0) \to \mathbb{R}^m$ *with disjoint images. We define*

$$[(M,g)] + [(N,h)] \;:=\; [(\varphi(M) \sqcup \psi(N), \varphi_*(g) \sqcup \psi_*(h))] \;\in\; \Omega_k^{\mathrm{th}}(\mathbb{R}^m).$$

(*The definition of the operation "+" is illustrated in Figure 1602.*)

(1) *The definition of "+" on* $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$ *is well-defined and commutative.*
(2) *The operation "+" defines a group structure on* $\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$. *Here the following two statements hold:*
   (a) *The neutral element is represented by the empty submanifold.*[1625]
   (b) *Let* $\rho\colon \mathbb{R}^m \to \mathbb{R}^m$ *be the reflection in a hyperplane. The inverse of* $[(M,g)]$ *is given by* $[(\rho(M), \rho_*(g))]$.

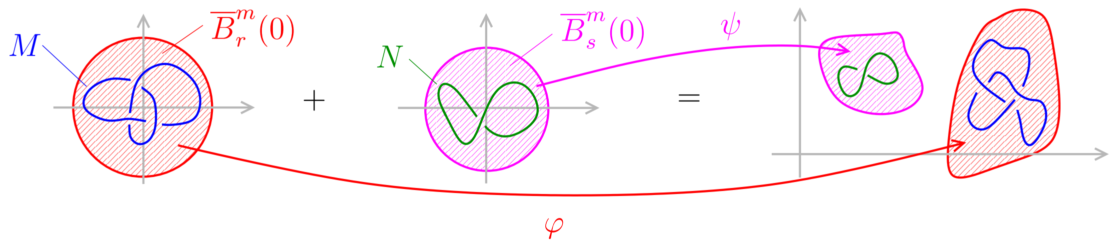


FIGURE 1602

PROOF.

(1) It follows from Theorem 8.36 and Lemma 115.7 that "+" is well-defined and commutative. Note that here we use that $m \geq 2$.
(2) It is basically obvious that "+" is associative and that the empty submanifold is a neutral element. Thus it remains to prove the statement regarding the inverse. It follows from Lemmas 115.7 and 18.5 that without loss of generality we can assume that the reflection $\rho\colon \mathbb{R}^m \to \mathbb{R}^m$ is the reflection in the $(x_m = 0)$-hyperplane. So let $(M, g\colon \overline{B}^{m-k} \times M \to \mathbb{R}^m)$ be a thickened k-dimensional submanifold of $\mathbb{R}^m$. We set $\epsilon = \frac{1}{4}$. It follows immediately from the discussion on page 2734 that we can assume that $g(\overline{B}^{m-k} \times M)$ is contained in the ball $\overline{B}_\epsilon^m(\underbrace{(0,\dots,0,2\epsilon)}_{\in\mathbb{R}^m})$. We consider the map

$$\begin{aligned}\varphi\colon \overline{B}^{m-k} \times (M \times [0,\pi]) \;&\to\; [0,1] \times \mathbb{R}^m \\ (v,(P,\gamma)) \;&\mapsto\; \begin{pmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & \mathrm{id}_{m-1} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 \\ g(v,P) \end{pmatrix}}_{\in\mathbb{R}^{m+1}}.\end{aligned}$$

Using that $g(\overline{B}^{m-k} \times M)$ is contained $\overline{B}_\epsilon^m((0,\dots,0,2\epsilon))$ we can make the following three observations:
(a) The image of φ does indeed lie in $[0,1] \times \mathbb{R}^m$.

[1625] It is worth remembering that the empty set is a submanifold of any dimension.

(b) The restriction to $\{0\} \times M \times [0,\pi]$ is a proper smooth embedding. Therefore we can now consider the compact orientable proper $(k+1)$-dimensional submanifold $W = \varphi(\{0\} \times M \times [0,\pi])$ of $[0,1] \times \mathbb{R}^m$.

(c) The map φ gives rise to a thickening for W.

It is now basically clear that we have constructed a thickened cobordism from the thickened submanifold $(M \sqcup \rho(M), g \sqcup \rho_*(g))$ to the empty submanifold. (We refer to Figure 1603 for an illustration.) But by the definition of "+" this implies that $(\rho(M), \rho_*(g))$ represents the inverse of (M,g). ■

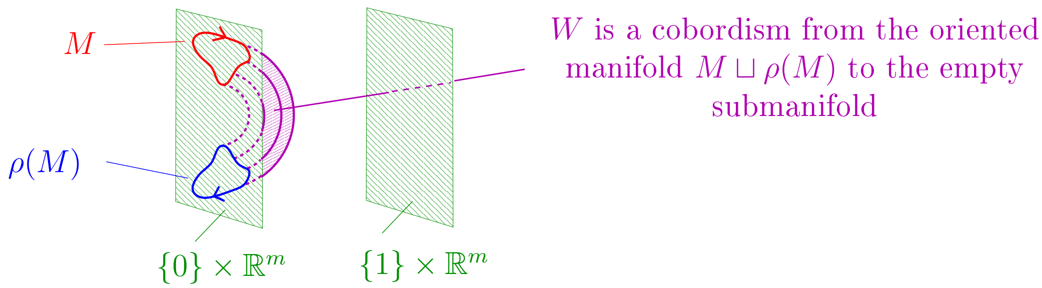


FIGURE 1603

Evidently our challenge now is to compute the groups $\Omega^{\text{th}}_k(\mathbb{R}^m)$ for suitable values of m and k. In general this might be quite difficult, but the following proposition gives a reasonably amusing calculation.

Proposition 115.10. *Let $m \geq 2$. The map*[1626]

$$\begin{array}{rcl} \varphi\colon \Omega^{\text{th}}_0(\mathbb{R}^m) & \to & \mathbb{Z} \\ [(\{P_1\}, g_1) \sqcup \dots \sqcup (\{P_k\}, g_k)] & \mapsto & \sum\limits_{i=1}^{k} \operatorname{sign}(g_i) \end{array}$$

is well-defined and it is an isomorphism.

SKETCH OF PROOF. First we need to show that $\varphi\colon \Omega^{\text{th}}_0(\mathbb{R}^m) \to \mathbb{Z}$ is well-defined. It suffices to show that for any *connected* thickened 1-dimensional cobordism $(Z,V) \subset [0,1] \times \mathbb{R}^m$ between two 0-dimensional thickened submanifolds (N_0, g_0) and (N_1, g_1) we have the equality $\varphi([(N_0,g_0)]) = \varphi([(N_1,g_1)])$. This statement follows from the classification of compact 1-dimensional smooth manifolds, see Theorem 7.5, and some orientation considerations. Since writing down the precise arguments regarding the orientations will only obfuscate the argument we refer to Figure 1604 for convincing pictures, and we leave it to the reader to fill in the details.

Next note that the map $\varphi\colon \Omega^{\text{th}}_0(\mathbb{R}^m) \to \mathbb{Z}$ is, basically by definition, a homomorphism. Finally note that it is clear that φ is an epimorphism. Thus it remains to show that φ is a monomorphism. A straightforward induction argument shows that it suffices to prove the following claim.

Claim. For points $P, Q \in \mathbb{R}^m$ and thickenings g and h of P and Q with $\operatorname{sign}(g) = -\operatorname{sign}(h)$ we have $[(P,g)] + [(Q,h)] = 0 \in \Omega^{\text{th}}_0(\mathbb{R}^m)$.

[1626] We refer to page 2728 for the definition of the sign of a thickening of a point.

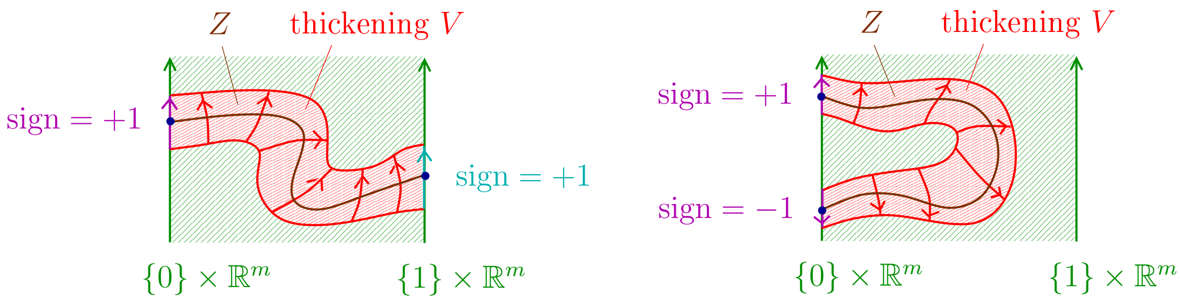


FIGURE 1604. Illustration for the proof of Proposition 115.10.

In the following let $\rho\colon \mathbb{R}^m \to \mathbb{R}^m$ be the reflection in a hyperplane. By the discussion on page 2734 we can assume, after possibly applying a translation, that $Q = \rho(P)$. Since ρ is orientation-reversing we see that $\operatorname{sign}(\rho_*(g)) = -\operatorname{sign}(g) = \operatorname{sign}(h)$. Now we calculate that

$$[(P,g)] + [(Q,h)] \;=\; [(P,g)] + [(\rho(P),h)] \underset{\substack{\uparrow \\ \text{by } \operatorname{sign}(h)=\operatorname{sign}(\rho_*(g)),\text{ Theorem 8.36 and Lemma 115.7}}}{=} [(P,g)] + [(\rho(P),\rho_*(g))] \underset{\substack{\uparrow \\ \text{by Proposition 115.9 (2b)}}}{=} 0 \;\in \Omega_0^{\text{th}}(\mathbb{R}^m).$$ ■

Considering the discussion on page 2733, the following theorem does not come as a big surprise. It justifies our interest in the groups $\Omega_k^{\text{th}}(\mathbb{R}^m)$.

Theorem 115.11. (Thom-Pontryagin Theorem) *Let $n,k \in \mathbb{N}_0$ with $n+k \geq 2$. The map*

$$\begin{array}{rcl} \Psi\colon \Omega_k^{\text{th}}(\mathbb{R}^{n+k}) & \to & \pi_{n+k}(S^n,*) \\ [(N,g)] & \mapsto & [\rho_{(N,g)}] \end{array}$$

is well-defined and it is an isomorphism.

SKETCH OF PROOF. Fortunately we have already done basically all the work:

(1) First we argue that Ψ is well-defined. Thus suppose that we are given thickened k-dimensional submanifolds (M,g) and (N,h) of $\mathbb{R}^{n+k}$ that are cobordant. We pick a thickened cobordism $(W, F\colon \overline{B}^n \times W \to [0,1]\times\mathbb{R}^{n+k})$ in $[0,1]\times\mathbb{R}^{n+k}$. The corresponding map

$$\rho_{(W,F)}\colon [0,1]\times\mathbb{R}^{n+k} \;\to\; S^n$$

has the property that it sends a neighborhood of $[0,1]\times\{\infty\}$ to $*$. Therefore it follows easily from Lemma 3.10 that the following map is continuous:

$$\begin{array}{rcl} S^{n+k}\times[0,1] = (\mathbb{R}^{n+k}\cup\{\infty\})\times[0,1] & \to & S^n \\ (P,t) & \mapsto & \begin{cases} \rho_{(W,F)}(t,P), & \text{if } P\in\mathbb{R}^{n+k}, \\ *, & \text{if } P=\infty. \end{cases} \end{array}$$

It follows from the above that this map is in fact a homotopy rel $*$ between the map $\rho_{(M,g)}\colon (S^{n+k},*) \to (S^n,*)$ and the map $\rho_{(N,h)}\colon (S^{n+k},*) \to (S^n,*)$. In particular we have $[\rho_{(M,g)}] = [\rho_{(N,h)}] \in \pi_{n+k}(S^n,*)$.

(2) It follows immediately from Lemma 115.5 that Ψ is a homomorphism.

(3) The statement of Proposition 115.6 is equivalent to saying that Ψ is an epimorphism.

(4) Finally it remains to show that Ψ is in fact a monomorphism. So suppose that we are given two thickened k-dimensional submanifolds (M,g) and (N,h) of $\mathbb{R}^{n+k}$ such that $[\rho_{(M,g)}] = [\rho_{(N,h)}] \in \pi_{n+k}(S^n,*)$, i.e. such that there exists a homotopy $F\colon S^{n+k}\times[0,1]\to S^n$ rel $*$ from $\rho_{(M,g)}$ to $\rho_{(N,h)}$.

The proof of Proposition 115.6 can be modified without too many problems to show that the homotopy F gives rise to a thickened cobordism between (M,g) and (N,h). We leave it to the reader to fill in the details. See also [**Kos93**, Theorem 9.5.5] or [**Bre93**, Chapter 16] for similar claims stated with more authority. ■

From our above results we obtain the following corollary. Of course we proved this result earlier in Corollary 53.6, using very different methods.

Corollary 115.12. *For any $m\geq 2$ the map*

$$\begin{array}{rcl}\mathbb{Z} & \to & \pi_m(S^m,*)\\ n & \mapsto & n\cdot[\mathrm{id}_{S^m}]\end{array}$$

is an isomorphism.

Proof. Let $P\in\mathbb{R}^m$ and let g be a thickening for the submanifold $\{P\}$. We consider the two maps

$$\begin{array}{rcl}\mathbb{Z} & \to & \Omega_0^{\mathrm{th}}(\mathbb{R}^m)\\ n & \mapsto & n\cdot[(\{0\},g\text{ with }\mathrm{sign}(g)=1)]\end{array}\quad\text{and}\quad\begin{array}{rcl}\Omega_0^{\mathrm{th}}(\mathbb{R}^m) & \to & \pi_m(S^m,*)\\ [(N,g)] & \mapsto & [\rho_{(N,g)}].\end{array}$$

The first map is an isomorphism by Proposition 115.10 and the second map is an isomorphism by the Thom-Pontryagin Theorem 115.11, applied to $k=0$ and $m\geq 2$. Thus the composition of the two maps is an isomorphism. Finally it follows from Lemma 115.1 that the composition of these two maps sends n to $n\cdot[\mathrm{id}_{S^m}]$. ■

115.4. **The calculation of the thickened cobordism group $\Omega_1^{\mathrm{th}}(\mathbb{R}^3)$.** Surely the most interesting calculation of a higher homotopy group that we have performed so far is that in Theorem 113.10 (2) we saw $\pi_3(S^2)\cong\mathbb{Z}$. Our goal now is to give a new proof of this fact using the Thom-Pontryagin Theorem 115.11. Thus we need to show that $\Omega_1^{\mathrm{th}}(\mathbb{R}^3)\cong\mathbb{Z}$.

We start out with the following slightly subtle definition.

Definition. Let $m\in\mathbb{N}$.

(1) Let $(M, v\colon \overline{B}^{m-k}\times\mathbb{R}^m)$ be a thickened $(m-k)$-dimensional submanifold of $\mathbb{R}^m$. The *natural orientation of M* is the unique orientation that turns v into an orientation-preserving map.[1627]
(2) Similarly we define the natural orientation of a thickened cobordism.

This definition leads us to the following convention.

Convention. In this section we equip any thickened submanifold and any thickened cobordism with the natural orientation.

[1627]Recall that M is assumed to be orientable, but a priori it does not come with an orientation. It is not difficult to see that there exists a unique orientation on M such that, if we equip $\overline{B}^{m-k}\times M$ with the product orientation, v becomes orientation-preserving.

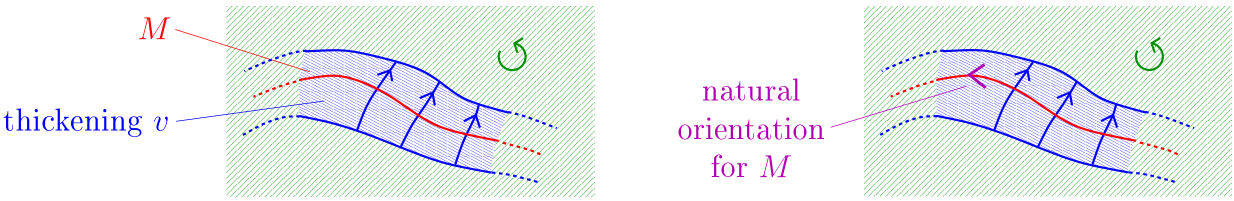


FIGURE 1605

Before we write down an explicit isomorphism we introduce the following definition which generalizes the definition of the self-linking number of a thickened knot that we introduced on page 2384.

Definition.

(1) Let $K = K_1 \sqcup \cdots \sqcup K_m$ and $L = L_1 \sqcup \cdots \sqcup L_n$ be two disjoint oriented links. We define the *linking number of K and L* as

$$\operatorname{lk}(K, L) \;=\; \sum_{i=1}^{m}\sum_{j=1}^{n} \operatorname{lk}(K_i, L_j).$$

(2) Let (L, g) be a thickened link. We write $L' = g(\{1\} \times L)$. We define the *self-linking number of* (L, g) as $\operatorname{slk}(L, g) := \operatorname{lk}(L, L')$.[1628]

The following elementary lemma is at times useful for determining self-linking numbers of thickened links.

Lemma 115.13. *If $(L, g) = (L_1 \sqcup \cdots \sqcup L_m, g_1 \sqcup \cdots \sqcup g_m)$ is a thickened link, then*

$$\operatorname{slk}(L, g) \;:=\; \sum_{i \neq j} \operatorname{lk}(L_i, L_j) + \sum_{i=1}^{m} \operatorname{slk}(L_i, g_i).$$

PROOF. We will prove the lemma in Exercise 115.2. ■

Now we can give the promised isomorphism.

Theorem 115.14. *The map*

$$\begin{array}{rcl} \Psi\colon \Omega_1^{\text{th}}(\mathbb{R}^3) & \to & \mathbb{Z} \\ [(L, g)] & \mapsto & \textit{self-linking number } \operatorname{slk}(L, g) \end{array}$$

is well-defined and it is an isomorphism. In particular we have $\pi_3(S^2) \cong \mathbb{Z}$.

Examples.

(1) We consider the trivial knot K with a thickening g of self-linking number $+1$. By Theorem 115.14 the thickened knot (K, g) defines a generator of $\Omega_1^{\text{th}}(\mathbb{R}^3)$. It follows from the isomorphism provided by the Thom-Pontryagin Theorem 115.11 that $\rho_{(K,g)}\colon S^3 \to S^2$ represents a generator of $\pi_3(S^2, *)$. In Lemma 115.4 we saw that $\rho_{(K,g)}$ represents the same element of $\pi_3(S^2, *)$ as the Hopf map $H\colon S^3 \to S^2$. Thus we see once again that the Hopf map represents a generator of $\pi_3(S^2)$.

[1628]Here we use the above convention to view L as an *oriented* submanifold of $\mathbb{R}^3$, and we equip $L' = g(\{1\} \times L)$ with the obvious corresponding orientation.

(2) We consider the Hopf link $L = L_1 \sqcup L_2$ with the orientations shown in Figure 1606. We equip $L = L_1 \sqcup L_2$ with a "trivial thickening" $g_1 \sqcup g_2$, i.e. where the thickening g_i of each component L_i has self-linking number 0. We see that

$$\operatorname{slk}(L_1 \sqcup L_2, g_1 \sqcup g_2) \underset{\substack{\uparrow \\ \text{Lemma 115.13}}}{=} \operatorname{lk}(L_1, L_2) + \operatorname{lk}(L_2, L_1) + \overbrace{\operatorname{slk}(L_1, g_1)}^{=0} + \overbrace{\operatorname{slk}(L_2, g_2)}^{=0} \underset{\substack{\uparrow \\ \text{pick your favorite method from Section 99.4}}}{=} 1 + 1 = 2.$$

This shows that the Hopf link, perhaps slightly counter-intuitively, gives rise to *twice* the generator of $\pi_3(S^2, *)$.

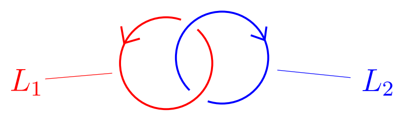


FIGURE 1606

Since we already have an independent proof of Theorem 115.14 we will take the liberty to cut a few corners in the new proof of the theorem. The new proof of Theorem 115.14 will build on the following four lemmas. The first two lemmas generalize Proposition 99.6 and Lemma 99.14 from knots to links. They are perhaps interesting in their own right.

Lemma 115.15. *Given any oriented link $L \subset S^3$ there exists a compact oriented connected surface $F \subset S^3$ such that $\partial F = L$.*[1629]

PROOF OF LEMMA 115.15. Let $L \subset S^3$ be an oriented link. A modest generalization of Proposition 99.6 shows that there exists a compact oriented surface $G \subset S^3$ with $\partial G = L$. In Exercise 115.5 we will show that we can turn G into a compact oriented *connected* surface F with $\partial F = L$.[1630] ■

Lemma 115.16. *If $K = K_1 \sqcup \cdots \sqcup K_m$ and $L = L_1 \sqcup \cdots \sqcup L_n$ are two oriented links, then the following statements hold:*

(1) *There exist compact oriented proper 2-dimensional submanifolds S and T of $\overline{B}^4$ with $\partial S = K$ and $\partial T = L$ such that S and T are transverse.*
(2) *Let S, T be two oriented proper 2-dimensional submanifolds of $\overline{B}^4$ with $\partial S = K$ and $\partial T = L$. If S and T are transverse, then*[1631]

$$\operatorname{lk}(K, L) \;=\; S \cdot_{\overline{B}^4} T.$$

[1629] Here the equality $\partial F = L$ is understood to be an equality of *oriented* submanifolds.

[1630] Frankly it can be slightly annoying to give a completely rigorous proof that such a surface F exists. This is one of the main reasons why we made the caveat above that we are going to cut a few corners in the argument.

[1631] Here $S \cdot_{\overline{B}^4} T$ denotes the algebraic intersection number of the oriented transverse surfaces S and T in the oriented smooth manifold $\overline{B}^4$. We refer to page 2271 for the definition of the algebraic intersection number.

Example. As on page 1734 we consider the Hopf link that is illustrated in Figure 1607 on the left. On the right of Figure 1607 we show a compact oriented connected surface whose boundary equals the Hopf link.

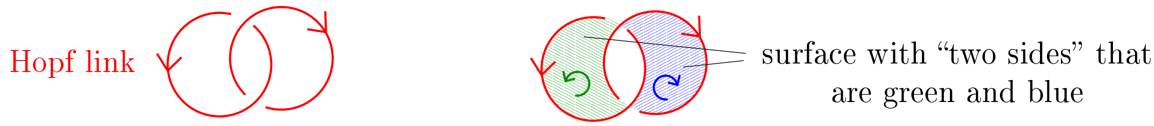


FIGURE 1607

PROOF OF LEMMA 115.16.

(1) The proof of this statement is virtually the same as the proof of Lemma 99.14, we just need to replace Proposition 99.6 by Lemma 115.15.
(2) We leave it to the reader to verify that the proof of Lemma 99.14 (2) can be modified to deal not only with knots but also with links. ■

The following two lemmas that we need for the proof of Theorem 115.14 are arguably more technical.

Lemma 115.17. *Let K be an oriented* knot *in $\mathbb{R}^3$ and let g and h be two thickenings for K in $\mathbb{R}^3$. If $\operatorname{slk}(K,g) = \operatorname{slk}(K,h)$, then $[(K,g)] = [(K,h)] \in \Omega_1^{\text{th}}(\mathbb{R}^3)$.*

PROOF OF LEMMA 115.17(∗). Since $\operatorname{slk}(K,g) = \operatorname{slk}(K,h)$ it follows from Lemma 99.1 that there exists a smooth isotopy $F\colon (\overline{B}^2 \times K) \times [0,1] \to \mathbb{R}^3$ between g and h which has compact support and which has the property that $F(0,k,t) = k$ for all $k \in K$ and all $t \in [0,1]$. We consider $W = [0,1] \times K$ with the thickening given by

$$\begin{array}{rcl} j\colon \overline{B}^2 \times ([0,1] \times K) & \to & [0,1] \times \mathbb{R}^3 \\ (v,(t,x)) & \mapsto & (t, F((v,x),t)). \end{array}$$

The thickened smooth manifold (W,j) is a thickened cobordism between (K,g) and (K,h). ■

Lemma 115.18. *Let W be a compact oriented proper connected 2-dimensional submanifold of $\mathbb{R}^3 \times [0,1)$ with boundary components $L_1, \dots, L_m$. Given $s_1, \dots, s_{m-1} \in \mathbb{Z}$ there exists a thickening h for W such that for $i = 1, \dots, m-1$ we have $\operatorname{slk}(L_i, h|_{L_i}) = s_i$.*

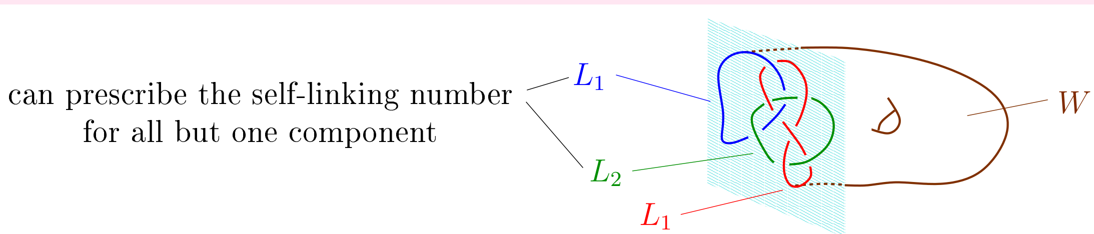


FIGURE 1608

PROOF. Evidently we can assume that $m \geq 1$. In this case W is a compact oriented proper connected 2-dimensional submanifold with non-empty boundary. Thus by Proposition 10.6

(3) there exists a thickening $f\colon \overline{B}^2 \times W \to \mathbb{R}^3 \times [0,1)$ for W. For $i = 1, \ldots, m-1$ we write $r_i := \operatorname{slk}(L_i, g|_{L_i})$.

By Lemma 23.12 there exists a map $\varphi\colon W \to L_1 \vee \cdots \vee L_{m-1}$[1632] that is the identity on each L_i. We pick a map $\psi\colon L_1 \vee \cdots \vee L_{m-1} \to \mathrm{SO}(2) = S^1$ with the following property: under an orientation-preserving identification $L_i = S^1$ the restriction of ψ to L_i is given by $z \mapsto z^{s_i - r_i}$. Now we consider the thickening

$$\begin{array}{rcl} h\colon \overline{B}^2 \times W & \to & W \\ (v, P) & \mapsto & f(\underbrace{(\overbrace{(\psi \circ \varphi)(P)}^{\in \mathrm{SO}(2)} \cdot v, P)}_{\in \overline{B}^2 \times W}). \end{array}$$

By Lemma 99.1 (1) we know that for any $i \in \{1, \ldots, m-1\}$ we added $s_i - r_i$ to the self-linking number of L_i. Thus the thickening h has all the desired properties. ■

Now we are in a position to provide the proof of Theorem 115.14.

Sketch of proof of Theorem 115.14.

(1) First we need to show that $\Psi\colon \Omega_1^{\mathrm{th}}(\mathbb{R}^3) \to \mathbb{Z}$ is well-defined. In the proof of this fact we borrow some ideas from the proof of Lemma 99.14 (2). So let (W, F) be a thickened cobordism between (K, g) and (L, h). We write $W' = F(\{1\} \times W)$, $K' = g(\{1\} \times K)$ and $L' = h(\{1\} \times L)$. By Lemma 115.16 (1) there exist oriented proper 2-dimensional submanifolds S and S' of $\overline{B}^4$ with $\partial S = K$ and $\partial S' = K'$ such that S and S' are transverse. Similarly there exist T, T' for L and L'.

We can make the identification $S^4 = (-\overline{B}^4) \cup [0,1] \times S^3 \cup \overline{B}^4$[1633] (See Figure 1609 for an illustration.) Using Proposition 8.22 we can arrange that both $(-S) \cup W \cup T$ and $(-S') \cup W' \cup T'$ are oriented submanifolds of the smooth manifold S^4. We calculate that

$$\begin{aligned} -\operatorname{slk}(K, g) + \operatorname{slk}(L, h) &\overset{\substack{\text{by definition of slk} \\ \downarrow}}{=} -\operatorname{lk}(K, K') + \operatorname{lk}(L, L') \overset{\substack{\text{Lemma 115.16 (2)} \\ \downarrow}}{=} (-S) \cdot_{\overline{B}^4} S' + T \cdot_{\overline{B}^4} T' \\ &= (-S) \cdot_{-\overline{B}^4} (-S') + T \cdot_{\overline{B}^4} T' \underset{\substack{\uparrow \\ \text{since the only intersections appear} \\ \text{between } S, S' \text{ and } T, T', \text{ here we use that} \\ W \text{ and } W' \text{ are disjoint}}}{=} (-S \cup W \cup T) \cdot_{S^4} (-S' \cup W' \cup T') \underset{\substack{\uparrow \\ \text{by Theorem 95.7 and} \\ \text{since } \mathrm{H}_2(S^4; \mathbb{Z}) = 0}}{=} 0. \end{aligned}$$

We have thus shown that the self-linking numbers of (K, g) and (L, h) agree. This shows that the map $\Psi\colon \Omega_1^{\mathrm{th}}(\mathbb{R}^3) \to \mathbb{Z}$ is indeed well-defined.

(2) It follows easily from Exercise 99.6 that Ψ is a homomorphism.

(3) By Lemma 99.1 there exists a knot in $\mathbb{R}^3$ with a thickening in $\mathbb{R}^3$ with self-linking number one, thus Ψ is an epimorphism.

[1632]Here, as a reminder, $L_1 \vee \cdots \vee L_{m-1}$ denotes the wedge of the topological spaces $L_1, \ldots, L_{m-1}$.

[1633]If the reader does not feel like making this identification, then we point out that we only need that the right-hand side is a smooth manifold and that the second homology vanishes.

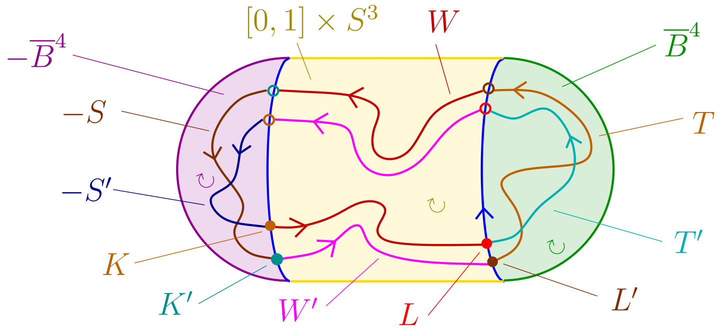


FIGURE 1609. Illustration for the proof of Theorem 115.14.

(4) It remains to show that $\Psi\colon \Omega_1^{\text{th}}(\mathbb{R}^3) \to \mathbb{Z}$ is a monomorphism. So suppose we are given a thickened m-component link $(L,g) = \bigsqcup\limits_{i=1}^{m} (L_i, g_i)$ with $\Psi([(L,g)]) = 0$.

Note that by Lemma 115.15 there exists a compact oriented connected surface W in $\mathbb{R}^3 \times [0,1)$ whose boundary equals $L \subset \mathbb{R}^3 = \mathbb{R}^3 \times \{0\}$. Next note that by Lemma 115.18 we can equip W with a thickening h such that for $i = 1,\dots,m-1$ we have $\operatorname{slk}(L_i, h|_{L_i}) = \operatorname{slk}(L_i, g_i)$.

Claim. We also have $\operatorname{slk}(L_m, h|_{L_m}) = \operatorname{slk}(L_m, g_m)$.

Note that (W,h) is a thickened cobordism between $(L, h|_L)$ and the empty link. In particular we obtain from (1) that $\operatorname{slk}(L, h|_L) = 0$. By our hypothesis on (L,g) we also have $\operatorname{slk}(L,g) = 0$. We had already arranged that for $i = 1,\dots,m-1$ we have $\operatorname{slk}(L_i, h|_{L_i}) = \operatorname{slk}(L_i, g_i)$. It now follows immediately from Lemma 115.13 that $\operatorname{slk}(L_m, h|_{L_m}) = \operatorname{slk}(L_m, g_m)$. ⊞

Now we are done, since

$$[(L,g)] \underset{\substack{\uparrow \\ \text{Lemma 115.17}}}{=} [(L, h|_g)] \underset{\substack{\uparrow \\ \text{since } (W,h) \text{ is a null-cobordism for } (L,h|_g)}}{=} 0 \in \Omega_1^{\text{th}}(\mathbb{R}^3).$$

■

115.5. **The thickened cobordism group $\Omega^{\text{th}}_{2n-1}(\mathbb{R}^{4n-1})$.** In Corollary 91.14 we had sketched a proof that given any $n \in \mathbb{N}$ there exists an epimorphism $\pi_{4n-1}(S^{2n}) \to \mathbb{Z}$. By the Thom-Pontryagin Theorem 115.11 this statement is equivalent to saying that there exists an epimorphism $\Omega^{\text{th}}_{2n-1}(\mathbb{R}^{4n-1}) \to \mathbb{Z}$.

In this section we will give an alternative proof that such an epimorphism exists. Since we already have a proof for this statement we will only prove the most salient statements. In other words, this section is stuffed with entertaining exercises for the reader.

First of all we work with the same convention as in the last section.

Convention. In this section we equip any thickened submanifold and any thickened cobordism with the natural orientation that we introduced on page 2739.

We continue with the following notation that is germane to this section.

Notation. Let $n \in \mathbb{N}$.

(1) Given a closed oriented m-dimensional submanifold K of S^{2m+1} we denote by
$$\beta_K\colon \mathrm{H}_m(K;\mathbb{Z}) \;\to\; \mathrm{H}^m(S^{2m+1}\setminus K;\mathbb{Z})$$
the inverse of the isomorphism given by the Alexander Duality Theorem 92.5.

(2) Given a compact oriented proper $(m+1)$-dimensional submanifold W of $[0,1]\times S^{2m+1}$ we denote by
$$\beta_W\colon \mathrm{H}_{m+1}(W,\partial W;\mathbb{Z}) \;\to\; \mathrm{H}^m(([0,1]\times S^{2m+1})\setminus W;\mathbb{Z})$$
the inverse of the isomorphism given by the Alexander Duality Theorem 92.5.

With this notation we can introduce the following definition.

Definition. Let $m \in \mathbb{N}$ and let K and L be two disjoint closed oriented m-dimensional submanifolds of S^{2m+1}. We define the *linking number of K and L* as
$$\mathrm{lk}(K,L) \;=\; \langle \beta_K([K]),[L]\rangle \;\in\; \mathbb{Z}$$
where we view the fundamental class $[L]$ of L as a homology class in $\mathrm{H}_m(S^{2m+1}\setminus K;\mathbb{Z})$.

Remark. The vigilant reader will not have failed to notice that in the 3-dimensional setting it is a priori not clear that the above definition of the linking number agrees with the definition given on page 2396. In Exercise 115.6 we will prove that in the 3-dimensional setting the two competing notions of linking numbers agree.

Proposition 115.19. *Let $n \in \mathbb{N}$ and let $* \in \overline{B}^{2n}$. The map*
$$\begin{array}{rcl}\Psi\colon \Omega^{\mathrm{th}}_{2n-1}(\mathbb{R}^{4n-1}) & \to & \mathbb{Z}\\ [(N,g\colon \overline{B}^{2n}\times N \to \mathbb{R}^{4n-1})] & \mapsto & \mathrm{lk}(N,g(*\times N))\end{array}$$
is a well-defined homomorphism.

Remark. Let $n \in \mathbb{N}$. We recall that on page 2217 we used the cup product to define the Hopf invariant
$$\mathrm{Hopf}\colon \pi_{4n-1}(S^{2n},*) \;\to\; \mathbb{Z}$$
which is, as shown in Proposition 91.7, in fact a homomorphism. The audacious reader can attempt to show that, under the above isomorphism $\Omega^{\mathrm{th}}_{2n-1}(\mathbb{R}^{4n-1}) \xrightarrow{\cong} \pi_{4n-1}(S^{2n},*)$, the Hopf invariant and the homomorphism $\Psi\colon \Omega^{\mathrm{th}}_{2n-1}(\mathbb{R}^{4n-1}) \to \mathbb{Z}$ from Proposition 115.19 agree up to sign.[1634]

Proof. In Exercise 115.7 we will verify that Ψ is a homomorphism. Now let us actually show that Ψ is well-defined. Thus suppose that we are given thickened $(2n-1)$-dimensional submanifolds (N_0,g_0) and (N_1,g_1) of $\mathbb{R}^{4n-1}$ that are cobordant via a thickened cobordism (W,G). We denote by $i_0\colon \{0\}\times S^{4n-1} \to [0,1]\times S^{4n-1}$ the inclusion. Similarly we define i_1. We also use i_0, i_1 in a rather obvious way for restrictions of the above inclusion maps to suitable subsets. It follows immediately from the Alexander Duality Theorem 92.5 that

[1634]Attempting to determine the correct sign with confidence might have negative effects on one's mental health.

we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}_{2n}(W,\partial W;\mathbb{Z}) \arrow[r, "\cong", "\beta_W"'] \arrow[d, "\partial"] & \mathrm{H}^{2n-1}(([0,1]\times S^{4n-1})\setminus W;\mathbb{Z}) \arrow[d, "i_0^*"', "i_1^*"] \\
\begin{array}{c}\mathrm{H}_{2n-1}(\{0\}\times N_0;\mathbb{Z})\\ \mathrm{H}_{2n-1}(\{1\}\times N_1;\mathbb{Z})\end{array} \arrow[r, "\begin{pmatrix}\beta_{N_0} & 0\\ 0 & \beta_{N_1}\end{pmatrix}", "\cong"'] & \begin{array}{c}\mathrm{H}^{2n-1}(\{0\}\times (S^{4n-1}\setminus N_0);\mathbb{Z})\\ \mathrm{H}^{2n-1}(\{1\}\times (S^{4n-1}\setminus N_1);\mathbb{Z})\end{array}
\end{tikzcd}$$

where all horizontal maps are isomorphisms and where the left-hand vertical map is the connecting homomorphism of the long exact sequence in homology of the pair $(W,\partial W)$.

It follows that

$$\begin{array}{rcl}
\operatorname{lk}(N_0,g_0(*\times N_0)) & = & \langle \beta_{N_0}([N_0]),[g_0(*\times N_0)]\rangle \overset{\substack{\text{follows from the above diagram and the fact}\\ \text{that } [\partial W]=[\{0\}\times N_0]+[\{1\}\times N_1]\text{, see Proposition 87.27}\\ \downarrow}}{=} \langle i_0^*(\beta_W([W])),[g_0(*\times N_0)]\rangle \\
& \underset{\substack{\uparrow\\ \text{Lemma 74.6 (3)}}}{=} & \langle \beta_W([W]),(i_0)_*([g_0(*\times N_0)])\rangle \underset{\substack{\uparrow\\ \text{it follows from } \partial(G(*\times W))=g_0(*\times N_0)\cup g_1(*\times N_1) \text{ together with}\\ \text{Corollary 87.28 (2) that } (i_0)_*([g_0(*\times N_0)])=(i_1)_*([g_1(*\times N_1)])}}{=} \langle \beta_W([W]),(i_1)_*([g_1(*\times N_1)])\rangle \\
& \underset{\substack{\uparrow\\ \text{same argument backwards}}}{=} & \operatorname{lk}(N_1,g_1(*\times N_1)).
\end{array}$$

We have thus shown that Ψ is indeed well-defined. ■

Of course an invariant is only interesting if we can show that it assumes non-trivial values. If $n=1$, then we already know[1635] that the trivial knot K together with the thickening g of self-linking number one satisfies $\Psi([(N,g)])=1$. In particular we see that $\Psi\colon \Omega_1^{\mathrm{th}}(\mathbb{R}^3)\to\mathbb{Z}$ is an epimorphism.

Recall that the combination of Propositions 91.7 and 91.13 shows that given any $n\in\mathbb{N}$ there exists a map $f\colon S^{4n-1}\to S^{2n}$ such that the Hopf invariant equals 2. The following proposition that can be viewed as an analogue of this fact.

Proposition 115.20. *For any $n\in\mathbb{N}$ there exists a thickened $(2n-1)$-dimensional submanifold (N,g) in $\mathbb{R}^{4n-1}$ with $\Psi([(N,g)])=2$.*

Sketch of proof. We consider the "trivial $(2n-1)$-dimensional knot"

$$K \;=\; S^{2n-1}\times\{0\} \;=\; \{(z_1,\dots,z_{2n},0,\dots,0)\in\mathbb{R}^{4n-1}\,|\,z_1^2+\dots+z_{2n}^2=1\}$$

together with the map[1636]

$$\begin{array}{rcl}
g\colon \overline{B}^{2n}\times K & \to & \mathbb{R}^{4n-1}\\
\big((x,\underbrace{(y_1,\dots,y_{2n-1})}_{=:y\in\mathbb{R}^{2n-1}}),\underbrace{(z,0)}_{\in K}\big) & \mapsto & (z+\tfrac14 x\cdot z,y).
\end{array}$$

[1635]Here we assume for convenience that we already know that the two different definitions of the linking pairing in the 3-dimensional setting agree.

[1636]This map g is basically just the "obvious" thickening of $S^{2n-1}\subset\mathbb{R}^{2n}$ extended to higher dimensions.

One can easily verify that, with a suitable orientation for K, the pair (K, g) is a thickened $(2n-1)$-dimensional submanifold of $\mathbb{R}^{4n-1}$. It is not particularly difficult to show that $[g(* \times K)] = 0 \in \mathrm{H}_{2n-1}(\mathbb{R}^{4n-1} \setminus K)$. Unfortunately this implies that $\mathrm{lk}(K, g(* \times K)) = 0$.

The first idea would be to modify the thickening g in a similar vein as in Lemma 99.1. This might work in principle, but at least your author does not know how to make this work in general.[1637] Instead we will recycle the example from page 2741 where we saw that the Hopf link gives rise to a non-trivial element in $\Omega_1^{\mathrm{th}}(\mathbb{R}^3)$. More precisely, we consider the diffeomorphism

$$\begin{array}{rcl} \Theta\colon \mathbb{R}^{4n-1} & \to & \mathbb{R}^{4n-1} \\ (\underbrace{x_1, \dots, x_{2n}}_{=x}, \underbrace{y_1, \dots, y_{2n-1}}_{=y}) & \mapsto & (y, x) + (0, \dots, 0, \underset{\substack{\uparrow \\ 2n\text{-th coordinate}}}{1}, 0, \dots, 0). \end{array}$$

Similar to the discussion on page 2741 one can now fairly easily show (voluntary exercise!) that $\Psi([K \sqcup \Theta(K), g \sqcup \Theta_*(g)]) = \pm 2$. Since Ψ is a homomorphism we see that there exists a thickened $(2n-1)$-dimensional submanifold (N, g) in $\mathbb{R}^{4n-1}$ with $\Psi([(N, g)]) = 2$. ■

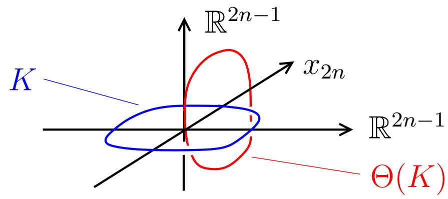


Figure 1610

Remark. In Theorem 110.11 we saw that, unless $n = 2, 4, 8$, there does not exist a map $f\colon S^{2n-1} \to S^n$ with $\mathrm{Hopf}(f) = 1$. If we believe for a second that the Hopf invariant corresponds to the above homomorphisms Ψ, then this shows that Proposition 115.20 is optimal for $n \notin \{1, 2, 4\}$.

In summary, the Thom-Pontryagin Theorem 115.11 together with Propositions 115.19 and 115.20 give us the following result, which is precisely the content of Corollary 91.14.

Proposition 115.21. *Given any* even $n \in \mathbb{N}$ *there exists an epimorphism* $\pi_{2n-1}(S^n) \to \mathbb{Z}$.

Exercises for Chapter 115.

Exercise 115.1. We consider the map

$$\begin{array}{rcl} f\colon (S^1 \times \overline{B}^2) \cup_{S^1 \times S^1} (\overline{B}^2 \times S^1) & \to & \overline{B}^2 / S^1 \\ P & \mapsto & \begin{cases} [Q], & \text{if } P = [(Q, z)] \text{ with } Q \in \overline{B}^2 \text{ and } z \in S^1, \\ [S^1], & \text{if } P = [(z, Q)] \text{ with } z \in S^1 \text{ and } Q \in \overline{B}^2. \end{cases} \end{array}$$

Show that f is null-homotopic.

[1637] Also the remark after the proof might temper one's optimism.

Exercise 115.2. Let $(L = L_1 \sqcup \cdots \sqcup L_m, g = g_1 \sqcup \cdots \sqcup g_m)$ be a thickened link. Show that

$$\operatorname{slk}(L,g) \ := \ \sum_{i \neq j} \operatorname{lk}(L_i, L_j) \ + \ \sum_{i=1}^{m} \operatorname{slk}(L_i, g_i).$$

Exercise 115.3. Let $m \geq 2$. The Thom-Pontryagin Theorem 115.11 implies in particular that $\Omega^{\text{th}}_{m-1}(\mathbb{R}^m) \cong \pi_1(S^m) = 0$. This implies that every thickened submanifold of $\mathbb{R}^m$ of codimension-one is null-cobordant. Prove this statement without making use of the Thom-Pontryagin Theorem 115.11.
Hint. Use Lemma 98.3.

Exercise 115.4. We consider the Hopf link H as shown in Figure 1611. Draw an explicit oriented connected compact surface whose boundary equals the oriented link H.
Remark. Note that the orientation of one of the components is different from the Hopf link that we had considered above in Figure 1607.

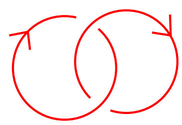


FIGURE 1611

Exercise 115.5. Let $L \subset S^3$ be an oriented link and let $G \subset S^3$ be a compact oriented surface with $\partial G = L$. We assume that G has no closed components.

(a) Show that $S^3 \setminus G$ is path-connected.
(b) Give an outline of an argument why there exists a compact oriented *connected* surface F with $\partial F = L$.

Exercise 115.6. Let K and J be two oriented disjoint knots in S^3. Show that the two definitions of the linking number $\operatorname{lk}(K, J)$ given on pages 2396 and 2745 agree.

Exercise 115.7. Show that the map

$$\begin{array}{rcl} \Psi \colon \Omega^{\text{th}}_{2n-1}(\mathbb{R}^{4n-1}) & \to & \mathbb{Z} \\ [(N, g \colon \overline{B}^{2n} \times N \to \mathbb{R}^{4n-1})] & \mapsto & \operatorname{lk}(N, g(* \times N)) \end{array}$$

considered in Proposition 115.19 is a homomorphism.

Exercise 115.8. In Section 115.5 we had just seen that given any even $n \in \mathbb{N}$ there exists an epimorphism $\Omega^{\text{th}}_{n-1}(\mathbb{R}^{2n-1}) \to \mathbb{Z}$. What about the group $\Omega^{\text{th}}_{n-1}(\mathbb{R}^{2n-1})$ if n is odd? Can we use linking numbers to define a homomorphism to $\mathbb{Z}$? Can we use them to find an epimorphism from $\Omega^{\text{th}}_{n-1}(\mathbb{R}^{2n-1})$ to $\mathbb{Z}$?

116. Thom-Pontryagin Theory II: The Freudenthal Suspension Theorem

116.1. **The framed cobordism group.** In Chapter 115 we introduced the thickened cobordism groups $\Omega_k^{\text{th}}(\mathbb{R}^{n+k})$ and we showed that they are isomorphic to $\pi_{n+k}(S^n,*)$. Furthermore we computed that $\Omega_0^{\text{th}}(\mathbb{R}^n)\cong\mathbb{Z}$ and $\Omega_1^{\text{th}}(\mathbb{R}^3)\cong\mathbb{Z}$. Finally we showed that given any $n\in\mathbb{N}$ there exists an epimorphism $\Omega_{2n-1}^{\text{th}}(\mathbb{R}^{4n-1})\to\mathbb{Z}$. These calculations gave us new proofs for Corollary 53.6, Theorem 113.10 (2) and Corollary 91.14.

Even though these new proofs were very pleasant, it is now time for some new results. In this section we will prove the Freudenthal Suspension Theorem and we will calculate the homotopy groups $\pi_{n+1}(S^n)$ for $n\geq 3$. For the subsequent calculations it is better though to work with the "framed cobordism groups" $\Omega_k^{\text{fr}}(\mathbb{R}^n)$ that we introduce in this section. As we will see, the framed cobordism groups are isomorphic to the thickened cobordism group, but at times they are easier to calculate since they allow us to reduce many issues to linear algebra.

First we introduce the following definitions in linear algebra.

Definition. Given $m,k\in\mathbb{N}_0$ we write

$$\begin{aligned}\mathrm{GL}(m,k) &:= \text{ set of } k\text{-tuples of linearly independent vectors of } \mathbb{R}^m\\ \mathrm{O}(m,k) &:= \text{ set of } k\text{-tuples of orthonormal vectors of } \mathbb{R}^m.\end{aligned}$$

We adopt the following conventions:

(1) Given $v\in\mathrm{GL}(m,k)$ we denote by $v_1,\dots,v_k$ the corresponding column vectors.
(2) Throughout we view both $\mathrm{GL}(m,k)$ and $\mathrm{O}(m,k)$ as subsets of the smooth manifold $\mathrm{M}(m\times k,\mathbb{R})=\mathbb{R}^{mk}$.

Now we can give the topological definition we are really interested in.

Definition. Let $m\in\mathbb{N}_0$ and let N be an m-dimensional submanifold of $\mathbb{R}^m$. (Later on we will only be interested in $N=\mathbb{R}^m$, $N=[0,1]\times\mathbb{R}^{m-1}$ and the case that $N=\overline{B}^m_r(P)$ is a closed m-dimensional ball.)

(1) Let M be a compact orientable proper k-dimensional submanifold of N.
   (a) Let $P\in M$. We say $v=(v_1,\dots,v_{m-k})\in\mathrm{GL}(m,m-k)$ is a *frame* of M at P if the vectors $v_1,\dots,v_{m-k}$, together with $\mathrm{T}_P M$[1638] spans $\mathbb{R}^m$.
   (b) A *framing for M* is a smooth[1639] map $v\colon M\to\mathrm{GL}(m,m-k)$ such that for each $P\in M$ the vectors $v_1(P),\dots,v_{m-k}(P)$ are a frame of M at P.
   (c) We say that a framing $v=(v_1,\dots,v_{m-k})$ is *perpendicular* if for each $P\in M$ the vectors $v_1(P),\dots,v_{m-k}(P)$ lie in the orthogonal complement of $\mathrm{T}_P M$, i.e. if they lie in the vector space

$$(\mathrm{T}_P M)^\perp \;=\; \{v\in\mathbb{R}^m\,|\,\langle v,w\rangle=0 \text{ for all } w\in\mathrm{T}_P M\}.$$

   If the framing is perpendicular and if all the vectors are orthonormal, then we that the framing is *orthonormal* .
(2) A *framed k-dimensional submanifold* of N is a pair (M,v) where M is a compact orientable proper k-dimensional submanifold of N and v is a framing for M.

(3) Let (M, v) be a framed k-dimensional submanifold of N and let $\varphi\colon N \to W$ be a smooth embedding of N into a smooth manifold of the same dimension. We consider the map

$$\begin{array}{rcl} \varphi_*(v)\colon \varphi(M) & \to & \mathrm{GL}(m, m-k) \\ P & \mapsto & (\varphi_*(v_1(\varphi^{-1}(P))), \ldots, \varphi_*(v_{m-k}(\varphi^{-1}(P)))). \end{array}$$

The pair $(\varphi(M), \varphi_*(v))$ is a framed submanifold of W.

Some of the above definitions are illustrated in Figure 1612.

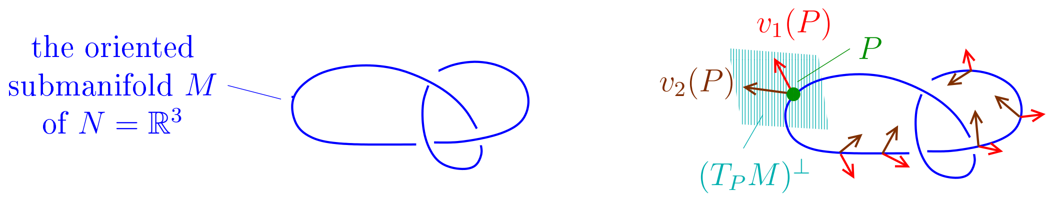


FIGURE 1612

Definition. Let $k, m \in \mathbb{N}_0$ and let (N_0, v_0) and (N_1, v_1) be two framed k-dimensional submanifolds of $\mathbb{R}^m$. A *framed cobordism between* (N_0, v_0) *and* (N_1, v_1) is a framed $(k+1)$-dimensional submanifold (W, V) of $[0,1] \times \mathbb{R}^m$ such that the following three conditions are satisfied:

(1) $\partial W = (\{0\} \times N_0) \cup (\{1\} \times N_1)$,
(2) $V|_{\{0\}\times N_0} = v_0$ under the identification $\{0\} \times \mathbb{R}^m = \mathbb{R}^m$,
(3) $V|_{\{1\}\times N_1} = v_1$ under the identification $\{1\} \times \mathbb{R}^m = \mathbb{R}^m$,

If such a framed cobordism exists, then we say that (N_0, v_0) and (N_1, v_1) are *cobordant.* The definition of a framed cobordism is illustrated in Figure 1613.

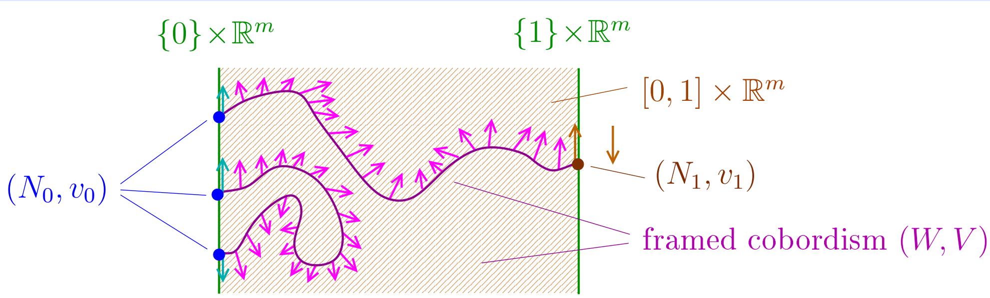


FIGURE 1613

Basically the same proof as in Lemma 115.8 gives us the following lemma.

Lemma 116.1. *The property of "being cobordant" is an equivalence relation on the set of framed k-dimensional submanifolds of $\mathbb{R}^m$.*

This lemma leads us, not surprisingly, to the following definition.

[1638]Here we use the isomorphism from Proposition 6.39 to identify $\mathrm{T}_P\, M$ with the vector subspace $\mathrm{V}_P\, M$ of $\mathbb{R}^m$.

[1639]Since we view $\mathrm{GL}(m, m-k)$ as a subset of the smooth manifold $\mathrm{M}(m \times (m-k), \mathbb{R}) = \mathbb{R}^{m\cdot(m-k)}$ it makes sense to say that a map $M \to \mathrm{GL}(m, m-k)$ is smooth.

Definition. We denote by $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$ the set of cobordism classes of framed k-dimensional submanifolds of $\mathbb{R}^m$. We refer to $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$ as a *framed cobordism group.*

The following proposition, which is the analogue of Proposition 115.9, justifies the name "framed cobordism group".

Proposition 116.2. *Let $k \in \mathbb{N}$ and let $m \in \mathbb{N}_{\geq 2}$. Let (M, v) and (N, w) be two framed k-dimensional submanifolds of $\mathbb{R}^m$. Since M and N are compact we can pick $r, s \in \mathbb{R}_{>0}$ with $M \subset B_r^m(0)$ and $N \subset B_s^m(0)$. Furthermore we pick two orientation-preserving smooth embeddings $\varphi\colon \overline{B}_r^m(0) \to \mathbb{R}^m$ and $\psi\colon \overline{B}_s^m(0) \to \mathbb{R}^m$ with disjoint images. We define*

$$[(M, v)] + [(N, w)] \;:=\; [(\varphi(M) \sqcup \psi(N), \varphi_*(v) \sqcup \psi_*(w))] \;\in\; \Omega_k^{\mathrm{fr}}(\mathbb{R}^m).$$

This operation "+" has the following properties:

(1) *The definition of "+" on $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$ is well-defined and commutative.*
(2) *The operation "+" defines a group structure on $\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$. The following statements hold:*
   (a) *The neutral element is represented by the empty submanifold.*
   (b) *Let $\rho\colon \mathbb{R}^m \to \mathbb{R}^m$ be the reflection in a hyperplane. The inverse of $[(M, v)]$ is given by $[(\rho(M), \rho_*(v))]$.*

Proof. The proof is basically identical to the proof of Proposition 115.9. We leave it to the reader to carry out the minor adjustments that are necessary to make the transition from thickened cobordisms groups to framed cobordism groups. ■

Lemma 116.3. *Let $m \in \mathbb{N}_{\geq 2}$ and $k \in \{0, \ldots, m\}$.*

(1) *Every framed k-dimensional submanifold (M, v) of $\mathbb{R}^m$ is cobordant to (M, w) where w is an orthonormal framing.*
(2) *Let (W, V) be a framed cobordism between two framed k-dimensional submanifolds (M_0, v_0) and (M_1, v_1) of $\mathbb{R}^m$. If v_0 and v_1 are orthonormal, then there exist an orthonormal framing V' for W such that (W, V') is a framed cobordism between (M_0, v_0) and (M_1, v_1).*

Proof (∗). We prove statement (1) of the lemma. The proof of statement (2) is a mild variation on the proof of statement (1). Thus let (M, u) be a framed k-dimensional submanifold of $\mathbb{R}^m$. Using the fact that being cobordant is transitive, the proof of the lemma can be broken up into two stages:

(1) First we show that there exists a perpendicular framing v such that (M, u) is cobordant to (M, v).
(2) Then we show that there exists an orthonormal framing w such that (M, v) is cobordant to (M, w).

We carry out the two steps:

(1) Let $(M, u\colon M \to \mathrm{GL}(m, m-k))$ be a framed k-dimensional submanifold of $\mathbb{R}^m$. Given $P \in M$ we write $U_P M := \operatorname{span}\{u_1(P), \ldots, u_k(P)\}$ and furthermore we write $Z_P M := (\mathrm{T}_P\, M)^\perp$. For each $P \in M$ the decomposition $\mathbb{R}^m = \mathrm{T}_P\, M \oplus Z_P M$ defines a projection of $\mathbb{R}^m$ onto $Z_P M$. We denote the restriction of this projection to $U_P M$

by $\rho_P\colon U_PM \to Z_PM$. By definition of a framing this map is an isomorphism. We consider the product cobordism $W = [0,1] \times M$ equipped with the framing[1640]

$$\begin{array}{rcl} V\colon W=[0,1]\times M & \to & \mathrm{GL}(m,m-k) \\ (t,P) & \mapsto & (u(P)\cdot(1-t)+\rho_P(u(P))\cdot t). \end{array}$$

Basically by definition this is a framed cobordism from (M,u) to (M,v) where $v=V_1$ is a perpendicular framing.

(2) In Lemma 8.8 (1) we showed that there exists a smooth deformation retraction $R\colon \mathrm{GL}(m,m-k)\times[0,1]\to \mathrm{O}(m,m-k)$ from $\mathrm{GL}(m,m-k)$ to $\mathrm{O}(m,m-k)$ which has the following extra property:

$(*)$ For any $v=(v_1,\dots,v_{m-k})\in \mathrm{GL}(m,m-k)$ and any $t\in[0,1]$ we have

$$\text{span of } R_t(v_1),\dots,R_t(v_{m-k}) \;=\; \text{span of } v_1,\dots,v_{m-k}.$$

Now let (M,v) be a framed k-dimensional submanifold. We consider the product cobordism $W=[0,1]\times M$ equipped with the framing

$$\begin{array}{rcl} V\colon W=[0,1]\times M & \to & \mathrm{GL}(m,m-k) \\ (t,P) & \mapsto & R_t(v(P)). \end{array}$$

It follows from the fact that R is smooth and that $(*)$ holds that this is a framed cobordism from (M,v) to (M,w) where $w=V_1$ is an orthonormal framing. ■

The following proposition shows that the two types of cobordism groups that we introduced in this and the last chapter are in fact isomorphic.

Proposition 116.4. *Let $m\in\mathbb{N}_{\geq 2}$ and let $k\in\{0,\dots,m\}$. The map*

$$\begin{array}{rcl} \Phi\colon \Omega_k^{\mathrm{th}}(\mathbb{R}^m) & \to & \Omega_k^{\mathrm{fr}}(\mathbb{R}^m) \\ {[(N,g\colon \overline{B}^{m-k}\times N)]} & \mapsto & \left[\begin{array}{l} N \text{ equipped with the framing that associates to } P\in N \\ \text{the frame } \mathrm{D}\, f_{(0,P)}(e_1),\dots,\mathrm{D}\, f_{(0,P)}(e_{m-k}) \end{array}\right] \end{array}$$

is well-defined and it is an isomorphism.

Convention. Let $n,k\in\mathbb{N}_0$ with $n+k\geq 2$. We use the isomorphisms from the Thom-Pontryagin Theorem 115.11 and Proposition 116.4 to make the identifications

$$\pi_{n+k}(S^n,*) \;=\; \Omega_k^{\mathrm{th}}(\mathbb{R}^{n+k}) \;=\; \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}).$$

Sketch of proof.

(1) It follows basically from the definitions that the framing associated to a thickened submanifold is indeed a framing. Applying the same procedure to a thickened cobordism between thickened submanifolds we obtain a framed cobordism between framed submanifolds. This shows that Φ is well-defined.
(2) It follows immediately from the definitions of the group structures that Φ is a homomorphism.

[1640] It takes a little bit of thought to show that this is indeed a smooth map.

(3) We sketch the construction of the inverse to Φ. Let $(N, \{v_1, \dots, v_{m-k}\})$ be a framed k-dimensional submanifold of $\mathbb{R}^m$. We consider the map

$$\begin{array}{rcl} g(v)\colon \mathbb{R}^{m-k} \times N & \to & \mathbb{R}^m \\ ((t_1, \dots, t_{m-k}), P) & \mapsto & P + t_1 \cdot v_1(P) + \dots + t_{m-k} \cdot v_{m-k}(P). \end{array}$$

The same argument as in the proof of Theorem 8.24 shows that there exists an $\epsilon > 0$ such that $g(v)$ restricts to a smooth embedding $\overline{B}^k_\epsilon(0) \times N \to \mathbb{R}^m$. After rescaling the ball we thus get a thickening $\overline{B}^k \times N \to \mathbb{R}^m$ of N. It is straightforward to see that the class $[(N, g(v)] \in \Omega^{\text{th}}_k(\mathbb{R}^m)$ does not depend on the choice of $\epsilon > 0$. Since the same approach also works for cobordisms we obtain a well-defined map

$$\Psi\colon \Omega^{\text{fr}}_k(\mathbb{R}^m) \to \Omega^{\text{th}}_k(\mathbb{R}^m).$$

It remains to show that Φ and Ψ are inverses of one another. It follows basically immediately from the definitions and Lemma 116.3 (1) that $\Phi \circ \Psi$ is the identity on $\Omega^{\text{fr}}_k(\mathbb{R}^m)$. The fact that $\Psi \circ \Phi$ is the identity on $\Omega^{\text{th}}_k(\mathbb{R}^m)$ follows fairly immediately from the uniqueness of tubular neighborhoods, see Proposition 10.7, together with Lemma 10.4. ■

Definition. Let N be an orientable k-dimensional submanifold of $\mathbb{R}^m$ and let $n \geq m$.

(1) We write $N_n := N \times \{0\} \subset \mathbb{R}^n$.

Now suppose we are given a framing $v\colon N \to \text{GL}(m, m-k)$ for N.

(2) We define

$$\begin{array}{rcl} v_n\colon N_n & \to & \text{GL}(n, n-k) \\ (P, 0) & \mapsto & \begin{pmatrix} v(P) & 0 \\ 0 & \text{id}_{n-m} \end{pmatrix}. \end{array}$$

It is clear that v_n is a framing for N_n. Note that $N_m = N$ and $v_m = v$.

(3) We refer to (N_n, v_n) as the *$(n-m)$-fold suspension of* (N, v). In particular we refer to (N_{m+1}, v_{m+1}) as *the suspension of* (N, v).

The same notation applies in an obvious way to (framed) submanifolds of $[0, 1] \times \mathbb{R}^m$.

The following lemma justifies the name "suspension" that we had used in the above definition.

Lemma 116.5. *Let $m \in \mathbb{N}_{\geq 2}$ and let $k \in \{0, \dots, m\}$.*

(1) *The map*

$$\begin{array}{rcl} \Omega^{\text{fr}}_k(\mathbb{R}^m) & \to & \Omega^{\text{fr}}_k(\mathbb{R}^{m+1}) \\ [(N, v)] & \mapsto & [\textit{the suspension of } (N, v)] \end{array}$$

is well-defined and it is a homomorphism. We refer to it as the suspension homomorphism.

(2) *Under the identification from the convention on page 2752 the suspension homomorphisms on $\pi_{n+k}(S^n, *)$ and $\Omega^{\text{fr}}_k(\mathbb{R}^{n+k})$ agree up to sign,*[1641] *in other words, the*

following diagram commutes up to sign:

$$\begin{tikzcd}
\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \arrow[r, "suspension"] \arrow[d, "\cong"] & \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1}) \arrow[d, "\cong"] \\
\pi_{n+k}(S^n,*) \arrow[r, "suspension"] & \pi_{n+k+1}(S^{n+1},*).
\end{tikzcd}$$

Proof.

(1) The first statement is almost obvious.
(2) We leave it to the reader to go through all the definitions to verify this statement. ■

116.2. **The Freudenthal Suspension Theorem.** In this section we will prove the following theorem that was first proved in 1937 by Hans Freudenthal [**Freu37**, p. 300].

Theorem 116.6. (Freudenthal Suspension Theorem) *Let $n, k \in \mathbb{N}_0$. We suppose that $n + k \geq 2$. The suspension homomorphism $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ from Lemma 40.19 is an isomorphism if $n > k+1$ and it is an epimorphism if $n = k+1$.*

Remark.

(1) The Freudenthal Suspension Theorem 116.6 can be reformulated as follows: given $n \in \mathbb{N}$ the suspension homomorphism $\pi_m(S^n) \to \pi_{m+1}(S^{n+1})$ is an isomorphism if $m < 2n-1$ and that it is an epimorphism if $m = 2n-1$. We consider two situations in slightly more detail:
  (a) The Freudenthal Suspension Theorem 116.6 says that the suspension homomorphism $\pi_3(S^2) \to \pi_4(S^3)$ is an epimorphism and it says that for any $n \geq 3$ the suspension homomorphism $\pi_{n+1}(S^n) \to \pi_{n+2}(S^{n+1})$ is an isomorphism. This special case will play an important role shortly.
  (b) The Freudenthal Suspension Theorem 116.6 almost recovers Theorems 113.11 (2) and 113.14 (2). Recall that these theorems stated that for $m \in \{0, \ldots, 5\}$ we have $\pi_m(S^3) \cong \pi_{m+1}(S^4)$ and that for $m \in \{0, \ldots, 13\}$ we have $\pi_m(S^7) \cong \pi_{m+1}(S^8)$.
(2) Let $n \in \mathbb{N}_0$. Recall that on page 695 we saw that the suspension $\Sigma(S^n)$ is homeomorphic to S^{n+1}. Thus we see that the Freudenthal Suspension Theorem 116.6 is a consequence of the following stronger statement: Let X be an $(n-1)$-connected topological space and let $* \in X$ be a good base point.[1642] Then the suspension homomorphism $\pi_{n+k}(X,*) \to \pi_{n+k+1}(\Sigma(X),*)$ is an isomorphism for all $n > k+1$ and it is an epimorphism for $n = k+1$. This statement is proved in most textbooks on homotopy theory, see e.g. [**Gra75**, Theorem 16.34], [**WhdG78**, Theorem VII.7.13] or [**DaK01**, Theorem 8.7 and Chapter 10.3]. If X is a CW-complex, then the statement is also proved in [**Hat02**, Corollary 4.24]. The proof of [**Hat02**, Corollary 4.24] is actually a corollary to the Blakers-Massey Theorem 111.17 that we had stated but not proved.

[1641] To be more precise, they might agree on the nose, but we will not spend valuable time on figuring out the sign.

[1642] We refer to page 604 for the definition of a good point.

It follows from Lemma 116.5 that the Freudenthal Suspension Theorem 116.6 is equivalent to the following theorem.

Theorem 116.7. *Let $n, k \in \mathbb{N}$.*

(1) *The suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is an epimorphism whenever $n \geq k+1$.*
(2) *The suspension homomorphism $\Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$ is a monomorphism whenever $n > k+1$.*

The basic behind the proof of Theorem 116.7 (1) is quite simple: given a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$ we want to use the "room provided by a large enough n" to show that it is cobordant to a framed submanifold coming from $\mathbb{R}^{n+k} = \mathbb{R}^{n+k} \times \{0\}$. This idea is for the most part carried out in the following two lemmas.

Lemma 116.8. *Let (M, v) be a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$. If $n \geq k+1$, then there exists a framed cobordism from (M, v) to a framed k-dimensional submanifold (N, w) of $\mathbb{R}^{n+k+1}$ with $N \subset \mathbb{R}^{n+k} \times \{0\}$.*

PROOF. Let $(M, v\colon M \to \mathrm{GL}(n+k+1, n+1))$ be a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$. We point out that by Lemma 116.3 (1) we can assume that v is an orthonormal framing.

It follows from $n \geq k+1$ and Lemma 11.7 that there exists a vector $\nu \in \mathbb{R}^{n+k+1}$ which satisfies the following condition:

(∗) the vector ν does not lie in any tangent space $T_P M$[1643] and ν is not secant to M, i.e. for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to ν.

By Lemma 3.32 (4) there exists a matrix $A \in \mathrm{SO}(n+k+1)$ with $A\nu = e_{n+k+1}$. By Lemma 8.5 there exists a diffeotopy of $\mathbb{R}^{n+k+1}$ from $\mathrm{id}_{\mathbb{R}^{n+k+1}}$ to the diffeomorphism given by multiplying by A. Therefore, by the obvious analogue of Lemma 115.7 for the framed cobordism group, we can assume without loss of generality that $\nu = e_{n+k+1}$.

Claim. The map

$$\begin{array}{rcl} \Psi\colon [0,1] \times M & \to & [0,1] \times \mathbb{R}^{n+k+1} \\ (t, \underbrace{(x_1, \ldots, x_{n+k+1})}_{\in M}) & \mapsto & (t, \underbrace{x_1, \ldots, x_{n+k}, (1-t) \cdot x_{n+k+1}}_{\in \mathbb{R}^{n+k+1}}) \end{array}$$

is a proper smooth embedding. The definition of Ψ is illustrated in Figure 1614.

It follows from the fact that for any two distinct points $x, y \in M$ the difference $x - y$ is not parallel to $\nu = e_{n+k+1}$ that the map is injective. Furthermore note that the map Ψ is the composition of the inclusion $i\colon [0,1] \times M \to [0,1] \times \mathbb{R}^{n+k+1}$ with the map

$$\begin{array}{rcl} \Theta\colon [0,1] \times \mathbb{R}^{n+k+1} & \to & [0,1] \times \mathbb{R}^{n+k+1} \\ (t, (x_1, \ldots, x_{n+k+1})) & \mapsto & (t, x_1, \ldots, x_{n+k}, (1-t) \cdot x_{n+k+1}). \end{array}$$

[1643] Here and throughout this proof we use Proposition 6.39 to view the tangent spaces of submanifolds of some $\mathbb{R}^m$ as vector subspaces of $\mathbb{R}^m$.

A straightforward calculation shows that

$$D\Theta_{(t,x_1,\dots,x_{n+k+1})} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{id}_{n+k} & 0 \\ -x_{n+k+1} & 0 & 1-t \end{pmatrix}.$$

It follows immediately from this calculation and the fact that e_{n+k+1} is not contained in any tangent space $\mathrm{T}_P M$ that $\Psi = \Theta \circ i$ is an immersion. Together with the injectivity of Θ, the fact that M is compact and the ever-popular Proposition 2.43 (2) we obtain that $\Psi = \Theta \circ i$ is in fact a smooth embedding. It is also clear that the smooth embedding is proper. ⊞

We set $W := \Psi([0,1]\times M)$. By the claim together with Proposition 8.1 (2) we know that W is a proper $(k+1)$-dimensional submanifold of $[0,1]\times\mathbb{R}^{n+k+1}$. Basically by definition W is a cobordism between $M \subset \mathbb{R}^{n+k+1}$ and some submanifold $N \subset \mathbb{R}^{n+k} \times \{0\}$.

It remains to extend the orthonormal framing $v\colon M \to \mathrm{GL}(n+k+1, n+1)$ of M to a framing of W. In fact, we obtain a framing of $W = \Psi([0,1]\times M)$ by using the "same vectors" as for M. More precisely, we consider the map

$$\begin{aligned} V\colon W = \Psi([0,1]\times M) &\;\to\; \mathrm{GL}(n+k+2, n+1) \\ \Psi(t,P) &\;\mapsto\; \begin{pmatrix} 0 \\ v(P) \end{pmatrix}. \end{aligned}$$

Evidently under the identification $\{0\}\times\mathbb{R}^{n+k+1} = \mathbb{R}^{n+k+1}$ we have $V|_M = v$. Thus it remains to show that V is indeed a framing. A short moment's thought shows that V is smooth. The definition of W and V are illustrated in Figure 1614. Therefore it suffices to

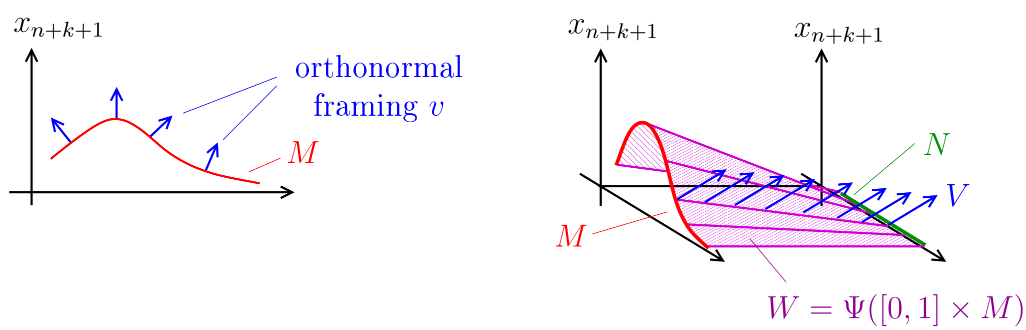


Figure 1614. Illustration for the proof of Lemma 116.8.

prove the following claim.

Claim. For any $t \in [0,1]$ and any $P \in M$ the matrix $V(\Psi(t,P)) \in \mathrm{GL}(n+k+2, n+1)$ is a frame for $W = \Psi([0,1]\times M)$ at $\Psi(t,P)$.

Let $t \in [0,1]$ and let $P \in M$. We point out that the columns of $V(\Psi(t,P))$ span an $(n+1)$-dimensional vector subspace of $\mathbb{R}^{n+k+2}$. Since $\mathrm{T}_{\Phi(t,P)}(W)$ is a vector subspace of the complementary dimension $k+1$ it suffices to show that the intersection of these two vector subspaces is trivial.

We note that

$$\underbrace{\mathrm{T}_{\Phi(t,P)}(W)\cap \begin{array}{c}\text{span of the}\\ \text{columns of}\\ V(\Psi(t,P))\end{array}}_{\subset\mathbb{R}^{1+n+k+1}} \underset{\substack{\text{by definition of } W=\Psi([0,1]\times M)=(\Theta\circ i)([0,1]\times M)\\ \text{and the calculation of } D\Theta}}{=} \underbrace{\begin{pmatrix} 1 & 0 & 0\\ 0 & \mathrm{id}_{n+k} & 0\\ -x_{n+k+1} & 0 & 1-t\end{pmatrix}\cdot\begin{pmatrix}\mathbb{R}\\ \oplus\\ \mathrm{T}_P(M)\end{pmatrix}\cap \begin{array}{c}\text{span of the}\\ \text{columns of}\\ V(\Psi(t,P))\end{array}}_{\subset\mathbb{R}^{1+n+k+1}}$$

$$\underset{\substack{\text{since the first row of } V \text{ is zero we can}\\ \text{“ignore” the first coordinate}}}{\cong} \underbrace{\begin{pmatrix}\mathrm{id}_{n+k} & 0\\ 0 & 1-t\end{pmatrix}\cdot\mathrm{T}_P(M)\cap \begin{array}{c}\text{span of the}\\ \text{columns of } v(P)\end{array}}_{\subset\mathbb{R}^{n+k+1}} \underset{\text{since } v \text{ is an orthonormal framing}}{=} \begin{pmatrix}\mathrm{id}_{n+k} & 0\\ 0 & 1-t\end{pmatrix}\cdot\mathrm{T}_P(M)\cap(\mathrm{T}_P\,M)^\perp.$$

Now let $u\in\mathrm{T}_P\,M$ be a non-zero vector. We write $u=\begin{pmatrix}u'\\ u_{n+k+1}\end{pmatrix}$ with $u'\in\mathbb{R}^{n+k}$ and $u_{n+k+1}\in\mathbb{R}$. Note that u' is non-zero since we had arranged that $e_{n+k+1}\notin\mathrm{T}_P\,M$. Now we calculate that

$$\left\langle \begin{pmatrix}\mathrm{id}_{n+k} & 0\\ 0 & 1-t\end{pmatrix}\begin{pmatrix}u'\\ u_{n+k+1}\end{pmatrix}, \underbrace{\begin{pmatrix}u'\\ u_{n+k+1}\end{pmatrix}}_{=u\in\mathrm{T}_P\,M}\right\rangle = \langle u',u'\rangle+(1-t)\cdot u_{n+k+1}^2 \underset{\text{since } u'\neq 0 \text{ and } (1-t)\geq 0}{>} 0.$$

But this shows that $\begin{pmatrix}\mathrm{id}_{n+k} & 0\\ 0 & 1-t\end{pmatrix}\cdot u$ is not orthogonal to $\mathrm{T}_P\,M$, in other words, this vector does not lie in $(\mathrm{T}_P\,M)^\perp$. ■

Lemma 116.9. *Let (M,v) be a framed k-dimensional submanifold in $\mathbb{R}^{n+k}\times\{0\}\subset\mathbb{R}^{n+k+1}$. If $n\geq k+1$, then there exists a framed cobordism to $(M,w=(w_1,\dots,w_{n+1}))$ where we have $w_{n+1}(P)=e_{n+k+1}$ for all $P\in M$.*

Proof. Let $(M,u=(u_1,\dots,u_{n+1})\colon M\to\mathrm{GL}(n+k+1,n+1))$ be a framed k-dimensional submanifold that is contained in $\mathbb{R}^{n+k}\times\{0\}\subset\mathbb{R}^{n+k+1}$. By Lemma 116.3 (1) and Lemma 116.1 we can assume that u is orthonormal. We write $e:=e_{n+k+1}$.

Claim. There exists an orthonormal framing $v=(v_1,\dots,v_{n+1})$ for M such that (M,u) is cobordant to (M,v) and such that for every $P\in M$ we have $v_{n+1}(P)\neq -e$.

Let $x\in M$. Since M is contained in $\mathbb{R}^{n+k}\times\{0\}$ we know that e is orthogonal to $\mathrm{T}_x\,M$. Therefore we can uniquely write $e=\sum_{i=1}^{n+1}\lambda_i(x)\cdot u_i(P)$. Since $\|e\|=1$ and since u is an orthonormal framing we actually have $\sum_{i=1}^{n+1}\lambda_i^2(P)=1$. Thus we obtain a map

$$\begin{aligned}\lambda\colon M &\to S^n\\ x &\mapsto (\lambda_1(x),\dots,\lambda_{n+1}(x))\end{aligned}$$

which is fairly easily seen to be smooth. If $(0,\dots,0,-1)\notin\lambda(M)$, then we can just take $u=v$ and we are done.[1644] If not we proceed as follows. Since $\dim(M)=k<n$ it follows from Proposition 6.62 that there exists a $\nu\in S^n$ with $\nu\notin\lambda(M)$. By Lemma 3.32 (4) there exists a matrix $A\in\mathrm{SO}(n+1)$ with $\nu=A\cdot(-e)\in S^n$. By Lemma 2.65 and Exercise 18.7 there exists a smooth path $\gamma\colon[0,1]\to\mathrm{SO}(n+1)$ from id to A. The desired framed cobordism is now given by the smooth manifold $[0,1]\times M$ together with the framing[1645]

$$\begin{array}{rcl}[0,1]\times M & \to & \mathrm{GL}(n+k+2,n+1)\\ (t,x) & \mapsto & \underbrace{\begin{pmatrix}0\\ u(x)\end{pmatrix}}_{\in\mathrm{GL}(n+k+2,n+1)}\cdot\gamma(t).\end{array}$$

⊞

As a final step we want to “rotate v_{n+1} into e”. To do this smoothly for every point we need to introduce two maps. Given $u\in\mathbb{R}^{n+1}\setminus\{0\}$ we write

$$\underbrace{\begin{array}{rcl}T_u\colon\mathbb{R}^{n+k+1} & \mapsto & \mathbb{R}^{n+k+1}\\ y & \to & 2\cdot\langle y,u\rangle\cdot u-y\end{array}}_{\text{reflection in the line }\mathbb{R}\cdot u}\quad\text{and}\quad\underbrace{\begin{array}{rcl}R_u\colon\mathbb{R}^{n+k+1} & \to & \mathbb{R}^{n+k+1}\\ y & \mapsto & T_e(T_u(y)).\end{array}}_{\text{orientation-preserving diffeomorphism}}$$

We will need the following elementary observation from linear algebra.

Claim. Let $w\in\mathbb{R}^{n+k+1}$ with $\|w\|=1$. Let $u=\frac{w+e}{\|w+e\|}$ be the normalized midpoint between w and e. Then $R_u(w)=e$.

We have

$$R_u(w)\;=\;T_e(T_u(w))\;=\;T_e\Big(2\cdot\Big\langle w,\tfrac{w+e}{\|w+e\|}\Big\rangle\cdot\tfrac{w+e}{\|w+e\|}-w\Big)\;\underset{\substack{\uparrow\\ \text{short calculation using }\langle w,w\rangle=\langle e,e\rangle}}{=}\;T_e(e)\;=\;e.$$

⊞

The above claim now allows us to come to the denouement of the argument. More precisely, given $t\in[0,1]$ and $x\in M$ we write

$$z(x)\;:=\;\underbrace{\frac{v_{n+1}(x)+e}{\|v_{n+1}(x)+e\|}}_{\text{note that }v_{n+1}(x)+e\neq0}\qquad\text{and}\qquad z_t(x)\;=\;\underbrace{\frac{(1-t)\cdot e+t\cdot z(x)}{\|(1-t)\cdot e+t\cdot z(x)\|}}_{\text{note that }t\cdot e+(1\text{-}t)\cdot z(x)\neq0}.$$

Note that $z_0=e$ and $R_{z_0}=\mathrm{id}$. Furthermore note that $z_1=z(x)$. Thus we obtain from the above claim that $R_{z_1(x)}(v_{n+1}(x))=e$. Finally we consider the framed cobordism that is given by the submanifold $Z=[0,1]\times M$ of $[0,1]\times\mathbb{R}^{n+k+1}$ together with the framing

$$\begin{array}{rcl}W\colon Z=[0,1]\times M & \to & \mathrm{GL}(n+k+2,n+1)\\ (t,x) & \mapsto & \begin{pmatrix}0\\ R_{z_t(x)}\circ v(x)\end{pmatrix}.\end{array}$$

[1644] Indeed, if $v_{n+1}(x)=-e$, then $e=-v_{n+1}(x)$, i.e. $\lambda(x)=(0,\dots,0,-1)$.

[1645] That this framing has the correct properties can be seen as follows. Let v be the new framing at $\{1\}\times M$. If we denote by $\lambda_u,\lambda_v\colon M\to S^n$ the corresponding maps, then one easily sees that for each $x\in M$ we have $\lambda_v(x)=A^{-1}\cdot\lambda_u(x)$. Thus the fact that ν does not lie in the image of λ_u implies that $-e=A^{-1}\nu$ does not lie in the image of λ_v.

We leave it to the reader to verify that the map W is indeed a framing for the cobordism $Z = [0,1] \times M \subset [0,1] \times \mathbb{R}^{n+k+1}$. The discussion above shows that $(Z = [0,1] \times M, W)$ is a framed cobordism from (M, v) to a framed smooth manifold $(M, w = w_1, \ldots, w_n, e)$. ■

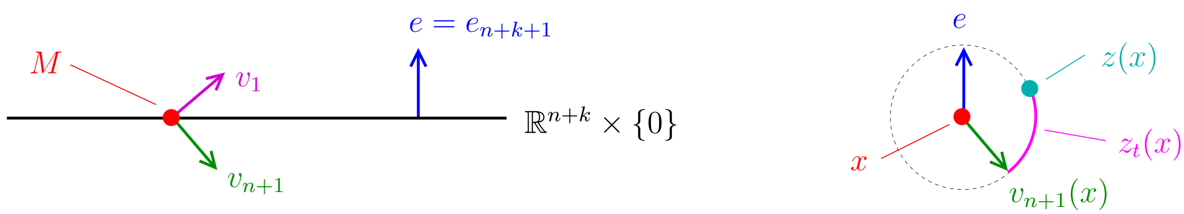


FIGURE 1615. Illustration for the proof of Lemma 116.9.

Now we turn to the actual proof of Theorem 116.7 and thus of the Freudenthal Suspension Theorem 116.6.

PROOF. Let $n, k \in \mathbb{N}_0$ with $n + k \geq 2$.

(1) Suppose that $n \geq k+1$. We want to show that the suspension homomorphism $\Omega_k^{\text{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\text{fr}}(\mathbb{R}^{n+k+1})$ is an epimorphism. Thus let (M, v) be a framed k-dimensional submanifold of $\mathbb{R}^{n+k+1}$. It follows immediately from Lemmas 116.8 and 116.9 that (M, v) is cobordant to a k-dimensional framed submanifold (N, w) such that $N \subset \mathbb{R}^{n+k} \times \{0\}$ and such that $w_{n+1}(P) = e_{n+k+1}$ for all $P \in N$. But this means that $[(M, v])] = [(N, w)]$ lies in the image of the suspension homomorphism. Thus the suspension homomorphism $\Omega_k^{\text{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\text{fr}}(\mathbb{R}^{n+k+1})$ is an epimorphism.

(2) Suppose that $n > k+1$. We want to show that the suspension homomorphism $\Omega_k^{\text{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\text{fr}}(\mathbb{R}^{n+k+1})$ is a monomorphism. Therefore let (M, v) and (N, w) be two framed k-dimensional submanifolds of $\mathbb{R}^{n+k}$. Suppose there exists a framed cobordism (W, V) in $[0,1] \times \mathbb{R}^{n+k+1}$ between the suspensions of (M, v) and (N, w). Since $n > k+1$, an argument as in Lemmas 116.8 and 116.9 shows that we can turn (W, V) into a cobordism (W', V') with $W' \subset [0,1] \times \mathbb{R}^{n+k} \times \{0\}$ and such that $V'(P) = e_{n+k+1}$ for all $P \in W'$. But this shows that (M, v) and (N, w) are already cobordant. Thus we have shown that for $n > k+1$ the suspension homomorphism $\Omega_k^{\text{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\text{fr}}(\mathbb{R}^{n+k+1})$ is a monomorphism. ■

Exercise for Chapter 116.

Exercise 116.1. In the proof of Proposition 117.4 we used the Freudenthal Suspension Theorem 116.6 to show that the map

$$\begin{array}{ccc} \mathbb{Z} & \to & \Omega_1^{\text{fr}}(\mathbb{R}^4) \\ [k] & \mapsto & k \cdot [(\mathrm{C}_4, h_4)] \end{array}$$

is an epimorphism. Use Exercise 11.5 to give an alternative argument, which does not rely on the Freudenthal Suspension Theorem 116.6.

117. Thom-Pontryagin Theory III: The groups $\Omega_1^{\text{fr}}(\mathbb{R}^{n+1})$ and $\Omega_2^{\text{fr}}(\mathbb{R}^{n+2})$

In this chapter we will calculate the framed cobordism groups $\Omega_1^{\text{fr}}(\mathbb{R}^{n+1})$ and $\Omega_2^{\text{fr}}(\mathbb{R}^{n+2})$ "by hand" for $n \geq 3$. By the Thom-Pontryagin Theorem 115.11 and Proposition 116.4 this means that we calculate the homotopy groups $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$ for $n \geq 3$ By now there might be more modern and efficient ways for computing these homotopy groups. But the geometric approach using Thom-Pontryagin theorem is still very charming, in particular it lets us use many techniques we have developed so far, and it gives us an opportunity to introduce a few new tricks. In particular we will get to introduce quadratic forms and the Arf invariant, which are purely algebraic objects, and are of great interest in the own right.

117.1. **The calculation of the framed cobordism groups $\Omega_1^{\text{fr}}(\mathbb{R}^{n+1})$ for $n \geq 3$.** In Theorem 113.10 (2) (and alternatively in Theorem 115.14) we saw that $\pi_3(S^2) \cong \mathbb{Z}$ and we saw that the Hopf map represents a generator of $\pi_3(S^2)$. Furthermore, by the Freudenthal Suspension Theorem 116.6 we know that for every $n \geq 3$ the suspension homomorphism $\pi_3(S^2) \to \pi_{n+1}(S^n)$ is an epimorphism. Thus we know that $\pi_{n+1}(S^n)$ is isomorphic to a quotient of $\mathbb{Z}$ and that $\pi_{n+1}(S^n)$ is generated by the $(n-2)$-fold suspension of the Hopf map. The goal of this section is to prove the following theorem which determines this quotient.

Theorem 117.1. *For every $n \geq 3$ there exists an isomorphism $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$.*

Remark.

(1) In Theorem 109.13 we already saw that for every $n \geq 3$ we have $\pi_{n+1}(S^n) \neq 0$. The proof of Theorem 117.1 below is independent of Theorem 109.13. In particular we will not make use of Steenrod operations.
(2) By the Thom-Pontryagin Theorem 115.11 and Proposition 116.4 the theorem is equivalent to showing that for every $n \geq 3$ we have an isomorphism $\Omega_1^{\text{fr}}(\mathbb{R}^{n+1}) \cong \mathbb{Z}_2$. This is exactly what we will do below. The approach that we take below to calculate $\Omega_1^{\text{fr}}(\mathbb{R}^{n+1})$ differs, to the best of my knowledge, from the other proofs in the literature. We refer to [**Pon59**, Chapter 14] for Pontryagin's original proof, which makes for challenging reading for the modern reader. A third, quite readable approach is presented in [**Put**].
(3) There are various alternative proofs of Theorem 117.1. The first proof was given by Hans Freudenthal [**Freu37**, p. 301] in 1937. Another proof is given in [**Hat**, p. 549] using "spectral sequences".
(4) The discussion preceding the theorem shows that the suspension of the Hopf map defines a non-trivial element in $\pi_4(S^3)$. This gives in particular an affirmative answer to Question 40.20.

As we had just mentioned above, proving Theorem 117.1 is equivalent to showing that for every $n \geq 3$ there exists an isomorphism $\Omega_1^{\text{fr}}(\mathbb{R}^{n+1}) \cong \mathbb{Z}_2$. We will perform this calculation in the remainder of this section.

We start out our calculation of $\Omega_1^{\text{fr}}(\mathbb{R}^{n+1})$ for $n \geq 3$ with the following definition.

Definition.

(1) We refer to $C := S^1$ as the *standard circle in* $\mathbb{R}^2$.

(2) We refer to the framing of $C \subset \mathbb{R}^2$ that is given by

$$\begin{array}{rcl} t\colon C & \to & \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} x \\ y \end{pmatrix} \end{array}$$

as the *trivial framing* t *of* C.

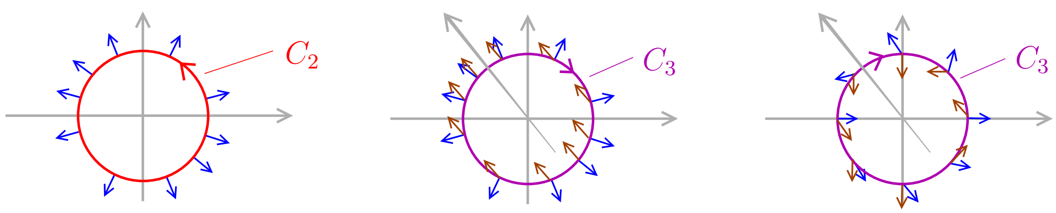
(3) Let $n \in \mathbb{N}_{\geq 2}$. We adopt the convention from page 2753 to define the *standard standard circle* C_n in $\mathbb{R}^n$. Furthermore, using the definition from page 2753 we define the *trivial framing* t_n *for* C_n.

(4) We refer to the framing of $\mathrm{C}_3 \subset \mathbb{R}^3$ that is given by the map[1646]

$$\begin{array}{rcl} h\colon \mathrm{C}_3 & \to & \mathbb{R}^3 \\ \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} & \mapsto & \underset{\substack{\uparrow \\ \text{don't overlook the } t_3}}{t_3} \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} = \begin{pmatrix} \cos(\varphi) & 0 \\ \sin(\varphi) & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \end{array}$$

as the *Hopf framing* h *of* C_3.

(5) Let $n \in \mathbb{N}_{\geq 3}$. As in (3) we use the convention from page 2753 to define the *Hopf framing* h_n *of* C_n. These definitions are illustrated in Figure 1616.



trivial framing t_2 trivial framing t_3 Hopf framing h_3

FIGURE 1616

Throughout this section we also use the following notation:

Notation.

(1) Given $g \in \mathbb{N}_0$ we denote by $\Sigma_{g,1}$ the surface of genus g minus one open disk. We pick once and for all an orientation for $\Sigma_{g,1}$.

(2) For any $n \in \mathbb{N}_{\geq 2}$ we denote by

$$\begin{array}{rcl} \rho\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ (x_1, x_2, x_3, \ldots, x_n) & \mapsto & (x_1, -x_2, x_3, \ldots, x_n) \end{array}$$

the reflection in the $(x_2 = 0)$-hyperplane.

(3) Given $n \in \mathbb{N}_0$ we make the identification $\mathbb{R}^n = \{0\} \times \mathbb{R}^n \subset [0,1] \times \mathbb{R}^n$.

The following two lemmas justify the names "trivial framing" and "Hopf framing".

[1646]We leave the task of showing that this map is smooth to the every-ready reader.

Lemma 117.2.

(1) *Given any $g \in \mathbb{N}_0$ there exists a proper smooth embedding $\psi\colon \Sigma_{g,1} \to [0,1] \times \mathbb{R}^2$ and a framing w for $\psi(\Sigma_{g,1})$ such that $\partial\psi(\Sigma_{g,1}) = \mathrm{C}_2$ and such that the restriction of w to C_2 equals t.*

(2) *For every $n \geq 2$ we have $[(C_n, t_n)] = 0 \in \Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$.*

Sketch of proof.

(1) By now the seasoned reader should be fairly convinced that there exists a proper smooth embedding $\psi\colon \Sigma_{g,1} \to [0,1] \times \mathbb{R}^2$ with $\psi(\partial\Sigma_{g,1}) = \mathrm{C}_2$ and such that the smooth embedding is a "product near the boundary". More precisely, such that for some $\epsilon \in (0,1)$ we have the equality $\psi(\Sigma_{g,1}) \cap ([0,\epsilon] \times \mathbb{R}^2) = [0,\epsilon] \times \mathrm{C}_2$. Since $\psi(\Sigma_{g,1})$ is an oriented connected submanifold of $[0,1] \times \mathbb{R}^2$ of codimension two we see that $\psi(\Sigma_{g,1})$ has precisely two normal vector fields of length one. We refer to Figure 1617 for an illustration.

It follows almost immediately from the fact that $\psi(\Sigma_{g,1})$ is a product near $\{0\} \times \mathbb{R}^2$ that one of the above two normal vectors fields agrees with t on $\{0\} \times S^1$.

(2) It follows from (1) that $[(\mathrm{C}_2, t_2)] = 0 \in \Omega_1^{\mathrm{fr}}(\mathbb{R}^2)$. From Lemma 116.5 we obtain that $[(C_n, t_n)] = 0 \in \Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$ for any $n \in \mathbb{N}_{\geq 2}$. ■

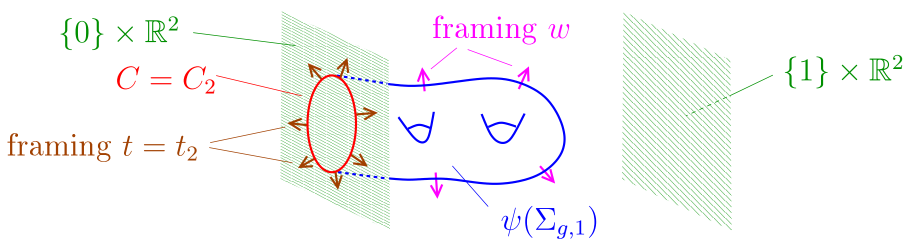


Figure 1617. Illustration of Lemma 117.2.

Lemma 117.3.

(1) *Under the isomorphism $\Omega_1^{\mathrm{fr}}(\mathbb{R}^3) \cong \pi_3(S^2, *)$ the element $[(\mathrm{C}_3, h_3)]$ corresponds, up to a sign, to the element represented by the Hopf map $S^3 \to S^2$.*

(2) *Let $n \geq 4$. Under the isomorphism $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \cong \pi_n(S^{n-1}, *)$ the element $[(C_n, h_n)]$ corresponds, up to a sign, to the element represented by the $(n-3)$-fold suspension of the Hopf map $S^3 \to S^2$.*

Proof. The first statement follows easily from Lemma 115.4. The second statement is a straightforward consequence of Lemma 116.5 (2). ■

The following proposition is the first key ingredient in the calculation of $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$ for $n \geq 4$.

Proposition 117.4. *Let $n \in \mathbb{N}_{\geq 4}$. The map*

$$\begin{array}{rcl} \mathbb{Z}_2 & \to & \Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \\ [k] & \mapsto & k \cdot [(C_n, h_n)] \end{array}$$

is well-defined and it is an epimorphism.

Our proof of Proposition 117.4 requires one definition and two lemmas.

Definition. Let $n \in \mathbb{N}_{\geq 2}$ and let $\gamma\colon S^1 \to \mathbb{R}^n$ be a smooth embedding. Furthermore let $f\colon \gamma(S^1) \to \mathrm{GL}(n, n-1)$ be a framing for $\gamma(S^1)$ and let $\alpha\colon S^1 \to \mathrm{GL}_+(n-1, \mathbb{R})$ be a smooth map. We define

$$\begin{array}{rcl} f(\alpha)\colon \gamma(S^1) & \to & \mathrm{GL}(n, n-1) \\ \gamma(z) & \mapsto & \underbrace{f(\gamma(z))}_{\in \mathrm{GL}(n,n-1)} \cdot \underbrace{\alpha(z).}_{\in \mathrm{GL}(n-1,\mathbb{R})} \end{array}$$

It follows easily from the fact that α takes values in $\mathrm{GL}_+(n-1, \mathbb{R})$ that $f(\alpha)$ is again a framing of $\gamma(S^1)$.

The first lemma that goes into the proof of Proposition 117.4 is really just a crucial example masquerading as a lemma.

Lemma 117.5. *Let $n \in \mathbb{N}_{\geq 3}$ and let $\gamma\colon S^1 \to C_n$ be the obvious diffeomorphism. We consider the loop*

$$\begin{array}{rcl} \alpha\colon S^1 & \to & \mathrm{GL}_+(n-1, \mathbb{R}) \\ z = e^{\mathrm{i}\varphi} & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-3} \end{pmatrix}. \end{array}$$

As above let t_n be the trivial framing for C_n and let h_n be the Hopf framing for C_n. We have the following two equalities of framings of C_n:

(1) *We have $t_n(\alpha) = h_n$.*
(2) *We have $t_n(\overline{\alpha}) = \rho_*(h_n)$.*[1647]

PROOF (∗). The first statement follows immediately from the definitions. We turn to the proof of the second statement. As we will see, the proof of the second statement is given by a slightly tedious calculation. To simplify the notation we only deal with the case $n = 3$.

[1647]Here $\overline{\alpha}$ is the "inverse of the loop α", i.e. it is the loop given by $\overline{\alpha}(e^{\mathrm{i}\varphi}) = \alpha(e^{-\mathrm{i}\varphi})$. As above $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ is the reflection in the $(x_2 = 0)$-hyperplane. Finally the framing $\rho_*(h_n)$ is defined as on page 2750.

At a given point $P = \gamma(e^{i\varphi}) = \begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\\ 0\end{pmatrix} \in \mathrm{C}_3$ we have

$$
\begin{aligned}
t_n(\overline{\alpha})\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\\ 0\end{pmatrix} \;&=\; t_n\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\\ 0\end{pmatrix}\cdot \underbrace{\alpha(e^{-i\varphi})}_{=\overline{\alpha}(e^{i\varphi})} \;=\; \begin{pmatrix}\cos(\varphi) & 0\\ \sin(\varphi) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos(-\varphi) & -\sin(-\varphi)\\ \sin(-\varphi) & \cos(-\varphi)\end{pmatrix}\\
&\underset{\substack{\uparrow\\ \text{since } \rho \text{ is reflection in the second coordinate}\\ \text{together with the symmetries of sin and cos}}}{=} \rho\left(\begin{pmatrix}\cos(-\varphi) & 0\\ \sin(-\varphi) & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}\cos(-\varphi) & -\sin(-\varphi)\\ \sin(-\varphi) & \cos(-\varphi)\end{pmatrix}\right) \underset{\substack{\uparrow\\ \text{definition of } h_n\text{, see page 2761}}}{=} \rho\left(h_n\begin{pmatrix}\cos(-\varphi)\\ \sin(-\varphi)\\ 0\end{pmatrix}\right)\\
&\underset{\substack{\uparrow\\ \text{definition of } \rho}}{=} \rho\left(h_n\left(\rho^{-1}\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\\ 0\end{pmatrix}\right)\right) \underset{\substack{\uparrow\\ \rho_*(h_n) \text{ as defined on page 2750}}}{=} \rho_*(h_n)\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\\ 0\end{pmatrix}.
\end{aligned}
$$

This concludes the mind-numbing verification of the second statement. ■

Lemma 117.6. *Let $n \in \mathbb{N}_{\geq 2}$ and let $\gamma\colon S^1 \to \mathbb{R}^n$ be a smooth embedding. Furthermore let $f\colon \gamma(S^1) \to \mathrm{GL}(n, n-1)$ be a framing for $\gamma(S^1)$. Finally let $\alpha, \beta\colon S^1 \to \mathrm{GL}_+(n-1, \mathbb{R})$ be two smooth maps. If α and β are smoothly homotopic, then*

$$[(\gamma(S^1), f(\alpha))] \;=\; [(\gamma(S^1), f(\beta))] \;\in\; \Omega_1^{\mathrm{fr}}(\mathbb{R}^n).$$

Proof of Lemma 117.6 (*). We write $M = \gamma(S^1)$. Since $\alpha, \beta\colon S^1 \to \mathrm{GL}_+(n-1, \mathbb{R})$ are smoothly homotopic we can pick a smooth homotopy $H\colon S^1 \times [0,1] \to \mathrm{GL}_+(n-1, \mathbb{R})$ with $H_0 = \alpha$ and $H_1 = \beta$. We consider the product cobordism $[0,1] \times M$ equipped with the framing

$$
\begin{aligned}
V\colon [0,1]\times M \;&\to\; \mathrm{GL}(n+1, n)\\
(t, \gamma(z)) \;&\mapsto\; \begin{pmatrix}1 & 0\\ 0 & f(\gamma(z))\cdot H(z,t)\end{pmatrix}
\end{aligned}
$$

It is clear that $([0,1]\times M, V)$ is a framed cobordism between $(\gamma(S^1), f(\alpha))$ and $(\gamma(S^1), f(\beta))$. ■

Now we can provide the proof of Proposition 117.4.

Proof of Proposition 117.4. Let $n \in \mathbb{N}_{\geq 4}$. We consider the following diagram

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r, "k\mapsto k\cdot[(\mathrm{C}_3, h_3)]", "\cong"'] \arrow[d, two heads] \arrow[dr, two heads, "k\mapsto k\cdot[(C_n, h_n)]"'] & \Omega_1^{\mathrm{fr}}(\mathbb{R}^3) \arrow[d, two heads, "\substack{(n\text{-}3)\text{-fold suspension}\\ \text{homomorphism}}"] \\
\mathbb{Z}_2 \arrow[r, dashed] & \Omega_1^{\mathrm{fr}}(\mathbb{R}^n).
\end{tikzcd}
$$

We make the following observations:

(1) It follows quite easily from Theorem 115.14, Lemma 117.3 and Proposition 116.4 that the top horizontal map is an isomorphism.
(2) The right vertical map is given by the $(n-3)$-fold application of the suspension homomorphism defined on page 2753. Note that the right vertical map is an epimorphism by the Freudenthal Suspension Theorem 116.7.
(3) The top right triangle commutes by definition of the suspension homomorphism and by definition of (C_n, h_n).

The above diagram and the above discussion show that it remains to prove the following claim.

Claim. The element $2 \cdot [(C_n, h_n)]$ is trivial in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$.

We consider the loop

$$\begin{array}{rcl} \alpha\colon S^1 & \to & \mathrm{GL}_+(n-1, \mathbb{R}) \\ z = e^{\mathrm{i}\varphi} & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-3} \end{pmatrix}. \end{array}$$

We have the following equalities in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$:

$$2 \cdot [(C_n, h_n)] \underset{\substack{\uparrow \\ \text{Lemma 117.5 (1)}}}{=} [(C_n, h_n)] + [(C_n, t_n(\alpha))] \underset{\substack{\uparrow \\ \text{since } n \geq 4 \text{ we know by Proposition 114.11 (2) that } \pi_1(\mathrm{GL}_+(n-1,\mathbb{R})) \cong \mathbb{Z}_2, \\ \text{therefore we see that } [\alpha]\cdot[\alpha] = 0 \in \pi_1(\mathrm{GL}_+(n-1,\mathbb{R})), \\ \text{this implies by Proposition 14.7 that } \alpha \text{ is path-homotopic to } \overline{\alpha}, \\ \text{in fact, by the Whitney Approximation Theorem 9.3} \\ \text{we see that } \alpha \text{ is actually smoothly homotopic to } \overline{\alpha}, \\ \text{by Lemma 117.6 this gives us the equality } [(C_n, t_n(\alpha))] = [(C_n, t_n(\overline{\alpha}))]}}{=} [(C_n, h_n)] + [(C_n, t_n(\overline{\alpha}))]$$

$$\underset{\substack{\uparrow \\ \text{Lemma 117.5 (2)}}}{=} [(C_n, h_n)] + [(\rho_*(\mathrm{C}_n), \rho_*(h_n))] \underset{\substack{\uparrow \\ \text{Proposition 116.2}}}{=} [(C_n, h_n)] + (-[(C_n, h_n)]) = 0.$$

■

Let $n \geq 4$. Note that by Proposition 117.4 we now know that there exists an epimorphism $\mathbb{Z}_2 \to \Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \cong \pi_n(S^{n-1})$. In Theorem 109.13 we showed that $\pi_{n+1}(S^n)$ is non-trivial. Therefore it follows that $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$, which means we have proved Theorem 117.1. The readers who studied Steenrod operations *and* who happen to be tired of Thom-Pontryagin theory can thus move on to next section.

For all others we provide a proof of Theorem 117.1 that does not rely on Theorem 109.13. More precisely, we prove the following proposition which is the main technical result of this section.

Proposition 117.7. *For any $n \in \mathbb{N}_{\geq 4}$ the map*

$$\begin{array}{rcl} \mathbb{Z}_2 & \to & \Omega_1^{\mathrm{fr}}(\mathbb{R}^n) \\ {[k]} & \mapsto & k \cdot [(C_n, h_n)] \end{array}$$

is an isomorphism.

As we had mentioned many times, Theorem 117.1 is an immediate consequence of the Thom-Pontryagin Theorem 115.11 and Proposition 116.4 together with Proposition 117.7.

PROOF. Let $n \in \mathbb{N}_{\geq 4}$. By Proposition 117.4 it remains to show that (C_n, h_n) represents a non-trivial element in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$.

We start out with a more general discussion. Given any $m \in \mathbb{N}_{\geq 2}$ we make the identification $\mathrm{C}_m = \{0\} \times \mathrm{C}_m \subset [0,1] \times \mathbb{R}^m$. The following claim applies to all dimensions.

Claim. Let $m \in \mathbb{N}_{\geq 2}$. Let F be a compact orientable connected proper 2-dimensional smooth manifold in $[0,1] \times \mathbb{R}^m$ with $\partial F = \mathrm{C}_m$. At most one of the two framings t_m and h_m can be extended to a framing of F.

Suppose there exist framings T and H on F with $T|_{\mathrm{C}_m} = t_m$ and $H|_{\mathrm{C}_m} = h_m$. It follows from Lemma 116.3 that we can assume that T and H are orthonormal. We consider the map[1648]

$$\begin{array}{rcl} \Psi\colon F & \to & \mathrm{GL}(m-1,\mathbb{R}) \\ P & \mapsto & \text{the unique matrix } \Psi(P) \in \mathrm{GL}(m-1,\mathbb{R}) \text{ with } T(P)\cdot\Psi(P) = H(P). \end{array}$$

We denote by $i\colon \mathrm{C}_m \to F$ the inclusion map and we denote by $\gamma\colon S^1 \to \mathrm{C}_m$ the usual diffeomorphism given by $\gamma(e^{\mathrm{i}\varphi}) = (\cos(\varphi), \sin(\varphi), 0, \dots, 0)$. We make the following key observation: it follows immediately from the definitions and Lemma 117.5 that the map $\Psi \circ i \circ \gamma\colon S^1 \to \mathrm{GL}(m-1,\mathbb{R})$ is precisely, on the nose, given by the map

$$\begin{array}{rcl} \alpha\colon S^1 & \to & \mathrm{GL}(m-1,\mathbb{R}) \\ e^{\mathrm{i}\varphi} & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{m-3} \end{pmatrix}. \end{array}$$

Now we consider the following diagram

$$\begin{tikzcd} \mathrm{H}_1(\partial F;\mathbb{Z}) \arrow[d, "i_*"'] & \mathrm{H}_1(S^1;\mathbb{Z}) \arrow[l, "\gamma_*"'] \arrow[d, "\alpha_*"'] & & \pi_1(S^1,1) \arrow[ll, "\text{Hurewicz isom.}"', "\cong"] \arrow[d, "\alpha_*"'] \\ \mathrm{H}_1(F;\mathbb{Z}) \arrow[r, "\Psi_*"] & \mathrm{H}_1(\mathrm{GL}(m-1,\mathbb{R})) & & \pi_1(\mathrm{GL}(m-1,\mathbb{R}),\mathrm{id}). \arrow[ll, "\cong"', "\text{Hurewicz isom.}"] \end{tikzcd}$$

We make the following observations:

(1) The left-hand square commutes by the above equality $\alpha = \Psi \circ i \circ \gamma$.
(2) The two horizontal maps to the right are the Hurewicz homomorphisms that we introduced on page 1314. The Hurewicz homomorphisms are isomorphisms by the Hurewicz Theorem 52.5 together with the fact that the fundamental groups on the right are both abelian.
(3) The right-hand square commutes by the naturality of the Hurewicz homomorphisms, see Proposition 52.2 (4).

[1648] Let $P \in F$. The fact that T and H are orthonormal means that the columns of $T(P)$ and $H(P)$ are bases for the same vector space, namely $(\mathrm{T}_P F)^\perp$. Therefore there exists a matrix $\Psi(P) \in \mathrm{GL}(m-1,\mathbb{R})$ such that $T(P)\cdot\Psi(P) = H(P)$. We leave it to the reader to verify that the map $P \mapsto \Psi(P)$ is continuous.

(4) By Proposition 114.11 (2) and (3) we know that the loop $\alpha\colon S^1 \to \mathrm{GL}(m-1,\mathbb{R})$ represents a non-trivial element in $\pi_1(\mathrm{GL}(m-1,\mathbb{R}),\mathrm{id})$. This implies that the right vertical map $\alpha_*\colon \pi_1(S^1,1) \to \pi_1(\mathrm{GL}(m-1,\mathbb{R}),\mathrm{id})$ is non-trivial.
(5) It follows from (1) – (4) that the left vertical map is also non-trivial. But this contradicts Lemma 48.8 (1). ⊞

Now we start out with the actual proof of the theorem. Thus let $n \in \mathbb{N}_{\geq 4}$. As we pointed out above, we need to show that (C_n, h_n) represents a non-trivial element in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$. Suppose that (C_n, h_n) represents in fact the trivial element in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^n)$. This means that there exists a framed cobordism $(F, v) = (F_n, v_n)$ in $[0,1] \times \mathbb{R}^n$ from (C_n, h_n) to the empty smooth manifold. In other words, $F = F_n$ is a compact orientable proper 2-dimensional submanifold of $[0,1] \times \mathbb{R}^n$ with $\partial F_n = C_n$ and $v = v_n$ is a framing with $v|_{\mathrm{C}_n} = h_n$. By throwing away all closed components of F_n we can assume that F_n is connected.

By the Surface Classification Theorem 23.4 we know that there exists a $g \in \mathbb{N}_0$ such that F is diffeomorphic to $\Sigma := \Sigma_{g,1}$. In other words, there exists a proper smooth embedding $\varphi\colon \Sigma \to [0,1] \times \mathbb{R}^n$ with $\varphi(\Sigma) = F_n$. Given any $m \geq n$ we denote by $\varphi_m\colon \Sigma \to [0,1] \times \mathbb{R}^m$ the smooth embedding that is given by the composition of the smooth embedding φ with the obvious smooth embedding $[0,1] \times \mathbb{R}^n \to [0,1] \times \mathbb{R}^m$.

By Lemma 117.2 (1) there exists a proper smooth embedding $\psi\colon \Sigma \to [0,1] \times \mathbb{R}^2$ and a framing $w_2 = w$ for $G_2 = G := \psi(\Sigma)$ such that $\partial\psi(\Sigma) = \mathrm{C}_2$ and such that the restriction of $w_2 = w$ to C_2 equals t_2. Given $m \geq 2$ we denote by $\psi_m\colon \Sigma \to [0,1] \times \mathbb{R}^m$ the smooth embedding that is given by the composition of the smooth embedding ψ with the obvious smooth embedding $[0,1] \times \mathbb{R}^2 \to [0,1] \times \mathbb{R}^m$.

Now we consider the two proper smooth embeddings $\psi_n, \varphi_n\colon \Sigma \to [0,1] \times \mathbb{R}^n$. We can arrange[1649] that $\psi_n|_{\partial\Sigma} = \varphi_n|_{\partial\Sigma}$.

It follows from Proposition 11.12 (2) and the Isotopy Extension Theorem 8.27 that there exists an $m \geq n$ and a diffeomorphism Ψ of the smooth manifold $[0,1] \times \mathbb{R}^m$ such that $\Psi(F_m) = G_m$ and such that the restriction of Ψ to $\{0\} \times \mathbb{R}^m$ is the identity. We refer to Figure 1618 for an illustration.

Since (F_m, v_m) is an extension of (C_m, h_m) and since the restriction of Ψ to $\{0\} \times \mathbb{R}^m$ is the identity we see that $(\Psi(F_m), \Psi_*(v_m)) = (G_m, \Psi_*(v_m))$ is an extension of (C_m, h_m). On the other hand we saw (G_m, w_m) is an extension of (C_m, t_m). Thus both h_m and t_m can be extended over G_m. But this contradicts the above claim. ■

In summary we have the following proposition.

Proposition 117.8.

[1649] This sentence contains a slight technical nuisance which we cleverly hide in a footnote. First of all, it should be pretty clear by now that Σ admits an orientation-reversing self-diffeomorphism. Thus, after possibly precomposing φ_n with such a diffeomorphism we can assume that $\psi_n^{-1} \circ \varphi_n$ is orientation-preserving. Using Corollary 30.2 and using the Collar Neighborhood Theorem 8.12 we can then arrange, after possibly precomposing φ_n by another diffeomorphism, that $\psi_n|_{\partial\Sigma} = \varphi_n|_{\partial\Sigma}$.

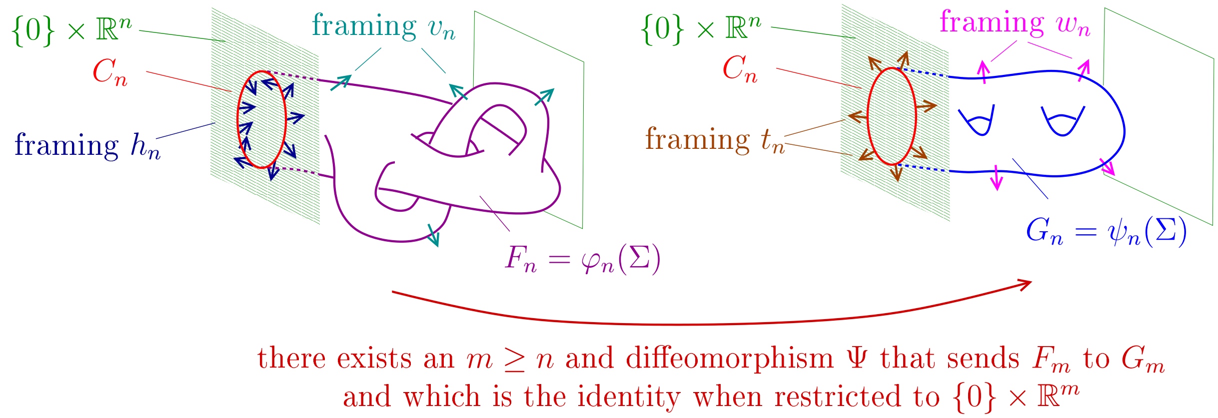


FIGURE 1618. Illustration for the proof of Proposition 117.7.

(1) *Let $n \geq 2$. The map*

$$\begin{array}{rcl} \chi_n\colon \Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1}) & \to & \mathbb{Z}_2 \\ [(C,v)] & \mapsto & \end{array}$$

is a well-defined epimorphism.

(2) *For $n = 2$ the map χ_2 agrees with the map*

$$\begin{array}{rcl} \Omega_1^{\mathrm{fr}}(\mathbb{R}^3) & \to & \mathbb{Z} & \xrightarrow{x\mapsto[x]} & \mathbb{Z}_2 \\ [(C,v)] & \mapsto & \mathrm{Lk}(C, C_+???????) & & \end{array}$$

(3) *For $n \geq 3$ the map $\chi_n\colon \Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1}) \to \mathbb{Z}_2$ is an isomorphism.*

(4) *Given any $n \geq 2$ the following diagram commutes:*

$$\begin{tikzcd} \Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+1}) \arrow[d] \arrow[dr, "\chi_n"] & \\ & \mathbb{Z}_2. \\ \Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+2}) \arrow[ur, "\chi_{n+1}"'] & \end{tikzcd}$$

117.2. **The calculation of the framed cobordism groups $\widetilde{\Omega}_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$ for $n \geq 3$.** In this section we will introduce and calculate a slight variation on the framed cobordism group $\widetilde{\Omega}_1^{\mathrm{fr}}(\mathbb{R}^{n+1})$. On its own this result is perhaps not all the interesting, but it will be a crucial ingredient in our calculation of $\widetilde{\Omega}_2^{\mathrm{fr}}(\mathbb{R}^{n+2})$ in the coming sections.

117.3. **The calculation of the framed cobordism groups $\Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2})$ for $n \geq 3$ I.** ++++++++ can we also do $\pi_4(S^2)$???????? Let $n \in \mathbb{N}_{\geq 3}$. Let $(F, v\colon \mathrm{GL}(n+2, n))$ be a closed oriented framed 2-dimensional submanifold of $\mathbb{R}^{n+2}$. Now let C be a closed oriented 1-dimensional submanifold of F. Given $P \in C$ we write

$$\begin{array}{rcl} u_1(P) & := & \text{the unique positive tangent vector of } \mathrm{V}_P\, C \text{ of length 1}, \\ u_2(P) & := & \text{the unique vector in } (\mathrm{V}_P\, C)^\perp \subset \mathrm{V}_P\, F \text{ of length 1 that has the property} \\ & & \text{that } (u_1(P), u_2(P)) \text{ form a positive basis for } \mathrm{V}_P\, F. \end{array}$$

We equip C with the framing v_C that at each point P is given by $v_C(P) \;:=\; (u_2(P), v(P)\}$. Note that (C, v_C) defines an element in $\Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+2})$. By Proposition 117.7 we know that we

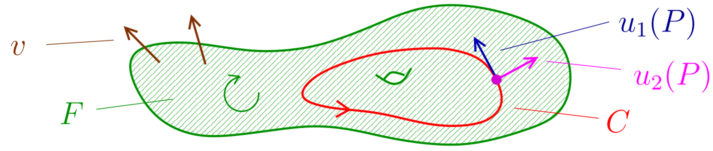


FIGURE 1619

have an isomorphism $\chi\colon \Omega_1^{\mathrm{fr}}(\mathbb{R}^{n+2}) \cong \mathbb{Z}_2$. Note that for knot in S^3 can be compute this explicitly as $\frac{1}{2}$ linking number ++++ if framing does indeed give a well-defined homomorphism in the previous section, then should say explicitly +++++++++

Lemma 117.9. *Let C and D be two closed oriented 1-dimensional submanifolds of a closed oriented framed 2-dimensional submanifold (F, v) of $\mathbb{R}^{n+2}$. If $[C] = [D] \in \mathrm{H}_1(F; \mathbb{Z})$, then $\chi([(C, v_C)]) = \chi([(D, v_D)])$.*

PROOF. Note that it follows from our hypothesis that $[C] = [D] \in \mathrm{H}_1(F; \mathbb{Z})$ together with Corollary 98.7 that there exists a compact oriented proper 2-dimensional submanifold $W \subset [0, 1] \times F$ such that $\partial W = (\{0\} \times C) \cup (\{1\} \times -D)$. Similar to the above we consider for a point $P \in C$ the vector

$u_2(P) \;:=\;$ the unique vector in $(\mathrm{V}_P W)^\perp \subset \mathrm{V}_P([0, 1] \times F)$ of length 1 that has the property that the orientation of W at P together with $u_2(M)$ form a positive basis for $\mathrm{V}_P F$.

We equip W with the framing v_W that at each point P is given by $v_W(P) \;:=\; (u_2(P), v(P)\}$. One can now easily verify that (W, v_W), viewed as a submanifold of $[0, 1] \times \mathbb{R}^{n+2}$ is a framed cobordism between (C, v_C) and (D, v_D). This implies that $\chi([(C, v_C)]) = \chi([(D, v_D)])$. ■

It follows from Lemma 117.9 that we obtain a well-defined map

$$\begin{aligned} q_{(F,v)}\colon \mathrm{H}_1(F; \mathbb{Z}_2) &\to \mathbb{Z}_2 \\ \sigma &\mapsto \chi([(\underset{\substack{\uparrow \\ \text{closed oriented 1-dimensional submanifold with } [C]=\sigma, \\ \text{this always exists by Corollary 70.14}}}{C}, v_C)]) \end{aligned}$$

On might hope that $q_{(F,v)}\colon \mathrm{H}_1(F; \mathbb{Z}_2) \to \mathbb{Z}_2$ is a homomorphism. In fact Pontryagin **[Pon38]** initially thought so. But it turns out that the situation is more complicated, namely it turns out that the following proposition holds.

Proposition 117.10. *The map $q_{(F,v)}\colon \mathrm{H}_1(F; \mathbb{Z}_2) \to \mathbb{Z}_2$ has the property, that for any $x, y \in \mathrm{H}_1(F; \mathbb{Z}_2)$ we have*

$$q(x + y) \;=\; q(x) + q(y) + Q_F(x, y)$$

where $Q_F\colon \mathrm{H}_1(F; \mathbb{Z}_2) \times \mathrm{H}_1(F; \mathbb{Z}_2) \to \mathbb{Z}_2$ denotes the intersection form that we introduced on page 2456.

PROOF. This requires some thought. Note that it suffices to deal with the case that F is standard in $\mathbb{R}^3$. ■

Proposition 117.10 shows that the map $q\colon \mathrm{H}_1(F;\mathbb{Z}_2) \to \mathbb{Z}_2$ has a peculiar behavior that we had not yet encountered. In the next, purely algebraic section, we will see that q is a non-degenerate quadratic form over $\mathbb{Z}_2 = \mathbb{F}_2$ and we will give a complete classification of non-degenerate quadratic forms over $\mathbb{Z}_2 = \mathbb{F}_2$. This will us then allow us to proceed with our actual topological problem. +++++++ need intersection form non-singular ++++++++++

117.4. **Interlude: Quadratic forms and the Arf invariant.**

Definition. Let $\mathbb{F}$ be a field.

(1) A *quadratic form over* $\mathbb{F}$ is a pair (V,q) where V is a finite dimensional $\mathbb{F}$-vector space and $q\colon V \to \mathbb{F}$ is a map which satisfies the following two conditions:
   (a) For any $x \in V$ and any $\lambda \in \mathbb{F}$ we have $q(\lambda \cdot x) = \lambda^2 \cdot q(v)$.
   (b) The map

$$\begin{array}{rcl} V \times V & \to & \mathbb{F} \\ (x,y) & \mapsto & q(x+y) - q(x) - q(y) \end{array}$$

   is a symmetric form over $\mathbb{F}$ in the sense of the definition on page 2410.

   We refer to the symmetric form in (b) as the *symmetric form associated to* q.

(2) We say that two quadratic forms (V,p) and (W,q) over $\mathbb{F}$ are *isometric* if there exists an isomorphism $\varphi\colon V \to W$ such that for all $v \in V$ we have $p(v) = q(\varphi(v))$.

Examples.

(1) Let $\mathbb{F}$ be a field, let V be a finite-dimensional vector space and let $\langle\,,\rangle\colon V \times V \to \mathbb{F}$ be a form. A straightforward calculation shows that $q(v) := \langle v, v\rangle$ defines a quadratic form. Indeed, it is clear, that (a) is satisfied, and for (b) we calculate that for any $x, y \in V$ we have

$$q(x+y) - q(x) - q(y) \;=\; \langle x+y, x+y\rangle - \langle x,x\rangle - \langle y,y\rangle \;=\; 2\cdot\langle x,y\rangle.$$

   We refer to q as the quadratic form associated to $\langle\,,\rangle$.

(2) If $\mathbb{F}$ is a field of characteristic $\neq 2$, then basically the same calculation as in (1) shows that q is the quadratic form associated to the form $(x,y) \mapsto \frac{1}{2}(q(x+y) - q(x) - q(y))$.

(3) Let (F,v) be a closed oriented framed 2-dimensional submanifold of $\mathbb{R}^n$. It follows from Propositions 117.10 and 101.21 that the map $q_{(F,v)}\colon \mathrm{H}_1(F;\mathbb{F}_2) \to \mathbb{F}_2$ that we introduced on page 2769 is a quadratic form over $\mathbb{F}_2$.

The above discussion shows that for fields of characteristic $\neq 2$ the classification of quadratic forms is essentially the same as the classification of forms. In the following we only consider the field $\mathbb{F}_2$.

Definition. Let (V,q) be a quadratic form over $\mathbb{F}_2$. We say q is *non-degenerate* if the associated symmetric form is non-singular in the sense of the definition on page 2411.

Example. We write $\mathcal{W} := \mathbb{F}_2^2$ and $x := (1,0)$ and $y := (0,1)$. One can easily verify that

$$\begin{array}{rcl} \langle\,,\rangle\colon \mathcal{W} \times \mathcal{W} & \to & \mathbb{F}_2 \\ (v,w) & \mapsto & \langle v,w\rangle = v^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} w \end{array}$$

is the unique non-singular symmetric form on $\mathcal{W}$. Next we consider the following two maps:

$$\begin{array}{rcl} \sigma_0\colon \mathcal{W} & \to & \mathbb{F}_2 \\ 0,x,y & \mapsto & 0 \\ x+y & \mapsto & 1 \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \sigma_1\colon \mathcal{W} & \to & \mathbb{F}_2 \\ 0 & \mapsto & 0 \\ x,y,x+y & \mapsto & 1. \end{array}$$

One can easily verify by hand that both maps define a quadratic form on $\mathcal{W}$ such that the associated symmetric form is the above form $\langle\, ,\, \rangle$. In particular both quadratic forms are non-degenerate. It is also straightforward to verify that these two maps are in fact the only two non-degenerate quadratic forms on $\mathcal{W}$. Finally note that σ_0 and σ_1 are not isometric for the rather silly reason that σ_0 sends three elements of $\mathcal{W}$ to 0, whereas σ_1 only sends one element of $\mathcal{W}$ to 0.

In the following our goal is to classify non-singular symmetric forms and non-degenerate quadratic forms over $\mathbb{F}_2$. It turns out that symmetric forms of $\mathbb{F}_2$ are not very interesting. In fact the following proposition holds:

Proposition 117.11. *Let V be a finite-dimensional vector space over $\mathbb{F}_2$.*

(1) *Every non-singular +++++ even +++++ symmetric form $\langle\, ,\, \rangle\colon V\times V\to\mathbb{F}_2$ admits a* symplectic basis*, i.e. there exists a basis $a_1,\dots,a_k,b_1,\dots,b_k$ for V such that the following conditions are satisfied:*
   (a) *For every $i,j\in\{1,\dots,k\}$ we have $\langle a_i,a_j\rangle=\langle b_i,b_j\rangle=0$.*
   (b) *For every $i,j\in\{1,\dots,k\}$ we have $\langle a_i,b_j\rangle=\delta_{ij}$.*
(2) *Any two non non-singular symmetric forms over V are isometric.*

Proof.

(1) The proposition follows immediately from Proposition 101.22. ■

Now let us turn to the study of non-degenerate quadratic forms over $\mathbb{F}_2$. We already encountered the non-isometric quadratic forms $\sigma_0,\sigma_1\colon V\to\mathbb{F}_2$. We obtain more quadratic forms by taking direct sums. But the following lemma says that this way we will produce some isometric quadratic forms.

Lemma 117.12.

(1) *The quadratic forms $\sigma_0\oplus\sigma_0$ and $\sigma_1\oplus\sigma_1$ on $\mathcal{W}\oplus\mathcal{W}$ are isometric.*
(2) *Let $m,n\in\mathbb{N}_0$. The quadratic form $m\cdot\sigma_0\oplus n\cdot\sigma_1$ is isometric to $(m+n)\cdot\sigma_0$ if n is even, and it is isometric to $(m+n-1)\cdot\sigma_0\oplus\sigma_1$ if n is odd.*
(3) *Any non-degenerate quadratic form (V,q) over $\mathbb{F}_2$ is isometric to a quadratic form $k\cdot\sigma_0$ or to a quadratic form $k\cdot\sigma_0\oplus\sigma_1$. (Note that we do not claim that this is an exclusive "or".)*

Proof.

(1) Let x_1,y_1,x_2,y_2 be the obvious basis of $\mathcal{W}\oplus\mathcal{W}=\mathbb{F}_2^2\oplus\mathbb{F}_2^2$. We consider the isomorphism $\varphi\colon\mathcal{W}\oplus\mathcal{W}\to\mathcal{W}\oplus\mathcal{W}$ that is determined by

$$\begin{array}{rcl} x_1 & \mapsto & x_1+x_2 \\ y_1 & \mapsto & y_1+x_2 \end{array} \qquad \begin{array}{rcl} x_2 & \mapsto & x_2+y_2+x_1+y_1 \\ y_2 & \mapsto & y_2+x_1+y_1. \end{array}$$

It is straightforward, albeit not particularly entertaining, to verify that φ defines an isometry from $(\mathcal{W} \oplus \mathcal{W}, \sigma_0 \oplus \sigma_0)$ to $(\mathcal{W} \oplus \mathcal{W}, \sigma_1 \oplus \sigma_1)$.

(2) This statement follows immediately from (1).

(3) Let (V, q) be a quadratic form. We denote by $\langle\, , \rangle$ the associated symmetric form. By Proposition 117.11 we know that there exists a symplectic basis $a_1, \dots, a_m, b_1, \dots, b_m$ for $(U, \langle\, , \rangle)$. Given $i = 1, \dots, m$ we denote by V_i the span of $\{a_i, b_i\}$ and write $q_i := q|_{V_i}$. It follows immediately from the fact that the V_i are orthogonal with respect to $\langle\, , \rangle$ that (V, q) is isometric to $(V_i, q_1) \oplus \dots \oplus (V_m, q_m)$. Note that each (V_i, q_i) is a non-degenerate quadratic form over a 2-dimensional vector space. It follows from the above discussion that each (V_i, q_i) is isometric to either $(\mathcal{W}, \sigma_0)$ or $(\mathcal{W}, \sigma_1)$. The desired statement now follows from (2). ■

If we want to classify quadratic forms over $\mathbb{F}_2$ we will need to figure out whether or not $m \cdot \sigma_0$ and $(m-1) \cdot \sigma_0 \oplus \sigma_1$ are isometric. Surprisingly the silly approach that worked for $m = 1$, still works in general. More precisely, let us first introduce the following notation.

Notation. Given a quadratic form (V, q) we set

$$N(q) \;:=\; \{v \in V \,|\, q(v) = 0\} \quad \text{and} \quad P(q) \;:=\; \{v \in W \,|\, q(v) = 1\}.$$

Now we can formulate the following lemma.

Lemma 117.13. *Given any $m \in \mathbb{N}$ we have*

$$\begin{array}{rcl} \#N(m \cdot \sigma_0) & > & \#P(m \cdot \sigma_0) \quad \textit{and} \\ \#N((m-1) \cdot \sigma_0 \oplus \sigma_1) & < & \#P((m-1) \cdot \sigma_0 \oplus \sigma_1). \end{array}$$

Proof. Given a quadratic form (V, q) over $\mathbb{F}_2$ we set

$$r(q) \;:=\; \#\{v \in V \,|\, q(v) = 0\} \,-\, \#\{v \in V \,|\, q(v) = 1\}.$$

The following claim contains the key calculation that we need to complete the proof of the lemma.

Claim. Let (V, q) be a quadratic form.

(1) We have $\#N(\sigma_0 \oplus q) = 3 \cdot \#N(q) + \#P(q)$ and $\#P(\sigma_0 \oplus q) = 3 \cdot \#P(q) + \#N(q)$.

(2) We have $r(\sigma_0 \oplus q) = 2 \cdot r(q)$.

Note that (2) is an immediate consequence of (1) and the simple observation that $r(q) = \#N(q) - \#P(q)$. Thus it remains to prove (1). Note that given $v \oplus w \in V \oplus \mathcal{W}$ we see that

$$\begin{aligned} (\sigma_0 \oplus q)(v \oplus w) = \sigma_0(v) + q(w) &= \begin{cases} 0, & \text{if } v \in N(\sigma_0) \text{ and } w \in N(q) \text{ or } v \in P(\sigma_0) \text{ and } w \in P(q) \\ 0, & \text{if } v \in N(\sigma_0) \text{ and } u \in P(q) \text{ or } v \in P(\sigma_0) \text{ and } w \in N(q). \end{cases} \\ &= \begin{cases} 0, & \text{if } v \in \{0, x, y\} \text{ and } w \in N(q) \text{ or } v = x + y \text{ and } w \in P(q) \\ 0, & \text{if } v \in \{0, x, y\} \text{ and } w \in P(q) \text{ or } v = x + y \text{ and } w \in N(q). \end{cases} \end{aligned}$$

It follows immediately that $P(q \oplus \sigma_0) = 3 \cdot P(q) + N(q)$ and that $N(q \oplus \sigma_0) = 3 \cdot N(q) + P(q)$.
⊞

Now iteratively applying the claim we see that

$$\begin{aligned} r(m\cdot\sigma_0) &= 2\cdot r((m-1)\cdot\sigma_0) &&= \dots = 2^{m-1}\cdot r(\sigma_0) = 2^{m-1}\cdot 2 &&> 0 \quad \text{and} \\ r((m-1)\cdot\sigma_0\oplus\sigma_1) &= 2\cdot r((m-1)\cdot\sigma_0\oplus\sigma_1) &&= \dots = 2^{m-1}\cdot r(\sigma_1) = 2^{m-1}\cdot(-2) &&< 0. \end{aligned}$$

This calculation immediately implies the desired result. ■

This leads us to the following definition.

Definition. Let (V,q) be a quadratic form over $\mathbb{F}_2$. We define its *Arf invariant* $\operatorname{Arf}(q)\in\mathbb{F}_2$ as follows:

$$\operatorname{Arf}(q) \;:=\; \begin{cases} 0, & \text{if } \#N(q) > \#P(q), \\ 1, & \text{if } \#P(q) > \#N(q). \end{cases}$$

Remark.

(1) The Arf invariant was introduced in 1941 in a slightly different form, which we will explain below, by Cahit Arf [**Arf41**].[1650]
(2) The Arf invariant is sometimes jokingly referred to as the *democratic invariant*: every element $u\in V$ gets to vote, and the majority decides that the value of the Arf invariant should be.

The following theorem now gives us the complete classification of non-degenerate quadratic forms over $\mathbb{F}_2$.

Theorem 117.14.

(1) *Let (V,q) be a non-degenerate quadratic form. We set $m := \dim(V)$.*
   (a) *(V,q) is isometric to $m\cdot\sigma_0$ if and only if $\operatorname{Arf}(q)=0$.*
   (b) *(V,q) is isometric to $(m-1)\cdot\sigma_0\oplus\sigma_1$ if and only if $\operatorname{Arf}(q)=1$.*
(2) *Two non-degenerate quadratic forms (V_1,q_1) and (V_2,q_2) over $\mathbb{F}_2$ are isometric if and only if $\dim(V_1)=\dim(V_2)$ and $\operatorname{Arf}(q_1)=\operatorname{Arf}(q_2)$.*
(3) *For any two non-degenerate quadratic forms (V_1,q_1) and (V_2,q_2) we have*

$$\operatorname{Arf}(q_1\oplus q_2) \;=\; \operatorname{Arf}(q_1)+\operatorname{Arf}(q_2).$$

Proof.

(1) This statement f is an immediate consequence of Lemmas 117.12 and 117.13.
(2) This statement follows immediately from (1).
(3) This statement follows easily from (1) and Lemma 117.12 (1). ■

The following proposition gives a neat calculation of the Arf invariant. In the next section we will use this proposition in our calculation of the Arf invariant.

Proposition 117.15. *Let (V,q) be a non-degenerate quadratic form over $\mathbb{F}_2$. Given any symplectic basis $a_1,\dots,a_n,b_1,\dots,b_n$ of the associated symmetric form we have the following equality:*

$$\operatorname{Arf}(q) \;=\; \sum_{i=1}^{n} q(a_i)\cdot q(b_i) \;\in\; \mathbb{F}_2.$$

[1650] To the best of my knowledge Cahit Arf and Alan Turing are the only two mathematicians of the 20th century who made it on a banknote.

FIGURE 1620

Remark.

(1) The reinterpretation of the Arf invariant given in Proposition 117.15 is actually the original definition given by Cahit Arf, see also Figure 1620.

(2) This reinterpretation of the Arf invariant has the great advantage that it also makes sense for quadratic forms over any field of characteristic 2. In this more general setting it is not entirely trivial to see that one actually obtains a well-defined invariant. We refer to [**Dye78**], [**Kapy69**, Theorem 27] and [**Cohn03**, Theorem 8.11.2] for details.

PROOF. Let (V,q) be a non-degenerate quadratic form over $\mathbb{F}_2$ and let $a_1,\dots,a_n,b_1,\dots,b_n$ be a symplectic basis of the associated symmetric form. As in the proof of Lemma 117.12 we denote by V_i the span of $\{a_i,b_i\}$, we write $q_i := q|_{V_i}$ and we point out that (V,q) is isometric to $(V_1,q_1)\oplus\cdots\oplus(V_n,q_n)$.

Note that each (V_i,q_i) is a non-degenerate quadratic form over a 2-dimensional vector space. It follows from the above discussion that each (V_i,q_i) is isometric to either $(\mathcal{W},\sigma_0)$ or $(\mathcal{W},\sigma_1)$. Now we perform the following calculation:

$$\begin{aligned}\sum_{i=1}^n q(a_i)\cdot q(b_i) &= \sum_{i=1}^n q_i(a_i)\cdot q(b_i) \underset{\substack{\uparrow\\ \text{by inspection of the } \sigma_i}}{=} \sum_{i=1}^n \begin{cases} 0, & \text{if } q_i \text{ is isometric to } \sigma_0\\ 1, & \text{if } q_i \text{ is isometric to } \sigma_1\end{cases} \underset{\substack{\uparrow\\ \text{by inspection of the } \sigma_i}}{=} \sum_{i=1}^n \operatorname{Arf}(q_i)\\ &\underset{\substack{\uparrow\\ \text{by Theorem 117.14 (3)}}}{=} \operatorname{Arf}(q_1\oplus\cdots\oplus q_n) = \operatorname{Arf}(q).\end{aligned}$$

■

This concludes our quick introduction to the Arf invariant. In the next section we will use the Arf invariant to calculate the framed cobordism groups $\Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2})$ for $n\geq 3$. Note though that the Arf invariant appears in many other settings of topology, e.g. in knot theory [**Pra07**, Chapter 6.2] and surgery theory [**Browd72**].

117.5. **The calculation of the framed cobordism groups $\Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2})$ for $n\geq 3$ II.** After this amusing purely algebraic section we now return to our actual goal, namely we still want to calculate the framed cobordism groups $\Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2})$ for $n\geq 3$.

Let (F,v) be a closed oriented framed 2-dimensional submanifold of $\mathbb{R}^n$. Recall that it follows from Propositions 117.10 and 101.21 that the map $q_{(F,v)}\colon \mathrm{H}_1(F;\mathbb{F}_2)\to\mathbb{F}_2$ that we introduced on page 2769 is a quadratic form over $\mathbb{F}_2$. We can now consider the corresponding Arf invariant $\operatorname{Arf}(F,v) := \operatorname{Arf}(q_{(F,v)})$.

Example. We consider the torus $F := S^1 \times S^1$ as a submanifold of $\mathbb{C}\times\mathbb{C} = \mathbb{R}^4$. We equip F with the framing

$$\begin{array}{rcl} v\colon S^1\times S^1 & \mapsto & \mathrm{GL}(4,2) \\ (w,z)=(e^{\mathrm{i}s},e^{\mathrm{i}t}) & \mapsto & \underbrace{\begin{pmatrix} \sin(s) & 0 \\ -\cos(s) & 0 \\ 0 & \sin(t) \\ 0 & -\cos(t) \end{pmatrix}}_{\substack{\text{columns are a basis}\\ \text{for } (\mathrm{T}_{(w,z)})^{\perp}}} \cdot \underbrace{\begin{pmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{pmatrix} \cdot \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}}_{\text{the columns get rotated by } e^{\mathrm{i}s} \text{ and } e^{\mathrm{i}t}} \end{array}$$

We equip $\mathrm{H}_1(S^1\times S^1;\mathbb{F}_2)$ with the "obvious basis" x and y given by the two submanifolds $\mathrm{C}_1 := S^1\times\{1\}$ and $\mathrm{C}_2 := \{1\}\times S^1$. One calculates by hand that $q_{(F,v)}(x) = 1$ and $q_{(F,v)}(y) = 1$. It follows from Proposition 117.10 that $q_{(F,v)}(x+y) = 1$. Thus we see that $\mathrm{Arf}(S^1\times S^1, v)) = 1$.

why does this correspond to the composition of the Hopf map with suspension of the Hopf map?

Proposition 117.16. *Let (F,v) be a closed oriented framed 2-dimensional submanifold of $\mathbb{R}^n$. If $[(F,v)] = 0 \in \Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2})$, then $\mathrm{Arf}([F,v)]) = 0$.*

Proof. Let (F,v) be a closed oriented framed 2-dimensional submanifold of $\mathbb{R}^{n+2}$ which represents the trivial element in $\Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2})$. By definition this means that there exists a compact oriented proper framed 3-dimensional submanifold (M,w) of $\mathbb{R}^{n+2}\times[0,1]$ such that $\partial M = \{0\}\times F$ and such that $w|_{\{1\}\times F} = v$. By Lemma **??**, Lemma **??** and Exercise **??** we know that there exists a symplectic basis $a_1,\dots,a_k,b_1,\dots,b_k$ of $\mathrm{H}_1(F;\mathbb{F}_2)$ such that $a_1,\dots,a_k \in \ker(\mathrm{H}_1(F;\mathbb{F}_2)\to\mathrm{H}_1(M;\mathbb{F}_2))$. By the long exact sequence we know that there exist $c_1,\dots,c_k \in \mathrm{H}_2(M,F;\mathbb{F}_2)$ with $\partial c_i = a_i$. By Proposition **??** there exist compact properly embedded submanifolds[1651] $G_1,\dots,G_k$ such that $[G_i]_{\mathbb{F}_2} = c_i$. But by ++++ these surfaces imply that $\chi(a_i) = 0$. Thus we see by Proposition 117.15 that $\mathrm{Arf}(F,v) = 0$. ■

Proposition 117.17.

(1) *Let $n \geq 2$. The map*

$$\begin{array}{rcl} \mathrm{Arf}\colon \Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2}) & \to & \mathbb{Z}_2 \\ [(F,v)] & \mapsto & \mathrm{Arf}(F,v) \end{array}$$

is a well-defined homomorphism.

(2) *Given any $n \geq 2$ the following diagram commutes:*

$$\begin{tikzcd} \Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2}) \arrow[d] \arrow[dr, "\mathrm{Arf}"] & \\ \Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+3}) \arrow[r, "\mathrm{Arf}"'] & \mathbb{Z}_2. \end{tikzcd}$$

(3) *For $n \geq 3$ the map $\mathrm{Arf}\colon \Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2}) \to \mathbb{Z}_2$ is an isomorphism.*

[1651] Note that we do not claim that the G_i are necessarily orientable.

Proof.

(1) By Proposition 117.16 we know that the map $[(F, v)] \mapsto \operatorname{Arf}(F, v)$ is well-defined. It follows easily from Theorem 117.14 that the map is actually a homomorphism.
(2) This statement follows easily from the definitions together with Proposition 117.8 (4).
(3) We just saw that there exists a framing v of $S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C} = \mathbb{R}^4$ such that $\operatorname{Arf}(S^1 \times S^1, v) = 1$. This shows that $\operatorname{Arf}\colon \Omega_2^{\mathrm{fr}}(\mathbb{R}^4) \to \mathbb{Z}_2$ is an epimorphism. Together with (2) we now see that for any $n \geq 2$ the map $\operatorname{Arf}\colon \Omega_2^{\mathrm{fr}}(\mathbb{R}^{n+2}) \to \mathbb{Z}_2$ is an epimorphism. ■

+++++ could define Arf invariant of a knot, as Arf invariant of any Seifert surface, arguments we give should show that zero for slice knot ++++++++

117.6. **Further results obtained using the Thom-Pontryagin approach.** In this final section we summarize a few more results that have been proved using the Thom-Pontryagin approach. First of all, using the Thom-Pontryagin method Pontryagin [**Pon50b**] proved in 1950 the following theorem.

Theorem 117.18. *For any $n \geq 2$ we have $\pi_{n+2}(S^n) \cong \mathbb{Z}_2$.*

Remark.

(1) Recall that by Theorem 113.10 (1) we have for every $n \geq 3$ we have an isomorphism $\pi_n(S^3) \cong \pi_n(S^2)$. Furthermore we know by Theorem 117.1 that for every $n \geq 3$ we have $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$. Putting these two results together shows that $\pi_4(S^2) \cong \mathbb{Z}_2$.
(2) In Theorem 110.2 we used Steenrod operations to show that $\pi_{n+2}(S^n) \neq 0$ for all $n \geq 2$.
(3) By the Freudenthal Suspension Theorem 116.6 we know that the suspension homomorphism $\pi_5(S^3) \to \pi_6(S^4)$ is an epimorphism and we know that for $n > k + 1$ the suspension homomorphism $\pi_{n+2}(S^n) \to \pi_{n+3}(S^{n+1})$ if an isomorphism for $n \geq 5$.
(4) Unfortunately (1), (2) and (3) are *not* strong enough to give an alternative proof of Theorem 117.18.

Proof.

(1) As mentioned before, the theorem was proved by Lev Pontryagin [**Pon50b, Pon59**]. See also [**Scor05**, p. 233] and [**WX10**, Chapter 2.5] for outlines of the proof. The most readable account of the Thom-Pontryagin approach to proving the theorem is surely [**Put**].
(2) At the same time as Pontryagin two very different proofs that $\pi_{n+2}(S^n) \cong \mathbb{Z}_2$ for $n \geq 2$ were given by George W. Whitehead [**WhdG50**] and Jean-Pierre Serre [**Ser53**, Proposition 10]. We also refer to [**Hat**, Theorem 1.40 (a)] for a proof using spectral sequences and Steenrod operations. Alternatively see also [**Hu59**, Theorem XI.15.2]. ■

Finally Rokhlin [**Rok52, Rok86a**] used the Thom-Pontryagin approach to prove the following theorem.

Theorem 117.19.

(1) *We have*[1652] $\pi_5(S^3) \cong \mathbb{Z}_2$, $\pi_6(S^3) \cong \mathbb{Z}_{12}$, $\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}_{12}$.
(2) *For any* $n \geq 5$ *we have* $\pi_{n+3}(S^n) \cong \mathbb{Z}_{24}$.

Remark.

(1) In Theorem 109.13 we used Steenrod operations to show the weaker statement that $\pi_{n+3}(S^n) \neq 0$ for all $n \geq 2$.
(2) Note that Theorem 117.19 (1) gives a negative answer to Question 91.15 (2).

Proof.

(a) The theorem was proved by Vladimir Rokhlin [**Rok52**][**Rok86a**, p. 21] in 1952, using the above Thom-Pontryagin approach.[1653]
(b) A somewhat different proof is sketched in [**WX10**, Chapter 2.6]. Some of the results can also be found in [**Ser53**, Proposition 10]. Other proofs are given in [**Hat**, Theorem 1.40 (b)] and [**Hu59**, Chapter XI.16]. ■

The results of the last two chapters summarize most (or all?) of the calculations of homotopy groups of spheres that have been obtained using the Thom-Pontryagin approach.

Exercises for Chapter 117.

Exercise 117.1. Let $n \in \mathbb{N}_{\geq 2}$. We consider the loops

$$\begin{array}{rcl} \alpha\colon S^1 & \to & \mathrm{GL}_+(n,\mathbb{R}) \\ z = e^{i\varphi} & \mapsto & \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-2} \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{rcl} \overline{\alpha}\colon S^1 & \to & \mathrm{GL}_+(n,\mathbb{R}) \\ z = e^{i\varphi} & \mapsto & \begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & \mathrm{id}_{n-2} \end{pmatrix} \end{array}.$$

Recall that in the proof of Lemma 117.6 we used the fact, proved in Proposition 114.11 (2), that $\pi_1(\mathrm{GL}_+(n-1,\mathbb{R})) \cong \mathbb{Z}_2$, to show that α is path-homotopic to $\overline{\alpha}$. Give a direct proof that the loop α is path-homotopic to the loop $\overline{\alpha}$.

[1652] In fact the first isomorphism $\pi_5(S^3) \cong \mathbb{Z}_2$ is a consequence of Theorem 113.10 and Theorem 117.18.

[1653] In fact Rokhlin [**Rok86b**] initially thought that for $n \geq 5$ the groups $\pi_{n+3}(S^n)$ are isomorphic to $\mathbb{Z}_{12}$. The key to fixing the mistake is Rokhlin's Theorem 102.5 regarding signatures of certain 4-dimensional smooth manifolds.

118. Summary of homotopy groups of spheres

The results from last chapter conclude our calculations of homotopy groups of spheres. Throughout the course we employed a long and varied list of tools to compute homotopy groups of spheres:

(1) Covering theory, see page 1068.
(2) The Cellular Approximation Theorem 38.13, see Proposition 40.10.
(3) The Simplicial Approximation Theorem 62.8, see Corollary 62.10 and Exercise 62.6.
(4) Homology theory and the Hurewicz Theorem 53.5, see Corollary 53.6.
(5) The cup product, see Corollary 91.14.
(6) Steenrod operations, see Theorems 109.13 and 110.2.
(7) The long exact sequence of a fibration, see Theorem 113.10.
(8) The Thom-Pontryagin Theory, see the Freudenthal Suspension Theorem 116.6 and Theorem 117.1.

In the following we summarize the results we have obtained on homotopy groups of spheres:

(1) On page 1068 we saw that $\pi_n(S^1) = 0$ for $n \geq 2$.
(2) Proposition 40.10, or alternatively Exercise 62.6, says that for any $n \in \mathbb{N}$ we have $\pi_1(S^n) = \cdots = \pi_{n-1}(S^n) = 0$.
(3) In Corollary 62.10 we showed that the homotopy groups of spheres are countable.
(4) Corollary 53.6 (or alternatively by Corollary 115.12) we know that for any $n \in \mathbb{N}$ we have $\pi_n(S^n) \cong \mathbb{Z}$.
(5) By Theorem 113.10 (1) we know that for every $n \geq 3$ we have an isomorphism $\pi_n(S^3) \cong \pi_n(S^2)$. In particular, together with (4) we know that $\pi_3(S^2) \cong \mathbb{Z}$. An alternative proof of the fact that $\pi_3(S^2) \cong \mathbb{Z}$ is given by the combination of the Thom-Pontryagin Theorem 115.11 and Theorem 115.14.
(6) In Corollary 91.14, and alternatively in Proposition 115.21 we saw that given any even $n \in \mathbb{N}$ there exists an epimorphism $\pi_{2n-1}(S^n) \twoheadrightarrow \mathbb{Z}$.
(7) By Theorem 113.11 (1) we know that for every $n \geq 2$ we have an isomorphism $\pi_n(S^4) \cong \pi_n(S^7) \oplus \pi_{n-1}(S^3)$. Furthermore, by Theorem 113.11 (1) we know that for every $n \geq 2$ we have an isomorphism $\pi_n(S^8) \cong \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)$.
(8) By the Freudenthal Suspension Theorem 116.6 we know that for $n = k+1$ we have an epimorphism $\pi_{n+k}(S^n) \twoheadrightarrow \pi_{n+k+1}(S^{n+1})$ and we know that for $n > k+1$ we have an isomorphism $\pi_{n+k}(S^n) \cong \pi_{n+k+1}(S^{n+1})$.
(9) By Theorem 113.11 (2) we know that $\pi_6(S^4) \cong \pi_5(S^3)$ and by Theorem 113.14 (3) we know that $\pi_{14}(S^8) \cong \pi_{13}(S^7)$.
(10) In Theorem 109.13 we proved the following three statements:
   (a) for any $n \geq 2$ the group $\pi_{n+1}(S^n)$ is non-trivial,
   (b) for any $n \geq 4$ the group $\pi_{n+3}(S^n)$ is non-trivial,
   (c) for any $n \geq 8$ the group $\pi_{n+7}(S^n)$ is non-trivial.
(11) By Theorem 117.1 we know that for every $n \geq 3$ we have $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$. This is a refinement of statement (10a). Note that it follows from this fact together with (5) that $\pi_4(S^2) \cong \mathbb{Z}_2$.

(12) In Theorem 110.2 we proved the following three statements:
  (a) for any $n \geq 2$ the group $\pi_{n+2}(S^n)$ is non-trivial,
  (b) for any $n \geq 4$ the group $\pi_{n+6}(S^n)$ is non-trivial,
  (c) for any $n \geq 8$ the group $\pi_{n+14}(S^n)$ is non-trivial.

We recall that Serre's Theorem 54.17 says that all homotopy groups of spheres are finitely generated abelian groups. In the discussion above we saw that some of the homotopy groups of spheres are infinite.

The following theorem can be summarized as saying that we have picked up all elements of infinite order. In particular the theorem gives a complete answer to Question 91.16. This theorem can also be viewed as a strengthening of Serre's Theorem 54.17. In fact, the strengthening is also due to Jean-Pierre Serre.

Theorem 118.1.
(1) *If n is odd, then $\pi_k(S^n)$ is finite for $k \geq n+1$.*
(2) *If n is even, then the following two statements hold:*
  (a) *The group $\pi_k(S^n)$ is finite for $k \geq n+1$ and $k \neq 2n-1$.*
  (b) *The group $\pi_{2n-1}(S^n)$ is isomorphic to the direct sum of $\mathbb{Z}$ with a finite group.*

Proof. The theorem was first proved by Serre [**Ser53**] using "spectral sequences". Expositions of the proof are also given in [**Hat**, Theorem 1.21] and [**Spa95**, p. 515f]. ■

The following table, obtained from [**Tod62**, Chapter XIV], shows the low-dimensional homotopy groups of low-dimensional spheres. The entries colored in blue have been proved in these lecture notes.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$
S^9	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$

Note that we showed in Theorem 113.10 (1) that the second and the third row, starting with π_3, are the same.

Recall that the Freudenthal Suspension Theorem 116.6 says that for any $n, k \in \mathbb{N}_0$ with $n > k+1$ the suspension homomorphism $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is an isomorphism. Thus we see that the groups $\pi_{n+k}(S^n)$ "stabilize" as $n \to \infty$. It is therefore perhaps more instructive to shift in the above diagram the i-th row leftward by $i-1$ terms. This way the diagonal lines become vertical. In other words we consider the following diagram where

π_{n+k} is short for $\pi_{n+k}(S^n)$.

n	π_{n+0}	π_{n+1}	π_{n+2}	π_{n+3}	π_{n+4}	π_{n+5}	π_{n+6}	π_{n+7}	π_{n+8}	π_{n+9}	π_{n+10}
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0
2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
3	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$
4	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$
5	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
6	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_8\times\mathbb{Z}_2\times\mathbb{Z}_9$
7	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8\times\mathbb{Z}_3\times\mathbb{Z}_2$
8	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$	$\mathbb{Z}_8^2\times\mathbb{Z}_2\times\mathbb{Z}_3^2$
9	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8\times\mathbb{Z}_6$
10	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}\times\mathbb{Z}_2^3$	$\mathbb{Z}_4\times\mathbb{Z}_6$
11	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$
11	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2\times\mathbb{Z}_3$

Here for convenience we colored the "diagonal" $\pi_n(S^{2n})$ in red, since the Freudenthal Suspension Theorem 116.6 implies that below the diagonal the groups on each column are the same.

We conclude this discussion of the precise homotopy groups of spheres with the following sobering fact:

Fact 118.2. *There is no sphere S^n with $n \geq 2$ for which we*[a] *know the isomorphism types of all homotopy groups.*

[a]Here "we" is defined as members of our species.

As we had mentioned above, the Freudenthal Suspension Theorem 116.6 says that homotopy groups of spheres "stabilize". This leads us to the following definition.

Definition. Let $k \in \mathbb{N}_0$. We refer to[1654]

$$\pi_k^s \;:=\; \varinjlim \pi_{n+k}(S^n)$$

as the *k-th stable homotopy group of spheres.*[1655]

Note that it follows from Theorem 118.1 that for any $k \geq 1$ the k-th stable homotopy group π_k^s is finite. The following table obtained from **[Wiki]** gives the stable homotopy groups up to $k = 63$. It seems like 64 is the first dimension for which we do not know the

[1654]Given $k \in \mathbb{N}_0$ the groups $\{\pi_{n+k}(S^n)\}_{n\in\mathbb{N}}$ together with the suspension homomorphisms form a direct system. Thus it makes sense to consider the direct limit.

[1655]Note that for any n, k with $n > k+1$ the Freudenthal Suspension Theorem 116.6, together with Lemma 76.2 (4) and Lemma 76.3, says that the natural map $\pi_{n+k}(S^n) \to \pi_k^s$ is an isomorphism.

corresponding stable homotopy groups.

n	0	1	2	3	4	5	6	7
π^s_{0+n}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{8\cdot 3}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{16\cdot 3\cdot 5}$
π^s_{8+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{2\cdot 3}$	$\mathbb{Z}_{8\cdot 9\cdot 7}$	0	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2\,\mathbb{Z}_{32\cdot 3\cdot 5}$
π^s_{16+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8\,\mathbb{Z}_2$	$\mathbb{Z}_2\,\mathbb{Z}_{8\cdot 3\cdot 11}$	$\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2\,\mathbb{Z}_{8\cdot 3}\,\mathbb{Z}_{16\cdot 9\cdot 5\cdot 7\cdot 13}$
π^s_{24+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\,\mathbb{Z}_3$	$\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{2\cdot 3}$	$\mathbb{Z}_2^2\,\mathbb{Z}_{64\cdot 3\cdot 5\cdot 17}$
π^s_{32+n}	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$	$\mathbb{Z}_4\,\mathbb{Z}_2^3$	$\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 27\cdot 7\cdot 19}$	$\mathbb{Z}_{2\cdot 3}$	$\mathbb{Z}_2^2\,\mathbb{Z}_3$	$\mathbb{Z}_4\,\mathbb{Z}_{2\cdot 3\cdot 5}$	$\mathbb{Z}_2^5\,\mathbb{Z}_3\,\mathbb{Z}_{16\cdot 3\cdot 25\cdot 11}$
π^s_{40+n}	$\mathbb{Z}_2^5\,\mathbb{Z}_{4\cdot 3}$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_{8\cdot 3\cdot 23}$	$\mathbb{Z}_8$	$\mathbb{Z}_2^3\,\mathbb{Z}_{16\cdot 9\cdot 5}$	$\mathbb{Z}_2^4\,\mathbb{Z}_3$	$\mathbb{Z}_2^3\,\mathbb{Z}_4\,\mathbb{Z}_3\,\mathbb{Z}_{32\cdot 9\cdot 5\cdot 7\cdot 13}$
π^s_{48+n}	$\mathbb{Z}_4\,\mathbb{Z}_2^4$	$\mathbb{Z}_2^2\,\mathbb{Z}_3$	$\mathbb{Z}_2^3\,\mathbb{Z}_3$	$\mathbb{Z}_4\,\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 3}$	$\mathbb{Z}_2^3\,\mathbb{Z}_3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_4\,\mathbb{Z}_2$	$\mathbb{Z}_3\,\mathbb{Z}_{16\cdot 3\cdot 5\cdot 29}$
π^s_{56+n}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\,\mathbb{Z}_{8\cdot 9\cdot 7\cdot 11\cdot 31}$	$\mathbb{Z}_4$	0	$\mathbb{Z}_2^2\,\mathbb{Z}_{4\cdot 3}$	$\mathbb{Z}_2^3\,\mathbb{Z}_{128\cdot 3\cdot 5\cdot 17}$

This concludes our discussion of the homotopy groups of spheres. We refer to [**May99b, WX10, IWX01, Wiki**] and [**Hat02**, Chapter 4] for a more detailed discussion.

Exercises for Chapter 118.

Exercise 118.1.

(a) Show that $\pi_4(\mathbb{H}\mathrm{P}^\infty) \cong \mathbb{Z}$.
(b) Show that $\pi_n(\mathbb{H}\mathrm{P}^\infty)$ is finite for $n \neq 4$.
(c) Show that $\pi_5(\mathbb{H}\mathrm{P}^\infty) \cong \mathbb{Z}_2$.

Remark. Use Proposition 113.9 together with the results of this chapter.

119. The CW-Approximation Theorem and the Whitehead Theorem

In the following chapters we will work a lot with CW-complexes. In this chapter we will first prove a hotchpotch of results on CW-complexes. Afterwards we will use these results to prove the CW-Approximation Theorem which often allows us to replace a topological space X by a CW-complex that is homotopy equivalent to X. Furthermore we will prove the Whitehead theorem which says that if a map $f\colon X \to Y$ between CW-complexes induces an isomorphism of all homotopy groups, then f is in fact a homotopy equivalence.

119.1. **Modifying CW-complexes.** As mentioned in the beginning of the chapter, CW-complexes will play a major role in this and the subsequent chapters. Thus, before we start out with proving the enticing theorems mentioned above, we intend to improve our understanding of CW-complexes.

The following, fairly self-evident convention will be in use in the coming chapters.

Convention.

(1) Given $n \in \mathbb{N}_0$ we write $* = (0, \dots, 0, 1) \in S^n$. We view S^n as a CW-complex with one 0-cell $\{*\}$ and one n-cell.
(2) We always form any wedge of spheres $\bigvee_{i \in I} S_i^{n_i}$ by identifying the points $*$. By Lemma 36.32 (4) we can view the wedge $\bigvee_{i \in I} S_i^{n_i}$ as a CW-complex in the obvious way, i.e. we have one 0-cell $\{*\}$ and for each $i \in I$ we have one n_i-cell.

Before we do anything else we will introduce the following definition which will accompany us in the near future.

Definition. Let $f\colon X \to Y$ be a map between topological spaces.

(1) We say the map $f\colon X \to Y$ is *0-connected* if $f_*\colon \pi_0(X) \to \pi_0(Y)$ is a surjection, in other words, each path-component of Y contains a point in $f(X)$.
(2) Let $k \in \mathbb{N}$. We say the map $f\colon X \to Y$ is *k-connected* if the following two conditions are satisfied:
    (a) the map $f_*\colon \pi_0(X) \to \pi_0(Y)$ is a bijection,
    (b) for every $x_0 \in X$ the induced map
$$f_*\colon \pi_i(X, x_0) \to \pi_i(Y, f(x_0)) \text{ is an } \begin{cases} \text{isomorphism for } i = 1, \dots, k-1, \\ \text{epimorphism for } i = k. \end{cases}$$

Remark. Let $f\colon X \to Y$ be a map between path-connected topological spaces and let $k \in \mathbb{N}$. It follows from Proposition 40.5 that to verify that f is k-connected it suffices to check the Condition (2b) for a single base point $x_0 \in X$.

Examples.

(1) The map $f\colon S^1 \vee S^1 \to S^1 \times S^1$ that is illustrated in Figure 1621 is easily seen to be 1-connected, but it is not 2-connected.

(2) Let X be a CW-complex and let $k \in \mathbb{N}_0$. Corollary 111.10 says that the inclusion $X^k \to X$ of the k-skeleton[1656] is a k-connected map. In fact, the previous example is of such type if we consider $X = S^1 \times S^1$ with the usual CW-structure, where the 1-skeleton is precisely $S^1 \vee S^1$.

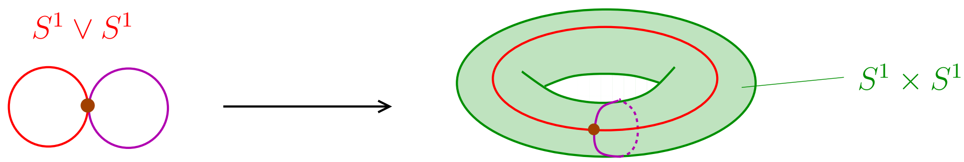


FIGURE 1621

The following is the main technical result which underpins many of the later results.

Proposition 119.1. *Let $k \in \mathbb{N}_0$, let A be a* CW*-complex and let $f\colon A \to X$ be a map to a topological space. If the map $f\colon A \to X$ is k-connected, then there exists* CW*-complex B and a map $g\colon B \to X$ such that the following statements hold:*

(1) *A is a subcomplex of B,*
(2) *the only extra cells of B are $(k+1)$-dimensional, in particular the k-skeleton of B equals the k-skeleton of A,*
(3) *the restriction of g to A equals f,*
(4) *the map $g\colon B \to X$ is $(k+1)$-connected.*

Remark. In most applications of Proposition 119.1 we start out with a k-dimensional CW-complex. In this case we obtain a $(k+1)$-dimensional CW-complex such that k-skeleton equals A and such that we can complete the map $f\colon A \to X$ to the following diagram:

$$\begin{tikzcd}
k\text{-dimensional CW-complex } A \arrow[r, "k\text{-connected}", "f"'] \arrow[d, hook, dashed] & \text{top. space } X \\
(k+1)\text{-dimensional CW-complex } B \text{ with } B^k = A\,. \arrow[ur, dashed, "g", "(k+1)\text{-connected}"'] &
\end{tikzcd}$$

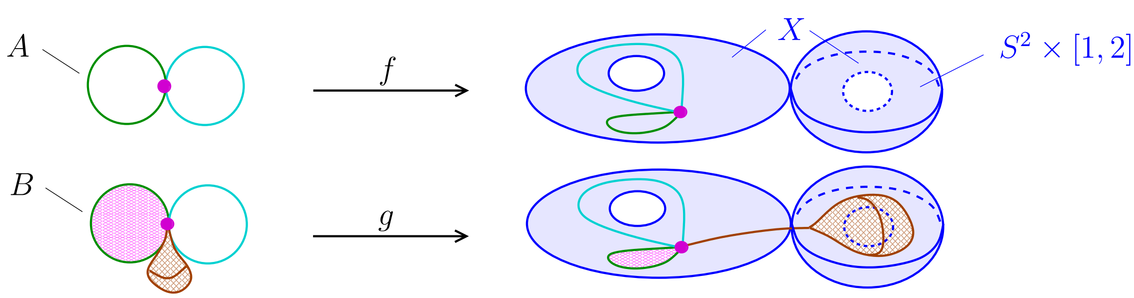


FIGURE 1622. Illustration of Proposition 119.1.

Before we turn to the proof of Proposition 119.1 let us first formulate and prove the following corollary.

[1656]Since we did not work with CW-complexes for a while we remind the reader that for a CW-complex X and $l \in \mathbb{N}_0$ we denote by X^l the l-skeleton i.e. the union of all cells of dimension $\leq l$.

Corollary 119.2. $(*)$ *Let A be a* CW*-complex and let $f\colon A \to X$ be a map to a topological space. There exists a* CW*-complex B and a map $g\colon B \to X$ such that the following statements hold:*

(1) *A is a subcomplex of B,*
(2) *the restriction of g to A equals f,*
(3) *the map $g\colon B \to X$ is k-connected for every $k \in \mathbb{N}_0$.*

Proof of Corollary 119.2 $(*)$. Let A be a CW-complex and let $f\colon A \to X$ be a map to a topological space. Without loss of generality we can and will assume that A and X are 0-connected. We set $B_0 := A$ and $g_0 := f$. We apply Proposition 119.1 to B_0 and g_0 and iterate this procedure to obtain a sequence of CW-complexes $\{B_n\}_{n\in\mathbb{N}_0}$ and maps $\{g_n\colon B_n \to X\}_{n\in\mathbb{N}_0}$ such that for each $n \in \mathbb{N}$ the following statements hold:

(a) A is a subcomplex of B_n and B_{n-1} is a subcomplex of B_n,
(b) the only cells of B_n that do not lie in B_{n-1} are n-dimensional,
(c) the restriction of g_n to A equals f,
(d) the map $g_n\colon B_n \to X$ is $(n+1)$-connected.

We set $B := \varinjlim B_n$ and we equip with the CW-structure given by Lemma 36.34. Furthermore we denote by $g\colon B \to X$ the map that is uniquely determined by the property that for each $n \in \mathbb{N}_0$ we have $g|_{B_n} = g_n$. We claim that (B, g) have the desired properties. In fact (1) and (2) are clear by construction. Now let $k \in \mathbb{N}_0$. For a fixed base point in B_{k+1} we consider the following commutative diagram

$$\begin{tikzcd}
\pi_k(B) \arrow[dddr, "g_*"'] & & \pi_k((k+1)\text{-skeleton of } B) \arrow[ll, "\cong"'] \arrow[rr, "="] \arrow[dr, "g_*"] & & \pi_k((k+1)\text{-skeleton of } B_{k+1}) \arrow[rr, "="] \arrow[dl, "g_*"'] & & \pi_k(B_{k+1}). \arrow[dlll, "\cong"', "g_*"] \\
& & & \pi_k(X) & & &
\end{tikzcd}$$

Note that by construction the $(k+1)$-skeleton of B equals the $(k+1)$-skeleton of B_{k+1}, thus the middle map is indeed the identity. It follows from Proposition 40.9 (1) that the outer horizontal maps are isomorphisms. Finally note that $g|_{B^{k+1}} = g_{k+1}$, which induces an isomorphism on π_k. Thus the left diagonal map is, as promised, an isomorphism. Keeping in mind the remark on page 2782 we see that the map $g\colon B \to X$ is k-connected for every $k \in \mathbb{N}_0$. ■

Now we turn to the proof of Proposition 119.1.

Proof of Proposition 119.1. To simplify the discussion we only deal with the case that A and X are 0-connected. We leave it to the reader to handle the case that at least one of A and Y is not 0-connected. In particular we will not deal with the case $k = 0$.

So in the following, let $k \in \mathbb{N}$, let A be a 0-connected CW-complex and let $f\colon A \to X$ be a k-connected map to a path-connected topological space. We pick a base point a_0 in the 0-skeleton of A. We write $x_0 = f(a_0)$. We note that in general the map $f\colon A \to X$ fails to be $(k+1)$-connected for the following two reasons:[1657]

(a) the map $\pi_k(A, a_0) \to \pi_k(X, x_0)$ is not a monomorphism,
(b) the map $\pi_{k+1}(A, a_0) \to \pi_{k+1}(X, x_0)$ is not an epimorphism.

Both issues will be fixed by attaching $(k+1)$-cells to A and by extending f to the $(k+1)$-cells in a suitable way. More precisely, we construct the CW-complex B and a map $g\colon B \to X$ as follows:

(a) As usual we denote by A^k the k-skeleton of A. Note that by Proposition 40.9 the map $\pi_k(A^k, x_0) \to \pi_k(A, a_0)$ is an epimorphism. This implies that we can pick a family $\{\varphi_i\colon (S^k, *) \to (A^k, a_0)\}_{i\in I}$ of maps that *go into the k-skeleton of A*, and that represent a generating set for $\ker(f_*\colon \pi_k(A, a_0) \to \pi_k(X, x_0))$.[1658] Note that by definition each map $f \circ \varphi_i\colon (S^k, *) \to (X, x_0)$ represents the trivial element in $\pi_k(X, x_0)$. Thus it follows from Lemma 40.4 that for each $i \in I$ where exists a map $F_i\colon \overline{B}^{k+1} \to X$ such that $F_i|S^k = f \circ \varphi_i$.
(b) Next we pick a family $\{\psi_j\colon (S^{k+1}, *) \to (X, x_0)\}_{j\in J}$ of maps that represent a generating set for $\pi_{k+1}(X, x_0)$.
(c) We set

$$B = \Big(A \sqcup \coprod_{i\in I} \overline{B}_i^{k+1}\Big)/\sim \;\vee\; \bigvee_{j\in J} S_j^{k+1} \qquad \text{with } \varphi_i(s) \sim s \text{ for } s \in S_i^k = \partial \overline{B}_i^{k+1}.$$

Note that B is a CW-complex in the following sense:
(i) We start out with the CW-complex that is given by the k-skeleton of A.
(ii) We attach one $(k+1)$-cell for each $i \in I$ with attaching map given by φ_i.
(iii) Afterwards we form the wedge with $\bigvee_{j\in J} S_j^{k+1}$. In other words, we attach one $(k+1)$-cell for each $j \in J$ where the attaching map is given by sending all points in $S_j^k = \partial \overline{B}_j^{k+1}$ to a_0.[1659]
(iv) Finally we attach the cells of A of dimension $\geq k+1$ as usual to the above.

(d) We extend the original map $f\colon A \to X$ to a map $g\colon B \to X$ by defining it to be F_i on each $\overline{B}_i^{k+1}$ and by defining it to be ψ_j on each S_j^{k+1}. It follows easily from Lemma 36.20 that this map $g\colon B \to X$ is indeed continuous.

By construction A is a subcomplex of B and the only extra cells of B are $(k+1)$-dimensional. Furthermore, by construction the restriction of g to A equals f. Thus it remains to show

[1657] Here and throughout the proof we will implicitly use the remark on page 2782 regarding k-connected maps.

[1658] A generating set for the kernel is for example given by taking all elements in the kernel.

[1659] Here we implicitly use the convention from page 2782 to view the wedge of spheres, and thus also B, as a CW-complex.

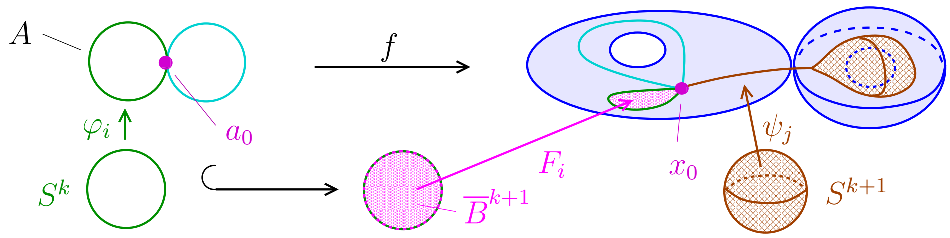


FIGURE 1623. Illustration for the proof of Proposition 119.1.

that the map $g\colon B \to X$ is $(k+1)$-connected. We denote by $K := A^k = B^k$ the k-skeleton of A which equals by construction the k-skeleton of B.

(a) First we show that the map $g\colon B \to X$ is k-connected. To do so we note that for any $r \in \mathbb{N}_0$ we have the following commutative diagram

$$\begin{tikzcd}
\pi_r(A,a_0) \arrow[rr, "i_*"] \arrow[dr, bend right, "f_*"'] & & \pi_r(B,a_0) \arrow[dl, bend left, "g_*"] \\
 & \pi_r(K,x_0) \arrow[ul, "i_*"] \arrow[ur, "i_*"'] \arrow[d, "f_*"] & \\
 & \pi_r(X,x_0) &
\end{tikzcd}$$

where i always denotes an inclusion map. Since K is the k-skeleton of A and B we obtain from Corollary 111.10 that the maps $K \to A$ and $K \to B$ are k-connected. By hypothesis the map $f\colon A \to X$ is k-connected. It follows almost immediately from this discussion and the commutative diagram that $g\colon B \to X$ is also k-connected. Note that this discussion also implies that the inclusion $A \to B$ is k-connected.

(b) Next we show that $g_*\colon \pi_k(B,a_0) \to \pi_k(X,x_0)$ is a monomorphism. Similar to (a) we consider the following commutative diagram:

$$\begin{tikzcd}
\pi_k(A,a_0) \arrow[rr, two heads, "i_*"] \arrow[dr, "f_*"'] & & \pi_k(B,a_0) \arrow[dl, "g_*"] \\
 & \pi_k(X,x_0). &
\end{tikzcd}$$

As we pointed out in (a), we know that the top map $i_*\colon \pi_k(A,a_0) \to \pi_k(B,a_0)$ is an epimorphism. Thus it suffices to show that

$$\ker(f_*\colon \pi_k(A,a_0) \to \pi_k(X,x_0)) \ \subset\ \ker(i_*\colon \pi_k(A,a_0) \to \pi_k(B,a_0)).$$

But this inclusion holds basically by definition of B. More precisely, it follows from the definition of B that for each $i \in I$ the map $\varphi_i\colon (S^k,*) \to (A,a_0)$ extends to a map $(\overline{B}^{k+1},*) \to (B,a_0)$. But by Lemma 40.4 we know that this implies that the maps $\varphi_i\colon (S^k,*) \to (A,a_0)$ represent elements in the kernel of i_*. On other hand, by definition, these maps represent a generating set for $\ker(f_*\colon \pi_k(A,a_0) \to \pi_k(X,x_0))$. The desired inclusion of kernels is now a consequence of Lemma 19.3.

(c) It remains to show that the map $g_*\colon \pi_{k+1}(B,a_0) \to \pi_{k+1}(X,x_0)$ is an epimorphism. In fact this is true almost by definition of B. More precisely, given $j \in J$ we have

$g_*([\mathrm{id}_{S_j^{k+1}}] = [g|_{S_j^{k+1}}] = [\psi_j]$. This shows that the classes $\{[\psi_j]\}_{j\in J}$ lie in the image of g_*. But since these classes generate $\pi_{k+1}(X, x_0)$ we obtain from Lemma 19.3 that g_* is an epimorphism. ■

The following lemma gives a simple but useful criterion for extending maps from a subcomplex to the whole CW-complex.

Lemma 119.3. *Let (X, A) be a pair of CW-complexes and let $f\colon A \to Y$ be a map to a path-connected topological space. Suppose the following condition holds:*

(∗) *For every $n\in\mathbb{N}$ such that $X\setminus A$ contains an n-cell we have the equality*[1660] $\pi_{n-1}(Y)=0$.

Then f can be extended to a map $\widetilde{f}\colon X \to Y$. This statement is illustrated in Figure 1624.

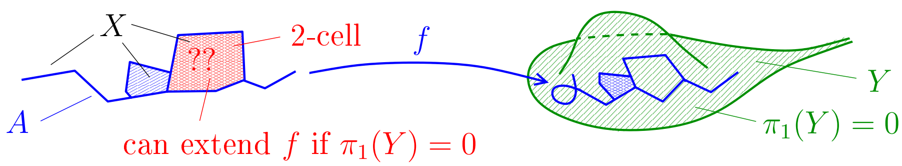


FIGURE 1624. Illustration of Lemma 119.3.

PROOF. We start out with the following claim.

Claim. Given any $n \in \mathbb{Z}_{\geq -1}$ we can extend $f\colon A \to Y$ to a map $A \cup X^n \to Y$.

We prove the claim by induction on $n \in \mathbb{Z}_{\geq -1}$. For $n = -1$ there is nothing to prove. So suppose the statement holds for some $n \in \mathbb{Z}_{n\geq -1}$. By the slightest of all abuses we denote the given extension $A \cup X^n \to Z$ again by f. We denote by $\{\Phi_i\colon S^n \to X^n\}_{i\in I}$ the family of the attaching maps of all $(n+1)$-dimensional cells of X that are not contained in A. If $I = \varnothing$, then there is nothing to do. On the other hand if $I \neq \varnothing$, then we know by the hypothesis (∗) that $\pi_n(Y) = 0$. It follows from Lemma 40.4 that for each $i \in I$ there exists a map $g_i\colon \overline{B}^{n+1} \to Y$ with $g_i|_{S^n} = f \circ \Phi_i\colon S^n \to Y$. We consider the following map:

$$\begin{array}{rcl} A \cup X^{n+1} = \Big((A \cup X^n) \sqcup \coprod\limits_{i\in I} \overline{B}_i^{n+1}\Big)/\sim & \to & Y \\ [P] & \mapsto & \begin{cases} f(P), & \text{if } P \in A \cup X^n, \\ g_i(P), & \text{if } P \in \overline{B}_i^{n+1}. \end{cases} \end{array}$$

It follows from $g_i|_{S^n} = f \circ \Phi_i$ that this map is well-defined. Furthermore we obtain from Lemma 3.22 that the map is continuous. ⊞

The sequence of maps from the claim defines a map $\widetilde{f}\colon X = A \cup X = \bigcup\limits_{n\in\mathbb{N}_0} (A \cup X^n) \to Y$. Since the map is continuous on each $A \cup X^n$ we obtain from Lemma 36.20 that $\widetilde{f}$ is also continuous. It now clear that this map $\widetilde{f}$ is the desired extension. ■

We conclude this section with the formulation of a special case Lemma 119.3.

Definition. We say a topological space Y is *weakly contractible* if[1661] $\pi_n(Y) = 0$ for all $n \in \mathbb{N}_0$.

[1660]Since Y is path-connected we know by Proposition 40.5 that we do not need to worry about base points.

Example. By Corollary 40.12 we know that the infinite-dimensional sphere S^∞ is weakly contractible.

The following corollary is an immediate consequence of Lemma 119.3.

Corollary 119.4. *Let (X, A) be a pair of CW-complexes and let $f\colon A \to Y$ be a map to a topological space. If Y is weakly contractible, then f can be extended to a map $\widetilde{f}\colon X \to Y$. In other words, we have the following situation*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y. \\ X \arrow[ur, dashed, "\widetilde{f}\text{ exists if } Y \text{ is weakly contractible}"'] & \end{tikzcd}$$

119.2. **Weak homotopy equivalences and the CW-Approximation Theorem.** The definition of a k-connected map between topological spaces naturally leads to the following definition which we had encountered once before on page 1569.

Definition. We say that a map $f\colon X \to Y$ between topological spaces is a *weak homotopy equivalence* if f is k-connected for all $k \in \mathbb{N}_0$.

The following lemma gives in particular a practical approach to showing that a given map is a weak homotopy equivalence.

Lemma 119.5. *Let $f\colon X \to Y$ be a map between topological spaces.*

(1) *If f is a homotopy equivalence, then it is also a weak homotopy equivalence.*
(2) *If $g\colon X \to Y$ is a map that is homotopic to f, then*
$$f \text{ is a weak homotopy equivalence} \iff g \text{ is a weak homotopy equivalence}$$
(3) *If X and Y are path-connected, then the following holds:*
$$f \text{ is a weak homotopy equivalence} \iff \begin{array}{c}\text{there exists an } x_0 \in X \text{ such that} \\ \text{for every } k \in \mathbb{N} \text{ the induced map} \\ f_*\colon \pi_k(X, x_0) \to \pi_k(Y, f(x_0)) \\ \text{is an isomorphism.}\end{array}$$

Proof.

(1) This statement is an immediate consequence of Proposition 40.7 (2).
(2) This statement follows immediately from Proposition 40.7 (1).
(3) This statement follows easily from the definitions, the fact that X and Y are path-connected and Proposition 54.9 (3). ■

As we just saw in Lemma 119.5, a homotopy equivalence is also a weak homotopy equivalence. But the following example shows that in general the converse does not hold.

[1661] Recall that by the definition on page 135 the statement $\pi_0(Y) = 0$ means that Y is *non-empty* and path-connected.

Example. We turn the "topologist's sine curve" from page 131 into the so-called "quasi-circle" X by connecting the vertical interval with the right-hand endpoint. This topological space is illustrated in Figure 1625. We make the following observations:

(1) One can easily show that the quasi-circle is path-connected, i.e. $\pi_0(X) = 0$.
(2) In Exercise 2.52 we showed that the topologist's sine curve is not path-connected. A similar argument, see Exercise 119.3 (a), shows that the image of every map $S^n \to X$ lies in a contractible subset of X. By Propositions 18.16 (2) and 40.7 (2) this implies that all homotopy groups $\pi_n(X)$, $n \geq 1$ of X are zero.
(3) Pick $x_0 \in X$. It follows from (1) and (2) that the inclusion $i\colon \{x_0\} \to X$ is a weak homotopy equivalence.
(4) In Exercise 119.3 (b) we will see, in contrast, that the inclusion i is not a homotopy equivalence.

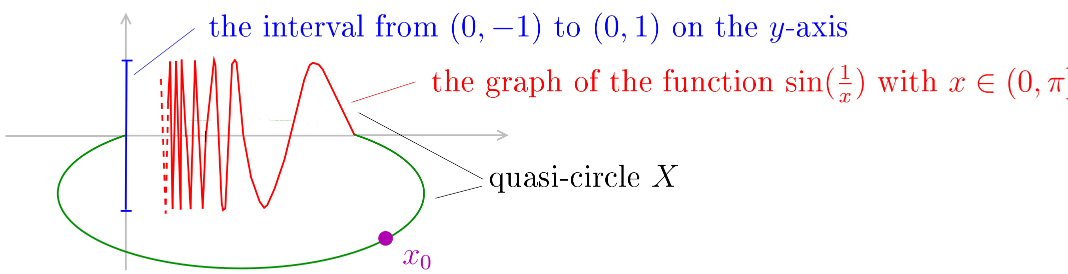


FIGURE 1625

We conclude this discussion of weak homotopies with the following proposition which relates weak homotopy equivalences to induced maps on homology groups.

Proposition 119.6. *Let $f\colon X \to Y$ be a map between topological spaces.*

(1) *If f is a weak homotopy equivalence, then for any abelian group G and any $n \in \mathbb{N}_0$ the induced maps $f_*\colon \mathrm{H}_n(X;G) \to \mathrm{H}_n(Y;G)$ and $f^*\colon \mathrm{H}^n(Y;G) \to \mathrm{H}^n(X;G)$ are isomorphisms.*
(2) *If X and Y are simply connected and if $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism for every $n \geq 2$, then f is a weak homotopy equivalence.*

In fact we can basically generalize (2) *to the following less readable statement:*

(2') *Suppose the following conditions are satisfied:*
  (a) *X and Y are both path-connected, both are locally path-connected and both are semi-locally simply connected.* (*As discussed in Corollary 29.9 this means that both topological spaces admit universal coverings.*)
  (b) *For some $x \in X$ the induced map $f_*\colon \pi_1(X, x)) \to \pi_1(Y, f(x))$ is an isomorphism.*

  We write $y = f(x)$ and we denote by $p\colon (\widetilde{X}, \widetilde{x}) \to (X, x))$ and $q\colon (\widetilde{Y}, \widetilde{y}) \to (Y, y)$ the universal covers of (X, x) and (Y, y). By (a) *and Proposition 29.2 we know that the map f lifts to a map $\widetilde{f}\colon (\widetilde{X}, \widetilde{x}) \to (\widetilde{Y}, \widetilde{y})$. We also demand the following:*
  (c) *The map $\widetilde{f}_*\colon \mathrm{H}_n(\widetilde{X}) \to \mathrm{H}_n(\widetilde{Y})$ is an isomorphism for every $n \geq 2$.*

  Then f is a weak homotopy equivalence.

Proof.

(1) Let $f\colon X \to Y$ be a weak homotopy equivalence. It follows easily from Lemmas 41.14 and Lemmas 73.14 that we can assume, without loss of generality, that X and Y are path-connected. Let $x_0 \in X$. By our hypothesis we know that the map $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism for all $n \in \mathbb{N}_0$. It follows from Theorem 111.14 (1), which in turn is a consequence of the Relative Hurewicz Theorem 111.13, that the induced maps $f_*\colon \mathrm{H}_n(X;\mathbb{Z}) \to \mathrm{H}_n(Y;\mathbb{Z})$ are isomorphisms for all $n \in \mathbb{N}_0$. The desired statement is now a consequence of Corollary 73.20.
(2) This statement is just a reformulation of Theorem 111.14 (2).
(3) By (b) together with Lemma 119.5 it remains to show that $f_*\colon \pi_n(X,x)) \to \pi_n(Y,y)$ is an isomorphism for every $n \geq 2$. Now let $n \geq 2$. We consider the following diagram:

$$\begin{tikzcd} \pi_n(\widetilde{X}, \widetilde{x}) \arrow[r, "\widetilde{f}_*"] \arrow[d, "p_*"'] & \pi_n(\widetilde{Y}, \widetilde{y}) \arrow[d, "q_*"] \\ \pi_n(X, x) \arrow[r, "f_*"] & \pi_n(Y, y). \end{tikzcd}$$

We make the following observations:
(a) The diagram commutes by definition of $\widetilde{f}$.
(b) It follows from Proposition 40.13 that the vertical maps are isomorphisms.
(c) By (2) and by our hypothesis (c) we know that $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ is a weak homotopy equivalence. This implies that the top horizontal map is an isomorphism.

It follows from the above that the bottom horizontal map is indeed an isomorphism.

■

In the remainder of this section we will study CW-approximations of topological spaces.

Definition. Let X be a topological space. A CW-*approximation* is a CW-complex Z together with a weak homotopy equivalence $f\colon Z \to X$.

Example. Let X be the quasi-circle. As we had just discussed, the inclusion of the CW-complex $\{x_0\}$, consisting of a single 0-cell, into the quasi-circle X is a CW-approximation of the quasi-circle X.

Before we state a general theorem we turn to the delicate topic of products of CW-complexes. More precisely, if X and Y are CW-complex with only countably many cells, then it follows from Proposition 36.23 that the product $X \times Y$ has a product CW-structure. But, as we pointed out in Lemma 36.24, in general this is not true for arbitrary CW-complexes. This leads us to the following lemma.

Lemma 119.7. (∗) *Let X and Y be two* CW*-complexes. We define the topological space $X \otimes Y$, whose underlying set is $X \times Y$, as on page 963. The identity map* $\mathrm{id}\colon X \otimes Y \to X \times Y$ *is a* CW*-approximation.*

Proof (∗). By Proposition 36.23 we know that $X \otimes Y$ is a CW-complex and we know that the identity map $\mathrm{id}\colon X \otimes Y \to X \times Y$ is continuous. In Proposition 36.26 we showed that the induced maps on fundamental groups are isomorphisms. Basically the same argument, with the obvious extension of Lemma 36.27 to higher dimensions, shows that the map $\mathrm{id}\colon X \otimes Y \to X \times Y$ is a weak homotopy equivalence. ■

The following theorem says that CW-approximations always exist.

Theorem 119.8. (CW-Approximation Theorem)

(1) *Every topological space X admits a* CW*-approximation $f\colon Z \to X$.*
(2) *Let $k \in \mathbb{N}_0$. If X is a k-connected topological space, then there exists a* CW*-approximation $f\colon Z \to X$ such that Z has a single 0-cell and such that Z has no cells in dimensions $1, \dots, k$.*

In both cases, given a point $x_0 \in X$ we can find f and $z_0 \in Z^0$ such that $f(z_0) = x_0$.

Proof of the CW-Approximation Theorem 119.8. Let X be a topological space.

(1) We need to show that X admits a CW-approximation. By considering each path-component of X separately we can assume that X is path-connected.

Thus let X be a path-connected topological space. We pick a point $x_0 \in X$. We set $Z_0 = \{x_0\}$ and we define $f_0\colon Z_0 \to X$ as the inclusion map. By iteratively appealing to Proposition 119.1 we obtain the following commutative diagram

$$\begin{tikzcd}
\text{0-dimensional CW-complex } Z_0 \arrow[r, hook, "f_0", "\text{0-connected}"'] \arrow[d, hook] & \text{topological space } X \\
\text{1-dimensional CW-complex } Z_1 \arrow[ur, bend right=10, "f_1", "\text{1-connected}"'] \arrow[d, hook] & \\
\text{2-dimensional CW-complex } Z_2 \arrow[uur, bend right=10, "f_2", "\text{2-connected}"'] \arrow[d, hook] & \\
\vdots &
\end{tikzcd}$$

such that the $(k-1)$-skeleton of each Z_k equals Z_{k-1}. We set $Z = \bigcup_{k \in \mathbb{N}_0} Z_k$ and we define $f\colon Z \to X$ in the obvious way. Note that by construction Z is a CW-complex such that for each $k \in \mathbb{N}_0$ the k-skeleton is given by Z_k. Furthermore it follows from Lemma 36.20 that f is continuous.

It remains to show that $f\colon Z \to X$ is indeed a weak homotopy equivalence. So let $k \in \mathbb{N}_0$. We consider the following commutative diagram

$$\begin{tikzcd}
\pi_k(Z_{k+1}, x_0) \arrow[rr, "\iota_*"] \arrow[dr, "f_{k+1*}"'] & & \pi_k(Z, x_0) \arrow[dl, "f_*"] \\
& \pi_k(X, x_0) &
\end{tikzcd}$$

where $\iota\colon Z_{k+1} \to Z$ denotes the inclusion map. The left-hand diagonal map is an isomorphism by construction of $f_{k+1}\colon Z_{k+1} \to X$. Furthermore the top horizontal map is an isomorphism by Proposition 40.9 and the fact that Z_{k+1} is the $(k+1)$-skeleton of Z. We deduce from the commutative diagram that the right-hand diagonal map is also an isomorphism.

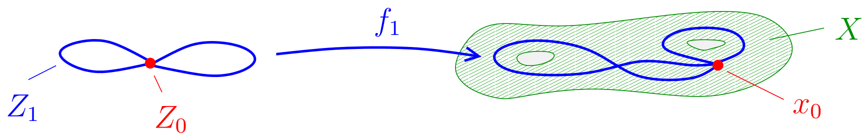


FIGURE 1626. Illustration for the proof of Theorem 119.8.

(2) Let $k \in \mathbb{N}_0$. If X is k-connected, then in the proof given in (1) we can start out with the CW-complexes $\{x_0\} = Z_0 = Z_1 = \cdots = Z_k$ and where $f_0 = f_1 = \cdots = f_k$ are the inclusion maps. Only beginning with f_{k+1} do we appeal to Proposition 119.1 to provide the further CW-complexes Z_j and maps $f_j \colon Z_j \to X$. For the remainder of the proof of (2) we now proceed as in (1). Since the k-skeleton of $Z = \bigcup\limits_{k \in \mathbb{N}_0} Z_k$ equals $Z_k = \{x_0\}$ we see that Z has the desired properties. ■

Examples.

(1) We consider the topological space $X = \{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\} \subset \mathbb{R}$ equipped with the subspace topology coming from $\mathbb{R}$. Note that X is not a CW-complex.[1662] Furthermore we consider $Z = \mathbb{N}_0 \subset \mathbb{R}$ with the discrete topology. This is a 0-dimensional CW-complex and the map

$$\begin{aligned} f \colon Z = \mathbb{N}_0 \;&\to\; X = \{0\} \cup \{\tfrac{1}{n} \,|\, n \in \mathbb{N}\} \subset \mathbb{R} \\ n \;&\mapsto\; \begin{cases} 0, & \text{if } n = 0, \\ \frac{1}{n}, & \text{otherwise} \end{cases} \end{aligned}$$

is easily seen[1663] to be a weak homotopy equivalence. Thus $f \colon Z \to X$ is a CW-approximation. We refer to Figure 1627 for an illustration.

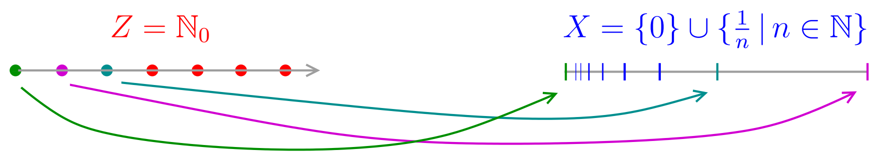


FIGURE 1627

(2) We consider the pseudocircle that we introduced on page 107. Recall that the pseudocircle is the set $X = \{a, b, c, d\}$ where the topology is given by the set

$$\mathcal{T} \;:=\; \{\varnothing, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}.$$

On page 107 we gave an explicit map $f \colon S^1 \to X$ that is also illustrated in Figure 1628. In Exercise 16.13 we showed that $f_* \colon \pi_1(S^1, 1) \to \pi_1(X, a)$ is an isomorphism. A slight variation on the definition of f also shows that X is path-connected. Furthermore, by the discussion on page 1068 and by Exercise 40.5 we know that $\pi_n(S^1, 1) = \pi_n(X, a) = 0$ for $n \geq 2$. Thus we obtain from Lemma 119.5 that f

[1662] Why is it not a CW-complex?
[1663] What is so easy about this?

is a weak homotopy equivalence. Since S^1 is a CW-complex we see that the map $f\colon S^1 \to X$ is a CW-approximation for the pseudocircle X.

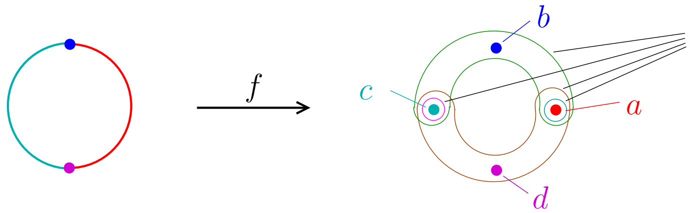


FIGURE 1628

119.3. **The Whitehead Theorem.** Usually, in algebraic topology one goes from topological situations to algebra using various functors. For example, any map between (pointed) topological spaces gives rise to a map between homology groups and homotopy groups. Unfortunately almost invariably in the passage from topology to algebra some information gets lost.

In Proposition 40.7 (2) we saw that a homotopy equivalence between topological spaces gives rise to isomorphisms of higher homotopy groups. The following theorem shows that, rather amazingly, in this particular case the converse holds, at least if we restrict ourselves to maps between CW-complexes.

Theorem 119.9. (Whitehead Theorem)[1664] *Let $f\colon X \to Y$ be a map between two connected* CW*-complexes.*[1665]

(1) *If $f\colon X \to Y$ is a weak homotopy equivalence, then f is a homotopy equivalence.*

In fact, there is a more refined version dealing with base points:

(2) *Let $x_0 \in X^0$ and $y_0 \in Y^0$ be base points in the 0-skeleta of X respectively Y. If $f\colon X \to Y$ is a weak homotopy equivalence with $f(x_0) = y_0$, then there exists a homotopy equivalence between the pointed topological spaces (X, x_0) and (Y, y_0).*

For inclusion maps we have the following very useful variation on the Whitehead Theorem 119.9.

Proposition 119.10. *Let Y be a connected* CW*-complex, let X be a connected subcomplex and let $x_0 \in X$ be a point. We denote by $i\colon X \to Y$ the inclusion map. If the induced maps $i_*\colon \pi_n(X, x_0) \to \pi_n(Y, x_0)$ are isomorphisms for all $n \in \mathbb{N}$, then X is a deformation retract of Y.*

Examples.

(1) Let Y be a simply connected 1-dimensional CW-complex and let x_0 be a point in the 0-skeleton of Y. By Lemma 40.15 (or alternatively Lemma 53.8) we know that $\pi_n(Y, x_0) = 0$ for all $n \in \mathbb{N}$. In other words, the inclusion map $\{x_0\} \to Y$ is a weak homotopy equivalence. Thus it follows from Proposition 119.10 that Y admits a deformation retraction to $\{x_0\}$, in particular Y is contractible.

[1664] As we pointed out on page 2637, the name "Whitehead Theorem" is sometimes also used for Theorem 111.14. Our naming convention agrees with [**Hat02**, Theorem 4.5] and [**Rot88**, p. 370].

[1665] Note that we do not assume that the map f is cellular.

For example, let Y be the universal covering of $S^1 \vee S^1$. We view $S^1 \vee S^1$ as a CW-complex in the obvious way. By Proposition 36.37 we can equip Y with the structure of a 1-dimensional CW-complex. By definition of a universal covering we know that Y is simply connected. It follows from the above discussion that any point x_0 in the 0-skeleton of Y is a deformation retract. We refer to Figure 1629 for an illustration.

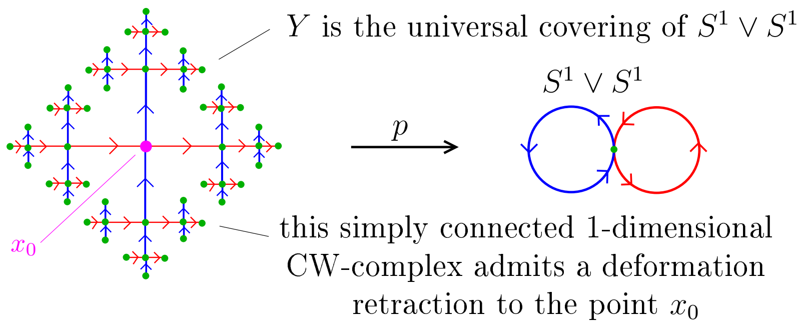


Figure 1629

(2) On page 2788 we pointed out that Corollary 40.12 says that S^∞ is weakly contractible. Thus it follows from the Whitehead Theorem 119.9 that S^∞ is contractible.[1666]

(3) We consider the knotted curve X in the solid torus $Y = \overline{B}^2 \times S^1$ that is shown in Figure 1630. It follows from Theorem 64.2 (3) together with Lemma 61.24 that we can equip Y with a CW-structure such that X is a subcomplex. It should be pretty clear that the inclusion induced map $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ is an isomorphism. Furthermore, since X and Y are homotopy equivalent to S^1 we see that the higher homotopy groups of X and Y vanish. In particular for every $n \in \mathbb{N}_{\geq 2}$ the induced maps $i_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ are isomorphisms. In other words we see that all the hypotheses of Proposition 119.10 are satisfied. This shows that X is a retract of Y. This answers the question that we had posed on page 484.

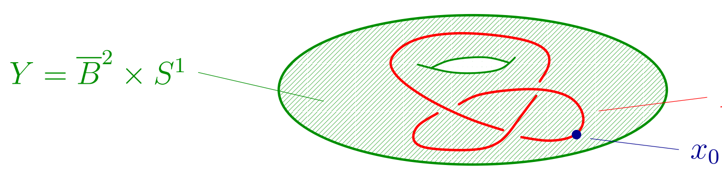


Figure 1630

Remark. One might be tempted to interpret the Whitehead Theorem 119.9 as saying that if two CW-complexes have isomorphic homotopy groups, then they are homotopy equivalent. But this statement is false.

(1) For example let us consider the lens spaces $L(5, 1)$ and $L(5, 2)$. Recall that both are quotients of S^3 by some action of $\mathbb{Z}_5$ and both are closed 3-dimensional smooth

[1666]In fact in Lemma 36.4 we gave an explicit homotopy from the identity of S^∞ to a constant map. By Lemma 18.12 this implies that S^∞ is contractible.

manifolds. In particular both are CW-complexes by Theorem 64.5.[1667] On page 1069 we saw that the homotopy groups of these two lens spaces are isomorphic. On the other hand we seen on page **??** that $L(5,1)$ and $L(5,2)$ are not homotopy equivalent.

(2) We consider the topological spaces S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$. Evidently S^2 and S^3 are CW-complexes. It follows from Lemma 36.6 and Proposition 36.23 that $S^3 \times \mathbb{C}\mathrm{P}^\infty$ is also a CW-complex. In Exercise 113.13 we saw that S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$ have isomorphic homotopic groups.[1668] On the other hand it follows immediately from the discussion on page 1263 and the Künneth Theorem 58.8 that the homology groups of S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$ are (dramatically) different. Thus the two CW-complexes are not homotopy equivalent.

Remark. The analogous statement of the Whitehead Theorem 119.9 does not hold if we drop the hypothesis that X and Y are CW-complexes, in other words, the statement does not hold for general topological spaces. We can provide two examples:

(1) On page 2789 we already discussed the fact that the quasi-circle is weakly contractible but not contractible. In other words, the quasi-circle is weakly homotopy equivalent to a point, but it is not homotopy equivalent to a point.

(2) Since weird topological spaces are so much fun, let us consider another example. More precisely, let us consider the following subset of $\mathbb{R}^2$ which is also illustrated in Figure 1631:

$$\begin{aligned} Y := \text{line segment from } (0,1) \text{ to } (0,-1) \;&\cup \bigcup_{n\in\mathbb{N}} \text{line segment from } (0,1) \text{ to } (\tfrac{1}{n},0) \\ &\cup \bigcup_{n\in\mathbb{N}} \text{line segment from } (0,-1) \text{ to } (-\tfrac{1}{n},0). \end{aligned}$$

We equip Y with the subspace topology. In Exercise 119.4 (a) we will see that the homotopy groups of Y are trivial. This implies that any inclusion $X = \{*\} \to Y$ is a weak homotopy equivalence. On the other hand we show in Exercise 119.4 (b) that Y is not contractible. The readers who are less inclined towards doing the exercises are referred to [**Mau80**, p. 301] instead.[1669]

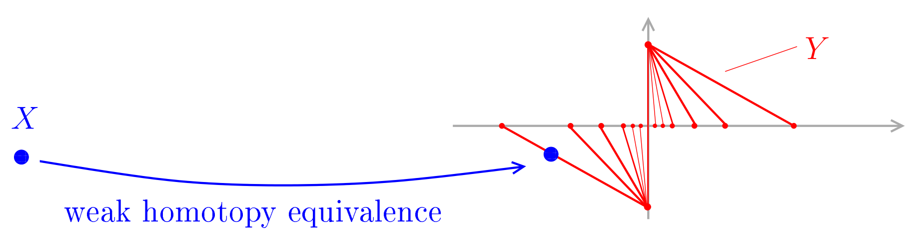


FIGURE 1631

[1667] It is also a moderately amusing exercise to find an explicit CW-structure on lens spaces. One could start either from the definition of a lens space or the reinterpretation given by Lemma **??**.

[1668] In fact this follows easily from the fact that $\pi_k(S^2) = \pi_k(S^3)$ for $k \geq 3$ (see Theorem 113.10) and the fact that $\pi_k(\mathbb{C}\mathrm{P}^\infty) = \mathbb{Z}$ for $k = 2$ and $\pi_k(\mathbb{C}\mathrm{P}^\infty) = 0$ for $k \neq 2$ (see Theorem 113.12 (2)).

[1669] This shows in particular that Y is not a CW-complex. Is there a more direct way for showing that Y is not a CW-complex?

The proof of the Whitehead Theorem 119.9 relies on the following lemma.

Lemma 119.11. *Let (X, A) be a pair of CW-complexes and let (Y, B) be a pair of topological spaces with $B \neq \varnothing$. We make the following assumption: For every $n \in \mathbb{N}_0$ such that $X \setminus A$ has at least one n-dimensional cell we assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f\colon (X, A) \to (Y, B)$ is homotopic rel A to a map $X \to B$.*

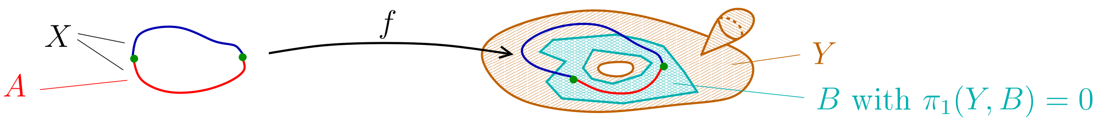


FIGURE 1632. Illustration of Lemma 119.11.

PROOF. We start out with the following claim.

Claim. Let $k \in \mathbb{N}$. Let $f\colon (X, A) \to (Y, B)$ be a map. If $f(X^{k-1}) \subset B$, then there exists a homotopy rel $A \cup X^{k-1}$ from f to a map g with $g(X^k) \subset B$.

If there are no k-cells in $X \setminus A$, then we have $X^k \subset X^{k-1} \cup A$, in particular we already have $f(X^k) \subset B$, and we are thus done. Now suppose that there exists at least one k-cell in $X \setminus A$. By our hypothesis this implies that $\pi_k(Y, B, y_0) = 0$ for every $y_0 \in B$. Let $\{\Phi_i\colon \overline{B}^k \to X\}_{i \in I}$ be the characteristic maps of the k-cells in $X \setminus A$. Given $i \in I$ we denote by $\phi_i\colon S^{k-1} \to X$ the corresponding attaching map. It follows from $f(X^{k-1}) \subset B$, from $\pi_k(Y, B, y_0) = 0$ and Proposition 111.8 (2) $\Rightarrow$ (1) that there exists a homotopy F_i rel S^{k-1} from the map $f \circ \Phi_i\colon \overline{B}^k \to Y$ to a map whose image is contained in B. It follows from Lemma 18.23, applied to $A \cup X^k = (A \cup X^{k-1}) \cup_{\sqcup_{i\in I} \phi_i} \left(\bigsqcup_{i \in I} \overline{B}^k_i\right)$ that these homotopies on all such cells, together with the constant homotopy on $A \cup X^{k-1}$, give rise to a homotopy H on $A \cup X^k$ rel $A \cup X^{k-1}$ with $\mathrm{H}_0|_{A \cup X^k} = f$ and with $\mathrm{H}_1(X^k) \subset B$. By the Homotopy Extension Theorem 38.1 we can extend this homotopy H to a homotopy $\widetilde{H}\colon X \times [0, 1] \to Y$ with $\widetilde{\mathrm{H}}_0 = f$. This homotopy has all the desired properties. ⊞

First suppose that X is a finite-dimensional CW-complex. It is clear that the hypothesis of the claim holds for $k = -1$ since, by definition, $X^{-1} = \varnothing$. Now we can apply the claim iteratively finitely many times. Combining these finitely many homotopies together as in Lemma 18.3 we obtain the desired homotopy.

In the case that X is infinite-dimensional we obtain an infinite sequence of homotopies. We can combine these homotopies using the rather tricky Lemma 38.11 and we obtain again the desired homotopy. ■

Now we are already in a position to provide the proof of Proposition 119.10.

PROOF OF PROPOSITION 119.10. Let Y be a connected CW-complex, let X be a connected subcomplex and let $x_0 \in X$. We denote by $i\colon X \to Y$ the inclusion map. We suppose that the induced maps $i_*\colon \pi_n(X, x_0) \to \pi_n(Y, x_0)$ are isomorphisms for all $n \in \mathbb{N}$. We need to show that X is a deformation retract of Y.

Recall that by our hypothesis Y is connected, in particular it is path-connected by Proposition 36.10 (7). Thus it follows from Proposition 54.9 that the induced maps

$i_*\colon \pi_n(X,x_0) \to \pi_n(Y,x_0)$ are in fact isomorphisms for all $x_0 \in X$. Next we note that it follows from our hypothesis and the long exact sequence of homotopy groups given by Proposition 111.4 that all relative homotopy groups $\pi_n(Y,X,x_0)$ are trivial for all $x_0 \in X$. This observation allows us apply Lemma 119.11 to the identity map $(Y,X) \to (Y,X)$. From Lemma 119.11 we obtain a homotopy H rel X from id_Y to a map whose image lies in X. But this means, by definition, that H is a deformation retraction from Y to X. ■

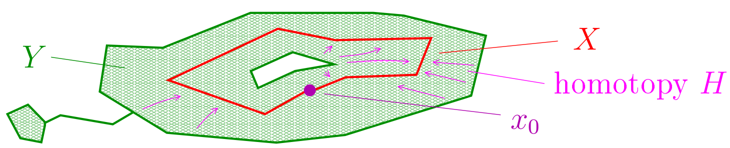


FIGURE 1633. Illustration for the proof of Proposition 119.10.

We will reduce the proof of the Whitehead Theorem 119.9 to Proposition 119.10. The trick hereby is to use the notion of a mapping cylinder that we introduced on page 697. For the reader's convenience we recall the definition.

Definition. Given a map $f\colon X \to Y$ between topological spaces we define the corresponding *mapping cylinder* to be the topological space

$$\mathrm{Cyl}(f) \;:=\; \mathrm{Cyl}(f\colon X \to Y) \;:=\; ((X \times [0,1]) \sqcup Y)/\sim \;\text{ where } (x,1) \sim f(x) \text{ for all } x \in X.$$

We illustrate the definition in Figure 1634.

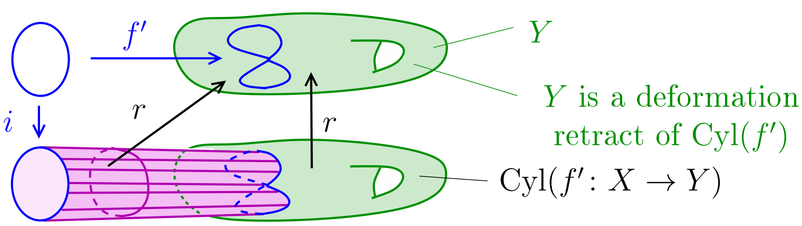


FIGURE 1634. Illustration for the proof of the Whitehead Theorem 119.9.

Now we can provide the actual proof of the Whitehead Theorem 119.9.

PROOF OF THE WHITEHEAD THEOREM 119.9. Let X and Y be two connected CW-complexes. Let $x_0 \in X$ be a base point. Furthermore let $f\colon X \to Y$ be a map such that the induced maps $f_*\colon \pi_n(X,x_0) \to \pi_n(Y,f(x_0))$ are isomorphisms for all $n \in \mathbb{N}$.

(1) By the Cellular Approximation Theorem 38.13 the map f is homotopic to a cellular map f'. We make the following two observations:
   (i) By Lemma 18.11 (3) the map f is a homotopy equivalence if and only if f' is a homotopy equivalence.
   (ii) By Lemma 119.5 (2) the map f is a weak homotopy equivalence if and only if f' is a weak homotopy equivalence.

As usual we denote by $\mathrm{Cyl}(f')$ the mapping cylinder of $f'\colon X \to Y$. We consider the following diagram of maps:

$$\begin{tikzcd} & Y \\ X \arrow[ur, "f'", "\text{weak homotopy equivalence}"'] \arrow[r, hook, "i"'] & \mathrm{Cyl}(f'). \arrow[u, "r"', "\simeq"] \end{tikzcd}$$

We make the following comments and observations:[1670]

(a) The map $r\colon \mathrm{Cyl}(f') \to Y$ is the obvious map given by the identity on Y and given by sending $[(x,t)]$ to $f'(x)$.
(b) The map $i\colon X \to \mathrm{Cyl}(f')$ is the obvious inclusion given by $x \mapsto [(x,0)]$.
(c) It is clear that the above diagram commutes.
(d) By (ii) and our hypothesis the map $f'\colon X \to Y$ is a weak homotopy equivalence.
(e) By Lemma 24.8 (2) we know that the map $r\colon \mathrm{Cyl}(f) \to Y$ is a homotopy equivalence. In particular we obtain from Lemma 119.5 (1) that r is a weak homotopy equivalence.
(f) It follows from (c), (d) and (e) that i is a weak homotopy equivalence.
(g) Since $f'\colon X \to Y$ is cellular we use Corollary 36.35 (3) to equip $\mathrm{Cyl}(f'\colon X \to Y)$ with a natural CW-structure such that $X \times \{0\} = i(X)$ is a subcomplex.
(h) It follows immediately from (f) and (g) together with Proposition 119.10 that the map $i\colon X \to \mathrm{Cyl}(f')$ is homotopy equivalence.
(i) We obtain from (e) and (h) and Lemma 18.11 (1) that f is indeed a homotopy equivalence.

(2) Now suppose we are given $x_0 \in X^0$ and $y_0 \in Y^0$ such that $f(x_0) = y_0$. Since f is cellular on the subcomplex $\{x_0\}$ we obtain from the Cellular Approximation Theorem 38.13 that the map f is homotopic *rel* x_0 to a cellular map f'. Note that we still have $f'(x_0) = y_0$. Using this input it is not difficult to convince oneself that the argument in (1) can be modified to give the desired homotopy rel $\{x_0\}$. We leave it to the punctilious reader to go through the details. ■

119.4. **Application of the Whitehead Theorem: Topological spaces dominated by CW-complexes.** We recall the following definition from page 2077.

Definition. We say that a topological space X is *dominated by a topological space* Y if there exist maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to the identity id_X.

The goal of this section is to prove the following result which we already mentioned in Section 85.5.

Proposition 119.12. *Let X be a topological space. If X is dominated by a* CW*-complex, then X is homotopy equivalent to a* CW*-complex.*

[1670]Some of the arguments below overlap with the proof of Lemma 24.8 (2). The reason for not appealing to Lemma 24.8 (2) is that in statement (2) of the Whitehead Theorem 119.9 we also want to do the case rel a fixed point.

Remark.

(1) An alternative proof for Proposition 119.12, that does not use the Whitehead Theorem 119.9, but that uses a mapping telescope construction similar to the telescope construction from page 1010, is given in [**Hat02**, Proposition A.11].
(2) In Section 85.5 we stated, without proofs but with detailed references, several generalizations of refinements of Proposition 119.12.

The following simply definition will prove convenient in the proof of Proposition 119.12.

Definition. Let $f\colon X \to Y$ be a map between topological spaces. A map $g\colon Y \to X$ is called a *homotopy left inverse of* f if $g \circ f$ is homotopic to id_X. Furthermore, a map $h\colon Y \to X$ is called a *homotopy right inverse of* f if $f \circ h$ is homotopic to id_Y.

In Exercise 18.18 we proved the following elementary lemma.

Lemma 119.13. *Let $f\colon X \to Y$ be a map between topological spaces. If f admits a homotopy left inverse g and a homotopy right inverse h, then f is in fact a homotopy equivalence and both g and h are homotopy inverses.*

Now it is fairly straightforward to provide a proof of Proposition 119.12.

PROOF OF PROPOSITION 119.12. Let X be a topological space that is dominated by a CW-complex Y. Recall that this means that there exist maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to the identity id_X. Without loss of generality we can and will assume that X and Y are path-connected. We pick a base point $y_0 \in Y$ and we write $x_0 := r(x_0)$. Since $r \circ i$ is homotopic to the identity we obtain from Proposition 40.7 (1) that the map $r_* \circ i_* = (r \circ i)_*\colon \pi_j(X, x_0) \to \pi_j(X, x_0)$ is an isomorphism for all $j \in \mathbb{N}_0$. In particular we see that $r_*\colon \pi_j(Y, y_0) \to \pi_j(X, x_0)$ is an epimorphism for all $j \in \mathbb{N}_0$.

By Corollary 119.2 there exists a CW-complex Z which contains Y as a subcomplex and a map $R\colon Z \to X$ that extends r and that is a weak homotopy equivalence. Thus we are now in the following setting:

$$\begin{tikzcd} X \arrow[r, shift left, "i"] & Y \arrow[l, shift left, "r"] \arrow[d, hook, "t"', "\text{subcomplex}"] \\ & Z. \arrow[ul, bend left, "R"', "\text{weak homotopy equivalence}"] \end{tikzcd}$$

Here denote by $t\colon X \to Z$ the inclusion map. (Note though that we do not claim that the diagram commutes.) It remains to prove the following claim.

Claim. The map $t \circ i\colon X \to Z$ is a homotopy equivalence.

Since $R \circ (t \circ i) = (R \circ t) \circ i = r \circ i$ is homotopic to id_X we know that R is homotopy left inverse of $t \circ i$. Furthermore, we obtain from Proposition 40.7 (1) that $R \circ (t \circ i)$ is a weak homotopy equivalence.[1671] Since R is a weak homotopy equivalence we see that $t \circ i$ is also a weak homotopy equivalence. Thus we see that $(t \circ i) \circ R\colon Z \to Z$ is a weak homotopy equivalence. Since Z is a CW-complex it follows from the Whitehead Theorem 119.9 that

[1671] Throughout the proof we use the characterization of weak homotopy equivalences given by Lemma 119.5 (3).

the map $(t\circ i)\circ R$ is a homotopy equivalence. This implies that there exists a map $k\colon Z\to Z$ such that $(t\circ i)\circ R\circ k$ is homotopic to id_Z. We have thus shown that $R\circ k$ is a homotopy right inverse for $t\circ i$. Since we already pointed out above that $t\circ i$ admits is a homotopy left inverse we obtain from Lemma 119.13 that $t\circ i$ is a homotopy equivalence. ■

119.5. **Application of the Whitehead Theorem: Highly connected CW-complexes.** In this section we want to discuss the following natural question.

Question 119.14. *Let X be a* CW*-complex which is n-connected for some $n\in\mathbb{N}_0$. Is X homotopy equivalent to a* CW*-complex Y which has one 0-cell and no cells in dimensions $1,\dots,n$?*

It turns out that we already dealt with the not entirely trivial case, that $n=0$. For the convenience of the reader we recall Proposition 39.11. Its precise statement gives us one possible approach to answering Question 119.14.

Proposition 39.11. *Let X be a 0-connected* CW*-complex.*

(1) *There exists a contractible 1-dimensional subcomplex T which contains all 0-cells.*[a]
(2) *Given any T as in (1) the quotient X/T is a* CW*-complex with precisely one 0-cell. Furthermore the projection $X\to X/T$ is a homotopy equivalence.*

[a]Such a subcomplex is basically what we called a spanning tree on page 229.

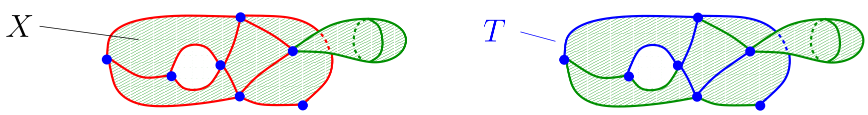


FIGURE 1635. Illustration of Proposition 39.11.

Proposition 39.11 suggests the following approach to answering Question 119.14: Given an n-connected CW-complex X we need to find a contractible subcomplex S that contains the n-skeleton X^n. The catch is, as the following example shows, in general such a subcomplex just does not exist.

Example. Let X be the 2-dimensional CW-complex corresponding to the presentation $\langle x\,|\,x^2=e,x^3=e\rangle$.[1672] This CW-complex is constructed by attaching two 2-cells to S^1 via the attaching maps $z\mapsto z^2$ and $z\mapsto z^3$. We refer to Figure 1636 for an illustration. We have

$$\pi_1(X)\underset{\substack{\uparrow\\ \text{Corollary 37.12}}}{\cong}\langle x\,|\,x^2=e,x^3=e\rangle\underset{\substack{\uparrow\\ \text{since }\langle x\rangle\cong\mathbb{Z}}}{\cong}\mathbb{Z}/\text{subgroup generated by 2 and 3}\underset{\substack{\uparrow\\ \text{since 2 and 3 are coprime}}}{=}0.$$

We have thus shown that X is 1-connected. In Exercise 119.9 we will see that X does not admit a contractible subcomplex S which contains the 1-skeleton of X and which has the property that X/S is homotopy equivalent to X.

In light of the example it comes perhaps as a surprise that Question 119.14 actually does have an affirmative answer:

[1672]Recall that on page 995 we showed that any finite presentation $\langle g_1,\dots,r_k\,|\,r_1,\dots,r_l\rangle$ naturally gives rise to a CW-complex with one 0-cell, k 1-cells and l 2-cells.

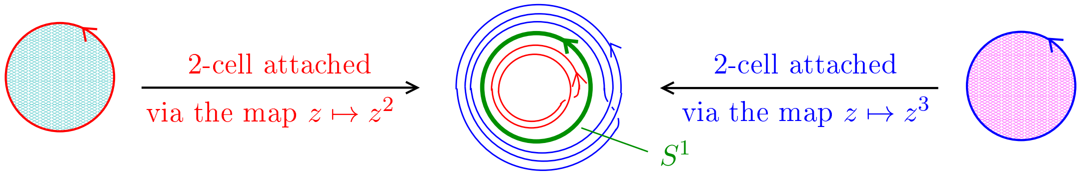


FIGURE 1636

Proposition 119.15. *Let X be a CW-complex, let $x_0 \in X$ and let $n \in \mathbb{N}_0$. If X is n-connected, then there exists a CW-complex Y which has one 0-cell y_0 and no cells in dimensions $1, \dots, n$ such that the pair (X, x_0) is homotopy equivalent to the pair (Y, y_0).*

PROOF. Let X be an n-connected CW-complex and let $x_0 \in X$. By the CW-Approximation Theorem 119.8 (2) there exists a CW-complex Y which has a single 0-cell $\{y_0\}$ and no cells in dimensions $1, \dots, n$ and a map $f\colon (Y, y_0) \to (X, x_0)$ which is a weak homotopy equivalence. Thus we can apply the Whitehead Theorem 119.9 (2) to the map f and we see that f is in fact a homotopy equivalence of pointed topological spaces. Well, that was quick. ■

Remark. The construction of the CW-complex Y in the proof of Proposition 119.15 is quite wasteful. In particular, without any refinement of the construction, it will almost invariably provide an infinite-dimensional CW-complex Y. It turns out that often one can do better. More precisely, in [**Hat02**, Proposition 4.C.1] the following statement is shown: given a finite n-connected CW-complex X there exists a CW-approximation $f\colon Y \to X$ where Y is a finite CW–complex with a single 0-cell and such that Y has no cells in dimensions $1, \dots, n$.

119.6. **Application of the Whitehead Theorem: Elementary collapses.** It is not always easy to determine whether two topological spaces are homotopy equivalent. The following definition is useful for systematizing the search for homotopy equivalences between CW-complexes.

Definition. Let X be a CW-complex. We say that a CW-complex Y is an *elementary expansion of X* if the following holds:

(1) $Y = X \cup e^{n-1} \cup e^n$ where e^{n-1} is an $(n-1)$-cell that is not contained in X and e^n is an n-cell that is not contained in X.
(2) There exists a map $\varphi\colon \overline{B}^n \to Y$ with the following properties:
   (a) The map φ is the characteristic map for the n-cell e^n.
   (b) The map[1673] $\overline{B}^{n-1} \xrightarrow{\cong} S^{n-1}_{\geq 0} \xrightarrow{\varphi} Y$ is the characteristic map for the $(n-1)$-cell e^{n-1}.
   (c) We have $\varphi(S^{n-1}_{\leq 0}) \subset X^{n-1}$.

Conversely we also say that the CW-complex X is an *elementary collapse of Y*. We refer to Figure 1637 for an illustration.

We continue with another definition.

[1673] Here we use the explicit homeomorphism $\overline{B}^{n-1} \xrightarrow{\cong} S^{n-1}_{\geq 0}$ from page 116.

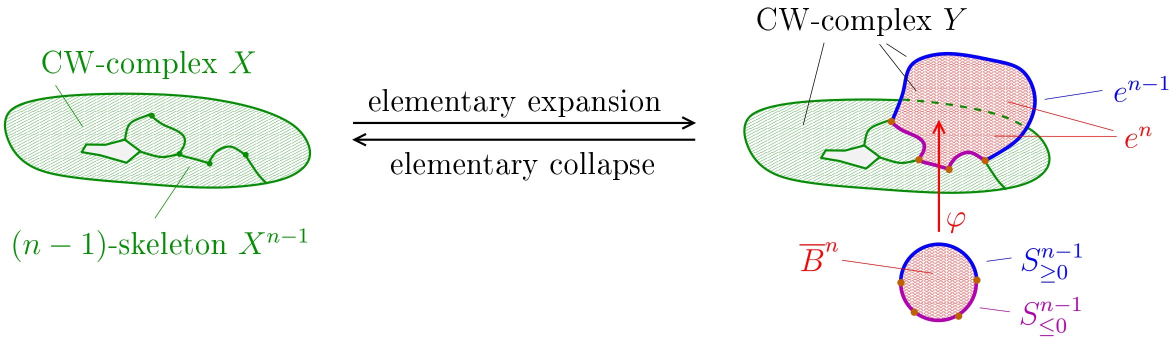


FIGURE 1637

Definition. Let X and Y be CW-complexes. We say X and Y are *simple homotopy equivalent* if there exists a sequence $X = X_0, X_1, \ldots, X_k = Y$ such that for $i = 1, \ldots, k$ one of the following three statements holds:

(1) X_i is an elementary expansion of X_{i-1}, or
(2) X_i is an elementary collapse of X_{i-1}, or
(3) X_i is isomorphic, as a CW-complex, to X_{i-1}.

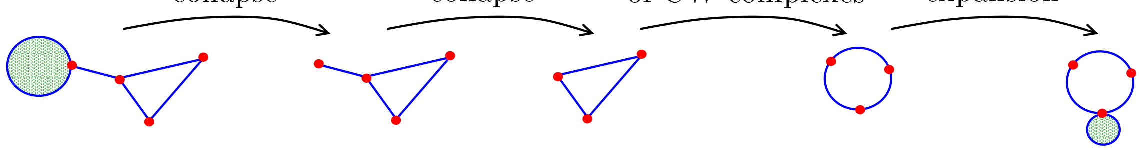


FIGURE 1638

The following lemma justifies the name "simple homotopy equivalent".

Lemma 119.16. *Let X and Y be* CW*-complexes. If X and Y are simple homotopy equivalent, then they are homotopy equivalent.*

PROOF (*). By Lemma 18.11 the notion of being homotopy equivalent is an equivalence relation. Thus it suffices to show that if a CW-complex X is an elementary collapse of a CW-complex Y, then X is homotopy equivalent to Y. In fact, by Lemma 18.14 it suffices to show that X is a deformation retract of Y. We saw on page 549 that there exists a deformation retraction from $\overline{B}^n$ to $S^{n-1}_{\leq 0}$. It follows from Lemma 18.23 that this deformation retraction, together with the constant homotopy on X, defines a deformation retraction from Y to X. ■

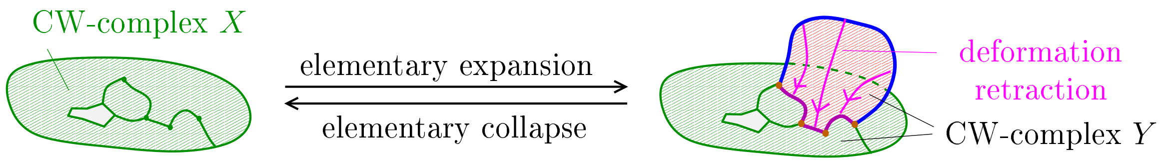


FIGURE 1639. Illustration for the proof of Lemma 119.16.

It is also interesting to consider an interesting special case.

Definition. A finite CW-complex is called *collapsible* if there exists a sequence of elementary collapses that abuts in a CW-complex consisting of a single point.

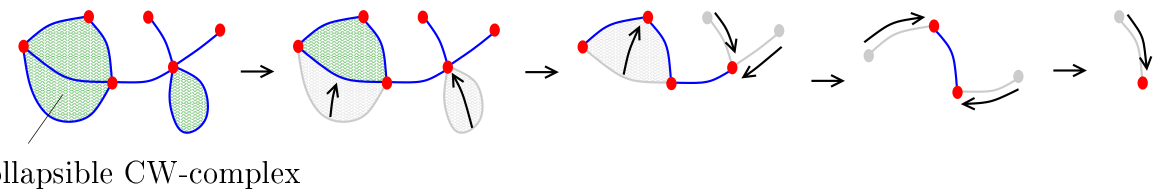


FIGURE 1640

The following lemma is an immediate consequence of Lemma 119.16.

Lemma 119.17. *Let X be a* CW*-complex. If X is collapsible, then X is contractible.*

It is natural to ask whether the converses of Lemmas 119.16 and 119.17 hold. More precisely, we have the following questions.

Question 119.18.

(1) *Let X be a finite* CW*-complex. If X is homotopy equivalent to a point, does it follow that X is collapsible?*
(2) *Let X and Y be finite* CW*-complexes.*
   (a) *If X and Y are homeomorphic, does it follow that X and Y are simple homotopy equivalent?*
   (b) *If X and Y are homotopy equivalent, does it follow that X and Y are simple homotopy equivalent?*

In the remainder of this chapter we will discuss Question 119.18 (1). We will answer Question 119.18 (2) when we discuss Chapman's Theorem **??** and Theorem **??**. To address Question 119.18 (1) it is useful to formulate the following corollary to the Whitehead Theorem 119.9.

Corollary 119.19. *Let X be a* CW*-complex. If X is simply connected and if we have $\mathrm{H}_n(X;\mathbb{Z}) = 0$ for all $n \geq 2$, then X is contractible, i.e. X is homotopy equivalent to a point.*

PROOF. We let $Y = \{*\}$ be the topological space that consists of a single point and let $f\colon X \to Y = \{*\}$ be the only map there is. It follows immediately from Proposition 119.6 that f is a weak homotopy equivalence. Thus the desired statement follows from the Whitehead Theorem 119.9 (1). ■

In the following we will use Corollary 119.19 to give not one, but two examples of CW-complexes that give a negative answer to Question 119.18 (1).

Examples.

(1) We equip S^1 with the usual CW-structure with one 0-cell $1 \in S^1$ and one 1-cell. Let $\gamma = \mathrm{id}\colon S^1 \to S^1$ be the identity map. The *dunce cap* is defined as the CW-complex X that is given by attaching one 2-cell to S^1 where the attaching map $S^1 \to S^1$ is given

by the loop $\gamma * \overline{\gamma} * \gamma$,[1674] i.e. by the loop that is given by the composition of γ, $\overline{\gamma}$ and γ. Using Proposition 37.11 (2) it is straightforward to show that X is simply connected. Furthermore it follows almost from Proposition 48.7 that the cellular homology, and thus by Proposition 48.4 also the singular homology, vanishes in dimensions ≥ 2. It follows from Corollary 119.19 that the dunce cap is contractible.[1675] On the other hand it is clear that the dunce cap is not collapsible since there is only one cell in dimensions 0, 1 and 2 and no two of them can be matched to allow for an elementary collapse.

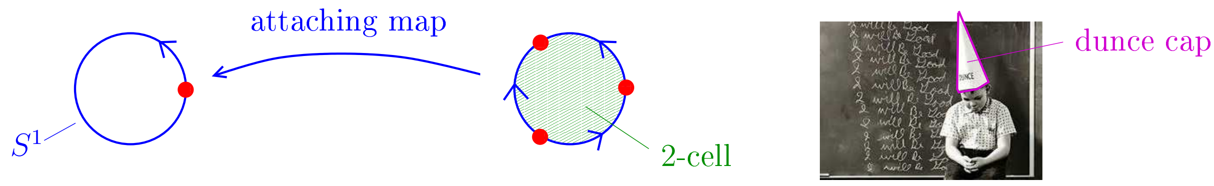


FIGURE 1641

(2) It is also possible to give an example of a CW-complex that can be visualized as a subset of $\mathbb{R}^3$. More precisely, consider the topological space X that is shown in Figure 1642 and in Figure 1643. It is commonly referred to as *Bing's house with two rooms*.[1676] We equip X with the fairly obvious CW-structure. In Exercise 119.11 we will see that X is simply connected and that $\mathrm{H}_n(X;\mathbb{Z}) = 0$ for $n \geq 2$. Thus, as in the previous example we can appeal to Corollary 119.19 to conclude that the house with two rooms is contractible.[1677] We leave the arduous task of showing that X is not collapsible to the reader.

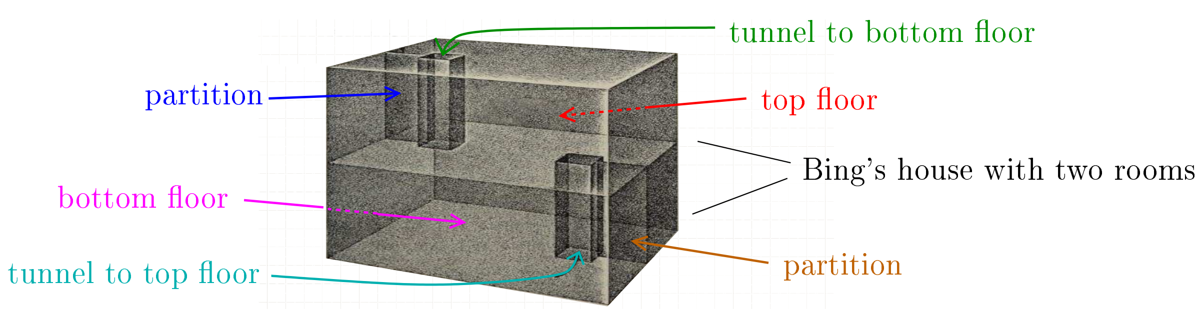


FIGURE 1642

Let X be the dunce cap. In 1963 Christopher Zeeman [**Ze63a**, Theorem 1] showed that $X \times [0,1]$, equipped with the product CW-structure, is in fact collapsible. This result prompted Zeeman in 1963 to propose the following conjecture [**Ze63a**, Conjecture 1].

[1674] The conscientious reader will notice that we had encountered this topological space in Exercise 38.2.

[1675] One could of course also prove this "by hand", but I am pretty sure that few people will enjoy doing this exercise. In fact I have not yet found a book which gives a down-to-earth proof that the dunce cap is contractible.

[1676] This example was given by R. H. Bing [**Bin64**, p. 109] in 1964. In fact the picture we show in Figure 1642 is a slightly modified version of Bing's original drawing.

[1677] A more direct argument for showing that Bing's house with two rooms is contractible is given on [**Hat02**, p. 4].

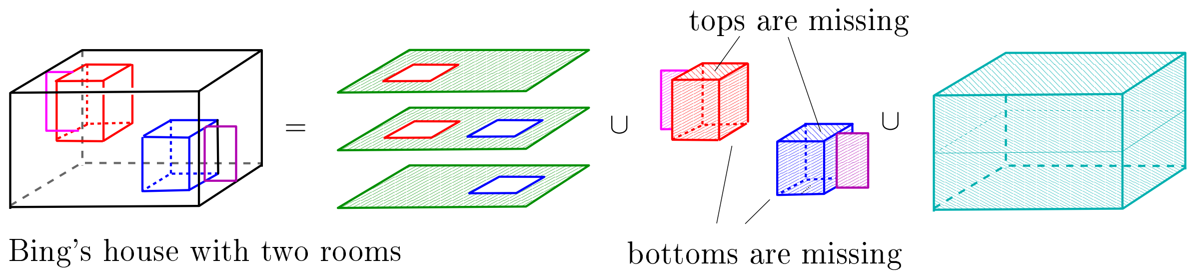


FIGURE 1643

Conjecture 119.20. (Zeeman Conjecture) *If X is a finite contractible 2-dimensional* CW*-complex, then $X \times [0,1]$ is collapsible.*

Like many other conjectures on CW-complexes, this conjecture sounds deceptively simple and elementary. In fact Zeeman was fully aware of the fact that the conjecture is almost certainly very difficult since he showed [**Ze63a**, Theorem 2] that a proof of the Zeeman Conjecture 119.20 would lead to a new proof of the Poincaré Conjecture in dimension three, which was finally proved in 2003 by Perelman using a heavy dose of analysis, see Theorem **??**. Furthermore, Sergey Matveev [**Matv87**] (see also [**Kupe17a**, Proposition 4.2] and [**Barm11**, Chapter 11]) showed that the Andrews-Curtis Conjecture 21.18 is a special case of the Zeeman Conjecture 119.20. We refer to [**MR93, Matv07, Kupe17a**] for more on the Zeeman Conjecture.

Exercises for Chapter 119.

Exercise 119.1. Let $X = \mathbb{R}$. We equip X with the CW-structure where the 0-skeleton is given by $\mathbb{Z}$. The proof of Proposition 119.10 provides a fairly explicit deformation retraction from $X = \mathbb{R}$ to $\{0\}$. What does this deformation retraction look like?

Exercise 119.2. Let X be the universal covering of $S^1 \vee S^1$. On page 2794 we saw that X admits a deformation retraction to $x_0 \in X$. What does an explicit deformation retraction of X to the point x_0 look like? We refer to Figure 1644 for an illustration.

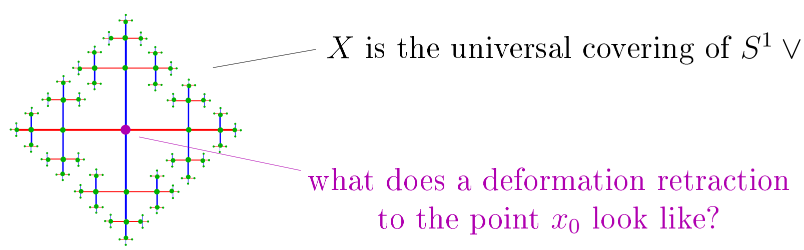


FIGURE 1644. Illustration of Exercise 119.2

Exercise 119.3. Let X be the quasi-circle that we introduced on page 2789,

(a) Show that the image of every map $S^n \to X$ lies in a contractible subset of X.

(b) Let x_0 be a point on X. Show that the inclusion $\{x_0\} \to X$ is not a homotopy equivalence.
Remark. By Lemma 18.12 this statement is equivalent to the statement that the inclusion $\{x_0\} \to X$ is not homotopic to the identity.

Exercise 119.4. We consider the following subset of $\mathbb{R}^2$:

$$\begin{aligned} X \;:=\; \text{line segment from } (0,1) \text{ to } (0,-1) \;&\cup \bigcup_{n\in\mathbb{N}} \text{line segment from } (0,1) \text{ to } (\tfrac{1}{n},0) \\ &\cup \bigcup_{n\in\mathbb{N}} \text{line segment from } (0,-1) \text{ to } (-\tfrac{1}{n},0). \end{aligned}$$

We equip X with the subspace topology coming from $\mathbb{R}^2$. We refer to Figure 1631 for an illustration.

(a) Show that all of the homotopy groups of X are trivial.
(b) Show that X is not contractible.
You might want to make use of Lemma 18.12 and Exercise 18.20.

Exercise 119.5. Let X and Y be two CW-complexes of dimension at most n. Furthermore let $f\colon X \to Y$ be a map that is $(n+1)$-connected. Show that f is a homotopy equivalence.

Exercise 119.6. Let $f\colon X \to Y$ be a map between path-connected CW-complexes. Let $x_0 \in X$. We assume that $f_*\colon \pi_k(X,x_0) \to \pi_k(Y,f(x_0))$ is the trivial map for every $k \in \mathbb{N}$. Does it follow that f is homotopic to a constant map?

Exercise 119.7.

(a) Give an explicit CW-approximation for the Cantor set, as defined on page 100.
(b) Give an explicit CW-approximation for the topologist's sine curve, as defined on page 131.

Exercise 119.8. Let X be a connected CW-complex and let $k \leq l$ be natural numbers. We suppose that $\pi_i(X) = \mathrm{H}_i(X;\mathbb{Z}) = 0$ for $i = k,\dots,l$. Does it follow that X is homotopy equivalent to a CW-complex which has no cells in dimensions $k,\dots,l$?
Remark. If $k = 1$, then this question is precisely Question 119.14.

Exercise 119.9. Let X be the 2-dimensional CW-complex corresponding to the presentation $\langle x \,|\, x^2 = e, x^3 = e\rangle$. We refer to Figure 1636 for an illustration. Show that X does not admit a contractible subcomplex S which contains the 1-skeleton of X and which has the property that X/S is homotopy equivalent to X.
Hint. Use the long exact sequence coming from Theorem 43.3.

Exercise 119.10. Show that the three CW-complexes shown in Figure 1645 are simple homotopy equivalent.
Remark. It is possibly less confusing to first consider Y and Z.

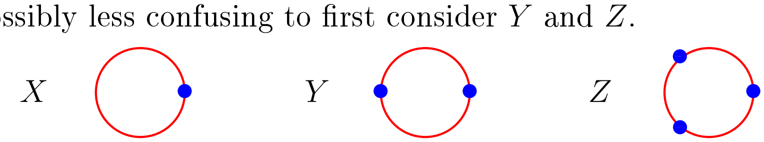


FIGURE 1645. Illustration of Exercise 119.10.

Exercise 119.11. Let X be Bing's house with two rooms, i.e. X is the CW-complex that we defined in Figure 1642.

(a) Show that X is simply connected.
(b) Show that $\mathrm{H}_n(X;\mathbb{Z}) = 0$ for $n \geq 2$.
Hint. Using Proposition 48.5 and using Euler characteristics will cut down the work considerably.
(c) Show that X is not collapsible.

Exercise 119.12. Let $n \in \mathbb{N}$ and furthermore let $f, g\colon S^n \to S^n$ be two maps. Show that if $f_* = g_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$, then f and g are homotopic.

Exercise 119.13.

(1) Let M be a non-empty connected n-dimensional smooth manifold. Furthermore let $\varphi\colon \overline{B}^n \to M$ be a smooth embedding. Show that there exists a homeomorphism $\varphi\colon M/\varphi(S^{n-1}) \to M \vee S^n$ such that the map $M \to M/\varphi(S^{n-1}) \xrightarrow{\varphi} M \vee S^n \to M$ is homotopic to the identity.
(2) We consider the map

$$\Theta\colon S^1 \times S^2 \xrightarrow{\text{projection}} (S^1 \times S^2) \vee S^3 \xrightarrow{\text{as in (1)}} (S^1 \times S^2) \vee S^3 \underset{\substack{\uparrow \\ \text{given by id on } S^1 \times S^2 \text{ and by the Hopf map } S^3 \to S^2 \text{ on } S^3}}{\longrightarrow} S^1 \times S^2.$$

Show the following statements:
(a) Θ induces the identity on all homology groups.
(b) Θ induces the identity on π_1 and π_2.
(c) Θ does not induce the identity on π_3.

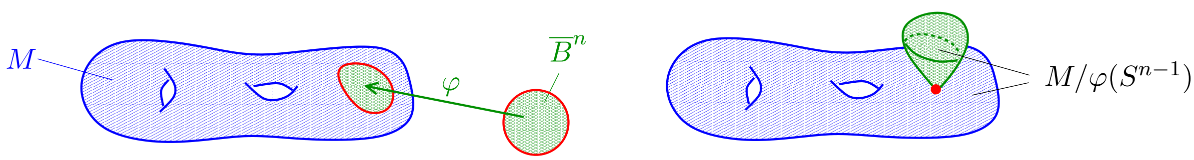


FIGURE 1646. Illustration for Exercise 119.13.

120. Eilenberg-Maclane Spaces

In this chapter we introduce Eilenberg-Maclane spaces. Even though the definition is quite straightforward, these topological spaces will henceforth play an essential role.

120.1. **The definition and basic examples of Eilenberg-Maclane Spaces.**

We recall the following definition from page 1251.

Definition. Let π be an abelian group and let $n \in \mathbb{N}$. We say that a non-empty topological space X is a *Moore space of type* $\mathrm{M}(\pi, n)$ if the following conditions are satisfied:

(1) X admits a CW-structure,
(2) for any $i \neq n$ we have $\widetilde{\mathrm{H}}_i(X;\mathbb{Z}) = 0$,
(3) $\mathrm{H}_n(X) \cong \pi$.
(4) If $n \geq 2$, then we demand that X is simply connected.

For the reader's convenience we recall the main result that we proved regarding Moore spaces.

Proposition 47.11. *Given any $n \in \mathbb{N}$ and given any abelian group π there exists an $(n+1)$-dimensional* CW*-complex that has no cells in dimensions $1, \dots, n-1$ and that is a Moore space of type $\mathrm{M}(\pi, n)$.*

In light of the topics discussed in this chapter it is perhaps helpful to give a quick recap of the proof of Proposition 47.11.

Sketch of proof. Since π is abelian it follows immediately from Lemma 57.16 (1) that there exists a free resolution of π of the following form:

$$0 \;\to\; \mathbb{Z}^{(A)} \;\xrightarrow{\varphi}\; \mathbb{Z}^{(B)} \;\xrightarrow{\rho}\; \pi \;\to\; 0.$$

We let $X := \bigvee_{a \in A} S^n_a$ and we let $Y := \bigvee_{b \in B} S^n_b$. We equip X and Y with the obvious CW-structures and we can make the identifications $\mathrm{H}_n(X;\mathbb{Z}) = \mathbb{Z}^{(A)}$ and $\mathrm{H}_n(Y;\mathbb{Z}) = \mathbb{Z}^{(B)}$. We construct a cellular map $h\colon X \to Y$ such that the induced map $\mathrm{H}_n(X;\mathbb{Z}) \to \mathrm{H}_n(Y;\mathbb{Z})$ agrees with φ under the above identifications. The mapping cone $\mathrm{Cone}(h\colon X \to Y)$ then has the desired properties. ■

The definition of an Eilenberg-Maclane space is almost identical to the definition of a Moore space, we just swap homology groups for homotopy groups:

Definition. Let π be a group and let $n \in \mathbb{N}$. Let X be a topological space.

(1) We say that X is an *Eilenberg-Maclane space of type* $\mathrm{K}(\pi, n)$ if the following conditions hold:
   (a) X admits a CW-structure,
   (b) for any $i \neq n$ we have $\pi_i(X) = 0$,
   (c) $\pi_n(X) \cong \pi$.[1678]
(2) Let $x_0 \in X$. We say (X, x_0) is a *pointed Eilenberg-Maclane space of type* $\mathrm{K}(\pi, n)$ if X is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ and if X admits a CW-structure such that x_0 lies in the 0-skeleton of X.

Remark.

(1) Our definition of an Eilenberg-Maclane space agrees with the definition in [**DaK01**, p. 177]. Conversely in [**WhdG78**, p. 239], [**Spa95**, p. 424], [**Bre93**, p. 488] and [**Hat02**, p. 365] it does not say (at least explicitly) that an Eilenberg-Maclane space needs to admit a CW-structure.
(2) If a topological space X is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$, then often in the literature one just says that X *is a* $\mathrm{K}(\pi, n)$. We will refrain from using this language.

It is perhaps initially not clear why the concept of an Eilenberg-Maclane space is useful, but at the very least it is a fun exercise to go through our long list of topological spaces that we have studied and to determine which ones are Eilenberg-Maclane spaces:

(1) We start with the trivial group $\{e\}$. Let $n \in \mathbb{N}$. In this case it is easy to find Eilenberg-Maclane spaces of type $\mathrm{K}(\{e\}, n)$. In fact it follows from Proposition 40.7 (2) that any contractible CW-complex is an Eilenberg-Maclane space of type $\mathrm{K}(\{e\}, n)$. Thus we see in particular that for any $k \in \mathbb{N}_0$ the open ball B^k and the closed ball $\overline{B}^k$ are Eilenberg-Maclane spaces of type $\mathrm{K}(\{e\}, n)$.
(2) Let $k \in \mathbb{N}$. It follows from Corollary 16.18 and the discussion on page 1068 that the k-dimensional torus $(S^1)^k$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}^k, 1)$.
(3) Let $g \in \mathbb{N}_{\geq 2}$ and let Σ_g be a surface of genus g. On page 1070 we saw that $\pi_i(\Sigma_g) = 0$ for $i \geq 2$. Thus we see that Σ_g is an Eilenberg-Maclane space of type $\mathrm{K}(\pi_1(\Sigma_g), 1)$.
(4) Let $k \in \mathbb{N}$ and let $F = \langle x_1, \ldots, x_k \rangle$ be the free group on k generators. It follows from the discussion on page 606 and Lemma 40.15 that the wedge $\bigvee_{i=1}^{k} S^1$ of k circles is an Eilenberg-Maclane space of type $\mathrm{K}(F, 1)$.
(5) It follows from Lemma 36.5 (3) together with Proposition 40.13 and Corollary 40.12 that the infinite real projective space $\varinjlim \mathbb{RP}^k =: \mathbb{RP}^\infty \cong S^\infty / x \sim -x$[1679] is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}_2, 1)$.

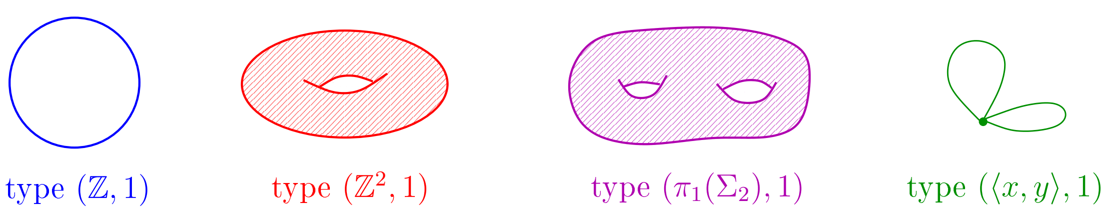


FIGURE 1647

[1678]It is perhaps worth recalling that $\pi_0(X) = 0$ means, by the definition on page 135, that X is non-empty and path-connected. In particular, by Proposition 54.9 (2) the isomorphism type of the homotopy groups $\pi_n(X)$, $n \in \mathbb{N}$, does not depend on the choice of the base point.

[1679]In Lemma 36.5 (4c) we showed that there exists a homeomorphism $\mathbb{RP}^\infty \cong S^\infty / x \sim -x$.

We have now obtained a short but pleasant list of Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, 1)$. It is harder to find examples of Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, n)$ with $n \geq 2$ for a non-trivial group π. It turns out that we had found one interesting example of that type:

(6) Recall that by Lemma 36.6 we know that $\mathbb{C}\mathrm{P}^\infty$ admits a CW-structure. It follows from Theorem 113.12 (2) that the infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$.

Remark.

(1) One might initially hope that perhaps $\mathbb{H}\mathrm{P}^\infty$ could be an Eilenberg-Maclane space of type $\mathrm{K}(\pi, 4)$. First note that by Lemma 60.9 (7) we know that $\mathbb{H}\mathrm{P}^\infty$ is indeed a CW-complex. Furthermore, in in Exercise 118.1 we obtained the following results:
   (a) There exists an isomorphism $\pi_4(\mathbb{H}\mathrm{P}^\infty) \cong \mathbb{Z}$.
   (b) The group $\pi_n(\mathbb{H}\mathrm{P}^\infty)$ is finite for $n \neq 4$, in particular $\pi_n(\mathbb{H}\mathrm{P}^\infty) \otimes \mathbb{Q} = 0$ for $n \neq 4$.
   (c) There exists an isomorphism $\pi_5(\mathbb{H}\mathrm{P}^\infty) \cong \mathbb{Z}_2$.
   So we see that "rationally" $\mathbb{H}\mathrm{P}^\infty$ looks like an Eilenberg-Maclane space, but it is just not an Eilenberg-Maclane space.

(2) In the following chapters we will give many examples of "perfectly reasonable" topological spaces that are Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, 1)$. For $n \geq 2$ it seems like $\mathbb{C}\mathrm{P}^\infty$ is basically[1680] the only Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ for some non-trivial group π which appears "in nature".

Our next goal is to find Eilenberg-Maclane spaces of type $\mathrm{K}(\mathbb{Z}_m, 1)$ for arbitrary m. For $m = 2$ we had just seen that $\mathbb{R}\mathrm{P}^\infty$ is such an Eilenberg-Maclane space. We recall that in Lemma 36.5 (3) we saw that we can make the identification $\mathbb{R}\mathrm{P}^\infty = S^\infty/\{\pm 1\}$ where the group $\{\pm 1\} \cong \mathbb{Z}_2$ acts in the obvious way on S^∞. The idea is to generalize this example. More precisely we want to find appropriate actions of any finite cyclic group on S^∞.

At this point it is convenient to recall that given coprime $p, q \in \mathbb{N}$ we defined on page 502 the lens spaces $L(p, q) = S^3/\mathbb{Z}_p$ as the quotient of S^3 by a suitable action of $\mathbb{Z}_p$ on S^3. If one ponders for a while about the definition of lens spaces one is led to the following lemma.

Lemma 120.1. *Let $m \in \mathbb{N}$ and let $r_1, r_2, \dots \in \mathbb{N}$ be natural numbers that are coprime to m.*

(1) *The map*[1681]

$$\begin{array}{rcl} \mathbb{Z}_m \times S^\infty & \to & S^\infty \\ ([k], \underbrace{(z_1, z_2, z_3, \dots)}_{\in \mathbb{C}^\infty}) & \mapsto & (e^{2\pi i k r_1/m} z_1, e^{2\pi i k r_2/m} z_2, \dots) \end{array}$$

defines a continuous and discrete action of $\mathbb{Z}_m$ on S^∞.

(2) *The quotient space $L(m, r_1, r_2, \dots) := S^\infty/\mathbb{Z}_m$ admits a* CW*-structure.*

(3) *The quotient space $L(m, r_1, r_2, \dots) := S^\infty/\mathbb{Z}_m$ is an Eilenberg-Maclane space of type* $\mathrm{K}(\mathbb{Z}_m, 1)$.

[1680] The slightly hesitant "basically" refers to the fact that out of $\mathbb{C}\mathrm{P}^\infty$ one can build a few other examples of Eilenberg-Maclane spaces, e.g. one can consider $\mathbb{C}\mathrm{P}^\infty \times \overline{B}^k$ or, as we will see shortly, the product $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}^2, 2)$.

On page **??** we saw, using the linking pairing, that lens spaces with isomorphic fundamental groups are not necessarily homotopy equivalent. We can not define a linking pairing on the topological spaces $L(m, r_1, r_2, \dots)$. Thus the following question arises:

Question 120.2. *How can one show that given $m \in \mathbb{N}$ the homotopy type of the topological spaces $L(m, r_1, r_2, \dots)$ depends on the choice of the parameters r_i?*

PROOF.

(1) The proof of this statement is quite similar to the task performed in Exercise 36.4. We leave it to the motivated reader to prove the desired statement. As always it takes a little effort to work rigorously with the topology of $S^\infty = \varinjlim S^n$.

(2) Given $n \in \mathbb{N}$ the action of $\mathbb{Z}_m$ on S^∞ restricts to an action on $S^{2n-1} \subset S^\infty$. In [**Hat02**, p. 144] or alternatively [**Shas14**, p. 196] an explicit CW-structure for $S^{2n-1}/\mathbb{Z}_m$ is constructed. These CW-structures have the property that for $k < l$ the $(2k-1)$-skeleton of $S^{2l-1}/\mathbb{Z}_m$ is precisely the CW-complex $S^{2k-1}/\mathbb{Z}_m$. It follows quite easily from Lemma 25.5 that the CW-structures on the $S^{2n-1}/\mathbb{Z}_m$ give rise to a CW-structure on $S^\infty/\mathbb{Z}_m$.

(3) (a) Let $n \geq 2$. We calculate that

$$\pi_n(L(m, r_1, r_2, \dots)) \cong \pi_n(S^\infty/\mathbb{Z}_m) \underset{\substack{\uparrow \\ \text{follows from from (1), Proposition 16.9 and} \\ \text{Proposition 40.13 and from the fact that } n \geq 2}}{\cong} \pi_n(S^\infty) \underset{\substack{\uparrow \\ \text{Corollary 40.12}}}{=} 0.$$

(b) It follows immediately from (1), Theorem 16.16 and Corollary 40.12 that we have isomorphisms $\pi_1(L(m, r_1, r_2, \dots)) = \pi_1(S^\infty/\mathbb{Z}_m) \cong \mathbb{Z}_m$. ■

The following proposition allows us to construct more Eilenberg-Maclane spaces out of given ones.

Proposition 120.3.

(1) *Let X be an Eilenberg-Maclane space of type $\mathrm{K}(A, n)$ and let Y be an Eilenberg-Maclane space of type $\mathrm{K}(B, n)$.*
   (a) *The CW-complex $X \otimes Y$, that is defined on page 963, is an Eilenberg-Maclane space of type $\mathrm{K}(A \times B, n)$.*
   (b) *If at least one of X or Y has only countably many cells, then the product $X \times Y$ is an Eilenberg-Maclane space of type $\mathrm{K}(A \times B, n)$.*

The remaining two statements only apply to the case $n = 1$.

(2) *Let X be an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$. We pick a base point $x_0 \in X$ and we pick an isomorphism $\varphi\colon \pi_1(X, x_0) \to \pi$. Let Γ be a subgroup of π. We denote by $\widetilde{X}$ the covering space of X corresponding to the subgroup[1682] $\varphi^{-1}(\Gamma) \subset \pi_1(X, x_0)$. Then $\widetilde{X}$ is an Eilenberg-Maclane space of type $\mathrm{K}(\Gamma, n)$.*

[1681] Here we view S^∞ as a subset of $\mathbb{C}^\infty$ via the identification $\mathbb{C}^\infty = \mathbb{R}^\infty$ from page 939.

(3) *Let X be an Eilenberg-Maclane space of type* $\mathrm{K}(A,1)$ *and let Y be an Eilenberg-Maclane space of type* $\mathrm{K}(B,1)$. *We pick* CW*-structures for X and Y. Then any wedge*[1683] $X \vee Y$ *is an Eilenberg-Maclane space of type* $\mathrm{K}(A * B, 1)$.

Remark. The restriction in Proposition 120.3 (1) on the CW-complexes X and Y is a major nuisance. Even though this condition on the number of cells is not an issue for the Eilenberg-Maclane spaces that we have considered above, the "canonical" Eilenberg-Maclane spaces that we will construct shortly do *not* satisfy this condition. Thus one should apply Proposition 120.3 (1) with great caution.

PROOF.

(1) Recall that by construction and by Proposition 36.23 we know $X \otimes Y$ is a CW-complex whose underlying set is $X \times Y$. Let $\mathrm{id}\colon X \otimes Y \to X \times Y$ be the identity map. We pick $x \in X$ and $y \in Y$. Given any $k \in \mathbb{N}_0$ we have the following isomorphisms

$$\pi_k(X \otimes Y, (x,y)) \underset{\substack{\uparrow \\ \text{Lemma 119.7}}}{\xrightarrow{\cong}} \pi_k(X \times Y, (x,y)) \underset{\substack{\uparrow \\ \text{Propositions 16.20 and 40.8}}}{\xrightarrow{\cong}} \pi_k(X,x) \times \pi_k(Y,y) \xrightarrow{\cong} \begin{cases} A \times B, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Since $X \otimes Y$ is a CW-complex we obtain from this discussion that $X \otimes Y$ is an Eilenberg-Maclane space of type $\mathrm{K}(A \times B, n)$.

If at least one of X or Y has only countably many cells, then we know by Proposition 36.23 that $X \otimes Y = X \times Y$, i.e. $X \times Y$ itself is an Eilenberg-Maclane space of type $\mathrm{K}(A \times B, n)$.

(2) First note that we know by Proposition 36.37 that $\widetilde{X}$ is a CW-complex. It now follows immediately from Proposition 40.13 that $\widetilde{X}$ is an Eilenberg-Maclane space of type $\mathrm{K}(\Gamma, n)$.

(3) Let $x \in X^0$ and $y \in Y^0$ be points in the 0-skeleta of X and Y. By Proposition 36.10 (5) we know that the points x and y are good. Thus it follows from Proposition 20.3 that we have isomorphisms $\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y) \cong A * B$. Furthermore we obtain basically immediately from Exercise 53.3 that all higher homotopy groups of $X \vee Y$ vanish. Finally we know by Lemma 36.32 (4) that $X \vee Y$ is a CW-complex. ■

The following lemma summarizes a few basic facts about Eilenberg-Maclane spaces.

Lemma 120.4. *Let $n \in \mathbb{N}$ and let π be a group. If X is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$, then given any abelian group G the following statements hold:*

(1) *We have* $\mathrm{H}_0(X;G) \cong G$ *and* $\mathrm{H}^0(X;G) \cong G$.

(2) *For* $i = 1, \dots, n-1$ *we have* $\mathrm{H}_i(X;G) = 0$ *and* $\mathrm{H}^i(X;G) = 0$.

[1682] Since X is a CW-complex we know from Proposition 36.10 (6) that X is locally simply connected. Therefore it follows from Proposition 29.5 that the covering exists and it follows from Proposition 29.8 that the covering corresponding to the subgroup $\varphi^{-1}(\Gamma) \subset \pi_1(X, x_0)$ is essentially unique.

[1683] Recall our convention from page 970: whenever we take the wedge of CW-complexes, we only use points in the 0-skeleton. The "any" now refers to the fact that, under this restriction, any wedge is in fact an Eilenberg-Maclane space.

(3) *Given any* $x_0 \in X$ *the Hurewicz homomorphism* $\pi_n(X,x_0)_{\mathrm{ab}} \to \mathrm{H}_n(X;\mathbb{Z})$ *is an isomorphism. In particular* $\mathrm{H}_n(X;\mathbb{Z}) \cong \pi_{\mathrm{ab}}$.
(4) *The evaluation map* $\mathrm{H}^n(X;G) \to \mathrm{Hom}(\mathrm{H}_n(X;\mathbb{Z}),G)$ *is an isomorphism.*

PROOF. The statements follow easily from the Hurewicz Theorems 52.5 and 53.5 and from the Universal Coefficient Theorems 57.19 and 75.13. The experienced reader will have no troubles with filling in the details. ■

120.2. **Killing higher homotopy groups.** In the subsequent section we will give a general construction of Eilenberg-Maclane spaces. To streamline the discussion we outsource a few preparations to the present section.

First we point out that the following, fairly self-evident convention will be in effect in the coming chapters.

Convention.

(1) Given $n \in \mathbb{N}_0$ we write $* = (0,\dots,0,1) \in S^n$. We equip S^n with the canonical CW-structure from page 935, i.e. we view S^n as a CW-complex with one 0-cell $\{*\}$ and one n-cell.
(2) We always form any wedge of spheres $\bigvee_{i\in I} S_i^{n_i}$ by identifying the points $* \in S_i^{n_i}$. By Lemma 36.32 (4) we can and will view the wedge $\bigvee_{i\in I} S_i^{n_i}$ as a CW-complex in the obvious way, i.e. we have one 0-cell $\{*\}$ and for each $i \in I$ we have one n_i-cell.
(3) By a slight abuse of notation we sometimes say X is a k-dimensional CW-complex if the highest dimension of a cell is $\le k$.

We start out with the following calculation.

Proposition 120.5. *Let I be a set, let $n \in \mathbb{N}$ and for each $i \in I$ let S_i^n be a copy of S^n. For each $i \in I$ we denote by $\varphi_i\colon S^n \to S_i^n$ the obvious homeomorphism. By a slight abuse of notation we denote by φ_i also the composition of φ_i with the canonical inclusion map* $S_i^n \to \bigvee_{i\in I} S_i^n$.

(1) *For any* $i \in \{1,\dots,n-1\}$ *we have* $\pi_i\Big(\bigvee_{i\in I} S_i^n, *\Big) = 0$.
(2) *If $n=1$, then the homomorphism*[1684]

$$\begin{array}{rcl} \overbrace{\langle \{x_i\}_{i\in I}\rangle}^{\text{free group}} & \to & \pi_1\Big(\bigvee_{i\in I} S_i^1, *\Big) \\ x_i & \mapsto & [\varphi_i] \end{array}$$

is an isomorphism.
(3) *If $n \ge 2$, then the homomorphism*[1685]

$$\begin{array}{rcl} \overbrace{\mathbb{Z}^{(I)} = \bigoplus_{i\in I} \mathbb{Z}\cdot i}^{\text{free abelian group}} & \to & \pi_n\Big(\bigvee_{i\in I} S_i^n, *\Big) \\ i & \mapsto & [\varphi_i] \end{array}$$

is an isomorphism.

PROOF.

(1) As we pointed out above, we can view $\bigvee_{i\in I} S_i^n$ as a CW-complex with one 0-cell and no other cells outside of dimension n. It follows from Proposition 40.9 that all homotopy groups in dimension $1,\dots,n-1$ vanish.
(2) In the case that I is a finite set, then, as we saw in the proof of Lemma 20.4, the statement follows easily from applying Proposition 20.3 altogether $k-1$ times. The general case is proved in Proposition 20.7.
(3) Let $n\in\mathbb{N}_{\geq 2}$. We consider the maps

$$\mathbb{Z}^{(I)} \;=\; \bigoplus_{i\in I}\mathbb{Z}\cdot i \;\underset{\substack{\uparrow\\ \text{given by } i\mapsto [S_i^n]}}{\xrightarrow{\cong}}\; \bigoplus_{i\in I}\mathrm{H}_n(S_i^n;\mathbb{Z}) \;\underset{\substack{\uparrow\\ \text{Proposition 47.9}}}{\xrightarrow{\cong}}\; \mathrm{H}_n\Big(\bigvee_{i\in I} S_i^n;\mathbb{Z}\Big) \;\underset{\substack{\uparrow\\ \text{by the Hurewicz Theorem 53.5}\\ \text{which we can apply by (1)}}}{\xleftarrow{\cong}}\; \pi_n\Big(\bigvee_{i\in I} S_i^n,*\Big).$$

It is straightforward to verify that this composition of isomorphisms is precisely the isomorphism under consideration. ■

We continue with the following two propositions which basically says that for a given CW-complex one can "kill" all homotopy groups beyond a chosen degree.

Proposition 120.6. *Let $k\in\mathbb{N}$, let X be a connected $(k+1)$-dimensional* CW*-complex*[1686] *and let $x_0\in X$ be a base point.*

(1) *There exists a $(k+2)$-dimensional* CW*-complex Y with the following properties:*
  (a) *the $(k+1)$-skeleton of Y agrees with X,*
  (b) *for $i=1,\dots,k$ the inclusion induced map $\pi_i(X,x_0)\to\pi_i(Y,x_0)$ is an isomorphism, and*
  (c) *we have $\pi_{k+1}(Y,x_0)=0$.*
(2) *The* CW*-complex Y in (1) can be chosen canonically.*[1687]
(3) *If X has countably many cells, then Y as in (1) can be chosen to have countably many cells.*[1688]

PROOF.

(1) We pick a set $\{\varphi_j\colon (S^{k+1},*)\to (X,x_0)\}_{j\in J}$ of maps that represent a generating set for the group $\pi_{k+1}(X,x_0)$. We set

$$Y \;=\; \Big(X \sqcup \bigsqcup_{j\in J}\overline{B}_j^{k+2}\Big)/\sim \qquad \text{with } \varphi_j(s)\sim s \text{ for } s\in S_j^{k+1}=\partial\overline{B}_j^{k+2}.$$

[1684] By Lemma 19.14 the given data does indeed determine a unique homomorphism.

[1685] Since $n\geq 2$ we know by Proposition 40.3 that the right-hand side is an abelian group. Thus it follows from Lemma 19.1 that the given data does indeed determine a unique homomorphism.

[1686] We follow the convention established on pages 931 and 2813, i.e. when we say "n-dimensional CW-complex" we allow for the possibility that the highest dimension of a cell is $<n$.

[1687] Here "canonical" is shorthand for saying that in the proof of the proposition we provide an explicit construction of Y does not depend on any choices.

[1688] Note though that we do *not* claim that we can find a canonical Y with countably many cells.

Put differently, Y is the $(k+2)$-dimensional CW-complex that is obtained from X by attaching one $(k+2)$-cell for each $j \in J$ with attaching map given by φ_j. We denote by $i\colon X \to Y$ the inclusion map. Now we verify that Y has all the desired qualities.

(a) By construction Y is a $(k+2)$-dimensional CW-complex such that the $(k+1)$-skeleton equals X.

(b) It follows immediately from Proposition 40.9 and (a) that the inclusion induced map $i_*\colon \pi_s(X, x_0) \to \pi_s(Y, x_0)$ is an isomorphism for $s = 1, \ldots, k$ and that the map $i_*\colon \pi_{k+1}(X, x_0) \to \pi_{k+1}(Y, x_0)$ is an epimorphism.

(c) Let $j \in J$. We have the following simple but useful commutative diagram:

$$\begin{tikzcd} S^{k+1} \arrow[rrr, "\varphi_j"] \arrow[d, hook] & & & X \arrow[d, "i"] \\ \overline{B}^{k+2} \arrow[r, "="] & \overline{B}_j^{k+2} \arrow[r, hook] & X \sqcup \coprod_{j\in J} \overline{B}_j^{k+2} \arrow[r, two heads] & Y. \end{tikzcd}$$

In other words, the map $i \circ \varphi_j\colon S^{k+1} \to Y$ extends to a map $\overline{B}^{k+2} \to Y$. By Lemma 40.4 this implies that $i_*([\varphi_j]) = [i \circ \varphi_j] = 0 \in \pi_{k+1}(Y, x_0)$.

(d) Since $\{[\varphi_j]\}_{j\in J}$ is by choice a generating set for $\pi_{k+1}(X, x_0)$ we obtain from (c) that $i_*\colon \pi_{k+1}(X, x_0) \to \pi_{k+1}(Y, x_0)$ has trivial image.

(e) It follows from (b) and (d) that $\pi_{k+1}(Y, x_0) = 0$.

We have thus verified that Y has all the desired properties.

(2) It is easy to come up with a canonical Y: in the construction provided in (1) we just need to consider the set of all maps $(S^{k+1}, *) \to (X, x_0)$. Evidently this set is canonical and evidently it represents a generating set for $\pi_{k+1}(X, x_0)$. The resulting CW-complex Y is the canonical CW-complex we desired.

(3) Finally suppose that X has countably many cells. It follows from the rather non-trivial Proposition 85.19 that $\pi_{k+1}(X, x_0)$ is countable. Therefore we can pick a countable family $\{\varphi_j\colon (S^{k+1}, *) \to (X, x_0)\}_{j\in J}$ of maps that represent a generating set for the group $\pi_{k+1}(X, x_0)$. The resulting CW-complex Y evidently has only countably many cells. ■

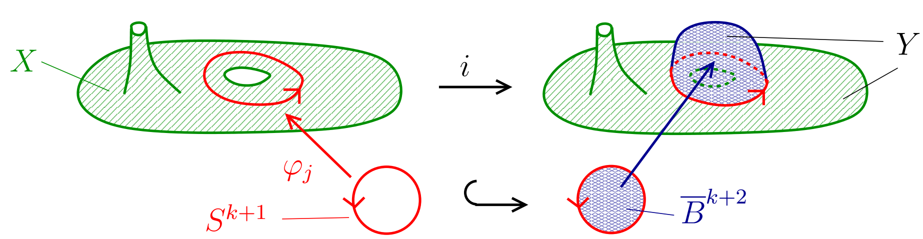


FIGURE 1648. Illustration for the proof of Proposition 120.6.

We conclude this technical section with the following proposition that is mostly just an iterated application of Proposition 120.6.

Proposition 120.7. *Let $k \in \mathbb{N}$, let Y be a connected $(k+1)$-dimensional* CW*-complex and let $y_0 \in Y$.*

(1) *There exists a* CW*-complex Z with the following three properties:*
  (a) *the $(k+1)$-skeleton of Z equals Y,*
  (b) *we have $\pi_i(Z) = 0$ for $i \geq k+1$,*
  (c) *for any $i \in \{1, \dots, k\}$ the inclusion induced map $\pi_i(Y, y_0) \xrightarrow{\cong} \pi_i(Z, y_0)$ is an isomorphism.*
(2) *The* CW*-complex Z in (1) can be chosen canonically.*
(3) *If Y has countably many cells, then Z as in (1) can be chosen to have countably many cells.*

Proof. Let $k \in \mathbb{N}$ and let Y be a connected $(k+1)$-dimensional CW-complex. Let $y_0 \in Y$.

(1) We define $Z_{k+1} = Y$. For $i = k+1, k+2, \dots$ we iteratively apply Proposition 120.6 to Z_i to obtain a sequence of CW-complexes $Z_{k+1}, Z_{k+2}, \dots$ with the following properties:
  (i) each Z_i is an i-dimensional CW-complex,
  (ii) for each $i \geq k+1$ the i-skeleton of Z_{i+1} is given by Z_i,
  (iii) for each $i \geq k+1$ we have $\pi_i(Z_{i+1}, y_0) = 0$.

  We set $Z := \varinjlim Z_i = \bigcup\limits_{i=k+1}^{\infty} Z_i$ and we equip it with the obvious CW-structure such that for each $i \geq k+1$ the i-skeleton of Z is given by Z_i. We claim that Z has the desired three properties:
  (a) It follows from (ii) that the $(k+1)$-skeleton of Z equals Y.
  (b) Let $i \geq k+1$. We have
$$\pi_i(Z, y_0) \underset{\substack{\uparrow \\ \text{Proposition 40.9}}}{\xleftarrow{\cong}} \pi_i((i+1)\text{-skeleton } Z^{i+1} \text{ of } Z, y_0) \underset{\substack{\uparrow \\ \text{since } Z^{i+1} = Z_{i+1}}}{=} \pi_i(Z_{i+1}, y_0) \underset{\substack{\uparrow \\ \text{by (iii)}}}{=} 0.$$
  (c) This statement follows immediately from (a) and Proposition 40.9.
(2) This statement follows immediately from Proposition 120.6 (2).
(3) This statement follows immediately from Proposition 120.6 (3). ■

120.3. **Existence of Eilenberg-Maclane Spaces.** Using the examples from the first section and using Proposition 120.3 we can "by hand" construct Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, 1)$ for many groups π. One can ask for which pairs (π, n) there exist corresponding Eilenberg-Maclane spaces. There is one obvious restriction: since homotopy groups $\pi_n(X)$ with $n \geq 2$ are abelian, see Proposition 40.3, we see that for $n \geq 2$ we can only hope to find an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ if π is abelian. Perhaps somewhat surprisingly this is the only restriction on the existence of Eilenberg-Maclane spaces.

Theorem 120.8. *Let $n \in \mathbb{N}$ and let π be a group. If $n \geq 2$, then we demand that π is abelian. There exists a* canonical *triple $(X, x_0, \varphi\colon \pi_n(X, x_0) \to \pi)$ with the following properties:*

(1) *X is an Eilenberg-Maclane space of type* $\mathrm{K}(\pi, n)$*.*

(2) *X is a* CW*-complex with one 0-cell* $\{x_0\}$ *and with no cells in dimensions* $1, \ldots, n-1$.
(3) $\varphi\colon \pi_n(X, x_0) \to \pi$ *is an isomorphism.*

Definition. We refer to the canonical CW-complex from Theorem 120.8 as *the canonical Eilenberg-Maclane space* $\mathrm{K}(\pi, n)$.

Remark. As we will see shortly, the canonical Eilenberg-Maclane spaces that we construct in the proof of Theorem 120.8 are monstrously large. More precisely, the CW-complex of type $\mathrm{K}(\pi, n)$ has uncountably many cells in each dimension $\geq n+1$. The cautionary remark from page 2812 about taking products of Eilenberg-Maclane spaces thus applies to the canonical Eilenberg-Maclane spaces.

After the discussion of the previous section the approach to producing an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ is presumably pretty clear:

(1) Start out with a CW-complex with one 0-cell, no cells in dimension $1, \ldots, n-1$ and "sufficiently" many n-cells to be able to "generate" the group π.
(2) Attach $(n+1)$-cells to chop down the n-th homotopy group to obtain the desired group π.
(3) Attach cells of dimension $\geq n+2$ to kill all homotopy groups in dimensions $\geq n+1$.

We basically know how to perform (1) and (3), so it remains to do (2) carefully. The combination of steps (1) and (2) is performed in the following proposition.

Proposition 120.9.

(1) *Let π be any group. There exists a canonical triple* $(Y, x_0, \varphi\colon \pi_1(Y, x_0) \to \pi)$ *with the following properties:*
  (a) *Y is a 2-dimensional* CW*-complex with one 0-cell* $\{x_0\}$,
  (b) $\varphi\colon \pi_1(Y, x_0) \to \pi$ *is an isomorphism.*
(2) *Let π be an abelian group π and let $n \in \mathbb{N}_{\geq 2}$. There exists a canonical triple* $(Y, x_0, \varphi\colon \pi_n(Y, x_0) \to \pi)$ *with the following properties:*
  (a) *Y is an $(n+1)$-dimensional* CW*-complex with one 0-cell* $\{x_0\}$ *and no cells in dimensions* $1, \ldots, n-1$,
  (b) $\varphi\colon \pi_n(Y, x_0) \to \pi$ *is an isomorphism.*

Remark. The first result is related to Corollary 37.12 which says that given any *finitely presented* group π there exists a *finite* connected 2-dimensional CW-complex Y with $\pi_1(Y) \cong \pi$.

Before we provide the proof of Proposition 120.9 we convince ourselves of the fact that this proposition, together with Proposition 120.7, does indeed imply Theorem 120.8.

Proof of Theorem 120.8. In the following we discuss the case $n \geq 2$. The proof for $n = 1$ is almost identical. Thus let $n \in \mathbb{N}_{\geq 2}$ and let π be an abelian group. By Proposition 120.9 (2) there exists a canonical triple $(Y, x_0, \varphi\colon \pi_n(Y, x_0) \to \pi)$ with the following properties:

(1) Y is an $(n+1)$-dimensional CW-complex with one 0-cell $\{0\}$ and no cells in dimensions $1, \ldots, n-1$,
(2) $\varphi\colon \pi_n(Y, x_0) \to \pi$ is an isomorphism.

We plug this data into Proposition 120.7 and we obtain a canonical CW-complex Z with the following three properties:

(3) the $(n+1)$-skeleton of Z equals Y (here we use that Y equals its own $(n+1)$-skeleton),
(4) we have $\pi_j(Z) = 0$ for $j \geq n+1$,
(5) the inclusion $i\colon Y \to Z$ induces an isomorphism $\pi_n(Y, x_0) \xrightarrow{\cong} \pi_n(Z, x_0)$.

Note that it follows from (1) and (3) together with Proposition 40.9 that the following holds:

(6) We have $\pi_j(Z, x_0) = 0$ for $j = 1, \dots, n-1$.

It follows immediately from all of the above statements that the canonical triple

$$\big(Z, x_0, \varphi \circ i_*^{-1}\colon \pi_n(Z, x_0) \xleftarrow[\cong]{i_*} \pi_1(Y, x_0) \xrightarrow[\cong]{\varphi} \pi\big)$$

has all the desired properties. ■

Now we turn to the proof of Proposition 120.9. The proof naturally splits into the two cases $n = 1$ and $n \geq 2$.

Proof of Proposition 120.9 (1). Let π be a group. Given $g \in \pi$ we write $S^1_g := S^1 \times \{g\}$, i.e. S^1_g is a copy of S^1. Let $* = (0, 1) \in S^1$ be the usual base point of S^1. We consider the wedge $X := \bigvee_{g \in \pi} S^1_g$ which is given by identifying all the points $(*, g) \in S^1_g$. We denote by $x_0 \in X$ the wedge point. We apply the convention from page 2813 to view X as a CW-complex in the obvious way.

By Proposition 120.5 (2) we have a canonical identification $\langle \{x_g\}_{g \in \pi} \rangle = \pi_1(X, x_0)$. Thus, by Lemma 19.14, there exists a unique homomorphism $\psi\colon \pi_1(X, x_0) = \langle \{x_g\}_{g \in \pi} \rangle \to \pi$ such that $\psi(x_g) = g$ for every $g \in \pi$. This homomorphism is, by construction, an epimorphism. We set $\Gamma = \ker(\psi)$.

Next let $\{\alpha_j\colon (S^1, *) \to (X, x_0)\}_{j \in J}$ be the family of all loops that represent an element in $\Gamma = \ker(\psi\colon \pi_1(X, x_0) \to \pi)$. We set

$$Y \;=\; \Big(X \sqcup \bigsqcup_{j \in J} \overline{B}^2_j\Big)/\sim \qquad \text{with } \alpha_j(s) \sim s \text{ for } s \in S^1_j = \partial \overline{B}^2_j.$$

Put differently, Y is the 2-dimensional CW-complex that is obtained from X by attaching one 2-cell for each $j \in J$ with attaching map given by α_j. By construction Y is a 2-dimensional CW-complex such that the 1-skeleton equals X. We denote by $i\colon X \to Y$ the inclusion map.

Claim. The map $i_*\colon \pi_1(X, x_0) \to \pi_1(Y, x_0)$ descends to a map

$$\varphi\colon \pi_1(X, x_0)/\Gamma \;\to\; \pi_1(Y, x_0)$$

and this map is an isomorphism.

The proof of the claim naturally breaks into several parts.

(1) Let $j \in J$. The same elementary argument as in the proof of Proposition 120.6 shows that the map $i \circ \alpha_j\colon S^1 \to Y$ extends to a map $\overline{B}^2 \to Y$. By Lemma 14.1 this

observation implies that $i_*([\alpha_j]) = [i \circ \alpha_j] = 0 \in \pi_1(X, x_0)$. Since $\{[\alpha_j]\}_{j \in J}$ contains every single element of Γ we see that $i_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ descends to a map

$$\varphi \colon \pi_1(X, x_0)/\Gamma \ \to \ \pi_1(Y, x_0).$$

(2) It follows from $X = Y^1$ and Proposition 40.9 that the map $i_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an epimorphism. This implies that φ is also an epimorphism.

(3) We still need to show that the map $\varphi \colon \pi_1(X, x_0)/\Gamma \to \pi_1(Y, y_0)$ is a monomorphism. At this stage this is equivalent to showing that $\ker(i_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)) = \Gamma$. Thus let $g \in \ker(i_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0))$. We start out with the following claim.

Claim. There exists a finite subset $\{i_1, \ldots, i_m\} \subset I$ such that g lies in the kernel of the inclusion induced map $\pi_1(X, x_0) \to \pi_1\big(\big(X \sqcup \big(\overline{B}^2_{i_1} \cup \cdots \cup \overline{B}^2_{i_m}\big)\big)/\sim, x_0\big)$.

By Lemma 25.9 there exists a compact subset K of Y such that g lies in the kernel of the inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup K, x_0)$. The claim now follows easily from Lemma 36.16 (2). ⊞

Let $I = \{i_1, \ldots, i_m\}$ be such a finite subset. It follows from Proposition 37.11 (2), applied altogether m times, together with Exercise 21.1 that we have the following commutative diagram

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r] \arrow[dr, two heads] & \pi_1\big(\big(X \sqcup \big(\overline{B}^2_{i_1} \cup \cdots \cup \overline{B}^2_{i_m}\big)\big)/\sim, x_0\big) \arrow[d] \\
& \pi_1(X, x_0)/\langle\langle [\alpha_{i_1}], \ldots, [\alpha_{i_m}] \rangle\rangle.
\end{tikzcd}$$

Thus we see that $g \in \langle\langle [\alpha_{i_1}], \ldots, [\alpha_{i_m}] \rangle\rangle$. Since $[\alpha_{i_1}], \ldots, [\alpha_{i_m}]$ lie by definition in Γ we see that $g \in \Gamma$.

We are now basically done. Indeed, the CW-complex Y is 2-dimensional, it has a single 0-cell x_0 and it is canonical. Furthermore the map $\psi \colon \pi_1(X, x_0) \to \pi$ induces an isomorphism $\overline{\psi} \colon \pi_1(X, x_0)/\Gamma \to \pi$. Thus we see that the canonical triple

$$\Big(Y, x_0, \pi_1(Y, x_0) \xleftarrow[\cong]{\varphi} \pi_1(X, x_0)/\Gamma \xrightarrow[\cong]{\overline{\psi}} \pi\Big)$$

has all the desired properties. ■

Now we turn to the proof of Proposition 120.9 (2).

Proof of Proposition 120.9 (2). Let π be an abelian group and let $n \geq 2$. If we do not care about a canonical construction, then we could just use Proposition 47.11 to obtain an $(n+1)$-dimensional CW-complex Y with one 0-cell and no cells in dimensions $1, \ldots, n-1$ and with $\mathrm{H}_n(Y; \mathbb{Z}) \cong \pi$. By the Hurewicz Theorem 53.5 this implies that $\pi_n(Y) \cong \pi$. Unfortunately this construction is not canonical.

Since we are really eager to have canonical constructions we need to provide a different argument. Evidently, in the following discussion, we will try for as long as possible to roll over ideas from the proof of Proposition 120.9 (1).

For each $g \in \pi$ we write $S^n_g := S^n \times \{g\}$. We consider $* = (0, \dots, 0, 1) \in S^n$. As in (1) we consider the wedge $X := \bigvee_{g \in \pi} S^n_g$ and we view X canonically as a CW-complex. We denote by $x_0 \in X$ the wedge point.

By Proposition 120.5 (3) we have a canonical identification $\pi_n(X, *) = \mathbb{Z}^{(\pi)}$. Since π is an abelian group we know from Lemma 19.1 that there exists a unique homomorphism $\psi\colon \pi_n(X,*) = \mathbb{Z}^{(\pi)} \to \pi$ with $\psi(g) = g$ for every $g \in \pi$. This homomorphism is, by construction, an epimorphism. We set $\Gamma = \ker(\psi)$.

Next let $\{\alpha_j\colon (S^n, *) \to (X, x_0)\}_{j \in J}$ be the family of all maps that represent an element in $\Gamma = \ker(\psi\colon \pi_n(X,*) \to \pi)$. As in the proof of (1) we set

$$Y \;=\; \Big(X \sqcup \bigsqcup_{j \in J} \overline{B}^{n+1}_j\Big)/\sim \qquad \text{with } \alpha_j(s) \sim s \text{ for } s \in S^n_j = \partial \overline{B}^n_j.$$

By construction Y is an $(n+1)$-dimensional CW-complex such that the n-skeleton equals X. We denote by $i\colon X \to Y$ the inclusion map.

Claim. The map $i_*\colon \pi_n(X, x_0) \to \pi_n(Y, x_0)$ descends to a map

$$\varphi\colon \pi_n(X, x_0)/\Gamma \;\to\; \pi_n(Y, x_0)$$

and this map is an isomorphism.

We consider the following diagram

$$\begin{tikzcd}
\Gamma \arrow[r, hook] & \pi_n(X,x_0) \arrow[r, "i_*"] \arrow[d, "="] & \pi_n(Y,x_0) \arrow[r] \arrow[d, "="] & 0 \\
\bigoplus_{j\in J} \mathbb{Z}\cdot a_j \arrow[u, two heads, "a_j \mapsto [\alpha_j]"] \arrow[r, "a_j \mapsto [\alpha_j]"] \arrow[d, "="] & \pi_n(X,x_0) \arrow[r, "i_*"] \arrow[d, "\cong"] & \pi_n(Y,x_0) \arrow[r] \arrow[d, "\cong"] & 0 \\
\bigoplus_{j\in J} \mathbb{Z}\cdot a_j \arrow[r, "a_j \mapsto \alpha_{j*}([S^n])"] & \mathrm{H}_n(X;\mathbb{Z}) \arrow[r, "i_*"] & \mathrm{H}_n(Y;\mathbb{Z}) \arrow[r] & 0.
\end{tikzcd}$$

We make the following clarification and observations:

(1) The squares at the top commute basically by definition. Note that the vertical map to the top left is an epimorphism by definition of the family $\{\alpha_j\}_{j \in J}$.
(2) The vertical maps between the second and the third row, which are located in the middle and to the right, are the Hurewicz homomorphisms from page 1324.
(3) It follows from the naturality of the Hurewicz homomorphism, see Lemma 53.1 (3), that the square to the bottom right commutes.
(4) It follows basically immediately from the definition of the Hurewicz homomorphism that the square to the bottom left commutes.
(5) By construction we know that X and Y have precisely one 0-cell and no cells in dimension $1, \dots, n-1$. This implies by Proposition 40.9 that X and Y are $(n-1)$-connected. Now we obtain from the Hurewicz Theorem 53.5 that the vertical maps in the middle and to the right are both isomorphisms.

(6) It follows from a modest generalization of Lemma 46.19 that the bottom horizontal sequence is exact.[1689]

It follows from all the information assembled above that the top horizontal sequence is also exact. But this fact is just a reformulation of the statement that we are supposed to prove in the claim. ⊞

As in the proof of (1) we are now basically done. Indeed, the CW-complex Y is $(n+1)$-dimensional, it has a single 0-cell x_0, it has no cells in dimensions $1, \ldots, n$ and it is canonical. Furthermore the map $\psi\colon \pi_n(X, x_0) \to \pi$ induces an isomorphism $\overline{\psi}\colon \pi_n(X, x_0)/\Gamma \to \pi$. Thus we see that the canonical triple

$$\Big(Y, x_0, \pi_n(Y, x_0) \xleftarrow[\cong]{\varphi} \pi_n(X, x_0)/\Gamma \xrightarrow[\cong]{\overline{\psi}} \pi\Big)$$

has all the desired properties. ■

We have now seen that given any group there exists a CW-complex that is an Eilenberg-Maclane space. But it is pretty clear that the construction provided by the proof of Theorem 120.8 is pretty wasteful, except for the trivial group the construction provides an infinite-dimensional CW-complex. In many cases, e.g. for the groups $\pi = \mathbb{Z}^m$, there are, as we have seen on page 2809, much simpler, finite-dimensional Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, 1)$. This point towards a common tension:

(1) On the one hand it is nice to have “small constructions” with few cells. For example it is much easier to visualize finite CW-complexes and it is usually much easier to calculate invariants for finite CW-complexes.
(2) On the other hand it is nice to have canonical constructions which do not depend on any choices. But such constructions are almost invariably humongous.

This situation is comparable to the discussion of free resolutions of abelian groups. On the one hand, for actual calculations one prefers “small” free resolutions. On the hand, say to give unambiguous definitions of Tor-groups and Ext-groups one needs to work with the canonical free resolution from page 1408 which is outrageously wasteful.

It is natural to ask whether “small” groups also admit “small” Eilenberg-Maclane spaces. For example the following question arises:

Question 120.10. *Let $m \in \mathbb{N}_{\geq 2}$. Can we find a finite-dimensional Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}_m, 1)$?*

In some instances one can also attempt to perform the three-step-process from page 2817 with more care to provide “smaller” Eilenberg-Maclane spaces. In the remainder of this section we want to discuss the construction of the Eilenberg-Maclane spaces $\mathrm{K}(\mathbb{Z}, n)$. For $n = 1$ everything that can be said had been said on page 2809.

We will now construct an explicit Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$ using the above procedure:

(1) We start with $X_2 := \mathbb{C}\mathrm{P}^1 = S^2$. We have $\pi_1(X_2 = \mathbb{C}\mathrm{P}^1) = 0$ and $\pi_2(X_2 = \mathbb{C}\mathrm{P}^1) \cong \mathbb{Z}$.

[1689] Lemma 46.19 was formulated for attaching a single $(n+1)$-cell, but basically the same argument also works for attaching an arbitrary family of $(n+1)$-cells.

(2) By Theorem 113.12 we know that $\pi_3(\mathbb{CP}^1) = [p]$ where $p\colon S^3 \to \mathbb{CP}^1$ denotes the Hopf map from page 2688.
(3) We set $X_4 := \overline{B}^4 \cup_{p\colon S^3 \to \mathbb{CP}^1} \mathbb{CP}^1$.
(4) By Lemma 91.2 (1) we know that X_4 is homeomorphic to $\mathbb{CP}^2$.[1690]
(5) By Theorem 113.12 we know that $\pi_3(\mathbb{CP}^2) = 0$ and[1691] $\pi_4(\mathbb{CP}^2) = 0$. Furthermore it tells us that $\pi_5(\mathbb{CP}^2) = [p]$ where $p\colon S^5 \to \mathbb{CP}^2$ denotes the Hopf map from page 2688.
(6) Similar to (3) we consider $X_6 := \overline{B}^6 \cup_{p\colon S^3 \to \mathbb{CP}^1} \mathbb{CP}^1$.
(7) As in (4), using Footnote 1690, we see that X_6 is homeomorphic to $\mathbb{CP}^3$.

Iterating this process we obtain the sequence $\mathbb{CP}^2, \mathbb{CP}^4, \dots$ and we end up with the infinite complex projective space $\mathbb{CP}^\infty$. This shows, once again, that $\mathbb{CP}^\infty$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$. A similar discussion, shows how to construct an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}_2, 1)$ starting with $X_2 = \mathbb{RP}^2$, by imitating the above approach one once again ends up with $\mathbb{RP}^\infty$.

For $n \geq 3$ there seem to be no straightforward models for $\mathrm{K}(\mathbb{Z}, n)$. But the following lemma gives us at least a little bit of extra control, which will play a crucial role at a later point.

Lemma 120.11. *Given any $n \in \mathbb{N}$ there exists an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, n)$ with precisely one 0-cell, precisely one n-cell, no cell of dimension $n+1$ and only countably many cells in dimensions $\geq n+2$.*

Remark. Let $n \in \mathbb{N}$. Note that the canonical Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, n)$ that we constructed in the proof of Theorem 120.8 is *not* of the form as described in Lemma 120.11. We will discuss this in more detail in Exercise 120.4.

PROOF. Let $n \in \mathbb{N}$. We consider $Y = S^n$. Recall that by Corollary 53.6 we know that $\pi_n(S^n) \cong \mathbb{Z}$. By Proposition 120.7 (3) there exists a CW-complex Z with the following four properties:

(1) the $(n+1)$-skeleton of Z equals $Y = S^n$, in particular Z has only one 0-cell, no cells in dimension $1, \dots, n-1$, one cell in dimension n and no cell in dimension $n+1$,
(2) we have $\pi_i(Z) = 0$ for $i \geq n+1$,

[1690] For the convenience of the reader we repeat the argument. For the convenience of the author we prove a slight generalization of Lemma 91.2 (1). Thus let $n \in \mathbb{N}$. We consider the maps

$$\begin{array}{rcl} f\colon \operatorname{Cone}(S^{2n+1}) & \to & \mathbb{CP}^{n+1} \\ [(z_0, \dots, z_n), r] & \mapsto & [(r \cdot z_0, \dots, r \cdot z_n, 1-r)] \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{CP}^n & \to & \mathbb{CP}^{n+1} \\ [z_0 : \dots : z_n] & \mapsto & [z_0 : \dots : z_n : 0]. \end{array}$$

These maps are continuous by Lemma 3.22 and Lemma 3.21 (3). It follows immediately from the definitions that these two maps define a map

$$\underbrace{\operatorname{Cone}(p\colon S^{2n+1} \to \mathbb{CP}^n)}_{=(\operatorname{Cone}(S^{2n+1}) \cup \mathbb{CP}^n)/\sim} \to \mathbb{CP}^{n+1}$$

which is easily seen to be a bijection. It follows from Lemma 3.22 that this map is continuous. Furthermore, using Proposition 2.43 (3) and Proposition 3.40 one sees that the map is in fact a homeomorphism.

[1691] Here we are lucky, we did everything to kill $\pi_3(\mathbb{CP}^2)$, but without any effort on our side we also managed to dispose of $\pi_4(\mathbb{CP}^2)$.

(3) for any $i \in \{1, \dots, n\}$ and any base point $x_0 \in S^n$ the inclusion induced map $\pi_i(Y, x_0) \xrightarrow{\cong} \pi_i(Z, x_0)$ is an isomorphism,

(4) the CW-complex Z has only countably many cells.

By Proposition 40.10 we know that $\pi_i(Y, x_0) = 0$ for $i = 1, \dots, n-1$. Thus Z is the desired Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, n)$. ■

120.4. **Uniqueness of Eilenberg-Maclane Spaces.** Next we turn to the question to what degree Eilenberg-Maclane spaces of a given type are unique. Evidently the homeomorphism type is not unique, since given an Eilenberg-Maclane space X we obtain Eilenberg-Maclane spaces of the same type by considering say the products $X \times \overline{B}^m$. Nonetheless the following theorem says that the homotopy type of Eilenberg-Maclane spaces of a given type are in fact unique:

Theorem 120.12. *Let $n \in \mathbb{N}$ and let π be an (abelian) group*[1692]. *The following two statements hold:*

(1) *Any two Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, n)$ are homotopy equivalent.*

(2) *If (X, x_0) and (Y, y_0) are two pointed Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, n)$ with $x_0 \in X^0$ and $y_0 \in Y^0$ and if $\varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism, then there exists a homotopy equivalence $f\colon (X, x_0) \to (Y, y_0)$ of pointed topological spaces with $f_* = \varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$.*

Before we turn to the proof of Theorem 120.12 it is worth, and fun, to discuss several immediate consequences thereof. For example, Theorem 120.12 settles Question 120.2 in a perhaps unexpected way: Given $m \in \mathbb{N}$ the homotopy type of the topological spaces $L(m, r_1, r_2, \dots)$ does *not* depend on the choice of the parameters r_i, since all of these topological spaces are Eilenberg-Maclane spaces of type $\mathrm{K}(\mathbb{Z}_m, 1)$, but these are therefore homotopy equivalent by Theorem 120.12.

For convenience we record the following corollary which is an immediate consequence of Theorem 120.12, Corollary 42.8 and the discussion on page 1401.

Corollary 120.13. *Let π be a group and let $n \in \mathbb{N}$. If X and Y are two Eilenberg-Maclane spaces of the type $\mathrm{K}(\pi, n)$, then for any $i \in \mathbb{N}$ and any abelian group G we have an isomorphism $\mathrm{H}_i(X; G) \cong \mathrm{H}_i(Y; G)$.*

Perhaps somewhat surprisingly we can use Corollary 120.13 to give a negative answer to Question 120.10 for $m = 2$.

Corollary 120.14. *There is no finite-dimensional Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}_2, 1)$.*

Proof. Let K be an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}_2, 1)$. Given any $i \in \mathbb{N}$ we calculate that

$$\mathrm{H}_i(K; \mathbb{Z}_2) \underset{\substack{\uparrow \\ \text{by Corollary 120.13}}}{\cong} \mathrm{H}_i(\text{any Eilenberg-Maclane space of type } \mathrm{K}(\mathbb{Z}_2, 1); \mathbb{Z}_2) \underset{\substack{\uparrow \\ \text{see page 2809}}}{\cong} \mathrm{H}_i(\mathbb{RP}^\infty; \mathbb{Z}_2) \underset{\substack{\uparrow \\ \text{see page 1403}}}{\cong} \mathbb{Z}_2.$$

[1692]Whenever we deal with Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, n)$ with $n \geq 2$ it is understood throughout these lecture notes that π is abelian.

It follows from the discussion on page 1402 that K is of dimension $> i$ for all $i \in \mathbb{N}$, in other words, K is infinite-dimensional. ■

Evidently this suggests that the answer to Question 120.10 is negative. But right now we lack the tools to calculate the homology groups of the quotients $S^\infty/\mathbb{Z}_m$ that we constructed in Lemma 120.1. We will return to this question on page **??**

The following proposition basically says that it is easy to construct maps from CW-complexes to Eilenberg-Maclane spaces. This statement is the key to proving Theorem 120.12. As we will see shortly, the proposition is of interest in its own right.

Proposition 120.15. *Let $n \in \mathbb{N}$ and let π be an (abelian) group. Furthermore let Y be an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ and let $y_0 \in Y$. Finally let X be a* CW*-complex and let $x_0 \in X$ be a point in the 0-skeleton X^0. If X is $(n-1)$-connected, then the following two statements hold:*

(1) *Let $\varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ be a homomorphism. There exists a map $f\colon X \to Y$ with $f(x_0) = y_0$ such that $f_* = \varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$.*
(2) *Suppose we are given two maps $f_0, f_1\colon X \to Y$ such that $f_0(x_0) = f_1(x_0) = y_0$. If $f_{0*} = f_{1*}\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$, then there exists a homotopy rel x_0 from f_0 to f_1.*

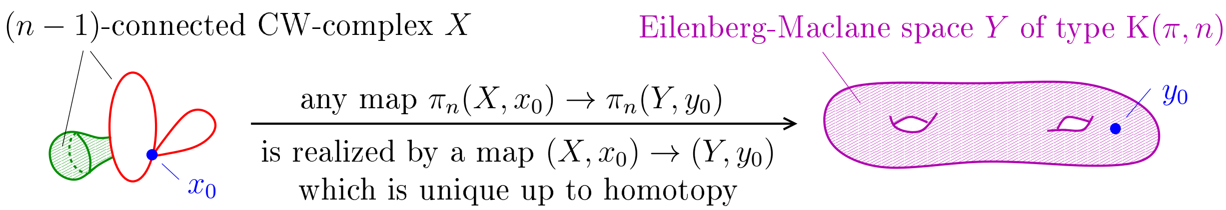


FIGURE 1649. Illustration for Proposition 120.15.

Remark. In Proposition 79.10, with $n = 1$ and $Y = S^1$, we proved a "smooth manifold version" of Proposition 120.15.

PROOF OF THEOREM 120.12 ASSUMING PROPOSITION 120.15. Let $n \in \mathbb{N}$ and let π be an (abelian) group. Statement (1) is basically an immediate consequence of Statement (2). Therefore it now remains to prove Statement (2). Thus let (X, x_0) and (Y, y_0) be two pointed Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, n)$ where $x_0 \in X^0$ and $y_0 \in Y^0$. Furthermore let $\varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ be an isomorphism. Since X is, as an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$, $(n-1)$-connected, we can appeal to Proposition 120.15 (1) to obtain a map $f\colon X \to Y$ with $f(x_0) = y_0$ such that $f_* = \varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$. It suffices to prove the following claim.

Claim. The map f is a homotopy equivalence $(X, x_0) \to (Y, y_0)$ of pointed topological spaces.

As above we obtain from Proposition 120.15 (1) the existence of a map $g\colon Y \to X$ with $g(y_0) = x_0$ such that $g_* = \varphi^{-1}\colon \pi_n(Y, y_0) \to \pi_n(X, x_0)$. Note that

$$(g \circ f)_* \;=\; \varphi^{-1} \circ \varphi \;=\; \mathrm{id}_{\pi_n(X, x_0)} \;=\; (\mathrm{id}_X)_*$$

and note that $(g \circ f)(x_0) = x_0 = (\mathrm{id}_X)(x_0)$. Thus it follows from Proposition 120.15 (2) that $g \circ f$ is homotopic rel x_0 to id_X. Basically the same argument shows that $f \circ g$ is homotopic rel y_0 to id_Y. But this means that g is indeed the desired homotopy inverse. ■

At this point it remains to prove Proposition 120.15. The proof of the proposition relies on the following two lemmas. The following lemma is the key ingredient in the proof of Proposition 120.15.

Lemma 120.16. *Let $n \in \mathbb{N}$. Let X be a* CW*-complex that consists of a single 0-cell x_0 and that otherwise contains only cells in dimensions n and $n+1$. Furthermore let (Y, y_0) be a pointed topological space and let $\gamma\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ be a homomorphism. There exists a map $h\colon (X, x_0) \to (Y, y_0)$ such that $h_* = \gamma$.*

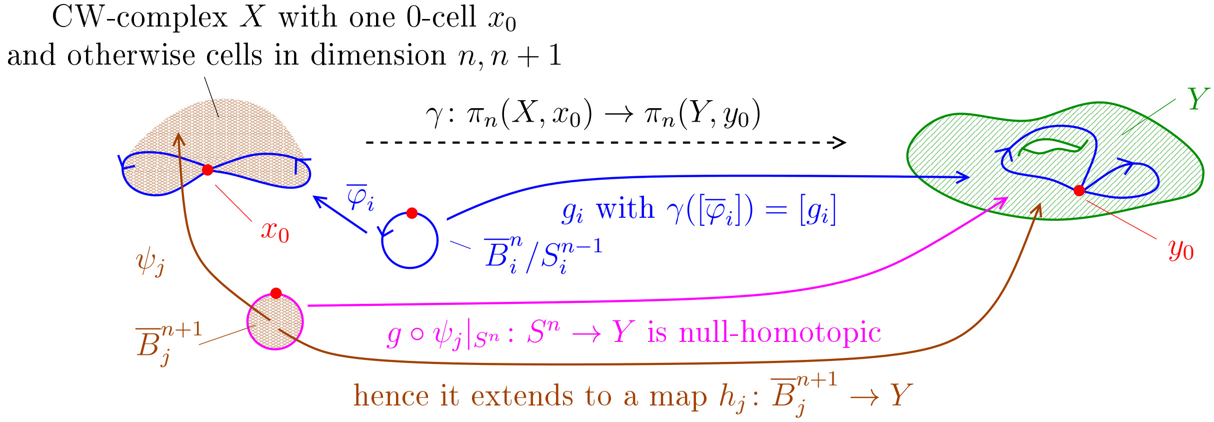


FIGURE 1650. Illustration for the proof of Lemma 120.16.

PROOF. We proceed in the following steps.

(1) We note that by our hypothesis we have $X^{n-1} = \{x_0\}$.
(2) Let $\{\varphi_i\colon (\overline{B}^n, S^{n-1}) \to (X^n, X^{n-1} = \{x_0\})\}_{i\in I}$ be the characteristic maps of the n-cells of the CW-complex X. Note that since $X^{n-1} = \{x_0\}$ each map φ_i descends to a map $\overline{\varphi}_i\colon \overline{B}^n/S^{n-1} \to X^n$.
(3) By Lemma 36.33 (2) the maps $\overline{\varphi}_i$ induce a homeomorphism $\Phi\colon \bigvee_{i\in I} \overline{B}^n_i/S^{n-1}_i \to X^n$. By a slight abuse of notation we denote by x_0 the wedge point of the wedge.
(4) Let $i \in I$. Note that the map $\overline{\varphi}_i$ defines an element in $\pi_n(X, x_0)$. We pick a map $g_i\colon \overline{B}^n/S^{n-1} \to (Y, y_0)$ with $\gamma([\overline{\varphi}_i]) = [g_i]$.
(5) We consider the map

$$\begin{array}{rcll} g\colon \bigvee_{i\in I} \overline{B}^n_i/S^{n-1}_i & \to & Y & \\ [x] & \mapsto & g_i([x]) & \text{if } x \in \overline{B}^n_i. \end{array}$$

(6) We consider the following diagram:

$$\begin{tikzcd}
\pi_n(X^n, x_0) \arrow[dr, "i_*"'] & & \pi_n\Big(\bigvee\limits_{i\in I} \overline{B}_i^n / S_i^{n-1}, x_0\Big) \arrow[ll, "\Phi_*"', "\cong"] \arrow[rr, "g_*"] & & \pi_n(Y, y_0). \\
& & \pi_n(X, x_0) \arrow[urr, "\gamma"'] & &
\end{tikzcd}$$

where the left diagonal map is induced by the inclusion $i\colon X^n \to X$.

(7) It follows immediately from the construction of g that if we send any of the classes $[\mathrm{id}\colon \overline{B}^n \to \overline{B}_i] \in \pi_n(X^n, x)$ on a journey through the diagram, then we obtain the same result. But by Proposition 120.5 (3) the classes $[\mathrm{id}\colon \overline{B}^n \to \overline{B}_i]$, $i \in I$, form a generating set for $\pi_n\Big(\bigvee\limits_{i\in I} \overline{B}_i^n / S_i^{n-1}, x_0\Big)$. Thus it follows that the diagram commutes.

(8) Let $* = (0, \dots, 0, 1) \in S^n \subset \overline{B}^{n+1}$ the usual base point. Let $\{\psi_j\colon \overline{B}^{n+1} \to X^{n+1}\}_{j\in J}$ be the characteristic maps of the $(n+1)$-cells of X. It follows from Lemma 3.32 that we can arrange that for every $j \in J$ we have $\psi_j(*) = x_0$

(9) Let $j \in J$. Note that

$$[g\circ\Phi^{-1}\circ\psi_j|_{S^n}] \;=\; (g_*\circ\Phi_*^{-1})(\overbrace{[\psi_j|_{S^n}]}^{\in\pi_n(X^n,x_0)}) \;\underset{\uparrow}{=}\; \gamma(\overbrace{[\psi_j|_{S^n}]}^{\in\pi_n(X,x_0)}) \;\underset{\uparrow}{=}\; \gamma([0]) \;\overset{\substack{\text{since } \gamma \text{ is a homomorphism}\\ \downarrow}}{=}\; 0 \in \pi_n(Y, y_0).$$

by the above commutative diagram   since $\psi_j|_{S^n}$ extends to a map $\overline{B}^{n+1} \to X$ we have $[\psi_j|_{S^n}] = 0 \in \pi_n(X, x_0)$ by Lemma 40.4

Thus we obtain from Lemma 40.4 that there exists a map $h_j\colon \overline{B}^{n+1} \to Y$ such that $h_j|_{S^n} = g \circ \Phi^{-1} \circ (\psi_j|_{S^n})$.

(10) Since for each $j \in J$ we have $h_j|_{S^n} = g \circ \Phi^{-1} \circ (\psi_j|_{S^n})$ we can use the maps h_j to extend $g \circ \Phi^{-1}$ to a map

$$h\colon X = \Big(X^n \cup \coprod_{j\in J} \overline{B}_j^{n+1}\Big)/\sim \;\to\; Y \qquad \text{where } \psi_j(x) \sim x \text{ for any } x \in S_j^n.$$

(11) We obtain the following diagram

$$\begin{tikzcd}
\pi_n(X^n, x_0) \arrow[rr, "(g\circ\Phi^{-1})_*"] \arrow[dr, "i_*"'] & & \pi_n(Y, y_0) \\
& \pi_n(X, x_0) \arrow[ur, "\gamma"] \arrow[ur, bend right, "h_*"'] &
\end{tikzcd}$$

(12) We already saw that the diagram commutes for γ. Since $h|_{X^n} = g \circ \Phi^{-1}$ we see that the diagram commutes also for the map h_*.

(13) The left diagonal map $i_*\colon \pi_n(X^n, x_0) \to \pi_n(X, x_0)$ is an epimorphism by Proposition 40.9.

(14) It follows from (12) and (13) that the maps γ and h_* from $\pi_n(X, x_0)$ to $\pi_n(Y, y_0)$ agree. ■

The second lemma is just a convenient special case of Lemma 119.3.

Lemma 120.17. *Let X be a* CW*-complex, let $n \in \mathbb{N}$ and let $f\colon A \to Y$ be a map from a subcomplex $A \subset X$ to a path-connected topological space with $X^n \subset A$.*

(1) *If $\pi_n(Y) = 0$, then we can extend f to a map $f\colon A \cup X^{n+1} \to Y$.*
(2) *If $\pi_i(Y) = 0$ for every $i \geq n$, then we can extend f to a map $f\colon X \to Y$.*

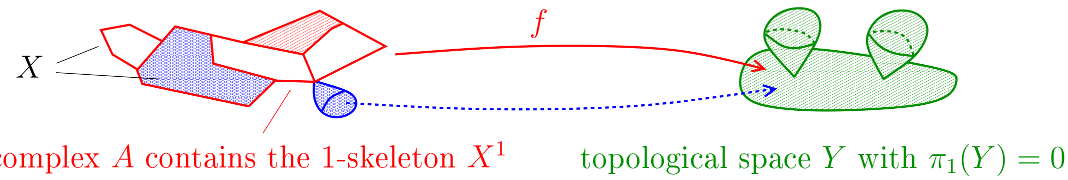


FIGURE 1651. Illustration of Lemma 120.17.

We break the proof of Proposition 120.15 into two parts. In the first part we deal with the situation that the $(n-1)$-skeleton consists just of a point. Afterwards we will discuss how the general case can be reduced to this special case.

PROOF OF PROPOSITION 120.15 IF $X^{n-1} = \{x_0\}$. Let $n \in \mathbb{N}$ and let π be an (abelian) group. Furthermore let Y be an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ and let $y_0 \in Y$. Finally let X be a CW-complex and let $n \in \mathbb{N}$. We assume that $X^{n-1} = \{x_0\}$.

(1) Let $\varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ be a homomorphism. We need to show that there exists a map $f\colon X \to Y$ with $f(x_0) = y_0$ such that $f_* = \varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$. We denote by $i\colon X^{n+1} \to X$ the inclusion map. By Proposition 40.9 we know that $i_*\colon \pi_n(X^{n+1}, x_0) \to \pi_n(X, x_0)$ is an isomorphism. The following diagram contains all the maps of interest to us:

$$\begin{tikzcd}
\pi_n(X^{n+1}, x_0) \arrow[d, "i_*"', "\cong"] \arrow[drr, dashed, bend left, "h_*"] & & \\
\pi_n(X, x_0) \arrow[rr, "\varphi"'] \arrow[rr, dashed, bend left, "f_*"] & & \pi_n(Y, y_0).
\end{tikzcd}$$

Since $X^{n-1} = \{x_0\}$ we can apply Lemma 120.16 to the CW-complex X^{n+1}. We obtain from Lemma 120.16 that there exists a map $h\colon X^{n+1} \to Y$ with $h(x_0) = y_0$ such that $h_* = \varphi \circ i_*\colon \pi_n(X^{n+1}, x_0) \to \pi_n(Y, y_0)$. Since Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ we know that $\pi_j(Y) = 0$ for $j \geq n+1$. Thus we obtain from Lemma 120.17 (2) that we can extend $h\colon X^{n+1} \to Y$ to a map $f\colon X \to Y$. It remains to show that $f_* = \varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$. This is indeed the case since

$$f_* \underset{\substack{\uparrow \\ \text{since } i_* \text{ is an isomorphism}}}{=} f_* \circ i_* \circ i_*^{-1} = (f \circ i)_* \circ i_*^{-1} \underset{\substack{\uparrow \\ \text{since } h \text{ extends } f}}{=} h_* \circ i_*^{-1} \underset{\substack{\uparrow \\ \text{since } h_* = \varphi \circ i_*}}{=} (\varphi \circ i_*) \circ i_*^{-1} = \varphi.$$

(2) Suppose we are given two maps $f_0, f_1\colon X \to Y$ such that $f_0(x_0) = f_1(x_0) = y_0$ and such that $f_{0*} = f_{1*}\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$. We need to show that there exists a homotopy rel x_0 from f_0 to f_1. We consider the product $X \times [0, 1]$ and we equip it with the product CW-structure coming from Proposition 36.23. Given $k \in \mathbb{N}_0$ we

write

$$Z^k = (X \times \{0\}) \cup \underbrace{(X^{k-1} \times [0,1])}_{k\text{-dimensional}} \cup (X \times \{1\}) \subset X \times [0,1].$$

The idea is define the homotopy on $X \times [0,1] = \bigcup\limits_{n \in \mathbb{N}_0} Z^n$ by breaking the task into three steps.

(a) First we note that the fact that $X^0 = \cdots = X^{n-1}$ implies that $Z^1 = \cdots = Z^n$. Now let $h\colon Z^1 = Z^n \to Y$ be the map that is given by f_0 on $X \times \{0\}$, f_1 on $X \times \{1\}$ and the constant map c_{x_0} on $\{x_0\} \times [0,1]$.

(b) We denote by $\{\varphi_i\colon \overline{B}_i^n \to X\}_{i \in I}$ the characteristic maps of the n-cells of X. Since $X^{n-1} = \{x_0\}$ we see that each φ_i descends to a map $\psi_i\colon \overline{B}_i^n / S_i^{n-1} \to X$. We denote by $* = [S_i^{n-1}]$ the obvious base point of the sphere $\overline{B}_i^n / S_i^{n-1}$. Now recall that by hypothesis we have $f_{0*} = f_{1*}\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$. This implies that for every $i \in I$ we have $[f_0 \circ \psi_i] = [f_1 \circ \psi_i] \in \pi_n(Y, y_0)$. In other words, for every $i \in I$ there exists a map $F_i\colon \overline{B}_i^n / S_i^{n-1} \times [0,1] \to Y$ with $F_{i0} = f_0 \circ \psi_i$, $F_{i1} = f_1 \circ \psi_i$ and $F_i(*, t) = y_0$ for all $t \in [0,1]$. We define

$$\begin{aligned} h_{n+1}\colon Z^{n+1} &\to Y \\ P &\mapsto \begin{cases} h_0(P), & \text{if } P \in Z^1 = Z^n, \\ F_i(Q,t), & \text{if } P = (\varphi_i(Q), t) \text{ with } Q \in \overline{B}_i^n. \end{cases} \end{aligned}$$

It follows immediately from the properties of the F_i that this map is well-defined. Furthermore, it follows almost immediately from Lemma 36.7 (4) that h_{n+1} is continuous.

(c) Finally we are in the following situation:

$$\begin{tikzcd} Z^{n+1} \arrow[r, "h_{n+1}"] \arrow[d, hook] & Y \\ X \times [0,1]. \arrow[ur, dashed, "H"'] & \end{tikzcd}$$

By definition of the CW-structure on $X \times [0,1]$ we know that the $(n+1)$-skeleton of $X \times [0,1]$ is contained in Z^{n+1}. In other words, any cell of $X \times [0,1]$ that is not contained in Z^{n+1} is of dimension $\geq n+2$. Since Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$ we know that $\pi_k(Y, y_0) = 0$ for $k \geq n+1$. Thus we obtain from Lemma 120.17 (2) that there exists a map $H\colon X \times [0,1] \to Y$ which makes the above diagram commute. Evidently this is the desired homotopy between f_0 and f_1. ■

Proof of Proposition 120.15 for the general case. Let X be a CW-complex and let $x_0 \in X$ be a point in the 0-skeleton X^0. We suppose that X is $(n-1)$-connected. By Proposition 119.15 we know that there exists a homotopy equivalence $f\colon (X, x_0) \to (\widetilde{X}, \widetilde{x}_0)$ where $\widetilde{X}$ is a CW-complex that has one 0-cell $\{\widetilde{x}_0\}$ and no cells in dimensions $1, \ldots, n$. By Proposition 40.7 we know that $f_*\colon \pi_n(X, x_0) \to \pi_n(\widetilde{X}, \widetilde{x}_0)$ is an isomorphism. By the

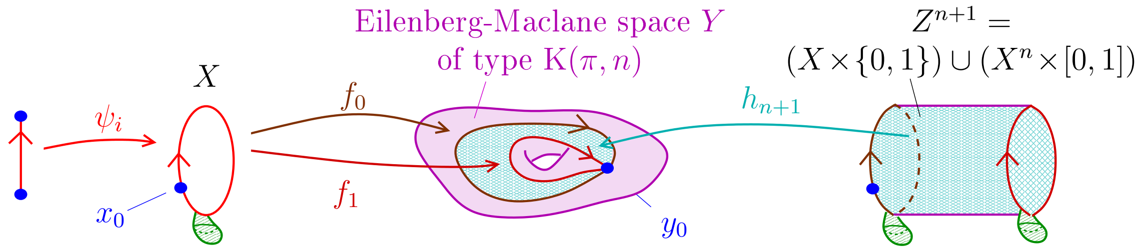


FIGURE 1652. Illustration for the proof of Proposition 120.15 (2).

above argument we know that the desired conclusions hold for $(\widetilde{X}, \widetilde{x}_0)$. It is now fairly elementary to see that the statements of Proposition 120.15 also hold for (X, x_0). ■

Before we move on we prove an important addendum to Proposition 18.16. To motivate its relevance we recall an earlier result: If $f_0, f_1\colon (X, x_0) \to (Y, y_0)$ are two maps between pointed topological spaces, then by Proposition 18.16 (1) the following two statements hold:

(1) If f_0 and f_1 are homotopic rel x_0, then $f_{0*} = f_{1*}$.
(2) If f_0 and f_1 are homotopic (with no constraints on the base point), then there exists an $\alpha \in \pi_1(Y, y_0)$ such that $f_{0*}(z) = \alpha \cdot f_{1*}(z) \cdot \alpha^{-1}$ for all $z \in \pi_1(Y, y_0)$.

One can view Proposition 120.15 (2) as a partial converse to (1). The following proposition is the analogous partial converse to (2).

Proposition 120.18. (∗) *Let π be a group, let Y be an Eilenberg-Maclane space of type* $K(\pi, 1)$ *and let $y_0 \in Y$. Furthermore let X be a 0-connected* CW*-complex and let $x_0 \in X$ be a point in the 0-skeleton X^0. Suppose we are given two maps $f_0, f_1\colon X \to Y$ such that $f_0(x_0) = f_1(x_0) = y_0$. If there exists an $\alpha \in \pi_1(Y, y_0)$ such that $f_{0*}(z) = \alpha \cdot f_{1*}(z) \cdot \alpha^{-1}$ for all $z \in \pi_1(Y, y_0)$, then there exists a homotopy*[1693] *from f_0 to f_1.*

PROOF (∗). The proof of the proposition is similar to the proof of Proposition 120.15 (2). In the following provide a sketch of the argument. We leave it to the reader to fill in the details. First of all we point out that by Proposition 119.15 we can assume that X has a single 0-cell $\{x_0\}$. Now we consider the product $X \times [0, 1]$ and we equip it with the product CW-structure coming from Proposition 36.23. Given $k \in \mathbb{N}_0$ we write

$$Z^k \;=\; (X \times \{0\}) \cup \underbrace{(X^{k-1} \times [0,1])}_{k\text{-dimensional}} \cup (X \times \{1\}) \;\subset\; X \times [0,1].$$

As above we define the homotopy on $X \times [0,1] = \bigcup\limits_{n \in \mathbb{N}_0} Z^n$ by breaking the task into three friendly steps.

(1) We pick a map $\beta\colon [0,1] \to Y$ that represents $\alpha \in \pi_1(Y, y_0)$. Let $h\colon Z^1 = Z^n \to Y$ be the map that is given by f_0 on $X \times \{0\}$, f_1 on $X \times \{1\}$ and the map β on $\{x_0\} \times [0,1] = [0,1]$.

[1693] Note that in contrast to Proposition 120.15 (2) it does *not* say "rel x_0".

(2) Given a 1-cell γ of X we denote by $\phi_\gamma\colon S^1 = \partial([0,1]^2) \to Z^1$ the attaching map of the product cell $\gamma \times [0,1]$. We have

$$\begin{aligned} h_*([\phi_\gamma]) &\overset{\substack{\text{by a generous interpretation of the notation}\\ \downarrow}}{=} [h|_{\gamma\times 0}]\cdot[h|_{\{x_0\}\times[0,1]}]\cdot\overline{[h|_{\gamma\times 1}]}\cdot\overline{[h|_{\{x_0\}\times[0,1]}]} \overset{\substack{\text{by definition of } h\\ \downarrow}}{=} [f_{0*}(\gamma)]\cdot[\beta]\cdot[f_{1*}(\gamma)]^{-1}\cdot[\beta]^{-1} \\ &\underset{\substack{\uparrow\\ \text{by hypothesis and choice of } \beta}}{=} \alpha\cdot[f_{1*}(\gamma)]\cdot\alpha^{-1}\cdot\alpha\cdot[f_{1*}(\gamma)]^{-1}\cdot\alpha^{-1} = e \in \pi_1(Y,y_0). \end{aligned}$$

It follows from Lemma 14.1 that the map $h\circ\phi_\gamma\colon \partial([0,1]^2) \to Y$ extends to a map $\psi_\gamma\colon [0,1]^2 \to Y$. We use this map to extend h over the 2-cell $\gamma\times[0,1]$. If we do this for all 1-cells of X we obtain the desired extension of h to a map $Z^2 \to Y$.

(3) We use Lemma 120.17 (2) and the fact that Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi,1)$ to extend the map h to a homotopy $F\colon X\times[0,1]\to Y$. ■

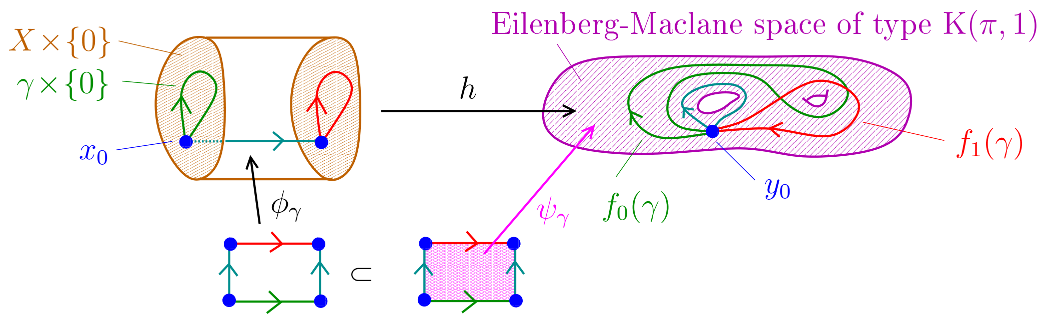


FIGURE 1653. Illustration for the proof of Proposition 120.18.

We conclude this section with a discussion of the statement of Proposition 120.15. It is not unreasonable to ask whether one really needs in Proposition 120.15 the hypothesis that X is $(n-1)$-connected. But we will now see that in neither of the two parts can we drop this condition.

(1) By Theorem 113.10 we know that there exists an isomorphism $\varphi\colon \pi_3(S^2,*) \xrightarrow{\cong} \mathbb{Z}$. Now let (Y,y_0) be any pointed Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z},3)$. At a very first glance one might hope that there exists a map $f\colon (S^2,*) \to (Y,y_0)$ such that $f_* = \varphi\colon \pi_3(S^2,*) \to \pi_3(Y,y_0) = \mathbb{Z}$. But this is not the case. Indeed, since Y is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z},2)$ we know that $\pi_2(Y,y_0)=0$. This means that any map $f\colon (S^2,*) \to (Y,y_0)$ is homotopic to the constant map. In particular any such map induces the zero map on π_3. Thus we cannot realize φ by a map $(S^2,*) \to (Y,y_0)$.

(2) On page 2810 we pointed out that the infinite complex projective space $\mathbb{CP}^\infty$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z},2)$. By the discussion on page 197 we can view $S^2 = \mathbb{CP}^1$ as a subset of $\mathbb{CP}^\infty$. We denote by $y_0 \in S^2$ the North Pole. We pick some point x_0 on the torus $S^1\times S^1$. We consider the following two maps from the torus $S^1\times S^1 \to S^2 \subset \mathbb{CP}^\infty$:

(a) The map f_0 is the constant map that sends all points to y_0.

(b) The map $f_1\colon S^1 \times S^1 \to \overline{B}^2/S^1 = S^2$ is a map that sends the points of a given embedded disk to $\overline{B}^2$ and sends all other points to the single point $[S^1] = y_0$.

(These two maps f_0 and f_1 are illustrated in Figure 1654.) Now we claim that we have $f_{0*} = f_{1*}\colon \pi_2(S^1 \times S^1, *) \to \pi_2(\mathbb{C}\mathrm{P}^\infty, *)$, but that f_0 and f_1 are, contrary to what one might expect, *not* homotopic. To prove this claim we recall that on page 1068 we saw that $\pi_2(S^1 \times S^1) = 0$. In particular both maps induce the same map on π_2. Nonetheless the two maps are not homotopic. This can be seen by studying the induced map on H_2. It is clear that f_0 induces the trivial map on H_2. On the other hand we saw in the proof of Proposition 69.3 that f_1 induces an isomorphism $\mathrm{H}_2(S^1 \times S^1; \mathbb{Z}) \to \mathrm{H}_2(S^2; \mathbb{Z})$. By the discussion on page 1263 we know that the inclusion induced map $\mathrm{H}_2(S^2; \mathbb{Z}) \to \mathrm{H}_2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$ is an isomorphism. Therefore we see that the map $f_{1*}\colon \mathrm{H}_2(S^1 \times S^1; \mathbb{Z}) \to \mathrm{H}_2(\mathbb{C}\mathrm{P}^\infty)$ is non-trivial. Thus we obtain from Proposition 42.5 that the maps f_0 and f_1 are in fact not homotopic.

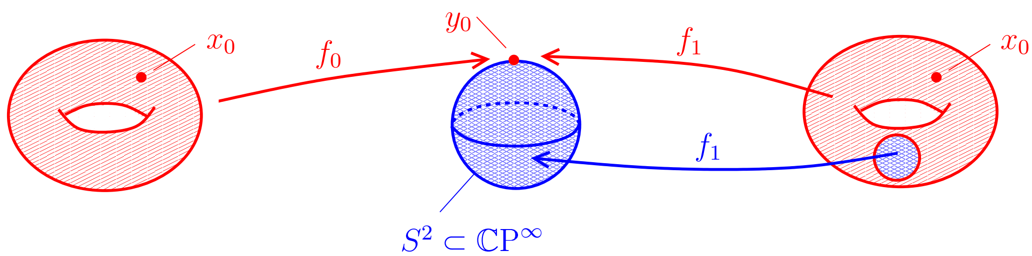


Figure 1654

At least in terms of invariants any Eilenberg-Maclane space $\mathrm{K}(\pi, n)$ is a particularly simple topological space. Therefore it seems not totally unreasonable to hope that given a CW-complex X one can perhaps classify the maps from X to $\mathrm{K}(\pi, n)$, say up to homotopy. The above examples show that the maps are *not* classified by $\mathrm{Hom}(\pi_n(X), \pi)$. But perhaps there is some other way? Let us record this as a question:

Question 120.19. *Let $n \in \mathbb{N}$ and let π be an (abelian) group. Let X be a* CW*-complex. Can we classify the maps $X \to \mathrm{K}(\pi, n)$ up to homotopy?*

120.5. **First applications of Eilenberg-Maclane spaces.** We recall the following definition from page 1944.

Definition. Let G be a group, let Y be a topological space and let $y_0 \in Y$ be a point with an identification $\pi_1(Y, y_0) \xrightarrow{=} G$. Given another topological space X and $x_0 \in X$ we say that a homomorphism $\varphi\colon \pi_1(X, x_0) \to G = \pi_1(Y, y_0)$ is *realized by a map* $g\colon X \to Y$ if $g(x_0) = y_0$ and if $g_* = \varphi$. If there is no danger of confusion we will drop the base points from the notation.

Next we recall the following question that we had formulated a while ago.

Questions 79.9 and 79.11. *Let X be a connected* CW*-complex.*[1694]

(0) *Can every homomorphism $\pi_1(X) \to \mathbb{Z} = \pi_1(S^1)$ be realized by a map $X \to S^1$?*
(1) *Let $g \geq 1$ and let Σ_g be the surface of genus $g \geq 1$. Can every homomorphism $\pi_1(X) \to \pi_1(\Sigma_g)$ be realized by a map $X \to \Sigma_g$?*
(2) *Can every homomorphism $\pi_1(X) \to \pi_1\Big(\bigvee\limits_{i=1}^{k} S^1\Big) = \langle x_1, \dots, x_k\rangle$ be realized by a map $X \to \bigvee\limits_{i=1}^{k} S^1$?*
(3) *Can every homomorphism $\pi_1(X) \to \mathbb{Z}_2 = \pi_1(\mathbb{R}\mathrm{P}^2)$ be realized by a map $X \to \mathbb{R}\mathrm{P}^2$?*

In Proposition 79.10 we had used de Rham cohomology to give an affirmative answer to (0) if X is a smooth manifold. Furthermore in Exercise 90.7 we gave, by considering the isomorphism $\pi_1(\mathbb{R}\mathrm{P}^3) \xrightarrow{\cong} \mathbb{Z}_2$, a negative answer to (3). Fortunately we can now give a complete answer to (0), (1) and (2).

Proposition 120.20.
(a) *The answer to (0), (1) and (2) is "yes". Furthermore, if in the formulation of (3) we replace $\mathbb{R}\mathrm{P}^2$ by $\mathbb{R}\mathrm{P}^\infty$, then the answer to (3) is also yes.*
(b) *If X is a smooth manifold, then the following two statements hold:*
(א) *In (0) and (1) we can find smooth realizations.*
(ב) *In (2) we can find a realization $f\colon X \to \bigvee\limits_{i=1}^{k} S^1_i$ which has the property that the restriction of f to the complement of f^{-1}(wedge point) is smooth.*

Proof.

(a) The desired statement follows immediately from Proposition 120.15 and the fact that on page 2809 we saw that S^1, $\bigvee\limits_{i=1}^{k} S^1$, Σ_g and $\mathbb{R}\mathrm{P}^\infty$ are Eilenberg-Maclane spaces.
(b) (א) This statement follows from the Whitney Approximation Theorem 9.3.
(ב) As we pointed out on page 404, this statement is a consequence of Proposition 9.7.

■

Using Proposition 120.20 we obtain the following pleasant proposition.

Proposition 120.21. *If M is a compact orientable connected n-dimensional smooth manifold, then the following two statements are equivalent:*
(1) *there exists an epimorphism $\pi_1(M) \to \langle x_1, \dots, x_k\rangle$,*
(2) *there exist disjoint connected non-empty $(n-1)$-dimensional submanifolds $F_1, \dots, F_k$ of M such that $M \setminus (F_1 \cup \dots \cup F_k)$ is connected.*

Sketch of a proof. First note that the "(2) $\Rightarrow$ (1)" implication is precisely the content of Proposition 98.8. Thus it remains to prove the "(1) $\Rightarrow$ (2)" implication. Therefore suppose that there exists an epimorphism $\pi_1(M) \to \langle x_1, \dots, x_k\rangle$. We consider the wedge

[1694] If M is a smooth manifold, then we saw in Theorem 64.2 and Lemma 61.24 that M admits a CW-structure.

$\bigvee_{i=1}^{k} S_i^1$. We denote by $*$ the wedge point and for $i = 1, \dots, k$ we write $J_i := S_i^1 \setminus \{*\}$. By Proposition 120.20 (b) we can realize this epimorphism by a map $f\colon M \to \bigvee_{i=1}^{k} S_i^1$ which is smooth on $M \setminus f^{-1}(\{*\})$. Let $i \in \{1, \dots, k\}$. It follows from Sard's Theorem 6.63 that there exists a regular value $z_i \in J_i$. We obtain from Lemma 9.8 that the preimage $G_i := f^{-1}(z_i)$ is an orientable proper $(n-1)$-dimensional submanifold of M. In Exercise 120.3 we will prove the following two statements:

(1) each G_i is non-empty,
(2) for each $i \in \{1, \dots, k\}$ we can find a component F_i of G_i such that $M \setminus (F_1 \cup \dots \cup F_k)$ is connected. ■

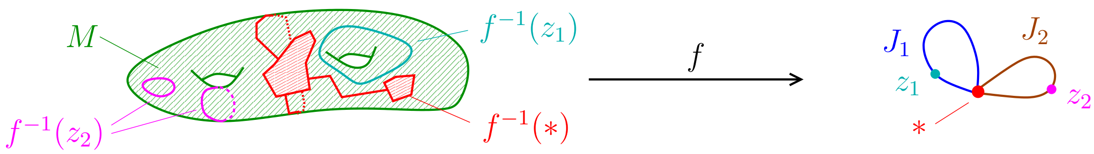


FIGURE 1655. Illustration for the proof of Proposition 120.21.

Now is a good time to recall the following question.

Question 98.9. *Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g.*

(1) *What is the maximal number $k = k(g) \in \mathbb{N}_0$ for which we can find k disjoint curves $F_1, \dots, F_k$ on Σ_g such that $\Sigma_g \setminus (F_1 \cup \dots \cup F_k)$ is connected?*
(2) *What is the maximal $l = l(g) \in \mathbb{N}_0$ for which there exists an epimorphism from $\pi_1(\Sigma_g)$ onto a free group of l generators?*

We make the following easy observations:

(1) As we already observed on page 2373, it follows immediately from peeking at Figure 1656 that $k(g) \geq g$.

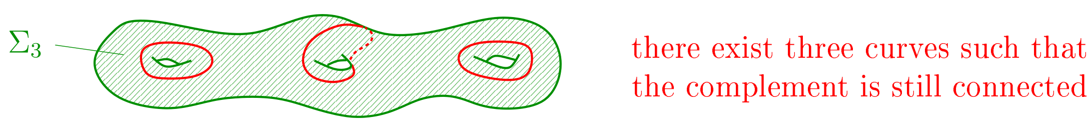


FIGURE 1656

(2) It follows from Lemma 21.13 that we have an epimorphism

$$\begin{array}{rcl} \overbrace{\langle x_1, y_1, \dots, x_g, y_g \,|\, x_1 y_1 x_1^{-1} y_1^{-1} \cdot \dots \cdot x_g y_g x_g^{-1} y_g^{-1} \rangle}^{\cong\, \pi_1(\Sigma_g) \text{ by Proposition 22.3}} & \to & \langle x_1, \dots, x_g \rangle \\ x_i & \mapsto & x_i \\ y_i & \mapsto & e \end{array}$$

from $\pi_1(\Sigma_g)$ onto a free group on g generators.

More interestingly, by Proposition 120.21 we now know that the two questions actually give the same answer. Even better, we can now give a complete answer:

Proposition 120.22. *Let $g \in \mathbb{N}$ and let Σ_g be the surface of genus g. If there exists an epimorphism Σ_g onto a free group on k generators, then $k \leq g$.*

Proof. Let $g \in \mathbb{N}$, let Σ_g be the surface of genus g and let $* \in \Sigma_g$ be a base point. Let $\varphi\colon \pi_1(\Sigma_g, *) \to F_k$ be an epimorphism onto a free group on k generators. We start out with the following preparations:

(1) By the discussion on page 2809 we know that $X := \bigvee\limits_{i=1}^{k} S^1$ is an Eilenberg-Maclane space of type $\mathrm{K}(F_k, 1)$. We pick $x_0 \in X$ and as usual we make the identification $\pi_1(X, x_0) = F_k$.
(2) It follows from Proposition 120.15 that there exists a map $f\colon (\Sigma_g, *) \to (X, x_0)$ with $f_* = \varphi\colon \pi_1(\Sigma_g, *) \to \pi_1(X, x_0) = F_k$. Note that the map $f_*\colon \pi_1(\Sigma_g, *) \to \pi_1(X, x_0)$ is an epimorphism.
(3) The Hurewicz Theorem 52.5, together with Proposition 21.20 (9), implies that the induced map $f_*\colon\ \mathrm{H}_1(\Sigma_g; \mathbb{Z}) \to \mathrm{H}_1(X; \mathbb{Z})$ is an epimorphism.
(4) In general, if $\alpha\colon A \to B$ is an epimorphism between two groups, then elementary algebra tells us that the induced map $\operatorname{Hom}(B, \mathbb{Z}) \to \operatorname{Hom}(A, \mathbb{Z})$ is a monomorphism. From this elementary observation, together with (3), we obtain that the induced map $f^*\colon\ \operatorname{Hom}(\mathrm{H}_1(X; \mathbb{Z}), \mathbb{Z}) \to \operatorname{Hom}(\mathrm{H}_1(\Sigma_g; \mathbb{Z}), \mathbb{Z})$ is a monomorphism.
(5) It follows immediately from (4) together with Proposition 75.18 that the induced map $f^*\colon\ \mathrm{H}^1(X; \mathbb{Z}) \to \mathrm{H}^1(\Sigma_g; \mathbb{Z})$ is also a monomorphism.
(6) Since X is a 1-dimensional CW-complex we obtain almost immediately from Proposition 74.4 that $\mathrm{H}^2(X; \mathbb{Z}) = 0$. In particular the cup product on $\mathrm{H}^1(X; \mathbb{Z})$ vanishes.

Now we consider the form

$$\begin{array}{rcl} \varphi\colon\ \mathrm{H}^1(\Sigma_g; \mathbb{Z}) \times \mathrm{H}^1(\Sigma_g; \mathbb{Z}) & \to & \mathbb{Z} \\ (\alpha, \beta) & \mapsto & \langle \alpha \cup \beta, [\Sigma_g] \rangle. \end{array}$$

Given $U \subset \mathrm{H}^1(\Sigma_g; \mathbb{Z})$ we write as usual

$$U^{\perp} \;=\; \{ v \in \mathrm{H}^1(\Sigma_g; \mathbb{Z}) \,|\, \varphi(u, v) = 0 \text{ for all } u \in U \}.$$

Now we see that

$$\begin{array}{rl} k & \overset{\substack{\text{since } X = \bigvee_{i=1}^{k} S^1 \\ \downarrow}}{=} \operatorname{rank}(\mathrm{H}^1(X; \mathbb{Z})) \overset{\substack{\text{since } i^* \text{ is a} \\ \text{monomorphism} \\ \downarrow}}{=} \operatorname{rank}(i^* \mathrm{H}^1(X; \mathbb{Z})) \overset{\substack{\text{by Proposition 90.2 we know that } \varphi \text{ is non-singular} \\ \text{hence we obtain the equality from Exercise 90.2} \\ \downarrow}}{=} \operatorname{rank}(\mathrm{H}^1(\Sigma_g; \mathbb{Z})) - \operatorname{rank}\big((i^* \mathrm{H}^1(X; \mathbb{Z}))^{\perp}\big) \\ & \underset{\substack{\uparrow \\ \text{it follows from (6) and Lemma 81.10} \\ \text{that } i^* \mathrm{H}^1(X; \mathbb{Z}) \subset (i^* \mathrm{H}^1(X; \mathbb{Z}))^{\perp}}}{\leq} \operatorname{rank}(\mathrm{H}^1(\Sigma_g; \mathbb{Z})) - \operatorname{rank}(i^* \mathrm{H}^1(X; \mathbb{Z})) \underset{\substack{\uparrow \\ \text{as above we have } \operatorname{rank}(i^*(\mathrm{H}^1(X; \mathbb{Z}))) = k \text{ and by the} \\ \text{discussion on page 1844 we know that } \operatorname{rank}(\mathrm{H}^1(\Sigma_g; \mathbb{Z})) = 2g}}{=} 2g - k. \end{array}$$

Thus we see that $k \leq 2g - k$, i.e. we see that $k \leq g$. ■

120.6. **Eilenberg–Maclane spaces and the social choice problem ❀.** In this section we give a fun, but perhaps not entirely serious application of Eilenberg-Maclane spaces. Let us recall the following definition from page 1347.

Definition. Let X be a topological space and let $n \in \mathbb{N}$. A *social choice of type* n is a map

$$f\colon X^n \;=\; \underbrace{X \times \cdots \times X}_{n \text{ times}} \;\to\; X$$

such that the following three axioms are satisfied:

(a) The map f is continuous.
(b) For every $x \in X$ we have $f(x,\dots,x) = x$.
(c) For every permutation $\sigma \in S_n$ and any $(x_1,\dots,x_n) \in X^n$ we have

$$f(x_{\sigma(1)},\dots,x_{\sigma(n)}) \;=\; f(x_1,\dots,x_n).$$

We say that a topological space is *social*, if for every $n \in \mathbb{N}$ it admits a social choice of type n.

We refer to Section 54.1 for a discussion why social topological spaces are interesting. Now let X be a topological space. The combination of Lemmas 54.4 and 54.5 shows that if X is social, then for every $k \in \mathbb{N}$ the homotopy group $\pi_k(X)$ is abelian and it is strongly divisible, i.e. given any $n \in \mathbb{N}$ the map $g \mapsto n \cdot g$ is an isomorphism. Furthermore, Proposition 54.2 together with the Whitehead Theorem 119.9 show that if X is in fact a CW-complex, with the property that in any dimension it has only finitely many cells, then X is contractible. Back in Section 54.1 this discussion lead us to the following question.

Question 54.7. *Does there exist a non-empty* CW*-complex that is social but that is not contractible?*

It turns out that we can now formulate a positive answer to Question 54.7.

Theorem 120.23.

(1) *Any* CW*-complex that is homotopy equivalent to the product of a family of Eilenberg-Maclane spaces of type* $\mathrm{K}(\mathbb{Q}, k_i)$*,* $i \in I$ *is social.*
(2) *Conversely, if* X *is a connected social* CW*-complex, then it is is homotopy equivalent to the product of a family of Eilenberg-Maclane spaces of type* $\mathrm{K}(\mathbb{Q}, k_i)$*,* $i \in I$*.*

Proof. This theorem is proved in [**EGH62**, p. 90] and [**Weinb04**, Theorem 1.3], see also [**Eckm04**, Theorems 4 and 7]. For the most part the proofs do not go much beyond what we have covered so far, except that they build on a result of Alexander Grothendieck [**Groth57**, Chapter V], see also [**HZ74**, p. 32] and [**Hirz69**, p. 254]. ■

Remark. As always there are various variations and generalizations of Theorem 120.23. For example by [**Weinb04**, Example 1.4] any Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}_2, 1)$, e.g. according to the discussion on page 2809 we could take $\mathbb{RP}^\infty$, has the property that given any *odd* $n \in \mathbb{N}$ it admits a social choice of type n.

120.7. **An extension of the Hurewicz Theorem 53.5.** In this last section we will use Eilenberg-Maclane spaces in a surprising way to obtain an addendum to the Hurewicz Theorem 53.5. To formulate the theorem it is convenient to introduce the following definition.

Definition. Given a group π and $n \in \mathbb{N}_0$ we define $\mathrm{H}_n(\pi) := \mathrm{H}_n(\mathrm{K}(\pi,1),\mathbb{Z})$ where $\mathrm{K}(\pi,1)$ denotes the canonical Eilenberg-Maclane space introduced in Theorem 120.8. We refer to $\mathrm{H}_n(\pi)$ as the *n-th homology group of the group* π.

Remark. The concept of the homology groups $\mathrm{H}_n(\pi)$ of a group π will play an important role in this section. Afterwards it takes a well-earned rest before it will take center stage in Chapter **??**.

The following lemma is an immediate consequence of Corollary 120.13.

Lemma 120.24. *Let π be a group and let $n \in \mathbb{N}_0$. For any Eilenberg-Maclane space X of type $\mathrm{K}(\pi, 1)$ we have $\mathrm{H}_n(\pi) \cong \mathrm{H}_n(X; \mathbb{Z})$.*

Example. Let $m \in \mathbb{N}$ and let $n \in \mathbb{N}_0$. We calculate that

$$\mathrm{H}_n(\text{the group } \mathbb{Z}^m) \underset{\substack{\uparrow \\ \text{by Lemma 120.24}}}{\cong} \mathrm{H}_n(\text{any Eilenberg-Maclane space } \mathrm{K}(\mathbb{Z}^m, 1)) \underset{\substack{\uparrow \\ \text{by the discussion on page 2809 we know that } (S^1)^m \\ \text{is an Eilenberg-Maclane space of type } \mathrm{K}(\mathbb{Z}^m,1)}}{\cong} \mathrm{H}_n((S^1)^m) \underset{\substack{\uparrow \\ \text{see page 1442}}}{\cong} \mathbb{Z}^{\binom{m}{n}}.$$

We continue with the following lemma.

Lemma 120.25. $(*)$ *Let (X, x_0) be a pointed 0-connected topological space. There exists a natural map*

$$\eta\colon\ \mathrm{H}_2(X; \mathbb{Z}) \ \to\ \mathrm{H}_2(\pi_1(X, x_0))$$

which has the following property: whenever (Y, y_0) is a pointed Eilenberg-Maclane space of type $\mathrm{K}(\pi_1(X, x_0), 1)$ and $f\colon (X, x_0) \to (Y, y_0)$ is a map of pointed topological spaces that induces an isomorphism of fundamental groups, then there exists an isomorphism $\mu\colon\ \mathrm{H}_2(Y; \mathbb{Z}) \to \mathrm{H}_2(\pi_1(X, x_0))$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}_2(X;\mathbb{Z}) \arrow[rr, "\eta"] \arrow[dr, "f_*"'] & & \mathrm{H}_2(\pi_1(X,x_0)) \\
& \mathrm{H}_2(Y;\mathbb{Z}). \arrow[ur, "\cong", "\mu"'] &
\end{tikzcd}$$

Proof $(*)$. Let (X, x_0) be a pointed 0-connected topological space. We write $\pi = \pi_1(X, x_0)$. Let $(Z, z_0, \varphi\colon \pi_1(Z, z_0) \to \pi)$ be the canonical triple introduced in Theorem 120.8. By the CW-Approximation Theorem 119.8 there exists a CW-approximation $f\colon (Y, y_0) \to (X, x_0)$. Note that by Proposition 119.6 we know that f induces isomorphisms on all homology groups.

By Proposition 120.15 (1) we know that there exists a map $g\colon (Y, y_0) \to (Z, z_0)$ such that $g_* = \varphi^{-1} \circ f_*\colon \pi_1(Y, y_0) \to \pi_1(Z, z_0)$. We set

$$\eta \ := \ g_* \circ (f_*)^{-1}\colon\ \mathrm{H}_2(X; \mathbb{Z}) \ \to\ \mathrm{H}_2(Z; \mathbb{Z}) =: \mathrm{H}_2(\pi_1(X, x_0)).$$

Using Proposition 120.15 (1) it is not difficult to verify that η has the desired properties. ■

Now we can formulate the following theorem.

Theorem 120.26. (Hurewicz Theorem) *Let $n \in \mathbb{N}$ and let X be an $(n-1)$-connected topological space and let $x_0 \in X$.*

(1) *If $n = 1$, then the sequence*

$$\pi_2(X, x_0) \xrightarrow{\Phi_{(X,x_0)}} \mathrm{H}_2(X; \mathbb{Z}) \xrightarrow{\eta} \mathrm{H}_2(\pi_1(X, x_0)) \to 0$$

is exact. This implies in particular that the cokernel of the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_2(X, x_0) \to \mathrm{H}_2(X; \mathbb{Z})$ is isomorphic to $\mathrm{H}_2(\pi_1(X))$.

(2) *If $n > 1$, then the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_{n+1}(X, x_0) \to \mathrm{H}_{n+1}(X; \mathbb{Z})$ is an epimorphism.*

Remark.

(1) We refer to [**WhdG78**, Theorem V.7.9] for a generalization of the above theorem.

(2) Let $n > 1$. Furthermore let X be an $(n-1)$-connected topological space. By the above Hurewicz Theorem 120.26 (2) we now know that the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_{n+1}(X, x_0) \to \mathrm{H}_{n+1}(X; \mathbb{Z})$ is an epimorphism. For $n \geq 3$ the kernel of the Hurewicz homomorphism is discussed in [**Hilt53**, p.105] and [**WhdJ50**, p. 72]. Furthermore, for the more tricky case $n = 2$ the kernel of the Hurewicz homomorphism is determined in [**WhdJ50**, pages 72, 76 and Theorem 20].

Example. Let X be a simply connected topological space and let $x_0 \in X$. By Theorem 120.26, applied to the case $n = 2$, the Hurewicz homomorphism $\pi_3(X, x_0) \to \mathrm{H}_3(X; \mathbb{Z})$ is in fact an epimorphism. It is natural to wonder whether Theorem 120.26 can be improved upon. For example it is a priori not unreasonable to ask whether the Hurewicz homomorphism $\pi_k(X, x_0) \to \mathrm{H}_k(X; \mathbb{Z})$ is an epimorphism for every $k \in \mathbb{N}_{\geq 3}$. But it turns out, we have a counterexample at hand, namely $X = \mathbb{C}\mathrm{P}^\infty$. By Theorem 113.12 we know that $\mathbb{C}\mathrm{P}^\infty$ is simply connected and that $\pi_k(\mathbb{C}\mathrm{P}^\infty) = 0$ for $k \geq 3$. But by the discussion on page 1263 we know that $\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}) \cong \mathbb{Z}$ for every even k. In particular the Hurewicz homomorphism $\pi_4(\mathbb{C}\mathrm{P}^\infty) \to \mathrm{H}_4(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$ is not an epimorphism.

The proof of Theorem 120.26 crucially relies on the following lemma.

Lemma 120.27. *Let π be an abelian group and let $n \in \mathbb{N}_{\geq 2}$. If Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$, then $\mathrm{H}_{n+1}(Y; \mathbb{Z}) = 0$.*

Examples.

(1) The statement of Lemma 120.27 is somewhat surprising since it is emphatically wrong for $n = 1$. For example by the discussion on page 2809 we know that $S^1 \times S^1$ is an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}^2, 1)$ and we have $\mathrm{H}_2(S^1 \times S^1; \mathbb{Z}) \neq 0$.

(2) As a reality check, let us consider the only Eilenberg-Maclane space for $n \geq 2$ that we understand, namely $\mathbb{C}\mathrm{P}^\infty$ which is by the discussion on page 2810 an Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$. By the calculation on page 1263 we do know that $\mathrm{H}_3(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$ is indeed trivial.

Proof of Lemma 120.27. Let $n \in \mathbb{N}_{\geq 2}$ and let π be an abelian group. It follows from Corollary 120.13 that it suffices to show that there exists a single Eilenberg-Maclane space Z of type $\mathrm{K}(\pi, n)$ with $\mathrm{H}_{n+1}(Z; \mathbb{Z}) = 0$.

By Proposition 47.11 we know that there exists an $(n+1)$-dimensional CW-complex that is a Moore space Y of type $\mathrm{K}(\pi, n)$. Recall that this means that

(1) Y is simply connected,
(2) $\mathrm{H}_n(Y;\mathbb{Z}) \cong \pi$,
(3) for any $i \neq n$ we have $\widetilde{\mathrm{H}}_i(Y;\mathbb{Z}) = 0$.

We pick a base point $y_0 \in Y$. By Corollary 53.7 we now know that (1), (2) and (3) imply that

(4) $\pi_i(Y, y_0) = 0$ for $i = 1, \dots, n-1$ and that $\pi_n(Y, y_0) \cong \pi$.

By Proposition 120.7 there exists a CW-complex Z with the following properties:

(5) The $(n+1)$-skeleton of Z equals Y.[1695]
(6) We have $\pi_i(Z) = 0$ for $i \geq n+1$.
(7) For $i = 1, \dots, n$ the inclusion $Y \to Z$ induces an isomorphism $\pi_i(Y, y_0) \xrightarrow{\cong} \pi_i(Z, y_0)$.

It follows from (4), (6) and (7) that Z is an Eilenberg-Maclane space of type $\mathrm{K}(\pi, n)$.

Since Y is the $(n+1)$-skeleton of the CW-complex Z we obtain from Lemma 48.1 (4) that the map $\mathrm{H}_{n+1}(Y;\mathbb{Z}) \to \mathrm{H}_{n+1}(Z;\mathbb{Z})$ is an epimorphism. Together with (3) this implies that $\mathrm{H}_{n+1}(Z;\mathbb{Z}) = 0$.[1696] We have thus found the desired Eilenberg-Maclane space Z. ■

Now we turn to the actual proof of Theorem 120.26. For convenience we break the proof into three parts:

(1) First we prove Theorem 120.26 (1) for 2-dimensional CW-complexes.
(2) Then we prove Theorem 120.26 (2) for $(n+1)$-dimensional CW-complexes.
(3) Finally we deal with the general case of Theorem 120.26.

Proof of Theorem 120.26 (1) for 2-dimensional CW-complexes. Let X be a 2-dimensional CW-complex that is 0-connected. Let $x_0 \in X$. By Proposition 120.7 there exists a CW-complex Y with the following properties:

(1) The 2-skeleton of Y equals X.
(2) We have $\pi_i(Y, x_0) = 0$ for $i \geq 2$.
(3) The inclusion map $i\colon X \to Y$ induces an isomorphism $i_*\colon \pi_1(X, x_0) \xrightarrow{\cong} \pi_1(Y, x_0)$.

It follows from (1), (2) and (3) that Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi_1(X, x_0), 1)$. Next we consider the following diagram:

$$\begin{tikzcd}
\pi_3(Y,X,x_0) \arrow[r, "\partial"] \arrow[d, two heads, "\Phi_{(Y,X,x_0)}"] & \pi_2(X,x_0) \arrow[r] \arrow[d, "\Phi_{(X,x_0)}"] & \overbrace{\pi_2(Y,x_0)}^{=0} \arrow[r] \arrow[d, "\Phi_{(Y,x_0)}"] & \pi_2(Y,X,x_0) \arrow[r, "\partial"] \arrow[d, "\Phi_{(Y,X,x_0)}"] & \pi_1(X,x_0) \arrow[r, "i_*", "\cong"'] & \pi_1(Y,x_0) \\
\mathrm{H}_3(Y,X;\mathbb{Z}) \arrow[r, "\partial"] & \mathrm{H}_2(X;\mathbb{Z}) \arrow[r, two heads] \arrow[d, "\eta"] & \mathrm{H}_2(Y;\mathbb{Z}) \arrow[r] \arrow[dl, "\mu", "\cong"'] & \underbrace{\mathrm{H}_2(Y,X;\mathbb{Z})}_{=0}. & & \\
 & \mathrm{H}_2(\pi_1(X,x_0)) & & & &
\end{tikzcd}$$

[1695] Here we use that Y is its own $(n+1)$-skeleton.

[1696] The reader will not have failed to notice that this proof shares some ideas with the proof of Lemma 120.11.

We make the following clarifications and observations:

(a) The horizontal sequences are the long exact sequences of homology and homotopy groups of the pair (Y, X), given by Corollary 43.16 and by Proposition 111.4.
(b) The vertical maps are the (relative) Hurewicz homomorphisms that we introduced on pages 1324 and 2633
(c) The diagram commutes by Lemma 111.11 (3).
(d) It follows from (2), (3) and the lower end of the upper long exact sequence that $\pi_j(Y, X) = 0$ for $j = 0, 1$. In other words, the pair (Y, X) is 2-connected.
(e) It follows from (d) and the Relative Hurewicz Theorem 111.13 that the left-hand vertical map is an epimorphism.[1697]
(f) The map η and the isomorphism μ are given by Lemma 120.25. In particular these maps turn the lower triangle into a commutative triangle.
(g) Since X is the 2-skeleton of Y we obtain from Lemma 48.1 that $\mathrm{H}_2(Y, X; \mathbb{Z}) = 0$.

From this data one can easily deduce that the sequence

$$\pi_2(X, x_0) \xrightarrow{\Phi_{(X,x_0)}} \mathrm{H}_2(X; \mathbb{Z}) \xrightarrow{\eta} \mathrm{H}_2(\pi_1(X, x_0)) \to 0$$

is exact. ■

Proof of Theorem 120.26 (2) for $(n+1)$-dimensional CW-complexes. Let $n \in \mathbb{N}_{\geq 2}$ and let X be an $(n+1)$-dimensional CW-complex that is $(n-1)$-connected. Let $x_0 \in X$. By Proposition 120.7 there exists a CW-complex Y with the following properties:

(1) The $(n+1)$-skeleton of Y equals X.
(2) We have $\pi_i(Y, x_0) = 0$ for $i \geq n+1$.
(3) For every $j \in \{1, \ldots, n\}$ the inclusion map $i\colon X \to Y$ induces an isomorphism $i_*\colon \pi_j(X, x_0) \xrightarrow{\cong} \pi_j(Y, x_0)$.

It follows from (1), (2) and (3) that Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi_n(X, x_0), n)$. Next we consider the following diagram:[1698]

$$\begin{tikzcd}
\pi_{n+2}(Y,X) \arrow[r, "\partial"] \arrow[d, "\cong"', "\Phi_{(X,Y)}"] & \pi_{n+1}(X) \arrow[r] \arrow[d, "\Phi_X"] & \overbrace{\pi_{n+1}(Y)}^{=0} \arrow[r] \arrow[d, "\Phi_Y"] & \pi_{n+1}(Y,X) \arrow[r, "\partial"] & \pi_n(X) \arrow[r, "i_*", "\cong"'] & \pi_n(Y) \arrow[r] & \ldots \\
\mathrm{H}_{n+2}(Y,X) \arrow[r, "\partial"] & \mathrm{H}_{n+1}(X) \arrow[r] & \underbrace{\mathrm{H}_{n+1}(Y).}_{=0 \text{ by Prop. } 120.27}
\end{tikzcd}$$

[1697]People with shaky memories, like your author, might be surprised that it does not say that the map is an isomorphism. The point is that the Relative Hurewicz Theorem 111.13 shows that the Hurewicz homomorphism $\pi_3(Y, X, x_0)' \to \mathrm{H}_3(Y, X; \mathbb{Z})$ is an isomorphism where $\pi_3(Y, X, x_0)'$ is the quotient of $\pi_3(Y, X, x_0)$ by a suitable action of $\pi_1(X, x_0)$. Thus the Hurewicz homomorphism $\pi_3(Y, X, x_0) \to \mathrm{H}_3(Y, X; \mathbb{Z})$ is an epimorphism, but in general it is not an isomorphism.

[1698]For evident space reasons we drop the base points from the discussion.

We make the following clarifications and observations:

(a) As in the previous proof the horizontal sequences are long exact sequences of the pair (Y, X). Furthermore the vertical maps are the (relative) Hurewicz homomorphisms and the diagram commutes by Lemma 111.11 (3).
(b) It follows from (2), (3) and the lower end of the upper long exact sequence that $\pi_j(Y, X) = 0$ for $j = 0, \ldots, n+1$. In other words, the pair (Y, X) is $(n+1)$-connected.
(c) It follows immediately from (a) and the Relative Hurewicz Theorem 111.13, together with the fact that $\pi_1(X, x_0) = 0$, that the left-hand vertical map is an isomorphism.
(d) Since Y is an Eilenberg-Maclane space of type $\mathrm{K}(\pi_n(X, x_0), n)$ and since $n \geq 2$ we obtain from Lemma 120.27 that $\mathrm{H}_{n+1}(Y) = 0$.

It follows from the above that the second vertical map $\pi_{n+1}(X) \to \mathrm{H}_{n+1}(X)$ is, as promised, an epimorphism. ■

Proof of the general Theorem 120.26. First we consider the case that $n \geq 2$. In the following let X be an $(n-1)$-connected topological space. Let $x_0 \in X$. By the CW-Approximation Theorem 119.8 there exists a CW-complex Y together with a weak homotopy equivalence $f\colon Y \to X$. Since f is a weak homotopy equivalence we see that Y itself is $(n-1)$-connected. We denote by $i\colon Y^{n+1} \to Y$ the inclusion of the $(n+1)$-skeleton Y^{n+1} into Y. We pick $y_1 \in Y^{n+1}$ and we write $x_1 = f(y_1)$. Since X is in particular path-connected we can pick a path γ from x_0 to x_1. We consider the following diagram

$$\begin{tikzcd}
\pi_{n+1}(Y^{n+1}, y_1) \arrow[r, "i_*"] \arrow[d, two heads, "\Phi_{(Y^{n+1}, y_1)}"] & \pi_{n+1}(Y, y_1) \arrow[r, "f_*", "\cong"'] \arrow[d, "\Phi_{(Y, y_1)}"] & \pi_{n+1}(X, x_1) \arrow[r, "\gamma_*", "\cong"'] \arrow[d, "\Phi_{(X, x_1)}"] & \pi_{n+1}(X, x_0) \arrow[d, "\Phi_{(X, x_0)}"] \\
\mathrm{H}_{n+1}(Y^{n+1}; \mathbb{Z}) \arrow[r, two heads, "i_*"] & \mathrm{H}_{n+1}(Y; \mathbb{Z}) \arrow[r, "f_*", "\cong"'] & \mathrm{H}_{n+1}(X; \mathbb{Z}) \arrow[r, "\mathrm{id}"] & \mathrm{H}_{n+1}(X; \mathbb{Z}).
\end{tikzcd}$$

We make the following clarifications and observations:

(1) The vertical maps are the Hurewicz homomorphisms.
(2) The horizontal map to the top right is the isomorphism that is induced by the path γ_*, see Proposition 40.5.
(3) The two squares to the left commute by Proposition 52.2.
(4) The square to the right commutes by Lemma 53.1 (2).
(5) We know from the above proof of Theorem 120.26 for $(n+1)$-dimensional CW-complexes, and the fact that Y^{n+1} is $(n-1)$-connected, that the left vertical map is an epimorphism.
(6) By Lemma 48.1 (4) we know that the bottom left horizontal map is an epimorphism.
(7) Since f is a weak homotopy equivalence we know that the top middle horizontal map is an isomorphism. Furthermore we obtain from Proposition 119.6 that the bottom middle horizontal map is also an isomorphism.
(8) It follows from all the information assembled above that the right vertical map is an epimorphism.

The proof for $n = 1$ is quite similar, with a few minor extra complications. We leave it to the reader to fill in the details. ■

Theorem 120.26 (2) says that given a pointed 0-connected topological space (X, x_0) there is a curious interplay between $\pi_2(X, x_0)$, $\mathrm{H}_2(X)$ and the second homology of $\pi_1(X, x_0)$. We want to get a better intuition for what is happening by considering two corollaries to Theorem 120.26 (2).

Corollary 120.28. *Let (X, x_0) be some pointed 0-connected topological space. If $\pi_1(X, x_0) \cong \mathbb{Z}$, then the Hurewicz homomorphism $\Phi_{(X,x_0)} \colon \pi_2(X, x_0) \to \mathrm{H}_2(X; \mathbb{Z})$ is an epimorphism.*

PROOF. In the discussion on page 2836 we saw that $\mathrm{H}_2(\pi_1(X, x_0)) \cong \mathrm{H}_2(\mathbb{Z}) \cong \mathrm{H}_2(S^1) = 0$. Thus it follows immediately from Theorem 120.26 (2) that the Hurewicz homomorphism $\pi_2(X, x_0) \to \mathrm{H}_2(X; \mathbb{Z})$ is an epimorphism. ■

The proof of Corollary 120.28 is in many ways beyond reproach, it is clear and rigorous. The only catch is that it is not very geometric and we do not really get a good understanding why the statement is true. To rectify this we will sketch an alternative, quite geometric, proof for Corollary 120.28 in Exercise 120.10.

We move on to the second corollary to Theorem 120.26 (2).

Corollary 120.29. *If X is a 0-connected topological space such that $\pi_1(X) \cong \mathbb{Z}^m$ for some $m \in \mathbb{N}_{\geq 2}$, then for any $x_0 \in X$ we have*

$$\operatorname{coker}\big(\Phi_{(X,x_0)} \colon \pi_2(X, x_0) \to \mathrm{H}_2(X; \mathbb{Z})\big) \underset{\substack{\uparrow \\ \text{Theorem 120.26 (1)}}}{\cong} \mathrm{H}_2(\pi_1(X)) \cong \mathrm{H}_2(\mathbb{Z}^m) \underset{\substack{\uparrow \\ \text{by the example on page 2836}}}{\cong} \mathbb{Z}^{\binom{m}{2}}.$$

In particular we see that $\mathrm{H}_2(X; \mathbb{Z})$ is non-trivial.

In the following we want to get a better understanding how this "extra" homology in $\mathrm{H}_2(X; \mathbb{Z})$ arises. To simplify the discussion let us consider the case $m = 2$. Thus let (X, x_0) be a pointed 0-connected topological space with $\pi_1(X, x_0) \cong \mathbb{Z}^2$. We proceed as follows:

(1) We pick two loops $\alpha, \beta \colon (S^1, *) \to (X, x_0)$ that represent a generating set for the group $\pi_1(X, x_0) \cong \mathbb{Z}^2$.
    We consider the loop $\varphi = \alpha * \beta * \overline{\alpha} * \overline{\beta} \colon (S^1, *) \to (X, x_0)$. Since $\pi_1(X, x_0)$ is abelian we see that the loop φ represents the trivial element in $\pi_1(X, x_0)$. By Lemma 14.1 this means that we can extend the loop to a map $\Phi \colon \overline{B}^2 \to X$.
(2) We make the "obvious" identification[1699] $\overline{B}^2 = [0, 1]^2$ which identifies the "quartercircles" on S^1 with the four sides of the square.
(3) Let $\sim$ be the usual equivalence relation on $[0, 1]^2$ given by $(x, 0) \sim (x, 1)$ and $(y, 0) \sim (y, 1)$. The quotient $T := [0, 1]^2/\sim$ is the 2-dimensional torus. We equip T with the base point $*$ that is represented by any of the vertices.
(4) The map $\Phi \colon \overline{B}^2 = [0, 1]^2 \to X$ descends to a map $\overline{\Phi} \colon T := [0, 1]^2/\sim \to X$.

This approach feels like we are creating surfaces ex nihilo. But the following lemma shows that miraculously this approach works.

[1699]The identification is given by rotation by $\frac{\pi}{4}$ followed by "outward" stretching as in the proof of Proposition 2.52.

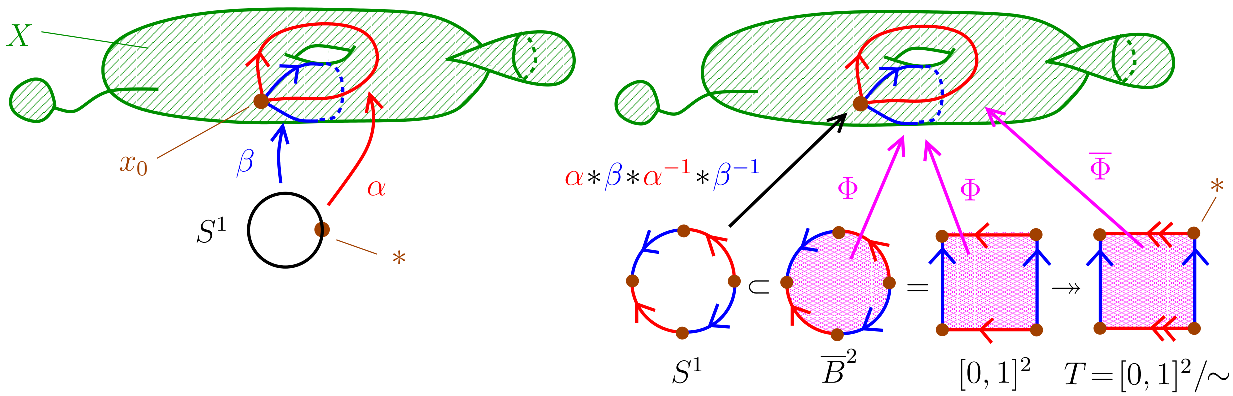


Figure 1657

Lemma 120.30. *We continue with the above notation. The map*

$$\begin{array}{rcl} \mu\colon \overbrace{\mathrm{H}_2(T;\mathbb{Z})}^{=\mathbb{Z}\cdot[T]} & \to & \operatorname{coker}(\pi_2(X,x_0) \to \mathrm{H}_2(X;\mathbb{Z})) \\ \sigma & \mapsto & [\overline{\Phi}_*(\sigma)] \end{array}$$

is an isomorphism.

Sketch of proof. As in the proof of Theorem 120.26 (2) we can assume that X is a 3-dimensional CW-complex. Furthermore, by the proof of Theorem 120.26 (2) we know that there exists an inclusion $i\colon X \to Y$ of X into an Eilenberg-Maclane space Y of type $\mathrm{K}(\mathbb{Z}^2, 1)$ such that the inclusion induced map $i_*\colon \pi_1(X,x_0) \to \pi_1(Y,x_0)$ is an isomorphism. We consider the following simple minded diagram

$$\begin{tikzcd} & X \arrow[dr, "i"] & \\ T \arrow[ur, "\overline{\Phi}"] \arrow[rr, "i\circ\overline{\Phi}"'] & & Y \end{tikzcd}$$

and we consider the induced diagram

$$\begin{tikzcd} & \pi_1(X,x_0) \arrow[dr, "i_*"] & \\ \pi_1(T,*) \arrow[ur, "\overline{\Phi}_*"] \arrow[rr, "(i\circ\overline{\Phi})_*"'] & & \pi_1(Y,y_0). \end{tikzcd}$$

We make the following observations:

(1) The map $\overline{\Phi}_*\colon \pi_1(T,*) \to \pi_1(X,x_0)$ is by design an epimorphism. Since both groups are isomorphic to $\mathbb{Z}^2$ we obtain from Lemma 19.8 (5) that $\overline{\Phi}_*$ is in fact an isomorphism.
(2) As we mentioned above, the map $i_*\colon \pi_1(X,x_0) \to \pi_1(Y,x_0)$ is an isomorphism.
(3) By (1) and (2) we know that $(i\circ\overline{\Phi})_* = i_*\circ\overline{\Phi}_*\colon \pi_1(T,*) \to \pi_1(Y,x_0)$ is an isomorphism. By the discussion on page 2809 and by the above we know that both T and Y are Eilenberg-Maclane spaces of type $\mathrm{K}(\mathbb{Z}^2,1)$. Thus we obtain from the Whitehead Theorem 119.9 that $i\circ\overline{\Phi}\colon T \to Y$ is in fact a homotopy equivalence.
(4) Since $i\circ\overline{\Phi}$ is a homotopy equivalence we obtain from Corollary 42.8 that the induced map $(i\circ\overline{\Phi})_* = i_*\circ\overline{\Phi}_*\colon \mathrm{H}_2(T;\mathbb{Z}) \to \mathrm{H}_2(Y;\mathbb{Z})$ is an isomorphism.

It remains to consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_2(T;\mathbb{Z}) \arrow[r, "\overline{\Phi}_*"] \arrow[rr, bend left, "\cong"] \arrow[dr, "\mu"'] & \mathrm{H}_2(X;\mathbb{Z}) \arrow[r, "i_*"] \arrow[d, two heads] & \mathrm{H}_2(Y;\mathbb{Z}) \\
& \operatorname{coker}(\pi_2(X,x_0) \to \mathrm{H}_2(X;\mathbb{Z})). \arrow[ur, "\cong"'] &
\end{tikzcd}$$

By the above discussion we know that the composition of the top two horizontal maps is an isomorphism and by Theorem 120.26, together with Lemma 120.25, we know that the diagonal map to the right is an isomorphism. It follows that the diagonal map to the left is also an isomorphism. ■

Exercises for Chapter 120.

Exercise 120.1. We consider the real projective plane $\mathbb{RP}^2 = \overline{B}^2/\sim$ where for $x \in S^1$ we have $x \sim -x$. Let $\varphi\colon \overline{B}^2 \to \mathbb{RP}^2$ be a smooth embedding. We consider the map

$$\begin{array}{rcl} f\colon \mathbb{RP}^2 & \to & \overline{B}^2/S^1 = S^2 = \mathbb{CP}^1 \subset \mathbb{CP}^\infty \\ P & \mapsto & \begin{cases} [Q], & \text{if } P = \varphi(Q) \text{ for some } Q \in \overline{B}^2, \\ [S^1], & \text{otherwise.} \end{cases} \end{array}$$

Is the map f homotopic to the constant map?

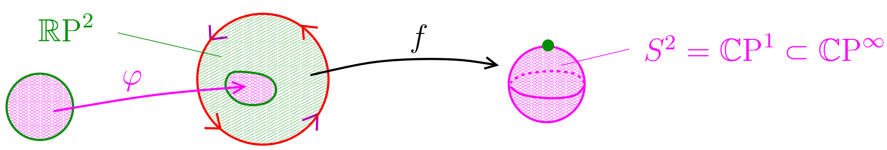


FIGURE 1658. Illustration of Exercise 120.1.

Exercise 120.2. Let X be a connected k-dimensional CW-complex and let $k \in \mathbb{N}$.

(a) Does there exist a CW-complex Y with $Y^k = X$ and $\pi_k(Y) = 0$?
(b) Does there exist a CW-complex Y with $Y^k = X$ and $\mathrm{H}_k(Y;\mathbb{Z}) = 0$?

Exercise 120.3. Finish the proof of Proposition 120.21.
Hint. Use Lemma 98.5.

Exercise 120.4. Let $n \in \mathbb{N}$. In Lemma 120.11 we showed that there exists an Eilenberg-Maclane space X of type $\mathrm{K}(\mathbb{Z}, n)$ with precisely one 0-cell, precisely one n-cell and no cell of dimension $n+1$. What is the minimal dimension in which the canonical Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, n)$ has more cells than X?

Exercise 120.5. Let $n \in \mathbb{N}$. We consider $n \cdot \mathbb{RP}^2$, i.e. we consider the connected sum of n copies of $\mathbb{RP}^2$.

(a) What is the maximal number $k \in \mathbb{N}_0$ for which we can find k disjoint curves $F_1, \dots, F_k$ on $n \cdot \mathbb{RP}^2$ such that $n \cdot \mathbb{RP}^2 \setminus (F_1 \cup \dots \cup F_k)$ is connected?

(b) What is the maximal $l \in \mathbb{N}_0$ for which there exists an epimorphism from $\pi_1(n \cdot \mathbb{R}\mathrm{P}^2)$ onto a free group of l generators?

Remark. By Proposition 22.7 we know that there exists an isomorphism

$$\pi_1(n \cdot \mathbb{R}\mathrm{P}^2) \;\cong\; \langle x_1, \dots, x_n \mid x_1^2 \cdot \dots \cdot x_n^2 \rangle.$$

Remark. This exercise is basically a variation on Question 98.9.

Exercise 120.6. In Lemma 81.9 we proved the following statement:

(∗) Let X be a topological space. For any $\phi \in \mathrm{H}^1(X;\mathbb{Z})$ we have $\phi \cup \phi = 0$.

Our goal is to give an alternative proof of this statement:

(a) Show that (∗) holds for $X = S^1$.
(b) Let X be a CW-complex. Use (a) and Proposition 120.15 to provide a proof of (∗) for CW-complexes.
(c) Use the CW-Approximation Theorem 119.8 and Proposition 119.6 to show, using (b), that (∗) holds for all topological spaces.

Exercise 120.7. Show that there exists an abelian group π and Moore spaces X and Y of type $\mathrm{K}(\pi, 1)$ that are not homotopy equivalent.
Hint. You could use the fundamental group to distinguish X and Y.
Remark. In Exercise 120.8 we will deal with Moore spaces of type $\mathrm{K}(\pi, n)$ with $n \geq 2$.

Exercise 120.8. Let π be an abelian group and let $n \in \mathbb{N}_{\geq 2}$. In this exercise we will show that any two Moore spaces of type $\mathrm{M}(\pi, n)$ are in fact homotopy equivalent. First note that by Proposition 47.11 there exists an $(n+1)$-dimensional CW-complex X that has no cells in dimensions $1, \dots, n-1$ and that is a Moore space of type $\mathrm{M}(\pi, n)$. It suffices to show that X is homotopy equivalent to any other Moore space Y of type $\mathrm{M}(\pi, n)$. This task is broken up into two steps:

(a) Use Lemma 120.16 to show that there exists a map $f\colon X \to Y$ that induces an isomorphism $f_*\colon \pi_n(X) \to \pi_n(Y)$.
(b) Use Lemma 53.1, Proposition 119.6 and the Whitehead Theorem 119.9 (1) to show that f is a homotopy equivalence.

Remark. Together with Exercise 120.7 this gives a complete answer to Question 47.14.

Exercise 120.9. Let $n \in \mathbb{N}$. We set $M = (S^1)^n$. Let $\varphi\colon \pi_1(M) \to \mathbb{Z}^n$ be an isomorphism. Show that it is not possible to extend φ over a bounding $(n+1)$-dimensional smooth manifold. More precisely, show that there is no pair $(W, \psi\colon \pi_1(W) \to \mathbb{Z}^n)$ where W is a compact orientable connected smooth manifold with $\partial W = M$ and where ψ is a homomorphism that satisfies $\varphi = \psi \circ i_*$ where $i\colon M \to W$ denotes the inclusion.
Remark. By Theorem 64.5 we know that any compact smooth manifold W admits a CW-structure.

Exercise 120.10. Let X be a 1-connected topological space with $\pi_1(X) \cong \mathbb{Z}$. By Corollary 120.28 we know that the Hurewicz homomorphism $\pi_2(X) \to \mathrm{H}_2(X;\mathbb{Z})$ is an epimorphism. The goal of this exercise is to sketch an alternative geometric proof. Thus let $\varphi \in \mathrm{H}_2(X;\mathbb{Z})$. By Proposition 70.3 there exists a surface Σ_g of some genus $g \in \mathbb{N}_0$ and a map $f\colon \Sigma_g \to X$ such that $f_*([\Sigma_g]) = \varphi \in \mathrm{H}_2(X;\mathbb{Z})$. If $g = 0$, then we are done. So suppose that $g \geq 1$. We proceed with the following steps:

(a) Show that there exists a curve $C \subset \Sigma_g$ with $[C] \neq 0 \in \mathrm{H}_1(\Sigma_g;\mathbb{Z})$ but such that $f_*([C]) = \pi_1(X)$.
Remark. You might want to use Corollary 70.17 at some point.
(b) Get inspired by Figure 1659.
(c) Complete the proof of the statement.

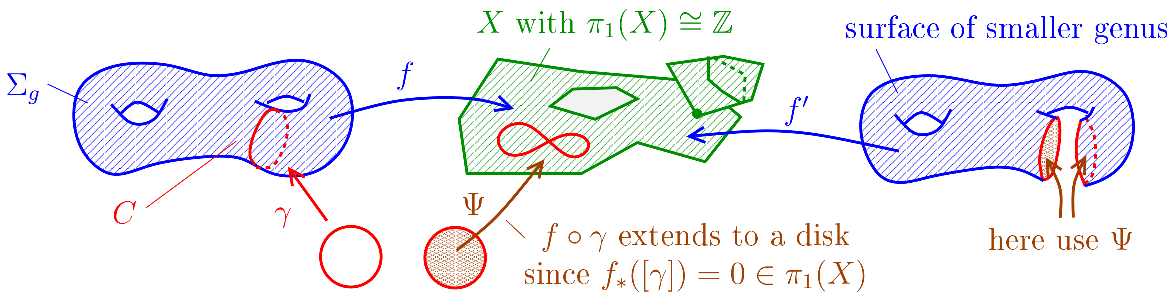


FIGURE 1659

Exercise 120.11. Let π be a group, let $\varphi\colon \pi \to \pi$ be an endomorphism and let K be an Eilenberg-Maclane space of type $\mathrm{K}(\pi,1)$. By Proposition 120.15 we know that there exists a map $f\colon (\mathrm{K}(\pi,1),*) \to (\mathrm{K}(\pi,1))$ such that $f_* = \varphi$. Now suppose that φ is an isomorphism. Can always find such a map f which is furthermore a homeomorphism?

Exercise 120.12. Let $F_n := \langle x_1,\dots,x_n\rangle$ the free group on n generators $x_1,\dots,x_n$ and let $\varphi\colon F_n \to F_n$ be an isomorphism. We consider the semidirect product $\pi := F_n \rtimes \mathbb{Z}$ as defined on page 718. Show that there exists a 2-dimensional Eilenberg-Maclane space of type $\mathrm{K}(\pi,1)$.

Hint. Note that $X := \bigvee_{i=1}^{n} S^1_i$ is an Eilenberg-Maclane space of type $\mathrm{K}(F_n,1)$. By Proposition 120.15 there exists a map $f\colon (X,*) \to (X,*)$ with $f_* = \varphi$. Now consider the mapping torus $\mathrm{Tor}(X,f)$. Use Proposition 24.28. It might also be useful to consider Lemma 24.26.

In this chapter we will see how one can use Eilenberg-Maclane spaces to define contravariant functors from the category of CW-complexes to the category of abelian groups. In this chapter and in Chapter **??** we will relate these functors to the cohomology functors.

Before we jump into the chapter we remind the reader that given topological spaces X and Y we denote by Y^X the set of all continuous maps from X to Y. Throughout this and the coming chapters we always equip Y^X with the compact-open topology that we introduced on page 234.

121.1. The homotopy category of topological spaces.

In this short section we recall some definitions and results regarding topological spaces and the homotopy category of topological spaces. This section partly serves as a warm-up for the discussion of pointed topological spaces in the following sections.

First we recall the following definition from page 543.

Definition. Given two topological spaces X and Y we define

$$[X,Y] := \underbrace{\{\text{continuous maps } X \to Y\}}_{=Y^X} / \sim \quad \text{where } f \sim g \text{ if there exists a homotopy between } f \text{ and } g.$$

We also recall the following definition from page 544.

Definition. The *homotopy category* $\mathcal{H}$ is defined to be the category where the objects are topological spaces and where the set of morphisms between two topological spaces X and Y is given by the set $[X,Y]$.[1700]

The following lemma is basically an immediate consequence of the definitions.

Lemma 121.1. *Let $f\colon X \to Y$ be a map between topological spaces. The corresponding morphism $[f] \in \operatorname{Mor}_{\mathcal{H}}(X,Y)$ is invertible in the sense of the definition on page 478 if and only if f is a homotopy equivalence.*

Proof. Note that by definition $[f] \in \operatorname{Mor}_{\mathcal{H}}(X,Y)$ is invertible if and only if there exists $[g] \in \operatorname{Mor}_{\mathcal{H}}(Y,X)$ with $[f \circ g] = [\mathrm{id}_Y]$ and $[g \circ f] = [\mathrm{id}_X]$. Now the statement of the lemma is indeed clear. ■

In the following we show how some constructions of topological spaces interact with the homotopy category $\mathcal{H}$. First we have following rather dull lemma.

Lemma 121.2. *Let X, Y and Z be topological spaces. The map*

$$\begin{array}{rcl} \chi\colon [X,Y] \times [X,Z] & \to & [X, Y \times Z] \\ ([f\colon X \to Y],[g\colon X \to Z]) & \mapsto & [f \times g\colon X \to Y \times Z] \end{array}$$

is well-defined.

Proof. Nothing can possibly go wrong in any attempt to prove this lemma. ■

[1700] The composition of $[f\colon X \to Y]$ and $[g\colon Y \to Z]$ is evidently defined to be $[g \circ f\colon X \to Z]$, by Lemma 18.7 this definition does not depend on the choice of representatives.

Lemma 121.3. *The maps*

$$\begin{array}{rcl} X & \mapsto & \textit{free loop space } \Omega(X) \;:=\; \{\varphi \in X^{[0,1]} \,|\, \varphi(0) = \varphi(1)\} \\ [f\colon X \to Y] & \mapsto & \left[\begin{array}{rcl} \Omega(f)\colon \Omega(X) & \to & \Omega(Y) \\ (\varphi\colon [0,1] \to X) & \mapsto & (f \circ \varphi\colon [0,1] \to Y) \end{array} \right] \end{array}$$

and

$$\begin{array}{rcl} X & \mapsto & \textit{suspension } \Sigma(X) := (X \times [-1,1])/\sim \\ [f\colon X \to Y] & \mapsto & \left[\begin{array}{rcl} \Sigma(f)\colon \Sigma(X) & \to & \Sigma(Y) \\ {[(x,t)]} & \mapsto & [(f(x),t)] \end{array} \right] \end{array}$$

both define covariant functors from $\mathcal{H}$ to itself.

Proof. There are only two statements one needs to think about: given a map $f\colon X \to Y$ one needs to show that $\Omega(f)$ and $\Sigma(f)$ are continuous, and given homotopic maps $f, g\colon X \to Y$ one needs to show that the corresponding maps on the right hand side are also homotopic. For the free loop spaces the statements follow from Lemma 5.5 (1) and (3). For suspensions these two statements follow immediately from Lemma 24.4 (1) and (4). ■

Just for kicks we also mention the following definition, which is reminiscent of the definition that we gave on page 1060. We will make use of this definition, with a slight twist, in the near future.

Definition. Let X be a topological space. We consider the wedge $\Sigma(X) \vee \Sigma(X)$ that is given by identifying the "South Pole $[X \times \{-1\}]$" on the left with the "North Pole $[X \times \{1\}]$" on the right. We refer to the map

$$\begin{array}{rcl} \Sigma(X) & \to & \Sigma(X) \vee \Sigma(X) \\ {[(x,t)]} & \mapsto & \begin{cases} [(x, 2t+1)] \text{ in the first term}, & \text{if } t \in [-1,0], \\ [(x, 2t-1)] \text{ in the second term}, & \text{if } t \in [0,1]. \end{cases} \end{array}$$

as the *pinching map*.[1701]

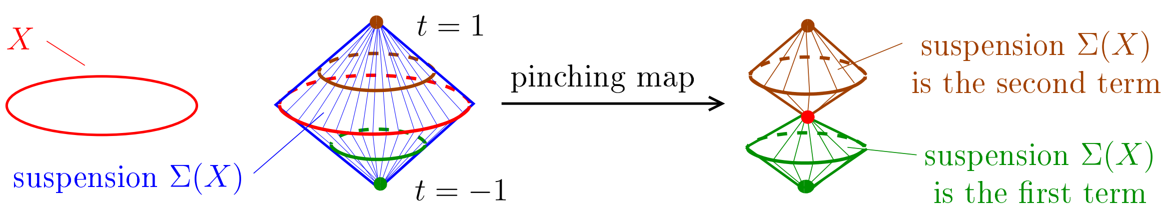


Figure 1660

In some instances the sets $[X, Y]$ have some extra structure. For example we have the following elementary but cute lemma.

[1701]It follows easily from Lemma 3.22 and Lemma 2.35 (2) that the pinching map is continuous.

Lemma 121.4. *Let X be a topological space and let G be an (abelian) topological group, e.g. we could take $G = S^1$. The set $[X,G]$ together with the multiplication map*

$$\begin{array}{rcl} \nu\colon [X,G]\times[X,G] & \to & [X,G] \\ [f],[g] & \mapsto & \left[\begin{array}{rcl} X & \to & G \\ x & \mapsto & f(x)\cdot g(x) \end{array}\right] \end{array}$$

is an (abelian) group.

Remark. Let X be a topological space and let G be a topological group. In Lemma 5.12 we showed, under the extra hypothesis that G is regionally compact, that the set G^X with the obvious multiplication map $G^X \times G^X \to G^X$ is in fact a topological group. We could equip $[X,G] = G^X/\sim$ with the quotient topology, but the discussion on page 2698 shows that in general the resulting multiplication map $\nu\colon [X,G]\times[X,G]\to[X,G]$ is no longer continuous. Thus in general we cannot view $[X,G]$ as a topological group.

PROOF. In fact the only thing one needs to verify is that the map ν is actually well-defined. But this follows easily from the fact that the group multiplication $G\times G\to G$ is continuous. We leave it to the reader to fill in the details. ■

There are several drawbacks to considering topological spaces which are not equipped with base points.

(1) Without a base point the homotopy groups $\pi_n(X)$ are at best defined up to isomorphism, but they are not defined "on the nose".
(2) The operation of taking wedges is not well-defined.

In the following section we will introduce the analogue of the homotopy category for pointed topological spaces. This will take care of the above issues, but it will require some adjustments at other places.

121.2. **The homotopy category of pointed topological spaces.** In the latter part of this chapter it will be essential that we always work with pointed topological spaces. To prepare for this discussion we need to rephrase all the concepts and results from the previous section in the language of pointed topological spaces.

We start out with a few definitions. Some of them we have encountered before on page 1005.

Definition.

(1) A *pointed topological space* is a pair (X,x_0) where X is a topological space and $x_0\in X$ is a point. We refer to x_0 as the *base point* of the pointed topological space.
(2) A map $f\colon (X,x_0)\to(Y,y_0)$ between pointed topological spaces is a map $f\colon X\to Y$ with $f(x_0)=y_0$.
(3) We say that two maps $f,g\colon (X,x_0)\to(Y,y_0)$ between pointed topological spaces are *homotopic* if there exists a homotopy $F\colon X\times[0,1]\to Y$ rel $\{x_0\}$ from f to g. We use this notion of "homotopic" to define the notion of a homotopy equivalence between pointed topological spaces.

(4) A *pointed* CW-*complex* is a pair (X, x_0) where X is a CW-complex and x_0 is a point in the 0-skeleton X^0 of X.
(5) Given two pointed topological spaces (X, x_0) and (Y, y_0) we write[1702]

$$\langle (X,x_0),(Y,y_0)\rangle \;:=\; \{\text{maps } (X,x_0)\to(Y,y_0)\}/\sim \qquad \begin{array}{c}\text{where } f\sim g \text{ if there exists a}\\ \text{homotopy in the above sense}\\ \text{between } f \text{ and } g.\end{array}$$

Given a map $f\colon (X,x_0)\to(Y,y_0)$ we denote its equivalence class by $[f]$ as usual.
(6) Let (X, x_0) and (Y, y_0) be two topological spaces. Let $0 \in \langle (X,x_0),(Y,y_0)\rangle$ be the element that is represented by constant map $X \to Y$ given by $x \mapsto y_0$.[1703] We refer to 0 as the *trivial element*.

The following lemma is the analogue of Lemma 18.7.

Lemma 121.5. *Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed topological spaces. The map*

$$\begin{array}{rcl}\langle (X,x_0),(Y,y_0)\rangle\times\langle (Y,y_0),(Z,z_0)\rangle &\to& \langle (X,x_0),(Z,z_0)\rangle\\ ([f],[g]) &\mapsto& [g\circ f]\end{array}$$

is well-defined.

Proof. The proof of this lemma is almost identical to the proof of Lemma 18.7 and it is left to the insomniac reader. ■

Lemma 121.5 shows that the following definition makes sense.

Definition. The *pointed homotopy category* $\mathcal{PHomTop}$ is the category where the objects are pointed topological spaces and where the set of morphisms between two pointed topological spaces (X, x_0) and (Y, y_0) is given by the set $\langle (X,x_0),(Y,y_0)\rangle$.

The following lemma is the analogue of Lemma 121.1.

Lemma 121.6. *Let (X, x_0) and (Y, y_0) be pointed topological spaces.*
(1) *Let $f\colon (X,x_0)\to(Y,y_0)$ be a map. The corresponding morphism*[1704]

$$[f] \;\in\; \operatorname{Mor}_{\mathcal{PHomTop}}((X,x_0),(Y,y_0)) \;=\; \langle (X,x_0),(Y,y_0)\rangle$$

is invertible if and only if f is a homotopy equivalence.
(2) *In the category $\mathcal{PHomTop}$ we have*[1705]

$$\langle (X,x_0),(Y,y_0)\rangle_{\mathrm{inv}} \;=\; \begin{array}{l}\textit{all elements of } \langle (X,x_0),(Y,y_0)\rangle \textit{ that are represented}\\ \qquad\qquad\qquad\textit{by a homotopy equivalence.}\end{array}$$

Proof. The first statement is, as in the case of Lemma 121.1, an immediate consequence of the definitions. The second statement is a reformulation of the first statement. ■

[1702] The notation $\langle X, Y\rangle$ goes back to [**Hat02**, p. 357], where it says “the notation $\langle \ldots \rangle$ is intended to suggest pointed homotopy classes”.
[1703] At the moment this notation might sound a little risqué. But it will turn out to be a useful notation.
[1704] We refer to page 478 for the definition of an invertible morphism.
[1705] On page 478 we defined $\operatorname{Mor}_{\mathcal{C}}(X, Y)_{\mathrm{inv}}$ to be the set of all invertible morphisms.

In the following we discuss the operations of taking products, wedges, loop spaces and suspensions in the context of the pointed homotopy category. We start out with the following convention.

Convention. Given pointed topological spaces (X, x_0) and (Y, y_0) we equip the product $X \times Y$ with the point (x_0, y_0).

For completeness we mention the following uninspiring analogue to Lemma 121.2.

Lemma 121.7. *Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed topological spaces. The map*

$$\begin{array}{rcl} \chi\colon \langle (X, x_0), (Y, y_0)\rangle \times \langle (X, x_0), (Z, z_0)\rangle & \to & \langle (X, x_0), (Y \times Z, (y_0, z_0))\rangle \\ ([f\colon X \to Y], [g\colon X \to Z]) & \mapsto & [f \times g\colon X \to Y \times Z] \end{array}$$

is well-defined and it is natural.

Proof. Conveniently enough the proof is identical to the proof of Lemma 121.2. ■

As discussed on page 559, one advantage of working with pointed topological spaces is that we have an unambiguous wedge operation.

Definition. Given a family $\{(X_i, x_i)\}_{i\in I}$ of pointed topological spaces we define the *wedge*

$$\bigvee_{i\in I} (X_i, x_i) \;:=\; \Big(\bigcup_{i\in I} X_i\Big)/\sim \qquad \text{where } x_i \sim x_j \text{ for any } i, j \in I.$$

We equip this wedge with the base point given by the wedge point.

Lemma 121.8. *Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed topological spaces. The map*

$$\begin{array}{rcl} \nu\colon \langle (X, x_0), (Z, z_0)\rangle \times \langle (Y, y_0), (Z, z_0)\rangle & \to & \langle (X, x_0) \vee (Y, y_0), (Z, z_0)\rangle \\ ([f\colon X \to Z], [g\colon Y \to Z]) & \mapsto & [f \vee g] \end{array}$$

is well-defined.

Proof. The statement of the lemma is, in contrast to Lemma 121.5 and 121.7, non-trivial. It is a reformulation of Lemma 18.27 which traces its ancestry to the formidable Theorem 5.16. ■

Now we would like to generalize the free loop space $\Omega(X)$ and the suspension $\Sigma(X)$ to pointed topological spaces. It is pretty clear what we do about loop spaces:

Definition. Given a pointed topological space (X, x_0) we consider the

$$\text{loop space } \Omega(X, x_0) \;:=\; \{f \in X^{[0,1]} \,|\, f(0) = f(1) = x_0\}$$

and we equip the loop space $\Omega(X, x_0)$ with the base point c_{x_0} which is just the constant path $[0,1] \to X$, $t \mapsto x_0$. Often we will suppress the base point from the notation, but throughout the subsequent chapters we will always view $\Omega(X, x_0)$ as a pointed topological space.

For completeness we write down explicitly the following fairly obvious lemma which is the analogue of the first statement of Lemma 121.3.

Lemma 121.9. *The maps*

$$\begin{array}{rcl} (X,x_0) & \mapsto & (\Omega(X,x_0),c_{x_0}) \\ [f\colon (X,x_0)\to(Y,y_0)] & \mapsto & \left[\begin{array}{rcl} \Omega(X,x_0) & \to & \Omega(Y,y_0) \\ (\gamma\colon[0,1]\to X) & \mapsto & (f\circ\gamma\colon[0,1]\to Y)\end{array}\right]\end{array}$$

define a covariant functor from the category $\mathcal{PHomTop}$ to itself.

PROOF. The lemma follows easily basically from Lemma 5.5 (1) and (3), one just needs to make a few minute modifications to factor in the base points. We leave this elementary task to the inexhaustible reader. ■

The loop spaces $\Omega(X,x_0)$ have one big advantage over the free loop spaces $\Omega(X)$. Given two loops in (X,x_0) we can form the product, as defined on page 465. As we had already hinted at in Lemma 114.22, this multiplication turns $\Omega(X,x_0)$ into a group "up to homotopy". More precisely we have the following definition and lemma which hark back to the happy days when our biggest worry was to define the fundamental group of a pointed topological space.

Definition. Let (X,x_0) be a pointed topological space. We consider the map

$$\begin{array}{rcl} \mu\colon \Omega(X,x_0)\times\Omega(X,x_0) & \to & \Omega(X,x_0) \\ (f,g) & \mapsto & f*g = \left(\begin{array}{rcl} [0,1] & \to & X \\ t & \mapsto & \begin{cases} f(2t), & \text{if } t\in[0,\frac{1}{2}], \\ g(2t-1), & \text{if } t\in(\frac{1}{2},1]\end{cases}\end{array}\right).\end{array}$$

For entertainment the definition of this map is illustrated in Figure 1661.

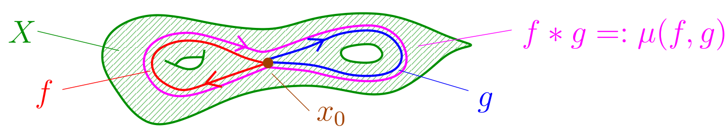


FIGURE 1661

The following lemma summarizes many important properties of μ.

Lemma 121.10. *Let (X,x_0) be a pointed topological space. The multiplication map*

$$\mu\colon \Omega(X,x_0)\times\Omega(X,x_0) \to \Omega(X,x_0)$$

has the following properties.

(1) *The map $\mu\colon \Omega(X,x_0)\times\Omega(X,x_0)\to\Omega(X,x_0)$ is continuous*[1706]*, it is base-point preserving and it is natural in (X,x_0).*[1707]
(2) *The two maps*

$$\begin{array}{rcl} \lambda\colon \Omega(X,x_0) & \to & \Omega(X,x_0) \\ f & \mapsto & \mu(c_{x_0},f)\end{array} \quad \text{and} \quad \begin{array}{rcl} \rho\colon \Omega(X,x_0) & \to & \Omega(X,x_0) \\ f & \mapsto & \mu(f,c_{x_0})\end{array}$$

between pointed topological spaces are both homotopic to the identity.

(3) *The map*

$$\begin{array}{rcl} \eta\colon \Omega(X,x_0) & \to & \Omega(X,x_0) \\ f & \mapsto & \overline{f} := \left(\begin{array}{rcl} [0,1] & \to & X \\ t & \mapsto & f(1-t) \end{array} \right) \end{array}$$

is a homeomorphism and base point preserving.

(4) *The map*

$$\begin{array}{rcl} \Omega(X,x_0) & \to & \Omega(X,x_0) \\ f & \mapsto & \mu(f,\overline{f}) \end{array}$$

between pointed topological spaces is homotopic to the constant map $f \mapsto c_{x_0}$.

(5) *The two maps*

$$\begin{array}{rcl} \Omega(X,x_0)\times\Omega(X,x_0)\times\Omega(X,x_0) & \to & \Omega(X,x_0) \\ (f,g,h) & \mapsto & \mu(\mu(f,g),h) \end{array}$$

and

$$(f,g,h) \mapsto \mu(f,\mu(g,h))$$

between pointed topological spaces are homotopic.

Remark.

(1) In many ways Lemma 121.10 is quite similar to Proposition 14.6. In fact it is not difficult to formulate and prove a result which combines both. As it is, the two results are stated at the appropriate level of generality for what they are aimed for.
(2) Let (X,x_0) be a pointed topological space. In Lemma 121.10 (2) we just showed that the multiplication map $\Omega(X,x_0)\times\Omega(X,x_0)\to\Omega(X,x_0)$ is continuous. The fundamental group $\pi_1(X,x_0) := \Omega(X,x_0)/\sim$ is a quotient of $\Omega(X,x_0)$, hence it can be equipped with the quotient topology. It turns out, see the discussion on page 2698, that the induced multiplication map $\pi_1(X,x_0)\times\pi_1(X,x_0)\to\pi_1(X,x_0)$ is in general *not* continuous.

Proof (∗). Let (X,x_0) be a pointed topological space.

(1) The statement that μ is continuous is precisely the content of Lemma 114.22 (1). The statements that μ is base-point preserving and natural follows basically immediately from the definitions.
(2) This statement is a reformulation of Lemma 114.22 (2).
(3) Evidently the map $\eta\colon \Omega(X,x_0)\to\Omega(X,x_0)$ is base point preserving. It follows immediately from Lemma 5.5 (5), applied to the homeomorphism $[0,1]\to[0,1]$ that

---

[1706]Recall that $\Omega(X,x_0)$ is equipped with the compact-open topology and $\Omega(X,x_0)\times\Omega(X,x_0)$ is correspondingly equipped with the product topology. Furthermore we view $\Omega(X,x_0)$ and $\Omega(X,x_0)\times\Omega(X,x_0)$ as pointed topological spaces.

[1707]Here "natural" means that if $f\colon (X,x_0)\to(Y,y_0)$ is a map between pointed topological spaces, then the following diagram commutes:

$$\begin{tikzcd} \Omega(X,x_0)\times\Omega(X,x_0) \arrow[r, "\mu"] \arrow[d, "f_*\times f_*"'] & \Omega(X,x_0) \arrow[d, "f_*"] \\ \Omega(Y,y_0)\times\Omega(Y,y_0) \arrow[r, "\mu"] & \Omega(Y,y_0). \end{tikzcd}$$

is given by $t \mapsto 1-t$, that the map $\eta\colon \Omega(X,x_0) \to \Omega(X,x_0)$ is continuous. Since $\eta \circ \eta = \mathrm{id}$ we see that η is in fact a homeomorphism.

(4) We consider the map

$$\begin{array}{rcl} H\colon \Omega(X,x_0)\times[0,1] & \to & \Omega(X,x_0) \\ (f,t) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ s & \mapsto & \begin{cases} f(2s\cdot t), & \text{if } s\in\left[0,\tfrac{1}{2}\right], \\ f(2t-2s\cdot t), & \text{if } s\in\left(\tfrac{1}{2},1\right] \end{cases} \end{array}\right). \end{array}$$

Again, using the argument in the proof of Lemma 114.22 (1) we see that H is continuous. It is now clear that H is a homotopy between the constant map $f \mapsto c_{x_0}$ and the map $f \mapsto \mu(f,\overline{f})$. We refer to Figure 1662 for an illustration.

(5) To simplify the notation we write $\Omega(X)$ instead of $\Omega(X,x_0)$. We consider the map

$$\begin{array}{rcl} H\colon (\Omega(X)\times\Omega(X)\times\Omega(X))\times[0,1] & \to & \Omega(X) \\ ((f,g,h),t) & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ s & \mapsto & \begin{cases} f(\frac{4s}{t+1}), & \text{if } s\in[0,\frac{t+1}{4}), \\ g(4s-t-1), & \text{if } s\in[\frac{t+1}{4},\frac{t+2}{4}), \\ h\left(\frac{4s-2-t}{2-t}\right), & \text{if } s\in[\frac{t+2}{4},1] \end{cases} \end{array}\right). \end{array}$$

The same argument as in the proof of Lemma 114.22 (1) shows that the map H is continuous. One can now easily verify that the map H defines a homotopy between $(f,g,h) \mapsto \mu(\mu(f,g),h)$ and $(f,g,h) \mapsto \mu(f,\mu(g,h))$. We refer to Figure 1662 for an illustration. ■

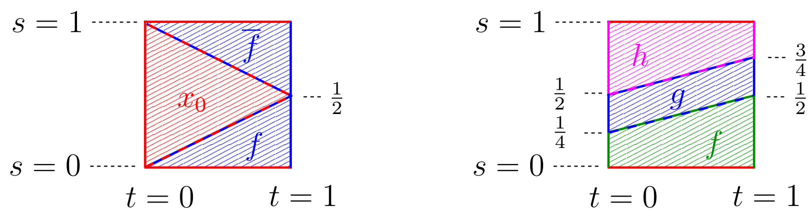


FIGURE 1662. Illustrations for the proofs of Lemma 121.10 (4) and (5).

The loop space construction is so much fun that one might as well apply it several times. This leads us to the following notation:

Notation. Let (X,x_0) be a pointed topological space. We define

$$\Omega^k(X,x_0) \;:=\; \Omega\cdots\Omega(\underbrace{\Omega(X,x_0)}_{\text{pointed topological space}}) \quad \text{where we apply the } \Omega\text{-functor } k \text{ times.}$$

In Figure 1663 we do our best at illustrating a loop in the pointed topological space $(\Omega(X,x_0),c_{x_0})$. What should become apparent is that such a loop is basically the same as a map $([0,1]^2,\partial[0,1]^2) \to (X,x_0)$. The following proposition makes this idea precise.

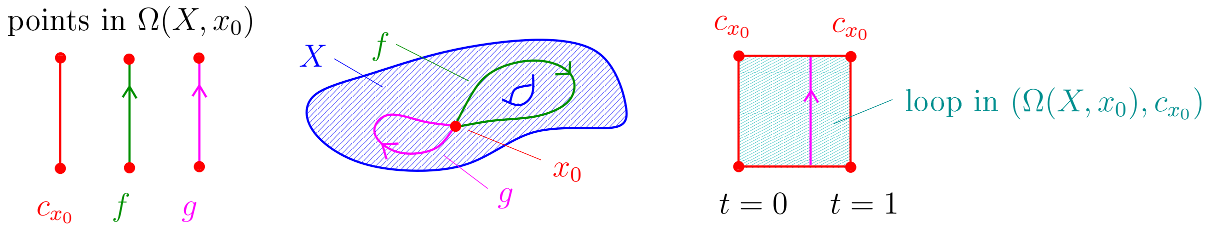


FIGURE 1663

Proposition 121.11. *Let (X,x_0) be a pointed topological space and let $k\in\mathbb{N}$. The map*

$$\begin{array}{rcl}\overbrace{\{F\in X^{[0,1]^k}\,|\,F(\partial[0,1]^k)=\{x_0\}\}}^{\text{subset of the topological space } X^{[0,1]^k}} & \to & \Omega^k(X,x_0)\\ F & \mapsto & (t_k\mapsto(t_{k-1}\mapsto\ldots\underbrace{(t_1\mapsto F(t_1,\ldots,t_k))}_{\in\Omega(X,x_0)}\ldots))\end{array}$$

is a homeomorphism.

PROOF (∗). To simplify the notation we only discuss the case $k=2$. The general case is treated the same way. We write $I=J=[0,1]$. We start out with the following subtle observation:

(∗) The inclusion $\Omega(X,x_0)\to X^I$ induces a map $\Omega(X,x_0)^J\to(X^I)^J$. By Lemma 5.5 (2) this map is an embedding. This means that it is legitimate to view the topological space $\Omega(\Omega(X,x_0),c_{x_0})$ as a subspace of $(X^I)^J$.

This preamble implies that it suffices to show that the map

$$\begin{array}{rcl}\Phi\colon\{F\in X^{I\times J}\,|\,F(\partial(I\times J))=\{x_0\}\} & \to & \Omega(\Omega(X,x_0),c_{x_0})\subset(X^I)^J\\ F & \mapsto & \left(\begin{array}{rcl} J & \to & \Omega(X,x_0)\subset X^I\\ t & \mapsto & \left(\begin{array}{rcl} I & \to & X\\ s & \mapsto & F(s,t)\end{array}\right)\end{array}\right)\end{array}$$

is a homeomorphism. First note that the map Φ is, basically by definition, a bijection. Next note that Φ is a restriction of the map

$$\begin{array}{rcl}\Psi\colon X^{I\times J} & \to & (X^I)^J\\ (F\colon I\times J\to X) & \mapsto & \left(\begin{array}{rcl} J & \to & X^I\\ t & \mapsto & (s\mapsto F(s,t))\end{array}\right).\end{array}$$

But it follows from the intricate Proposition 5.7, and the fact that $I=J=[0,1]$ are regionally compact and Hausdorff, that the map Ψ is a homeomorphism. Thus we have shown that our original map Φ is also a homeomorphism. ■

Now let us turn towards the suspension operation. Given a pointed topological space (X,x_0) ideally we would like to equip the suspension $\Sigma(X)$ with a base point in a "systematic" way. The obvious idea is to fix some $t\in[-1,1]$ once and for all and to equip every suspension $\Sigma(X)$ with the base point represented by $[(x_0,t)]$. As we will see later on, we really really want the pinching map to be a map of pointed topological spaces. But if one thinks about it, and if one keeps an eye on Figure 1664, one see that for every single choice of fixed t the pinching map (and note that for the wedge operation we also have to use our specified base points) will fail to be base point preserving.

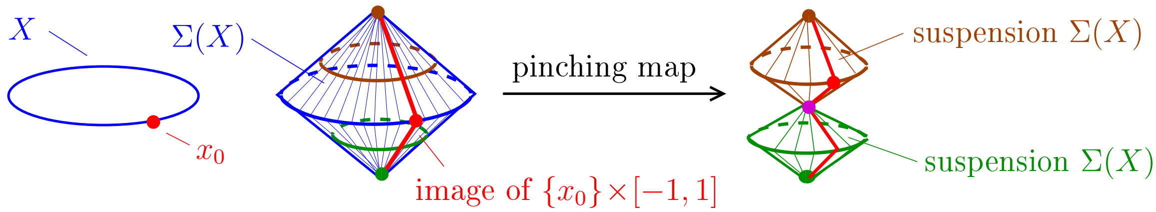


FIGURE 1664

There is a rather "brutal" solution to the dilemma: We solve the conundrum of which $[(x_0,t)]$ to choose by declaring all these points to be the same. More precisely, we consider the following modified version of the suspension.

Definition. Given a pointed topological space (X,x_0) we defined the *reduced suspension* of (X,x_0) to be the topological space[1708]

$$S(X,x_0) \;:=\; \Sigma(X)/[\{x_0\}\times[-1,1]] \underset{\substack{\uparrow \\ \text{strictly speaking we are using Lemma 3.28 (1)}}}{=} (X\times[-1,1])/(X\times\{-1,1\}\cup\{x_0\}\times[-1,1]).$$

We always equip $S(X,x_0)$ with the base point $s_{x_0} := [\{x_0\}\times[-1,1]]$. We refer to Figure 1665 for an illustration.

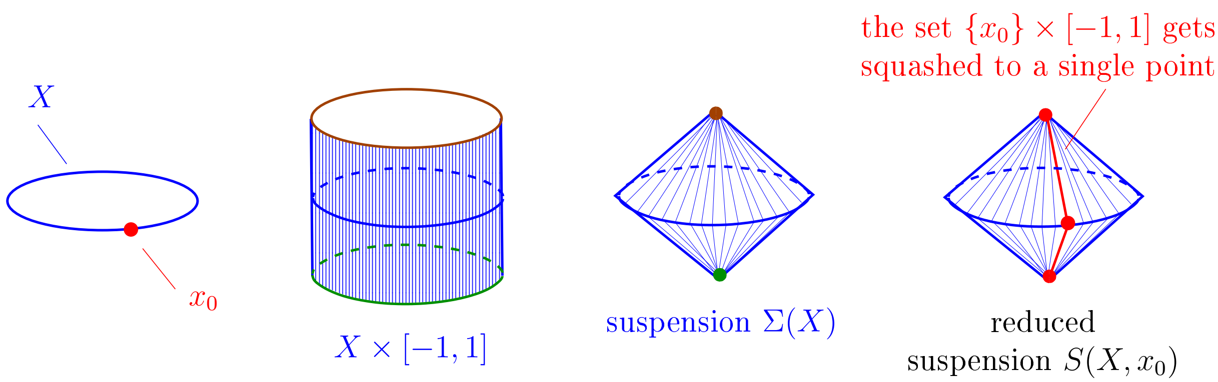


FIGURE 1665

The following lemma summarizes the key properties of the reduced suspension. The lemma can partly be viewed as the "pointed" version of the second part of Lemma 121.3.

Lemma 121.12.

[1708]Note that given a topological space X we denote the usual suspension by $\Sigma(X)$ and given a pointed topological space we denote its reduced suspension by $S(X,x_0)$. Rather confusingly this notation is the opposite of the notation used in [**Hat02**].

(1) *The maps*[1709]

$$\begin{array}{rcl} (X,x_0) & \mapsto & (S(X,x_0), s_{x_0}) \\ (f\colon (X,x_0)\to (Y,y_0)) & \mapsto & \left(\begin{array}{rcl} S(f)\colon S(X,x_0) & \to & S(Y,y_0) \\ {[(x,t)]} & \mapsto & [(f(x),t)] \end{array}\right) \end{array}$$

define a covariant functor from the category $\mathcal{PHomTop}$ to itself.

(2) *The obvious projection map $\Sigma(X)\to S(X,x_0)$ defines a natural transformation between the functors $(X,x_0)\mapsto\Sigma(X)$ and $(X,x_0)\mapsto S(X,x_0)$. In other words, in plain English, given a map $f\colon (X,x_0)\to(Y,y_0)$ between pointed topological spaces the following diagram commutes:*

$$\begin{array}{ccc} \Sigma(X) & \xrightarrow{\Sigma(f)} & \Sigma(Y) \\ \downarrow & & \downarrow \\ S(X,x_0) & \xrightarrow{S(f)} & S(Y,y_0). \end{array}$$

(3) *Given any pointed topological space (X,x_0) the pinching map*

$$\begin{array}{rcl} \omega\colon S(X,x_0) & \to & S(X,x_0)\vee S(X,x_0) \\ {[(x,t)]} & \mapsto & \begin{cases} [(x,2t+1)] \text{ in the first term}, & \text{if } t\in[-1,0], \\ [(x,2t-1)] \text{ in the second term}, & \text{if } t\in[0,1]. \end{cases} \end{array}$$

is continuous and base point preserving.

(4) *If (X,x_0) is a pointed CW-complex, then the following two statements hold:*
   (a) *The reduced suspension $S(X,x_0)$ has a natural CW-structure such that the obvious projection map $\Sigma(X)\to S(X,x_0)$ is a cellular map and such that $\{s_{x_0}\}$ is a 0-cell. In particular $(S(X,x_0),s_{x_0})$ is a pointed CW-complex.*
   (b) *Given any $t\in[-1,1]$ the projection map $\Sigma(X)\to S(X,x_0)$ is a homotopy equivalence of the pointed topological spaces $(\Sigma(X),[(x_0,t)])\to(S(X,x_0),s_{x_0})$.*

Convention. Given a pointed CW-complex (X,x_0) we always equip the reduced suspension $S(X,x_0)$ with the CW-structure coming from Lemma 121.12 (4a).

Proof.

(1) The first statement is proved basically the same way as we proved Lemma 24.4. Note that the proof of the statement that if $f,g\colon (X,x_0)\to(Y,y_0)$ are homotopic implies that $S(f),S(g)$ are homotopic is somewhat delicate, see the proof of Lemma 24.4 (4).
(2) This statement follows almost immediately from the definitions.
(3) This statement follows easily from Lemma 3.22.
(4) The last statement is slightly more subtle. We proceed as follows:
   (a) By Proposition 36.23 (3b) we can equip $X\times[-1,1]$ with the product CW-structure.
   (b) As Corollary 36.35 we equip the suspension $\Sigma(X)=(X\times[-1,1])/\sim$ with the quotient CW-structure.

[1709] It follows from Lemma 3.22 that the map $S(f)$ is actually continuous.

(c) This CW-structure has the property that the image of $\{x_0\} \times [-1,1]$ in $\Sigma(X)$ is a subcomplex which we now denote by A.

(d) It follows from Lemma 36.32 that the quotient $\Sigma(X)/A = S(X,x_0)$ admits a natural CW-structure which has the convenient property that the projection map $\Sigma(X) \to \Sigma(X)/A = S(X,x_0)$ is cellular and which has the property that s_{x_0} is a 0-cell.

By Theorem 39.1 we know that the inclusion map $i\colon A \to X$ is a cofibration. Since $[(x_0,t)]$ is a deformation retract of A we obtain from Lemma 39.9 (3) that the obvious projection map $\Sigma(X) \to S(X,x_0)$ is a homotopy equivalence of the pointed topological spaces $(\Sigma(X),[(x_0,t)]) \to (S(X,x_0),s_{x_0})$. ∎

The following innocuous looking proposition is one of the main building stones for the subsequent discussion.

Proposition 121.13. *Let (X,x_0) and (Y,y_0) be pointed topological spaces. The map*

$$\Upsilon\colon \langle (S(X,x_0),s_{x_0}),(Y,y_0)\rangle \;\to\; \langle (X,x_0),(\Omega(Y,y_0),c_{y_0})\rangle$$

$$\left[\begin{array}{rcl} f\colon S(X,x_0) & \to & Y \\ {[(x,t)]} & \mapsto & f([(x,t)]) \end{array}\right] \mapsto \left[\begin{array}{rcl} X & \to & \Omega(Y,y_0) \\ x & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & Y \\ t & \mapsto & f([(x,2t-1)]) \end{array}\right) \end{array}\right]$$

is a bijection. Furthermore this bijection is natural in (X,x_0) and (Y,y_0).[1710]

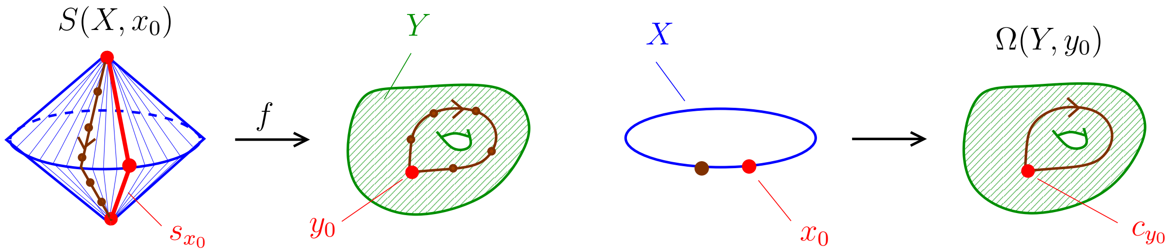


FIGURE 1666. Illustration of Proposition 121.13.

Remark. Recall that we denote by *PHomTop* the pointed homotopy category that we introduced on page 2849. In the language of the definition on page 2673 the proposition says that the functors

$$F\colon \mathit{PHomTop} \to \mathit{PHomTop} \qquad\qquad G\colon \mathit{PHomTop} \to \mathit{PHomTop}$$

which are defined by

$$\begin{array}{rcl} (X,x_0) & \mapsto & S(X,x_0) \\ {[f]} & \mapsto & [S(f)] \end{array} \quad \text{and} \quad \begin{array}{rcl} (X,x_0) & \mapsto & \Omega(X,x_0) \\ {[f]} & \mapsto & [\Omega(f)] \end{array}$$

are adjoint.

[1710]Put somewhat more precisely: if we are given a pointed topological space (Y,y_0), then the two maps $(X,x_0) \mapsto \langle (S(X,x_0),s_{x_0}),(Y,y_0)\rangle$ and $(X,x_0) \mapsto \langle (X,x_0),(\Omega(Y,y_0),c_{y_0})\rangle$ define contravariant functors from the category of pointed topological spaces to the category of sets. The map Υ defines a natural isomorphism between these two functors. The analogous statement holds with the roles of (X,x_0) and (Y,y_0) reversed.

PROOF. Let (X,x_0) and (Y,y_0) be pointed topological spaces. First we consider the map

$$\overbrace{\{\text{set of maps } X\times[-1,1]\to Y\}}^{=Y^{X\times[-1,1]}} \;\to\; \overbrace{\{\text{set of maps } X\to Y^{[0,1]}\}}^{=(Y^{[0,1]})^X}$$
$$\left(\begin{array}{rcl} f\colon X\times[-1,1] & \to & Y \\ (x,t) & \mapsto & f(x,t)\end{array}\right) \;\mapsto\; \left(\begin{array}{rcl} X & \to & Y^{[0,1]} \\ x & \mapsto & \left(\begin{array}{rcl}[0,1] & \to & Y \\ t & \mapsto & f(x,2t-1)\end{array}\right)\end{array}\right).$$

It follows from Proposition 5.6 and the fact that $[0,1]\cong[-1,1]$ is regionally compact that this map is well-defined[1711] and that it is a bijection. Using Lemma 3.24 it is not hard to see that the given map induces a bijection

$$\{\text{set of maps } (S(X,x_0),s_{x_0})\to(Y,y_0)\} \;\to\; \{\text{set of maps } (X,x_0)\to(\Omega(Y,y_0),c_{y_0})\}.$$

Basically the same approach also deals with homotopies. More precisely, it follows from Proposition 5.6 and the fact that $[0,1]\cong[-1,1]$ is regionally compact that the "obvious" map

$$Y^{[-1,1]\times(X\times[0,1])} \;\to\; \big(Y^{[0,1]}\big)^{X\times[0,1]}$$

is a bijection. Using Proposition 18.20 one can now show that this bijection induces a bijection between homotopies of maps $(S(X,x_0),s_{x_0})\to(Y,y_0)$ and homotopies of maps $(X,x_0)\to(\Omega(Y,y_0),c_{y_0})$. In summary this shows that the map

$$\Upsilon\colon \langle (S(X,x_0),s_{x_0}),(Y,y_0)\rangle \;\to\; \langle (X,x_0),(\Omega(Y,y_0),c_{y_0})\rangle$$

is indeed a bijection. We leave it to the reader to fill in the details.

Finally we point out that it follows immediately from the definitions that Υ is natural in (X,x_0) and (Y,y_0). ■

121.3. **Eilenberg-Maclane spaces give rise to invariants of topological spaces.** Ideally we prefer to work, for as long as possible, with CW-complexes. Alas not all interesting constructions of new topological spaces necessarily turn CW-complexes into CW-complexes. For example, if (X,x_0) is a pointed CW-complex it is not clear whether the loop space $\Omega(X,x_0)$ is again a CW-complex.[1712]

This leads us to the following definition.

Definition. We denote by $\mathcal{W}$ the class of all pointed topological spaces (X,x_0) such that there exists a pointed CW-complex (Y,y_0) which is homotopy equivalent to (Y,y_0).

The following theorem was first proved by Milnor in 1958.

Theorem 121.14. *If $(X,x_0)\in\mathcal{W}$, then for any $k\in\mathbb{N}$ the iterated loop space $\Omega^k(X,x_0)$ also lies in $\mathcal{W}$.*

[1711] Here "well-defined" means that the map $X\to Y^{[0,1]}$ that we associate to $f\colon X\times[-1,1]\to Y$ is indeed continuous.

[1712] I could not find an example where $\Omega(X,x_0)$ does not admit a CW-structure and I could not find a theorem that says that $\Omega(X,x_0)$ always admits a CW-structure.

PROOF. If (X, x_0) is a pointed CW-complex, then [**Miln59**, Corollary 3] implies that the pointed topological space $(\Omega(X, x_0), c_{x_0})$ lies in $\mathcal{W}$. (Alternatively we refer to [**FrPi90a**, Corollary 5.3.7] for a proof of a very similar statement.) More generally, if $(X, x_0) \in \mathcal{W}$, then it follows from Lemma 5.5 (5) and the above that $(\Omega(X, x_0), c_{x_0})$ again lies in $\mathcal{W}$. Iterating this argument we also obtain the statement for every $k \geq 2$. ■

This leads us to the following definition:

Definition. Let $n \in \mathbb{N}$ and let G be an (abelian) group[1713].

(1) A *weak pointed Eilenberg-Maclane space of type* $\mathrm{K}(G, n)$ is a pair $(X, x_0) \in \mathcal{W}$ such that for any $i \neq n$ we have $\pi_i(X, x_0) = 0$ and such that $\pi_n(X, x_0) \cong G$.
(2) A *polarized Eilenberg-Maclane space*[1714] *of type* $\mathrm{K}(G, n)$ is a triple (X, x_0, α) where (X, x_0) is a weak pointed Eilenberg-Maclane space of type $\mathrm{K}(G, n)$ and α is an isomorphism $\pi_n(X, x_0) \xrightarrow{\cong} G$.

Examples.

(1) We equip S^1 with the base point $* = (1, 0)$. Let $\varphi\colon \pi_1(S^1, *) \xrightarrow{\cong} \mathbb{Z}$ be the standard isomorphism from page 1068. We see that $(S^1, *, \varphi)$ is a polarized Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 1)$.
(2) We equip $\mathbb{C}\mathrm{P}^\infty$ with the base point $* = [1 : 0 : 0 \dots]$. Let $\psi\colon \pi_2(\mathbb{C}\mathrm{P}^\infty, *) \xrightarrow{\cong} \mathbb{Z}$ be the usual isomorphism from Theorem 113.12 (2). We see that $(\mathbb{C}\mathrm{P}^\infty, *, \psi)$ is a polarized Eilenberg-Maclane space of type $\mathrm{K}(\mathbb{Z}, 2)$.

Now we turn to more general constructions of polarized Eilenberg-Maclane spaces.

Examples. Let $n \in \mathbb{N}$ and let G be an (abelian) group.

(1) We denote by $\mathrm{K}(G, n)$ the canonical Eilenberg-Maclane space of type $\mathrm{K}(G, n)$ that was introduced in Theorem 120.8. We denote its canonical base point by $*$. Furthermore we recall that the canonical Eilenberg-Maclane space $\mathrm{K}(G, n)$ comes with a canonical isomorphism $\varphi_G\colon \pi_n(\mathrm{K}(G, n), *) \xrightarrow{\varphi} G$.
(2) Let (X, x_0) be a pointed topological space. By Corollary 112.15 we know that for any $k \in \mathbb{N}$ the map

$$\begin{array}{rcl} \partial_{k+1}\colon \pi_{k+1}(X, x_0) & \to & \pi_k(\Omega(X, x_0), c_{x_0}) \\ \left[[0,1]^{k+1} \xrightarrow{f} X\right] & \mapsto & \left[\begin{array}{rcl} [0,1]^k & \to & \Omega(X, x_0) \\ x & \mapsto & \left(\begin{array}{rcl} [0,1] & \to & X \\ t & \mapsto & f(x,t) \end{array} \right) \end{array} \right] \end{array}$$

is a natural isomorphism. This observation, together with Theorem 121.14, shows that if (X, x_0, α) is a polarized Eilenberg-Maclane space of type $\mathrm{K}(G, n+1)$, then $(\Omega(X, x_0), c_{x_0}, \alpha \circ \partial_{n+1}^{-1})$ is a polarized Eilenberg-Maclane space of type $\mathrm{K}(G, n)$.

[1713] As explained in Footnote 1692, whenever we deal with Eilenberg-Maclane spaces of type $\mathrm{K}(\pi, n)$ with $n \geq 2$ it is understood that π is abelian.

[1714] Our language now allows for the slightly awkward possibility that a polarized Eilenberg-Maclane space is not an Eilenberg-Maclane space since the underlying topological space might not be a CW-complex. We are confident though that this will not become an issue.

The discussion above shows that given any $k \in \mathbb{N}_0$ we can and will view $\Omega^k(\mathrm{K}(G, n+k))$ as a polarized Eilenberg-Maclane of type $\mathrm{K}(G,n)$.

The following lemma shows that in the based homotopy category all polarized Eilenberg-Maclane spaces of a given type are essentially the same.

Lemma 121.15. *Let (K, k_0, φ) and (L, l_0, ψ) be two polarized Eilenberg-Maclane spaces of the same type. There exists a unique class*[1715] $\Xi = \Xi_{(K,k_0),(L,l_0)} \in \langle (K, k_0), (L, l_0)\rangle_{\mathrm{inv}}$ *such that the induced map $\Xi_* \colon \pi_n(K, k_0) \to \pi_n(L, l_0)$ agrees with $\psi^{-1} \circ \varphi$. When there is no danger of confusion we just write $\Xi = \Xi_{(K,k_0),(L,l_0)}$.*

Proof. Let $n \in \mathbb{N}$, let G be an (abelian) group and let (K, k_0, φ) and (L, l_0, ψ) be two polarized Eilenberg-Maclane spaces of type $\mathrm{K}(G,n)$. Since (K, k_0) and (L, l_0) lie in $\mathcal{W}$ we know that there exist pointed CW-complexes (X, x_0) and (Y, y_0) together with homotopy equivalences $\kappa \colon (X, x_0) \to (K, k_0)$ and $\lambda \colon (Y, y_0) \to (L, l_0)$ of pointed topological spaces. We consider the following diagram

$$\begin{tikzcd}
\langle (K, k_0), (L, l_0)\rangle_{\mathrm{inv}} \arrow[r, "{[f]\mapsto f_*}"] \arrow[d, "\kappa^*"'] & \mathrm{Hom}(\pi_n(K, k_0), \pi_n(L, l_0)) \arrow[d, "\kappa^*"'] \\
\langle (X, x_0), (L, l_0)\rangle_{\mathrm{inv}} \arrow[r, "{[f]\mapsto f_*}"] \arrow[d, "\lambda_*"] & \mathrm{Hom}(\pi_n(X, x_0), \pi_n(L, l_0)) \arrow[d, "\lambda_*"] \\
\langle (X, x_0), (Y, y_0)\rangle_{\mathrm{inv}} \arrow[r, "{[f]\mapsto f_*}"] & \mathrm{Hom}(\pi_n(X, x_0), \pi_n(Y, y_0)).
\end{tikzcd}$$

It follows almost immediately from the definitions that this diagram commutes. We obtain from Lemmas 121.6 and 15.1 that the vertical maps to the left are bijections. Furthermore we obtain from Proposition 40.7 that the vertical maps to the right are bijections. It follows from Theorem 120.12 and Proposition 120.15 that any isomorphism in $\mathrm{Hom}(\pi_n(X, x_0), \pi_n(Y, y_0))$ is realized by a unique element in $\langle (X, x_0), (Y, y_0)\rangle_{\mathrm{inv}}$. By the diagram this information propagates to the top. ■

Notation. Let $n \in \mathbb{N}$, let G be an abelian group and let $k \in \mathbb{N}_0$. From the discussion on page 2859 and Lemma 121.15 we obtain a canonical element $\Xi \in \langle \mathrm{K}(G,n), \Omega^k\,\mathrm{K}(G, n+k)\rangle_{\mathrm{inv}}$. By Lemma 15.1 we know that the map

$$\begin{array}{rcl}
\Xi_* \colon \langle (X, x_0), \overbrace{\mathrm{K}(G,n)}^{\substack{\text{viewed as a pointed}\\ \text{topological space}}} \rangle & \to & \langle (X, x_0), \overbrace{\Omega^k\,\mathrm{K}(G, n+k)}^{\substack{\text{viewed as a pointed}\\ \text{topological space}}} \rangle \\
{[g]} & \mapsto & \Xi \circ [g]
\end{array}$$

is a bijection. Sometimes we use the bijection Ξ_* to identify the left-hand side with the right-hand side. We make the analogous identification if (X, x_0) is the target instead of the domain.

We formulate the next example as a lemma. Since we will not directly make use of it we take a few liberties in its formulation.

[1715] To lessen the burden on the reader's eyes we do not include φ and ψ in the notation.

Lemma 121.16. *We equip S^1 with the base point $* = (1,0)$ and we equip $\mathbb{C}\mathrm{P}^\infty$ with the base point $[1:0:0\dots]$. As on page 2859 we consider the polarized Eilenberg-Maclane spaces $(S^1,*,\varphi)$ of type $\mathrm{K}(\mathbb{Z},1)$ and we consider the polarized Eilenberg-Maclane space $(\mathbb{C}\mathrm{P}^\infty,*,\psi)$ of type $\mathrm{K}(\mathbb{Z},2)$. We have the following homeomorphisms and inclusions maps*

$$\begin{array}{ccccccccc} [0,1]^2/\partial[0,1]^2 & \xrightarrow{\cong} & \overline{B}^2/S^1 & \xrightarrow{\cong} & S^2 & \xrightarrow{\cong} & \mathbb{C}\mathrm{P}^1 & \hookrightarrow & \mathbb{C}\mathrm{P}^\infty. \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ & \textit{page 126} & & \textit{page 182} & & \textit{page 197} & & \textit{page 944} & \end{array}$$

We use these maps to view $[0,1]^2/\partial([0,1]^2)$ as a subset of $\mathbb{C}\mathrm{P}^\infty$.[1716] *The map*

$$\begin{array}{rcl} \Xi\colon S^1 & \to & \Omega(\mathbb{C}\mathrm{P}^\infty,*) \\ e^{\mathrm{i}s} & \mapsto & \left(\begin{array}{rcl} [0,1] & \mapsto & [0,1]^2/\partial([0,1]^2)\subset \mathbb{C}\mathrm{P}^\infty \\ t & \mapsto & [(s,t)] \end{array}\right) \end{array}$$

represents the class $\Xi \in \langle (S^1,*),\Omega(\mathbb{C}\mathrm{P}^\infty,*)\rangle$ given by Lemma 121.15.

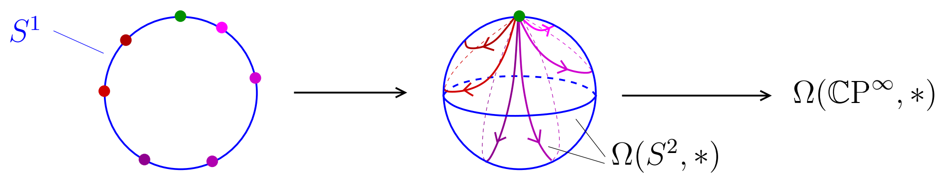


FIGURE 1667. Illustration of Lemma 121.16.

PROOF (*). Let $i\colon [0,1]^2/\partial([0,1]^2) \to \mathbb{C}\mathrm{P}^\infty$ be the above inclusion map. It follows immediately from the definition of $\Xi\colon S^1 \to \Omega(\mathbb{C}\mathrm{P}^\infty,*)$ that the map Ξ factors through a map $\eta\colon S^1 \to \Omega([0,1]^2/\partial([0,1]^2),*)$. We consider the following diagram:

$$\begin{tikzcd} \pi_2([0,1]^2/\partial([0,1]^2),*) \arrow[r, "\partial_2", "\cong"'] \arrow[d, "i_*"] & \pi_1(\Omega([0,1]^2/\partial([0,1]^2),*)) \arrow[d, "\Omega(i)_*"] & \\ \pi_2(\mathbb{C}\mathrm{P}^\infty,*) \arrow[r, "\partial_2", "\cong"'] & \pi_1(\Omega(\mathbb{C}\mathrm{P}^\infty,*)). & \pi_1(S^1,*) \arrow[ul, "\eta_*"'] \arrow[l, "\Xi_*"] \end{tikzcd}$$

We make the following observations:

(1) The horizontal maps are defined in Corollary 112.15. Furthermore note that Corollary 112.15 tells us that the horizontal maps are isomorphisms and that the square commutes.
(2) The triangle commutes since $\Xi = \Omega(i)\circ\eta$.
(3) The left-hand vertical map is an isomorphism by Corollary 53.6.
(4) By Corollary 53.6 we know that $\pi_2([0,1]^2/\partial([0,1]^2),*) = \mathbb{Z}\cdot \mathrm{id}_{[0,1]^2}$ and by Proposition 16.17 (1) we know that $\pi_1(S^1,*) = \mathbb{Z}\cdot\mathrm{id}_{[0,1]}$.
(5) It follows easily from the definition of ∂_2, see Corollary 112.15, that ∂_2 and η_* send their respective generators to the same element in $\pi_1(\Omega([0,1]^2/\partial([0,1]^2),*))$. Since

[1716] Note that by the definition of the above maps the point $\partial([0,1]^2)$ corresponds to the point $* = [1\colon 0\colon 0\colon \dots] \in \mathbb{C}\mathrm{P}^\infty$.

all groups are isomorphic to $\mathbb{Z}$ and since ∂_2 is an isomorphism we see that η_* is an isomorphism.

(6) We obtain from the above data that Ξ_* is an isomorphism. It is now basically clear that Ξ represents the desired element in $\langle (S^1,*), \Omega(\mathbb{C}\mathrm{P}^\infty,*)\rangle$. ■

So far it seems like we have been rambling. But the following amazing definition puts us finally on an interesting track.

Definition. Let $n \in \mathbb{N}$ and let G be an abelian group. Furthermore let $X = (X, x_0)$ be a pointed topological space. To simplify the notation we suppress the base point x_0 throughout the subsequent discussion.[1717] We refer to the following map as the *multiplication map* on $\langle X, \mathrm{K}(G,n)\rangle$:

$$
\begin{array}{rcl}
 & \overset{\text{since } G \text{ is abelian the Eilenberg-Maclane space } \Omega\,\mathrm{K}(G,n+1) \text{ exists for every } n}{\text{thus we can use the identification } \Xi_* \text{ from page 2860}} & \\
\big\langle X, \mathrm{K}(G,n)\big\rangle \times \big\langle X, \mathrm{K}(G,n)\big\rangle & \overset{\downarrow}{=} & \big\langle X, \Omega\,\mathrm{K}(G,n+1)\big\rangle \times \big\langle X, \Omega\,\mathrm{K}(G,n+1)\big\rangle \\
 & & \big\downarrow \text{map } \chi \text{ from Lemma 121.7} \\
 & & \big\langle X, \Omega\,\mathrm{K}(G,n+1) \times \Omega\,\mathrm{K}(G,n+1)\big\rangle \\
 & & \big\downarrow \mu_* \text{ defined on page 2851} \\
 & & \big\langle X, \Omega\,\mathrm{K}(G,n+1)\big\rangle \underset{\uparrow}{=} \big\langle X, \mathrm{K}(G,n)\big\rangle. \\
 & & \text{again using the identification } \Xi_* \text{ from page 2860}
\end{array}
$$

Remark.

(1) It is worth stressing what we just we did. In the definition of the multiplication on $\langle (X,x_0), \mathrm{K}(G,n)\rangle$ we used two essential facts:
   (a) We could replace $\mathrm{K}(G,n)$ by the loop space $\Omega(\mathrm{K}(G,n+1))$.
   (b) The loop space $\Omega(\mathrm{K}(G,n+1))$ comes with a multiplication.
   Later on we will use these two ideas in a more general context.

(2) It is perhaps worth pointing out that in general topological spaces are *not* even homotopy equivalent to loop spaces. For example we know by Lemma 114.22 together with Theorem 114.21, the discussion on page 2720 and Exercise 114.8 that S^n with $n \neq 1,3,7$ and $\mathbb{C}\mathrm{P}^n$ with $n \in \mathbb{N}$ are *not* homotopy equivalent to loop spaces.

The following proposition says that the above definition is actually meaningful.

Proposition 121.17. *Let $n \in \mathbb{N}$, let G be an abelian group and let (X,x_0) be a pointed topological space. The above multiplication turns the set $\langle (X,x_0), \mathrm{K}(G,n)\rangle$ into a group. Furthermore the neutral element of the group $\langle (X,x_0), \mathrm{K}(G,n)\rangle$ is given, not surprisingly, by 0.*[1718]

[1717] It is though essential that all maps we consider are in fact base point preserving. In particular, recall that $\mathrm{K}(G,n)$ and $\Omega(\mathrm{K}(G,n))$ are pointed topological spaces.

[1718] Recall that on page 2849 we defined $0 \in \langle (X,x_0),(Y,y_0)\rangle$ to be the class that is represented by the constant map $X \to Y$, $x \mapsto y_0$.

Proof. Let $n \in \mathbb{N}$, let G be an abelian group and let (X, x_0) be a pointed topological space. Recall that we have the identification

$$\Xi_*\colon \langle (X,x_0), \mathrm{K}(G,n)\rangle \;=\; \langle (X,x_0), \Omega\,\mathrm{K}(G,n+1)\rangle$$

from page 2860. Note that $\Xi_*(0) = 0$. Thus the proposition follows from the next lemma. ■

Lemma 121.18. *Let (K, k_0) be a pointed topological space. The map*

$$\begin{tikzcd}
\langle (X,x_0), \Omega(K,k_0)\rangle \times \langle (X,x_0), \Omega(K,k_0)\rangle \arrow[d, "\text{map } \chi \text{ from Lemma 121.7}"] \\
\langle (X,x_0), \Omega(K,k_0) \times \Omega(K,k_0)\rangle \arrow[d, "\mu_*"] \\
\langle (X,x_0), \Omega(K,k_0)\rangle.
\end{tikzcd}$$

defines a group structure. The group structure has the following properties:

(1) *The neutral element is given by* 0.
(2) *We consider the map*[1719]

$$\begin{array}{rrcl} \eta\colon & \Omega(K,k_0) & \to & \Omega(K,k_0) \\ & f & \mapsto & \overline{f}. \end{array}$$

The inverse of $[f] \in \langle (X,x_0), \Omega(K,k_0)\rangle$ is given by $\eta_*([f])$.

Proof. As we will see shortly, this statement follows fairly easily from Lemma 121.10. In particular we will see that the statements have preciously little to do with (X, x_0). In an effort to lighten the notation we henceforth write X instead of the more accurate (X, x_0) and similarly we write K instead of (K, k_0). This time we denote by $c \in \Omega K$ the constant path at $k_0 \in K$. We consider the maps

$$\begin{array}{rrcl} \rho\colon & \Omega K & \to & \Omega K \times \Omega K \\ & f & \mapsto & (f, c) \end{array} \qquad \begin{array}{rrcl} \lambda\colon & \Omega K & \to & \Omega K \times \Omega K \\ & f & \mapsto & (c, f) \end{array} \qquad \text{and} \qquad \begin{array}{rrcl} \sigma\colon & \Omega K & \to & \Omega K \times \Omega K \\ & f & \mapsto & (f, \overline{f}). \end{array}$$

Now we turn to the actual proofs of the various statements.

(1) Let us first show that $0 = [c]$ is a neutral element in $\langle X, \Omega K\rangle$. We consider the following diagram

$$\begin{tikzcd}
\langle X, \Omega K\rangle \arrow[r, "{[f]\mapsto([f],[c])}"] \arrow[dr, "\rho_*"] \arrow[ddr, "\mathrm{id}_*"'] & \langle X, \Omega K\rangle \times \langle X, \Omega K\rangle \arrow[d, "\chi"] \arrow[dd, bend left=60, "\text{multiplication}"] \\
& \langle X, \Omega K \times \Omega K\rangle \arrow[d, "\mu_*"] \\
& \langle X, \Omega K\rangle.
\end{tikzcd}$$

[1719] In Lemma 121.10 (3) we showed that η is continuous.

The upper triangle commutes basically by definition of the maps. Now let us turn to the lower triangle. It follows from Lemma 121.10 (2) that the map $\mu \circ \rho \colon \Omega K \to \Omega K$ is homotopic to the identity rel the base point. In other words, we have the equality $[\mu \circ \rho] = [\mathrm{id}] \in \langle \Omega K, \Omega K \rangle$. Thus it follows, if you want from Lemma 15.3, that the triangle does indeed commute.

Now we start out with $[f] \in \langle X, \Omega K \rangle$ at the top left. Since going right-down is precisely right-multiplication by $[c] = 0$ we see that 0 is indeed a right-neutral element with respect to the product structure. Basically the same argument, with ρ replaced by λ shows that 0 is also a left-neutral element.

(2) We consider the following diagram:

$$\begin{tikzcd}
\langle X, \Omega K \rangle \arrow[r, "{[f] \mapsto ([f], \eta_*([f]))}"] \arrow[dr, "\sigma_*"] \arrow[ddr, "0"'] & \langle X, \Omega K \rangle \times \langle X, \Omega K \rangle \arrow[d, "\chi"] \arrow[dd, bend left, "\text{multiplication}"] \\
& \langle X, \Omega K \times \Omega K \rangle \arrow[d, "\mu_*"] \\
& \langle X, \Omega K \rangle.
\end{tikzcd}$$

As in (1) we obtain from Lemma 121.10 (4) that the diagram commutes. It follows that $\eta_*([f])$ is a right-inverse to $[f]$. The same argument shows that it is also a left-inverse to $[f]$.

(3) Using Lemma 121.10 (5) one can show, with the same approach as above, that the product structure on $\langle X, \Omega K \rangle$ is associative. Filling in the details is the content of Exercise 121.3. ■

The reader of the previous proof might ask the following question:

Question 121.19. *Why do we actually consider the sets $\langle (X, x_0), \mathrm{K}(G, n) \rangle$ instead of the sets $\langle (X, x_0), \Omega\, \mathrm{K}(G, n+1) \rangle$? After all, we just showed that the group structure is really defined on the latter set.*

The group structure on $\langle (X, x_0), \mathrm{K}(G, n) \rangle$ is certainly rather mysterious. The following lemma hopefully elucidates the group structure in the case $\mathrm{K}(\mathbb{Z}, 1)$.

Lemma 121.20. *Let $\Xi \colon S^1 \to \Omega(\mathbb{C}\mathrm{P}^\infty, *)$ be the homotopy equivalence defined in Lemma 121.16. Given any topological space X we have the following commutative diagram where the horizontal maps are bijections:*

$$\begin{tikzcd}
{[X, S^1] \times [X, S^1]} \arrow[r, "\Xi_* \times \Xi_*"] \arrow[d, "\nu"] & {[X, \Omega \mathbb{C}\mathrm{P}^\infty] \times [X, \Omega \mathbb{C}\mathrm{P}^\infty]} \arrow[d, "\mu_*"] \\
{[X, S^1]} \arrow[r, "\Xi_*"] & {[X, \Omega \mathbb{C}\mathrm{P}^\infty].}
\end{tikzcd}$$

Here the left vertical map is given by Lemma 121.4.

Proof. The statement of the lemma has nothing to do with X. In fact, it suffices to show that the following map agrees up to homotopy:

$$\begin{tikzcd} S^1 \times S^1 \arrow[r, "\Xi\times\Xi"] \arrow[d, "(z,w)\mapsto z\cdot w"] & (\Omega\mathbb{C}\mathrm{P}^\infty,*)\times(\Omega\mathbb{C}\mathrm{P}^\infty,*) \arrow[d, "\mu"] \\ S^1 \arrow[r, "\Xi"] & \Omega(\mathbb{C}\mathrm{P}^\infty,*). \end{tikzcd}$$

We outsource this verification to Exercise 121.1. ■

Perhaps the first question that arises when we introduce a group is whether the group abelian. It turns out that in this case we can answer this question in the affirmative.

Proposition 121.21. *Let $n\in\mathbb{N}$ and let G be an abelian group. For every pointed topological space (X,x_0) the group $\langle (X,x_0),\mathrm{K}(G,n)\rangle$ is abelian.*

Proof. Let $n\in\mathbb{N}$ and let G be an abelian group. We write $L=\mathrm{K}(G,n+1)$ and we write $M=\mathrm{K}(G,n+2)$. Finally let (X,x_0) be a pointed topological space. We make the following preparations.

(1) By the discussion on page 2859 we know that $\mathrm{K}(G,n)$, ΩL and $\Omega(\Omega M)$ are polarized Eilenberg-Maclane spaces of type $\mathrm{K}(G,n)$. Thus we obtain from Lemma 121.15 homotopy equivalences $f\colon(\mathrm{K}(G,n),*)\to(\Omega L,*)$ and $g\colon(\Omega L,*)\to(\Omega(\Omega M),*)$.

(2) We define

$$\square M \;:=\; \{f\colon[0,1]^2\to M \mid f(\partial[0,1]^2)=\{*\}\}.$$

As always we equip $\square M$ with the compact-open topology. Furthermore let

$$\begin{array}{rcl}\Theta\colon \square M & \to & \Omega(\Omega M)\\ (f\colon[0,1]^2\to M) & \mapsto & (t\mapsto(s\mapsto f(s,t)))\end{array}$$

be the *homeomorphism* from Proposition 121.11.

(3) We consider the map

$$\begin{array}{rcl}\alpha\colon \square M\times\square M & \to & \square M\\ (f,g) & \mapsto & \left(\begin{array}{rcl}[0,1]^2 & \to & M\\ (s,t) & \mapsto & \begin{cases} f(s,2t), & \text{if } t\in[0,\frac12],\\ g(s,2t-1), & \text{if } t\in(\frac12,1]\end{cases}\end{array}\right).\end{array}$$

The same argument as in the proof of Lemma 114.22 (1) shows that α is continuous.

(4) Since the pointed topological space (X,x_0) is really just a bystander we will abbreviate it to X.

Some of the above objects are illustrated in Figure 1668.

Claim. The following diagram commutes:

$$\begin{tikzcd} \langle X,\mathrm{K}(G,n)\rangle\times\langle X,\mathrm{K}(G,n)\rangle \arrow[rr, "\text{group multiplication}"] \arrow[d, "(\Theta^{-1}\circ g\circ f)_*\times(\Theta^{-1}\circ g\circ f)_*"] & & \langle X,\mathrm{K}(G,n)\rangle \arrow[d, "(\Theta^{-1}\circ g\circ f)_*"] \\ \langle X,\square M\rangle\times\langle X,\square M\rangle \arrow[r, "\chi"] & \langle X,\square M\times\square M\rangle \arrow[r, "\alpha_*"] & \langle X,\square M\rangle. \end{tikzcd}$$

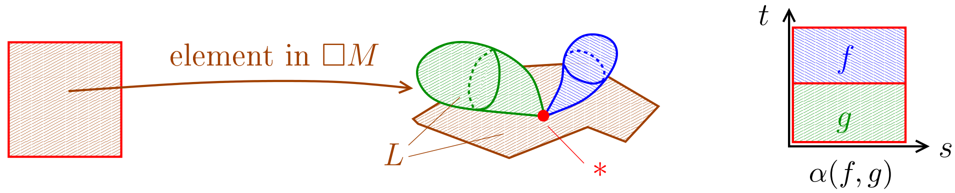


FIGURE 1668

Furthermore the vertical maps are bijections.

To show that the above diagram commutes we expand it into an even larger diagram. More precisely, we consider the following diagram:

$$\begin{tikzcd}
\langle X, \mathrm{K}(G,n)\rangle \times \langle X, \mathrm{K}(G,n)\rangle \arrow[rr, "\text{group multiplication}"] \arrow[d, "f_*\times f_*"] & & \langle X, \mathrm{K}(G,n)\rangle \arrow[d, "f_*"] \\
\langle X, \Omega L\rangle \times \langle X, \Omega L\rangle \arrow[r, "\chi"] \arrow[d, "g_*\times g_*"] & \langle X, \Omega L \times \Omega L\rangle \arrow[r, "\mu_*"] \arrow[d, "(g_*\times g_*)_*"] & \langle X, \Omega L\rangle \arrow[d, "g_*"] \\
\langle X, \Omega(\Omega M)\rangle \times \langle X, \Omega(\Omega M)\rangle \arrow[r, "\chi"] \arrow[d, "\Theta_*^{-1}\times\Theta_*^{-1}"] & \langle X, \Omega(\Omega M) \times \Omega(\Omega M)\rangle \arrow[r, "\mu_*"] \arrow[d, "(\Theta_*^{-1}\times\Theta_*^{-1})_*"] & \langle X, \Omega(\Omega M)\rangle \arrow[d, "\Theta_*^{-1}"] \\
\langle X, \square M\rangle \times \langle X, \square M\rangle \arrow[r, "\chi"] & \langle X, \square M \times \square M\rangle \arrow[r, "\alpha_*"] & \langle X, \square M\rangle.
\end{tikzcd}$$

We make the following observations:

(1) By Lemma 15.1 we know that the vertical maps between the first and second and between the second and the third row are bijections.
(2) Since Θ is a homeomorphism we see that the bottom vertical maps are also bijections.
(3) By definition of the group multiplication on $\langle X, \mathrm{K}(G,n)\rangle$ the top rectangle commutes.
(4) It follows immediately from the definition of χ that the two smaller rectangles on the left commute.
(5) It follows from Lemma 121.10 (1) that the middle right rectangle commutes.
(6) It follows immediately from the definition of the maps μ, α and Θ that

$$\alpha \circ (\Theta \times \Theta) \;=\; \Theta \circ \mu \colon \Omega(\Omega M) \times \Omega(\Omega M) \;\to\; \square M.$$

This equality implies that the lower right rectangle commutes. ⊞

The claim can be summarized as saying that it suffices to show that the bottom multiplication map on $\langle X, \square M\rangle$ is commutative. Next we consider the swap map

$$\begin{array}{rcl} \sigma\colon \square M \times \square M & \to & \square M \times \square M \\ (f,g) & \mapsto & (g,f). \end{array}$$

We obtain the following diagram:

$$\begin{tikzcd}
\langle X, \Box M\rangle \times \langle X, \Box M\rangle \arrow[r, "\chi"] \arrow[d, "(a,b)\mapsto(b,a)"] & \langle X, \Box M \times \Box M\rangle \arrow[r, "\alpha_*"] \arrow[d, "\sigma_*"] & \langle X, \Box M\rangle \arrow[d, "\mathrm{id}"] \\
\langle X, \Box M\rangle \times \langle X, \Box M\rangle \arrow[r, "\chi"] & \langle X, \Box M \times \Box M\rangle \arrow[r, "\alpha_*"] & \langle X, \Box M\rangle
\end{tikzcd}$$

We make the following observations:

(1) The statement that the multiplication on $\langle X, \Box M\rangle$ is commutative is, by definition, equivalent to the statement that the large rectangle commutes.

(2) It follows immediately from the definitions of the various maps that the smaller rectangle on the left commutes. Thus it suffices to show that the smaller rectangle on the right commutes.

In particular, since $\alpha_* \circ \sigma_* = (\alpha \circ \sigma)_*$ it suffices to prove the following claim.[1720]

Claim. The maps α and $\alpha \circ \sigma$ from the pointed topological space $\Box M \times \Box M$ to the pointed topological space $\Box M$ are homotopic.

In Figure 1669 we indicate how one can define a homotopy from α to $\alpha \circ \sigma$. We leave it to the punctilious reader to write down a precise formula for a homotopy[1721]

$$(\Box M \times \Box M) \times [0, 1] \;\to\; \Box M.$$

As in the proof of Lemma 121.10 one can use the "⇒"-direction of Proposition 5.6 to show that the map provided by the reader is actually continuous. ■

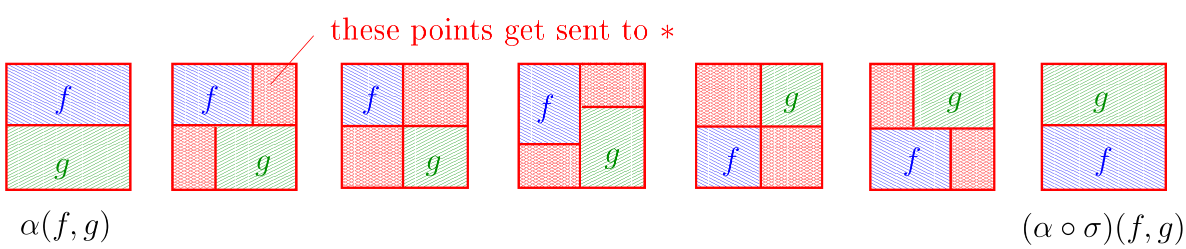


FIGURE 1669

Remark. As the reader with a well-developed long-term memory will have noticed, the proof of Proposition 121.21 eventually is very similar to the proof of Proposition 40.3. In particular Figure 1669 looks almost the same as Figure 684.

121.4. **Alternative description of the group structure.** Once again let $n \in \mathbb{N}$, let G be an abelian group and let (X, x_0) be a pointed topological space. In this section we give an alternative description of the group structure on

$$\langle (X, x_0), \mathrm{K}(G, n)\rangle \;\underset{\substack{\uparrow \\ \text{bijection from page 2860}}}{\xrightarrow{\Xi}}\; \langle (X, x_0), \Omega\, \mathrm{K}(G, n+1)\rangle \;\underset{\substack{\uparrow \\ \text{bijection from Proposition 121.13}}}{\xrightarrow{\Upsilon}}\; \langle S(X, x_0), \mathrm{K}(G, n+1)\rangle.$$

[1720] Thus once again X plays no role in the argument.

[1721] For example this could be done using the maps introduced in the proof of Lemma 40.2.

Before we state the relevant proposition we recall some of the objects that we have introduced recently:

(1) On page 2856 we introduced the pinching map
$$\omega\colon S(X,x_0) \;\to\; S(X,x_0)\vee S(X,x_0)$$
where $S(X,x_0)$ denotes the reduced suspension.

(2) In Lemmas 121.7 and 121.8 we introduced the maps
$$\begin{array}{rcl}\chi\colon \langle (X,x_0),(Y,y_0)\rangle\times\langle (X,x_0),(Z,z_0)\rangle & \to & \langle (X,x_0),(Y\times Z,(y_0,z_0))\rangle \\ ([f\colon X\to Y],[g\colon X\to Z]) & \mapsto & [f\times g\colon X\to Y\times Z]\end{array}$$
and
$$\begin{array}{rcl}\nu\colon \langle (X,x_0),(Z,z_0)\rangle\times\langle (Y,y_0),(Z,z_0)\rangle & \to & \langle (X,x_0)\vee(Y,y_0),(Z,z_0)\rangle \\ ([f\colon X\to Z],[g\colon Y\to Z]) & \mapsto & [f\vee g]\end{array}$$

The following proposition shows that instead of the map μ on the loop space we can also use the pinching map on the reduced suspension to define the group structure.

Proposition 121.22. *Let (X,x_0) and (K,k_0) be pointed topological spaces. The following diagram commutes:*

$$\begin{tikzcd}
\langle (X,x_0),\Omega(K,k_0)\rangle\times\langle (X,x_0),\Omega(K,k_0)\rangle \arrow[r, "\Upsilon\times\Upsilon"] \arrow[d, "\chi"] & \langle S(X,x_0),(K,k_0)\rangle\times\langle S(X,x_0),(K,k_0)\rangle \arrow[d, "\nu"] \\
\langle (X,x_0),\Omega(K,k_0)\times\Omega(K,k_0)\rangle \arrow[d, "\mu_*"] & \langle S(X,x_0)\vee S(X,x_0),(K,k_0)\rangle \arrow[d, "\omega^*"] \\
\langle (X,x_0),\Omega(K,k_0)\rangle \arrow[r, "\Upsilon"] & \langle S(X,x_0),(K,k_0)\rangle.
\end{tikzcd}$$

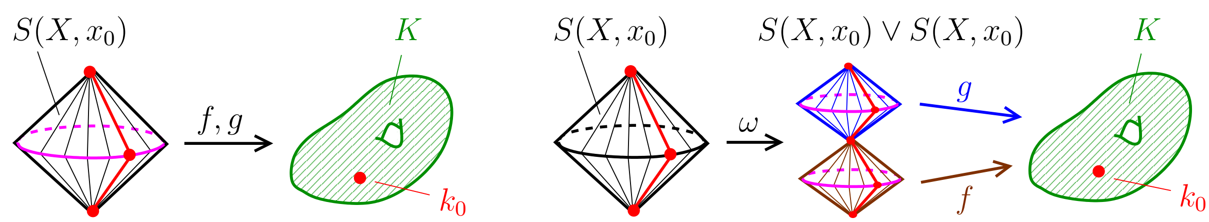


FIGURE 1670. Illustration of Proposition 121.22.

Remark. The multiplication map on $\langle S(X,x_0),(K,k_0)\rangle$ is arguably more geometric surely more appealing to some readers. Nonetheless, any attempt to prove that this multiplication map defines an (abelian) group structure promises to be notationally somewhat messy.

PROOF. Let $f,g\colon (X,x_0)\to\Omega(K,k_0)$ be two maps. One can easily verify that
$$(\Upsilon\circ\mu_*\circ\chi)([f],[g]) = \left[\begin{array}{l} S(X,x_0)\;\to\;(K,k_0) \\ \quad [(x,t)]\;\mapsto\begin{cases} f(t+1), & \text{if } t\in[-1,0], \\ g(t), & \text{if } t\in[0,1]\end{cases}\end{array}\right] = (\omega^*\circ\nu)(\Upsilon([f]),\Upsilon([g])).$$
This shows that the diagram does indeed commute. ■

121.5. **Functoriality.** Let $n \in \mathbb{N}$. In this section we will see, perhaps not surprisingly, that the abelian groups $\langle (X, x_0), \mathrm{K}(G, n)\rangle$ are functorial in (X, x_0) and G. To make this precise we need to introduce the following piece of notation.

Definition. Let $n \in \mathbb{N}$ and let $\varphi\colon G \to H$ be a homomorphism between two (abelian) groups. We denote by $[\varphi] \in \langle \mathrm{K}(G, n), \mathrm{K}(H, n)\rangle$ the canonical homotopy class of maps defined in Proposition 120.15.[1722]

We have the following lemma.

Lemma 121.23. *Let $n \in \mathbb{N}$. The maps*

$$\begin{array}{rcl} G & \mapsto & \mathrm{K}(G, n) \\ (\varphi\colon G \to H) & \mapsto & [\varphi] \in \langle \mathrm{K}(G, n), \mathrm{K}(H, n)\rangle \end{array}$$

define a covariant functor from the category of (abelian) groups to the pointed homotopy category.

PROOF. The fact that the maps define a covariant functor is an immediate consequence of the uniqueness statement in Proposition 120.15. ■

If one thinks about this lemma for a second, then one realizes that this lemma is actually quite astounding. So far we spent almost all our time trying to define functors from topological spaces to (abelian) groups in the hope of being able to use algebra to understand topological spaces. But now we have functors which go the other way around. In principle this will allow us to use tools from topology to gain a better understanding of groups. In fact we already saw a glimpse of this phenomenon in Proposition 120.22. Unfortunately at this very moment we cannot develop this theme. But in Chapter **??** this idea will come back with a vengeance.

After this short digression we can formulate the following proposition.

Proposition 121.24. *Let $n \in \mathbb{N}$.*

(1) *Let G be an abelian group. The maps*

$$\begin{array}{rcl} (X, x_0) & \mapsto & \langle (X, x_0), \mathrm{K}(G, n)\rangle \\ (f\colon (X, x_0) \to (Y, y_0)) & \mapsto & \left(\begin{array}{rcl} \langle (Y, y_0), \mathrm{K}(G, n)\rangle & \to & \langle (X, x_0), \mathrm{K}(G, n)\rangle \\ [\gamma] & \mapsto & [\gamma \circ f] \end{array} \right) \end{array}$$

define a contravariant functor from the category of pointed topological spaces to the category of abelian groups.

(2) *Let (X, x_0) be a pointed topological space. The maps*

$$\begin{array}{rcl} G & \mapsto & \langle (X, x_0), \mathrm{K}(G, n)\rangle \\ (\varphi\colon G \to H) & \mapsto & \left(\begin{array}{rcl} \langle (X, x_0), \mathrm{K}(G, n)\rangle & \to & \langle (X, x_0), \mathrm{K}(H, n)\rangle \\ [\gamma] & \mapsto & [\varphi \circ \gamma] \end{array} \right) \end{array}$$

[1722]More precisely, by Proposition 120.15 there exists a unique $[f] \in \langle \mathrm{K}(G, n), \mathrm{K}(H, n)\rangle$ such that the map $f_*\colon G = \pi_n(\mathrm{K}(G, n)) \to \pi_n(\mathrm{K}(H, n)) = H$ agrees with φ. Here we used that the canonical Eilenberg-Maclane spaces $\mathrm{K}(J, n)$ are pointed and that they come with an identification $\pi_n(\mathrm{K}(J, n)) = J$.

define a covariant functor from the category of abelian groups to the category of abelian groups.

PROOF. Let $n \in \mathbb{N}$.

(1) We fix an abelian group G. First note that it follows from Lemma 15.4 that the assignment $(X, x_0) \to \langle (X, x_0), \mathrm{K}(G, n)\rangle$ is a contravariant functor from the category of pointed topological spaces to the category of sets. Thus it remains to show that this assignment defines actually a functor to the category of abelian groups. More precisely, it remains to show that for any map $f \in \langle (X, x_0), (Y, y_0)\rangle$ the induced map $\langle (Y, y_0), \mathrm{K}(G, n)\rangle \to \langle (X, x_0), \mathrm{K}(G, n)\rangle$ is actually a group homomorphism. But that is basically obvious since the above definition of the group structure on $\langle (X, x_0), \mathrm{K}(G, n)\rangle$ has nothing to do with X. We leave it to the skeptical reader to fill in the details.

(2) Let (X, x_0) be a pointed topological space. By Lemmas 15.3 and Lemma 121.23 we know that the assignment $G \mapsto \langle (X, x_0), \mathrm{K}(G, n)\rangle$ defines a covariant functor from the category of abelian groups to the category of sets. As in (1) it remains to show that the assignment actually defines a functor to the category of abelian groups. This requires a little more effort than (1). The argument breaks up into two parts, namely we have to deal with the "horizontal" and the "vertical" aspects of the definition on page 2862.

(a) Recall that in Corollary 112.15 we gave an explicit *natural* isomorphism

$$\alpha_k \colon \pi_k(K, k_0) \xrightarrow{\cong} \pi_{k-1}(\Omega(K, k_0), c_{x_0}) \qquad \text{for every } k \in \mathbb{N}.$$

It follows from the naturality of these isomorphisms, and the precise definition of the maps Ξ on page 2860, that for a given homomorphism $\varphi \colon G \to H$ of (abelian) groups we obtain a commutative diagram

$$\begin{array}{ccc} \langle (X, x_0), \mathrm{K}(G, n)\rangle & \xrightarrow[\Xi_*]{=} & \langle (X, x_0), \Omega\,\mathrm{K}(G, n+1)\rangle \\ \downarrow \varphi_* & & \downarrow \varphi_* \\ \langle (X, x_0), \mathrm{K}(H, n)\rangle & \xrightarrow[\Xi_*]{=} & \langle (X, x_0), \Omega\,\mathrm{K}(H, n+1)\rangle. \end{array}$$

This takes care of the "horizontal" part.

(b) It follows from Lemma 121.7, Lemma 121.10 (1) and Lemma 121.23 that the "vertical" part of the definition on page 2862 is functorial in φ. ■

As the reader will know by now, base points are something of a nuisance. Fortunately in Proposition 38.7 we already saw that on many occasions we can stop worrying about base points, the following proposition nicely complements Proposition 38.7.

Proposition 121.25. *Let (X, x_0) and (Y, y_0) be two pointed CW-complexes. If Y is path-connected, if $\pi_1(Y, y_0)$ is abelian and if $\pi_i(Y, y_0) = 0$ for $i \geq 2$,*[1723] *then the natural map $\langle (X, x_0), (Y, y_0)\rangle \to [X, Y]$ is also a bijection.*

[1723] In other words, (Y, y_0) is a pointed Eilenberg-Maclane space of type $\mathrm{K}(\pi, 1)$ where $\pi \cong \pi_1(Y, y_0)$ is an abelian group.

Remark. We equip S^1, S^2 and $S^1 \vee S^2$ with the usual base point $*$. In Exercise 121.2 we will show that the map $[(S^2,*),(S^1 \vee S^2,*)] \to [S^2, S^1 \vee S^2]$ is not an injection. This shows that in Proposition 121.25 we cannot just drop the hypothesis that $\pi_i(Y,y_0)=0$ for $i \geq 2$.

PROOF. Let (Y,y_0) be a pointed path-connected CW-complex such that the fundamental group $\pi_1(Y,y_0)$ is abelian and such that $\pi_i(Y,y_0)=0$ for $i \geq 2$. By Proposition 38.7 (1) it remains to show that the map $\langle (X,x_0),(Y,y_0)\rangle \to [X,Y]$ is injective. Despite initial appearances, the argument will be quite different from the argument provided in the proof of Proposition 38.7 (1) (2). So let $f_0, f_1\colon (X,x_0) \to (Y,y_0)$ be two maps that are homotopic as maps $X \to Y$. By Proposition 18.16 (1) we know that there exists a $\gamma \in \pi_1(Y,y_0)$ such that $f_{0*}(g) = \gamma \cdot f_{1*}(g) \cdot \gamma^{-1} \in \pi_1(Y,y_0)$ for every $g \in \pi_1(X,x_0)$. But since $\pi_1(Y,y_0)$ is abelian we see that $f_{0*}=f_{1*}$. Since by hypothesis we have $\pi_i(Y,y_0)=0$ for $i \geq 2$ we can appeal to Proposition 120.15 (2) which now shows that there exists a homotopy from f_0 to f_1 rel x_0. ■

In the following discussion we will use Proposition 121.24 (1) to dispose of the base points.

Definition. Let $n \in \mathbb{N}$ and let G be an abelian group.

(1) Let X be a connected CW-complex. We pick a point $x_0 \in X^0$. By Propositions 38.7 and 121.25[1724] the natural map

$$\begin{array}{rcl} \Psi_{(X,x_0)}\colon \langle (X,x_0),(\mathrm{K}(G,n),*)\rangle & \to & [X,\mathrm{K}(G,n)] \\ [f] & \mapsto & [f] \end{array}$$

is a bijection. We use this bijection, together with the group structure on the left-hand side defined in Proposition 121.17, to turn the right-hand side $[X,\mathrm{K}(G,n)]$ into a group. In Corollary 121.26 below we will see that this definition does not depend on the choice of x_0.

(2) Now let X be any CW-complex. Let $\{X_i\}_{i\in I}$ be the set of its components. We have

$$[X,\mathrm{K}(G,n)] \underset{\substack{\uparrow \\ \text{by Lemma 3.3 (2)}}}{=} \prod_{i\in I} [X_i,\mathrm{K}(G,n)]$$

We define the group structure on each $[X_i,\mathrm{K}(G,n)]$ as in (1) and for $i \neq j$ we define the multiplication on $[X_i,\mathrm{K}(G,n)] \times [X_j,\mathrm{K}(G,n)]$ to be the trivial map. One easily verifies that this defines an abelian group structure on $[X,\mathrm{K}(G,n)]$.

The following corollary to Proposition 121.24 (1) shows that this definition of the group structure on $[X,\mathrm{K}(G,n)]$ does not depend on the choice of the base point $x_0 \in X^0$.

Corollary 121.26. *Let $n \in \mathbb{N}$, let G be an abelian group and let X be a connected* CW-*complex. The definition of the group structure on $[X,\mathrm{K}(G,n)]$ does not depend on the choice of $x_0 \in X^0$.*

[1724] Note that here we use that G is an abelian group, even for $n=1$.

Proof (∗). Let $n \in \mathbb{N}$, let G be an abelian group and let X be a connected CW-complex. Now let $x_0, x_1 \in X$ be two points in the 0-skeleton of X. Since X is connected we know by Proposition 38.10 that there exists a map $f\colon X \to X$ with the following properties:

(a) The map f is homotopic to the identity.
(b) We have $f(x_0) = x_1$.
(c) The map f defines a homotopy equivalence $(X, x_0) \to (X, x_1)$ of pointed topological spaces.

We consider the following diagram

$$\begin{tikzcd}
\langle (X,x_1),(\mathrm{K}(G,n),*)\rangle \arrow[rr, "f^*", "{[g]\mapsto[g\circ f]}"'] \arrow[d, "\Psi_{(X,x_1)}"', "{[g]\mapsto[g]}"] & & \langle (X,x_0),(\mathrm{K}(G,n),*)\rangle \arrow[d, "\Psi_{(X,x_0)}"', "{[g]\mapsto[g]}"] \\
{[X,\mathrm{K}(G,n)]} \arrow[rr, "f^*", "{[g]\mapsto[g\circ f]}"'] & & {[X,\mathrm{K}(G,n)].}
\end{tikzcd}$$

We make the following observations:

(1) It follows basically immediately from the definitions that the diagram commutes.
(2) It follows from (c) and Lemmas 15.4 and 15.5 that the top horizontal map is a bijection.
(3) By Proposition 121.24 (1) we know that top horizontal map is in fact an isomorphism.
(4) By (a) we know that the bottom map is in fact the identity.

It follows immediately from the above that it does not matter whether we use $\Psi_{(X,x_1)}$ or $\Psi_{(X,x_0)}$ to define the group structure on $[X, \mathrm{K}(G, n)]$. ■

121.6. **Basic calculations.** In this section we want to do a few bare-hands calculation of our new invariant $\langle (X, x_0), \mathrm{K}(G, n)\rangle$. Before we start with the calculations it is useful to introduce the following notation.

Notation. Let $n \in \mathbb{N}$ and let G be an abelian group. Furthermore let (X, x_0) be a pointed topological space. For the purpose of this section we write

$$h^n((X,x_0);G) \;:=\; \langle (X,x_0),\mathrm{K}(G,n)\rangle.$$

Before we state the second main calculation it is helpful to introduce the following convention.

Convention. Let G be an abelian group. We make the identifications

$$\mathrm{H}_1(\mathrm{K}(G,1);\mathbb{Z}) \underset{\substack{\uparrow\\ \text{Hurewicz Theorem 52.5}}}{=} \pi_1(\mathrm{K}(G,1),*)_{\mathrm{ab}} \underset{\substack{\uparrow\\ \text{from Theorem 120.8}}}{=} G_{\mathrm{ab}} \underset{\substack{\uparrow\\ \text{since } G \text{ is abelian}}}{=} G.$$

Lemma 121.27. *Let G be an abelian group. Let X be a connected CW-complex and let $x_0 \in X^0$ be a point. The map*

$$\begin{array}{rcl} h^1((X,x_0);G) & \to & \mathrm{Hom}(\mathrm{H}_1(X;\mathbb{Z}),G) \\ {[f\colon (X,x_0)\to(\mathrm{K}(G,1),*)]} & \to & (f_*\colon\ \mathrm{H}_1(X;\mathbb{Z})\to \mathrm{H}_1(\mathrm{K}(G,1);\mathbb{Z})=G) \end{array}$$

is a natural isomorphism.

PROOF.

(1) It follows from Proposition 120.15 and the Hurewicz Theorem 52.5 that the given map is a bijection.

(2) It follows immediately from the definitions that the map is natural in X.

(3) It remains to show that the given map is a homomorphism. We write $Y = \mathrm{K}(G, 2)$. We make the usual identification $\langle X, \mathrm{K}(G, 1)\rangle = \langle X, \Omega Y\rangle$. Furthermore we use the natural isomorphism

$$\begin{array}{rcl} \operatorname{Hom}(\mathrm{H}_1(X;\mathbb{Z}), G) & \to & \operatorname{Hom}(\pi_1(X, x_0), \pi_1(\Omega Y)) \\ \varphi & \mapsto & (\pi_1(X, x_0) \to \pi_1(X, x_0)_{\mathrm{ab}} = \mathrm{H}_1(X;\mathbb{Z}) \xrightarrow{\varphi} G = \pi_1(\Omega Y)) \end{array}$$

to identify the group on the left with the group on the right. Next we consider the following diagram

$$\begin{tikzcd}
\langle X, \Omega Y\rangle \times \langle X, \Omega Y\rangle \arrow[r, "{([f],[g]) \mapsto (f_*, g_*)}"] \arrow[dd, "{([f],[g]) \mapsto [f \times g]}"] & \operatorname{Hom}(\pi_1(X), \pi_1(\Omega Y)) \times \operatorname{Hom}(\pi_1(X), \pi_1(\Omega Y)) \arrow[d, "{(\varphi,\psi) \mapsto (g \mapsto (\varphi(g), \psi(g)))}"] \\
 & \operatorname{Hom}(\pi_1(X), \pi_1(\Omega Y) \times \pi_1(\Omega Y)) \arrow[d] \\
\langle X, \Omega Y \times \Omega Y\rangle \arrow[r] \arrow[d, "\mu_*"] & \operatorname{Hom}(\pi_1(X), \pi_1(\Omega Y \times \Omega Y)) \arrow[d, "\mu_*"] \\
\langle X, \Omega Y\rangle \arrow[r] & \operatorname{Hom}(\pi_1(X), \pi_1(\Omega Y))
\end{tikzcd}$$

We make the following clarifications and observations:

(a) The upper rectangle commutes basically by definition.

(b) The lower square also commutes basically by definition. Slightly more precisely, let $[f] \in \langle X, \Omega Y \times \Omega Y\rangle$. We consider the maps $X \xrightarrow{f} \Omega Y \times \Omega Y \xrightarrow{\mu} \Omega Y$. By functoriality we have $(\mu \circ f)_* = \mu_* \circ f_*$. But this equality just means that in the lower square it does not matter whether we take the down-right route or the right-down route.

(c) It follows from Lemma 114.28 that the right vertical maps give precisely the group multiplication on $\operatorname{Hom}(\pi_1(X), \pi_1(\Omega Y))$.

This shows that the given map is indeed a group homomorphism. ■

121.7. **Product structures** $(*)$**.** Let (X, x_0) be a pointed CW-complex and let $m, n \in \mathbb{N}$. Our goal in this section is to define a potentially interesting map

$$\langle (X, x_0), \mathrm{K}(\mathbb{Z}, m)\rangle \times \langle (X, x_0), \mathrm{K}(\mathbb{Z}, n)\rangle \;\to\; \langle (X, x_0), \mathrm{K}(\mathbb{Z}, m+n)\rangle.$$

The key to defining such a map is the following humble construction.

Definition. Given two pointed topological spaces (X, x_0) and (Y, y_0) the *smash product* is defined as

$$X \wedge Y \;=\; (X \times Y)/(X \vee Y) \quad \text{where we use Exercise 18.30 to make the identification } X \vee Y := (X \times \{y_0\}) \times (\{x_0\} \times Y).$$

We equip $X \wedge Y$ with the base point given by $[X \vee Y]$.

Remark. As we discussed in Exercise 36.12 and on page 965, in the category of pointed topological spaces the smash product is actually the categorical product.

In Exercise 18.30 we already showed that $S^m \wedge S^n$ is homeomorphic to S^{m+n}. For our purpose it is useful to have a canonical homeomorphism that we can all agree upon. This is given in the following lemma.

Lemma 121.28. *Given $m, n \in \mathbb{N}$ there exists a canonical homeomorphism*

$$S^m \wedge S^n \xrightarrow{\cong} S^{m+n}.$$

Convention. We will use the canonical homeomorphism from Lemma 121.28 to make the identification $S^m \wedge S^n = S^{m+n}$.

PROOF. We write $I = [0,1]$. Given any $k \in \mathbb{N}_0$ we have the following identifications of pointed topological spaces:

$$(I^k/\partial I^k, [\partial I^k]) \underset{\substack{\uparrow \\ \text{page 126}}}{\xrightarrow{\cong}} (\overline{B}^k/S^{k-1}, [S^{k-1}]) \underset{\substack{\uparrow \\ \text{page 182}}}{\xrightarrow{\cong}} (S^k, *).$$

Let $m, n \in \mathbb{N}_0$. Using the above identification we obtain the following homeomorphism

$$S^m \wedge S^n \xrightarrow{\cong} I^m/\partial I^m \wedge I^n/\partial I^n \overset{\substack{\text{homeomorphism by Lemma 3.28 (1)} \\ \downarrow}}{\underset{[([x],[y])] \mapsto [(x,y)]}{\xrightarrow{\cong}}} (I^{m+n})/(I^m \times \partial I^n \cup \partial I^m \times I^n) = I^{m+n}/\partial I^{m+n} \xleftarrow{\cong} S^{m+n}.$$

We have thus given an explicit homeomorphism between $S^m \wedge S^n$ and S^{m+n}. ■

The following lemma is basically just a mild reformulation of Lemma 120.11.

Lemma 121.29. *Let $k \in \mathbb{N}$. There exists a polarized Eilenberg-Maclane space $(Y_k, *, \varphi)$ of type $\mathrm{K}(\mathbb{Z}, k)$ with the following properties:*

1. *The k-skeleton of Y_k equals S^k with the usual* CW*-structure defined on page 2813.*
2. *The* CW*-complex Y_k has no cell of dimension $k+1$.*
3. *The* CW*-complex Y_k has only countably many cells in dimensions $\geq k+2$.*

Now we turn to the actual definition of the product structure.

Definition. Let $m, n \in \mathbb{N}$. In the following discussion all topological spaces are viewed as pointed objects.

1. We consider the polarized Eilenberg-Maclane spaces Y_m and Y_n that we introduced in Lemma 121.29. Recall that from Lemma 121.15 we obtain canonical elements $\Xi_m \in \langle \mathrm{K}(\mathbb{Z}, m), Y_m \rangle_{\text{inv}}$ and $\Xi_n \in \langle \mathrm{K}(\mathbb{Z}, n), Y_n \rangle_{\text{inv}}$.
2. Since Y_m and Y_n are CW-complexes with *countably* many cells we can equip the product $Y_m \times Y_n$ with the CW-structure defined in Lemma 36.32.
3. We denote by $p\colon Y_m \times Y_n \to Y_m \wedge Y_n = (Y_m \times Y_n)/\sim$ the obvious projection.
4. It follows from the explicit description of the CW-structure defined in Lemma 36.32 that $Y_m \wedge Y_n = (Y_m \times Y_n)/\sim$ has precisely one cell in dimension 0, no cells in

dimension $1,\ldots,m+n-1$, one cell in dimension $m+n$ and no cell in dimension $m+n+1$. Furthermore it follows immediately from the definitions that the map

$$S^{m+n} \;=\; S^m \wedge S^n \;\to\; Y_m \wedge Y_n$$

defines a homeomorphism onto the $(m+n+1)$-skeleton of $Y_m \wedge Y_n$. In particular we obtain from Proposition 40.9 that Y_{m+n} is $(m+n-1)$-connected and we obtain that the map $\varphi\colon \mathbb{Z} = \pi_{m+n}(S^{m+n}) \to \pi_{m+n}(Y_m \wedge Y_n)$ is an isomorphism.

(5) Since $Y_m \times Y_n$ is an $(m+n-1)$-connected CW-*complex* we obtain from Proposition 120.15 a unique class $\mu \in \langle Y_m \wedge Y_n, \mathrm{K}(\mathbb{Z}, m+n)\rangle$ that induces the preferred isomorphism $\varphi^{-1}\colon \pi_{m+n}(Y_m \wedge Y_n, *) \xrightarrow{\cong} \mathbb{Z}$.

After these preparations we consider the following diagram

$$\begin{tikzcd}
\langle X, Y_m\rangle \times \langle X, Y_n\rangle \arrow[r, "\chi"] & \langle X, Y_m \times Y_n\rangle \arrow[r, "p_*"] & \langle X, Y_m \wedge Y_n\rangle \arrow[dr, "\mu_*"] & \\
\langle X, \mathrm{K}(\mathbb{Z}, m)\rangle \times \langle X, \mathrm{K}(\mathbb{Z}, n)\rangle \arrow[u, "\Xi_{m*}\times\Xi_{n*}"] \arrow[rrr, dashed, "\text{definition of cup product } \cup"] & & & \langle X, \mathrm{K}(\mathbb{Z}, m+n)\rangle.
\end{tikzcd}$$

We refer to the bottom horizontal map, which we defined through the upper path of maps, as the *cup product*.

Theorem 121.30. *Let $X = (X, x_0)$ be a pointed* CW-*complex and let $m, n \in \mathbb{N}$. The cup product*

$$\cup\colon \langle X, \mathrm{K}(\mathbb{Z}, m)\rangle \times \langle X, \mathrm{K}(\mathbb{Z}, n)\rangle \;\to\; \langle X, \mathrm{K}(\mathbb{Z}, m+n)\rangle$$

is bilinear. Furthermore the cup product is associative and graded commutative in the sense that for $\varphi \in \langle X, \mathrm{K}(\mathbb{Z}, m)\rangle$ and $\psi \in \langle X, \mathrm{K}(\mathbb{Z}, n)\rangle$ we have

$$\varphi \cup \psi \;=\; (-1)^{m\cdot n} \cdot \psi \cup \varphi.$$

Proof. We will not make use of this theorem, thus we will also not provide a proof. Instead we refer to [**Stro11**, Theorem 21.61] for the proof. Note that [**Stro11**, Theorem 21.61] does not explicitly say that X needs to be a CW-complex, but implicitly the discussion on [**Stro11**, Chapter 21] seems to assume that the setting is a convenient category in the sense of Theorems 36.28 and 36.30. We will discuss this issue in more detail right after this proof. ∎

The above definition of the cup product is in a way rather awkward, it is esthetically wrong to introduce the CW-complexes Y_m and Y_n. We conclude this section with a longish but hopefully quite interesting discussion which explains why we introduced the CW-complexes Y_m and Y_n and how we can get around introducing them.

Definition.

(1) Let A and B be CW-complexes. We denote by $A \otimes B$ the product CW-complex that we defined on page 963. Recall that the underlying set is just $A \times B$.

(2) Let (A, a_0) and (B, b_0) be pointed CW-complexes. We write

$$A \oslash B \;:=\; (A \otimes B)/(A \times \{b_0\} \cup \{a_0\} \times B).$$

We equip $A \owedge B$ with the CW-structure given by the CW-structure on $A \otimes B$ and the quotient structure defined in Lemma 36.32.

(3) We denote by $\eta\colon A \otimes B \to A \times B$ the identity map. Note that η is continuous by Proposition 36.26 (1).

Now let $X = (X, x_0)$ be a pointed topological space and let $m, n \in \mathbb{N}$. We introduce the following notation:

(1) Let $r \in \mathbb{N}$. We write $K_r = \mathrm{K}(\mathbb{Z}, r)$. We consider the polarized Eilenberg-Maclane spaces Y_r that we introduced in Lemma 121.29 and we denote by $\Xi_r \in \langle K_r, Y_r \rangle_{\mathrm{inv}}$ the canonical element given by Lemma 121.15.

(2) Given topological spaces A and B we denote by $p\colon A \times B \to A \wedge B$ the obvious projection. Similarly, given CW-complexes A and B we denote by $p\colon A \otimes B \to A \owedge B$ the obvious projection.

We consider the following diagram

$$\begin{tikzcd}
\langle X, Y_m\rangle \times \langle X, Y_n\rangle \arrow[r, "\chi"] & \langle X, Y_m \times Y_n\rangle \arrow[r, "p_*"] & \langle X, Y_m \wedge Y_n\rangle \arrow[dr, "\mu_*"] & \\
\langle X, K_m\rangle \times \langle X, K_n\rangle \arrow[u, "\Xi_{m*} \times \Xi_{n*}"'] \arrow[r, "\chi", "(f,g)\mapsto f\times g"'] \arrow[d, dashed, "(f,g)\mapsto f \widetilde{\times} g"] & \langle X, K_m \times K_n\rangle \arrow[r, "p_*"] & \langle X, K_m \wedge K_n\rangle \arrow[r, "???"] & \langle X, K_{m+n}\rangle. \\
\langle X, K_m \widetilde{\times} K_n\rangle \arrow[r, dashed, "\Theta_*"'] & \langle X, K_m \otimes K_n\rangle \arrow[u, "\eta_*"] \arrow[r, "p_*"] & \langle X, K_m \owedge K_n\rangle \arrow[ur, "\nu_*"'] &
\end{tikzcd}$$

We make the following observations:

(1) Since K_m and K_n have uncountably many cells the product $K_m \times K_n$ is not a CW-complex.

(2) It follows from (1) that $K_m \wedge K_n$ has no obvious CW-structure. Thus there is no clear path to constructing a map $K_m \wedge K_n \to K_{m+n}$.[1725]

(3) It follows from (2) that the "direttissima" approach to defining the cup product by going straight from left to right runs into a major obstacle.

(4) Since Y_m and Y_n have countably many cells we know, by Lemmas 36.32 and 36.32 that $Y_m \times Y_n$ and $Y_m \wedge Y_n$ are CW-complexes. As discussed above, we obtain from Proposition 120.15 a canonical class $\mu \in \langle Y_m \wedge Y_n, \mathrm{K}(\mathbb{Z}, m+n)\rangle$.

(5) One way of getting around (3) is to use (4), i.e. we take the "upper route" in the diagram. This is how in the above we defined the cup product.

(6) As for $Y_m \wedge Y_n$ one sees that $K_m \owedge K_n$ is $(m+n-1)$-connected and that we have a canonical map $\pi_{m+n}(K_m \owedge K_n) \to \mathbb{Z}$. Since $K_m \owedge K_n$ is a CW-complex we obtain from Proposition 120.15 a canonical element $\nu \in \langle K_m \owedge K_n, K_{m+n}\rangle$.

(7) Unfortunately the issue in (1) also means that the map $\eta\colon K_m \otimes K_n \to K_m \times K_n$ is not a homeomorphism. Thus there is no obvious way how we can take the lower route in the diagram.

[1725] In fact it is also not immediately clear whether $K_m \wedge K_n$ is say $(m+n-1)$-connected.

Now we switch to the category of CGWH-spaces that we introduced on page 967. In particular we now assume that X is a CGWH-space. Recall that by Theorem 36.30 (2) we know that every CW-complex is a CGWH-space.

(8) Given two CGWH-spaces A and B we know by Theorem 36.30 that there exists a product *in the category of* CGWH-*spaces*. As on page 968 we denote this categorical product by $A \widetilde{\times} B$ and we denote by $p\colon A \widetilde{\times} B \to A$ and $q\colon A \widetilde{\times} B \to B$ the two projections maps.

(9) Let $f\colon X \to K_m$ and $g\colon X \to K_n$ be maps. Since X, K_m and K_n are CGWH-spaces we know, by definition of the product in a category, that there exists a unique map $f \widetilde{\times} g\colon X \to K_m \widetilde{\times} K_n$ such that $f = p \circ (f \widetilde{\times} g)$ and $g = q \circ (f \widetilde{\times} g)$.

(10) Since K_m and K_n are CW-complexes we know by Theorem 36.30 (5) that the identity map $\Theta\colon K_m \widetilde{\times} K_n \to K_m \otimes K_n$ is actually a homeomorphism.[1726]

(11) It follows from (9) and (10) that we have a "lower route" from $\langle X, K_m\rangle \times \langle X, K_n\rangle$ to $\langle X, K_{m+n}\rangle$. This gives us an approach to defining the cup product without introducing the slightly artificial Y_m and Y_n.

(12) It follows easily from the definitions that in the diagram the square to the lower left commutes. One can now see reasonably easily that the two definitions of the cup product agree.

The second approach to defining the cup product is much more natural than the initial approach. In particular, using a similar circle of ideas one can define a cup product

$$\langle X, \mathrm{K}(R,m)\rangle \times \langle X, \mathrm{K}(R,n)\rangle \to \langle X, \mathrm{K}(R,m+n)\rangle$$

for any commutative ring R. We refer to [**Stro11**, Chapter 21.4] for details.

Exercises for Chapter 121.

Exercise 121.1. We consider the following diagram:

$$\begin{tikzcd}
S^1 \times S^1 \arrow[r, "\Xi\times\Xi"] \arrow[d, "(z,w)\mapsto z\cdot w"] & (\Omega\mathbb{C}\mathrm{P}^\infty, *) \times (\Omega\mathbb{C}\mathrm{P}^\infty, *) \arrow[d, "\mu"] \\
S^1 \arrow[r, "\Xi"] & \Omega(\mathbb{C}\mathrm{P}^\infty, *).
\end{tikzcd}$$

The map $\Xi\colon S^1 \to \Omega(\mathbb{C}\mathrm{P}^\infty, *)$ is defined in Lemma 121.16 and the map μ is defined on page 2851. Show that the diagram commutes up to homotopy.

Exercise 121.2. We equip S^1, S^2 and $S^1 \vee S^2$ with the usual base points which in each case we call $*$. Show that the map $\langle (S^2,*), (S^1\vee S^2, *)\rangle \to [S^2, S^1 \vee S^2]$ is not an injection.

Exercise 121.3. Let (X, x_0) and (K, k_0) be pointed topological spaces. Show that the multiplication on $\langle (X,x_0), (K,k_0)\rangle$ defined in Lemma 121.18 is actually associative.

[1726] For our purpose we only need the weaker result that $\Theta\colon K_m \widetilde{\times} K_n \to K_m \otimes K_n$ is continuous.

Exercise 121.4. Let (X, x_0) and $(K, *)$ be pointed topological spaces. We consider the double loop space $\Omega^2(K,*) = \Omega(\Omega(K,*))$. By the construction on page 2862 we have a binary operation

$$\circ\colon \langle (X,x_0), \Omega(\Omega(K,*))\rangle \times \langle (X,x_0), \Omega(\Omega(K,*))\rangle \to \langle (X,x_0), \Omega(\Omega(K,*))\rangle.$$

In Proposition 121.17 we showed in particular that this binary operation is unital. Provide another binary unital operation "$\otimes$" on $\langle (X,x_0), \Omega(\Omega(K,*))\rangle$ such that the conditions of the Eckmann-Hilton Theorem 114.27 are satisfied.

Remark. Once we have shown that we can apply the Eckmann-Hilton Theorem 114.27 we see that "$\circ$" defines a group structure on $\langle (X,x_0), \Omega^2(K,*)\rangle$ and we obtain that this group is abelian. Thus this provides an alternative approach to proving Proposition 121.21 and to proving some of the statements of Proposition 121.17.

Remark. Coming up with an idea for defining "$\otimes$" is not difficult. Providing a complete proof that Condition (b) of the Eckmann-Hilton Theorem 114.27 is satisfied might be more painful. Especially once one starts wondering why certain maps are actually continuous.

Exercise 121.5. Let $n \in \mathbb{N}$ and let G be an abelian group. Let $k \in \mathbb{N}$. We denote by $*$ the usual base point of S^k. Compute the groups $\langle (S^k, *), \mathrm{K}(G,n)\rangle$.

Remark. Once one unravels the definitions one can fairly easily write down *bijections* to familiar groups. The only catch is that one really needs to show that these bijections are in fact group isomorphisms. The Eckmann-Hilton Theorem 114.27 might once again ride to the rescue.

Exercise 121.6. Let (X, x_0) be a pointed topological space. We equip S^1 with the base point $* = (0,1)$. Show that the smash product $X \wedge S^1$ is homeomorphic to the reduced suspension $S(X, x_0)$.

Remark. Personally I like Lemma 3.28 (1).

Exercise 121.7. To simplify the notation, given a pointed topological space (Y, y_0) we now write $SY = S(Y, y_0)$ and $\Omega Y = \Omega(Y, y_0)$.

(a) Let (Y, y_0) be a point topological space. Show that the set $\pi_0(\Omega Y)$ has a natural group structure.

(b) Let (X, x_0) be a point topological space.

  (1) Show that there exists a natural isomorphism from $\pi_0(\Omega(SX))$ to the free group on the set $\pi_0(X)$.

  (2) Let (X, x_0) be a point topological space. Show that there exists a natural isomorphism from $\pi_0(\Omega^2(S^2X))$ to the free *abelian* group on the set $\pi_0(X)$.

  (3) How does the sequence continue?

Index

Nomenclature

$(M\#N, K\#L)$	connected sum of pairs (M,K) and (N,L)
$*\mathbb{C}\mathrm{P}^2$	fake $\mathbb{C}\mathrm{P}^2$ with non-trivial Kirby-Siebenmann invariant
$-M$	smooth manifold equipped with the opposite orientation
1_K	multiplicatively neutral element in $\mathrm{H}^*_{\mathrm{simp}}(K;R)$
1_X	multiplicatively neutral element in $\mathrm{H}^*(X;R)$
$[K]$	fundamental class of a closed oriented pseudomanifold
$[K]_{\mathbb{F}_2}$	$\mathbb{F}_2$-fundamental class of a closed pseudomanifold
$[M]$	fundamental class of a compact oriented smooth manifold
$[M]$	fundamental class of a compact oriented topological manifold
$[M]^*$	dual fundamental class of a compact oriented topological manifold
$[M]_{\mathbb{F}_2}$	$\mathbb{F}_2$-fundamental class of a compact smooth manifold
$[M]_{\mathbb{R}}$	real fundamental class of a compact oriented smooth manifold
$[M]_{\mathbb{F}_2}$	$\mathbb{F}_2$ fundamental class of a compact oriented smooth manifold
$[M]_{\mathbb{F}_2}$	$\mathbb{F}_2$-fundamental class of a compact topological manifold
$[S^n]$	standard generator of $\mathrm{H}_n(S^n)$
$[X,Y]$	maps $X \to Y$ up to homotopy
$[X,Y]$	set of homotopy classes of maps $X \to Y$
$[x,y]$	commutator $xyx^{-1}y^{-1}$
$[z_0,\dots,z_k]$	map $\Delta^k \to \mathbb{R}^n$ defined by $z_0,\dots,z_k \in \mathbb{R}^n$
$\#S$	cardinality of a set
$\alpha * \beta$	product of paths
$\int_M \omega$	integral of a differential form ω
$\mathcal{AbGr}$	category of abelian groups
$\mathcal{AbsGraph}$	category of abstract graphs
$\mathcal{CGWH}$	category of CGWH-spaces
$\mathcal{ChCplx}$	category of chain complexes
$\mathcal{CRing}$	category of commutative rings
$\mathcal{CW}$	category of CW-complexes
$\mathcal{FullCW}$	full category of CW-complexes
$\mathcal{FullCW}$	full category of CW-complexes
$\mathcal{FullPairCW}$	full category of pairs of CW-complexes
$\mathcal{FullPCW}$	full category of pointed CW-complexes
$\mathcal{Gr}$	category of groups
$\mathcal{GradAb}$	category of S-graded abelian groups
$\mathcal{GradMod}_R$	category of S-graded R-modules

$\mathcal{GrRing}$	category of graded rings
$\mathcal{HomTop}$	homotopy category of topological spaces
$\mathcal{MapTop}$	category of maps between topological spaces
$\mathcal{PairTop}$	category of pairs of topological spaces
$\mathcal{PSmMfd}$	category of pointed smooth manifolds
$\mathcal{PTop}$	category of pointed topological spaces
$\mathcal{Ring}$	category of rings
$\mathcal{Set}$	category of sets
$\mathcal{SimpCplx}$	category of abstract simplicial complexes
$\mathcal{SimpCplx}$	category of simplicial complexes
$\mathcal{SmMfd}$	category of smooth manifolds
$\mathcal{Top}$	category of topological spaces
$\mathcal{TopGr}$	category of topological groups
$\mathcal{UndirAbsGraph}$	category of undirected abstract graphs
$\mathcal{Vec}_{\mathbb{K}}$	category of $\mathbb{K}$-vector spaces
$\chi(X)$	Euler characteristic of a CW-complex
$\chi(X)$	Euler characteristic of a topological space
$\mathrm{C}^*(X,A;G)$	relative cochain complex
$\mathrm{C}^*_n(X;\mathbb{R})$	bounded cochain complex
$\mathrm{C}^*_{\mathrm{smooth}}(M;G)$	smooth singular cochain complex
Δ^n	standard n-simplex
$\varnothing$	the empty set
ϵ_X	augmentation map $\mathrm{C}_0(X) \to \mathbb{Z}$
$\Gamma^A_R(M)$	set of all R-sections along A
$\mathring{\Delta}^n$	$\Delta^n \setminus \partial\Delta^n$
$\mathrm{St}(\mathring{K}, s)$	open star of the simplex s
$\mathbb{L}$	long line
$\mathbb{L}^+$	open long ray
$\Lambda(\varphi)$	Lefschetz number of a map φ
$\Lambda(\varphi,\mathbb{F})$	$\mathbb{F}$-Lefschetz number of a map φ
$\langle(X,x_0),(Y,y_0)\rangle$	maps $(X,x_0) \to (Y,y_0)$ up to homotopy
$\langle(X,x_0),(Y,y_0)\rangle$	maps $(X,x_0) \to (Y,y_0)$ up to homotopy
$\langle X \mid R\rangle$	presentation with generating set X and relator set R
$\langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle$	presentation with generators $x_1,\dots,x_k$ and relators $r_1,\dots,r_l$
$\langle\ ,\ \rangle$	Kronecker pairing
$\langle\ ,\ \rangle$	Kronecker pairing with R-coefficients
◺	planar n-simplex
я	map from ordinary to relative homology
ж	map from reduced to ordinary homology
ж	map from reduced to relative homology
ю	map from relative homology to homology of the quotient
$\mathbb{RP}^n$	real projective space
$\mathbb{CP}^\infty$	infinite-dimensional complex projective space

$\mathbb{CP}^n$	complex projective space
$\mathbb{HP}^\infty$	infinite-dimensional quaternionic projective space
$\mathbb{HP}^n$	quaternionic projective space
$\mathbb{N}$	the natural numbers $1, 2, \ldots$
$\mathbb{N}_0$	the natural numbers $0, 1, 2, \ldots$
$\mathbb{OP}^1$	octonionic projective line
$\mathbb{OP}^2$	octonionic projective plane
$\mathbb{RP}^\infty$	infinite-dimensional real projective space
$\mathbb{Z}^{(S)}$	free abelian group generated by the set S
$\mathbb{Z}_n$	$\mathbb{Z}/n\mathbb{Z}$
$\mathcal{N}_n$	unoriented cobordism group
$\mathcal{P}(X)$	power set of the set X
$\Omega(X)$	free path space
$\Omega(x, x_0)$	loop space
$\Omega_k^{\mathrm{fr}}(\mathbb{R}^m)$	framed cobordism group
$\Omega_k^{\mathrm{th}}(\mathbb{R}^m)$	thickened cobordism group
Ω_n	n-th cobordism group
$\mathrm{Sp}(2g, \mathbb{Z})$	symplectic group
$\mathrm{Aut}(\pi)$	automorphism group of a group π
$\mathrm{cat}(X)$	Lusternik-Schnirelmann category of a topological space X
$\mathrm{cl}(X)$	cup length of a topological space X
$\mathrm{Cone}(K)$	simplicial cone on an abstract simplicial complex K
$\mathrm{Cone}(X)$	cone of a topological space X
$\mathrm{C}_{\mathrm{c}}^*(X; G)$	chain complex with compact support
$\mathrm{C}^*_{\mathrm{simp},\leq}(K; G)$	simplicial cochain complex
$\mathrm{C}^*_{\mathrm{simp}}(K; G)$	simplicial cochain complex
$\mathrm{C}_*^{\mathrm{CW}}(X)$	cellular chain complex
$\mathrm{C}_*^{\mathrm{CW}}(X)$	cellular cochain complex
$\mathrm{C}_*^{\mathrm{han}}(M)$	handle chain complex
$\mathrm{C}_*^{\mathrm{simp},\leq}(K)$	simplicial chain complex of an ordered abstract simplicial complex
$\mathrm{C}_*^{\mathrm{simp}}(K)$	simplicial chain complex of an abstract simplicial complex
$\mathrm{C}_n(X)$	n-th singular chain group of X
$\mathrm{C}_n(X, A)$	relative singular chain group
$\mathrm{C}_n^{\mathrm{simp},\leq}(K)$	n-th simplicial chain group of an ordered abstract simplicial complex K
$\mathrm{C}_n^{\mathrm{simp}}(K)$	n-th simplicial chain group of the abstract simplicial complex K
$\mathrm{C}^*_{\mathrm{dR}}(M)$	de Rham cochain complex
$\deg(\Phi)$	degree of an automorphism of a group isomorphic to $\mathbb{Z}$
$\deg(f)$	degree of a map $f\colon S^n \to S^n$
$\deg(f, x)$	local degree of a map f at a point x
$\mathrm{diam}(A)$	diameter of a subset A of a metric space
$\mathrm{D}_W M$	double of a topological (smooth) manifold M along W
$\mathrm{D} f_P$	differential of a smooth map $f\colon M \to N$ at $P \in M$
$\mathrm{D} M$	double of a topological (smooth) manifold M

$\mathrm{ev}\colon \mathrm{H}^n(C;G) \to \mathrm{Hom}(\mathrm{H}_n(C),G)$	evaluation map
$\mathrm{ev}\colon \mathrm{H}^n(X;\mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_n(X;\mathbb{F}),\mathbb{F})$	evaluation map with $\mathbb{F}$-coefficients
$\mathrm{Ext}(H,G)$	G-Ext-group of H
$\mathrm{F}\,H$	maximal torsion-free quotient of H
$\mathrm{F}\,H$	maximal torsion-free quotient of an abelian group
$\mathrm{genus}(M)$	genus of a surface M
$\mathrm{GL}_+(n,\mathbb{R})$	real matrices of positive determinant
$\mathrm{Homeo}(X)$	group of self-homeomorphism of X
$\mathrm{H}^k_{\mathrm{simp},\leq}(K;G)$	k-th simplicial cohomology group
$\mathrm{H}^k_{\mathrm{simp}}(K;G)$	k-th simplicial cohomology group
$\mathrm{H}^n(X,A;G)$	n-th relative singular cohomology
$\mathrm{H}^n_{\mathrm{CW}}(X)$	n-th cellular cohomology group
$\mathrm{H}^n_{\mathrm{smooth}}(M;G)$	n-th smooth singular cohomology group
$\mathrm{H}_k^{\mathrm{han}}(M)$	n-th handle homology group
$\mathrm{H}_n(\pi)$	n-th homology group of a group π
$\mathrm{H}_n(X)$	n-th singular homology group
$\mathrm{H}_n(X;G)$	singular homology with G-coefficients
$\mathrm{H}_n^{\mathrm{CW}}(X)$	n-th cellular homology group
$\mathrm{H}_n^{\mathrm{lf}}(X;G)$	locally finite homology
$\mathrm{H}_n^{\mathrm{simp},\leq}(K)$	n-th simplicial homology group of an ordered abstract simplicial complex
$\mathrm{H}_n^{\mathrm{simp}}(K)$	n-th simplicial homology group of an abstract simplicial complex
$\mathrm{H}_n^{\mathrm{simp}}(K;G)$	simplicial homology with G-coefficients
$\mathrm{H}_{\mathbb{H}}$	quaternionic Hopf map
$\mathrm{H}_{\mathbb{O}}$	octonionic Hopf map
$\mathrm{H}^n_{\mathrm{b}}(X;\mathbb{R})$	n-th bounded cohomology group
$\mathrm{H}^n_{\mathrm{c}}(X;G)$	cohomology with compact support
$\mathrm{H}^k_{\mathrm{dR}}(M,A)$	relative de Rham cohomology groups
$\mathrm{H}^n_{\mathrm{dR}}(M)$	n-th de Rham cohomology group
$\mathrm{index}(f,g,P)$	index of an intersection point of two maps f and g
$\mathrm{index}(f,z)$	index of a map f at a fixed point z
$\mathrm{index}(v,x)$	index of a vector field v at a zero x
$\mathrm{Inn}(\pi)$	inner automorphism group of a group π
$\mathrm{Isom}(M,g)$	group of isometries of the Riemannian manifold (M,g)
$\mathrm{Isom}^+(M,g)$	group of orientation-preserving isometries of the Riemannian manifold (M,g)
$\mathrm{K}(\pi,n)$	canonical Eilenberg-Maclane space of type $\mathrm{K}(\pi,n)$
$\lim^1$	lim one of an inverse system of abelian groups
$\mathrm{link}(K,s)$	link of the simplex s
$\mathrm{lk}(K,J)$	linking number of oriented knots K and J
$\mathrm{MCG}(M)$	mapping class group of a smooth manifold
$\mathrm{MCG}(M)$	orientation-preserving mapping class group of a smooth manifold
$\mathrm{mesh}(Y)$	the largest diameter of a simplex
$\mathrm{Mor}_{\mathcal{C}}(X,Y)_{\mathrm{inv}}$	set of invertible morphisms
$\mathrm{M}(\pi,n)$	Moore space of type $\mathrm{M}(\pi,n)$

$\mathrm{M}(\varphi)$	algebraic mapping cone of $\varphi\colon C_* \to D_*$
$\mathrm{Out}(\pi)$	outer automorphism group of a group π
O	infinite orthogonal group
PD_M	Poincaré duality isomorphism
$\mathrm{rank}(A)$	rank of an abelian group A
$\mathrm{rank}(A)$	rank of an abelian group
$\mathrm{rank}(M)$	rank of a module over a commutative domain R.
$\mathrm{sd}(K)$	barycentric subdivision of the simplicial complex K
$\mathrm{sign}(M)$	signature of a $4n$-dimensional topological manifold
$\mathrm{Spin}(n)$	spin group
Sp	infinite symplectic group
$\mathrm{Sp}(n)$	compact symplectic group
$\mathrm{St}(K,s)$	star of the simplex s
$\mathrm{Tor}(A)$	torsion subgroup of an abelian group A
$\mathrm{Tor}(H,G)$	G-torsion of H
$\mathrm{Tor}(X,f)$	mapping torus of $f\colon X \to X$
$\mathrm{tr}(\varphi)$	trace of the endomorphism φ
$\mathrm{T}_P\, M$	tangent space of a smooth manifold M at P
U	infinite unitary group
$\mathrm{Vect}^n_{\mathbb{C}}(X)$	set of isomorphism classes of n-dimensional $\mathbb{C}$-vector bundles over X
$\mathrm{Vect}^n_{\mathbb{R}}(X)$	set of isomorphism classes of n-dimensional $\mathbb{R}$-vector bundles over X
$\mathrm{Vect}^{n,+}_{\mathbb{R}}(X)$	set of isomorphism classes of oriented n-dimensional $\mathbb{R}$-vector bundles over X
$\mathrm{V}_P\, M$	visual tangent space of a submanifold of some $\mathbb{R}^n$
$\mathrm{sign}(\sigma)$	sign of the permutation σ
$\overline{\mathbb{L}}_{\geq 0}$	closed long ray
$\overline{\mathbb{L}}$	long ray
$\overline{B}^n$	n-dimensional closed ball in $\mathbb{R}^n$
$\overline{B}^n_r(y)$	closed r-ball around $y \in \mathbb{R}^n$
$\partial\Delta^n$	all points on Δ^n where at least one coordinate is zero
$\partial_0 X$	$\partial_0 X = \partial X \setminus \partial M$ of a submanifold X of a manifold M
∂_0	∂_0 of a submanifold with corner
$\partial_1 X$	$\partial_1 X = \partial X \cap \partial M$ of a submanifold X of a manifold M
∂_1	∂_1 of a submanifold with corner
$\partial_c M$	corner set of a submanifold M with corner
$\Phi^{\leq}_s$	characteristic map in a totally ordered simplicial complex
$\pi_0(X)$	the set of path-components of a topological space X
$\pi_1(X,x_0)$	fundamental group of (X,x_0)
π^s_k	stable homotopy group
$\pi_n(X,A,x_0)$	relative homotopy group
$\pi_n(X,x_0)$	n-th homotopy group of (X,x_0)
$\Sigma(K)$	simplicial suspension of an abstract simplicial complex K
$\Sigma(X)$	Suspension of X

$\Sigma(X)$	suspension of a topological space X
Σ_g	surface of genus g
$\Sigma_{g,n}$	surface of genus g minus n open disks
$\theta_{\mathbb{R}}$	the canonical singular cochain in $\mathrm{C}^1;\mathbb{R})$
$\theta_{\mathbb{R}}$	the canonical singular cochain in $\mathrm{C}^1;\mathbb{Z})$
$\varprojlim$	inverse limit of an inverse system
$\varinjlim$	direct limit of a direct system
$\varphi \cup \psi$	cup product
$\varphi \cap \sigma$	cap product on (co-) homology
$\varphi \cup \psi$	relative cup product on cohomology
$\varphi \cup \psi$	relative cup product
$\varphi \times \psi$	cross product on cohomology
$\wedge^k V^*$	vector space of alternating k-forms on V
$\widetilde{C}_*(X)$	reduced chain complex
$\widetilde{p}_I(M)$	I-Pontryagin number of a smooth manifold M
$\widetilde{X}_{x_0}$	canonical universal covering corresponding of (X, x_0)
$\widetilde{X}_{x_0}$	universal covering with respect to $x_0 \in X$
$\widetilde{X}^{\Gamma}_{x_0}$	canonical covering corresponding to $\Gamma \subset \pi_1(X, x_0)$
$\widetilde{\mathrm{H}}_n(X)$	n-th reduced homology group
$A \otimes B$	tensor product of two abelian groups A and B
$A \otimes_{\mathbb{K}} B$	tensor product of two real vector spaces A and B
B^n	n-dimensional open ball in $\mathbb{R}^n$
$B^n_r(y)$	open r-ball around $y \in \mathbb{R}^n$
$b_n(X)$	the n-th Betti number of a topological space
$b_n(X, A)$	the n-th Betti number of a pair of topological space
$C(X, x_0)$	reduced cone of (X, x_0)
$C_* \otimes D_*$	tensor product of chain complexes
c_y	the constant map that sends all points to y
$d(A, B)$	distance between two subsets of a metric space
E_8	the 8×8-matrix E_8
$f \cdot g$	algebraic intersection of two maps that intersect nicely
H	Hopf map
H_n	upper half-space in $\mathbb{R}^n$
i^n_j	the j-th face map $\Delta^{n-1} \to \Delta^n$
K^{inv}	inverse of a knot
K^{mir}	mirror image of a knot K
K^{mir}	mirror of a knot
K^{rev}	reverse of a knot
$K_1 * \cdots * K_m$	simplicial join of abstract simplicial complexes
$L(p, q)$	lens space
$M \# N$	connected sum of smooth manifolds M and N
$M \# N$	connected sum of topological manifolds M and N

$M \#_b M'$	boundary connected sum of two smooth manifolds M and N
$M \setminus\setminus F$	the smooth manifold M cut along a codimension-one submanifold F
$M \setminus\setminus F$	the smooth manifold M cut along a codimension-one submanifold F
$M^{\mathrm{h}}(\varphi)$	handle attachment to M along φ
$M^{\mathrm{h}}(\varphi)$	handle attachment to M along φ
$M^{\mathrm{s}}(\varphi)$	surgery on M along φ
$N \rtimes_\varphi \mathbb{Z}_k$	semidirect product of groups N and $\mathbb{Z}_k$
$N(\mathcal{U})$	nerve complex of a cover $\mathcal{U}$
N_g	non-orientable surface of genus g
$N_{g,n}$	non-orientable surface of genus g minus n open disks
$p(E)$	total Pontryagin class of a vector bundle E
$P(x, x_0)$	path space
$p_i(E)$	i-th Pontryagin class of a vector bundle E
$p_I(M)$	I-Pontryagin class of a smooth manifold M
$p_i(M, \mathbb{Q})$	i-th rational Pontryagin class of a smooth manifold
$Q^{\mathrm{as}}(M)$	asymmetric intersection pairing
$Q^{\mathrm{as}}_{M,A,B}$	asymmetric intersection pairing
$Q^{\mathrm{red}}(M)$	reduced intersection form
Q_M	intersection form of an even dimensional topological manifold
$Q_M^{\mathbb{F}_2}$	$\mathbb{F}_2$-value intersection form of an even dimensional topological manifold
$S(X, x_0)$	reduced suspension
$S^3_r(L)$	Dehn surgery along the framed link (L, r)
$S^3_r(K)$	Dehn surgery along K
S^∞	infinite-dimensional sphere
S^n	n-dimensional sphere in $\mathbb{R}^{n+1}$
$S^n_r(y)$	n-dimensional sphere of radius r around y
$S^n_{\geq 0}$	upper hemisphere of S^n
$S^n_{\leq 0}$	lower hemisphere of S^n
$s^\dagger$	dual chain of a simplex s
$X * X$	join of topological spaces X and Y
$X * Y * Z$	join of topological spaces X, Y and Z
X/A	quotient of topological space X by subset A
$X \otimes Y$	product of two CW-complexes
$X \vee Y$	wedge of the topological spaces X and Y
$X \wedge Y$	smash product of X and Y
$X \widetilde{\times} Y$	product in the category of CGWH-spaces
X^Y	set of continuous maps $Y \to X$
$Y \cup_\varphi Z$	Y glued to Z via φ
$Y \cup_Z X$	pushout of $Y \leftarrow Z \to X$

Notation

Statements, results and conjectures

Calculations