Topology

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Part I

General topology and basics of category theory

0. Preliminaries

In this preliminary chapter we introduce some concepts and results from set theory and linear algebra which we will use throughout these notes, but which otherwise have little to do with topology itself. The reader might want to skip this preliminary chapter and directly move to the next chapter, where we actually start with topology.

0.1. Basic set theory.

Topology builds on top of set theory. We will not go into proper set theory, but whatever we are doing is covered by ZFC, the Zermelo-Frankel set theory with the Axiom of Choice, which was developed by Ernst Zermelo and Adolf Fraenkel [Zer08; Fra22; Zer30]. We refer the reader to [Cie97] for a detailed introduction. In these notes we mostly employ the language of "naive set theory". In particular, when we describe sets we sometimes use creative notation and language. The hope is that the language is chosen in such a way that any good-willed reader will understand what is meant.

In this section we recall a few basic definitions and results from set theory which will frequently use throughout these notes. Usually we will not refer to the statements explicitly, unless it helps to clarify an argument.

First we introduce the following notation which should mostly be standard.

Notation.

(1) We write $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$ and $\mathbb{N} := \{1, 2, 3, \dots\}$.
(2) We denote by $\varnothing$ the empty set.
(3) Given a set X we write $A \subset X$ if A is a subset of X and we write $A \subsetneq X$ if A is a subset of X with $A \neq X$.
(4) Given a set X and a subset $A \subset X$ we denote by $X \setminus A := \{x \in X \,|\, x \notin A\}$ the complement of A in X.
(5) Given a set X we denote by $\mathcal{P}(X)$ its power set, i.e. $\mathcal{P}(X)$ is the set of all subsets of X.

Definition. A family of subsets of a set X consists of a set I and a map $f\colon I \to \mathcal{P}(X)$.

Remark.

(1) When it comes to families of subsets we often use suggestive notation, e.g. we might write "let $\{U_i\}_{i\in I}$ be a family of subsets of X", which just means that I is a set and the map $I \to \mathcal{P}(X)$ is denoted by $i \mapsto U_i$.
(2) When we write "let $U_1, \dots, U_k$ be subsets" or "let A, B be subsets", then at times we will think of these as families of subsets with index sets $\{1, \dots, k\}$ respectively $\{1, 2\}$.

Convention. Let X be a set. We will view any subset $\mathcal{U} \subset \mathcal{P}(X)$ as a family of subsets of X via the index set $I = \mathcal{U}$ and the map $I = \mathcal{U} \to \mathcal{P}(X)$ given by $U \mapsto U$. This convention might sound silly, but it will be useful on many occasions.

We continue with two of the most important definitions of set theory.

Definition. Let X be a set.

(1) Let $\{U_i\}_{i\in I}$ be a family of subsets of X. We write

$$\bigcup_{i\in I} U_i = \{x \in X \mid \text{ there exists an } i \in I \text{ with } x \in X_i\} \quad \text{(Union)}$$
$$\bigcap_{i\in I} U_i = \{x \in X \mid \text{ for every } i \in I \text{ we have } x \in X_i\} \quad \text{(Intersection)}.$$

In particular, if $I = \varnothing$, then $\bigcup_{i\in I} U_i = \varnothing$ and $\bigcap_{i\in I} U_i = X$.[1]

(2) As mentioned above we view subsets $U_1, \ldots, U_k$ of X as a family of subsets. We write $U_1 \cup \cdots \cup U_k := \bigcup_{i\in\{1,\ldots,k\}} U_i$ and the same way for the intersection.

(3) We say that two subsets A and B of X are disjoint if $A \cap B = \varnothing$.

The very first lemma of these lecture notes says that intersection is distributive with respect to union.

Lemma 0.1. Let X be a set and let $\{U_i\}_{i\in I}$ be a family of subsets of X. Given any subset A of X we have

$$A \cap \left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} (A \cap U_i).$$

Proof. The lemma follows easily from the definitions and elementary logic. ■

Notation. Let $\{U_i\}_{i\in I}$ be a family of subsets of a given set X. If the U_i are pairwise disjoint, i.e. if for any $i \neq j \in I$ we have $U_i \cap U_j = \varnothing$, then at times we write

$$\bigsqcup_{i\in I} U_i := \bigcup_{i\in I} U_i.$$

We move on to the famous de Morgan's Laws from elementary set theory.

Lemma 0.2. (De Morgan's Laws) Let X be a set and let $\{A_i\}_{i\in I}$ be a family of subsets of X. If given a subset $B \subset X$ we denote its complement by $B^c := X \setminus B$, then

$$\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c \quad \text{and} \quad \left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c.$$

Or equivalently, in a more cumbersome notation, we have

$$X \setminus \left(\bigcap_{i\in I} A_i\right) = \bigcup_{i\in I} (X \setminus A_i) \quad \text{and} \quad X \setminus \left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I} (X \setminus A_i).$$

Proof. The lemma follows again easily from the definitions and elementary logic. ■

Next we study maps between sets and their effect on subsets.

Notation. Let $f\colon X \to Y$ be a map between sets.

(1) Given a subset $A \subset X$ we write $f(A) := \{f(x) \mid x \in A\}$.

(2) Given a subset $A \subset Y$ we write $f^{-1}(A) := \{x \in X \mid f(x) \in A\}$. Furthermore, given $y \in Y$ we write $f^{-1}(y) := f^{-1}(\{y\})$.[2]

The next lemma concerns the interaction between operations on sets and maps between sets.

[1] This simple observation will be useful at times.

[2] Purists might decry this notation, but it does make notation much more readable.

Lemma 0.3. (Image-Preimage Lemma) Let $f\colon X \to Y$ be a map between two sets X and Y. Then the following statements hold:

(1)	(a)	for every subset $U \subset X$ we have	U	$\subset$	$f^{-1}(f(U))$
	(b)	furthermore, if f is injective[3], then we have	U	$=$	$f^{-1}(f(U))$
(2)	(a)	for every subset $V \subset Y$ we have	V	$\supset$	$f(f^{-1}(V))$
	(b)	furthermore, if f is surjective, then we have	V	$=$	$f(f^{-1}(V))$
(3)		for every family $\{U_i\}_{i\in I}$ of subsets of X we have	$f\Big(\bigcup_{i\in I} U_i\Big)$	$=$	$\bigcup_{i\in I} f(U_i)$
(4)		for every family $\{U_i\}_{i\in I}$ of subsets of X we have	$f\Big(\bigcap_{i\in I} U_i\Big)$	$\subset$	$\bigcap_{i\in I} f(U_i)$
(5)	(a)	for every subset $U \subset X$ we have	$f(X \setminus U)$	$\supset$	$f(X) \setminus f(U)$
	(b)	furthermore, if $f^{-1}(f(U)) = U$, then we have	$f(X \setminus U)$	$=$	$f(X) \setminus f(U)$
(6)		for every family $\{V_i\}_{i\in I}$ of subsets of Y we have	$f^{-1}\Big(\bigcup_{i\in I} V_i\Big)$	$=$	$\bigcup_{i\in I} f^{-1}(V_i)$
(7)		for every family $\{V_i\}_{i\in I}$ of subsets of Y we have	$f^{-1}\Big(\bigcap_{i\in I} V_i\Big)$	$=$	$\bigcap_{i\in I} f^{-1}(V_i)$
(8)		for every subset $V \subset Y$ we have	$f^{-1}(Y \setminus V)$	$=$	$X \setminus f^{-1}(V)$.

Proof. The lemma follows again from the definitions and elementary logic. ∎

Definition. Let $\{X_i\}_{i\in I}$ be a family of sets. The product of the family $\{X_i\}_{i\in I}$ is defined as the set[4] [5]

$$\prod_{i\in I} X_i := \Big\{ f\colon I \to \bigcup_{i\in I} X_i \;\Big|\; \text{for every } i \in I \text{ we have } f(i) \in X_i \Big\}.$$

Furthermore, given a set X and a set I we write

$$X^I := \prod_{i\in I} X.$$

We can now state the Axiom of Choice, which is discussed in greater detail in [Cie97, p. 15] and in [Jec73].

Axiom of Choice 0.1. For every family of sets $\{X_i\}_{i\in I}$, such that each I is non-empty and such that each X_i is non-empty, the product $\prod_{i\in I} X_i$ is non-empty.

As we mentioned in the beginning, in these lecture notes we will work with ZFC theory, in particular we will use the Axiom of Choice, even though for the most part it is for our purposes rather irrelevant.

For finitely many sets we adopt the following convenient notation.

Notation.

[3] As a totally irrelevant side remark we would like to point out that the fundamental terms "surjective", "injective" and "bijective" are actually fairly recent additions to the mathematical language. They were first introduced on page 80 of Nicolas Bourbaki's "Théorie des ensembles" first published in 1954, see [Bou54]. The reader who has not read the biography of the great mathematician Nicolas Bourbaki is strongly encouraged to do so.

[5] This definition is now possibly beyond the realm of "naive set theory", since it is perhaps a little unclear what the union of arbitrary sets is supposed to mean. But the Zermelo-Fraenkel axioms allow for unions of arbitrary sets.

(1) Given finitely many sets $X_1, \ldots, X_n$ we write

$$X_1 \times \cdots \times X_n \ := \ \{(x_1, \ldots, x_n) \mid \text{ for } i = 1, \ldots, n \text{ we have } x_i \in X_i\}.$$

This set has an obvious bijection to the product $\prod_{i=1}^{n} X_i$, and we will freely go back and forth between these two notations.

(2) Given a set X and given $n \in \mathbb{N}$ we write

$$X^n \ := \ \underbrace{X \times \cdots \times X}_{n \text{ times}}.$$

We turn to the product of not just finitely many sets but the product of a family of sets. We conclude this section with one last piece of notation.

Notation.

(1) If $\{A_i\}_{i \in I}$ is a family of sets, then we define the disjoint union as the set

$$\bigsqcup_{i \in I} A_i \ := \ \bigcup_{i \in I} (A_i \times \{i\}).$$

(2) If $I = \{1, \ldots, k\}$, then we write $A_1 \sqcup \cdots \sqcup A_k := \bigsqcup_{i=1}^{k} A_i$.

(3) We use obvious variations on (1) and (2), for example given sets A and B we refer to

$$A \sqcup B \ := \ (A \times \{1\}) \cup (B \times \{2\})$$

as the disjoint union of A and B

Remark. Let $\{U_i\}_{i \in I}$ be a family of subsets of a given set X such that for any $i \neq j \in I$ we have $U_i \cap U_j = \varnothing$. The expression $\bigsqcup_{i \in I} U_i$ now describes two different sets, namely it describes the subset of X defined on page 19 and it describes the disjoint union as defined above. There is a natural bijection between these two sets and thus we will freely go back and forth between these notations. In practice should not pose any problems.

0.2. **Relations and equivalence relations on sets.** We start out with recalling the notion of a relation.

Definition. A relation on a set S is a subset of $S \times S$.

The definition of a relation in its full generality is pretty useless. It becomes more interesting once we introduce several adjectives.

Definition. Let S be a set and let $R \subset S \times S$ be a relation on S.

(1) The relation is called reflexive, if for every $s \in S$ we have $(s, s) \in R$.
(2) The relation is called transitive, if for every $(s, t), (t, u) \in R$ we also have $(s, u) \in R$.
(3) The relation is called symmetric, if for every (s, t) in R we also have $(t, s) \in R$.

We adopt the following notation.

Notation. Let S be a set and let $R \subset S \times S$ be a relation on S. Given $x, y \in S$ we write $x \, R \, y$ if and only if $(x, y) \in R$.[6]

[6]Most relations that we are working with are denoted by symbols like "$\sim$", "$\leq$", "$\cong$" or "$\simeq$".

Example. We consider the set $\mathbb{R}$ together with the relation

$$\leq \ := \ \{(x,y) \in \mathbb{R}^2 \,|\, y - x \text{ is not negative}\}.$$

One can easily verify that this relation $\leq$ is reflexive and transitive, but that it is not symmetric. Note that given $x, y \in \mathbb{R}$ the above expression "$x \leq y$" has the usual meaning that we are all used to.

Most of the relations that we will consider are going to be reflexive and transitive. There is a big difference though in the roles of symmetric and (possibly) non-symmetric relations. Let us consider the symmetric ones first. We will turn to the (possibly) non-symmetric relations in the next section.

Definition. An equivalence relation on a set S is a relation that is reflexive, transitive and symmetric.

Convention. Equivalence relations on a set S are almost invariably denoted by "$\sim$" or some variation thereof.

To practice our conventions and notations, the three properties of an equivalence relation "$\sim$" on a set S are precisely the following three statements:

(1) for any $s \in S$ we have $s \sim s$ (reflexivity),
(2) for any $s, t, u \in S$ with $s \sim t$ and $t \sim u$ we have $s \sim u$ (transitivity),
(3) for any $s, t \in S$ with $s \sim t$ we also have $t \sim s$ (symmetry).

We continue with the following definition.

Definition. Let S be a set together with an equivalence relation "$\sim$". An equivalence class of $(S, \sim)$ is a subset C of S with the following three properties:

(a) the subset is non-empty,
(b) given any two $s, t \in C$ we have $s \sim t$, and
(c) given $s \in C$ and $t \in S$ with $s \sim t$ we have $t \in C$.

The following lemma summarizes the key properties of equivalence classes.

Lemma 0.4. (Equivalence Class Lemma) Let S be a set together with an equivalence relation "$\sim$".

(1) Every element of S is contained in a unique equivalence class, namely

$$[s] \ := \ \{t \in S \,|\, t \sim s\}.$$

(2) Two equivalence classes are either disjoint or the same.

Proof. Surely the reader will have seen the proof of the lemma before. Nonetheless we give the proof, in particular to highlight the fact that we do indeed need the symmetry condition.

(1) Let $s \in S$. Since $\sim$ is reflexive we see that $s \in [s]$. Thus it remains to show that $[s] := \{t \in S \,|\, t \sim s\}$ is indeed an equivalence class.
  (a) We already showed that $s \in [s]$, in particular we see that $[s]$ is non-empty.
  (b) Let $t, t' \in [s]$. By definition this means that $t \sim s$ and $t' \sim s$. By symmetry we $s \sim t'$. By transitivity we deduce from $t \sim s$ and $s \sim t'$ that $t \sim t'$.
  (c) Let $t \in [s]$ and suppose that $t \sim u$. Since by definition we have $s \sim t$ we obtain from transitivity that $s \sim u$, i.e. we see that $u \in [s]$.

(2) Let C and D be two equivalence classes. We suppose that $C \cap D \neq \varnothing$. Thus there exists an $x \in C \cap D$. We need to show that $C = D$. Thus let $d \in D$. By Condition (2) of an equivalence class applied to D we know that $x \sim d$. By condition (1) of an equivalence class, applied to C, we see that $d \in C$. Thus we have shown that $D \subset C$. Evidently the same argument shows that $C \subset D$. ■

Throughout these notes we adopt the following standard notation.

Notation. Let S be a set together with an equivalence relation "$\sim$".
(1) We denote by $S/\sim$ the set of equivalence classes.
(2) Given $s \in S$ we denote by $[s]$ the unique equivalence class that contains $[s]$.
(3) We refer to the map

$$\begin{array}{rcl} p\colon X & \to & X/\sim \\ x & \mapsto & [x] \end{array}$$

as the natural projection.

Example. Let $n \in \mathbb{N}$. We consider the set $\mathbb{Z}$ together with the equivalence relation that is given by

$$x \sim y \quad :\Longleftrightarrow \quad (x-y) \text{ is divisible by } n.$$

There are exactly n equivalence classes, namely

$$\begin{array}{rcl} [0] & = & \{\dots, -2n, -n, 0, n, 2n, 3n, \dots\} \\ [1] & = & \{\dots, 1-2n, 1-n, 1, 1+n, 1+2n, 1+3n, \dots\} \\ & \vdots & \\ [n-1] & = & \{\dots, -2n-1, -n-1, -1, n-1, 2n-1, 3n-1, \dots\}. \end{array}$$

To the despair of many of my cultured friends we write $\mathbb{Z}_n := \mathbb{Z}/\sim$.

Lemma 0.5. (Quotient-Factorization Lemma) Let $\sim$ be an equivalence relation on a set X. We denote by $p\colon X \to X/\sim$ the natural projection. Let $f\colon X \to Y$ be a map to some set Y with the property that $f(x) = f(y)$ whenever $x \sim y$. Then the map

$$\begin{array}{rcl} g\colon X/\sim & \to & Y \\ [x] & \mapsto & f(x) \end{array}$$

is well-defined and it is the unique map $g\colon X/\sim \to Y$, such that $f = g \circ p$, i.e. such that the following diagram of maps commutes:

$$\begin{array}{ccc} X & \xrightarrow{p} & X/\sim \\ & \searrow^{f} & \downarrow{\scriptstyle \exists!\, g} \\ & & Y. \end{array}$$

Proof. Let $\sim$ be an equivalence relation on a set X and let $f\colon X \to Y$ be a map with the property that $f(x) = f(y)$ whenever $x \sim y$. Let a be an element of $X/\sim$. Since equivalences classes are by definition non-empty there exists an $x \in X$ with $a = [x]$. We set

$$g(a) \;:=\; f(x).$$

Evidently this definition does not depend on the choice of x. The map g satisfies $f = g \circ p$. Since the projection $X \to X/\sim$ is surjective we see that g is uniquely determined by this property. ■

We will now see that every relation gives rise to an equivalence relation.

Definition. Let S be a set and let R be a relation on S. We say $x, y \in S$ are equivalent, if there exists a sequence $x_0, \dots, x_k$ of elements in S such that $x = x_0$, $x_k = y$ and such that for each $i \in \{0, \dots, k-1\}$ we have $x_i \, R \, x_{i+1}$ or $x_{i+1} \, R \, x_i$. It is straightforward to verify that "$\sim$" defines an equivalence relation on S. We refer to it as the equivalence relation generated by the relation R.

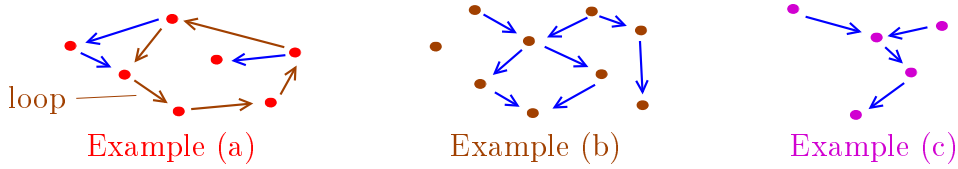
Example. We consider the set $S = \mathbb{R}$ and the relation that is defined by $x \, R \, y$ if and only if $y = x + 1$. The equivalence relation generated by R is the equivalence relation that is given by $x \sim y$ if and only if $x - y \in \mathbb{Z}$.

Convention. At times we are a little sloppy in our notation and we do not distinguish in our notation between a relation and the equivalence relation it generates.

0.3. **Orders on sets.** In this section we turn our gaze towards relations which are not required to be symmetric. We have the following harmless definition.

Definition.
(1) A relation "$\leq$" on a set S is called antisymmetric if $a \leq b$ and $b \leq a$ implies $a = b$.
(2) A relation that is reflexive, transitive and antisymmetric is called a partial order.
(3) A set together with a partial order is called a partially ordered set.
(4) Let "$\leq$" be a partial order on a set A. Given $a, b \in A$ we write $a < b$ if $a \leq b$ and if $a \neq b$.

Example.
(1) Let X be a subset of $\mathbb{R}$ and denote by $\leq$ the usual "less or equal" relation on the real numbers. Then $(X, \leq)$ is a partially ordered set.
(2) We equip $\mathbb{R} \cup \{\pm\infty\}$ with the relation $\leq$ that is given by the usual relation on $\mathbb{R}$ together with $-\infty \leq a \leq \infty$ for any $a \in \mathbb{R}$. This turns $\mathbb{R} \cup \{\pm\infty\}$ into a partially ordered set.
(3) Let M be a set. Recall that we denote by $\mathcal{P}(M)$ the power set of M, i.e. the set of all subsets of M. Then "being a subset", i.e. the relation given by "$\subset$", is a partial order on $\mathcal{P}(M)$.
(4) Let X be a set. The trivial partial order $\leq_{\text{tr}}$ is the partial order where $x \leq_{\text{tr}} y$ if and only if $x = y$.
(5) We consider the three reflexive transitive relations that are illustrated in the figure below. Here it is understood that $P \leq Q$ if there exists a "directed path" from Q to P.
   (a) The example on the left has a "loop". Given any two points P, Q on the loop we have $P \leq Q$ and $Q \leq P$, hence this example is not a partial order.
   (b) This example is a partial order.
   (c) This example is also a partial order.



The following construction lets us build new examples of partially ordered sets out of given ones.

Definition. Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. Given (a, b) and (a', b') in $A \times B$ we define[7]

$$(a, b) < (a', b') \quad :\Longleftrightarrow \quad \text{either } (a < a') \text{ or } (a = a' \text{ and } b < b')$$

and we define $(a, b) \leq (a', b')$ if either $(a, b) = (a', b')$ or $(a, b) < (a', b')$. One can easily verify that "$\leq$" is a partial order on $A \times B$. We refer to this partial order on $A \times B$ as the lexicographic order on $A \times B$.

0.4. **The cardinality of sets.** In this section we introduce the notion of the cardinality of a set.

Definition.

(1) We say that two given sets S and T are equinumerous if there exists a bijection $\varphi\colon S \to T$.[8]

(2) Given a set S we define the cardinality $\#S$ of S as the equinumerosity equivalence class of S in the class of all sets[9]. Put differently we have

$$\#S \ := \ \text{class of all sets } T \text{ that are equinumerous with } S.$$

(3) We say a set is finite if there exists an $n \in \mathbb{N}_0$ such that $\#S = \#\{1, \ldots, n\}$, otherwise we say that the set is infinite

(4) If a set S is finite, then we write $\#S = n$ for the unique $n \in \mathbb{N}_0$ with $\#S = \#\{1, \ldots, n\}$. If S infinite, then sometimes we write $\#S = \infty$[10]. At times we also write $|S| \in \mathbb{N}_0 \cup \{\infty\}$ instead of $\#S \in \mathbb{N}_0 \cup \{\infty\}$.

(5) Let S be a set. If $\#S = n$ for some $n \in \mathbb{N}_0$ or if $\#S = \#\mathbb{N}$, then we say that S is countable, otherwise we say that S is uncountable.

Example. We have $\#(\mathbb{R} \setminus \{0\}) = \#\mathbb{R}$. Indeed a bijection is given by

$$\begin{array}{ccc} \mathbb{R} \setminus \{0\} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} x, & \text{if } x \in \mathbb{R} \setminus \mathbb{N}, \\ x - 1, & \text{if } x \in \mathbb{N}. \end{cases} \end{array}$$

Remark. Another way of putting the definition of cardinality is that given two sets S and T we have $\#S = \#T$ if and only if there exists a bijection from S to T.

The following lemma gets used on many occasions in these lecture notes.

Lemma 0.6. (Countability Lemma)

[7]Note that if $a < a'$, then we do not demand that $b \leq b'$.

[8]It follows immediately from the definitions that being equinumerous defines an equivalence relation on the class of all sets.

[9]We defined the notion of an equivalence relation on a set, but not on a class. You can ask any of your professors, as long as it is not me, what this now means.

[10]This notation is a little risque. The sets $\mathbb{N}$ and $\mathbb{R}$ are both infinite, but in Corollary 0.8 we will see that $\mathbb{N}$ and $\mathbb{R}$ are not equinumerous. Thus we write $\#\mathbb{N} = \infty$ and $\#\mathbb{R} = \infty$, but $\#\mathbb{N} \neq \#\mathbb{R}$.

(1) Every subset of a countable set is again countable.
(2) Let S be a countable set and let $f\colon S \to T$ be a map. The image $f(S)$ is again countable.
(3) The product of finitely many countable sets is again countable.
(4) The disjoint union of countably many countable sets is again countable.
(5) Let T be a set and let S be a subset such that $T \setminus S$ is countable. Then S is countable if and only if T is countable.
(6) If S is a finite set and if T is a countable set, then the set of maps from S to T is also countable.

Example. We consider the map

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{N} & \to & \mathbb{Q} \\ (m,n) & \mapsto & \frac{m}{n}. \end{array}$$

Evidently this map is surjective. It follows from the Countability Lemma 0.6 (2) and (3) that $\mathbb{Q}$ is countable.

Sketch of proof.

(1) Let T be a subset of a countable set S. After excluding the trivial case that S is finite and the case that T is finite one sees that one has to prove the following: given any infinite subset T of $S = \mathbb{N}$ there exists a bijection $\varphi\colon \mathbb{N} \to T$. Such a bijection is given by defining $\varphi(i)$ as the i-th smallest element of T.
(2) Let S be a countable set and let $f\colon S \to T$ be a map. We only really need to consider the case that $S = \mathbb{N}$ and that $f(S)$ is infinite. In this case the bijection $\varphi\colon \mathbb{N} \to f(S)$ is given by defining $\varphi(1) = f(1)$ and by defining iteratively

$$\varphi(i+1) \ := \ \left\{ f(k) \,\middle|\, \begin{array}{c} k \in \mathbb{N} \text{ is the smallest number such} \\ \text{that } f(k) \notin \{\varphi(1), \ldots, \varphi(i)\} \end{array} \right\}.$$

(3) By induction and by ignoring the rather dull case of finite sets it remains to show that $\mathbb{N} \times \mathbb{N}$ is countable. This can be shown by the well-known argument sketched in the figure below.

An alternative, more formal argument would be to consider the map $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ that is given by $(k,l) \mapsto (2^k \cdot 3^l)$. This map is injective, thus it follows from (1) that $\mathbb{N} \times \mathbb{N}$ is countable.
(4) Let $\{S_i\}_{i \in I}$ be a countable family of countable subsets. It is straightforward to define an injective map $\bigsqcup_{i \in I} S_i \to \mathbb{N} \times \mathbb{N}$. Thus it follows from (1) and (3) that the disjoint union $\bigsqcup_{i \in I} S_i$ is countable.
(5) Let T be a set and let S be a subset such that $T \setminus S$ is countable. If S is countable, then it follows from (4) that $T = (T \setminus S) \cup S$ is countable. On the other hand if T is countable, then it follows from (1) that S is countable.

(6) Let $S = \{v_1, \dots, v_n\}$ be a finite set and let T be a countable set. We have an obvious bijection between T^n and the set of maps from S to T. Thus it follows from (4) that the set of maps from S to T is countable. ■

We move on to the next definition.

Definition. Let S and T be two sets.
(1) We write $\#S \leq \#T$ if there exists an injection $\varphi\colon S \to T$.
(2) We write $\#S < \#T$ if $\#S \leq \#T$ but $\#S \neq \#T$.

Example. It follows from the Countability Lemma 0.6 (1) that a set S is countable if and only if $\#S \leq \#\mathbb{N}$.

The following theorem was first proved by Georg Cantor [Can92]. The ratio of significance of the statement to length of the proof seems to be unbeatable.

Theorem 0.7. (Cantor's Theorem) Recall that given a set X we denote by $\mathcal{P}(X)$ its power set, i.e. the set of all subsets of X. For every set X we have $\#X < \#\mathcal{P}(X)$.

Proof. First note that the map

$$\begin{array}{ccc} X & \to & \mathcal{P}(X) \\ x & \mapsto & \{x\} \end{array}$$

is evidently injective, hence $\#X \leq \#\mathcal{P}(X)$. Now let $f\colon X \to \mathcal{P}(X)$ be a map. We have to show that it cannot be surjective. We consider the set

$$Y \;:=\; \{x \in X \,|\, x \notin f(x)\}.$$

We claim that Y is not in the image of f. Indeed, if there did exist a $z \in X$ with $f(z) = Y$, then we would have

$$z \in Y \quad \underset{\text{by definition of } Y}{\overset{\uparrow}{\iff}} \quad z \notin f(z) \quad \underset{\text{since } f(z) = Y}{\overset{\uparrow}{\iff}} \quad z \notin Y.$$

We have thus obtained a contradiction. ■

Corollary 0.8. The set $\mathbb{R}$ of real numbers is uncountable.

Proof. [11] We consider the maps

$$\begin{array}{ccc} f\colon \mathcal{P}(\mathbb{N}) & \to & \{\mathbb{Z}_2\}^{\mathbb{N}} \\ X & \mapsto & \left(\begin{array}{ccc} \mathbb{N} & \to & \mathbb{Z}_2 \\ k & \mapsto & \begin{cases} 1, & \text{if } k \in X \\ 0, & \text{otherwise} \end{cases} \end{array} \right) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} g\colon \{\mathbb{Z}_2\}^{\mathbb{N}} & \to & \mathbb{R} \\ f & \mapsto & \sum_{k=1}^{\infty} \frac{f(k)}{3^k}. \end{array}$$

One can easily verify that $f\colon \mathcal{P}(\mathbb{N}) \to \{\mathbb{Z}_2\}^{\mathbb{N}}$ is a bijection and that $g\colon \{\mathbb{Z}_2\}^{\mathbb{N}} \to \mathbb{R}$ is injective. Thus it follows from Cantor's Theorem 0.7 together with the Countability Lemma 0.6 (1) that $\mathbb{R}$ is uncountable. ■

Now we can formulate the final result of this section.

Theorem 0.9. (Bernstein-Schröder Theorem)[12] Let S and T be two sets. Then

$$\#S \leq \#T \quad \text{and} \quad \#T \leq \#S \quad \Longrightarrow \quad \#S = \#T.$$

[11] There are many proofs that the real numbers are uncountable. Perhaps the most beautiful and astounding proof that $\mathbb{Q}$ is countable is given via the amazing Farey sequence, see [Hat21, Chapter 3].

Example. Let U be a non-empty subset of $\mathbb{R}$ which contains an open interval (a, b) with $a < b$. Then $\#U = \#\mathbb{R}$. Indeed, the inclusion $U \to \mathbb{R}$ is of course injective, thus we see that $\#U \leq \#\mathbb{R}$. Furthermore the map $\varphi\colon (a, b) \to \mathbb{R}$ given by $x \mapsto \frac{x}{(x-a)(x-b)}$ defines a bijection between (a, b) and $\mathbb{R}$. Thus $\mathbb{R} \xrightarrow{\varphi^{-1}} (a, b) \to U$ is also an injection. This means that $\#\mathbb{R} \leq \#U$. Therefore it follows from the Bernstein-Schröder-Theorem 0.9 that there exists a bijection from U to $\mathbb{R}$.

Proof. Let $f\colon A \to B$ and $g\colon B \to A$ be two injective maps between two sets A and B. Without loss of generality we can assume that A and B are disjoint.[13] We denote by $\sim$ the equivalence relation on $A \sqcup B$ that is generated by $a \sim f(a)$ for $a \in A$ and by $b \sim g(b)$ for $b \in B$. Note that if $a \in A$ lies in the image of g, then by the injectivity of g there exists a unique element $g^{-1}(a) \in B$ and we have $a \sim g^{-1}(a)$. The same remark works with the roles of f and g reversed.

Claim. For each equivalence class I there exists a bijection $A \cap I \to B \cap I$.

Proof. So let I be an equivalence class and let $a \in A \cap I$.[14] The subset $I \subset A \sqcup B$ consists of all elements equivalent to a. The equivalence class I is then of one of the following three types:[15]

			A	B	A	B	A	B		
(1)	a'	$\dots$	$f^{-1}(g^{-1}(a))$,	$g^{-1}(a)$,	a,	$f(a)$,	$g(f(a))$,	$f(g(f(a))$,	$\dots$	with $a' \in A \setminus g(B)$
(2)	b'	$\dots$	$f^{-1}(g^{-1}(a))$,	$g^{-1}(a)$,	a,	$f(a)$,	$g(f(a))$,	$f(g(f(a))$,	$\dots$	with $b' \in B \setminus f(A)$
(3)	$\dots$	$\dots$	$f^{-1}(g^{-1}(a))$,	$g^{-1}(a)$,	a,	$f(a)$,	$g(f(a))$,	$f(g(f(a))$,	$\dots$	

In case (1) the map f defines a bijection $A \cap I \to B \cap I$, in case (2) the map g^{-1} defines a bijection $A \cap I \to B \cap I$ and in case (3) both f and g^{-1} define a bijection $A \cap I \to B \cap I$.⊞

Recall that by the Equivalence Class Lemma 0.4 we know that equivalence classes always give a disjoint decomposition of a set. In our case, if I_j, $j \in J$ denote the set of equivalence classes of $\sim$ on $A \sqcup B$, then we have

$$A \sqcup B = \bigsqcup_{j \in J} I_j \quad \text{and therefore also} \quad A = \bigsqcup_{j \in J} (A \cap I_j) \text{ and } B = \bigsqcup_{j \in J} (B \cap I_j).$$

Since for each $j \in J$ we have a bijection $A \cap I_j \to B \cap I_j$ we get a bijection $A \to B$. ■

0.5. The Cauchy-Schwarz Inequality.

[12] The theorem was initially stated by Georg Cantor [Can87] in 1887 without a proof and it was first proved by Felix Bernstein [Ber05] and Ernst Schröder [**Schro1896**] in 1896. Sometimes it is also called the Cantor-Bernstein-Schröder Theorem. The proof we provide is attributed to Julius König [Kön06].

[13] If A and B are not disjoint, then we replace A by $A' = A \times \{0\}$ and B by $B' = B \times \{1\}$, then A' and B' are disjoint, since any two elements in A' and B' now have different second coordinate.

[14] There always exists an $a \in A \cap I$. Indeed, I is by definition non-empty. If it contains an element in A we are done. Otherwise it contains an element in $b \in B$, but then it also contains $g(b) \in A$, since $b \sim g(b)$.

[15] Note that we do not claim that the elements in the given sequences are distinct, for example $f\colon A \to B$ could be a bijection and $g = f^{-1}$ its inverse, then the equivalence class consists only of the two elements a and $g(a)$.

Definition. Let R be a commutative ring and let V and W be R-modules.

(1) A pairing on $V \times W$ is a bilinear map

$$\varphi\colon V \times W \ \to \ R,$$

were by bilinear we mean that for all $v, v' \in v$ and for all $w, w' \in W$ we have the equalities $\varphi(v + v', w) = \varphi(v, w) + \varphi(v', w)$ and $\varphi(v, w + w') = \varphi(v, w) + \varphi(v, w')$.

(2) A pairing is called R-bilinear if for all $\lambda \in R$ and all $v \in V$ and $w \in W$ we have $\varphi(\lambda \cdot v, w) = \lambda \cdot \varphi(v, w)$ and $\varphi(v, \lambda \cdot w) = \varphi(v, w) \cdot \lambda$.

(3) An R-bilinear pairing $\varphi\colon V \times V \to S$ is called symmetric if $\varphi(v, w) = \varphi(w, v)$ for all $v, w \in V$.

We now specialize to the case that the ring R are the real numbers.

Definition. Let V be a real vector space.

(1) We say a pairing φ is positive-definite if $\varphi(v, v) > 0$ for all $v \in V \setminus \{0\}$.

(2) We say a pairing φ is positive-semidefinite if $\varphi(v, v) \geq 0$ for all $v \in V \setminus \{0\}$.

Example. Let $n \in \mathbb{N}_0$. The standard pairing on $\mathbb{R}^n$ is given by

$$\begin{array}{rcl} \langle -, - \rangle\colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R} \\ (v, w) & \mapsto & \langle v, w \rangle := \sum_{i=1}^{n} v_i \cdot w_i. \end{array}$$

Here we use the standard convention that for $v \in \mathbb{R}^n$ we denote by $v_1, \ldots, v_n$ its coordinates. This standard pairing on $\mathbb{R}^n$ is evidently $\mathbb{R}$-bilinear, symmetric and positive-definite.

Proposition 0.10. (Cauchy-Schwarz Inequality) Let V be a real vector space and let

$$\begin{array}{rcl} \varphi\colon V \times V & \to & \mathbb{R}_{\geq 0} \\ (v, w) & \mapsto & \varphi(v, w) \end{array}$$

be an $\mathbb{R}$-bilinear form.

(1) If φ is positive semidefinite, then for any $v, w \in V$ we have

$$\varphi(v, w)^2 \ \leq \ \varphi(v, v) \cdot \varphi(w, w).$$

(2) Now suppose that φ is positive-definite. In this equality in (1) holds if and only if v and w are linearly dependent.

Example. Let $v_1, \ldots, v_n \in \mathbb{R}^n$ and let $w_1, \ldots, w_n \in \mathbb{R}$. It follows immediately from the Cauchy-Schwarz Inequality 0.10, applied to the standard pairing on $\mathbb{R}^n$, that

$$(v_1 \cdot w_1 + \cdots + v_n \cdot w_n)^2 \ \leq \ (v_1^2 + \cdots + v_n^2) \cdot (w_1^2 + \cdots + w_n^2).$$

Proof. First we just assume that φ is positive semidefinite. Let $v, w \in V$. Given $t \in \mathbb{R}$ we consider

$$f(t) \ := \ \varphi(v + t \cdot w, v + t \cdot w) \ \underset{\text{since } \varphi \text{ is } \mathbb{R}\text{-bilinear and symmetric}}{=} \ \varphi(v, v) + 2t \cdot \varphi(v, w) + t^2 \cdot \varphi(w, w).$$

Since φ is positive semidefinite we see that f assumes only non-negative values. Since f is a polynomial function we obtain from elementary analysis that f assumes a minimum, which in turn implies that there exists a $t_0 \in \mathbb{R}$ with $f'(t_0) = 0$. It follows from the second description of $f(t)$ that

$$f'(t) \ = \ 2 \cdot \varphi(v, w) + 2t \cdot \varphi(w, w).$$

This implies that $t_0 \cdot \varphi(w,w) = -\varphi(v,w)$. Now note that

$$\overbrace{\varphi(v+t_0\cdot w, v+t_0\cdot w)}^{\geq 0}\cdot\overbrace{\varphi(w,w)}^{\geq 0} = \varphi(w,w)\cdot\varphi(v,v) + 2\overbrace{t_0\cdot\varphi(w,w)}^{=-\varphi(v,w)}\cdot\varphi(v,w) + \overbrace{t_0^2\cdot\varphi(w,w)\cdot\varphi(w,w)}^{=\varphi(v,w)^2}$$
$$= \varphi(w,w)\cdot\varphi(v,v) - \varphi(v,w)\cdot\varphi(v,w).$$
$$\uparrow$$
follows from $t_0\cdot\varphi(w,w) = -\varphi(v,w)$

Since φ is positive semidefinite we see that $\varphi(v + t_0 \cdot w, v + t_0 \cdot w) \cdot \varphi(w,w) \geq 0$. Thus follows from the above little calculation above that $\varphi(v,w)^2 \leq \varphi(v,v) \cdot \varphi(w,w)$.

Now suppose that $\varphi(v,w)^2 = \varphi(v,v) \cdot \varphi(w,w)$. This implies, by the above calculation, that $\varphi(v + t_0 \cdot w, v + t_0 \cdot w) \cdot \varphi(w,w) = 0$. Since φ is positive-definite this means that $v + t_0 \cdot w = 0$ or $w = 0$. In either case, v, w are linearly dependent. ■

Finally on some occasions we will need a variation of the Cauchy-Schwarz Inequality 0.10 where we work with pairings over $\mathbb{C}$. This leads us to the following definitions.

Definition. Let V be a complex vector space.

(1) A pairing $\varphi\colon V \times V \to \mathbb{C}$ is called $\mathbb{C}$-sesquilinear if for all $\lambda \in \mathbb{C}$ and all $v, w \in V$ we have $\varphi(\lambda \cdot v, w) = \lambda \cdot \varphi(v,w)$ and $\varphi(v, \lambda \cdot w) = \varphi(v,w) \cdot \overline{\lambda}$.

(2) A $\mathbb{C}$-sesquilinear pairing $\varphi\colon V \times V \to \mathbb{C}$ is called hermitian if $\varphi(v,w) = \overline{\varphi(w,v)}$ for all $v, w \in V$.

(3) We say a pairing φ is positive-definite if $\varphi(v,v) \in \mathbb{R}_{>0}$ for all $v \in V \setminus \{0\}$.

(4) We say a pairing φ is positive-semidefinite if $\varphi(v,v) \in \mathbb{R}_{\geq 0}$ for all $v \in V \setminus \{0\}$.

Example. Let $n \in \mathbb{N}_0$. The standard hermitian pairing on $\mathbb{C}^n$ is given by

$$\begin{array}{rccl} \langle -,- \rangle\colon \mathbb{C}^n \times \mathbb{C}^n & \to & \mathbb{R} \\ (v,w) & \mapsto & \langle v,w \rangle := \sum_{i=1}^{n} v_i \cdot \overline{w}_i. \end{array}$$

This standard pairing on $\mathbb{C}^n$ is evidently $\mathbb{C}$-sesquilinear, hermitian and positive-definite.

Notation. Given $z = x + y\mathrm{i} \in \mathbb{C}$ with $x, y \in \mathbb{R}$ we write $|z| := \sqrt{x^2 + y^2} = \sqrt{z \cdot \overline{z}}$.

Proposition 0.11. (Cauchy-Schwarz Inequality) Let V be a complex vector space and let

$$\begin{array}{rccl} \varphi\colon V \times V & \to & \mathbb{R}_{\geq 0} \\ (v,w) & \mapsto & \varphi(v,w) \end{array}$$

be a hermitian form.

(1) If φ is positive semidefinite, then for any $v, w \in V$ we have

$$|\varphi(v,w)|^2 \leq \varphi(v,v) \cdot \varphi(w,w).$$

(2) Now suppose that φ is positive-definite. In this case equality in (1) holds if and only if v and w are linearly dependent.

Proof. The proof is very similar to the proof of the above Cauchy-Schwarz Inequality 0.10. We leave it to the reader to make the necessary minor modifications. ■

0.6. **The Gram-Schmidt Orthogonalization Process.** The following proposition should be, perhaps in a slightly simplified version, familiar from a course on linear algebra.

Proposition 0.12. (Gram-Schmidt Orthogonalization Process) Let V be a real vector space and let $\langle\ ,\ \rangle\colon V\times V\to\mathbb{R}$ be a positive-definite symmetric $\mathbb{R}$-bilinear[16] form. Given $v\in V$ we write $\|v\|:=\sqrt{\langle v,v\rangle}$. Let $v_1,\dots,v_n$ be a basis of V. We consider the following sequence of n-tuples of vectors in V:[17]

$$
\begin{aligned}
&u_{0,1},..,u_{0,n}:=v_1,..,v_n\\
&u_{1,1},..,u_{1,n}:=\frac{v_1}{\|v_1\|},\|v_1\|\cdot v_2,v_2,..,v_n\\
&\qquad\vdots\qquad\qquad\vdots\\
&u_{i,1},..,u_{i,n}:=u_{i-1,1},..,u_{i-1,i-1},\frac{z}{\|z\|},\|z\|\cdot v_{i+1},v_{i+2},..,v_n \text{ with } z:=v_i-\sum_{j=1}^{i-1}\langle v_i,u_{i-1,j}\rangle\cdot u_{i-1,j}\\
&\qquad\vdots\qquad\qquad\vdots\\
&u_{n,1},..,u_{n,n}:=u_{n-1,1},..,u_{n-1,n-1},\frac{z}{\|z\|} \qquad\qquad \text{with } z:=v_n-\sum_{j=1}^{n-1}\langle v_n,u_{n-1,j}\rangle\cdot u_{n-1,j}.
\end{aligned}
$$

This sequence has the following properties:

(1) For each $i\in\{1,\dots,n\}$ the vectors $\{u_{i,1},\dots,u_{i,n}\}$ form a basis of V.
(2) For each $i\in\{1,\dots,n-1\}$ the base change matrix from the basis $v_1,\dots,v_n$ to the basis $u_{i,1},\dots,u_{i,n}$ has determinant 1.
(3) The base change matrix from the basis $v_1,\dots,v_n$ to the basis $u_{n,1},\dots,u_{n,n}$ has positive determinant.
(4) For each $i\in\{1,\dots,n\}$ the vectors $u_{i,1},\dots,u_{i,i}$ are orthonormal.
(5) If $v_1,\dots,v_n$ are orthonormal, then the sequence of n-tuples of vectors is constant, i.e. for any $i,j\in\{1,\dots,n\}$ we have $u_{i,j}=v_j$.

The analogous statement also holds if we are dealing with a positive-definite hermitian $\mathbb{C}$-sesquilinear form over a complex vector space.

Proof. The lemma follows from elementary arguments which we leave to the readers who suffer from linear algebra withdrawal syndrome. ■

0.7. **Group actions.** The following definition might also have appeared in an earlier math course.

Definition. Let X be a set and let G be a group with trivial element e.

(1) An action of G on X is a map

$$
\begin{array}{rcl}
G\times X & \to & X\\
(g,x) & \mapsto & g\cdot x
\end{array}
$$

[16] Recall that according, to the definition that we gave on page 29, we say that a symmetric $\mathbb{R}$-bilinear form $\langle -,-\rangle\colon V\times V\to\mathbb{R}$ is positive-definite if for every $v\neq 0$ we have $\langle v,v\rangle>0$.

[17] Our formulation is more complicated than the standard formulation of the Gram-Schmidt Orthogonalization Process. The reason is that we want to have control over the base change matrices. This is important when we show, in the proof of the Matrix Group-π_0-Proposition 2.23 (2), that $\mathrm{SL}(n,\mathbb{R})$ is path-connected.

with the following properties:

$$\begin{array}{rcll} e \cdot x & = & x, & \text{for all } x \in X, \\ g \cdot (h \cdot x) & = & (g \cdot_G h) \cdot x, & \text{for all } x \in X \text{ and } g, h \in G. \end{array}$$

group action; multiplication in the group G

(2) The action is called free, if $g \cdot x = x$ for some $x \in X$ implies that $g = e$.
(3) We say G acts transitively, if for every x and y in X there exists a $g \in G$ with $g \cdot x = y$.

Remark. Sometimes what we call an "action" is more precisely referred to as a "left-action". A "right-action" is exactly the same as a left-action except that the second property gets replaced by $g \cdot (h \cdot x) = (h \cdot_G g) \cdot x$. Or, more intuitively, a right-action is a map $X \times G \to X$ such that we have $(x \cdot h) \cdot g = x \cdot (h \cdot_G g)$. Basically all the statements and definitions that we give for left-actions carry over, with minuscule modifications, to the setting of right-actions.

Examples.

(1) Given any group G the map

$$\begin{array}{rcl} G \times G & \to & G \\ (g, h) & \mapsto & ghg^{-1} \end{array}$$

defines an action of G on itself. Let G be a non-trivial group. Considering $h = e$ we see that this action is not free and it is also straightforward to see that the action is not transitive.

(2) Let $\mathbb{F}$ be a field and let V be a k-dimensional vector space over $\mathbb{F}$. We denote by B the set of all ordered bases of V. The map

$$\begin{array}{rcl} \mathrm{GL}(k, \mathbb{F}) \times B & \to & B \\ \big((a_{ij})_{i,j=1,\dots,k}, (v_1, \dots, v_k)\big) & \mapsto & \Big(\sum_{i=1}^{k} a_{i1} v_i, \dots, \sum_{i=1}^{k} a_{ik} v_i\Big) \end{array}$$

is a transitive action. It is an amusing linear algebra exercise to figure out under what circumstances this action is free.

(3) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is evidently free but it is not transitive.

We conclude this section on group actions with the following elementary lemma.

Lemma 0.13. (Group Action-Equivalence Relation Lemma) Let X be a set and let G be a group that acts on X. Then

$$x \sim y \quad :\Longleftrightarrow \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y$$

is an equivalence relation on X.

Proof. The statement follows easily from the definitions. ∎

Definition. Let X be a set and let G be a group that acts on X. By the Group Action-Equivalence Relation Lemma 0.13 we know that

$$x \sim_G y \quad :\Longleftrightarrow \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y$$

is an equivalence relation on X. We denote by[18]

$$X/G \;:=\; X/\sim_G .$$

the set of equivalence classes.

0.8. **Normed vector spaces.** We continue with another definition from linear algebra.

Definition. Let V be a real vector space. A norm on V is a map $\|-\|\colon V \to \mathbb{R}_{\geq 0}$ that has the following three properties:

(1) For every $v \in V$ we have $\|v\| = 0 \iff v = 0$.
(2) For every $v \in V$ and $\lambda \in \mathbb{R}$ we have $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$.
(3) For every $v, w \in V$ we have $\|v + w\| \leq \|v\| + \|w\|$.

Lemma 0.14. Let V be a vector space. If $\langle -, - \rangle\colon V \times V \to \mathbb{R}_{\geq 0}$ is a symmetric positive definite $\mathbb{R}$-linear pairing, then the map

$$\begin{array}{rcl} V \times V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \|v\| := \sqrt{\langle v, v \rangle} \end{array}$$

is a norm.

Proof. It is straightforward to verify that the first two properties of a norm are satisfied. It remains to prove the third property. Given $v, w \in \mathbb{R}^n$ we see that

$$\begin{array}{rcl} \|v+w\|^2 & = & \langle v+w, v+w \rangle \overset{\text{follows from the } \mathbb{R}\text{-bilinearity of } \langle -,- \rangle}{\overset{\downarrow}{=}} \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \overset{\text{since } \langle w, v \rangle = \langle v, w \rangle}{\overset{\downarrow}{=}} \langle v, v \rangle + 2 \cdot \langle v, w \rangle + \langle w, w \rangle \\ & \underset{\uparrow}{\leq} & \langle v, v \rangle + 2 \cdot \sqrt{\langle v, v \rangle \cdot \langle w, w \rangle} + \langle w, w \rangle = \left(\sqrt{\langle v, v \rangle} + \sqrt{\langle w, w \rangle} \right)^2 = (\|v\| + \|w\|)^2. \end{array}$$

by the Cauchy-Schwarz Inequality 0.10 we have $\langle v, w \rangle^2 \leq \langle v, v \rangle \cdot \langle w, w \rangle$

By taking square roots on both sides we obtain that $\|v + w\| \leq \|v| + \|w\|$. ∎

Definition. The standard pairing on $\mathbb{R}^n$ gives rise to the Euclidean norm

$$\begin{array}{rcl} \|-\|_{\text{Eucl}}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ v = (v_1, \ldots, v_n) & \mapsto & \sqrt{\langle v, v \rangle} = \sqrt{v_1^2 + \cdots + v_n^2}. \end{array}$$

Usually, when there is no source of confusion, we just write $\|v\|$ instead of $\|v\|_{\text{Eucl}}$.

Example. We will now give more examples of norms $\mathbb{R}^n$. One can easily show that the following maps are norms and that for $n \geq 2$ they do *not* arise from a positive definite $\mathbb{R}$-bilinear pairing:

$$\underbrace{\begin{array}{rcl} \|-\|_{\max}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x_1, \ldots, x_n) & \mapsto & \max\{|x_1|, \ldots, |x_n|\} \end{array}}_{\text{maximum norm}} \quad \text{and} \quad \underbrace{\begin{array}{rcl} \|-\|_{\text{Manh}}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x_1, \ldots, x_n) & \mapsto & \sum_{i=1}^{n} |x_i|. \end{array}}_{\text{Manhattan norm}}$$

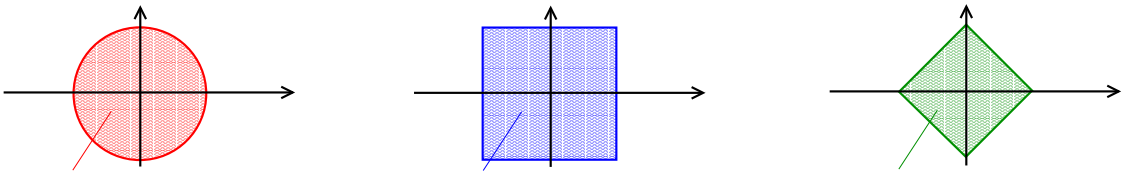
Since the human brain is equipped with a great image processor it is usually helpful to turn an abstract definition into an image. In the present context the key to doing so is given by the following definition.

[18] It is arguably more logical to write $G \backslash X$ since we are dealing with a left-action and to reserve the notation X/G for right-actions. We stick to X/G since it is closer to our language "X mod G" and since we will not deal with right-actions at all.

Definition. Given a norm $\|-\|\colon V \to \mathbb{R}_{\geq 0}$ we define

$$\text{norm ball of } \|-\| \;:=\; \{v \in V \,|\, \|v\| \leq 1\}.$$

Example. In the figure below we show the norm balls of the Euclidean norm, the maximum norm and the Manhattan norm.



norm ball of $\|-\|_{\text{Eucl}}$ norm ball of $\|-\|_{\max}$ norm ball of $\|-\|_{\text{Manh}}$

0.9. **Metric spaces.** Finally we recall the notion of a metric space which is most likely familiar from an earlier course on real analysis.

Definition. A metric space is a pair (X, d) consisting of a set X and a metric d on X, i.e. a map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \,|\, x \geq 0\} \\ (x, y) & \mapsto & d(x, y) \end{array}$$

with the following properties:

(1)		$d(x,y)$	$=$	$0 \iff x = y$	
(2)	for all $x, y \in X$ we have	$d(x,y)$	$=$	$d(y,x)$	(symmetry)
(3)	for all $x, y, z \in X$ we have	$d(x,z)$	$\leq$	$d(x,y) + d(y,z)$	(triangle inequality).

Sometimes we refer to $d(x, y)$ as the distance between x and y.

The following lemma shows that any norm on a vector space V give rise to a metric on V.

Lemma 0.15. Let V be a real vector space and let $\|-\|\colon V \to \mathbb{R}_{\geq 0}$ be a norm. Then

$$\begin{array}{rcl} V \times V & \to & \mathbb{R}_{\geq 0} \\ (v, w) & \mapsto & d(v, w) := \|v - w\| \end{array}$$

is a metric on V.

Proof. The three properties of a metric follow easily from the three properties of a norm. For example, given $v, w \in V$ we see that

$$d(v,w) \;=\; \|v-w\| \;=\; \|(-1)\cdot(w-v)\| \underset{\uparrow}{=} |-1|\cdot\|w-v\| \;=\; \|w-v\| \;=\; d(w,v).$$

property (2) of a norm

The other two properties of a metric are verified in a similar fashion. ∎

It follows from Lemma 0.15 that the Euclidean norm $\|-\|_{\text{Eucl}}$ gives rise to a metric, called the Euclidean metric, which is perhaps the most frequently used example of a metric:

Definition. Given $x, y \in \mathbb{R}^n$ we set

$$d(x,y) \;:=\; \|x-y\|_{\text{Eucl}} \quad \text{where} \quad \|v\|_{\text{Eucl}} = \|(v_1,\ldots,v_n)\|_{\text{Eucl}} := \sqrt{v_1^2 + \cdots + v_n^2}\,.$$

We refer to the map

$$\begin{array}{rcl} d\colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & d(x,y) \end{array}$$

as the Euclidean metric on $\mathbb{R}^n$.[19] Usually we consider $\mathbb{R}^n$ as a metric space with respect to this Euclidean metric, unless we say something else.

Of course there are many other examples of metrics on $\mathbb{R}^n$. In the following we will see two examples, the first one corresponds to the Manhattan norm and the other one is an example of a metric that does not come from a norm.

Example. Let $n \in \mathbb{R}^n$. We fix a point $P \in \mathbb{R}^n$. Besides the ever-popular Euclidean metric on $\mathbb{R}^n$ we can also consider the following two metrics on $\mathbb{R}^n$:

$$\begin{array}{rcl} d_{\text{Manh}}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & \sum_{i=1}^{n} |x_i - y_i| \end{array} \quad \& \quad \begin{array}{rcl} d_{\text{SNCF}}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & \begin{cases} \|x-y\|, & \text{if } x,y,P \text{ lie on a line,} \\ \|x-P\|+\|y-P\|, & \text{otherwise.} \end{cases} \end{array}$$

The metric d_{Manh} is called the Manhattan metric since it measures the distance between two points if one can go only vertically (avenues) and horizontally (streets). The metric d_{SNCF} is called the SNCF metric since it models the distance in France where every train between two points necessarily has to go through Paris.

Manhattan metric

SNCF-metric

Not surprisingly there are many other examples of metric spaces. The last definition shows that we can define a metric on any set.

Definition. Let X be a set. The map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y \end{cases} \end{array}$$

defines a metric on X, it is called the trivial metric.

[19] It follows from Lemma 0.14 together with Lemma 0.15 that this is indeed a metric on $\mathbb{R}^n$.

1. Topological spaces: basic definitions and properties

In this chapter we give the definition of topological spaces, we provide many examples and we discuss a long list of basic properties. Many of these results might well be familiar to the reader. Nonetheless we attempt to give a fairly complete list of basic statements so that later on we have a firm common foundation to work with.

1.1. Topological spaces.

In this section we give the definition of a topological space and we discuss some very basic definitions and statements. Presumably most readers will have seen this material. Throughout these lecture notes, in the proofs, we strive for giving precise references to earlier results. The main exceptions are the some of the basic lemmas from this and the following chapter, which we will use without referring to them explicitly. The following definition will keep us occupied for the rest of these notes.

Definition.

(1) Let X be a set. A topology on X is a subset $\mathcal{T}$ of the power set $\mathcal{P}(X)$[20] with the following properties:
   (a) the empty set and the entire set X lie in $\mathcal{T}$,
   (b) the intersection of *finitely many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$,
   (c) the union of *arbitrarily many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$.
   The sets in $\mathcal{T}$ are called open with respect to $\mathcal{T}$. If $\mathcal{T}$ is understood from the context, then we just say open.
(2) A topological space is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a topology on X.

Remark.

(1) Usually we suppress the topology from the notation, e.g. usually we just write "let X be a topological space".
(2) If X is a topological space, then the properties of a topology imply immediately the following three statements:
   (a) the empty set and the entire set X are open,
   (b) the intersection of *finitely many* open subsets is again open,
   (c) the union of *arbitrarily many* open subsets is again open.

Examples.

(1) Let X be a set, then $\mathcal{T} = \{\varnothing, X\}$ is a topology on X. This topology is sometimes called the trivial topology on X.
(2) Let X be a set. Then $\mathcal{T} = \mathcal{P}(X)$ is also a topology on X. In other words, $\mathcal{T}$ is the topology such that every subset of X is open. This topology is usually referred to as the discrete topology on X. Sometimes we say that a topological space is discrete, if the topology is indeed the discrete topology.
(3) Let X be a set. We define $\mathcal{T} \subset \mathcal{P}(X)$ as follows:

$$U \in \mathcal{T} \iff \text{either } U = \varnothing \text{ or } U \text{ is the complement of a finite subset of } X.$$

[20] In more humble words, $\mathcal{T}$ is a set of subsets of X.

It follows easily from de Morgan's Laws 0.2, that $\mathcal{T}$ is a topology on X. We refer to it as the cofinite topology. Sometimes this topology is also called the finite-complement topology. In a very similar fashion one can define the cocountable topology.[21]

(4) Let X be the empty set. Together with $\mathcal{T} = \{\varnothing\}$ this is a topological space, called the empty topological space. It is a never ending source of counterexamples to carelessly formulated statements.

(5) (a) Let $X = \{*\}$ be a set with a single element $*$. It follows immediately from the definition of a topology there is a unique topology on $X = \{*\}$, namely the topology given by $\mathcal{T} = \{\varnothing, \{*\}\}$. Together with the empty set this is the only case where the topology is already determined by the set. We refer to any topological space consisting of a single point as a singleton.

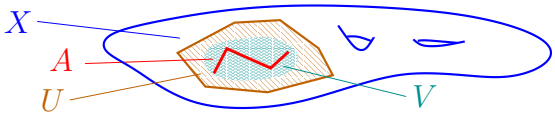
(b) Given a set $X = \{a, b\}$ with two elements it is straightforward to see that there exist four distinct topologies on X. More precisely, there is the trivial and the discrete topology, and there are the two topologies where precisely one of the subsets $\{a\}$ and $\{b\}$ is open.

Not surprisingly the number of topologies on a finite set grows rapidly with the cardinality of the set, we refer to [**KR1970**; BM02] for details.

Definition. Let X be a topological space.

(1) Let $A \subset X$ be a subset. We say $U \subset X$ is a neighborhood of A if there exists an open subset V such that $A \subset V \subset U$. We say U is an open neighborhood of A, if U is furthermore open. This definition is illustrated in the figure below.

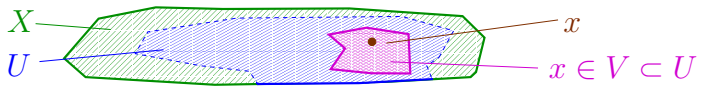
(2) An (open) neighborhood of a point $x \in X$ is an (open) neighborhood of $\{x\}$.



Example. If $X = \mathbb{R}$ and $A = [0, 2)$, then $U = (-1, 3]$ and $V = (-2, \infty)$ are neighborhoods of A in X.

Using the previous definition we can now state the following lemma which gives a useful criterion for showing that a given subset is open.

Lemma 1.1. (JH-Openness Criterion[22]) Let X be a topological space and let $U \subset X$ be a subset. If for every $x \in U$ there exists a neighborhood V of x that is contained in U, then U is open.



Proof. Let X be a topological space and let $U \subset X$ be a subset. Suppose that for every $x \in U$ there exists a neighborhood V_x of x that is contained in U. By definition of a

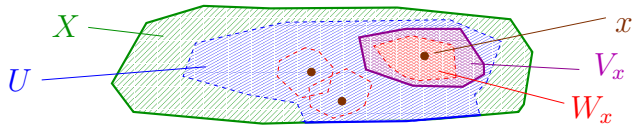
[21] To verify that the axioms are satisfied one will need the Countability Lemma 0.6.

[22] A student, let's call him JH, pointed out to me in an earlier version of these notes, that I was implicitly frequently using the statement of this lemma.

neighborhood there exists for each $x \in U$ an open subset W_x with $x \in W_x \subset V_x$. We see that

$$U = \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} W_x \underset{\text{since each } W_x \text{ is contained in } U}{\subset} U.$$

It follows that $U = \bigcup_{x \in U} W_x$, i.e. U is the union of open subsets of X, thus by the third property of a topology, U itself is open. ■



Definition. Let X be a topological space. We say a subset A of X is closed if the complement $X \setminus A$ is open.

Example. By an *interval* we mean any subset of $\mathbb{R}$ of the form

$$\begin{array}{rcll} (a,b) & := & \{x \in \mathbb{R} \,|\, a < x < b\} & \text{where } a, b \in \mathbb{R} \cup \{\pm\infty\} \\ (a,b] & := & \{x \in \mathbb{R} \,|\, a < x \leq b\} & \text{where } a \in \mathbb{R} \cup \{-\infty\} \text{ and } b \in \mathbb{R} \text{ with } a < b \\ [a,b) & := & \{x \in \mathbb{R} \,|\, a \leq x < b\} & \text{where } a \in \mathbb{R} \text{ and } b \in \mathbb{R} \cup \{\infty\} \text{ with } a < b \\ [a,b] & := & \{x \in \mathbb{R} \,|\, a \leq x \leq b\} & \text{where } a, b \in \mathbb{R} \text{ with } a \leq b. \end{array}$$

One can easily verify that the intervals of the form (a,b), $(-\infty,b)$ and (a,∞) with $a,b \in \mathbb{R}$ are open subsets of $\mathbb{R}$ and that the intervals of the form $[a,b]$, $(-\infty,b]$ and $[a,\infty)$ with $a,b \in \mathbb{R}$ are closed subsets of $\mathbb{R}$. Note that $\varnothing$ and $\mathbb{R}$ are also intervals, they are both open and closed in $\mathbb{R}$.

The following lemma mirrors the remark on page 36.

Lemma 1.2. (Closed Subset-Lemma) Let X be a topological space.
(1) The empty set and the entire set X are closed.
(2) The intersection of *arbitrarily many* closed subsets is again closed.
(3) The union of *finitely many* closed subsets is again closed.

Proof. The lemma is an immediate consequence of the definition of a topology and de Morgan's Laws 0.2. ■

Remark. Let X be a set. If we specify what we mean by closed subsets such that the three properties of the Closed Subset-Lemma 1.2 are satisfied, then it follows once again from de Morgan's Laws 0.2 that the complements of the closed subsets define a topology.

1.2. **The metric topology.** Recall that on page 34 we introduced the notion of a metric space. We will now see that metric spaces give rise to topological spaces. To do so we will need the following notation.

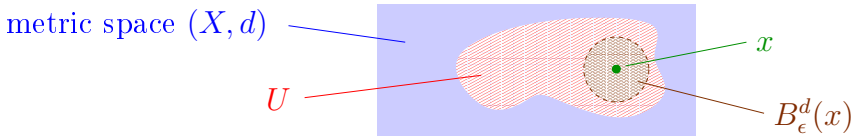
Notation. Let (X,d) be a metric space. Given $x \in X$ and $r \in \mathbb{R}$ we write

$$B_r^d(x) := \{y \in X \,|\, d(x,y) < r\}$$

If the metric d is understood from the context, then sometimes we drop it from the notation.

The following lemma shows that metric spaces give rise to topological spaces.

Lemma 1.3. (Metric-Topology Lemma) Let (X, d) be a metric space. A subset U of X is called open if for every $x \in U$ there exists an $\epsilon > 0$ such that $B^d_\epsilon(x)$ is contained in U. These open subsets define a topology on X.



Proof. We proved this lemma in Analysis IV, alternatively see [Fri23]. ■

Definition. Let (X, d) be a metric space.

(1) We refer to the topology introduced in the Metric-Topology Lemma 1.3 as the metric topology.

(2) Unless we say something else we always equip (X, d) with the metric topology.

Example. Let X be a set and let $d\colon X \times X \to \mathbb{R}_{\geq 0}$ be the trivial metric that we introduced on page 35. It is straightforward to see that the corresponding metric topology is the discrete topology.

Convention. Let $n \in \mathbb{N}$.

(1) Unless we say something else we consider $\mathbb{R}^n$ as a metric space with the Euclidean metric given by

$$d(x, y) := \|x - y\| \quad \text{where} \quad \|z\| = \|(z_1, \ldots, z_n)\| := \sqrt{z_1^2 + \cdots + z_n^2}.$$

Furthermore, unless we say something else we equip $\mathbb{C}^n$ with the metric given by[23]

$$d(x, y) := \|x - y\| \quad \text{where} \quad \|z\| = \|(z_1, \ldots, z_n)\| := \sqrt{z_1 \cdot \overline{z_1} + \cdots + z_n \cdot \overline{z_n}}.$$

(2) In particular, unless we say explicitly otherwise, we equip $\mathbb{R}^n$ with the topology arising from the Euclidean metric and we $\mathbb{C}^n$ with the topology arising from the above metric. We call these the Euclidean topologies on $\mathbb{R}^n$ and $\mathbb{C}^n$.

Example. Let $n \in \mathbb{R}^n$ and let $P \in \mathbb{R}^n$. On page 35 we introduced the following metrics:

$$\begin{aligned} d_{\text{Manh}}\colon \mathbb{R}^n &\to \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \sum_{i=1}^{n} |x_i - y_i| \end{aligned} \quad \& \quad \begin{aligned} d_{\text{SNCF}}\colon \mathbb{R}^n &\to \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \begin{cases} \|x - y\|, & \text{if } x, y, P \text{ lie on a line,} \\ \|x - P\| + \|y - P\|, & \text{otherwise.} \end{cases} \end{aligned}$$

One can easily show that the Manhattan metric is equivalent to the Euclidean metric and we that the SNCF-metric is not equivalent to the Euclidean metric.

1.3. Basic definitions.

We proceed with the next definition.

The following proposition gives us important examples of closed and open subsets.

Proposition 1.4. Let (X, d) be a metric space, let $y \in X$ and let $r \in \mathbb{R}$.

(1) the set $B^d_r(y) := \{x \in X \mid d(x, y) < r\}$ is open in X,

[23] It follows from the observation that $(x + \mathrm{i}y) \cdot \overline{x + \mathrm{i}y} = (x + \mathrm{i}y) \cdot (x - \mathrm{i}y) = x^2 + y^2$ that the metric on $\mathbb{C}^n$ is the Euclidean metric on $\mathbb{C}^n = \mathbb{R}^{2n}$ in slight disguise. In particular it is indeed a metric.

(2) the set $\overline{B}_r^d(y) := \{x \in X \mid d(x,y) \leq r\}$ is closed in X,
(3) the set $S_r^d(y) := \{x \in X \mid d(x,y) = r\}$ is closed in X.

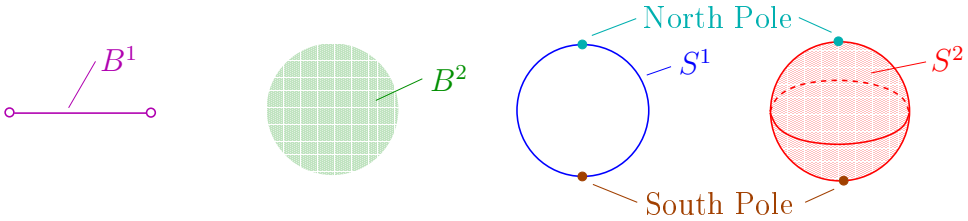
Proof. We proved this proposition in Analysis II. ■

The most frequent application of Proposition 1.4 is to the metric space $\mathbb{R}^n$. Thus we introduce the following notation.

Notation. Let $n \in \mathbb{N}_0$. Given $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$ we write[24] [25]

$$\begin{array}{rcll} B_r^n(y) & := & \{x \in \mathbb{R}^n \mid \|x-y\| < r\} & \text{called open } n\text{-ball} \\ \overline{B}_r^n(y) & := & \{x \in \mathbb{R}^n \mid \|x-y\| \leq r\} & \text{called closed } n\text{-ball} \\ S_r^{n-1}(y) & := & \{x \in \mathbb{R}^n \mid \|x-y\| = r\} & \text{called } (n-1)\text{-sphere.} \end{array}$$

We adopt the following conventions:
(1) A 2-ball is often called a disk and a 1-sphere is often referred to as a circle.
(2) If $y = 0$, then we drop y from the notation and if $r = 1$, then we drop r from the notation.
(3) Sometimes we write rB^n instead of $B_r^n(0)$ and sometimes we write $r\overline{B}^n$ instead of $\overline{B}_r^n(0)$.
(4) We refer to the point $(0,\ldots,0,1) \in S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ as the North Pole of S^n and we refer to the point $(0,\ldots,0,-1) \in S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ as the South Pole of S^n.

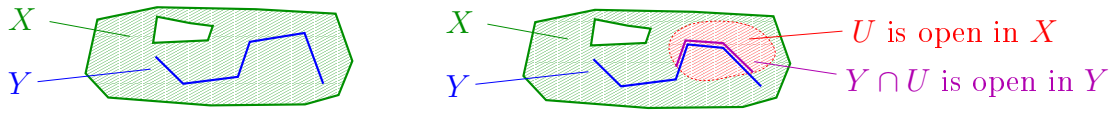


1.4. **The subspace topology.** The next definition gives us a rich source of new examples of topological spaces.

Definition. Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$ be a subset.
(1) It follows from elementary set theory, see Lemma 0.1, that

$$\mathcal{S} := \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y. We refer to $\mathcal{S}$ as the subspace topology on Y induced by Y.
(2) If Y is indeed equipped with the subspace topology, then sometimes we refer to Y as a subspace of X.



[25] These definitions also make sense for $r < 0$, in that case all of the sets are the empty set. Also note that for $n = 0$ we obtain that $S_r^{n-1}(y) = S_r^{-1}(y) = \varnothing$.

Convention. Unless we say something else, we consider each subset Y of $\mathbb{R}^n$ always as a topological space with respect to the subspace topology.

1.5. **Covers of topological spaces.** On many occasions it will be useful to "break" a topological space into smaller pieces. This leads us to the following definition.

Definition. Let X be a topological space.

(1) A cover[26] of a subset $A \subset X$ is a family[27] $\{U_i\}_{i\in I}$ of subsets of X with $A \subset \bigcup_{i\in I} U_i$.
(2) We say a cover $\{U_i\}_{i\in I}$ is open if each U_i is open. Similarly we define a closed cover.
(3) We say a cover $\{U_i\}_{i\in I}$ is finite if I is finite. Otherwise we say that the cover is infinite.

The following lemma is crucial for many arguments.

Lemma 1.5. (Cover-Open Subset Lemma) Let X be a topological space and let M be a subset of X.

(1) Let $\{U_i\}_{i\in I}$ be an *arbitrary* open cover of X. The following two statements hold:
  (a) M is open if and only if each intersection $M \cap U_i$ is open in U_i,
  (b) M is closed if and only if each intersection $M \cap U_i$ is closed in U_i.
(2) Let $\{A_i\}_{i\in I}$ be a *finite* closed cover of X. The following two statements hold:
  (a) M is open if and only if each intersection $M \cap A_i$ is open in A_i,
  (b) M is closed if and only if each intersection $M \cap A_i$ is closed in A_i.

Remark. Note that in the Cover-Open Subset Lemma 1.5 we allow an arbitrary family of open subsets but we only allow a finite family of closed subsets. One can easily see that this restriction is necessary. For example, consider $X = \mathbb{R}$ and the infinite family $\{A_i\}_{i\in I}$ of closed subsets of $X = \mathbb{R}$ given by all subsets consisting of a single point. Let $M \subset \mathbb{R}$ be a subset. For each $i \in I$ the intersection $M \cap A_i$ is either the empty set or all of A_i, in particular the intersection $M \cap A_i$ is closed in A_i. But of course there is no reason why M should be a closed subset of $\mathbb{R}$.

Proof. The proof is rather elementary. The curious reader is referred to [Fri23] for details. ■

1.6. **The interior and the closure of a subset.**

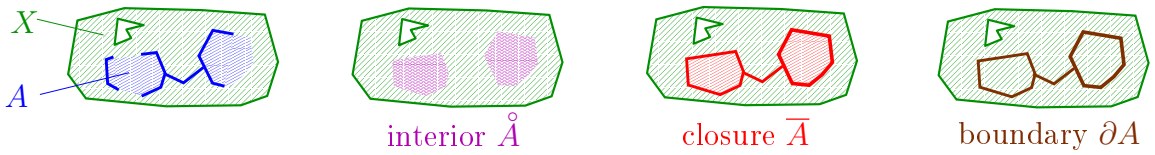
Definition. Let X be a topological space and let A be a subset of X. We make the following two definitions:[28]

$$\text{interior } \mathring{A} := \begin{array}{c}\text{union of all open subsets of } X \\ \text{that are contained in } A\end{array} \qquad \text{closure } \overline{A} := \begin{array}{c}\text{intersection of all closed} \\ \text{subsets of } X \text{ that contain } A\end{array}$$

Sometimes, to avoid confusion, we will also write $\operatorname{int}(A)$ for the interior of a subset A. Furthermore we define the boundary of A in X as $\partial A := \overline{A} \setminus \mathring{A}$.

[27] What we call a "cover" is often also called a "covering". We reserve the name "covering" for a completely different concept that we will introduce later.

[27] Recall that according to the definition on page 18 a family of subsets of a given set X consists of a set I and a map $f\colon I \to \mathcal{P}(X)$. We also recall the convention from page 18 which says that we will view any subset $\mathcal{U} \subset \mathcal{P}(X)$ as a family of subsets of X via the index set $I = \mathcal{U}$ and the map $I = \mathcal{U} \to \mathcal{P}(X)$ given by $U \mapsto U$.



In the following we will state general properties of the interior, the closure and the boundary of a subset. Evidently the point is not to memorize the following lemmas. Instead they are meant as a resource to turn to, when one is dealing with these objects. Therefore it is warmly recommended to move on to the next section and to return to the present section when the need arises.

Lemma 1.6. (Interior-Closure Lemma) Let X be a topological space and let A be a subset of X.

(1) The interior $\mathring{A}$ is an open subset of X.
(2) The closure $\overline{A}$ of A is a closed subset of X.

Proof.

(1) The union of arbitrarily many open subsets is, by definition of a topology, again an open subset. Thus the interior $\mathring{A}$ is an open subset of X.
(2) In the Closed Subset-Lemma 1.2 we saw that the intersection of arbitrarily many closed subsets is a closed subset. It follows immediately that the closure $\overline{A}$ is a closed subset of X. ■

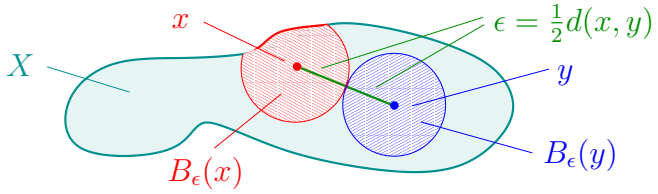
1.7. The Hausdorff property.

Definition. We say a topological space X is Hausdorff, if given any two distinct points $x \neq y$ there exist disjoint open neighborhoods U of x and V of y.

The following proposition, which very likely is well-known to most readers, says that topological spaces arising from metric spaces are Hausdorff.

Proposition 1.7. (Metric Space-Hausdorff Proposition) Let (X, d) be a metric space. The corresponding topological space is Hausdorff.

Proof. Let $x, y \in X$ be two different points. We set $\epsilon := \frac{1}{2}d(x, y)$. It follows from Proposition 1.4 that $U := B_\epsilon(x)$ and $V := B_\epsilon(y)$ are neighborhoods of x respectively y. An elementary argument using the triangle inequality shows that U and V are disjoint. ■

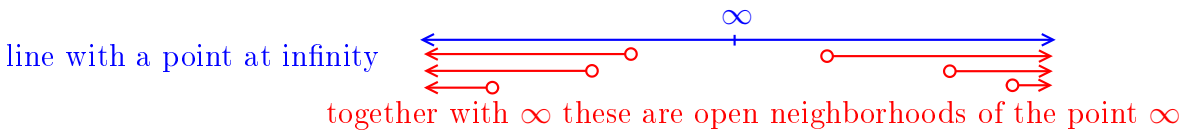


To get some practice for the notion of the Hausdorff property it is convenient to introduce two more examples of topological spaces, which do not come from a metric space. Both new examples are of the type "$\mathbb{R}^n$ together with an extra point".

Definition. Let $n \in \mathbb{N}$. We consider the set $X := \mathbb{R}^n \cup \{\infty\}$, i.e. X consists of $\mathbb{R}^n$ and an extra point ∞. We say $U \subset X$ is open, if both of the following two conditions are satisfied:

(1) for each point $x \in U \cap \mathbb{R}^n$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset U$,
(2) if $\infty \in U$, then there exists a $C > 0$ such that $\{x \in \mathbb{R}^n \mid \|x\| > C\} \subset U$.

It is straightforward to see that this defines indeed a topology on X. We refer to $\mathbb{R}^n \cup \{\infty\}$ as $\mathbb{R}^n$ with a point at infinity. For $n = 1$ we refer to $\mathbb{R} \cup \{\infty\}$ also as the line with a point at infinity.[29]

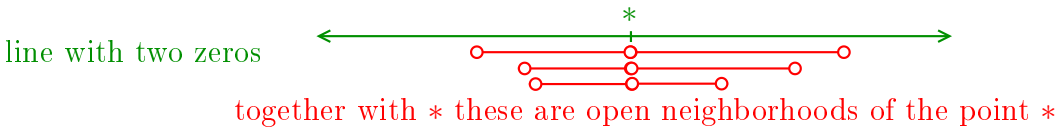


The above example has a mischievous twin brother, which is often a counterexample to carelessly formulated statements.

Definition. We consider the set $X := \mathbb{R} \cup \{*\}$, i.e. X consists of $\mathbb{R}$ and an extra point $*$. We say $U \subset X$ is open, if the following two conditions are satisfied:

(a) for each point $x \in U \cap \mathbb{R}$ there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$,
(b) if $* \in U$, then there exists an $\epsilon > 0$ such that $(-\epsilon, 0) \cup (0, \epsilon) \subset U$.

One can easily verify that this is indeed a topology on X. We refer to this topological space as the line with two zeros.[30]



Example. Even though the "line with two zeros" looks at first glance similar to the "line with a point at infinity", they are strikingly different topological spaces. For example it is a fun exercise to show that "$\mathbb{R}^n$ with a point at infinity" is Hausdorff and that "the line with two zeros" is not Hausdorff.

The following lemma gives us a modest but still quite useful consequence of being Hausdorff.

Lemma 1.8. (Points-in-Hausdorff Lemma) Let X be a topological space. If X is Hausdorff, then for every $x \in X$ the corresponding subset $\{x\}$ is closed.

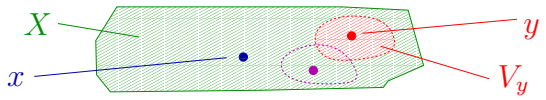
Proof. Let X be a topological space which is Hausdorff. Let $x \in X$. Since X is Hausdorff we know in particular that for every $y \neq x$ there exists an open neighborhoods V_y of y that

[29] It is straightforward to see that the subspace topology on $\mathbb{R}^n \subset \mathbb{R}^n \cup \{\infty\}$ is the usual topology on $\mathbb{R}^n$.
[30] It is straightforward to see that the subspace topology on $\mathbb{R}^n \subset \mathbb{R}^n \cup \{*\}$ is the usual topology on $\mathbb{R}^n$.

is disjoint from x. Note that

$$X \setminus \{x\} \;=\; \bigcup_{y \in X\setminus\{x\}} \{y\} \;\underset{\substack{\uparrow \\ \text{since } y \in V_y}}{\subset}\; \bigcup_{y \in X\setminus\{x\}} V_y \;\underset{\substack{\uparrow \\ \text{since } x \notin V_y}}{\subset}\; X \setminus \{x\}.$$

Thus we see that the inclusions are equalities. Since each V_y is open we see that $X \setminus \{x\}$ is the union of open subsets, in particular $X \setminus \{x\}$ is open. In other words, $\{x\}$ is a closed subset of X. ■



1.8. **Compact topological spaces.** In this section we introduce the notion of compactness. Usually it takes a while till one appreciates the brilliance of the definition.

Definition. We say a topological space X is compact if for each open cover $\{U_i\}_{i\in I}$ of X there exist finitely many indices $i_1, \ldots, i_k \in I$ such that

$$X \;=\; U_{i_1} \cup \cdots \cup U_{i_k}.$$

Remark. The introduction of the notion of compact space by Pavel Alexandrov and Pavel Urysohn [AU29] in 1929 shows that often in mathematics the really clever thing is not necessarily the proofs, but coming up with just the right definition. It is difficult to exaggerate the role of the notion of compactness in topology and analysis.

Example.

(1) Every finite topological space (regardless of the topology) is compact, since there exist only finitely many distinct open subsets. This simple example can sometimes be quite useful.
(2) We consider the topological space $X = \mathbb{R}$. The sets $\{(-n, n)\}_{n\in\mathbb{N}}$ form an open cover of $X = \mathbb{R}$, but evidently we cannot cover $X = \mathbb{R}$ by finitely many of these sets. This shows that $X = \mathbb{R}$ is not compact.

In the following we will often use the following two naming conventions.

Convention. Let P be a property of a topological space (e.g. being discrete, Hausdorff, compact).

(1) If X is a topological space that satisfies P we might just say that "X is a P-space". For example, a Hausdorff space is a topological space that is Hausdorff.
(2) We say that a subset Y of a topological space X has the property P if Y, equipped with the subspace topology, has the property P.

Before we give some more interesting examples of compact spaces we want to discuss a few properties of compact spaces and compact subsets. The following lemma might be the lemma that gets most often cited in these notes.

Lemma 1.9. (Compact-Closed Lemma) Let X be a topological space.

(1) If X is compact, then any closed subset of X is also compact.
(2) Let $A \subset X$ be a compact subset. If X is Hausdorff, then A is a closed subset of X.

Example.

(1) Let X be a topological space. As we pointed out on page 44, any finite subset of X is compact. Thus if X is Hausdorff, then it follows from the Compact-Closed Lemma 1.9 (2) that any finite subset of X, in particular any point in X, is a closed subset of X. Thus we have found a new proof for the Points-in-Hausdorff Lemma 1.8 (1).

(2) Let X be a topological space, let A be a closed subset and let K be a compact subset of X. It follows from the Compact-Closed Lemma 1.9 (1) that $A \cap K$ is also compact. This simple observation will get used frequently.

(3) We consider the topological space $X = \{0, 1\}$ that is equipped with the trivial topology $\{\varnothing, X\}$. The subset $A = \{0\}$ is evidently compact, but it is not a closed subset of X. This shows that in the formulation of Compact-Closed Lemma 1.9 (2) we cannot drop the hypothesis that X is Hausdorff.

Proof. We proved this lemma in Analysis II and Analysis IV. Alternatively see [Fri23]. ■

1.9. **The Heine-Borel Theorem.** We start out this section with the following almost self-explanatory definition.

Definition. Let (X, d) be a metric space. A subset A of X is called bounded if there exists a $C \geq 0$ such that for all $x_0, x_1 \in A$ we have $d(x_0, x_1) \leq C$.

Example. Let (X, d) be a metric space, let $y \in X$ and let $r \geq 0$. It follows easily from the triangle inequality that the balls $B_r^d(y)$, $\overline{B}_r^d(y)$ and the sphere $S_r^d(y)$ are bounded with $C = 2r$.

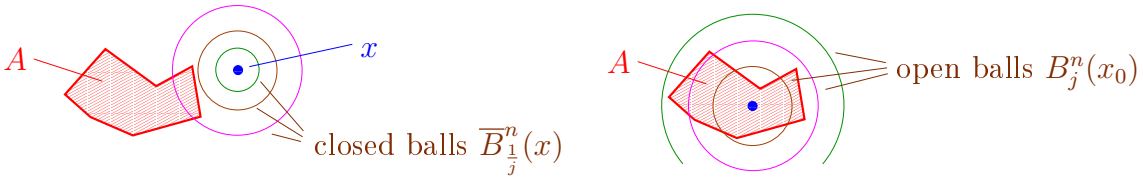
Lemma 1.10. Let (X, d) be a metric space and let $A \subset X$ be a subset. If A is compact, then A is closed and it is bounded.

Proof. Let (X, d) be a metric space and let $A \subset X$ be a subset.

(1) We want to show that $X \setminus A$ is open. Let $x \in X \setminus A$. We need to show that there exists an $\epsilon > 0$ with $B_\epsilon^d(x) \subset X \setminus A$. Given $j \in \mathbb{N}$ we consider the subset $U_j = X \setminus \overline{B}_{\frac{1}{j}}^d(x)$. By Proposition 1.4 we know that the U_j are open. Evidently these open subsets cover A. Since A is compact there exist $j_1, \ldots, j_k \in \mathbb{N}$ with $A \subset U_{j_1} \cup \cdots \cup U_{j_k}$. We set $\epsilon := \min\{\frac{1}{j_1}, \ldots, \frac{1}{j_k}\}$. We obtain that $A \subset X \setminus \overline{B}_\epsilon^d(x)$. Thus we have found the desired ϵ.

(2) We need to show that A is bounded. If A is empty, then there is nothing to show. Now suppose that A is non-empty. We pick $x_0 \in A$. It follows from the triangle inequality that it suffices to show that there exists an $R \geq 0$ with $A \subset \overline{B}_R^d(x_0)$. We consider the open subsets $U_j = B_j^d(x_0)$ with $j \in \mathbb{N}$. Evidently these sets cover A. Since A is compact we obtain that there exist $j_1, \ldots, j_k \in \mathbb{N}$ such that $A \subset U_{j_1} \cup \cdots \cup U_{j_k}$. We set $R := \max\{j_1, \ldots, j_k\}$. We obtain that $A \subset B_R^d(x_0) \subset \overline{B}_R^d(x_0)$. We have thus found the desired R. ■

The ever-popular Heine-Borel Theorem says that for the metric space $X = \mathbb{R}^n$ mercifully the converse to Lemma 1.10 holds.



Theorem 1.11. (Heine-Borel Theorem) A subset of $\mathbb{R}^n$ is compact if and only if it is bounded and closed.

Proof. We proved this theorem in Analysis II. Alternatively see [Fri23]. ■

Example.

(1) Let $y \in \mathbb{R}^n$ and $r \geq 0$. It follows from the Heine-Borel Theorem 1.11 and Proposition 1.4 that the closed ball $\overline{B}^n_r(y)$ and the sphere $S^{n-1}_r(y)$ are compact.

(2) It is an amusing exercise to show, using the Heine-Borel Theorem 1.11, that "$\mathbb{R}^n$ with a point at infinity" is compact.

(3) It is an equally entertaining exercise to show that the "line with two zeros" is not compact.

(4) Since the concepts of being bounded and closed are initially much more intuitive than being compact it is not uncommon that many, especially students, like to replace "compact" by "bounded and closed". But this equality only holds for subsets of $\mathbb{R}^n$, it does not hold for other metric spaces. For example, the set $[0,1] \cap \mathbb{Q}$ is a bounded and closed subset of the metric space $(\mathbb{Q}, d(x,y) = |x-y|)$ it is not compact.

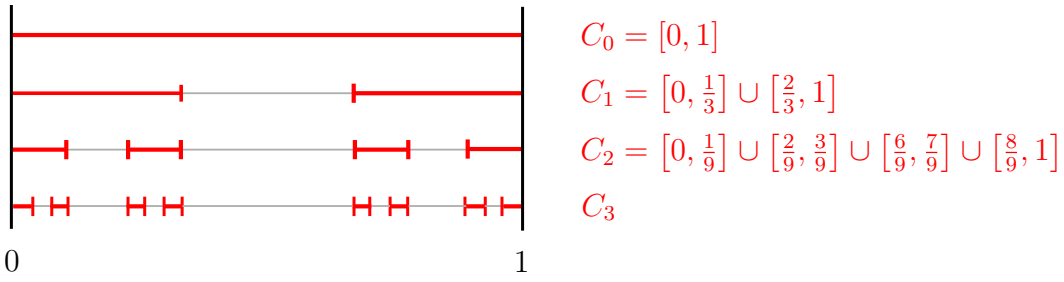
Our final example deserves to be introduced as a proper definition.

Definition. We set $C_0 := [0,1]$. Iteratively we now define sets $C_2, C_3, \ldots$ as follows:

$$C_i := \underbrace{\{\tfrac{1}{3} \cdot x \,|\, x \in C_{i-1}\}}_{=\frac{1}{3}\cdot C_{i-1}} \cup \underbrace{\{\tfrac{2}{3} + \tfrac{1}{3} \cdot x \,|\, x \in C_{i-1}\}}_{=\frac{2}{3}+\frac{1}{3}\cdot C_{i-1}}.$$

Note that each C_i consists of 2^i closed intervals of length $\frac{1}{3^i}$ and that we obtain C_i from C_{i-1} by removing the "open middle third" of each interval. We define

$$\text{Cantor set } C := \bigcap_{i \in \mathbb{N}_0} C_i.$$



Lemma 1.12. The Cantor set is compact.

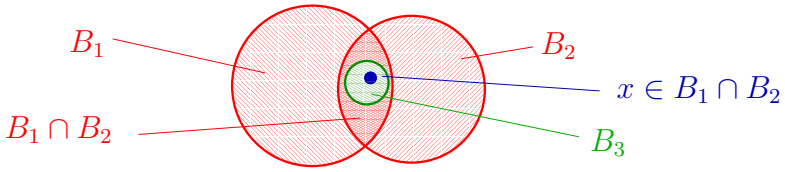
Proof. Each C_i is the union of 2^i closed intervals, hence each C_i is closed. It follows from the Closed Subset-Lemma 1.2 that C itself is a closed subset of $\mathbb{R}$. It is evidently bounded, hence the Heine-Borel Theorem 1.11 implies that the Cantor set is compact. ■

1.10. **Bases of a topology.** In this section we will introduce the notion of a (sub-) basis of a topology. These notions make it possible to define topologies and to describe a given topology. Nonetheless, the topic is somewhat dry and it is best not to spend too much time on this section.

Definition. Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$ be a subset of the power set of X. We say $\mathcal{B}$ has the basis property if the following two conditions are satisfied:

(B1) For every $x \in X$ there exists a $B \in \mathcal{B}$ with $x \in B$.

(B2) Given any $B_1, B_2 \in \mathcal{B}$ and given any $x \in B_1 \cap B_2$ there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.



Example. Let (X, d) be a metric space. It follows from an elementary argument, see e.g. the proof of Proposition 1.4 (1), that the set of all open balls

$$\mathcal{B} \;=\; \{\text{all balls } B_r(x) \text{ with } r > 0 \text{ and } x \in X\} \;\subset\; \mathcal{P}(X)$$

satisfies (B1) and (B2).

The following lemma justifies our sudden interest in this new definition.

Lemma 1.13. (Topology-from-Basis Lemma) Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. We set

$$\mathcal{T}(\mathcal{B}) \;:=\; \{V \subset X \,|\, \text{for every } x \in V \text{ there exists a } B \in \mathcal{B} \text{ with } x \in B \subset V\}.$$

If $\mathcal{B}$ has the basis property, then $\mathcal{T}(\mathcal{B})$ is a topology on X.

Proof. We proved the lemma in Analysis IV. Alternatively see [Fri23]. ∎

Definition. Let X be a set.

(1) If $\mathcal{B} \subset \mathcal{P}(X)$ has the basis property, then we refer to $\mathcal{T}(\mathcal{B})$ as the topology on X generated by $\mathcal{B}$.

(2) Conversely, if $\mathcal{T}$ is a given topology on X, if $\mathcal{B} \subset \mathcal{P}(X)$ has the basis property and if $\mathcal{T} = \mathcal{T}(\mathcal{B})$, then we say that $\mathcal{B}$ is a basis of the topology $\mathcal{T}$ of X.

Examples.

(1) If (X, d) is a metric space, then the topology generated by the set of all open balls, is by definition precisely the metric topology we already introduced on page 39.

(2) Let $n \in \mathbb{N}$ and let $\mathbb{R}^n \cup \{\infty\}$ be the topological space defined on page 43. It follows immediately from the definition of the topology that a basis for the topological space $\mathbb{R}^n \cup \{\infty\}$ is given by

$$\mathcal{B} \;=\; \left\{B_r^n(x) \,\middle|\, r > 0 \text{ and } x \in \mathbb{R}^n\right\} \cup \left\{\{\infty\} \cup (\mathbb{R}^n \setminus \overline{B}_C^n(0)) \,\middle|\, C \in \mathbb{R}\right\}.$$

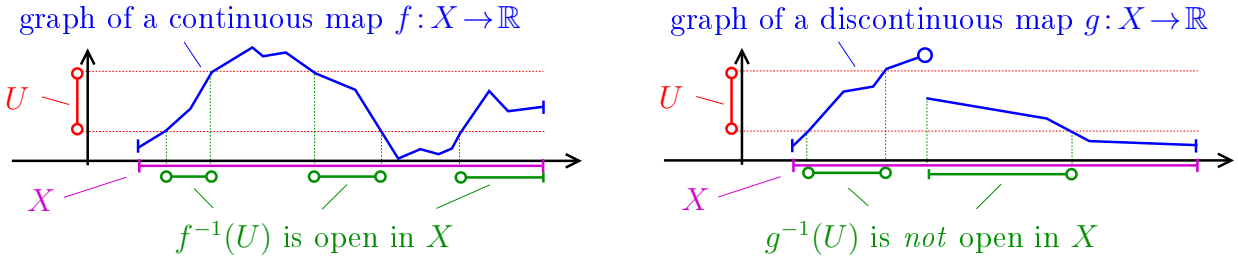
2. Continuity

2.1. Continuous maps. The following is the key definition in the context of topological spaces.

Definition. We say a map $f\colon X \to Y$ between two topological spaces X and Y is continuous, if for each open subset U in Y the preimage $f^{-1}(U)$ is open in X. Otherwise the map is called discontinuous.

Examples.

(1) For subspaces $X \subset \mathbb{R}$ and maps $\mathbb{R} \to \mathbb{R}$ we see in the figure below that the above definition of continuous matches our intuition.



(2) Let X and Y be two topological spaces. If Y is equipped with the trivial topology or if X is equipped with the discrete topology, then every map $f\colon X \to Y$ is continuous.

(3) We say a map $X \to Y$ between two sets is *constant* if for any $x_1, x_2 \in X$ we have $f(x_1) = f(x_2)$. Any constant map $f\colon X \to Y$ between two topological spaces is continuous since the preimage of an open subset of Y is either the empty set or X. This simple example will come in handy on many occasions.

Lemma 2.1. (Basics-of-Continuity Lemma)

(1) If $f\colon X \to Y$ and $g\colon Y \to Z$ are two continuous maps between topological spaces, then the composition $g \circ f\colon X \to Z$ is also continuous.

(2) Let X be a topological space and let A be a subset of X. If we equip A with the subspace topology, then the inclusion map $i\colon A \to X$ is continuous.

(3) Let $f\colon X \to Y$ be a continuous map between topological spaces and let A be a subset of X. If we equip A with the subspace topology, then the restriction of f to the map $f|_A\colon A \to Y$ is also continuous.

(4) Let $f\colon X \to Y$ be a continuous map. If $Z \subset Y$ is a subset with $f(X) \subset Z$, then the map $f\colon X \to Z$ is also continuous.

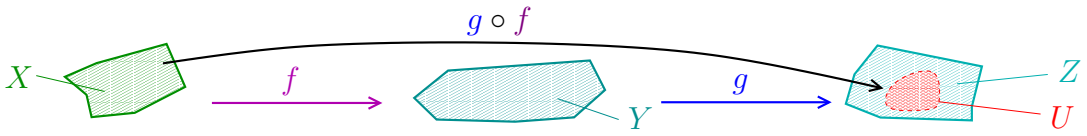
Proof.

(1) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two continuous maps between topological spaces. We need to show that the composition $g \circ f\colon X \to Z$ is also continuous. Let $U \subset Z$ be an open subset. Then

$$(g \circ f)^{-1}(U) \;=\; f^{-1}(g^{-1}(U)) \;\underset{\uparrow}{=}\; f^{-1}(\text{the open subset } g^{-1}(U)) \;\underset{\uparrow}{=}\; \text{open}.$$

since g is continuous and U is open; since f is continuous

(2) This statement follows immediately from the definitions.



(3) This statement is an immediate consequence of (1) and (2) since the restriction of f to A is precisely the same as $f \circ i$ where $i\colon A \to X$ denotes the inclusion.

(4) For clarity we denote by $g\colon X \to Z$ the map given by $x \mapsto f(x)$. Let $U \subset Z$ be an open subset. We need to show that $g^{-1}(U)$ is an open subset of X. By definition of the subspace topology there exists an open subset $V \subset X$ with $U = Z \cap V$. Thus we see that

$$g^{-1}(U) \;=\; g^{-1}(Z \cap V) \underset{\substack{\uparrow \\ \text{definition of } g}}{=} f^{-1}(Z \cap V) \underset{\substack{\uparrow \\ \text{since } f(X) \subset Z}}{=} f^{-1}(V) \underset{\substack{\uparrow \\ \text{since } f \text{ is continuous}}}{=} \text{open subset of } X.$$

■

The following proposition makes it much easier to show that a given map is in fact continuous.

Proposition 2.2. (Basis-Continuity Proposition) Let $f\colon X \to Y$ be a map between topological spaces. Let $\mathcal{C}$ be a basis for the topology of Y. Then the following holds

$$f \text{ is continuous} \iff \text{for each } C \in \mathcal{C} \text{ the preimage } f^{-1}(C) \text{ is open in } X.$$

Proof. We proved the proposition in Analysis IV. Alternatively see [Fri23]. ■

At times it is also useful to have a characterization of continuous maps in terms of closed subsets.

Lemma 2.3. (Continuity-via-Closed Subsets Lemma) A map $f\colon X \to Y$ between two topological spaces X and Y is continuous if and only if for each closed subset A in Y the preimage $f^{-1}(A)$ is closed in X.

Proof. This lemma is a straightforward consequence of the Image-Preimage Lemma 0.3 (8) and the definitions. We leave it to the reader to fill in the details. ■

By the Basics-of-Continuity Lemma 2.1 (3) we know that the restriction of a continuous map to any subspace is still continuous. The following lemma can be viewed as a converse: if "enough" restrictions of a given map are continuous, then the original map is also continuous.

Proposition 2.4. (Pasting Proposition) Let $f\colon X \to Y$ be a map between topological spaces.

(1) If there exists an open cover $\{W_i\}_{i\in I}$ of X such that each restriction $f|_{W_i}$ is continuous, then f itself is continuous.

(2) If there exists a finite closed cover $\{A_i\}_{i\in I}$ of X such that each restriction $f|_{A_i}$ is continuous, then f itself is continuous.

In practice often the following variation on (2) gets used:

(2′) If X is Hausdorff and if there exists a locally finite cover $\{A_i\}_{i\in I}$ of X such that each A_i is compact and such that each restriction $f|_{A_i}$ is continuous, then f itself is continuous.

Remark.

(1) In practice the Pasting Proposition 2.4 (2) often gets used in the following way: Suppose we are given two topological spaces X and Y and closed subsets $A_1, \ldots, A_m$ of X such that $X = A_1 \cup \cdots \cup A_m$. Furthermore we are given continuous maps $f_i \colon A_i \to Y$, $i = 1, \ldots, m$. If the f_i agree on the overlaps, then the map

$$\begin{array}{ccl} X & \to & Y \\ P & \mapsto & f_i(P) \quad \text{if } P \in A_i \end{array}$$

is well-defined and it follows from the Pasting Proposition 2.4 (2) that it is continuous.

(2) Note that in the Pasting Proposition 2.4 (1) we are allowed to deal with arbitrarily many open subsets whereas in the Pasting Proposition 2.4 (2) we are restricted to finitely many closed subsets. It is clear that the Pasting Proposition 2.4 (2) cannot be generalized to arbitrarily many closed subsets, after all, the restriction of a map $f \colon \mathbb{R} \to \mathbb{R}$ to any one-point subset $\{x\} \subset \mathbb{R}$ is continuous, but clearly not every map $f \colon \mathbb{R} \to \mathbb{R}$ is continuous.

Proof. Statements (1) and (2) of the Pasting Proposition follow fairly easily from the Cover-Open Subset Lemma 1.5. Statement (2′) follows from Statement (2) and the Compact-Closed Lemma 1.9 (2). We refer to [Fri23] for details. ■

The notion of continuity is of course already familiar from earlier analysis courses. The following proposition shows that for maps between metric spaces the earlier "ϵ–δ"–definition of continuity agrees with the above definition of continuity.

Proposition 2.5. (Metric Continuity Proposition) Let $f \colon X \to Y$ be a map between metric spaces. Then the following holds:

$$f \text{ is continuous} \quad \Longleftrightarrow \quad \underset{x \in X}{\forall} \; \underset{\epsilon > 0}{\forall} \; \underset{\delta > 0}{\exists} \; \underset{x' \in B_\delta(x)}{\forall} \; d(f(x), f(x')) < \epsilon.$$

Proof. We proved the Metric Continuity Proposition 2.5 in Analysis II. Alternatively see [Fri23]. ■

Example. Let $n \in \mathbb{N}$. For each $i \in \{1, \ldots, n\}$ the projection

$$\begin{array}{ccl} \mathbb{R}^n & \to & \mathbb{R} \\ (x_1, \ldots, x_n) & \mapsto & x_i \end{array}$$

is continuous since the "$\epsilon - \delta$"-condition is satisfied with $\delta = \epsilon$.

The final lemma of this section shows that we can add and multiply continuous real-valued maps.

Lemma 2.6. (Sum-Product-Continuity Lemma) Let X be a topological space. If $f \colon X \to \mathbb{R}$ and $g \colon X \to \mathbb{R}$ are continuous maps, then the maps

$$\begin{array}{llll} X \to \mathbb{R} & X \to \mathbb{R} & X \to \mathbb{R} & X \to \mathbb{R} \\ x \mapsto f(x) + g(x) & x \mapsto f(x) \cdot g(x) & x \mapsto \max\{f(x), g(x)\} & x \mapsto \min\{f(x), g(x)\} \end{array}$$

are also continuous. The obvious analogues of the first two statements also hold if we replace $\mathbb{R}$ by $\mathbb{C}$.

Example. Let A be a real $(m \times n)$–matrix. It follows easily from the previous example, the Basics-of-Continuity Lemma 2.1 (1) and the Sum-Product-Continuity Lemma 2.6 that the following map is continuous:

$$\begin{array}{rcl} f\colon \mathbb{R}^n & \to & \mathbb{R}^m \\ v & \mapsto & A \cdot v. \end{array}$$

Proof. For brevity's sake we outsource the proof to [Fri23]. ■

2.2. **Open and closed maps.** We continue with the following straightforward definition.

Definition. Let $f\colon X \to Y$ be a continuous map between topological spaces.
(1) We say f is open if the image of every open subset of X is open in Y.
(2) We say f is closed if the image of every closed subset of X is closed in Y.

Examples.

(1) The projection $\mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x$ is easily seen to be an open map, but it is not a closed map.[31]
(2) On the other hand the inclusion $i\colon \mathbb{R} \to \mathbb{R}^2$, $i(x) = (x, 0)$ is evidently not an open map, but it is a closed map.

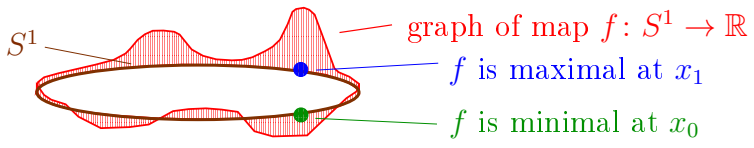
Lemma 2.7. (Open-Closed Inclusion Lemma) Let X be a topological space.
(1) If $U \subset X$ is an open subset, then the inclusion map $i\colon U \to X$ is open.
(2) If $A \subset X$ is a closed subset, then the inclusion map $i\colon A \to X$ is closed.

Proof.

(1) Suppose that V is an open subset of U. By definition of the subspace topology there exists an open subset $W \subset X$ with $V = U \cap W$. But then $i(V) = V = U \cap W$ is the intersection of two open subsets of X, thus it is an open subset of X.
(2) The proof of this statement is analogous to the proof of Statement (1). ■

2.3. **The Compact Image Lemma.** The first statement of the following lemma is perhaps the most important feature of compact spaces.

Lemma 2.8. (Compact Image Lemma)
(1) Let $f\colon X \to Y$ be a continuous map. If X is compact, then $f(X)$ is also compact.
(2) Let $f\colon X \to \mathbb{R}$ be a continuous map. If X is compact and non-empty, then f assumes its maximum and its minimum, i.e. there exist $x_0, x_1 \in X$ such that
$$f(x_0) \leq f(x) \leq f(x_1) \quad \text{for all } x \in X.$$
(3) Let $f\colon X \to Y$ be a continuous map. If X is compact and if Y is Hausdorff, then $f(X)$ is a closed subset of Y.



[31] Why is not a closed map?

Proof.

(1) Let $\{U_i\}_{i\in I}$ be an open cover of the topological space $f(X)$. We need to show that $f(X)$ is contained in the union of finitely many U_i's. We have

$$X \underset{\uparrow}{=} f^{-1}(f(X)) \subset f^{-1}\Big(\bigcup_{i\in I} U_i\Big) \underset{\uparrow}{=} \bigcup_{i\in I} f^{-1}(U_i).$$

Image-Preimage Lemma 0.3 (1) — Image-Preimage Lemma 0.3 (6)

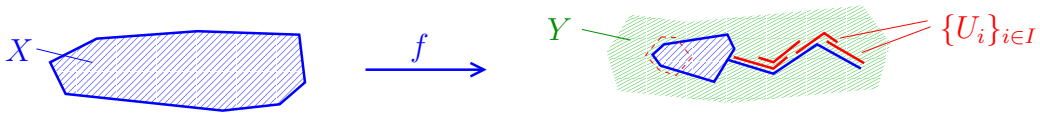
By hypothesis the map $f\colon X \to Y$ is continuous. It follows from the elementary Basics-of-Continuity Lemma 2.1 that the map $f\colon X \to f(X)$ is also continuous, in particular each $f^{-1}(U_i)$ is an open subset of X. This observation together with the above discussion shows that $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of X. Since X is compact there exist $i_1, \dots, i_k \in I$ such that

$$X = f^{-1}(U_{i_1}) \cup \cdots \cup f^{-1}(U_{i_k}).$$

Therefore

$$\begin{aligned} f(X) &= f\left(f^{-1}(U_{i_1}) \cup \cdots \cup f^{-1}(U_{i_k})\right) \\ &\underset{\uparrow}{=} f\left(f^{-1}(U_{i_1})\right) \cup \cdots \cup f\left(f^{-1}(U_{i_k})\right) \underset{\uparrow}{\subset} U_{i_1} \cup \cdots \cup U_{i_k}. \end{aligned}$$

Image-Preimage Lemma 0.3 (3) — Image-Preimage Lemma 0.3 (2)



(2) It follows from (1) and the Heine-Borel Theorem 1.11 that $f(X) \subset \mathbb{R}$ is bounded and closed. Since X is non-empty we thus see that $y_0 := \inf(f(X))$ and $y_1 := \sup(f(X))$ exist and they lie in $f(X)$. Any choice of points $x_0 \in f^{-1}(y_0)$ and $x_1 \in f^{-1}(y_1)$ have the desired properties.

(3) This statement is an immediate consequence of the first statement and of the Compact-Closed Lemma 1.9 (1). ■

2.4. **Homeomorphisms.** We introduce the main objects of interest of this section.

Definition.

(1) We say a map $f\colon X \to Y$ between two topological spaces X and Y is a homeomorphism if the following three properties are satisfied:
   (a) f is continuous,
   (b) f is bijective,
   (c) the inverse map $f^{-1}\colon Y \to X$ is also continuous.
   If there exists a homeomorphism between X and Y we say that X and Y are homeomorphic and sometimes we write $X \cong Y$.

(2) We say a map $f\colon X \to Y$ between two topological spaces X and Y is an embedding if $f\colon X \to f(X)$, where $f(X)$ is equipped with the subspace topology, is a homeomorphism. We say the embedding is closed if $f(X)$ is a closed subset of Y and we say the embedding is open if $f(X)$ is an open subset of Y.

Remark.

(1) If two topological spaces are homeomorphic, then they have the same topological properties, i.e. they share all properties that are defined purely in terms of the topology.

For example, if X and Y are homeomorphic, then X is Hausdorff if and only if Y is Hausdorff, X is compact if and only if Y is compact and so on.

(2) It follows from the Open-Closed Inclusion Lemma 2.7 that an open (respectively closed) embedding is also an open (respectively closed) map.

Before we turn to our first examples we introduce some standard notation.

Notation. Given $n_0 \in \mathbb{N}$ we write

$$\begin{array}{lll} S^n_{\geq 0} & := & \{(x_1,\ldots,x_{n+1}) \in S^n \mid x_{n+1} \geq 0\} \quad \text{the upper hemisphere} \\ S^n_{\leq 0} & := & \{(x_1,\ldots,x_{n+1}) \in S^n \mid x_{n+1} \leq 0\} \quad \text{the lower hemisphere} \\ S^n_{=0} & := & \{(x_1,\ldots,x_{n+1}) \in S^n \mid x_{n+1} = 0\} \quad \text{the equator.} \end{array}$$

Similarly we define

$$\begin{array}{lll} B^n_{\geq 0} & := & \{(x_1,\ldots,x_n) \in B^n \mid x_n \geq 0\} \\ B^n_{\leq 0} & := & \{(x_1,\ldots,x_n) \in B^n \mid x_n \leq 0\} \end{array}$$

Finally, the same way we also define $\overline{B}^n_{\geq 0}$ and $\overline{B}^n_{\leq 0}$.

Examples.

(1) The maps

$$\begin{array}{rcl} \varphi\colon \overline{B}^n & \to & S^n_{\geq 0} \\ x & \mapsto & \left(x, \sqrt{1-\|x\|^2}\right) \end{array} \quad \text{and} \quad \begin{array}{rcl} p\colon S^n_{\geq 0} & \to & \overline{B}^n \\ (x_1,\ldots,x_n,x_{n+1}) & \mapsto & (x_1,\ldots,x_n) \end{array}$$

are easily seen to be continuous and inverses to one another. Thus both maps are homeomorphisms. The same way we see that the lower hemisphere is also homeomorphic to the closed ball $\overline{B}^n$.

(2) Let $n \in \mathbb{N}$. It follows easily from the definitions of the standard metrics on $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ that the map

$$\begin{array}{rcl} \mathbb{R}^{2n} & \to & \mathbb{C}^n \\ (x_1,y_1,\ldots,x_n,y_n) & \mapsto & (x_1+\mathrm{i}y_1,\ldots,x_n+\mathrm{i}y_n) \end{array}$$

is an isometry of metric spaces. Recall that on page 39 we defined the standard topology on $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ to be the metric topology corresponding to the standard metrics. Thus we see that the above map is a homeomorphism. We will use this homeomorphism to make the identification $\mathbb{C}^m =_i \mathbb{R}^{2m}$.[32]

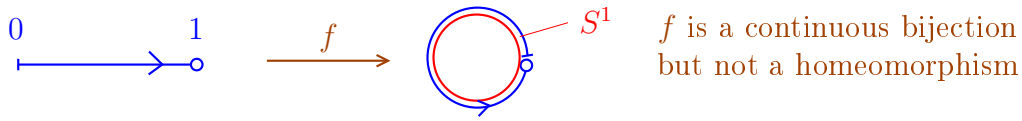
(3) We consider the map

$$\begin{array}{rcl} f\colon [0,1) & \to & S^1 \\ t & \mapsto & \exp(2\pi \mathrm{i} t). \end{array}$$

The map f is a continuous bijection but it is not a homeomorphism. One can see this by either directly showing that f^{-1} is not continuous or by noting that $[0,1)$ is not compact whereas S^1 is compact. The importance of this example cannot be overestimated. It is an unfortunate fact of life that all too often one writes down a continuous map $f\colon X \to Y$ between topological spaces, checks that it is a bijection, and walks away

[32]By an "identification" we mean in these notes that we picked once and for all a homeomorphism. If in the following we write $\mathbb{C}^m =_i \mathbb{R}^{2m}$, then this means that we use the given homeomorphism to go back and forth between these two topological spaces.

under the warm, but mistaken, impression that one now has found a homeomorphism.



In many situations one is given an explicit continuous bijection $f\colon X \to Y$ between two topological spaces and one would like to show that f is in fact a homeomorphism. Unfortunately usually it is very painful to write down the explicit inverse map $f^{-1}\colon Y \to X$, let alone to verify that it is continuous. Fortunately we have several results which often help us in circumventing this difficulty.

Lemma 2.9. (Open-Injective Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces. We assume that f is open or that f is closed.
(1) If f is an injection, then f is an open (respectively closed) embedding.
(2) If f is a bijection, then f is a homeomorphism.

Proof. Let $f\colon X \to Y$ be a continuous map between topological spaces.

(2) We assume that f is a bijection. Furthermore we now assume that f is open. We want to show that f is a homeomorphism. It remains to show that $g := f^{-1}\colon Y \to X$ is continuous. Thus let $U \subset X$ be an open subset. We need to show that $g^{-1}(U)$ is an open subset of Y. But this is obvious, since $g^{-1}(U) = f(U)$ is an open subset since we assume that f is an open map.

Now suppose that the map f is closed. The proof is almost identical to the above argument, except that now we have to use the continuity criterion provided by the Continuity-via-Closed Subsets Lemma 2.3.

(1) We assume that f is an injection and we assume that f is open. We write $Z := f(X)$. Since $f\colon X \to Y$ is open it follows basically immediately from the definition of the subspace topology that $f\colon X \to Z$ is also open. Thus we obtain from (2) that f is an embedding. Furthermore, since f is open we see that f is an open embedding. The proof in the case that f is closed is basically identical. ■

The next proposition is certainly one of the most frequently used results in topology.

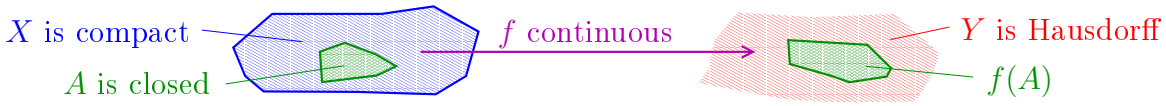
Proposition 2.10. (Compact-Hausdorff Proposition) Let $f\colon X \to Y$ be a continuous map between topological spaces. If X is compact and if Y is Hausdorff, then the following statements hold:
(1) The map $f\colon X \to Y$ is closed.
(2) If f is an injection, then f is a closed embedding.
(3) If f is a bijection, then f is a homeomorphism.

Remark. On page 53 we gave an example of a continuous bijective map $f\colon [0,1) \to S^1$ that is not a homeomorphism. This shows that we cannot drop the condition that X is compact from the proposition. But what about the "Hausdorff condition"? It is a nice exercise to come up with an example of a bijective continuous map $f\colon X \to Y$ between two topological spaces where X is compact but such that f is nonetheless not a homeomorphism.

Proof. Let $f\colon X \to Y$ be a continuous map between topological spaces. We assume that X is compact and that Y is Hausdorff.

(1) Let A be a closed subset of X. We need to show that $f(A)$ is a closed subset of Y. In fact we see that

$$\begin{array}{ccccccc} A \subset X \text{ closed} & \Longrightarrow & A \text{ is compact} & \Longrightarrow & f(A) \text{ is compact} & \Longrightarrow & f(A) \subset Y \text{ is closed.} \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{by the Compact-Closed} & & \text{by the Compact Image} & & \text{by the Compact-Closed Lemma 1.9 (2)} & \\ & \text{Lemma 1.9 since } X \text{ is compact} & & \text{Lemma 2.8 (1)} & & \text{since } Y \text{ is Hausdorff} & \end{array}$$



(2) If f is an injection, then it follows immediately from (1) together with the Open-Injective Lemma 2.9 (1) that f is a closed embedding.

(3) This follows immediately from (2). ∎

We formulate our first application of the Compact-Hausdorff Proposition 2.10 (3) as a lemma.

Lemma 2.11. (Stereographic Projection Lemma) Let $n \in \mathbb{N}$. We consider the map

$$\begin{array}{rcl} \Phi\colon S^n & \to & \mathbb{R}^n \cup \{\infty\} \\ (x_1, \ldots, x_{n+1}) & \mapsto & \begin{cases} \left(\frac{x_1}{1-x_{n+1}}, \ldots, \frac{x_n}{1-x_{n+1}}\right), & \text{if } x_{n+1} < 1, \\ \infty, & \text{if } x_{n+1} = 1. \end{cases} \end{array}$$

where we equip $\mathbb{R}^n \cup \{\infty\}$ with the topology that we introduced on page 43. We refer to Φ as the stereographic projection. This map has the following properties:

(1) The map Φ sends the North Pole $N = (0, \ldots, 0, 1) \in S^n$ to ∞.

(2) For any $P \in S^n$ that does not equal the North Pole N the point $\Phi(P) \in \mathbb{R}^n$ is the unique point such that the ray emanating from N and that goes through P intersects the plane $\mathbb{R}^n \times \{0\}$ in $(\Phi(P), 0)$.

(3) For any $(v_1, \ldots, v_n, 0) \in S^n \cap (\mathbb{R}^n \times \{0\})$ we have $\Phi(v_1, \ldots, v_n, 0) = (v_1, \ldots, v_n)$.

(4) The map Φ is a homeomorphism.

(5) (a) The restriction of Φ to a map $S^n \setminus \{N\} \to \mathbb{R}^n$ is also a homeomorphism, where the inverse is given by

$$\begin{array}{rcl} \mathbb{R}^n & \to & S^n \setminus \{N\} \\ x & \mapsto & \left(x \cdot \frac{2}{\|x\|^2+1}, \underbrace{\frac{\|x\|^2-1}{\|x\|^2+1}}_{=1-\frac{2}{\|x\|^2+1}}\right). \end{array}$$

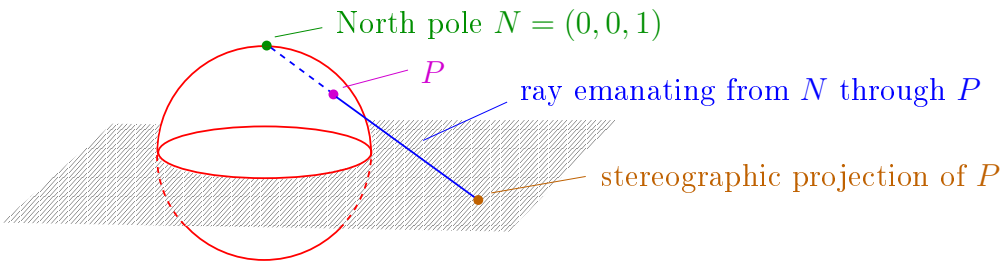
(b) The restriction of Φ to a map $S^n_{\leq 0} \to \overline{B}^n$ is a homeomorphism.

Identification. Let $n \in \mathbb{N}$. On many occasions we will use the homeomorphism from the Stereographic Projection Lemma 2.11 to make the identification $S^n =_i \mathbb{R}^n \cup \{\infty\}$. In particular we have the identification $S^2 =_i \mathbb{C} \cup \{\infty\}$.

Proof.

(1) This statement follows immediately from the definition of Φ.

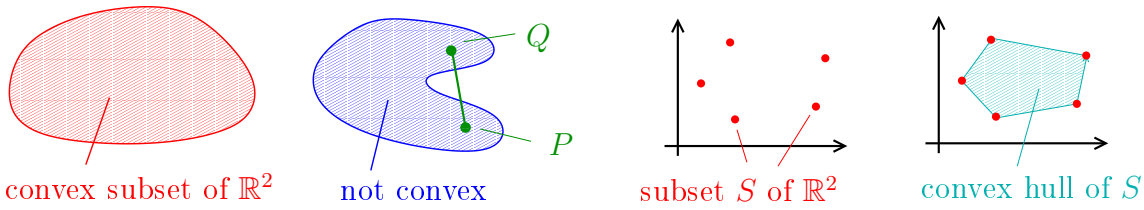
(2) This statement follows from an elementary calculation which we leave to the reader.



(3) This statement follows immediately from the definition of Φ.
(4) Using the Subbasis-Continuity Proposition 2.2 one can easily show that Φ is continuous. We leave it to the reader to verify that Φ is a bijection. Note that S^n is compact by the Heine-Borel Theorem 1.11. Furthermore note that on page 43 we pointed out that $\mathbb{R}^n \cup \{\infty\}$ is Hausdorff. Hence it follows from the Compact-Hausdorff Proposition 2.10 (3) that Φ is a homeomorphism.
(5) It follows almost immediately from (4) that the two maps are homeomorphism. One can easily verify that the inverse of Φ is indeed given by the map we provided. ■

2.5. **Convex subsets.** We start out with the definition of a convex subset.

Definition.
(1) We say that a subset $A \subset \mathbb{R}^n$ is convex, if for every two points P and Q in A the segment $\overline{PQ} := \{t \cdot P + (1-t) \cdot Q \,|\, t \in [0,1]\}$ also lies in A.
(2) Given a subset S of $\mathbb{R}^k$ the convex hull of S is defined as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. Since the intersection of convex sets is again convex we see that the convex hull of S is a convex subset of $\mathbb{R}^n$.



Examples.
(1) For any point $x \in \mathbb{R}^n$ and any $\epsilon > 0$ the open ball $B^n_\epsilon(x)$ is convex. Indeed, let $P, Q \in B^n_\epsilon(x)$. Then for any $t \in [0,1]$ we have

$$\begin{aligned} \|t\cdot P + (1-t)\cdot Q - x\| &= \|t\cdot(P-x) + (1-t)\cdot(Q-x)\| \\ &\underset{\uparrow}{\leq} t\cdot\|P-x\| + (1-t)\cdot\|Q-x\| \underset{\uparrow}{<} t\cdot\epsilon + (1-t)\cdot\epsilon = \epsilon. \end{aligned}$$

triangle inequality and homogeneity of $\|-\|$ — since $P, Q \in B^n_\epsilon(x)$

We have thus shown that $t \cdot P + (1-t) \cdot Q \in B^n_\epsilon(x)$. Basically the same way one shows that the closed ball $\overline{B}^n_\epsilon(x)$ is also convex.
(2) It is straightforward to verify that any hyperrectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is convex.
(3) We leave it as an exercise to show that the convex hull of points $P_1, \ldots, P_n \in \mathbb{R}^k$ is given by the set

$$\left\{ \sum_{i=1}^n t_i \cdot P_i \,\middle|\, t_1, \ldots, t_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^n t_i = 1 \right\}.$$

(4) One of the most useful properties of convex subsets is that the intersection of arbitrarily many convex subsets is, basically by definition, again convex.

The following proposition gives a useful criterion for showing that subsets of $\mathbb{R}^n$ are homeomorphic to an open or to a closed ball.

Proposition 2.12. (Convex-to-Ball Proposition)

(1) Let A be a bounded *open* convex subset of $\mathbb{R}^n$ and let $Q \in A$.[33] We consider the map

$$\begin{array}{rcl} \rho_A \colon A \setminus \{Q\} & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \sup\left\{ \|r \cdot x\| \,\middle|\, r \in \mathbb{R}_{>0} \text{ and } r \cdot (x-Q) + Q \in A \right\}. \end{array}$$

This map is well-defined in the sense that the supremum exists for each $x \in A \setminus \{Q\}$. Furthermore the map ρ_A is continuous and the map

$$\begin{array}{rcl} f_Q \colon A & \to & B^n \\ x & \mapsto & \begin{cases} (x-Q) \cdot \frac{1}{\rho_A(x)}, & \text{if } x \neq Q, \\ 0, & \text{if } x = Q. \end{cases} \end{array}$$

is a homeomorphism with $f_Q(Q) = 0$.

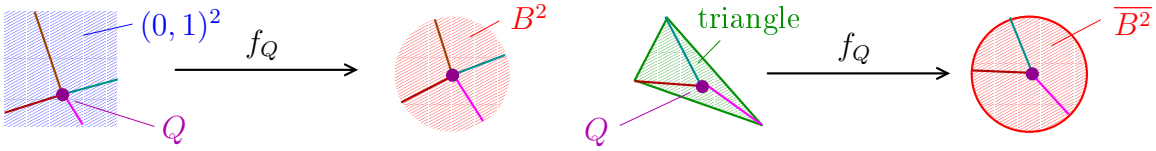
(2) Let A be a bounded *closed*[34] convex subset of $\mathbb{R}^n$. If Q is a point in the interior[35] of A, then the same expression as in (1) defines a homeomorphism[36]

$$f_Q \colon A \to \overline{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

with the following properties:

(a) the homeomorphism f_Q sends Q to the origin,

(b) the homeomorphism f_Q restricts to a homeomorphism $\partial A \to S^{n-1}$.



The proof of the Convex-to-Ball Proposition 2.12 relies on the following lemma.

Lemma 2.13. Let A be a bounded closed convex subset of $\mathbb{R}^n$ such that the origin 0 lies in the interior of A.

(1) The map $\rho_A \colon A \setminus \{0\} \to \mathbb{R}_{\geq 0}$ defined in the Convex-to-Ball Proposition 2.12 is continuous.

(2) The map

[33] The statement becomes slightly more readable if one sets $Q = 0$.

[34] Of course "bounded and closed" is in this instance, by the Heine-Borel Theorem 1.11 equivalent to "compact". In this instance we prefer to talk of "bounded closed" to stress the analogy to Statement (1) of the proposition.

[35] Note that the interior might well be empty, i.e. such a point Q might not exist.

[36] This says in particular that if A is a bounded closed convex subset of $\mathbb{R}^n$ with *non-empty interior*, then A is homeomorphic to $\overline{B}^n$.

$$\begin{array}{rccl} f_0\colon A & \to & \overline{B}^n & \\ x & \mapsto & \begin{cases} x \cdot \frac{1}{\rho_A(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases} \end{array}$$

is continuous and it satisfies $f_0^{-1}(S^{n-1}) = \partial A$.

Proof of Lemma 2.13. We start out with the following claim.

Claim. The map $\rho_A\colon A \setminus \{0\} \to \mathbb{R}$ is defined, it is continuous, it is bounded from above and it is bounded from below by 1.

Proof. Since A is bounded it follows that ρ_A is defined and that it is bounded from above. It follows from the definition of ρ_A that ρ_A is bounded from below by 1. Therefore it remains to show that ρ_A is continuous. Now let $x \in A \setminus \{0\}$ and let $\epsilon > 0$. It suffices to show the following two statements:

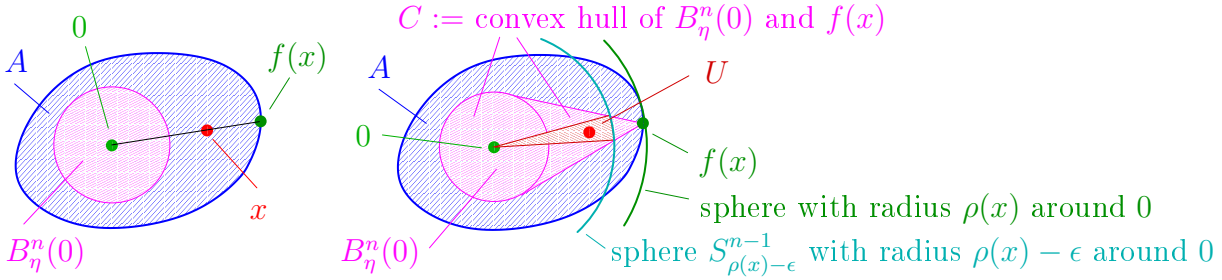
(1) there exists a neighborhood U of $x \in A \setminus \{0\}$ with $\rho_A(y) > \rho_A(x) - \epsilon$ for all $y \in U$,
(2) there exists a neighborhood V of $x \in A \setminus \{0\}$ with $\rho_A(y) < \rho_A(x) + \epsilon$ for all $y \in V$.

We first show the existence of U. After possibly replacing ϵ by $\min\{\frac{1}{2}\rho_A(x), \epsilon\}$ we can suppose that $\epsilon \in (0, \rho_A(x))$.

Since A is closed we have $f_0(x) \in A$. Since 0 lies in the interior of A there exists an $\eta > 0$ such that $B^n_\eta(0) \subset A$. Since A is convex, the convex hull C of $B^n_\eta(0) \cup \{f_0(x)\}$ is still contained[37] in A. Furthermore, since $B^n_\eta(0)$ is open it is straightforward to see that $C' := C \setminus \{f_0(x)\}$ is an open subset of $\mathbb{R}^n$. We denote by $S^{n-1}_{\rho_A(x)-\epsilon}$ the sphere of radius $\rho_A(x) - \epsilon$ around 0. The point $x \cdot \frac{\rho_A(x)-\epsilon}{\|x\|}$ lies on $S^{n-1}_{\rho_A(x)-\epsilon}$ and it lies in C'. Since C' is open there exists an open neighborhood U' on $S^{n-1}_{\rho_A(x)-\epsilon}$ that is contained in C'. We set

$$U \;:=\; \{r \cdot z \,|\, z \in U' \text{ and } r \in (0,1)\}.$$

This is an open neighborhood of x and for any $y \in U$ we have $(\rho_A(x)-\epsilon)\cdot y \in A$, i.e. for any $y \in A$ we have $\rho_A(y) \geq \rho_A(x) - \epsilon$. Thus we have shown that the desired neighborhood U exists. The existence of V is proved in a very similar way. We refer to [Ber09, Chapter 11.3] for full details. ⊞



From the continuity of ρ_A it follows that f is continuous on $A \setminus \{0\}$. From the fact that ρ_A is bounded it follows easily that f_Q is continuous in 0. Finally one can easily verify, using the convexity of A, that $f_0^{-1}(S^{n-1}) = \partial A$. We leave it to the reader to fill in the details. ■

Proof of the Convex-to-Ball Proposition 2.12.

[37] Why is that the case?

(2) Let A be a bounded closed convex subset of $\mathbb{R}^n$ and let Q be a point in the interior of A. After a translation we can assume that $Q = 0$ is the origin, in particular we can assume that the origin 0 lies in the interior of A. By Lemma 2.13 we know that the map $\rho_A \colon A \setminus \{0\} \to \mathbb{R}_{\geq 0}$ is defined and continuous. Next we consider the map

$$\begin{array}{rccl} f_0 \colon A & \to & \overline{B}^n & \\ x & \mapsto & \begin{cases} x \cdot \frac{1}{\rho_A(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{array}$$

It is straightforward to verify that this map is bijective[38] and by Lemma 2.13 we know that f_0 is continuous and that $f_0^{-1}(S^{n-1}) = \partial A$. Note that our hypothesis on A and the Heine-Borel Theorem 1.11 imply that A is compact. Thus we obtain from the Compact-Hausdorff Proposition 2.10 (3) that f_0 is in fact a homeomorphism. Finally note that it follows from $f_0^{-1}(S^{n-1}) = \partial A$ that f_0 restricts to a homeomorphism $\partial A \to S^{n-1}$.

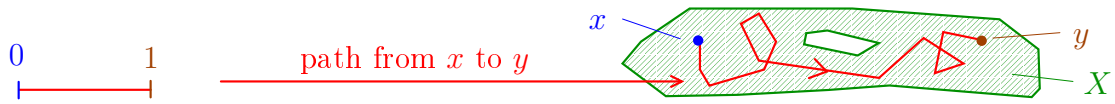
(1) We denote by $C = \overline{A}$ the closure of A. Since A is open and convex it follows fairly easily that $A = \mathring{C}$. One verifies fairly easily that C is convex. It is clear that C is bounded and it follows from the Interior-Closure Lemma 1.6 that C is closed. One verifies easily that the map f_0 defined for $C = \overline{A}$ restricts to the map f_0 defined for A. Thus it follows from (2) that the map $f_0 \colon C = \overline{A} \to \overline{B}^n$ is a homeomorphism and it restricts to a homeomorphism $f_0 \colon \mathring{C} \to B^n$. Since $\mathring{C} = A$ we get the desired homeomorphism. ■

2.6. **The two notions of connected spaces.** In this section we will see that there are two notions of connectedness of a topological space. As we will see, for most "reasonable" topological spaces the two notions mercifully coincide. We start out with the more obvious definition.

Definition.

(1) Let X be a topological space. A path is a continuous map $p \colon [0,1] \to X$. Often we say that p is a path from $p(0)$ to $p(1)$.

(2) We say that a topological space X is path-connected if given any two points x and y there exists a path from x to y.



Before we discuss examples let us immediately provide the other, less obvious definition of connectedness.

Definition. We say that a topological space X is connected if the following holds[39]

$$\text{if } X = U \sqcup V \text{ is the disjoint union of open subsets } U, V \quad \Longrightarrow \quad X = U \text{ or } X = V.$$

A topological space which is not connected is called disconnected.

---

[38] It follows easily from the convexity of A, the hypothesis that $\mathring{A} \neq \varnothing$ and the definition of f_0 that for each $v \in S^{n-1}$ the map f_0 restricts to a bijection on the "ray defined by v", i.e. it restricts to a bijection

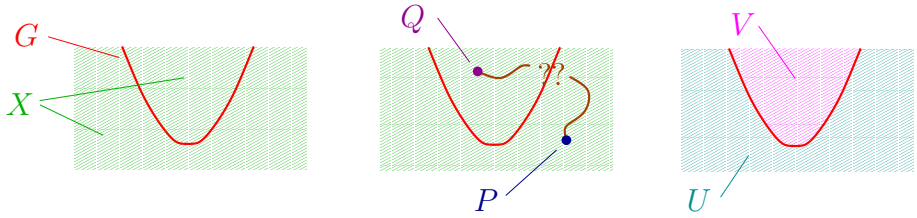
$$A \cap \{r \cdot v \,|\, r \in \mathbb{R}_{>0}\} \;\to\; \overline{B}^n \cap \{r \cdot v \,|\, r \in \mathbb{R}_{>0}\}.$$

[39] Recall that we write $X = U \sqcup V$ if X is the disjoint union of U and V, i.e. if $X = U \cup V$ and if $U \cap V = \varnothing$.

Example.

(1) It follows basically from the definitions that any convex subset of some $\mathbb{R}^n$ is path-connected. But it is much less clear whether say $\mathbb{R}^n$ is connected in the above sense.

(2) Let $G \subset \mathbb{R}^2$ be the graph of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$. As in the figure below we consider the complement $X = \mathbb{R}^2 \setminus G$. It is pretty clear that X is not path-connected, surely there is no path from a point P below G to a point above G. But how can one show the non-existence of such a path?[40] But it is clear that X is not connected, since X is the disjoint union of the two open non-empty sets $U := \{(x, y) \in \mathbb{R}^2 \,|\, y < x^2\}$ and $V := \{(x, y) \in \mathbb{R}^2 \,|\, y > x^2\}$.



(3) It follows from the Intermediate Value Theorem that the rational numbers $\mathbb{Q}$ are not path-connected. But $\mathbb{Q}$ is also not connected, for example we can write the topological space $\mathbb{Q}$ as the disjoint union of the two open non-empty subsets $\{x \in \mathbb{Q} \,|\, x^2 < 2\}$ and $\{x \in \mathbb{Q} \,|\, x^2 > 2\}$.

So we see that both notions of connectedness have their advantages and disadvantages. It is often easy to show that a given topological space is path-connected but it can be a pain to show that a given topological space is not path-connected. With connectedness it is precisely the other way round.

It is basically obvious that any interval is path-connected. But as we will see, it takes some effort to show that the interval $[0, 1]$ is connected.

Proposition 2.14. (Interval-Connected Proposition) The interval $[0, 1]$ is connected.

Proof. We proved the proposition in Analysis II, alternatively see [Fri23]. ■

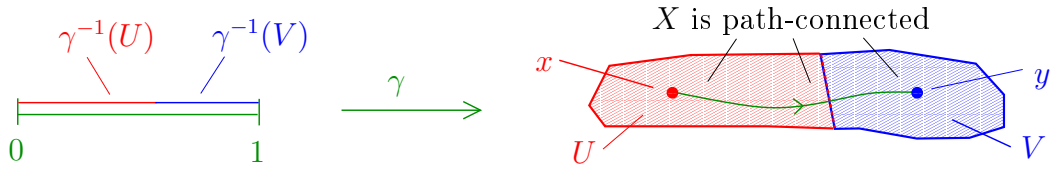
Corollary 2.15. (Path-Connected Implies Connected Corollary) Every path-connected topological space is also connected.

Proof. Let X be a topological space that is path-connected. Suppose that X is not connected. This means that there exist disjoint open non-empty subsets U and V with $X = U \cup V$. Since U and V are non-empty we can find points $x \in U$ and $y \in V$. Since X is path-connected there exists a path $\gamma\colon [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\gamma^{-1}(U) \cup \gamma^{-1}(V)$ is a decomposition of $[0, 1]$ into two disjoint open non-empty subsets. But that is a contradiction to the Interval-Connected Proposition 2.14. ■

Lemma 2.16. (Image-Connected Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces.

(1) If X is path-connected, then $f(X)$ is also path-connected.

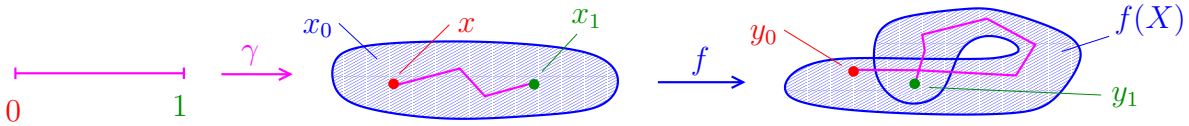
[40] The fact that such a path does not exist can be shown, with some dexterity, using real analysis. How?



(2) If X is connected, then $f(X)$ is also connected.

Proof.

(1) We suppose that X is path-connected. Let $y_0 = f(x_0)$ and $y_1 = f(x_1)$ be two points in $f(X)$. Since X is path-connected there exists a path $\gamma\colon [0,1] \to X$ from x_0 to x_1. But then $f \circ \gamma\colon [0,1] \to f(X)$ is a path from $f(x_0)$ to $f(x_1)$.



(2) We suppose that $f(Y)$ is disconnected. This means that there exist disjoint open non-empty subsets U and V of $f(Y)$ with $f(Y) = U \cup V$. By the Image-Preimage Lemma 0.3 we know that $f^{-1}(U)$ and $f^{-1}(V)$ are also disjoint and we know that $X = f^{-1}(U) \cup f^{-1}(V)$. By definition of a continuous map $f^{-1}(U)$ and $f^{-1}(V)$ are open. Finally, since $U, V \subset f(X)$ we see that $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty. Thus we have shown that X is also disconnected. ■

In the following lemma we show that most of the topological spaces that we had encountered so far are path-connected and connected. We will use this lemma on numerous occasions without explicitly referring to it.

Lemma 2.17. (Ball-Sphere-Connected Lemma)

(1) Let $n \geq 1$. Every open ball $B_r^n(y)$, every closed ball $\overline{B}_r^n(y)$ and $\mathbb{R}^n$ are path-connected and connected.

(2) Every sphere $S_r^n(y)$ with $n \geq 1$ is path-connected and connected.

Proof. By the Path-Connected Implies Connected Corollary 2.15 it suffices to show that all these topological spaces are path-connected.

(1) It follows from the example on page 56 that all balls (open or closed) and $\mathbb{R}^n$ are convex. But as we already pointed out on page 60, convex subsets are basically by definition path-connected.

(2) We leave it to the reader to show that spheres of dimension ≥ 1 are path-connected. ■

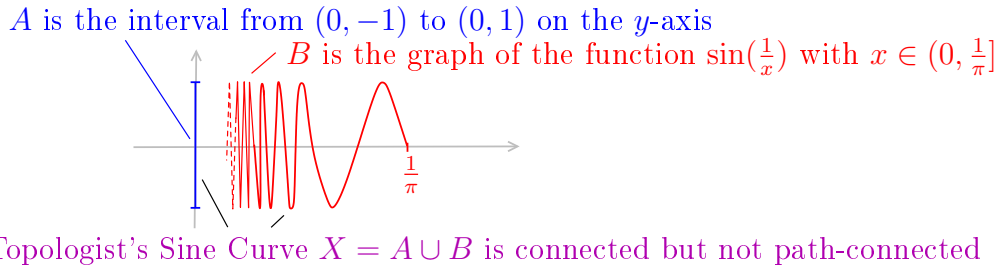
For better or worse, the converse to the Path-Connected Implies Connected Corollary 2.15 is not always correct.

Example. The *Topologist's Sine Curve* is defined as the following topological space

$$X \;:=\; \big\{(0,y)\,\big|\,y \in [-1,1]\big\} \cup \big\{(x,\sin(\tfrac{1}{x}))\,\big|\,x \in (0,\tfrac{1}{\pi}]\big\} \;\subset\; \mathbb{R}^2,$$

equipped with the subspace topology. In other words, the Topologist's Sine Curve is the union of the graph of $x \mapsto \sin(\frac{1}{x})$ for $x \in (0,\pi]$ and an interval on the y-axis. It is a very

instructive exercise to show that X is connected and it is an equally instructive exercise to show that X is *not* path-connected.



Remark. The fact that an example of a topological space that is connected but not path-connected is so "wild" indicates that perhaps for most "reasonable" spaces the two notions of connectedness and path-connectedness agree. In fact in basically all examples of topological spaces that we encounter throughout these notes the two notions of connectedness will agree.

We continue with the following elementary but useful lemma.

Lemma 2.18. (Connected-to-Discrete Lemma) Every continuous map $f\colon X \to Y$ from a connected space X to a discrete space is constant.

Remark. One could argue that the Connected-to-Discrete Lemma 2.18 is the lemma in topology that is most relevant to real life. More precisely, an immediate consequence of the Connected-to-Discrete Lemma 2.18 is that any *non-constant* map $(a,b) \to A$ from an interval to a finite set is not continuous. This is of huge significance since humanity likes to squeeze a real-parameter (e.g. time, intelligence, performance) into a finite set of values (e.g. historical stages, grades in school and exams, hired-or-fired). Any such function is non-continuous and hence very problematic.

Proof. We need to show that for any $x \in X$ we have $f(\{x\}) = f(X)$. Since Y is a discrete space we know that $\{f(x)\}$ and $Y \setminus \{f(x)\}$ are open subsets of Y. The preimages $U := f^{-1}(f(x))$ and $V := f^{-1}(Y \setminus \{f(x)\})$ are also open and they are evidently disjoint. Since U is non-empty and since X is connected we see that $V = \varnothing$. But this means that f is constant. ∎

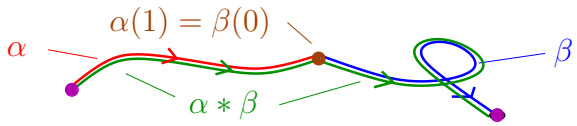
Before we continue we introduce the concatenation of two paths.

Definition. Let X be a topological space and let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be two paths with $\alpha(1) = \beta(0)$. We define the concatenation of α and β as the path $\alpha * \beta$ which is given by[41]

$$\begin{array}{rccl} \alpha * \beta\colon [0,1] & \to & X & \\ t & \mapsto & \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta((2t-1)), & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases} \end{array}$$

The concatenation of paths will play a major role throughout these notes, especially when we introduce the fundamental group of a pointed topological space. In a more humble

[41] It follows from the Pasting Proposition 2.4 (2) and our hypothesis $\alpha(1) = \beta(0)$ that $\alpha * \beta\colon [0,1] \to X$ is a continuous map.



fashion it also plays a minor role in the proof of the following lemma, which gives a criterion for the union of (path-) connected subsets to be (path-) connected.

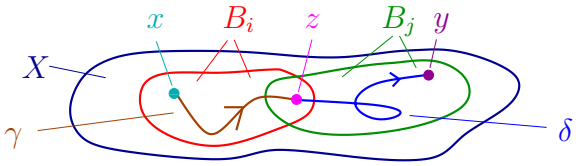
Lemma 2.19. (Union Connected Lemma) Let X be a topological space and let $\{B_i\}_{i\in I}$ be a family of subsets such that the intersection $\bigcap_{i\in I} B_i$ is non-empty.

(1) If each B_i, $i \in I$ is path-connected, then $\bigcup_{i\in I} B_i$ is path-connected.

(2) If each B_i, $i \in I$ is connected, then $\bigcup_{i\in I} B_i$ is connected.

Proof. Let X be a topological space and let $\{B_i\}_{i\in I}$ be a family of subsets such that the intersection $\bigcap_{i\in I} B_i$ contains a point z. We write $Z := \bigcup_{i\in I} B_i$.

(1) We suppose that each B_i, $i \in I$ is path-connected. So let $x, y \in Z$. There exists an $i \in I$ with $x \in B_i$ and there exists a $j \in I$ with $y \in B_j$. Since B_i and B_j are path-connected we can pick a path $\gamma\colon [0,1] \to B_i$ from x to z and we can pick a path $\delta\colon [0,1] \to B_j$ from z to y. Note that the concatenation of the paths γ and δ is a path from x to y.



(2) We suppose that each B_i, $i \in I$ is connected. We write $Z = U \sqcup V$ where U and V are open subsets. Without loss of generality we can assume that $z \in U$. For each $i \in I$ we have a decomposition $B_i = (U \cap B_i) \sqcup (V \cap B_i)$ into disjoint open subsets (here we use the definition of the subspace topology). Since B_i is connected and since $U \cap B_i \neq \varnothing$ we see that $B_i = U \cap B_i$. Thus we have shown that

$$U \;=\; \bigcup_{i\in I} (B_i \cap U) \;=\; \bigcup_{i\in I} B_i \;=\; Z. \qquad \blacksquare$$

2.7. **The (path-) components of a topological space.** This section is a follow-up on the previous section. Let us start with the following simple-minded definition.

Definition. Let x and y be two points in a topological space X. We say that the points x and y are path-equivalent, if there exists a continuous map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Lemma 2.20. (Path-Equivalence Lemma) For every topological space X the notion of path-equivalence is indeed an equivalence relation.

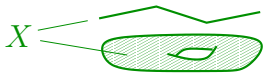
Proof. All properties of an equivalence relation are trivial except possibly for transitivity. The fact that path-equivalence is transitive can be shown by concatenating paths. More precisely, if $\alpha\colon [0,1] \to X$ is a path from $P = \alpha(0)$ to $Q = \alpha(1)$ and if $\beta\colon [0,1] \to X$ is a

path from $Q = \beta(0)$ to $R = \beta(1)$, then the concatenation $\alpha * \beta \colon [0,1] \to X$ is a path from P to R. ■

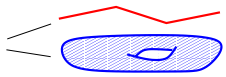
The Path-Equivalence Lemma 2.20 allows us to introduce the following definition.

Definition. Let X be a topological space.

(1) We call the path-equivalence classes of points the path-components of X.
(2) We denote by $\pi_0(X)$ the set of path-components of X.[42] Furthermore we write $\pi_0(X) = 0$ if X is non-empty and if it consists of a single path-component.



the two path-components of X



Example. It is a straightforward consequence of the Intermediate Value Theorem that each point in $\mathbb{Q}$ defines a path-component of $\mathbb{Q}$.

Lemma 2.21. (Path-Component Lemma) Let X be a topological space.

(1) As a set, X is the disjoint union of its path-components.
(2) Each path-component of X is path-connected.
(3) Each path-component C of X is a maximal path-connected subset, i.e. if $C \subsetneq D$, then D is not path-connected.
(4) Let A be a subset of X.
  (a) If no point in A is path-equivalent to a point in $X \setminus A$, then A is the union of path-components of X.
  (b) If A is path-connected, then A is contained in a path-component of X.
  (c) If A is path-connected and if no point in A is path-equivalent to a point in $X \setminus A$, then A is a path-component of X.

Proof. The lemma follows immediately from the definitions and basic properties of equivalence relations, see the Equivalence Class Lemma 0.4. We leave it to the reader to fill in the few remaining details. ■

Example. We consider the topological space $\mathbb{R} \setminus \{0\}$. The subsets $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are evidently path-connected. It follows from the Intermediate Value Theorem that no point in $\mathbb{R}_{<0}$ is path-equivalent to a point in $\mathbb{R}_{>0}$. Thus it follows from the Path-Component Lemma 2.21 (4c) that $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are the path-components of $\mathbb{R} \setminus \{0\}$.

2.8. The path-components of matrix groups.

In this section we will discuss the (path-) connectedness of certain matrix groups over $\mathbb{R}$ and over $\mathbb{C}$. First we introduce the main players.

Notation. Given a commutative ring R we write

$$\begin{aligned} \mathrm{GL}(n,R) &:= \{A \in \mathrm{M}(n \times n, R) \mid \det(A) \text{ is a multiplicative unit in } R\} \\ \mathrm{SL}(n,R) &:= \{A \in \mathrm{M}(n \times n, R) \mid \det(A) = 1\}. \end{aligned}$$

[42] The notation "$\pi_0(X)$" might look rather odd at the moment, but it will become clearer later on, when we introduce the fundamental groups and the higher homotopy groups, why this is a perfectly sensible notation. At that point even the weird definition of "$\pi_0(X) = 0$" will start making more sense.

For $\mathbb{K} = \mathbb{R}$ we have the following extra definitions:

$$\begin{aligned} \mathrm{GL}_+(n,\mathbb{R}) &:= \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid \det(A) > 0\} \\ \mathrm{GL}_-(n,\mathbb{R}) &:= \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid \det(A) < 0\} \\ \mathrm{O}(n) &:= \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid A^T \cdot A = \mathrm{id}\} \\ \mathrm{SO}(n) &:= \mathrm{O}(n) \cap \mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid A^T \cdot A = \mathrm{id} \text{ and } \det(A) = 1\}. \end{aligned}$$

Furthermore for $\mathbb{K} = \mathbb{C}$ we have the following extra definitions:

$$\begin{aligned} \mathrm{U}(n) &:= \{A \in \mathrm{M}(n\times n,\mathbb{C}) \mid \overline{A}^T \cdot A = \mathrm{id}\} \\ \mathrm{SU}(n) &:= \mathrm{U}(n) \cap \mathrm{SL}(n,\mathbb{C}) = \{A \in \mathrm{M}(n\times n,\mathbb{C}) \mid \overline{A}^T \cdot A = \mathrm{id} \text{ and } \det(A) = 1\}. \end{aligned}$$

The following convention explains in what sense these matrix groups are topological spaces.

Identification. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Given $m, n \in \mathbb{N}$ we make the identification

$$\mathrm{M}(m\times n,\mathbb{K}) =_i \mathbb{K}^{m\cdot n}$$

by "juxtaposing" the rows of the matrix. We equip $\mathbb{K}^{m\cdot n}$ and thus also $\mathrm{M}(m\times n,\mathbb{K})$ with the standard topology that we introduced on page 39. We equip each subset of $\mathrm{M}(m\times n,\mathbb{K})$ with the corresponding subspace topology that we introduced on page 40.

Lemma 2.22. (Matrix Group Lemma) Let $n \in \mathbb{N}$.

(1) $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}_+(n,\mathbb{R})$ and $\mathrm{GL}_-(n,\mathbb{R})$, are open subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(2) $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$, are closed subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(3) $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are compact.
(4) $\mathrm{GL}(n,\mathbb{C})$ is an open subset of $\mathrm{M}(n\times n,\mathbb{C})$.
(5) $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$, are closed subsets of $\mathrm{M}(n\times n,\mathbb{C})$.
(6) $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are compact.

Proof. We consider the following two maps

$$\det\colon \begin{array}{ccc} \mathrm{M}(n\times n,\mathbb{R}) & \to & \mathbb{R} \\ A & \mapsto & \det(A) \end{array} \quad \text{and} \quad \varphi\colon \begin{array}{ccc} \mathrm{M}(n\times n,\mathbb{R}) & \to & \mathrm{M}(n\times n,\mathbb{R}) \\ A & \mapsto & A^T \cdot A. \end{array}$$

It follows from the Leibniz formula and the rather elementary the Sum-Product-Continuity Lemma 2.6 that both maps are continuous.

(1) It follows from the above that $\mathrm{GL}(n,\mathbb{R}) = \det^{-1}(\mathbb{R}\setminus\{0\})$, $\mathrm{GL}_+(n,\mathbb{R}) = \det^{-1}((0,\infty))$ and $\mathrm{GL}_-(n,\mathbb{R}) = \det^{-1}((-\infty,0))$ are open subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(2) Note that it follows from the above and the Continuity-via-Closed Subsets Lemma 2.3 that $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(\{1\})$, $\mathrm{O}(n) = \varphi^{-1}(\{\mathrm{id}\})$ and $\mathrm{SO}(n) = \mathrm{SL}(n,\mathbb{R}) \cap \mathrm{O}(n,\mathbb{R})$ are closed subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(3) Note that each column of an orthogonal matrix has norm one. This implies that $\mathrm{O}(n) \subset \mathbb{R}^{n^2}$ is a bounded subset of $\mathrm{M}(n\times n,\mathbb{R}) = \mathbb{R}^{n^2}$. It follows from this observation together with (2) and the Heine-Borel Theorem 1.11 that $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are compact.

The proofs of Statements (4), (5) and (6) are almost identical to the above proofs. ■

The following proposition will hopefully quench the reader's thirst for information on the (path-) connectedness of the matrix groups.

Proposition 2.23. (Matrix Group-π_0-Proposition) Let $n \in \mathbb{N}$.

	topological space	path-components
(1)	$\mathrm{GL}(n,\mathbb{R})$	path-components are $\mathrm{GL}_+(n,\mathbb{R})$ and $\mathrm{GL}_-(n,\mathbb{R})$
(2)	$\mathrm{SL}(n,\mathbb{R})$	path-connected
(3)	$\mathrm{O}(n)$	path-components are $\mathrm{SO}(n)$ and $\mathrm{O}(n) \cap \mathrm{GL}_-(n,\mathbb{R})$
(4)	$\mathrm{GL}(n,\mathbb{C})$	path-connected
(5)	$\mathrm{SL}(n,\mathbb{C})$	path-connected
(6)	$\mathrm{U}(n)$	path-connected
(7)	$\mathrm{SU}(n)$	path-connected.

Proof. The proof is provided in [Fri23]. ■

We turn to the proof of Proposition 2.23 for the matrix groups that are defined over $\mathbb{C}$:

Proof of Proposition 2.23 (4), (5), (6) and (7).

Claim 4.

(1) Given any $A \in \mathrm{GL}(n,\mathbb{C})$ there exists a path $\delta\colon [0,1] \to \mathrm{GL}(n,\mathbb{C})$ with $\gamma(0) = A$ and $\gamma(1) \in \mathrm{SL}(n,\mathbb{C})$.
(2) If $A \in \mathrm{U}(n)$, then we can find such a path that takes values in $\mathrm{U}(n)$.

Proof. Let $A \in \mathrm{GL}(n,\mathbb{C})$. We pick $r \in \mathbb{R}_{>0}$ and $\varphi \in \mathbb{R}$ with $\det(A) = r \cdot \exp(\mathrm{i}\varphi)$. The map

$$\begin{array}{rcl} \delta\colon [0,1] & \to & \mathrm{GL}(n,\mathbb{C}) \\ t & \mapsto & r^{-t/n} \cdot \exp(-\mathrm{i}\varphi \cdot t/n) \cdot A \end{array}$$

is a path from A to a matrix in $\mathrm{SL}(n,\mathbb{C})$. If $A \in \mathrm{U}(n)$, then $r = 1$ and we see that the path takes values in $\mathrm{U}(n)$. ⊞

Using Claim 4 one sees that it suffices to show that $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SU}(n)$ are connected. This is achieved by making straightforward modifications to the Claims 2 and 3 that we considered in the proof of the real case. We leave it to the reader to fill in the details. ■

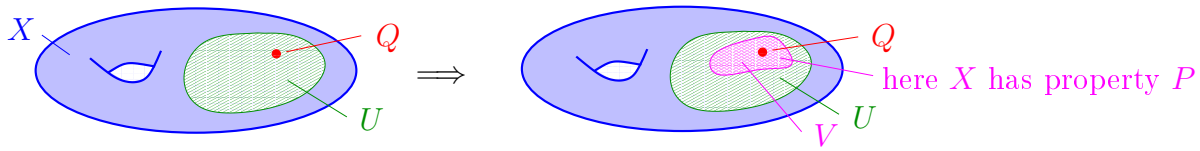
2.9. Local and regional properties.

As so often, let us start out with a definition.

Definition. Let P be a property of topological spaces and let X be a topological space.

(1) We say X is regionally P if given any $Q \in X$ and any neighborhood U of Q there exists a neighborhood V of Q that is contained in U that has the property P.
(2) We say X is locally P if given any $Q \in X$ and any neighborhood U of Q there exists an *open* neighborhood V of Q that is contained in U that has the property P.

Note that by definition a topological space that is locally P is also regionally P.



Warning. Our usage of the expression "locally P" is non-standard. In fact most textbooks, see e.g. [Hat02; Bre93; Rot88], define "locally P" as what we call "regionally P". (Note though that, as is apparent from [Mun75, p. 161], in [Mun75] our notion of "locally P" is being used.) The property that we now call "locally P" does not seem to have an established

name. For our purpose it is useful to give names to both concepts since both have their uses and justifications. Later on we will also introduce notion of a local property of a map, in this context most books implicitly or explicitly seem to work with open neighborhoods. Thus it seems reasonable to us to use the word "local" whenever we deal with open neighborhoods and to rename the concept when any neighborhood is allowed.

Examples.

(1) Every open subset of $\mathbb{R}^n$ is locally path-connected. Indeed, suppose A is an open subset of $\mathbb{R}^n$. Let $Q \in A$ and let U be a neighborhood of Q in A. Then there exists an $\epsilon > 0$ such that $B^n_\epsilon(x) \subset U$. The ϵ-ball $B^n_\epsilon(x)$ is of course path-connected.

(2) The topological space $X = (-2, 1) \cup [3, 6] \subset \mathbb{R}$ with the usual topology is not path-connected but it is locally path-connected.[43] On the other hand $X = \mathbb{Q}$ is not even locally path-connected.

(3) The line with two zeros, i.e. the topological space $\mathbb{R} \cup \{*\}$ with the topology from page 43, is not Hausdorff, but one can easily show that $\mathbb{R} \cup \{*\}$ is locally Hausdorff.

The following lemma is the key to showing that for "reasonable" topological spaces the notions of being connected and being path-connected are actually the same.

Lemma 2.24. (Locally Path-Connected Lemma) Let X be a topological space. If X is locally path-connected, then X is connected if and only if X is path-connected.

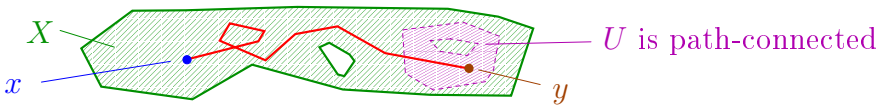
Proof. First note that if X is path-connected, then we know by the Path-Connected Implies Connected Corollary 2.15 that X is connected. This gives us the cheap "if"-direction.

We now assume that X is connected. We start out with the following claim.

Claim 1. Each path-component of X is open.

Proof. Let P be a path-component. We pick $x \in X$. To show that P is open it suffices by the JH-Openness Criterion 1.1 to show that each $y \in P$ admits a neighborhood that is contained in P.

Thus let $y \in P$. We pick a path $p\colon [0, 1] \to X$ with $p(0) = x$ and $p(1) = y$. Since X is locally path-connected there exists an open neighborhood U of y that is path-connected. It follows from the Image-Connected Lemma 2.16 (1) together with the Union Connected Lemma 2.19 that $p([0, 1]) \cup U$ is path-connected. It follows from the Path-Component Lemma 2.21 (4b) that $p([0, 1]) \cup U \subset P$. Finally note that since U is open we know that $p([0, 1]) \cup U$ is a neighborhood of x. ⊞



Claim 2. Each path-component of X is closed.

Proof. By Claim 1 we now know that each path-component is open. By the Path-Component Lemma 2.21 (1) we know that the complement of a path-component is the union of path-components. Since the union of open subsets is open we see that each path-component is also closed. ⊞

[43] Why is that?

Since X is connected we obtain from Claim 1 and Claim 2 that each path-component equals X, i.e. X is path-connected. ■

In the following we discuss the notion of being regionally compact.

Examples.

(1) Every open subset of $\mathbb{R}^n$ is regionally compact. Indeed, suppose A is an open subset of $\mathbb{R}^n$. Let $Q \in A$ and let U be a neighborhood of Q in A. Then there exists an $\epsilon > 0$ such that $B^n_\epsilon(x) \subset U$. The closed ϵ-ball $\overline{B}^n_{\frac{\epsilon}{2}}(x)$ is a neighborhood of x and it is compact by the Heine-Borel Theorem 1.11.

(2) Similar to (1) one can easily show that every interval is regionally compact. In fact later on we will mostly need the fact that the interval $[0, 1]$ is regionally compact.

Next we turn from properties of topological spaces to properties of maps.

Definition. Let P be a property of maps between topological spaces and let $f\colon X \to Y$ be a map between topological spaces.

(1) We say that $f\colon X \to Y$ is locally P at $x \in X$ if there exists an open neighborhood U of x and an open neighborhood V of $f(x)$ such that $f\colon U \to V$ has the property P.

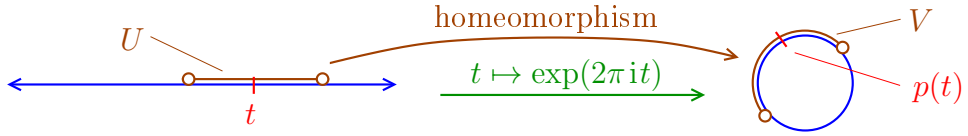
(2) We say that f is locally P if it is locally P at any $x \in X$.

Example.

(1) The map

$$\begin{array}{rcl} p\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(2\pi \mathrm{i} t) \end{array}$$

is a local homeomorphism. Indeed, let $t \in \mathbb{R}$. We set $U = (t - \frac{1}{4}, t + \frac{1}{4})$ and $V = p(U)$. The restriction of p to $p\colon U \to V$ is continuous and a bijection. Furthermore it is not difficult to show that p is an open map. It follows from the Open-Injective Lemma 2.9 (2) that $p\colon U \to V$ is a homeomorphism.



(2) The statement of the Pasting Proposition 2.4 (1) can be reformulated as saying that a map between two topological spaces is continuous, if it is locally continuous.

3. Categories, functors and natural transformations

In the last two chapters we introduced the concept of topological spaces, we studied several natural examples and we discussed various basic concepts. We will now take a short break from discussing topological spaces.

In this chapter we introduce the notion of a category, of a functor and of a natural transformation that many readers will already have encountered in different courses. In the subsequent chapter we will introduce the notions of (co-) products and (co-) limits in categories. These notions might initially appear to be rather abstract, but they play an essential role in topology and they are regularly used in many branches of pure mathematics. These concepts will also guide us for how to proceed with the discussion of topological spaces.

3.1. **Definition and examples of categories.** We start out with the definition of a category. As we will see, the notion of a category is extremely versatile and encompasses many situations.

Definition. A category $\mathcal{C}$ consists of the following data:

(1) A class[44] $\mathrm{Ob}(\mathcal{C})$, the elements of which are called the objects of the category.

(2) For each pair (X, Y) of objects there exists a set[45] [46] $\mathrm{Mor}_{\mathcal{C}}(X, Y)$, the elements of which are called the morphisms from X to Y.

(3) For any three objects X, Y and Z we are provided a map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{C}}(X,Y) \times \mathrm{Mor}_{\mathcal{C}}(Y,Z) & \to & \mathrm{Mor}_{\mathcal{C}}(X,Z) \\ (f,g) & \mapsto & g \circ f, \end{array}$$

called composition map, that satisfies the following axioms:

(A) (associativity): For every $f \in \mathrm{Mor}_{\mathcal{C}}(W,X), g \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ and $h \in \mathrm{Mor}_{\mathcal{C}}(Y,Z)$ we have

$$(h \circ g) \circ f \;=\; h \circ (g \circ f) \;\in\; \mathrm{Mor}_{\mathcal{C}}(W,Z).$$

(B) (identity): For every object X there exists a morphism $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{C}}(X,X)$ with the property that

(i) $\mathrm{id}_X \circ f = f$ for all $W \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Mor}_{\mathcal{C}}(W,X)$, and

(ii) $f \circ \mathrm{id}_X = f$ for all $Y \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$.

Before we give a long list of examples let us first state a lemma which will look familiar from a basic algebra course.

---

[44] As many "working mathematicians" the author has a slightly guilty conscience for not knowing what a "set" and what a "class" is. Therefore the reader is referred to [Cie97; FP85] for details. Truth be told, most mathematicians learn from logicians at some point when to say "class" instead of "set", but many mathematicians do not know exactly what they are doing at that point.

[45] One could ask why $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ has to be a set, why not just a class? To the best of my knowledge the axioms on morphisms that we write down also make sense if $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ is only required to be a class. Here's a long discussion which gives a satisfactory answer to people in category theory:

`http://mathoverflow.net/questions/48810/why-need-the-morphisms-to-form-a-set`

I myself do not understand the answer.

[46] If the category is clear from the context, then we just write $\mathrm{Mor}(X,Y)$ instead of $\mathrm{Mor}_{\mathcal{C}}(X,Y)$.

Lemma 3.1. (Identity-Uniqueness Lemma) Let $\mathcal{C}$ be a category. For every $X \in \mathrm{Ob}(\mathcal{C})$ there exists a *unique* morphism $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{C}}(X,X)$ with the property that

(i) $\mathrm{id}_X \circ f = f$ for all $W \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Mor}_{\mathcal{C}}(W,X)$,
(ii) $f \circ \mathrm{id}_X = f$ for all $X \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$.

Proof. The existence of id_X is given by property (3B) of a category. Now suppose that $I \in \mathrm{Mor}_{\mathcal{C}}(X,X)$ has the same property. Then we have

$$\mathrm{id}_X \underset{\substack{\uparrow \\ \text{property (i) of } I \text{ applied to } f=\mathrm{id}_X}}{=} I \circ \mathrm{id}_X \underset{\substack{\uparrow \\ \text{property (ii) of } \mathrm{id}_X \text{ applied to } f=I}}{=} I.$$

■

The Identity-Uniqueness Lemma 3.1 allows us to introduce the following definition.

Definition. Let $\mathcal{C}$ be a category and let $X \in \mathrm{Ob}(\mathcal{C})$. We refer to the morphism that is uniquely defined in the Identity-Uniqueness Lemma 3.1 as the identity morphism id_X.

In the following we introduce all kinds of categories. Many of those will be faithful companions throughout these notes.

Definition.

(a) We refer to the category $\mathcal{Set}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Set}) &:= \text{all sets}, \\ \mathrm{Mor}(X,Y) &:= \text{all maps from } X \text{ to } Y, \end{aligned}$$

and together with the usual composition of maps as the category of sets.

(b) We refer to the category $\mathcal{Top}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Top}) &:= \text{all topological spaces}, \\ \mathrm{Mor}(X,Y) &:= \text{all continuous maps from } X \text{ to } Y, \end{aligned}$$

with the usual composition of maps[47] as the category of topological spaces.

(c) Let $\mathbb{K}$ be a field. We refer to the category $\mathcal{Vec}_{\mathbb{K}}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Vec}_{\mathbb{K}}) &:= \text{all } \mathbb{K}\text{-vector spaces}, \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}_{\mathbb{K}}(X,Y) = \text{all } \mathbb{K}\text{-homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of homomorphisms as the category of $\mathbb{K}$-vector spaces.

(d) We refer to the category $\mathcal{Group}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Group}) &:= \text{all groups}, \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}(X,Y) = \text{all group homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of group homomorphisms as the category of groups. Similarly we define the category $\mathcal{AbGroup}$ of abelian groups, the category $\mathcal{Ring}$ of rings and the category $\mathcal{CRing}$ of commutative rings.

(e) A pair of sets is a pair (X,A), where X is a set and A is a subset of X. We refer to the category $\mathcal{PairSet}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{PairSet}) &:= \text{all pairs of sets}, \\ \mathrm{Mor}((X,A),(Y,B)) &:= \text{all maps } f \text{ from } X \text{ to } Y \text{ with } f(A) \subset B \end{aligned}$$

with the usual composition of maps as the category of pairs of sets. With the same approach we could also introduce the category of pairs of topological spaces and so on.

The name "morphisms" suggests that morphisms are necessarily maps. But as we will see in the following two examples, this is not necessarily always the case.

Example.

(f) Let G be an arbitrary group. We consider the category $\mathcal{C}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}) &:= \text{the set with the unique element } *, \\ \mathrm{Mor}(*,*) &:= G. \end{aligned}$$

We define

$$\begin{aligned} \mathrm{Mor}(*,*) \times \mathrm{Mor}(*,*) &\to \mathrm{Mor}(*,*) \\ (f,g) &\mapsto g \circ f := \underbrace{g \cdot f}_{\text{multiplication in } G} \end{aligned}$$

via the group structure on $\mathrm{Mor}(*,*) = G$. The axioms of a category can be deduced easily from the group axioms of G.

Our last example deserves being a proper definition again.

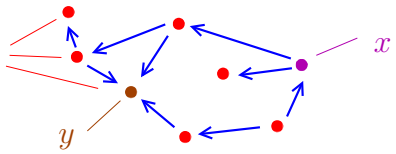
Definition.

(g) Let $(P, \leq)$ be a partially ordered set, as defined on page 24. We fix a set $\{*\}$ consisting of a single element. We define the order category $\mathit{Order}_{(P,\leq)}$ to be the category that is given by

$$\begin{aligned} \mathrm{Ob}(\mathit{Order}_{(P,\leq)}) &= P \\ \mathrm{Mor}(x,y) &= \begin{cases} \{*\}, & \text{if } x \leq y, \\ \varnothing, & \text{otherwise.} \end{cases} \end{aligned}$$

together with the only possible way to compose morphisms.[48]

partially ordered set where $a \leq b$ if there exists a directed path from a to b



$\mathrm{Mor}(x,y) = \{*\}$
$\mathrm{Mor}(y,x) = \varnothing$
$\mathrm{Mor}(x,x) = \{*\}$
$\mathrm{Mor}(y,y) = \{*\}$

Given a category $\mathcal{C}$ we now introduce its doppelgänger, namely the corresponding opposite category $\mathcal{C}^{\mathrm{op}}$. The idea is very simple, we keep the objects, but we "invert the direction of the morphisms". This simple concept can be very convenient in future discussions and gives, almost for free, a rich source of new categories.

Definition. Let $\mathcal{C}$ be a category. The opposite category $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ is defined as the category where

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) &:= \mathrm{Ob}(\mathcal{C}) \\ \mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(X,Y) &:= \mathrm{Mor}_{\mathcal{C}}(Y,X) \end{aligned}$$

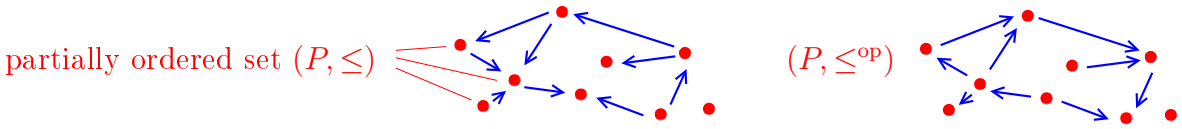
with the composition given by $f \circ_{\mathcal{C}^{\mathrm{op}}} g := g \circ_{\mathcal{C}} f$.

[47] By the Basics-of-Continuity Lemma 2.1 (1) we know that the composition of continuous maps is indeed continuous.

[48] For example, we have $\mathrm{Mor}(x,y) \neq \varnothing$ and $\mathrm{Mor}(y,z) \neq \varnothing$ precisely if $x \leq y$ and $y \leq z$. But since we are dealing with a partial order we have $x \leq z$, which implies that $\mathrm{Mor}(x,z) = \{*\}$. So the composition of the unique morphism in $\mathrm{Mor}(x,z)$ and the unique morphism in $\mathrm{Mor}(y,z)$ is the unique morphism in $\mathrm{Mor}(x,z)$.

Example.

(1) Let $(P, \leq)$ be a partially ordered set. We consider the corresponding order category $\mathit{Order}_{(P,\leq)}$. The opposite of the order category $\mathit{Order}_{(P,\leq)}$ is the order category corresponding to the partially ordered set $(P, \leq^{\mathrm{op}})$ where $x \leq^{\mathrm{op}} y$ if and only if $y \leq x$.



(2) For any category $\mathcal{C}$ we evidently have $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$.

3.2. **Isomorphisms.** In this section we introduce the notion of an isomorphism, which generalizes the notion that is well-known from linear algebra.

Definition. Let $\mathcal{C}$ be a category and let X, Y be two objects in $\mathcal{C}$.

(1) Let $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ be a morphism and let $g \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$.
  (a) We say g is left inverse of f if $g \circ f = \mathrm{id}_X$.
  (b) We say g is right inverse of f if $f \circ g = \mathrm{id}_Y$.
  (c) We say g is an inverse of f if g is a left and a right inverse of f.
(2) A morphism $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ is called invertible, or an isomorphism, if it admits an inverse.
(3) Two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ are called isomorphic, if there exists an isomorphism in $\mathrm{Mor}_{\mathcal{C}}(X, Y)$.
(4) Given an object $X \in \mathrm{Ob}(\mathcal{C})$ we refer to the class of all objects that are isomorphic to X as the isomorphism type of X.

Example.

(1) In the category of vector spaces the invertible morphisms are precisely the isomorphisms which are familiar from linear algebra.
(2) In the category of sets the isomorphisms are precisely the bijections.
(3) In the category of topological spaces the isomorphisms are precisely the homeomorphisms.

Lemma 3.2. (Inverse Uniqueness Lemma) Let $\mathcal{C}$ be a category.

(1) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$. If f admits a left inverse $\widetilde{g}$ and a right inverse g, then they agree, i.e. $g = \widetilde{g}$. In particular f is invertible.
(2) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathrm{Ob}(\mathcal{C})$. If f admits an inverse, then the inverse is unique.

Definition. Let $\mathcal{C}$ be a category, let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)_{\mathrm{inv}}$ be an invertible morphism. By the Inverse Uniqueness Lemma 3.2 there exists a *unique* morphism $g \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. We refer to this morphism as the inverse of f and usually we denote it by f^{-1}.

Proof of Lemma 3.2.

(1) Let $g \in \mathrm{Mor}_{\mathcal{C}}(Y,X)$ be a morphism with $f \circ g = \mathrm{id}_Y$ and let $\widetilde{g} \in \mathrm{Mor}_{\mathcal{C}}(Y,X)$ be a morphism with $\widetilde{g} \circ f = \mathrm{id}_X$. We have the following equalities of morphisms:

$$g \underset{\substack{\uparrow\\ \text{defining}\\ \text{property of } \mathrm{id}_X}}{=} \mathrm{id}_X \circ g \underset{\substack{\uparrow\\ \text{defining}\\ \text{property of } \widetilde{g}}}{=} (\widetilde{g} \circ f) \circ g \underset{\substack{\uparrow\\ \text{associativity of ``}\circ\text{''}}}{=} \widetilde{g} \circ (f \circ g) \underset{\substack{\uparrow\\ \text{defining}\\ \text{property of } g}}{=} \widetilde{g} \circ \mathrm{id}_Y \underset{\substack{\uparrow\\ \text{defining}\\ \text{property of } \mathrm{id}_Y}}{=} \widetilde{g}.$$

(2) Suppose we have $g, \widetilde{g} \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$ and with $f \circ \widetilde{g} = \mathrm{id}_Y$ and $\widetilde{g} \circ f = \mathrm{id}_X$. It follows immediately from (1) that $g = \widetilde{g}$. ■

Lemma 3.3. (Isomorphism Lemma) Let $\mathcal{C}$ be a category.

(1) Let $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(Y,Z)$ be morphisms. If two morphisms out of $f, g, g \circ f$ are invertible, then so is the third and we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

(2) "Being isomorphic" is an equivalence relation in the class of objects of $\mathcal{C}$.[49]

Proof. The two statements are basically trivial. We leave it to the nervous reader to fill in the details. ■

3.3. **Monomorphisms and epimorphisms.** In the context of categories the following definition is very natural.

Definition. Let $\mathcal{C}$ be a category.

(1) A morphism $g \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ is called a monomorphism if for every two morphisms f_1, f_2 in $\mathrm{Mor}_{\mathcal{C}}(W,X)$ with $g \circ f_1 = g \circ f_2$ we also have $f_1 = f_2$.

(2) A morphism $g \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ is called an epimorphism if for every two morphisms h_1, h_2 in $\mathrm{Mor}_{\mathcal{C}}(Y,Z)$ with $h_1 \circ g = h_2 \circ g$ we also have $h_1 = h_2$.

Of course the names "monomorphism" and "epimorphism" sound familiar. We will now see that in many, but perhaps not all settings, the new notions coincide with the familiar notions.

Proposition 3.4. (Monomorphism Proposition) Suppose we are in one of the following categories:

(1) The category *Group* of groups.

(2) The category *AbGroup* of abelian groups.

(3) The category $\mathcal{V}ec_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces, where $\mathbb{K}$ is a fixed field.

(4) The category *Ring* of rings.

If $\varphi\colon G \to H$ is a monomorphism in any of the above categories, then φ, viewed as a map between sets, is injective.

Proof. Let $\varphi\colon G \to H$ be a monomorphism in one of the first three categories. We set $K := \ker(\varphi) := \{g \in G \mid \varphi(g) = e\}$, where e is the trivial element of the group or the zero

[49] The perspicacious reader might notice that here things get logically a little dicey. More precisely, on page 21 we had recalled the notion of an equivalence relation. But we had only defined the notion of an "equivalence relation" on a set and "all topological spaces" do not form a set they "only" form a class. One can also define the notion of an "equivalence relation" on a class, see e.g. [FP85, Kapitel 2]. But we do not want to get into the discussion "what is a class?". As a cheap way out we interpret the second statement of the lemma as saying that reflexivity, symmetry and transitivity hold for "homotopy equivalence" with the obvious interpretation.

in the vector space. Note that in the first three categories we listed K is again an object of the category and φ is injective if and only if K is trivial.

We consider the trivial homomorphisms $\alpha\colon K \to G$ and we consider the inclusion homomorphism $\beta\colon K \to G$. Note that $\varphi \circ \alpha = \varphi \circ \beta$. Since φ is a monomorphism we see that $\alpha = \beta$. But this implies that K is trivial.

Finally we turn to the category *Ring* of rings. Note that the argument above does not apply since the multiplicatively neutral element never lies in the kernel, which implies that the kernel of a ring homomorphism is not a ring. Therefore we have to work a little harder. Let $\varphi\colon A \to B$ be a monomorphism in the category *Ring* of rings and let $a_1, a_2 \in A$ with $\varphi(a_1) = \varphi(a_2)$. We consider the polynomial ring $\mathbb{Z}[x]$ and we consider the two homomorphisms $f_1, f_2\colon \mathbb{Z}[x] \to A$ that are defined by $f_1(x) = a_1$ and $f_2(x) = a_2$. Then $(\varphi \circ f_1)(x) = (\varphi \circ f_2)(x)$ and hence $\varphi \circ f_1 = \varphi \circ f_2\colon \mathbb{Z}[x] \to B$. Since φ is a monomorphism we see that $f_1 = f_2$ and thus $x = y$. ■

Proposition 3.5. (Epimorphism Proposition) Suppose we are in one of the following categories:

(1) The category *Group* of groups.
(2) The category *AbGroup* of abelian groups.
(3) The category $\mathcal{V}ec_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces, where $\mathbb{K}$ is a fixed field.

If $\varphi\colon G \to H$ is an epimorphism in any of the above categories, then φ, viewed as a map between sets, is surjective.

Example. The perspicacious reader will have noticed that in the Epimorphism Proposition 3.5 we do not say anything about the category of rings. In fact one can easily show that in the category *Ring* of rings the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism.

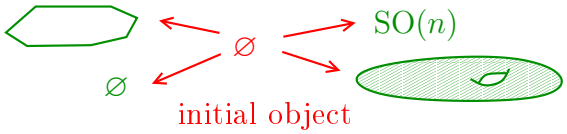
3.4. **Initial and terminal objects.** In this very short section we study the following definition.

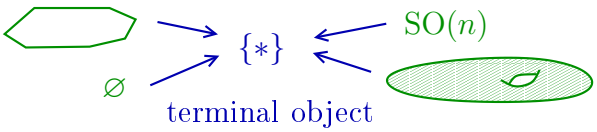
Definition. Let $\mathcal{C}$ be a category.

(1) An object $X \in \mathrm{Ob}(\mathcal{C})$ is called initial if for each object $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism from X to Y.
(2) An object $X \in \mathrm{Ob}(\mathcal{C})$ is called terminal if for each object $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism from Y to X.

Example.

(1) In the category *Set* of sets an initial object is given by the empty set and a terminal object is given by any set consisting of a single element. Similarly in the category *Top* of topological spaces an initial object is given by the empty topological space and a terminal object is given by a topological space consisting of a single element.





(2) In the category *Group* of groups any group consisting of a single element is an initial and also a terminal object. The same applies to the category *AbGroup* of abelian groups and the analogous statement holds in the category $\mathcal{V}ec_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$.

(3) Let X be an initial object of a category $\mathcal{C}$. It follows easily from the definitions that it is also a terminal object of the opposite category $\mathcal{C}^{\mathrm{op}}$.

Lemma 3.6. (Initial-Terminal Object Uniqueness Lemma) Let $\mathcal{C}$ be a category.
(1) If an initial object exists, then it is unique up to isomorphism.
(2) If a terminal object exists, then it is unique up to isomorphism.

Proof. The statements follow easily from the definition. ∎

3.5. **Covariant functors.** In this and the following section we introduce the all-important notions of covariant and contravariant functors.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of a map

$$F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$$

and for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ of a map

$$F\colon \mathrm{Mor}_{\mathcal{C}}(X,Y) \to \mathrm{Mor}_{\mathcal{D}}(F(X),F(Y)),$$

such that the following axioms are satisfied:
(F1) For all $X \in \mathrm{Ob}(\mathcal{C})$ we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
(F2) For all $\phi \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ and all $\psi \in \mathrm{Mor}_{\mathcal{C}}(Y,Z)$ the following equality holds

$$F(\psi \circ \phi) = F(\psi) \circ F(\phi) \quad \in \mathrm{Mor}_{\mathcal{D}}(F(X),F(Z)).$$

Convention. If the covariant functor F is understood from the context, then given a morphism $\phi \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ we often write ϕ_* instead of $F(\phi)$. The first axiom (F1) then becomes $(\mathrm{id}_X)_* = \mathrm{id}_{F(X)}$ and the second axiom (F2) becomes $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

Examples.

(1) The map

$$\begin{aligned} F\colon \mathrm{Ob}(\mathcal{V}ec_{\mathbb{R}}) &\to \mathrm{Ob}(\mathcal{S}et) \\ (V,+,\cdot) &\mapsto V, \end{aligned}$$

together with the maps

$$\begin{aligned} \mathrm{Mor}_{\mathcal{V}ec_{\mathbb{R}}}(V,W) &\to \mathrm{Mor}_{\mathcal{S}et}(V,W) \\ (\varphi\colon V \to W) &\mapsto (\varphi\colon V \to W) \end{aligned}$$

define a covariant functor from the category $\mathcal{V}ec_{\mathbb{R}}$ of real vector spaces to the category $\mathcal{S}et$ of sets. This example can be generalized to many contexts, for example there is a similar covariant functor from the category of topological spaces to the category of sets. Functors of this type are colloquially called *forgetful functor*.

(2) Given any category there is the identity functor with the obvious definition. Funnily enough this functor will soon, namely on page 79, play an interesting role.

(3) Let $\mathcal{T}op$ be the category of topological spaces and let $\mathcal{S}et$ be the category of sets. One can easily show that the map

$$\begin{aligned} F\colon \mathrm{Ob}(\mathcal{T}op) &\to \mathrm{Ob}(\mathcal{S}et) \\ X &\mapsto \pi_0(X) := \text{set of path-components of } X, \end{aligned}$$

and the maps

$$\begin{aligned} \mathrm{Mor}_{\mathcal{T}op}(X,Y) &\to \mathrm{Mor}_{\mathcal{S}et}(\pi_0(X),\pi_0(Y)) \\ f &\mapsto \left(\begin{array}{ccc} \pi_0(X) & \to & \pi_0(Y) \\ C & \mapsto & \text{unique path-component} \\ & & \text{of } Y \text{ that contains } f(C) \end{array} \right) \end{aligned}$$

define a covariant functor from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{Set}$ of sets.

(4) The maps
$$\begin{array}{rcl} X & \mapsto & (X, \text{discrete topology}) \\ f & \mapsto & f \end{array}$$
define a covariant functor from the category $\mathcal{Set}$ of sets to the category $\mathcal{Top}$ of topological spaces. Evidently we can also replace the discrete topology by the trivial topology.

We state our next example of a covariant functor as a lemma.

Lemma 3.7. (Fixed Domain-Covariant Functor Lemma) Let $\mathcal{D}$ be a category and let $\mathcal{Set}$ be the category of sets. Furthermore let $W \in \mathrm{Ob}(\mathcal{D})$. The map
$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{D}) & \to & \mathrm{Ob}(\mathcal{Set}) \\ X & \mapsto & \mathrm{Mor}_{\mathcal{D}}(W, X) \end{array}$$
together with the maps
$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{D}}(X, Y) & \to & \mathrm{Mor}_{\mathcal{Set}}(\mathrm{Mor}_{\mathcal{D}}(W, X), \mathrm{Mor}_{\mathcal{D}}(W, Y)) \\ \phi & \mapsto & \begin{pmatrix} \mathrm{Mor}_{\mathcal{D}}(W, X) & \to & \mathrm{Mor}_{\mathcal{D}}(W, Y) \\ g & \mapsto & \phi \circ g \end{pmatrix} \end{array}$$
is a covariant functor from the category $\mathcal{D}$ to the category $\mathcal{Set}$ of sets.

Proof. The statement follows easily from the axioms of a category. ∎

Example. The example of a covariant functor in the Fixed Domain-Covariant Functor Lemma 3.7 comes with many variations, in particular if $\mathrm{Mor}_{\mathcal{D}}(W, X)$ has more structure than that of a set. For instance let W be a real vector space. Then the above map
$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) & \to & \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ V & \mapsto & \underbrace{\mathrm{Hom}_{\mathbb{R}}(W, V)}_{\text{real vector space}}, \end{array}$$
and the maps
$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(U, V) & \to & \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(F(U), F(V)) \\ (\phi\colon U \to V) & \mapsto & \begin{pmatrix} \mathrm{Hom}_{\mathbb{R}}(W, U) & \to & \mathrm{Hom}_{\mathbb{R}}(W, V) \\ g & \mapsto & \phi \circ g \end{pmatrix} \end{array}$$
define a covariant functor from the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces to itself.

3.6. **Contravariant functors.** In this section we introduce the confusing evil cousin of the covariant functor, namely the contravariant functor. We will not make use of this concept in the course Algebraic Topology I. It will play a role later in Algebraic Topology III, when we consider cohomology groups. We only include this discussion for "general culture".

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A contravariant functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of a map
$$F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$$
and for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ of a map
$$\mathrm{Mor}_{\mathcal{C}}(X, Y) \to \mathrm{Mor}_{\mathcal{D}}(F(Y), F(X))$$
such that the following axioms are satisfied:

(F1) For all $X \in \mathrm{Ob}(\mathcal{C})$ we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

(F2) For all $\phi \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ and all $\psi \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$ we have
$$F(\psi \circ \phi) = F(\phi) \circ F(\psi) \quad \in \mathrm{Mor}_{\mathcal{D}}(F(Z), F(X)).$$

The definition of a contravariant functor looks very much the same as the definition of a covariant functor. The difference is that a contravariant functor "reverses the direction of morphisms" and the orders of $F(\psi)$ and $F(\phi)$ in the second axiom (F2) are swapped. In fact, it follows immediately from the definitions that a contravariant functor $\mathcal{C} \to \mathcal{D}$ is the same as a covariant functor $\mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ to the opposite category.

Convention. If the contravariant functor F is understood from the context, then given a morphism $\phi \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ we often write ϕ^* instead of $F(\phi)$. The first axiom then becomes $(\mathrm{id}_X)^* = \mathrm{id}_{F(X)}$ and the second axiom becomes $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

We state our first example as a lemma, namely we have the following analogue of the Fixed Domain-Covariant Functor Lemma 3.7.

Lemma 3.8. (Fixed Target-Contravariant Functor Lemma) Let $\mathcal{D}$ be a category and let $\mathcal{Set}$ be the category of sets. Furthermore let $W \in \mathrm{Ob}(\mathcal{D})$. The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{D}) & \to & \mathrm{Ob}(\mathcal{Set}) \\ X & \mapsto & \mathrm{Mor}_{\mathcal{D}}(X, W) \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{D}}(X, Y) & \to & \mathrm{Mor}_{\mathcal{Set}}(\mathrm{Mor}_{\mathcal{D}}(Y, W), \mathrm{Mor}_{\mathcal{D}}(X, W)) \\ f & \mapsto & \begin{pmatrix} \mathrm{Mor}_{\mathcal{D}}(Y, W) & \to & \mathrm{Mor}_{\mathcal{D}}(X, W) \\ g & \mapsto & g \circ f \end{pmatrix} \end{array}$$

is a contravariant functor from the category $\mathcal{D}$ to the category $\mathcal{Set}$ of sets.

Proof. Again the statement follows easily from the axioms of a category. ∎

Example. The Fixed Target-Contravariant Functor Lemma 3.8 can be viewed as the mother of all contravariant functors. As in the case of the Fixed Domain-Covariant Functor Lemma 3.7 it spawns many variations if $\mathrm{Mor}_{\mathcal{D}}(X, W)$ has more structure than just being a set.

(1) The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{Set}) & \to & \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ X & \mapsto & \mathrm{Map}(X, \mathbb{R}) := \underbrace{\text{set of maps } X \to \mathbb{R}}_{\text{vector space in the obvious way}}, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Set}}(X, Y) & \to & \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(\mathrm{Map}(Y, \mathbb{R}), \mathrm{Map}(X, \mathbb{R})) \\ (\phi\colon X \to Y) & \mapsto & \begin{pmatrix} \mathrm{Map}(Y, \mathbb{R}) & \to & \mathrm{Map}(X, \mathbb{R}) \\ f & \mapsto & f \circ \phi \end{pmatrix} \end{array}$$

is a contravariant functor from the category $\mathcal{Set}$ of sets to the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces.

(2) The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) & \to & \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ V & \mapsto & V^* := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) =: \text{dual vector space of } V, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(U, V) & \to & \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(V^*, U^*) \\ (\phi\colon U \to V) & \mapsto & \begin{pmatrix} \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) & \to & \mathrm{Hom}_{\mathbb{R}}(U, \mathbb{R}) \\ f & \mapsto & f \circ \phi \end{pmatrix} \end{array}$$

is a contravariant functor from the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces to itself.

3.7. **Natural transformations.** In this section we introduce the notion of a natural transformation between two functors. This topic might eventually look even more esoteric, but once one keeps one's eyes open one realizes that the mathematical world is stuffed with natural transformations and natural isomorphisms. They just suffer from the sad fate that often they do not recognized as such.
After this long preamble, let us give the definition of natural transformations.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two covariant functors.

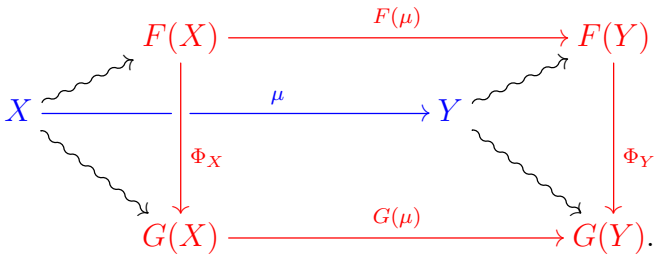
(1) A natural transformation between the functors F and G assigns to each $X \in \mathrm{Ob}(\mathcal{C})$ a morphism $\Phi_X\colon F(X) \to G(X)$ in $\mathcal{D}$ such that for each morphism $\mu\colon X \to Y$ in $\mathcal{C}$ the following diagram of morphisms in $\mathcal{D}$ commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\mu)} & F(Y) \\ {\scriptstyle \Phi_X}\downarrow & & \downarrow{\scriptstyle \Phi_Y} \\ G(X) & \xrightarrow{G(\mu)} & G(Y). \end{array}$$

(2) If for every $X \in \mathrm{Ob}(\mathcal{C})$ the morphism $\Phi_X\colon F(X) \to G(X)$ is an isomorphism, then we refer to the natural transformation as a natural isomorphism. In this case, given $X \in \mathrm{Ob}(\mathcal{C})$ we often say that $F(X)$ and $G(X)$ are naturally isomorphic.[50]

The same way we can of course also define the notion of a natural transformation and a natural isomorphism between two contravariant functors.

Remark. The notion of a natural transformation can also be summarized in the following diagram:



Here the wiggly arrows are not morphisms but they are just meant to remind us which objects a functor associates to the objects X and Y.

Now we will see that we have encountered natural transformations and natural isomorphisms much more often than one might have initially thought.

Example.

(1) Let $\mathbb{K}$ be a field. We consider the category $\mathcal{V}ec_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$. Given a $\mathbb{K}$-vector space V we write $V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ and given $f\colon V \to W$ we denote by $f^*\colon W^* \to V^*$ the induced map. We consider the following two covariant functors from

[50] In the literature the words "natural isomorphism" and "canonical isomorphism" are often used indiscriminately, but whereas the expression "natural isomorphism" has a precise meaning, the expression "canonical isomorphism" is not a well-defined mathematical concept. However, in the lecture notes we will on occasions write that "some object exists canonically". This means that in the proof we show the existence of the object by giving an explicit construction that does not rely on any choices.

$\mathcal{Vec}_{\mathbb{K}}$ to $\mathcal{Vec}_{\mathbb{K}}$:

$$\begin{array}{rcl} \mathrm{id}\colon V & \mapsto & V \\ (f\colon V\to W) & \mapsto & (f\colon V\to W) \end{array} \quad \text{and} \quad \underbrace{\begin{array}{rcl} F\colon V & \mapsto & (V^*)^* \\ (f\colon V\to W) & \mapsto & ((f^*)^*\colon (V^*)^*\to (W^*)^*). \end{array}}_{\text{this is a covariant functor}}$$

For each $\mathbb{K}$-vector space V we denote by Φ_V the homomorphism

$$\begin{array}{rcl} V & \to & (V^*)^* \\ v & \mapsto & \left(\begin{array}{rcl} V^* & \to & \mathbb{K} \\ (\varphi\colon V\to\mathbb{K}) & \mapsto & \varphi(v) \end{array}\right). \end{array}$$

It is straightforward to verify that these maps define a natural transformation from the identity functor $\mathrm{id}\colon \mathcal{Vec}_{\mathbb{K}} \to \mathcal{Vec}_{\mathbb{K}}$ to the functor $F\colon \mathcal{Vec}_{\mathbb{K}} \to \mathcal{Vec}_{\mathbb{K}}$. If we restrict ourselves to the category $\mathcal{Vec}_{\mathbb{K}}^{\dim<\infty}$ of finite-dimensional $\mathbb{K}$-vector spaces, then the above maps show that any finite-dimensional $\mathbb{K}$-vector space V is naturally isomorphic to its double dual $(V^*)^*$.

(2) We consider the covariant functors

$$\begin{array}{rcl} F\colon \mathcal{Ring} & \to & \mathcal{Ring} \\ R & \mapsto & \mathbb{Z} \\ (f\colon R\to S) & \mapsto & (\mathrm{id}\colon \mathbb{Z}\to\mathbb{Z}) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{id}\colon \mathcal{Ring} & \to & \mathcal{Ring} \\ R & \mapsto & R \\ (f\colon R\to S) & \mapsto & (f\colon R\to S). \end{array}$$

For each commutative ring R we consider the ring homomorphism

$$\begin{array}{rcl} \Phi_R\colon \mathbb{Z} & \to & R \\ n & \mapsto & n\cdot 1_R. \end{array}$$

By our definition of a ring homomorphism it follows that for each ring homomorphism $f\colon R\to S$ we have $f(1_R)=1_S$. This implies that the following diagram commutes:

$$\begin{array}{ccc} F(R)=\mathbb{Z} & \xrightarrow{F_*(f)=\mathrm{id}} & \mathbb{Z}=F(S) \\ {\scriptstyle\Phi_R}\downarrow & & \downarrow{\scriptstyle\Phi_S} \\ \mathrm{id}(R)=R & \xrightarrow{\mathrm{id}_*(f)=f} & S=\mathrm{id}(S). \end{array}$$

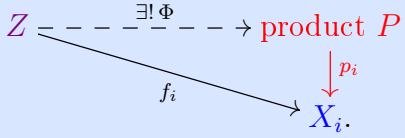
But this shows that the ring homomorphisms Φ_R define a natural transformation from the covariant functor F to the identity functor id. Thus, given a commutative ring, we refer to this ring homomorphism $\mathbb{Z}\to R$ as the *natural ring homomorphism*.

4. Products and Coproducts – Limits and colimits

In this chapter we will continue with our general discussion of categories. First we will introduce the notion of products and coproducts. Afterwards we will discuss the even more general notions of limits and colimits. These notions will guide us in our exploration of topological spaces, which we will resume in the next chapter.

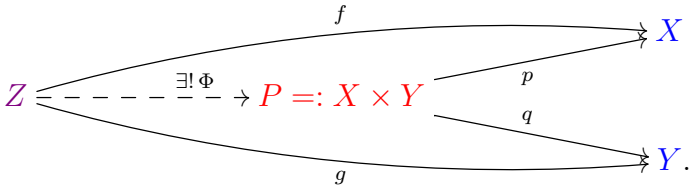
4.1. Products.

Definition. Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. A product of $\{X_i\}_{i\in I}$ in the category $\mathcal{C}$ is an object $P \in \mathrm{Ob}(\mathcal{C})$ together with a family of morphisms $\{p_i\colon P \to X_i\}_{i\in I}$ such that the following universal property is satisfied: Whenever we are given a $Z \in \mathrm{Ob}(\mathcal{C})$ and a family of morphisms $\{f_i\colon Z \to X_i\}_{i\in I}$ there exists a unique morphism $\Phi\colon Z \to P$ such that for each $i \in I$ we have $f_i = p_i \circ \Phi$, i.e. such that the following diagram commutes:[51]



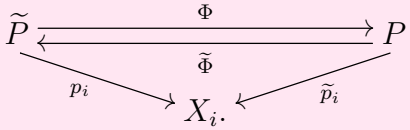
Example. Let $\mathcal{C}$ be a category.

(1) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ be two objects. A product of X and Y in the category $\mathcal{C}$ is some object $P \in \mathrm{Ob}(\mathcal{C})$ together with two morphisms $p\colon P \to X$ and $q\colon P \to Y$ such that the following universal property is satisfied: Whenever we are given an object $Z \in \mathrm{Ob}(\mathcal{C})$ and two morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ there exists a unique morphism $\Phi\colon Z \to P$ such that $f = p \circ \Phi$ and $g = q \circ \Phi$. The situation is summarized in the following diagram:



(2) If we take the empty family of objects in a category $\mathcal{C}$, then it follows easily from the definitions that a corresponding product is the same as a terminal object in the category $\mathcal{C}$.

Lemma 4.1. (Product Uniqueness Lemma) Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. If we are given products $(P, \{p_i\colon P \to X_i\}_{i\in I})$ and $(\widetilde{P}, \{\widetilde{p}_i\colon \widetilde{P} \to X_i\}_{i\in I})$ then there exist uniquely determined isomorphisms $\Phi\colon \widetilde{P} \to P$ and $\widetilde{\Phi}\colon P \to \widetilde{P}$ which are inverses of one another and such that for each $i \in I$ the following diagram commutes:



Sketch of proof. The two morphisms $\Phi\colon \widetilde{P} \to P$ and $\widetilde{\Phi}\colon P \to \widetilde{P}$ come from the universal properties of P and $\widetilde{P}$. By comparing $\mathrm{id}_P\colon P \to P$ and $\Phi \circ \widetilde{\Phi}\colon P \to P$ one sees, by the uniqueness property, that $\mathrm{id}_P = \Phi \circ \widetilde{\Phi}$ and similarly one sees that $\mathrm{id}_{\widetilde{P}} = \widetilde{\Phi} \circ \Phi$, i.e. Φ and $\widetilde{\Phi}$ are isomorphisms. ■

Notation. Let $\mathcal{C}$ be a category. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ such that a product exists.

(1) We usually denote the product by $\prod_{i\in I} X_i$ and we refer to the morphisms $\prod_{i\in I} X_i \to X_i$ as natural projections. Furthermore, given morphisms $f_i\colon Z \to X_i$ we denote the corresponding morphism $Z \to \prod_{i\in I} X_i$ by $\prod_{i\in I} f_i$.

(2) If $I = \{1, \dots, k\}$, then often we denote the product by $X_1 \times \cdots \times X_k$. Furthermore given morphisms $f_i\colon Z \to X_i$, $i = 1, \dots, k$ we denote the corresponding morphism $Z \to X_1 \times \cdots \times X_k$ by $(f_1, \dots, f_k)$.

Lemma 4.2. (Product-of-Sets-and-Groups Lemma)

(1) Let *Set* be the category of sets and let $\{X_i\}_{i\in I}$ be a family of sets. The product in *Set* is precisely the product

$$\prod_{i\in I} X_i := \left\{ f\colon I \to \bigcup_{i\in I} X_i \;\middle|\; \text{for every } i \in I \text{ we have } f(i) \in X_i \right\}$$

as defined on page 20, together with the obvious maps to the X_i that are given by $f \mapsto f(i)$.

(2) Let *Group* be the category of groups and let $\{X_i\}_{i\in I}$ be a family of groups. The product in *Group* is the set

$$\prod_{i\in I} X_i = \left\{ f\colon I \to \bigcup_{i\in I} X_i \;\middle|\; \text{for every } i \in I \text{ we have } f(i) \in X_i \right\}$$

as in (1), which we now view as a group by the obvious "entry wise" multiplication, together with the obvious homomorphisms to X_i.

(3) The same approach as in (2) also applies to the category *AbGroup* of abelian groups and the category $\mathcal{V}ec_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$.

Proof.

(1) So let $Z \in \mathrm{Ob}(\mathcal{S}et)$ and let $\{f_i\colon Z \to X_i\}_{i\in I}$ be a family of maps between sets. We define

$$\begin{array}{rcl} \Phi\colon Z & \to & \prod_{i\in I} X_i \\ z & \mapsto & \begin{pmatrix} I & \to & \bigcup_{i\in I} X_i \\ i & \mapsto & f_i(z) \end{pmatrix}. \end{array}$$

It follows immediately from the definition that for each $i \in I$ we have $f_i = p_i \circ \Phi$. It remains to show that Φ is unique. So let $\Psi\colon Z \to \prod_{i\in I} X_i$ be a continuous map such that for each $i \in I$ we have $f_i = p_i \circ \Psi$. We need to show that $\Phi = \Psi$, i.e. we need to show that for each $z \in Z$ and $i \in I$ we have $\Phi(z)(i) = \Psi(z)(i) \in X_i$. But this follows from the observation that $\Phi(z)(i) = p_i(\Phi(z)) = f_i(z) = p_i(\Psi(z)) = \Psi(z)(i) \in X_i$.

(2) The same argument as in (1) also works for the category *Group* of groups.

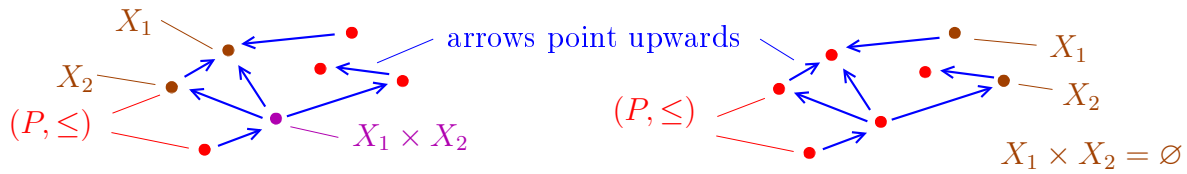
(3) The argument of (2) also works in (3). ■

After these examples one might assume that one gets the hang of the product in categories. The following examples might rob the reader of this illusion.

Example.

(1) Let $(P, \leq)$ be a partially ordered set. We consider the corresponding order category $\mathcal{O}rder_{(P,\leq)}$ as defined on page 71. Given a family $\{X_i\}_{i\in I}$ of objects in $\mathcal{O}rder_{(P,\leq)}$, i.e. given a family $\{X_i\}_{i\in I}$ of elements of P, one can easily show that[52]

$$\prod_{i\in I} X_i := \begin{cases} \inf(\{X_i\}_{i\in I}), & \text{i.e. a maximal element among all lower bounds on } \{X_i\}_{i\in I} \\ \varnothing, & \text{if no such infimum exists.} \end{cases}$$



(2) We introduce a new category. A pointed set is a pair (X, x_0), where X is a set and x_0 is a point in X. We refer to the category $\mathcal{P}\mathcal{S}et$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{P}\mathcal{S}et) &:= \text{all pointed sets,} \\ \mathrm{Mor}((X, x_0), (Y, y_0)) &:= \text{all maps } f \text{ from } X \text{ to } Y \text{ with } f(x_0) = y_0 \end{aligned}$$

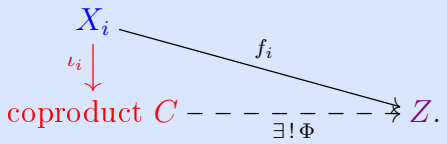
with the usual composition of maps as the category $\mathcal{P}\mathcal{S}et$ of pointed sets. Now let (X, x_0) and (Y, y_0) be two pointed sets. It is an instructive exercise to show that the product of (X, x_0) and (Y, y_0) in the category $\mathcal{P}\mathcal{S}et$ is given by the set

$$(X \times Y)/\underbrace{((X \times \{y_0\}) \cup (\{x_0\} \times Y))}_{\substack{\text{the points in this subset get} \\ \text{identified to a single point } *}} \quad \text{together with the point} \quad *$$

and the obvious morphisms to (X, x_0) and (Y, y_0).

4.2. **Coproducts.** We now reverse the direction of the morphisms in the definition of a product and we obtain a new interesting definition.

Definition. Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. A coproduct of $\{X_i\}_{i\in I}$ in the category $\mathcal{C}$ is an object $C \in \mathrm{Ob}(\mathcal{C})$ together with a family of morphisms $\{\iota_i \colon X_i \to C\}_{i\in I}$ such that the following property is satisfied: If we are given a $Z \in \mathrm{Ob}(\mathcal{C})$ and a family of morphisms $\{f_i \colon X_i \to Z\}_{i\in I}$ there exists a unique morphism $\Phi \colon C \to Z$ such that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$, i.e. such that the following diagram commutes:[53]

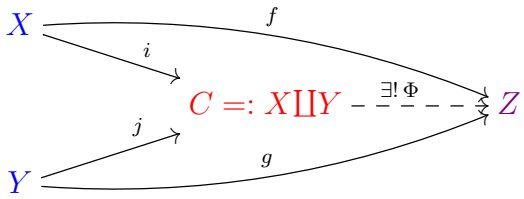


Remark. Note that by definition coproducts in $\mathcal{C}$ are products in the opposite category $\mathcal{C}^{\mathrm{op}}$. So we did not introduce a new concept at all.

[52] In fact it is a very good exercise to provide a proof.

Example. Let $\mathcal{C}$ be a category.

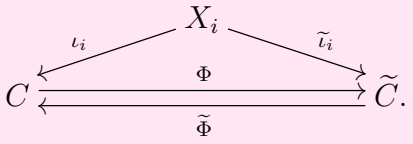
(1) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ be two objects. A coproduct of X and Y in the category $\mathcal{C}$ is some object $C \in \mathrm{Ob}(\mathcal{C})$ together with two morphisms $i\colon X \to C$ and $j\colon Y \to C$ such that the following universal property is satisfied: Whenever we are given an object $Z \in \mathrm{Ob}(\mathcal{C})$ and two morphisms $f\colon X \to Z$ and $g\colon Y \to Z$ there exists a unique morphism $\Phi\colon C \to Z$ such that $f = \Phi \circ i$ and $g = \Phi \circ j$. The situation is summarized in the following diagram:



(2) If we take the empty family of objects in $\mathcal{C}$, then it follows easily from the definitions that the corresponding coproduct is an initial object in the category $\mathcal{C}$.

(3) Basically by definition coproducts in a category $\mathcal{C}$ correspond to products in the opposite category $\mathcal{C}^{\mathrm{op}}$.

Lemma 4.3. (Coproduct Uniqueness Lemma) Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. If we are given two coproducts $(C, \{\iota_i\colon X_i \to C\}_{i\in I})$ and $(\widetilde{C}, \{\widetilde{\iota}_i\colon X_i \to \widetilde{C}\}_{i\in I})$, then there exist unique isomorphisms $\Phi\colon C \to \widetilde{C}$ and $\widetilde{\Phi}\colon \widetilde{C} \to C$ which are inverses of one another and such that for each $i \in I$ the following diagram commutes:



Proof. The lemma is just a reformulation of the Product Uniqueness Lemma 4.1 since coproducts in $\mathcal{C}$ are products in the opposite category $\mathcal{C}^{\mathrm{op}}$. Alternatively the statement also follows easily from the definitions. ∎

Notation. Let $\mathcal{C}$ be a category. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ such that a coproduct exists. Often[54] we denote the coproduct by $\coprod_{i\in I} X_i$ and we refer to the morphisms $X_i \to \coprod_{i\in I} X_i$ as natural injections.[55]

Lemma 4.4. (Coproduct-of-Sets-and-Abelian Groups Lemma)

(1) Let $\mathcal{S}et$ be the category of sets and let $\{X_i\}_{i\in I}$ be a family of sets. The coproduct in $\mathcal{S}et$ is the disjoint union

[54] In fact, as we will see shortly, in many settings we will see that the coproduct already has an established different notation.

[54] The name *natural injection* is somewhat unfortunate since in general for a morphism it makes no sense to say that it is injective. Even if there is a reasonable notion of injective, the natural injection does not need to be an injection. On the other hand, in all examples we ever care about the natural injection will be an injective map.

$$\bigsqcup_{i\in I} X_i \;:=\; \bigcup_{i\in I}(X_i\times\{i\}),$$

as defined on page 21, together with the natural injections of the X_i.

(2) Let $\mathcal{AbGroup}$ be the category of *abelian* groups and let $\{X_i\}_{i\in I}$ be a family of groups. The coproduct in $\mathcal{AbGroup}$ is given by

$$\bigoplus_{i\in I} X_i \;:=\; \left\{ (g_i)_{i\in I} \in \prod_{i\in I} X_i \;\middle|\; \text{all but finitely many } g_i \text{ are trivial} \right\}$$

and the homomorphisms

$$\begin{array}{rcl} X_i & \to & \bigoplus\limits_{i\in I} X_i \\ x & \mapsto & \left(\begin{array}{rcl} I & \to & \bigcup\limits_{i\in I} X_i \\ j & \mapsto & \begin{cases} x_i, & \text{if } i=j, \\ 0_{X_j}, & \text{if } i\neq j. \end{cases} \end{array} \right) \end{array}$$

(3) The same approach as in (2) also applies to the category $\mathcal{Vec}_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$.

Notation. Let $\{X_i\}_{i\in I}$ be objects in the category $\mathcal{AbGroup}$ of abelian groups. We write

$$\bigoplus_{i\in I} X_i \;:=\; \left\{ (g_i)_{i\in I} \in \prod_{i\in I} X_i \;\middle|\; \text{all but finitely many } g_i \text{ are trivial} \right\}$$

and we refer to it as the direct sum of the X_i. We proceed similarly with objects in the category $\mathcal{Vec}_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$.

Proof.

(1) So let $Z \in \mathrm{Ob}(\mathcal{Set})$ and let $\{f_i\colon X_i \to Z\}_{i\in I}$ be a family of maps between sets. We define

$$\begin{array}{rcl} \Phi\colon \bigsqcup\limits_{i\in I} X_i & \to & Z \\ (z_i, i) & \mapsto & f_i(z_i). \end{array}$$

It follows immediately from the definition that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$. It remains to show that Φ is unique. So let $\Psi\colon \bigsqcup\limits_{i\in I} X_i \to Z$ be a continuous map such that for each $i \in I$ we have $f_i = \Psi\circ\iota_i$. We need to show that $\Phi = \Psi$, i.e. we need to show that for each $i \in I$ and each $x_i \in X_i$ we have $\Phi((x_i, i)) = \Psi((x_i, i)) \in Z$. But this follows from the observation that $\Phi((x_i, i)) = \Phi(\iota_i(x_i)) = f_i(z) = \Psi(\iota_i(x_i)) = \Psi((x_i, i)) \in Z$.

(2) Let a $Z \in \mathrm{Ob}(\mathcal{AbGroup})$ and let $\{f_i\colon X_i \to Z\}_{i\in I}$ be a family of homomorphisms. We set[55]

$$\begin{array}{rcl} \Phi\colon \left\{ (g_i)_{i\in I} \in \prod\limits_{i\in I} X_i \;\middle|\; \text{all but finitely many } g_i \text{ are trivial} \right\} & \to & Z \\ (g_i)_{i\in I} & \mapsto & \sum\limits_{\substack{i\in I \text{ with} \\ g_i \neq e}} f_i(g_i). \end{array}$$

Note that this map makes sense since we take the product only of finitely elements of Z. Furthermore note that this map is a homomorphism since we work with the category of abelian groups.

It follows immediately from the definition that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$. It remains to show that Φ is unique. So let Ψ be another homomorphism such that

[55] We use addition notation for the abelian groups.

for each $i \in I$ we have $f_i = \Psi \circ \iota_i$. We need to show that $\Phi = \Psi$. So suppose we are given $(g_i)_{i\in I} \in \prod_{i\in I} X_i$ such that there exist only finitely many $i_1, \dots, i_k \in I$ such that the corresponding group element is non-trivial. Then

$$\Phi((g_i)_{i\in I}) = \Phi\Big(\sum_{r=1}^{k} \iota_{i_r}(g_{i_r})\Big) = \sum_{r=1}^{k} \Phi(\iota_{i_r}(g_{i_r})) = \sum_{r=1}^{k} f_{i_r}(g_{i_r}) \underset{\text{same argument backwards}}{\overset{\uparrow}{=}} \Psi((g_i)_{i\in I}).$$

(3) This statement is proved the same way as we proved (2). ■

4.3. **Limits.** In this section and the following section we introduce the very general notion of a limit and of a colimit, which in particular generalize the notion of a product and a coproduct. Before we can introduce these new concepts we need to provide the following definition.

Definition. Let $(P, \leq)$ be a partially ordered set and let $\mathcal{C}$ be a category. A diagram in $\mathcal{C}$ over $(P, \leq)$ is a covariant functor from the order category $\mathit{Order}_{(P,\leq)}$ to $\mathcal{C}$.

Remark. In plain English a diagram F in a category $\mathcal{C}$ over a partially ordered set $(P, \leq)$ is the following data:

(1) for each $x \in P$ we are given an object $F(x) \in \mathrm{Ob}(\mathcal{C})$,
(2) for each $x < y$ we are given a morphism $F(x) \to F(y)$,[56]
(3) if $x < y < z$, then the following diagram needs to commute:

$$\begin{array}{ccc} F(x) & \longrightarrow & F(z). \\ & \searrow \quad \nearrow & \\ & F(y) & \end{array}$$

Example.

(1) We consider the partially ordered set $(P, \leq_{\mathrm{tr}})$ that is given by a set P with the trivial partial order $\leq_{\mathrm{tr}}$, i.e. $x \leq_{\mathrm{tr}} y$ if and only if $x = y$. In this setting a diagram over $(P, \leq_{\mathrm{tr}})$ is the same data as a family of objects $\{X_i\}_{i\in P}$.
(2) We consider the partially ordered set $(\mathbb{N}, \leq)$. In this setting a diagram over $(\mathbb{N}, \leq)$ is the same data as a sequence

$$X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} X_3 \xrightarrow{\varphi_3} \dots$$

of morphisms $\varphi_i \colon X_i \to X_{i+1}$. For $i < j$ the morphisms $X_i \to X_j$ are understood to be given by $\varphi_{j-1} \circ \dots \circ \varphi_{i+1} \circ \varphi_i$.

In the following we will introduce the notion of a limit and of a pullback. For the time being we will not make use of these notions. We only include them here for completeness. We will make use of the opposite notions of a colimit and a pushout, that we will introduce in the next section.

Definition. Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. A limit of F consists of an object $L \in \mathrm{Ob}(\mathcal{C})$ together with

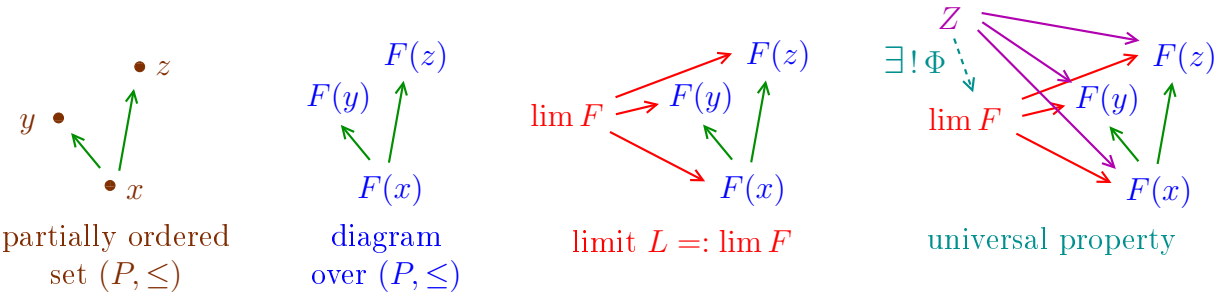
[56] Note that for $x \in P$ we have $x \leq x$ and the morphism $* \in \mathrm{Mor}(x, x)$ is the identity morphism. Since F is a covariant functor we see that $F(*) = \mathrm{id}_{F(x)}$. This is the reason why we only need to specify morphisms for $x < y$.

morphisms $f_x\colon L \to F(x)$ for each $x \in P$ such that the following two conditions are satisfied:

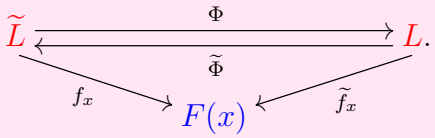
(1) For any $x \leq y$ we have $f_y = F(x \to y) \circ f_x\colon L \to F(y)$.

(2) The following universal property holds: If Z is an object in $\mathcal{C}$ and if we are given morphisms $g_x\colon Z \to F(x)$ for each $x \in P$ such that (1) is satisfied, then there exists a unique morphism $\Phi\colon Z \to L$ such that $g_x = f_x \circ \Phi$ for all $x \in P$.

If a limit exists, then often we denote it by $\lim F$. (We will see in the Limit Uniqueness Lemma 4.5 below that limits, if they exist, are essentially unique.)



Lemma 4.5. (Limit Uniqueness Lemma) Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. Suppose that we are given two limits $(L, \{f_x\colon L \to F(x)\}_{x \in P})$ and $(\widetilde{L}, \{\widetilde{f}_x\colon \widetilde{L} \to F(x)\}_{x \in P})$. Then there exist uniquely determined isomorphisms $\Phi\colon \widetilde{L} \to L$ and $\widetilde{\Phi}\colon L \to \widetilde{L}$ which are inverses of one another and such that for each $x \in P$ the following diagram commutes:



Proof. As in the case of the Product Uniqueness Lemma 4.1 the statement follows easily from the definitions. ∎

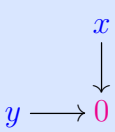
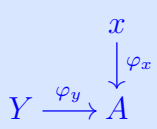
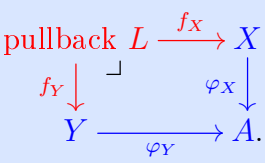
Example.

(1) If we consider a trivial partially ordered set $(P, \leq_{\mathrm{tr}})$, then the limit is by definition precisely the product in the sense of the definition on page 80.

(2) We consider $(\mathbb{N}, \leq)$ and we consider a diagram in the category $\mathcal{Set}$ of sets where each $\varphi_i\colon X_i \to X_{i+1}$ is an inclusion of subsets of a given set Y. One can easily show that in this case the limit is the first set X_1.

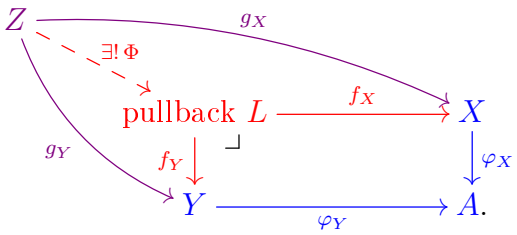
We conclude our discussion of limits with a particularly frequently used special case.

Definition. We consider the set $P = \{0, x, y\}$ with the partial order that is given by the trivial relations $0 \leq 0, x \leq x, y \leq y$ and the non-trivial relations $x \leq 0$ and $y \leq 0$:[57]

the partially ordered set  gives rise to the diagram  the corresponding limit is called a pullback 

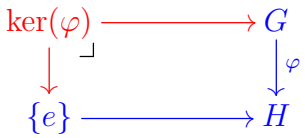
Here the symbol "⌟" indicates that L is the pullback of the diagram.

Remark. The following diagram summarizes the definition of a pullback:



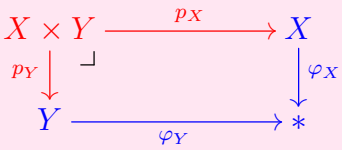
(1) We start out with a pullback diagram.
(2) The pullback L comes with morphisms $f_X \colon L \to X$ and $f_Y \colon L \to Y$ such that the rectangle commutes.
(3) If we are given an object Z of $\mathcal{C}$ with morphisms $g_X \colon Z \to X$ and $g_Y \colon Z \to Y$ such that the big outer quadrilateral commutes, then there exists a unique morphism $\Phi \colon Z \to L$ that makes the diagram commute.

Example. We consider the following pullback diagram in the category of groups:

one can easily show that the corresponding pullback is given by 

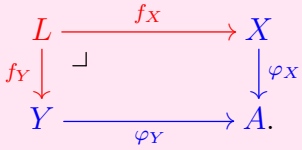
Lemma 4.6. (Pullback Lemma) Let $\mathcal{C}$ be a category.

(1) Suppose that $\mathcal{C}$ admits a terminal object $*$. Then the pullback of the diagram



is given by the product $X \times Y$.

(2) Let



[57] We do not need to specify the morphisms from the pullback L to A since it is agrees with $\varphi_X \circ f_X = \varphi_Y \circ f_Y$.

be a pullback.

(a) If φ_X is an isomorphism, then so is the "opposite" morphism f_Y.

(b) If φ_X is a monomorphism, then so is the "opposite" morphism f_Y.

The analogous statements of course also hold with the roles of X and Y swapped.

(3) If we are given pullbacks

$$\begin{array}{ccc} L & \xrightarrow{f_X} & X \\ {\scriptstyle f_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle \varphi_X} \\ Y & \xrightarrow{\varphi_Y} & A \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{\theta} & Z \\ {\scriptstyle \varphi_X}\downarrow & \lrcorner & \downarrow{\scriptstyle \psi_Z} \\ A & \xrightarrow{\psi_A} & B \end{array} \quad \text{then the "composition"} \quad \begin{array}{ccc} L & \xrightarrow{\theta\circ f_X} & Z \\ {\scriptstyle f_y}\downarrow & \lrcorner & \downarrow{\scriptstyle \psi_Z} \\ Y & \xrightarrow{\psi_A\circ\varphi_Y} & B \end{array}$$

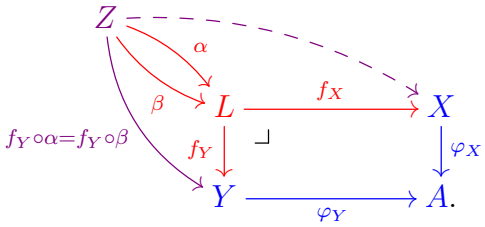
is also a pullback.

Proof.

(1) This statement follows easily from the definitions.

(2) (a) This statement also follows easily from the definitions.

(b) We assume that φ_X is a monomorphism. We need to show that f_Y is also a monomorphism. Thus let $\alpha\colon Z \to L$ and $\beta\colon Z \to L$ be two morphisms such that $f_Y \circ \alpha = f_Y \circ \beta$.



We need to show that $\alpha = \beta$. We proceed as follows:

(i) We assume that $f_Y \circ \alpha = f_Y \circ \beta$.

(ii) By (i) we also have $\varphi_Y \circ f_Y \circ \alpha = \varphi_Y \circ f_Y \circ \beta$.

(iii) Since the diagram commutes we get $\varphi_X \circ f_X \circ \alpha = \varphi_X \circ f_X \circ \beta\colon Z \to A$.

(iv) Since φ_X is a monomorphism we see that $f_X \circ \alpha = f_X \circ \beta\colon Z \to X$.

(v) Since we dealing with a pullback there exists a *unique* morphism $\Phi\colon Z \to L$ such that $f_Y \circ \Phi = f_Y \circ \alpha = f_Y \circ \beta$ and such that $f_X \circ \Phi = f_X \circ \alpha = f_X \circ \beta$.

(vi) From (i), (iv) and (v) we obtain that $\alpha = \Phi = \beta$.

(3) This statement follows from purely formal arguments which do not become clearer by writing them down. Thus we leave it to the reader to go through the argument. ■

4.4. **Colimits.** In this section we introduce the notion of a colimit, which is very similar to the notion of a limit, except that the directions of the morphisms are reversed.

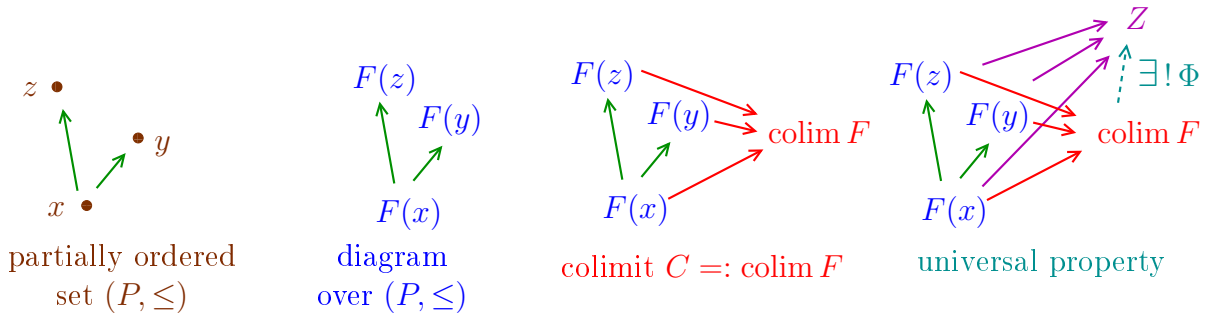
Definition. Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. A colimit of F consists of an object $C \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $f_x\colon F(x) \to C$ for each $x \in P$ such that the following two conditions are satisfied:

(1) For any $x \leq y$ we have $f_x = f_y \circ F(x \to y)\colon F(x) \to C$.

[57] It is a "meta theorem" that in many categories two morphisms in a pullback diagram, that are opposite of one another, have similar properties.

(2) The following universal property holds: If Z is an object in $\mathcal{C}$ and if we are given morphisms $g_x\colon F(x) \to Z$ for each $x \in P$ such that (1) is satisfied, then there exists a unique morphism $\Phi\colon C \to Z$ such that $g_x = \Phi \circ f_x$ for all $x \in P$.

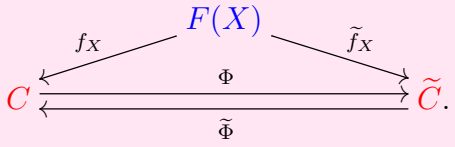
If a colimit exists, then often we denote it by $\operatorname{colim} F$.



Remark. Note that by definition colimits in $\mathcal{C}$ are limits in the opposite category $\mathcal{C}^{\mathrm{op}}$. So we did not introduce a new concept at all.

Not surprisingly we have the following analogue of the Limit Uniqueness Lemma 4.5.

Lemma 4.7. (Colimit Uniqueness Lemma) Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. Suppose that we are given two colimits $(C, \{f_x\colon F(x) \to C\}_{x\in P})$ and $(\widetilde{C}, \{\widetilde{f}_x\colon F(X) \to \widetilde{C}\}_{x\in P})$. Then there exist uniquely determined isomorphisms $\Phi\colon C \to \widetilde{C}$ and $\widetilde{\Phi}\colon \widetilde{C} \to C$ which are inverses of one another and such that for each $x \in P$ the following diagram commutes:



Proof. As always the proof follows easily from the definitions. ∎

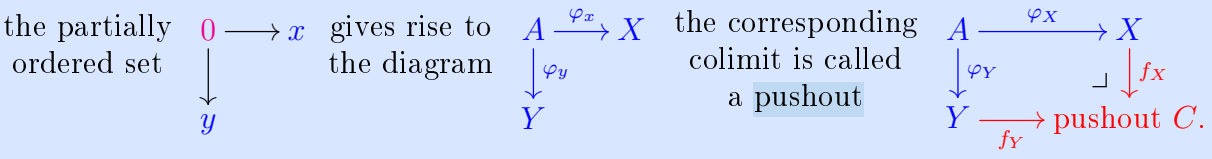
Example.

(1) If we consider a trivial partially ordered set $(P, \leq_{\mathrm{tr}})$, then the colimit is by definition precisely the coproduct, as defined on page 82.

(2) We consider $(\mathbb{N}, \leq)$ and we consider a diagram in the category $\mathcal{S}et$ of sets where each $\varphi_i\colon X_i \to X_{i+1}$ is an inclusion of subsets of a given set Y. We leave it to the reader to show that in this case the colimit is the union of the X_i.

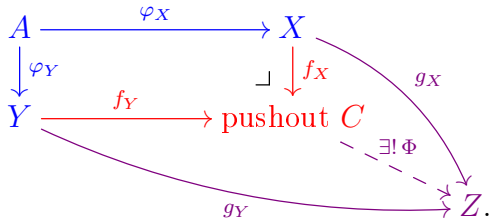
Next we introduce the analogue of the pullback.

Definition. We consider the set $P = \{0, x, y\}$ with the partial order that is given by the trivial relations $0 \leq 0, x \leq x, y \leq y$ and the non-trivial relations $0 \leq x$ and $0 \leq y$:



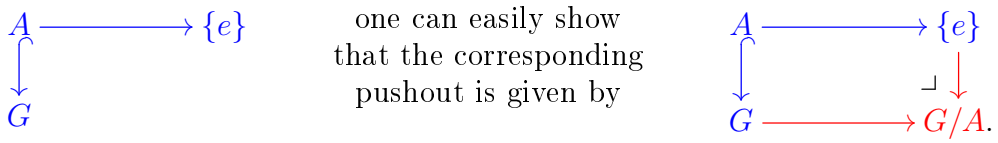
Here the symbol "⌟" indicates that C is the pushout of the diagram.

Remark. The following diagram summarizes the definition of a pushout:



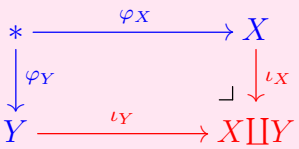
(1) We start out with a pushout diagram.
(2) The pushout C comes with morphisms $f_X\colon X \to C$ and $f_Y\colon Y \to C$ such that the rectangle commutes.
(3) If we are given an object Z with morphisms $g_X\colon X \to Z$ and $g_Y\colon Y \to Z$ such that the big outer quadrilateral commutes, then there exists a unique morphism $\Phi\colon C \to Z$ that makes the diagram commute.

Example. Let G be a group and let A be a *normal* subgroup of G. We consider the following pushout diagram in the category of groups:

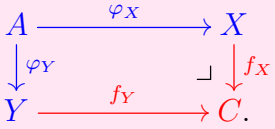


We conclude this section with the following analogue of the Pullback Lemma 4.6.

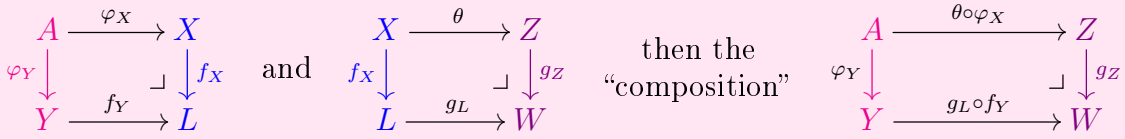
Lemma 4.8. (Pushout Lemma) Let $\mathcal{C}$ be a category.
(1) If $\mathcal{C}$ has an initial object $*$, then the pushout of



is given by the coproduct $X \coprod Y$.
(2) Let



be a pushout.
(a) If φ_X is an isomorphism, then so is the "opposite" morphism f_Y.
(b) If φ_X is an epimorphism, then so is the "opposite" morphism f_Y.
The analogous statements of course also hold with the roles of X and Y swapped.
(3) If we are given pushouts



is also a pushout.

Proof.

(1) This statement follows easily from the definitions.
(2) (a) This statement also follows easily from the definitions.
    (b) See exercise sheet 11.
(3) This statement follows from purely formal arguments which do not become clearer by writing them down. Thus we leave it to the reader to go through the argument. ■

5. Products and disjoint unions of topological spaces

In Chapter 4 we introduced the concepts of (co-) products and more generally (co-) limits in categories. In this and the following chapter we will see how these concepts play out in the category of topological spaces.

5.1. The product of topological spaces.

In this section we will see that the category of topological spaces has products. To do so it is convenient to recall the following definition from page 20.

Definition. Let $\{X_i\}_{i\in I}$ be a family of sets. The product of these sets is defined as the set

$$\prod_{i\in I} X_i := \left\{ f\colon I \to \bigcup_{i\in I} X_i \,\middle|\, \text{for every } i\in I \text{ we have } f(i)\in X_i \right\}.$$

Sometimes we also denote an element of $\prod_{i\in I} X_i$ by $(x_i)_{i\in I}$.

Proposition 5.1. (Topological-Product Proposition) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. The following statements hold:

(0) The set

$$\mathcal{B} := \left\{ \prod_{i\in I} U_i \subset \prod_{i\in I} X_i \,\middle|\, \begin{array}{ll} (1) & \text{each } U_i \text{ is open in } X_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}$$

has the basis property that we introduced on page 47. Thus it follows from the Topology-from-Basis Lemma 1.13 that $\mathcal{B}$ defines a topology on $\prod_{i\in I} X_i$.

(1) (a) For any $i\in I$ the projection

$$\begin{array}{rccc} p_i\colon & \prod_{i\in I} X_i & \to & X_i \\ & f & \mapsto & f(i) \end{array}$$

is continuous with respect to the topology from (0).

(b) Given a topological space Z and given a family $\{f_i\colon Z\to X_i\}_{i\in I}$ of continuous maps the map

$$\begin{array}{rccc} F\colon Z & \to & \prod_{i\in I} X_i & \\ z & \mapsto & \left(\begin{array}{ccc} I & \to & \bigcup_{i\in I} X_i \\ i & \mapsto & f_i(z) \end{array} \right). \end{array}$$

is continuous with respect to the topology from (0).

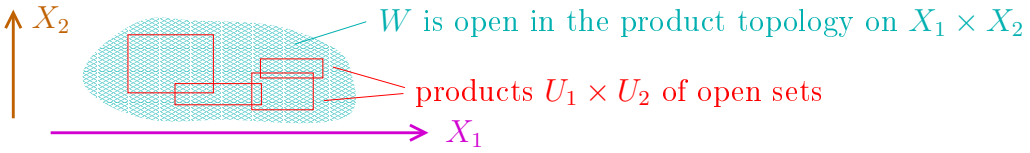
(∗) A map $F\colon Z\to \prod_{i\in I} X_i$ from some topological space Z is continuous if and only if each map $p_i\circ F\colon Z\to X_i$ is continuous.

(2) The topological space $\prod_{i\in I} X_i$ together with the natural projections $p_j\colon \prod_{i\in I} X_i \to X_j$ is a product (in the sense of the definition on page 80) of $\{X_i\}_{i\in I}$ in the category $\mathcal{Top}$ of topological spaces.

Example.

(1) It is worth considering the case that we will use most frequently, namely the case that we are given finitely many topological spaces $X_1,\dots,X_k$. In this case the product topology on $X_1\times\cdots\times X_k$ is by definition the topology on $X_1\times\cdots\times X_k$ that is generated by

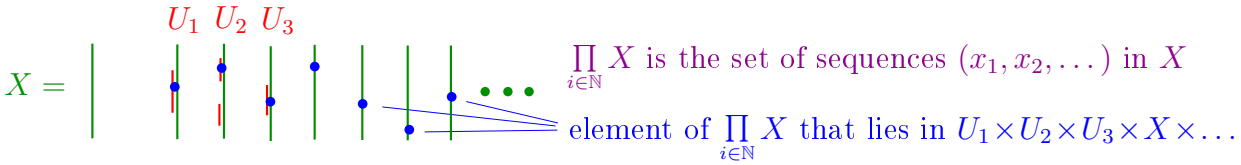
$$\mathcal{B} = \{U_1\times\cdots\times U_k \,|\, \text{each } U_i \text{ is open in } X_i\}.$$



(2) Let X be a topological space. We consider the product

$$X^{\mathbb{N}} = X \times X \times \ldots = \text{Maps}(\mathbb{N} \to X) = \text{set of sequences of points in } X.$$

A basis of the topology is given by the sets of the form $U_1 \times \cdots \times U_k \times X \times \ldots$ where $U_1, \ldots, U_k \subset X$ are open. In the figure below we try to illustrate this definition.



Proof of Proposition 5.1. Let $\{X_i\}_{i\in I}$ be a family of topological spaces.

(0) We need to verify that

$$\mathcal{B} := \left\{ \prod_{i\in I} U_i \subset \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \text{ is open in } X_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}$$

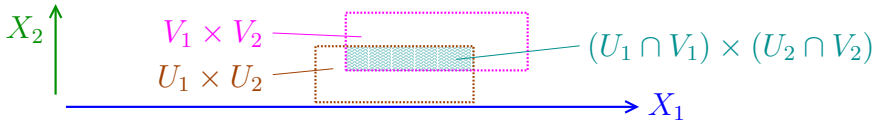
satisfies the conditions (B1) and (B2) formulated on page 47.

(B1) Evidently we have $\prod_{i\in I} X_i \in \mathcal{B}$. It follows immediately that $\mathcal{B}$ satisfies (B1).

(B2) Let $\prod_{i\in I} U_i$ and $\prod_{i\in I} V_i$ be two sets in $\mathcal{B}$. Then

$$\prod_{i\in I} \underbrace{U_i}_{\substack{\text{only finitely} \\ \text{many are} \neq X_i}} \cap \prod_{i\in I} \underbrace{V_i}_{\substack{\text{only finitely} \\ \text{many are} \neq X_i}} = \prod_{i\in I} \underbrace{(U_i \cap V_i)}_{\substack{\text{only finitely} \\ \text{many are} \neq X_i}}$$

is again a set in $\mathcal{B}$. Thus $\mathcal{B}$ also satisfies (B2).



(1) (a) It follows immediately from the definitions that the projection maps p_i are continuous with respect to the given topology.

(b) Suppose we are given a topological space Z and continuous maps $f_i \colon Z \to X_i$. We denote by $F \colon Z \to \prod_{i\in I} X_i$ the map that is given by $z \mapsto (i \mapsto f_i(z))$. We claim that F is continuous. By the Subbasis-Continuity Proposition 2.2 it suffices to verify that the preimage of any set in the basis $\mathcal{B}$ is open. So let $\prod_{i\in I} U_i$ be a set in $\mathcal{B}$. By definition there exists a finite set $J \subset I$ such that $U_i = X_i$ for any $i \notin J$. Then

$$F^{-1}\Big(\prod_{i\in I} U_i\Big) = \bigcap_{i\in I} f_i^{-1}(U_i) = \bigcap_{j\in j} \underbrace{f_j^{-1}(U_j)}_{\text{open in } Z} = \begin{array}{c}\text{intersection of } \textit{finitely} \\ \text{many open subsets of } Z.\end{array}$$

This concludes the proof that F is continuous.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) Let Z be a topological space and let $\{f_i\colon Z \to X_i\}_{i\in I}$ be a family of continuous maps. By the definition of a product on page 80 we need to show that there exists a unique continuous map $\Phi\colon Z \to \prod_{i\in I} X_i$ such that $f_i = p_i \circ \Phi$ for all $i \in I$. To do so, first note that it follows from the Product-of-Sets-and-Groups Lemma 4.2 that the map

$$\begin{array}{rcl} \Phi\colon Z & \to & \prod_{i\in I} X_i \\ z & \mapsto & (f_i(z))_{i\in I} \end{array}$$

is the unique map (of sets) $\Phi\colon Z \to \prod_{i\in I} X_i$ such that for each $i \in I$ we have the equality $f_i = p_i \circ \Phi$. In (1b) we just showed that this map is fortunately continuous. These two observations give us the desired result. ■

Definition. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. We refer to the topology on the set $\prod_{i\in I} X_i$ that we defined in the Topological-Product Proposition 5.1 (0) as the product topology on $\prod_{i\in I} X_i$.

Example.

(1) It follows from the Topological-Product Proposition 5.1 (1a) and (1b) that given any topological space X the

diagonal map $\begin{array}{rcl} d\colon X & \to & X\times X \\ x & \mapsto & (x,x) \end{array}$ and the projection $\begin{array}{rcl} p\colon X\times X & \to & X \\ (x,y) & \mapsto & x \end{array}$

are continuous. Since $p \circ d = \mathrm{id}_X$ we see that $d\colon X \to X \times X$ is an embedding. This fact will come in handy on many occasions.

(2) Let $\{f_i\colon X_i \to Y_i\}_{i\in I}$ be a family of continuous maps between topological spaces. It follows from the Topological-Product Proposition 5.1 (1) together with the Basics-of-Continuity Lemma 2.1 (1) that the map

$$\begin{array}{rcl} \prod_{i\in I} f_i\colon \prod_{i\in I} X_i & \to & \prod_{i\in I} Y_i \\ (x_i)_{i\in I} & \mapsto & (f(x_i))_{i\in I} \end{array}$$

is continuous.

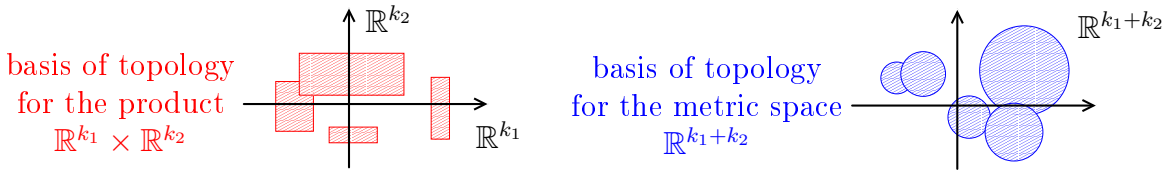
The following lemma gives us a reinterpretation of the product topology in some of the most frequently studied cases.

Lemma 5.2. (Product Metric Lemma) Let $k_1, \dots, k_m \in \mathbb{N}_0$. The map

$$\begin{array}{rcl} \overbrace{\mathbb{R}^{k_1+\cdots+k_m}}^{\text{Euclidean topology}} & \to & \overbrace{\mathbb{R}^{k_1}\times\cdots\times\mathbb{R}^{k_m}}^{\text{product of Euclidean topologies}} \\ (x_1,\dots,x_{k_1},\dots,x_1,\dots,x_{k_m}) & \mapsto & ((x_1,\dots,x_{k_1}),\dots,(x_1,\dots,x_{k_m})) \end{array}$$

is a homeomorphism.

Proof. The proof, which is not very difficult, can be found in [Fri23]. ■



Identification. Given $k_1, \ldots, k_m \in \mathbb{N}_0$ we will use the Product Metric Lemma 5.2 to identify the product topological space $\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_m}$ with the topological space $\mathbb{R}^{k_1+\cdots+k_m}$. We proceed the same way with "$\mathbb{R}$" replaced by "$\mathbb{C}$".

Example. It follows almost immediately from the definition of the topology on the vector space $\mathrm{M}(r \times s, \mathbb{R}) = \mathbb{R}^{r \cdot s}$ on page 65, the example on page 50, the Sum-Product-Continuity Lemma 2.6 and the Product Metric Lemma 5.2 that for any $k, l, m \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{M}(k \times l, \mathbb{R}) \times \mathrm{M}(l \times m, \mathbb{R}) & \to & \mathrm{M}(k \times m, \mathbb{R}) \\ ((a_{ij})_{i,j}, (b_{rs})_{r,s}) & \mapsto & \left(\sum\limits_{j=1}^{l} a_{ij} \cdot b_{js} \right)_{i,s} \end{array}$$

is continuous. The same also applies to matrices over $\mathbb{C}$.

After these examples let us turn to the study of the more formal properties of the product topology. On several occasions we will use, consciously or unconsciously, the following lemma.

Lemma 5.3. (Product Topology-Basics Lemma)

(1) Let $\{X_i\}_{i \in I}$ be a family of topological spaces.

(a) For each $i \in I$ let A_i be a subspace of X_i.

(i) The subspace topology of $\prod\limits_{i \in I} A_i \subset \prod\limits_{i \in I} X_i$ agrees with product topology on $\prod\limits_{i \in I} A_i$.

(ii) The natural map

$$\prod_{i \in I} A_i \ \to \ \prod_{i \in I} X_i$$

given by the inclusions $A_i \to X_i$ is an embedding, i.e. it is a homeomorphism onto its image.[58]

(b) If $J \subset I$ is a subset such that for each $j \in J$ the set X_j consists of a single element x_j, then the map

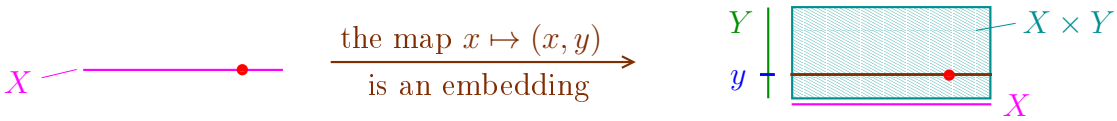
$$\begin{array}{rcl} \prod\limits_{i \in I \setminus J} X_i & \to & \prod\limits_{i \in I} X_i \\ (f \colon I \setminus J \to \bigcup\limits_{i \in I \setminus J} X_i) & \mapsto & \left(\begin{array}{lcl} I & \to & \bigcup\limits_{i \in I} X_i \\ i & \mapsto & \begin{cases} x_i, & \text{if } i \in J, \\ f(i), & \text{if } i \notin J \end{cases} \end{array} \right) . \end{array}$$

is a homeomorphism.

(2) If $\{X_i\}_{i \in I}$ is a family of topological spaces and if for each $i \in I$ we are given a basis $\mathcal{B}_i$ of the topological space X_i, then

$$\mathcal{C} \ = \ \left\{ \prod_{i \in I} U_i \ \subset \ \prod_{i \in I} X_i \ \middle| \ \begin{array}{ll} (1) & \text{each } U_i \neq X_i \text{ is contained in } \mathcal{B}_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}$$

is a basis for $\prod\limits_{i \in I} X_i$.



Proof of Lemma 5.3.

(1) (a) (i) This statement follows easily from the observation that if we take the basis of the topology of the product $\prod_{i\in I} X_i$ from the Topological-Product Proposition 5.1 (0) and if we intersect each basis element with $\prod_{i\in I} A_i$ we obtain precisely the basis of the product $\prod_{i\in I} A_i$ from the Topological-Product Proposition 5.1 (0).

(ii) The given map is clearly an injection. It follows from the Topological-Product Proposition 5.1 (0) that this map is continuous and it follows from (i) that this map is a homeomorphism onto its image.

(b) It follows from the Topological-Product Proposition 5.1 that the given map is continuous. It has an obvious inverse, which is also continuous by the Topological-Product Proposition 5.1.

(2) First note that by definition the sets in $\mathcal{C}$ are open. Next let $W \subset \prod_{i\in I} X_i$ be an open subset and let $(x_i)_{i\in I} \in W$. By definition of the product topology there exist open subsets $V_i \subset X_i$, $i \in I$ such that $X_i = V_i$ for all but finitely many $i \in I$ and such that $(x_i)_{i\in I} \subset \prod_{i\in I} V_i \subset W$. Since $\mathcal{B}_i$ is a basis for X_i we see that for each i with $V_i \neq X_i$ there exists a $U_i \in \mathcal{B}_i$ with $x \in U_i \subset V_i$. For all other i we set $U_i := V_i = X_i$. It follows that

$$(x_i)_{i\in I} \in \prod_{i\in I} U_i, \subset \prod_{i\in I} V_i \subset W.$$

We deduce that the given set $\mathcal{C}$ is indeed a basis for $\prod_{i\in I} X_i$. ■

5.2. **Properties of products of topological spaces I.** Throughout these notes we will almost invariably work with the product of only finitely many topological spaces. In this section we will therefore discuss properties of the product of *finitely* many topological spaces. This simplifies some of the arguments and it makes the notation arguably more natural. In the following section we will, for completeness sake, also discuss some of the properties of products of arbitrarily many topological spaces.
The next proposition deals with some of our favorite properties of topological spaces.

Proposition 5.4. (Product Topology-Properties Proposition) Let $X_1, \ldots, X_k$ be non-empty[59] topological spaces.
(1) The product $X_1 \times \cdots \times X_k$ is compact if and only if each X_i is compact.
(2) The product $X_1 \times \cdots \times X_k$ is (path)-connected if and only if each X_i is (path)-connected.
(3) The product $X_1 \times \cdots \times X_k$ is Hausdorff if and only if each X_i is Hausdorff.

[58] If I is finite and if each $A_i \subset X_i$ is open (closed), then it follows easily from the definition of the product topology that the natural map $\prod_{i\in I} A_i \to \prod_{i\in I} X_i$ is an open (closed) embedding.

[59] The "if"-directions also hold if one the X_i is the empty space, since in this case the product is the empty space.

Proof. We proved the proposition in Analysis IV. Alternatively see [Fri23]. ■

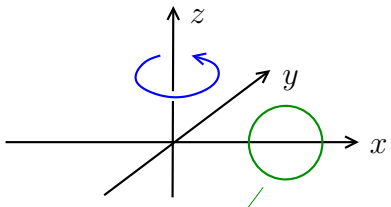
Definition.
(1) We refer to $(S^1)^n$ as the n-dimensional torus. For $n = 2$ we just say the torus.
(2) We refer to $S^1 \times \overline{B}^2$ as the solid torus and we refer to $S^1 \times B^2$ as the open solid torus.
We deduce from the Product Topology-Properties Proposition 5.4 that the n-dimensional torus $(S^1)^n$ and the solid torus $S^1 \times \overline{B}^2$ are compact.

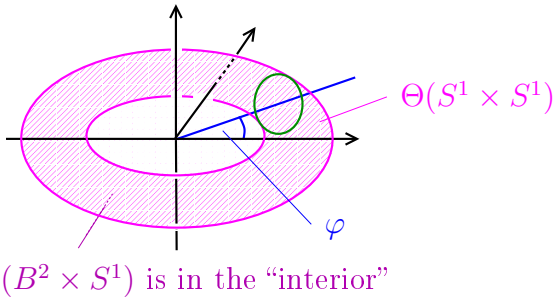
The following lemma allows us to view (solid) tori as subspaces of $\mathbb{R}^3$.

Lemma 5.5. (Torus Embedding Lemma) The map

$$\begin{aligned} \Theta\colon \overline{B}^2 \times S^1 &\to \mathbb{R}^3 \\ ((x,y),\exp(\mathrm{i}\varphi)) &\mapsto \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 1+\frac{1}{3}x \\ 0 \\ -\frac{1}{3}y \end{pmatrix}}_{\substack{\text{describes circle} \\ \text{in } xz\text{-plane}}} \end{aligned}$$

is an embedding. The restrictions of Θ to $S^1 \times S^1$ and $B^2 \times S^1$ are also embeddings.





Proof. The map $\Theta\colon \overline{B}^2 \times S^1 \to \mathbb{R}^3$ can be written as the composition of the maps

$$\begin{aligned} \overline{B}^2 \times S^1 \to \mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\}) \xleftarrow{\cong} \{(x,y,a,b) \in \mathbb{R}^4 \mid a^2+b^2 > 0\} &\to \mathbb{R}^3 \\ (x,y,a,b) &\mapsto \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1+\frac{1}{3}x \\ 0 \\ -\frac{1}{3}y. \end{pmatrix} \end{aligned}$$

The first map is continuous by the Product Topology-Properties Proposition 5.4 (1), the second map is a homeomorphism by the Product Metric Lemma 5.2 and the last map is continuous by the Sum-Product-Continuity Lemma 2.6. This shows that Θ is continuous.

An elementary argument shows that the restriction of Θ to $\overline{B}^2 \times S^1$ is injective. By the Product Topology-Properties Proposition 5.4 (1) we know that the solid torus $\overline{B}^2 \times S^1$ is compact. Thus we obtain from the Compact-Hausdorff Proposition 2.10 (3) that the restriction of Θ to $\Theta\colon \overline{B}^2 \times S^1 \to \Theta(S^1 \times \overline{B}^2)$ is an embedding. But then it follows that the restrictions of Θ to $S^1 \times S^1$ and $B^2 \times S^1$ are embeddings. ■

5.3. **Properties of products of topological spaces II.** The generalization of the Product Topology-Properties Proposition 5.4 to the product of arbitrarily many topological

spaces also holds, see [Fri23] for details. We will not really make use of it, but for "general culture" let us formulate the following generalization of the Product Topology-Properties Proposition 5.4 (1).

Proposition 5.6. (Product Topology-Properties Proposition) Let $\{X_i\}_{i\in I}$ be a family of non-empty topological spaces.

(1) The product $\prod_{i\in I} X_i$ is compact if and only if each X_i is compact.

(2) The product $\prod_{i\in I} X_i$ is (path)-connected if and only if each X_i is (path)-connected.

(3) The product $\prod_{i\in I} X_i$ is Hausdorff if and only if each X_i is Hausdorff.

Proof. The proof of all statements of the Product Topology-Properties Proposition 5.6 is basically the same as the proof of the corresponding statements of the Product Topology-Properties Proposition 5.4, *except*, for the "if"-direction of Proposition 5.6 (1) and the "if"-direction of Proposition 5.6 (2) in the connected case. We outsource the proof the "if"-direction of Proposition 5.6 (2) to [Fri23]. The proof of the "if"-direction of Proposition 5.6 (1) will be discussed below. ■

We now deal with the "if"-direction of the Product Topology-Properties Proposition 5.6 (1). This proof is now considerably harder and thus the proposition gets upgraded to a theorem:

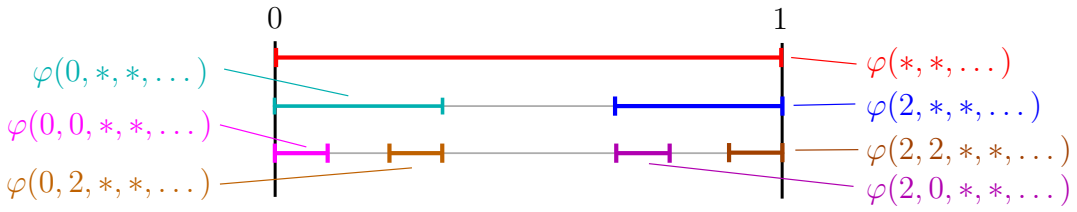
Theorem 5.7. (Tychonoff's Theorem) The product of arbitrarily many compact spaces is again a compact space.

Proof. Let $\{X_i\}_{i\in I}$ be a family of topological space. We will prove Tychonoff's Theorem with $I = \mathbb{N}$ in exercise sheet 2, with copious hints. The proof of the general case builds on the Axiom of Choice. We refer to [Fri23] for references to proofs in the literature. ■

Example. We equip $\{0, 2\}$ with the discrete topology. We consider the map

$$\begin{aligned} \varphi\colon \prod_{i\in\mathbb{N}} \{0,2\} &\to C = \text{Cantor set from page 46} \\ (f\colon \mathbb{N} \to \{0,2\}) &\mapsto \sum_{i=1}^{\infty} \frac{f(i)}{3^i}. \end{aligned}$$

It is elementary to see that φ takes values in C. It is a refreshing exercise, left to the enthusiastic reader, to show that φ is a bijection. The left-hand side is compact by Tychonoff's Theorem 5.7 and the right-hand is evidently Hausdorff. We leave it to the reader to verify, using the Subbasis-Continuity Proposition 2.2 and the JH-Openness Criterion 1.1, that φ is continuous. Thus it follows from the Compact-Hausdorff Proposition 2.10 (3) that φ is a homeomorphism.



5.4. **The pullback of topological spaces.** On page 87 we introduced the general notion of pullbacks in a category. In this section we show that the category of topological spaces always has pullbacks. It is rather reassuring to hear that this is the case, even though the pullback will only really play a role much later when we study bundles. Thus this section can also be skipped without any major loss.

Proposition 5.8. (Topological-Pullback Proposition) Let $p\colon X \to B$ be a continuous map of topological spaces, let C be a topological space and let $f\colon C \to B$ be a continuous map. We consider the topological space

$$f^*X \;:=\; \{(c,x) \in C \times X \mid f(c) = p(x)\}$$

and the maps $\varphi\colon f^*X \to C$, $(c,x) \mapsto c$ and $\psi\colon f^*X \to X$, $(c,x) \mapsto x$.

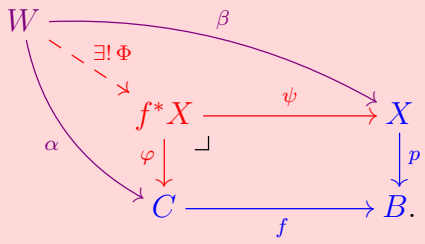
The following statements hold:

(1) (a) The maps $\varphi\colon f^*X \to C$ and $\psi\colon f^*X \to X$ are continuous.

(b) Suppose we are given a topological space W and two continuous maps $\alpha\colon W \to C$ and $\beta\colon W \to X$ such that $f \circ \beta = p \circ \alpha\colon W \to B$. The map

$$\Phi\colon W \to f^*X,\quad w \mapsto (\alpha(w), \beta(w)).$$

is continuous. In particular we obtain the following commutative diagram:



(∗) A map $\Phi\colon W \to f^*X$ is continuous if and only if the maps $\varphi \circ \Phi\colon W \to C$ and $\psi \circ \Phi\colon W \to X$ are continuous.

(2) The topological space f^*X with the natural maps $\varphi\colon f^*X \to C$ and $\psi\colon f^*X \to X$ is the pullback, in the sense of the definition on page 87, in the category $\mathcal{T}op$ of topological spaces.

Proof.

(1) (1) This statement follows from the Topological-Product Proposition 5.1 (1a) and the Basics-of-Continuity Lemma 2.1 (3).

(2) This statement follows from the Topological-Product Proposition 5.1 (1b) and the Basics-of-Continuity Lemma 2.1 (4).

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) It remains to show uniqueness of Φ. This follows immediately from the definitions and the fact that a point in $C \times X$ is uniquely determined by its projections. ∎

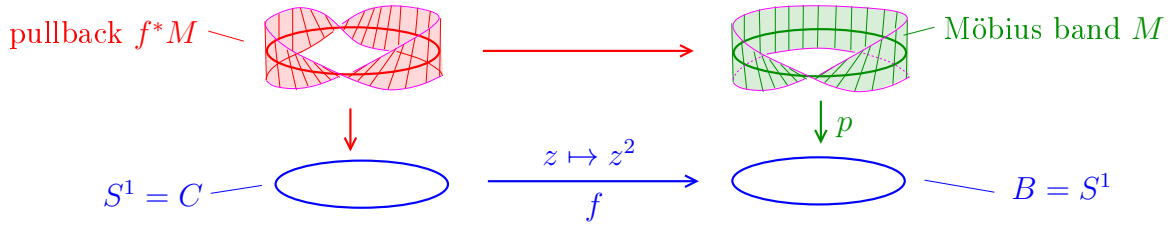
Example.

(1) Let $p\colon X \to B$ be a continuous map of topological spaces and let $C \subset B$ be a subspace. We denote by $\iota\colon C \to B$ the inclusion. One can easily verify that the maps

$$\begin{array}{rcl} p^{-1}(C) & \to & \iota^* X \\ x & \mapsto & (p(x), x) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \iota^* X & \to & p^{-1}(C) \\ (c, x) & \mapsto & x \end{array}$$

are continuous and inverses of one another. Thus we see that $\iota^* X$ is naturally homeomorphic to $p^{-1}(C)$.

(2) In the figure below we show the projection $p\colon M \to S^1$ from the Möbius band to S^1 and we see the map $f\colon C = S^1 \to S^1 = B$ that is given by $f(z) = z^2$. It is a wonderful exercise to convince oneself that the pullback f^*M is a "doubly twisted band" and thus homeomorphic to $S^1 \times [-1, 1]$.



For completeness we also study our favorite three properties of topological spaces for pullbacks. We will not really make use of this result.

Proposition 5.9. (Pullback-Properties Proposition) Let $p\colon X \to B$ be a continuous map of topological spaces, let C be a topological space and let $f\colon C \to B$ be a continuous map.

(1) If X and C are compact and if X, B, C are Hausdorff, then the pullback f^*X is also compact.

(3) If X and C Hausdorff, then the pullback f^*X is also Hausdorff.

Proof.

(1) We consider the map

$$\begin{array}{rcl} \mu\colon C \times X & \to & B \times B \\ (c, x) & \mapsto & (f(c), p(x)). \end{array}$$

It follows from the Topological-Product Proposition 5.1 (1) that μ is continuous. We denote by $\Delta := \{(b, b) \mid b \in B\} \subset B \times B$ the diagonal. Since B is Hausdorff it follows from an elementary exercise that $\Delta \subset B \times B$ is a closed subset. By definition of the pullback we have $f^*X = \mu^{-1}(\Delta) \subset C \times X$. Since X and C are compact and Hausdorff we know by the Product Topology-Properties Proposition 5.4 (1) and (3) that $X \times C$ is compact and Hausdorff. It follows from this discussion and the Compact-Closed Lemma 1.9 (2) that f^*X is indeed compact.

(3) This statement follows easily from the Product Topology-Properties Proposition 5.4 (3). ■

5.5. **The disjoint union of topological spaces.** In this section we will discuss the coproduct in the category of topological spaces. Before we get to topological spaces it is worth recalling the following definitions from page 21.

Definition. Let $\{A_i\}_{i\in I}$ be a family of sets.

(1) We define the disjoint union as the set

$$\bigsqcup_{i\in I} A_i := \bigcup_{i\in I} (A_i \times \{i\}).$$

As usual we take some liberties with the notation if we have finitely many sets.

(2) Given $j \in I$ we refer to the map $A_j \to \bigsqcup_{i\in I} A_i$ given by $a \mapsto (a, j)$ as the natural injection.

In the Coproduct-of-Sets-and-Abelian Groups Lemma 4.4 we saw that the disjoint union, together with the obvious natural injections, gives us the coproduct in the category of sets. In a second we will see that a very similar statement holds in the context of the category of topological spaces. But before we get to this point, let us introduce the following convention.

Convention. Let $\{A_i\}_{i\in I}$ be a family of sets. For any $i \in I$ we use the natural injection $A_i \to \bigsqcup_{i\in I} A_i$ to think of A_i as a subset of the disjoint union.

The following proposition is the analogue of the Topological-Product Proposition 5.1.

Proposition 5.10. (Topological-Coproduct Proposition) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. We consider the disjoint union $\bigsqcup_{i\in I} X_i$.

(0) (a) We say $U \subset \bigsqcup_{i\in I} X_i$ is open if and only if for each $i \in I$ the intersection $X_i \cap U$ is an open subset of X_i. This defines a topology on $\bigsqcup_{i\in I} X_i$. We will work with this topology throughout the proposition.

(b) A subset $A \subset \bigsqcup_{i\in I} X_i$ is closed if and only if for each $i \in I$ the intersection $X_i \cap A$ is a closed subset of X_i.

(1) (a) Each natural injection $\iota_j \colon X_j \to \bigsqcup_{i\in I} X_i$ is continuous. In fact it is an embedding that is open and closed.[60]

(b) If Z is a topological space and if $\{f_i \colon X_i \to Z\}_{i\in I}$ is a family of continuous maps, then the map

$$\begin{array}{ccc} \bigsqcup_{i\in I} X_i & \to & Z \\ (x, i) & \mapsto & f_i(x) \end{array}$$

is continuous.

(∗) A map $f \colon \bigsqcup_{i\in I} X_i \to Z$ to a topological space Z is continuous if and only if for each $i \in I$ the map $f \circ \iota_i \colon X_i \to Z$ is continuous.

(2) The topological space $\bigsqcup_{i\in I} X_i$ together with the natural injections ι_i is a coproduct of $\{X_i\}_{i\in I}$ in the category $\mathcal{T}op$ of topological spaces.

Proof.

(0) (a) This statement follows easily from the definitions.
(b) This statement follows from an elementary argument.

(1) (a) It follows immediately from writing down the definitions that each natural injection $X_j \to \bigsqcup_{i\in I} X_i$ is continuous, and that it is open and closed. It follows from the Open-Injective Lemma 2.9 that the natural injection is an embedding that is open and closed.

(b) Let $\{f_i\colon X_i \to Z\}_{i\in I}$ be a family of continuous maps. We consider the corresponding map $F\colon \bigsqcup_{i\in I} X_i \to Z$. Let $U \subset Z$ be an open subset. For each $i \in I$ we have $F^{-1}(U) \cap X_i = f_i^{-1}(U)$. This shows that $F^{-1}(U)$ is open, i.e. F is continuous.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) Let Z be a topological space and let $\{f_i\colon X_i \to Z\}_{i\in I}$ be a family of continuous maps. By the definition of the coproduct on page 82 we have to show that there exists a *unique* continuous map $\Phi\colon \bigsqcup_{i\in I} X_i \to Z$ such that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$. In (1b) we defined such a map Φ and we saw that it is continuous. It remains to prove uniqueness. The proof of uniqueness is basically identical to the argument in the proof of the Coproduct-of-Sets-and-Abelian Groups Lemma 4.4 (1) where we dealt with the coproduct in the category of sets. ■

Definition. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. We refer to the topology on $\bigsqcup_{i\in I} X_i$ that we introduced in the Topological-Coproduct Proposition 5.10 as the disjoint union topology. We refer to this topological space as the disjoint union of the topological spaces X_i.[61]

The following lemma plays the role of the Product Topology-Properties Proposition 5.4.

Lemma 5.11. (Disjoint Union Topology-Properties Lemma) Let $\{X_i\}_{i\in I}$ be a family of topological spaces.

(1) The following two statements are equivalent:
(a) The disjoint union $\bigsqcup_{i\in I} X_i$ is compact.
(b) Each X_i is compact and there are only finitely many $i \in I$ such that $X_i \neq \varnothing$.

(2) (a) Each X_i is an open and closed subset of $\bigsqcup_{i\in I} X_i$.
(b) The path-components of the disjoint union $\bigsqcup_{i\in I} X_i$ are precisely the path-components of the X_i.

(3) The disjoint union $\bigsqcup_{i\in I} X_i$ is Hausdorff if and only if each X_i is Hausdorff.

[60] In most cases this statement allows us to safely view each X_j as a closed and open subspace of the disjoint union $\bigsqcup_{i\in I} X_i$.

[61] It would perhaps make more sense to refer to $\bigsqcup_{i\in I} X_i$ as the coproduct of the X_i. But we prefer to stick to the more classical name.

Proof. We prove the statements of the lemma in a different order than they were stated.

(2) (a) Let $i \in I$. Note that for $j \neq i$ we have $X_i \cap X_j = \varnothing$, which is open and closed. Thus it follows from the Topological-Coproduct Proposition 5.10 (0a) and (0b) that X_i is an open and closed subset.

(b) This statement follows from (3a) and the Path-Component Lemma 2.21 (4).

(1) We first prove the "(b)⇒(a)"-direction. Thus we assume that each X_i is compact and that there are only finitely many $i_1, \ldots, i_k \in I$ such that the corresponding topological space is non-empty. Let $\{U_j\}_{j\in J}$ be an open cover of $\bigsqcup_{i\in I} X_i$. Since each X_i is compact we can find for each $i \in I$ a finite subset $J_i \subset J$ such that $\bigsqcup_{j\in J_i} (U_j \cap X_i) = X_i$. It is now clear that $X = \bigcup_{r\in\{1,\ldots,k\}} X_i = \bigcup_{r\in\{1,\ldots,k\}} \bigsqcup_{j\in J_{i_r}} U_j$.

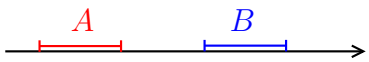
We turn to the proof of the "(a)⇒(b)"-direction. Thus we assume that $\bigsqcup_{i\in I} X_i$ is compact. It follows from (2a) together with the Compact-Closed Lemma 1.9 (1) that each X_i is compact. Next note that by (2a) we also know that the $\{X_i\}_{i\in I}$ form an open cover of the compact topological space $\bigsqcup_{i\in I} X_i$. Thus we see that there exists a finite subset $J \subset I$ with $\bigsqcup_{j\in J} X_j = \bigsqcup_{i\in I} X_i$. In particular $X_i = \varnothing$ for $i \notin J$.

(3) This statement follows easily from the definitions and (2a). ■

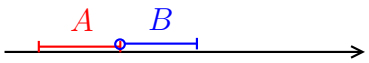
Example. Let X be a topological space and let $A, B \subset X$ be subsets of X. It follows from the Topological-Coproduct Proposition 5.10 (1b) that the obvious map

$$\underbrace{A \sqcup B}_{\text{disjoint union topology}} \quad \to \quad \underbrace{A \cup B}_{\text{subspace topology}}$$

is continuous. If A and B are disjoint, then this map is also a bijection. To be more specific, let $X = \mathbb{R}$. For $A = [0, 1]$ and $B = [2, 3]$ one can show by hand that the map is open, thus a homeomorphism. On the other hand for $A = [0, 1]$ and $(1, 2]$ the above map is certainly not a homeomorphism, for example $A \sqcup B$ is not compact whereas $A \cup B = [0, 2]$ is compact. In [Fri23] we formulate a lemma which gives a criterion for showing that $A \sqcup B$ is homeomorphic to $A \cup B$.



$A \sqcup B$ is homeomorphic to $A \cup B \subset \mathbb{R}$



$A \sqcup B$ is *not* homeomorphic to $A \cup B \subset \mathbb{R}$

6. Quotients of topological spaces

In Chapter 5 we introduced the product and disjoint union (vulgo coproduct) of topological spaces. In this chapter we introduce the very flexible notion of a quotient of a topological space by an equivalence relation. This will allow us to introduce some of our favorite topological spaces, namely surfaces and projective spaces, and it will allow us to study the pushout in the category of topological spaces.

6.1. **The quotient topology.** On page 21 we introduced the notion of an equivalence relation. We recall the following notation from page 23.

Notation. Let $\sim$ be an equivalence relation on a set X.

(1) Given $x \in X$ we denote by $[x]$ the unique equivalence class that contains x.

(2) We refer to the map
$$\begin{array}{rccc} p\colon & X & \to & X/\!\sim \\ & x & \mapsto & [x] \end{array}$$
as the natural projection.

(3) Let $f\colon X \to Y$ be a map between sets such that $f(x) = f(y)$ whenever $x \sim y$. By the Quotient-Factorization Lemma 0.5 the map
$$\begin{array}{rccc} g\colon & X/\!\sim & \to & Y \\ & [x] & \mapsto & f(x) \end{array}$$
is well-defined and it is the unique map $g\colon X/\!\sim \to Y$ such that $f = g \circ p$, i.e. such that the following diagram of maps commutes:
$$\begin{array}{ccc} X & \xrightarrow{p} & X/\!\sim \\ & \searrow^{f} & \downarrow g \\ & & Y. \end{array}$$
We refer to this map g as the induced map.

In this chapter we introduce and study the quotient of topological space by an equivalence relation. This notion is, in contrast to the (co-) product, not a categorical notion. Nonetheless there are structural similarities and the following proposition can be viewed as the analogue of the Topological-Product Proposition 5.1 and of the Topological-Coproduct Proposition 5.10.

Proposition 6.1. (Topological-Quotient Proposition) Let $\sim$ be an equivalence relation on a topological space X. We denote by $p\colon X \to X/\!\sim$ the natural projection from X onto the set of equivalence classes $X/\!\sim$. The following statements hold:

(0) (a) The set
$$\mathcal{T} := \{U \subset X/\!\sim \mid p^{-1}(U) \text{ is open in } X\}$$
is a topology on $X/\!\sim$. We use this topology throughout this proposition.[62]

(b) A subset $A \subset X/\!\sim$ is closed if and only if the preimage $p^{-1}(A)$ is closed in X.

(1) (a) The natural projection $p\colon X \to X/\!\sim$ is continuous.

(b) Let Y be a topological space and let $f\colon X \to Y$ be a continuous map with the property that $f(x) = f(y)$ whenever $x \sim y$. The induced map
$$\begin{array}{rccc} g\colon & X/\!\sim & \to & Y \\ & [x] & \mapsto & f(x) \end{array}$$

is also continuous.

(∗) A map $f\colon X/\sim \to Y$ is continuous if and only if the map $f \circ p\colon X \to Y$ is continuous.

(2) If $\mathcal{S}$ is a topology on $X/\sim$ such that $(X/\sim, \mathcal{S})$ has properties (1a) and (1b), then $\mathcal{S}$ equals the topology from (0).

Proof. Let $\sim$ be an equivalence relation on a topological space X.

(0) (a) This statement follows easily from the Image-Preimage Lemma 0.3 (6) and (7).

(b) This statement follows easily from the Image-Preimage Lemma 0.3 (8).

(1) (a) It follows immediately from the definition of the topology on $X/\sim$ that the natural projection $p\colon X \to X/\sim$ is continuous.

(b) Let $f\colon X \to Y$ be a continuous map such that $f(x) = f(y)$ whenever $x \sim y$. We need to show that the induced map $g\colon X/\sim \to Y$ is continuous as well. Let $U \subset Y$ be open. We need to show that $g^{-1}(U) \subset X/\sim$ is open. By definition of the quotient topology on $X/\sim$ we need to show that $p^{-1}(g^{-1}(U))$ is open in X. We observe that

$$p^{-1}(g^{-1}(U)) \;=\; (g\circ p)^{-1}(U) \;\underset{\text{by the commutative diagram}}{\overset{\uparrow}{=}}\; f^{-1}(U) \;\underset{\text{since } f \text{ continuous}}{\overset{\uparrow}{=}}\; \text{open subset}\,.$$

We have thus shown that g is continuous.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) Let $\mathcal{S}$ be a topology on $X/\sim$ such that the following conditions are satisfied:

(a′) The natural projection $p\colon X \to (X/\sim, \mathcal{S})$ is continuous.

(b′) Whenever we are given a topological space Y and a continuous map $f\colon X \to Y$ with the property that $f(x) = f(y)$ whenever $x \sim y$, then the corresponding induced map $(X/\sim, \mathcal{S}) \to Y$ is also continuous.

We apply (b) and (a′) to obtain that the identity map $\Phi\colon (X/\sim, \mathcal{T}) \to (X/\sim, \mathcal{S})$ is continuous. This implies that $\mathcal{S} \subset \mathcal{T}$. Similarly we apply (a) and (b′) to obtain that the identity map $\Psi\colon (X/\sim, \mathcal{S}) \to (X/\sim, \mathcal{T})$ is continuous, which tells us that $\mathcal{T} \subset \mathcal{S}$. In summary we see that $\mathcal{S} = \mathcal{T}$. ■

The following definition will not come as a surprise.

Definition. Let $\sim$ be an equivalence relation on a topological space X.

(1) We refer to the topology on $X/\sim$ from the Topological-Quotient Proposition 6.1 (0a) as the quotient topology and we refer to $X/\sim$ as a quotient space of X.

(2) Let Y be a topological space and let $f\colon X \to Y$ be a continuous map with the property that $f(x) = f(y)$ whenever $x \sim y$.

(a) We refer to the corresponding map $X/\sim \to Y$ as the induced map.

(b) At times we denote the induced map by $\overline{f}$. Sometimes, when there is no danger of confusion we denote the induced map just by f itself.

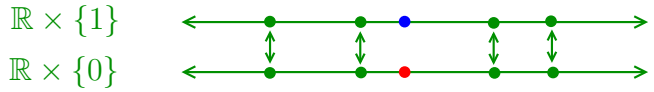
(c) Sometimes we also say that the map $f\colon X \to Y$ descends to a map $X/\sim \to Y$.

As before we care about compactness, connectedness and the Hausdorff property of topological spaces. We will see shortly that compactness and connectedness is preserved by going

[62] In other words, $U \subset X/\sim$ is open if and only if $p^{-1}(U) \subset X$ is open.

to quotients. The Hausdorff property is more tricky and we start out with the following cautionary example:

Example. We consider the topological space $X = (\mathbb{R}\times\{0\})\cup(\mathbb{R}\times\{1\})$ and the equivalence relation that is generated by $(x,0) \sim (x,1)$ for all $x \neq 0$. We denote by $p\colon X \to X/\sim$ the natural projection. One can easily show that $p((0,0))$ and $p((0,1))$ cannot be separated by open neighborhoods. One can also easily show that $X/\sim$ is homeomorphic to the line with two zeros, which is not Hausdorff. This shows that the quotient of a Hausdorff space is not necessarily Hausdorff.



Definition. Let X be a topological space and let $\sim$ be an equivalence relation on X. Recall that an equivalence relation is, by the definition on page 21, a subset of $X\times X$ with suitable properties. We say that the relation $\sim$ is closed if $\sim$ is a closed subset of $X\times X$.

The following lemma is the analogue of the Product Topology-Properties Proposition 5.4 and the Disjoint Union Topology-Properties Lemma 5.11.

Proposition 6.2. (Quotient Topology-Properties Proposition) Let $\sim$ be an equivalence relation on a topological space X.

(1) If X is compact, then $X/\sim$ is also compact.
(2) If X is (path)-connected, then $X/\sim$ is also (path)-connected.
(3) If X is Hausdorff and compact and if $\sim$ is a closed equivalence relation on X, then $X/\sim$ is Hausdorff.

Proof. By the Topological-Quotient Proposition 6.1 (1a) we know that the natural projection $X \to X/\sim$ is continuous. The first two statements of the lemma thus follow from the Compact Image Lemma 2.8 and the Image-Connected Lemma 2.16. The proof of the third statement is non-trivial but dull, we refer to [Fri23] for details. ∎

A careful reading of these notes shows that the following lemma gets used on many occasions without giving it its due credit.

Lemma 6.3. (Twice Quotient Lemma) Let $f\colon X \to Y$ be a continuous map between two topological spaces and let $\sim_X$ and $\sim_Y$ be equivalence relations on X respectively Y. If $x_1 \sim_X x_2$ implies that $f(x_1) \sim_Y f(x_2)$, then the map

$$\begin{array}{ccc} X/\sim_X & \to & Y/\sim_Y \\ [x] & \mapsto & [f(x)] \end{array}$$

is well-defined and continuous.

Proof. First note that it follows from the Topological-Quotient Proposition 6.1 (1a) that the map $X \xrightarrow{f} Y \xrightarrow{y\mapsto[y]} Y/\sim_Y$ is continuous. Next note that it follows from our hypothesis on f and the Topological-Quotient Proposition 6.1 (1b) that this map descends to a continuous map $X/\sim_X \to Y/\sim_Y$. ∎

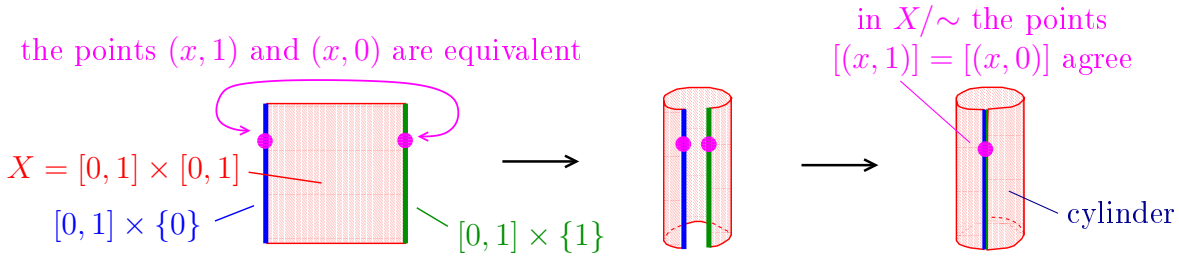
6.2. **Examples: Surfaces I.** In this section, and in its sister Section 6.6, we introduce some of my favorite topological spaces, namely surfaces. In the following we use the word "surface" in a colloquial manner.

(1) We consider $X = [0,1] \times [0,1] \subset \mathbb{R}^2$ and the equivalence relation that is generated by

$$(x,0) \sim (x,1) \quad \text{for all } x \in [0,1].$$

We will give two descriptions of this topological space, there first one is slightly informal, the second one is more rigorous.

(a) The quotient space $X/\sim$ is obtained from the square $X = [0,1]\times[0,1]$ by identifying each point on the upper edge with the corresponding point on the lower edge. In other words, "we glue the upper edge to the lower edge". This interpretation is illustrated in the figure below.



(b) We consider the map

$$\begin{array}{rcl} X/\sim & \to & S^1 \times [0,1] \\ [(s,t)] & \mapsto & (\exp(2\pi\,\mathrm{i}t), s). \end{array}$$

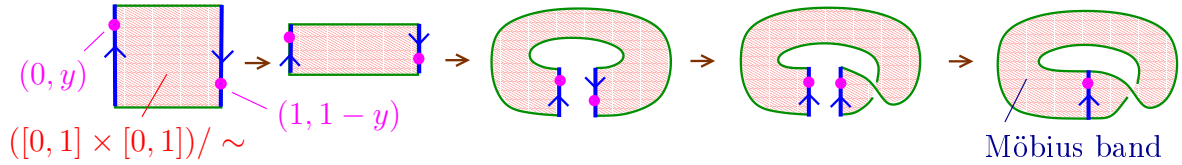
It follows from the Topological-Product Proposition 5.1 (1b) and the Topological-Quotient Proposition 6.1 (1b) that this map is continuous. One can easily verify that the map is a bijection. By the Quotient Topology-Properties Proposition 6.2 (1) we know that $X/\sim$ is compact and by the Product Topology Properties Proposition 5.4 (3) we know that $S^1 \times [0,1]$ is Hausdorff. Thus we obtain from the Compact-Hausdorff Proposition 2.10 (3) that the above map $X/\sim \to S^1 \times [0,1]$ is a homeomorphism. We refer to $X/\sim$ and also to $S^1 \times [0,1]$ as the cylinder or the annulus.

(2) We consider $X = [0,1] \times [0,1] \subset \mathbb{R}^2$, this time with the equivalence relation that is generated by

$$(0,y) \sim (1,1-y) \quad \text{for all } y \in [0,1].$$

We refer to the quotient $X/\sim$ as the Möbius band. Again we give two descriptions:

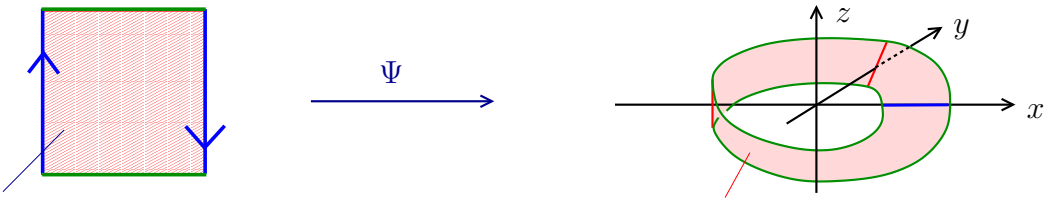
(a) The quotient space $X/\sim$ is obtained from the square X by gluing the edge on the left to the edge on the right, but this time "we glue the edges with a twist".



(b) The name "Möbius band" conjures up a certain picture. We will now see that our name is justified. We consider the map

$$\begin{array}{rcl} \Psi\colon ([0,1]\times[0,1])/\sim & \to & \mathbb{R}^3 \\ [(\varphi,r)] & \mapsto & \underbrace{\begin{pmatrix} \cos(2\pi\cdot\varphi) & -\sin(2\pi\cdot\varphi) & 0 \\ \sin(2\pi\cdot\varphi) & \cos(2\pi\cdot\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 2+(r-\frac{1}{2})\cdot\sin(\pi\cdot\varphi) \\ 0 \\ (r-\frac{1}{2})\cdot\cos(\pi\cdot\varphi) \end{pmatrix}}_{\substack{\text{describes interval} \\ \text{that makes half a twist} \\ \text{as we go around the } z\text{-axis}}} \end{array}$$

Similar to (1b) one can see that Ψ is an embedding. Thus the Möbius band, defined abstractly as above, is indeed homeomorphic to a subspace of $\mathbb{R}^3$ that looks like what we had in mind.



$M = ([0,1]\times[0,1])/(0,y)\sim(1,1-y)$ Möbius band viewed as a subspace of $\mathbb{R}^3$

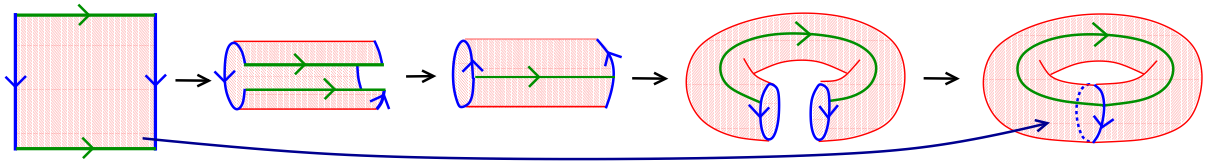
(3) We consider again $X = [0,1]\times[0,1]\subset\mathbb{R}^2$, but this time with the equivalence relation that is generated by

$$(x,0)\sim(x,1) \quad \text{for all } x\in[0,1]$$

and by

$$(0,y)\sim(1,y) \quad \text{for all } y\in[0,1].$$

(a) We obtain the quotient space $X/\sim$ by gluing the top edge to the bottom edge and the left-hand edge to the right-hand edge, each time without adding a twist. The figure below illustrates that the quotient space $X/\sim$ is indeed a torus.



by gluing of the opposite edges of a square we obtain a torus

(b) As in (1b) one can write down an obvious homeomorphism $X/\sim\to S^1\times S^1$. Thus we can make the identification $([0,1]\times[0,1])/\sim =_i S^1\times S^1$.

(4) We consider $X = [0,1]\times[0,1]\subset\mathbb{R}^2$, this time with the equivalence relation that is generated by

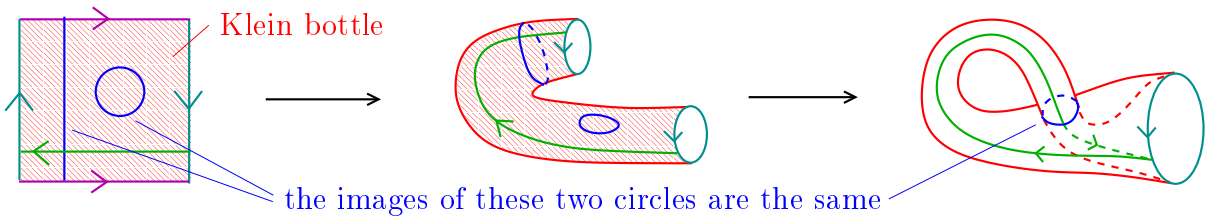
$$(x,0) \sim (x,1) \quad \text{for all } x\in[0,1]$$

and by

$$(0,y) \sim (1,1-y) \quad \text{for all } y\in[0,1].$$

The resulting topological space is called the Klein bottle.

(a) We obtain the quotient space $X/\sim$ by gluing the upper edge to the lower edge and by gluing the edge on the left to the edge on the right with a twist. In the figure below we show the Klein bottle to the left. Furthermore we indicate a map from the Klein bottle to $\mathbb{R}^3$. This particular map is not injective.



(b) Let $0 < b < a$ be real numbers. It is elementary, but painful, to show that the map

$$\begin{array}{rcl} ([0,1]\times[0,1])/\sim & \to & \mathbb{R}^4 \\ [(x,y)] & \mapsto & \begin{pmatrix} (a+b\cdot\cos(y\cdot 2\pi))\cdot\cos(x\cdot 2\pi) \\ (a+b\cdot\cos(y\cdot 2\pi))\cdot\sin(x\cdot 2\pi) \\ b\cdot\sin(y\cdot 2\pi)\cdot\cos(x\cdot\pi) \\ b\cdot\sin(y\cdot 2\pi)\cdot\sin(x\cdot\pi) \end{pmatrix} \end{array}$$

is an embedding. Thus we can view the Klein bottle as a subspace of $\mathbb{R}^4$.

We will introduce more surfaces a little later in Section 6.6.

6.3. **Quotients by subspaces.** In this section we will study another important source of quotient spaces.

Definition. Let X be a topological space and let $A \subset X$ be a subset.

(1) First we suppose that A is a non-empty subset. For $P, Q \in X$ we define

$$P \sim_A Q \quad :\Longleftrightarrow \quad P = Q \text{ or } P, Q \text{ both lie in } A.$$

This is easily seen to be an equivalence relation on X. We write $X/A := X/\sim_A$ and we always equip X/A with the quotient topology. We refer to X/A as the quotient of X by A. Note that $A \subset X$ is an equivalence class, thus it defines a point X/A. In an attempt to avoid confusion we write $[A]$ for the point in X/A that is given by the equivalence class A.

(2) If $A = \varnothing$ is the empty set, then we define $X/\varnothing := X \sqcup \{\varnothing\}$, i.e. $X/\varnothing$ equals the disjoint union of the topological space X with the topological space consisting of a single point $\varnothing$. For consistency we denote this point by $[A] = [\varnothing]$ as well.[63][64]

Remark. Let us attempt to digest the definition of X/A. As a set X/A corresponds precisely to the points in $X \setminus A$ together with the point $[A]$. Furthermore, by definition of the quotient topology a set $U \subset X/A$ is open if and only if $p^{-1}(U)$ is an open subset of X. Note that given any subset $Z \subset X/A = (X \setminus A) \cup \{[A]\}$ we have[65]

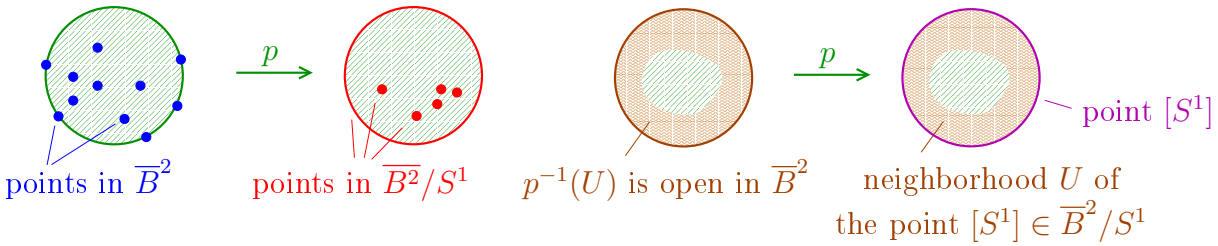
$$p^{-1}(Z) \;=\; \begin{cases} Z, & \text{if } [A] \notin Z, \\ Z \cup A, & \text{if } [A] \in Z. \end{cases}$$

[63] In particular $\varnothing/\varnothing$ is the topological space consisting of the single point $[\varnothing]$.

[64] The reader will have spotted that the case $A = \varnothing$ is rather ad hoc. On page 120 we will give a less ad hoc interpretation. In general the case $A = \varnothing$ is a major nuisance. It is conceivable that later on in proofs we will subconsciously suppress this special case. Whenever this occurs we leave it to the reader to deal with the case $A = \varnothing$ separately.

[65] In an attempt to lighten the notation, given $x \in X \setminus A$ we do not distinguish between the point $x \in X \setminus A$ and the equivalence class $\{x\} \subset X$.

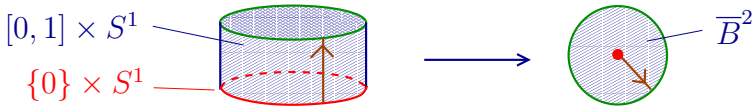
This shows that the open subsets in X/A correspond precisely to the open subsets of X that lie in $X \setminus A$ and the open subsets of X that contain A.



Lemma 6.4. (Cylinder Quotient-Ball Lemma) The map

$$\begin{array}{rcl} ([0,1] \times S^n)/(\{0\} \times S^n) & \to & \overline{B}^{n+1} \\ [(r,v)] & \mapsto & r \cdot v \end{array}$$

is a homeomorphism.



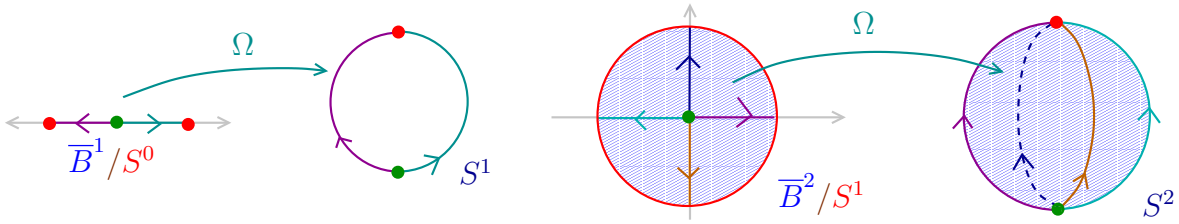
Proof. As in the proof of the Torus Embedding Lemma 5.5 it follows from the Topological-Quotient Proposition 6.1 (1b) that the given map is continuous. It is basically clear that the map is a bijection. The left-hand side is compact by the Topological-Quotient Proposition 6.1 (1b). It follows from the Compact-Hausdorff Proposition 2.10 (3) that the map is a homeomorphism. ■

Lemma 6.5. (Ball Quotient-Sphere Lemma) Let $n \in \mathbb{N}$.[66] The map[67]

$$\begin{array}{rcl} \Omega\colon \overline{B}^n/S^{n-1} & \to & S^n \\ [r \cdot v] & \mapsto & \underbrace{\begin{pmatrix} 0 & (-1)^{n-1} & 0 \\ 0 & 0 & \mathrm{id}_{n-1} \\ -1 & \dots & 0 \end{pmatrix}}_{\in \mathrm{SO}(n+1)} \cdot \underbrace{\begin{pmatrix} \cos(\pi \cdot r) \\ \sin(\pi \cdot r) \cdot v \end{pmatrix}}_{\in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}} \end{array} \quad \text{with } r \in [0,1] \text{ and } v \in S^{n-1}$$

is a homeomorphism.

Since $\overline{B}^0/S^{-1} = \overline{B}^0/\varnothing = \overline{B}^0 \cup \{*\}$ this lemma, with a couple of minor changes, also makes sense for $n = 0$.



[67] The weird signs come from the fact that we want the map $B^n \to S^n$ to be orientation-preserving, whatever that means.

Identification. We use the homeomorphism from the Ball Quotient-Sphere Lemma 6.5 to make the identification $S^n =_i \overline{B}^n/S^{n-1}$.

Proof of Lemma 6.5. It follows quite easily from the Cylinder Quotient-Ball Lemma 6.4 (2), the Topological-Quotient Proposition 6.1 (1b), the Quotient Topology-Properties Proposition 6.2 (1) and the Compact-Hausdorff Proposition 2.10 (3) that the map $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ is a homeomorphism. ■

6.4. **Group actions on topological spaces.** We recall the following definitions from pages 31 and 32.

Definition. Let X be a set and let G be a group with trivial element e.

(1) An action of G on X is a map

$$\begin{array}{rcl} G \times X & \to & X \\ (g,x) & \mapsto & g \cdot x \end{array}$$

with the following properties:

$$\begin{array}{rcll} e \cdot x & = & x, & \text{for all } x \in X, \\ g \cdot (h \cdot x) & = & (g \cdot_G h) \cdot x, & \text{for all } x \in X \text{ and } g,h \in G. \end{array}$$

group action; multiplication in the group G

(2) The action is called free, if $g \cdot x = x$ for some $x \in X$ implies that $g = e$.

(3) If X is a topological space, then we say that the action is continuous, if for every $g \in G$ the map

$$\begin{array}{rcl} X & \to & X \\ x & \mapsto & g \cdot x \end{array}$$

is continuous.

(4) Let X be a topological space and let G be a group that acts on X. By the Group Action-Equivalence Relation Lemma 0.13 we know that

$$x \sim_G y \quad :\Longleftrightarrow \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y$$

is an equivalence relation on X. We set $X/G := X/\sim_G$ and we view X/G as a topological space equipped with the quotient topology.

We formulate our first example as a lemma.

Lemma 6.6. (Torus-as-Quotient Lemma)

(1) The map

$$\begin{array}{rcl} \mathbb{Z}^n \times \mathbb{R}^n & \to & \mathbb{R}^n \\ (z,v) & \mapsto & z+v \end{array}$$

defines a free and continuous action of the group $(\mathbb{Z}^n,+)$ on the topological space $X = \mathbb{R}^n$.

(2) The map

$$\begin{array}{rcl} \mathbb{R}^n/\mathbb{Z}^n & \to & (S^1)^n \\ [(t_1,\dots,t_n)] & \mapsto & \big(\exp(2\pi \mathrm{i} t_1),\dots,\exp(2\pi \mathrm{i} t_n)\big) \end{array}$$

is a homeomorphism.

Proof of Lemma 6.6. We start out with the following observations:

(1) This statement follows easily from the definitions.

(2) (a) The map is continuous by the Topological-Product Proposition 5.1 (1b) and the Topological-Quotient Proposition 6.1 (1b). Furthermore it is straightforward to see that it is a bijection.

(b) The obvious map $[0,1]^n \to \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is continuous and it is easily seen to be surjective. By the Heine-Borel Theorem 1.11 we know that $[0,1]^n$ is compact. It follows from the Compact Image Lemma 2.8 that $\mathbb{R}^n/\mathbb{Z}^n$ is compact.

(c) From the Product Topology-Properties Proposition 5.4 (3) we obtain that $(S^1)^n$ is Hausdorff.

It follows from the above together with the Compact-Hausdorff Proposition 2.10 (3) that the given map $\mathbb{R}^n/\mathbb{Z}^n \to (S^1)^n$ is indeed a homeomorphism. ■

Example. If X is a Hausdorff space and if G acts continuously on X, then it is unfortunately not necessarily true that the quotient space X/G itself is Hausdorff. For example, consider the obvious action of $G = (\mathbb{Q}, +)$ on $X = \mathbb{R}$. It is a moderately insightful exercise to verify that the quotient space $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.

This example leads us to the following definition.

Definition. Let X be a topological space and let G be a group that acts on X. We say that G acts properly if for every two points[68] x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \,|\, gU \cap V \neq \varnothing\}$ is finite.[69]

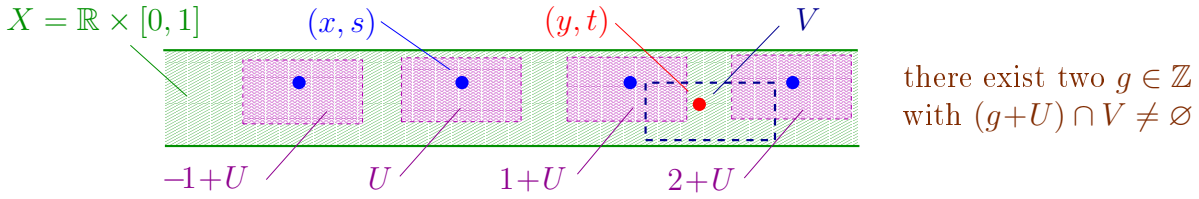
Examples.

(1) The action of any finite group is proper by definition.

(2) We consider the action of the group $G = \mathbb{Z}$ on $X = \mathbb{R} \times [0,1]$ that is given by

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [0,1]) & \to & \mathbb{R} \times [0,1] \\ (n, (x,y)) & \mapsto & (x+n, y). \end{array}$$

This action is proper. For example if we are given by $(x,s), (y,t) \in \mathbb{R} \times [0,1]$, then $U := \{(a,b) \in \mathbb{R} \times [0,1] \,|\, |a-x| < 1\}$ and $V := \{(a,b) \in \mathbb{R} \times [0,1] \,|\, |a-y| < 1\}$ have the desired properties. With some artistic freedom this situation is shown in the figure below.



(3) Similar to (2) one can easily verify that the action of $G = (\mathbb{Z}^n, +)$ on $X = \mathbb{R}^n$ is proper.

(4) One can easily show that the obvious action of $G = (\mathbb{Q}, +)$ on $X = \mathbb{R}$ is not proper.

Proposition 6.7. (Group Action-Hausdorff Quotient Proposition) Let G be a group that acts on a topological space X. If X is Hausdorff and if the action is proper and continuous, then the quotient space X/G is also Hausdorff.

[68] Note that we allow $x = y$.

[69] The notion of a "proper action" is defined in different ways by different authors.

Proof. We outsource the uninspiring proof to [Fri23]. ■

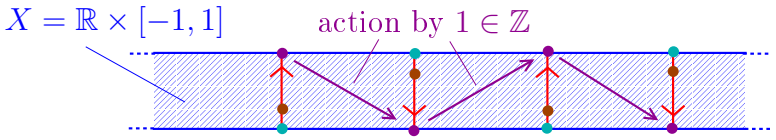
Instead of dealing with the proof of the above proposition, let us look at an example:

Lemma 6.8. (Möbius Band-as-Quotient Lemma)

(1) The map
$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n,(x,y)) & \mapsto & (x+n,(-1)^n \cdot y) \end{array}$$
defines an action of the group $(\mathbb{Z},+)$ on the topological space $X = \mathbb{R} \times [-1,1]$ that is free, continuous and proper.

(2) As on page 107 let $([0,1] \times [0,1])/\sim$ be the Möbius band, where $(0,y) \sim (1,1-y)$. The map
$$\begin{array}{rcl} ([0,1] \times [0,1])/\sim & \to & (\mathbb{R} \times [-1,1])/\sim \\ [(x,y)] & \mapsto & [(x,2y-1)] \end{array}$$
is a homeomorphism.

(3) The Möbius band $M = ([0,1] \times [0,1])/\sim$ is compact and Hausdorff.



Identification. We will use the homeomorphism from the Möbius Band-as-Quotient Lemma 6.8 to identify the Möbius band $([0,1] \times [0,1])/\sim$ with $(\mathbb{R} \times [-1,1])/\sim$.

Proof of Lemma 6.8.

(1) It is clear that the action is free and continuous. The fact that the action is proper is proved as in the above example.

(2) It follows from the Twice Quotient Lemma 6.3 that the map is well-defined and continuous. One can easily verify that the map is a bijection. By the Quotient Topology-Properties Proposition 6.2 (1) we know that $([0,1] \times [0,1])/\sim$ is compact. Furthermore it follows from (1) and the Group Action-Hausdorff Quotient Proposition 6.7 that $(\mathbb{R} \times [0,1])/\sim$ is Hausdorff. Thus it follows from the Compact-Hausdorff Proposition 2.10 (3) that the map $([0,1] \times [0,1])/\sim \to (\mathbb{R} \times [0,1])/\sim$ is a homeomorphism.

(3) We showed in the proof of (2) that the Möbius band is compact and Hausdorff. ■

6.5. **Projective spaces.** In this section we introduce two related families of topological spaces, namely the real and complex projective spaces, which initially might look artificial, but which will play an essential role throughout these notes.

Definition.

(1) The map
$$\begin{array}{rcl} (\mathbb{R} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\}) & \to & \mathbb{R}^{n+1} \setminus \{0\} \\ (z,P) & \mapsto & z \cdot P \end{array}$$
defines a continuous action of the multiplicative group $(\mathbb{R} \setminus \{0\},\cdot)$ on the topological space $\mathbb{R}^{n+1} \setminus \{0\}$. The quotient space $(\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\})$ is called the n-dimensional real projective space $\mathbb{R}\mathrm{P}^n$. For $n = 2$ we sometimes refer to $\mathbb{R}\mathrm{P}^2$ as the real projective plane.

(2) In (1) we can replace "$\mathbb{R}$" by "$\mathbb{C}$" and we obtain the n-dimensional complex projective space $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$.

Remark. Informally speaking we can view $\mathbb{R}\mathrm{P}^n$ as the set of lines in in $\mathbb{R}^{n+1}$ through the origin, or equivalently as the set of 1-dimensional vector subspaces of $\mathbb{R}^{n+1}$.

In our treatments of real and complex projective spaces we will often use the following notation.

Notation. For $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ we denote by $[x_0 : \ldots : x_n]$ the corresponding point in $\mathbb{R}\mathrm{P}^n$. We also use the same notation for points in complex projective spaces.

The following lemma gives us a convenient alternative description of projective spaces.

Lemma 6.9. (Projective Space-Sphere Quotient Lemma) Let $n \in \mathbb{N}_0$.

(1) The map

$$\begin{array}{rcl} \{\pm 1\} \times S^n & \to & S^n \\ (w, (x_1, \ldots, x_{n+1})) & \mapsto & (w \cdot x_1, \ldots, w \cdot x_{n+1}) \end{array}$$

defines a continuous action of the group $\{\pm 1\}$ on S^n. The maps

$$\begin{array}{rcl} S^n/\{\pm 1\} & \to & \mathbb{R}\mathrm{P}^n \\ [x] & \mapsto & [x] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{R}\mathrm{P}^n & \to & S^n/\{\pm 1\} \\ [x] & \mapsto & \left[\frac{x}{\|x\|}\right] \end{array}$$

are inverses of one another and they are homeomorphisms.

(2) The map

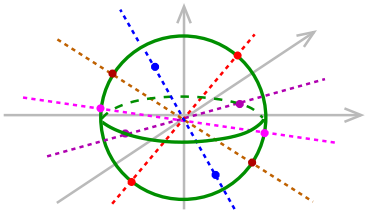
$$\begin{array}{rcl} S^1 \times \overbrace{S^{2n+1}}^{\subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}} & \to & S^{2n+1} \\ (w, (z_1, \ldots, z_{n+1})) & \mapsto & (w \cdot z_1, \ldots, w \cdot z_{n+1}) \end{array}$$

defines a continuous action of the group $(S^1, \cdot)$ on S^{2n+1}. The maps

$$\begin{array}{rcl} S^{2n+1}/S^1 & \to & \mathbb{C}\mathrm{P}^n \\ [z] & \mapsto & [z] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{C}\mathrm{P}^n & \to & S^{2n+1}/S^1 \\ [z_1 : \ldots : z_{n+1}] & \mapsto & \left[\frac{(z_1, \ldots, z_{n+1})}{\sqrt{|z_1|^2 + \ldots + |z_{n+1}|^2}}\right] \end{array}$$

are inverses of one another and they are homeomorphisms.

$S^2/\{\pm 1\}$ is the "set of pairs of antipodal points on S^2"



$\mathbb{R}\mathrm{P}^2 = (\mathbb{R}^3 \setminus \{0\}/(\mathbb{R} \setminus \{0\})$ is the "set of lines through 0"

Identification. We use the homeomorphisms from the Projective Space-Sphere Quotient Lemma 6.9 to make the identifications $\mathbb{R}\mathrm{P}^n =_i S^n/\{\pm 1\}$ and $\mathbb{C}\mathrm{P}^n =_i S^{2n+1}/S^1$.

Proof of Lemma 6.9. It follows easily from the Twice Quotient Lemma 6.3 that all four maps are continuous. It is easy to verify that in (1) and (2) the maps are inverses of one another. ■

The following lemma shows that real and complex projective spaces have some of our favorite properties.

Proposition 6.10. (Projective Spaces-Compact+Hausdorff Proposition) The real projective and the complex projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ are compact, path-connected and Hausdorff.

Example. Let $n \in \mathbb{N}_0$ and let $m \leq n$. We consider the maps

$$\begin{array}{rcl} i\colon \mathbb{R}\mathrm{P}^m & \to & \mathbb{R}\mathrm{P}^n \\ [x_0 \colon \ldots \colon x_m] & \mapsto & [x_0 \colon \ldots \colon x_m \colon 0 \colon \ldots \colon 0] \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 \colon \ldots \colon z_m] & \mapsto & [z_0 \colon \ldots \colon z_m \colon 0 \colon \ldots \colon 0]. \end{array}$$

It follows from the Twice Quotient Lemma 6.3, the Projective Spaces-Compact+Hausdorff Proposition 6.10 and the Compact-Hausdorff Proposition 2.10 (2) that both maps are embeddings.

Proof. Let $n \in \mathbb{N}_0$. By the Projective Space-Sphere Quotient Lemma 6.9 it suffices to show that $S^n/\{\pm 1\}$ and S^{2n+1}/S^1 are compact, path-connected and Hausdorff. In fact it follows immediately from the fact that S^n and S^{2n+1} are compact and path-connected together with the Quotient Topology-Properties Proposition 6.2 (1) and (2) that both quotients are compact and path-connected. Furthermore note that it follows from the Group Action-Hausdorff Quotient Proposition 6.7 that $S^n/\{\pm 1\}$ is Hausdorff. In [Fri23] it is shown, using the Quotient Topology-Properties Proposition 6.2 (3), that S^{2n+1}/S^1 is also Hausdorff. ■

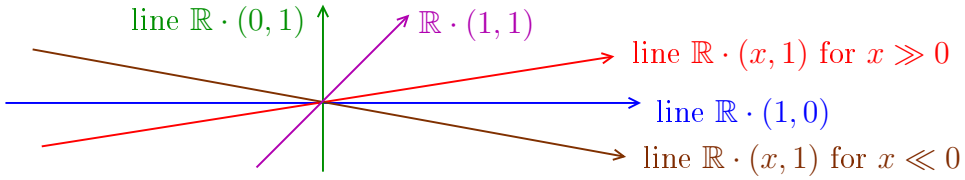
In the following lemma we give alternative descriptions of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^1$ that will be used on many occasions. For technical reasons the lemma is split into three parts.

Lemma 6.11. (Projective Space-Avatar Lemma)

(1) The maps

$$\begin{array}{rcl} \overbrace{\mathbb{C} \cup \{\infty\}}^{=_i S^2} & \to & \mathbb{C}\mathrm{P}^1 \\ z & \mapsto & \begin{cases} [z:1], & \text{if } z \in \mathbb{C}, \\ [1:0], & \text{if } z = \infty \end{cases} \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{C}\mathrm{P}^1 & \to & \overbrace{\mathbb{C} \cup \{\infty\}}^{=_i S^2} \\ [z_0 : z_1] & \mapsto & \begin{cases} z_0 \cdot z_1^{-1}, & \text{if } z_1 \neq 0, \\ \infty, & \text{if } [z_0 : z_1] = [1:0] \end{cases} \end{array}$$

are inverses of one another and both are homeomorphisms. Similarly we see that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to $\mathbb{R} \cup \{\infty\}$, which in turn is homeomorphic to S^1.



Lemma 6.11. (Projective Space-Avatar Lemma)

(2) Let $n \in \mathbb{N}_0$. If we denote by $\sim$ the equivalence relation on $\overline{B}^n$ that is generated by $P \sim -P$ for $P \in S^{n-1} = \partial \overline{B}^n$, then the maps[70]

$$\begin{array}{rcl} \overline{B}^n/\sim & \to & \mathbb{R}\mathrm{P}^n \\ [x] & \mapsto & [(x, \sqrt{1 - \|x\|^2})] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{R}\mathrm{P}^n =_i S^n/\{\pm 1\} & \xrightarrow{\cong} & \overline{B}^n/\sim \\ \underbrace{[(v, x_{n+1})]}_{v \in \mathbb{R}^n,\, x_{n+1} \in \mathbb{R}} & \mapsto & \begin{cases} [v], & \text{if } x_{n+1} \geq 0, \\ [-v], & \text{if } x_{n+1} \leq 0 \end{cases} \end{array}$$

are homeomorphisms that are inverses of one another.

Lemma 6.11. (Projective Space-Avatar Lemma)

[70] Note that the map to the left is in fact well-defined even if $x_{n+1} = 0$.

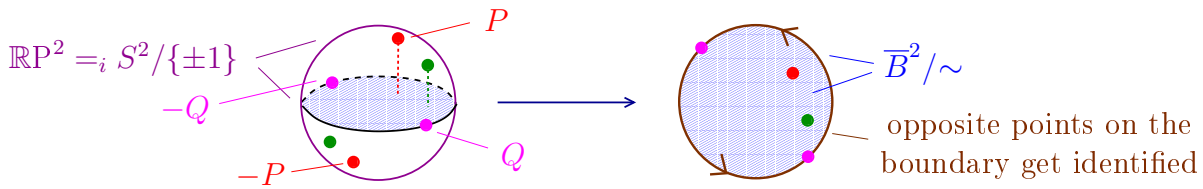


Illustration of the homeomorphism $\mathbb{R}\mathrm{P}^2 \to \overline{B}^2/\sim$

(3) As in (2) let $\sim$ be the equivalence relation on $\overline{B}^3$ that is generated by $x \sim -x$ for $x \in S^2$. The map

$$\begin{array}{rcl} F\colon \overline{B}^3/\sim & \to & \mathrm{SO}(3) \\ [\underbrace{r \cdot v}_{r \in [0,1] \text{ and } v \in S^2}] & \mapsto & \text{matrix in SO(3) which represents rotation by the angle } \pi \cdot r \\ & & \text{around the axis } \mathbb{R}_{\geq 0} \cdot v \text{ where the rotation} \\ & & \text{direction is given by the right-hand rule} \end{array}$$

is a homeomorphism. In particular SO(3) is homeomorphic to $\mathbb{R}\mathrm{P}^3$.

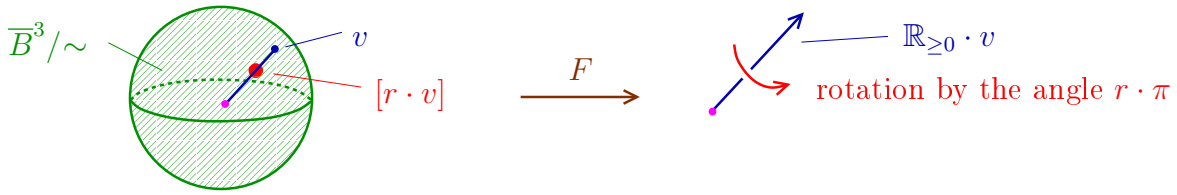


Illustration of the homeomorphism $\overline{B}^3/\sim \to \mathrm{SO}(3)$

Identification.

(1) We will use the homeomorphism from the Projective Space-Avatar Lemma 6.11 (1) to make the identification $S^1 =_i \mathbb{R}\mathrm{P}^1$.

(2) Recall that in the Stereographic Projection Lemma 2.11 (1) we gave an explicit homeomorphism $S^2 \to \mathbb{R}^2 \cup \{\infty\}$. Combining this homeomorphism with the homeomorphism from the Projective Space-Avatar Lemma 6.11 (1) we obtain a homeomorphism $S^2 \to \mathbb{C}\mathrm{P}^1$ which we use to make the identification $S^2 =_i \mathbb{C}\mathrm{P}^1$. Note that under this identification the North Pole $(0,0,1) \in S^2$ corresponds to the point $[1:0] \in \mathbb{C}\mathrm{P}^1$.

(3) We will use the homeomorphism from the Projective Space-Avatar Lemma 6.11 (2) to make the identification $\mathbb{R}\mathrm{P}^n =_i \overline{B}^n/\sim$.

Proof of Lemma 6.11.

(1) One can easily verify that the given maps are inverses of one another. Next we show that the given map $\mathbb{C}\mathrm{P}^1 \to \mathbb{C} \cup \{\infty\}$ is continuous. First note that it follows from the Topological-Quotient Proposition 6.1 (1b) that it suffices to show that the corresponding map $\Theta\colon \mathbb{C}^2 \setminus \{(0,0\} \to \mathbb{C} \cup \{\infty\}$ is continuous. Next note that for any $C \in \mathbb{R}$ we have

$$\Theta^{-1}(\text{complement of } \overline{B}^n_C) \;=\; \{(z_0, z_1) \in \mathbb{C}^2 \setminus \{(0,0)\} \,|\, |z_0| > |z_1|\}$$

which is an open subset of $\mathbb{C}^2 \setminus \{(0,0)\}$. It follows quite easily from this discussion together with the Subbasis-Continuity Proposition 2.2 and together with the discussion of the basis of $\mathbb{R}^2 \cup \{\infty\}$ on page 47 that the map Θ is continuous. By the Projective

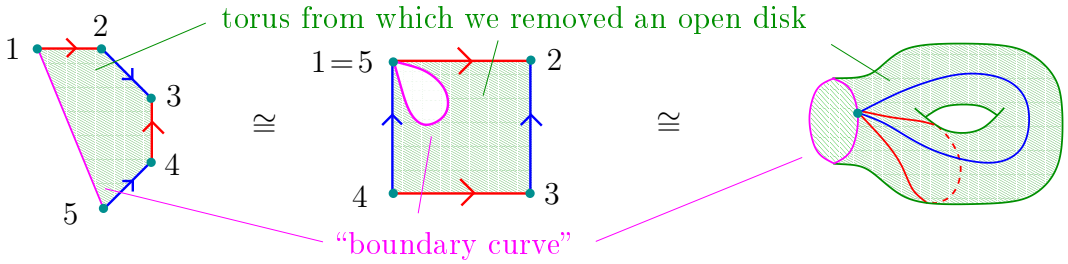
Spaces-Compact+Hausdorff Proposition 6.10 we know that each $\mathbb{C}\mathrm{P}^1$ is compact. We leave it to the reader to verify that $\mathbb{R}^2 \cup \{\infty\}$ is Hausdorff. Thus it follows from the Compact-Hausdorff Proposition 2.10 (3) that the map $\mathbb{C}\mathrm{P}^1 \to \mathbb{C} \cup \{\infty\}$ is a homeomorphism.

(2) It is elementary to verify that the given continuous maps are inverses of one another, in particular they are bijections. We still need to show that the maps are continuous.
   (a) We consider the map $S^n/\{\pm 1\} \to \overline{B}^n/\sim$. By the Topological-Quotient Proposition 6.1 (1b) it suffices to show that the corresponding map $S^n \to \overline{B}^n/\sim$ is continuous. One can easily verify that the restrictions of the map $S^n \to \overline{B}^n/\sim$ to the closed subsets $S^n_{\geq 0}$ and $S^n_{\leq 0}$ are continuous. Thus it follows from the Pasting Proposition 2.4 (2) that the map $S^n \to \overline{B}^n/\sim$ is continuous.
   (b) It follows almost immediately from the Twice Quotient Lemma 6.3 that the given map $\overline{B}^n/\sim \to \mathbb{R}\mathrm{P}^n$ is continuous.

(3) The proof that $F\colon \overline{B}^3/\sim \to \mathrm{SO}(3)$ is a homeomorphism is given in [Fri23]. ■

6.6. **Examples: Surfaces II.** In Section 6.2 we already introduced some surfaces. It is time to introduce a few more.

Example. We start out with a slightly informal discussion of an example. In the figure below we see to the left a pentagon, where we identify two pairs of sides. In the figure we illustrate that this topological space is homeomorphic to the torus minus an open disk.



The above example motivates our next example:

Example. We introduce the following notation:

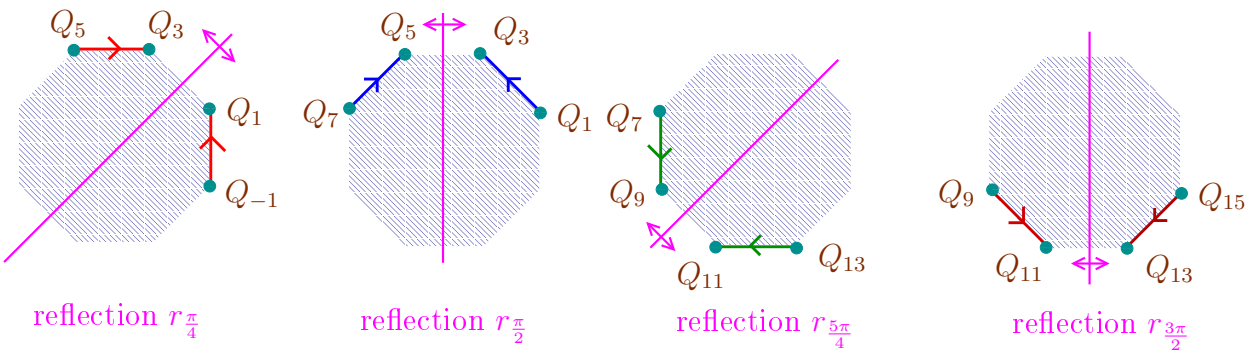
(1) Let E_8 be the regular octagon in $\mathbb{C}$ with the vertices $Q_k = \exp(2\pi \mathrm{i}k/16)$, $k = 1, 3, \ldots, 15$, i.e. E_8 is the convex hull of these eight points in $\mathbb{R}^2$.
(2) Given two points $A, B \in \mathbb{R}^2 = \mathbb{C}$ let $\overline{AB}$ be the Euclidean segment from A to B.
(3) Given $\varphi \in \mathbb{R}$ let $r_\varphi\colon \mathbb{C} \to \mathbb{C}$ be the reflection in the Euclidean line $\{t \cdot \exp(\mathrm{i}\varphi) \,|\, t \in \mathbb{R}\}$.

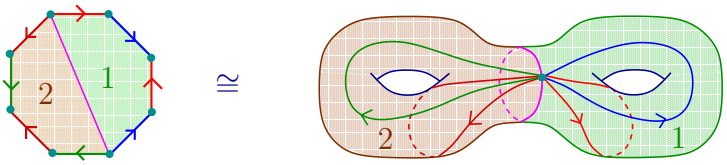
We denote by $\sim$ the equivalence relation on E_8 that is generated by

$$P \in \overline{Q_{2k-1}Q_{2k+1}} \;\sim\; r_{2\pi(2k+2)/16}(P) \;\in\; \overline{Q_{2k+3}Q_{2k+5}}$$

for $k = 0, 1, 4, 5$. We consider the quotient space $E_8/\sim$.
Note that $E_8/\sim$ is given by two copies of the previous example glued together along the "boundary curves". In other words, $E_8/\sim$ is homeomorphic to the actual "physical" surface of genus 2 in $\mathbb{R}^3$.

The above example leads us to the following definition.



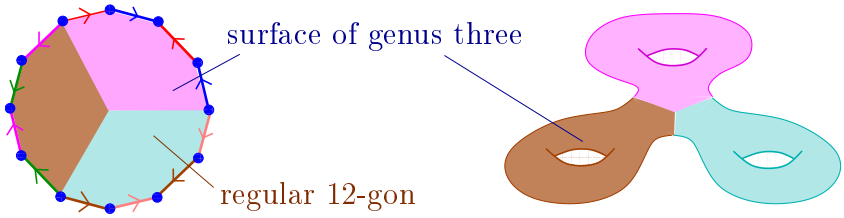


two copies of torus minus an open disk glued along the boundary

Definition.

(1) Let $g \in \mathbb{N}$. We take a regular $4g$-gon E_{4g} in $\mathbb{C}$ and for $j = 0, \dots, g-1$ we use reflections to identify the $(4j+1)$-st edge with the $(4j+3)$-rd edge and the $(4j+2)$-nd edge with the $(4j+4)$-th edge. We refer to the quotient space $E_{4k}/\sim$ as the surface of genus g.

(2) We refer to the sphere S^2 as the surface of genus 0.



Example. On page 108 we saw that the surface of genus 1 is homeomorphic to the torus $S^1 \times S^1$.

We move on to a closely related definition.

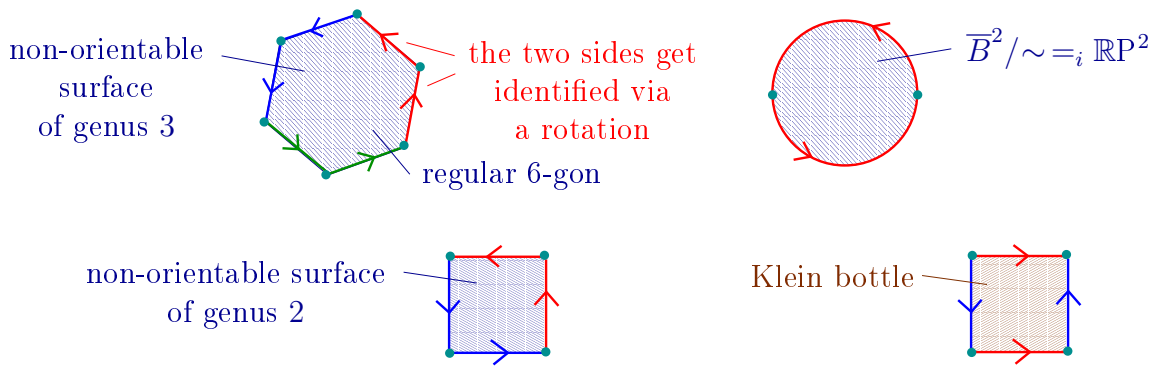
Definition.

(1) Given $g \in \mathbb{N}_{\geq 2}$ we consider the regular $2g$-gon E_{2g} with the identification of the boundary segments shown in the figure below. We refer to this topological space as the non-orientable surface N_g of genus g.

(2) We refer to the real projection place $\mathbb{R}\mathrm{P}^2$ as the non-orientable surface of genus 1.

Example. We leave it to the reader to show that the non-orientable surface of genus 2 is homeomorphic to the Klein bottle.

6.7. **The pushout in the topological category.** We conclude our long list of various types of quotients of topological spaces with a discussion of pushouts in the category of topological spaces. The following proposition is formulated in the tradition of the



Topological-Product Proposition 5.1,f the Topological-Coproduct Proposition 5.10 and the Topological-Quotient Proposition 6.1.

Proposition 6.12. (Topological-Pushout Proposition) Let $f\colon A \to Y$ and $g\colon A \to Z$ be two continuous maps between topological spaces. We consider the topological space

$$Y \cup_A Z \;:=\; (Y \sqcup Z)/\sim \qquad \text{where } f(a) \sim g(a) \text{ for all } a \in A.$$

and the maps

$$\begin{array}{rcl} i\colon Y & \to & Y \cup_A Z \\ y & \mapsto & [y] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} j\colon Z & \to & Y \cup_A Z \\ z & \mapsto & [z] \end{array}$$

The following statements hold:

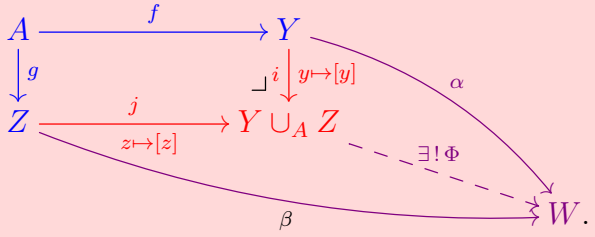
(0) A subset $U \subset Y \cup_A Z$ is open if and only if $i^{-1}(U) \subset Y$ and $j^{-1}(U) \subset Z$ are open. The same statement holds for "closed" instead of "open".

(1) (a) The maps $i\colon Y \to Y \cup_A Z$ and $j\colon Z \to Y \cup_A Z$ are continuous.

(b) Suppose we are given two continuous maps $\alpha\colon Y \to W$ and $\beta\colon Z \to W$ to some topological space W such that $\alpha \circ f = \beta \circ g\colon A \to W$. The map

$$\begin{array}{rcl} \Phi\colon Y \cup_A Z & \to & W \\ [P] & \mapsto & \begin{cases} \alpha(y) & \text{if } P = [y] \text{ for some } y \in Y, \\ \beta(z) & \text{if } P = [z] \text{ for some } z \in Z \end{cases} \end{array}$$

is well-defined and it is the unique map $h\colon Y \cup_A Z \to W$ with $\Phi \circ i = \alpha$ and $\Phi \circ j = \beta$. Furthermore, if α and β are continuous, then Φ is also continuous. In particular we obtain the following commutative diagram:



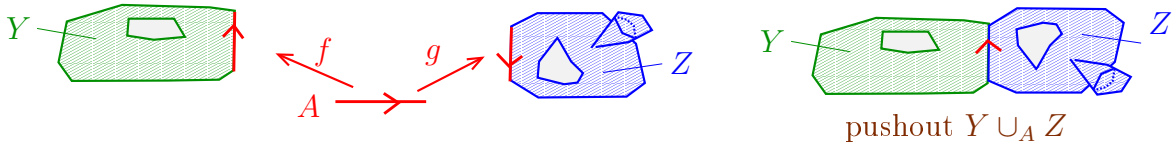
(∗) A map $f\colon Y \cup_A Z \to W$ to some topological space W is continuous if and only if the maps $f \circ i\colon Y \to W$ and $f \circ j\colon Z \to W$ are continuous.

(2) The topological space $Y \cup_A Z$ with the natural maps $Y \to Y \cup_A Z$ and $Z \to Y \cup_A Z$ is the pushout, in the sense of the definition on page 89, in the category *Top* of topological spaces.

Definition. Let $f\colon A \to Y$ and $g\colon A \to Z$ be two continuous maps between topological spaces. We refer to

$$Y \cup_A Z \;:=\; (Y \sqcup Z)/\!\sim \qquad \text{where } f(a) \sim g(a) \text{ for all } a \in A$$

as the pushout of f and g and we refer to the maps $i\colon Y \to Y \cup_A Z$ and $j\colon Z \to Y \cup_A Z$ as the two natural maps.



Proof.

(0) This statement follows from the Topological-Coproduct Proposition 5.10 (0) and the Topological-Quotient Proposition 6.1 (0).

(1) (a) This statement follows immediately from (0). Alternatively this statement follows from the Topological-Coproduct Proposition 5.10 (1a) and the Topological-Quotient Proposition 6.1 (1a).

(b) This existence of the continuous map follows immediately from the Topological-Coproduct Proposition 5.10 (1b) and the Topological-Quotient Proposition 6.1 (1b). We leave the proof of the uniqueness to the reader.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) This statement follows easily from the definitions and Statement (1). We leave it to the reader to fill in the details. ■

Example. Let X be a topological space, let $A \subset X$ be a subspace, let $f\colon A \to \{*\}$ be the constant map to the topological space consisting of a single point and let $g\colon A \to X$ be the inclusion map. Using the uniqueness statement of the Quotient-Factorization Lemma 0.5 and using the Topological-Quotient Proposition 6.1 one can show that the quotient X/A, as defined on page 109, is a pushout of the pushout diagram. In summary we have the following pushout diagram:

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & \{*\} \\ {\scriptstyle g}\downarrow & & \downarrow \\ X & \xrightarrow{\;x\mapsto[x]\;} & X/A. \end{array}$$

From the uniqueness statements for pushouts, see the Colimit Uniqueness Lemma 4.7, it follows that X/A is homeomorphic to the pushout $X \cup_A \{*\}$.[71]

The following proposition is the analogue of the Disjoint Union Topology-Properties Lemma 5.11 and the Quotient Topology-Properties Proposition 6.2.

[71] Note that this even works if $A = \varnothing$.

Proposition 6.13. (Pushout-Properties Proposition) Let $f\colon A \to Y$, $g\colon A \to Z$ be two continuous maps between topological spaces. We consider the corresponding pushout $Y \cup_A Z := (Y \sqcup Z)/\sim$.

(1) If Y and Z are compact, then so is the pushout $Y \cup_A Z$.
(2) If Y and Z are (path-) connected and if A is non-empty, then the pushout $Y \cup_A Z$ is also (path-) connected.
(3) If A, Y and Z are compact, if Y and Z are Hausdorff and if $f\colon A \to Y$ and $g\colon A \to Z$ are injective, then $Y \cup_A Z$ is also Hausdorff.

Proof.

(1) If Y and Z are compact, then it follows from the Disjoint Union Topology-Properties Lemma 5.11 (1) and the Quotient Topology-Properties Proposition 6.2 (1) that $Y \cup_A Z$ is compact.
(2) If Y and Z are (path-) connected and if A is non-empty, then it follows easily from the Topological Pushout Proposition 6.12 Image-Connected Lemma 2.16 and the Union Connected Lemma 2.19 that $Y \cup_A Z$ is also (path-) connected.
(3) Since f and g are injective we see that the relation on $Y \sqcup Z$ is given by

$$\{(f(a), g(a)) \,|\, a \in A\} \cup \{(w, w) \,|\, w \in Y \sqcup Z\} \subset (Y \sqcup Z) \times (Y \sqcup Z).$$

It follows from the Compact Image Lemma 2.8 and the Compact-Closed Lemma 1.9 (2), together with the hypotheses that A, Y, Z are compact and Y, Z are Hausdorff, that this is a closed subset. It now follows from the Quotient Topology-Properties Proposition 6.2 (3) that $Y \cup_A Z$ is Hausdorff. ∎

Example. Let $n \in \mathbb{N}_0$. We consider the two obvious inclusion maps $S^{n-1} \to \overline{B}^n_\pm := \overline{B}^n$ and the corresponding pushout $\overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_-$. As in the Stereographic Projection Lemma 2.11 we consider for $\epsilon = -1, 1$ the homeomorphisms[72]

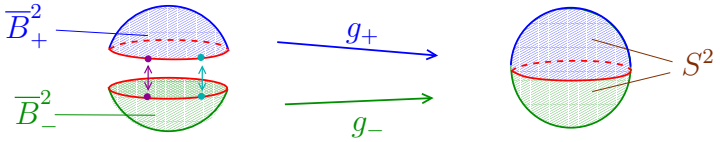
$$\begin{array}{rcl} g_\epsilon\colon \overline{B}^n_\epsilon & \to & S^n_{\geq 0} \\ x = (x_1, \ldots, x_n) & \mapsto & \left(((-1)^{n+1} \cdot x_1, \ldots, x_n) \cdot \frac{2}{\|x\|^2+1}, \epsilon \cdot \frac{\|x\|^2-1}{\|x\|^2+1}\right). \end{array}$$

Note that $g_+ = g_-$ on S^{n-1}. Thus it follows from the Topological Pushout Proposition 6.12 (1b) that the map

$$\begin{array}{rcl} \overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_- & \to & S^n \\ [P] & \mapsto & \begin{cases} g_+(P), & \text{if } P \in \overline{B}^n_+, \\ g_-(P), & \text{if } P \in \overline{B}^n_- \end{cases} \end{array}$$

[72] We put in the superfluous looking sign $(-1)^{n+1}$ for the following reason: once we have introduced orientations of smooth manifolds and once we have introduced the standard orientations of $\overline{B}^n$ and S^n this sign assures us that the map $g_+\colon \overline{B}^n \to S^n$ is orientation-preserving and it means that $g_-\colon \overline{B}^n \to S^n$ is orientation-reversing.

is continuous. One can easily verify that it is a bijection. It follows from the Pushout-Properties Proposition 6.13 (1) combined with the Compact-Hausdorff Proposition 2.10 (3) that the map is a homeomorphism.[73][74]



Identification. We use the homeomorphism from the above example to make the identification $\overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_- =_i S^n$.

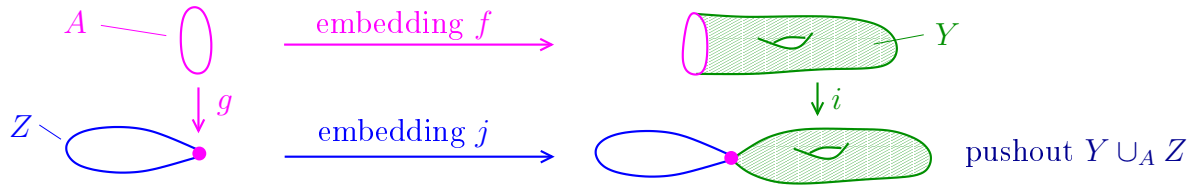
Lemma 6.14. (Topological Pushout-Maps Lemma) Let $f\colon A \to Y$ and $g\colon A \to Z$ be two continuous maps between topological spaces. We obtain the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & & \lrcorner \downarrow{\scriptstyle i} \; {\scriptstyle y\mapsto[y]} \\ Z & \xrightarrow[z\mapsto[z]]{j} & Y \cup_A Z. \end{array}$$

The following statements hold:

(1) If f is a homeomorphism, then the "opposite map" j is a homeomorphism.
(2) If f is injective, then the "opposite map" j is injective. The same statement holds for "surjective" and "bijective".[75]
(3) If f is an embedding, then the "opposite map" j is an embedding.

By symmetry the statements in (1), (2) and (3) hold also for g and its "opposite map" i.



Proof. Throughout the proof we denote by $p\colon Y \sqcup Z \to Y \cup_A Z$ the natural projection.

(1) This statement is a special case of the Pushout Lemma 4.8 (2).

[73] In [Lip14, Chapter 2] and [Pet79] the argument is made that the universe in Dante's Divina Commedia consists of two 3-balls (corresponding to heaven and hell) that are glued along the boundary. In other words, Dante describes the 3-sphere.

[74] We could also have worked with the maps

$$\begin{array}{rcl} h_\pm\colon \overline{B}^n = \overline{B}^n_\pm & \to & S^n \\ (x_1,\dots,x_n) & \mapsto & \big((-1)^n \cdot x_1, x_2, \dots, x_n, \pm\sqrt{1 - x_1^2 - \dots - x_n^2}\big). \end{array}$$

The advantage of our maps $g_\pm$ is that they are not only homeomorphisms but they are actually "diffeomorphisms".

[75] This statement has nothing to do with topological spaces, the statement also holds for sets.

(2) First we assume that $f\colon A \to Y$ is injective. We need to show that $j\colon Z \to Y \cup_A Z$ is injective. Let $z \in Z$. Note that

$$j^{-1}(j(z)) = \{z' \in Z \mid j(z') = j(z)\} = \begin{array}{c}\text{set of elements in } Z \text{ equivalent to } z \\ \text{with respect to } \sim \text{ on } Y \sqcup Z\end{array}$$
$$= \bigcup_{k\in\mathbb{N}_0} \underbrace{g(f^{-1}(f(g^{-1}(\dots g(f^{-1}(f(g^{-1}}_{\text{applied } k \text{ times}}(\{z\}))))\dots))))$$
$$\overset{\text{by the Image-Preimage Lemma 0.3 (1b), since } f \text{ is injective}}{=} \bigcup_{k\in\mathbb{N}_0} g(g^{-1}(\dots g(g^{-1}(\{z\}))\dots)) \overset{\text{Image-Preimage Lemma 0.3 (2a)}}{=} \{z\}.$$

This shows that the map j is injective.

Now we assume that $f\colon A \to Y$ is surjective. We need to show that $j\colon Z \to Y \cup_A Z$ is surjective. Recall that by definition $Y \cup_A Z := (Y \sqcup Z)/\sim$ where $f(a) \sim g(a)$ for all $a \in A$. Let $[w] \in Y \cup_A Z$. If $w \in Z$, then evidently $j(w) = [w]$. Now consider the case that $w \in Y$. Since $f\colon A \to Y$ is surjective there exists an $a \in A$ with $f(a) = w$. But now we see that $j(g(a)) = i(f(a)) = i(w) = [w]$.

(3) We suppose that $f\colon A \to Y$ is an embedding. We need to show that $j\colon Z \to Y \cup_A Z$ is an embedding. By the Topological Pushout Proposition 6.12 (1a) we know that j is continuous and by (2) we know that $j\colon Z \to Y \cup_A Z$ is injective. By the Open-Injective Lemma 2.9 (2) and by definition of the subspace topology on $j(Z)$ it remains to prove the following claim.

Claim. If U is an open subset of Z, then there exists an open subset W of $Y \cup_A Z$ such that $j(U) = W \cap j(Z)$.

Proof. Let U be an open subset of Z. Since g is continuous we know that $g^{-1}(U)$ is an open subset of A. Since $f\colon X \to Y$ is an embedding we know that there exists an open subset $V \subset Y$ with $f(g^{-1}(U)) = f(A) \cap V$. We set $W := p(U \sqcup V)$. An elementary argument as in (2), that uses that f is injective, shows that $p^{-1}(W) = U \sqcup V$. Thus it follows from the definition of the quotient topology and the disjoint union topology that W is indeed an open subset of $Y \cup_A Z$. Basically by construction we have $j(U) = W \cap j(Z)$. ∎

7. Graphs

In this chapter we will first introduce the purely combinatorial concept of an abstract graph. Afterwards we introduce the corresponding topological realizations. These notions are of interest for various reasons. Abstract graphs are ubiquitous in mathematics and at times they will also play a useful role in our discussions. The corresponding topological realizations form fairly simple, but nonetheless interesting examples of topological spaces.

7.1. Abstract graphs.

We start out with the following rather abstract definition.

Definition. An abstract graph G is a quadruple (V, E, i, t) where V is a set, E is a set and $i, t\colon E \to V$ are maps from E to V.

(1) The elements of V are called the vertices of G and the elements of E are called the edges of G.
(2) Given $e \in E$ the point $i(e)$ is called the initial point of e and $t(e)$ is called the terminal point of e. We refer to $i(e)$ and $t(e)$ as the vertices of e.
(3) We say an edge $e \in E$ is a loop if $i(e) = t(e)$.
(4) We say G is finite (countable) if both V and E are finite (countable).

Example. We take $V = \{A, B, C\}$ and $E = \{e, f, g, h\}$. Furthermore let $i\colon E \to V$ and $t\colon E \to V$ be the maps that are given by the following table:

	e	f	g	h
i	A	B	B	A
t	B	C	C	A

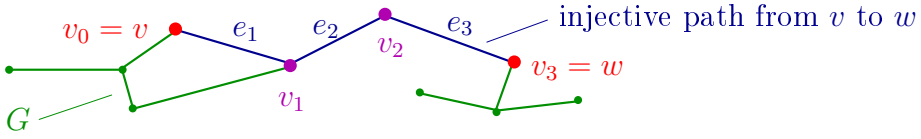
The quadruple (V, E, i, t) is an abstract graph. We can try to visualize such an abstract graph by drawing a point for each vertex and by drawing an arrow for each edge e that points from the initial point $i(e)$ to the endpoint $t(e)$.

The above figure already indicates that abstract graphs give rise to topological spaces. The following definition is already somewhat topology-like.

Definition. Let $G = (V, E, i, t)$ be an abstract graph.

(1) Let $v, w \in V$. A path of length k from v to w is a sequence of edges $e_1, \ldots, e_k$ such that there exist vertices $v = v_0, v_1, \ldots, v_k = w$ such that for each $i \in \{1, \ldots, k\}$ the vertices of e_i are v_{i-1}, v_i. We say that the path $e_1, \ldots, e_k$ is injective if the vertices $v_0, \ldots, v_k$ are distinct.
(2) We say G is connected if for every pair of vertices $v, w \in V$ there exists a path from v to w. One can easily show that if G is connected, then any two vertices can also be connected via an injective path.

Example. The above abstract graph is connected, for example if we take $v = A$ and $w = C$, then $v_0 = A$, $v_1 = B$ and $v_2 = C$ and $e_0 = e$ and $e_1 = f$ have the desired property.



Definition.

(1) Let
$$\begin{aligned} G &= (V, E, i\colon E \to V, t\colon E \to V) \\ \text{and} \quad \widetilde{G} &= (\widetilde{V}, \widetilde{E}, \widetilde{i}\colon \widetilde{E} \to \widetilde{V}, \widetilde{t}\colon \widetilde{E} \to \widetilde{V}) \end{aligned}$$
be two abstract graphs in the sense of the definition on page 124. We define a morphism $G \to \widetilde{G}$ of abstract graphs to be a pair $(\alpha\colon V \to \widetilde{V}, \beta\colon E \to \widetilde{E})$ of maps such that for each $e \in E$ we have $\widetilde{i}(\beta(e)) = \alpha(i(e))$ and $\widetilde{t}(\beta(e)) = \alpha(t(e))$.

(2) We refer to the category $\mathcal{AbsGraph}$ with
$$\begin{aligned} \mathrm{Ob}(\mathcal{AbsGraph}) &:= \text{all abstract graphs,} \\ \mathrm{Mor}(G, \widetilde{G}) &:= \text{all maps from } G \text{ to } \widetilde{G} \text{ in the above sense} \end{aligned}$$
with the usual composition of maps as the category of abstract graphs.

Example. The maps
$$\begin{aligned} (V, E, i\colon E \to V, t\colon E \to V) &\mapsto (V, E, t\colon E \to V, i\colon E \to V) \\ (f\colon G \to \widetilde{G}) &\mapsto (f\colon G \to \widetilde{G}) \end{aligned}$$
define a covariant functor from the category $\mathcal{AbsGraph}$ to itself. The effect on each abstract graph is that it swaps the roles of the initial points and the endpoints. Pictorially speaking the functor "inverts the orientations of the arrows".

The following definition is almost self explanatory.

Definition. Let $G = (V, E, i, t)$ be an abstract graph. A subgraph of $G = (V, E, i, t)$ is an abstract graph $G' = (V', E', i', t')$ such that $V' \subset V$ and $E' \subset E$ are subsets such that $i' = i|_{E'}$ and $t' = t|_{E'}$. Since i' and t' are determined by V' and E' we often suppress both of them from the notation.

Example.

(1) Every $v \in V$ gives rise to the subgraph with a single vertex v and no edge.

(2) Let $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{i}, \widetilde{t})$ be an abstract graph and let $G = (V, E, i, t)$ be a subgraph. It follows immediately from the definitions that the inclusions $V \to \widetilde{V}$ and $E \to \widetilde{E}$ define a map $\iota\colon G \to \widetilde{G}$ of abstract graphs in the above sense.

7.2. **The topological realization of an abstract graph.** Surely the reader is bothered by the absence of topology. We will change this sorry state of affairs by introducing the topological realization of a graph. The idea is, basically, that the topological realization of an abstract graph $G = (V, E, i, t)$ is given by one point for each vertex $v \in V$ and one interval for each edge $e \in E$ with endpoints given by $i(e)$ and $t(e)$. The definition below is the formalization of that idea.

Definition. Let $G = (V, E, i, t)$ be an abstract graph.

(1) We define the topological realization of G as the topological space

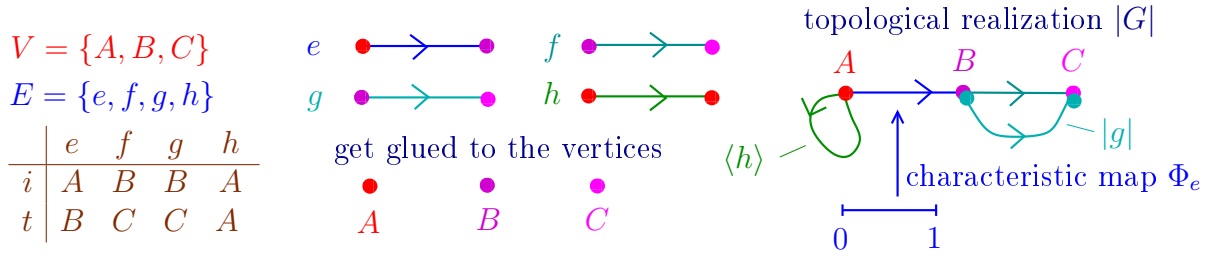
$$|G| \;:=\; \big(\underset{\uparrow}{V} \sqcup \bigsqcup_{e\in E} [0,1]\big)/\sim \qquad \begin{array}{l}\text{where for any } e\in E \text{ we have}\\ (e,0)\sim i(e) \text{ and } (e,1)\sim t(e).\end{array}$$

here V is equipped with the *discrete* topology

(2) Let $e \in E$.

(a) We refer to the map $\Phi_e\colon [0,1] \xrightarrow{x\mapsto(e,x)} \{e\}\times[0,1] \to |G|$ as the characteristic map of the edge e.

(b) We write $|e| = \Phi_e([0,1])$ and we write $\langle e\rangle := \Phi_e((0,1))$.



The following lemma gives us a practical way to understand the topology on the topological realization of an abstract graph.

Lemma 7.1. (Abstract Graph-Topology Lemma) Let $G = (V, E, i, t)$ be an abstract graph.

(1) For each $e \in E$ the characteristic map $\Phi_e\colon [0,1] \to |G|$ is continuous.

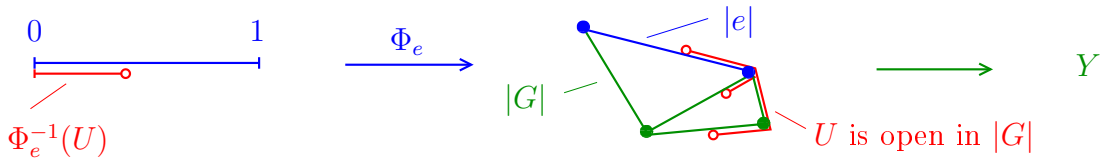
(2) Let $U \subset |G|$ be a subset.

(a) $U \subset |G|$ is open $\iff$ for each $e \in E$ the preimage $\Phi_e^{-1}(U)$ is open in $[0,1]$.

(b) $U \subset |G|$ is closed $\iff$ for each $e \in E$ the preimage $\Phi_e^{-1}(U)$ is closed in $[0,1]$.

(3) Let $f\colon |G| \to X$ be a map to some topological space X.

f is continuous $\iff$ for each $e \in E$ the map $f \circ \Phi_e\colon [0,1] \to X$ is continuous.



Example. Let $G = (V, E, i, t)$ be an abstract graph. The following statements follow easily from the Abstract Graph-Topology Lemma 7.1:

(1) The obvious map $V \to |G|$ is injective and the image is a discrete subset of $|G|$.[76]

(2) Let $e \in E$. Let $\Phi_e\colon [0,1] \to |G|$ be the characteristic map.

(a) $|e|$ is a closed subset of $|G|$ and $\langle e\rangle$ is an open subset of $|G|$.

(b) If $i(e) \neq t(e)$, then Φ_e restricts to a homeomorphism $[0,1] \to |e|$.

(c) If $i(e) = t(e)$, then Φ_e induces a homeomorphism $[0,1]/0\sim 1 \to |e|$.

[76]Here are the details why $V \subset |G|$ is discrete. Let $v \in V$. We consider $U := |G| \setminus (V \setminus \{v\})$. Using the Abstract Graph-Topology Lemma 7.1 (2a) one can easily verify that U is an open neighborhood of $v \in |G|$. By construction U does not contain any other point of $V \subset |G|$. This shows that $V \subset |G|$ is discrete.

Proof of Lemma 7.1.

(1) This statement follows immediately from the Topological-Coproduct Proposition 5.10 (1a) and the Topological-Quotient Proposition 6.1 (1a).

(2) (a) This statement follows easily from the Topological-Coproduct Proposition 5.10 (0a) and the Topological-Quotient Proposition 6.1 (0a) and the observation that any subspace of a topological space consisting of a single point is open.

(b) This statement follows easily from the Topological-Coproduct Proposition 5.10 (0b) and the Topological-Quotient Proposition 6.1 (0b) and the observation that any subspace of a topological space consisting of a single point is closed.

(3) This statement follows immediately from (1) and (2a). ■

The following lemma makes the unsurprising assertion that the above assignment of a topological space to an abstract simplicial complex is functorial.

Lemma 7.2. (Topological Realization–Functor Lemma) The following two statements hold:

(1) Let $G = (V, E, i, t)$ and $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{i}, \widetilde{t})$ be two abstract graphs. Furthermore let $f := (\alpha\colon V \to \widetilde{V}, \beta\colon E \to \widetilde{E})$ be a map between the two abstract graphs. We consider the following map between the topological realizations of G and $\widetilde{G}$:

$$\begin{array}{ccc} & \text{where } (e,0)\sim i(e) \text{ and } (e,1)\sim t(e) & \text{similarly } (e,0)\sim \widetilde{i}(e) \text{ and } (e,1)\sim \widetilde{t}(e) \\ |f|\colon |G| = \Big(V \sqcup \bigsqcup_{e\in E}[0,1]\Big)/\!\sim & \to & |\widetilde{G}| = \Big(\widetilde{V} \sqcup \bigsqcup_{e\in \widetilde{E}}[0,1]\Big)/\!\sim \\ [x] & \mapsto & \begin{cases} \alpha(x), & \text{if } x \in V, \\ (\beta(e), t), & \text{if } x = (e,t) \in E \times [0,1]. \end{cases} \end{array}$$

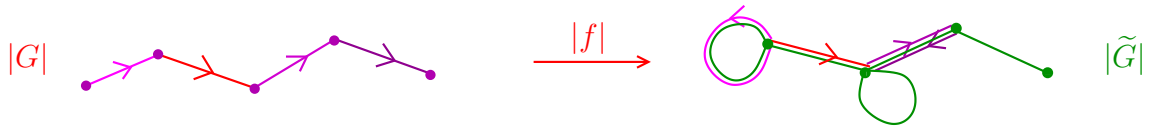
This map $|f|\colon |G| \to |\widetilde{G}|$ is well-defined and continuous.

(2) The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{AbsGraph}) & \to & \mathrm{Ob}(\mathcal{Top}) \\ G & \mapsto & |G|, \\ \mathrm{Mor}_{\mathcal{AbsGraph}}(G,\widetilde{G}) & \to & \mathrm{Mor}_{\mathcal{Top}}(|G|,|\widetilde{G}|) \\ f & \mapsto & |f| \end{array}$$

and the maps

form a covariant functor from the category $\mathcal{AbsGraph}$ of abstract graphs to the category $\mathcal{Top}$ of topological spaces.



Example. Let $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{i}, \widetilde{t})$ be an abstract graph and let $G = (V, E, i, t)$ be a subgraph. The inclusions $V \to \widetilde{V}$ and $E \to \widetilde{E}$ define a map $\iota\colon G \to \widetilde{G}$ of abstract graphs in the sense of the definition on page 125. Using the Topological Realization–Functor Lemma 7.2 (1), the Abstract Graph-Topology Lemma 7.1 (2a) and the Open-Injective Lemma 2.9 one can easily show that the induced map $|\iota|\colon |G| \to |\widetilde{G}|$ between the topological realizations is a closed embedding. In the following we will sometimes use this observation as an excuse to view $|G|$ as a closed subset of $|\widetilde{G}|$.

Proof of Lemma 7.2.

(1) It follows almost immediately from the definitions that the map is well-defined. Furthermore it follows easily from the Abstract Graph-Topology Lemma 7.1 (1) and (3) that the map is continuous.
(2) This statement follows easily from the definitions. ■

The following lemma can be viewed as the analogue of the Product Topology-Properties Proposition 5.4 and the Disjoint Union Topology-Properties Lemma 5.11.

Lemma 7.3. (Topological Realization–Properties Lemma) Let $G = (V, E, i, t)$ be an abstract graph.

(1) (a) Every compact subspace of $|G|$ is contained in the topological realization of a finite subgraph.
    (b) $|G|$ is compact if and only if G is finite.
(2) The following statements are equivalent:
    (a) G is connected.
    (b) Given any two distinct vertices v, w there exists an embedding $\gamma\colon [0,1] \to |G|$ with $\gamma(0) = v$, $\gamma(1) = w$ and such that $\gamma([0,1]) \subset |G|$ is the union of edges.
    (c) The topological realization $|G|$ is path-connected.
    (d) The topological realization $|G|$ is connected.
(3) $|G|$ is Hausdorff.

Proof.

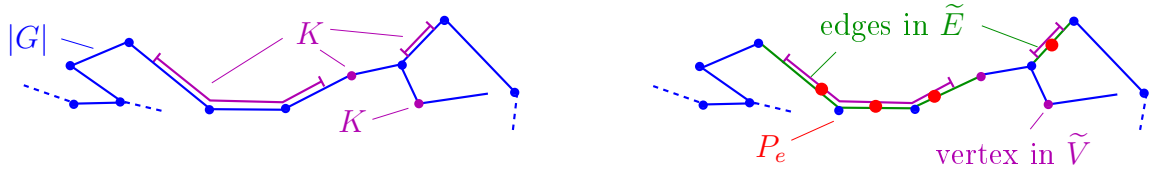
(1) (a) Let $K \subset |G|$ be a compact subspace. We set

$$\begin{aligned} \widetilde{E} &:= \{e \in E \,|\, K \cap \langle e\rangle \neq \varnothing\} \\ \widetilde{V} &:= \{v \in V \,|\, v \in K \text{ and there is no } e \in \widetilde{E} \text{ with } v \in |e|\}. \end{aligned}$$

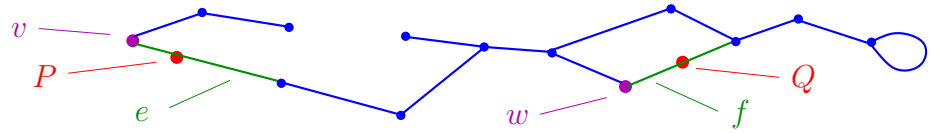
We do the following:
- For each $e \in \widetilde{E}$ we pick a $P_e \in K \cap \langle e\rangle$ and we set $A_e := \langle e\rangle$.
- For each $v \in \widetilde{V}$ we set $B_v := \{v\} \cup \bigcup_e \langle e\rangle$, where we run over all $e \in E$ such that v is a vertex of e.
- We set $C := (|G| \setminus \bigcup_{e\in\widetilde{E}} \{P_e\}) \setminus \widetilde{V}$.

It follows easily from the Topological Realization–Properties Lemma 7.3 that each A_e, B_v and C is an open subset of $|G|$. Note that $\{A_e\}_{e\in\widetilde{E}}$, $\{B_v\}_{v\in\widetilde{V}}$ and C form an open cover of K. Since K is compact we see that we can cover K by finitely many of these open subsets. But each A_e contains a point of K, namely P_e, that does not lie in any of the other sets in the open cover. Similarly we see that each B_v contains a point in K, namely v, that does not lie in any of the other sets in the open cover. This shows that $\widetilde{E}$ and $\widetilde{V}$ are finite. Now we see that K is contained in the topological realization of the finite abstract graph that is formed by $\widetilde{E}$, the vertices of the $e \in \widetilde{E}$ and by $\widetilde{V}$.

(b) If $|G|$ is compact, then it follows from (a) that G is finite. Next we assume that $G = (V, E, i, t)$ is finite. It follows from the Compact Image Lemma 2.8 that each $|e|$ is compact. Since there are only finitely edges we see that $|G|$ is compact.



(2) (a)⇒(b). Let $v \neq w \in V$. Since we assume that G is connected there exists an injective path $e_1, \dots, e_k$ from v to w. Recall that this means that there exists distinct vertices $v = v_0, \dots, v_k = w$ such that the vertices of each e_i are v_{i-1}, v_i. We obtain a map $\gamma\colon [0,1] \to |G|$ with $\gamma(0) = v$ such that on each subinterval $[\frac{i-1}{n}, \frac{i}{n}]$ the map γ is just a reparametrization of the characteristic map Φ_{e_i}, which now goes from v_{i-1} to v_i. It follows from the Abstract Graph-Topology Lemma 7.1 (1) and the Pasting Proposition 2.4 (2) that γ is continuous. Using the Abstract Graph-Topology Lemma 7.1 (2b) one can easily show that γ is a closed embedding.
(b)⇒(c). Let $P, Q \in |G|$. We pick edges e, f with $P \in |e|$ and $Q \in |f|$ and we pick vertices v of e and w of f. Since the characteristic maps of e and f are continuous one sees that there exists a path from P to v and a path from Q to w. By (b) there exists also a path from v to w. It follows from the Path-Equivalence Lemma 2.20 that there exists a path from P to Q.



(c)⇒(d). This follows from the Path-Connected Implies Connected Corollary 2.15.
(d)⇒(a). It follows easily from the definitions that G can be written as the disjoint union of subgraphs $\{G_i\}_{i\in I}$ such that each G_i is connected. It follows that $|G| = \bigsqcup_{i\in I} |G_i|$ and it follows easily from the Abstract Graph-Topology Lemma 7.1 (2a) that each $|G_i| \subset |G|$ is open. Since $|G|$ is connected we see that $\#I = 1$, i.e. G is connected.
(3) We leave it to the reader to prove this statement. ■

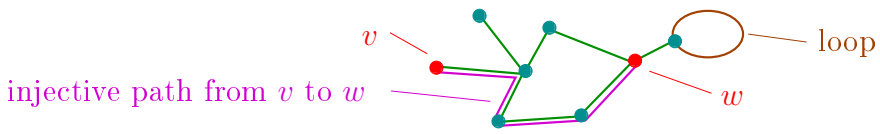
We conclude this section with the following definition.

Definition. We say a topological space X is a topological graph if there exists an abstract graph G such that X is homeomorphic to $|G|$.

7.3. **Trees.** In this and the following section we again leave the realm of topology and we stay strictly in the world of abstract graphs. For the reader's convenience we recall the following definition from page 124.

Definition. Let G be an abstract graph.
(1) An edge is called a loop if it has precisely one vertex.
(2) Let $v, w \in V$. A path of length k from v to w is a sequence of edges $e_1, \dots, e_k$ such that there exist vertices $v = v_0, v_1, \dots, v_k = w$ such that for each $i \in \{1, \dots, k\}$ the vertices of e_i are v_{i-1}, v_i. We say that the path $e_1, \dots, e_k$ is injective if the vertices $v_0, \dots, v_k$ are distinct.

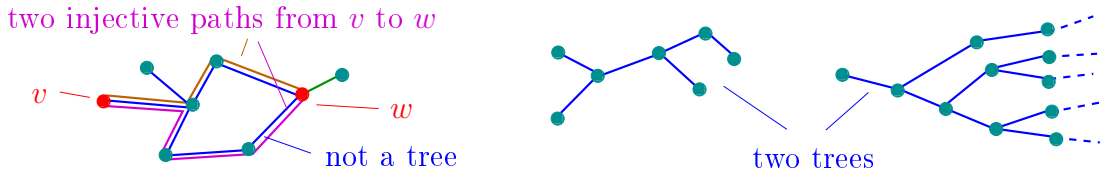
Now we can introduce the following basic but surprisingly useful class of abstract graphs.



Definition. Let G be an abstract graph.

(1) We say G is a tree if G is non-empty, if G does not contain a loop, and if given any $v, w \in V$ there exists a unique injective path from v to w.

(2) A subgraph that is a tree is called a subtree.



Lemma 7.4. (Tree–Basics Lemma)

(1) Any connected non-empty subgraph of a tree is again a tree.

(2) Let G be a graph. Let $T = (V, E)$ be a subtree of G and let $\{e_i\}_{i \in I}$ be a family of edges of G with the following properties:
   (a) The edge e_i has two vertices, one vertex t_i of e_i lies in T and the other vertex v_i does not lie in T.
   (b) The vertices v_i are distinct.

   Then $(V \cup \bigcup_{i \in I} \{v_i\}, E \cup \bigcup_{i \in I} \{e_i\})$ is also a subtree of G.

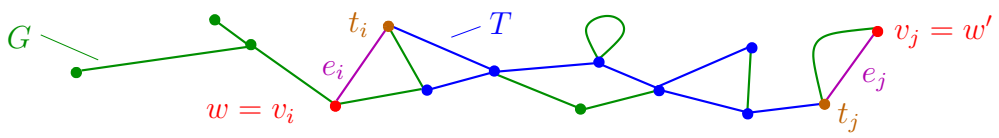
(3) Let G be a graph. If $T_1 \subset T_2 \subset \dots$ is a sequence of subtrees of G, then $\bigcup_{i \in \mathbb{N}} T_i$ is also a subtree of G.

Proof.

(1) This statement follows easily from the definitions.

(2) We set $W := V \cup \bigcup_{i \in I} \{v_i\}$ and $F := E \cup \bigcup_{i \in I} \{e_i\}$. We claim that (W, F) is a subtree. Since none of the e_i's is a loop we see that (W, F) has no loops. Next let $w, w' \in W$ be two vertices. We need to show that there exists a unique injective path from w to w'. There are several cases to distinguish.

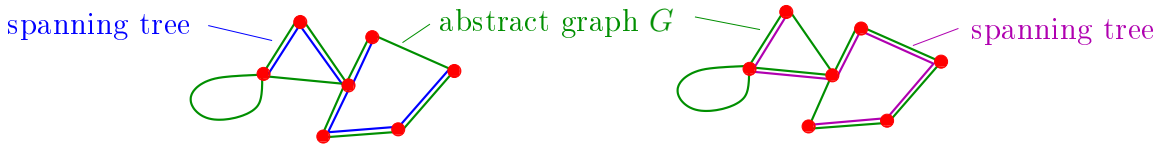
First assume that there exist $i, j \in I$ with $w = v_i$ and $w' = v_j$. Since (V, E) is a subtree we know that there exists an injective path in (V, E) from t_i to t_j. Together with e_i and e_j these form an injective path in (W, F) from $w = v_i$ to $w' = v_j$. We need to show that this is the unique injective path in (W, F) from $w = v_i$ to $w' = v_j$. Thus let $f_1, \dots, f_k$ be an injective path in (W, F) from w to w'. It follows from our hypothesis (a) and (b) that $f_1 = e_i$ and $f_k = e_j$ and that $f_2, \dots, f_{k-1}$ is an injective path which cannot involve any of the other e_l. Thus it is an injective path in (V, E), which is unique since T is a subtree. All other cases are treated in a similar fashion.



(3) We leave the proof of this statement as an instructive exercise to the reader. ■

Now we provide a few more definitions on abstract graphs.

Definition. Let G be an abstract graph. A spanning tree for G is a subtree of G that contains all vertices of G.

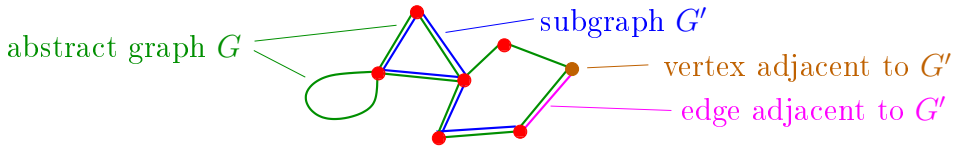


Remark. The example of the figure above shows that an abstract graph usually has many different spanning trees.

Definition. Let G be an abstract graph and let G' be a subgraph.

(1) We say an edge e of G is adjacent to G' if e does not lie in G' and if at least one vertex of e lies in G'.

(2) We saw a vertex v of G is adjacent to G' if v does not lie in G' and if v is a vertex of an edge that is adjacent to G'.



Proposition 7.5. (Spanning Tree–Characterization Proposition) Let G be a connected abstract graph and let $T \subset G$ be a tree. The following statements are equivalent:

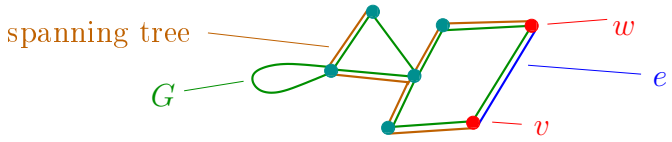
(1) T is a spanning tree.

(2) T is maximal in the sense that if T is contained in a subtree T' of G, then $T = T'$.[77]

(3) T does not admit an adjacent vertex.

Proof.

(1)⇒(2). Let T be a spanning tree. We claim that T is maximal. Let H be a subgraph of G with $T \subsetneq H$. We need to show that H cannot be a tree. Since T contains all vertices and since $T \subsetneq H$ we see that there exists an edge e in H that does not lie in T. If e is a loop, then we know already that H is not a tree. Thus it remains to consider the case that e is not a loop. This means that e has two vertices v and w. Since T is connected there exists an injective path from v to w in T. On the other hand e is an injective path from v to w in H. Thus we see that there exist two injective paths in H from v to w. Thus H is not a tree.

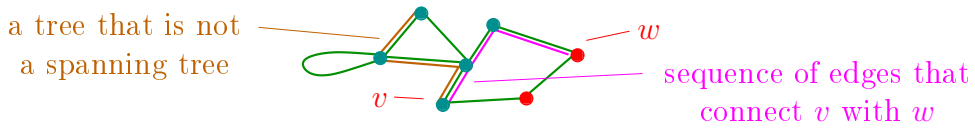


(2)⇒(3). Let $T = (W, F)$ be a subtree of G that admits an adjacent vertex x. We need to show that T is not a maximal subtree. Let f be an edge such that one vertex is x and

[77] The equivalence of (1) and (2) explains why a spanning tree is often also called a *maximal tree*.

such that the other vertex lies in W. It follows from the Tree–Basics Lemma 7.4 (2) that $(W \cup \{x\}, F \cup \{f\})$ is a subtree of G that contains T.

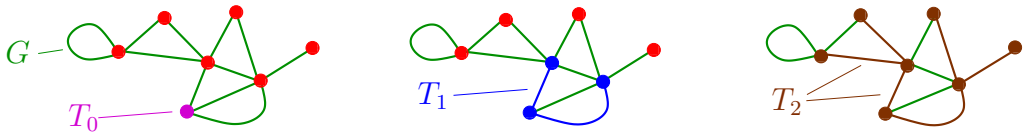
(3)⇒(1). Let T be a subtree of G. We will show that if T is not a spanning tree, then T admits an adjacent vertex. Thus suppose that there exists a vertex w that does not lie in T. Let v be a vertex in T. Since G is connected there exists a path $e_1, \dots, e_n$ from v to w. By definition this means that there exist vertices $v = v_0, \dots, v_n = w$ such that the vertices of each e_i are v_{i-1}, v_i. The first vertex that does not lie in T is adjacent to T. ■



We conclude this section with the following proposition.

Proposition 7.6. (Spanning Tree–Existence Proposition) Every connected non-empty abstract graph admits a spanning tree.

Proof. Let $G = (V, E)$ be a connected non-empty abstract graph. Since G is non-empty we can pick a vertex $v \in V$. We set $T_0 := \{v\}$ and view T_0 as a subtree of G. Now we construct iteratively a sequence of subtrees $T_0 \subset T_1 \subset \dots$ as follows: Suppose we have a subtree $T_i = (V_i, E_i)$. We denote by W_i the set of all vertices of G that are adjacent to T_i. For each $w \in W_i$ we pick an edge e_w that is adjacent to T_i and such that w is a vertex of e_w.[78] We set $V_{i+1} := V_i \cup W_i$ and $E_{i+1} := E_i \cup \bigcup_{w \in W_i} \{e_w\}$. It follows from the Tree–Basics Lemma 7.4 (2) that T_{i+1} is also a tree.



We set $T := \bigcup_{i \in \mathbb{N}_0} T_i$. It follows from the Tree–Basics Lemma 7.4 (3) that T is a tree. We claim that T is a spanning tree. By the (3)⇒(1)-Implication of the Spanning Tree–Characterization Proposition 7.5 it suffices to show that there is no vertex of G that is adjacent to T. In other words, we have to show that if there is an edge e of G with one vertex v in T, then the other vertex w also lies in T. If we are given such an edge e with one vertex v in T, then there exists an $i \in \mathbb{N}_0$ with $v \in T_i$, but then it follows from the definition of T_{i+1} that $w \in T_{i+1} \subset T$. ■

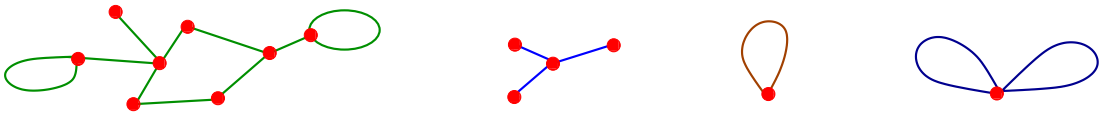
7.4. **The Euler characteristic.** The following essential definition introduces a concept that will stay with us, in many different guises, throughout these lecture notes.

Definition. Let $G = (V, E)$ be a finite abstract graph. The Euler characteristic of G is defined as

$$\chi(G) := \#V - \#E = \text{number of vertices} - \text{number of edges}.$$

The following proposition gives a characterization of finite trees in terms of the Euler characteristic.

[78] Note that here we secretly use the Axiom of Choice 0.1.



Euler characteristic of G equals $\chi(G) = 8 - 10 = -2$

$\chi = 1$ $\chi = 0$ $\chi = -1$

Proposition 7.7. (Tree Characterization Proposition) Let $G = (V, E)$ be a finite abstract graph. The following statements are equivalent:

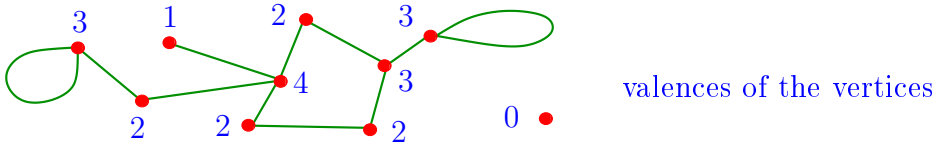
(1) G is a tree.
(2) G is connected and $\chi(G) = 1$.

The proof of the Tree Characterization Proposition 7.7 rests on a lemma which is of independent interest. To formulate the lemma we need to introduce the following natural definition.

Definition. If $G = (V, E, i, t)$ is an abstract graph, then the valence of a vertex v is defined as

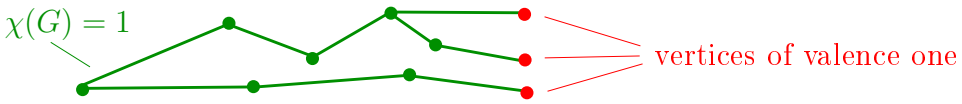
$$\text{valence}(v) \;:=\; \#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\}.$$

In a slightly less formal way, the valence of a vertex v is the number of edges at v, but hereby we have to count any loop at v twice, since it "goes twice into v".



valences of the vertices

Lemma 7.8. (Tree–Valence One Edge Lemma)[79] Let $G = (V, E)$ be a finite abstract graph. If $\chi(G) = 1$ and if G has at least one edge, then G admits at least two vertices of valence one.



Proof of Lemma 7.8. Let $G = (V, E, i, t)$ be a finite abstract graph with $\chi(G) = 1$ that has at least one edge.

(a) Since G has at least one edge and since it is by definition connected, we see that the abstract graph G does not have a vertex of valence 0.

(b) We calculate that

$$\sum_{\text{vertex } v} \underbrace{\text{valence}(v)}_{\geq 1 \text{ by (a)}} \;=\; \sum_{\text{vertex } v} (\#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\})$$
$$\overset{\uparrow}{=} \; 2 \cdot \#\text{ edges} \;\overset{\uparrow}{=}\; 2 \cdot \#\text{vertices} - 2.$$

each edge contributes 2 to the sum — by hypothesis we have $\chi(G) = 1$

By (a) the equality can only hold if there are at least two vertices with valence one. ■

Proof of the Tree Characterization Proposition 7.7. Let $G = (V, E)$ be a finite abstract graph.

(2)⇒(1). We now assume that G is connected and that $\chi(G) = 1$. We need to show that G is a tree. We will do so by inducting on the number of edges of G. If G has no edge, then it follows from $\chi(G) = 1$ that G has one vertex. Clearly this implies that G is a tree.

Now suppose that the statement holds for all graphs with n edges. Let $G = (V, E)$ be a connected graph with $n+1$ edges and with $\chi(G) = 1$. It follows from the Tree–Valence One Edge Lemma 7.8 that G admits a vertex v of valence one. Let e be the corresponding edge. Note that $G' = (V \setminus \{v\}, E \setminus \{e\})$ is connected and that

$$\chi(G') \;=\; (\#V - 1) - (\#E - 1) \;=\; \#V - \#E \;=\; \chi(G) \;=\; 1.$$

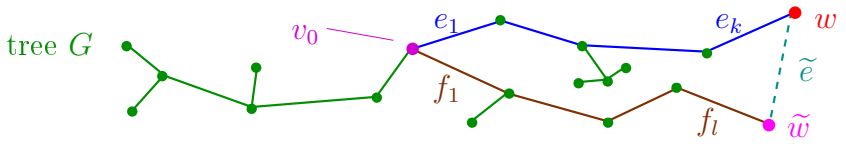
By induction G' is a tree. But then it follows from the Tree–Basics Lemma 7.4 (2) that G itself is also a tree.

(1)⇒(2). We suppose that G is a tree. If follows immediately from the definition of a tree that G is connected. We still need to show that $\chi(G) = 1$. We prove this statement by induction on the number of edges. If there is no edge then it follows from the fact that trees are non-empty and connected that there is precisely one vertex, thus $\chi(G) = 1$.

Next suppose that the statement holds if we have n edges. Let G be a tree with $n+1$ edges. We pick a vertex $v_0 \in V$. Let $w \in V$ be a vertex of maximal distance to v_0.

Claim. There exists precisely one edge e that has w as a vertex.

Proof. Let $e_1, \dots, e_k$ be a path of minimal length from v_0 to w. Since the path has minimal length we see that it is injective. Evidently w is a vertex of e_k. Suppose there was a second edge $\widetilde{e}$ for which w is a vertex. As we mentioned before, a tree does not contain a loop. This implies that the edge $\widetilde{e}$ has a second vertex besides w. We denote this second vertex by $\widetilde{w}$. Since G is connected and since w has maximal distance from v_0 there exists an injective path $f_0, \dots, f_l$ from v_0 to $\widetilde{w}$ of length $\leq k$. Thus this path cannot be the path $e_1, \dots, e_k, \widetilde{e}$, which has length $k+1$.



We have thus found two different injective paths from v_0 to $\widetilde{w}$, namely $e_1, \dots, e_k, \widetilde{w}$ and $f_1, \dots, f_l$. But this contradicts the fact that we started out with a tree. ⊞

We can now consider the subgraph G' that is given by $V \setminus \{w\}$ and $E \setminus \{e\}$. One can easily show that this graph is again connected, thus it follows from the Tree–Basics Lemma 7.4 (1) that G' is a tree. We now see that

$$\chi(G) \;=\; \#V - \#E \;=\; (\#V - 1) - (\#E - 1) \;=\; \chi(G') \;\underset{\text{by induction hypothesis}}{=}\; 1.$$

■

7.5. **Quotients of abstract graphs.** In this section we will define and discuss quotients of abstract graphs and of the corresponding topological realizations.

Definition. Let $G = (V, E, i, t)$ be an abstract graph and let $H = (W, F)$ be a non-empty subgraph.

(1) We define $\sim$ to be the equivalence relation generated by $v \sim v'$ if v and v' are both vertices in H. We denote by $*$ the equivalence class defined by the vertices in H.
(2) We denote by $\pi\colon V \to V/\sim$ the map given by $v \mapsto [v]$.
(3) Finally we define the abstract graph

$$G/H \;:=\; (V/\sim, E \setminus F, \pi \circ i\colon E \setminus F \to V/\sim, \pi \circ t\colon E \setminus F \to V/\sim).$$

We refer to G/H as the quotient of the abstract graph G by the abstract graph H.

The following lemma summarizes a few properties of the abstract graph G/H.

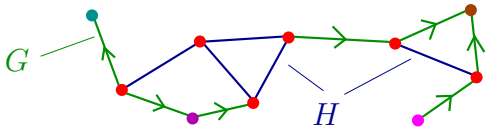
Lemma 7.9. (Abstract Graph–Quotient Lemma) Let $G = (V, E, i, t)$ be an abstract graph and let $H = (W, F)$ be a non-empty subgraph. We continue with the above notation.

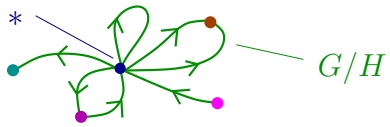
(1) If G is finite, then we have $\chi(G/H) = \chi(G) - \chi(H) + 1$.
(2) We consider the map

$$\begin{array}{ccc} & \text{where } (e,0)\sim i(e) \text{ and } (e,1)\sim t(e) & \text{similarly } (e,0)\sim \pi(i(e)) \text{ and } (e,1)\sim \pi(t(e)) \\ p\colon |G| = (V \sqcup (E\times[0,1]))/\!\sim & \to & |G/H| = (V/\sim \sqcup ((E\setminus F)\times[0,1]))/\!\sim \end{array}$$

$$[x] \mapsto \begin{cases} [*], & \text{if } x \in |H|, \\ [\pi(x)], & \text{if } x \in V, \\ [(e,t)], & \text{if } x = (e,t) \text{ with } e \in E\setminus F \text{ and } t \in [0,1]. \end{cases}$$

The induced map $\overline{p}\colon |G|/|H| \to |G/H|$ is well-defined and it is a homeomorphism.





Proof.

(1) We calculate that

$$\chi(G/H) = \#(V/\sim) - \#(E\setminus F) = (\#V - \#W + 1) - (\#E - \#F) = \chi(G) - \chi(H) + 1.$$

(2) First note that it follows easily from the Abstract Graph-Topology Lemma 7.1 (3) and the Topological-Quotient Proposition 6.1 (1b) that the map $\overline{p}\colon |G|/|H| \to |G/H|$ is continuous. One can easily verify that the map is a bijection. We refer to [Fri23] for the proof that the map is actually a homeomorphism. ∎

[79] It follows from the discussion on page 127 that it is legitimate to view $|H|$ as a subspace of $|G|$.

8. Homotopies

Loosely speaking two continuous maps $f, g\colon X \to Y$ between topological spaces are called homotopic if one can be deformed "continuously" into the other. In the subsequent chapter we will use to this notion of homotopic maps to explain what it means for two topological spaces to be "homotopy equivalent".

8.1. **Homotopic maps.**

In its most basic form a homotopy is a continuous deformation of continuous maps between two topological spaces. To make this notion precise it is convenient to introduce the following notation.

Notation. Let X and Y be two topological spaces and let $I \subset \mathbb{R}$ be a subset. Given a continuous map $F\colon X \times I \to Y$ and given $t \in I$ we denote by F_t the continuous map[80]

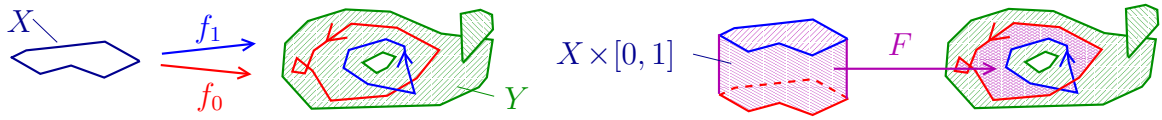
$$\begin{array}{rccc} F_t\colon & X & \to & Y \\ & x & \mapsto & F(x,t). \end{array}$$

Now we move on to the actual definition we are interested in.

Definition. Let X and Y be two topological spaces.

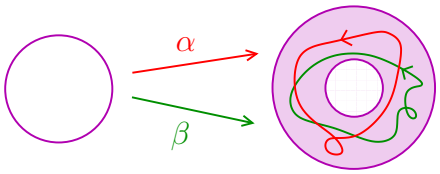
(1) Let $f_0, f_1\colon X \to Y$ be two continuous maps. A homotopy between the maps f_0 and f_1 is a continuous map $F\colon X \times [0,1] \to Y$ with $F_0 = f_0$ and $F_1 = f_1$. If there exists a homotopy between the maps f_0 and f_1, then we say that f_0 and f_1 are homotopic and sometimes we write $f_0 \simeq f_1$. If every F_t is an embedding, then we call F an isotopy.

(2) A continuous map $f\colon X \to Y$ is called null-homotopic if it is homotopic to a constant map.

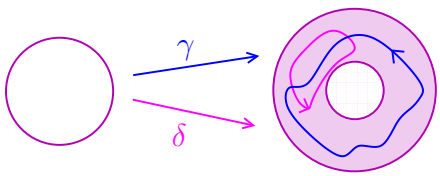


Examples.

(1) In the figure below we show four maps α, β, γ and δ from the circle S^1 to the annulus $Y = B_2(0) \setminus B_1(0)$. The maps α and β are homotopic whereas the maps γ and δ look as if they are not homotopic.



the maps α and β are homotopic



the maps γ and δ appear not to be homotopic

[80] Note that it follows from the Product Topology-Basics Lemma 5.3 (1∗) that the map F_t is indeed continuous.

(2) Given an orthogonal matrix $A \in \mathrm{O}(n)$ we denote by $\rho(A)$ the map $S^{n-1} \to S^{n-1}$ that is given $x \mapsto A \cdot x$.
   (a) Let $\gamma\colon [0,1] \to \mathrm{SO}(n)$ be a path from some matrix A to some matrix B. The map[81]

$$\begin{array}{rcl} S^{n-1} \times [0,1] & \to & S^{n-1} \\ (x,t) & \mapsto & \gamma(t) \cdot x \end{array}$$

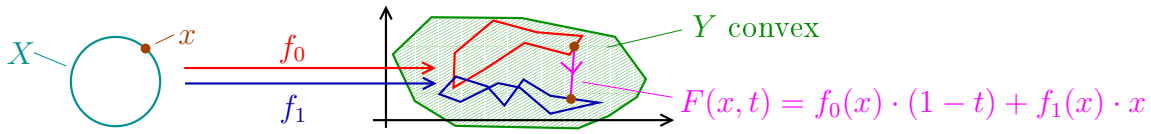
   is a homotopy from $\rho(A)$ to $\rho(B)$.
   (b) Let $A \in \mathrm{O}(n)$. If $\det(A) = 1$, i.e. if $A \in \mathrm{SO}(n)$, then the Matrix Group-π_0-Proposition 2.23 (3) provides us with a path $\gamma\colon [0,1] \to \mathrm{SO}(n)$ from $\gamma(0) = A$ to $\gamma(1) = \mathrm{id}$. It follows from (a) that $\rho(A)\colon S^{n-1} \to S^{n-1}$ is homotopic to the identity.
(3) It follows immediately from the definitions that a topological space X is path-connected if and only if any two maps $\{*\} \to X$ are homotopic.

The next example of a homotopy is written down as a lemma.

Lemma 8.1. (Convex Homotopy Lemma) Let X be a topological space. Given any two continuous maps $f_0, f_1\colon X \to Y$ to a *convex* subset $Y \subset \mathbb{R}^n$ the map

$$\begin{array}{rcl} F\colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & \underbrace{f_0(x) \cdot (1-t) + f_1(x) \cdot t}_{\in Y \text{ since } Y \text{ is convex}} \end{array}$$

is a homotopy between f_0 and f_1.



Proof. Basically by definition we have $F_0 = f_0$ and $F_1 = f_1$. But we are not done yet, the conscientious reader will not have failed to notice that we need to show that the map F is actually continuous. The map F can be written as the composition of the following two maps:

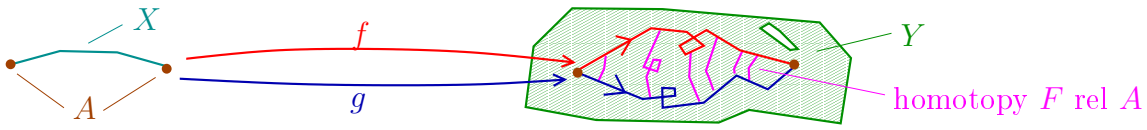
$$X \times [0,1] \xrightarrow{(x,t)\mapsto(f_0(x),f_1(x),t)} \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \xrightarrow{(x,y,t)\mapsto(x\cdot(1-t)+y\cdot t)} \mathbb{R}^n.$$

It follows from the Topological-Product Proposition 5.1 (1) that the map to the left is continuous. As in the proof of the Torus Embedding Lemma 5.5 we see that the map to the right is continuous. ∎

Often it is more interesting to consider homotopies that fix a given subset. This leads us to the following definition.

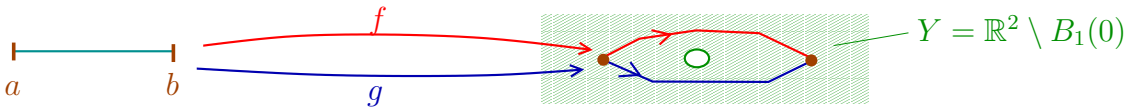
Definition. Let X and Y be topological spaces and let $A \subset X$ be a subset.
(1) We say a continuous map $F\colon X \times [0,1] \to Y$ is a homotopy rel A if for all $a \in A$ and $s, t \in [0,1]$ we have $F(a,s) = F(a,t)$.
(2) We say that two continuous maps $f, g\colon X \to Y$ with $f|_A = g|_A$ are homotopic rel A if there exists a homotopy F rel A with $F_0 = f$ and $F_1 = g$.

[81] The argument of the proof of the Torus Embedding Lemma 5.5 shows that the map $S^{n-1} \times [0,1] \to S^{n-1}$ is continuous.

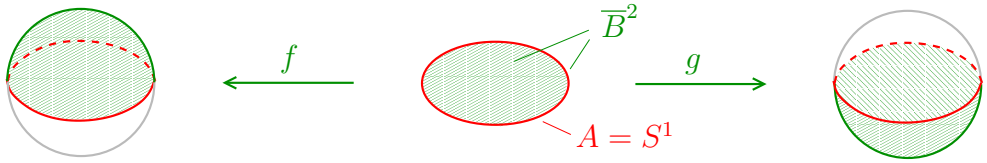


Examples.

(1) Let $f, g\colon [a,b] \to Y = \mathbb{R}^2 \setminus B_1(0)$ be the two maps from an interval to the topological space Y that are shown in the figure below. Note that $f(a) = g(a)$ and $f(b) = g(b)$. One can easily show that any two continuous maps from an interval to a path-connected space are homotopic. On other hand it looks rather unlikely that the two maps are homotopic rel $\{a, b\}$, i.e. it does not look like we can deform f into g while keeping the endpoints of the interval fixed.



(2) We consider the two maps $f, g\colon \overline{B}^2 \to S^2$ that are shown in the figure below. These maps agree on S^1. It is fairly easy to see that the two maps are homotopic but it seems rather unlikely that they are homotopic rel S^1. Unfortunately, as in (1), right now we lack the tools for showing the latter statement.

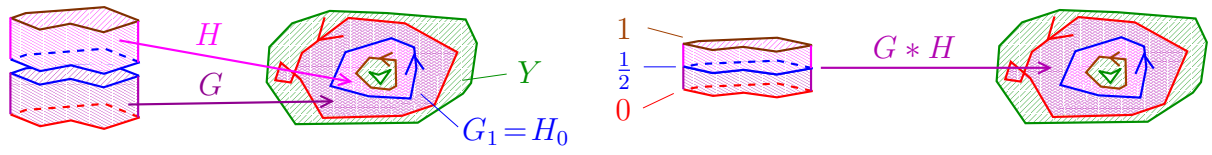


After these examples we return to developing the general theory of homotopies. On many occasions it will be useful to combine two homotopies. This leads us to the following lemma.

Lemma 8.2. (Homotopy Combination Lemma) Let X and Z be topological spaces. Let $A \subset X$ be a (possibly empty) subset of X and let $G, H\colon X \times [0,1] \to Z$ be two homotopies rel A. If $G_1 = H_0$, then the map

$$\begin{array}{rcl} G * H\colon X \times [0,1] & \to & Z \\ (x,t) & \mapsto & \begin{cases} G(x,2t), & \text{if } t \in [0,\frac{1}{2}], \\ H(x,2t-1), & \text{if } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

is a homotopy rel A from G_0 to H_1.



Proof. It follows from the Pasting Proposition 2.4 (2) together with the Product Topology-Basics Lemma 5.3 (1a) that the map is continuous. ■

Lemma 8.3. (Homotopy-Equivalence Relation Lemma) Let X and Y be topological spaces. Then "homotopy of maps" forms an equivalence relation on the set of continuous maps from X to Y.

Proof. We have to show that "homotopy" satisfies the three properties of an equivalence relation. In the following let $f, g, h\colon X \to Y$ be three continuous maps.

(1) The "constant homotopy" given by $F(x,t) := f(x)$ is a homotopy between f and f.
(2) Let F be a homotopy from f to g. Then it is clear that the map F' that is defined by $F'(x,t) := F(x,1-t)$ is a homotopy from g to f.
(3) Let F be a homotopy from f to g and let G be a homotopy from g to h. We consider the map

$$\begin{array}{rcl} F * G\colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & \begin{cases} F(x,2t), & \text{if } t \in \left[0,\frac{1}{2}\right], \\ G(x,2t-1), & \text{if } t \in \left(\frac{1}{2},1\right]. \end{cases} \end{array}$$

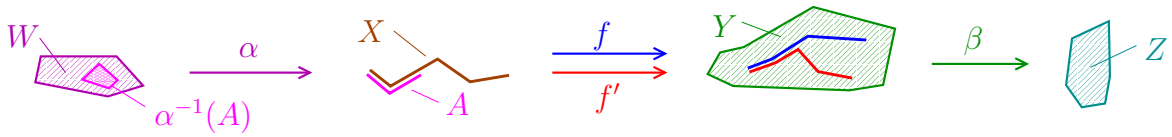
By the Homotopy Combination Lemma 8.2 (2) this is a homotopy from f to h. ■

Lemma 8.4. (Homotopy Composition Lemma)

(1) Let $f, f'\colon X \to Y$, let $\alpha\colon W \to X$ and $\beta\colon Y \to Z$ be continuous map between topological spaces. Furthermore let $A \subset X$ be a (possibly empty) subset. The following conclusion holds:

$$\begin{array}{rcl} f \text{ is homotopic to } f' \text{ rel } A & \implies & f \circ \alpha \text{ is homotopic to } f' \circ \alpha \text{ rel } \alpha^{-1}(A) \\ f \text{ is homotopic to } f' \text{ rel } A & \implies & \beta \circ f \text{ is homotopic to } \beta \circ f' \text{ rel } A \end{array}$$

(2) Let X, Y and Z be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets. We suppose that $f, f'\colon X \to Y$ and $g, g'\colon Y \to Z$ are continuous maps such that f and f' are homotopic rel A and such that g and g' are homotopic rel B. If $f(A) = f'(A) \subset B$, then $g \circ f$ is homotopic to $g' \circ f'$ rel A.

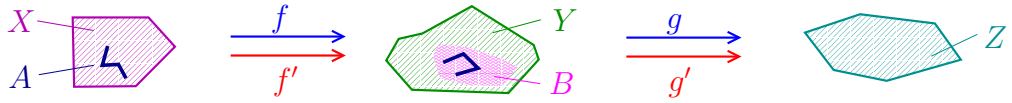


Proof.

(1) This statement is basically obvious, we just need to precompose and postcompose a homotopy $H\colon X \times [0,1] \to Y$ between f and g by $\alpha \times \mathrm{id}_{[0,1]}\colon W \times [0,1] \to X \times [0,1]$ respectively by $\beta\colon Y \to Z$.
(2) Note that:

$$g \circ f \underset{\text{both follow from (1)}}{\simeq} g' \circ f \underset{\text{both follow from (1)}}{\simeq} g' \circ f'.$$

Furthermore one can easily verify that the "rel" business also works out. ■



8.2. Homotopies on quotients.

Theorem 8.5. (Homotopy Quotient Theorem) Let X and Y be topological spaces. Furthermore let $\sim_X$ be an equivalence relation on X and let $\sim_Y$ be an equivalence relation on Y. If $F\colon X\times[0,1]\to Y$ is a homotopy such that $F(x,t)\sim_Y F(x',t)$ whenever $x\sim_X x'$, then the map[82]

$$\begin{array}{rcl}\varphi\colon (X/\sim_X)\times[0,1] & \to & Y/\sim_Y\\ ([x],t) & \mapsto & [F(x,t)]\end{array}$$

is continuous.

Example. Let $n\in\mathbb{N}$ and let $\varphi\colon[0,1]\to \mathrm{GL}(n,\mathbb{R})$ be a continuous map. It follows from the argument of the proof of the Torus Embedding Lemma 5.5 that the map

$$\begin{array}{rcl}F\colon (\mathbb{R}^n\setminus\{0\})\times[0,1] & \to & \mathbb{R}^n\setminus\{0\}\\ (P,t) & \mapsto & \varphi(t)\cdot P\end{array}$$

is continuous. Next recall that $\mathbb{R}\mathrm{P}^{n-1}=(\mathbb{R}^n\setminus\{0\})/(\mathbb{R}\setminus\{0\})$. It follows from the Homotopy Quotient Theorem 8.5 that the map

$$\begin{array}{rcl}\mathbb{R}\mathrm{P}^{n-1}\times[0,1] & \to & \mathbb{R}\mathrm{P}^{n-1}\\ ([P],t) & \mapsto & [\varphi(t)\cdot P]\end{array}$$

is continuous. The same conclusion also holds if we replace "$\mathbb{R}$" by "$\mathbb{C}$".

Proof of Theorem 8.5. The result is called a theorem, since the proof is indeed rather tricky. We refer to [Fri23] for details. Below we will deal with a special case. ∎

Proof of Theorem 8.5 if X is compact and if $X/\sim_X$ is Hausdorff. We now assume that X is compact and that the quotient space $X/\sim_X$ is Hausdorff. On $X\times[0,1]$ we consider the equivalence relation $\sim$ given by

$$(x,t)\sim(x',t') \iff x\sim_X x' \text{ and } t=t'.$$

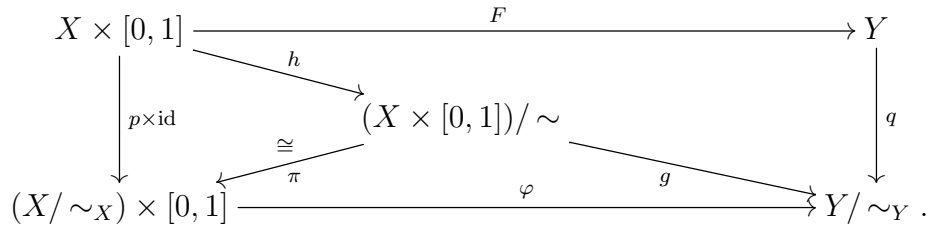
We denote by $h\colon X\times[0,1]\to(X\times[0,1])/\sim$ the natural projection. We consider the maps

$$\begin{array}{rcl}p\times\mathrm{id}\colon X\times[0,1] & \to & X/\sim_X\times[0,1]\\ (x,t) & \mapsto & ([x],t)\end{array}\qquad\text{and}\qquad\begin{array}{rcl}\pi\colon (X\times[0,1])/\sim & \to & X/\sim_X\times[0,1]\\ [(x,t)] & \mapsto & ([x],t).\end{array}$$

By the Topological-Quotient Proposition 6.1 (1a) the projection $p\colon X\to X/\sim_X$ is continuous. This shows that the map to the left is continuous. By the Topological-Quotient Proposition 6.1 (1b) this implies that the induced map π to the right is also continuous. One can easily verify that π is a bijection. By hypothesis X is compact. By the Product Topology-Properties Proposition 5.4 (1) this implies that $X\times[0,1]$ is compact. It follows from the Topological-Quotient Proposition 6.1 (1a) and the Compact Image Lemma 2.8 that $(X\times[0,1])/\sim$ is compact. Next recall that by hypothesis $X/\sim_X$ is Hausdorff. By the Product Topology-Properties Proposition 5.4 (3) this implies that $X/\sim_X\times[0,1]$ is Hausdorff. In summary it follows from the Compact-Hausdorff Proposition 2.10 (3) that π is a homeomorphism.

[82] It follows from our hypothesis on F that the map φ is well-defined.

Now we consider the following diagram:



We make the following clarifications and observations:

(1) We denote by $q\colon Y \to Y/\sim_Y$ the natural projection.
(2) First note that the map $q \circ F\colon X \times [0,1] \to Y/\sim_Y$ factors by our hypothesis through the map $h\colon X \times [0,1] \to (X \times [0,1])/\sim$. It follows again from the Topological-Quotient Proposition 6.1 (1b) that the induced map $g\colon (X \times [0,1])/\sim \to Y/\sim_Y$ is continuous.
(3) The map $\varphi\colon (X/\sim_X) \times [0,1] \to Y/\sim_Y$ equals the map $g \circ \pi^{-1}$. We just showed that the two maps π and g are continuous. Thus φ is continuous. ■

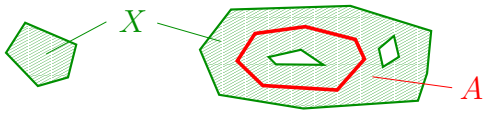
9. Homotopy equivalences and homotopy equivalent topological spaces

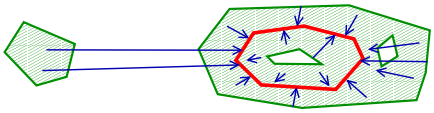
This chapter is a direct continuation of the previous chapter. In the last chapter we defined what it means for *maps* between topological spaces to be *homotopic*. Now in this chapter we will use this notion to define what it means for *topological spaces* to be *homotopy equivalent*. This notion is rather subtle and - at least for this author - not entirely intuitive.

9.1. Deformation retractions.

Before we introduce the notion of a homotopy equivalence we want to introduce the easier notions of a (deformation) retraction.

Definition. Let X be a topological space. We say a subset $A \subset X$ is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a continuous map with $r(a) = a$ for all $a \in A$.





retraction r from X to A

Let us recall the following definition from page 136.

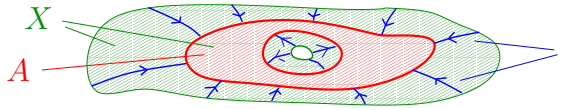
Definition. Let X and Y be two topological spaces.

(1) Given a continuous map $F\colon X \times [0,1] \to Y$ and given $t \in [0,1]$ we denote by F_t the continuous map
$$\begin{array}{rccc} F_t\colon & X & \to & Y \\ & x & \mapsto & F(x,t). \end{array}$$
(2) A continuous map $F\colon X \times [0,1] \to Y$ map is called a homotopy from F_0 to F_1.

Next we introduce an interesting special case of a homotopy.

Definition. Let X be a topological space and let A be a subset.

(1) A deformation retraction from X to A is a homotopy $F\colon X \times [0,1] \to X$ which has the following three properties:
   (a) $F_0 = \mathrm{id}_X$,
   (b) F is a homotopy rel A, i.e. for all $a \in A$ and all $t \in [0,1]$ we have $F(a,t) = a$,
   (c) $F_1(X) = A$.
(2) We say A is a deformation retract of X if there exists a deformation retraction from X to A. We say a point $a \in X$ is a deformation retract of X if the subset $\{a\}$ is a deformation retract of X.



deformation retraction from X to A

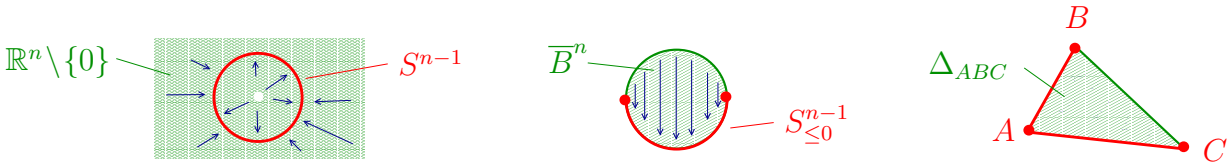
Examples. Let $n \in \mathbb{N}$.

(1) The map[83]
$$\begin{array}{rccc} & (\mathbb{R}^n \setminus \{0\}) \times [0,1] & \to & \mathbb{R}^n \setminus \{0\} \\ & (v,t) & \mapsto & v \cdot (1 - t + t \cdot \frac{1}{\|v\|}) \end{array}$$
is a deformation retraction from $\mathbb{R}^n \setminus \{0\}$ to the sphere S^{n-1}.

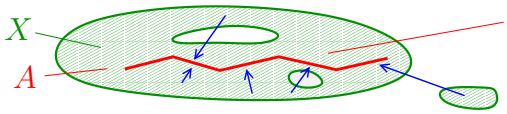
(2) We leave it to the reader to prove the following two facts:
   (a) The hemisphere $S^{n-1}_{\leq 0}$ is a deformation retract of $\overline{B}^n$.

[83] The argument of the proof of the Torus Embedding Lemma 5.5 shows that this map is continuous.

(b) Given any three non-collinear points $A, B, C \in \mathbb{R}^n$ the triangle Δ_{ABC} admits a deformation retraction to the union of any two sides of the triangle.



(3) If $F\colon X \times [0,1] \to X$ is a deformation retraction from X to A, then $F_1\colon X \to A$ is by definition a retraction from X to A. But, as we see in the figure below, not every retract is a deformation retract.

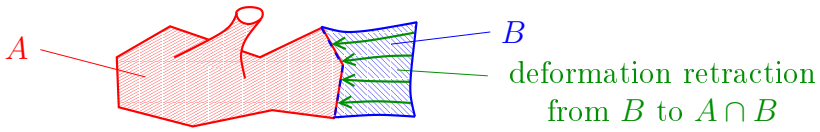


A is a retract of X
if it was a deformation retract, any point $x \in X$
would be connected to A via the path $R(x,t)$

(4) Let X be a topological space. Suppose that we are given a decomposition $X = A \cup B$ into two *closed* subsets. If $A \cap B$ is a deformation retract of B, then A is a deformation retract of X. Indeed, if $F\colon B \times [0,1] \to B$ is a deformation retraction from B to $A \cap B$, then it follows from the Homotopy Combination Lemma 8.2 and our hypothesis that A and B are closed subsets of X, that the map

$$\begin{array}{rcl} G\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x \in A, \\ F(x,t), & \text{if } x \in B \end{cases} \end{array}$$

is continuous. It is now clear that G is the desired deformation retraction from X to A. We will use this remark on several occasions.



We continue with an example of a deformation retraction which is so interesting, that it gets formulated as a proposition.

Proposition 9.1. (Matrix Groups Deformation Retract Proposition) Let $n \in \mathbb{N}_0$. There exists a deformation retraction

$$R\colon \mathrm{GL}(n,\mathbb{R}) \times [0,1] \to \mathrm{GL}(n,\mathbb{R})$$

from $\mathrm{GL}(n,\mathbb{R})$ to $\mathrm{O}(n)$ such that $R_1\colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{O}(n)$ is the map given by the Gram-Schmidt Orthonormalization Process 0.12. The analogous statement holds if we replace the pair $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{O}(n)$ by one of the following four pairs:

(a) $\mathrm{GL}_+(n,\mathbb{R})$ and $\mathrm{SO}(n)$
(b) $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SO}(n)$
(c) $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{U}(n)$
(d) $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SU}(n)$.

Proof of Proposition 9.1. For $i = 1, \dots, n$ we consider the following subspaces of $\mathrm{GL}(n, \mathbb{R})$:

$$\begin{aligned} X(i) &:= \{(v_1 \dots v_n) \in \mathrm{GL}(n, \mathbb{R}) \,|\, v_1, \dots, v_{i-1} \text{ are orthonormal and } v_i \perp v_1, \dots, v_{i-1}\} \\ Y(i) &:= \{(v_1 \dots v_n) \in \mathrm{GL}(n, \mathbb{R}) \,|\, v_1, \dots, v_i \text{ are orthonormal}\}. \end{aligned}$$

In the proof of the Matrix Group Path-Component Proposition 2.23, we implicitly already showed that the $2k$ steps of the Gram-Schmidt Orthogonalization Process define deformation retractions from $X(i)$ to $Y(i)$ and $Y(i)$ to $X(i+1)$. Since $Y(1) = \mathrm{GL}(n, \mathbb{R})$ and since $Y(n) = \mathrm{O}(n)$ we can combine these $2k-1$ deformation retractions to obtain a deformation retraction

$$R\colon \mathrm{GL}(n, \mathbb{R}) \times [0, 1] \;\to\; \mathrm{GL}(n, \mathbb{R})$$

from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{O}(n)$. Furthermore note that it follows from the Gram-Schmidt Orthonormalization Process 0.12 (2) and (3) that this deformation retraction R restricts to a deformation retraction from $\mathrm{GL}_+(n, \mathbb{R})$ to $\mathrm{SO}(n)$ and from $\mathrm{SL}(n, \mathbb{R})$ to $\mathrm{SO}(n)$.

Finally note that the complex cases are treated in a very similar fashion. ■

In the following we will see that deformation retractions descend to quotients under suitable hypotheses.

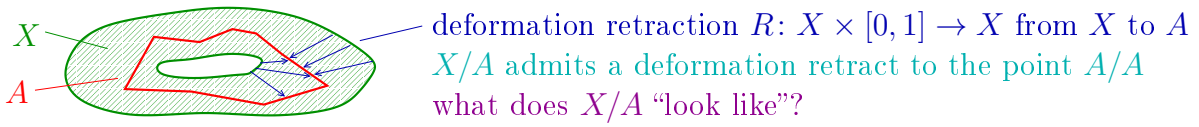
Theorem 9.2. (Deformation Retraction-Quotient Theorem) Let X be a topological space and let $A \subset X$ be a subset.

(1) Let "$\sim$" be an equivalence relation on X and let $R\colon X \times [0, 1] \to X$ be a deformation retraction from X to A. If "$\sim$" is compatible with the deformation retraction R in the sense that $R(x, t) \sim R(x', t)$ whenever $x \sim x'$, then the map

$$\begin{aligned} \overline{R}\colon (X/\sim) \times [0, 1] &\to X/\sim \\ ([x], t) &\mapsto [R(x, t)] \end{aligned}$$

is a deformation retraction from $X/\sim$ to $A/\sim$.

(2) Let X be a topological space and let A be a subset of X. If A is a deformation retract of X, then the point A/A is a deformation retract of X/A.



Proof.

(1) It follows from the Homotopy Quotient Theorem 8.5 that $\overline{R}$ is continuous. It is now clear that $\overline{R}$ is a deformation retraction from $X/\sim$ to $A/\sim$.

(2) Let $r\colon X \times [0, 1] \to X$ be a deformation retraction from X to A. We consider the equivalence relation "$\sim_A$" on X that is generated by the condition that all points in A are equivalent. Note that by definition of a deformation retraction the homotopy r is compatible with the equivalence relation "$\sim_A$". Therefore it follows from (1) that the map

$$\begin{aligned} (X/\sim_A) \times [0, 1] &\to X/\sim_A \\ ([x], t) &\mapsto [r(x, t)]. \end{aligned}$$

is a deformation retraction from $X/\sim_A = X/A$ to $A/\sim_A = A/A$. ■

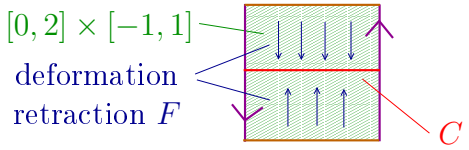
Example. We consider $X = [0, 2] \times [-1, 1]$ together with the equivalence relation "$\sim$" that is generated by

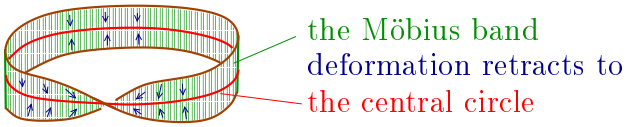
$$(0, y) \sim (2, -y) \quad \text{for all } y \in [-1, 1].$$

It is straightforward to see that $X/\sim$ is homeomorphic to the Möbius band as defined on page 107. We consider the maps

$$\begin{array}{rcl} F\colon X\times[0,1] & \to & X \\ ((x,y),t) & \mapsto & (x,y\cdot(1-t)) \end{array} \quad \text{and} \quad \begin{array}{rcl} \overline{F}\colon (X/\sim)\times[0,1] & \to & X/\sim \\ ([(x,y)],t) & \mapsto & [(x,y\cdot(1-t))]. \end{array}$$

Note that F is a deformation retraction from X to $C := [0,2]\times\{0\}$. One can easily verify that F is compatible with the equivalence relation "$\sim$". Thus we obtain from the Deformation Retraction-Quotient Theorem 9.2 (1) that $\overline{F}$ is a deformation retraction from the Möbius band to $C/\sim$. Note that $C/\sim$ is homeomorphic to S^1.





9.2. **Homotopy equivalent topological spaces.** For the reader's convenience we recall the following definition from page 136.

Definition. Let X and Y be two topological spaces. We say that two maps $f_0, f_1\colon X\to Y$ are homotopic, written $f_0\simeq f_1$, if there exists a continuous map $F\colon X\times[0,1]\to Y$ with $F_0=f_0$ and with $F_1=f_1$.

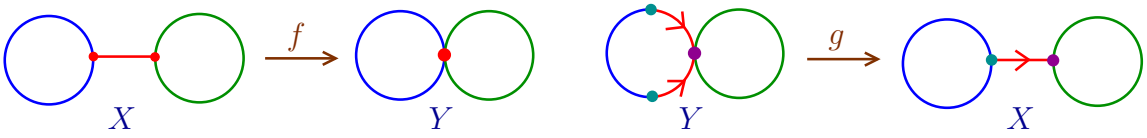
The following definition is one of the key definitions in topology. Unfortunately it usually takes some time to fully digest it.

Definition. Let X be and Y be topological spaces.

(1) A continuous $f\colon X\to Y$ is a homotopy equivalence between X and Y, if f admits a homotopy inverse, i.e. a continuous map $g\colon Y\to X$ with $g\circ f\simeq \mathrm{id}_X$ and $f\circ g\simeq \mathrm{id}_Y$.
(2) If there exists a homotopy equivalence between X and Y, then we say that X and Y are homotopy equivalent.

Example.

(1) Basically by definition a homeomorphism is a homotopy equivalence.
(2) In the figure we see two subspaces X and Y of $\mathbb{R}^2$ and we indicate continuous maps $f\colon X\to Y$ and $g\colon Y\to X$. With some effort one can show that $g\circ f\simeq \mathrm{id}_X$ and $f\circ g\simeq \mathrm{id}_Y$. Thus f and g are homotopy equivalences and X and Y are homotopy equivalent.



The following lemma shows that in the previous section we already acquired a long list of homotopy equivalences.

Lemma 9.3. (Deformation Retract–Homotopy Equivalence Lemma) Let X be a topological space and let A be a subset. If $F\colon X\times[0,1]\to A$ is a deformation retraction

from X to A, then the inclusion map

$$\begin{array}{rcl} i\colon A & \to & X \\ a & \mapsto & a \end{array} \qquad \text{and the map} \qquad \begin{array}{rcl} r\colon X & \to & A \\ x & \mapsto & F(x,1) \end{array}$$

are homotopy inverses of one another. In particular both maps are homotopy equivalences and in particular the deformation retract A is homotopy equivalent to the total space X.

Example. It follows from the Deformation Retract–Homotopy Equivalence Lemma 9.3 and the discussions on page 142 and 145 that the sphere S^{n-1} is homotopy equivalent to $\mathbb{R}^n \setminus \{0\}$ and that the Möbius band is homotopy equivalent to S^1.

Proof. By definition of a deformation retraction the map $r \circ i = F_1 \circ i\colon A \to A$ is the identity id_A. Thus it remains to show that $i \circ r\colon X \to X$ is homotopic to the identity id_X. A homotopy between id_X and $i \circ r$ is given by $F\colon X \times [0,1] \to X$. ■

The name "homotopy equivalence" already suggests that this is an equivalence relation on the class of topological spaces. It is elementary, but slightly messy, to prove this by hand. Instead of proving this by hand we will first reformulate the notion of being homotopy equivalent in a more categorical language, and then the statement will become trivial.

9.3. **The homotopy category.** In this section we introduce the homotopy category. Initially this might look like a detour, but in fact it simplifies and clarifies the discussion of homotopy equivalences.

Definition. Given two topological spaces X and Y we set

$$[X,Y] := \{\text{continuous maps } X \to Y\}/\sim$$

where $f \sim g$ if there exists a homotopy between f and g. Sometimes, given a continuous map $f\colon X \to Y$ we denote by $[f]$ the equivalence class it represents in $[X,Y]$.

Example. It follows easily from the definitions that given a topological space the set $[\{*\}, X]$ is in bijection to $\pi_0(X)$, i.e. in bijection to the set of path-components of X.

Definition. We define the homotopy category $\mathcal{HomTop}$ of topological spaces by[84]

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{HomTop}) & := & \text{all topological spaces,} \\ \mathrm{Mor}(X,Y) & := & [X,Y] \end{array}$$

with the composition[85]

$$\begin{array}{rcl} \mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z) & \to & \mathrm{Mor}(X,Z) \\ ([f],[g]) & \mapsto & [g \circ f]. \end{array}$$

The following lemma changes subtly our point on view on homotopy equivalences.

[84] Note that in this category the morphisms are not maps but they are equivalence classes of maps.

[85] It follows from the Homotopy Composition Lemma 8.4 (2) that this composition map is well-defined. It follows immediately from the definitions that this composition satisfies the two conditions on a category, as defined on page 69, namely associativity and existence of an identity morphism.

Lemma 9.4. (Homotopy Equivalence-Reformulation Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces. The following statements are equivalent:

(1) The element $[f] \in \mathrm{Mor}_{\mathcal{HomTop}}(X, Y)$ is an isomorphism in the homotopy category $\mathcal{HomTop}$.
(2) f is a homotopy equivalence.

In particular the following statements are equivalent:

(1′) X and Y are isomorphic in the homotopy category $\mathcal{HomTop}$.
(2′) X and Y are homotopy equivalent.

Proof. First we show the "(1)⇒(2)"-direction. Let $f\colon X \to Y$ be a continuous map. If $[f]$ is an isomorphism in the category $\mathcal{HomTop}$, then there exists a morphism $\varphi \in \mathrm{Mor}_{\mathcal{HomTop}}(Y, X)$ with $[f] \circ \varphi = \mathrm{id}_Y$ and $\varphi \circ [f] = \mathrm{id}_X$. We pick a continuous map $g\colon Y \to X$ with $\varphi = [g]$. Then we see that $[f \circ g] = [f] \circ [g] = [f] \circ \varphi = \mathrm{id}_Y = [\mathrm{id}_Y]$. But this means that $f \circ g \simeq \mathrm{id}_Y$. The same argument shows that $g \circ f \simeq \mathrm{id}_Y$. Thus we have shown that f is a homotopy equivalence.

The argument for "(2)⇒(1)" follows almost immediately from the definitions. Finally note that the equivalence of (1′) and (2′) is an immediate consequence of the equivalence of (1) and (2). ∎

Lemma 9.5. (Homotopy-Equivalence Basics Lemma)

(1) The relation "homotopy equivalence" is an equivalence relation on the class of all topological spaces.
(2) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces. If f is a homotopy equivalence and if g is homotopic to f, then g is also a homotopy equivalence.

Proof. All the statements follow easily from the Homotopy Equivalence-Reformulation Lemma 9.4 and the Isomorphism Lemma 3.3. We leave it to the reader to fill in the details. ∎

Example. We now have three methods for showing that topological spaces are homotopy equivalent:

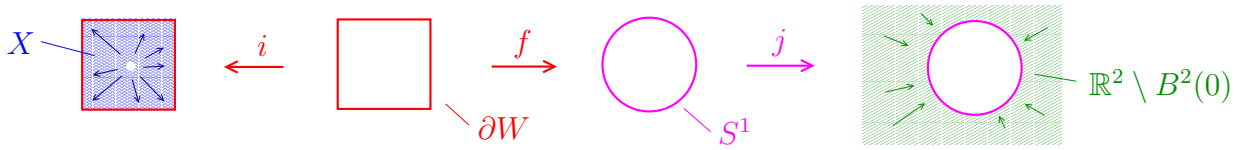
(1) Homeomorphisms are almost by definition homotopy equivalences.
(2) The Homotopy-Equivalence Basics Lemma 9.5 (1) says that being homotopy equivalent is an equivalence relation.
(3) The Deformation Retract–Homotopy Equivalence Lemma 9.3 says that if A is a deformation retract of some topological space X, then the inclusion $i\colon A \to X$ is a homotopy equivalence.

The combination of these three methods can be very powerful: We set $W = [-1, 1] \times [-1, 1]$. We want to show that the topological spaces $X := W \setminus \{0\}$ and $Y := \mathbb{R} \setminus B^2(0)$ are homotopy equivalent. To do so we consider the maps

$$X = W \setminus \{0\} \xhookleftarrow{\ i\ } \partial W \xrightarrow[f]{x \mapsto \frac{x}{\|x\|}} S^1 \xhookrightarrow{\ j\ } \mathbb{R}^2 \setminus B^2(0).$$

As on page 142 one can easily show that ∂W is a deformation retract of $X = W \setminus \{0\}$ and that S^1 is a deformation retract of $\mathbb{R}^2 \setminus B^2(0)$. Furthermore f is a homeomorphism. It

follows from the combination of (1), (2) and (3) that $X = W \setminus \{0\}$ is homotopy equivalent to $Y = \mathbb{R}^2 \setminus B^2(0)$.



We recall the following definitions from pages 64 and 75.

Definition. Let X be a topological space.

(1) We denote by $\pi_0(X)$ the set of path-components of X.

(2) Given a continuous map $f\colon X \to Y$ we write

$$\begin{array}{rcl} f_*\colon \pi_0(X) & \to & \pi_0(Y) \\ C & \mapsto & \text{unique path-component of } Y \text{ that contains } f(C). \end{array}$$

Lemma 9.6. (Homotopy-π_0-Lemma)

(1) The maps

$$\begin{array}{rcl} X & \mapsto & \pi_0(X) \\ [f\colon X \to Y] & \mapsto & [f_*\colon \pi_0(X) \to \pi_0(Y)] \end{array}$$

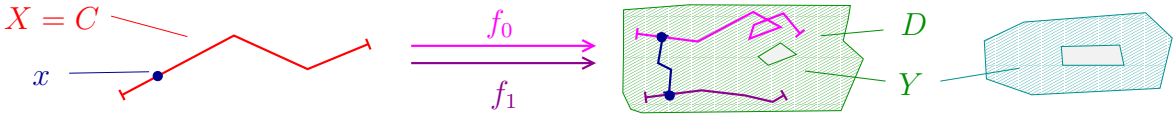
form a covariant functor from the homotopy category *HomTop* to the category *Set* of sets.

(2) (a) Let $f\colon X \to Y$ be a continuous map between topological spaces. If f is a homotopy equivalence, then f induces a bijection $f_*\colon \pi_0(X) \to \pi_0(Y)$.

(b) If X and Y are homotopy equivalent topological spaces, then X is path-connected if and only if Y is path-connected.

Proof.

(1) We only really need to show that the functor is well-defined. More precisely, we need to show that if $f_0, f_1\colon X \to Y$ are homotopic, then $f_{0*} = f_{1*}\colon \pi_0(X) \to \pi_0(Y)$. Thus let C be a path-component of X and let D be the unique path-component of Y that contains $f_0(C)$. Since f_0 and f_1 are homotopic we see that for every $x \in X$ the points $f_0(x)$ and $f_1(x)$ are path-equivalent in Y. But that means, by definition of a path-component as a path-equivalence class, that $f_1(C) \subset D$. This implies that $f_{1*}(C) = D = f_{0*}(C)$.



(2) (a) This statement follows from (1), the Homotopy Equivalence-Reformulation Lemma 9.4 and the fact that functors send isomorphisms to isomorphisms.

(b) This statement is a special case of (a). ■

9.4. **Contractible topological spaces.** We turn to an important special case of homotopy equivalences:

Definition. Let X be a topological space. We say X is contractible, if X is homotopy equivalent to a topological space which consists of precisely one point.

Remark. Note that the empty topological space is *not* contractible.

The following straightforward lemma gives an easy criterion for showing that a topological space is contractible.

Lemma 9.7. (Point Deformation Retract Lemma) If a topological space X admits a deformation retraction to a point $x_0 \in X$, then X is homotopy equivalent to $\{x_0\}$, in particular X is contractible.

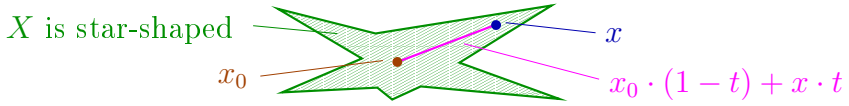
Proof. This lemma is just a special case of the Deformation Retract–Homotopy Equivalence Lemma 9.3. ■

Example.

(1) We say a subset X of $\mathbb{R}^n$ is star-shaped if there exists a point $x_0 \in X$ such that for every $x \in X$ the segment from x_0 to x lies in X. Basically by the same argument as in the Convex Homotopy Lemma 8.1 we see that the map

$$\begin{array}{rcl} f\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & \underbrace{x \cdot (1-t) + x_0 \cdot t}_{\in X, \text{ since } X \text{ is star-shaped}} \end{array}$$

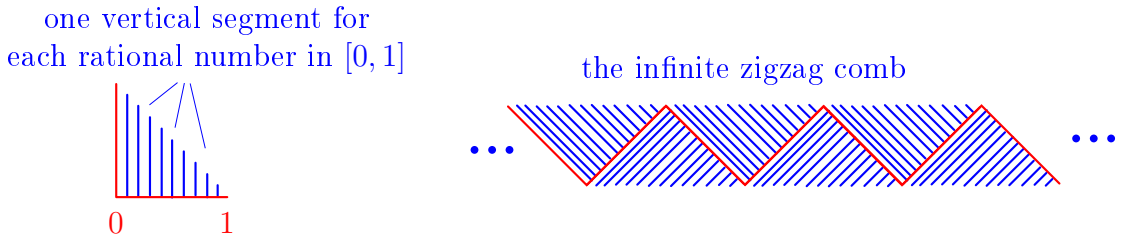
is deformation retraction from X to the point x_0. It follows from the Point Deformation Retract Lemma 9.7 that X is contractible.



(2) Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$. We just learned that the topological spaces $\mathbb{R}^n$ and $\mathbb{R}^k$ are contractible. It follows from the Homotopy-Equivalence Basics Lemma 9.5 (2) that $\mathbb{R}^n$ and $\mathbb{R}^k$ are homotopy equivalent.

In the Point Deformation Retract Lemma 9.7 we saw that if a topological space X admits a deformation retraction to a point $x_0 \in X$, then X is contractible. The following lemma shows that in general the converse of this statement does *not* hold.

Lemma 9.8. (Zigzag Comb-Lemma) The "infinite zigzag comb" shown in the figure below is contractible, but it does not admit a deformation retraction to a single point.



Proof. We leave it to the reader to provide the proof. ■

Part II

Fundamental groups and higher homotopy groups

10. How can we show that two topological spaces are (not) homeomorphic?

In the previous chapters we collected many examples of topological spaces. In many cases it was relatively straightforward to see that two topological spaces are homeomorphic. For example we wrote down an explicit homeomorphism from $\mathbb{R}^n/\mathbb{Z}^n$ to $(S^1)^n$. Sometimes it is rather painful to explicitly write down a homeomorphism, for example it was not particularly easy to find a homeomorphism from the open cube $(0,1)^n$ to the open ball $B^n = \{x \in \mathbb{R}^n \,|\, \|x\| < 1\}$. Nonetheless, if two topological spaces are homeomorphic, then usually one can write down an explicit homeomorphism.

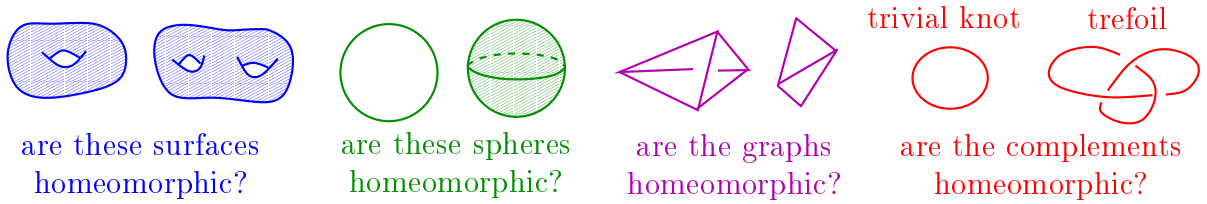
On the other hand, how can we tell that two topological spaces are not homeomorphic? There are a couple of trivial criteria: being compact, connected, Hausdorff are properties of topological spaces that are shared by homeomorphic topological spaces. This implies for example that the closed cube $[0,1]^n$ is not homeomorphic to the open cube $(0,1)^n$, since the former topological space is compact and the latter is not.

But these criteria are of no use when it comes to studying more subtle examples and questions. For example they do not suffice to address the following questions:

Question 10.1.

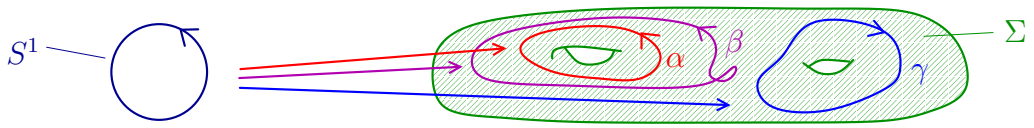
(1) Can surfaces with different genera be homeomorphic?
(2) Given $k \neq l \in \mathbb{N}$, is it possible that the spheres S^k and S^l are homeomorphic?
(3) Are the two graphs shown below homeomorphic?
(4) Is the complement of the trefoil homeomorphic to the complement of the trivial knot?

In each case we suspect that the answer is no. Certainly we do not succeed in finding a homeomorphism, but how can we show that this is not due to lack of imagination, but that there is indeed no homeomorphism?



Here is another type of question that we cannot really address right now.

Question 10.2. We consider the maps $S^1 \to \Sigma$ from S^1 to the surface Σ that are shown in the figure below. The curves α and β can be deformed into one another. But it does not look like one can deform α into γ. But how can one prove the non-existence of a deformation?



The reader might have encountered some ideas for tackling these questions in one or both of the following topics:

(a) Differential forms on smooth manifolds, in particular de Rham cohomology, as developed say in [**BoT1982**; Lee02].

(b) Complex analysis, see e.g. [Lan99; Jän11].

First let's talk about approach (a). For the reader who has seen differential forms and de Rham cohomology we recall that to each smooth manifold M and each $k \in \mathbb{Z}_{\geq 0}$ we can assign the de Rham cohomology group $\mathrm{H}^k_{\mathrm{dR}}(M)$. It follows more or less from the definition that diffeomorphic smooth manifolds have isomorphic de Rham cohomology groups. This approach has several disadvantages:

(1) The de Rham cohomology groups are only defined for smooth manifolds.
(2) The de Rham cohomology groups can only be used to show that two smooth manifolds are not *diffeomorphic*, a priori they cannot be used to show that two smooth manifolds are not *homeomorphic*.
(3) At least just starting from the definition it is not clear how one can calculate the de Rham cohomology groups of a given smooth manifold.

Later on we will introduce "singular cohomology" which will take care of all of these three issues.

We turn to the second approach from complex analysis. We recall some definitions that are familiar to the reader who has taken a course on complex analysis.

Definition.

(1) Given a subset U of $\mathbb{C}$, we define a path in U to be a continuous map $\gamma\colon [a,b] \to U$. If $\gamma(a) = \gamma(b)$, then we say that γ is a loop.
(2) Two paths $\gamma_0, \gamma_1\colon [a,b] \to U$ with the same starting point $P := \gamma_0(a) = \gamma_1(a)$ and the same endpoint $Q := \gamma_0(b) = \gamma_1(b)$ are called path-homotopic in U, if there exists a homotopy rel $\{a,b\}$ from γ_0 to γ_1. In other words, if there exists a continuous map

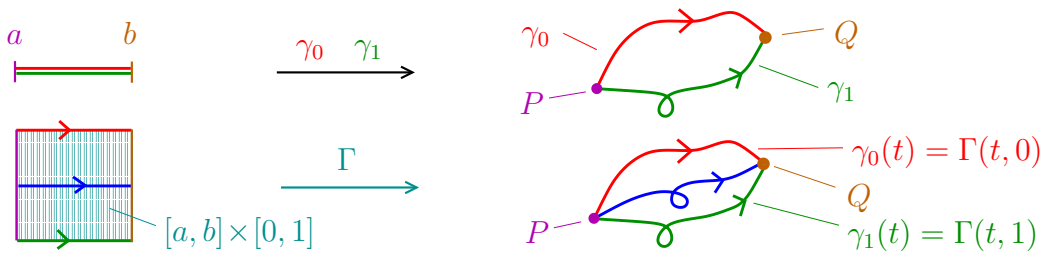
$$\begin{array}{rcl} \Gamma\colon [a,b]\times[0,1] & \to & U \\ (t,s) & \mapsto & \Gamma(t,s), \end{array}$$

with the following properties:

(a) for every $t \in [a,b]$ we have $\Gamma(t,0) = \gamma_0(t)$ and $\Gamma(t,1) = \gamma_1(t)$,
(b) for every $s \in [0,1]$ we have $\Gamma(a,s) = P$ and $\Gamma(b,s) = Q$.

In the figure below we illustrate a path-homotopy between two paths.

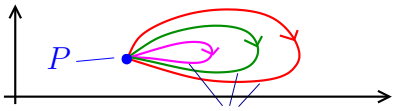
(3) A loop $\gamma\colon [a,b] \to U$ is called null-homotopic if it is path-homotopic to the constant path given by $\delta(t) := \gamma(a)$ for all $t \in [a,b]$.
(4) We say U is simply connected if U is non-empty, if U is path-connected and if each loop is null-homotopic.



Example. Every loop in $\mathbb{C}$ is null-homotopic. Indeed, we let $\gamma\colon [0,1] \to \mathbb{C}$ be a loop with starting and endpoint P. Then

$$\begin{array}{rccl} \Gamma\colon & [a,b]\times[0,1] & \to & \mathbb{C} \\ & (t,s) & \mapsto & \gamma(t)\cdot(1-s)+P\cdot s \end{array}$$

is a path-homotopy between the loop γ and the constant path at P. In particular $\mathbb{C}$ is simply connected.



path-homotopy from γ to a constant path

Definition. Let $U \subset \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be a continuous map. Furthermore let $\gamma\colon [a,b] \to U$ a smooth path. We define the path integral of f along γ as follows:

$$\int_\gamma f(z)\,dz \;:=\; \int_{t=a}^{t=b} \underbrace{f(\gamma(t))}_{\in\mathbb{C}} \cdot \underbrace{\gamma'(t)}_{\in\mathbb{C}}\,dt \;\in \mathbb{C}.$$

product of complex numbers

Examples.

(1) Let $U = \mathbb{C}\setminus\{0\}$, let $f(z) = \frac{1}{z}$ and let $\gamma\colon [0,1] \to \mathbb{C}\setminus\{0\}$ be the loop that is given by $\gamma(t) = \exp(2\pi\,\mathrm{i}\,t)$, i.e. γ is a loop that "goes once around the origin". Then

$$\int_\gamma f(z)\,dz \;=\; \int_{t=0}^{t=1} f(\gamma(t))\cdot\gamma'(t)\,dt \;=\; \int_{t=0}^{t=1} \frac{1}{\exp(2\pi\,\mathrm{i}\,t)}\cdot\exp(2\pi\,\mathrm{i}\,t)\cdot 2\pi\,\mathrm{i}\,dt \;=\; 2\pi\,\mathrm{i}.$$

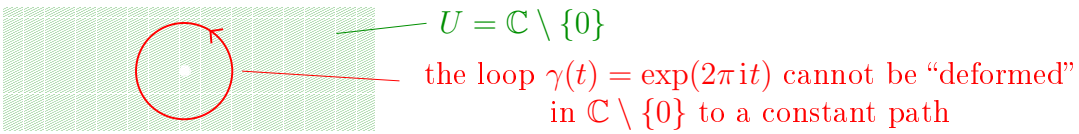
(2) If γ is a constant path, then $\gamma'(t) \equiv 0$. This shows that the path-integral along a constant path is zero.

Theorem 10.3. (Cauchy Theorem)[86] Let $U \subset \mathbb{C}$ be an open subset, let $f\colon U \to \mathbb{C}$ be a holomorphic function and let $\gamma, \delta\colon [a,b] \to U$ be two smooth paths. If γ and δ are path-homotopic, then

$$\int_\gamma f(z)\,dz \;=\; \int_\delta f(z)\,dz.$$

Proof. This theorem is proved in complex analysis. ∎

Corollary 10.4. The loop $\gamma\colon [0,1] \to \mathbb{C}\setminus\{0\}$ given by $\gamma(t) = \exp(2\pi\,\mathrm{i}\,t)$ is *not* path-homotopic to a constant path, i.e. γ is not null-homotopic.



[86] We will not really make use of the Cauchy Theorem 154. But it motivates us to study path-homotopies in general topological spaces.

Proof. In the above example we saw that $\int_\gamma \frac{1}{z}\,dz \neq 0$ and we saw that for a constant loop δ in $\mathbb{C}\setminus\{0\}$ we have $\int_\delta \frac{1}{z}\,dz = 0$. Thus it follows from the Cauchy Theorem 10.3 that the loop γ is not path-homotopic to a constant loop. ■

Example. If two subsets of $\mathbb{C}$ are homeomorphic, then evidently either both are simply connected or none is. It follows from the discussion on page 154 and Corollary 10.4 that $\mathbb{R}^2 = \mathbb{C}$ and $\mathbb{R}^2 \setminus \{(0,0)\} = \mathbb{C}\setminus\{0\}$ are not homeomorphic.

This second approach to studying topological spaces, namely through path-homotopies of loops, also has several disadvantages:

(1) A priori these notions are defined only for subsets of $\mathbb{C}$.
(2) This approach does not allow us to show that the two subsets $U = \mathbb{C}\setminus\{0\}$ and $V = \mathbb{C}\setminus\{0,1\}$ are not homeomorphic, since neither is simply connected.

The first objection is of course easy to rectify, since the definition of being simply connected makes sense for any topological space. But if we work with more general situations a new problem arises: how can we show, without using complex analysis, that a topological space is not simply connected? In the following chapters we will address all these issues.

Regarding the second objection, here is one idea for distinguishing $U = \mathbb{C}\setminus\{0\}$ and $V = \mathbb{C}\setminus\{0,1\}$: naively one could say that V has "more" loops that are not null-homotopic than U. In the Chapter 11 we will in particular make precise what "more" means.

11. The fundamental group

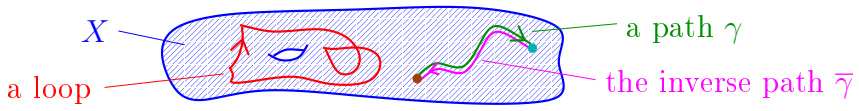
11.1. **Path-homotopy.** In this chapter we extend some of the ideas and notions about paths that we introduced in the previous section to the more general setting of paths in topological spaces. The following definition is a slight generalization of the concept of a path that we initially introduced on page 59.

Definition. Let X be a topological space.

(1) We define a path in X as a continuous map $\gamma\colon [a,b] \to X$. We call $\gamma(a)$ the starting point of γ and we call $\gamma(b)$ the endpoint of γ. Often we say that γ is a path from $\gamma(a)$ to $\gamma(b)$.

(2) A loop is a path for which the starting point and the endpoint coincide.

(3) Given a path $\gamma\colon [a,b] \to X$ we refer to the path $\overline{\gamma}\colon [a,b] \to X$ that is defined by $\overline{\gamma}(t) = \gamma(b+a-t)$ as the inverse path of γ.



Definition. Let X be a topological space.

(1) Let $\gamma_0, \gamma_1\colon [a,b] \to X$ be two paths with the same starting point P and the same endpoint Q. A path-homotopy between the paths γ_0 and γ_1 is a homotopy rel $\{a,b\}$ between γ_0 and γ_1. In other words, spelling out the definition on page 137, a path-homotopy is a continuous map

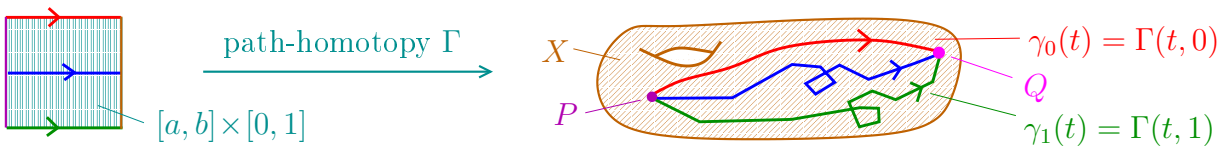
$$\begin{array}{rccc} \Gamma\colon [a,b]\times[0,1] & \to & X \\ (t,s) & \mapsto & \Gamma(t,s), \end{array}$$

with the following properties:

(a) for every $t \in [a,b]$ we have $\Gamma(t,0) = \gamma_0(t)$ and $\Gamma(t,1) = \gamma_1(t)$,
(b) for every $s \in [0,1]$ we have $\Gamma(a,s) = P$ and $\Gamma(b,s) = Q$.

(2) If there exists a path-homotopy between two paths γ_0 and γ_1, then we say that γ_0 and γ_1 are path-homotopic, and we write $\gamma_0 \simeq_{\mathrm{p}} \gamma_1$.

(3) We say a loop $\gamma\colon [a,b] \to X$ is null-homotopic, if it is path-homotopic to the constant path δ given by $\delta(t) := \gamma(a)$ for all $t \in [a,b]$.



Example.

(A) Let C be a convex subset of $\mathbb{R}^n$ and let $\gamma_0, \gamma_1\colon [a,b] \to C$ be two paths in C with same starting point and the same endpoint. As in the Convex Homotopy Lemma 8.1 we see that

$$\begin{array}{rcl} \Gamma\colon [a,b]\times[0,1] & \to & C \\ (t,s) & \mapsto & \underbrace{\gamma_0(t)\cdot(1-s)+\gamma_1(t)\cdot s}_{\in C,\text{ since } C \text{ is convex}} \end{array}$$

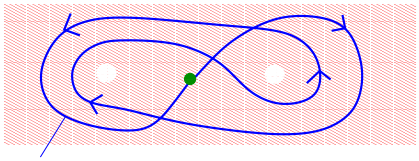
is a path-homotopy between γ_0 and γ_1.[87]

(B) Let $X = \mathbb{C}\setminus\{0\} = \mathbb{R}^2\setminus\{(0,0)\}$. We consider the loop

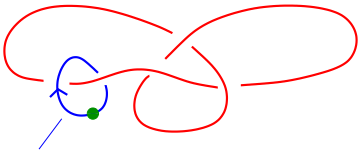
$$\begin{array}{rcl} \gamma\colon [0,1] & \to & X \\ t & \mapsto & \exp(2\pi i t) = (\cos(2\pi t), \sin(2\pi t)). \end{array}$$

In Corollary 10.4 we showed that the path is not null-homotopic in X. We will soon give a different proof of this statement.

(C) We consider the two loops in the figure below. In both cases it is hard to see how these loops could possibly be path-homotopic to a constant path, but it is also difficult to find an argument why that should not be possible.[88] We will return to this example on page 265.



loop in $\mathbb{C}\setminus\{0,1\}$

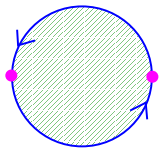


loop in the complement of a knot

(D) We consider the real projective space $\mathbb{R}\mathrm{P}^2 =_i \overline{B}^2/\!\sim$ where $P \sim -P$ for $P \in S^1 = \partial\overline{B}^2$. Note that

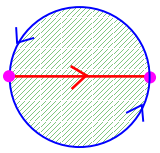
$$\begin{array}{rcl} \gamma\colon [0,1] & \to & \overline{B}^2/\!\sim \\ t & \mapsto & [(2t-1,0)] \end{array}$$

is a loop in $\mathbb{R}\mathrm{P}^2 = \overline{B}^2/\!\sim$.[89] But is it null-homotopic?



$\overline{B}^2/\!\sim\ =_i \mathbb{R}\mathrm{P}^2$

loop in $\overline{B}^2/\!\sim\ =_i \mathbb{R}\mathrm{P}^2$



Our first lemma of this chapter will provide us with a rich source of path-homotopies. To formulate the lemma we need the following definition.

Definition. Let $\gamma\colon [a,b] \to X$ be a path in a topological space X and let $\varphi\colon [a,b] \to [a,b]$ be a continuous map with $\varphi(a) = a$ and $\varphi(b) = b$. We refer to $\gamma\circ\varphi\colon [a,b] \to X$ as a reparametrization of γ.

The following lemma says that reparametrizing does not change the path-homotopy class of a path.

[87] We had already illustrated this fact in the figure on page 154.

[88] For example the given loop in $\mathbb{C}\setminus\{0,1\}$ is actually null-homotopic in $\mathbb{C}\setminus\{0\}$ and it is null-homotopic in $\mathbb{C}\setminus\{1\}$. In particular our trick of using Corollary 10.4 does not work in this case.

[89] Why is it a loop?

Lemma 11.1. (Reparametrization-Homotopy Lemma) Let $\gamma\colon [a,b] \to X$ be a path in a topological space X. If a path $\delta\colon [a,b] \to X$ is obtained from γ by a reparametrization, then γ and δ are path-homotopic.

Proof. Let $\gamma\colon [a,b] \to X$ be a path in a topological space and let $\varphi\colon [a,b] \to [a,b]$ be a continuous map with $\varphi(a) = a$, $\varphi(b) = b$. A path-homotopy between γ and $\gamma \circ \varphi$ is given by[90]

$$\begin{array}{rcl} P\colon [a,b] \times [0,1] & \to & X \\ (t,s) & \mapsto & \gamma(t \cdot s + \varphi(t) \cdot (1-s)). \end{array}$$

■

The following lemma is a special case of the Homotopy-Equivalence Relation Lemma 8.3.

Lemma 11.2. (Path-Homotopy–Equivalence Relation Lemma) Let X be a topological space and let $x, y \in X$. Then "path-homotopy" is an equivalence relation on the set of paths $\gamma\colon [a,b] \to X$ from x to y.

Proof. This lemma is just the special case of the Homotopy-Equivalence Relation Lemma 8.3 with $X = [a,b]$ and $A = \{a,b\}$. ■

The Path-Homotopy–Equivalence Relation Lemma 11.2 allows us to make the following definition.

Definition. Given a path $f\colon [a,b] \to X$ in a topological space X we denote by $[f]$ the equivalence class of f with respect to the equivalence relation which is given by path-homotopies. We call $[f]$ the path-homotopy class of f.

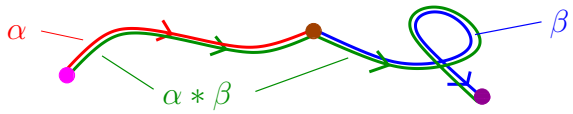
We now recall the following definition from page 62.

Definition. Let X be a topological space and let $\alpha\colon [a,b] \to X$ and $\beta\colon [c,d] \to X$ be two paths with $\alpha(b) = \beta(c)$. We define the concatenation of α and β as the path $\alpha * \beta$ which is given by

$$\begin{array}{rcl} \alpha * \beta\colon [0,1] & \to & X \\ t & \mapsto & \begin{cases} \alpha(a + 2t(b-a)), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(c + (2t-1)(d-c)), & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases} \end{array}$$

Similarly we also define the concatenation of a finite number of paths.

The concatenation of two paths is thus given by "first running along α" and then "running along β". Here we reparametrized the path so that the domain of $\alpha * \beta$ is $[0,1]$. This convention might look slightly odd, but later on we will mostly work with paths that are defined on the interval $[0,1]$.



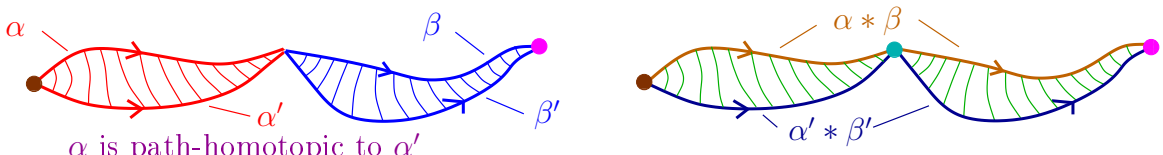
Lemma 11.3. (Path-Homotopy–Concatenation Lemma) Let X be a topological space. Let $\alpha, \alpha'\colon [a,b] \to X$ and $\beta, \beta'\colon [c,d] \to X$ be paths with $\alpha(b) = \alpha'(b) = \beta(c) = \beta'(c)$. Then

$$\alpha \simeq_{\mathrm{p}} \alpha' \quad \text{and} \quad \beta \simeq_{\mathrm{p}} \beta' \quad \Longrightarrow \quad \alpha * \beta \simeq_{\mathrm{p}} \alpha' * \beta'.$$

Put differently, if $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$, then $[\alpha * \beta] = [\alpha' * \beta']$.

Proof. We leave it to the reader to provide the fairly elementary proof. ■

[90] Where do we actually use the hypothesis that $\varphi(a) = a$ and $\varphi(b) = b$?



α is path-homotopic to α' and β is path-homotopic to β'

then $\alpha * \beta$ is path-homotopic to $\alpha' * \beta'$

Definition. Let $[\alpha]$ and $[\beta]$ be two equivalence classes of paths in a topological space X with $\alpha(1) = \beta(0)$. We define the[91] product of the equivalence classes $[\alpha]$ and $[\beta]$ as

$$[\alpha] \cdot [\beta] \;:=\; [\alpha * \beta].$$

The following proposition says that the product of equivalence classes of paths shares many properties of the product structure of a group.

Proposition 11.4. (Path-Homotopy Product Proposition) Let X be a topological space. Given any three paths $\alpha, \beta, \gamma\colon [0,1] \to X$ the following hold:

(1) If $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$, then[92]

$$[\alpha] \cdot ([\beta] \cdot [\gamma]) \;=\; ([\alpha] \cdot [\beta]) \cdot [\gamma].$$

(2) For $x \in X$ let e_x be the constant path that is given by $e_x(t) := x$. The following equality holds:

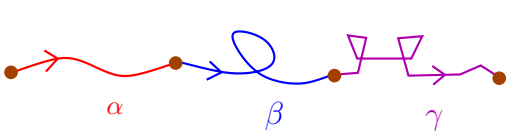
$$[e_{\alpha(0)}] \cdot [\alpha] \;=\; [\alpha] \;=\; [\alpha] \cdot [e_{\alpha(1)}].$$

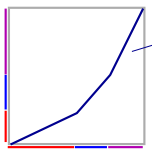
(3) We denote by $\overline{\alpha}$ the inverse path. Then the following equalities hold:

$$[\alpha] \cdot [\overline{\alpha}] \;=\; [e_{\alpha(0)}] \quad \text{and} \quad [\overline{\alpha}] \cdot [\alpha] \;=\; [e_{\alpha(1)}].$$

Proof. Let $\alpha, \beta, \gamma\colon [0,1] \to X$ be three paths in a topological space X.

(1) Suppose that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$. We consider the map $\varphi\colon [0,1] \to [0,1]$ that is shown in the figure below. It follows immediately from the definitions that $(\alpha * (\beta * \gamma)) \circ \varphi = (\alpha * \beta) * \gamma$. Thus it follows from the Reparametrization-Homotopy Lemma 11.1 that $\alpha * (\beta * \gamma) \simeq_{\mathrm{p}} (\alpha * \beta) * \gamma$, which is what we need to show.





graph of $\varphi\colon [0,1] \to [0,1]$ which is used to reparametrize the path $\alpha * (\beta * \gamma)$ to obtain the path $(\alpha * \beta) * \gamma$

(2) We consider the map

$$\begin{aligned} \varphi\colon [0,1] &\to [0,1] \\ t &\mapsto \begin{cases} 0, & \text{if } t \in [0, \tfrac{1}{2}], \\ 2t-1, & \text{if } t \in [\tfrac{1}{2}, 1]. \end{cases} \end{aligned}$$

Then $\quad e_{\alpha(0)} * \alpha \;\underset{\text{by definition}}{=}\; \alpha \circ \varphi \;\underset{\text{Reparametrization-Homotopy Lemma 11.1}}{\simeq_{\mathrm{p}}}\; \alpha.$

[91] According to the Path-Homotopy–Concatenation Lemma 11.3 this definition does not depend on the choice of the representatives α and β of the equivalence classes.

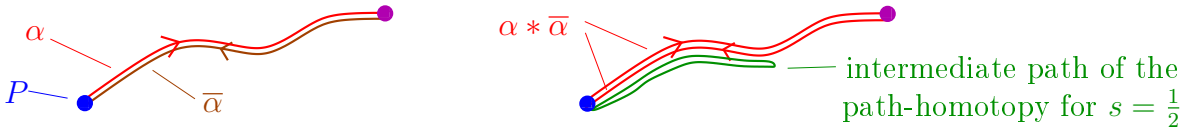
[92] Put differently, the paths $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ are path-homotopic. Note that these paths are similar in the sense that both paths "run through" the points on α, β and γ, but at different speeds. In general the paths are not equal, for example we have $(\alpha * (\beta * \gamma))(\frac{1}{2}) = \alpha(1)$ and $((\alpha * \beta) * \gamma)(\frac{1}{2}) = \beta(1)$.

This shows that $[e_{\alpha(0)}] \cdot [\alpha] = [e_{\alpha(0)} * \alpha] = [\alpha]$. Almost the same argument also implies that $[\alpha] = [\alpha] \cdot [e_{\alpha(1)}]$.

(3) We write $P = \alpha(0)$. Note that $\alpha * \overline{\alpha}$ is a loop in P. We consider the map

$$\begin{array}{rcl} F\colon [0,1]\times[0,1] & \to & X \\ (t,s) & \mapsto & \begin{cases} \alpha(2t\cdot s), & \text{if } t\in[0,\frac{1}{2}], \\ \alpha(2s-2t\cdot s), & \text{if } t\in[\frac{1}{2},1]. \end{cases} \end{array}$$

It follows from the Homotopy Combination Lemma 8.2 (1) that this map is indeed continuous. For $s = 0$ we obtain the constant path at P and for $s = 1$ we obtain the path $\alpha * \overline{\alpha}$. For any $s \in [0,1]$ we furthermore have $F(0,s) = F(1,s) = P$. Thus F is a path-homotopy from the constant path to $\alpha * \overline{\alpha}$. This implies that $[e_{\alpha(0)}] = [\alpha * \overline{\alpha}] = [\alpha] \cdot [\overline{\alpha}]$. Almost the same argument shows that $[\overline{\alpha} * \alpha] = [e_{\alpha(1)}]$. ■



11.2. **The fundamental group of a pointed topological space.** In the Path-Homotopy Product Proposition 11.4 we saw that the concatenation of path-homotopy classes of paths shares many properties of the multiplication in a group. Nonetheless the path-homotopy classes do not form a group, since the concatenation of two paths is not defined, unless the endpoint of the first path agrees with the starting point of the second path.

We will resolve this problem by restricting our attention to loops that start and end at a fixed base point.

Definition. Let X be a topological space and let $x_0 \in X$ be a point. A loop in (X, x_0) is a path $f\colon [0,1] \to X$[93] with $f(0) = f(1) = x_0$, i.e. x_0 is the starting as well as the endpoint of f.

Proposition 11.5. (π_1–Group Proposition) Let X be a topological space and $x_0 \in X$. The set

$$\pi_1(X,x_0) \;:=\; \{\text{path-homotopy classes of loops in } (X,x_0)\}$$

together with the product map $[\alpha]\cdot[\beta] := [\alpha * \beta]$ forms a group where the neutral element is represented by the constant path at x_0 and where the inverse of $[\gamma]$ is given by $[\overline{\gamma}]$.

Proof. It is clear that the concatenation of any two loops in (X, x_0) is defined. According to the Path-Homotopy–Concatenation Lemma 11.3 the concatenation of paths descends to a well-defined map

$$\begin{array}{rcl} \pi_1(X,x_0)\times\pi_1(X,x_0) & \to & \pi_1(X,x_0) \\ ([\alpha],[\beta]) & \mapsto & [\alpha]\cdot[\beta] := [\alpha*\beta]. \end{array}$$

The Path-Homotopy Product Proposition 11.4 (1)–(3) gives us the following:

(1) The above product map satisfies the associativity axiom.
(2) The path-homotopy class of the constant loop $[e_{x_0}]$ is a neutral element.
(3) The inverse of the path-homotopy class of a loop $\gamma\colon [0,1] \to X$ in x_0 is given by the path-homotopy class of the inverse loop defined by $\overline{\gamma}(t) := \gamma(1-t)$, $t \in [0,1]$. ■

[93] Note that part of the definition of "loop in (X, x_0)" is that the domain is the interval $[0,1]$.

Definition. Let (X, x_0) be a pointed topological space. We call

$$\pi_1(X, x_0) \;:=\; \{\text{path-homotopy classes of loops in } (X, x_0)\}$$

the fundamental group of X with respect to the base point x_0. If $\pi_1(X, x_0)$ is the trivial group, then we usually write $\pi_1(X, x_0) = 0$.[94]

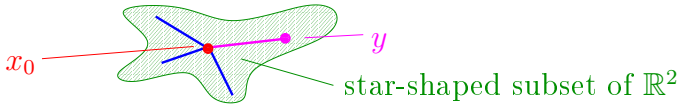
One of our main goals in the intermediate future will be to determine the fundamental group of the topological spaces that we introduced in the earlier chapters, for example spheres, surfaces, the projective spaces $\mathbb{R}\mathrm{P}^n$ and so on. We start with the simplest example.

Example.

(1) We say that a subset $X \subset \mathbb{R}^n$ is star-shaped with respect to $x_0 \in X$ if for all $y \in X$ the segment $\{x_0 \cdot (1-t) + y \cdot t \,|\, t \in [0,1]\}$ lies in X. As on page 156 we see that all loops in (X, x_0) are null-homotopic. More precisely, if $\gamma\colon [0,1] \to \mathbb{R}^n$ is a loop in (X, x_0), then

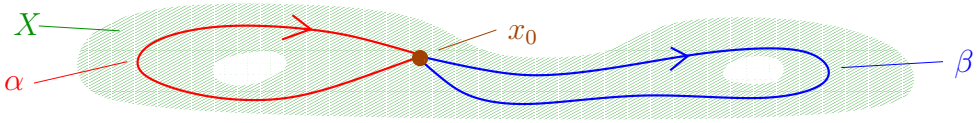
$$\begin{array}{rccc} F\colon [0,1] \times [0,1] & \to & X \\ (t, s) & \mapsto & \gamma(t) \cdot (1-s) + x_0 \cdot s \end{array}$$

is a path-homotopy in X between the loop γ and the constant loop e_{x_0}. Thus it follows that $\pi_1(X, x_0) = 0$.



(2) Let X be a convex subset of $\mathbb{R}^n$. (For example open and closed balls and open and closed rectangles are convex.) Note that X is star-shaped with respect to any $x_0 \in X$, thus we obtain from (1) that $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

(3) The π_1–Group Proposition 11.5 says that $\pi_1(X, x_0)$ is a group, but it is not clear whether the group is necessarily abelian. In fact, in the figure below we see two loops α, β in (X, x_0) where it is not clear at all whether $[\alpha] \cdot [\beta] \stackrel{?}{=} [\beta] \cdot [\alpha] \in \pi_1(X, x_0)$.



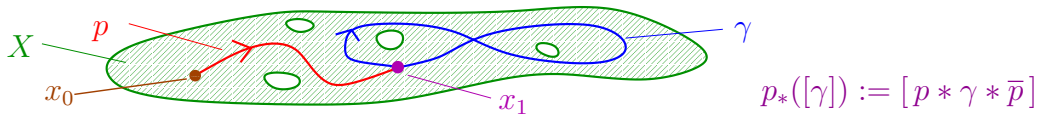
11.3. **Change of base points.** The definition of the fundamental group of a topological space X relies on the choice of a base point x_0. The following proposition shows that, at least for path-connected topological spaces, the choice of the base point does not affect the isomorphism type of the fundamental group.

Proposition 11.6. (Change-of-Base Point Proposition) Let X be a topological space, let x_0 and x_1 be two points in X and let $p\colon [0,1] \to X$ be a path from x_0 to x_1. The map

$$\begin{array}{rccc} p_*\colon \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ [\gamma] & \mapsto & [p * \gamma * \overline{p}\,] \end{array}$$

is well-defined and it is a group isomorphism.

[94] This notation is commonly used, even though it is not entirely logical: we use the multiplicative notation for the product structure on the group $\pi_1(X, x_0)$, and it would therefore make more sense to write $\pi_1(X, x_0) = 1$ if the group $\pi_1(X, x_0)$ is trivial.



Proof. Let X be a topological space and let x_0 and x_1 be two points in X that are connected via a path $p\colon [0,1] \to X$ from x_0 to x_1. Let γ be a loop in (X, x_1). Then $p * \gamma * \overline{p}$ is evidently a loop in (X, x_0). From the Path-Homotopy–Concatenation Lemma 11.3 and the Path-Homotopy Product Proposition 11.4 it follows that the path-homotopy class of $p*\gamma*\overline{p}$ depends only on the choice of the path-homotopy class of γ, i.e. the map

$$\begin{array}{rcl} \Phi\colon \pi_1(X,x_1) & \to & \pi_1(X,x_0) \\ [\gamma] & \mapsto & [p*\gamma*\overline{p}\,] \end{array}$$

is well-defined. Furthermore for two loops γ, δ in (X, x_1) we have

$$\begin{array}{rcl} \Phi([\gamma]) \cdot \Phi([\delta]) & = & [p*\gamma*\overline{p}\,] \cdot [p*\delta*\overline{p}\,] \\ & \underset{\uparrow}{=} & [p*\gamma*e_{x_1}*\delta*\overline{p}\,] \end{array} \quad \begin{array}{rcl} & = & [p*\gamma*\overline{p}*p*\delta*\overline{p}] \\ & \underset{\uparrow}{=} & [p*(\gamma*\delta)*\overline{p}\,] = \Phi([\gamma*\delta]) = \Phi([\gamma]\cdot[\delta]). \end{array}$$

the two statements follow from the Path-Homotopy Concatenation Proposition 11.4 (3) and (2) together with the Path-Homotopy–Concatenation Lemma 11.3

Thus we have shown that Φ is a group homomorphism. Furthermore Φ is even a group isomorphism, since an inverse map is given by[95]

$$\begin{array}{rcl} \Psi\colon \pi_1(X,x_0) & \to & \pi_1(X,x_1) \\ [\delta] & \mapsto & [\overline{p}*\delta*p]. \end{array}$$

■

We obtain the following corollary:

Corollary 11.7. (Base Point Independence Corollary) Let X be a topological space and let $x_0, x_1 \in X$. If X is path-connected, then the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.

Proof. This statement follows easily from the Change-of-Base Point Proposition 11.6. ■

Example. Let X be a star-shaped subset of $\mathbb{R}^n$. Above we saw that there exists an $x_0 \in X$ such that $\pi_1(X, x_0) = 0$. It now follows from the Base Point Independence Corollary 11.7 that $\pi_1(X, x_1) = 0$ for any point $x_1 \in X$. Put differently, the fundamental group of every star-shaped subset of $\mathbb{R}^n$ with respect to any base point is trivial.

If X is a path-connected topological space, then the Base Point Independence Corollary 11.7 states that the isomorphism type of the fundamental group does not depend on the choice of the base point. This leads us to the following definition.

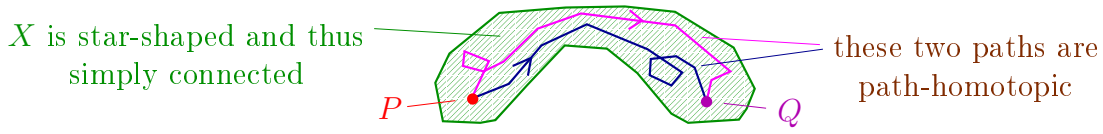
Definition. Let X be a path-connected non-empty topological space.
(1) We denote by $\pi_1(X)$ the isomorphism type of the fundamental group.
(2) If $\pi_1(X) = 0$, then we say that X is simply connected.[96]

Example. On page 161 we saw that star-shaped subsets of $\mathbb{R}^n$ are simply connected.

[95] Why is $\Phi \circ \Psi$ the identity on $\pi_1(X, x_0)$ and why is $\Psi \circ \Phi$ the identity on $\pi_1(X, x_1)$?

[96] It is straightforward to see that for subsets of $\mathbb{C}$ we obtain the same notion of "simply connectedness" as on page 153.

Lemma 11.8. (Paths-in-Simply Connected Space Lemma) Let X be a topological space and let P, Q be two points in X. If X is simply connected, then any two paths $\alpha, \beta\colon [0,1] \to X$ from P to Q are path-homotopic.



Proof. We leave the proof as an exercise to the reader. ■

In the following two sections we will first calculate the isomorphism type of the fundamental groups of the circle S^1 and then we will calculate the isomorphism type of the fundamental groups of the spheres S^n with $n \geq 2$. Both calculations require some effort. But the effort will pay off in multiple ways:

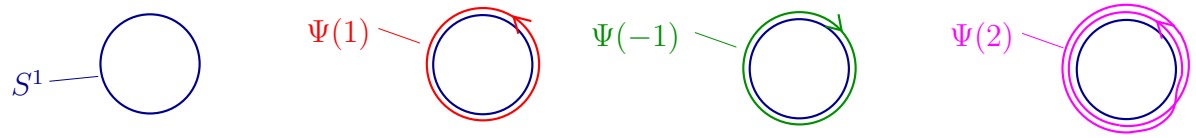
(1) Using these calculations we can already prove several interesting results, like the Fundamental Theorem of Algebra and (with a little bit of extra theory) the Borsuk-Ulam Theorem for S^2.
(2) The techniques we will employ in these special cases will give us the ideas for tackling much more general computational problems later on.

11.4. **The fundamental group of the circle.** The goal of this section is to prove the following important theorem, which gives us our first calculation of a non-trivial fundamental group.

Theorem 11.13. (Circle-π_1-Theorem) The map

$$\begin{array}{rcl} \Psi\colon \mathbb{Z} & \to & \pi_1(S^1, 1) \\ n & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & S^1 \\ t & \mapsto & \exp(2\pi \mathrm{i} \cdot n \cdot t) \end{array}\right] \end{array}$$

is an isomorphism of groups.



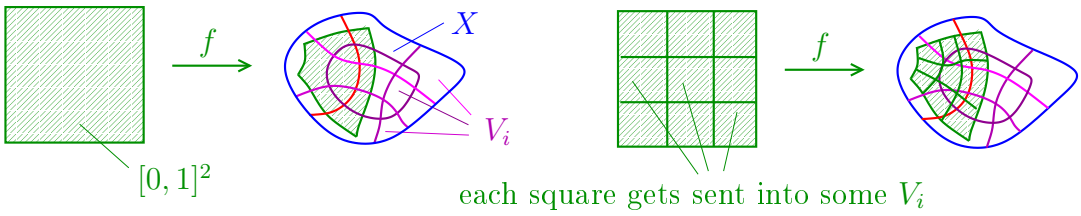
The proof of the Circle-π_1-Theorem 11.13 requires several preparations, which are actually interesting in their own right. In particular, as many other calculations of fundamental groups the proof of the Circle-π_1-Theorem 11.13 makes use of the Lebesgue Lemma:

Lemma 11.9. (Lebesgue Lemma)

(1) Let $K \subset \mathbb{R}^n$ be a compact subset and let $\{U_i\}_{i \in I}$ be an open cover of K. There exists a $\delta > 0$ such that for every subset A of K with diameter $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ with $A \subset U_i$.
(2) Let $f\colon [0,1]^m \to X$ be a continuous map from the cube $[0,1]^m$ to a topological space X and let $\{V_i\}_{i \in I}$ be an open cover of X. Then there exists an $N \in \mathbb{N}$ such that for any $a_1, \ldots, a_m \in \{0, \ldots, N-1\}$ there exists an $i \in I$ such that

$$f\left(\left[\tfrac{a_1}{N}, \tfrac{a_1+1}{N}\right] \times \cdots \times \left[\tfrac{a_m}{N}, \tfrac{a_m+1}{N}\right]\right) \subset V_i.$$

Pictorially the Lebesgue Lemma 11.9 (2) says that if $f\colon [0,1]^m \to X$ is a continuous map, and if we are given an open cover of X, then we can always find a small grid on the cube, such that each cube of the grid gets mapped into one of the open subsets V_i of the given open cover of X.



Proof. We proved Statement (1) in Analysis III. Statement (2) is a straightforward consequence of Statement (1). Full details can also be found in [Fri23]. ■

The next definition is the key tool to studying the fundamental group of S^1:

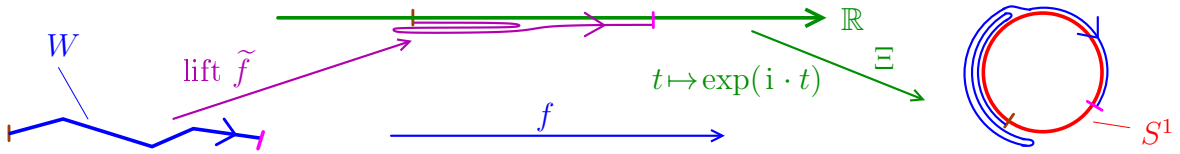
Definition.

(1) We work with the map

$$\begin{array}{rcl} \Xi\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(\mathrm{i}\cdot t). \end{array}$$

(2) Let W be a topological space and let $f\colon W \to S^1$ be a continuous map. A lift of f is a continuous map $\widetilde{f}\colon W \to \mathbb{R}$ such that $\Xi \circ \widetilde{f} = f$, in other words, such that for all $w \in W$ we have

$$f(w) \;=\; \exp\big(\mathrm{i}\cdot \widetilde{f}(w)\big).$$



Example. For $n \in \mathbb{Z}$ we consider the map $f\colon [0,1] \to S^1$ with $f(t) := \exp(2\pi\,\mathrm{i}\cdot n\cdot t)$. A lift of f is given by the map

$$\begin{array}{rcl} \widetilde{f}\colon [0,1] & \to & \mathbb{R} \\ t & \mapsto & 2\pi\cdot n\cdot t. \end{array}$$

This lift is not unique though since we can "shift" the lift by any $k\cdot 2\pi$ with $k \in \mathbb{Z}$.

Lemma 11.10. (Lift Uniqueness Lemma) Let W be a connected topological space, let $f\colon W \to S^1$ be a continuous map and let $\widetilde{\varphi}, \widetilde{\psi}\colon W \to \mathbb{R}$ be two lifts of f. If $\widetilde{\varphi}$ and $\widetilde{\psi}$ agree at some point $w_0 \in W$, then they agree everywhere.

Proof. We consider the map

$$\begin{array}{rcl} \theta\colon W & \to & \mathbb{R} \\ w & \mapsto & \theta(w) := \widetilde{\varphi}(w) - \widetilde{\psi}(w). \end{array}$$

By hypothesis we have $\theta(w_0) = 0$. Next note that for each $w \in W$ we have

$$\underbrace{\exp\big(\mathrm{i}\cdot(\widetilde{\varphi}(w) - \widetilde{\psi}(w))\big)}_{=\exp(\mathrm{i}\cdot\theta(w))=\Xi(\theta(w))} \;=\; \exp\big(\mathrm{i}\cdot\widetilde{\varphi}(w)\big)\cdot\exp\big(\mathrm{i}\cdot\widetilde{\psi}(w)\big)^{-1} \;\underset{\text{since } \widetilde{\varphi},\widetilde{\psi}\colon W\to\mathbb{R} \text{ are lifts of } f}{=}\; f(w)\cdot f(w)^{-1} \;=\; 1.$$

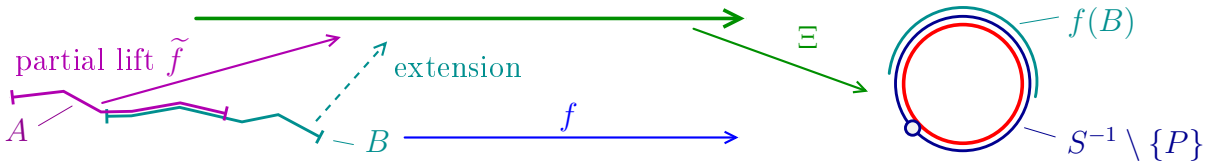
This shows that the continuous map $s \mapsto \theta(w) := \widetilde{\varphi}(w) - \widetilde{\psi}(w)$ takes values in the discrete subspace $2\pi \cdot \mathbb{Z} = \Xi^{-1}(1) \subset \mathbb{R}$. Since W is connected it follows from the Connected-to-Discrete Lemma 2.18 that the map θ is constant. Since the map θ vanishes at w_0 we see that it is zero everywhere. This implies that $\widetilde{\varphi} = \widetilde{\psi}$. ∎

Our next goal is to prove the existence of lifts of continuous maps $X \to S^1$ in suitable settings. The key tool for constructing lifts is the following lemma.

Lemma 11.11. (Lift-Extension Lemma) Let X be a topological space. We suppose that we have a decomposition $X = A \cup B$ such that A and B are closed subsets. Let $f\colon X \to S^1$ be a continuous map and let $\widetilde{f}\colon A \to \mathbb{R}$ be a lift of $f|_A$. We suppose that the following statements hold:

(1) There exists a $P \in S^1$ with $f(B) \subset S^1 \setminus \{P\}$.
(2) The intersection $A \cap B$ is connected and non-empty.

Then there exists a lift $\widetilde{f}\colon X \to \mathbb{R}$ of f that agrees with the given lift on A.

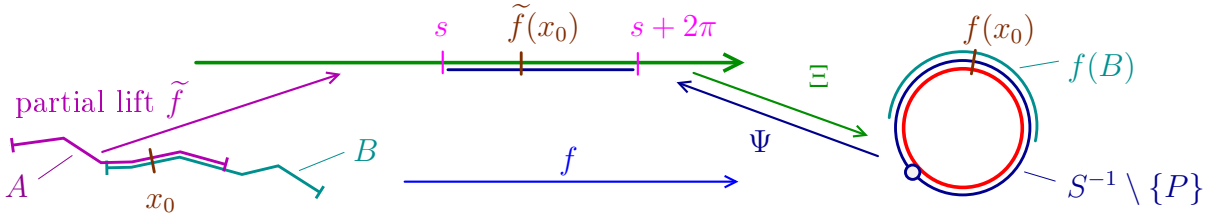


Proof. We start with a few preparations:

(1) We pick $x_0 \in A \cap B$ and we pick $s \in \mathbb{R}$ with $\Xi(s) = P$.
(2) Since $f(x_0) \in f(A \cap B) \subset f(B) \subset S^1 \setminus \{P\}$ and since $\widetilde{f}\colon A \to \mathbb{R}$ is a lift of $f\colon A \to S^1$ we see that $\widetilde{f}(x_0) \notin \Xi^{-1}(P)$.
(3) It follows from (2) that there exists an $m \in \mathbb{Z}$ such that $\widetilde{f}(x_0) \in (s+m\cdot 2\pi, s+m\cdot 2\pi+2\pi)$.
(4) One can easily verify that the restriction of Ξ to $(s+m\cdot 2\pi, s+m\cdot 2\pi+2\pi) \to S^1 \setminus \{P\}$ is a homeomorphism. We denote by Ψ the inverse.

We set

$$\begin{array}{rccl} \widetilde{f}\colon & X & \to & \mathbb{R} \\ & x & \mapsto & \begin{cases} \widetilde{f}(x), & \text{if } x \in A, \\ \Psi(\underbrace{f(x)}_{\in S^1\setminus\{P\}}), & \text{if } x \in B. \end{cases} \end{array}$$



Note that on $A \cap B$ we have a priori two different definitions of $\widetilde{f}$. But both definitions are lifts of $f|_{A\cap B}$ and by design they agree on x_0. By hypothesis the intersection $A \cap B$ is connected. Thus we obtain from the Lift Uniqueness Lemma 11.10 that the two maps actually agree on the overlap $A \cap B$. This shows that $\widetilde{f}\colon X \to \mathbb{R}$ is well-defined.

Next note that since A and B are closed it follows from the Pasting Proposition 2.4 that this map $\widetilde{f}\colon X \to \mathbb{R}$ is continuous. It is now clear that it is a lift of $f\colon X \to \mathbb{R}$. ∎

Lemma 11.12. (Lift-from-S^1-to-$\mathbb{R}$ Lemma) Given any continuous map $f\colon [0,1] \to S^1$ and given any $\widetilde{y}_0 \in \mathbb{R}$ with $f(0) = \exp(\mathrm{i}\cdot\widetilde{y}_0) =: \Xi(\widetilde{y}_0)$ there exists a *unique* lift $\widetilde{f}\colon [0,1] \to \mathbb{R}$ of f with $\widetilde{f}(0) = \widetilde{y}_0$.

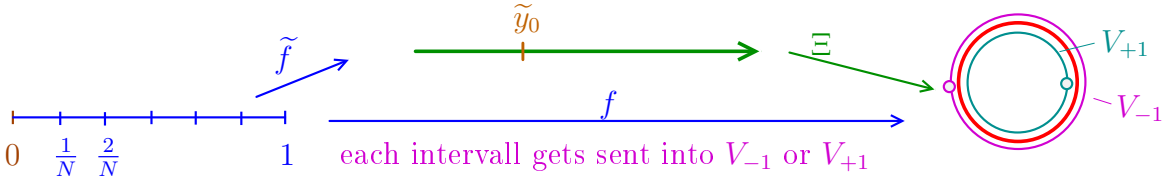
Proof of Lemma 11.12. Let $f\colon [0,1] \to S^1$ be a continuous map and let $\widetilde{y}_0 \in \mathbb{R}$ such that $f(0) = \exp(\mathrm{i}\cdot\widetilde{y}_0)$. We need to show that there exists a *unique* lift $\widetilde{f}\colon [0,1] \to \mathbb{R}$ of f with $\widetilde{f}(0) = \widetilde{y}_0$.

The uniqueness statement follows immediately from the Lift Uniqueness Lemma 11.10. Thus it remains to show existence.

The idea is quite simple: We want to apply the Lift-Extension Lemma 11.11. To do so we need to decompose the interval $[0,1]$ into smaller intervals, such that on each subinterval the image of f misses at least one point in S^1.

We set $V_{\pm 1} := S^1 \setminus \{\pm 1\}$. By the Lebesgue Lemma 11.9 there exists an $N \in \mathbb{N}$ such that for every $i \in \{0,\dots,N-1\}$ the image $f([\frac{i}{N}, \frac{i+1}{N}])$ is contained in V_{-1} or V_{+1}. It remains to prove the following claim.

Claim. For each $i \in \{0,\dots,N\}$ there exists a lift $\widetilde{f}\colon [0, \frac{i}{N}] \to \mathbb{R}$ of f with $\widetilde{f}(0) = \widetilde{y}_0$.



Proof. For $i = 0$ we just set $\widetilde{f}(0) := \widetilde{y}_0$. Now suppose that we have already defined $\widetilde{f}$ on some interval $[0, \frac{i}{N}]$ with $i < N$. Note that by construction there exists an $\epsilon \in \{\pm 1\}$ with $f([\frac{i}{N}, \frac{i+1}{N}]) \in V_\epsilon = S^1 \setminus \{\epsilon\}$. Thus it follows from the Lift-Extension Lemma 11.11[97] that we can extend the lift $\widetilde{f}$ to the interval $[0, \frac{i+1}{N}]$. ∎

We can now formulate and prove the full version of the Circle-π_1-Theorem 11.13.

Theorem 11.13. (Circle-π_1-Theorem) Let $z_0 = \exp(2\pi\,\mathrm{i}\cdot t_0) \in S^1$. Given any continuous map $\gamma\colon [0,1] \to S^1$ with $\gamma(0) = z_0$ we denote by $\widetilde{\gamma}\colon [0,1] \to \mathbb{R}$ the unique lift[98] with $\widetilde{\gamma}(0) = t_0 \in \mathbb{R}$. The maps

$$\begin{array}{rcl} \Psi\colon \mathbb{Z} & \to & \pi_1(S^1, z_0) \\ n & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & S^1 \\ t & \mapsto & z_0\cdot\exp(2\pi\,\mathrm{i}\cdot n\cdot t) \end{array}\right] \end{array} \quad \text{and} \quad \begin{array}{rcl} \Theta\colon \pi_1(S^1, z_0) & \to & \mathbb{Z} \\ [\gamma\colon [0,1] \to S^1] & \mapsto & \frac{1}{2\pi}\cdot(\widetilde{\gamma}(1) - \widetilde{\gamma}(0)) \end{array}$$

are well-defined, they are isomorphisms and they are inverses of one another.

Proof. To simplify the notation we only consider the case that $z_0 = 1$ and $t_0 = 0$. The following claim shows that $\Theta\colon \pi_1(S^1, 1) \to \mathbb{Z}$ is well-defined.

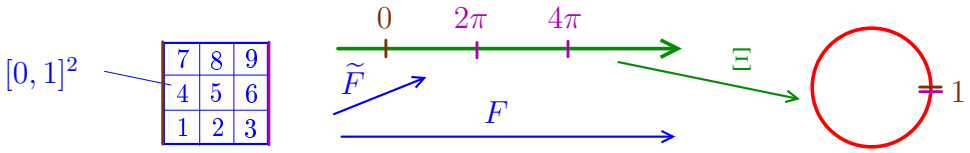
Claim 1. Let $\alpha, \beta\colon [0,1] \to S^1$ be two loops in $(S^1, 1)$. If α and β are path-homotopic, then $\widetilde{\alpha}(1) = \widetilde{\beta}(1) \in \mathbb{R}$.

[97] Here we use that the intersection of two adjacent intervals is connected and non-empty.

Proof. Let $F\colon [0,1]\times[0,1]\to S^1$ be a path-homotopy between α and β. First we want to show that F lifts to a map $[0,1]\times[0,1]\to\mathbb{R}$. As in the proof of the Lift-from-S^1-to-$\mathbb{R}$ Lemma 11.12 we set $V_{\pm1} := S^1\setminus\{\pm1\}$. By the Lebesgue Lemma 11.9 there exists an $N\in\mathbb{N}$ such that for any $i,j\in\{0,\dots,N-1\}$ there exists an $\epsilon\in\{\pm1\}$ with

$$f\big(\big[\tfrac{i}{N},\tfrac{i+1}{N}\big]\times\big[\tfrac{j}{N},\tfrac{j+1}{N}\big]\big)\ \subset\ V_\epsilon.$$

The same way as we lifted the map $[0,1]\to S^1$ to $\mathbb{R}$ by lifting the map "intervall-by-intervall" starting from the left, we now lift the map $F\colon[0,1]^2\to S^1$ to $\mathbb{R}$ by lifting the map "square-by-square" starting from the "bottom left".[99] We leave it to the reader to fill in the details.



Finally note that the fact that F is a path-homotopy means that $F(1,t)=1$ for all $t\in[0,1]$. This in turn implies that $\widetilde{F}(1,t)\in 2\pi\cdot\mathbb{Z}$ for all $t\in[0,1]$. It follows from the Connected-to-Discrete Lemma 2.18 the map $t\mapsto\widetilde{F}(1,t)$ is constant. Therefore we have shown that $\widetilde{\alpha}(1)=\widetilde{F}(1,0)=\widetilde{F}(1,1)=\widetilde{\beta}(1)$. ⊞

Next note that it follows almost immediately from the definitions that $\Psi\colon\mathbb{Z}\to\pi_1(S^1,1)$ is a homomorphism. Furthermore note that it follows easily from the definitions[100] that $\Theta\circ\Psi\colon\mathbb{Z}\to\mathbb{Z}$ is the identity. Thus it remains to prove the following claim.

Claim 2. The map $\Theta\colon\pi_1(S^1,1)\to\mathbb{Z}$ is a bijection.

Proof. Since $\Theta\circ\Psi=\mathrm{id}_{\mathbb{Z}}$ we see that Θ is surjective. It remains to show that Θ is injective. So suppose we are given $[\alpha],[\beta]\in\pi_1(S^1,1)$ with $\Theta([\alpha])=\Theta([\beta])$, i.e. with $\widetilde{\alpha}(1)=\widetilde{\beta}(1)$. Note that the maps $\widetilde{\alpha},\widetilde{\beta}\colon[0,1]\to\mathbb{R}$ have the same starting point and the same endpoint in $\mathbb{R}$. *Since* $\mathbb{R}$ *is convex* we see, as in the Convex Homotopy Lemma 8.1, that

$$\begin{array}{rcl} F\colon[0,1]\times[0,1] & \to & \mathbb{R}\\ (t,s) & \mapsto & \widetilde{\alpha}(t)\cdot(1-s)+\widetilde{\beta}(t)\cdot s\end{array}$$

is a path-homotopy in $\mathbb{R}$ between $\widetilde{\alpha}$ and $\widetilde{\beta}$. But this implies that

$$\begin{array}{rcl} \Xi\circ F\colon[0,1]\times[0,1] & \to & S^1\\ (t,s) & \mapsto & \Xi(\widetilde{\alpha}(t)\cdot(1-s)+\widetilde{\beta}(t)\cdot s)\end{array}$$

is a path-homotopy in S^1 between $\alpha=\Xi\circ\widetilde{\alpha}$ and $\beta=\Xi\circ\widetilde{\beta}$. Thus $[\alpha]=[\beta]\in\pi_1(S^1,1)$. ■

11.5. **The fundamental group of spheres.** In the Circle-π_1-Theorem 11.13 we just calculated the fundamental group of S^1. In this section we will deal with the higher-dimensional spheres.

[99] The key observation is that if for any square that we add to the domain the intersection with the previous domain is always connected and non-empty.

[100] To be slightly more precise, this follows from the fact that the lift of the loop $t\mapsto\exp(2\pi\mathrm{i}\cdot n\cdot t)$ is given by $t\mapsto 2\pi\cdot n\cdot t$.

Proposition 11.14. (Sphere-π_1-Proposition) For any $n \geq 2$ the sphere S^n is simply connected, i.e. we have $\pi_1(S^n) = 0$.

In the proof of the Sphere-π_1-Proposition 11.14 we will need the following lemma.

Lemma 11.15. (Sphere-Minus-Point Lemma) For every $P \in S^n$ the topological space $S^n \setminus \{P\}$ is simply connected.

Proof of Lemma 11.15. Recall that on page 161 we saw that $\mathbb{R}^n$ is simply connected. Thus it suffices to show that $S^n \setminus \{P\}$ is homeomorphic to $\mathbb{R}^n$.

Recall that in the Stereographic Projection Lemma 2.11 we showed that there exists a homeomorphism $f\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ with $f^{-1}(\infty) = N := (0, \ldots, 0, 1)$. Thus f restricts to a homeomorphism $f\colon S^n \setminus \{N\} \to \mathbb{R}^n$. Now let P be any other point on S^n. There exists an orthogonal matrix A with $A \cdot P = N$. This shows that there exists a homeomorphism $g\colon S^n \to S^n$ with $g(P) = N$. Then $f \circ g\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ restricts to a homeomorphism $S^n \setminus \{P\} \to \mathbb{R}^n$. ■

Now we can turn to the proof of the Sphere-π_1-Proposition 11.14.

Here is a very short "proof": Let $x_0 \in S^n$ and let $\gamma\colon [0,1] \to S^n$ be a loop in x_0. We have to show that γ is null-homotopic. We pick a point P that does not lie in the image of γ. By the Sphere-Minus-Point Lemma 11.15 the topological space $S^n \setminus \{P\}$ is simply connected, hence γ is null-homotopic in $S^n \setminus \{P\}$, in particular it is null-homotopic in S^n.

This sounds convincing, except for one crucial gap: we did not justify why there exists a point P that does not lie in the image of γ. In fact such a point does not need to exist, since there are surjective continuous maps from an interval to any S^n.

Proof of the Sphere-π_1-Proposition 11.14. Let $n \geq 2$. We write

$$N := (0, \ldots, 0, 1) \quad \text{"North Pole"} \quad \text{and} \quad U := S^n \setminus \{N\},$$

and also

$$S := (0, \ldots, 0, -1) \quad \text{"South Pole"} \quad \text{and} \quad V := S^n \setminus \{S\}.$$

We fix a base point $x_0 \in U \cap V$. In the following we will only use the following three facts about U and V:

(I) U and V form an open cover of S^n.

(II) By the Sphere-Minus-Point Lemma 11.15 we know that U and V are simply connected.

(III) Our hypothesis $n \geq 2$ implies that the intersection $U \cap V = S^n \setminus \{N, S\}$ is path-connected.[101]

It follows from (II) together with the Path-Homotopy–Equivalence Relation Lemma 11.2 that it suffices to show that any loop in (S^n, x_0) is path-homotopic to a loop in U. Thus let $\gamma\colon [0,1] \to S^n$ be a loop in (S^n, x_0).

Claim 1. There exists a subdivision

$$0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = 1,$$

such that the following hold:

(1) for each $i \in \{0, \ldots, k-1\}$ we have $\gamma([t_i, t_{i+1}]) \subset U$ or $\gamma([t_i, t_{i+1}]) \subset V$, and

[101] This can be seen as follows: $S^n \setminus \{N, S\}$ is homeomorphic to $S^{n-1} \times (-1, 1)$, which in turn is clearly path-connected.

(2) for each $i \in \{1, \dots, k-1\}$ we have $\gamma(t_i) \in U \cap V$.

Proof. By (I) we can apply the Lebesgue Lemma 11.9 to $S^n = U \cup V$ and to $\gamma\colon [0,1] \to S^n$. Thus there exists a $k \in \mathbb{N}$ such that for any $i \in \{0, \dots, k-1\}$ the image $\gamma([\frac{i}{k}, \frac{i+1}{k}])$ lies in U or in V. In particular the subdivision $t_i = \frac{i}{k}$, $i = 0, \dots, k$ satisfies condition (1).

Now suppose that there exists an $i \in \{1, \dots, k-1\}$ with $\gamma(t_i) \notin U$. Then neither $\gamma([t_{i-1}, t_i])$ nor $\gamma([t_i, t_{i+1}])$ are contained in U. So by (1) both have to be contained in V, in particular we have $\gamma([t_{i-1}, t_{i+1}]) \subset V$. But then we can remove t_i from the subdivision. The same way we deal with the case that $\gamma(t_i) \notin V$. Iterating this process we can assume that for each every i we have $\gamma(t_i) \in U \cap V$. ⊞

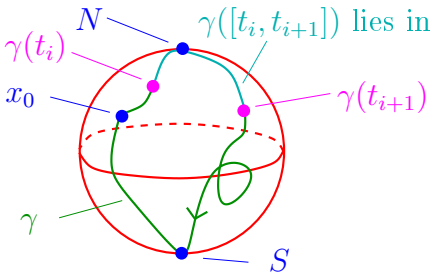


illustration for Claim 1

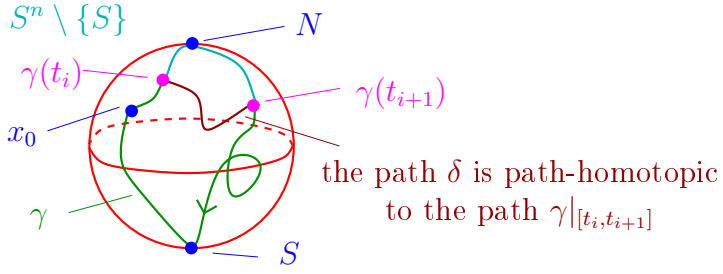


illustration for the proof of Claim 2

As mentioned before, it now, suffices to prove the following claim.

Claim 2. The loop $\gamma\colon [0,1] \to S^n$ in (S^n, x_0) is path-homotopic to a path in U.

Proof. We pick a subdivision of $[0,1]$ as in Claim 1. Suppose there exists an $i \in \{0, \dots, k-1\}$ such that $\gamma([t_i, t_{i+1}])$ is not contained in U. By the choice of our subdivision we know that this implies that $\gamma([t_i, t_{i+1}])$ lies entirely in V. Since $U \cap V$ is path-connected and since $\gamma(t_i)$ and $\gamma(t_{i+1})$ both lie in $U \cap V$ we can pick a path $\delta\colon [t_i, t_{i+1}] \to U \cap V$ that connects $\gamma(t_i)$ and $\gamma(t_{i+1})$. Since the paths $\gamma|_{[t_i,t_{i+1}]}$ and δ both lie in the simply connected set V they are path-homotopic by the Paths-in-Simply Connected Space Lemma 11.8. In the loop γ we now replace $\gamma|_{[t_i,t_{i+1}]}$ by δ. It is straightforward to see that this way we obtain a new loop that is path-homotopic to the original loop and that stays within $U \cap V \subset U$ on the interval $[t_i, t_{i+1}]$. Iterating this procedure gives us the desired path-homotopy to a loop that is contained in U. ■

11.6. **Alternative description of fundamental groups.** In this short section we will present a slightly different way to think about fundamental groups, namely we will see that we can think of fundamental groups also as defined via continuous maps $S^1 \to X$. We start out with the following identification.

Identification. Note that the map

$$\begin{array}{rcl} [0,1]/0 \sim 1 & \to & S^1 \\ [t] & \mapsto & \exp(2\pi \mathrm{i} \cdot t) \end{array}$$

is a homeomorphism. We use it to make the identification $S^1 =_i [0,1]/0 \sim 1$.

This identification allows us to introduce the following convention:

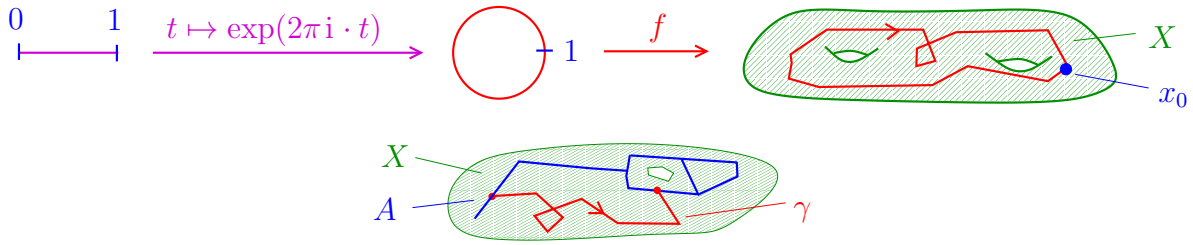
Convention. Let X be a topological space and let $x_0 \in X$.

(1) Using the above identification, using the Topological-Quotient Proposition 6.1 (1b) and using the Homotopy Quotient Theorem 8.5 we see that we can make the natural identification

$$\pi_1(X, x_0) \ =_i \ \{\text{all continuous maps } (S^1, 1) \to (X, x_0)\}/\sim$$

where $f \sim g$ if there exists a homotopy $F\colon S^1 \times [0,1] \to X$ such that for all $t \in [0,1]$ we have $F_t(1) = x_0$.

(2) We refer to continuous maps $(S^1, 1) \to (X, x_0)$ also as loops in (X, x_0).



12. Applications of fundamental groups

In this short chapter we present a few applications of the results on fundamental groups that we proved in the previous chapter:

(1) We will prove the Fundamental Theorem of Algebra.
(2) We prove the Borsuk-Ulam Theorem for S^2

We will not use any of the results of this chapter later on. The reader who is eager to move forward with the development of the theory can safely skip this chapter.

12.1. The Fundamental Theorem of Algebra.

As a first application of our methods we give a proof of the Fundamental Theorem of Algebra that does not rely on complex analysis:

Theorem 12.1. (Fundamental Theorem of Algebra) Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$.

Proof. We start out with some notation: Let $q(z)$ be a polynomial with complex coefficients. If $q(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| = r$, then we can consider the path

$$\begin{array}{rccl} \gamma_r^{q(z)}\colon [0,1] & \to & S^1 \\ s & \mapsto & \dfrac{q(r\cdot\exp(2\pi\,\mathrm{i}\cdot s))}{\left|q(r\cdot\exp(2\pi\,\mathrm{i}\cdot s))\right|}\cdot\dfrac{|q(r)|}{q(r)}. \end{array}$$

Note that $\gamma_r^{q(z)}(0) = \gamma_r^{q(z)}(1) = 1$, i.e. $\gamma_r^{q(z)}$ is a loop in $(S^1, 1)$.

Now we start out with the actual proof of the theorem. Thus let

$$p(z) \;=\; a_n\cdot z^n + a_{n-1}\cdot z^{n-1} + \cdots + a_1\cdot z + a_0$$

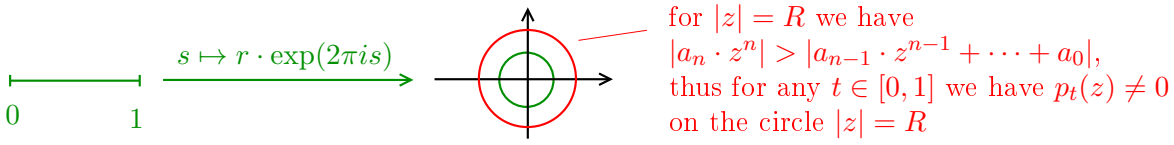
be a polynomial with $a_n \neq 0$. We assume that $p(z)$ has no zeros in $\mathbb{C}$. We need to show that $n = 0$. For $t \in [0,1]$ we also consider the polynomial

$$p_t(z) \;:=\; a_n\cdot z^n + t\cdot(a_{n-1}\cdot z^{n-1} + \cdots + a_1\cdot z + a_0).$$

Note that $p_1(z) = p(z)$ and $p_0(z) = a_n\cdot z^n$. Since $a_n \neq 0$ there exists an $R > 0$ such that[102]

$$|a_n\cdot z^n| \;>\; |\underbrace{a_{n-1}\cdot z^{n-1} + \cdots + a_1\cdot z + a_0}_{=0 \text{ if } n=0}|$$

for any $z \in \mathbb{C}$ with $|z| = R$. This inequality implies that for any $t \in [0,1]$ and any $z \in \mathbb{C}$ with $|z| = R$ we have $p_t(z) \neq 0$, i.e. $p_t(z)$ has no zeros on the circle $|z| = R$.



[102] In fact we can write it down explicitly: it follows easily from the triangle inequality that any $R \in \mathbb{R}$ with $R\cdot|a_n| > |a_{n-1}| + \cdots + |a_0|$ will work.

We obtain the following sequence of path-homotopic loops:

$$\text{constant loop at } 1 \overset{\text{follows from the definition}}{=} \gamma_0^{p(z)} \underset{\text{maps } \gamma_r^{p(z)} \text{ with } r\in[0,R] \text{ are defined, since the polynomials } p(z) \text{ has no zeros, they form a path-homotopy from } \gamma_0^{p(z)} \text{ to } \gamma_R^{p(z)}}{\simeq_{\mathrm{p}}} \gamma_R^{p(z)} = \gamma_R^{p_1(z)} \underset{\text{maps } \gamma_R^{p_t(z)} \text{ with } t\in[0,1] \text{ are defined, since polynomials } p_t(z) \text{ have no zeros on } |z|=R, \text{ and they form a path-homotopy from } \gamma_R^{p_1(z)} \text{ to } \gamma_R^{p_0(z)}}{\simeq_{\mathrm{p}}} \gamma_R^{p_0(z)} \overset{\text{follows from } p_0(z)=a_nz^n}{=} \text{the loop } \begin{array}{l}[0,1]\to S^1\\ t\mapsto \exp(2\pi i n t).\end{array}$$

This shows that the constant loop is path-homotopic to the loop on the right hand side. By the Circle-π_1-Theorem 11.13 this is only possible if $n=0$. ■

12.2. **The 2-dimensional Borsuk-Ulam Theorem.** The following theorem goes back to the Polish mathematicians Karol Borsuk (1905-1982) and Stanislaw Ulam (1909-1984).[103] It was first proved in 1932.

Theorem 12.2. (Borsuk-Ulam Theorem) For every continuous map $f\colon S^2\to\mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x)=f(-x)$.

Example.

(1) It follows from the Borsuk-Ulam Theorem 12.2 that there exists a pair of antipodal points on earth with the same temperature and the same barometric pressure.
(2) The Borsuk-Ulam Theorem 12.2 implies in particular that there is no injective continuous map from $S^2\to\mathbb{R}^2$.

Proof. We prove the theorem by contradiction. So suppose there exists a continuous map $f\colon S^2\to\mathbb{R}^2$ such that $f(x)\neq f(-x)$ for all $x\in S^2$. This allows us to define the map

$$\begin{array}{rcl} g\colon S^2 & \to & S^1\\ x & \mapsto & \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}.\end{array}$$

Note that $g(-x)=-g(x)$ for all $x\in S^2$. Now let η be the loop that is defined by

$$\begin{array}{rcl} \eta\colon [0,1] & \to & S^2\\ s & \mapsto & \eta(s)=(\cos(2\pi s),\sin(2\pi s),0)\end{array}$$

and let $h=g\circ\eta\colon[0,1]\to S^1$ be the composition of η with g. The argument now breaks up into two parts:

(1) By the Sphere-π_1-Proposition 11.14 the loop $\eta\colon[0,1]\to S^2$ is null-homotopic in S^2.[104] Note that this implies that $h=g\circ\eta$ is null-homotopic in S^1.
(2) As on page 164 we consider the map $\Xi\colon\mathbb{R}\to S^1$ that is given by $t\mapsto\exp(it)$. We note that for any $z,w\in\mathbb{R}$ we have

$$(*)\qquad \Xi(z)=-\Xi(w) \iff z=w+(2q+1)\cdot\pi \quad \text{for some } q\in\mathbb{Z}.$$

We pick a $c\in\mathbb{R}$ such that $h(0)=\Xi(c)$. By the Lift-from-S^1-to-$\mathbb{R}$ Lemma 11.12 we can lift the path $h\colon[0,1]\to S^1$ to a path $\widetilde{h}\colon[0,1]\to\mathbb{R}$ with $\widetilde{h}(0)=c$. We make the following observations:

(a) It follows immediately from the definition of g that $g(-x)=-g(x)$ for all $x\in S^2$.

[103] Ulam became much more famous through his work on the hydrogen bomb.
[104] In fact since η is given explicitly one can also write down an explicit path-homotopy from η to a constant loop.

(b) It follows from (a) that $g(\eta(s+\frac{1}{2})) = -g(\eta(s))$ for all $s \in [0, \frac{1}{2}]$.

(c) It follows from (b) and the fact that $\Xi \circ \widetilde{h} = h = g \circ \eta$ that we have the equality $\Xi(\widetilde{h}(s+\frac{1}{2})) = -\Xi(\widetilde{h}(s))$ for all $s \in [0, \frac{1}{2}]$.

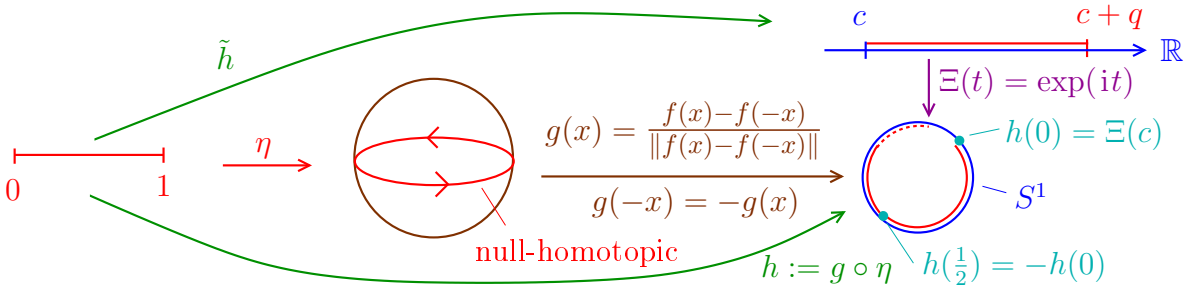
(d) It follows from (c) and $(*)$ that for each $s \in [0, \frac{1}{2}]$ there exists a $q(s) \in \mathbb{Z}$ with $\widetilde{h}(s+\frac{1}{2}) = \widetilde{h}(s) + (2q(s)+1) \cdot \pi$.

(e) The map $q\colon [0, \frac{1}{2}] \to \mathbb{Z}$ is the difference of two continuous maps, hence continuous. It follows from the Connected-to-Discrete Lemma 2.18 that it is constant. We denote this constant value by q.

It now follows that

$$\widetilde{h}(1) \underset{\text{by (e)}}{=} \widetilde{h}\big(\tfrac{1}{2}\big) + (2q+1)\cdot\pi \underset{\text{by (e)}}{=} \widetilde{h}(0) + (2q+1)\cdot\pi + (2q+1)\cdot\pi = \widetilde{h}(0) + (4q+2)\cdot\pi \underset{\text{since } q\in\mathbb{Z} \text{ we see that } 4q+2\neq 0}{\neq} \widetilde{h}(0).$$

Since $\widetilde{h}(1) \neq \widetilde{h}(0)$ we obtain the loop $h = g \circ \eta\colon [0,1] \to S^1$ is not null-homotopic. But this contradicts (1). ■



13. Functoriality of fundamental groups

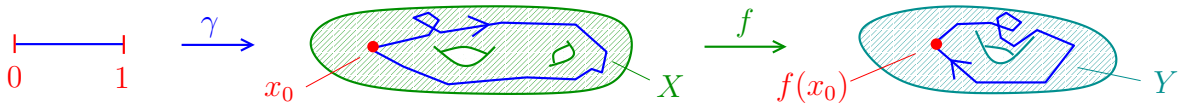
To unlock the full power of fundamental groups we need to view the fundamental group as a covariant functor.

13.1. The fundamental group as functor.

Definition. Let $f\colon X \to Y$ be a continuous map between topological spaces and let $x_0 \in X$ be a base point. If $\gamma\colon [0,1] \to X$ is a loop in the point x_0, then $f \circ \gamma\colon [0,1] \to Y$ is a loop in the point $f(x_0)$. It follows from the Homotopy Composition Lemma 8.4 that

$$\begin{array}{rcl} f_*\colon \pi_1(X,x_0) & \to & \pi_1(Y,f(x_0)) \\ [\gamma] & \mapsto & [f\circ\gamma] \end{array}$$

is a well-defined map, i.e. $f_*([\gamma])$ is independent of the choice of the representative of the path-homotopy class. We call f_* the induced map.



Example.

(1) If $f\colon X \to Y$ is a constant map, then the induced map $f_*\colon \pi_1(X,x_0) \to \pi_1(Y,f(x_0))$ is the trivial group homomorphisms.

(2) Let $n \in \mathbb{Z}$. We consider the map $f\colon S^1 \to S^1$ that is given by $z \mapsto z^n$. We want to understand the map $f_*\colon \pi_1(S^1,1) \to \pi_1(S^1,1)$. Here is one possible answer: It follows immediately from the definitions that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(S^1,1) & \xrightarrow{f_*} & \pi_1(S^1,1) \\ {\scriptstyle k\mapsto[t\mapsto\exp(2\pi\mathrm{i}kt)]}\uparrow\cong & & \cong\uparrow{\scriptstyle k\mapsto[t\mapsto\exp(2\pi\mathrm{i}kt)]} \\ \mathbb{Z} & \xrightarrow{g\mapsto n\cdot g} & \mathbb{Z}. \end{array}$$

(Here the vertical isomorphisms are given by the Circle-π_1-Theorem 11.13.)

The induced maps look they should define a covariant functor. We just need to figure out what the categories are supposed to be. This leads us to the following definition.

Definition. A pointed topological space is a pair (X,x_0), where X is a topological space and x_0 is a point in X. We refer to the category $\mathcal{PTop}$ with

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PTop}) & := & \text{all pointed topological spaces,} \\ \mathrm{Mor}((X,x_0),(Y,y_0)) & := & \text{all continuous maps } f \text{ from } X \text{ to } Y \text{ with } f(x_0)=y_0 \end{array}$$

with the usual composition of maps as the category of pointed topological spaces.

Proposition 13.1. (π_1-Functor Proposition)

(1) Let $\mathcal{PTop}$ be the category of pointed topological spaces and let $\mathcal{Group}$ be the category of groups. The map

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PTop}) & \to & \mathrm{Ob}(\mathcal{Group}) \\ (X,x_0) & \mapsto & \pi_1(X,x_0) \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}((X,x_0),(Y,y_0)) & \to & \mathrm{Mor}(\pi_1(X,x_0),\pi_1(Y,y_0)) \\ (f\colon (X,x_0)\to(Y,y_0)) & \mapsto & \left(\begin{array}{rcl} f_*\colon \pi_1(X,x_0) & \to & \pi_1(Y,y_0) \\ [\gamma] & \mapsto & [f\circ\gamma] \end{array}\right) \end{array}$$

form a covariant functor from the category *PTop* of pointed topological spaces to the category *Group* of groups.

(2) If $f\colon X\to Y$ is a homeomorphism between two topological spaces, then for any $x_0\in X$ the induced map $f_*\colon \pi_1(X,x_0)\to\pi_1(Y,f(x_0))$ is a group isomorphism.

Example. On page 53 we saw that the upper and lower hemispheres $S^n_{\geq 0}$ and $S^n_{\leq 0}$ are homeomorphic to the closed ball $\overline{B}^n$. It follows from the π_1-Functor Proposition 13.1 and the discussion on page 161 that the hemispheres are simply connected.

Proof. The second statement follows immediately from the first statement. Thus it remains to prove the first statement.

Let $f\colon (X,x_0)\to(Y,y_0)$ be a continuous map between pointed topological spaces. It follows easily from the definitions that for any two loops γ and δ in (X,x_0) we have $(f\circ\gamma)*(f\circ\delta)=f\circ(\gamma*\delta)$. But this means that $f_*\colon\pi_1(X,x_0)\to\pi_1(Y,y_0)$ is a group homomorphism. Furthermore one can easily verify that

$$\begin{array}{rcll} (\mathrm{id}_X)_* & = & \mathrm{id}_{\pi_1(X,x_0)}, & \text{for all pointed pairs } (X,x_0), \\ (g\circ f)_* & = & g_*\circ f_*, & \text{for all continuous maps } f\colon(X,x_0)\to(Y,y_0), g\colon(Y,y_0)\to(Z,z_0). \end{array}$$

But this means precisely that the given maps define a covariant functor from the category *PTop* of pointed topological spaces to the category *Group* of groups. ∎

In many applications we will want to know whether a given map between topological spaces induces an isomorphism of fundamental groups. The following proposition, which can be viewed as an addendum to the Change-of-Base Point Proposition 11.6, says that in most settings the property of being an isomorphism is independent of the choice of base point.

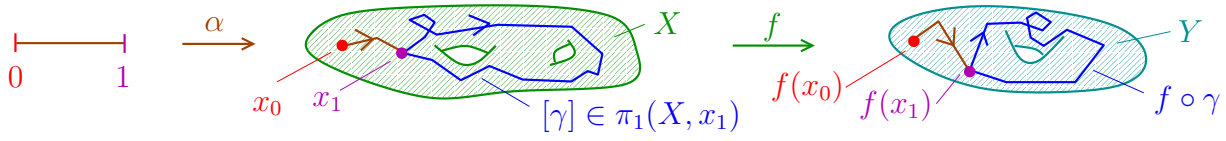
Proposition 13.2. (Change-of-Base Point Proposition) Let $f\colon X\to Y$ be a continuous map between topological spaces and let $x_0\in X$.

(1) If $\alpha\colon[0,1]\to X$ is a path from x_0 to some point x_1, then the following diagram commutes

$$\begin{array}{ccc} \pi_1(X,x_1) & \xrightarrow{f_*} & \pi_1(Y,f(x_1)) \\ \cong\downarrow{\scriptstyle \alpha_*} & & \cong\downarrow{\scriptstyle (f\circ\alpha)_*} \\ \pi_1(X,x_0) & \xrightarrow{f_*} & \pi_1(Y,f(x_0)). \end{array}$$

(The vertical maps are the isomorphisms that we introduced in the Change-of-Base Point Proposition 11.6 (1).)

(2) If $f_*\colon\pi_1(X,x_0)\to\pi_1(Y,f(x_0))$ is an isomorphism (epimorphism, monomorphism) for some base point $x_0\in X$, then for any other base point x_1 in the same path component of X the map $f_*\colon\pi_1(X,x_1)\to\pi_1(Y,f(x_1))$ is also an isomorphism (epimorphism, monomorphism).



Proof.

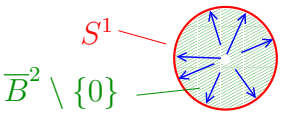
(1) This statement follows immediately from the definitions.
(2) This statement follows from (1) and the fact that by the Change-of-Base Point Proposition 11.6 (1) the vertical maps are isomorphisms. ■

13.2. **Retractions and fundamental groups.** In this section we will see that the functoriality of the fundamental groups can be helpful for showing the non-existence of retractions. Let us recall the following definition from page 142.

Definition. Let X be a topological space and let $A \subset X$ be a subspace. We say A is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a continuous map with $r(a) = a$ for all $a \in A$.

Examples.

(1) The circle $S^1 \times \{1\}$ is a retract of the torus $S^1 \times S^1$, in fact a retraction is given by $r(z, w) = (z, 1)$.
(2) On page 142 we saw that S^{n-1} is a retract of $\overline{B}^n \setminus \{0\}$.



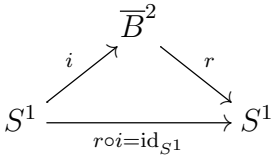
$r(x) := \frac{x}{\|x\|}$ defines a retraction from $\overline{B}^2 \setminus \{0\}$ to S^1

(3) The set $S^0 = \{-1, 1\}$ is not a retract of $\overline{B}^1 = [-1, 1]$. Indeed, since $[-1, 1]$ is connected and since S^0 is discrete it follows from the Connected-to-Discrete Lemma 2.18 that any continuous map from $[-1, 1]$ to S^0 has to be constant. Thus there does not exist a continuous map $f\colon [-1, 1] \to S^0$ with $f(-1) = -1$ and $f(1) = 1$.

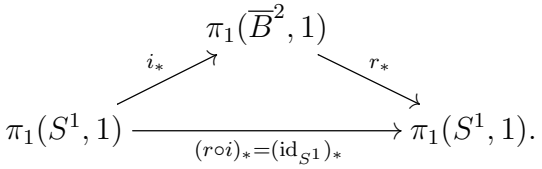
Now we can push last example up by one dimension.

Proposition 13.3. (Circle–Non Retract Proposition) The circle S^1 is not a retract of $\overline{B}^2$.

Proof. Let $i\colon S^1 \to \overline{B}^2$ be the inclusion map. We now suppose there exists a retraction $r\colon \overline{B}^2 \to S^1$. By definition of a retraction this means that $r \circ i = \mathrm{id}_{S^1}$. We obtain the following commutative diagram of continuous maps between topological spaces



which induces



It follows from the functoriality of fundamental groups that $r_* \circ i_* = (r \circ i)_*$, i.e. the diagram on the right-hand side commutes. By the functoriality we also know that $(\mathrm{id}_{S^1})_*$ is the identity of $\pi_1(S^1, 1)$.

We saw on page 161 that $\pi_1(\overline{B}^2, 1)$ is the trivial group whereas we saw in the Circle-π_1-Theorem 11.13 that $\pi_1(S^1, 1)$ is not the trivial group. The lower horizontal map in the second diagram is thus an isomorphism of a non-trivial group, but the upper map factors through the trivial group, i.e. the composition of i_* and r_* cannot be an isomorphism of a non-trivial group. Thus we have obtained a contradiction. ■

13.3. **The fundamental groups of homotopy equivalent topological spaces.** Now we will see that homotopy equivalent topological spaces have isomorphic fundamental groups. On numerous occasions this fact will greatly simplify the calculation of fundamental groups.

Proposition 13.4. (Homotopy-π_1-Proposition)

(1) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces and let $x_0 \in X$. If f and g are homotopic, then there exists a path $\alpha\colon [0,1] \to Y$ from $f(x_0)$ to $g(x_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} & \xrightarrow{g_*} & \pi_1(Y, g(x_0)) \\ \pi_1(X, x_0) & & \Big\downarrow{\scriptstyle \alpha_*}\ {\scriptstyle [\gamma]\mapsto[\alpha*\gamma*\overline{\alpha}]} \\ & \xrightarrow[f_*]{} & \pi_1(Y, f(x_0)). \end{array}$$

(Here the vertical map is the isomorphism given by the Change-of-Base Point Proposition 11.6 (1).) In particular

$$f_*\colon \pi_1(X,x_0) \to \pi_1(Y, f(x_0)) \text{ is an isomorphism} \iff g_*\colon \pi_1(X,x_0) \to \pi_1(Y, g(x_0)) \text{ is an isomorphism.}$$

The analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism".

(2) (a) Let $f\colon X \to Y$ be a continuous map between topological spaces and let $x_0 \in X$. If f is a homotopy equivalence, then the induced map

$$f_*\colon \pi_1(X, x_0) \ \to\ \pi_1(Y, f(x_0))$$

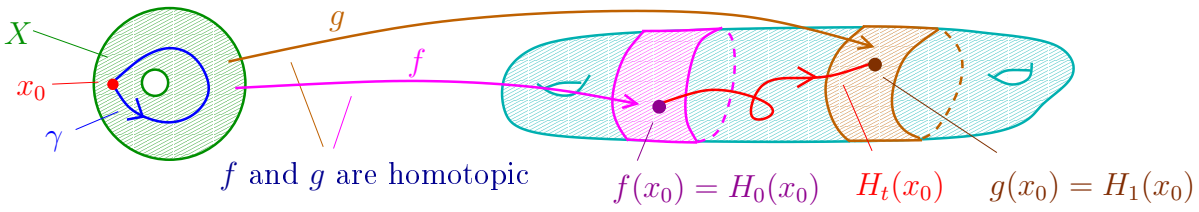
is an isomorphism.

(b) Let X be a topological space and let $A \subset X$ be a subspace. We denote by $i\colon A \to X$ the inclusion. If A is a deformation retract of X, then for any $a_0 \in A$ the induced map

$$i_*\colon \pi_1(A, a_0) \ \to\ \pi_1(X, a_0)$$

is an isomorphism.

(3) If X is a contractible topological space, then for any $x_0 \in X$ we have $\pi_1(X, x_0) = 0$.



Example. For any $n \geq 2$ and any $Q \in \mathbb{R}^n$ we have

$$\pi_1(\mathbb{R}^n \setminus \{Q\}) \ \underset{\text{follows from the } \pi_1\text{-Functor Proposition 13.1 (2) since } \mathbb{R}^n \setminus \{Q\} \text{ is homeomorphic to } \mathbb{R}^n \setminus \{0\}}{\cong}\ \pi_1(\mathbb{R}^n \setminus \{0\}) \ \underset{\text{from the Homotopy-}\pi_1\text{-Proposition 13.4 (2b)}}{\cong}\ \pi_1(S^{n-1}) \ \underset{\text{by the Circle-}\pi_1\text{-Theorem 11.13 and the Sphere-}\pi_1\text{-Proposition 11.14}}{\cong}\ \begin{cases} \mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{if } n \geq 3. \end{cases}$$

on page 142 we saw that S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$, thus we get an isomorphism

Proof.

(1) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces, let $x_0 \in X$ and let $H\colon X \times [0,1] \to Y$ be a homotopy between f and g. Let

$$\begin{array}{rcl} \alpha\colon [0,1] & \to & Y \\ t & \mapsto & H(x_0,t) \end{array}$$

be the path with which the homotopy H connects the two points $H(x_0,0) = f(x_0)$ and $H(x_0,1) = g(x_0)$.

Claim. The maps

$$\begin{array}{rcl} f_*\colon \pi_1(X,x_0) & \to & \pi_1(Y,f(x_0)) \\ [\gamma] & \mapsto & [f\circ\gamma] \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi_1(X,x_0) & \to & \pi_1(Y,f(x_0)) \\ [\gamma] & \mapsto & [\alpha * (g\circ\gamma) * \overline{\alpha}] \end{array}$$

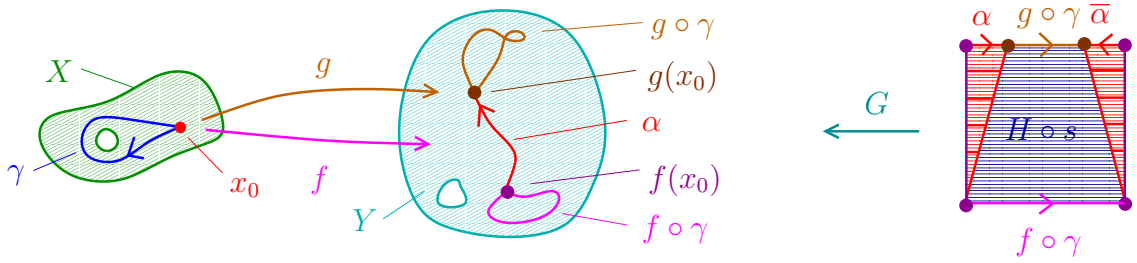
are identical.

Proof. Let $\gamma\colon [0,1] \to X$ be a loop in (X,x_0). We need to show that the loops $f\circ\gamma$ and $\alpha * (g\circ\gamma) * \overline{\alpha}$ are path-homotopic. We consider the map

$$\begin{array}{rcl} G\colon [0,1]\times[0,1] & \to & Y \\ (r,t) & \mapsto & \begin{cases} \alpha(4r), & \text{if } r \in \left[0, \frac{1}{4}t\right], \\ H\left(\gamma\left(\frac{r-\frac{1}{4}t}{1-\frac{1}{2}t}\right), t\right), & \text{if } r \in [\frac{1}{4}t, 1-\frac{1}{4}t], \\ \overline{\alpha}(4r-3), & \text{if } r \in \left[1-\frac{1}{4}t, 1\right]. \end{cases} \end{array}$$

Using the Pasting Proposition 2.4 (2) it is straightforward to verify that G is indeed a continuous map. Furthermore it follows easily from the definition that the map G satisfies $G_0(t) = G_1(t) = f(x_0)$ for all $t \in [0,1]$ and that it satisfies

$$G(r,0) = (f\circ\gamma)(r) \quad \text{and} \quad G(r,1) = (\alpha * (g\circ\gamma) * \overline{\alpha})(r) \quad \text{for all } r \in [0,1].$$



This observation shows that G is in fact a path-homotopy between the loops $f\circ\gamma$ and $\alpha * (g\circ\gamma) * \overline{\alpha}$[105] in $(Y, f(x_0))$. ⊞

As in the statement of the proposition we consider the following diagram:

$$\begin{array}{ccc} & \overset{g_*}{\nearrow} & \pi_1(Y,g(x_0)) \\ \pi_1(X,x_0) & & \Big\downarrow{\scriptstyle \alpha_*\ [\gamma]\mapsto[\alpha*\gamma*\overline{\alpha}]} \\ & \underset{f_*}{\searrow} & \pi_1(Y,f(x_0)). \end{array}$$

[105] To be precise, G is a homotopy between $f\circ\gamma$ and a suitable parametrization of $\alpha * (g\circ s) * \overline{\alpha}$.

We make the following observations:
(i) It follows from the claim that the diagram commutes.
(ii) By the Change-of-Base Point Proposition 11.6 (1) we know that the vertical map is an isomorphism.

It follows from (i) and (ii) that f_* is an isomorphism if and only if g_* is an isomorphism. Evidently the analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism".

(2) (a) Let $f\colon X \to Y$ be a continuous map between topological spaces which is a homotopy equivalence. Let $x_0 \in X$ be a point. We pick a homotopy inverse $g\colon Y \to X$ of f. By definition of a homotopy inverse this means that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Next note that it follows from (1) and the functoriality of fundamental groups that the induced maps

$$\begin{aligned} \alpha := g_* \circ f_* = (g \circ f)_*\colon \pi_1(X, x_0) &\to \pi_1(X, g(f(x_0)) \\ \beta := f_* \circ g_* = (f \circ g)_*\colon \pi_1(Y, f(x_0)) &\to \pi_1(Y, f(g(f(x_0)))) \end{aligned}$$

are isomorphisms. This shows that f_* has the left inverse $\alpha^{-1} \circ g_*$ and it has the right inverse $g_* \circ \beta^{-1}$. It follows from the Inverse Uniqueness Lemma 3.2 (1) that f_* is an isomorphism. Similarly we see that g_* is an isomorphism.

(b) This statement follows immediately from (a) and the Deformation Retract–Homotopy Equivalence Lemma 9.3.

(3) This statement is just a frequently used special case of the second statement. ∎

In an exercise sheet we already saw that the topological spaces $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. Now we can prove a similar statement in one dimension higher.

Lemma 13.5. ($\mathbb{R}^2$-$\mathbb{R}^3$-Non Homeomorphism–Lemma) The topological spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic.

Proof. Let us suppose that there exists a homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^3$. Let $P \in \mathbb{R}^2$ be a point. We put $Q := f(P)$. Note that f restricts to a homeomorphism from $\mathbb{R}^2 \setminus \{P\}$ to $\mathbb{R}^3 \setminus \{Q\}$. But then we obtain that

$$\mathbb{Z} \underset{\text{page 177}}{\cong} \pi_1(\mathbb{R}^2 \setminus \{P\}) \underset{\text{since } f \text{ is a homeomorphism}}{\cong} \pi_1(\mathbb{R}^3 \setminus \{Q\}) \underset{\text{page 177}}{=} 0.$$

Thus we obtained a contradiction. ∎

We conclude this section with one more calculation of a fundamental group.

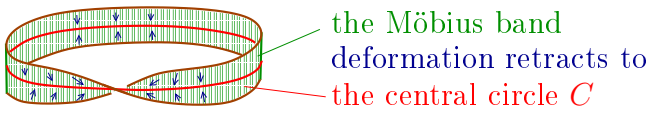
Corollary 13.6. (Möbius Band-π_1-Corollary) We have

$$\pi_1(\text{Möbius band }) \cong \mathbb{Z}.$$

Proof. On page 145 we saw that the Möbius band admits a deformation retraction onto the "central circle" C. Thus we see that

$$\pi_1(\text{Möbius band }) \underset{\text{Homotopy-}\pi_1\text{-Proposition 13.4}}{\cong} \pi_1(C) \cong \pi_1(S^1) \underset{\text{Circle-}\pi_1\text{-Theorem 11.13}}{\cong} \mathbb{Z}.$$

∎



13.4. **The fundamental group of the product of two topological spaces.** In Chapters 5 and 6 we got to know several ways to construct new (pointed) topological spaces out of given (pointed) topological spaces. It is natural to ask how these constructions affect the fundamental groups. In this section we consider the product of two pointed topological spaces.

Before we state the relevant proposition it is worth recalling the definition of the direct product of two groups.

Definition. Let G and H two groups. We can form the cartesian product

$$G \times H \;:=\; \{(g,h) \,|\, g \in G \text{ and } h \in H\}.$$

This is again a group with the group structure given by

$$(g_1,h_1)\cdot(g_2,h_2) \;:=\; (g_1 \cdot g_2, h_1 \cdot h_2)$$

for $g_1, g_2 \in G$ and $h_1, h_2 \in H$. We call the group $G \times H$ the direct product of the groups G and H.

Now we can formulate the promised proposition.

Proposition 13.7. (Product-π_1-Proposition) Let $n \in \mathbb{N}$.

(1) Let $(X_i, x_i)_{i\in I}$ be a family of pointed topological spaces. For each $j \in I$ we denote by $p_j \colon \prod_{i\in I} X_i \to X_j$ the natural projection. The induced natural homomorphism[106]

$$\begin{array}{rcl} \Phi \colon \pi_1\Big(\prod_{i\in I} X_i, (x_i)_{i\in I}\Big) & \to & \prod_{i\in I} \pi_1(X_i,x_i) \\ [f] & \mapsto & (p_{i*}([f]))_{i\in I} \end{array}$$

is an isomorphism.

(2) If we are given a finite family $(X_1,x_1),\ldots,(X_k,x_k)$ of pointed topological spaces, then the inverse to the map from (1) is given by the natural homomorphism

$$\begin{array}{rcl} \Psi \colon \pi_1(X_1,x_1)\times\cdots\times\pi_1(X_k,x_k) & \to & \pi_1(X_1\times\cdots\times X_k,(x_1,\ldots,x_k)) \\ (g_1,\ldots,g_k) & \mapsto & \iota_{1*}(g_1)\cdot\ldots\cdot\iota_{k*}(g_k) \end{array}$$

where $\iota_i \colon X_i \to X_1\times\cdots\times X_k$ is given by $y \mapsto (x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_k)$.

Example.

(1) For $n \in \mathbb{N}$ we obtain from the Product-π_1-Proposition 13.7 together with the Circle-π_1-Theorem 11.13 that $\pi_1((S^1)^n) \cong \mathbb{Z}^n$.

(2) From the Product-π_1-Proposition 13.7 together with the Circle-π_1-Theorem 11.13 we also obtain that $\pi_1((S^1)^{\mathbb{N}}) \cong \mathbb{Z}^{\mathbb{N}}$. This result is interesting, since by Tychonoff's Theorem 5.7 we know that $(S^1)^{\mathbb{N}}$ is compact. Since $\mathbb{Z}^{\mathbb{N}}$ is uncountable we have shown that there exists a compact topological space with uncountable fundamental group.

[106] The projections p_i induce homomorphisms on fundamental groups which in turn, by the Product-of-Sets-and-Groups Lemma 4.2, induce the given homomorphism.

Remark.

(1) The Product-π_1-Proposition 13.7 can be summarized as saying that the π_1-functor turns a product in the category of pointed topological spaces into a product in the category of groups. Put more succinctly: the fundamental group preserves products.

(2) In most cases we apply the Product-π_1-Proposition 13.7 to the case of two pointed topological spaces (A, a_0) and (B, b_0). Let us spell out this special case. We consider the projections

$$\begin{array}{rcl} p\colon A\times B & \to & A \\ (a,b) & \mapsto & a \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon A\times B & \to & B \\ (a,b) & \mapsto & b \end{array}$$

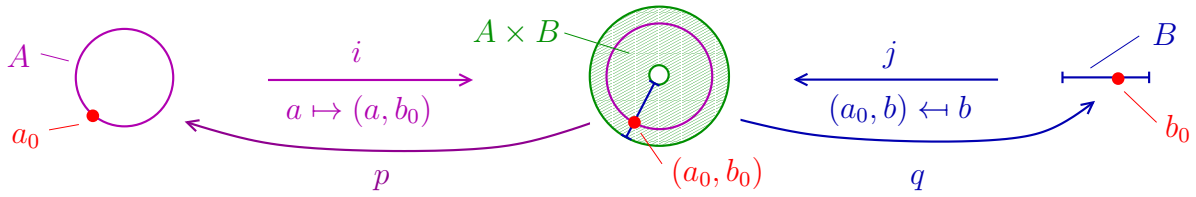
and the inclusions

$$\begin{array}{rcl} i\colon A & \to & A\times B \\ a & \mapsto & (a,b_0) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon B & \to & A\times B \\ b & \mapsto & (a_0,b). \end{array}$$

The maps

$$\begin{array}{rcl} \Phi\colon \pi_1(A\times B,(a_0,b_0)) & \to & \pi_1(A,a_0)\times\pi_1(B,b_0) \\ \gamma & \mapsto & (p_*(\gamma),q_*(\gamma)) \\ i_*(\alpha)\cdot j_*(\beta) & \leftarrow\!\shortmid & (\alpha,\beta) \end{array}$$

are both natural[107] isomorphisms.



Proof.

(1) First we introduce some general notation:

- Given two pairs of topological spaces (W, A) and (X, B) we denote by $C((W, A), (X, B))$ the set of all continuous maps $f\colon W \to X$ with $f(A) \subset B$.
- We say two continuous maps $f, g \in C((W, A), (X, B))$ are *homotopic* if there exists a homotopy $H\colon W \times [0, 1] \to X$ with $H_0 = f$ and $H_1 = g$ such that for all $t \in [0, 1]$ we have $H_t(A) \subset B$.
- We denote by $[(W, A), (X, B)]$ the set of all homotopy classes of continuous maps $(W, A) \to (X, B)$.
- By definition we have $\pi_1(Z, z_0) = [([0, 1], \{0, 1\}), (Z, \{z_0\})]$ for any pointed topological space (Z, z_0).

Let $(X_i, x_i)_{i\in I}$ be a family of pointed topological spaces. We set $x := (x_i)_{i\in I}$. It follows from the Topological-Product Proposition 5.1 (1) that the map

$$\begin{array}{rcl} C\Big(([0,1],\{0,1\}),\Big(\prod_{i\in I} X_i, x\Big)\Big) & \to & \prod_{i\in I} C(([0,1],\{0,1\}),(X_i,x_i)) \\ f & \mapsto & (p_i\circ f)_{i\in I} \end{array}$$

[107] Here "natural" means the following. If $f\colon (A, a_0) \to (\widetilde{A}, \widetilde{a}_0)$ and $g\colon (B, b_0) \to (\widetilde{B}, \widetilde{b}_0)$ are continuous maps of pointed topological spaces, then it follows easily from the definitions that the following diagrams commute:

$$\begin{array}{ccc} \pi_1(A\times B,(a_0,b_0)) & \xrightarrow{\Phi} & \pi_1(A,a_0)\times\pi_1(B,b_0) \\ \downarrow{\scriptstyle (f\times g)_*} & & \downarrow{\scriptstyle f_*\times g_*} \\ \pi_1(\widetilde{A}\times\widetilde{B},(\widetilde{a}_0,\widetilde{b}_0) & \xrightarrow{\Phi} & \pi_1(\widetilde{A},\widetilde{a}_0)\times\pi_1(\widetilde{B},\widetilde{b}_0)) \end{array} \quad \text{and} \quad \begin{array}{ccc} \pi_1(A,a_0)\times\pi_1(B,b_0) & \xrightarrow{\Psi} & \pi_1(A\times B,(a_0,b_0)) \\ \downarrow{\scriptstyle f_*\times g_*} & & \downarrow{\scriptstyle (f\times g)_*} \\ \pi_1(\widetilde{A},\widetilde{a}_0)\times\pi_1(\widetilde{B},\widetilde{b}_0) & \xrightarrow{\Psi} & \pi_1(\widetilde{A}\times\widetilde{B},(\widetilde{a}_0,\widetilde{b}_0)). \end{array}$$

is a bijection. The analogous statement holds if we replace the pair $([0,1],\{0,1\})$ by the pair $([0,1]\times[0,1],\{0,1\}\times[0,1])$. This shows that the map

$$\begin{array}{rcl}\pi_1\Big(\prod_{i\in I}X_i,x\Big)=\Big[([0,1],\{0,1\}),\Big(\prod_{i\in I}X_i,x\Big)\Big] & \to & \prod_{i\in I}[([0,1],\{0,1\}),(X_i,x_i)]=\prod_{i\in I}\pi_1(X_i,x_i)\\ [f] & \mapsto & \underbrace{[(p_i\circ f)_{i\in I}]}_{=p_{i*}(f)}\end{array}$$

is a bijection. But this is precisely the homomorphism that we are considering.

(2) To simplify the notation we just work with two pointed topological spaces (X,x_0) and (Y,y_0) and we denote the two inclusions by $i\colon X\to X\times Y$ and $j\colon Y\to X\times Y$. Since by (1) we already know that Φ is an isomorphism it remains to prove the following claim.

Claim. We have $\Phi\circ\Psi=\mathrm{id}$ as self-maps of $\pi_1(X,x_0)\times\pi_1(Y,y_0)$.

Proof. Given $(\alpha,\beta)\in\pi_1(X,x_0)\times\pi_1(Y,y_0)$ we obtain the following equalities in the group $\pi_1(X,x_0)\times\pi_1(Y,y_0)$:

$$\begin{array}{rcl}(\Phi\circ\Psi)(\alpha,\beta) & \overset{\text{definition of }\Psi}{=} & \Phi(i_*(\alpha)\cdot j_*(\beta)) \overset{\text{definition of }\Phi}{=} (p_*(i_*(\alpha)\cdot j_*(\beta)),q_*(i_*(\alpha)\cdot j_*(\beta))) \overset{\text{since } p_* \text{ and } q_* \text{ are homomorphisms}}{=}\\ & = & (\underbrace{p_*(i_*(\alpha))}_{=(p\circ i)_*=\mathrm{id}_*}\cdot\underbrace{p_*(j_*(\beta))}_{=(p\circ j)_*=0},\underbrace{q_*(i_*(\alpha))}_{=(q\circ i)_*=0}\cdot\underbrace{q_*(j_*(\beta))}_{=(q\circ j)_*=\mathrm{id}_*}) = (\alpha,\beta).\end{array}$$

■

We record one interesting application of the Product-π_1-Proposition 13.7 as a corollary:

Corollary 13.8. (Torus-π_1-Corollary) Let $n\in\mathbb{N}$.

(1) We consider the n-dimensional torus $(S^1)^n=S^1\times\cdots\times S^1$. Given $j\in\{1,\dots,n\}$ we consider the loop

$$\begin{array}{rcl}v_j\colon[0,1] & \to & S^1\times\cdots\times S^1\\ t & \mapsto & (1,\dots,1,\underbrace{\exp(2\pi\,\mathrm{i}\cdot t)}_{\uparrow},1,\dots,1).\end{array}$$

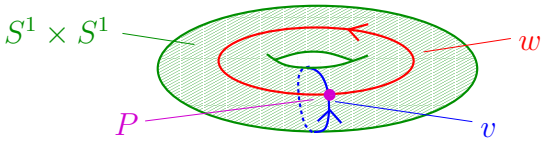
j-th coordinate

The map

$$\begin{array}{rcl}\mathbb{Z}^n & \mapsto & \pi_1(S^1\times\cdots\times S^1,(1,\dots,1))\\ (r_1,\dots,r_n) & \mapsto & [v_1]^{r_1}\cdot\ldots\cdot[v_n]^{r_n}\end{array}$$

is an isomorphism.

(2) For $n\geq 2$ the n-dimensional torus is not homeomorphic to the sphere S^n.



the map $\begin{array}{rcl}\mathbb{Z}^2 & \to & \pi_1(S^1\times S^1,P)\\ (r,s) & \mapsto & [v]^r\cdot[w]^s\end{array}$ is an isomorphism

Proof.

(1) The statement follows immediately from the Product-π_1-Proposition 13.7 and the explicit isomorphism $\Psi\colon\mathbb{Z}\to\pi_1(S^1,1)$ given in the Circle-π_1-Theorem 11.13.

(2) This statement follows from the π_1-Functor Proposition 13.1 (2), the above calculation of $\pi_1((S^1)^n)$ and the fact, shown in the Sphere-π_1-Proposition 11.14 that for $n\geq 2$ we have $\pi_1(S^n)=0$. ■

14. Coverings and fundamental groups

Arguably our greatest success so far in the theory of fundamental groups was the calculation of $\pi_1(S^1)$ in the Circle-π_1-Theorem 11.13. The calculation relied on a detailed study of the map $\Xi\colon \mathbb{R} \to S^1$ that is given by $\Xi(t) = \exp(it)$. In particular this setup had two interesting features:

(1) Given any $z \in S^1$ we could find an open neighborhood U (called $V_\pm$) such that the preimage of the open neighborhood consists of disjoint open subsets $\{U_i\}_{i\in\mathbb{Z}}$ of $\mathbb{R}$ such that for each $i \in \mathbb{Z}$ the map Ξ restricts to a homeomorphism $U_i \to U$.
(2) $\mathbb{R}$ is simply connected.

The idea now is to generalize the above approach of calculating $\pi_1(S^1, 1)$ to more general settings. The key to doing so is the concept of a "covering" $p\colon X \to B$ between topological spaces. In this chapter we will introduce coverings, we will discuss many examples and we will relate coverings to fundamental groups.

In the following chapter we will use the results of this chapter to calculate in particular the fundamental groups of the real projective spaces and the Klein bottle.

14.1. **Group actions.** The main topic of this chapter are "coverings". As we will see shortly, a main source of "coverings" is the already familiar concept of group actions. In this section we consider group actions in their own right. In particular we will give a new description of the Klein bottle and we will introduce new, fascinating topological spaces, namely the "lens spaces". In the subsequent sections we turn to "coverings" and their interaction with groups actions and fundamental groups.

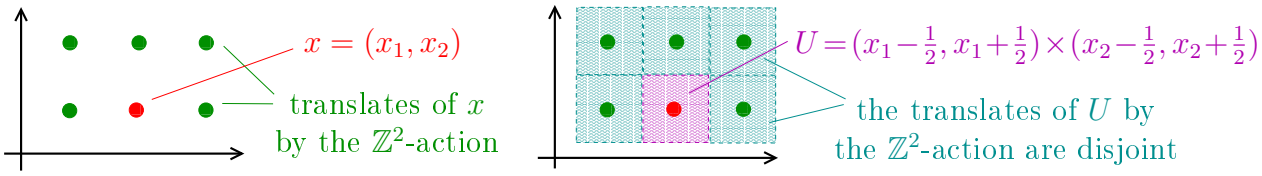
We recall two definitions about actions from page 32 and we introduce one new definition.

Definition. Let G be a group that acts continuously on a topological space X.

(1) We say G acts freely if $g \cdot x = x$ for some $g \in G$ and some $x \in X$ implies that $g = e$.
(2) We G acts properly if for every two points[108] x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \,|\, g \cdot U \cap V \neq \varnothing\}$ is finite.
(3) We say G acts discretely, if for each $x \in X$ there exists an open neighborhood U of x so that[109] [110] [111]
$$U \cap g \cdot U = \varnothing \quad \text{for all } g \neq e.$$

Examples.

(1) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is discrete. Indeed, for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the open subset $U := (x_1 - \frac{1}{2}, x_1 + \frac{1}{2}) \times \cdots \times (x_n - \frac{1}{2}, x_n + \frac{1}{2})$ has the desired property.



[110] Note that if G acts discretely, then it also acts freely.
[111] It follows easily from the properties of an action that if $U \cap g \cdot U = \varnothing$ for all $g \neq e$, then $g \cdot U \cap k \cdot U = \varnothing$ for all $k \neq g$. On several occasions we will make use of this observation.

(2) The action

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n,(x,y)) & \mapsto & (x+n,(-1)^n \cdot y) \end{array}$$

of the group $\mathbb{Z}$ on the topological space $\mathbb{R} \times [-1,1]$ is discrete. Indeed, for any (x,y) in X the open neighborhood $U := (x-\frac{1}{2}, x+\frac{1}{2}) \times [-1,1]$ has the desired property.

(3) The map

$$\begin{array}{rcl} \{\pm 1\} \times S^n & \to & S^n \\ (\epsilon, P) & \mapsto & \epsilon \cdot P \end{array}$$

defines an action of the group $\{\pm 1\}$ on S^n that is discrete. Indeed, for any $P \in S^n$ the open hemisphere $U := \{Q \in S^n \mid \underbrace{P \cdot Q}_{\text{scalar product}} > 0\}$ has the desired property.

(4) Evidently every discrete action is free. But the converse does not hold. For example the action of the group $(\mathbb{R},+)$ on $\mathbb{R}$ by addition is clearly free but one can easily verify that it is not discrete.

We consider our last two types of examples in greater detail.

Lemma 14.1. (Klein Bottle-as-Quotient Lemma) We consider the following two self-homeomorphisms of $\mathbb{R}^2$:

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1,1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x,y+1). \end{array}$$

We denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. This means that

$$G := \begin{array}{l} \text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written as} \\ \text{a finite composition of the maps } A, B, A^{-1} \text{ and } B^{-1}. \end{array}$$

The group G acts by definition on $\mathbb{R}^2$.

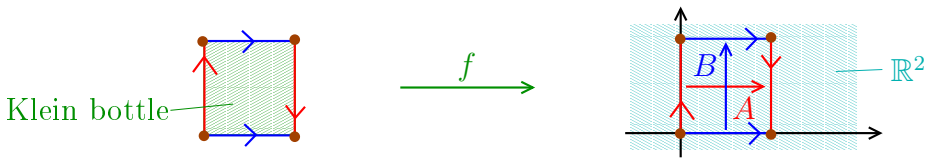
(1) The elements A and B of G do not commute, in particular the group G is non-abelian.

(2) The action of G on $\mathbb{R}^2$ is proper and discrete.

(3) We consider the Klein bottle $K = ([0,1] \times [0,1])/\!\sim$ as defined on page 108. The map

$$\begin{array}{rcl} f\colon K = ([0,1] \times [0,1])/\!\sim & \to & \mathbb{R}^2/G \\ [(x,y)] & \mapsto & [(x,y)] \end{array}$$

is a homeomorphism.



Proof. In the group G we use the product symbol "$\cdot$" to indicate the group structure instead of the "$\circ$" symbol.

(1) We calculate that

$$(A \cdot B)(0,0) \;=\; A(0,1) \;=\; (1,0) \;\neq\; (1,2) \;=\; B(1,1) \;=\; (B \cdot A)(0,0).$$

This shows that A and B do not commute in G. This shows that G is not abelian.

(2) We start out with the following claim.

Claim 1. Given $\epsilon, \eta \in \{-1, 1\}$ we have $A^\epsilon \cdot B^\eta = B^{-\eta} \cdot A^\epsilon$.

Proof. First let $\epsilon = 1$. We perform the following calculation:

$$\begin{aligned}(A\cdot B)(x,y) &= A(x,y+1) = (x+1,-y) &&= B^{-1}(x+1,1-y) &&= (B^{-1}\cdot A)(x,y),\\ (A\cdot B^{-1})(x,y) &= A(x,y-1) = (x+1,2-y) &&= B(x+1,1-y) &&= (B\cdot A)(x,y).\end{aligned}$$

We just showed that $A \cdot B^\eta = B^{-\eta} \cdot A$. By taking inverses on both sides we get the promised equalities $A^{-1} \cdot B^\eta = B^{-\eta} \cdot A^{-1}$ for $\epsilon = -1$. ⊞

Claim 2.

(a) Any $g \in G$ is of the form $A^k \cdot B^l$ for some $k, l \in \mathbb{Z}$.

(b) For $k, l \in \mathbb{Z}$ we have

$$(A^k \cdot B^l)(x,y) = \begin{cases} (x+k, y+l), & \text{if } k \text{ is even,} \\ (x+k, 1-y-l), & \text{if } k \text{ is odd.} \end{cases}$$

Proof.

(a) Let $g \in G$. By definition we can write g as a product of powers of A and B. Using the equality $A^\epsilon \cdot B^\eta = B^{-\eta} \cdot A^\epsilon$ from Claim 1 we can swap the order of powers of A with powers of B, at the price of changing the exponent of B. This means that we can bring all $A^{\pm 1}$ to the left and get the desired result.

(b) This statement follows from an elementary straightforward calculation. ⊞

Claim 3. The group G acts properly on $\mathbb{R}^2$.

Proof. Let $P, Q \in \mathbb{R}^2$. We set $U := B^2_{\frac{1}{2}}(P)$ and $V := B^2_{\frac{1}{2}}(Q)$. It follows easily from Claim 2 (a) and (b) that the set $\{g \in G \,|\, g \cdot U \cap V \neq \varnothing\}$ is finite. ⊞

Claim 4. The group G acts discretely on $\mathbb{R}^2$.

Proof. Let $(x, y) \in \mathbb{R}^2$. We set $U := (x - \frac{1}{2}, x + \frac{1}{2}) \times (y - \frac{1}{2}, y + \frac{1}{2})$. It follows easily from Claim 2 (a) and (b) that for any non-trivial $g \in G$ we have $U \cap g \cdot U = \varnothing$. ⊞

(3) Using Claim 2 from (2) one can easily show that the map $f\colon ([0,1] \times [0,1])/\sim \to \mathbb{R}^2/G$ is a bijection. It follows from the Twice Quotient Lemma 6.3 that f is continuous. By the Quotient Topology-Properties Proposition 6.2 (1) we know that $([0,1] \times [0,1])/\sim$ is compact. By (2) we know that the action of G is proper. Thus it follows from the Group Action-Hausdorff Quotient Proposition 6.7 that $\mathbb{R}^2/\sim$ is Hausdorff. In summary it follows from the Compact-Hausdorff Proposition 2.10 (3) that f is a homeomorphism. ■

We turn to our final example of a group acting on a topological space.

Lemma 14.2. (Lens Space-Discrete Action Lemma) We consider the 3-dimensional sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$. Let $p \in \mathbb{Z} \setminus \{0\}$ and let $q \in \mathbb{Z}$ with $\gcd(p, q) = 1$. We set $\xi := \exp(2\pi \mathrm{i}/p)$.

(1) The map

$$\begin{aligned}\mathbb{Z}_p \times S^3 &\to S^3\\ (k + p\mathbb{Z}, (z_1, z_2)) &\mapsto (\xi^k \cdot z_1, \xi^{q\cdot k} \cdot z_2)\end{aligned}$$

defines a smooth, orientation-preserving and discrete action of $\mathbb{Z}_p$ on S^3.

(2) The quotient space $S^3/\mathbb{Z}_p$ is a closed 3-dimensional topological manifold that has a unique structure of an oriented smooth manifold such that the projection $S^3 \to S^3/\mathbb{Z}_p$ is a local diffeomorphism and orientation-preserving.

Proof.

(1) It is straightforward to verify that the map $\mathbb{Z}_p \times S^3 \to S^3$ does indeed define an action of the group $\mathbb{Z}_p$ on the topological space S^3. Evidently the action is smooth and a direct calculation shows that the action is orientation-preserving.

Claim. The action is discrete.

Proof. Let $(z_1, z_2) \in S^3$.

(a) If $z_1 \neq 0$, then we set $U := \{(u_1, v_1) \in S^3 \,|\, u_1 = z_1 \cdot \exp(2\pi \mathrm{i} t) \text{ with } t \in (-\frac{1}{2p}, \frac{1}{2p})\}$.

(b) If $z_1 = 0$, then we set $U := \{(u_1, v_1) \in S^3 \,|\, u_2 = z_2 \cdot \exp(2\pi \mathrm{i} t) \text{ with } t \in (-\frac{1}{2p}, \frac{1}{2p})\}$.

One can easily verify that in both cases U is an open neighborhood of (z_1, z_2). Next note that $\xi^k = 1$ if and only if $k \in p \cdot \mathbb{Z}$ and note that our hypothesis $\gcd(p, q) = 1$ implies that $\xi^{k \cdot q} = 1$ if and only if $k \in p \cdot \mathbb{Z}$. Using these two observations one can easily show that all non-trivial translates of U are disjoint. ⊞

(2) We refer to [Fri23] for details. ■

Definition. Let $p \in \mathbb{Z} \setminus \{0\}$ and let $q \in \mathbb{Z}$ with $\gcd(p, q) = 1$.[112] We refer to

$$L(p, q) \;:=\; S^3/\mathbb{Z}_p,$$

with the action from the previous lemma, as a lens space.[113]

Example. We consider two special cases of lens spaces.

(1) For $p = 1$ we only have the action of the trivial group on S^3 and the quotient space $L(1, q)$ is therefore just the original sphere S^3.

(2) For $p = 2$ we have q is odd, so for $k = 0$ we have $\exp(2\pi \mathrm{i} k/2) = \exp(2\pi \mathrm{i} kq/2) = 1$ and for $k = 1$ we have $\exp(2\pi \mathrm{i} k/2) = \exp(2\pi \mathrm{i} kq/2) = -1$. Put differently, the action of $\mathbb{Z}_2$ on S^3 is the same as the action of $\mathbb{Z}_2 = \{\pm 1\}$ that we had used in the definition of the real projective space $S^2/\sim_i \mathbb{R}\mathrm{P}^3$. This shows that $L(2, q) = S^3/\sim_i \mathbb{R}\mathrm{P}^3$.

For $p \in \mathbb{N}$ and $q \in \mathbb{N}$ with $\gcd(p, q) = 1$ we just introduced a new topological space, namely the lens space $L(p, q)$. It is natural to ask: for which parameters $(p, q), (\widetilde{p}, \widetilde{q})$ are these topological spaces homeomorphic? We will now provide a partial answer:

Lemma 14.3. (Lens Spaces-Diffeomorphisms Lemma) Let $p \in \mathbb{N}$ and furthermore let $q, r \in \mathbb{Z}$ with $\gcd(p, q) = \gcd(p, r) = 1$. If $q \equiv \pm r^{\pm 1} \bmod p$,[114] then there exists a diffeomorphism $L(p, q) \to L(p, r)$.

Proof. Let $p \in \mathbb{N}$ and let $q, r \in \mathbb{Z}$ with $\gcd(p, q) = \gcd(p, r) = 1$. We pick $s \in \mathbb{Z}$ with $r \cdot s \equiv 1 \bmod p$. We suppose that $q \equiv \pm r \bmod p$ or that $q \equiv \pm s \bmod p$.

Claim. The map

[112] By definition we have $L(p, q + k \cdot p) = L(p, q)$. It follows that any lens space is of the form $L(p, q)$ with $q \in \{0, \ldots, p - 1\}$.

[113] The name "lens space" comes from an alternative description of these smooth manifolds.

[114] By our hypothesis we have $\gcd(p, r) = 1$. This implies that there exists a multiplicative inverse of r modulo p, i.e. there exists an s with $r \cdot s = 1 \bmod p$. We write $r^{-1} := s$.

$$\begin{array}{rcl} \Phi\colon S^3 & \to & S^3 \\ (z_1, z_2) & \mapsto & \begin{cases} (z_1, z_2), & \text{if } q \equiv r \bmod p, \\ (z_1, \overline{z_2}), & \text{if } q \equiv -r \bmod p, \\ (z_2, z_1), & \text{if } q \equiv s \bmod p, \\ (z_2, \overline{z_1}), & \text{if } q \equiv -s \bmod p \end{cases} \end{array}$$

descends to a well-defined map $\overline{\Phi}\colon L(p,q) \to L(p,r)$ and this map is a bijection.

Proof. We set $\xi := \exp(2\pi\,\mathrm{i}/p)$. If $q \equiv \pm r \bmod p$, then one can easily verify that Φ descends to a well-defined map $\overline{\Phi}\colon L(p,q) \to L(p,r)$. Now let us consider the case that $q \equiv s \bmod p$. Given $(z_1, z_2), (\widetilde{z_1}, \widetilde{z_2}) \in S^3$ we see that :

$$\begin{array}{rcl} [(z_1, z_2)] = [(\widetilde{z_1}, \widetilde{z_2})] \in L(p,q) & \Longrightarrow & \text{there exists } k \in \mathbb{Z} \text{ with } (\xi^k \cdot z_1, \xi^{q\cdot k} \cdot z_2) = (\widetilde{z_1}, \widetilde{z_2}) \\ & \overset{rs\equiv 1 \bmod p}{\Longrightarrow} & \text{we have } (\xi^{k\cdot r\cdot s} \cdot z_1, \xi^{q\cdot k} \cdot z_2) = (\widetilde{z_1}, \widetilde{z_2}) \\ & \overset{s\equiv q \bmod p}{\Longrightarrow} & \text{we have } (\xi^{k\cdot r\cdot q} \cdot z_1, \xi^{q\cdot k} \cdot z_2) = (\widetilde{z_1}, \widetilde{z_2}) \\ & \Longrightarrow & \text{with } l = k \cdot q \text{ we have } (\xi^{r\cdot l} \cdot z_1, \xi^{l} \cdot z_2) = (\widetilde{z_1}, \widetilde{z_2}) \\ & \Longrightarrow & \text{with } l = k \cdot q \text{ we have } (\xi^{l} \cdot z_2, \xi^{r\cdot l} \cdot z_1) = (\widetilde{z_2}, \widetilde{z_1}) \\ & \Longrightarrow & [(z_2, z_1)] = [(\widetilde{z_2}, \widetilde{z_1})] \in L(p,r) \\ & \Longrightarrow & [\Phi(z_1, z_2)] = [\Phi(\widetilde{z_1}, \widetilde{z_2})] \in L(p,r). \end{array}$$

This argument shows that we get a well-defined map $\overline{\Phi}\colon L(p,q) \to L(p,r)$. Evidently the map $\overline{\Phi}\colon L(p,q) \to L(p,r)$ is surjective. Furthermore, running the above argument backwards one see that the map $\overline{\Phi}\colon L(p,q) \to L(p,r)$ is injective. In summary we see that the map $\overline{\Phi}\colon L(p,q) \to L(p,r)$ is a bijection. Finally we note that the case $r \equiv -s \bmod p$ is dealt with in a similar fashion. ⊞

One can now easily verify that the induced map $\overline{\Phi}\colon L(p,q) \to L(p,r)$ is a diffeomorphism. We refer to [Fri23] for full details. ■

14.2. **Coverings of topological spaces.** After this short excursion into group actions on topological spaces we turn to the main topic of this chapter, namely coverings:

Definition. Let $p\colon X \to B$ be a continuous map between topological spaces.

(1) An open subset $U \subset B$ is called uniformly covered, if $p^{-1}(U) \subset X$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ of X with the property such that for each $i \in I$ the map $p|_{V_i}\colon V_i \to U$ is a homeomorphism.[115]

(2) We say the map $p\colon X \to B$ is a[116] covering, if p is surjective and if for every $b \in B$ there exists an open neighborhood U of b which is uniformly covered.

(3) Sometimes we call X a covering space of B.

[115] Let $b \in B$. It follows immediately from the definition that each $x \in p^{-1}(b)$ is contained in precisely one V_i. This implies that we can arrange that the index set is $p^{-1}(b)$. We will often make use of this observation.

[116] On page 41 we introduced the notion of a "cover". As we mentioned on page 41, such a "cover" is sometimes also called a "covering". Now we introduce the notion of a "covering". Rather maddeningly, in the literature this notion often also gets called a "cover". Fortunately, from the context, it is usually clear what is meant by "cover" and "covering". These issues do not arise in other languages, e.g. in German and French what we call "cover" is referred to as "Überdeckung" and "recouvremement" and what we call "covering" is referred to as "Überlagerung" and "revêtement".

Example.

(1) The map

$$\begin{array}{rccl} p\colon & \mathbb{R} & \to & S^1 \\ & \varphi & \mapsto & \exp(\mathrm{i}\varphi) \end{array}$$

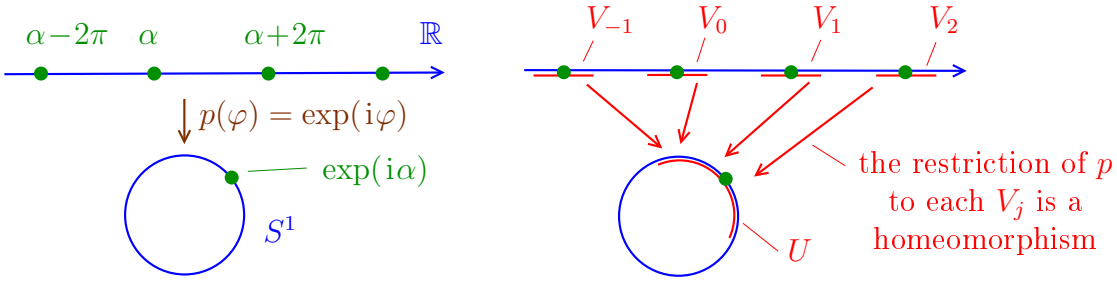
is a covering. Indeed, let $P = \exp(\mathrm{i}\alpha)$ be a point in S^1. We pick the open neighborhood

$$U := \left\{ \exp(\mathrm{i}\varphi) \,\middle|\, \varphi \in \left(\alpha - \tfrac{\pi}{4}, \alpha + \tfrac{\pi}{4}\right)\right\}.$$

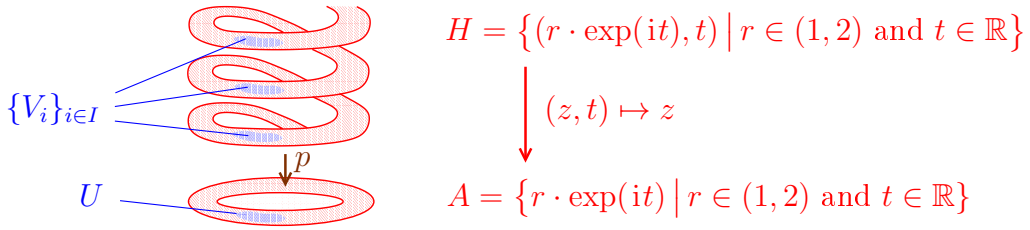
Then

$$p^{-1}(U) = \bigsqcup_{j\in\mathbb{Z}} \underbrace{\left(\alpha - \tfrac{\pi}{4} + 2\pi j, \alpha + \tfrac{\pi}{4} + 2\pi j\right)}_{=:V_j},$$

and for each $j \in \mathbb{Z}$ the restriction of p to $V_j \to U$ is a homeomorphism.



(2) In the figure below we illustrate another covering, which gives a good pictorial way to think about coverings.



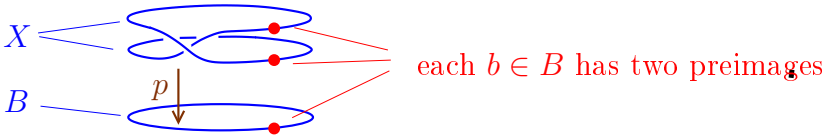
Lemma 14.4. (Covering–Preimage Lemma) Let $p\colon X \to B$ be a covering of topological spaces. If B is a connected topological space, then for any two points P and Q in B we have[117]

$$\#p^{-1}(P) = \#p^{-1}(Q).$$

Proof. The proof of the lemma is left as an exercise to the reader. ∎

Example.

(1) We consider the map $p\colon X \to B$ that is shown in the figure.[118] This map is a covering and every $b \in B$ has precisely two preimages.



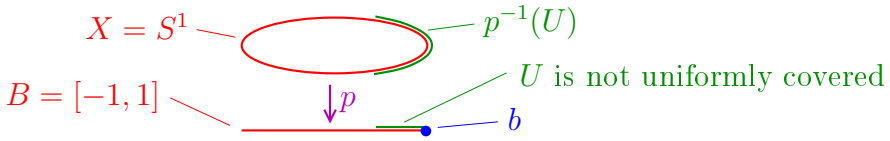
[117] Recall that given a set S we denote by $\#S$ the cardinality of S, as defined on page 25. In particular $\#p^{-1}(P) = \#p^{-1}(Q)$ means that there exists a bijection between the sets $p^{-1}(P)$ and $p^{-1}(Q)$.

[118] A mathematically precise description would be to consider the Möbius band $([0,1] \times [0,1])/\sim$ and to consider the map from the boundary to the central circle that is given by $[(x,y)] \mapsto [(x, \frac{1}{2})]$.

(2) The map

$$\begin{array}{rcl} p\colon S^1 & \to & [-1,1] \\ (x,y) & \mapsto & x. \end{array}$$

is *not* a covering since we see that the conclusion of the Covering–Preimage Lemma 14.4 does not hold. In fact one can show that $b = 1 \in [-1,1]$ does not admit a neighborhood that is uniformly covered.



Definition. Let $p\colon X \to B$ be a covering of a connected non-empty topological space.

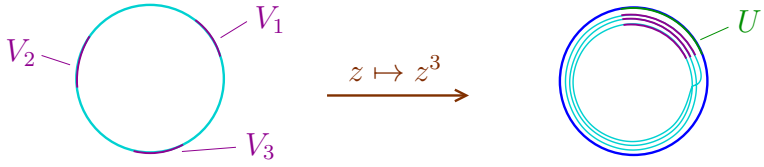
(1) We define the degree $[X : B]$ of p as the cardinality of $\#p^{-1}(b)$ for some $b \in B$.[119]

(2) If the degree is a finite number n, then we say that p is an n-fold covering. Otherwise we say that p is an infinite covering.

(3) If $p\colon X \to B$ is a covering over a possible disconnected topological space, then we say that p is a finite covering if for each component C of B the covering $p\colon p^{-1}(C) \to C$ is a finite covering in the sense of (2).

Examples.

(1) As usual we consider $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ as a subset of $\mathbb{C}$. Given any $n \in \mathbb{N}$ the map $p\colon S^1 \to S^1$ given by $p(z) := z^n$ is an n-fold covering.[120]



(2) We consider the map

$$\begin{array}{rcl} p\colon \overbrace{(\mathbb{R}\times[-1,1])/(x,y)\sim(x+1,y)}^{\text{annulus}} & \to & \overbrace{(\mathbb{R}\times[-1,1])/(x,y)\sim(x+\frac{1}{2},-y)}^{\text{Möbius band}} \\ [(x,y)] & \mapsto & [(x,y)]. \end{array}$$

It is straightforward to show that p is a 2-fold covering. Thus we can view the annulus as a 2-fold covering of the Möbius band.

(3) If in the previous example we replace the interval $[-1,1]$ by the circle $[-1,1]/-1 \sim 1$, then on the left-hand side we obtain the torus and on the right-hand side we obtain the Klein bottle. This shows that there exists a 2-fold covering from the torus to the Klein bottle.

---

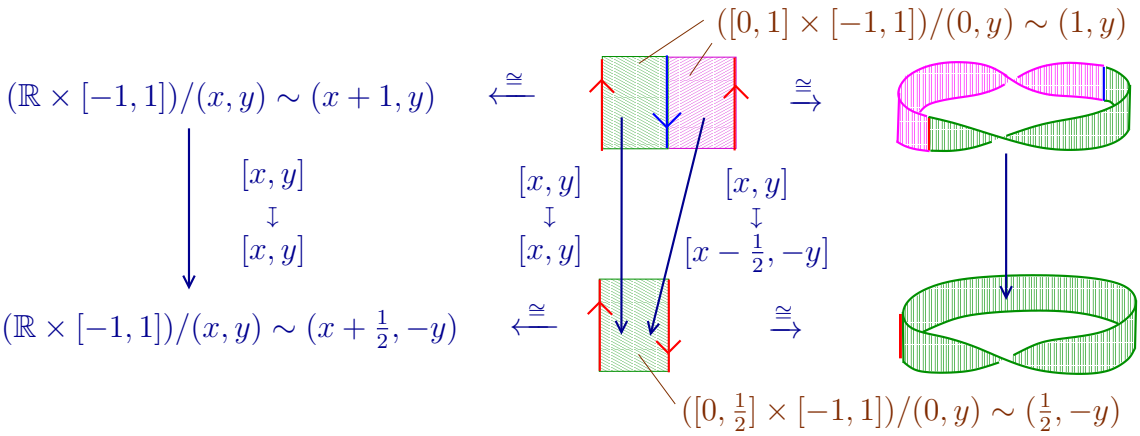
[119] By the Covering–Preimage Lemma 14.4 this definition does not depend on the choice of b.

[120] This can be seen as follows: Let $z = \exp(2\pi \mathrm{i}\varphi)$ be a point in S^1. We set

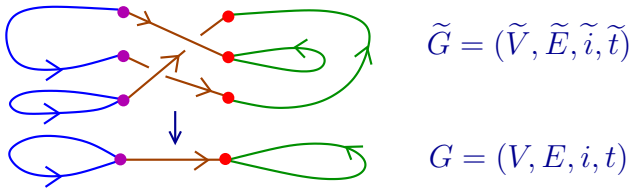
$$\begin{array}{rcl} U & := & \big\{\exp(2\pi\mathrm{i}\psi) \,\big|\, \psi \in (\varphi - \tfrac{1}{4}, \varphi + \tfrac{1}{4})\big\}. \\ p^{-1}(U) & = & \bigsqcup_{j=1}^{n} \underbrace{\big\{\exp(2\pi\mathrm{i}\psi) \,\big|\, \psi \in (\tfrac{1}{n}(\varphi - \tfrac{1}{4}) + \tfrac{j}{n}, \tfrac{1}{n}(\varphi + \tfrac{1}{4}) + \tfrac{j}{n})\big\}}_{=:V_j}. \end{array}$$

Note that

One can easily verify that the restriction of p to each V_j is a homeomorphism.



(4) In the figure below we sketch three coverings of topological graphs. The coverings are hereby given by sending edges of a certain color, with the indicated direction, to the edge of the same color with the indicated direction.[121]



The following proposition connects the notion of discrete actions to the theory of coverings.

Proposition 14.5. (Action–Covering Proposition) Let X be a topological space together with a discrete and continuous action by a group G.

(1) The natural projection $p\colon X\to X/G$ is a covering where the degree is given by $\#G$.

(2) Let H be a subgroup of G. The natural projection

$$\begin{array}{rcl} p\colon X/H & \to & X/G \\ [x] & \mapsto & [x] \end{array}$$

is a covering of degree $[G:H]$[122].

Example. The Action–Covering Proposition 14.5 (1), applied to the examples on page 183, shows that

$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ x & \mapsto & [x]=x+\mathbb{Z}^n \end{array} \quad \text{and} \quad \begin{array}{rcl} p\colon S^n & \to & \mathbb{R}\mathrm{P}^n=_i S^n/x\sim -x \\ x & \mapsto & [x] \end{array}$$

is an infinite respectively 2-fold covering.

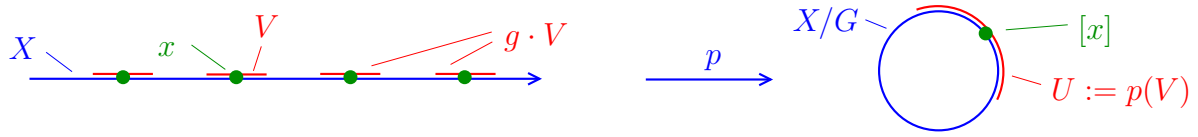
Proof of Proposition 14.5. In the following we prove Statement (1). We leave it to the reader to generalize the proof to the more general setting of Statement (2).

Let G be a group that acts discretely and continuously on a topological space X. We want to show that the projection $p\colon X\to X/G$ is a covering. Evidently the map is

[121] It is a good exercise to convince oneself, that the indicated maps are in fact coverings.

[122] Here $[G:H]$ denotes the index of the subgroup $H\subset G$, i.e. the cardinality of G/H. We define this notion in detail on page 199.

surjective. Next let $[x] \in X/G$. Since the group acts discretely there exists, by definition, an open neighborhood V of $x \in X$ such that $V \cap g{\cdot}V = \varnothing$ for all $g \neq e$. We put $U := p(V)$.



Claim. U is a uniformly covered open neighborhood of $[x]$.

Proof. Using the fact that G acts continuously one can easily show that $p\colon X \to X/G$ is an open map. Thus it follows that the set U is an open neighborhood of $[x] = p(x)$.

Next note that the preimage $p^{-1}(U)$ is the disjoint union of the subsets $g \cdot V, g \in G$. Since G acts continuously we know that each $g \cdot V$ is again open. The restriction of the map p to any $g \cdot V$ is a continuous and bijective map $g \cdot V \to U$. Since $p\colon X \to X/G$ is open and since $g \cdot V \subset X$ is open we see that the restriction of p to $g \cdot V$ is also open. It follows from the Open-Injective Lemma 2.9 that the map $p|_{g\cdot V}\colon g \cdot V \to U$ is in fact a homeomorphism. Thus we have shown that U is uniformly covered. ⊞

Since this holds for any point in X/G we have shown that the projection $p\colon X \to X/G$ is a covering. Finally note that $[X : X/G] = p^{-1}([x]) = \#G$. ■

Example. Let $X = \mathbb{R} \times [-1, 1]$ and $G = \mathbb{Z}$. Once again we consider the discrete action

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n, (x, y)) & \mapsto & \big(x + \frac{1}{2}n, (-1)^n \cdot y\big). \end{array}$$

Let $H = 2\mathbb{Z}$. According to Proposition 14.5 (2) the projection $p\colon X/H \to X/G$ is a 2-fold covering. Put differently, all the horizontal maps in the following commutative diagram are coverings:

$$\begin{array}{ccc} (\mathbb{R}\times[-1,1])/2\mathbb{Z} & \longrightarrow & (\mathbb{R}\times[-1,1])/\mathbb{Z} \\ \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\ (\mathbb{R}\times[-1,1])/(x,y) \sim (x+\frac{1}{2}2k, (-1)^{2k}\cdot y) & \longrightarrow & (\mathbb{R}\times[-1,1])/(x,y) \sim (x+\frac{1}{2}k, (-1)^k\cdot y) \\ \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\ \underbrace{(\mathbb{R}\times[-1,1])/(x,y) \sim (x+k,y)}_{=\text{the annulus}} & \longrightarrow & \underbrace{(\mathbb{R}\times[-1,1])/(x,y) \sim (x+\frac{1}{2}k, (-1)^k\cdot y)}_{=\text{Möbius band}}. \end{array}$$

Thus we recover the result from page 189 that the annulus is a 2-fold covering of the Möbius band.

14.3. **Basic properties of coverings.** The following lemma summarizes some basic facts about coverings.

Lemma 14.6. (Covering–Basics Lemma) Given any covering $p\colon X \to B$ of topological spaces the following statements hold:

(1) The map $p\colon X \to B$ is open.[123]
(2) For every $b \in B$ the preimage $p^{-1}(b)$ is a discrete subset of X.
(3) If B is Hausdorff, then so is X.

(4) The following two statements are equivalent:
  (a) X is compact.
  (b) p is a finite covering and B is compact.

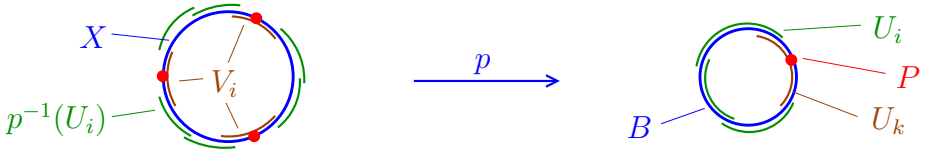
Proof. Let $p\colon X \to B$ be a covering of topological spaces.

(1) We will not really make use of this statement, thus we refer to [Fri23] for the proof.
(2) We will not really make use of this statement, thus we refer to [Fri23] for the proof.
(3) This statement is relatively easy to prove. We leave it to the reader to fill in the details.
(4) We first prove the (a)⇒(b)-direction. Thus we assume that X is compact. By definition the covering $p\colon X \to B$ is surjective and continuous. Thus it follows from the Compact Image Lemma 2.8 that B is also compact.

It remains to show that the covering has finite degree.[124] By definition we can cover the topological space B by open subsets that are uniformly covered. Since B is compact we can cover B by finitely many open subsets $U_1, \dots, U_k$ that are uniformly covered. Without loss of generality we can assume that U_k is not contained in $U_1 \cup \dots \cup U_{k-1}$.[125] Now let P be a point in $U_k \setminus (U_1 \cup \dots \cup U_{k-1})$. Furthermore let V_i, $i \in I$ be the disjoint open subsets of X such that $p^{-1}(U_k)$ is the union of the V_i's and such that each $p|_{V_i}\colon V_i \to U_k$ is a homeomorphism. For $i \in I$ we denote by $Q_i \in V_i$ the unique point with $p(Q_i) = P$. Now we consider the open cover

$$X \;=\; p^{-1}(U_1) \cup \ldots \cup p^{-1}(U_{k-1}) \cup \bigcup_{i\in I} V_i.$$

Note that for $i \in I$ the point Q_i is contained only in V_i. This means that we cannot remove any of the open subsets V_i, $i \in I$ from the open cover. Since X is compact we see that I is finite. But $\#I$ is precisely the degree of the covering. Thus we have shown that the covering has finite degree.



We turn to the proof of the (b)⇒(a)-direction. Let $p\colon X \to B$ be an n-fold covering of a compact space B. We need to show that X is compact. Thus let $\{U_i\}_{i\in I}$ be an open cover of X.

We start out with some preparations: Let $b \in B$. We pick an open neighborhood M of b that is uniformly covered. Recall that by definition this means that $p^{-1}(M)$ is the disjoint union of open subsets $\{N_x\}_{x\in p^{-1}(b)}$ with the following two properties:

- For each $x \in p^{-1}(b)$ we have $x \in N_x$.
- The restriction of p to each N_x is a homeomorphism.

[123] Recall that this means that the image of each open subset in X is open in B.
[124] If B is Hausdorff the argument is quite simple: Let $b \in B$. By the Points-in-Hausdorff Lemma 1.8 we know that $\{b\} \subset B$ is a closed subset. Since p is continuous we see that $p^{-1}(b)$ is a closed subset of X. By (2) we know that $p^{-1}(b)$ is a discrete subset. Since X is compact it now follows from an elementary argument that $[X : B] := p^{-1}(b)$ is finite.
[125] Indeed, if U_k was contained in $U_1 \cup \dots \cup U_{k-1}$, then we could remove U_k from the collection of open subsets and we continue with the collection $U_1, \dots, U_{k-1}$.

For each $x \in p^{-1}(b)$ we pick an $i_x \in I$ with $x \in U_{i_x}$. We set

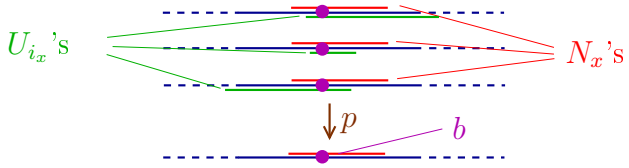
$$W_b := \bigcap_{x \in p^{-1}(b)} p(U_{i_x} \cap N_x).$$

Claim. The preimage $p^{-1}(W_b)$ is contained in the union of $n = \#p^{-1}(b)$ subsets of our open cover $\{U_i\}_{i \in I}$.

Proof. Let $x \in p^{-1}(b)$. We denote by $p_x := p|_{N_x} \colon N_x \to M$ the homeomorphism that is given by restricting p to N_x. We write $F := p^{-1}(b)$. We see that:

$$p^{-1}(W_b) = \bigcup_{y \in F} p_y^{-1}\Big(\bigcap_{x \in F} p_x(U_{i_x} \cap N_x)\Big) = \bigcup_{y \in F} \overbrace{\bigcap_{x \in F} \underbrace{p_y^{-1}(p_x(U_{i_x}))}_{= U_{iy} \text{ if } y = x} \cap \underbrace{p_y^{-1}(p_x(N_x))}_{= N_y}}^{\subset U_{i_y} \cap N_y} \subset \bigcup_{y \in F} U_{i_y}.$$

since $p^{-1}(W_b) \subset \bigcup_{y \in F} N_y$ — Image-Preimage Lemma 0.3 (7)

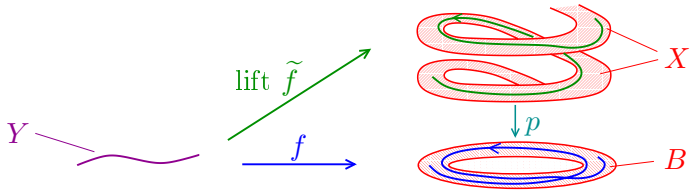


By (1) each $p(U_{i_x} \cap N_x)$ is an open subset of B and since $p^{-1}(b)$ is a finite set we see that W_b is the intersection of finitely many open subsets of B, thus W_b is an open subset of B. Furthermore b is contained in each $p(U_{i_x} \cap N_x)$, so W_b is in fact an open neighborhood of $b \in B$. Since B is compact we can cover B by finitely many of these open subsets, i.e. there exist $b_1, \ldots, b_k \in B$ with $B = W_{b_1} \cup \cdots \cup W_{b_k}$. The preimages $p^{-1}(W_{b_1}), \ldots, p^{-1}(W_{b_k})$ cover X. Note that by the above claim each preimage $p^{-1}(W_{b_i})$ is contained in the union of $n = \#p^{-1}(b_i)$ subsets of our open cover $\{U_i\}_{i \in I}$. Summarizing we have shown that we can cover X by $n \cdot k$ sets out of the collection $\{U_i\}_{i \in I}$. ∎

14.4. **Lifts of maps.** The following definition generalizes the notion of a lift that we introduced on page 164.

Definition. Let $p \colon X \to B$ be a covering and let $f \colon Y \to B$ be a continuous map between topological spaces. A lift of f to X is a continuous map $\widetilde{f} \colon Y \to X$ such that $p \circ \widetilde{f} = f$, i.e. such that the following diagram of maps commutes:

$$\begin{array}{ccc} & & X \\ & \overset{\widetilde{f}}{\nearrow} & \big\downarrow p \\ Y & \xrightarrow[f]{} & B. \end{array}$$



Similar to the discussion in Chapter 11.4 we want to prove the existence and uniqueness of lifts of paths $p\colon [0,1] \to B$ or more generally, of continuous maps $[0,1]^m \to B$. We start out with the following proposition which is a generalization of the Lift Uniqueness Lemma 11.10.

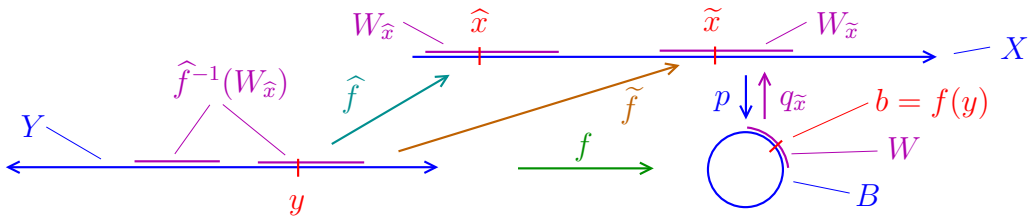
Proposition 14.7. (Lift Uniqueness Lemma) Let $p\colon X \to B$ be a covering and let Y be a connected topological space. Furthermore let $f\colon Y \to B$ be a continuous map and let $\widetilde{f}, \widehat{f}\colon Y \to X$ be two lifts of f. If $\widetilde{f}$ and $\widehat{f}$ agree at some point $y_0 \in Y$, then they agree everywhere.

Proof. We set
$$\begin{aligned} U &:= \{y \in Y \mid \widetilde{f}(y) = \widehat{f}(y)\} \quad \subset Y \\ V &:= \{y \in Y \mid \widetilde{f}(y) \neq \widehat{f}(y)\} \quad \subset Y. \end{aligned}$$
Note that by hypothesis $U \neq \varnothing$. We want to show that $U = Y$. Since Y is connected it suffices to prove that U and V are open subsets of Y. By the JH-Openness Criterion 1.1 it suffices to prove the following claim.

Claim. Every $y \in Y$ admits an open neighborhood N such that $N \subset U$ or $N \subset V$.

Proof. Let $y \in Y$. We set $b := f(y)$ and we set $\widetilde{x} := \widetilde{f}(y)$ and $\widehat{x} := \widehat{f}(y)$. Since $p\colon X \to B$ is a covering we know that b admits an open neighborhood W that is uniformly covered, i.e. we know that we can write $p^{-1}(W) = \bigsqcup_{x \in p^{-1}(b)} W_x$ such that for each $x \in p^{-1}(b)$ the following statements hold:

(1) W_x is an open neighborhood of $x \in X$.
(2) The map $p|_{W_x}\colon W_x \to W$ is a homeomorphism. We denote by $q_x\colon W \to W_x$ the inverse.



Note that on $\widetilde{f}^{-1}(W_{\widetilde{x}})$ we have
$$\widetilde{f} = \underbrace{q_{\widetilde{x}} \circ p|_{W_{\widetilde{x}}}}_{=\mathrm{id}_{W_{\widetilde{x}}}} \circ \widetilde{f} = q_{\widetilde{x}} \circ \underbrace{p|_{W_{\widetilde{x}}} \circ \widetilde{f}}_{=f} = q_{\widetilde{x}} \circ f\colon \widetilde{f}^{-1}(W_{\widetilde{x}}) \to W_{\widetilde{x}}.$$
Similarly we see that on $\widehat{f}^{-1}(W_{\widehat{x}})$ we have $\widehat{f} = q_{\widehat{x}} \circ f$. The claim follows from the following observations:

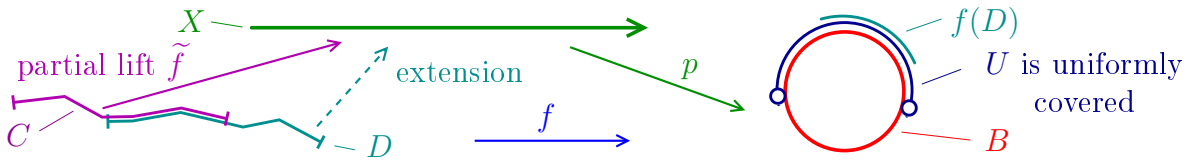
- If $y \in U$, i.e. if $\widetilde{x} = \widehat{x}$, then we see that $\widetilde{f} = \widehat{f}$ on $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}})$, i.e. we see that $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}}) \subset U$.
- If $y \notin U$, i.e. if $\widetilde{x} \neq \widehat{x}$, then we see that the images of $\widetilde{f}$ and $\widehat{f}$ on $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}})$ are disjoint, thus we see that $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}}) \subset V$. ∎

As in Section 11.4 our next goal is to prove the existence of lifts in suitable settings. The key tool for constructing lifts is the following generalization of the Lift-Extension Lemma 11.11.

Lemma 14.8. (Lift-Extension Lemma) Let $p\colon X \to B$ be a covering and let Y be a topological space. We suppose that we have a decomposition $Y = C \cup D$ such that C and D are closed subsets. Let $f\colon Y \to B$ be a continuous map and let $\widetilde{f}\colon C \to X$ be a lift of $f|_C$. We suppose that the following statements hold:

(1) The image $f(D)$ is contained in a uniformly covered open subset of B.
(2) The intersection $C \cap D$ is connected and non-empty.

Then there exists a lift $\widetilde{f}\colon Y \to X$ of f that agrees with the given lift on C.

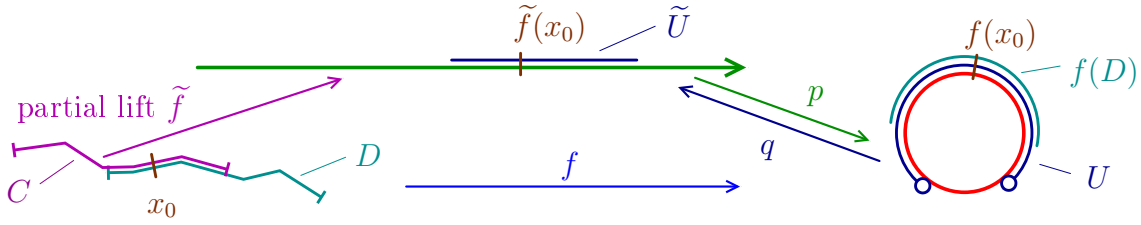


Proof. We start with a few preparations:

(a) Since $C \cap D = \varnothing$ we can pick $x_0 \in C \cap D$.
(b) We pick a uniformly covered open subset U of B that contains $f(D)$.
(c) Since U is uniformly covered open subset it thus follows from $p(\widetilde{f}(x_0)) = f(x_0) \in f(D)$ that there exists an open neighborhood $\widetilde{U}$ of $\widetilde{f}(x_0)$ such that $p\colon \widetilde{U} \to U$ is a homeomorphism.
(d) We denote by $q\colon U \to \widetilde{U}$ the inverse of $p\colon \widetilde{U} \to U$.

We set

$$\begin{array}{rccl} \widetilde{f}\colon & Y & \to & X \\ & x & \mapsto & \begin{cases} \widetilde{f}(x), & \text{if } x \in C, \\ q(\underbrace{f(x)}_{\in U}), & \text{if } x \in D. \end{cases} \end{array}$$



Note that on $C \cap D$ we have a priori two different definitions of $\widetilde{f}$. But both definitions are lifts of $f|_{C\cap D}$ and by design they agree on x_0. By hypothesis the intersection $C \cap D$ is connected. Thus we obtain from the Lift Uniqueness Lemma 11.10 that the two maps actually agree on the overlap $C \cap D$. This shows that $\widetilde{f}\colon Y \to X$ is well-defined.

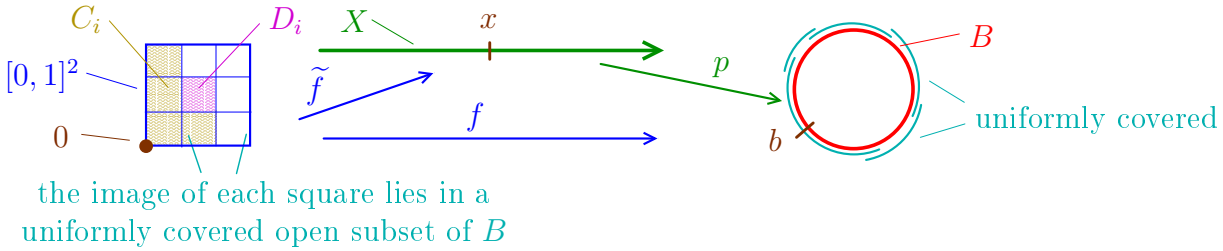
Finally note that since C and D are closed it follows from the Pasting Proposition 2.4 that this map $\widetilde{f}\colon Y \to X$ is continuous. It is now clear that it is a lift of $f\colon Y \to X$. ■

The following lemma is in particular a generalization of the Lift-from-S^1-to-$\mathbb{R}$ Lemma 11.12.

Lemma 14.9. (Lift-Existence Lemma) Let $p\colon X \to B$ be a covering of topological spaces. Given any continuous map $f\colon [0,1]^m \to B$ and given any $x \in X$ with $p(x) = f(0)$ there exists a unique lift $\widetilde{f}\colon [0,1]^m \to X$ with $\widetilde{f}(0) = x$.

Proof. [126] Let $p\colon X \to B$ be a covering of topological spaces, let $f\colon [0,1]^m \to B$ be a continuous map and let $x \in X$ with $p(x) = f(0)$. Since $p\colon X \to B$ is a covering there exists an open cover of B by uniformly covered open subsets of B. It follows from the Lebesgue Lemma 11.9 that there exists an $N \in \mathbb{N}$ such that for any $(i_1, \ldots, i_m) \in \{0, \ldots, N-1\}^m$ the image $f([\frac{i_1}{N}, \frac{i_1+1}{N}] \times \cdots \times [\frac{i_m}{N}, \frac{i_m+1}{N}])$ is contained in a uniformly covered open subset of B. The idea is now to lift the map $f\colon [0,1]^m \to B$ as follows:

(1) We set $\widetilde{f}(0) := x$.
(2) We apply the Lift-Extension Lemma 14.8 iteratively to the N^m cubes of $[0,1]^m$ starting from the "bottom left".



We now carry out the technical details. We introduce the following notation:

(a) We write $I := \{0, \ldots, N-1\}^m$. We give I the lexicographic order.
(b) Given $i := (i_1, \ldots, i_m) \in I$ we write $D_i := [\frac{i_1}{N}, \frac{i_1+1}{N}] \times \cdots \times [\frac{i_m}{N}, \frac{i_m+1}{N}]$.
(c) Given $i \in I$ we write $C_i := \bigcup_{h<i} D_h$.

We set $\widetilde{f}(0) := x$. Now we use the Lift-Extension Lemma 14.8 to add the cubes $\{D_i\}_{i \in I}$ in lexicographic order to the domain of $\widetilde{f}$. This is possible since the image of each D_i is contained in a uniformly covered open subset of B and since given any $i \in I$ the intersection $C_i \cap D_i$ is easily seen to be non-empty and connected. ■

14.5. **Lifts of paths.** We now want to apply the results from the previous sections to paths and path-homotopies. It is convenient to introduce the following definition.

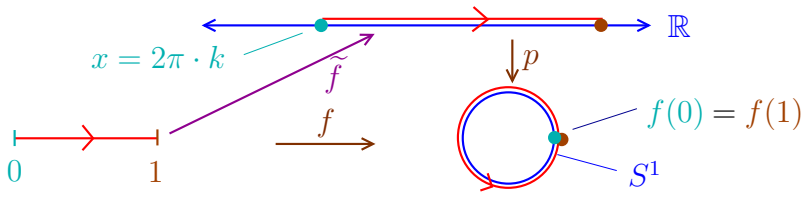
Definition. Let $p\colon X \to B$ be a covering and let $f\colon [0,1] \to B$ be a path. Let $x \in X$ be a point with $p(x) = f(0)$. A path $\widetilde{f}\colon [0,1] \to X$ is called a lift of f to the starting point x, if $\widetilde{f}$ is a lift of f with $\widetilde{f}(0) = x$.

Example. Again we consider the covering $p\colon \mathbb{R} \to S^1$ that is given by $t \mapsto \exp(\mathrm{i}t)$. We consider the path $f\colon [0,1] \to S^1$ that is given by $t \mapsto \exp(2\mathrm{i}t)$. Given any $k \in \mathbb{Z}$ the map $\widetilde{f}\colon [0,1] \to \mathbb{R}$ given by $t \mapsto 2\pi \cdot k + 2t$ is a lift of f to the starting point $2\pi \cdot k$. Note that the path $f\colon [0,1] \to S^1$ is a loop but the lift $\widetilde{f}\colon [0,1] \to \mathbb{R}$ is not a loop.

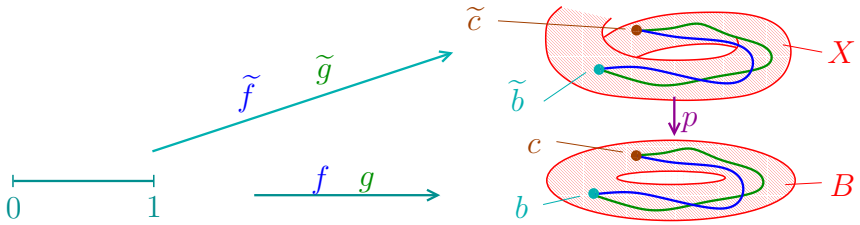
Proposition 14.10. (Path-Lifting Proposition) Let $p\colon X \to B$ be a covering.

(1) Let $f\colon [0,1] \to B$ be a path. Given any $\widetilde{b} \in X$ with $p(\widetilde{b}) = f(0)$ there exists a unique lift $\widetilde{f}\colon [0,1] \to X$ of f to the starting point $\widetilde{b}$.

[126] The proof of the proposition is a generalization of the proof of Claim 1 in the proof of the Circle-π_1-Theorem 11.13.



(2) Let $f, g\colon [0,1] \to B$ be two paths and let $\widetilde{f}, \widetilde{g}\colon [0,1] \to X$ be two lifts with the same starting point. If f and g are path-homotopic, then the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree and the paths $\widetilde{f}$ and $\widetilde{g}$ are path-homotopic.



Proof.

(1) This statement follows immediately from the Lift-Existence Lemma 14.9.

(2) We set $b := f(0) = g(0)$ and $c := f(1) = g(1)$. Furthermore let $\widetilde{b} := \widetilde{f}(0) = \widetilde{g}(0)$ be the common starting point of the lifts. Let

$$\begin{array}{rcl} H\colon [0,1] \times [0,1] & \to & B \\ (t,s) & \mapsto & H(t,s) \end{array}$$

be a path-homotopy between the paths f and g. By the Lift-Existence Lemma 14.9 we can lift H to a continuous map $\widetilde{H}\colon [0,1] \times [0,1] \to X$ with $\widetilde{H}(0,0) = \widetilde{f}(0)$.

Claim. The map $\widetilde{H}\colon [0,1] \times [0,1] \to X$ is a path-homotopy from $\widetilde{f}$ to $\widetilde{g}$. In particular the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree.

Proof.

(a) For each $s \in [0,1]$ we have $p(\widetilde{H}(0,s)) = H(0,s) = b$, thus the map $s \mapsto \widetilde{H}(0,s)$ takes values in $p^{-1}(b)$. By the Covering–Basics Lemma 14.6 we know that $p^{-1}(b)$ is a discrete subspace. It follows from the Connected-to-Discrete Lemma 2.18 that the map $s \mapsto \widetilde{H}(0,s)$ is constant. Similarly we see that the $s \mapsto \widetilde{H}(1,s)$ is constant. Thus we have shown that $\widetilde{H}$ is a path-homotopy.

(b) Since $\widetilde{H}_0$ and $\widetilde{f}$ are both lifts of $H_0 = f$ to the starting point $\widetilde{b}$ we obtain from the Lift Uniqueness Proposition 14.7 that $\widetilde{H}_0 = \widetilde{f}$.

(c) It follows from (a) that $\widetilde{H}_1$ and $\widetilde{g}$ are both lifts of $H_1 = g$ to the starting point $\widetilde{b}$. Thus we obtain again from the Lift Uniqueness Proposition 14.7 that $\widetilde{H}_1 = \widetilde{g}$.

In summary, we have shown that $\widetilde{H}$ is a path-homotopy from $\widetilde{f}$ to $\widetilde{g}$. ■

Proposition 14.11. (Covering-π_1-Monomorphism Proposition) Let $p\colon X \to B$ be a covering. Given any $x_0 \in X$ the induced map $p_*\colon \pi_1(X, x_0) \to \pi_1(B, p(x_0))$ is a monomorphism.

Example. In the figure below we see one of the coverings from page 190. Even though we cannot compute the fundamental groups of X and B it is clear that the fundamental group of X is "bigger" in a suitable sense than the fundamental group of B. But counter intuitively the Covering-π_1-Monomorphism Proposition 14.11 shows that there exists a monomorphism, namely p_*, from $\pi_1(X, x_0)$ into $\pi_1(B, p(x_0))$.

Proof. We set $b_0 := p(x_0)$. We see that

$$[f] \in \ker(p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)) \implies [p \circ f] = e \in \pi_1(B, b_0) \implies p \circ f \simeq_{\mathrm{p}} \mathrm{const}_{b_0}$$
$$\implies f \simeq_{\mathrm{p}} \mathrm{const}_{x_0} \implies [f] = e \in \pi_1(X, x_0).$$

this follows from the Path-Lifting Proposition 14.10 (2), since f is the lift of $p \circ f$ to the starting point x_0 and const_{x_0} is the lift of const_{b_0} to the starting point x_0 ■

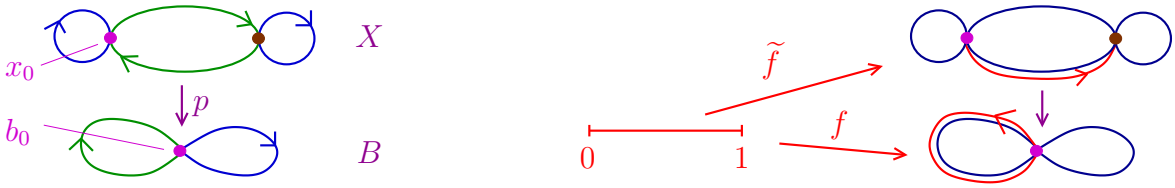
Example. By the Sphere-π_1-Proposition 11.14 we know that $\pi_1(S^2) = 0$ whereas we saw in the Torus-π_1-Corollary 13.8 that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ is non-trivial. It follows from the Covering-π_1-Monomorphism Proposition 14.11 that there is no covering map from $S^1 \times S^1$ to S^2.

Proposition 14.12. (Loop–Lifting Proposition) Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. Let $f\colon [0,1] \to B$ be a loop in (B, b_0). We denote by $\widetilde{f}\colon [0,1] \to X$ the lift[127] of f to the starting point x_0. The following holds:

$$\widetilde{f} \text{ is a loop in } (X, x_0) \iff [f] \text{ lies in } p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0).$$

In particular, if $\widetilde{f}$ is not a loop in (X, x_0), then $[f]$ is a non-trivial element of $\pi_1(B, b_0)$.

Example. We consider the pointed topological space (B, b_0) and the loop $f\colon [0,1] \to B$ that are shown in the figure below. We also show a covering $p\colon X \to B$ together with the lift $\widetilde{f}$ of f to the starting point x_0. Since $\widetilde{f}$ is not a loop it follows from the Loop–Lifting Proposition 14.12 that $[f] \in \pi_1(B, b_0)$ is non-trivial.



Proof. Let f be a loop in (B, b_0). We denote by $\widetilde{f}$ the lift of f to the starting point x_0.

(⇒) If $\widetilde{f}$ is a loop in (X, x_0), then it defines an element in $\pi_1(X, x_0)$. Therefore we see that $[f] = [p \circ \widetilde{f}] = p_*([\widetilde{f}]) \in p_*(\pi_1(X, x_0))$.

(⇐) We assume that $[f]$ lies in $p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0)$. We make two observations:

- The hypothesis that $[f] \in p_*(\pi_1(X, x_0))$ implies that there exists a loop g in (X, x_0) with $[f] = p_*([g]) = [p \circ g]$. In particular we see that f and $p \circ g$ are path-homotopic.

[127] Note that such a lift exists, and is unique, by the Path-Lifting Proposition 14.10 (1).

- Note that $\widetilde{f}$ is a lift of f and that g is a lift of $p \circ g$ and note that these two lifts have the same starting point x_0.

It follows from these two observations and the Path-Lifting Proposition 14.10 (2) that the endpoint of $\widetilde{f}$ agrees with the endpoint of $\widetilde{g}$, which is x_0. Thus we have shown that $\widetilde{f}$ is a loop in (X, x_0). ■

The following definition should be familiar from a (linear) algebra course.

Definition. Let G be a group and let H be a subgroup of G.

(1) We get an equivalence relation $\sim$ on G by setting $g_1 \sim g_2$ if there exists an $h \in H$ with $g_1 \cdot h = g_2$. We denote by G/H the set of equivalence classes.
(2) If in (1) we use left multiplication by $h \in H$, then also get an equivalence relation and we denote the set of equivalence classes by $H\backslash G$. Note that the map $[g] \mapsto [g^{-1}]$ defines a bijection $H\backslash G \to G/H$.
(3) We write $[G : H] := \#G/H = \#H\backslash G$.

Proposition 14.13. (Covering-Degree Proposition) Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. If X is path-connected, then[128]

$$\underbrace{[X : B]}_{\substack{\text{degree as defined} \\ \text{on page 189}}} = [\pi_1(B, b_0) : p_*(\pi_1(X, x_0))].$$

Example. Let $n \geq 2$.

(1) On page 190 we saw that we have a 2-fold covering $p\colon S^n \to S^n/\{\pm 1\} =_i \mathbb{R}\mathrm{P}^n$.
(2) It follows from the Covering-Degree Proposition 14.13 that $p_*(\pi_1(S^n))$ is subgroup of index two of $\pi_1(S^n/\{\pm 1\})$.
(3) Since $n \geq 2$ we know by the Sphere-π_1-Proposition 11.14 that $\pi_1(S^n) = 0$.
(4) It follows from (1)–(3) that $\pi_1(S^n/\{\pm 1\})$ is a group with two elements.
(5) It is elementary to see that (4) implies that $\pi_1(\mathbb{R}\mathrm{P}^n) = \pi_1(S^n/\{\pm 1\}) \cong \mathbb{Z}_2$.

In the next chapter we will return to this calculation in a more systematic fashion.

As we will see shortly, the Covering-Degree Proposition 14.13 follows almost immediately from the following lemma.

Lemma 14.14. (Covering–Bijection Lemma) Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. If X is path-connected, then the map

$$\begin{aligned} \Theta\colon p^{-1}(b_0) &\to p_*(\pi_1(X, x_0))\backslash\pi_1(B, b_0) \\ y_0 &\mapsto [p \circ (\underbrace{\text{path } f \text{ in } X \text{ from } x_0 \text{ to } y_0}_{\text{exists since } X \text{ is path-connected}})] \end{aligned}$$

is well-defined and it is a bijection.

[128] Recall that the left-hand side and the right-hand side are defined as cardinalities of sets.

Proof of Lemma 14.14. Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. We assume that X is path-connected.

(1) First we show that the map Θ is well-defined. Thus let $f, g\colon [0,1] \to X$ be two paths from x_0 to y_0. Note that

$$\underbrace{[p \circ f]}_{\in \pi_1(B,b_0)} = [p \circ f * \underbrace{p \circ \overline{g} * p \circ g}_{\text{null-homotopic by the Path-Homotopy Product Proposition 11.4}}] = \underbrace{[p \circ (\underbrace{f * \overline{g}}_{\text{loop in } (X,x_0) \text{ since the endpoint of } f \text{ equals the starting point of } \overline{g}})] \cdot [p \circ g]}_{} = p_*(\underbrace{[f * \overline{g}]}_{\in \pi_1(X,x_0)}) \cdot \underbrace{[p \circ g]}_{\in \pi_1(B,b_0)}.$$

(2) Next we show that the map Θ is surjective. Thus let $[f] \in \pi_1(B, b_0)$. By the Path-Lifting Proposition 14.10 (1) there exists a lift $\widetilde{f}$ of f to the starting point x_0. Note that $p(\widetilde{f}(1)) = f(1) = b_0$. Thus $\widetilde{f}(1) \in p^{-1}(b_0)$. Evidently $\widetilde{f}$ is a path in X from x_0 to $\widetilde{f}(1)$. Thus we see that $\Theta(\widetilde{f}(1)) = [p \circ \widetilde{f}] = [f]$.

(3) Finally we show that the map Θ is injective. Suppose we have $y_0, z_0 \in p^{-1}(b_0)$ such that $\Theta(y_0) = \Theta(z_0)$.

(a) We pick a path $f_0\colon [0,1] \to X$ from x_0 to y_0 and we pick a path $g_0\colon [0,1] \to X$ from x_0 to z_0.

(b) The fact that $\Theta(y_0) = \Theta(z_0) \in p_*(\pi_1(X,x_0))\backslash\pi_1(B,b_0)$ implies that there exists a loop $h\colon [0,1] \to X$ in (X,x_0) such that $[p \circ h] \cdot [p \circ f_0] = [p \circ g_0]$.

(c) Note that $h * f_0$ is defined since the endpoint of h equals x_0, which is the starting point of f_0.[129] Now we see that

$$[p \circ (h * f_0)] = [(p \circ h) * (p \circ f_0)] = [p \circ h] \cdot [p \circ f_0] = [p \circ g_0].$$

(d) By (c) we know that $p \circ (h * f_0) \simeq_{\mathrm{p}} p \circ g_0$.

(e) Since $h * f_0$ is a lift of $p \circ (h * f_0)$ to the starting point x_0 and since g_0 is a lift of $p \circ g_0$ to the starting point x_0 we obtain from (d) and the Path-Lifting Proposition 14.10 (2) that the endpoints of $h * f_0$ and g_0 agree. But these are precisely y_0 and z_0. ∎

Proof of Proposition 14.13. Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. We assume that X is path-connected. We see that

$$[X : B] \underset{\text{definition of degree}}{=} \#p^{-1}(b_0) \underset{\text{Covering–Bijection Lemma 14.14}}{=} \#p_*(\pi_1(X,x_0))\backslash\pi_1(B,b_0) \underset{\text{definition of index}}{=} [\pi_1(B,b_0) : p_*(\pi_1(X,x_0))].$$

∎

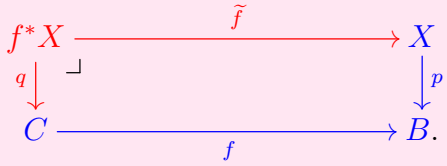
[129] This is the reason why we work with right cosets.

14.6. **Pullbacks of coverings.** We conclude this chapter with a lemma on pullbacks of coverings. This lemma will play a minor role here and there, but it can also be skipped at a first reading.

Lemma 14.15. (Pullback-Covering Lemma) Let $p\colon X \to B$ be a continuous map of topological spaces, let C be a topological space and let $f\colon C \to B$ be a continuous map. As in the Topological-Pullback Proposition 5.8 we consider the pullback

$$f^*X \;:=\; \{(c,x) \in C \times X \mid f(c) = p(x)\}$$

which gives us the following commutative diagram:



If $p\colon X \to B$ is in fact a covering, then the following statements hold:

(1) The map $q\colon f^*X \to C$ is also a covering.

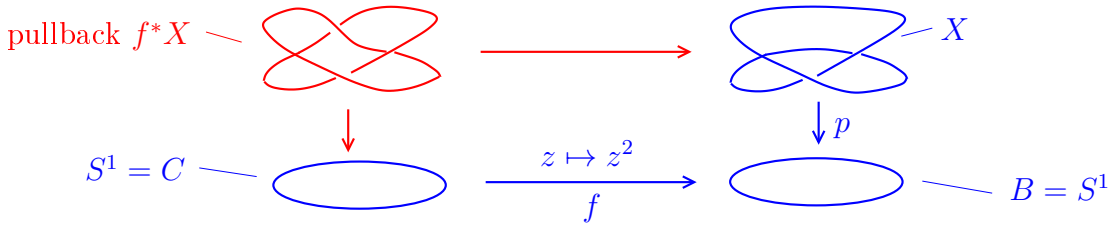
(2) If B and C are connected, then

$$\deg(q\colon f^*X \to C) \;=\; \deg(p\colon X \to B).$$

(3) Let $c \in C$. We write $b = f(c)$ and we pick[130] $x \in X$ such that $p(x) = b$. The following equality holds:

$$q_*(\pi_1(f^*X, (c,x))) \;=\; f_*^{-1}(p_*(\pi_1(X,x))) \;\subset\; \pi_1(C,c).$$

Example. Similar to the discussion on page 100 we consider the covering $p\colon X \to B = S^1$ that is shown in the figure below and we consider the map $f\colon C = S^1 \to S^1 = B$ that is given by $f(z) = z^2$. The pullback covering $f^*X \to C = S^1$ is a product, i.e. there exists a homeomorphism $f^*X \cong S^1 \times \{-1,1\}$ that commutes with the projections to S^1.



Proof. Let $p\colon X \to B$ be a covering. We write $Z := f^*X$ and $z := (c,x)$. Finally we denote by $q\colon f^*X \to C$ the map given by $(c,x) \mapsto c$.

(1) First we show that $q\colon f^*X \to C$ is surjective. Thus let $c \in C$. Since $p\colon X \to B$ is in particular surjective we see that there exists an $x \in X$ with $p(x) = f(c)$. Note that $(c,x) \in f^*X$. Since $q(c,x) = c$ we have shown that q is surjective.

Next let $c \in C$. We need to show that c admits a uniformly covered neighborhood. Since p is a covering the point $f(c) \in B$ admits a uniformly covered neighborhood U. This means that $p^{-1}(U) = \bigsqcup_{i\in I} A_i$ such that each A_i is open in X and such that each $p|_{A_i}\colon A_i \to U$ is a homeomorphism.

[130] Recall that covering maps are in particular surjective, in particular such x exists.

We set $V := f^{-1}(U)$. We claim that V is uniformly covered. Given $i \in I$ we consider $B_i := \{(c,x) \in f^*X \mid c \in V, x \in U_i\}$. Since $B_i = i^*X \cap (V \times X_i)$ we see that B_i is an open subset of i^*X. Finally one can easily verify that each restriction $q|_{B_i} \colon B_i \to V$ is a homeomorphism.

(2) Since B and C are connected it makes sense to speak of the degrees of p and q. For any $c \in C$ we have a bijection

$$q^{-1}(c) \;=\; \{(c,x) \in C \times X \mid f(c) = p(x)\} \xrightarrow{\text{bijection}} p^{-1}(f(c)) \;=\; \deg(p).$$

This shows that the degree of q equals the degree of p.

(3) We need to show that $q_*(\pi_1(Z,z)) = f_*^{-1}(\pi_1(X,x))$. We show the two inclusions "$\subset$" and "$\supset$" separately.

"$\subset$" Let $\widetilde{f} \colon Z \to X$ be the map given by $(c,x) \to x$. By construction we have $f \circ q = \widetilde{f} \circ p$. But this implies immediately that $f_*(q_*(\pi_1(Z,z)))$ is contained in $p_*(\pi_1(X,x))$.

"$\supset$" Let $\gamma \colon [0,1] \to C$ be a loop in (C,c) with $[f \circ \gamma] = f_*([\gamma]) \in p_*(\pi_1(X,x))$. We write $g := f \circ \gamma$. Let $\widetilde{g} \colon [0,1] \to X$ be the lift with $\widetilde{g}(0) = x$. Since $[g] \in p_*(\pi_1(X,x))$ we obtain from the Loop–Lifting Proposition 14.12 that $\widetilde{g}$ is actually a loop. Thus the map $\alpha \colon [0,1] \to C \times X$ that is given by $t \mapsto (\gamma(t), \widetilde{g}(t))$ is a loop. By construction the loop takes values in Z and evidently we have $q_*([\alpha]) = [\gamma]$. This shows that $[\gamma] \in q_*(\pi_1(Z,z))$. ■

15. Group actions and fundamental groups

In this chapter we will build on the techniques and results of the last chapter to calculate the fundamental groups of many topological spaces, in particular of the real projective spaces, lens spaces and the Klein bottle. We will also consider mapping tori and their fundamental groups.

15.1. The main theorem on group actions and fundamental groups.

The following theorem will allow us to determine many fundamental groups. This proposition is the reward for the many rather theoretical results of the previous chapter.

Theorem 15.1. (Action–π_1-Theorem) Let X be a topological space and let G be a group that acts continuously and discretely[131] on X. We choose an $x_0 \in X$. We denote by $p\colon X \to X/G$ the natural projection.

(1) If X is simply connected, then the map

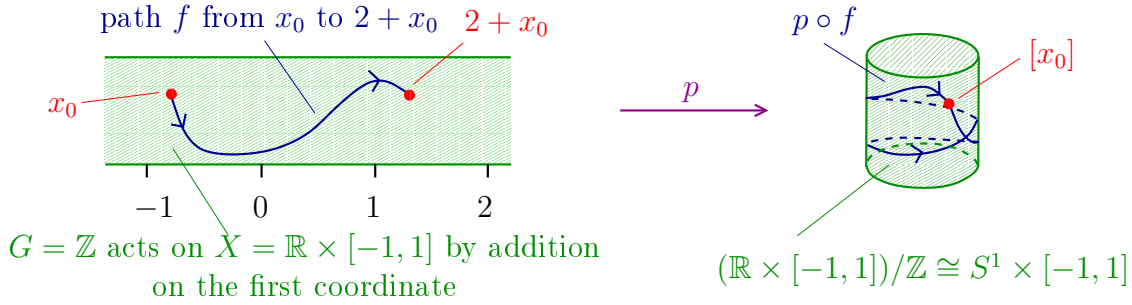
$$\begin{array}{rcl} G & \to & \pi_1(X/G, [x_0]) \\ g & \mapsto & [p \circ (\text{path in } X \text{ from } x_0 \text{ to } g \cdot x_0)] \end{array}$$

is a well-defined isomorphism of groups. In particular we have $\pi_1(X/G) \cong G$.

(2) [132] If X is path-connected (but not necessarily simply connected), then $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(X/G, [x_0])$ and the map

$$\begin{array}{rcl} G & \to & \pi_1(X/G, [x_0])/p_*(\pi_1(X, x_0)) \\ g & \mapsto & [p \circ (\text{path in } X \text{ from } x_0 \text{ to } g \cdot x_0)] \end{array}$$

is a well-defined isomorphism of groups.



Before we provide the proof of the Action–π_1-Theorem 15.1 it is perhaps motivating to consider a few basic examples. In the following examples we will see that the Action–π_1-Theorem 15.1 can be used to recover some of our calculations, namely we can use it to calculate the fundamental groups of tori and the Möbius band. After the proof we will give more interesting applications.

Example.

(1) Let $n \in \mathbb{N}$. It follows from the discussion on page 161 that $\mathbb{R}^n$ is simply connected. We saw on page 183 that the usual continuous action of $(\mathbb{Z}^n, +)$ on $\mathbb{R}^n$ is discrete. It

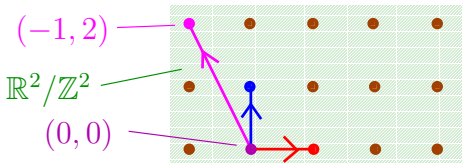
[131] Recall that we say that G acts discretely, if for each $x \in X$ there exists an open neighborhood U such that $U \cap g \cdot U = \varnothing$ for all $g \neq e$.

[132] Statement (2) is a generalization of Statement (1). For many, but not all, applications Statement (1) is good enough.

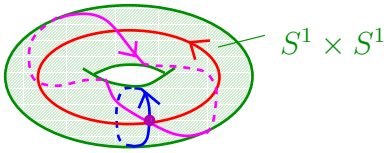
follows from the Action–π_1-Theorem 15.1 (1) that the map

$$\begin{array}{rcl} \mathbb{Z}^n & \to & \pi_1(\mathbb{R}^n/\mathbb{Z}^n, [(0)]) \\ v & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & \mathbb{R}^n/\mathbb{Z}^n \\ t & \mapsto & [t \cdot v] \end{array}\right] \end{array}$$

is an isomorphism. Using the homeomorphism $(S^1)^n \to \mathbb{R}^n/\mathbb{Z}^n$ from the Torus-as-Quotient Lemma 6.6 this also gives a new calculation of the fundamental group of $(S^1)^n$. In particular we obtain a new proof of the Circle-π_1-Theorem 11.13 and the Torus-π_1-Corollary 13.8.



which corresponds to



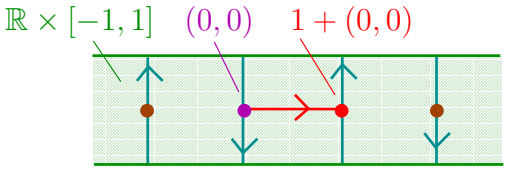
(2) We consider $X = \mathbb{R} \times [-1,1]$ with the continuous $\mathbb{Z}$-action

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n, (x,y)) & \mapsto & (x+n, (-1)^n \cdot y). \end{array}$$

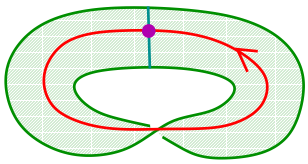
On page 183 we saw that this action is discrete. Note that on page 161 we saw that $\mathbb{R} \times [-1,1]$ is simply connected. It follows from the Action–π_1-Theorem 15.1 (1) that the map

$$\begin{array}{rcl} \mathbb{Z} & \to & \pi_1((\mathbb{R} \times [-1,1])/\mathbb{Z}, ([(0,0)]) \\ n & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & (\mathbb{R} \times [-1,1])/\mathbb{Z} \\ t & \mapsto & [(n \cdot t, 0)] \end{array}\right] \end{array}$$

is an isomorphism. Recall that in the Möbius Band-as-Quotient Lemma 6.8 we saw that $(\mathbb{R} \times [-1,1])/\mathbb{Z}$ is homeomorphic to the Möbius band. Thus we obtain a new proof of the Möbius Band-π_1-Corollary 13.6.



which corresponds to



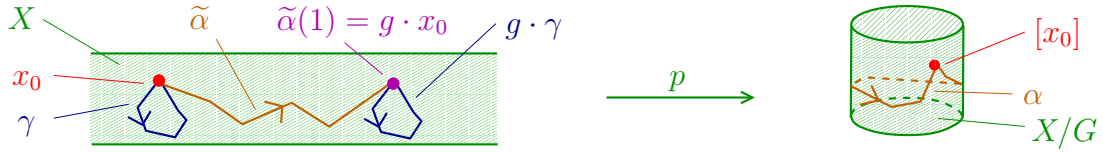
Now we finally provide the proof of the Action–π_1-Theorem 15.1

Proof of the Action–π_1-Theorem 15.1. Note that Statement (1) is just a user-friendly special case of Statement (2). Thus it suffices to prove Statement (2). Let X be a path-connected topological space and let G be a group that acts continuously and discretely on X. Note that it follows from the Action–Covering Proposition 14.5 that the natural projection $p\colon X \to X/G$ is a covering. Finally let $x_0 \in X$. We start out with the following claim.

Claim 1. The image $p_*(\pi_1(X, x_0)) \subset \pi_1(X/G, [x_0])$ is a normal subgroup.

Proof. Let $[\gamma] \in \pi_1(X, x_0)$ and let $[\alpha] \in \pi_1(X/G, [x_0])$. We denote by $\widetilde{\alpha}\colon [0,1] \to X$ the lift of α with the starting point x_0. Note that $p(\widetilde{\alpha}(1)) = \alpha(1) = [x_0]$. This shows that there exists a $g \in G$ with $\widetilde{\alpha}(1) = g \cdot x_0$. We denote by $g \cdot \gamma$ the loop in X that is given by

$t \mapsto g \cdot \gamma(t)$. Note that we can form the loop $\widetilde{\alpha} * g \cdot \gamma * \overline{\widetilde{\alpha}}$ since the endpoints always match the starting point.

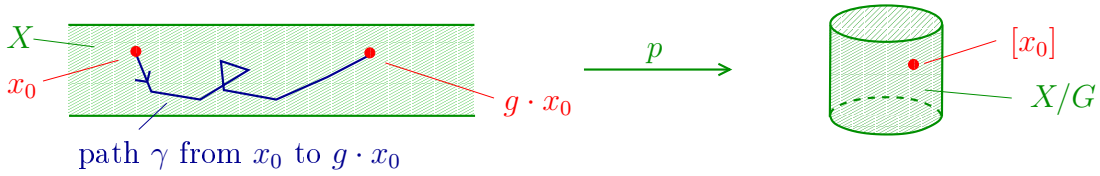


Now we see that

$$\begin{array}{rcl} [\alpha] \cdot p_*([\gamma]) \cdot [\alpha]^{-1} & = & [p \circ \widetilde{\alpha}] \cdot [p \circ \gamma] \cdot [p \circ \overline{\widetilde{\alpha}}] \\ & = & [p \circ (\underbrace{\widetilde{\alpha} * g \cdot \gamma * \overline{\widetilde{\alpha}}}_{\text{defined since the endpoints always match the starting points}})] \end{array} \quad \begin{array}{rcl} & & \\ \overset{\text{since } p(g \cdot x) = p(x) \in X/G \text{ for all } x \in X}{=} & [p \circ \widetilde{\alpha}] \cdot [p \circ (g \cdot \gamma)] \cdot [p \circ \overline{\widetilde{\alpha}}] & \\ = & p_*([\underbrace{\widetilde{\alpha} * g \cdot \gamma * \overline{\widetilde{\alpha}}}_{\text{loop in } (X, x_0)}]) \in p_*(\pi_1(X, x_0)). & \end{array}$$

Thus we have shown that $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(X/G, [x_0])$. ⊞
Similar to the Covering–Bijection Lemma 14.14 we consider the map

$$\begin{array}{rcl} \Phi\colon G & \to & \pi_1(X/G, [x_0])/p_*(\pi_1(X, x_0)) \\ g & \mapsto & [p \circ (\text{path in } X \text{ from } x_0 \text{ to } g \cdot x_0)]. \end{array}$$



Claim 2. The map Φ is well-defined and it is a bijection.

Proof. We consider the maps

$$\begin{array}{rcccl} G & \to & p^{-1}([x_0]) & \to & \pi_1(X/G, [x_0])/p_*(\pi_1(X, x_0)) \\ g & \mapsto & g \cdot x_0 & & \\ & & y_0 & \mapsto & [p \circ (\text{path in } X \text{ from } x_0 \text{ to } y_0)]. \end{array}$$

The map to the left is clearly a bijection. The claim now follows from the observation that we showed in the Covering–Bijection Lemma 14.14 that the map to the right is well-defined and that it is a bijection. ⊞
It remains to prove the following claim.

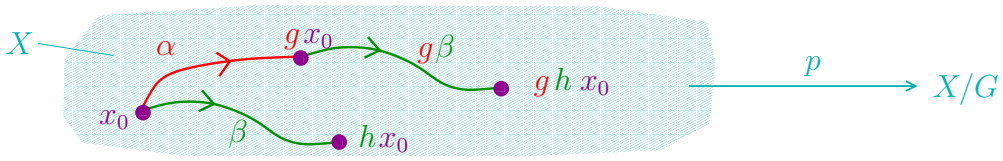
Claim 3. The map $\Phi\colon G \to \pi_1(X/G, [x_0])/p_*(\pi_1(X, x_0))$ is a group homomorphism.

Proof. Let $g, h \in G$. We pick a path α from x_0 to $g \cdot x_0$ and we pick a path β from x_0 to $h \cdot x_0$. Then $g \cdot \beta$[133] is a path from $g \cdot x_0$ to $(g \cdot (h \cdot x_0)) = gh \cdot x_0$ and $\alpha * g \cdot \beta$ is a path from x_0 to $gh \cdot x_0$. It follows that

$$\begin{array}{rcl} \Phi(g \cdot h) & = & [p \circ (\alpha * g \cdot \beta)] \\ & & \uparrow \\ & & \text{since } \alpha * g \cdot \beta \text{ is a path from } x_0 \text{ to } gh \cdot x_0 \end{array} \quad \begin{array}{rcl} & & \\ \overset{\text{since } p \circ (\gamma * \delta) = p \circ \gamma * p \circ \delta}{=} & [(p \circ \alpha) * (p \circ g \cdot \beta)] & \\ & & \end{array} \quad \begin{array}{rcl} & & \\ \overset{\text{since } p(g \cdot x) = p(x) \in X/G \text{ for all } x \in X}{=} & [(p \circ \alpha) * (p \circ \beta)] & \\ = & [p \circ \alpha] \cdot [p \circ \beta] & = \Phi(g) \cdot \Phi(h). \\ & \uparrow & \\ & \text{definition of } \cdot \text{ in } \pi_1(X/G, [x_0]) & \end{array}$$

■

[133] Here $g \cdot \beta$ denotes the path $[0, 1] \to X$ that is given by $t \mapsto g \cdot \beta(t)$.



15.2. **Applications of the Action–π_1-Theorem 15.1.** Using the Action–π_1-Theorem 15.1 we can now determine the fundamental groups of many of the topological spaces that we have encountered so far.

Proposition 15.2. (Lens Space-π_1-Proposition)

(1) Let $p \in \mathbb{Z} \setminus \{0\}$ and let $q \in \mathbb{Z}$ with $\gcd(p,q) = 1$. If $L(p,q) = S^3/\mathbb{Z}_p$ is a lens space as defined on page 186, then
$$\pi_1(L(p,q)) \;\cong\; \mathbb{Z}_p.$$

(2) Let $p, \widetilde{p} \in \mathbb{Z} \setminus \{0\}$ and let $q, \widetilde{q} \in \mathbb{Z}$ with $\gcd(p,q) = 1$ and $\gcd(\widetilde{p}, \widetilde{q}) = 1$. If $L(p,q)$ is homotopy equivalent (e.g. homeomorphic) to $L(\widetilde{p}, \widetilde{q})$, then $p = \widetilde{p}$.

Proof.

(1) First we note that by the Sphere-π_1-Proposition 11.14 we know that S^3 is simply connected. Next note that in the Lens Space-Discrete Action Lemma 14.2 we showed that the action of $\mathbb{Z}_p$ on S^3 is discrete. Thus the statement follows from the Action–π_1-Theorem 15.1 (1).

(2) This statement follows from (1) and the Homotopy-π_1-Proposition 13.4 (2a). ■

We move on to the only slightly less bewildering real projective spaces. The following proposition can be viewed as a refinement of the argument on page 199.

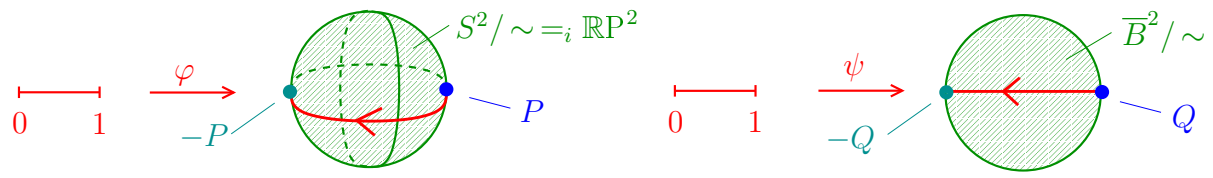
Proposition 15.3. (Real Projective Space–π_1-Proposition) Let $n \in \mathbb{N}_{\geq 2}$.[134]

(1) We consider the real projective space $\mathbb{R}\mathrm{P}^n =_i S^n/\{\pm 1\}$.
  (a) $\pi_1(\text{real projective space } \mathbb{R}\mathrm{P}^n) \;=_i\; \pi_1(S^n/\{\pm 1\}) \;\cong\; \mathbb{Z}_2$.
  (b) We set $P := (1, 0, \ldots, 0) \in S^n$. The unique non-trivial element of the fundamental group $\pi_1(\mathbb{R}\mathrm{P}^n, [P]) =_i \pi_1(S^n/\{\pm 1\}, [P])$ is represented by the loop
$$\begin{array}{rcl} \varphi\colon [0,1] & \to & \mathbb{R}\mathrm{P}^n =_i S^n/\{\pm 1\} \\ t & \mapsto & [(\cos(\pi t), \sin(\pi t), 0, \ldots, 0)]. \end{array}$$

(2) Let "$\sim$" be the equivalence relation on $\overline{B}^n$ that is generated by $x \sim -x$ for $x \in S^{n-1}$.
  (a) $\pi_1(\overline{B}^n/\!\sim) \cong \mathbb{Z}_2$.
  (b) With $Q = (1, 0, \ldots, 0) \in \overline{B}^n$ the unique non-trivial element of $\pi_1(\overline{B}^n/\!\sim), [Q])$ is represented by the loop
$$\begin{array}{rcl} \psi\colon [0,1] & \to & \overline{B}^n/\!\sim \\ t & \mapsto & [2t-1, 0, \ldots, 0)]. \end{array}$$

Proof.

[134] Recall that in the Projective Space-Avatar Lemma 6.11 we showed that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to S^1. Thus it follows from the Circle-π_1-Theorem 11.13 that $\pi_1(\mathbb{R}\mathrm{P}^1) \cong \mathbb{Z}$.



(1) These two statements follows almost immediately from the following observations:
  (i) We consider the usual continuous action of $\{\pm 1\}$ on S^n. On page 184 we saw that the action of $\{\pm 1\}$ on S^n is discrete. We denote by $p\colon S^n \to S^n/\{\pm 1\}$ the projection.
  (ii) Since $n \geq 2$ we obtain from the Sphere-π_1-Proposition 11.14 that the sphere S^n is simply connected.
  (iii) Note that it follows from (i) and (ii) together with the Action–π_1-Theorem 15.1 (1) that the map

$$\begin{array}{rcl} \{\pm 1\} & \to & \pi_1(S^n/\sim) \\ \epsilon & \mapsto & [p \circ (\text{path in } S^n \text{ from } P \text{ to } \epsilon \cdot P)] \end{array}$$

  is an isomorphism. Since $t \mapsto (\cos(\pi t), \sin(\pi t), 0, \ldots, 0)$ is precisely a path from P to $-P$ we see that φ represents the non-trivial element of $\pi_1(S^n/\{\pm 1\}, [P])$.
  (iv) Recall that in the Projective Space-Sphere Quotient Lemma 6.9 we saw that the inclusion $S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ induces a homeomorphism $S^n/\{\pm 1\} \to \mathbb{R}\mathrm{P}^n$.
(2) In the Projective Space-Avatar Lemma 6.11 (2) we showed that the map

$$\begin{array}{rcl} f\colon \mathbb{R}\mathrm{P}^n =_i S^n/\{\pm 1\} & \xrightarrow{\cong} & \overline{B}^n/\sim \\ \underbrace{[(v, x_{n+1})]}_{v \in \mathbb{R}^n,\, x_{n+1} \in \mathbb{R}} & \mapsto & \begin{cases} [v], & \text{if } x_{n+1} \geq 0, \\ [-v], & \text{if } x_{n+1} \leq 0 \end{cases} \end{array}$$

is a homeomorphism with $f([P]) = [Q]$. It follows from this observation together with the results in (1) that $\pi_1(\overline{B}^n/\sim) \cong \mathbb{Z}_2$ and that the unique non-trivial of $\pi_1(\overline{B}^n/\sim, [Q])$ is presented by the loop $f \circ \varphi$, i.e. by the loop $t \mapsto [(\cos(\pi t), \sin(\pi t), 0, \ldots, 0)]$. It is straightforward to see that this loop is path-homotopic to the loop ψ. ■

Proposition 15.4. (SO(3)–π_1**-Proposition**)
(1) $\pi_1(\mathrm{SO}(3), \mathrm{id}) \cong \mathbb{Z}_2$.
(2) The unique non-trivial element is represented by the loop

$$\begin{array}{rcl} \gamma\colon [0,1] & \to & \mathrm{SO}(3) \\ t & \mapsto & \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi t) & \sin(2\pi t) \\ 0 & -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix}}_{\text{rotation around the } x\text{-axis}} \end{array}$$

at the base point $\mathrm{id} \in \mathrm{SO}(3)$.

Proof.
(1) (a) Let $\sim$ be the equivalence relation on $\overline{B}^3$ that is generated by $x \sim -x$ for $x \in S^2$.
  (b) We set $Q := (1, 0, 0) \in \overline{B}^3$.

(c) In the Projective Space-Avatar Lemma 6.11 (3) we saw that the map

$$\begin{array}{rcl} f\colon \overline{B}^3/\!\sim & \to & \mathrm{SO}(3) \\ [\underbrace{r\cdot v}_{r\in[0,1] \text{ and } v\in S^2}] & \mapsto & \text{matrix in SO(3) which represents rotation by the angle } \pi\cdot r \\ & & \text{around the axis } \mathbb{R}_{\geq 0}\cdot v \text{ where the rotation} \\ & & \text{direction is given by the right-hand rule} \end{array}$$

is a homeomorphism. Note that $f(0) = \mathrm{id}$.

(d) In the Real Projective Space–π_1-Proposition 15.3 (2) we showed that $\pi_1(\overline{B}^3/\!\sim, [Q]) \cong \mathbb{Z}_2$.

(e) It follows from (c) and (d) that $\pi_1(\mathrm{SO}(3), f([Q])) \cong \mathbb{Z}_2$.

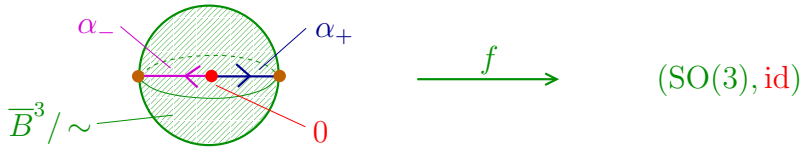
(2) We still need to show that γ represents the unique non-trivial element of $\pi_1(\mathrm{SO}(3), \mathrm{id})$.

(f) Given $\epsilon \in \{-1, 1\}$ let $\alpha_\epsilon\colon [0,1] \to \overline{B}^3$ be the path that is given by $t \mapsto [t \cdot Q \cdot \epsilon]$.

(g) In the Real Projective Space–π_1-Proposition 15.3 (2) we showed that the unique non-trivial element of $\pi_1(\overline{B}^3/\!\sim, [Q])$ is represented by the loop $\overline{\alpha}_+ * \alpha_-$.

(h) Note that α_+ is a path from 0 to Q. It follows from (d) and the Change-of-Base Point Proposition 11.6 that the unique non-trivial element of $\pi_1(\overline{B}^3/\!\sim, 0)$ is represented by $\alpha_+ * (\overline{\alpha}_+ * \alpha_-) * \overline{\alpha}_+ \simeq_{\mathrm{p}} \alpha_- * \overline{\alpha}_+$.

(i) It follows from (c) and (h) and the fact that $f(0) = \mathrm{id}$ that the unique non-trivial element of $\pi_1(\mathrm{SO}(3), \mathrm{id})$ is represented by $\varphi := f \circ (\overline{\alpha}_+ * \alpha_-)$.



(j) The loop that we gave in the statement of the proposition is precisely $\overline{\varphi}$. But if φ represents a non-trivial element of $\pi_1(\mathrm{SO}(3), \mathrm{id})$, then so does $\overline{\varphi}$. ■

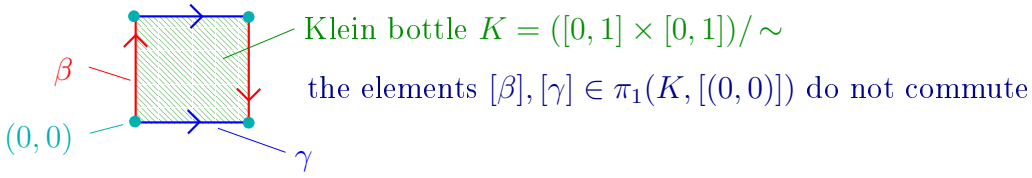
Proposition 15.5. (Klein Bottle–π_1-Proposition) As in the Klein Bottle-as-Quotient Lemma 14.1 we consider the group G that is given by all self-homeomorphisms of $\mathbb{R}^2$ that can be written as concatenations of the two self-homeomorphisms

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1) \end{array}$$

and their inverses.

(1) The fundamental group of the Klein bottle $K = ([0,1]\times[0,1])/\!\sim$ is isomorphic to G.

(2) The fundamental group of the Klein bottle K is non-abelian, in particular the two elements $[\beta]$ and $[\gamma]$ shown in the figure below do not commute.



Proof.

(1) In the Klein Bottle-as-Quotient Lemma 14.1 we showed the following two facts:
  (a) The group G acts discretely on $\mathbb{R}^2$.
  (b) The inclusion induced map $f\colon ([0,1]\times[0,1])/\sim \to \mathbb{R}^2/G$ is a homeomorphism.
  It follows from (a) together with the Action–π_1-Theorem 15.1 (1) that the map

$$\begin{array}{rcl}\Theta\colon G & \to & \pi_1(\mathbb{R}^2/G,[(0,0)]) \\ g & \mapsto & \left[\begin{array}{rcl}[0,1] & \to & \mathbb{R}^2/G \\ t & \mapsto & [t\cdot \underbrace{g(0,0)}_{\in\mathbb{R}^2}]\end{array}\right]\end{array}$$

  is an isomorphism. If we combine this calculation with (b) we see that the fundamental group of the Klein bottle is isomorphic to G.

(2) (a) We set $C := B^{-1}\cdot A$. An easy calculation shows that the two homeomorphisms $B(x,y)=(x,y+1)$ and $C(x,y)=(x+1,-y)$ do not commute.
  (b) It follows from (a) that $(f_*)^{-1}(\Theta(B)),(f_*)^{-1}(\Theta(C)) \in \pi_1(K,[(0,0)])$ do not commute.
  (c) We consider the two loops β,γ in $(K,[(0,0)])$ that are shown in the above figure. Note that it follows easily from the definitions that $(f_*)^{-1}(\Theta(B)) = [\beta]$ and that $(f_*)^{-1}(\Theta(C)) = [\gamma]$.
  (d) It follows from (b) and (c) that $[\beta]$ and $[\gamma]$ do not commute. ■

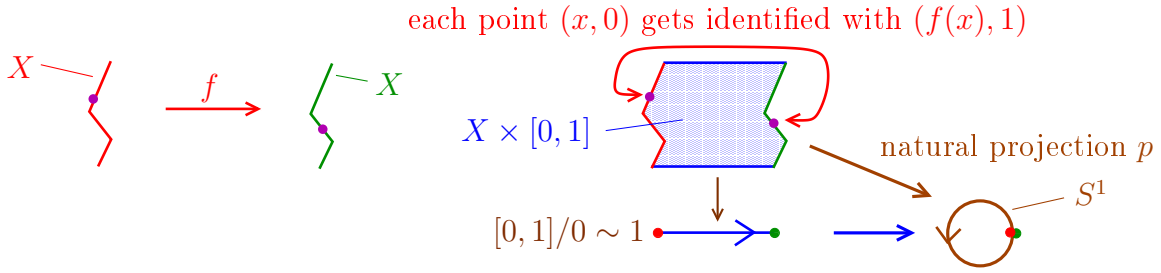
15.3. **The fundamental group of mapping tori.** We start out with the following definition which we will use on several occasions.

Definition. Let X be a topological space and let $f\colon X\to X$ be a homeomorphism. We refer to

$$\mathrm{Tor}(X,f) \;:=\; (X\times[0,1])/\sim \qquad \text{where } (x,0)\sim(f(x),1) \text{ for all } x\in X,$$

as the mapping torus of (X,f). We introduce two maps:

$$\underbrace{\begin{array}{rcl}X & \to & \mathrm{Tor}(X,f) \\ x & \mapsto & [(x,0)]\end{array}}_{\text{the natural injection}} \quad\text{and}\quad \underbrace{\begin{array}{rcl}\mathrm{Tor}(X,f) & \to & S^1 \\ [(x,t)] & \mapsto & \exp(2\pi\,\mathrm{i}t).\end{array}}_{\text{the natural projection}}$$



Examples.

(1) Let X be a topological space. In [Fri23] it is shown that the map

$$\begin{array}{rcl}\Phi\colon \mathrm{Tor}(X,\mathrm{id}) & \to & X\times S^1 \\ [(x,t)] & \mapsto & (x,\exp(2\pi\,\mathrm{i}t))\end{array}$$

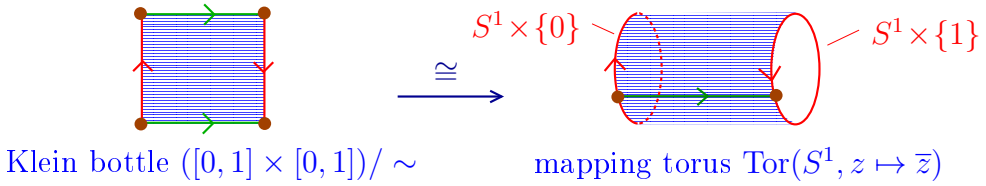
[134] It follows from the Topological-Quotient Proposition 6.1 (1b) that the natural projection is indeed continuous.

is a homeomorphism.

(2) We consider the homeomorphism $f\colon S^1 \to S^1$ that is given by $z \mapsto \overline{z}$. We also consider the Klein bottle $([0,1]\times[0,1])/\sim$, as defined on page 108. It is not difficult to show that the map

$$\begin{array}{rcl} ([0,1]\times[0,1])/\sim & \to & \mathrm{Tor}(S^1, z\mapsto\overline{z}) \\ [(x,y)] & \mapsto & [(\exp(2\pi\,\mathrm{i}\,x), y)] \end{array}$$

is a homeomorphism.



Our goal in this section is to determine the fundamental groups of mapping tori. Before we can do so we need to introduce a new concept from group theory.

Definition. Let N be a group, let $k \in \mathbb{N}_0$ and let $\varphi\colon N \to N$ be an isomorphism such that $\varphi^k = \mathrm{id}$.[135] We define the semidirect product of N and $\mathbb{Z}_k$ with respect to φ as the group[136] $N \rtimes_\varphi \mathbb{Z}_k$ where the underlying set is given by the direct product $N \times \mathbb{Z}_k$,[137] but where the group multiplication[138] is given by

$$(h,m)\cdot(h',m') := (h\cdot\varphi^m(h'), m+m').$$

If φ is understood from the context, then we drop it from the notation, i.e. we write $N \rtimes \mathbb{Z}_k$ instead of $N \rtimes_\varphi \mathbb{Z}_k$.

Examples.

(1) If N is a group and if $\varphi = \mathrm{id}_N$, then $N \rtimes_{\mathrm{id}} \mathbb{Z} = N \times \mathbb{Z}$ is the direct product of N and $\mathbb{Z}$.

(2) Let $N = \mathbb{Z}_n$ for some $n \in \mathbb{N}_0$. We denote by $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ the isomorphism that is given by multiplication by -1. Clearly this isomorphism satisfies $\varphi^2 = \mathrm{id}$, thus we can form the semidirect product $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$, which is called the n-th dihedral group D_n. It has $2n$ elements. As an example, the third dihedral group D_3 is isomorphic to the permutation

[135] Note that this condition is automatically satisfied if $k = 0$, i.e. for any isomorphism φ of any group we always have $\varphi^0 = \mathrm{id}$.

[136] One can easily show that $N = \{(h,0) \mid h \in N\} \subset N \rtimes \mathbb{Z}_k$ is a normal subgroup of $N \rtimes \mathbb{Z}_k$. Put differently, we have $N \lhd N \rtimes \mathbb{Z}_k$. This is the reason for the symbol "$\rtimes$" in the notation of the semidirect product.

[137] In particular, if N is finite and $k \in \mathbb{N}$, then the cardinality of $N \rtimes \mathbb{Z}_k$ is given by $|N| \cdot k$.

[138] It is an elementary argument to show that $N \rtimes_\varphi \mathbb{Z}_k$ with the given multiplication is indeed a group.

group S_3 via the isomorphism [139]

$$\begin{array}{rcl}\Psi\colon \mathbb{Z}_3 \rtimes \mathbb{Z}_2 & \to & S_3 \\ (h,m) & \mapsto & (1\,2\,3)^h \cdot (1\,2)^m.\end{array}$$

The following lemma shows that many groups can be written as a semidirect product $N \rtimes \mathbb{Z}$.

Lemma 15.6. (Semidirect Product Lemma) Let G be a group, let $\xi\colon G \to \mathbb{Z}$ be an epimorphism and let $t \in G$ with $\xi(t) = 1$. We write $N := \ker(\xi)$ and we denote by φ the isomorphism

$$\begin{array}{rcl}\varphi\colon N & \to & N \\ h & \mapsto & t \cdot h \cdot t^{-1}.\end{array}$$

Then the map[140]

$$\begin{array}{rcl}\Theta\colon G & \to & N \rtimes_\varphi \mathbb{Z} \\ g & \mapsto & \big(g \cdot t^{-\xi(g)}, \xi(g)\big)\end{array}$$

is an isomorphism of groups.

Proof. It is straightforward to verify that Θ is injective and surjective. It remains to show that Θ is a homomorphism: given $g, h \in G$ we have indeed

$$\begin{aligned}\Theta(g\cdot h) \;=\; (g\cdot h\cdot t^{-\xi(g\cdot h)}, \xi(g\cdot h)) \;&=\; (g\cdot t^{-\xi(g)}\cdot \underbrace{t^{\xi(g)}\cdot h\cdot t^{-\xi(h)}}_{=\varphi^{\xi(g)}(h)}\cdot t^{-\xi(g)}, \xi(g)+\xi(h)) \\ &=\; (g\cdot t^{-\xi(g)}, \xi(g))\cdot(h\cdot t^{-\xi(h)}, \xi(h)) \;=\; \Theta(g)\cdot\Theta(h). \qquad \blacksquare\end{aligned}$$

The following proposition shows that the fundamental group of a mapping torus $\mathrm{Tor}(X, f)$ is isomorphic to some semidirect product $\pi_1(X) \rtimes \mathbb{Z}$.

Proposition 15.7. (Mapping Torus-π_1-Proposition) Let (X, x_0) be a pointed path-connected topological space, let $f\colon X \to X$ be a homeomorphism and let γ be a path from x_0 to $f(x_0)$. We denote by $\gamma_*\colon \pi_1(X, f(x_0)) \to \pi_1(X, x_0)$ the isomorphism given by the Change-of-Base Point Proposition 11.6 and we define

$$\varphi := \gamma_* \circ f_*\colon \pi_1(X, x_0) \xrightarrow[\cong]{f_*} \pi_1(X, f(x_0)) \xrightarrow[\cong]{\gamma_*} \pi_1(X, x_0).$$

Let $i\colon X \to \mathrm{Tor}(X, f)$ be the natural inclusion and let $p\colon \mathrm{Tor}(X, f) \to S^1$ be the natural projection. There exists a natural isomorphism

$$\Theta\colon \pi_1(\mathrm{Tor}(X, f), [(x_0, 0)]) \xrightarrow{\cong} \pi_1(X, x_0) \rtimes_\varphi \mathbb{Z}$$

---

[139] First note that a straightforward calculation shows that $(1\,2)^{-1}\cdot(1\,2\,3)\cdot(1\,2) = (1\,3\,2) = (1\,2\,3)^{-1}$. This implies that also for any $m \in \mathbb{Z}_2$ and $s \in \mathbb{Z}$ we have $(1\,2)^{-m}\cdot(1\,2\,3)^s\cdot(1\,2)^m = (1\,2\,3)^{s(-1)^m}$. Now for any $(h, m), (h', m') \in \mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$ we have

$$\begin{aligned}\Psi(h,m)\Psi(h',m') &= (1\,2\,3)^h\cdot(1\,2)^m\cdot(1\,2\,3)^{h'}\cdot(1\,2)^{m'} = (1\,2\,3)^h\cdot(1\,2)^m\cdot(1\,2)^{-m}\cdot(1\,2\,3)^{(-1)^m h'}\cdot(1\,2)^m\cdot(1\,2)^{m'} \\ &= (1\,2\,3)^{h+(-1)^m h'}\cdot(1\,2)^{m+m'} = \Psi(h+(-1)^m h', m+m') = \Psi((h,m)\cdot(h',m')).\end{aligned}$$

We have thus shown that Ψ is a homomorphism. It is now straightforward to verify that Ψ is in fact an isomorphism.

[140] Note that $\xi(g\cdot t^{-\xi(g)}) = \xi(g) + \xi(t)(-\xi(g)) = 0$, i.e. $g\cdot t^{-\xi(g)}$ does indeed lie in $N = \ker(\xi)$.

such that the following diagram commutes[141]

$$\begin{array}{ccccc}
\pi_1(X,x_0) & \xrightarrow{i_*} & \pi_1(\mathrm{Tor}(X,f),[(x_0,0)]) & \xrightarrow{p_*} & \pi_1(S^1,1) \\
\downarrow{\scriptstyle \mathrm{id}} & & \cong\downarrow\Theta & & \downarrow\cong \\
\pi_1(X,x_0) & \xrightarrow{g\mapsto(g,0)} & \pi_1(X,x_0)\rtimes_\varphi\mathbb{Z} & \xrightarrow{(g,n)\mapsto n} & \mathbb{Z}.
\end{array}$$

(Here the vertical isomorphism to the right is the standard isomorphism from the Circle-π_1-Theorem 11.13.)

Proof. Let X be a topological space and let $f\colon X\to X$ be a homeomorphism. One can easily show, that the following two statements hold:

(1) The map
$$\begin{array}{rcl} \mathbb{Z}\times(X\times\mathbb{R}) & \to & X\times\mathbb{R} \\ (n,(x,t)) & \mapsto & (f^n(x),t+n) \end{array}$$
defines a continuous, proper and discrete action of the group $\mathbb{Z}$ on $X\times\mathbb{R}$.

(2) The map
$$\begin{array}{rccl} \Psi\colon & \mathrm{Tor}(X,f) & \to & (X\times\mathbb{R})/\mathbb{Z} \\ & [(x,t)] & \mapsto & [(x,t)] \end{array}$$
is a homeomorphism.

We introduce the following notation:

(a) We denote by $q\colon X\times\mathbb{R}\to(X\times\mathbb{R})/\mathbb{Z}$ the natural projection.

(b) We denote by $j\colon X\to X\times\mathbb{R}$ the inclusion that is given by $x\mapsto(x,0)$. Using the fact that $\{0\}$ is a deformation retract of $\mathbb{R}$ and using the Deformation Retract–Homotopy Equivalence Lemma 9.3 one can easily show that $j\colon X\to X\times\mathbb{R}$ is a homotopy equivalence.

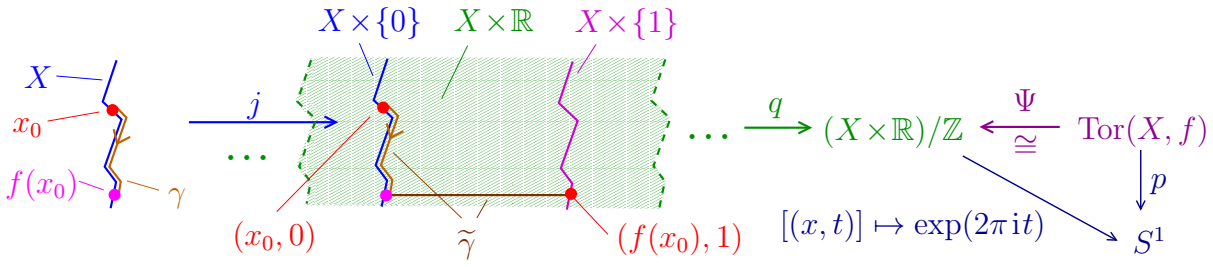
(c) We denote by $\widetilde{\gamma}\colon[0,1]\to X\times\mathbb{R}$ the path from $(x_0,0)$ to $(f(x_0),1))$ that is given by the concatenation of the path $i\circ\gamma$ and the "horizontal" path $t\mapsto(x_0,t)$.

It follows from (1) and the Action–π_1-Theorem 15.1 (2) that the map

$$\begin{array}{rcl} \mathbb{Z} & \to & \pi_1((X\times\mathbb{R})/\mathbb{Z},[(x_0,0)])/q_*(\pi_1(X\times\mathbb{R},(x_0,0))) \\ n & \mapsto & [(q\circ\widetilde{\gamma})^n] \end{array}$$

is an isomorphism. This map gives rise to an epimorphism $\xi\colon\pi_1((X\times\mathbb{R})/\mathbb{Z},[(x_0,0)])\to\mathbb{Z}$ with kernel $q_*(\pi_1(X\times\mathbb{R},(x_0,0)))$. Note that $\xi([q\circ\widetilde{\gamma}])=1$. From the Semidirect Product Lemma 15.6 we obtain an explicit isomorphism

$$\Theta\colon\pi_1((X\times\mathbb{R})/\mathbb{Z},[(x_0,0)]) \to q_*(\pi_1(X\times\mathbb{R},(x_0,0)))\rtimes\mathbb{Z}$$



[141] Note that both maps on the bottom are group homomorphisms.

In summary we get the following diagram:

$$\begin{array}{ccccc}
\pi_1(X,x_0) & \xrightarrow{i_*} & \pi_1(\mathrm{Tor}(X),f),[(x_0,0)]) & \xrightarrow{q_*} & \pi_1(S^1,1) \\
\cong\downarrow j_* & & \cong\downarrow \Psi_* & & \downarrow \mathrm{id} \\
\pi_1(X\times\mathbb{R},(x_0,0)) & \longrightarrow & \pi_1((X\times\mathbb{R})/\mathbb{Z},[(x_0,0)]) & \xrightarrow[[(x,t)]\mapsto \exp(2\pi i t)]{\text{induced by}} & \pi_1(S^1,1) \\
\downarrow \mathrm{id} & & \cong\downarrow \Theta & & \uparrow n\mapsto[t\mapsto\exp(2\pi i n)] \\
\pi_1(X\times\mathbb{R},(x_0,0)) & \xrightarrow{g\mapsto(g,0)} & q_*(\pi_1(X\times\mathbb{R},(x_0,0)))\rtimes\mathbb{Z} & \xrightarrow{(g,n)\mapsto n} & \mathbb{Z} \\
\cong\uparrow j_* & & \cong\uparrow \text{induced by } \psi_*{}^{142} & & \cong\downarrow \mathrm{id} \\
\pi_1(X,x_0) & \xrightarrow{g\mapsto(g,0)} & \pi_1(X,x_0)\rtimes_\varphi\mathbb{Z} & \xrightarrow{(g,n)\mapsto n} & \mathbb{Z}.
\end{array}$$

It follows fairly easily from the definitions that the diagram commutes.[143] Therefore the composition of the three middle vertical isomorphisms gives us the promised isomorphism $\pi_1(\mathrm{Tor}(X,f),[(x_0,0)]) \to \pi_1(X,x_0)\rtimes_\varphi\mathbb{Z}$. ■

We conclude this section with a discussion of several examples.

Example.

(1) Let $K = \mathrm{Tor}(S^1, f(z) := \overline{z})$ be the Klein bottle. It follows from the discussion on page 174 that the automorphism $f_* =: \varphi$ of $\pi_1(S^1) = \mathbb{Z}$ is multiplication by -1. It follows from the Mapping Torus-π_1-Proposition 15.7 that

$$\pi_1(\text{Klein bottle}) \;\cong\; \mathbb{Z}\rtimes_\varphi\mathbb{Z}.$$

(2) Let X be a path-connected topological space. We calculate that

$$\pi_1(X\times S^1) \;\cong\; \pi_1(\mathrm{Tor}(X,\mathrm{id}_X)) \underset{\substack{\uparrow\\ \text{by the Mapping Torus-}\pi_1\text{-Proposition 15.7}}}{\cong} \pi_1(X)\rtimes_{\mathrm{id}}\mathbb{Z} = \pi_1(X)\times\mathbb{Z} = \pi_1(X)\times\pi_1(S^1).$$

So in this special case we obtain the same result as in the Product-π_1-Proposition 13.7.

(3) Let $A\in \mathrm{GL}(n,\mathbb{Z})$. We leave it as an exercise to the reader to prove the following two statements:
- The map $f(A)\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ given by $[v]\mapsto[Av]$ is a homeomorphism.
- We use the Action–π_1-Theorem 15.1 to make the identification $\pi_1(\mathbb{R}^n/\mathbb{Z}^n,0)=\mathbb{Z}^n$. We will see that the induced map $f(A)_*$ on $\pi_1(\mathbb{R}^n/\mathbb{Z}^n,0)=\mathbb{Z}^n$ is given by multiplication by A.

It follows from these two facts together with the Mapping Torus-π_1-Proposition 15.7 that $\pi_1(\mathrm{Tor}(\mathbb{R}^n/\mathbb{Z}^n, f(A))) \cong \mathbb{Z}^n\rtimes_A\mathbb{Z}$.

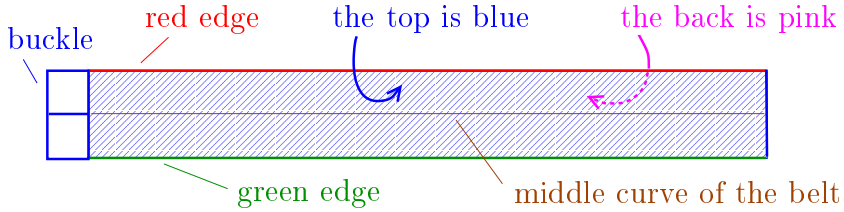
[142] It follows from the Covering-π_1-Monomorphism Proposition 14.11 that $\psi_*\colon \pi_1(X,x_0)\to q_*(\pi_1(X\times\mathbb{R},(x_0,0)))$ is an isomorphism.

[143] Perhaps the most opaque statement is that the two semidirect products match up. But this also follows once one writes down the definitions in detail.

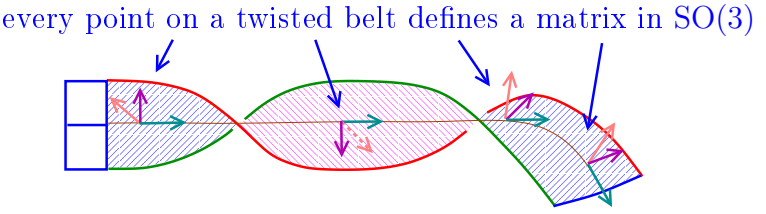
16. Applied topology

16.1. **The belt trick.** In this section we will see that one can visualize the non-trivial element in $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$ using a belt. We proceed as follows.

(1) First of all note that a matrix in SO(3) is precisely given by an ordered orthonormal basis of $\mathbb{R}^3$ that obeys the "right hand rule", i.e. by a choice of an ordered set of three orthogonal vectors v_1, v_2, v_3 of length 1 in $\mathbb{R}^3$ with $\det(v_1\ v_2\ v_3) = 1$.
(2) We equip SO(3) with the base point given by the identity matrix, i.e. by the standard basis $\{e_1, e_2, e_3\}$.
(3) Now suppose we are given a belt with a buckle, a red edge and a green edge, with a blue top side and a pink lower side.



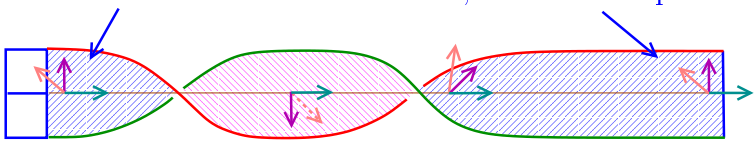
(4) Any point on a twisted belt defines a triple of orthogonal vectors:
  (a) The first vector lies in the plane defined by the belt and points away from the buckle.
  (b) The second vector also lies in the plane defined by the belt and points towards the red edge.
  (c) The third is orthogonal to the belt and it is attached to the top side of the belt.
(5) In the remainder of the discussion we fix the buckle in the xy-plane such that it points in the x-direction, so that to a point at the buckle we assign the identity matrix.
(6) We say that the end of the belt is in *standard position* if it lies in a plane parallel to the plane of the buckle and if it points into the x-direction. Put differently, the end of the belt is in standard position precisely if the matrix assigned to the endpoint is the identity matrix.
(7) When we walk on the middle curve of the belt from the buckle towards the other end of the belt, then we obtain a path $[0,1] \to \mathrm{SO}(3)$. In particular, if the end of the belt is in standard position, then the path defined by the belt is in fact a loop in SO(3) at the base point id.



(8) When we hold the belt tightly without twisting it, then we assign the identity matrix to every point on the belt, i.e. the belt defines the constant loop $t \mapsto \mathrm{id} \in \mathrm{SO}(3)$.

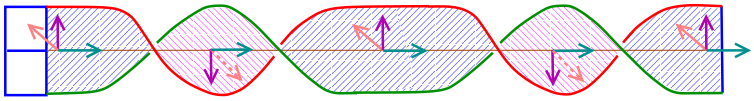
(9) Now we give the belt, that is still fixed at the buckle, a full twist. As in (7) this defines a loop $\gamma\colon [0,1] \to \mathrm{SO}(3)$, in fact it is straightforward to see that this loop is precisely the loop that we considered in the SO(3)–π_1-Proposition 15.4 (2). In particular we have visualized the non-trivial element $[\gamma] \in \pi_1(\mathrm{SO}(3), \mathrm{id}) \in \mathbb{Z}_2$.

both points define the standard basis of $\mathbb{R}^3$, i.e. the base point id $\in$ SO(3)



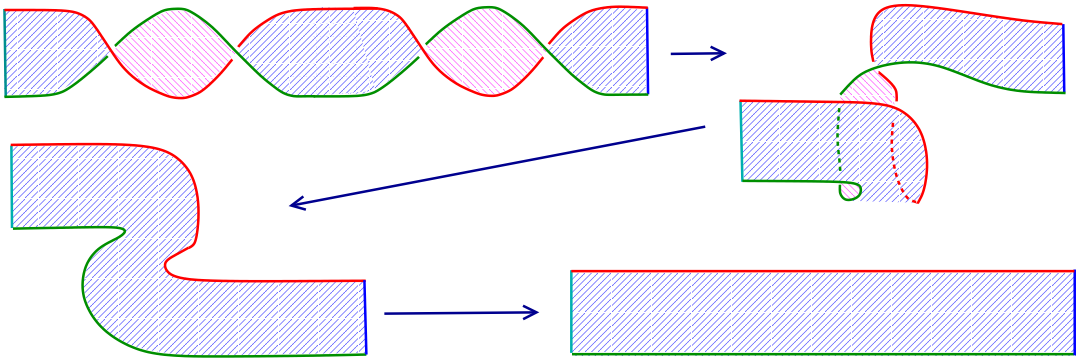
the path $\gamma\colon [0,1] \to \mathrm{SO}(3)$ defined by the matrices along the middle curve is a loop and thus defines an element $[\gamma] \in \pi_1(\mathrm{SO}(3), \mathrm{id})$

(10) The loop $\gamma * \gamma\colon [0,1] \to \mathrm{SO}(3)$ is given by performing another full twist along the belt. It follows from the SO(3)–π_1-Proposition 15.4 that $[\gamma * \gamma] = [\gamma]^2$ is trivial in $\pi_1(\mathrm{SO}(3), \mathrm{id}) \cong \mathbb{Z}_2$, i.e. $\gamma * \gamma$ is null-homotopic.



the double full twist defines the loop $\gamma * \gamma$ which represents $[\gamma]^2 \in \pi_1(\mathrm{SO}(3), \mathrm{id})$

(10) One can see that the loop $\gamma * \gamma$ is null-homotopic as follows: one can untwist the belt while keeping the end in standard position, put differently, we can find a path-homotopy from the loop $\gamma * \gamma$ to the trivial loop. This homotopy is illustrated in the figure below.

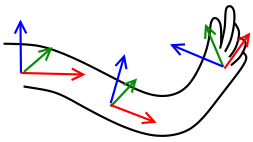


16.2. **The plate trick.** The plate trick is a variation on the belt trick. It describes the same phenomenon as the belt trick. The plate trick has the great advantage that the only tool it requires is the body of a volunteer. The disadvantage is that it is perhaps mathematically slightly less compelling.

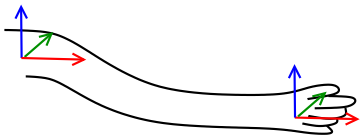
The basic idea is as in the belt trick. Namely we want to describe paths and loops in SO(3). This time we think of a path from the shoulder to ones left hand as a path in SO(3):

(a) the first vector lies on the arm and it points from the shoulder to the hand,
(b) the second vector also lies on the arm and it points from one side to the other,
(c) the third vector points away from the body.

If the fingers of the left hands are parallel to the shoulder, then we see a loop in SO(3).



the path from the shoulder
to the hand defines a path in SO(3)

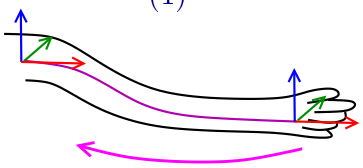


hand parallel to the shoulder
defines a loop in SO(3)

Now you, or your volunteer, does the following:

(1) You stretch your left arm and hands outwards such that the fingers are parallel to the shoulder.
(2) You rotate the left arm counterclockwise. The physiology of most arms will force the forearm to go under the arm.
(3) One continues till one does a 2π turn and the left arm is outstretched again. At this time we are describing a loop β in SO(3) which is non-constant since the arm is twisted. (If it doesn't hurt slightly, then the experiment is done incorrectly).
(4) Now one rotates the left arm still counterclockwise. The usual physiology will force the movement of the forearm to go over the arm.
(5) Again one continues till one has done another 2π turn. Up to a homotopy, keeping the end points fixed, the arm now describes $\beta * \beta$. But the arm turns out to be untwisted (which feels much better). This illustrates that β represents an element of order 2 in $\pi_1(\mathrm{SO}(3))$.

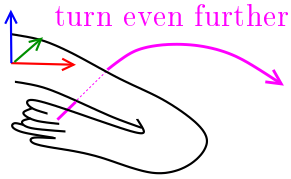
(1)



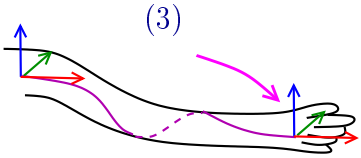
turn arm this direction

(2)

turn even further

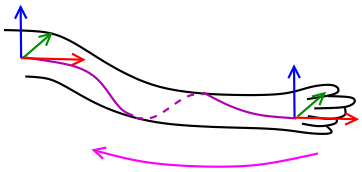


(3)



we obtain a loop β in SO(3),
it is not the standard loop
since the arm is twisted (it hurts!)

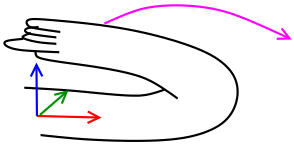
(3)



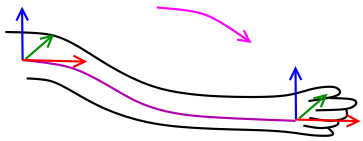
turn arm the same
direction even further

(4)

turn even further



(5)



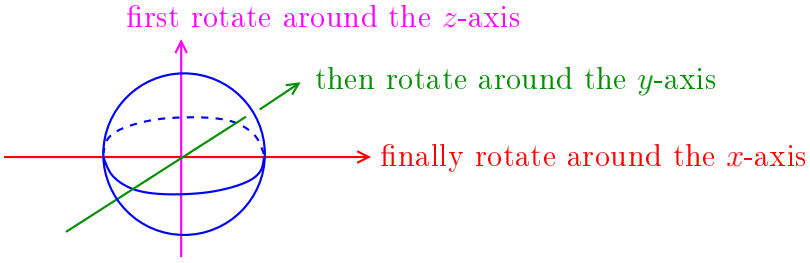
we obtain, up to a homotopy
the loop β^2, since the arm
is untwisted we se that β represents
an element of order 2 in $\pi_1(\mathrm{SO}(3))$

There are many more or less convincing illustrations and explanations of the plate trick. The most charming way of doing might be this one:

`http://ariwatch.com/VS/Algorithms/PlateTrick.htm`

16.3. **Topological robotics.** In this section we want to discuss a "real-life" application of the SO(3)–π_1-Proposition 15.4. Let us assume that we are given a solid ball and we are allowed to perform the following operations:

(1) first we can rotate the ball around the z-axis by an angle $\gamma \in S^1$,
(2) then we can rotate the ball around the y-axis by an angle $\beta \in S^1$, finally
(3) we can rotate the ball around the x-axis by an angle $\alpha \in S^1$.



Using these three operations, in that order, we can rotate the ball into any position. More precisely, we have the following lemma:

Lemma 16.1. The map

$$\begin{array}{rcl} \Phi\colon S^1\times S^1\times S^1 & \to & \mathrm{SO}(3) \\ (\alpha,\beta,\gamma) & \mapsto & \begin{array}{c}\text{rotation by } \alpha\\ \text{around the } x\text{-axis}\end{array} \circ \begin{array}{c}\text{rotation by } \beta\\ \text{around the } y\text{-axis}\end{array} \circ \begin{array}{c}\text{rotation by } \gamma\\ \text{around the } z\text{-axis}\end{array} \end{array}$$

is a surjection.[144]

Sketch of proof. Let $A \in \mathrm{SO}(3)$ be matrix with columns v_1, v_2, v_3. After a rotation around the z-axis we can assume that v_1 lies in the xz-plane. After a rotation around the y-axis we can then arrange that v_1 lies in fact on the x-axis, i.e. we can arrange that $v_1 = e_1$. Since $v_1 = e_1, v_2, v_3$ is an orthonormal basis we see that v_2, v_3 lie in the yz-plane. After a rotation around the x-axis we can finally arrange that $v_2 = e_2$ and $v_3 = e_3$. ∎

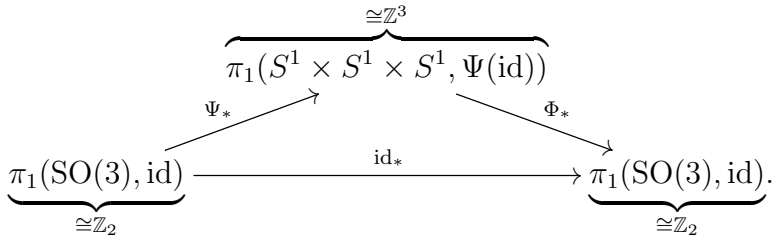
This lemma gives rise to the following practical problem: given a matrix $P \in \mathrm{SO}(3)$, can we continuously find parameters $\alpha(P), \beta(P), \gamma(P) \in S^1$ such that these parameters give back the matrix P?
It turns out that the answer is negative. More precisely, we have the following proposition.

Proposition 16.2. There is no continuous map $\Psi\colon \mathrm{SO}(3) \to S^1\times S^1\times S^1$ such that $\Phi\circ\Psi = \mathrm{id}_{\mathrm{SO}(3)}$.

Remark. This proposition falls into the area of "topological robotics", more information on this example and on topological robotics can be found in [Ghr14, p. 76]. Other aspects of topological robotics are discussed in the monograph [Far08].

[144] The map is also continuous. Why is that?

Proof. Let us suppose there exists a continuous map $\Psi\colon \mathrm{SO}(3) \to S^1 \times S^1 \times S^1$ such that $\Phi \circ \Psi = \mathrm{id}_{\mathrm{SO}(3)}$. Then we get the following commutative diagram



By the SO(3)–π_1-Proposition 15.4 we have that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$ and by the Torus-π_1-Corollary 13.8 we know that $\pi_1(S^1 \times S^1 \times S^1) \cong \mathbb{Z}^3$. Since $\mathbb{Z}_2$ is a finite group and since $\mathbb{Z}^3$ is torsion-free we see that the homomorphism $\Psi_*\colon \pi_1(\mathrm{SO}(3)) \to \pi_1(S^1 \times S^1 \times S^1)$ is the trivial map. But that is not possible since the horizontal map is an isomorphism of a non-trivial group. ■

17. Wedges and topological graphs

In the previous chapters we used coverings and group actions to calculate fundamental groups of many topological spaces, e.g. of the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n \cong (S^1)^n$, of the real projective spaces $\mathbb{R}\mathrm{P}^n$, of the Klein bottle and the matrix group $\mathrm{SO}(3)$. There are still many topological spaces that elude us, for example so far we cannot calculate the fundamental groups of the surfaces of genus $g \geq 2$ or of topological graphs.

The next big goal will be to formulate and prove the Seifert–van Kampen Theorem which will allow us to do many more calculations. In this intermediate chapter we will introduce the wedge of pointed topological spaces, which is the same as the coproduct of pointed topological spaces. Furthermore we will discuss topological graphs and we will relate topological graphs to wedges of circles. Later in the Seifert–van Kampen Theorem we will then learn how to calculate fundamental groups of wedges and of topological graphs.

17.1. **The wedge of pointed topological spaces.** In Section 5.5 we introduced the disjoint union of topological spaces and we saw that the disjoint union is the coproduct in the category of topological spaces. In this section we now consider analogous constructions and results in the context of pointed topological spaces.

Definition. Let $\{(X_i, x_i)\}_{i\in I}$ be a non-empty family of pointed topological spaces.

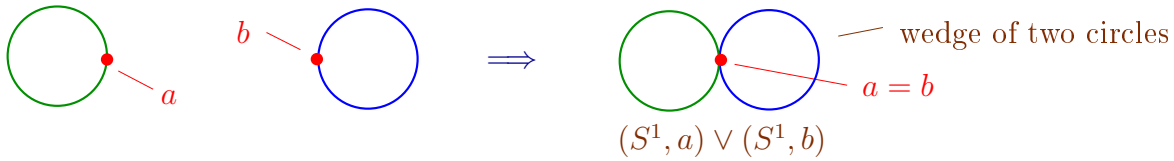
(1) We refer to[145]
$$\bigvee_{i\in I}(X_i, x_i) \;:=\; \Big(\coprod_{i\in I} X_i\Big)\Big/\sim \qquad \text{where } x_i \sim x_j \text{ for all } i,j \in I$$
as the wedge of the pointed topological spaces (X_i, x_i). We refer to the point given by the identifications as the wedge point. If we do not say anything else it will be denoted by $*$. We equip the wedge with this base point to view the wedge as a pointed topological space.

(2) Given $j \in I$ we refer to the map
$$\begin{array}{rccc} \iota_j\colon & X_j & \to & \bigvee_{i\in I}(X_i, x_i) \\ & x & \mapsto & [x] \end{array}$$
as the natural injection.

We complete this definition by defining the wedge of the empty family of pointed topological spaces to be the one-point topological space $\{*\}$.

Example.

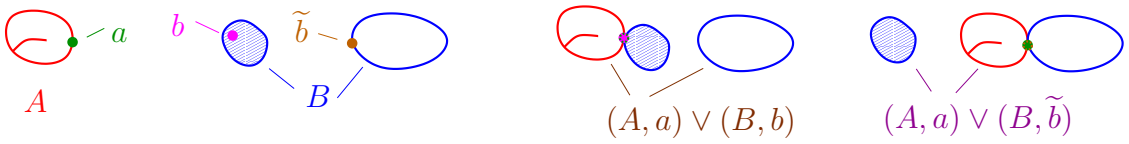
(1) The wedge $(A, a) \vee (B, b)$ of two pointed topological spaces (A, a) and (B, b) should be viewed as the result of gluing A and B along the points a and b.



(2) In general the homeomorphism type of the wedge depends on the choice of base points. This becomes particularly clear if we consider disconnected spaces. For example it is

[145] We use obvious adaptions of the notation for finite families of pointed topological spaces.

pretty clear that the wedges $(A,a)\vee(B,b)$ and $(A,a)\vee(B,\widetilde{b})$ considered in the figure below are not homeomorphic.



The following proposition can be viewed as an analogue of the Topological-Coproduct Proposition 5.10.

Proposition 17.1. (Wedge Proposition) Let $\{(X_i,x_i)\}_{i\in I}$ be a family of pointed topological spaces. We consider the wedge

$$\bigvee_{i\in I}(X_i,x_i) \;:=\; \Big(\bigsqcup_{i\in I}X_i\Big)/\!\sim \qquad \text{where } x_i\sim x_j \text{ for all } i,j\in I.$$

(0) A subset $U\subset\bigvee_{i\in I}(X_i,x_i)$ is open (closed) if and only if for each $i\in I$ the preimage $\iota_i^{-1}(U)$ is an open (closed) subset of X_i.

(1) (a) Let $j\in J$. The natural injection $\iota_j\colon X_j\to\bigvee_{i\in I}(X_i,x_i)$ is continuous and it is an embedding.[146]

(b) If (Z,z_0) is a pointed topological space and if $\{f_i\colon(X_i,x_i)\to(Z,z_0)\}_{i\in I}$ is a family of continuous maps, then the map

$$\begin{aligned}\bigvee_{i\in I}f_i\colon\bigvee_{i\in I}(X_i,x_i) &\to Z\\ [(x,i)] &\mapsto f_i(x)\end{aligned}$$

is continuous and it sends the wedge point to z_0.

(∗) A map $f\colon\bigvee_{i\in I}(X_i,x_i)\to Z$ is continuous if and only if each $f\circ\iota_i\colon X_i\to Z$ is continuous.

(2) The pointed topological space $\Big(\bigvee_{i\in I}(X_i,x_i),*\Big)$ together with the natural injections ι_i is a coproduct of $\{(X_i,x_i)\}_{i\in I}$ in the category $\mathcal{PTop}$ of pointed topological spaces, that we defined on page 174.

Proof. Let $\{(X_i,x_i)\}_{i\in I}$ be a family of pointed topological spaces.

(0) Both statements follow easily from the Topological-Coproduct Proposition 5.10 (0) together with the Topological-Quotient Proposition 6.1 (0).

(1) (a) Let $j\in I$. It follows immediately from the Topological-Quotient Proposition 6.1 (1a) and the Topological-Coproduct Proposition 5.10 (1a) that the natural injection $\iota_j\colon X_j\to X:=\bigvee_{i\in I}(X_i,x_i)$ is continuous. It also pretty clear that this map is injective. It remains to prove the following claim.

[146] In most cases this statement allows us to safely view each X_j as a subspace of the wedge $\bigvee_{i\in I}(X_i,x_i)$.

Claim. The map $\iota_j \colon X_j \to \iota_j(X_j)$ is open.

Proof. Let $U \subset X_j$ be an open subset. First suppose that $x_j \notin U$. Using (0) one can now easily verify that $\iota_j(U)$ is in fact open in X, in particular it is open in $\iota_j(X_j)$. Finally we suppose that $x_j \in U$. We set $V := \iota_j(U) \cup \bigcup_{i \neq j} \iota_i(X_i)$. Using (0) one can again easily verify that V is an open subset of X. Furthermore it follows easily from the definitions that $\iota_j(U) = \iota_j(X_j) \cap V$. Thus we have shown that $\iota_j(U)$ is indeed open in $\iota_j(X_j)$.

(b) This statement follows immediately from the Topological-Quotient Proposition 6.1 (1b) and the Topological-Coproduct Proposition 5.10 (1b).

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) This statement follows easily from (1a) and (1b). We leave it to the reader to fill in the few remaining details. ■

Example.

(1) Let $(X_i, x_i)_{i \in I}$ be a family of pointed topological spaces. Let $J \subset I$. Using the Wedge Proposition 17.1 one can easily show that the natural map $\bigvee_{j \in J} (X_j, x_j) \to \bigvee_{i \in I} (X_i, x_i)$ is an embedding. Furthermore we see that a retraction from $\bigvee_{i \in I} (X_i, x_i)$ to the subspace $\bigvee_{j \in J} (X_j, x_j)$ is given by sending all points in X_i with $i \neq j$ to the wedge point.

(2) Let $G = (V = \{v\}, E, i, t)$ be an abstract graph with a *single* vertex. We consider the bijection

$$f \colon |G| = \Big(V \sqcup \bigsqcup_{e \in E} [0,1]\Big)/\sim \;\to\; \bigvee_{e \in E} (S^1_e, 1)$$
$$[x] \;\mapsto \begin{cases} *, & \text{if } x = v, \\ \exp(2\pi \mathrm{i} t) \in S^1_e, & \text{if } x = (e,t) \text{ with } e \in E,\ t \in [0,1]. \end{cases}$$

It follows from the Abstract Graph-Topology Lemma 7.1 (3) that this map is continuous. Using the Wedge Proposition 17.1, and using the fact that the map $[0,1]/0 \sim 1 \to S^1$ given by $[t] \mapsto \exp(2\pi \mathrm{i} t)$ is a homeomorphism, one can show that the inverse f^{-1} is also continuous. In summary, the above map f is a homeomorphism.

The following lemma plays the role of the Disjoint Union Topology-Properties Lemma 5.11. It is entirely possible that later on we use the lemma below without giving it credit.

Lemma 17.2. (Wedge–Properties Lemma) Let $\{(X_i, x_i)\}_{i \in I}$ be a family of pointed topological spaces.

(1) If I is finite and if each X_i is compact, then $\bigvee_{i \in I} (X_i, x_i)$ is compact.

(2) If each X_i is (path)-connected, then the wedge $\bigvee_{i \in I} (X_i, x_i)$ is (path)-connected.

(3) The wedge $\bigvee_{i \in I} (X_i, x_i)$ is Hausdorff if and only if each X_i is Hausdorff.

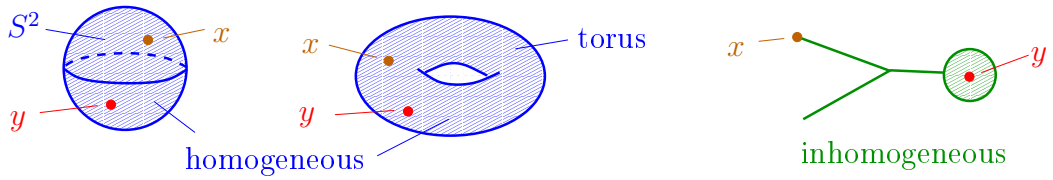
Proof.

(1) This statement follows immediately from the Disjoint Union Topology-Properties Lemma 5.11 (1) together with the Quotient Topology-Properties Proposition 6.2 (1).

(2) This statement follows immediately from the Union Connected Lemma 2.19 together with the Wedge Proposition 17.1 (1a).
(3) This statement can be proved easily using the Wedge Proposition 17.1 (0a). ■

On page 220 we saw that in general the homeomorphism type of the wedge of two topological spaces depends on the choice of the base points. Fortunately we will now see that in many settings the choice of base point is largely irrelevant.

Definition. We say a topological space X is homogeneous if given any two points $x, y \in X$ there exists a homeomorphism $f\colon X \to X$ with $f(x) = y$.



Example.

(1) It is clear that the topological space $\mathbb{R}^n$ is homogeneous. One can also easily show "by hand" that the sphere S^n is homogeneous.
(2) In [Fri23] it is shown that every connected topological manifold with empty boundary is homogeneous.

Lemma 17.3. (Wedging Point–Irrelevance Lemma) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. If each X_i is homogeneous, then different choices of base points still lead to homeomorphic wedges.

Proof. Suppose that for each $i \in I$ we have base points $x_i, y_i \in X_i$. Since each X_i is homogeneous we know that for each $i \in I$ there exists a homeomorphism $f_i\colon X_i \to X_i$ with $f_i(x_i) = y_i$. It follows from the Wedge Proposition 17.1 (1) that the maps

$$\bigvee_{i\in I}(X_i, x_i) \;\to\; \bigvee_{i\in I}(X_i, y_i) \qquad \text{and} \qquad \bigvee_{i\in I}(X_i, y_i) \;\to\; \bigvee_{i\in I}(X_i, x_i)$$

induced by the maps f_i and by the maps f_i^{-1} are continuous. Clearly they are inverses of one another, so they are homeomorphisms. ■

This lemma leads us to the following slightly sloppy definition.

Definition. Let $\{X_i\}_{i\in I}$ be a family of non-empty homogeneous topological spaces (e.g. a family of spheres). For each $i \in I$ we pick a point $x_i \in X_i$ and we write

$$\bigvee_{i\in I} X_i \;:=\; \bigvee_{i\in I}(X_i, x_i).$$

By the Wedging Point–Irrelevance Lemma 17.3 the homeomorphism type of $\bigvee_{i\in I} X_i$ is well-defined.

17.2. **Abstract graphs and wedges of circles.** In this section we have another look at abstract graphs and their topological incarnations. First we recall the following definitions from pages 130 and 133.

Definition. Let $G = (V, E)$ be an abstract graph.

(1) If G is finite, i.e. if V and E are finite, then we refer to

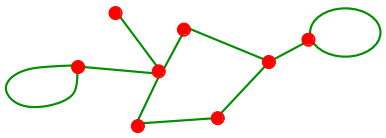
$$\chi(G) \;:=\; \#V \,-\, \#E \;=\; \text{number of vertices} \,-\, \text{number of edges}$$

as the Euler characteristic of G.

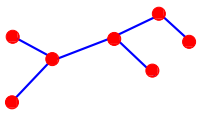
(2) We say G is a tree if G is non-empty, if it does not contain a loop and if given any $v, w \in V$ there exists a unique injective path from v to w. If G is finite, then it follows from the Tree Characterization Proposition 7.7 that G is a tree if and only if G is connected and if $\chi(G) = 1$.

(3) The valence of a vertex v is defined as

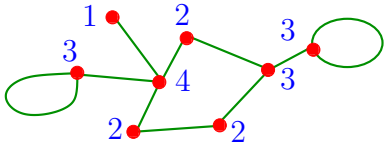
$$\text{valence}(v) \;:=\; \#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\}.$$



Euler characteristic of G equals $\chi(G) = 8 - 10 = -2$



example of a tree



valences of the vertices

Proposition 17.4. (Finite Tree–Contractible Proposition) Let T be an abstract graph. If T is a finite tree, then the following hold:

(1) The topological realization $|T|$ admits a deformation retraction to a vertex, in particular it is contractible.

(2) The fundamental group $\pi_1(|T|)$ is trivial.

Proof. Note that Statement (2) is an immediate consequence of Statement (1) together with the Homotopy-π_1-Proposition 13.4 (3). Thus it suffices to prove Statement (1).

We will prove that any tree admits a deformation retraction to a vertex. We will prove this statement by induction on the number of edges. If a tree has zero edges, then it follows from the connectedness of T that T consists of precisely one vertex. So we are done with this special case.

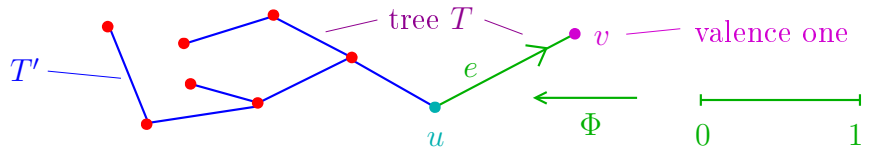
Now suppose that we know that all trees with n edges admit a deformation retraction to a vertex. Let $T = (V, E)$ be a tree with $n + 1$ edges. By the Tree–Valence One Edge Lemma 7.8 we know that T admits a vertex v of valence one. We denote by e the unique edge with v as an endpoint. We set $T' = (V \setminus \{v\}, E \setminus \{e\})$. Note that T' is a non-empty connected abstract subgraph of T. It follows from the Tree–Basics Lemma 7.4 (1) that T' is again a tree.

Claim. $|T'|$ is a deformation retract of $|T|$.

Proof. We denote by u the other vertex of e. Since $u \neq v$ we obtain from the Abstract Graph-Topology Lemma 7.1, see also the discussion on page 126, that there exists a closed embedding $\Phi\colon [0,1] \to |T|$ with $\Phi(0) = u$ and $\Phi(1) = v$. We consider the map

$$\begin{aligned} H\colon |T| \times [0,1] &\to |T| \\ (x,t) &\mapsto \begin{cases} x, & \text{if } x \in |T'|, \\ \Phi(t \cdot y), & \text{if } x = \Phi(y) \text{ for some } y \in [0,1]. \end{cases} \end{aligned}$$

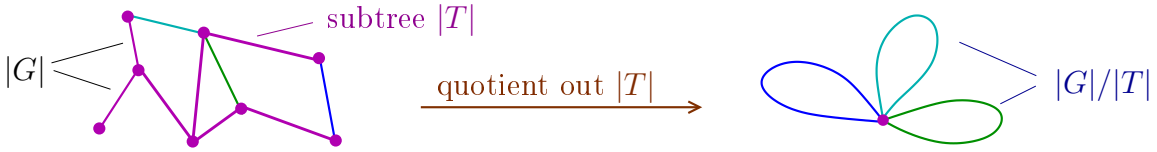
It follows from the Abstract Graph-Topology Lemma 7.1 that $|T'|$ and $|e| = \Phi([0,1])$ are closed subsets of $|T|$. It thus follows from the Homotopy Combination Lemma 8.2 that the map H is continuous. It is now clear that H is a deformation retraction from $|T|$ to $|T'|$. ⊞



We have thus shown that $|T'|$ is a deformation retract of $|T'|$. By our induction hypothesis $|T'|$ admits a deformation retraction to a vertex $v \in T'$. It follows that $|T|$ also admits a deformation retraction to the vertex v. Finally it follows from this fact together with the Point Deformation Retract Lemma 9.7 that T is contractible. ■

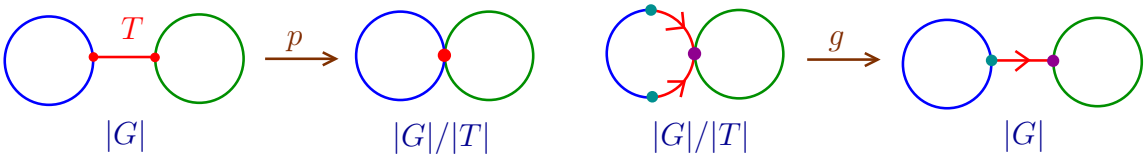
We conclude this discussion of abstract and topological graphs with the following proposition and a corresponding corollary.

Proposition 17.5. (Graph-mod-Tree Proposition) Let G be an abstract graph. For every finite subtree T of G the projection $|G| \to |G|/|T|$ is a homotopy equivalence.



Proof. We outsource the proof to the next section. ■

Example. We consider the abstract graph G and the subtree T that are shown in the figure below. It follows from the Graph-mod-Tree Proposition 17.5 that the projection $p\colon |G| \to |G|/|T|$ is a homotopy equivalence. On page 145 we already saw that the map $g\colon |G|/|T| \to |G|$ that is also sketched in the figure is a homotopy inverse to p. The Graph-mod-Tree Proposition 17.5 can be viewed as a notationally slightly heroic generalization of this example.



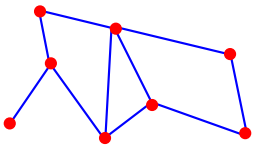
Corollary 17.6. (Graph-Homotopy Type Corollary)

(1) Let $G = (V, E)$ be a finite abstract graph. If G is connected and non-empty, then the topological realization $|G|$ is homotopy equivalent to the wedge of $1 - \chi(G)$ circles.

(2) Let G, G' be two finite abstract connected non-empty graphs. If $\chi(G) = \chi(G')$, then the topological realizations $|G|$ and $|G'|$ are homotopy equivalent.

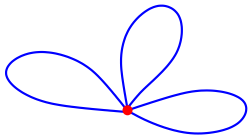
Proof. Let G be a finite abstract graph.

(1) By the Spanning Tree–Existence Proposition 7.6 (1) we know that G admits a spanning tree T. Next note that by the Graph-mod-Tree Proposition 17.5 we know that the map



topological graph of Euler characteristic $8-10=-2$

is homotopy equivalent to



wedge of three circles

$p\colon |G| \to |G|/|T|$ is a homotopy equivalence. By the Abstract Graph–Quotient Lemma 7.9 (2) we know that $|G|/|T|$ is homeomorphic to $|G/T|$. Thus it remains to show that $|G/T|$ is homeomorphic to the wedge of $1-\chi(G)$ circles. We make the following observation:

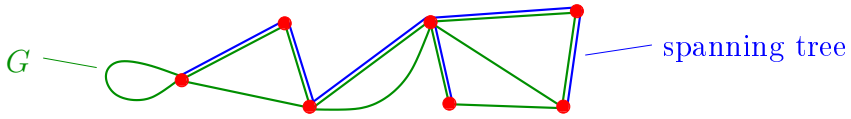
(a) Note that by definition the spanning tree T contains all vertices of G. In other words, the abstract graph G/T contains only a single vertex.

(b) Note that

$$\chi(G/T) \underset{\text{Abstract Graph–Quotient Lemma 7.9 (1)}}{=} \chi(G)-\chi(T)+1 \underset{\text{Tree Characterization Proposition 7.7}}{=} \chi(G)-1+1 = \chi(G).$$

(c) By (a) and (b) we know that G/T has $1-\chi(G)$ edges.

(d) It follows from (a) and (c), together with the discussion on page 221 that $|G/T|$ is indeed homeomorphic to the wedge of $1-\chi(G)$ circles.



(2) The last statement is an immediate consequence of (1) and the Homotopy-Equivalence Basics Lemma 9.5. ■

17.3. **Proof of the Graph-mod-Tree Proposition 17.5.** The sole purpose of this section is to provide the following proof:

Proof of the Graph-mod-Tree Proposition 17.5. Let $G=(V,E,i,t)$ be an abstract graph and let $T=(W,F)$ be a finite subtree of G. We consider the quotient abstract graph G/T given by identifying all vertices in T and by removing all edges in T. We consider the map

$$p\colon \underbrace{|G| = (V \sqcup (E\times[0,1]))/\!\sim}_{\text{where } (e,0)\sim i(e) \text{ and } (e,1)\sim t(e)} \to \underbrace{|G/T| = (V/\!\sim \sqcup((E\setminus F)\times[0,1]))/\!\sim}_{\text{similarly } (e,0)\sim \rho(i(e)) \text{ and } (e,1)\sim \rho(t(e))}$$

$$[x] \mapsto \begin{cases} [*], & \text{if } x\in |T|,\\ [x], & \text{if } x\in V,\\ [(e,t)], & \text{if } x=(e,t) \text{ with } e\in E\setminus F \text{ and } t\in[0,1]. \end{cases}$$

One can easily show that the induced map $p\colon |G|/|T| \to |G/T|$ is a homeomorphism. Note that p fits into the following commutative diagram

$$\begin{array}{ccc} |G| & \xrightarrow{\quad p \quad} & |G/T| \\ & {\scriptstyle x\mapsto[x]}\searrow \quad \nearrow & \\ & |G|/|T|. & \end{array}$$

We need to show that the projection $|G| \to |G|/|T|$ to the left is a homotopy equivalence. As we just mention, the diagonal map to the right is a homeomorphism. Thus it suffices to show that the horizontal map $p\colon |G| \to |G/T|$ is a homotopy equivalence.

Note that by the Finite Tree–Contractible Proposition 17.4 we know that $|T|$ admits a deformation retraction $R\colon |T| \times [0,1] \to |T|$ to some vertex $t_0 \in T$. Put differently, there exists a homotopy $R\colon |T| \times [0,1] \to |T|$ such that $R_1 = \mathrm{id}_{|T|}$ and such that $R_0 = \mathrm{const}_{t_0}$ is the constant map. We extend R somewhat by setting $R(v,t) := v$ for all vertices v that do not lie in $|T|$.

Let f be an edge of G that is not contained in T.

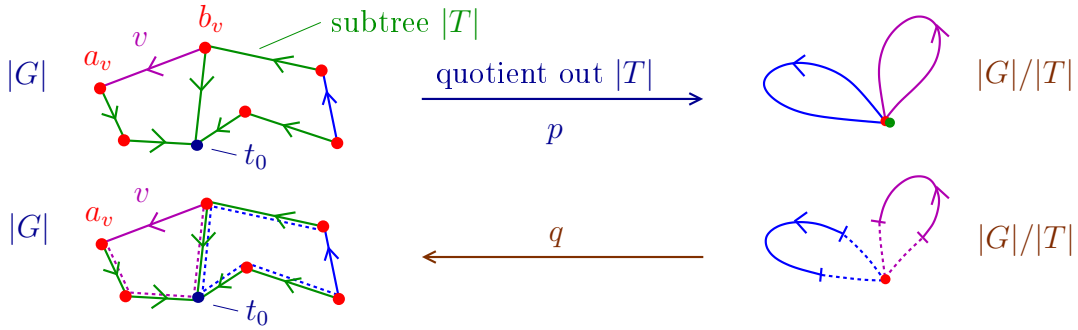
- If $i(f) = t(f)$, then it follows from the discussion on page 126 that the characteristic map of f gives rise to a continuous map $\Phi_f\colon [-2,2] \to |G|$ that induces a homeomorphism $[-2,2]/-2 \sim 2 \to |f|$.
- If $i(f) \neq t(f)$, then the same discussion on page 126 shows that there exists a continuous map $\Phi_f\colon [-2,2] \to |G|$ which restricts to a homeomorphism $[-2,2] \to |f|$.[147]

We write $a_f = \Phi_f(-2)$ and $b_f = \Phi_f(2)$.

Next we consider the quotient graph G/T. Let $p\colon |G| \to |G/T|$ be the projection. For every edge f of G/T, i.e. for every $f \in E \setminus F$, we denote by $\Psi_f\colon [-2,2] \to |G/T|$ the map $p \circ \Phi_f$. We consider the map

$$q\colon |G/T| \to |G|$$
$$x \mapsto \begin{cases} R(a_f, -s-1), & \text{if } x = \Psi_f(s) \text{ for some } s \in [-2,-1], \\ \Psi_f(2 \cdot s), & \text{if } x = \Psi_f(s) \text{ for some } s \in [-1,1], \\ R(b_f, s-1), & \text{if } x = \Psi_f(s) \text{ for some } s \in [1,2]. \end{cases}$$

Using the Abstract Graph-Topology Lemma 7.1 one can show that q is continuous.



It remains to prove the following claim.

Claim. The maps p and q are homotopy inverses of one another.

Proof. The proof of the claim is fairly elementary, just notationally a little messy. First we consider the following map

$$H\colon |G/T| \times [0,1] \to |G/T|$$
$$(x,t) \mapsto \begin{cases} *, & \text{if } x = \Psi_f(s) \text{ for some } s \in [-2,-1-t], \\ \Psi_f(\frac{2s}{1+t}), & \text{if } x = \Psi_f(s) \text{ for some } s \in [-1-t,1+t], \\ *, & \text{if } x = \Psi_f(s) \text{ for some } s \in [1+t,2]. \end{cases}$$

[147] In other words, the map Φ_f is just a rescaling of the characteristic maps of f.

Using the Abstract Graph-Topology Lemma 7.1 and the Homotopy Combination Lemma 8.2 one can easily show that H is continuous. It follows immediately from the definitions that $H_0 = p \circ q$ and $H_1 = \mathrm{id}_{|G/T|}$. Next we consider the following map

$$\begin{aligned} \widetilde{H} \colon |G| \times [0,1] &\to |G| \\ (x,t) &\mapsto \begin{cases} R(x,t), & \text{if } x \in T, \\ R(a_f, s+2+t), & \text{if } x = \Phi_f(s) \text{ for some } s \in [-2,-1-t], \\ \Phi_f(\frac{2s}{1+t}), & \text{if } x = \Phi_f(s) \text{ for some } s \in [-1-t,1+t], \\ R(b_f, -s+2+t), & \text{if } x = \Phi_f(s) \text{ for some } s \in [1+t,2]. \end{cases} \end{aligned}$$

Again, using the Abstract Graph-Topology Lemma 7.1 and the Homotopy Combination Lemma 8.2 one can show that $\widetilde{H}$ is continuous. Furthermore it follows again immediately from the definitions that $\widetilde{H}_0 = q \circ p$ and $\widetilde{H}_1 = \mathrm{id}_{|G|}$. ■

17.4. **Outlook.** Summarizing we have now finally managed to calculate the fundamental groups of the following topological spaces:

(1) the circle S^1 and all spheres S^n with $n \geq 2$,
(2) the n-dimensional torus $(S^1)^n \cong \mathbb{R}^n/\mathbb{Z}^n$,
(3) the Klein bottle,
(4) the real projective spaces $\mathbb{R}\mathrm{P}^n$,
(5) the lens spaces $L(p,q)$,
(6) trees,
(7) products of the above.

Nonetheless, the list of topological spaces for which we cannot yet calculate the fundamental group is still pretty long, it includes

(1) topological graphs,
(2) the complement of more than one point in $\mathbb{C}$, e.g. $\mathbb{C} \setminus \{0,1\}$,
(3) surfaces of genus ≥ 2,
(4) knot complements,
(5) wedges of topological spaces.

In the following two chapters we will first build up our knowledge of group theory. Then we will state and prove the Seifert-van Kampen Theorem in various flavors which will help us with calculating the open fundamental groups.

18. Basics of group theory I: Abelian groups

In the Product-π_1-Proposition 13.7 we showed that for any two pointed topological spaces (X, x_0) and (Y, y_0) we have an isomorphism $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$. In more formal terms, we showed the following:

(∗) The fundamental group of the product of two pointed topological spaces is isomorphic to the product of the fundamental groups of the two pointed topological spaces.

In the Wedge Proposition 17.1 we just determined the coproduct of two pointed topological spaces. It is natural to ask, whether in (∗) we can replace product by coproduct.

To answer this question we first need to understand the coproduct in the category of groups. Also, more generally it is worthwhile to build up our understanding of groups.

Since general groups are quite difficult we will first build up our understanding of *abelian* groups. More precisely, in this chapter we study the coproduct in the category of abelian groups, we introduce free abelian groups, we study the rank of (free) abelian groups and we will state and prove the classification of finitely generated abelian groups.

In the next chapter we will then see to what degrees these concepts and results on abelian groups generalize to the context of (potentially non-abelian) groups.

18.1. The direct sum of abelian groups.

We start out by recalling the following definitions from the Product-of-Sets-and-Groups Lemma 4.2 and from page 84.

Definition. Let $\{X_s\}_{s\in S}$ be a family of abelian groups.

(1) We define the product of the X_s to be the group[148]

$$\prod_{s\in S} X_s := \left\{ f\colon S \to \bigcup_{s\in S} X_s \;\middle|\; \text{for every } s \in S \text{ we have } f(s) \in X_s \right\},$$

where the group structure is given by the obvious "component wise multiplication".

(2) We define the direct sum of the X_s to be the group

$$\bigoplus_{s\in S} X_s := \left\{ (g_s)_{s\in S} \in \prod_{s\in S} X_s \;\middle|\; \text{all but finitely many } g_s \text{ are trivial} \right\}.$$

(3) Given $t \in S$ we refer to

$$\begin{array}{rccl} \iota_t\colon X_t & \to & \displaystyle\bigoplus_{s\in S} X_s & \\ x & \mapsto & \left(\begin{array}{lcl} S & \to & \displaystyle\bigcup_{s\in S} X_s \\ s & \mapsto & \begin{cases} x, & \text{if } s = t, \\ 0_{X_t}, & \text{if } s \neq t. \end{cases} \end{array} \right) & \end{array}$$

as the natural homomorphism.

By the Product-of-Sets-and-Groups Lemma 4.2 we know that $\prod_{s\in S} X_s$ is the product of the X_s in the category $\mathcal{AbGroup}$ of abelian groups and by the Coproduct-of-Sets-and-Abelian Groups Lemma 4.4 we know that $\bigoplus_{s\in S} X_s$ is the coproduct of the X_s in the category $\mathcal{AbGroup}$ of abelian groups. For the reader's convenience we recall this second statement in greater detail:

[148] Note that if $\{X_s\}_{s\in S}$ is the empty family of abelian groups, then there is precisely one such map, thus the product and the direct sum are the trivial group.

Lemma 18.1. (Coproduct-of-Abelian Groups Lemma) Let $\{X_s\}_{s\in S}$ be a family of abelian groups.

(1) Let Z be an abelian group and let $\{f_s\colon X_s \to Z\}_{s\in S}$ be a family of homomorphisms. The map

$$\begin{array}{rcl} \bigoplus_{s\in S} f_s\colon \bigoplus_{s\in S} X_s & \to & Z \\ (g_s)_{s\in S} & \mapsto & \sum_{\substack{s\in S \text{ with} \\ g_s \neq e}} f_s(g_s) \end{array}$$

is the unique homomorphism[149] such that for each $s \in S$ we have $f_s = \Phi \circ \iota_s$.

(2) The coproduct in the category *AbGroup* of *abelian* groups of $\{X_s\}_{s\in S}$ is given by $\bigoplus_{s\in S} X_s$ together with the above natural homomorphisms $\iota_t\colon X_t \to \bigoplus_{s\in S} X_s$.

Proof. This statement is basically the content of the Coproduct-of-Sets-and-Abelian Groups Lemma 4.4 (2). But of course one can also prove it easily by hand. ■

Convention.

(1) Let $\{X_s\}_{s\in S}$ be a family of abelian groups. For $t \in S$ we use the natural injection $\iota_s\colon X_t \to \bigoplus_{s\in S} X_s$ from the Coproduct-of-Abelian Groups Lemma 18.1 to view X_t as a subgroup of the direct sum $\bigoplus_{s\in S} X_s$.

(2) Let G be an abelian group and let $\{X_s\}_{s\in S}$ be a family of subgroups. The inclusions $X_s \to G$ define a natural homomorphism

$$\bigoplus_{s\in S} X_s \to G.$$

If this natural homomorphism is a monomorphism, then we refer to the image also as the direct sum of the X_s.[150]

Example. Let G be an abelian group and let A and B be two subgroups. If $A \cap B = \{0\}$ then the natural homomorphism $A \oplus B \to G$ is easily seen to be a monomorphism. Thus we can view the direct sum $A \oplus B$ as a subgroup of G.

Theorem 18.2. (Chinese Remainder Theorem) If $r, s \in \mathbb{N}$ are coprime, then the map

$$\begin{array}{rcl} \Phi\colon \mathbb{Z}_{r\cdot s} & \to & \mathbb{Z}_r \times \mathbb{Z}_s = \mathbb{Z}_r \oplus \mathbb{Z}_s \\ [k] & \mapsto & ([k],[k]) \end{array}$$

is an isomorphism.

Proof. Since $r, s \in \mathbb{N}$ are coprime there exist $a, b \in \mathbb{Z}$ with $a \cdot r + b \cdot s = 1$. We consider the map

$$\begin{array}{rcl} \Psi\colon \mathbb{Z}_r \times \mathbb{Z}_s & \to & \mathbb{Z}_{r\cdot s} \\ ([e]_{\mathbb{Z}_r}, [f]_{\mathbb{Z}_s}) & \mapsto & [e\cdot b\cdot s + f\cdot a\cdot r]_{\mathbb{Z}_{rs}}. \end{array}$$

[149] Note that in the verification of that assertion that Φ is a homomorphism it is essential that Z is abelian.

[150] Sometimes in the literature this image is called the *internal direct sum*. In practice it becomes a mess to distinguish in the notation between the "direct sum" and the "internal direct sum", thus we will not try to distinguish the two concepts.

One can easily verify that this map is well-defined. We claim that it is an inverse to Ψ. Indeed, we have

$$\Psi(\Phi([k]_{\mathbb{Z}_{rs}})) = \Psi([k]_{\mathbb{Z}_r},[k]_{\mathbb{Z}_s}) = [k\cdot b\cdot s + k\cdot a\cdot r]_{\mathbb{Z}_{rs}} = [k\cdot \underbrace{(b\cdot s + a\cdot r)}_{=1}]_{\mathbb{Z}_{rs}} = [k]_{\mathbb{Z}_{rs}}$$

and we have

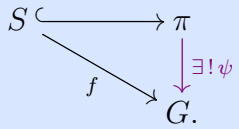
$$\begin{aligned}\Phi(\Psi([e]_{\mathbb{Z}_r},[f]_{\mathbb{Z}_r})) &= \Phi([e\cdot b\cdot s + f\cdot a\cdot r]_{\mathbb{Z}_{rs}})\\ &= ([e\cdot b\cdot s + f\cdot a\cdot r]_{\mathbb{Z}_r},[e\cdot b\cdot s + f\cdot a\cdot r]_{\mathbb{Z}_s})\\ &= ([e\cdot \underbrace{b\cdot s}_{=1-a\cdot r}]_{\mathbb{Z}_r},[f\cdot \underbrace{a\cdot r}_{=1-b\cdot s}]_{\mathbb{Z}_s}) = ([e]_{\mathbb{Z}_r},[f]_{\mathbb{Z}_s}).\end{aligned}$$

■

18.2. **Free abelian groups.** We start out with the following definition, which in principle can be modified to many different settings.

Definition.

(1) We say that a group π is a free abelian group on a subset $S\subset\pi$, if the following universal property holds: for any *map* $f\colon S\to G$ of the set S to an abelian group G there exists a unique *homomorphism* $\psi\colon\pi\to G$ that makes the following diagram commute



We call S a basis of the free abelian group π.

(2) We say that a group π is free abelian if it is a free abelian group on some subset S.

As always, it is easy to prove a suitable uniqueness of an object that is described by a universal property:

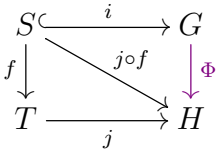
Lemma 18.3. (Free Abelian Group–Uniqueness Lemma) Let G be a free abelian group on a subset $S\subset G$ and let H be a free abelian group on a subset $T\subset H$. Given any bijection $f\colon S\to T$ there exists a unique isomorphism $\psi\colon G\to H$ such that $\psi(s)=f(s)$ for all $s\in S$.

Example.

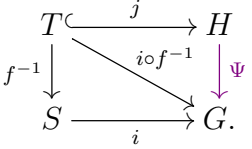
(1) Let $m\in\mathbb{N}_0$. One can easily verify that $\mathbb{Z}^m$ is a free abelian group on the subset $e_1,\dots,e_m$ given by the m standard basis vectors of $\mathbb{Z}^m$. It follows from the Free Abelian Group–Uniqueness Lemma 18.3 that if G is a free abelian group on a finite set S, then G is isomorphic to $\mathbb{Z}^{\#S}$.

(2) Let π be a free abelian group with basis S. If $\Phi\colon\pi\to\pi$ is a homomorphism such that $\Phi(s)=s$ for all $s\in S$, then it follows immediately from the uniqueness statement of the above definition, that $\Phi=\mathrm{id}$.

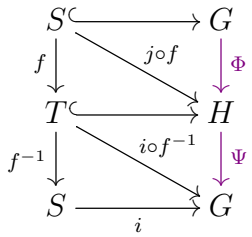
Proof of Lemma 18.3. We denote by $i\colon S\to G$ and $j\colon T\to H$ the inclusion maps. Furthermore we consider the maps $f\colon S\to T$ and $f^{-1}\colon T\to S$. Since G is a free abelian group on the subset $S\subset G$ and since H is a free abelian group on the subset $T\subset H$ we see that there exist homomorphisms $\Phi\colon G\to H$ and $\Psi\colon H\to G$ that make the following diagrams commute:



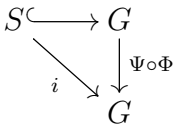
and



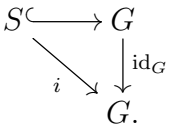
In particular we get the commutative diagram



which simplifies to



and of course we also have the commutative diagram



It follows from the uniqueness part of the universal property applied to the two diagrams on the right that $\Psi \circ \Phi = \mathrm{id}_G$. The same way one shows that $\Phi \circ \Psi = \mathrm{id}_H$. In particular Φ and Ψ are isomorphisms. ■

To prove existence of free abelian groups it is useful to specialize the definition of direct sums from the previous section to the case that $X_s = \mathbb{Z}$ for all $s \in S$. This leads us to the following definition, which will stay with us throughout these notes.

Definition. Let S be a set.

(1) We define

$$\mathbb{Z}^{(S)} := \bigoplus_{s \in S} \mathbb{Z} = \{\text{all maps } S \to \mathbb{Z} \text{ that are zero for all but finitely many } s \in S\}.$$

(2) Let $s \in S$. The map

$$\begin{array}{rcl} S & \to & \mathbb{Z} \\ t & \mapsto & \begin{cases} 1, & \text{if } t = s, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

lies in $\mathbb{Z}^{(S)}$ and it is also called s. Thus we can view S as a subset of $\mathbb{Z}^{(S)}$.

(3) Given elements $s_1, \dots, s_k \in S$ and $n_1, \dots, n_k \in \mathbb{Z}$ we obtain the corresponding element $n_1 \cdot s_1 + \cdots + n_k \cdot s_k$ of $\mathbb{Z}^{(S)}$. Every element of $\mathbb{Z}^{(S)}$ can be written in a unique way as such a linear combination of finitely many elements of S. Indeed, for $f \in \mathbb{Z}^{(S)}$ we can uniquely write[151]

$$f = \sum_{s \in S \text{ with } f(s) \neq 0} f(s) \cdot s.$$

The following lemma will not really be a surprise to the alert reader.

Lemma 18.4. (Free Abelian Group–Existence Lemma) Let S be a set. The group $\mathbb{Z}^{(S)}$ is a free abelian group on the subset $S \subset \mathbb{Z}^{(S)}$.

The Free Abelian Group–Existence Lemma 18.4 suggests the following definition.

Definition. Let S be a set. We refer to $\mathbb{Z}^{(S)}$ as *the* free abelian group generated by S.

Proof of Lemma 18.4. Let $f\colon S \to G$ be a map to an abelian group. We consider the map[152]

[151] It is easy to see that both sides define the same map $S \to \mathbb{Z}$, so they are the same element of $\mathbb{Z}^{(S)}$.

[152] Here we assume that G is written as an additive group. If G is a multiplicative group we would need to replace $n_i \cdot \phi(s_i)$ by $\phi(s_i)^{n_i}$ and we would need to replace the sum by a product.

$$\begin{array}{rcl} \psi\colon \mathbb{Z}^{(S)} & \to & G \\ \underbrace{\sum_{i=1}^{k} n_i \cdot s_i}_{\substack{\text{we saw above that} \\ \text{every element of } \mathbb{Z}^{(S)} \text{ is} \\ \text{uniquely of that form}}} & \mapsto & \sum_{i=1}^{k} n_i \cdot f(s_i). \end{array}$$

It is straightforward to verify that ψ is in fact a homomorphism, that ψ has the desired property and that ψ is the unique such homomorphism. ■

The following lemma gives a typical application of free abelian groups.

Lemma 18.5. (Free Abelian Group–Splitting Lemma) Let $\varphi\colon G \to A$ be an epimorphism between abelian groups.

(1) If A is a free abelian group, then there exists a right inverse $\sigma\colon A \to G$ of φ, this means that there exists a homomorphism $\sigma\colon A \to G$ with $\varphi \circ \sigma = \mathrm{id}_A$.[153]

(2) If $\sigma\colon A \to G$ is a right inverse of φ, then $G = \ker(\varphi) \oplus \sigma(A)$.[154]

Proof. Let $\varphi\colon G \to A$ be an epimorphism between abelian groups.

(1) We assume that A is a free abelian group. We pick a basis B for the free abelian group A. Since $\varphi\colon G \to A$ is an epimorphism we can choose for each $b \in B$ an element $\widetilde{b} \in G$ with $\varphi(\widetilde{b}) = b$.

Since A is a free abelian group on B there exists a unique homomorphism $\sigma\colon A \to G$ with $\sigma(b) = \widetilde{b}$ for all $b \in B$. For each $b \in \widetilde{B}$ we have $(\varphi \circ \sigma)(b) = \varphi(\widetilde{b}) = b$. It follows from the discussion on page 230 that $\varphi \circ \sigma = \mathrm{id}_A$.

(2) Let $\sigma\colon A \to G$ be a right inverse of φ. We start out with the following claim:

Claim. We have $\ker(\varphi) \cap \sigma(A) = \{0\}$.

Proof. If $g = \sigma(a) \in \ker(\varphi) \cap \sigma(A)$, then

$$g = \sigma(a) = (\sigma \circ \mathrm{id}_A)(a) \underset{\substack{\uparrow \\ \text{since } \varphi\circ\sigma = \mathrm{id}_A}}{=} \sigma(\varphi(\sigma(a))) \underset{\substack{\uparrow \\ \text{since } g = \sigma(a)}}{=} \sigma(\varphi(g)) \underset{\substack{\uparrow \\ \text{since } g \in \ker(\varphi)}}{=} \sigma(0) = 0.$$ ⊞

As we discussed on page 229, it follows from the above claim that the natural homomorphism

$$\begin{array}{rcl} \Theta\colon \ker(\varphi) \oplus \sigma(A) & \to & G \\ g \oplus h & \mapsto & g + h \end{array}$$

is a monomorphism. It remains to show that Θ is an epimorphism. Let $g \in G$. Note that

$$\varphi(g - \sigma(\varphi(g))) = \varphi(g) - \varphi(\sigma(\varphi(g))) \underset{\substack{\uparrow \\ \text{since } \varphi\sigma = \mathrm{id}_A}}{=} \varphi(g) - \varphi(g) = 0.$$

This shows that $g - \sigma(\varphi(g)) \in \ker(\varphi)$. It follows that $g \in \mathrm{im}(\sigma) \oplus \ker(\varphi)$. ■

Theorem 18.6. (Basis Cardinality Theorem) Let F be an abelian group. If F is free on two subsets S and T, i.e. if S and T are bases for F, then $\#S = \#T$.

[153] Note that any such right inverse $\sigma\colon A \to G$ is in particular a monomorphism.

[154] Here we use the convention from page 229 to view the direct sum $\ker(\varphi) \oplus \sigma(A)$ as a subgroup of G.

Proof. The proof of this statement is basically the same as the proof that in a vector space any two bases have the same cardinality. This proof is given in detail in [Loe14, Chapter 16.7]. ■

The Basis Cardinality Theorem 18.6 allows us to make the following definition.

Definition. If F is a free abelian group on a subset S, then we define[155] [156]

$$\operatorname{rank}(F) := \#S \quad \in \mathbb{N}_0 \cup \{\infty\}.$$

Example.

(1) The rank of the free abelian group $\mathbb{Z}^m$ is m.
(2) Let G be a free abelian group on a subset S and let H be a free abelian group on a subset T. It follows easily from the definitions and the Coproduct-of-Abelian Groups Lemma 18.1, that $G \oplus H$ is a free abelian group of rank $\operatorname{rank}(G) + \operatorname{rank}(H) \in \mathbb{N}_0 \cup \{\infty\}$.

We conclude this section with the following non-trivial theorem.

Theorem 18.7. (Free Abelian Subgroup Theorem) Let F be a free abelian group. Every subgroup H of F is a free abelian subgroup with $\operatorname{rank}(H) \leq \operatorname{rank}(F)$.

Proof. We only provide the proof in the special case that the rank of F is finite. The general case is proved in [Fri23]. Thus let F be an abelian group of rank n and let H be a subgroup of F. We need to show that H is free of rank $\leq n$.

We prove the statement by induction on n. If $n = 1$, then $F \cong \mathbb{Z}$. Since $\mathbb{Z}$ is a PID (principal ideal domain) we know that every subgroup of $\mathbb{Z}$ is of the form $k \cdot \mathbb{Z}$ for some $k \in \mathbb{N}_0$. Thus we see that every subgroup is a free abelian group of rank ≤ 1.

Now we suppose that the statement holds for all free abelian groups of rank $< n$ with $n \in \mathbb{N}_{\geq 2}$. Let F be a free abelian group of rank n and let $H \subset F$ be a subgroup. Let $t_1, \ldots, t_n$ be a basis for F. Let $F_n \subset F$ be the free abelian group on $t_1, \ldots, t_{n-1}$. We set $H_n := H \cap F_n$. By induction H_n is a free abelian group of rank $\leq n - 1$. It remains to prove the following claim.

Claim. Either $H = H_n$ or there exists an $x \in H \setminus H_n$ with[157] $H = H_n \oplus \mathbb{Z} \cdot x$.

Proof. We consider the following sequence of homomorphisms:

$$\begin{array}{ccccccc} H/H_n & = & H/(H \cap F_n) & \overset{\cong}{\to} & (H + F_n)/F_n & \subset & F/F_n & \overset{\cong}{\leftarrow} & \mathbb{Z}. \\ & & [h] & \mapsto & [h] & & [m \cdot t_n] & \leftarrow\!\shortmid & m. \end{array}$$

If the resulting homomorphism $H/H_n \to \mathbb{Z}$ is trivial, then $H = H_n$, and we are done. Now suppose that the homomorphism $\varphi\colon H/H_n \to \mathbb{Z}$ is non-trivial. Then there exists an $r \in \mathbb{N}$ such that the map $\varphi\colon H/H_n \to \mathbb{Z} \cdot r$ is an epimorphism. It follows from the Free Abelian Group–Splitting Lemma 18.5, applied to the free abelian group $\mathbb{Z} \cdot r$, that $H = H_n \oplus \mathbb{Z} \cdot x$ for some $x \in H \setminus H_n$. ■

[155] Here we denote by $\#S$ the cardinality of the set S, as defined on page 25.
[156] For simplicity we only view the cardinalities as elements of $\mathbb{N}_0 \cup \{\infty\}$. In other words, for the rank of an abelian group we do not distinguish between the different infinite cardinalities. This simplification is good enough for our purpose and it prevents us from having to deal with thorny set theoretic issues.
[157] It follows from the above example, that $\operatorname{rank}(H_n \oplus \mathbb{Z} \cdot x) = \operatorname{rank}(H_n) + \operatorname{rank}(\mathbb{Z} \cdot x) = n - 1 + 1 = n$.

18.3. **Finitely generated abelian groups.** In this section we will introduce the notion of a "finitely generated abelian group" and we will see that these are among the rare classes of mathematical objects which are completely classified.

Definition.

(1) Given an abelian group G we say that $S \subset G$ is a generating set if the natural homomorphism $\Xi_S \colon \mathbb{Z}^{(S)} \to G$ is an epimorphism.
(2) We call an abelian group finitely generated if it admits a finite generating set.

Lemma 18.8. (Finite Generation Lemma) Let $\varphi \colon A \to B$ be a homomorphism of abelian groups.

(1) If φ is an epimorphism and if A is finitely generated, then so is B.
(2) If φ is a monomorphism and if B is finitely generated, then so is A.

Proof.

(1) Let $\varphi \colon A \to B$ be an epimorphism of abelian groups. If S is a finite generating set for A, then clearly the finite set $\varphi(S)$ is a generating set for B.
(2) We only need to show that if B is a finitely generated group, then so is every subgroup A of B. Let T be a finite generating set for B. We denote by $\Xi_T \colon \mathbb{Z}^{(T)} \to B$ the natural epimorphism. Note that $\Xi_T^{-1}(A)$ is a subgroup of $\mathbb{Z}^{(T)}$. We can summarize the maps in the following commutative diagram:

$$\begin{array}{ccc} \Xi_T^{-1}(A) & \hookrightarrow & \mathbb{Z}^{(T)} \\ {\scriptstyle \Xi_T}\downarrow & & \downarrow{\scriptstyle \Xi_T} \\ A & \hookrightarrow & B. \end{array}$$

By the Free Abelian Subgroup Theorem 18.7 we know that $\Xi_T^{-1}(A)$ is a free abelian group of rank $\leq \#T$. Let S be a basis for $\Xi_T^{-1}(A)$. It follows immediately from the definitions that $\Xi_T(S)$ is a generating set for A. Since $\#\Xi_T(S) \leq \#S \leq \#T$ we see that A is finitely generated. ∎

Lemma 18.9. (Matrix Quotient Lemma)

(1) Let H be an abelian group with finite generating set $S = \{s_1, \ldots, s_m\}$. There exists an $n \in \mathbb{N}_0$ and an $m \times n$–matrix A over $\mathbb{Z}$ such that the map

$$\begin{array}{rcl} \mathbb{Z}^m / A \cdot \mathbb{Z}^n & \to & H \\ \left[\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \right] & \mapsto & \sum_{i=1}^{m} v_i \cdot s_i \end{array}$$

is a well-defined isomorphism.
(2) If $A \in \mathrm{M}(n \times n, \mathbb{Z})$ is a diagonal matrix with entries $d_1, \ldots, d_n \in \mathbb{Z}$, then

$$\mathbb{Z}^n / A \cdot \mathbb{Z}^n \cong \bigoplus_{i=1}^{n} \mathbb{Z}_{d_i}.$$

(3) Let A be an $m \times n$–matrix over $\mathbb{Z}$. For matrices $P \in \mathrm{GL}(m, \mathbb{Z})$ and $Q \in \mathrm{GL}(n, \mathbb{Z})$ the map

$$\begin{array}{ccc} \mathbb{Z}^m/A\cdot\mathbb{Z}^n & \cong & \mathbb{Z}^m/PAQ\cdot\mathbb{Z}^n \\ [v] & \mapsto & [P\cdot v] \end{array}$$

is well-defined and it is an isomorphism.

Proof.

(1) Let H be an abelian group with finite generating set $S=\{s_1,\dots,s_m\}$. By definition the natural homomorphism $\Xi_S\colon \mathbb{Z}^{(S)}\to H$ is an epimorphism. Let $\Theta\colon \mathbb{Z}^m\to\mathbb{Z}^{(S)}$ be the natural isomorphism given by $\Theta(e_i)=s_i$. Note that $\Xi_S\circ\Theta\colon \mathbb{Z}^m\to H$ is also an epimorphism. By the Free Abelian Subgroup Theorem 18.7 we know that $\ker(\Xi_s)\subset\mathbb{Z}^m$ is a finitely generated free abelian subgroup. Let $a_1,\dots,a_n\in\mathbb{Z}^m$ be a basis for this free abelian group. We set $A:=(a_1\ \dots\ a_n)\in \mathrm{M}(m\times n)$. One can easily verify that this matrix has the desired properties.

(2) This statement follows easily from the definitions.

(3) First note that for $Q\in\mathrm{GL}(n,\mathbb{Z})$ we have $Q\cdot\mathbb{Z}^n=\mathbb{Z}^n$. It follows that the given map is well-defined. An inverse is given by $[w]\mapsto[P^{-1}\cdot w]$. This shows that the given map is an isomorphism. ■

Theorem 18.10. (Smith Normal Form Theorem) Let A be an $m\times n$–matrix over $\mathbb{Z}$. There exist two matrices $P\in\mathrm{GL}(m,\mathbb{Z})$ and $Q\in\mathrm{GL}(n,\mathbb{Z})$ and $d_1,\dots,d_t\in\mathbb{N}$ such that[158]

$$P\cdot A\cdot Q \;=\; \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & d_t & 0 \\ 0 & 0 & 0 & 0_{(m-t)\times(n-t)} \end{pmatrix} \;\in\; \mathrm{M}(m\times n,\mathbb{Z}).$$

The matrix on the right is sometimes called a Smith normal form of A.

Proof. We postpone the proof to the next section. Alternatively we refer to [New72, Chapter II.15] and [Nor12, Chapter 1] for a proof. ■

Now we can state and prove one of the best-loved theorems in mathematics.

Theorem 18.11. (Abelian Group-Classification Theorem) Let H be a finitely generated abelian group. There exist prime powers[159] $q_1,\dots,q_k$ such that

$$H \;\cong\; \mathbb{Z}^r\oplus\bigoplus_{i=1}^{k}\mathbb{Z}_{q_i}.$$

Furthermore r is uniquely determined and the prime powers $q_1,\dots,q_k$ are uniquely determined up to permutation.

Sketch of proof. [160] Let H be a finitely generated abelian group. We first prove the existence of the promised isomorphism. Note that it follows from the Matrix Quotient Lemma 18.9 (1), the Smith Normal Form Theorem 18.10 and the Matrix Quotient Lemma 18.9 (2) and (3) that there exist $r\in\mathbb{N}$ and $d_1,\dots,d_t\in\mathbb{N}$ such that

$$H \;\cong\; \mathbb{Z}^r\oplus\bigoplus_{i=1}^{t}\mathbb{Z}_{d_i}.$$

[158] Here the last row and column of zeros can be arbitrarily large.

[159] By a prime power we mean any natural number of the form p^s where p is a prime and $s\in\mathbb{N}$.

[160] We provide a sketch of a proof, full details are for example given in [Hun80, Chapter II.2].

The existence of the promised decomposition follows from from the Chinese Remainder Theorem 18.2 which tells us, that each summand $\mathbb{Z}_{d_i}$ is isomorphic to the direct sum of quotients of $\mathbb{Z}$ by prime powers.

It remains to prove the uniqueness of the prime power q_i up to permutation. One can do so by introducing the following invariant of abelian groups: Given a prime p, given $k \in \mathbb{N}$ and given an abelian group V we consider

$$T_{p,k}(V) \;:=\; \#\{v \in V \,|\, p^k \cdot v = 0\}.$$

These numbers are clearly isomorphism invariants of abelian groups. By studying these invariants one can easily prove that the q_i are unique up to permutation. We leave it to the reader to fill in the details. ■

18.4. **Appendix: Proof of the Smith Normal Form Theorem 18.10.** In this appendix we will provide a sketch of the proof of the Smith Normal Form Theorem 18.10.

Proof of the Smith Normal Form Theorem 18.10. First note that if an $m\times n$–matrix P is obtained from an $m \times n$–matrix Q by swapping two rows, then P is obtained by left-multiplying Q by an $m \times m$–matrix of the form[161]

$$\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Similarly if P is obtained from an $m \times n$–matrix Q by adding the k-th row to the l-th row, then P is obtained by left-multiplying Q by an $m \times m$–matrix of the form

$$\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}$$

where the extra 1 sits in the $(1,k)$-entry. Both these $m \times m$-matrices lie in $\mathrm{GL}(m,\mathbb{Z})$. If we do the same operations for columns, then this corresponds to right multiplication by the same type of matrices in $\mathrm{GL}(n,\mathbb{Z})$ as above.

Now let A be an $m \times n$–matrix over $\mathbb{Z}$. It follows from the above discussion that it suffices to show that we can turn A into the desired form using row and column operations.

If A is the zero matrix, then there is nothing to show. So suppose that A is not the zero matrix. We perform the following steps:

(a) After swapping rows and columns we can arrange that the absolute value of the $(1,1)$-entry is less or equal than the absolute value of all other entries. After possibly multiplying the first column by -1 we can assume that the $(1,1)$-entry v_{11} is positive.

[161] Here is the reality check whether that is really correct: we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

(b) By adding and subtracting the first column from the other columns we can arrange that all entries in the first row, except for the $(1,1)$-entry, lie between 0 and $v_{11}-1$.
(c) The same way as in (2) we can arrange that all entries in the first column, except for the $(1,1)$-entry, lie between 0 and $v_{11}-1$.

Now we have to distinguish two cases:

(1) If there exists outside of the $(1,1)$-entry another entry in the first row or column that is non-zero, then we start again with (1). Note that the new entry $(1,1)$-entry will be less than before. So this process will come to a halt after finitely many steps.
(2) If all entries in the first row and column, except for the $(1,1)$-entry are zero, then

$$A = \begin{pmatrix} v_{11} & 0 \\ 0 & A' \end{pmatrix}$$

for some $(m-1)\times(n-1)$–matrix A' and we perform the same operations on the smaller matrix A'. ■

19. Basics of group theory II: The free and amalgamated product of groups

19.1. **The free product of groups.** In the Coproduct-of-Abelian Groups Lemma 18.1 we saw that the coproduct of a family $\{X_s\}_{s\in S}$ of *abelian* groups in the category $\mathcal{AbGroup}$ of *abelian* groups is given by the direct sum $\bigoplus_{s\in S} X_s$. In this section we want to determine the coproduct of any family of groups in the category $\mathcal{Group}$ of all groups. The following definition is the key to doing so.

Definition. Let $\{X_s\}_{s\in S}$ be a family of groups. (After possibly replacing the family $\{X_s\}_{s\in S}$ by the family $\{X_s \times \{s\}\}_{s\in S}$ we can and will assume that the groups X_s are pairwise distinct.)

(1) A reduced word in $\{X_s\}_{s\in S}$ is a finite sequence $(x_1, \dots, x_m)$ such that the following conditions are satisfied:
   (a) each x_i lies in one of the groups X_s,[162]
   (b) no x_i is the neutral element of any X_s,
   (c) any two consecutive x_j's lie in two different groups.
   Note that we also allow the "empty sequence" ().

(2) We denote by $\ast_{s\in S} X_s$ the set of all reduced words.

(3) Let $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ be two elements in $\ast_{s\in S} X_s$. We define the product of these two elements as follows:[163]
   (a) First suppose that x_m and y_1 lie in different groups. Then we define
   $$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \;:=\; (x_1, \dots, x_m, y_1, \dots, y_n).$$
   (b) Now suppose that x_m and y_1 lie in the same group. Let $j \in \{1, \dots, \min\{m, n\}\}$ be the maximal element such that x_{m+1-i} and y_i lie in the same group and such that $x_{m+1-i} = y_i^{-1}$ for $i = 1, \dots, j$. If $j < \min\{m, n\}$, then we set[164]
   $$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \;:=\; (x_1, \dots, \underbrace{x_{m-j} \cdot y_{j+1}}_{\neq e}, \dots, y_n).$$
   If $j = m$, we set $\quad (x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \;:=\; (y_{m+1}, \dots, y_n).$
   We proceed similarly if $j = n$.

(4) We refer to $\ast_{s\in S} X_s$ together with this product structure as the free product of the X_s. For a finite family $X_1, \dots, X_k$ of groups we denote the free product by $X_1 \ast \cdots \ast X_k$.

---

[162] Since the groups X_s are disjoint we know that for each x_i there exists a unique $s \in S$ with $x_i \in X_s$.

[163] A more informal way of stating the product formula is as follows: given two reduced words $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ we put them next to another to obtain $(x_1, \dots, x_m, y_1, \dots, y_n)$ and we do the following:
   (a) We delete any occurrence of a subsequence of the form a, a^{-1} for $a \in X_s$.
   (b) If there exists a subsequence of the form a, b with $a, b \in X_s$ and $a \neq b^{-1}$, then we replace it by $a \cdot b \in X_s$.

This way we end up with a reduced word. The problem with this informal definition is that it is not entirely clear that mixing (a) and (b) we have a well-defined procedure.

[164] Since x_m and y_1 lie in the same group, the elements x_{m-j} and y_{j+1} also lie in the same group, so it makes sense to consider $x_{m-j} \cdot y_{j+1}$.

To consider our first example of a free product we introduce one new definition.

Definition. Let t be a symbol. We define

$$\langle t\rangle \;:=\; \{\dots, t^{-2}, t^{-1}, e, t, t^2, \dots\}.$$

The set $\langle t\rangle$ admits an obvious group structure that is given by $t^i \cdot t^j := t^{i+j}$. This group $\langle t\rangle$ is called the infinite cyclic group generated by t. The map $\mathbb{Z} \to \langle t\rangle$, $n \mapsto t^n$ is evidently an isomorphism.

Example. We consider the free product $\langle s\rangle * \langle t\rangle$. Examples of elements in this free product are (s^3, t^{-5}, s^2) and $(s^{-2}, t^4, s^2, t^{-1})$. The product of these two elements is given by

$$(s^3, t^{-5}, s^2) \cdot (s^{-2}, t^4, s^2, t^{-1}) \;=\; (s^3, \underbrace{t^{-5} \cdot t^4}_{=t^{-1}}, s^2, t^{-1}).$$

Note that $(s) \cdot (t) = (s,t) \neq (t,s) = (t) \cdot (s)$. Thus we see that the free product $\langle s\rangle * \langle t\rangle$ is very much non-abelian.

We have the following important lemma, which allows us to view each free product of groups as a group in its own right.

Lemma 19.1. (Free Product Lemma) Let $\{X_s\}_{s\in S}$ be a family of groups.

(1) The free product $\ast_{s\in S} X_s$ with the above product is again a group.

(2) The neutral element of $\ast_{s\in S} X_s$ is given by the empty sequence $()$.

(3) The inverse of an element $(x_1, \dots, x_m) \in \ast_{s\in S} X_s$ is given by $(x_m^{-1}, \dots, x_1^{-1}) \in \ast_{s\in S} X_s$.

Proof. It is clear that the empty sequence is a neutral element. It follows immediately from the definition of the product that the inverse of an element $(x_1, \dots, x_m) \in \ast_{s\in S} X_s$ is given by $(x_m^{-1}, \dots, x_1^{-1})$. It remains to show that the multiplication satisfies associativity.

In principle one can show associativity naively "by hand", but there are surprisingly many cases one needs to distinguish, so that the proof becomes rather painful.

We will therefore follow a different approach. We write $W := \ast_{s\in S} X_s$ and we denote by $(\mathrm{Bij}(W), \circ)$ the group of bijections $W \to W$, where the group structure is given by composition of maps. For $g \in X_s$ we consider the map[165]

$$\begin{aligned} \lambda_g \colon W &\to W \\ (x_1, \dots, x_m) &\mapsto \begin{cases} (g, x_1, x_2, \dots, x_m), & \text{if } x_1 \notin X_s, \\ (g \cdot x_1, x_2, \dots, x_m), & \text{if } x_1 \in X_s \text{ and } g \neq x_1^{-1}, \\ (x_2, \dots, x_m), & \text{if } g = x_1^{-1}. \end{cases} \end{aligned}$$

We make the following two observations:

(a) If $() \in W$ is the empty word, then $\lambda_{()} = \mathrm{id}$.

(b) It is straightforward to see that for any $g, g' \in X_s$ we have $\lambda_g \circ \lambda_{g'} = \lambda_{g\cdot g'} \colon W \to W$.

[165] It is straightforward to see that the sequences on the right do indeed lie in $\ast_{s\in S} X_s$.

It follows from (a) and (b) that for any $g \in X_s$ we have $\lambda_{g^{-1}} \circ \lambda_g = \mathrm{id}_W$. This shows in particular that each λ_g is a bijection. Next we consider the map

$$\begin{array}{rcl} \lambda\colon W & \to & \mathrm{Bij}(W) \\ (x_1, \ldots, x_n) & \mapsto & \lambda_{x_1} \circ \cdots \circ \lambda_{x_n}. \end{array}$$

This map is injective since $\lambda(x_1, \ldots, x_n)$ applied to the empty word $()$ gives the word $(x_1, \ldots, x_n)$. Thus different elements of W define different bijections of W.

Using the above two observations (a) and (b) it is straightforward to verify that the map $\lambda\colon W \to \mathrm{Bij}(W)$ satisfies $\lambda(x \cdot y) = \lambda_x \circ \lambda_y$. Put differently, λ defines an injective map $\lambda\colon (W, \cdot\,) \to (\mathrm{Bij}(W), \circ)$ that preserves the product structure. But since $(\mathrm{Bij}(W), \circ)$ satisfies the associativity law we now see that $(W, \cdot\,)$ also satisfies the associativity law. ■

Example. Let G and H be two groups. We consider the corresponding free product $G * H$. Let $g \in G$ and $h \in H$ be two non-trivial elements. Note that in $G * H$ we have $(g) \cdot (h) = (g, h)$ and $(h) \cdot (g) = (h, g)$. These are different elements of the free product $G * H$. This shows, similar to the discussion on page 239, that the free product of at least two non-trivial groups is non-abelian.

The following lemma, which can be viewed as an analogue of the Coproduct-of-Abelian Groups Lemma 18.1, summarizes the main properties of the free product of groups.

Lemma 19.2. (Coproduct-of-Groups Lemma) Let $\{X_s\}_{s \in S}$ be a family of groups.

(1) (a) Let $t \in S$. The map

$$\begin{array}{rcl} \iota_t\colon X_t & \to & \ast_{s \in S} X_s \\ x & \mapsto & \begin{cases} (x), & \text{if } x \neq e, \\ (), & \text{if } x = e \end{cases} \end{array}$$

is a monomorphism.

(b) Let Z be a group and let $\{f_s\colon X_s \to Z\}_{s \in S}$ be a family of homomorphisms. The map

$$\begin{array}{rcl} \Phi\colon \ast_{s \in S} X_s & \to & Z \\ (x_1, \ldots, x_m) & \mapsto & \underbrace{f_{s_1}(x_1) \cdot \ldots \cdot f_{s_m}(x_m)}_{\text{where } x_i \in X_{s_i}} \end{array}$$

is a homomorphism and it is the unique homomorphism such that for each $s \in S$ we have $f_s = \Phi \circ \iota_s$.

(2) The coproduct in the category *Group* of groups of $\{X_s\}_{s \in S}$ is given by $\ast_{s \in S} X_s$ together with the homomorphisms $\iota_t\colon X_t \to \ast_{s \in S} X_s$ from (1a).

Proof.

(1) (a) Let $t \in T$. It is clear that the map $\iota_t\colon X_t \to \ast_{s \in S} X_s$ is injective and it follows immediately from the definition of the group structure on $\ast_{s \in S} X_s$ that the map is a group homomorphism.

(b) Let Z be a group and let $\{f_s\colon X_s \to Z\}_{s \in S}$ be a family of homomorphisms. One can easily verify that the given map $\Phi\colon \ast_{s \in S} X_s \to Z$ is a homomorphism and that it satisfies $f_s = \Phi \circ \iota_s$ for each $s \in S$. It remains to show the uniqueness of Φ.

Any element in $\ast_{s\in S} X_s$ is a product of elements of the form $\iota_t(x)$ with $x \in X_t$. It follows that any homomorphism $\delta\colon \ast_{s\in S} X_s \to Z$ is already uniquely determined by the values of δ on all elements of the form $\iota_t(x)$ with $x \in X_t$. Put differently, there is at most one homomorphism $\delta\colon \ast_{s\in S} X_s \to Z$ with $\delta(\iota_t(x)) = f_t(\iota_t(x)) = (f_t \circ \iota_t)(x)$ for all $t \in T$ and all $x \in X_t$. This implies the uniqueness of Φ.

(2) This statement follows immediately from (1a) and (1b) and the definition of the coproduct on page 82. ∎

Convention. Let $\{X_s\}_{s\in S}$ be a family of groups.

(1) Let $t \in T$. We refer to

$$\begin{aligned} \iota_t\colon X_t &\to \ast_{s\in S} X_s \\ x &\mapsto \begin{cases} (x), & \text{if } x \neq e, \\ (), & \text{if } x = e \end{cases} \end{aligned}$$

as the natural monomorphism $\iota_t\colon X_t \to \ast_{s\in S} X_s$.

(2) For $t \in S$ we use the natural monomorphism $\iota_s\colon X_t \to \ast_{s\in S} X_s$ to view X_t as a subgroup of the free product $\ast_{s\in S} X_s$.

(3) Let $(x_1, \dots, x_m) \in \ast_{s\in S} X_s$. Since $(x_1, \dots, x_m) = (x_1)\cdot\dots\cdot(x_m)$ we can use (2) to denote this element by $x_1 \cdot \ldots \cdot x_m$.

Example. We consider again the free product $\langle s\rangle * \langle t\rangle$. Using the above convention we now write $s^3 \cdot t^{-5} \cdot s^2 = (s^3, t^{-5}, s^2)$ and $s^{-2} \cdot t^4 \cdot s^2 \cdot t^{-1} = (s^{-2}, t^4, s^2, t^{-1})$. The product of these two elements is given by very reasonable looking

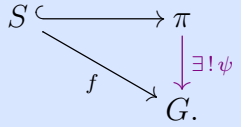
$$(s^3 \cdot t^{-5} \cdot s^2) \cdot (s^{-2} \cdot t^4 \cdot s^2 \cdot t^{-1}) \;=\; s^3 \cdot \underbrace{t^{-5} \cdot t^4}_{=t^{-1}} \cdot s^2 \cdot t^{-1}.$$

Also note that the discussion on page 239 shows that $s \cdot t \neq t \cdot s$, in particular we see that the free product $\langle s\rangle * \langle t\rangle$ is non-abelian.

19.2. **Free groups.** In Section 18.2 we introduced the notion of a free *abelian* group. In this section we will modify the definition to introduce the notion of a free group. We start out with the following definition which is the analogue of the definition on page 230.

Definition.

(1) We say that a group π is a free group on a subset $S \subset \pi$, if the following universal property holds: for any *map* $f\colon S \to G$ of the set S to a group G there exists a unique *homomorphism* $\psi\colon \pi \to G$ that makes the following diagram commute



We call S a basis of the free group π.

(2) We say that a group π is free if it is a free group on some subset S.

Lemma 19.3. (Free Group–Uniqueness Lemma) Let G be a free group on a subset $S \subset G$ and let H be a free group on a subset $T \subset H$. Given any bijection $f\colon S \to T$ there exists a unique isomorphism $\psi\colon G \to H$ such that $\psi(s) = f(s)$ for all $s \in S$.

Proof. The proof of this lemma is basically identical to the proof of the Free Abelian Group–Uniqueness Lemma 18.3. ■

Definition. For a set S we write $$\langle S\rangle \ := \ \mathop{\ast}_{s\in S} \langle s\rangle.$$

The lemma below allows us to call $\langle S\rangle$ the free group on the set S. The map $S \to \langle S\rangle$ that is given by $s \mapsto (s)$ is an injection and we use it to view S as a subset of $\langle S\rangle$.

The following lemma is now the analogue of the Free Abelian Group–Existence Lemma 18.4.

Lemma 19.4. (Free Group–Existence Lemma) Let S be a set. The group $\langle S\rangle$ is a free group on the subset $S \subset \langle S\rangle$.

Proof. Let G be a group and let $f\colon S \to G$ be a map. For $s \in S$ the map

$$\begin{array}{rcl} \psi_s\colon \langle s\rangle & \to & G \\ s^n & \mapsto & f(s)^n \end{array}$$

is evidently the unique homomorphism $\psi_s\colon \langle s\rangle \to G$ with $\psi_s(s) = f(s)$. It now follows from the Coproduct-of-Groups Lemma 19.2 (2) that the map

$$\begin{array}{rcl} \psi\colon \langle S\rangle = \mathop{\ast}_{s\in S} \langle s\rangle & \to & G \\ s_{i_1}^{n_1} \cdot \ldots \cdot s_{i_k}^{n_k} & \mapsto & f(s_{i_1})^{n_1} \cdot \ldots \cdot f(s_{i_k})^{n_k} \end{array}$$

is the unique homomorphism $\langle S\rangle \to G$ with $\psi(s) = \psi_s(s) = f(s)$ for $s \in S$. ■

The Free Group–Existence Lemma 19.4 motivates the following definition.

Definition.

(1) For a set S we refer to $\langle S\rangle = \mathop{\ast}_{s\in S} \langle s\rangle$ as *the* free group on the set S.

(2) Let $s_1, \ldots, s_k$ be distinct elements of a set. We refer to

$$\langle s_1, \ldots, s_k\rangle \ := \ \langle s_1\rangle * \cdots * \langle s_k\rangle$$

as the free group on the generators $s_1, \ldots, s_k$. Note that each element of the free group $\langle s_1, \ldots, s_k\rangle$ is uniquely of the form

$$s_{r_1}^{n_1} \cdot \ldots \cdot s_{r_m}^{n_m}$$

with $r_i \neq r_{i+1}$ for $i = 1, \ldots, m-1$ and $s_i \neq 0$ for $i = 1, \ldots, m$.

19.3. **The abelianization of a group.** It is pretty clear that abelian groups are much easier to handle than non-abelian groups. In this section we will get to know the abelianization of a group, which allows us in many settings to turn questions about arbitrary groups into questions about abelian groups.

Definition. Let G be a group.

(1) Given $x, y \in G$ we refer to $[x, y] := x \cdot y \cdot x^{-1} \cdot y^{-1}$ as the commutator of x and y.

(2) We refer to

$$[G,G] \;:=\; \left\{ \prod_{i=1}^{n} [x_i, y_i] \;\middle|\; x_1, y_1, \ldots, x_n, y_n \in G \right\}$$

as the commutator subgroup of G.

Remark. Note that in general not every element of the commutator subgroup is a commutator. For example, if $G = \langle a, b, c, d \rangle$ is the free group on four generators a, b, c, d, then we have $[a, b] \cdot [c, d] \in [G, G]$. But one can easily show that $[a, b] \cdot [c, d]$ is not a commutator in G.

Lemma 19.5. (Commutator Subgroup Lemma) Let G be a group.

(1) The commutator subgroup $[G, G]$ is a normal subgroup of G.
(2) The group $G_{\text{ab}} := G/[G, G]$ is abelian.
(3) Let $\alpha\colon G \to H$ be a homomorphism to an abelian group. There exists a unique homomorphism $\beta\colon G_{\text{ab}} = G/[G, G] \to H$ such that the following diagram commutes:[166]

$$\begin{array}{ccc} G & & \\ {\scriptstyle g \mapsto [g]}\downarrow & \searrow^{\alpha} & \\ G_{\text{ab}} = G/[G,G] & \xrightarrow[\exists!\,\beta]{} & H \text{ abelian}. \end{array}$$

This homomorphism β is given by $\beta([g]) = \alpha(g)$.

Proof. Let G be a group.

(1) First we show that $[G, G]$ is a subgroup of G:
- Evidently the subset $[G, G]$ is closed under products.
- With $n = 0$ we see that $e \in [G, G]$.
- The inverse of a commutator is again a commutator; indeed given $x, y \in G$ we have

$$[x, y]^{-1} \;=\; (xyx^{-1}y^{-1})^{-1} \;=\; yxy^{-1}x^{-1} \;=\; [y, x].$$

It follows easily that $[G, G]$ is closed under taking inverses.

We have thus shown that $[G, G]$ is a subgroup of G. It remains to show that $[G, G]$ is a normal subgroup of G. The key to doing so is the following claim:

Claim. A conjugate of a commutator is again a commutator.

Proof. Given $x, y, g \in G$ we have

$$\begin{aligned} g[x, y]g^{-1} = gxyx^{-1}y^{-1}g^{-1} &= gxg^{-1}gyg^{-1}gx^{-1}g^{-1}gy^{-1}g^{-1} \\ &= (gxg^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1} = [gxg^{-1}, gyg^{-1}]. \end{aligned}$$ ⊞

Given $x_1, \ldots, x_n, y_1, \ldots, y_n \in G$ and given $g \in G$ we now see that

$$g \cdot \left(\prod_{i=1}^{n} [x_i, y_i] \right) g^{-1} \;\underset{\text{since } g(ab)g^{-1} = gag^{-1} \cdot gbg^{-1}}{\uparrow}\; \prod_{i=1}^{n} g \cdot [x_i, y_i] \cdot g^{-1} \;\underset{\text{by the claim}}{\uparrow}\; \prod_{i=1}^{n} [gx_ig^{-1}, gy_ig^{-1}] \;\in\; [G, G].$$

We have thus shown that $[G, G]$ is a normal subgroup.

[166] Another way of formulating this statement is to say that for any group G and any abelian group H the map

$$\begin{array}{ccc} \text{Hom}(G_{\text{ab}}, H) & \to & \text{Hom}(G, H) \\ (\alpha\colon G_{\text{ab}} \to H) & \mapsto & \left(G \xrightarrow{g \,\mapsto\, [g]} G_{\text{ab}} \xrightarrow{\alpha} H\right) \end{array}$$

is a bijection.

(2) For any two elements $x, y \in G$ we have

$$xy\,[G,G] \;=\; yx \cdot x^{-1}y^{-1}xy\,[G,G] \;=\; yx \cdot \underbrace{[x^{-1},y^{-1}]}_{\in [G,G]}\,[G,G] \;=\; yx\,[G,G].$$

This shows that $G_{\mathrm{ab}} = G/[G,G]$ is indeed abelian.

(3) Let G be a group and let $\alpha\colon G \to H$ be a homomorphism to an abelian group. Using the fact that H is abelian one can easily verify that $[G,G]$ lies in the kernel of α. The desired statement now follows immediately from the universal property of quotient groups. ■

Definition. Let G be a group.

(1) We call $G_{\mathrm{ab}} := G/[G,G]$ the abelianization of G.

(2) We denote by $\Phi_G\colon G \to G_{\mathrm{ab}} = G/[G,G],\ g \mapsto [g]$ the natural projection.

(3) Let $\alpha\colon G \to H$ be a homomorphism. By the Commutator Subgroup Lemma 19.5 (3), applied to the abelian group H_{ab} and the homomorphism $\Phi_H \circ \alpha\colon G \to H_{\mathrm{ab}}$, there exists a unique homomorphism $\alpha_*\colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & H \\ {\scriptstyle \Phi_G}\downarrow & & \downarrow{\scriptstyle \Phi_H} \\ G_{\mathrm{ab}} & \xrightarrow{\alpha_*} & H_{\mathrm{ab}}. \end{array}$$

We refer to α_* as the induced homomorphism. Note that α_* is given by $\alpha_*([g]) = [\alpha(g)]$.

Example. Given $k \in \mathbb{N}$ we denote by S_k the permutation group on k elements and we denote by $A_k \subset S_k$ the alternating group, i.e. the group of all permutations with positive sign. In basically every standard algebra book (see [Rob95, p. 69] or [Bog08, Theorem 11.3]) it is shown that for $k \geq 5$ the group A_k is simple, i.e. the only normal subgroups of A_k are the trivial group and the group A_k itself. We just saw in the Commutator Subgroup Lemma 19.5 (1) that the commutator subgroup $[A_k, A_k]$ is normal. Since A_k is not abelian we cannot have $[A_k, A_k] = \{e\}$. Thus we see that $[A_k, A_k] = A_k$, which implies that the abelianization of A_k is the trivial group.

Proposition 19.6. (Abelianization Functor Proposition)

(1) The maps $G \mapsto G_{\mathrm{ab}} = G/[G,G]$, $(\alpha\colon G \to H) \mapsto (\alpha_*\colon G_{\mathrm{ab}} \to H_{\mathrm{ab}})$ define a covariant functor from the category *Group* of groups to the category *AbGroup* of abelian groups.

(2) If $\varphi\colon G \to H$ is an isomorphism of groups, then the induced map $\varphi_*\colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}$ is also an isomorphism. In particular, isomorphic groups have isomorphic abelianizations.

Proof.

(1) This statement follows easily from the definitions.

(2) This statement follows from (1) and the fact that functors turn isomorphisms into isomorphisms. ■

The following lemma will be used on many occasions.

Lemma 19.7. (Abelianization-Epimorphism Lemma) If $\alpha\colon G \to H$ is an epimorphism, then the induced map $\alpha_*\colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}$ is also an epimorphism.

Proof. By definition of α_* we have the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & H \\ {\scriptstyle \Phi_G}\downarrow & & \downarrow{\scriptstyle \Phi_H} \\ G_{\mathrm{ab}} & \xrightarrow{\alpha_*} & H_{\mathrm{ab}}. \end{array}$$

By hypothesis $\alpha\colon G \to H$ is an epimorphism. It follows that $\Phi_H \circ \alpha\colon G \to H_{\mathrm{ab}}$ is an epimorphism, which implies that α_* is an epimorphism. ∎

The following proposition allows us to calculate the abelianization in several basic situations.

Proposition 19.8. (Abelianization Calculation Proposition)

(1) If π is abelian, then the natural homomorphism $\Phi_\pi\colon \pi \to \pi_{\mathrm{ab}} = \pi/[\pi,\pi]$ is an isomorphism.

(2) Let $\{X_s\}_{s\in S}$ be a family of groups. We have natural homomorphisms

$$\begin{array}{rcl} \left(\underset{s\in S}{*} X_s\right)_{\mathrm{ab}} & \to & \bigoplus_{s\in S} (X_s)_{\mathrm{ab}} \\ [(x_1,\dots,x_k)] & \mapsto & \sum_{i=1}^{k} [x_i] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \bigoplus_{s\in S} (X_s)_{\mathrm{ab}} & \to & \left(\underset{s\in S}{*} X_s\right)_{\mathrm{ab}} \\ \sum_{i=1}^{k} [x_i] & \mapsto & [x_1 \cdot \ldots \cdot x_k]. \end{array}$$

These are inverses of one another, in particular they are isomorphisms.

(3) Let S be a set. The map

$$\begin{array}{rcl} \langle S\rangle_{\mathrm{ab}} & \to & \mathbb{Z}^{(S)} \\ [s_{i_1}^{n_1} \cdot \ldots \cdot s_{i_k}^{n_k}] & \mapsto & \sum_{i=1}^{k} n_i \cdot s_i \end{array}$$

is a natural isomorphism.

Proof of the Abelianization Calculation Proposition 19.8.

(1) If π is abelian, then every commutator is trivial, thus $[\pi,\pi] = \{e\}$. This implies that the natural homomorphism $\Phi_\pi\colon \pi \to \pi/[\pi,\pi]$ is an isomorphism.

(2) It is worth explaining where the two maps in the proposition come from.

(a) By the Coproduct-of-Abelian Groups Lemma 18.1 (1a) we have for each $s \in S$ a natural homomorphism $X_s \to (X_s)_{\mathrm{ab}} \to \bigoplus_{s\in S}(X_s)_{\mathrm{ab}}$. By the Coproduct-of-Groups Lemma 19.2 these give rise to a homomorphism $\underset{s\in S}{*} X_s \to \bigoplus_{s\in S} (X_s)_{\mathrm{ab}}$. Since the target is abelian we obtain from the Commutator Subgroup Lemma 19.5 (3) an induced homomorphism $\left(\underset{s\in S}{*} X_s\right)_{\mathrm{ab}} \to \bigoplus_{s\in S} (X_s)_{\mathrm{ab}}$.

(b) By the Coproduct-of-Groups Lemma 19.2 (1a) we have for each $s \in S$ a natural homomorphism $X_s \to \underset{s\in S}{*} X_s$. This homomorphism induces a homomorphism $(X_s)_{\mathrm{ab}} \to \left(\underset{s\in S}{*} X_s\right)_{\mathrm{ab}}$. By the Coproduct-of-Abelian Groups Lemma 18.1 (1b) these homomorphisms gives rise to a homomorphism $\bigoplus_{s\in S} (X_s)_{\mathrm{ab}} \to \left(\underset{s\in S}{*} X_s\right)_{\mathrm{ab}}$.

If one goes through the explicit descriptions of the homomorphisms one sees that the above two homomorphisms are precisely the homomorphisms we wrote down in the proposition. This shows that the maps of the proposition are well-defined and that they are homomorphisms. Now that we know that the given maps are well-defined it is pretty clear that they are inverses of one another.

(3) This statement follows immediately from the statements (1) and (2) together with the observation that by definition we have $\langle S\rangle = \ast_{s\in S}\langle s\rangle$ and $\mathbb{Z}^{(S)} = \bigoplus_{s\in\mathbb{Z}}\mathbb{Z}$. ■

We conclude this section with a proposition which be viewed as an analogue of the Basis Cardinality Theorem 18.6 for free abelian groups.

Proposition 19.9. (Basis Cardinality Proposition) Let G be a group that is free on a subset S and let H be a group that is free on a subset $T \subset H$. If G and H are isomorphic, then $\#S = \#T$.

Proof. It follows immediately from the Free Group–Uniqueness Lemma 19.3 that we can assume that $G = \langle S\rangle$ and that $H = \langle T\rangle$.

(a) Let $\alpha\colon \langle S\rangle \to \langle T\rangle$ be an isomorphism.

(b) It follows from the Abelianization Functor Proposition 19.6 (2) that $\alpha_*\colon \langle S\rangle_{\mathrm{ab}} \to \langle T\rangle_{\mathrm{ab}}$ is also an isomorphism.

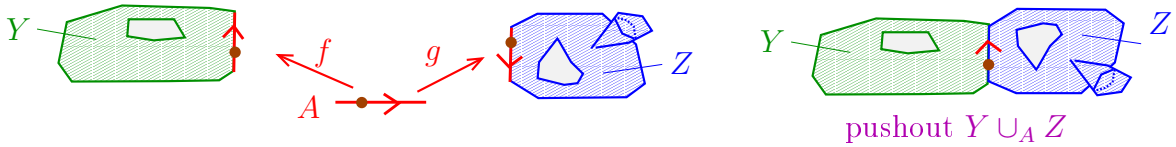
(c) It follows from (b) and the Abelianization Calculation Proposition 19.8 (3) that $\mathbb{Z}^{(S)}$ and $\mathbb{Z}^{(T)}$ are isomorphic.

(d) It follows from (c) and the Basis Cardinality Theorem 18.6 that $\#S = \#T$. ■

19.4. **The amalgamated product of groups.** Suppose we are given two continuous maps $f\colon (A, a_0) \to (Y, y_0)$ and $g\colon (A, a_0) \to (Z, z_0)$ between pointed topological spaces. Note that by the same argument as in the Topological Pushout Proposition 6.12 we see that the topological space

$$Y \cup_A Z \;:=\; (Y \sqcup Z)/\!\sim \qquad \text{where } f(a) \sim g(a) \text{ for all } a \in A$$

together with the base points $[y_0] = [z_0]$ is the pushout of the two maps in the category of pointed topological spaces.



In Section 6.7 we saw that the pushout is a useful tool to construct and describe (pointed) topological spaces. Two questions now arise:

(1) Does the fundamental group preserve pushouts?

(2) What on earth is the pushout in the category of groups?

In the remainder of this section we will answer Question (2). We will return to Question (1) in the next chapter, when we will formulate and prove the Seifert–van Kampen Theorem.

The construction of the pushout in the category of groups will be formally quite similar to the above explicit description of the pushout in the category of pointed topological spaces. We first take a coproduct, afterwards we take a quotient. One problem arises: given a group G and a subgroup A the quotient G/A is in general not a group. Only if A is a normal subgroup, then G/A is in fact a group. This leads us to the following definition.

Definition. Let G be a group and let $A \subset G$ be a subset. We define

$$\text{subgroup of } G \text{ normally generated by } A \;:=\; \langle\langle A \rangle\rangle \;:=\; \bigcap_{\substack{A \subset H \lhd G \\ H \text{ normal subgroup}}} H,$$

i.e. $\langle\langle A \rangle\rangle$ is the intersection of all normal subgroups H of G that contain A. This is again a normal[167] subgroup of G.

Lemma 19.10. (Normal Closure–Factorization Lemma) Let G be a group and let $A \subset G$ be a subset. Let $\alpha\colon G \to \Gamma$ be a group homomorphism such that $\alpha(a) = e \in \Gamma$ for all $a \in A$. Then there exists a unique group homomorphism $\beta\colon G/\langle\langle A \rangle\rangle \to \Gamma$ such that the following diagram commutes:

$$\begin{array}{ccc} G & & \\ \downarrow & \searrow^{\alpha} & \\ G/\langle\langle A \rangle\rangle & \xrightarrow[\exists!\,\beta]{} & \Gamma. \end{array}$$

Proof. Let G be a group, let $A \subset G$ be a subset and let $\alpha\colon G \to \Gamma$ be a group homomorphism such that $\alpha(a) = e \in \Gamma$ for all $a \in A$. This means that $A \subset \ker(\alpha)$. Since $\ker(\alpha)$ is a normal subgroup of G it follows immediately from the definition of $\langle\langle A \rangle\rangle$ that $\langle\langle A \rangle\rangle \subset \ker(\alpha)$. The lemma follows from the universal property of the quotient group. ∎

The following proposition can be viewed as an analogue of the Topological Pushout Proposition 6.12 and the above discussion of the pushout in the category of pointed topological spaces.

Proposition 19.11. (Group Pushout Proposition) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. We consider the group

$$A *_G B \;:=\; (A * B)/\langle\langle \{\alpha(g) \cdot \beta(g)^{-1}\}_{g \in G} \rangle\rangle$$

and the maps

$$\begin{array}{rcl} \mu\colon A & \to & A *_G B \\ a & \mapsto & [(a)] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \nu\colon B & \to & A *_G B \\ b & \mapsto & [(b)] \end{array}$$

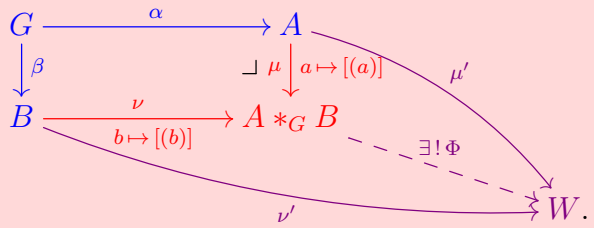
The following statements hold:

(1) (a) The maps $\mu\colon A \to A *_G B$ and $\nu\colon B \to A *_G B$ are homomorphisms.

(b) Suppose we are given two homomorphisms $\mu'\colon A \to W$ and $\nu'\colon B \to W$ to some group W such that $\mu' \circ \alpha = \nu' \circ \beta\colon G \to W$. By the Coproduct-of-Groups Lemma 19.2 the homomorphisms μ' and ν' induce a homomorphism $A * B \to W$. This homomorphism descends to a homomorphism

$$\Phi\colon A *_G B \;\to\; W.$$

This homomorphism is the unique homomorphism $\Phi\colon A *_G B \to W$ with $\Phi \circ \mu = \mu'$ and $\Phi \circ \nu = \nu'$. In particular we obtain the following commutative diagram:



(2) The group $A *_G B$ with the natural homomorphisms $\alpha\colon A \to A *_G B$ and $\beta\colon B \to A *_G B$ is the pushout, in the sense of the definition on page 89, in the category *Group* of groups.

Definition. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. We write

$$A *_G B \;:=\; (A * B)/\langle\langle \{\alpha(g) \cdot \beta(g)^{-1}\}_{g \in G} \rangle\rangle$$

and we refer to $A *_G B$ as the amalgamated product of A and B with amalgam G.[168] We refer to the obvious homomorphisms $A \to A *_G B$ and $B \to A *_G B$ as the natural homomorphisms.

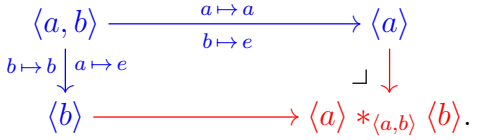
Proof.

(1) (a) The statement follows from the Coproduct-of-Groups Lemma 19.2 (1a).
(b) This statement follows easily from the Coproduct-of-Groups Lemma 19.2 (1b). together with the Normal Closure–Factorization Lemma 19.10.

(2) This statement follows easily from the definitions and Statement (1). We leave it to the reader to fill in the details. ■

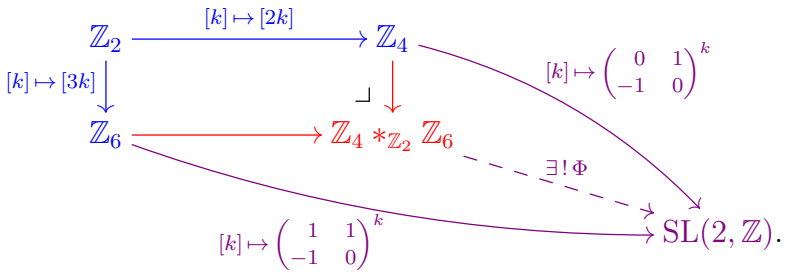
Example.

(1) We consider the following pushout diagram



It follows easily from the universal property of a pushout or from the explicit description of the amalgamated product that $\langle a \rangle *_{\langle a,b \rangle} \langle b \rangle = \{e\}$.

(2) We consider the following diagram:



[168] The notation is a little misleading since the homomorphisms α and β do not get mentioned in the notation, even though they play of course an important role. In particular different choices of α and β can lead to non-isomorphic amalgamated products.

Since the two maps from $\mathbb{Z}_2$ to $\mathrm{SL}(2,\mathbb{Z})$ agree we obtain from the Group Pushout Proposition 6.12 (1b) a unique homomorphism $\Phi\colon \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 \to \mathrm{SL}(2,\mathbb{Z})$ that makes the diagram commute. More surprisingly, this homomorphism is in fact an isomorphism. We refer to [Ros94, p. 218] and [Ser80, p. 35] for a proof.

The following lemma summarizes some formal properties of the amalgamated product.

Lemma 19.12. (Amalgamated Product Lemma)

(1) For two groups A and B and the trivial group $G = \{e\}$, the corresponding amalgamated product is the usual free product, i.e.

$$A *_{\{e\}} B \ = A * B.$$

(2) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be group homomorphisms. We consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & A \\ {\scriptstyle\beta}\downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & A *_G B. \end{array}$$

The following two statements hold:

(a) If the left vertical map β is an isomorphism, then the right vertical map is also an isomorphism and the inverse $A *_G B \to A$ is given by the identity on A and $\beta^{-1}\colon B \to G$.

(b) If the left vertical map β is an epimorphism, then the right vertical map is also an epimorphism.

The analogous statements of course hold with the roles of "A" and "B" swapped.

Proof. As we will now see, all statements follow from formal properties of pushouts:

(1) This statement follows from the fact that $\{e\}$ is the initial object of the category of groups and together with the Pushout Lemma 4.8.

(2) These two statements follows from the Pushout Lemma 4.8 (2a) and (2b). ∎

We conclude this discussion of amalgamated products with the following which is specific to the group theoretic setting.[169]

Lemma 19.13. (Amalgamated Product–Quotient Lemma) If $\alpha\colon G \to A$ is a group homomorphism and if $\beta\colon G \to \{e\}$ is the trivial homomorphism, then the projection $A \to A/\langle\langle \alpha(G) \rangle\rangle$ induces a natural isomorphism

$$A *_G \{e\} \ \xrightarrow{\cong}\ A/\langle\langle \alpha(G) \rangle\rangle.$$

Proof. The statement follows immediately from the definition of the amalgamated product and the natural isomorphism $A * \{e\} \xrightarrow{\cong} A$. ∎

[169] Note thought that the lemma is analogous to the discussion on page 120 where we discussed the pushout in the category of topological spaces where one of the two maps is given by the unique map $A \to \{*\}$ to the terminal object in the category $\mathcal{T}op$ of topological spaces.

20. The Seifert–van Kampen Theorem

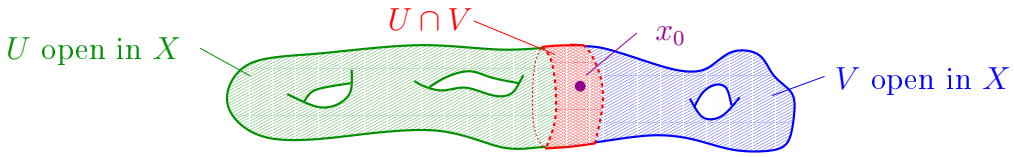
We are still lacking the computation of fundamental groups of even fairly simple topological spaces, for example we do not know the fundamental groups of wedges of circles, of topological graphs and of surfaces. In this chapter we will take a new approach to calculating fundamental groups, namely we will formulate and prove the Seifert–van Kampen Theorem which allow us to calculate the fundamental group of a given topological space in terms of the fundamental groups of suitable subspaces and in terms of inclusion induced maps. Using the Seifert–van Kampen Theorem we will be able to calculate many of the missing fundamental groups.

20.1. **The formulation of the general Seifert–van Kampen Theorem.** The idea of the Seifert–van Kampen theorem is to reduce the calculation of the fundamental group of a topological space X to the determination of fundamental groups of "simpler subsets" of X.

More precisely: Let X be a topological space and let $U, V \subset X$ be two subsets such that $X = U \cup V$ and such that $U \cap V \neq \varnothing$. We choose a base point $x_0 \in U \cap V$ and we consider the following inclusion maps and the corresponding induced homomorphisms of the fundamental groups:

$$\begin{array}{ccc} U \cap V & \xrightarrow{\ i\ } & U \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ V & \xrightarrow[\ l\ ]{} & X = U \cup V \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \pi_1(U \cap V, x_0) & \xrightarrow{\ i_*\ } & \pi_1(U, x_0) \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle k_*} \\ \pi_1(V, x_0) & \xrightarrow[\ l_*\ ]{} & \pi_1(X, x_0) = \pi_1(U \cup V, x_0). \end{array}$$

It is not difficult to show that if U and V are *open* subsets of X, then the diagram to the left is a pushout. The following theorem now says that if U and V are open and if $U \cap V$ is path-connected, then the diagram to the right is also a pushout.

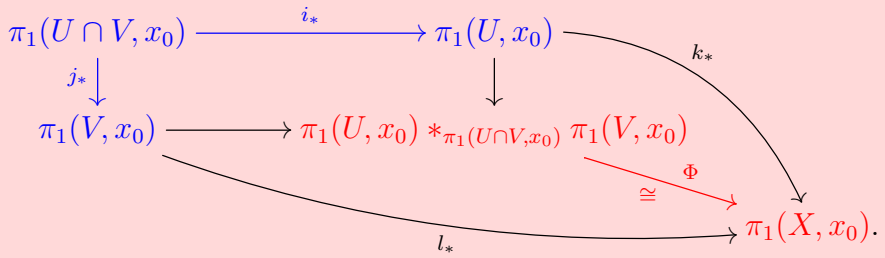


Theorem 20.1. (Seifert–van Kampen Theorem)[170] Let X be a topological space and let $X = U \cup V$ be a decomposition of X in two *open* subsets U and V such that $U \cap V$ is non-empty and *path-connected*. Let $x_0 \in U \cap V$. We denote by $i\colon U \cap V \to U$ and $j\colon U \cap V \to V$ the inclusions.

(1) (a) The inclusions $k\colon U \to X$ and $l\colon V \to X$ induce an isomorphism

$$\Phi\colon \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) \xrightarrow{\ \cong\ } \pi_1(X, x_0)$$

which has the property that the following diagram commutes:

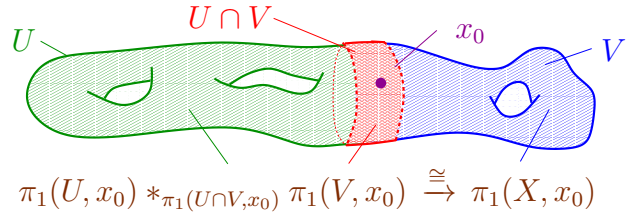


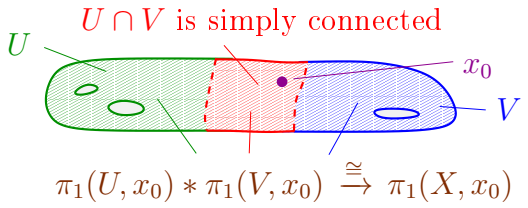
Here the "undecorated" maps are natural homomorphism to the amalgamated product.

(b) The group $\pi_1(X, x_0)$ is a pushout.

(2) If $U \cap V$ is simply connected, then the inclusions $U \to X$ and $V \to X$ induce an isomorphism

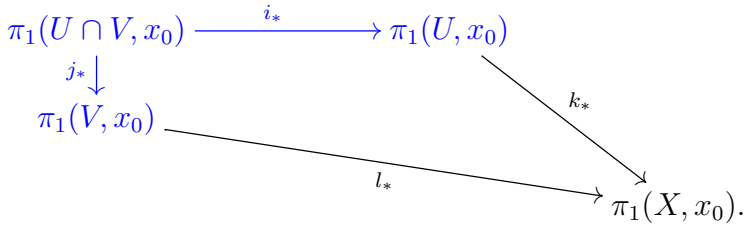
$$\Phi \colon \pi_1(U, x_0) * \pi_1(V, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$$





Summary of proof. First note that Statement (2) is a special case of Statement (1a). Furthermore note that Statement (1b) follows from Statement (1a) and the Group Pushout Proposition 6.12 (2),

Thus it remains to prove Statement (1a). Note that the various inclusions of topological spaces give rise to the following commutative diagram:

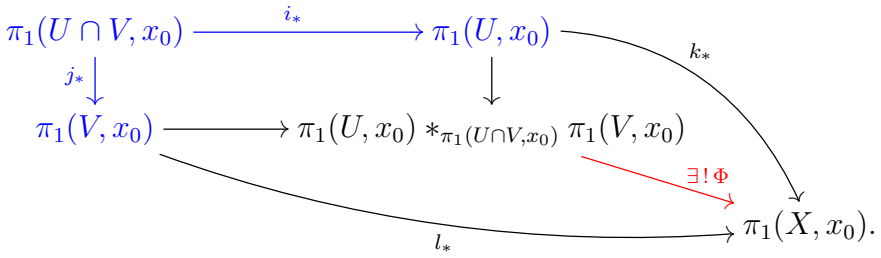


By the Group Pushout Proposition 6.12 (1b) there exists a unique homomorphism

$$\Phi \colon \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \to \pi_1(X, x_0)$$

[170] The theorem was proved independently by Herbert Seifert (1907-1996) and Egbert van Kampen (1908-1942).

such that the following diagram commutes:



It remains to show that Φ is surjective and that Φ is injective. The plan is as follows:

(a) In Section 20.2 we will see that Φ is surjective.

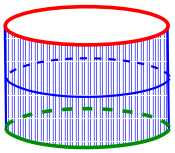
(b) In Section 20.4 we will show that Φ is injective by showing that there exists a map $\Psi\colon \pi_1(X, x_0) \to \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0)$ with $\Psi \circ \Phi = \mathrm{id}$. To construct the map Ψ it is convenient to work with an alternative description of free and amalgamated products. We will explain this alternative point of view in Section 20.3. ■

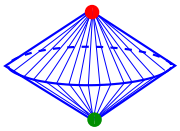
As we outlined above, we will provide the longish proof of the Seifert–van Kampen Theorem 20.1 in the next three sections. But let us first consider an example to get a feel for the Seifert–van Kampen Theorem 20.1.

Definition. Let X be a non-empty topological space. We define the suspension of X to be the topological space

$$\Sigma(X) \;:=\; (X \times [-1,1])/\sim$$

where we identify all points in $X \times \{-1\}$ to a single point and we identify all points in $X \times \{1\}$ to a single point.





As a first application of the Seifert–van Kampen Theorem 20.1 we can now prove the following lemma.

Lemma 20.2. (Suspension-π_1-Lemma) If X is a path-connected non-empty topological space, then its suspension $\Sigma(X) = (X \times [-1,1])/\sim$ is simply connected.

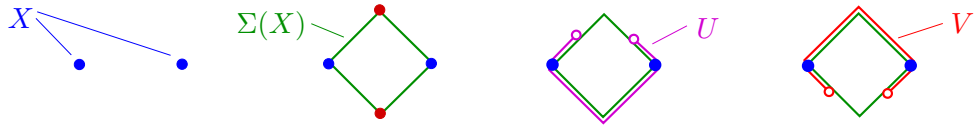
Example. One can easily show that the suspension $\Sigma(S^n)$ of the n-dimensional sphere is homeomorphic to the $(n+1)$-dimensional sphere S^{n+1}.

(1) By the Ball-Sphere-Connected Lemma 2.17 we know that for $n \geq 1$ the sphere S^n is path-connected. Thus it follows from the Suspension-π_1-Lemma 20.2 that for $n \geq 1$ we have $\pi_1(S^{n+1}) = 0$. In other words, we have obtained a new proof of the Sphere-π_1-Proposition 11.14.

(2) The example $n = 0$ together with the Circle-π_1-Theorem 11.13 shows that in the Suspension-π_1-Lemma 20.2 we cannot drop the hypothesis that X is path-connected.

Proof. Let X be a path-connected non-empty topological space. In the following we denote by $p\colon X \times [-1,1] \to \Sigma(X)$ the natural projection. We consider the two subsets

$U = p(X \times [-1, \frac{1}{2}))$ and $V = p(X \times (-\frac{1}{2}, 1])$. It follows from the definition of the quotient topology on $\Sigma(X)$ that U and V are open subsets of $\Sigma(X)$. We pick a base point x on $X = X \times \{0\} \subset \Sigma(X)$.



We calculate that

$$\pi_1(\Sigma(X), x) \underset{\uparrow}{\cong} \pi_1(U, x) *_{\pi_1(U\cap V, x)} \pi_1(V, x) \underset{\uparrow}{=} \begin{array}{c}\text{amalgamated product}\\ \text{of trivial groups}\end{array} = \text{trivial group.}$$

by the Seifert–van Kampen -Theorem 20.1 which we can apply since $U \cap V \cong X \times (-\frac{1}{2}, \frac{1}{2})$ is path-connected

one can easily show that U and V are contractible, hence their fundamental groups are trivial by the Homotopy-π_1-Proposition 13.4 (3)

We have thus shown that $\Sigma(X)$ is indeed simply connected. ■

In the following three sections we provide the proof of the Seifert–van Kampen-Theorem 20.1. Afterwards we will consider many more applications of the Seifert–van Kampen-Theorem 20.1.

20.2. Proof of the Seifert–van Kampen-Theorem 20.1: Surjectivity.

Proof of the Seifert–van Kampen-Theorem 20.1: Surjectivity. Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and such that $U \cap V$ is path-connected and non-empty. We choose a base point $x_0 \in U \cap V$. We need to show that the map

$$\Phi\colon \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \;\to\; \pi_1(X, x_0)$$

from page 251 is surjective. We consider the following commutative diagram of homomorphisms:

$$\begin{array}{ccc} \pi_1(U, x_0) * \pi_1(V, x_0) & & \\ \downarrow & \overset{\Xi}{\searrow} & \\ \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) & \xrightarrow[\Phi]{} & \pi_1(X, x_0) \end{array}$$

where the horizontal homomorphism Ξ and the left diagonal homomorphism are given by the Coproduct-of-Groups Lemma 19.2 (1b). It remains to prove that the horizontal homomorphism $\Xi\colon \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ is surjective. Clearly it suffices to prove the following claim.

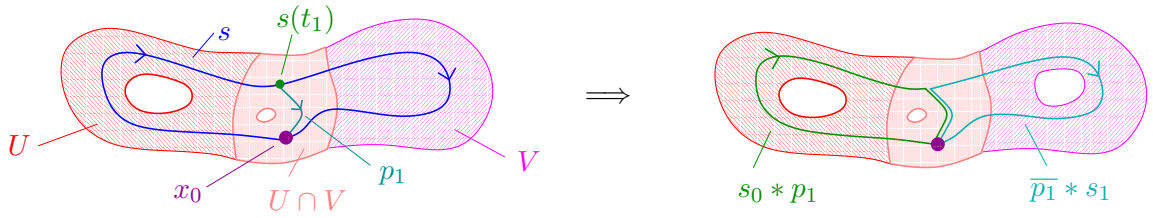
Claim. Every loop $s\colon [0, 1] \to X = U \cup V$ in x_0 is path-homotopic to a loop which is the concatenation of finitely many loops, each of which lies in (U, x_0) or in (V, x_0).

Proof. Let $s\colon [0, 1] \to X = U \cup V$ be a loop in x_0. Since U and V are *open* we can use the Lebesgue Lemma 11.9 to find a subdivision

$$0 = t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} = 1$$

of the interval $[0, 1]$ such that the image of each $s_i := s|_{[t_i, t_{i+1}]}$ lies in U or in V. The same argument as in the proof of the Sphere-π_1-Proposition 11.14 shows that we can assume

that each $s(t_i)$ lies in $U \cap V$.[171] The problem is that even though the paths s_i lie in U respectively in V, they are not necessarily loops in x_0. But since $U \cap V$ is path-connected we can find for each $i \in \{1, \dots, k\}$ a path p_i in $U \cap V$ that connects $s(t_i) \in U \cap V$ to the base point $x_0 \in U \cap V$.



Now we have the following path-homotopies:

$$s \overset{\text{by the Reparametrization–Homotopy Lemma 11.1, since } s \text{ is a reparametrization of } s_0 * \cdots * s_k}{\simeq_{\mathrm{p}}} s_0 * \cdots * s_k \overset{\text{since by the Path-Homotopy Product Proposition 11.4 (3) we know that each } p_i * \overline{p_i} \text{ is null-homotopic}}{\simeq_{\mathrm{p}}} s_0 * p_1 * \overline{p_1} * s_1 * p_2 * \overline{p_2} * s_2 * p_3 * \overline{p_3} * \ldots * \overline{p_k} * s_k$$

$$\simeq_{\mathrm{p}} \underbrace{(\, s_0 * p_1 \,)}_{\substack{\text{loop in } (U,x_0) \\ \text{or in } (V,x_0)}} * \underbrace{(\, \overline{p_1} * s_1 * p_2 \,)}_{\substack{\text{loop in } (U,x_0) \\ \text{or in } (V,x_0)}} * \underbrace{(\, \overline{p_2} * s_2 * p_3 \,)}_{\substack{\text{loop in } (U,x_0) \\ \text{or in } (V,x_0)}} * (\overline{p_3} * \ldots * (\overline{p_k} * s_k).$$

Thus we have shown that s is path-homotopic to a concatenation of loops, each of which lies in (U, x_0) or in (V, x_0). ■

20.3. **Interlude: An alternative definition of the free product of groups.** In our proof of the injectivity of the map in the Seifert–van Kampen-Theorem 20.1 it is convenient to work with a different model for the free and the amalgamated product of two groups. We will present this alternative approach in this purely algebraic section. We return to the proof of the Seifert–van Kampen-Theorem 20.1 in the next section.

Definition. Let A and B be two groups.

(1) We set[172]
$$S(A,B) \;:=\; \{\text{all finite sequences } (g_1, \dots, g_k) \text{ with each } g_i \text{ in } A \text{ or } B\}.$$
We equip $S(A,B)$ with the product structure given by
$$(x_1, \dots, x_k) \cdot (y_1, \dots, y_l) \;:=\; (x_1, \dots, x_k, y_1, \dots, y_l).$$
This turns $S(A,B)$ into a monoid.[173]

[171] Indeed, if $s(t_i)$ lies in $U \setminus V$, then $s([t_{i-1}, t_i])$ and $s([t_i, t_{i+1}])$ cannot lie in V, hence they both lie in U. This implies that $s_{([t_{i-1}, t_{i+1}])}$ lies in U, and therefore we can remove t_i from the subdivision. We proceed similarly if $s(t_i)$ lie in $V \setminus U$.

(2) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. We denote by $\sim_G$ the equivalence relation on $S(A,B)$ that is generated by the relations

(a) $$(x_1,\dots,x_l,e,x_{l+2},\dots,x_k) \sim_G (x_1,\dots,x_l,x_{l+2},\dots,x_k)$$

(b) $$(x_1,\dots,x_l,x_{l+1},\dots,x_k) \sim_G (x_1,\dots,x_l\cdot x_{l+1},\dots,x_k).$$
$\uparrow$ if both x_l and x_{l+1} lie in A or both lie in B

(c) $$(x_1,\dots,x_l,\alpha(g),x_{l+2},\dots,x_k) \sim_G (x_1,\dots,x_l,\beta(g),x_{l+2},\dots,x_k).$$
$\uparrow$ for any $g \in G$

If $G=\{e\}$, then we just write $\sim$ instead of $\sim_G$.

(3) The monoid structure on $S(A,B)$ descends to a well-defined map

$$\begin{array}{rcl} S(A,B)/\sim_G \times S(A,B)/\sim_G & \to & S(A,B)/\sim_G \\ ([(g_1,\dots,g_k)],[h_1,\dots,h_l]) & \mapsto & [(g_1,\dots,g_k,h_1,\dots,h_l)]. \end{array}$$

One can easily verify that this defines a group structure on $S(A,B)/\sim_G$.[174]

Proposition 20.3. (Alternative–Amalgamated Product Proposition) Let A and B be two groups and let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. The map[175]

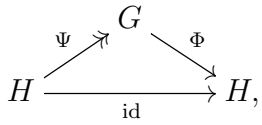
$$\begin{array}{rcl} \Phi\colon S(A,B)/\sim_G & \to & A *_G B = (A*B)/\langle\langle\{\alpha(g)\cdot\beta(g)^{-1}\}_{g\in G}\rangle\rangle \\ [\underbrace{(x_1,\dots,x_k)}_{\in S(A,B)}] & \mapsto & [\underbrace{x_1\cdot\ldots\cdot x_k}_{\in A*B}] \end{array}$$

is well-defined and it is a group isomorphism.

In the proof of the Alternative–Amalgamated Product Proposition 20.3 we will need the following little lemma.

Auxiliary Lemma 20.4. Let $\Phi\colon G \to H$ and $\Psi\colon H \to G$ be two group homomorphisms such that $\Phi\circ\Psi = \mathrm{id}_H$ and such that Ψ is an epimorphism. Then Φ is an isomorphism.

The lemma can be summarized as follows: If we are given a commutative diagram of the form



then Φ is an isomorphism.

Proof of Lemma 20.4. It follows from $\Phi\circ\Psi = \mathrm{id}_H$ that Φ is an epimorphism. It remains to show that Φ is a monomorphism. So let $g \in \ker(\Phi)$. Since Ψ is an epimorphism there

[172]The definition of $S(A,B)$ is similar to the definition of $A*B$, but now we allow that elements of the sequence are trivial and we allow that consecutive elements lie in the same group.

[173]Loosely speaking a monoid is like a group, except that we do not demand the existence of inverses. In our case the neutral element is again represented by the empty sequence ( ).

[174]Since we have a monoid structure on $S(A,B)$ we see that this multiplication defines a monoid structure on $S(A,B)/\sim_G$. Furthermore it follows from (2a) and (2b) that the inverse of $[(g_1,\dots,g_k)]$ is given by $[(g_k^{-1},\dots,g_1^{-1})]$. Thus we have in fact a group structure.

[175]Recall that we use the natural monomorphisms from the Coproduct-of-Groups Lemma 19.2 (1a) to view A and B as subgroups of $A*B$. If we want to be more precise, then we could say that Φ is given by $[(x_1,\dots,x_k)] \mapsto [(x_1)\cdot\ldots\cdot(x_k)]$.

exists an $h \in H$ with $\Psi(h) = g$. Since $\Phi \circ \Psi = \mathrm{id}_H$ we see that $h = \Phi(\Psi(h)) = \Phi(g) = e$. Thus $h = e$ and thus $g = \Psi(h) = \Psi(e) = e$. ■

Proof of Proposition 20.3. We start out with the following claim.

Claim 1. The map
$$\begin{array}{rcl} \Phi\colon S(A,B)/\!\sim_G & \to & A *_G B = (A * B)/\langle\langle\{\alpha(g)\cdot\beta(g)^{-1}\}_{g\in G}\rangle\rangle \\ [(x_1,\dots,x_k)] & \mapsto & [x_1\cdot\ldots\cdot x_k] \end{array}$$
is well-defined, it is a homomorphism and it is surjective.

Proof. We consider the map
$$\begin{array}{rcl} \widetilde{\Phi}\colon S(A,B) & \to & A *_G B \\ (x_1,\dots,x_k) & \mapsto & [x_1\cdot\ldots\cdot x_k]. \end{array}$$

As above we view $S(A,B)$ as a monoid. With this structure it is clear that $\widetilde{\Phi}$ is a monoid homomorphism. Since $A * B$ is a subset of $S(A,B)$ we see that $\widetilde{\Phi}$ is surjective.

It remains to show that $\widetilde{\Phi}$ descends to a well-defined map on $S(A,B)/\!\sim_G$. The first two relations in the definition of $\sim_G$ do not pose a problem. But let us consider relation (c). Let $w = (x_1,\dots,x_l)$, $\widetilde{w} = (x_{l+2},\dots,x_k)$ in $S(A,B)$ and let $g \in G$. Note that we have the following equality in $A *_G B$:

$$\widetilde{\Phi}(\underbrace{(w,\alpha(g),\widetilde{w})}_{\in S(A,B)}) \overset{\text{since } \widetilde{\Phi} \text{ is a monoid homomorphism}}{\downarrow} = \widetilde{\Phi}(w)\cdot\alpha(g)\cdot\widetilde{\Phi}(\widetilde{w}) = \widetilde{\Phi}(w)\cdot\overbrace{\alpha(g)\cdot\beta(g)^{-1}}^{=e\in A*_G B}\cdot\beta(g)\cdot\widetilde{\Phi}(\widetilde{w})$$
$$= \widetilde{\Phi}(w)\cdot\beta(g)\cdot\widetilde{\Phi}(\widetilde{w}) = \widetilde{\Phi}(\underbrace{(w,\beta(g),\widetilde{w})}_{\in S(A,B)}).$$

We have thus verified that $\widetilde{\Phi}$ factors through the relation given by (c). ⊞

Claim 2. The map
$$\begin{array}{rcl} \Psi\colon A *_G B & \to & S(A,B)/\!\sim_G \\ \underbrace{[(x_1,\dots,x_k)]}_{\in A*B} & \mapsto & \underbrace{[(x_1,\dots,x_k)]}_{\in S(A,b)} \end{array}$$
is well-defined and it is a homomorphism.

Proof. First we consider the map
$$\begin{array}{rcl} \widetilde{\Psi}\colon A * B & \to & S(A,B)/\!\sim_G \\ (x_1,\dots,x_k) & \mapsto & [(x_1,\dots,x_k)]. \end{array}$$

One can easily verify that this is in fact a group homomorphism. It follows immediately from the definitions that for any $g \in G$ we have

$$\widetilde{\Psi}(\alpha(g)\cdot\beta(g)^{-1}) = [(\alpha(g),\beta(g)^{-1})] \underset{\uparrow}{=} [(\beta(g),\beta(g)^{-1})] \underset{\uparrow}{=} [(\beta(g)\cdot\beta(g)^{-1})] \underset{\uparrow}{=} [()] = e.$$
follows from part (c), (b) and (a) of the definition of $\sim_G$

Thus it follows from the Normal Closure–Factorization Lemma 19.10 that $\widetilde{\Psi}$ descends to a well-defined homomorphism $\Psi\colon A *_G B \to S(A,B)/\!\sim_G$. ⊞

It is clear that $\Phi \circ \Psi = \mathrm{id}_{A*_G B}$. It is not so obvious that $\Psi \circ \Phi = \mathrm{id}_{S(A,B)/\sim_G}$. In fact we will proceed in a slightly different fashion: to show that Φ is an isomorphism it suffices, according to Lemma 20.4, to show that $\Psi\colon A *_G B \to S(A,B)/\!\sim_G$ is surjective. This in turn follows from the following claim.

Claim 3. Any element in $S(A,B)$ is equivalent, via $\sim_G$, to an element in $A * B$.

Proof. We prove the claim by induction on the length of a sequence in $S(A,B)$. Clearly the claim holds for the unique sequence in $S(A,B)$ of length 0. Suppose we already know the claim for all sequences of length $\leq k-1$. Let $(x_1,\dots,x_k) \in S(A,B)$ be a sequence of length k. If $(x_1,\dots,x_k)$ is a reduced word then by definition it lies in $A * B$ and we are done. Now we suppose that $(x_1,\dots,x_k)$ is not reduced. Then, by the definition on page 238, one of the following occurs:

(1) there exists an $i \in \{1,\dots,k\}$ with $x_i = e$, or
(2) there exists an $i \in \{1,\dots,k-1\}$ with $x_i, x_{i+1} \in A$, or
(3) there exists an $i \in \{1,\dots,k-1\}$ with $x_i, x_{i+1} \in B$.

In the first case the sequence is equivalent to $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_k)$. In the second and third case the sequence is equivalent to $(x_1,\dots,x_{i-1}\cdot x_{i+1},\dots,x_k)$. In all three cases we have thus shown that our original sequence is equivalent to a sequence of length $k-1$. So we are done by our induction hypothesis. ∎

Remark. Let A and B be two groups. We have now seen that there are two ways to think about the free product $A * B$. We can either take the original approach by viewing elements of $A * B$ as reduced words. Or we can use the Alternative–Amalgamated Product Proposition 20.3 to identify $A * B$ with the group $S(A,B)/\sim$. Both approaches to defining the free product of A and B have their advantages:

(1) In the group $S(A,B)/\sim$ it is easier to write down the product of two elements, we just need to juxtapose sequences of elements in A and B.
(2) On the other hand in the group $A * B$ it is trivial to check whether two elements are the same. For example for two non-trivial elements $a \in A$ and $b \in B$ the elements (a,b) and (b,a) of $A * B$ are by definition different. But from the definition of $S(A,B)/\sim$ it is not immediately clear that $[(a,b)]$ and $[(b,a)]$ are indeed different.

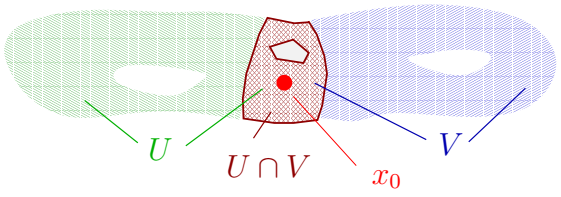
20.4. **Proof of the Seifert–van Kampen-Theorem 20.1: Injectivity.** We return to the proof of the Seifert–van Kampen-Theorem 20.1. We recall the setup. Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and with $U \cap V \neq \varnothing$ such that $U \cap V$ is non-empty. We choose a base point $x_0 \in U \cap V$. We write $A := \pi_1(U,x_0)$, $B := \pi_1(V,x_0)$ and $G := \pi_1(U \cap V, x_0)$. On page 251 we defined a homomorphism

$$\Phi\colon A *_G B = \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) \to \pi_1(X,x_0).$$

In Section 20.2 we showed that this homomorphism is surjective. It remains to show that this homomorphism is injective. This is what we will do now:

Proof of the Seifert–van Kampen-Theorem 20.1: Injectivity. We want to show that the map $\Phi\colon A *_G B = \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) \to \pi_1(X,x_0)$ which is induced by the inclusions $i\colon U \to X$ and $j\colon V \to X$ is injective.
To prove injectivity of Φ it is enough to show that there exists a map $\Psi\colon \pi_1(X,x_0) \to A *_G B$ such that $\Psi \circ \Phi = \mathrm{id}_{A*_G B}$. We will prove the existence of such a map Ψ in three steps:

(1) First we construct a map $\overline{\Psi}\colon \{\text{loops in } (X,x_0)\} \to A *_G B$.



(2) Then we show that $\overline{\Psi}$ descends to a well-defined map

$$\begin{array}{rcl} \Psi\colon \pi_1(X,x_0) & \to & A *_G B \\ [g] & \mapsto & \overline{\Psi}(g). \end{array}$$

(3) Finally we show that $\Psi \circ \Phi = \mathrm{id}_{A*_G B}$.

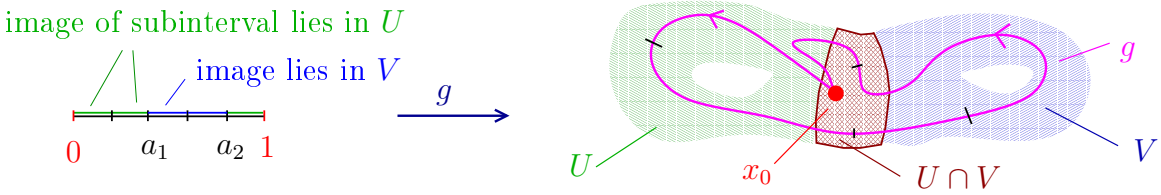
For the remainder of the proof we will identify $A *_G B$ with $S(A,B)/\sim_G$ using the isomorphism given by the Alternative–Amalgamated Product Proposition 20.3.

Step (1). Let $g\colon [0,1] \to X$ be a loop in (X, x_0). We apply the Lebesgue Lemma 11.9 to the map $g\colon [0,1] \to X$ and the open cover $X = U \cup V$. We obtain an N such that for every $i \in \{0, \ldots, N-1\}$ the image of the interval $[\frac{i}{N}, \frac{i+1}{N}]$ under the map g lies in U or in V. We introduce a few definitions.

(a) We say a subinterval of $[0,1]$ can be *colored green* if the image of the subinterval under the map g lies in U and we say an interval can be *colored blue* if the image of the subinterval lies in V. Note that the image of a subinterval lies in $U \cap V$ if and only if the subinterval can be colored green as well as blue.

(b) A *coloring* is a map $\alpha\colon \{0, \frac{1}{N}, \ldots, 1 - \frac{1}{N}\} \to \{\text{blue}, \text{green}\}$ such that each subinterval $[i, i + \frac{1}{N}]$ can be colored by $\alpha(i)$.

(c) Given a coloring α we denote by $0 < a_1 < \cdots < a_{k-1} < 1$ the numbers where the coloring of the subintervals changes from one color to another. Note that for each a_i the intervals $[a_i - \frac{1}{N}, a_i]$ and $[a_i, a_i + \frac{1}{N}]$ can be colored in two different colors, which implies that $g(a_i) \in U \cap V$.

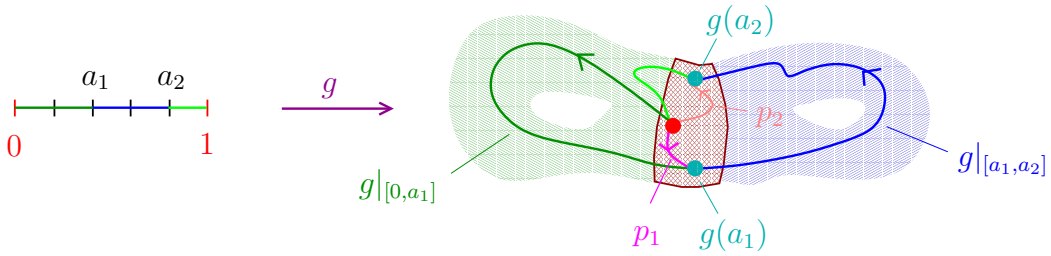


(d) A *basing* for a coloring α is a collection of paths $p_1, \ldots, p_{k-1}$ in $U \cap V$ such that each path p_i goes from x_0 to $g(a_i)$.[176]

[176]Such basing exists since each $g(a_i)$ lies in $U \cap V$ and since $U \cap V$ is path-connected.

Given a coloring α and a basing $p_1, \dots, p_{k-1}$ we set[177] [178]

$$\overline{\Psi}(g) := [\underbrace{[g|_{[0,a_1]} * \overline{p}_1]}_{\substack{\text{loop in } (U,x_0) \text{ or} \\ \text{loop in } (V,x_0)}}, \underbrace{[p_1 * g|_{[a_1,a_2]} * \overline{p}_2]}_{\substack{\text{loop in } (U,x_0) \text{ or} \\ \text{loop in } (V,x_0)}}, \dots, \underbrace{[p_{k-1} * g|_{[a_{k-1},1]}]}_{\substack{\text{loop in } (U,x_0) \text{ or} \\ \text{loop in } (V,x_0)}}] \in S(A,B)/\sim_G = A *_G B.$$



Our next goal is to prove the following claim.

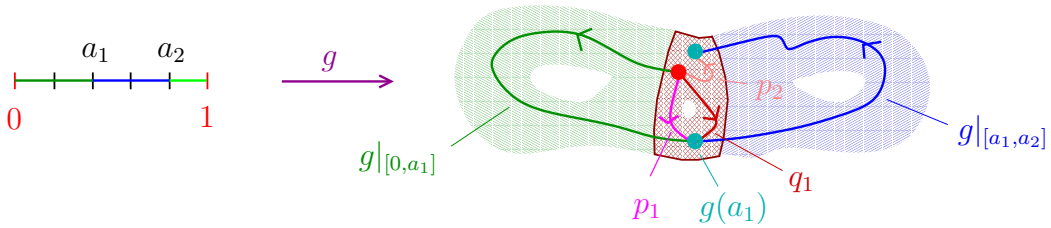
Claim 1. The definition of $\overline{\Psi}(g) \in S(A,B)/\sim_G$ does not depend on any of the choices made.

Proof. We need to prove the following three statements:

(a) Given N and a coloring the definition does not depend on the choice of the basings.
(b) Given N the definition does not depend on the choice of the coloring.
(c) The definition does not depend on the choice of N.

We now prove these three statements:

(a) We first fix a coloring α and we show that the definition of $\overline{\Psi}(g)$ does not depend on the choice of the basing. Thus let us pick a different path q_1 from x_0 to $g(a_1)$.



[177] Each path $p_m * g|_{[a_m,a_{m+1}]} * \overline{p}_{m+1}$ is a loop in (U,x_0) or (V,x_0), so each path represents an element in $\pi_1(U,x_0)$ or $\pi_1(V,x_0)$. Thus we obtain a sequence of elements in $\pi_1(U,x_0)$ and $\pi_1(V,x_0)$. The equivalence class of this sequence defines an element in $S(A,B)/\sim_G = A *_G B$.
[178] This definition is almost the same as in the proof on page 253.

In this case we have the following equality in $S(A,B)/\sim_G$:

$$\begin{aligned}
&\text{definition of } \overline{\Psi}(g) \text{ using the basing } p_1, p_2, \dots, p_{k-1} \;=\\
&\quad= \big[[g|_{[0,a_1]} * \overline{p}_1],\ [p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&\quad= \big[[g|_{[0,a_1]} * \overline{p}_1 * \overbrace{(p_1 * \overline{q}_1) * (\underbrace{q_1 * \overline{p}_1}_{\text{loop in } U\cap V})}^{\text{null-homotopic loop}}],\ [p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&\quad= \big[[g|_{[0,a_1]} * \overline{p}_1 * (p_1 * \overline{q}_1)],\ [(q_1 * \overline{p}_1) * p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&\qquad\uparrow\\
&\qquad\text{note that } q_1 * \overline{p}_1 \text{ is a loop in } U\cap V, \text{ thus it defines elements in } G = \pi_1(U\cap V, x_0),\\
&\qquad\text{thus we get this equality in } S(A,B)/\sim_G\\
&\quad= \big[[g|_{[0,a_1]} * (\underbrace{\overline{p}_1 * p_1}_{\text{null-homotopic}}) * \overline{q}_1)],\ [q_1 * (\underbrace{\overline{p}_1 * p_1}_{\text{null-homotopic}}) * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&\quad= \big[[g|_{[0,a_1]} * \overline{q}_1],\ [q_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&\quad= \text{definition of } \overline{\Psi}(g) \text{ using the basing } q_1, p_2, \dots, p_{k-1}.
\end{aligned}$$

Basically the same argument (with slightly more messy notation) shows that the choices of the other paths p_i are also irrelevant.

(b) Now we show that the definition does not depend on the choice of the coloring. So suppose we have two colorings α and β. We can turn one coloring into the other by a finite sequence of color swaps of a single subinterval. Therefore it suffices to consider the case that there exists a single $i \in \{0, \frac{1}{N}, \dots, 1-\frac{1}{N}\}$ where the values for $\alpha(i)$ and $\beta(i)$ differ.

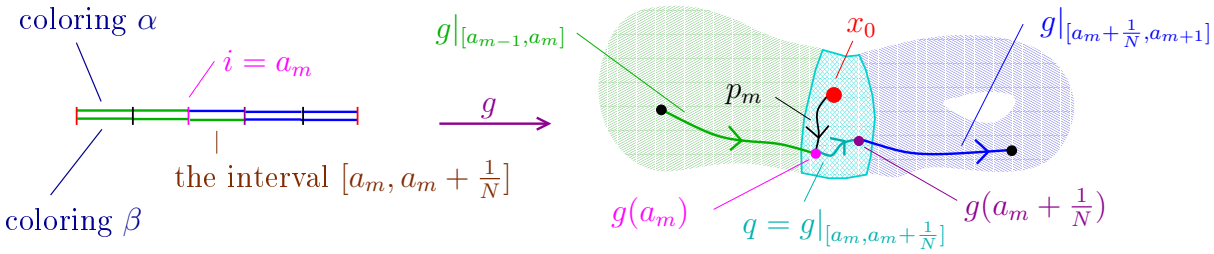
We consider the case, sketched in the figure below, that $\alpha(i)$ = blue, $\beta(i)$ = green and such that the common coloring to the left is green and the common coloring to the right is blue. All other cases are dealt with in a similar way.[179]

Let $0 = a_0 < a_1 < \cdots < a_k = 1$ be the decomposition corresponding to the coloring α. We pick a basing $p_1, \dots, p_{k-1}$ for α. There exists a unique m with $a_m = i$. The corresponding decomposition for the coloring β is given by $0 = a_0 < a_1 < \cdots < a_m + \frac{1}{N} < a_{m+1} < a_k = 1$. We write $q = g_{[a_m, a_m+\frac{1}{N}]}$. Since $[a_m, a_m + \frac{1}{N}]$ can be colored green and blue we see that q lies actually in $U \cap V$. We have

$$\begin{aligned}
&\text{definition of } \overline{\Psi}(g) \text{ using the coloring } \alpha \text{ and the basing } p_1, \dots, p_{k-1} \;=\\
&\quad= \big[[g|_{[0,a_1]} * \overline{p}_1], \dots, [p_{m-1} * g|_{[a_{m-1},a_m]} * \overline{p}_m], [p_m * g|_{[a_m,a_{m+1}]} * \overline{p}_{m+1}], \dots\big]\\
&\quad= \big[[g|_{[0,a_1]} * \overline{p}_1], \dots, [p_{m-1} * g|_{[a_{m-1},a_m+\frac{1}{N}]} * \overline{q} * \overline{p}_m], [p_m * q * g|_{[a_m+\frac{1}{N},a_{m+1}]} * \overline{p}_{m+1}], \dots\big]\\
&\qquad\uparrow\\
&\qquad\text{since } g_{[a_{m-1},a_m]} \simeq_{\mathrm{p}} g_{[a_{m-1},a_m+\frac{1}{N}]} * \overline{q} \text{ and } g_{[a_m,a_m+\frac{1}{N}]} \simeq_{\mathrm{p}} q * g_{[a_m+\frac{1}{N},a_{m+1}]}\\
&\quad= \text{definition of } \overline{\Psi}(g) \text{ using the coloring } \beta \text{ and basing } p_1, \dots, p_{m-1}, p_m * q, p_{m+1}, \dots, p_{k-1}.
\end{aligned}$$

(c) First suppose that instead of dividing $[0,1]$ into N intervals we divide it into $m \cdot N$ intervals for some $m \in \mathbb{N}$. A coloring for the subdivision into N subintervals then also gives rise to a coloring into $m \cdot N$ subintervals, and it follows immediately from the definitions that we end up with the same element of $A *_G B$. Now if we had chosen a

[179] In fact the only case which requires a slightly modified argument is that the coloring to the left and right are the same, i.e. either both are blue or both are green. We leave it to the reader to perform that argument.



different N' instead, then we can replace N and N' by $N \cdot N'$ and by the argument we just made we obtain the same element of $A *_G B$. This concludes the proof of (c). ⊞

Step (2). We need to prove the following claim.

Claim 2. If $g, g' \colon [0,1] \to X$ are two path-homotopic loops in (X, x_0), then $\overline{\Psi}(g) = \overline{\Psi}(g')$.

Proof. Let $g, g' \colon [0,1] \to X$ be two path-homotopic loops in (X, x_0). We pick a path-homotopy

$$\begin{array}{rcl} H \colon [0,1] \times [0,1] & \to & X \\ (t,s) & \mapsto & H(t,s) \end{array}$$

from the loop g to the loop g'. This time we apply the Lebesgue Lemma 11.9 to the map $H \colon [0,1] \times [0,1] \to X$ and the open cover $X = U \cup V$. We obtain an $N \in \mathbb{N}$ such that for every choice of $i, j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$ the image of the square $[i, i + \frac{1}{N}] \times [j, j + \frac{1}{N}]$ under the map H lies in U or in V. Given $a \in [0,1]$ we introduce the two paths

$$\underbrace{\begin{array}{rcl} H_a \colon [0,1] & \to & X \\ t & \mapsto & (t,a) \end{array}}_{\text{“horizontal path”}} \quad \text{and} \quad \underbrace{\begin{array}{rcl} V_a \colon [0,1] & \to & X \\ t & \mapsto & (a,t). \end{array}}_{\text{“vertical path”}}$$

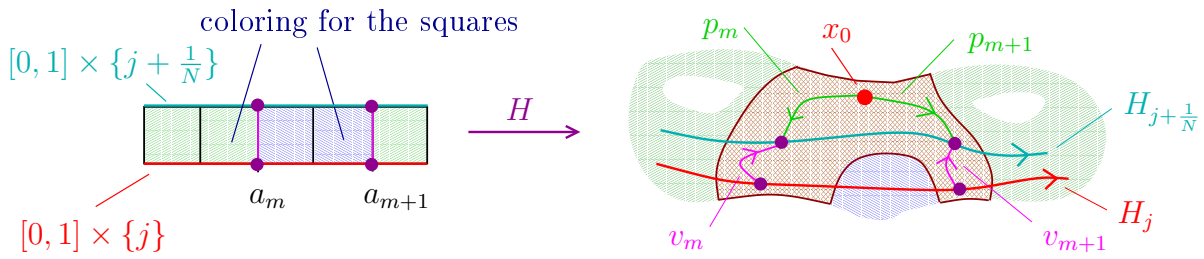
Note that each H_a is a loop in x_0 and that $H_0 = g$ and $H_1 = g'$. Therefore it suffices to show that for any $j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$ we have $\overline{\Psi}(H_j) = \overline{\Psi}(H_{j+\frac{1}{N}})$. So let $j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$.

As above we say that a subset of $[0,1] \times [0,1]$ can be *colored green* if the image of the subset under the map H lies in U and we say the subset can be *colored blue* if the image lies in V. We pick a map $\alpha \colon \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\} \to \{\text{blue}, \text{green}\}$ such that for each i the value $\alpha(i)$ is a color with which the square $[i, i + \frac{1}{N}] \times [\frac{j}{N}, j + \frac{1}{N}]$ can be colored. Note that α also defines a coloring for the loops H_j and $H_{j+\frac{1}{N}}$.

We define $0 = a_0 < a_1 < \cdots < \dots a_k = 1$ as above for the coloring α. We pick a basing p_i, $i = 1, \dots, k-1$ for the loop H_j. For $i = 0, \dots, k$ we write $v_i = V_{a_i}|_{[j, j+\frac{1}{N}]}$. Put differently, v_j connects the point $H_j(a_i)$ to the corresponding point $H_{j+\frac{1}{N}}(a_i)$. Now we have

$$\begin{array}{rcl} \overline{\Psi}(H_j) & = & \overline{\Psi}(H_j) \text{ defined using the coloring } \alpha \text{ and the basing } p_1, \dots, p_{k-1} \\ & = & \big[\dots\big] * \big[p_m * H_j|_{[a_m, a_{m+1}]} * \overline{p}_{m+1}\big] * \big[\dots\big] \\ & = & \big[\dots\big] * \big[p_m * v_m * \overline{v}_m * H_j|_{[a_m, a_{m+1}]} * v_{m+1} * \overline{v}_{m+1} * \overline{p}_{m+1}\big] * \big[\dots\big] \\ & \uparrow & \\ \multicolumn{3}{l}{\text{since each } v_m * \overline{v}_m \text{ is null-homotopic} \qquad\qquad} \\ & = & \big[\dots\big] * \big[p_m * v_m * H_{j+\frac{1}{N}}|_{[a_m, a_{m+1}]} * \overline{v}_{m+1} * \overline{p}_{m+1}\big] * \big[\dots\big] \\ & \uparrow & \\ \multicolumn{3}{l}{\text{since } H \text{ gives rise to path-homotopies from } \overline{v}_m * H_j|_{[a_m, a_{m+1}]} * v_{m+1} \text{ to } H_{j+\frac{1}{N}}|_{[a_m, a_{m+1}]} \text{ for } m = 0, \dots, k-1} \\ & = & \overline{\Psi}(H_{j+\frac{1}{N}}) \text{ defined using } \alpha \text{ and the basing } p_1 * v_1, \dots, p_{k-1} * v_{k-1}. \end{array}$$

This concludes the proof of the claim and thus of part (2) of our three step program. ⊞



Summarizing we now showed that the map $\overline{\Psi}$ descends to a well-defined map

$$\begin{array}{rcl}\Psi\colon \pi_1(X,x_0) & \to & A *_G B \\ [g] & \mapsto & \big[\overline{\Psi}(g)\big].\end{array}$$

Step (3). It remains to prove the following claim.

Claim 3. We have $\Psi\circ\Phi = \mathrm{id}_{A*_G B}$.

Proof. So let $[y_1]\cdot\ldots\cdot[y_k]\in A*_G B$ where $y_1,\ldots,y_k$ are loops in (U,x_0) and (V,x_0). Then

$$(\Psi\circ\Phi)([y_1]\cdot\ldots\cdot[y_k]) \overset{\text{definition of } \Phi\colon A*_G B=\pi_1(U,x_0)*\pi_1(V,x_0)\to\pi_1(X,x_0)}{\downarrow} = \Psi([y_1*\cdots*y_k]) = \big[\overline{\Psi}(y_1*\cdots*y_k)\big] = [y_1*\cdots*y_k]$$
$$= [y_1]\cdot\ldots\cdot[y_k].$$

$\uparrow$ by definition of $\overline{\Psi}$, where we use $N=k$ and where we can use the constant paths as basings since $y_1,\ldots,y_k$ are already loops in (U,x_0) and (V,x_0)

This concludes the proof of the claim and thus also the proof of the Seifert–van Kampen-Theorem 20.1 ■

20.5. **Fundamental groups of wedges.** We recall the following definition from page 219.

Definition. Let $\{(X_i,x_i)\}_{i\in I}$ be a non-empty family of pointed topological spaces.

(1) We refer to

$$\bigvee_{i\in I}(X_i,x_i) := \Big(\bigsqcup_{i\in I} X_i\Big)\Big/\sim \quad \text{where } x_i\sim x_j \text{ for all } i,j\in I$$

as the wedge of the pointed topological spaces (X_i,x_i). We refer to the point given by the identifications as the wedge point and we denote it by $*$.

(2) Let $j\in I$. We refer to

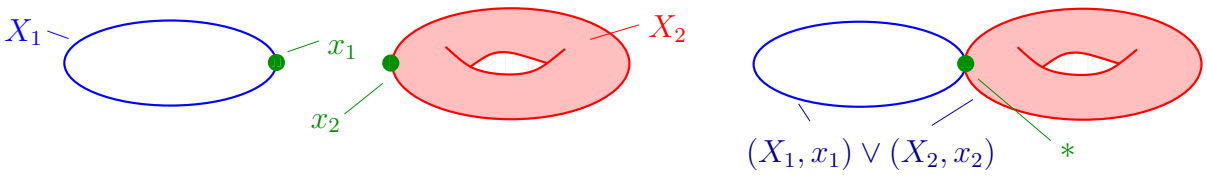
$$\begin{array}{rcl}\iota_j\colon X_j & \to & \bigvee_{i\in I}(X_i,x_i) \\ x & \mapsto & [x]\end{array}$$

as the natural inclusion. Note that in the Wedge Proposition 17.1 we showed that each natural inclusion is an embedding, thus we can view each X_j as a subspace of the wedge.

The wedge of the empty family of pointed topological spaces is defined as the one-point topological space $\{*\}$.

Let $(X_i,x_i)_{i\in I}$ be a family of pointed topological spaces. The natural inclusions ι_{i*} induce a homomorphism

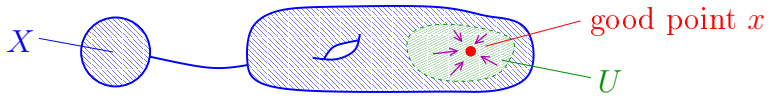
$$\underbrace{\ast_{i\in I}\pi_1(X_i,x_i)}_{\substack{\text{coproduct of the}\\ \text{fundamental groups}}} \to \pi_1\Big(\underbrace{\bigvee_{i\in I}(X_i,x_i)}_{\substack{\text{coproduct of}\\ \text{the } (X_i,x_i)}},*\Big).$$



It seems reasonable to expect that this homomorphism is an isomorphism. For simplicity let us restrict ourself for a second to the case that we are dealing with just two topological spaces X_1 and X_2. It is now very tempting to apply the Seifert–van Kampen-Theorem 20.1 to the decomposition $X_1 \vee X_2 = X_1 \cup X_2$. Unfortunately this approach hits a snag: in the Seifert–van Kampen Theorem we have to deal with decompositions of a topological space X into *open* subsets. But in most cases X_1 and X_2 will certainly not be open subspaces of $X_1 \vee X_2$.

The idea now is to replace each X_i by an *open* subset U_i which is "close" to X_i. This idea leads us to the following somewhat technical definition.

Definition. We say a point x in a topological space X is good, if the following two conditions are satisfied:
(1) $\{x\}$ is a closed subset of X[180].
(2) There exists an open neighborhood U of x such that x is a deformation retract of U.



Remark. What we call a "good point" is sometimes called a "non-degenerate point", see e.g. [Lee00, p. 212]. On the other hand, the term "non-degenerate point" is defined in a subtly different way, see e.g. [Spa95, p. 380] or [Bre93, p. 435]. Also note that a point $x \in X$ is good if the pair $(X, \{x\})$ is good in the sense of [Hat02, p. 114].

The following lemma gives us two important types of examples of good points, which will take care of most examples that we will have to deal with.

Lemma 20.5. (Good Point-Lemma)
(1) Every point in a topological manifold (e.g. on a sphere) is good.
(2) Let (X_i, x_i) be a family of pointed topological spaces. If each $x_i \in X_i$ is good, then the wedge point $*$ is a good point in $\bigvee_{i \in I} (X_i, x_i)$.

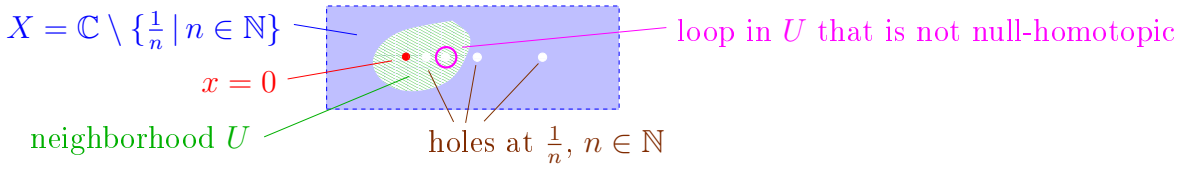
Proof.

[180]Recall that if X is Hausdorff, then we know by the Points-in-Hausdorff Lemma 1.8 (1) that any subset consisting of a single point is closed. Thus the first condition in the definition of a good point is often satisfied "for free".

(1) This follows easily from the fact that topological manifolds are by definition Hausdorff and that they are locally homeomorphic to open (half-) balls.

(2) Let (X_i, x_i) be a family of pointed topological spaces such that each $x_i \in X_i$ is good. It follows easily from the Wedge Proposition 17.1 (0) that the wedge point $*$ defines a closed subset of the wedge. For each $i \in I$ let U_i be an open neighborhood of $x_i \in X_i$ such that x_i is a deformation retract of U_i. We can think of $U := \bigvee_{i\in I} U_i$ as an open neighborhood of the wedge point $*$. One can easily show that U admits a deformation retraction to $*$. ■

Non-Example. We consider $X = \mathbb{C} \setminus \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Morally speaking the point $x = 0$ is not good, "since any open neighborhood of 0 has non-trivial fundamental group". We leave leave it to the reader to turn this slogan into a rigorous proof.



We can now state the main result of this section:

Proposition 20.6. (Wedge-π_1-Proposition) Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces. We suppose that each x_i is a good point in X_i. Given $j \in I$ we denote by $\iota_j \colon X_j \to \bigvee_{i\in I} X_i$ the natural inclusion. We denote by $*$ the wedge point of $\bigvee_{i\in I}(X_i, x_i)$. The map

$$\ast_{i\in I} \pi_1(X_i, x_i) \;\to\; \pi_1\Big(\bigvee_{i\in I}(X_i, x_i), *\Big),$$

which is induced by the inclusions ι_i together with the Coproduct-of-Groups Lemma 19.2 (1b), is an isomorphism.

Proof. We postpone the proof of the Wedge-π_1-Proposition 20.6 to the next section. ■

Using the Wedge-π_1-Proposition 20.6 we can now provide the long overdue calculation of the fundamental groups of the wedge of circles:

Lemma 20.7. (Wedge-of-Circles–π_1-Lemma) Let S be a set. The map[181]

$$\begin{array}{rcl} \langle S\rangle & \to & \pi_1\Big(\bigvee_{s\in S}(S^1, 1), *\Big) \\ s & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & \bigvee_{s\in S}(S^1, 1) \\ t & \mapsto & [\exp(2\pi \mathrm{i} t), s] \end{array}\right] \end{array}$$

that is determined by

is an isomorphism.

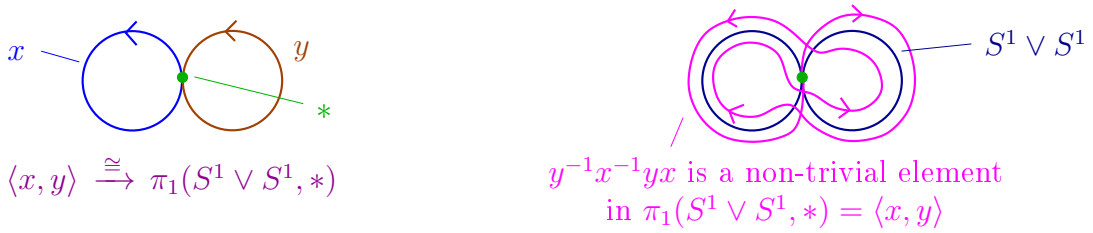
Proof. We consider the following maps:

$$\langle S\rangle \underset{\text{(a)}}{=} \ast_{s\in S} \langle s\rangle \xrightarrow[\text{(b)}]{\cong} \ast_{s\in S} \pi_1(S^1, 1) \xrightarrow[\text{(c)}]{\cong} \pi_1\Big(\bigvee_{s\in S}(S^1, 1), *\Big)$$

We now clarify what we mean by the above maps:

(a) By the definition on page 242 we have $\langle S\rangle = \ast_{s\in S} \langle s\rangle$ where $\langle s\rangle = \{\ldots, s^{-1}, e, s, s^2, \ldots\}$.

[181] Here $\langle S\rangle$ denotes the free group on the set S as defined on page 242.



(b) It follows from the Circle-π_1-Theorem 11.13 that for each $s \in S$ we have an isomorphism from the infinite cyclic group $\langle s \rangle$ to $\pi_1(S^1, 1)$ by sending the generator s of $\langle s \rangle$ to the path-homotopy class of the loop given by $t \mapsto \exp(2\pi i t)$.

(c) By the Good Point-Lemma 20.5 (or alternatively by a completely elementary argument) we know that $1 \in S^1$ is a good point. The isomorphism is thus given by the Wedge-π_1-Proposition 20.6.

We have now seen that all of the above maps are indeed isomorphisms. The composition is precisely the map that we claimed was an isomorphism. ■

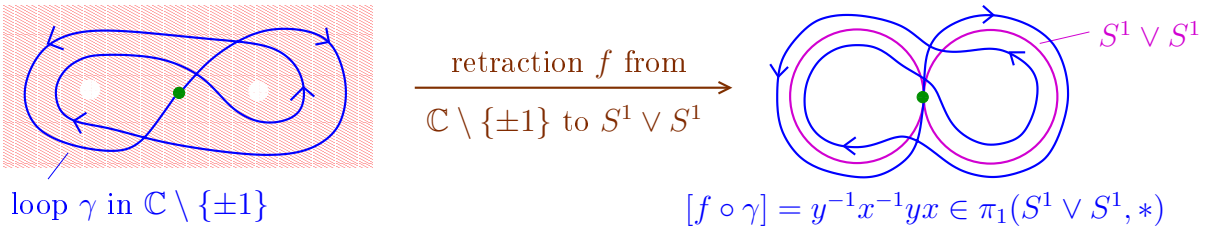
Example. We consider the loop γ from page 157 in $\mathbb{C} \setminus \{\pm 1\}$ that is also illustrated in the figure below. One can easily show that there exists a deformation retraction

$$f\colon \mathbb{C} \setminus \{\pm 1\} \;\to\; S^1 \vee S^1 \;\overset{\text{we use the homeomorphism from page 157 to make this identification}}{\downarrow =}\; \underbrace{\{z \in \mathbb{C} \mid |z+1| = 1\} \cup \{z \in \mathbb{C} \mid |z-1| = 1\}}_{\text{two circles that meet at the origin}}.$$

Note that

$$f_*([\gamma]) \;=\; [f \circ \gamma] \;=\; y^{-1}x^{-1}yx \;\in \langle x, y \rangle \;\underset{\text{we use the Wedge-of-Circles–}\pi_1\text{-Lemma 20.7 to make this identification}}{\uparrow =}\; \pi_1(S^1 \vee S^1, *),$$

which is non-trivial. Thus we see that $f \circ \gamma$ is not null-homotopic in $S^1 \vee S^1$. This implies that γ is not null-homotopic in $\mathbb{C} \setminus \{\pm 1\}$.



20.6. **Proof of the Wedge-π_1-Proposition 20.6.** In this section we provide the proof of the Wedge-π_1-Proposition 20.6. We proceed as follows:

(1) In the Finite Wedge-π_1-Lemma 20.8 we first deal with the case of the wedge of two pointed topological spaces. By applying this lemma iteratively we get the desired statement also for the wedge of finitely many pointed topological spaces.

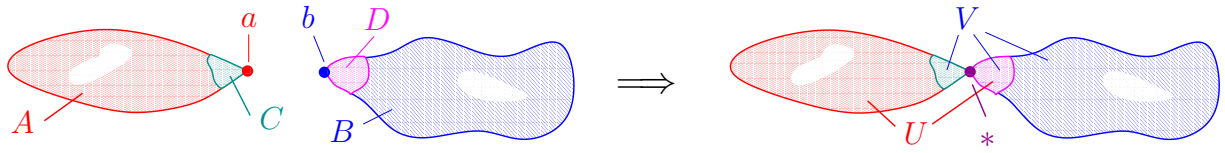
(2) We reduce the general case of the Wedge-π_1-Proposition 20.6 via a compactness argument to the case of the wedge of finitely many pointed topological spaces.

Lemma 20.8. (Finite Wedge-π_1-Lemma) Let A and B be two path-connected topological spaces and let $a \in A$ and $b \in B$ be good points. The inclusion maps induce an isomorphism[182]

$$\pi_1(A, a) * \pi_1(B, b) \;\overset{\cong}{\to}\; \pi_1((A, a) \vee (B, b), *).$$

Proof. Let A and B be two path-connected topological spaces and let $a \in A$ and $b \in B$ be good points. All wedges below will be with respect to these two base points, thus, to simplify the notation we will drop them from the notation. In this spirit we write $X = A \vee B$. As usual we denote by $*$ the wedge point in $A \vee B$. By the Wedge Proposition 17.1 (1b) we can view A and B as subspaces of $X = A \vee B$ in the obvious way.

Since a is a good point we can pick an open neighborhood C in A of a that deformation retracts to a. We can also pick an open neighborhood D of b in B that deformation retracts to b. We consider $U := A \vee D$ and $V := B \vee C$.



Next we consider the following diagram

$$\begin{array}{ccc} \pi_1(A,a) * \pi_1(B,b) & \longrightarrow & \pi_1(X,*) \\ \downarrow & & \downarrow \mathrm{id} \\ \pi_1(U,*) * \pi_1(V,*) & \longrightarrow & \pi_1(X,*). \end{array}$$

We make the following observations:

(1) The vertical and horizontal maps are all induced by the various inclusions of topological spaces. In particular the diagram commutes.
(2) We can view $U = A \vee D$ and $V = B \vee C$ as open subsets of $X = A \vee B$.
(3) One can easily show that A is a deformation retract of $U = A \vee D$.
(4) As in (3) we see that B is a deformation retract of $V = B \vee C$.
(5) As in (3), we see that $A \cap B = \{*\}$ is a deformation retract of $U \cap V = C \vee D$. It follows from the Homotopy-π_1-Proposition 13.4 (2b) that $U \cap V$ is simply connected.
(6) By (2) and (5) we can apply the Seifert–van Kampen-Theorem 20.1 to the decomposition $A \vee B = U \cup V$ and we obtain that the bottom horizontal map of the above diagram is an isomorphism.
(7) It follows from (3), (4) and the Deformation Retract–Homotopy Equivalence Lemma 9.3 that the inclusions $A \to U = A \cup D$ and $B \to V = B \cup C$ are homotopy equivalences.
(8) It follows from (7) together with the Homotopy-π_1-Proposition 13.4 (2) that the two inclusion induced maps $\pi_1(A,a) \to \pi_1(U,*)$ and $\pi_1(B,b) \to \pi_1(V,*)$ are isomorphisms. This implies that the left vertical map in the above commutative diagram is an isomorphism.
(9) We obtain from (1), (6) and (8) that the top horizontal map is, as desired, also an isomorphism. ■

Proof of the Wedge-π_1-Proposition 20.6. Let $\{(X_i, x_i)\}_{i \in I}$ be a family of pointed topological spaces such that each x_i is a good point in X_i. We use these points to form the wedge $\bigvee_{i \in I} X_i$. Recall that the fact that x_i is good means that

(1) $\{x_i\}$ is a closed subset of X_i and

[182] In the definition of a "good point" we demanded that the one-point subset given by the point is closed. Strictly speaking we do not need this fact in the proof of the Finite Wedge-π_1-Lemma 20.8.

(2) that there exists an open neighborhood U_i of x_i in X_i such that x_i is a deformation retract of U_i.

We need to show that the homomorphism

$$\Phi\colon \mathop{\ast}_{i\in I} \pi_1(X_i, x_i) \;\to\; \pi_1\Big(\bigvee_{i\in I} X_i, *\Big)$$

that is induced by the natural inclusions ι_i is an isomorphism. We will show that the homomorphism Φ is surjective and injective. In both cases the trick is to reduce the statement to the case of the Finite Wedge-π_1-Lemma 20.8.

Claim 1. The map

$$\Phi\colon \mathop{\ast}_{i\in I} \pi_1(X_i, x_i) \;\to\; \pi_1\Big(\bigvee_{i\in I} X_i\Big)$$

is injective.

Proof. Given a finite subset $J \subset I$ we consider the following diagram:

$$\begin{array}{ccc} \mathop{\ast}_{i\in J} \pi_1(X_i, x_i) & \xrightarrow{\;\cong\;} & \pi_1\Big(\bigvee_{i\in J} X_i, *\Big) \\ \downarrow & & \downarrow{\scriptstyle \iota_*} \\ \mathop{\ast}_{i\in I} \pi_1(X_i, x_i) & \longrightarrow & \pi_1\Big(\bigvee_{i\in I} X_i, *\Big). \end{array}$$

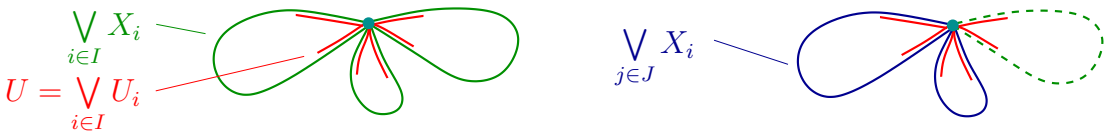
We make the following clarifications and observations:

(1) The maps are induced by the various obvious inclusion. It follows easily from the definitions that the diagram commutes.
(2) Since J is finite we know by the Finite Wedge-π_1-Lemma 20.8, applied finitely many times, that the top horizontal homomorphism is an isomorphism.
(3) Note that as on page 221 we see that $\bigvee_{i\in J} X_i \subset \bigvee_{i\in I} X_i$ is a retract. It follows from the functoriality of fundamental groups that the right vertical map is a monomorphism.
(4) Note that it follows easily from the explicit description of a free product, that given any $g = (g_1, \dots, g_m) \in \mathop{\ast}_{i\in I} \pi_1(X_i, x_i)$ there exists a finite subset $J \subset I$ such that g lies in the image of the left vertical map.
(5) The claim follows from the above observations. ⊞

The proof that Φ is also surjective requires more of an effort.

Claim 2. We set $U := \bigcup_{i\in I} U_i$. Given any compact subset $K \subset \bigvee_{i\in I} X_i$ there exists a finite subset $J \subset I$ with $K \subset U \cup \bigvee_{i\in J} X_i$.

Proof. Given $i \in I$ we set $V_i := X_i \setminus \{x_i\}$. Since x_i is a good point we see that V_i is an open subset of X_i. Using the Wedge Proposition 17.1 (0) one can now easily see that V_i is in fact an open subset of $\bigvee_{i\in I} X_i$. Evidently the V_i together with U form an open cover of $\bigvee_{i\in I} X_i$. Since K is compact we see that there exists a finite subset $J \subset I$ with $K \subset U \cup \bigcup_{j\in J} V_j$. But this set is precisely $U \cup \bigvee_{i\in J} X_i$. ⊞



Claim 3. The map

$$\Phi\colon \mathop{\ast}_{i\in I} \pi_1(X_i, x_i) \;\to\; \pi_1\Big(\bigvee_{i\in I} X_i, *\Big)$$

is surjective.

Proof. Given a finite subset $J \subset I$ we consider the following commutative diagram

$$\begin{array}{ccccc}
\mathop{\ast}_{j\in J} \pi_1(X_j, *) & \xrightarrow[\Phi_J]{\cong} & \pi_1\Big(\bigvee_{i\in J} X_i, *\Big) & \xrightarrow{\cong} & \pi_1\Big(U \cup \bigvee_{i\in J} X_i, *\Big) \\
\downarrow & & \downarrow & \swarrow & \\
\mathop{\ast}_{i\in I} \pi_1(X_i, x_i) & \xrightarrow{\Phi} & \pi_1\Big(\bigvee_{i\in I} X_i, *\Big). & &
\end{array}$$

We make the following observations and comments:

(1) The maps are either the obvious algebraic inclusion maps or they are the obvious maps induced by inclusions of topological spaces. It follows almost immediately from the definitions that the diagram commutes.

(2) Since J is finite we obtain, from applying the Finite Wedge-π_1-Lemma 20.8 finitely many times, that the top horizontal map is an isomorphism.

(3) Note that $\bigvee_{i\in J} X_i$ is a deformation retract of $U \cup \bigvee_{i\in J} X_i$. Thus it follows from the Homotopy-π_1-Proposition 13.4 (2b) that the top diagonal map is an isomorphism.

Now we turn to the actual proof that the above map Φ is surjective. So suppose we are given some $[\gamma] \in \pi_1\Big(\bigvee_{i\in J} X_i, *\Big)$. It follows from the Compact Image Lemma 2.8 together with Claim 2 that there exists a finite subset $J \subset I$ such that the image of γ is contained in $U \cup \bigvee_{i\in J} X_i$. But considering the above diagram, and using the fact that the top horizontal map is an isomorphism we see that $[\gamma]$ actually lies in the image of the bottom horizontal map Φ. ∎

20.7. **Fundamental groups of topological graphs.** Now we have all the tools to calculate the fundamental groups of the topological realizations of finite abstract graphs.

Proposition 20.9. (Graph-π_1-Proposition)

(1) Let $G = (V, E, i, t)$ be a finite connected non-empty abstract graph and let T be a spanning tree[183] of G. Let $e_1, \dots, e_n$ be the edges not contained in T. We pick some $v \in V$. For $j = 1, \dots, n$ we pick a path[184] $\alpha_j\colon [0,1] \to |T|$ from v to $i(e_j)$, we denote by $\beta_j\colon [0,1] \to |e_j| \subset |G|$ the obvious path from $i(e_j)$ to $t(e_j)$, and we pick a path $\gamma_j\colon [0,1] \to |T|$ from $t(e_j)$ to v. The map

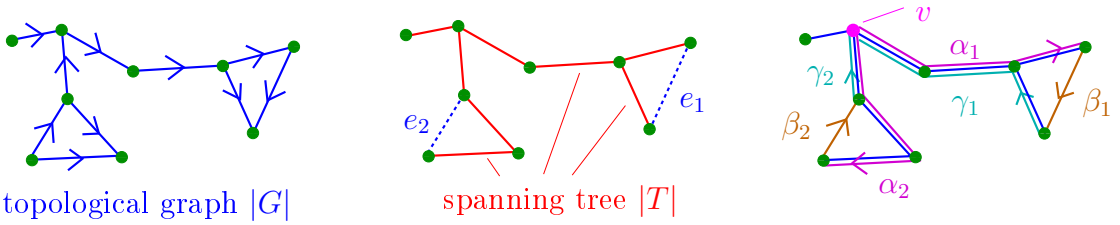
$$\begin{array}{rcl} \Theta_G\colon \langle x_1, \dots, x_n\rangle & \to & \pi_1(|G|, v) \\ x_j & \mapsto & [\alpha_j * \beta_j * \gamma_j] \end{array}$$

that is given by

is an isomorphism.

(2) Let X be a finite connected non-empty topological graph[185] with v vertices and e edges. Then
$$\pi_1(X) \;\cong\; \text{free group on } 1-\chi(X) = 1+e-v \text{ generators.}$$

(3) Two finite connected topological graphs are homotopy equivalent if and only if they have the same Euler characteristic.



Proof.

(1) We consider the maps

isomorphism induced by the homeomorphism on page 221

$$\langle x_1,\dots,x_n\rangle \xrightarrow{\Theta_G} \pi_1(|G|,[v]) \to \pi_1(|G|/|T|,v) \xleftarrow{\cong} \pi_1\Big(\bigvee_{j=1}^{n} S^1_j,*\Big) \xleftarrow{\cong} \langle y_1,\dots,y_n\rangle.$$

isomorphism by the Graph-Homotopy Type Corollary 17.6 (1) and the Homotopy-π_1-Proposition 13.4 (2)

isomorphism given by the Wedge-of-Circles–π_1-Lemma 20.7

It follows fairly easily from the explicit descriptions of the various maps that for each $j \in \{1,\dots,n\}$ the images of x_j and y_j in $\pi_1(|G|/|T|,v)$ are represented by path-homotopic loops, in particular the images of x_j and y_j in $\pi_1(|G|/|T|,[v])$ are the same. It follows from the above and the fact that all other maps are isomorphisms that Θ_G itself is indeed an isomorphism.

(2) Basically by definition we only need to prove the statement if $X = |G|$ is the topological realization of a finite connected non-empty abstract graph $G = (V,E,i,t)$. By the Spanning Tree–Existence Proposition 7.6 we know that G admits a spanning tree $T = (W,F)$. By definition T contains all vertices of V. Furthermore, since T is a tree we know by the Tree Characterization Proposition 7.7 that $\#W - \#F = 1$. Thus it follows that there are precisely $\#E - \#F = e - (\#W - 1) = e - v + 1$ edges of G that are not contained in T. The desired statement now follows from (1).

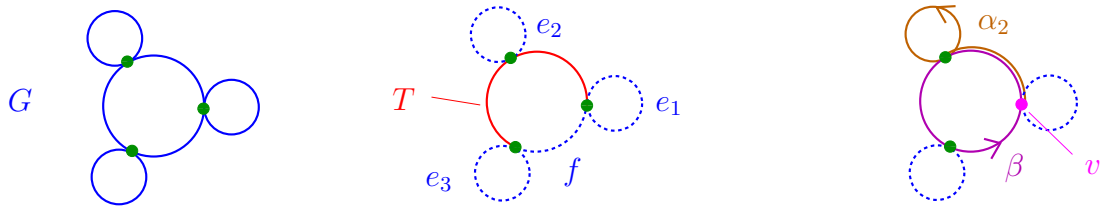
(3) Let X and X' be two finite connected topological graphs. If $\chi(X) = \chi(X')$, then we already saw in the Graph-Homotopy Type Corollary 17.6 (2) that X and X' are homotopy equivalent. On the other hand, if $\chi(X) \neq \chi(X')$, then it follows from (1) together with the Basis Cardinality Proposition 19.9 that X and X' are not homotopy equivalent. ■

Example. We look at the graph G shown in the figure below. We also show a spanning tree T and a "base vertex v". We make the following observations:

[183] Recall that according to the definition on page 131 a spanning tree for G is a subtree of G that contains all vertices of G.

[183] A topological graph is a topological space that is homeomorphic to the topological realization of an abstract graph.

[184] Note that it follows from the Abstract Graph-Topology Lemma 7.1 that $|T|$ is path-connected.

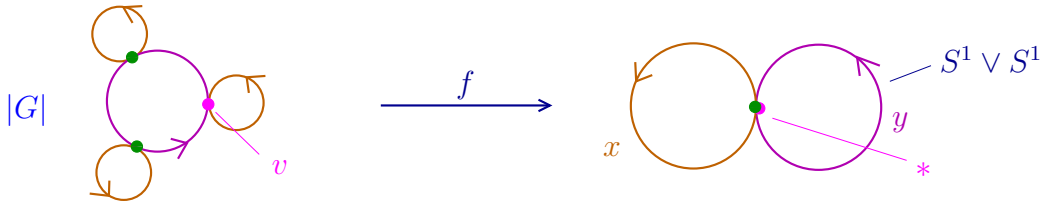


(1) The three small loops e_1, e_2, e_3 and the edge f are not contained in T.
(2) Each loop e_i gives rise to a loop α_i with base point v, by traveling along T, going once around e_i and going back to v along T.
(3) The edge f gives rise to a loop β by going once around the central circle.

It follows from the Graph-π_1-Proposition 20.9 that the map

$$\begin{array}{rcl} \langle a_1, a_2, a_3, b\rangle & \to & \pi_1(|G|, v) \\ a_i & \mapsto & [\alpha_i] \\ b & \mapsto & [\beta] \end{array}$$

is an isomorphism. Next we do something much more interesting: We consider the map $f\colon |G| \to S^1 \vee S^1$ shown in the figure below.



One can easily verify that f is a covering of degree 3. We use the Wedge-of-Circles–π_1-Lemma 20.7 to make the usual identification $\pi_1(S^1 \vee S^1, *) = \langle x, y\rangle$. By working directly with the definition of f_* one can see that f_* is given by

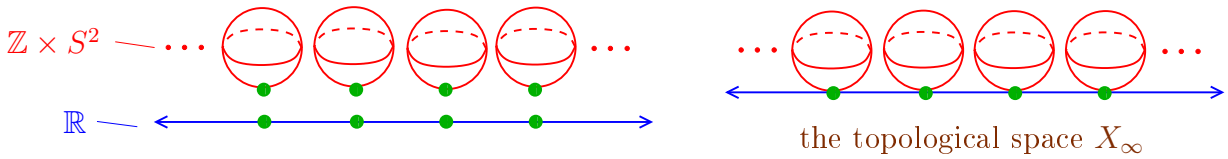
$$\begin{array}{rcl} f_*\colon \langle a_1, a_2, a_3, b\rangle = \pi_1(|G|, v) & \to & \pi_1(S^1 \vee S^1, *) = \langle x, y\rangle \\ a_i & \mapsto & y^{i-1} \cdot x \cdot y^{-(i-1)} \\ b & \mapsto & y^3. \end{array}$$

Since $f\colon |G| \to S^1 \vee S^1$ is a covering of degree 3 we obtain from the Covering-π_1-Monomorphism Proposition 14.11 together with the Covering-Degree Proposition 14.13 that the induced map f_* is a monomorphism and that the image of f_* is a subgroup of index three. In summary, we have now shown that the free group $\langle x, y\rangle$ on two generators contains the subgroup $f_*(\langle a_1, a_2, a_2, b\rangle) = \langle x, yxy^{-1}, y^2xy^{-2}, y\rangle$ which is a free group on four generators. Thus we see that the free group of rank two can contain "larger" free groups. This is in complete contrast to the situation in free abelian groups which we studied in the Free Abelian Subgroup Theorem 18.7.

20.8. **Exhaustions of topological spaces.** In many situations one is dealing with non-compact topological spaces where one would be tempted to apply the Seifert–van Kampen-Theorem 20.1 or the Wedge-π_1-Proposition 20.6 "infinitely many times", which does not really make sense. For example let us consider the topological space

$$X_\infty \;=\; \big(\mathbb{R} \sqcup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n, (0, 0, -1)).$$

The topological space X consists of an infinite line where to each $n \in \mathbb{Z}$ we attach a 2-dimensional sphere. We would like to show that X_∞ is simply connected. The following lemma lets us get around the issue of "infinite operations".



Proposition 20.10. (Exhaustion–Simply Connected Proposition) Let X be a topological space and let

$$X_0 \subset X_1 \subset X_2 \subset X_3 \subset \ldots$$

be a sequence of subspaces of X such the following three conditions are satisfied:

(1) We have $\bigcup_{i \in \mathbb{N}_0} X_i = X$.

(2) For each $i \in \mathbb{N}$ we have $X_i \subset \mathring{X}_{i+1}$.

(3) Each X_i is simply connected.

Then for each $x_0 \in X$ we have $\pi_1(X, x_0) = 0$.

The proof of the Exhaustion–Simply Connected Proposition 20.10 rests on the following lemma which will come in handy on several occasions.

Lemma 20.11. (Compact Subspace–Exhaustion Lemma) Let X be a topological space and let

$$X_0 \subset X_1 \subset X_2 \subset X_3 \subset \ldots$$

be a sequence of subspaces of X such the following two conditions are satisfied:

(1) We have $\bigcup_{i \in \mathbb{N}_0} X_i = X$.

(2) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.

Given any compact subset K of X there exists an $i \in \mathbb{N}_0$ with $K \subset X_i$.

Proof of the Compact Subspace–Exhaustion Lemma 20.11. We start out with the following little claim.

Claim. We have $\bigcup_{i \in \mathbb{N}} \mathring{X}_i = X$.

Proof. We have the following equalities and inclusions

$$X \underset{\text{by hypothesis (1)}}{=} \bigcup_{i \in \mathbb{N}} X_i \underset{\text{by hypothesis (2)}}{=} \bigcup_{i \in \mathbb{N}} \mathring{X}_i \subset X.$$

We see that the inclusion to the right is in fact an equality. ⊞

Now let $K \subset X$ be a compact subspace. Recall that by the Interior-Closure Lemma 1.6 (1) we know that interiors are always open subsets. It follows from the compactness of K and the claim that K is contained in the union of finitely many of the $\mathring{X}_i$. It follows from our hypothesis on the X_i that K is in fact contained in a single X_i. ■

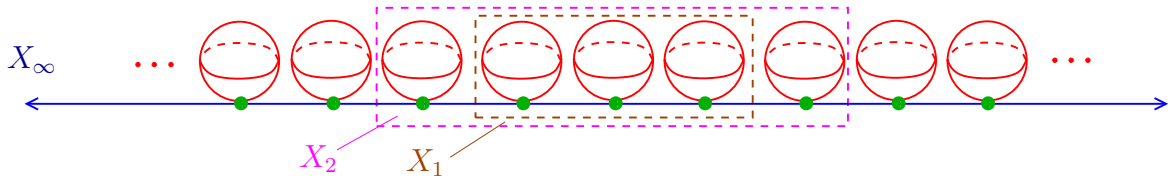
Proof of the Exhaustion–Simply Connected Proposition 20.10. Let $x_0 \in X$ and let $f\colon [0,1] \to X$ be a loop in x_0. By the Compact Image Lemma 2.8 we know that $f([0,1])$ is a compact subset of X. By the Compact Subspace–Exhaustion Lemma 20.11

we know that $f([0,1])$ is contained in a single X_i. It follows from Hypothesis (3) that f is null-homotopic in X_i. It is clear that this implies that f is also null-homotopic in X. ■

Example. We return to the topological space X_∞ from the beginning of the section. Given $k \in \mathbb{N}$ we set

$$X_k \;=\; \big((-k-1, k+1) \sqcup (\{-k, \dots, k\} \times S^2)\big) \,/\, n \sim (n, (0,0,-1)).$$

Using the Wedge-π_1-Proposition 20.6 one can easily show that each X_k is simply connected. It follows from the Exhaustion–Simply Connected Proposition 20.10 that X_∞ is simply connected.



Using the Exhaustion–Simply Connected Proposition 20.10 we can now give a characterisation of arbitrary trees in terms of their fundamental groups.

Proposition 20.12. (Tree–π_1-Proposition) Let $T = (V, E, \varphi)$ be a non-empty connected undirected abstract graph. The following statements are equivalent:[185]

(1) T is a tree.
(2) $\pi_1(|T|)$ is the trivial group.

Proof. Let $T = (V, E, \varphi)$ be a non-empty connected undirected abstract graph.

We start out with a proof of the (1)⇒(2)-direction. Since T is non-empty we can pick a vertex $v \in V$. We consider the subgraph given by $T_0 := \{v\}$. Iteratively we build a sequence of subgraphs $T_0, T_1, T_2, \dots$ of T as follows: given T_i we add all edges (and the corresponding vertices) that have at least one vertex in common with T_i. We make the following observations:

(a) One can easily verify that each T_i is connected.
(b) It follows from (a) and the Tree–Basics Lemma 7.4 (1) that each T_i is a tree.
(c) It follows from (b) and Finite Tree–Contractible Proposition 17.4 that each T_i is simply connected.
(d) One can easily verify that for each $i \in \mathbb{N}_0$ we have $T_i \subset \mathring{T}_{i+1}$.
(e) We have $T = \bigcup_{i \in \mathbb{N}_0} T_i$.

It follows from the above together with the Exhaustion–Simply Connected Proposition 20.10 that T is simply connected.

We turn to the proof of the (2)⇒(1)-direction. By the Spanning Tree–Existence Proposition 7.6 we know that T admits a spanning tree S. It remains to show that if $S \neq T$, then $\pi_1(|T|)$ is non-trivial. Thus suppose that there exists an edge $e \in T \setminus S$. We denote by $|e| \subset |T|$ the corresponding subspace. By definition there exist $v, w \in V$ with $\varphi(e) = \{v, w\}$.[186] By definition of the topological realization there exists a continuous map $\alpha\colon [0,1] \to |e|$ with $\alpha(0) = v$ and $\alpha(1) = w$ such that α restricts to a homeomorphism $(0,1) \to |e| \setminus \{v, w\}$. Since S is a spanning tree we know that $v, w \in S$. Also note that

[186] Note that it is possible that $v = w$.

trees are connected. Thus it follows from the Abstract Graph-Topology Lemma 7.1 that $|S|$ is connected. Therefore there exists a continuous map $\beta\colon [0,1] \to |S|$ with $\beta(0) = v$ and $\beta(1) = w$. Next we consider the map

$$\begin{array}{rcl} f\colon |T| & \to & S^1 \\ x & \mapsto & \begin{cases} 1, & \text{if } x \notin \alpha(0), \\ \exp(2\pi \mathrm{i} t), & \text{if } x = \alpha(t). \end{cases} \end{array}$$

It follows from the Topological-Pushout Proposition 6.12 that f is continuous.

Note that $\alpha * \beta$ is a loop in v and note that $f \circ (\alpha * \beta)\colon [0,1] \to S^1$ is path-homotopic to the loop $t \mapsto \exp(2\pi \mathrm{i} t)$. It follows from these observations together with the Circle-π_1-Theorem 11.13 that $f_*([\beta * \alpha]) \in \pi_1(S^1, 1)$ is non-trivial. But this shows that $\pi_1(|T|, v)$ is non-trivial. ■

21. Presentations of groups

In the Seifert–van Kampen Theorem 20.1 we just learned how to express fundamental groups of suitable topological spaces as amalgamated products. But it is not always clear in what sense we calculated a fundamental group and what such a description as an amalgamated product is trying to tell us.

In this chapter we introduce the notion of a (finite) presentation of a group. We will see in this and the following chapters that we can describe (fundamental) groups as finitely presented groups, i.e. we can give a finite description of these groups.

21.1. **Generating sets of groups.** On page 238 we introduced the notion of a "reduced word", which begs the question, what is a "word". Here is the answer.

Definition. Let G be a group and let $S \subset G$ be a subset.

(1) A word in S is a finite sequence $(s_1^{\epsilon_1}, \dots, s_k^{\epsilon_k})$ with $s_1, \dots, s_k \in S$ and $\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}$. We refer to k as the length of the word.

(2) A word $(s_1^{\epsilon_1}, \dots, s_k^{\epsilon_k})$ defines an element $s_1^{\epsilon_1} \cdot \ldots \cdot s_k^{\epsilon_k}$ in G. In casual language we also refer to an expression of the form $s_1^{\epsilon_1} \cdot \ldots \cdot s_k^{\epsilon_k}$ as a word.

Definition.

(1) Let G be a group and let $S \subset G$ be a subset. We denote by $\Xi_S \colon \langle S \rangle \to G$ the natural homomorphism (given by the Free Group–Existence Lemma 19.4) from the free group $\langle S \rangle$ to G that is given by $s \mapsto s$.

(2) Given a group G we say that $S \subset G$ is a generating set if the natural homomorphism $\Xi_S \colon \langle S \rangle \to G$ is an epimorphism.

(3) We say that a group G is finitely generated if it admits a finite generating set. Otherwise we say that the group is infinitely generated.

Lemma 21.1. (Generating Set–Characterization Lemma) Let G be a group and let $S \subset G$ be a subset. The following statements are equivalent:

(1) S is a generating set for G.

(2) Every element of G can be represented by a word in S, i.e. we have

$$G = \left\{ s_1^{\epsilon_1} \cdot \ldots \cdot s_k^{\epsilon_k} \mid s_1, \dots, s_k \in S \text{ and } \epsilon_1, \dots, \epsilon_k \in \{-1, 1\} \right\}.$$

(3) If H is a subgroup of G that contains S, then $H = G$.

Proof.
(1)⇒(2): This statement follows immediately from the definitions and the trivial observation that for $s \in S$ and $n = \epsilon \cdot m$ with $\epsilon \in \{-1, 1\}$ and $m \in \mathbb{N}$ we have $s^{\epsilon \cdot m} = (s^{\epsilon})^m$.
(2)⇒(3): This statement follows from the observation that if $H \subset G$ is subgroup that contains S, then H also contains all words in S.
(3)⇒(1): This statement follows from the observation that the image of the natural homomorphism $\Xi_S \colon \langle S \rangle \to G$ is a subgroup of G that contains S. ∎

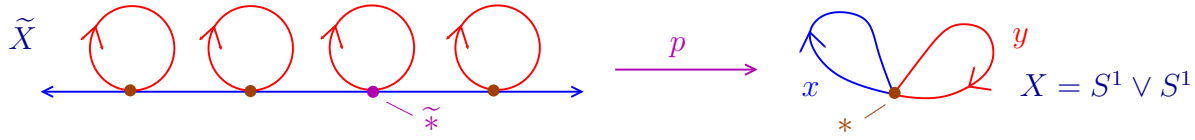
Example. It follows from the Abelianization-Epimorphism Lemma 19.7 together with the Abelianization Calculation Proposition 19.8 (2) that any finitely generated abelian group in the sense of page 234 is also a finitely generated group in the above sense.

Lemma 21.2. (Generating Set–Lemma)
(1) If G is a free group on a subset S, then S is a generating set for G.
(2) If $\varphi\colon G \to H$ is an epimorphism between two groups and if S is a generating set for G, then $\varphi(S)$ is a generating set of H. In particular, if G is finitely generated, then so is H.

Proof. Both statements follow easily from the definitions. ■

Example. First recall that it follows from the Wedge-of-Circles–π_1-Lemma 20.7 that we can make the identification $\pi_1(S^1 \vee S^1, *) = \langle x, y\rangle$. In the figure below we show a covering $p\colon \widetilde{X} \to S^1 \vee S^1$. It follows from the Covering-π_1-Monomorphism Proposition 14.11 that $\pi_1(\widetilde{X}, \widetilde{*})$ is isomorphic to a subgroup of $\pi_1(S^1 \vee S^1, *) = \langle x, y\rangle$.

Using the Graph-π_1-Proposition 20.9 one can show that $\pi_1(\widetilde{X}, \widetilde{*})$ admits epimorphisms onto free groups of any rank $n \in \mathbb{N}$. Thus it follows from the Generating Set–Lemma 21.2 that $\pi_1(\widetilde{X}, \widetilde{*})$ is infinitely generated.



We have thus shown that there exists a finitely generated group that admits an infinitely generated subgroup. This is in contrast to the situation in *abelian* groups, where in the Finite Generation Lemma 18.8 (2) we showed that every subgroup of a finitely generated *abelian* subgroup is again finitely generated.

Lemma 21.3. (Generating Set Properties–Lemma) Let G be a group and let S be a generating set for G.
(1) If $\varphi\colon G \to G$ is a homomorphism such that $\varphi(s) = s$ for all $s \in S$, then $\varphi = \mathrm{id}$.
(2) If for all $s, t \in S$ we have $s \cdot t = t \cdot s$, then G is abelian.
(3) If $\varphi\colon H \to G$ is a group homomorphism with $S \subset \varphi(H)$, then φ is an epimorphism.

Proof.
(1) Let $\varphi\colon G \to G$ be a homomorphism such that $\varphi(s) = s$ for every $s \in S$. We consider $H := \{g \in G \,|\, \varphi(g) = g\}$. It is straightforward to verify that H is a subgroup of G. Since H contains S it follows from the Generating Set–Characterization Lemma 21.1 (1)⇒(3) that $H = G$, i.e. $\varphi = \mathrm{id}$.
(2) We assume that for all $s, t \in S$ we have $s \cdot t = t \cdot s$. We want to show that G is abelian. We start out with the following claim.

Claim. For all $s, t \in S$ and all $\mu, \nu \in \{-1, 1\}$ we have $s^\mu \cdot t^\nu = t^\nu \cdot s^\mu$.

Proof. First we consider the case $\mu = 1$ and $\nu = -1$:

$$s \cdot t^{-1} \;=\; \underbrace{t^{-1} \cdot t}_{=e} \cdot s \cdot t^{-1} \;\underset{\text{since } s\cdot t = t\cdot s}{=}\; t^{-1} \cdot s \cdot t \cdot t^{-1} \;=\; t^{-1} \cdot s.$$

By swapping the roles of s and t we see that the above also takes care of the case $\mu = -1$ and $\nu = 1$. Thus it remains to deal with the case $\mu = -1$ and $\nu = -1$. By

hypothesis we have $t \cdot s = s \cdot t$. Taking inverses we see that $(t \cdot s)^{-1} = (s \cdot t)^{-1}$. But this means precisely that $s^{-1} \cdot t^{-1} = t^{-1} \cdot s^{-1}$. ⊞

Now let $g, h \in G$. By the Generating Set–Characterization Lemma 21.1 (1)⇒(2) we can write $g = s_1^{\epsilon_1} \cdot \ldots \cdot s_k^{\epsilon_k}$ and $h = t_1^{\tau_1} \cdot \ldots \cdot t_l^{\tau_l}$ for some $s_1, \ldots, s_k, t_1, \ldots, t_l \in S$ and $\epsilon_1, \ldots, \epsilon_k, \tau_1, \ldots, \tau_l \in \{-1, 1\}$. By applying the above claim altogether $k \cdot l$ times we see that $g \cdot h = h \cdot g$.

(3) Let $\varphi\colon H \to G$ be a group homomorphism with $S \subset \varphi(G)$. Note that $\varphi(H)$ is a subgroup of G. Since $S \subset \varphi(G)$ it thus follows from the Generating Set–Characterization Lemma 21.1 (1)⇒(3) that $\varphi(H) = G$. ■

21.2. **Presentations of groups.** In this section we introduce presentations of groups. Before we do so it is convenient to recall the following definition from page 247.

Definition. Let G be a group and let $S \subset G$ be a subset. We define

$$\text{subgroup of } G \text{ normally generated by } S \;:=\; \langle\langle S \rangle\rangle \;:=\; \bigcap_{S \subset H \lhd G} H.$$

Now we can provide the main definition of this section.

Definition. Let X be a set and let R be a subset of the free group $\langle X \rangle$ generated by the set X. We write

$$\langle X \mid R \rangle \;:=\; \langle X \rangle \,/\, \langle\langle R \rangle\rangle.$$

If $X = \{x_1, \ldots, x_k\}$ and $R = \{r_1, \ldots, r_l\}$, then we use the convenient shorthand

$$\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle \;:=\; \langle \{x_1, \ldots, x_k\} \,|\, \{r_1, \ldots, r_l\} \rangle.$$

On many occasions we will use that the Generating Set–Lemma 21.2 implies that X is a generating set of $\langle X \,|\, R \rangle$.

Examples.

(1) For $k \in \mathbb{N}_0$ we have

$$\langle g \mid g^k \rangle \;=\; \langle g \rangle / \langle\langle g^k \rangle\rangle \;\underset{\uparrow}{=}\; \langle g \rangle / \{g^{k \cdot n} \mid n \in \mathbb{Z}\} \;\cong\; \mathbb{Z}_k.$$

this follows easily from the observation that $\langle g \rangle$ is abelian

(2) The group $\langle x_1, \ldots, x_k \mid \rangle$ with $R = \varnothing$ is just the free group on $x_1, \ldots, x_k$.

Convention. Given a group $\langle X \,|\, R \rangle$ and given $g \in \langle X \rangle$ we denote the images of the g in the quotient $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ again by g.

As we will now see, finite presentations of groups are useful for finding group homomorphisms. To formulate the corresponding lemma we need the following definition.

Definition. Let $r \in \langle x_1, \ldots, x_k \rangle$. Furthermore let G be a group and let $g_1, \ldots, g_k$ be elements of G. We say that $g_1, \ldots, g_k$ satisfy the relation r if the homomorphism[187] $\langle x_1, \ldots, x_k \rangle \to G$ that is given by $x_i \mapsto g_i$, $i = 1, \ldots, k$ sends r to the trivial element.[188]

[187] The fact that there exists a unique such homomorphism is a consequence of the Free Group–Existence Lemma 19.4.

[188] More colloquially speaking: $g_1, \ldots, g_k$ satisfy a relation r if replacing the x_i by g_i in the word r_i returns the trivial element in G.

Example.

(1) We consider the word $r = [x_1, x_2] = x_1 x_2 x_1^{-1} x_2^{-1} \in \langle x_1, x_2 \rangle$. If a group G is abelian, then any two elements $g_1, g_2 \in G$ satisfy the relation $x_1 x_2 x_1^{-1} x_2^{-1}$.

(2) Let G be a finite group with d elements. It follows from Lagrange's Theorem, see e.g. [KS04, p. 8], that every element in G satisfies the relation $x^d \in \langle x \rangle$.

Lemma 21.4. (Free Abelian Group-Presentation Lemma) Let $m \in \mathbb{N}$. There exists a unique isomorphism[189]

$$\alpha \colon \langle x_1, \dots, x_m \mid [x_i, x_j] = e \text{ for all } 1 \leq i < j \leq m \rangle \xrightarrow{\cong} \mathbb{Z}^m$$

which sends each x_i to the i-th standard basis vector e_i of $\mathbb{Z}^m$.

Proof. We set $X = \{x_1, \dots, x_m\}$ and $R = \{[x_i, x_j] \mid 1 \leq i < j \leq m\}$. Since $\mathbb{Z}^m$ is abelian we obtain from the Free Group–Existence Lemma 19.4 together with the Normal Closure–Factorization Lemma 19.10 that there exists a unique homomorphism $\alpha \colon \langle X \mid R \rangle \to \mathbb{Z}^m$ with $\alpha(x_i) = e_i$. Since the image of α contains the generating set $\{e_1, \dots, e_m\}$ of $\mathbb{Z}^m$ we obtain from the Generating Set Properties–Lemma 21.3 (3) that α is an epimorphism. Thus it remains to show that α is a monomorphism.

Claim. The group $\langle X \mid R \rangle$ is abelian.

Proof. Recall that it follows from the Generating Set–Lemma 21.2 that $X = \{x_1, \dots, x_m\}$ is a generating set for $\langle X \mid R \rangle$. Let $i < j$. We have the following equality in $\langle X \mid R \rangle$:

$$x_j \cdot x_i \;=\; \underbrace{x_i \cdot x_j \cdot x_i^{-1} \cdot x_j^{-1}}_{=[x_i, x_j] = e \in \langle X \mid R \rangle} \cdot x_j \cdot x_i \;=\; x_i \cdot x_j.$$

This shows that the elements of the generating set commute. Thus it follows from the Generating Set Properties–Lemma 21.3 (2) that the group $\langle X \mid R \rangle$ is abelian. ⊞

As we mentioned on page 230 (2), the group $\mathbb{Z}^m$ is free abelian on the set $e_1, \dots, e_m$. We just learned in the above claim that $\langle X \mid R \rangle$ is abelian. This implies in particular that there exists a unique homomorphism $\beta \colon \mathbb{Z}^m \to \langle X \mid R \rangle$ with $\beta(e_i) = x_i$. Note that for each $i \in \{1, \dots, m\}$ we have $(\beta \circ \alpha)(x_i) = \beta(\alpha(x_i)) = \beta(e_i) = x_i$. Thus it follows from the Generating Set Properties–Lemma 21.3 (1) that $\beta \circ \alpha = \mathrm{id}$. We have thus shown that $\alpha \colon \langle X \mid R \rangle \to \mathbb{Z}^m$ is a monomorphism. ■

We continue with the following definitions.

Definition.

(1) A presentation is a pair (X, R) consisting of a set X together with a subset $R \subset \langle X \rangle$. We call the elements of X the generators and we call the elements of R the relators[190] of the presentation.

(2) Let π be a group. A presentation for π is a presentation (X, R) together with an isomorphism

$$\langle X \mid R \rangle \xrightarrow{\cong} \pi.$$

Very often one just says $\langle X \mid R \rangle$ is a presentation for π and the explicit isomorphism is suppressed from the notation.

[189] On page 242 we introduced for two elements g, h of a group G the commutator $[x, y] := xyx^{-1}y^{-1}$.

(3) We say a group π is finitely presented, if π admits a presentation with finitely many generators and finitely many relators.

Example.

(1) Every group admits a presentation. Indeed, let π be a group. Then we set $X := \pi$, i.e. X is the set of elements of π. By the Free Group–Existence Lemma 19.4 there exists a unique homomorphism $\alpha\colon \langle X\rangle \to \pi$ with $\alpha(g) = g$ for all $g \in X = \pi$. We set $R := \ker(\alpha)$. Then

$$\langle X \mid R\rangle \;=\; \langle X\rangle/\langle\langle R\rangle\rangle \;=\; \langle X\rangle/\langle\langle \ker(\alpha)\rangle\rangle \;\underset{\substack{\uparrow\\ \text{since } R=\ker(\alpha) \text{ is a normal subgroup}}}{=}\; \langle X\rangle/\ker(\alpha) \;\xrightarrow[\alpha]{\cong}\; G.$$

(2) Using a modification of the argument in (1) one sees that a group π is finitely generated, in the sense of the definition on page 274, if and only if π admits a presentation $\langle X \mid R\rangle \to \pi$ such that X is finite.

(3) We had just seen in the Free Abelian Group-Presentation Lemma 21.4 that

$$\mathbb{Z}^n \;\cong\; \langle x_1, \dots, x_n \mid [x_i, x_j] = e \text{ with } 1 \le i < j \le n\rangle.$$

This shows that $\mathbb{Z}^n$ is a finitely presented group.

(4) There exist groups that are finitely generated but not finitely presented. In fact it follows easily from the Countability Lemma 0.6 that there are only countably many isomorphism types of finitely presented groups. On the other hand it is shown in [Rob95, Chapter 14] that there are uncountably many isomorphism types of finitely generated groups.[191][192]

Let us recall the following definition from page 244.

Definition. Let π be a group. The commutator subgroup of π is defined as

$$[\pi,\pi] \;:=\; \left\{ \prod_{i=1}^{n} [x_i, y_i] \,\middle|\, x_1, y_1, \dots, x_n, y_n \in \pi \right\}$$

We call $\pi_{\mathrm{ab}} := \pi/[\pi,\pi]$ the abelianization of π.

Proposition 21.5. (Abelianization-of-Presentation Proposition) Suppose we are given a finitely presented group $\pi = \langle x_1, \dots, x_m \mid r_1, \dots, r_n\rangle$. We denote by

$$\Phi\colon \langle x_1, \dots, x_m\rangle \;\to\; \mathbb{Z}^m$$

the homomorphism that is given by $\Phi(x_i) = e_i$. There exists a unique isomorphism[193]

$$\Psi\colon \underbrace{\pi/[\pi,\pi]}_{=\pi_{\mathrm{ab}}} \;\xrightarrow{\cong}\; \mathbb{Z}^m/\operatorname{span}(\Phi(r_1), \dots, \Phi(r_n))$$

[190] The relators are often also called relations.

[191] The idea is that in a free group $\langle x_1, \dots, x_k\rangle$ there are uncountably many infinite sequences of elements $\{r_1, r_2, \dots\}$. The difficulty is to show that there exist uncountably many sequences that give raise to non-isomorphic groups.

[192] Note that this also shows that any free group on ≥ 2 generators contains uncountably many different subgroups.

such that the following diagram commutes

$$\begin{array}{ccc}
\langle x_1,\dots,x_m\rangle & \xrightarrow[\Phi]{x_i\mapsto e_i} & \mathbb{Z}^m \\
\downarrow & & \\
\pi = \langle x_1,\dots,x_m\rangle/\langle\langle r_1,\dots,r_n\rangle\rangle & & \downarrow \\
\downarrow & & \\
\pi_{\mathrm{ab}} = \pi/[\pi,\pi] & \xrightarrow[\cong]{\Psi} & \mathbb{Z}^m/\operatorname{span}(\Phi(r_1),\dots,\Phi(r_n)).
\end{array}$$

Example. We consider the group $\pi = \langle x, y \mid x^2y^{-2}x, y^{-1}x^3yx\rangle$.

Let $\Phi\colon \langle x,y\rangle \to \mathbb{Z}^2$ be the epimorphism given by $\Phi(x) = (1,0)$ and $\Phi(y) = (0,1)$. It follows from the Abelianization-of-Presentation Proposition 21.5 that the abelianization of π is given by

$$\pi_{\mathrm{ab}} = \mathbb{Z}^2/\operatorname{span}(\underbrace{\Phi(x^2y^{-2}x)}_{=\binom{3}{-2}}, \underbrace{\Phi(y^{-1}x^3yx)}_{=\binom{4}{0}}) = \mathbb{Z}^2/\begin{pmatrix}3 & 4\\ -2 & 0\end{pmatrix}\mathbb{Z}^2 \cong \mathbb{Z}_8.$$

$\uparrow$ can be calculated using the Smith normal form of the matrix

Proof. It is straightforward to see that there exists a unique homomorphism

$$\Psi\colon \pi/[\pi,\pi] \to \mathbb{Z}^m/\operatorname{span}(\Phi(r_1),\dots,\Phi(r_n))$$

with $\Psi([x_i]) = [e_i]$, i.e. that makes the diagram in the proposition commute. It remains to show that Ψ is an isomorphism. We write $F = \langle x_1,\dots,x_m\rangle$ and we write $G = \langle y_1,\dots,y_n\rangle$. We consider the following diagram

$$\begin{array}{ccc}
G_{\mathrm{ab}} = G/[G,G] & \xrightarrow[\cong]{[y_i]\mapsto e_i} & \mathbb{Z}^n \\
{\scriptstyle [y_i]\mapsto [r_i]}\downarrow & & \downarrow{\scriptstyle e_i\mapsto \Phi(r_i)} \\
F_{\mathrm{ab}} = F/[F,F] & \xrightarrow[\cong]{[x_i]\mapsto e_i} & \mathbb{Z}^m \\
\downarrow & & \downarrow \\
\pi_{\mathrm{ab}} = \pi/[\pi,\pi] & \xrightarrow{\Psi} & \mathbb{Z}^m/\operatorname{span}(\Phi(r_1),\dots,\Phi(r_n)).
\end{array}$$

Note that the diagram commutes. It follows from the Abelianization Calculation Proposition 19.8 (4) that the top two horizontal maps are isomorphisms. The bottom vertical maps are clearly epimorphisms and it is clear that the images of the top vertical maps lie in the kernel of the bottom vertical maps. It now follows from an elementary "diagram chase" that the bottom horizontal map is also an isomorphism.[194] We leave the details to the reader. ■

[193] By $\operatorname{span}(v_1,\dots,v_n)$ we mean the set of all linear combinations $\sum_{i=1}^{n} \lambda_i \cdot v_i$ with $\lambda_1,\dots,\lambda_n \in \mathbb{Z}$.

[194] More precisely, we use the following fact: suppose we are given a commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\cong} & A' \\
\downarrow & & \downarrow \\
B & \xrightarrow{\cong} & B' \\
\downarrow & & \downarrow \\
C & \longrightarrow & C'
\end{array}$$

of group homomorphisms where the bottom vertical maps are epimorphisms, where the top two horizontal maps are isomorphisms and where the kernels of the bottom vertical maps are the images of the top vertical

21.3. **Presentations of groups and constructions of groups.** In this section we will see how various familiar constructions of groups interact with presentations.

Lemma 21.6. (Quotient Presentation–Lemma) Let $G = \langle X \mid R \rangle$ be a group and let $\varphi\colon G \to H$ be an epimorphism. If $S \subset \langle X \rangle$ is a subset, such that $\ker(\varphi) = \langle\langle \{[s] \mid s \in S\} \rangle\rangle$, then

$$\begin{array}{rcl} \langle X \mid R \cup S \rangle & \to & H \\ x & \mapsto & \varphi([x]) \end{array}$$

is a presentation for H.

Proof. We consider the following diagram:

$$\begin{array}{ccccc} \langle X \rangle & \xrightarrow{\quad\mathrm{id}\quad} & \langle X \rangle & & \\ \downarrow & & \downarrow & & \\ & & \langle X \mid R \rangle & & \\ & & \downarrow & & \\ \langle X \mid R \cup S \rangle & \longrightarrow & \langle X \mid R \rangle / \ker(\varphi) & \xrightarrow{[g] \mapsto \varphi(g)} & H. \end{array}$$

Note that the identity homomorphism on the top (in both directions) descends to homomorphisms (in both directions) between the groups on the bottom left. It follows from the Generating Set Properties–Lemma 21.3 (1) that the two homomorphisms are inverses of one another, in particular the homomorphism from the left to the center is an isomorphism. The lemma now follows from the observation that the homomorphism to the bottom right is an isomorphism, since φ is an epimorphism. ■

Before we give our next example, and before we formulate further results on propositions let us introduce the following intuitive notation.

Notation. Given a set X we sometimes write relator in the form $a = b$ for $a, b \in \langle X \rangle$, this just means that the relator is given by $a \cdot b^{-1} \in \langle X \rangle$.

Example. Let A be a finitely generated abelian group. By the Abelian Group-Classification Theorem 18.11 there exist $a_1, \ldots, a_n \in \mathbb{N}_0$ such that $A \cong \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_n}$. In other words, we can write A as a quotient of $\mathbb{Z}^n$. Using the Abelianization-of-Presentation Proposition 21.5 and the Quotient Presentation–Lemma 21.6 one can now easily verify that

$$A \cong \bigoplus_{i=1}^{n} \mathbb{Z}_{a_i} \cong \langle x_1, \ldots, x_n \mid [x_i, x_j] = e \text{ for } 1 \leq i < j \leq n \text{ and } x_i^{a_i} = e \text{ for } i = 1, \ldots, n \rangle.$$

Thus we have shown that every finitely generated abelian group is also finitely presented.

The following lemma gives us a convenient way to find a presentation of the amalgamated product of two groups.

Lemma 21.7. (Amalgamated Product–Presentation Lemma) Let $A = \langle X \mid R \rangle$ and $B = \langle Y \mid S \rangle$ be two groups with $X \cap Y = \varnothing$. Furthermore let G be a group with generating set Z and let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. There is a natural

---

maps. Then the bottom horizontal map is also an isomorphism. This general statement can be proved by a "diagram chase".

isomorphism

$$\langle X \sqcup Y \mid R \sqcup S \sqcup \{\alpha(z) = \beta(z) \mid z \in Z\}\rangle \xrightarrow{\cong} A *_G B.$$

In particular A and B are finitely presented and if G is finitely generated, then $A *_G B$ is finitely presented.

Example.

(1) The case $G = \{e\}$ in the Amalgamated Product–Presentation Lemma 21.7, together with the Amalgamated Product Lemma 19.12 (1), gives us presentations of free products of two groups.

(2) Let $A = \langle a\rangle$ and let $B = \langle b\rangle$ be two infinite cyclic groups and let $G = \langle x, y \mid [x, y]\rangle$ be a group that is isomorphic to $\mathbb{Z}^2$. We consider the homomorphism $\alpha\colon G \to A$ that is given by $\alpha(x) = a$ and $\alpha(y) = e$ and we consider the homomorphism $\beta\colon G \to B$ that is given by $\beta(x) = e$ and $\beta(y) = b$. Then, similar to the discussion on page 248, we see that

$$A *_G B \underset{\uparrow}{=} \langle a, b \mid \alpha(x){=}\beta(x), \alpha(y){=}\beta(y)\rangle = \langle a, b \mid a{=}e, e{=}b\rangle = \langle a, b \mid a, b\rangle = \{e\}.$$

the Amalgamated Product–Presentation Lemma 21.7

This shows in particular that the amalgamated product of two non-trivial groups can in fact be trivial.

Proof of the Amalgamated Product–Presentation Lemma 21.7. We first consider the setting of the free product $\langle X \mid R\rangle * \langle Y \mid S\rangle$. We consider the following diagram:

$$\begin{array}{ccccc} \langle X \sqcup Y\rangle & \longrightarrow & \langle X\rangle * \langle Y\rangle & \longrightarrow & \langle X \sqcup Y\rangle \\ \downarrow & & \downarrow & & \downarrow \\ \langle X \sqcup Y \mid R \cup S\rangle & \xrightarrow{\alpha} & \langle X \mid R\rangle * \langle Y \mid S\rangle & \xrightarrow{\beta} & \langle X \sqcup Y \mid R \cup S\rangle. \end{array}$$

We make the following observations and clarifications:

(1) The top horizontal maps are given by the universal properties of free groups and free products.

(2) The vertical maps are (induced by) the natural projections.

(3) The bottom left horizontal map is induced by the Normal Closure–Factorization Lemma 19.10.

(4) It follows from the Normal Closure–Factorization Lemma 19.10 that the homomorphism $\langle X\rangle \to \langle X \sqcup Y \mid R \cup S\rangle$ descends to a homomorphism starting at $\langle X \mid R\rangle$. The same way we obtain a homomorphism starting at $\langle Y \mid S\rangle$. These two homomorphisms give us the bottom right horizontal homomorphism.

(5) Using the Generating Set–Lemma 21.2 and the Generating Set Properties–Lemma 21.3 (1) one can easily verify that $\beta \circ \alpha = \mathrm{id}$ and $\alpha \circ \beta = \mathrm{id}$.

We have thus proved the desired statement for the case of a free product. The case of a non-trivial amalgamated product follows easily from the above discussion together with the Quotient Presentation–Lemma 21.6. ∎

21.4. **Tietze transformations.** We have now seen that many groups can be described by finite presentations. The question arises, to what degree are finite presentations unique? To answer this question we introduce the following definition.

Definition. Let
$$\phi\colon \langle x_1,\dots,x_k \,|\, r_1,\dots,r_l\rangle \xrightarrow{\cong} \pi$$
be a presentation of a group π. We obtain new presentations of π as follows:

(T1) For any element s of $\langle\langle r_1,\dots,r_l\rangle\rangle \subset \langle X\rangle$ we obtain the presentation
$$\langle x_1,\dots,x_k \mid r_1,\dots,r_l, s\rangle \underset{\uparrow}{=} \langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle \xrightarrow{\phi} \pi.$$
since $s \in \langle\langle r_1,\dots,r_l\rangle\rangle$ the groups are identically

(T2) For any $x \notin \{x_1,\dots,x_k\}$ and any two words s,t in $x_1,\dots,x_k$ we obtain the presentation[195]
$$\begin{aligned} \langle x_1,\dots,x_k, x \mid r_1,\dots,r_l, s\cdot x\cdot t\rangle &\to \langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle \xrightarrow{\phi} \pi. \\ x_i &\mapsto x_i \\ x &\mapsto s^{-1}\cdot t^{-1} \end{aligned}$$

The above two methods, and their inverses, of obtaining new presentations out of a given one are called Tietze transformations.[196]

Example. We want to show that the group $\langle a,b,c \,|\, (ab)^2ab^2\rangle$ is a free group on two generators. Borrowing an argument from [MKS04, p. 50] we see this as follows:
$$\begin{aligned} \langle a,b,c \,|\, (ab)^2ab^2\rangle &\overset{(T2),(T2)}{\cong} \langle a,b,c,x,y \,|\, (ab)^2ab^2, x=ab, y=ab^2\rangle \\ &\overset{(T1)}{\cong} \langle a,b,c,x,y \,|\, (ab)^2ab^2, x=ab, y=ab^2, x^2y\rangle \\ &\overset{(T1)^{-1}}{\cong} \langle a,b,c,x,y \,|\, x=ab, y=ab^2, x^2y\rangle \\ &\overset{(T1),(T1)}{\cong} \langle a,b,c,x,y \,|\, x=ab, y=ab^2, x^2y, b=x^{-1}y, a=xy^{-1}x\rangle \\ &\overset{(T2)^{-1},(T2)^{-1}}{\cong} \langle c,x,y \,|\, x=xy^{-1}x\cdot x^{-1}y, y=xy^{-1}x(x^{-1}y)(x^{-1}y), x^2y\rangle \\ &\overset{(T2)^{-1}}{\cong} \langle c,x,y \,|\, x^2y\rangle \overset{(T1),(T1)^{-1}}{\cong} \langle c,x,y \,|\, y=x^{-2}\rangle \overset{(T2)^{-1}}{\cong} \langle c,x \,|\, \rangle. \end{aligned}$$
As we can see, working with Tietze transformations leads too long and unreadable arguments.

The above Tietze transformations show that from a given finite presentation we can obtain infinitely many new finite presentations. This fact is not that surprising. What is much more interesting is the following theorem which was first proved by Heinrich Tietze [Tie08] in 1908.

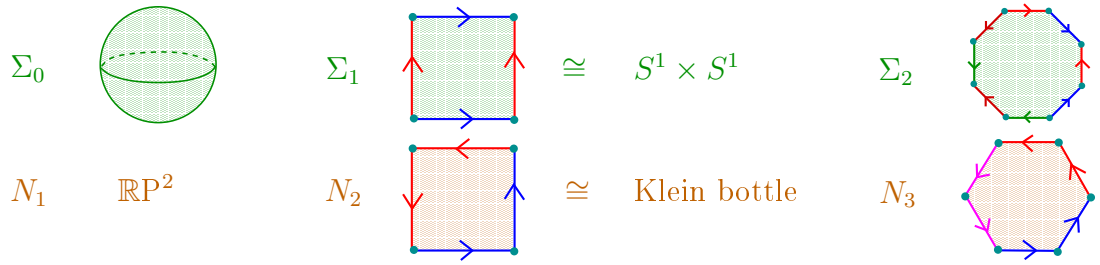
Theorem 21.8. (Tietze Theorem) Any two finite presentations for a given group are related by a finite sequence of Tietze transformations.

Proof. We outsource the proof to [Fri23]. ■

[196] First note that the map to the left is well-defined. This can be seen using the observation that under the assignment $x_i \mapsto x_i$ and $x \mapsto s^{-1}\cdot t^{-1}$ the relation $s\cdot x\cdot t$ gets sent to $s\cdot s^{-1}\cdot t^{-1}\cdot t = e$. The map to the left is also an isomorphism. In fact, as one can verify using the Generating Set Properties–Lemma 21.3 (1), an inverse is given by $x_i \mapsto x_i$.

22. The classification of compact 2-dimensional manifolds

Given $g \in \mathbb{N}_0$ we introduced on page 118 the surface Σ_g of genus g and given $g \in \mathbb{N}$ we introduced on page 118 the non-orientable surface N_g of genus g.[197]

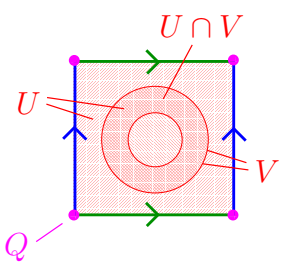


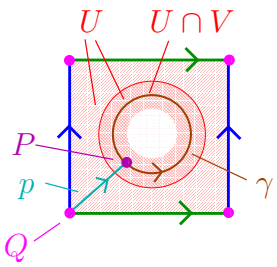
In this chapter we will first use the Seifert–van Kampen Theorem 20.1 to provide the long overdue calculation of the fundamental groups of the (non-orientable) surfaces Σ_g and N_g. Afterwards, in Section 22.5 we will use these calculations to give the classification of compact 2-dimensional smooth and topological manifolds.

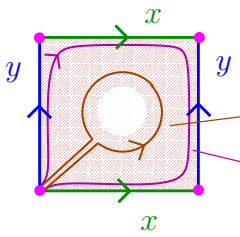
22.1. **The fundamental groups of the surfaces Σ_g.** We want to use the Seifert–van Kampen-Theorem 20.1 to determine the fundamental groups of the surfaces Σ_g of genus $g \geq 2$. Before we do so we consider the torus, the Klein bottle and the real projective plane to gain some confidence in our newly acquired methods.

(1) We first consider the torus $T = ([0,1] \times [0,1])/\sim$. We will use the following notation:
   (a) We denote by Q the point on T corresponding to one of the four vertices of the square.
   (b) We set
$$U = T \setminus \overline{B_{\frac{1}{8}}(\tfrac{1}{2},\tfrac{1}{2})} \subset T \quad \text{and } V = B_{\frac{1}{4}}(\tfrac{1}{2},\tfrac{1}{2}) \subset T,$$
   i.e. U is the complement of a closed disk and V is an open disk. Note that $T = U \cup V$ and note that $U \cap V$ is an open annulus, i.e. homeomorphic to $S^1 \times (0,1)$.
   (c) We denote by $i\colon U \cap V \to U$ the inclusion.
   (d) We let $P = (\frac{1}{3},\frac{1}{3}) \in U \cap V$ and we denote by p the direct path from Q to P.
   (e) We pick a loop γ in $(U \cap V, P)$ that "goes counterclockwise around the annulus". Since $U \cap V$ has an obvious deformation retraction to a circle we obtain from the Circle-π_1-Theorem 11.13 together with the Homotopy-π_1-Proposition 13.4 (2b) that γ represents a generator of the infinite cyclic group $\pi_1(U \cap V, P) \cong \mathbb{Z}$.
   (f) We denote by x and y the two loops in (T, Q) corresponding to the horizontal and the vertical edge of the square. By a slight abuse of notation we denote the corresponding elements in $\pi_1(T, Q)$ by x and y as well.

[197] In these notes, the notion of "(non-orientable) surface of genus g" is properly defined. At times we will also use the word "surface" in a colloquial sense, even though as a standalone term we never introduced it.





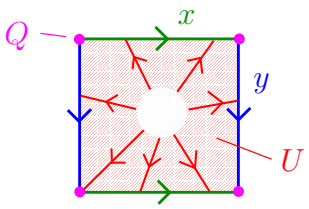


the loop $p * \gamma * \overline{p}$ is path-homotopic to the loop $xyx^{-1}y^{-1}$

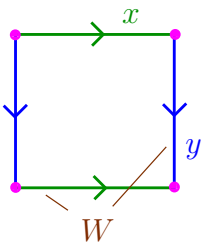
Now we see that

$$\begin{array}{ccccc}
\pi_1(T,P) & \underset{\uparrow}{\cong} & \pi_1(U,P) *_{\pi_1(U\cap V,P)} \pi_1(V,P) & \underset{\uparrow}{=} & \pi_1(U,P) *_{\pi_1(U\cap V,P)} \{e\} \\
 & & \text{Seifert–van Kampen-Theorem 20.1} & & V \text{ is a disk, hence } \pi_1(V,P) = \{e\} \\
 & \underset{\uparrow}{=} & \pi_1(U,P)/\langle\langle i_*(\pi_1(U\cap V,P))\rangle\rangle & \underset{\uparrow}{=} & \pi_1(U,P)/\langle\langle [\gamma]\rangle\rangle \\
 & & \text{by the Amalgamated Product–Quotient Lemma 19.13} & & \text{since } \pi_1(U\cap V,P) = \langle[\gamma]\rangle \\
 & & \text{we have } A *_G \{e\} = A/\langle\langle \alpha(G)\rangle\rangle & & \\
 & \underset{\uparrow}{\overset{\cong}{\to}} & \pi_1(U,Q)/\langle\langle [p*\gamma*\overline{p}]\rangle\rangle & \underset{\uparrow}{=} & \pi_1(U,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle \quad = \quad (*). \\
 & & \text{by the Change-of-Base Point Proposition 11.6 (1)} & & \text{since } p*\gamma*\overline{p} \text{ is path-homotopic to } xyx^{-1}y^{-1} \\
 & & \text{the path } p \text{ gives rise to an isomorphism } p_* & &
\end{array}$$

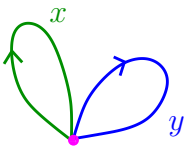
One can easily show that $W := \partial([0,1]\times[0,1])/\sim$ is a deformation retract of U[198] and that W is homeomorphic to the wedge of two circles[199]. By the Wedge-of-Circles–π_1-Lemma 20.7 we can therefore make the identification $\pi_1(W,P) = \langle x, y\rangle$.



deformation retraction from U to W



$\cong$



Now we continue with the above calculation of $\pi_1(T,P)$:

$$\begin{array}{ccccc}
(*) & = & \pi_1(U,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle & \underset{\uparrow}{\overset{\cong}{\to}} & \pi_1(W,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle \\
 & & & & \text{by the Homotopy-}\pi_1\text{-Proposition 13.4 (2b), since } W \text{ is a deformation retract of } U \\
 & \underset{\uparrow}{=} & \langle x,y\rangle/\langle\langle xyx^{-1}y^{-1}\rangle\rangle & = \langle x,y \mid [x,y]\rangle & \underset{\uparrow}{\cong} \mathbb{Z}^2. \\
 & & \text{by the Wedge-of-Circles–}\pi_1\text{-Lemma 20.7} & & \text{by the Free Abelian Group-Presentation Lemma 21.4} \\
 & & \text{since } W \text{ is the wedge of the circles } x \text{ and } y & &
\end{array}$$

---

[198] We consider the map

$$\begin{array}{rcl} r\colon U & \to & \partial([0,1]\times[0,1]) \\ x & \mapsto & \text{unique point in } \mathbb{R}_{>0}\cdot x \cap \partial([0,1]\times[0,1]). \end{array}$$

A deformation retraction is given by

$$\begin{array}{rcl} U\times[0,1] & \to & U \\ ([P],t) & \mapsto & [P\cdot(1-t) + r(P)\cdot t]. \end{array}$$

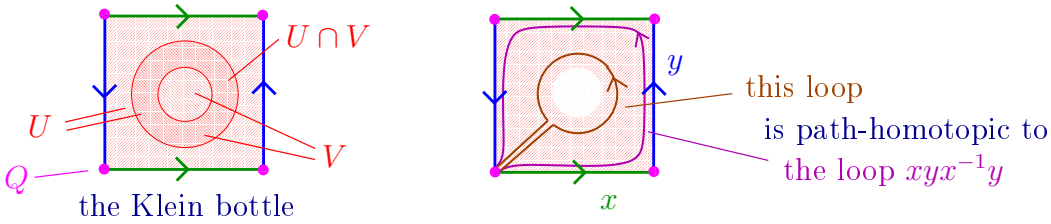
(This map is continuous by the popular Homotopy Quotient Theorem 8.5.)

[199] A homeomorphism is given by sending the point on W that corresponds to the vertices to the wedge point of the two circles, and by sending the edges that get identified to the same circle. This idea can be made precise using the Twice Quotient Lemma 6.3 and the Compact-Hausdorff Proposition 2.10 (3).

Thus we have reproved the fact, first shown in the Torus-π_1–Corollary 13.8, that the fundamental group of the torus is isomorphic to the free abelian group $\mathbb{Z}^2$.

(2) We turn to the Klein bottle $K = ([0,1] \times [0,1])/\sim$ with base point $Q = [(0,0)]$. The approach to calculating the fundamental group of the Klein bottle is almost identical to the above approach to calculating the fundamental group of the torus. We just replace the situation illustrated in the figure above by the situation illustrated in the figure below. The same argument as above shows that [200]

$$\pi_1(K,Q) \;\cong\; \langle x, y \mid xyx^{-1}y \rangle.$$



(3) Next we consider again the real projective space $\mathbb{R}\mathrm{P}^2$. Here we use the description of $\mathbb{R}\mathrm{P}^2$ from the Projective Space-Avatar Lemma 6.11 (2) as

$$\mathbb{R}\mathrm{P}^2 \;=_i\; \overline{B}^2/\sim \qquad \text{where for } z \in S^1 \text{ we have } z \sim -z.$$

As in the previous two examples we decompose $\mathbb{R}\mathrm{P}^2$ into an open disk V and the complement of a closed disk U such that $U \cap V$ is an annulus. We consider the base point $Q = [1] = [-1] \in \mathbb{R}\mathrm{P}^2$ and we denote by α the loop

$$\begin{array}{rccc} \alpha\colon [0,1] & \to & \mathbb{R}\mathrm{P}^2 \\ t & \mapsto & [\exp(\mathrm{i}\pi t)]. \end{array}$$

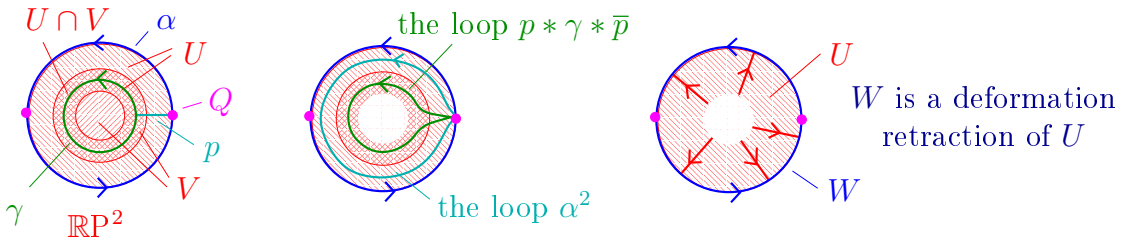
By a rather slight abuse of notation we denote by α also the corresponding element in $\pi_1(\mathbb{R}\mathrm{P}^2, Q)$. Furthermore we denote by W the image of α in $\mathbb{R}\mathrm{P}^2$. One can easily verify that W is homeomorphic to S^1. It follows from the Circle-π_1-Theorem 11.13 that α is a generator of $\pi_1(W,Q)$. With similar arguments as in the torus case, we see that

$$\pi_1(\mathbb{R}\mathrm{P}^2, Q) \;\underset{\text{by the Seifert–van Kampen-Theorem 20.1 and the Amalgamated Product–Quotient Lemma 19.13}}{\cong}\; \pi_1(U,Q)/\langle\langle p * \gamma * \overline{p} \rangle\rangle \;\underset{\text{by the Homotopy-}\pi_1\text{-Proposition 13.4 (2b), since } W \text{ is a deformation retract of } U}{\cong}\; \pi_1(W,Q)/\langle\langle p * \gamma * \overline{p} \rangle\rangle \;\underset{\text{since } \pi_1(W,Q) = \langle\alpha\rangle \text{ and } [p * \gamma * \overline{p}] = \alpha^2}{\cong}\; \langle\alpha\rangle/\langle\langle \alpha^2 \rangle\rangle \;\cong\; \mathbb{Z}_2.$$

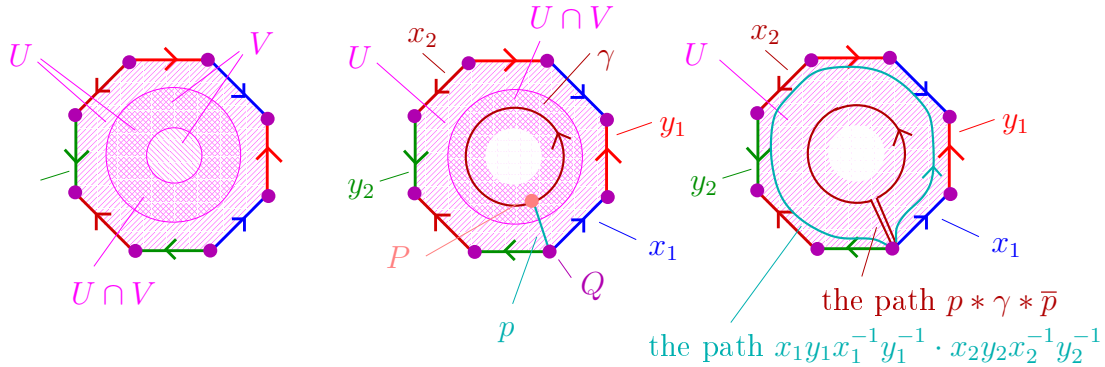
In particular we obtain the same result, as we should, as in the Real Projective Space–π_1-Proposition 15.3.

(4) Now that we have gained some confidence that the approach of determining fundamental groups using the Seifert–van Kampen-Theorem 20.1 does indeed give the right fundamental groups we turn to the surfaces of higher genus that we could not handle with our previous methods.

[200] Using this presentation and using the Abelianization-of-Presentation Proposition 21.5 we see that the abelianization of $\pi_1(K,Q)$ is given by $\pi_1(K,Q)_{\mathrm{ab}} \;\cong\; \langle x, y \mid xyx^{-1}y \rangle_{\mathrm{ab}} \;\cong\; \mathbb{Z}^2/\begin{pmatrix} 0 \\ 2 \end{pmatrix} \cdot \mathbb{Z} \;\cong\; \mathbb{Z} \oplus \mathbb{Z}_2$.



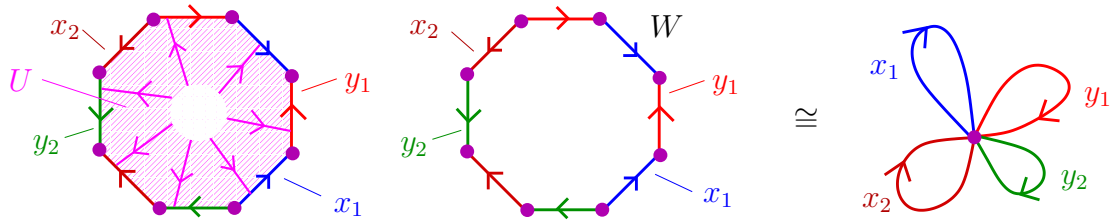
So let us consider the surface $F = E_8/\sim$ of genus 2. As in the torus case we write $F = U \cup V$ where V is an open disk and where U is the complement of a closed disk such that $U \cap V$ is an annulus. Similar to the torus case we write $W = \partial E_8/\sim$. As in the torus case we see that W is a deformation retraction of U and that W is homeomorphic to the wedge of four circles corresponding to x_1, y_1, x_2, y_2. Finally almost the same



argument as in the torus case shows that

$$\begin{aligned}
\pi_1(E_8/\sim, P) &\cong \pi_1(U, Q)/\langle\langle \overbrace{x_1 y_1 x_1^{-1} y_1^{-1} \cdot x_2 y_2 x_2^{-1} y_2^{-1}}^{\text{path-homotopic to } p*\gamma*\overline{p}} \rangle\rangle \\
&\qquad \overbrace{\qquad\qquad\qquad\qquad}^{=[x_1,y_1]\cdot[x_2,y_2]} \\
&\cong \pi_1(W, Q)/\langle\langle [x_1, y_1] \cdot [x_2, y_2] \rangle\rangle \\
&\cong \langle x_1, y_1, x_2, y_2 \rangle / \langle\langle [x_1, y_1] \cdot [x_2, y_2] \rangle\rangle \;=\; \langle x_1, y_1, x_2, y_2 \mid [x_1, y_1] \cdot [x_2, y_2] \rangle.
\end{aligned}$$

To summarize: we have finally succeeded in calculating the fundamental group of the surface of genus 2.



Proposition 22.1. (Surface-π_1-Proposition)

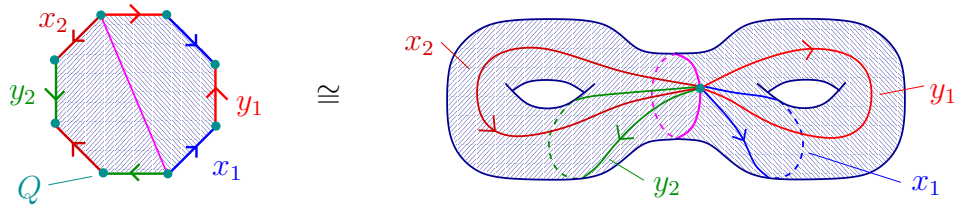
(1) Let $g \in \mathbb{N}$. Let Q be the point in $\Sigma_g = E_{4g}/\sim$ corresponding to the vertex of E_{4g}. We have an isomorphism[201]

$$\pi_1(\text{surface } \Sigma_g \text{ of genus } g, Q) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \rangle$$

where the $x_1, y_1, \ldots, x_g, y_g$ correspond to edges of E_{4g}. Furthermore we have

$$\pi_1(\text{surface } \Sigma_g \text{ of genus } g, Q)_{\text{ab}} \;\cong\; \mathbb{Z}^{2g}.$$

(2) For $g \neq h$ the surfaces of genus g and h are not homotopy equivalent, in particular they are not homeomorphic.



Proof.

(1) Let $g \in \mathbb{N}_0$. For $g = 0$ we obtain the fundamental group from the Sphere-π_1-Proposition 11.14 and for $g = 1$ we obtain the presentation from the discussion on page 284. For $g \geq 2$ the presentation follows from a straightforward generalization of the calculation preceding the proposition. Finally the stated abelianization of the fundamental group is an immediate consequence of the Abelianization-of-Presentation Proposition 21.5.

(2) For $g \neq h$ we obtain from the Abelian Group-Classification Theorem 18.11 that the abelianizations of the fundamental groups of the surfaces of genus $g \neq h$ are not isomorphic. It follows from the Homotopy-π_1-Proposition 13.4 that the surfaces are not homotopy equivalent. ■

22.2. **The fundamental groups of the non-orientable surfaces N_g.** In this section we will discuss the fundamental groups of the non-orientable surfaces N_g of some genus $g \in \mathbb{N}$. The following proposition is the analogue of the Surface-π_1-Proposition 22.1.

Proposition 22.2. (Surface-π_1-Proposition II)

(1) For any $g \in \mathbb{N}$ we have

$$\pi_1(\text{non-orientable surface } N_g \text{ of genus } g) \;\cong\; \langle x_1, \ldots, x_g \mid x_1^2 \cdot \ldots \cdot x_g^2 \rangle$$

and

$$\pi_1(\text{non-orientable surface } N_g \text{ of genus } g)_{\text{ab}} \;\cong\; \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2.$$

(2) For $g \neq h$ the non-orientable surfaces of genus g and h are not homeomorphic.

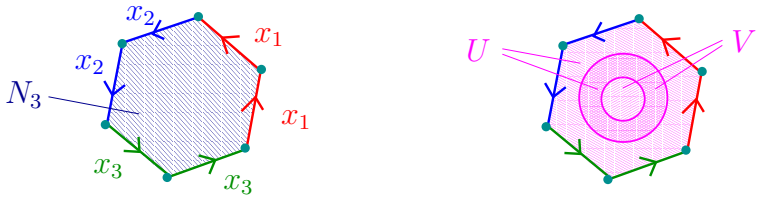
Proof.

(1) Let $g \in \mathbb{N}$. Recall that by the definition on page 118 the non-orientable surface of genus one is the real projective plane. In the Real Projective Space–π_1-Proposition 15.3 (or alternatively above on page 285) we saw that $\pi_1(\mathbb{R}\text{P}^2) \cong \mathbb{Z}_2 \cong \langle x \mid x^2 \rangle$. The case $g \geq 2$ follows from a straightforward modification of the calculation of the fundamental group

[201] This statement also makes sense for $g = 0$.

of the surface of genus 2 that we gave on page 286. Using the figure below the reader should have no troubles with assembling a full argument. Finally note that

$$\pi_1(N_g)_{\mathrm{ab}} \underset{\text{by (1)}}{\cong} \langle x_1,\dots,x_g \mid x_1^2\cdot\ldots\cdot x_g^2\rangle_{\mathrm{ab}} \underset{\text{by the Abelianization-of-Presentation Proposition 21.5}}{\cong} (\mathbb{Z}^g)/\mathbb{Z}\cdot\begin{pmatrix}2\\ \vdots\\ 2\end{pmatrix} \cong \mathbb{Z}^{g-1}\oplus\mathbb{Z}_2.$$



(2) For $g \neq h$ we obtain from the above calculation and from the Abelian Group-Classification Theorem 18.11 that the abelianizations of the fundamental groups of the non-orientable surfaces of genus g and h are not isomorphic. Hence the non-orientable surfaces of genus $g \neq h$ are not homeomorphic. ■

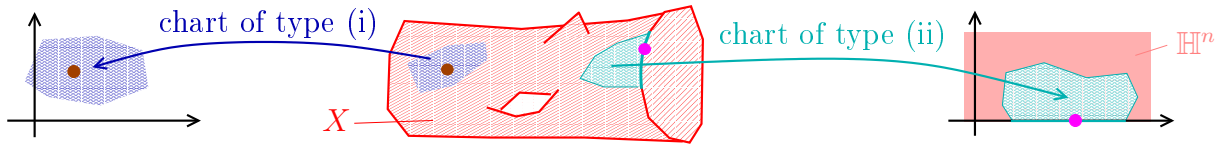
22.3. **Interlude: The boundary of 2-dimensional topological manifolds.** In the next section we will study the fundamental groups of 2-dimensional topological manifolds with boundary. Before we get there we need to prove the subtle statement that the boundary of a 2-dimensional topological manifold is what you think it should be.

Definition. Let X be a topological space.

(a) An n-dimensional chart for X at a point $x \in X$ is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of x and V is one of the following:
  (i) V is an open subset of $\mathbb{R}^n$ or
  (ii) V is an open subset of the upper half-space $\mathbb{H}^n := \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and $\Phi(x)$ lies on $\partial\mathbb{H}^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n = 0\}$.
  In the former case we say that Φ is a chart of type (i), in the latter case we say that Φ is a chart of type (ii).

(b) We say X is an n-dimensional topological manifold if every point admits a chart and if X is Hausdorff and second countable.

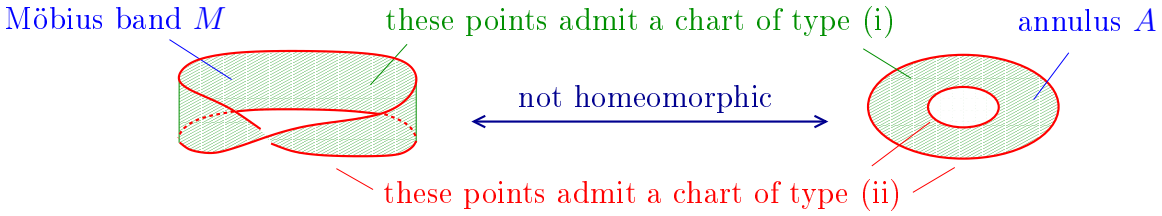
(c) We say that a point x on a topological manifold X is a boundary point if x does not admit a chart of type (i). We denote by ∂X the set of all boundary points of X.



Proposition 22.3. (Topological Surface–Boundary Proposition) Let F be a 2-dimensional topological manifold. Every point on F admits either a chart of type (i) or a chart of type (ii).

Remark. We will prove the high-dimensional analogue of the above proposition in algebraic topology II, once we have introduced and studied homology groups.

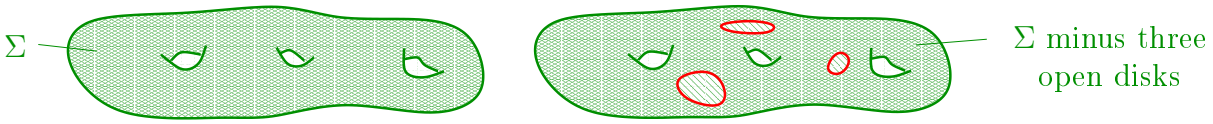
Example. In the figure below we show the Möbius band M and the annulus A. The Topological Surface–Boundary Proposition 22.3 says that the boundary of the Möbius band M and the annulus A, *as a topological manifold*, is exactly what we think it should be, namely precisely the points that "obviously" admit a chart of type (ii). In particular we see that ∂M consists of one component whereas ∂A consists of two components. Since homeomorphisms of topological manifolds induce homeomorphisms of their boundaries and since ∂M is not homeomorphic to ∂A we see that the Möbius band M is not homeomorphic to the annulus A.



Proof. We outsource the proof of the Topological Surface–Boundary Proposition 22.3 (which uses the fundamental group) to [Fri23]. ■

22.4. **The fundamental groups of surfaces with boundary.** As mentioned above, one goal of any classification theorem is to provide a "complete" list of objects. We already have a good list of closed surfaces. Now we need to add surfaces with non-empty boundary to our arsenal.

Definition. Let Σ be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$. We pick smooth embeddings $\varphi_1, \dots, \varphi_n \colon \overline{B}^2 \to \Sigma$ with disjoint images. We refer to $\Sigma \setminus (\varphi_1(B^2) \cup \dots \cup \varphi_n(B^2))$ as Σ minus n open disks.



The following lemma captures some of the main features of the above definition.

Lemma 22.4. (Surface-Minus-Open Disks Lemma) Let Σ be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$.

(1) The topological space Σ minus n open disks is a compact connected 2-dimensional topological manifold with n boundary components, which are given by the boundaries of the closed disks whose interiors were removed.
(2) Σ minus n open disks is a submanifold of the smooth manifold Σ.
(3) The diffeomorphism type of the smooth manifold Σ minus n open disks does not depend on the choice of the disks.

Proof. The lemma follows from basic facts about smooth manifolds. For example (3) follows from the fact, proved in the Smooth Ball Embedding Theorem of [Fri23], that any two choices of smooth embeddings $\overline{B}^2 \to \Sigma$ are smoothly isotopic (up to a possible reflection of $\overline{B}^2$.) We refer to [Fri23] for details. ■

The Surface-Minus-Open Disks Lemma 22.4 allows us to introduce the following notation:

Notation. Given $g, n \in \mathbb{N}_0$ we write

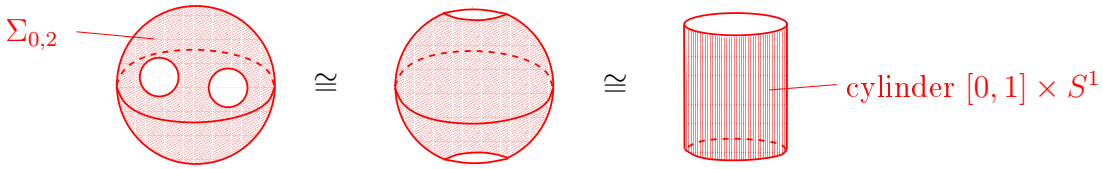
$$\Sigma_{g,n} \quad := \quad \text{the surface of genus } g \text{ minus } n \text{ open disks.}$$

Furthermore for $g \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we write

$$N_{g,n} \quad := \quad \text{the non-orientable surface of genug } g \text{ minus } n \text{ open disks.}$$

Example.

(1) The surface $\Sigma_{0,2}$ is the result of removing the interiors of two disjoint closed disks from the 2-sphere. It is pretty elementary to show that the resulting smooth manifold is diffeomorphic to the "cylinder" $[0,1] \times S^1$.



(2) We saw in some exercise that $\mathbb{R}\mathrm{P}^2$ minus an open disk is diffeomorphic to the Möbius band, i.e. $N_{1,1}$ is diffeomorphic to the Möbius band.

The following lemma shows that the fundamental groups of our new friends are rather dull and relatively uninformative.

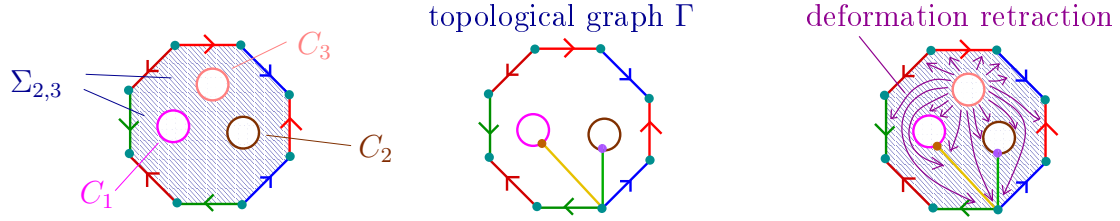
Lemma 22.5. (Surface-with-Boundary–π_1-Lemma)

(1) Given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}$ the surface $\Sigma_{g,n}$ admits a deformation retract to a topological graph of Euler characteristic $2 - 2g - n$. In particular $\pi_1(\Sigma_{g,n})$ is a free group on $2g + n - 1$ generators.

(2) Given $k \in \mathbb{N}$ and $n \in \mathbb{N}$ the surface $N_{k,n}$ admits a deformation retract to a topological graph of Euler characteristic $2-k-n$. In particular $\pi_1(N_{k,n})$ is a free group on $k+n-1$ generators.

Sketch of proof.

(1) We denote by $C_1, \ldots, C_n$ the boundary components of $\Sigma_{g,n}$. Let Γ be the topological space that is given by the wedge of $2g$-circles (which is obtained from identifying as usual the edges of the $4g$-gon pairwise), together with the boundary components $C_1, \ldots, C_{n-1}$ and together with edges $e_1, \ldots, e_{n-1}$ that connect the wedge point to $C_1, \ldots, C_{n-1}$. The deformation retraction from $\Sigma_{g,n}$ to Γ is given by pushing the remaining boundary component C_n "inward". We illustrate this in the figure below. (This is one of the rare occasions where we provide a "proof by picture". It is of course possible to give a completely rigorous proof, but one does not gain anything from doing so.)



We equip Γ with the obvious structure of a topological graph which has $n = 1+(n-1)$ vertices and $2g + 2(n-1)$ edges. Note that $\chi(\Gamma) = 2 - 2g - n$. Finally we see that

$$\pi_1(\Sigma_{g,n}) \underset{\uparrow}{\cong} \pi_1(\Gamma) \cong \pi_1\Big(\begin{array}{c}\text{connected topological graph } \Gamma \\ \text{with } \chi(\Gamma) = 2-2g-n\end{array}\Big) \underset{\uparrow}{\cong} \begin{array}{c}\text{free group on} \\ 2g+n-1 \text{ generators}\end{array}$$

by the above together with the Homotopy-π_1-Proposition 13.4 (2b) — by the Graph-π_1-Proposition 20.9

(2) The proof of this statement is almost identical to the proof of the first statement. ■

In the following we will see that we can "recover" the closed surfaces Σ_g and N_g from any $\Sigma_{g,n}$ respectively any $N_{g,n}$. The key to doing so is the following definition.

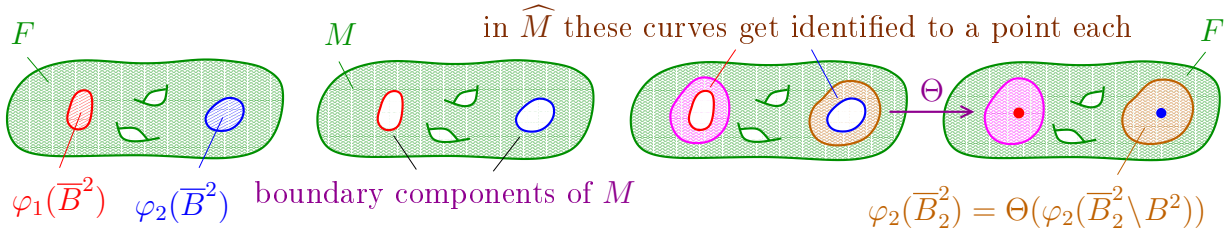
Definition. Let M be a topological manifold. Let "$\sim$" be the equivalence relation on M that is generated by $x \sim y$ whenever x and y lie on the same component of ∂M. We write $\widehat{M} := M/\sim$.

The following lemma shows in particular that $\widehat{\Sigma_{g,n}}$ is homeomorphic to Σ_g and that $\widehat{N_{g,n}}$ is homeomorphic to N_g.

Lemma 22.6. (Surface Recovery Lemma) Let F be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$. If we set M to be F minus n open disks, then $\widehat{M}$ is homeomorphic to F.

Proof. By the Surface-Minus-Open Disks Lemma 22.4 we can work with any smooth embeddings. Thus let $\varphi_1, \ldots, \varphi_n \colon \overline{B}^2_2 \to F$ be smooth embeddings of the closed disks *of radius* 2 with disjoint images. We need to show that for $M := F \setminus (\varphi_1(B^2) \cup \cdots \cup \varphi_n(B^2))$ there exists a homeomorphism $\widehat{M} \to F$. We consider the following map:[202]

$$\begin{array}{rcl}\Theta \colon \widehat{M} & \to & F \\ [P] & \mapsto & \begin{cases} \varphi_i(Q \cdot 2 \cdot (\|Q\| - 1)), & \text{if } P = \varphi_i(Q) \text{ for some } i \in \{1, \ldots, n\} \text{ and } Q \in \overline{B}^2_2 \setminus B^2, \\ [P], & \text{otherwise.} \end{cases}\end{array}$$



It follows from the Pasting Proposition 2.4 (2) and the Topological-Quotient Proposition 6.1 (1b) that Θ is continuous. Note that by the Compact Image Lemma 2.8 we know that $\widehat{M}$ is compact. Now it follows almost immediately from the Compact-Hausdorff Proposition 2.10 (3) that the given map $\Theta \colon \widehat{M} \to F$ is indeed a homeomorphism. ■

[202] Note that discretely we are using the Surface-Minus-Open Disks Lemma 22.4 (1) which says that the boundary of M, viewed as a topological manifold, is indeed what we think it should be, namely it is given by $\varphi_1(S^1) \sqcup \cdots \sqcup \varphi_n(S^1)$.

22.5. **The Surface Classification Theorem.** One of the goals of mathematicians is to classify objects of one's favorite category $\mathcal{C}$. We could for example consider the category of (finitely presented) groups, the category of (finite-dimensional) $\mathbb{K}$-vector spaces or the category of (compact) n-dimensional topological (or smooth) manifolds. But what does "classify" mean? One interpretation would be the following:

(I) Provide a collection $\{X_i\}_{i\in I}$ of objects with the following two properties:
  (a) Any other object of $\mathcal{C}$ is isomorphic to some X_i.
  (b) If $i \neq j$, then X_i and X_j are not isomorphic.

(II) Provide an algorithm which determines, given an object in $\mathcal{C}$, to which X_i it is isomorphic to.

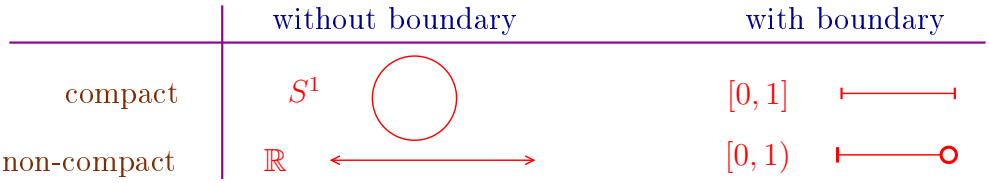
For example in the category of finite-dimensional $\mathbb{K}$-vector spaces a complete list is given by $\{\mathbb{K}^i\}_{i\in\mathbb{N}_0}$ and the calculation of the dimension gives us (II). Furthermore the Abelian Group-Classification Theorem 18.11 gives a classification of the objects in the category of finitely generated abelian groups.

In these notes we are mostly interested in the classification of connected n-dimensional smooth and topological manifolds. The following theorem gives the classification of 1-dimensional topological manifolds.

Theorem 22.7. (Topological 1-Dimensional Manifold Classification Theorem) Let M be a connected non-empty 1-dimensional topological manifold. The homeomorphism type of M is given by the following table

	without boundary	with non-empty boundary
compact	the circle S^1	the closed interval $[0,1]$
non-compact	the real line $\mathbb{R}$	the half-open interval $[0,1)$.

An analogous statement holds for the diffeomorphism type of smooth manifolds.



Proof. The proof is relatively elementary, but it is nonetheless not totally obvious. For example it is not entirely clear where in the proof one uses that topological manifolds are Hausdorff and second countable. We provide the proof in [Fri23]. ■

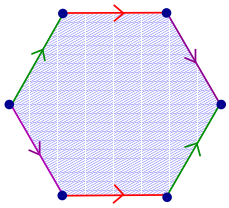
We turn our gaze to the 2-dimensional setting. We will state the classification theorem for compact connected non-empty 2-dimensional topological and smooth manifolds and we will prove (I b) and (II) in the above setting. For the proof of (I a) we will need some patience though, we will only provide it in a later topology course.

Theorem 22.8. (Surface Classification Theorem) Every compact connected non-empty 2-dimensional topological manifold is homeomorphic to either

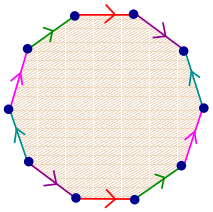
(1) the surface $\Sigma_{g,n}$ for *unique* $g, n \in \mathbb{N}_0$ or
(2) to the surface $N_{k,n}$ for *unique* $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Furthermore, if we are given a smooth manifold, then we can upgrade "homeomorphic" to "diffeomorphic".

Example. At least for closed orientable connected 2-dimensional smooth manifolds it is perhaps tempting to dismiss the Surface Classification Theorem 22.8 (1) as "obvious". After all, what other closed orientable connected 2-dimensional smooth manifold could there possibly be? It is thus instructive to consider the figure below. It shows a hexagon and a decagon where *opposite sides* are identified. The quotient spaces are closed orientable 2-dimensional smooth manifolds. To which of the standard surfaces Σ_g are they diffeomorphic to?



hexagon with opposite sides identified



dodecagon with opposite sides identified

The Surface Classification Theorem 22.8 is an immediate consequence of the following three theorems.

Theorem 22.9. (Diffeomorphism-to-Standard Surfaces Theorem) Let Σ be a compact connected non-empty 2-dimensional smooth manifold.

(1) If Σ is orientable, then Σ is diffeomorphic to the surface $\Sigma_{g,n}$ for some $g, n \in \mathbb{N}_0$.
(2) If Σ is non-orientable, then Σ is diffeomorphic to the surface $N_{k,n}$ for some $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Theorem 22.10. (Surface-Smooth Structure Existence Theorem) Every compact 2-dimensional topological manifold admits a smooth structure.

Theorem 22.11. (Surfaces-Non Homeomorphic Theorem) The topological spaces $\Sigma_{g,m}$ and $N_{h,n}$ are pairwise non-homeomorphic.

We now turn to the discussion of the proofs of the three theorems.

Proof of the Diffeomorphism-to-Standard Surfaces Theorem 22.9. We will prove this theorem later in a few semesters time using "Morse theory". The impatient student can also look up the proof in [Fri23]. ■

Proof of the Surface-Smooth Structure Existence Theorem 22.10. This theorem is technically quite tricky. A proof is for example given in [Hat13, Theorem A]. ■

Since we did not really prove the first two theorems we should at least provide full details for the third theorem. The idea is to find a complete set of invariants to distinguish the $\Sigma_{g,m}$ and $N_{h,n}$ pairwise. The first invariant we want to use is of course the (abelianization) of the fundamental group. But since this invariant is not enough we have to look for further invariants. One idea would be to use the number of boundary components. But even that is not good enough: In the following figure we see two compact connected 2-dimensional topological manifolds such that the fundamental groups are both isomorphic to the free group on two generators and such that both have one boundary component.

It is rather unlikely though that they are homeomorphic.[203] This example strongly suggests that the fundamental group and the number of boundary components is not enough to distinguish all examples.

Proof of the Surfaces-Non Homeomorphic Theorem 22.11. Let M be a topological manifold. As before let "$\sim$" be the equivalence relation on M that is generated by $x \sim y$ whenever x and y lie on the same component of ∂M. As before we write $\widehat{M} := M/\sim$.

Claim. For $g \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we have the following table of invariants:

X	Σ_g	$\Sigma_{g,m}$ with $m \geq 1$	N_k	$N_{k,n}$ with $n \geq 1$
isomorphism type of $\pi_1(X)_{\text{ab}}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g+m-1}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{k+n-1}$
number of boundary components	0	m	0	n
isomorphism type of $\pi_1(\widehat{X})_{\text{ab}}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$.

Proof. The number of boundary components can be determined using the Topological Surface–Boundary Proposition 22.3. The abelianizations of the fundamental groups of X are given by the Surface-π_1-Propositions 22.1 and 22.2 and the Surface-with-Boundary–π_1-Lemma 22.5. Finally the abelianizations of the fundamental of $\widehat{X}$ are given by the above calculation and the fact, shown in the Surface Recovery Lemma 22.6, that $\widehat{\Sigma_{g,m}} \cong \Sigma_g$ and $\widehat{N_{k,n}} \cong N_k$. ⊞

It follows from the second and third row that the $\Sigma_{g,m}$, $g \in \mathbb{N}_0, m \in \mathbb{N}_0$ and $N_{k,n}$, $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$ are pairwise non-homeomorphic. ■

[203] As a smooth manifold X is orientable and Y is non-orientable. But as of right now we do not know whether orientability is preserved under homeomorphisms.

23. The Seifert–van Kampen Theorem applied to Smooth Manifolds

In this chapter we will first prove a variation on the Seifert–van Kampen-Theorem 20.1 which is more suitable to smooth manifolds. We will use this variation to prove that every finitely presented group occurs as the fundamental group of a closed orientable smooth manifold of dimension ≥ 4.

23.1. **The Seifert-van Kampen Theorem for smooth manifolds.** For many applications the following theorem, which is a variation on the Seifert–van Kampen-Theorem 20.1, is very useful.

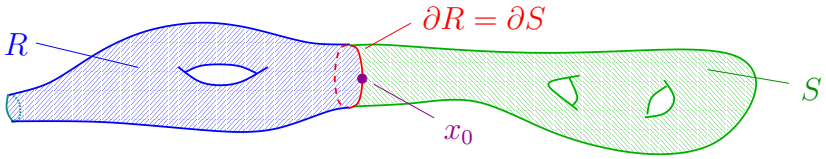
Theorem 23.1. (Manifold-Seifert–van Kampen Theorem) Let M be an m-dimensional smooth or topological manifold and let $R, S \subset M$ be two codimension zero submanifolds such that the following hold:

(1) $M = R \cup S$.
(2) R and S are closed subsets of M. (Note that by the Compact-Closed Lemma 1.9 (2) this condition is satisfied if R and S are compact.)
(3) $R \cap S$ is a single component of ∂R and it is a single component of ∂S.

For every base point $x_0 \in R \cap S$ the inclusion induced map

$$\pi_1(R, x_0) *_{\pi_1(R\cap S, x_0)} \pi_1(S, x_0) \ \to \ \pi_1(M, x_0)$$

is an isomorphism.



Example. Given $n \in \mathbb{N}_0$ we denote as usual by $S^n_{\geq 0}$ the upper hemisphere of S^n and we denote by $S^n_{\leq 0}$ the lower hemisphere of S^n. Clearly the two hemispheres are closed subsets and one can show easily that both are codimension-zero submanifolds of S^n. We denote by $S^n_{=0} = S^n_{\geq 0} \cap S^n_{\leq 0}$ the "equator" of S^n, which is of course homeomorphic to S^{n-1}. Given any $n \in \mathbb{N}_{\geq 2}$ and any $P \in S^n_{=0}$ we can perform the following calculation:

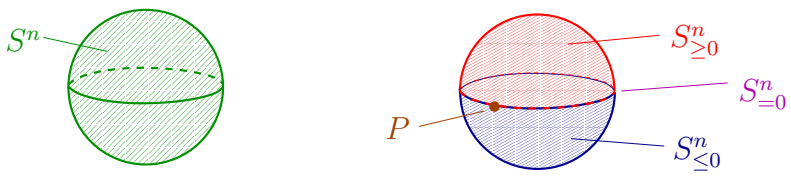
by page 53 the hemispheres are homeomorphic to $\overline{B}^n$, hence the fundamental groups are trivial

$$\pi_1(S^n, P) \ \underset{\uparrow}{\cong} \ \pi_1(S^n_{\leq 0}, P) *_{\pi_1(S^n_{=0}, P)} \pi_1(S^n_{\geq 0}, P) \ \underset{\downarrow}{\cong} \ \{e\} *_{\pi_1(S^{n-1}, P)} \{e\} \ \underset{\uparrow}{\cong} \ \{e\}.$$

by the Manifold-Seifert–van Kampen Theorem 23.1, note that we use that our hypothesis $n \in \mathbb{N}_{\geq 2}$ implies that S^{n-1} is path-connected

by definition or by the Amalgamated Product–Quotient Lemma 19.13

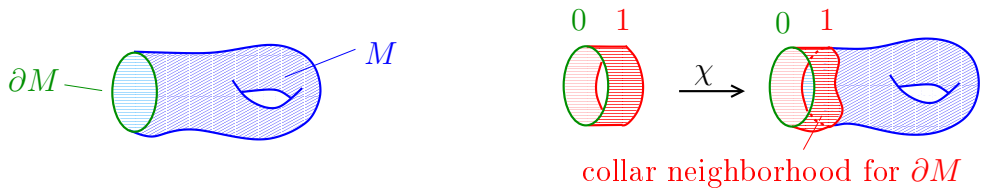
This calculation gives another proof of the Sphere-π_1-Proposition 11.14.[204]



[204] This argument is at the end quite similar to the proof of the Suspension-π_1-Lemma 20.2.

Of course the idea is to prove the Manifold-Seifert–van Kampen-Theorem 23.1 by reducing it to the Seifert-van Kampen-Theorem 20.1. But note that we cannot apply the Seifert-van Kampen-Theorem 20.1 directly, since R and S are not open. We will apply the same trick as in the proof of the Wedge-π_1-Proposition 20.6, namely we will expand R and S to open subsets U and V to which we can apply the Seifert-van Kampen-Theorem 20.1. The key to carrying out this idea is the following theorem.

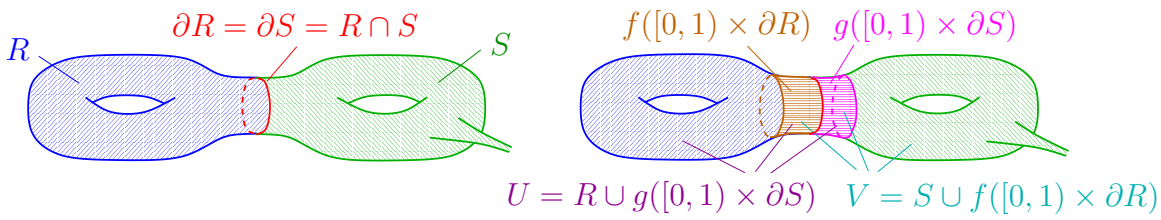
Theorem 23.2. (Collar Neighborhood Theorem) Given any smooth or topological manifold M and given any union R of components of ∂M there exists an embedding $\chi\colon [0,1] \times R \to M$ with the following two properties:
(1) For every $P \in R$ we have $\chi(0, P) = P$.
(2) The image of $[0,1) \times R$ is an open subset of M.
The image of χ is sometimes called a collar neighborhood of R.



Proof. The proof is provided in [Fri23]. ■

Proof of Theorem 23.1. Let M be an m-dimensional smooth or topological manifold. Furthermore let $R, S \subset M$ be two codimension-zero submanifolds such that the following hold:
(1) $M = R \cup S$.
(2) R and S are closed subsets of M.
(3) $R \cap S$ is a component of ∂R and it is a component of ∂S.
In fact, to simplify the notation a little bit we assume that $R \cap S = \partial R = \partial S$. Let $x_0 \in R \cap S$. We pick embeddings $f\colon [0,1] \times \partial R \to R$ and $g\colon [0,1] \times \partial S \to S$ that are provided by the Collar Neighborhood Theorem 23.2. We set

$$U := R \cup \underbrace{g([0,1) \times \partial S)}_{\subset S} \quad \text{and} \quad V := S \cup \underbrace{f([0,1) \times \partial R)}_{\subset R}.$$



We consider the following diagram

$$\begin{array}{ccc} \pi_1(R, x_0) *_{\pi_1(R\cap S, x_0)} \pi_1(S, x_0) & \longrightarrow & \pi_1(M, x_0) \\ \downarrow & & \downarrow \mathrm{id} \\ \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) & \longrightarrow & \pi_1(M, x_0). \end{array}$$

The following steps are similar to the ones we performed in the proof of the Wedge-π_1-Proposition 20.6.

(a) Since the vertical maps are induced by inclusions we see that the diagram commutes.

(b) We claim that U is an open subset of $M = R \cup S$. Since R and S are by our hypothesis (2) closed subsets of M we obtain from the Cover-Open Subset Lemma 1.5 (2a) that it suffices to show that $U \cap R$ is open in R and that $U \cap S$ is open in S. Evidently $U \cap R = R$ is open in R and $U \cap S = g([0,1) \times \partial S)$ is open in S by the second property of g, as stated in the Smooth Collar Neighborhood Theorem 23.2.

(c) We want to show that R is a deformation retract of U. To do we consider the following map:

$$\begin{array}{rcl} U \times [0,1] & \to & U \\ (u,t) & \mapsto & \begin{cases} u, & \text{if } u \in R, \\ g(s \cdot t, y), & \text{if } u = g(s,y) \in g([0,1) \times \partial S). \end{cases} \end{array}$$

It remains to show that this map is continuous. By the Homotopy Combination Lemma 8.2 (1) it suffices to show that R and $g([0,1) \times \partial S)$ are closed subsets of U. We see this as follows: By our hypothesis R and S are closed subsets of M. It follows that $R = R \cap U$ and $g([0,1) \times \partial S) = S \cap U$ are closed subsets of U.

(d) The obvious analogues of (b) and (c) hold for V and S.

(e) Similar to (c) one can show that $R \cap S$ is a deformation retract of $U \cap V$.

(f) We consider the following diagram:

$$\begin{array}{ccccc} \pi_1(R, x_0) & \longleftarrow & \pi_1(R \cap S, x_0) & \longrightarrow & \pi_1(S, x_0) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1(U, x_0) & \longleftarrow & \pi_1(U \cap V, x_0) & \longrightarrow & \pi_1(V, x_0). \end{array}$$

Here all maps are induced by inclusions. It follows that the diagram commutes. Furthermore we obtain from (c), (d), (e) and the Homotopy-π_1-Proposition 13.4 (2) that the vertical maps are all isomorphisms. It follows that the induced map between amalgamated products, shown in the initial diagram to the left, is an isomorphism.

(g) By (b) and (e) we can apply the Seifert–van Kampen-Theorem 20.1 to the decomposition $M = U \cup V$ and we know that the bottom horizontal map of the initial commutative diagram is an isomorphism.

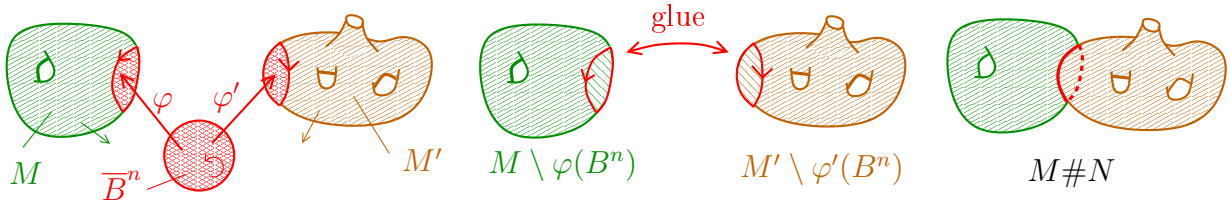
(h) Since the vertical maps are isomorphisms we see that the top horizontal map of the initial diagram is also an isomorphism. ■

23.2. **Connected sum of smooth manifolds.** The following construction gives us a very general approach to constructing new examples of smooth manifolds.

Definition. Let $n \in \mathbb{N}$. Let M and M' be two connected non-empty n-dimensional smooth manifolds. Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$ be smooth embeddings. We define the connected sum of the smooth manifolds M and M' as

$$M \# M' := \big(M \setminus \varphi(B^n)\big) \sqcup \big(M' \setminus \varphi'(B^n)\big) / \sim \quad \text{where } \varphi(P) \sim \varphi'(P) \text{ for all } P \in S^{n-1}.$$

Example. Note that the connected sum of g tori is diffeomorphic to the surface Σ_g of genus g. Furthermore note that the connected sum of g copies of $\mathbb{R}\mathrm{P}^2$ is diffeomorphic to the non-orientable surface N_g of genus g.



Example. Note that $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ is homeomorphic to the Klein bottle K. Just for fun and for practice we provide the direct calculation $\pi_1(K) \cong \pi_1(\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2)$ using the Manifold-Seifert–van Kampen Theorem 23.1. In order to distinguish the two copies of $\mathbb{R}\mathrm{P}^2 =_i \overline{B}^2/\sim$ we write $P = \mathbb{R}\mathrm{P}^2$ and $Q = \mathbb{R}\mathrm{P}^2$. We view $\frac{1}{2} \in \overline{B}^2 \subset \mathbb{C}$ as a common base point. We have

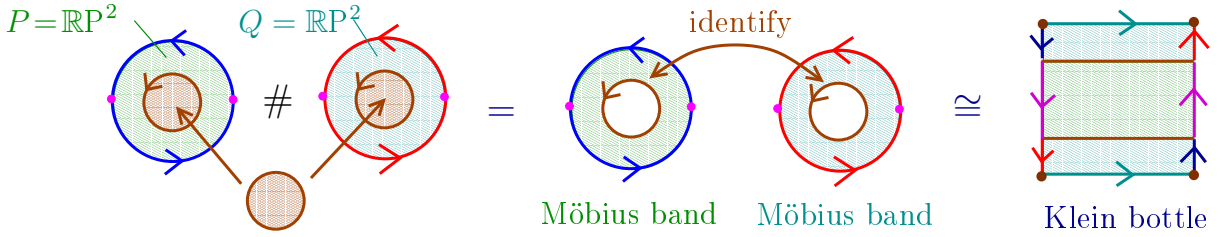
$$\pi_1(\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2, \tfrac{1}{2}) \;= \pi_1(P\#Q, \tfrac{1}{2}) \;\underset{\uparrow}{\cong}\; \pi_1\big(P \setminus \tfrac{1}{2}B^2, \tfrac{1}{2}\big) *_{\pi_1(\frac{1}{2}S^1, \frac{1}{2})} \pi_1\big(Q \setminus \tfrac{1}{2}B^2, \tfrac{1}{2}\big).$$

by the definition of $P\#Q$ and the Manifold-Seifert–van Kampen Theorem 23.1

We pick a generator g for $\pi_1(\frac{1}{2}S^1, \frac{1}{2})$. In the calculation of $\pi_1(\overline{B}^2/\sim)$ on page 285 we saw that $\pi_1(P \setminus \frac{1}{2}B^2, \frac{1}{2}) \cong \langle x\rangle$ and that the inclusion induced map $\pi_1(\frac{1}{2}S^1) \to \pi_1(P \setminus \frac{1}{2}B^2, \frac{1}{2})$ is given by $g \mapsto x^2$. Evidently the same holds for Q with $\pi_1(Q \setminus \frac{1}{2}B^2) \cong \langle y\rangle$. Summarizing we see that

$$\begin{aligned}
\pi_1(K) \;\cong\; \pi_1(\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2, \tfrac{1}{2}) \;&\cong\; \pi_1\big(P \setminus \tfrac{1}{2}B^2\big) *_{\pi_1(\frac{1}{2}S^1, \frac{1}{2})} \pi_1\big(Q \setminus \tfrac{1}{2}B^2, \tfrac{1}{2}\big) \\
&\cong \text{ amalgamated product of } \begin{pmatrix} \langle g\rangle \to \langle x\rangle \\ g \mapsto x^2 \end{pmatrix} \text{ and } \begin{pmatrix} \langle g\rangle \to \langle y\rangle \\ g \mapsto y^2 \end{pmatrix} \\
&\underset{\uparrow}{\cong}\; \langle x, y \mid x^2 = y^2\rangle \;=\; \langle x, y \mid x^2 \cdot y^{-2}\rangle.
\end{aligned}$$

the Amalgamated Product–Presentation Lemma 21.7



We now turn to the study of the fundamental groups of the connected sum of two smooth manifolds of dimension ≥ 3.

Proposition 23.3. (Connected Sum-π_1-Proposition) Let $n \geq 3$ and let M and M' be two connected non-empty n-dimensional smooth manifolds. Then

$$\pi_1(M\#M') \;\cong\; \pi_1(M) * \pi_1(M').$$

Example. Let $m \geq 2$. We consider the connected sum of k copies of $S^1 \times S^m$. We see that:

$$\pi_1((S^1\times S^m)\#\ldots\#(S^1\times S^m)) \underset{\uparrow}{\cong} \pi_1(S^1\times S^m) *\ldots* \pi_1(S^1\times S^m) \underset{\uparrow}{\cong} \overbrace{\langle g_1\rangle *\ldots* \langle g_k\rangle}^{=\langle g_1,\ldots,g_k\rangle}.$$

by the Connected Sum-π_1-Proposition 23.3, since $m \geq 2$

by the Product-π_1-Proposition 13.7 we know that $\pi_1(S^1 \times S^m) \cong \langle g_i\rangle$

Proof of the Connected Sum-π_1-Proposition 23.3. Let M and M' be two connected non-empty n-dimensional smooth manifolds, and let $\varphi\colon \overline{B}^n \to M$ and $\varphi'\colon \overline{B}^n \to M'$ be two smooth embeddings. Recall that the connected sum of M and M' is defined as

$$M\#M' := \big(\underbrace{(M \setminus \varphi(B^n))}_{=:X} \sqcup \underbrace{(M' \setminus \varphi'(B^n))}_{=:X'}\big) / \sim \quad \text{where } \varphi(P) \sim \varphi'(P) \text{ for } P \in S^{n-1}.$$

We pick a base point x_0 that lies in the intersection of the images of X and X' in $M\#M'$.

Claim. The inclusion induced maps $\pi_1(X, x_0) \to \pi_1(M, x_0)$ and $\pi_1(X', x_0) \to \pi_1(M', x_0)$ are isomorphisms.

Proof. By symmetry it suffices to show that the inclusion induced map $\pi_1(X, x_0) \to \pi_1(M, x_0)$ is an isomorphism. Note that X and $\varphi(\overline{B}^n)$ are n-dimensional submanifolds of M and that they are closed subsets of M. Note that $X \cap \varphi(\overline{B}^n) = \varphi(S^{n-1})$. From the Manifold-Seifert–van Kampen Theorem 23.1 we obtain the following pushout diagram:

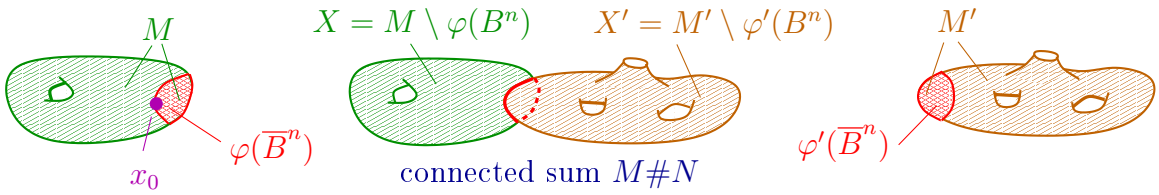
$$\begin{array}{ccc} \pi_1(\varphi(S^{n-1}), x_0) & \longrightarrow & \pi_1(X, x_0) \\ \downarrow & & \lrcorner \downarrow \\ \pi_1(\varphi(\overline{B}^n), x_0) & \longrightarrow & \pi_1(M, x_0). \end{array}$$

Since $n \geq 3$ we obtain from the Sphere-π_1-Proposition 11.14 that $\pi_1(\varphi(S^{n-1}), x_0) = \{e\}$. Thus the vertical map to the left is an isomorphism. It follows from the Amalgamated Product Lemma 19.12 (2a) that the right vertical map is also an isomorphism. ⊞

Now we see that

$$\pi_1(M\#M', x_0) \underset{\text{by the Manifold-Seifert–van Kampen Theorem 23.1, since } X \cap X' \cong S^{n-1} \text{ is simply connected}}{\overset{\cong}{\longleftarrow}} \pi_1(X, x_0) * \pi_1(X', x_0) \underset{\text{by the above claim}}{\overset{\cong}{\longrightarrow}} \pi_1(M, x_0) * \pi_1(M', x_0).$$

■



23.3. **Fundamental groups of high-dimensional smooth manifolds.** Later we will see that the fundamental group of a compact connected smooth manifold is necessarily finitely presented. In this section we want to study the converse: which finitely presented groups can occur as fundamental groups of compact smooth manifolds?

First note that it follows from the Topological 1-Dimensional Manifold Classification Theorem 22.7 that only the trivial group and $\mathbb{Z}$ occur as the fundamental group of a compact connected 1-dimensional topological manifold.

Similarly it follows from the Surface Classification Theorem 22.8, the Surface-π_1-Propositions 22.1 and 22.2 and the Surface-with-Boundary–π_1-Lemma 22.5 that only "few" finitely presented groups occur as the fundamental group of compact connected 2-dimensional topological manifolds. In contrast we will now see that once we go to dimension ≥ 4

any finitely presented group occurs as the fundamental group of a closed orientable smooth manifold.

Proposition 23.4. (Manifold-π_1-Realization Proposition) Let π be a finitely presented group. Given any $n \geq 4$ there exists a closed orientable connected non-empty n-dimensional smooth manifold M with $\pi_1(M) \cong \pi$.

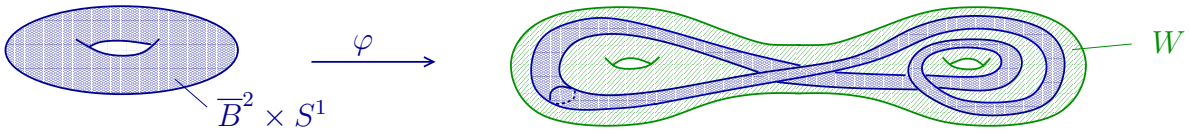
The idea of the proof is as follows. Given a finite presentation $\langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$ we start out with the connected sum $X = (S^1 \times S^{n-1}) \# \dots \# (S^1 \times S^{n-1})$ of k copies of $S^1 \times S^{n-1}$. Above we showed that the fundamental group of X is isomorphic to the free group $\langle g_1, \dots, g_k \rangle$. We will modify X by removing "solid tori $\overline{B}^{n-1} \times S^1$" and gluing in copies of $S^{n-2} \times \overline{B}^2$ to modify the fundamental group.

To carry out this idea we need to formulate and prove two lemmas. The first lemma is actually quite interesting in its own right.

Lemma 23.5. (Drilling Out–π_1–Lemma) Let W be an n-dimensional smooth manifold. Let $k \in \mathbb{N}$ and let $\varphi \colon \overline{B}^{n-k} \times S^k \to W \setminus \partial W$ be a smooth embedding.

(1) If $n \geq k+3$, then for every base point $w_0 \in W \setminus \varphi(B^{n-k} \times S^k)$ the inclusion induced map $\pi_1(W \setminus \varphi(B^{n-k} \times S^k), w_0) \to \pi_1(W, w_0)$ is an isomorphism.

(2) If $n = k+2$, then for every base point $w_0 \in W \setminus \varphi(B^{n-k} \times S^k)$ the inclusion induced map $\pi_1(W \setminus \varphi(B^{n-k} \times S^k), w_0) \to \pi_1(W, w_0)$ is an epimorphism.



Proof of the Drilling Out–π_1–Lemma 23.5. First note that it follows from the Change-of-Base Point Proposition 13.2 that it suffices to prove the statement for some base point $w_0 \in \varphi(S^{n-k-1} \times S^k)$. We consider the following pushout diagram:

$$\begin{array}{ccc} \pi_1(\varphi(S^{n-k-1} \times S^k), w_0) & \longrightarrow & \pi_1(W \setminus \varphi(B^{n-k} \times S^k), w_0) \\ \downarrow & & \lrcorner \downarrow \\ \pi_1(\varphi(B^{n-k} \times S^k), w_0) & \longrightarrow & \pi_1(W, w_0) \end{array}$$

It follows from the Manifold-Seifert–van Kampen Theorem 23.1 that this is a pushout diagram.

If $n \geq k+3$, then $n-k-1 \geq 2$. Thus we obtain from the Sphere-π_1-Proposition 11.14 that $\pi_1(S^{n-k-1})$ is trivial. It follows from the Product-π_1-Proposition 13.7 that the left vertical map is an isomorphism[205], thus it follows from the Amalgamated Product Lemma 19.12 (2a) that the right vertical map is – as promised – an isomorphism.

If $n = k+2$, then the left vertical map is an epimorphism. It follows from the Amalgamated Product Lemma 19.12 (2b) that the right vertical map is also an epimorphism. ■

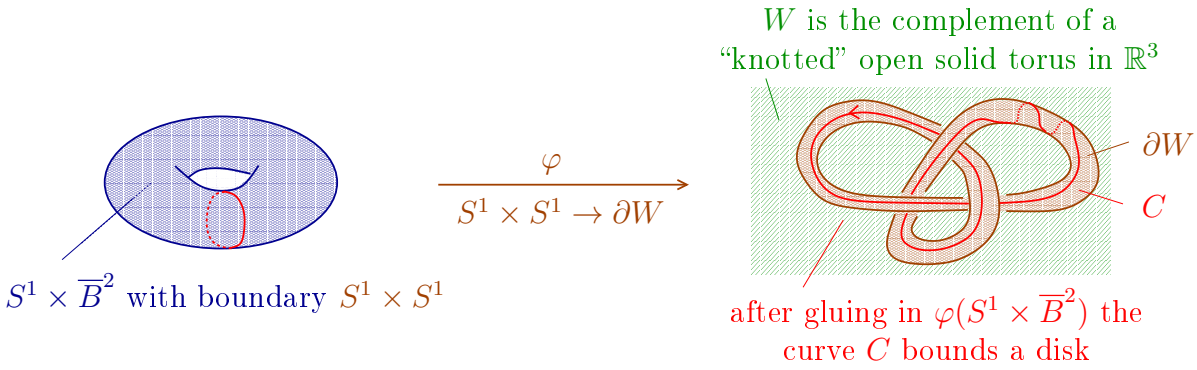
[205] Note that this argument also applies for $k = 1$

Lemma 23.6. (Fill-in-Solid Torus-π_1-Lemma) Let W be an n-dimensional smooth manifold. Let $\varphi\colon S^{n-2}\times S^1 \to \partial W$ be a diffeomorphism onto a component of ∂W. We set

$$X \;:=\; (W \sqcup (S^{n-2}\times \overline{B}^2))/\sim \qquad \text{where } \varphi(x,y)\sim(x,y) \text{ for } (x,y)\in S^{n-2}\times S^1.$$

We pick $x\in S^{n-2}$ and we set $C := \varphi(\{x\}\times S^1)\subset \partial W$. If $n\geq 3$, then for any $w_0\in W$ the inclusion induced map $\pi_1(W,w_0)\to\pi_1(X,w_0)$ induces an isomorphism

$$\pi_1(W,w_0)/\langle\langle C\rangle\rangle \;\xrightarrow{\cong}\; \pi_1(X,w_0).$$



Proof of the Fill-in-Solid Torus-π_1-Lemma 23.6. We will only need the case $n\geq 4$. Thus in the following we will provide the (slightly easier) proof of the special case $n\geq 4$. We leave it to the reader to modify the argument below to deal with the case $n=3$. Note that it follows from the Change-of-Base Point Proposition 13.2 that it suffices to prove the statement for some base point $w_0\in\varphi(S^{n-2}\times S^1)$. Next note that we have the following two isomorphisms:

$$\pi_1(W,w_0)/\langle\langle C\rangle\rangle \;\underset{\uparrow}{\xrightarrow{\cong}}\; \pi_1(W,w_0)*_{\pi_1(S^{n-2}\times S^1,w_0)}\pi_1\big(S^{n-2}\times\overline{B}^2,w_0\big) \;\underset{\uparrow}{\xrightarrow{\cong}}\; \pi_1(X,w_0).$$

an isomorphism by the Amalgamated Product–Quotient Lemma 19.13, here we use that $\pi_1(S^{n-2}\times\overline{B}^2,w_0)=0$ and that $\pi_1(S^{n-2}\times S^1,w_0)=\langle C\rangle$, note that for the latter statement we used that $n\geq 4$

an isomorphism by the Manifold-Seifert–van Kampen Theorem 23.1

Since the horizontal maps are induced by the inclusions we see that the inclusion induced map $\pi_1(W,w_0)/\langle\langle C\rangle\rangle\to\pi_1(X,w_0)$ is indeed an isomorphism. ■

Now we are fully equipped for proving the Manifold-π_1-Realization Proposition 23.4.

Proof of the Manifold-π_1-Realization Proposition 23.4. Throughout the proof we will mostly ignore base points. But at the end of the proof we will argue, why being a little sloppy with base points in between does not affect the outcome.

Let $\pi=\langle g_1,\dots,g_k\,|\,r_1,\dots,r_l\rangle$ be a finitely presented group. Furthermore let $n\geq 4$. We consider

$$X \;:=\; \underbrace{(S^1\times S^{n-1})\#\dots\#(S^1\times S^{n-1})}_{k \text{ copies of } S^1\times S^{n-1}}.$$

Note that X is a closed orientable connected n-dimensional smooth manifold. As on page 298 we see that

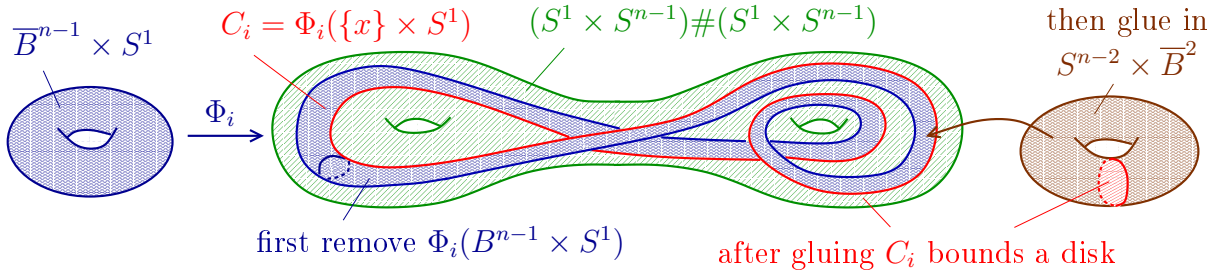
$$\pi_1(\overbrace{(S^1\times S^{n-1})\#\ldots\#(S^1\times S^{n-1})}^{=X}) \underset{\text{by the Connected Sum-}\pi_1\text{-Proposition 23.3, since } n\geq 3}{\cong} \pi_1(S^1\times S^{n-1})*\ldots*\pi_1(S^1\times S^{n-1}) \underset{\text{by the Product-}\pi_1\text{-Proposition 13.7 and since } n\geq 3 \text{ we know that } \pi_1(S^1\times S^{n-1})\cong\mathbb{Z}}{\cong} \overbrace{\langle g_1\rangle *\ldots* \langle g_k\rangle}^{=\langle g_1,\ldots,g_k\rangle}.$$

We use this isomorphism to make the identification $\pi_1(X,x_0)=\langle g_1,\ldots,g_k\rangle$. Next we pick loops $b_1,\ldots,b_l\colon (S^1,*)\to(X,x_0)$ that represent $r_1,\ldots,r_l\in\pi_1(X,x_0)=\langle g_1,\ldots,g_k\rangle$.

Since $\dim(X)=n\geq 3$ we can apply the Circle–Smooth Embedding Theorem from [Fri23] and we obtain that the map $b_1\sqcup\cdots\sqcup b_l\colon S^1\sqcup\cdots\sqcup S^1\to X$ is in fact homotopic to a smooth embedding $c_1\sqcup\cdots\sqcup c_l\colon S^1\sqcup\cdots\sqcup S^1\to X$. It follows from the Smooth Embedding Theorem and the Tubular Neighborhood Theorem from [Fri23] that there exist smooth embeddings $\Phi_i\colon \overline{B}^{n-1}\times S^1\to X$, $i=1,\ldots,l$ with disjoint images such that for each $i\in\{1,\ldots,l\}$ and each $y\in S^1$ we have $\Phi_i(0,y)=c_i(y)$. Next we consider

$$Y \;:=\; X\setminus\bigcup_{i=1}^{l}\Phi_i(B^{n-1}\times S^1) \qquad \text{and} \qquad Z \;:=\; \Big(Y\sqcup\bigsqcup_{i=1}^{l}(S^{n-2}\times\overline{B}^2)\Big)\Big/\sim$$

where $\Phi_i(x,y)\sim(x,y)$ for $x\in S^{n-2}$ and $y\in S^1$



Finally we see that

$$\begin{array}{ccccc}
\pi_1(Z) & \underset{\text{the Fill-in-Solid Torus-}\pi_1\text{-Lemma 23.6 applied } l \text{ times, hereby we set } C_i:=\Phi_i(\{x\}\times S^1) \text{ for some } x\in S^{n-2}}{\overset{\cong}{\leftarrow}} & \pi_1(Y)/\langle\!\langle C_1,\ldots,C_l\rangle\!\rangle & \underset{\text{the Drilling Out–}\pi_1\text{–Lemma 23.5 applied } k \text{ times}}{\overset{\cong}{\leftarrow}} & \pi_1(X)/\langle\!\langle C_1,\ldots,C_l\rangle\!\rangle \\
 & \underset{\text{by the above, the } x_i \text{ are necessary since we lost track of the base points}}{\cong} & \langle g_1,\ldots,g_k\rangle/\langle\!\langle x_1r_1x_1^{-1},\ldots,x_lr_lx_l^{-1}\rangle\!\rangle & = & \underbrace{\langle g_1,\ldots,g_k\rangle/\langle\!\langle r_1,\ldots,r_l\rangle\!\rangle}_{=\langle g_1,\ldots,g_k\,|\,r_1,\ldots,r_l\rangle}.
\end{array}$$

We have thus found the promised smooth manifold. ■

24. Decision problems

In the last chapters we developed several techniques for determining the fundamental groups of topological spaces. In particular we managed to find finite presentations for the fundamental groups of large classes of topological spaces, e.g. topological graphs, surfaces and mapping tori. In fact we now have enough tools to determine, with enough effort and perseverance, a presentation for the fundamental groups of most "nice" spaces.

The question that now arises is, what information can we extract from a finite presentation of a group? Here are a couple of natural questions.

Question 24.1.

(1) Triviality Problem: Does there exist an algorithm that can decide whether or not a given finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ represents the trivial group?
(2) Isomorphism Problem: Does there exist an algorithm that can decide whether or not two given finite presentations $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and $\Gamma = \langle h_1, \dots, h_m \mid s_1, \dots, s_n \rangle$ represent isomorphic groups?
(3) Word Problem: Does there exist an algorithm that can determine, given
   (a) a finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and
   (b) a word w in $g_1, \dots, g_k$, i.e. an element $w \in \langle g_1, \dots, g_k \rangle$,
   whether or not w represents the trivial element in the group π?
(4) Conjugacy Problem: Does there exist an algorithm that can determine, given
   (a) a finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and
   (b) and two words v and w in $g_1, \dots, g_k$,
   whether or not v and w represent conjugate elements in the group π?

Remark.

(1) These above questions were first posed by Max Dehn [Deh11] in 1911.
(2) The above questions are formulated in a slightly informal way. After all, what is an "algorithm" supposed to be? The questions are made more precise in [CZ93, Chapter 7], [Sti82] and [**MillC1992**].

On page 282 we introduced the Tietze transformations on presentations. In the Tietze Theorem 21.8 we showed that any two finite presentations for a given group are in fact related by a finite sequence of Tietze transformations. In particular, if $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ is a presentation for the trivial group, then we can turn the presentation after finitely many Tietze transformations into the trivial presentation $\langle | \rangle$. The problem is that at times it can be very hard to find the correct Tietze transformations.

Example. We consider the presentation

$$\pi \;=\; \langle a, b, c \mid a^3, b^3, c^4, ac = ca^{-1}, aba^{-1} = bcb^{-1} \rangle.$$

This is a presentation of the trivial group as can be seen as follows:[206] [207]

(1) from $aba^{-1} \sim bcb^{-1}$ we get $(aba^{-1})^3 \sim (bcb^{-1})^3$,
(2) since $a^{-1}a \sim 1$ and $b^{-1}b \sim 1$ it follows that $(aba^{-1})^3 \sim ab^3a^{-1}$ and $(bcb^{-1})^3 \sim bc^3b^{-1}$,

[206] We leave it to the reader to show that these steps (1)-(7) can be realized by Tietze transformations.
[207] For elements g, h of $\langle a, b, c \rangle$ we write $g \sim h$ if they represent the same element in π.

(3) we have $b^3 \sim 1$, hence $ab^3a^{-1} \sim 1$, therefore we get from (1) and (2) that $bc^3b^{-1} \sim 1$ hence $c^3 \sim 1$,
(4) we have $c^3 \sim 1$ but we also have the relation $c^4 \sim 1$, so $c \sim 1$,
(5) since $aba^{-1} \sim bcb^{-1}$ it follows that $aba^{-1} \sim 1$ and so $b \sim 1$,
(6) from $ac \sim ca^{-1}$ and $c \sim 1$ it follows that $a \sim a^{-1}$, hence $a^2 \sim 1$,
(7) together with $a^3 \sim 1$ we obtain $a \sim 1$.
(8) we now showed that a, b, c are all equivalent to 1, thus we see that the group defined by the presentation is the trivial group.

This example shows that if a presentation corresponds to the trivial group, then it can be surprisingly complicated to verify this statement.

Now suppose we are given a finite presentation and we have to decide whether or not it is a presentation of the trivial group. If we are given a finite presentation of the trivial group and if we systematically run through all finite sequences of Tietze transformations, then sooner or later we end up with the empty presentation.

But what happens if after many attempts at modifying the presentation using Tietze transformations we still do not have the trivial presentation? Does that mean that the presentation corresponds to a non-trivial group? Or does it just mean that we did not try long enough?

So the real question is, does there exist an algorithm, that given a presentation which defines a *non-trivial* group, actually verifies that the group is *non-trivial*?

Unfortunately the answer to this question and in fact all of the above questions is quite sobering.

Theorem 24.2. (Non-Solvability Theorem) The answer to all four questions raised in Question 24.1 is no.

Remark. Here "no" means that one can show that no such algorithm exists. This is a much stronger statement than saying that we do not know of an example of such an algorithm.

Proof.

(1) Sergei Adyan [Ady55] and Michael Rabin [**Rabi1985**] showed in 1955 that the answer to the first question is no. Alternatively see [**MillC1992**] for a proof.
(2) An algorithm that performs (2) would also give an algorithm for determining whether or not a given presentation is isomorphic to the trivial group with the empty presentation $\langle | \rangle$. But since the answer to (1) is no, it follows that the answer to (2) is also no.
(3) If there did exist an algorithm that dealt with the third problem, then we could apply it in particular to the generators. Since a presentation presents the trivial group if and only if all generators are trivial, it follows that an algorithm for (3) would also give an algorithm for (1). Since the latter cannot exist, the former cannot exist either.

    Historically, it was first proved by Sergey Novikov [Nov55] and William Boone [Boo58] that (3) has a negative answer before a negative answer to (1) was given.
(4) An element in a group is trivial if and only if it is conjugate to the trivial element. Hence the fact that (3) has a negative answer leads to a negative answer to (4). ■

Remark.

(1) There are many other non-existence results in the vein of the Non-Solvability Theorem 24.2. For example, [**MillC1992**] says the following:
   (a) it is undecidable whether a finitely presented group is free,
   (b) it is undecidable whether a finitely presented group is abelian,
   (c) it is undecidable whether a finitely presented group is nilpotent,
   (d) it is undecidable whether a finitely presented group is solvable,
   (e) it is undecidable whether a finitely presented group is simple,
   (f) it is undecidable whether a finitely presented group is torsion-free.

(2) The statement of the Non-Solvability Theorem 24.2 that the answer to all four questions of Question 24.1 is no, is of course somewhat disappointing. But this raises the question whether there are interesting classes of groups for which the answer is actually yes. For example Mikhail Gromov [Gro87][208] introduced the notion of a "word hyperbolic group". Examples of word hyperbolic groups are given by
   (a) finite groups,
   (b) free groups, fundamental groups of surfaces of genus ≥ 2,
   (c) fundamental groups of hyperbolic smooth manifolds of any dimension,
   (d) fundamental groups of "Riemannian manifolds with negative sectional curvature".

   In fact in a precise sense a "random" finitely presented group is word hyperbolic, see e.g. [Oll05] for more details. If we restrict ourselves to the class of all word hyperbolic groups, then it is known that the answer to all four questions posed in Question 24.1 turns out to be yes. We refer to [BH99; Löh17; DG11; Sel95] for an introduction to word hyperbolic groups and for details on the various algorithms that deal with the problems in the setting of word hyperbolic groups.

Of course our main goal is not to classify groups, but our goal is to understand "nice" topological spaces. For example ideally we would like to classify compact topological and smooth manifolds.

Given $n \in \mathbb{N}$ we ideally would like to classify closed orientable connected n-dimensional smooth manifolds up to diffeomorphism. For $n = 1$ we did so in the Smooth 1-Dimensional Manifold Classification Theorem 22.7 and for $n = 2$ we did so in the Surface Classification Theorem 22.8. In particular we saw that in these dimensions only "few groups" appear as fundamental groups.

We also proved the following result which shows that in dimension ≥ 4 the situation is dramatically different.

Proposition 23.4. (Manifold-π_1-Realization Proposition) Given any finitely presented group π and any $n \geq 4$ there exists a closed orientable connected n-dimensional smooth manifold M with $\pi_1(M) \cong \pi$.

The Non-SolvabilityTheorem 24.2 and the Manifold-π_1-Realization Proposition 23.4 are at the heart of the proof of the following theorem.

[208] Mikhail Gromov (1943-) is a French-Russian mathematician famous for his many contributions to geometry and group theory. He was the first person to view groups as "geometric objects", giving rise to the subject of "geometric group theory".

Theorem 24.3. (Manifold–Non-Classification Theorem) Let $n \geq 4$.

(1) There is no algorithm that can decide whether or not two closed orientable n-dimensional smooth manifolds are homeomorphic.
(2) There is no algorithm that can decide whether or not two closed orientable n-dimensional smooth manifolds are diffeomorphic.

Proof. The first statement was first proved by Aleksandr Markov [Mar58]. We refer to [BHP68, Theorem 1] and [Hak73] for a detailed discussion of the proofs of both statements. In particular these references also explain in more detail how one can describe a closed smooth manifold with finite data. We also refer to [Sti93, Chapter 9.4.2], [CZ93, Chapter 7.2] and [McA21] for more information and variations on the above results. ■

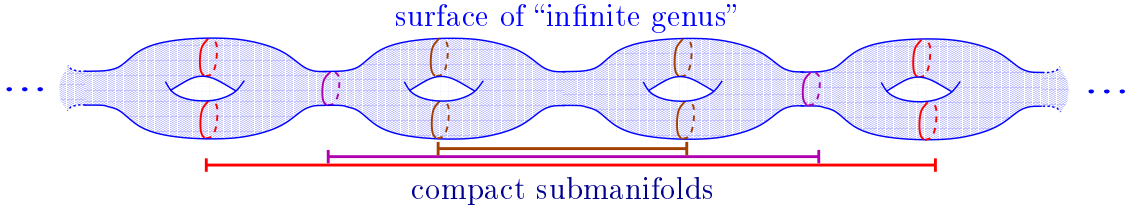
This leaves us with the case of 3-dimensional smooth manifolds. It turns out that in this dimension there does in fact exist such an algorithm.

Theorem 24.4. (Smooth 3-Manifold Recognition Theorem) There exists an algorithm that can decide whether or not two closed orientable connected 3-manifolds are diffeomorphic.

Proof. The proof of this theorem is considerably more complicated than in the 1-dimensional and the 2-dimensional case. The proof that such an algorithm exists is due to the work of many mathematicians, first and foremost William Thurston [Thu82] and Grigori Perelman. The proof of the theorem was completed about 2003. It is impossible to mention here all the work that goes into proving Theorem 24.4. We refer to [AFW, Theorem 4.27] for details and precise references. ■

25. Direct limits

The goal of this and the next chapter is to determine the fundamental group of "unbounded spaces", e.g. we would like to determine the fundamental group of the surface of "infinite genus" illustrated in the figure below.[209] One idea might be to view the surface of "infinite genus" as a "limit" of a sequence of compact surfaces. But then we need to make precise what we mean by a "limit of a sequence of topological spaces" and "limit of a sequence of (fundamental) groups". We will introduce these notions in this chapter. In the next chapter we will use this language of limits to calculate, among many other things, the fundamental group of the surface of infinite genus.



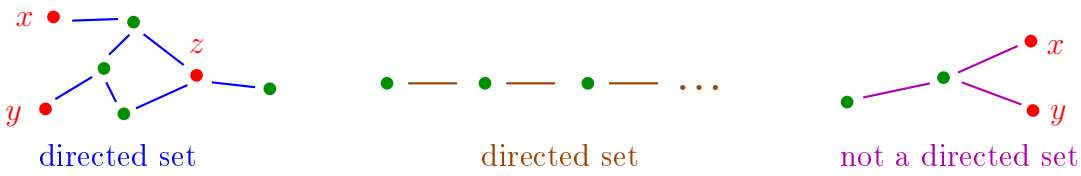
25.1. **Directed sets.** The following definition partly recalls and extends some notions that we introduced on page 21.

Definition.

(1) Let X be a set and let $\leq$ be a relation on X.
  (a) The relation is called reflexive if for every $x \in X$ we have $x \leq x$.
  (b) The relation is called transitive if for every $x, y, z \in X$ we have
$$x \leq y \text{ and } y \leq z \implies x \leq z.$$
  (c) The is called antisymmetric if $x \leq y$ and $y \leq x$ implies $x = y$.
  (d) A relation that is reflexive, transitive and antisymmetric is called a partial order.
(2) A directed set is a set X together with a partial order $\leq$ that has the extra property, that for any $x, y \in X$ there exists a $z \in X$ with $x \leq z$ and $y \leq z$.

Examples.

(1) We consider the three partial orders that are illustrated in the figure below. Here it is understood that $P \leq Q$ if P is connected to Q via a path that goes everywhere from left to right. We see that the two partial orders to the left form directed sets whereas the one on the right does not form a directed set.



(2) Let X be a subset of $\mathbb{R}$. If we denote by $\leq$ the usual "less or equal" relation on the real numbers, then $(X, \leq)$ is a directed set. In particular $(\mathbb{N}, \leq)$ is a directed set.

[209] Actually we also need to make precise what we mean by the surface of "infinite genus".

25.2. Direct systems and direct limits.

Definition. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. A direct system in the category $\mathcal{C}$ over $(I \leq)$ is a diagram (in the sense of the definition on page 85) in $\mathcal{C}$ over the directed set $(I, \leq)$. In other words, a direct system in $\mathcal{C}$ over $(I, \leq)$ is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ji} \colon X_i \to X_j\}_{i \leq j}$ such that the following two conditions are satisfied:

(1) $f_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$,

(2) $f_{ki} = f_{kj} \circ f_{ji}$ for all $i, j, k \in I$ with $i \leq j \leq k$.

For the most part we are interested in the following special case of a direct system:

Definition. Let $\mathcal{C}$ be a category and let $w \in \mathbb{Z}$. A sequence in the category $\mathcal{C}$ consists of a family $\{X_i\}_{i \in \mathbb{Z}_{\geq w}}$ of objects and a family of morphisms $\{\varphi_i \colon X_i \to X_{i+1}\}_{i \in \mathbb{Z}_{\geq w}}$. Evidently we write a sequence as follows:

$$X_w \xrightarrow{\varphi_w} X_{w+1} \xrightarrow{\varphi_{w+1}} X_{w+2} \xrightarrow{\varphi_{w+2}} \dots$$

A sequence of morphisms as above gives rise to a direct system over the directed set $(\mathbb{Z}_{\geq w}, \leq)$ by considering for $i \leq j$ the morphism $f_{ji} := \varphi_{j-1} \circ \dots \circ \varphi_{i+1} \circ \varphi_i \colon X_i \to X_j$.

Examples.

(A) In the category of abelian groups we consider the sequence

$$\mathbb{Z} \xrightarrow{v \mapsto (v,0)} \mathbb{Z}^2 \xrightarrow{v \mapsto (v,0)} \mathbb{Z}^3 \xrightarrow{v \mapsto (v,0)} \dots$$

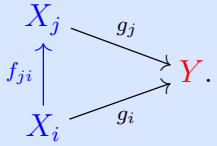
(B) In the category of groups we consider the sequence

$$\langle x_1 \rangle \hookrightarrow \langle x_1, x_2 \rangle \hookrightarrow \langle x_1, x_2, x_3 \rangle \hookrightarrow$$

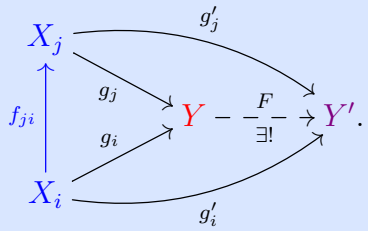
of free groups where the morphisms are the obvious inclusion homomorphisms.

Definition. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. Suppose we are given a direct system $(\{X_i\}_{i \in I}, \{f_{ji \colon X_i \to X_j}\}_{i \leq j})$ in the category $\mathcal{C}$. A direct limit[210] of the direct system is a colimit[211], in the sense of the definition on page 88, of the direct system viewed as a diagram. In other words, a direct limit is an object Y in $\mathcal{C}$ together with a family of morphisms $\{g_i \colon X_i \to Y\}_{i \in I}$ in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i \leq j$ we have $g_j \circ f_{ji} = g_i \colon X_i \to Y$, i.e. for all $i \leq j$ the following diagram commutes

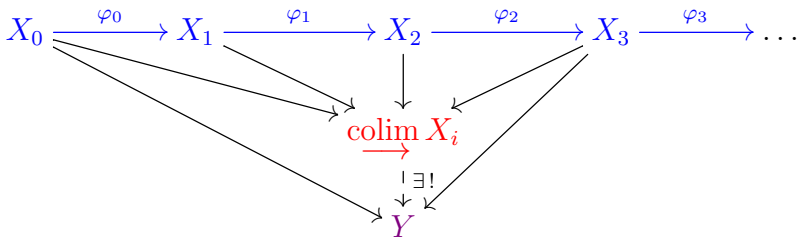


(2) If we are given another object Y' and a family of morphisms $\{g'_i \colon X_i \to Y'\}_{i \in I}$ that satisfy (1), then there exists a unique morphism $F \colon Y \to Y'$ such that for all $i \in I$ we have $g'_i = F \circ g_i \colon X_i \to Y'$. In particular for all $i \leq j$ the following diagram commutes



By the Colimit Uniqueness Lemma 4.7 we know that if a direct limit exists, then it is unique in the usual sense. We denote the direct limit, if it exists, by $\varinjlim X_i$ or sometimes by $\varinjlim_I X_i$ or $\varinjlim_i X_i$. [212]

Remark. In the case of a sequence $X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$ of morphisms the notion of a direct limit can be summarized in the following diagram:



More precisely, a direct limit $\varinjlim X_i$, if it exists, comes with morphisms $X_i \to \varinjlim X_i$ which form commutative diagrams, furthermore given morphisms $X_i \to Y$ which form commutative diagrams, we get a unique morphism $\varinjlim X_i \to Y$ which makes everything commute.

Examples. We discuss the direct limits of the direct systems we had discussed above.

(A) As on page 308 we consider in the category of abelian groups the sequence of monomorphisms $\mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \dots$ that is given by the inclusion maps $\varphi_i(v) := (v, 0)$. We claim that

$$\varinjlim \mathbb{Z}^i \;=\; \mathbb{Z}^{(\mathbb{N})} \;=\; \{(x_1, x_2, \dots) \mid x_i \in \mathbb{Z} \text{ but only finitely many } x_i\text{'s are non-zero}\},$$

where the maps $g_j \colon \mathbb{Z}^j \to \varinjlim \mathbb{Z}^i = \mathbb{Z}^{(\mathbb{N})}$, $j \in \mathbb{N}$ are the obvious monomorphisms. Indeed, if we are given an abelian group Y' and homomorphisms $g'_j \colon \mathbb{Z}^j \to Y'$ that satisfy $g'_{j+1} \circ \varphi_j = g'_j$ for all $j \in \mathbb{N}_0$, then we set

$$\begin{aligned} F \colon \varinjlim \mathbb{Z}^i \;=\; \mathbb{Z}^{(\mathbb{N})} &\to Y' \\ (x_1, \dots, x_j, 0, \dots) &\mapsto g'_j(x_1, \dots, x_j). \end{aligned}$$

[210] Somewhat confusingly a "direct limit" is often also called a "inductive limit".

[211] It is somewhat unfortunate that what historically is called a direct limit is actually a colimit. It is important to remember this, since as a general principle colimits tend to commute with colimits, but colimits do not commute with limits. For example in [Fri23] it is shown that in general colimits do not commute with products.

[212] Here we ignore the maps f_{ji} and g_i in the notation.

It is straightforward to see that this definition is well-defined, i.e. independent of the choice of x_j.[213] It is clear that for all $i \in \mathbb{N}_0$ we have $g_i' = F \circ g_i$. Furthermore, since any $x \in \mathbb{Z}^{(\mathbb{N})}$ is contained in the image of some g_j we see that F is the unique homomorphism that makes the diagram (2) commute.

Thus we have shown that the direct limit is the free abelian group on an infinite countable generating set.

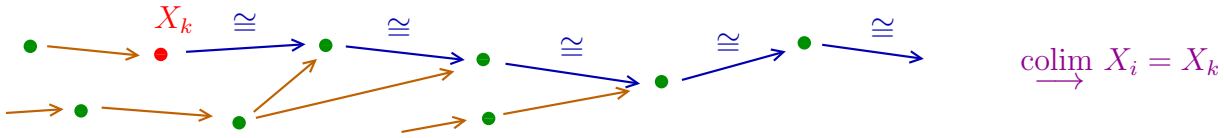
(B) As on page 308 we consider in the category of groups the sequence of monomorphisms $\langle x_1\rangle \to \langle x_1, x_2\rangle \to \dots$. Similar to the argument in (A) one can show that

$$\operatorname*{colim}_{\longrightarrow} \langle x_1, x_2, \dots, x_i\rangle \;=\; \langle x_1, x_2, \dots\rangle \;=\; \text{free group on the generators } x_1, x_2, x_3, \dots$$

where the maps $F_j \to \operatorname*{colim}_{\longrightarrow} F_i = \langle x_1, x_2, \dots\rangle$, $j \in \mathbb{N}$ are the obvious homomorphisms. Put differently, the direct limit is the free group on an infinite countable generating set.

The following lemma gives a simple but frequently applicable approach to calculating a direct limit.

Lemma 25.1. (Direct Limit–Stabilization Lemma) Let $(I, \leq)$ be a directed set, let $\mathcal{C}$ be a category and let $(\{X_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ be a direct system in the category $\mathcal{C}$. Suppose that there exists a $k \in I$ such that for all $l \geq k$ the morphism $f_{lk}\colon X_k \to X_l$ is in fact an isomorphism. Then $\operatorname*{colim}_{\longrightarrow} X_i$ exists and it is naturally isomorphic to X_k. More precisely, a direct limit is given by $\operatorname*{colim}_{\longrightarrow} X_i := X_k$ and for each $l \in I$ we consider the morphism $f_{mk}^{-1} \circ f_{ml}\colon X_l \to X_k$ where $m \in I$ is chosen such that $m \geq k$ and $m \geq l$.



Example.

(1) If $(I, \leq)$ is a finite directed set, then there exists a maximal element $i_{\max}$. It follows from the Direct Limit–Stabilization Lemma 25.1 that the direct limit is given by $X_{i_{\max}}$.

(2) We have

$$\operatorname*{colim}_{\longrightarrow} \left(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot(-1)} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{\cdot(-1)} \mathbb{Z} \xrightarrow{\mathrm{id}} \dots\right) \underset{\uparrow}{\cong} \mathbb{Z}.$$

by the Direct Limit–Stabilization Lemma 25.1

Proof. We leave the proof as an exercise to the reader. ∎

In the following we prove the existence of direct limits in some of our favorite categories. We start out with the category of sets, which in its own might not be so interesting, but which will serve as a guide for other categories.

[213] Note that we did not demand that $x_j \neq 0$.

Proposition 25.2. (Sets–Direct Limit Existence Proposition) Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in the category $\mathcal{Set}$ of sets.

(1) The direct limit is given by the set

$$\varinjlim X_i \;:=\; \Big(\bigsqcup_{i\in I} X_i\Big)/\sim \quad \text{where}^{214}\ x \sim f_{ji}(x) \text{ for all } i \leq j,\ x \in X_i$$

together with the natural maps from each X_i into $\varinjlim X_i$.

(2) If all of the maps $f_{ji}\colon X_i \to X_j$ are inclusion maps, i.e. if for each $i \leq j$ we have $X_i \subset X_j$ and $f_{ji}\colon X_i \to X_j$ is the inclusion, then we can also take

$$\varinjlim X_i \;:=\; \bigcup_{i\in I} X_i$$

together with the natural maps from each X_i into $\bigcup_{i\in I} X_i$.

Proof. Both statements follows easily from the definitions and elementary arguments. For peace of mind we carry out the argument in detail for (2).

(2) We now assume that all of the maps $f_{ji}\colon X_i \to X_j$ are inclusion maps. For $i \in I$ we denote by $g_i\colon X_i \to \bigcup_{i\in I} X_i$ the natural inclusion. Since we only deal with inclusion maps it is clear that for any $i \leq j$ we have $g_j \circ f_{ji} = g_i$. Now suppose that we are given a set Y' and maps $g'_i\colon X_i \to Y'$ such that for all $i \leq j$ we have $g'_j \circ f_{ji} = g'_i\colon X_i \to Y$.

Claim. The map

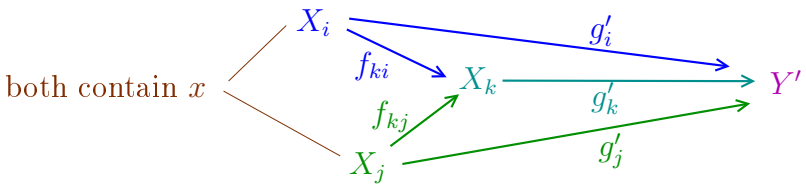
$$\begin{aligned} F\colon \bigcup_{i\in I} X_i &\to Y' \\ x &\mapsto g'_i(x) \qquad \text{if } x \in X_i \end{aligned}$$

is well-defined.

Proof. Let $x \in \bigcup_{i\in I} X_i$. Suppose that $x \in X_i$ and $x \in X_j$. Since we are dealing with a directed set there exists a $k \in I$ with $k \geq i$ and $k \geq j$. Now see that

$$g'_i(x) = (g'_k \circ f_{ki})(x) = g'_k(f_{ki}(x)) \underset{\uparrow}{=} g'_k(x) \underset{\uparrow}{=} g'_k(f_{kj}(x)) = (g'_k \circ f_{kj})(x) = g'_j(x).$$

since f_{ki} and f_{kj} are inclusion maps



It follows immediately from the claim that the map $F\colon \bigcup_{i\in I} X_i \to Y'$ has the property that for all $i \in I$ we have $g'_i = F \circ g_i\colon X_i \to Y'$.

It remains to proof the uniqueness of F. Let $x \in \bigcup_{i\in I} X_i$. We pick $i \in I$ with $x \in X_i$. We necessarily have $F(x) = (F \circ g_i)(x) = g'_i(x)$.

[214] Of course, as always, we mean by "$\sim$" the equivalence relation that is generated, in the sense of the definition on page 24, by the relation $x \sim f_{ji}(x)$.

(1) This statement follows easily from a modification of the argument in (2), using basic observations about disjoint unions and using the Quotient-Factorization Lemma 0.5. We leave the details to the reader. ■

Next we turn to our favorite algebraic categories. The Sets–Direct Limit Existence Proposition 25.2 suggests that one can construct direct limits as quotients of coproducts. This is indeed the case.

Proposition 25.3. (Algebraic–Direct Limit Existence Proposition)

(1) Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be one of the following two categories:
  (a) the category *Group* of groups,
  (b) the category *AbGroup* of abelian groups.
  Let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in $\mathcal{C}$. A direct limit is given by[215]

$$\text{(a)} \quad \varinjlim X_i \;=\; \Big(\mathop{\ast}_{i\in I} X_i\Big)\big/\,\langle\langle f_{ji}(x)\cdot x^{-1} \mid \; i\leq j \text{ and } x\in X_i\rangle\rangle,$$

$$\text{(b)} \quad \varinjlim X_i \;=\; \Big(\bigoplus_{i\in I} X_i\Big)\big/\,\langle f_{ji}(x) - x \mid \; i\leq j \text{ and } x\in X_i\rangle,$$

  with the obvious algebraic structure. Alternatively in each of the two categories a direct limit can be chosen whose underlying set is given by

$$\varinjlim X_i \;=\; \Big(\bigsqcup_{i\in I} X_i\Big)\big/\sim \quad \text{where } x\sim f_{ji}(x) \text{ for all } i\leq j,\ x\in X_i.$$

  In particular in either of the two categories any direct system admits a direct limit.

(2) Suppose we are in the setting of (1). If all of the homomorphisms $f_{ji}\colon X_i \to X_j$ are inclusion maps, then as underlying set we can always take

$$\varinjlim X_i \;=\; \bigcup_{i\in I} X_i.$$

Proof. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be one of the two given categories. Furthermore let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in $\mathcal{C}$.

(1) The first statement for the category of groups follows easily from the Coproduct-of-Groups Lemma 19.2. Similarly we see that the first statement for the category of abelian groups follows from the Coproduct-of-Abelian Groups Lemma 18.1.

  Now we turn to the second statement. We set

$$\varinjlim X_i \;:=\; \Big(\bigsqcup_{i\in I} X_i\Big)\big/\sim \quad \text{where } x\sim f_{ji}(x) \text{ for all } i\leq j,\ x\in X_i$$

  and we denote by $g_j\colon X_j \to \varinjlim X_i$, $j\in I$, the obvious maps. If we work in the category of (abelian) groups, then $\varinjlim X_i = \big(\bigsqcup_{i\in I} X_i\big)/\sim$ is naturally an (abelian) group as follows: Let $a, b \in \varinjlim X_i = \big(\bigsqcup_{i\in I} X_i\big)/\sim$. This means that there exist $i, j \in I$ and $\widetilde{a}\in X_i$ and $\widetilde{b}\in X_j$ with $a = [\widetilde{a}]$ and $b = [\widetilde{b}]$. By definition of a directed set

[215] Here given a group G and a subset A of G we denote by $\langle\langle A\rangle\rangle \subset G$ the subgroup normally generated by A, see page 247 and $\langle A\rangle$ denotes the subgroup generated by A. If G is abelian, then basically be definition we have $\langle\langle A\rangle\rangle = \langle A\rangle$.

there exists a $k \in I$ with $i \leq k$ and $j \leq k$. We set

$$a \cdot b \;:=\; \big[\underbrace{f_{ki}(\widetilde{a})}_{\in X_k} \cdot \underbrace{f_{kj}(\widetilde{b})}_{\in X_k}\big] \;\in\; \varinjlim X_i.$$

It follows easily from the definition of a direct system that this definition of the product structure is independent of the choice of k and that this defines indeed an (abelian) group structure on $\varinjlim X_i$.

(2) This statement follows easily from the definitions. ■

Example. We consider the two direct systems that are given by $\{\mathbb{Z}^i\}_{i\in\mathbb{N}}$ and $\{\mathbb{Z}^i \oplus 0\}_{i\in\mathbb{N}}$ where $\mathbb{Z}^i \oplus 0 \subset \mathbb{Z}^\infty$. We have the following isomorphisms:

$$\varinjlim \mathbb{Z}^i \quad \underset{\substack{\uparrow\\ \text{since the two direct systems}\\ \text{are isomorphic}}}{\cong} \quad \varinjlim (\mathbb{Z}^i \oplus 0) \quad \underset{\substack{\uparrow\\ \text{Algebraic–Direct Limit Existence}\\ \text{Proposition 25.3}}}{\cong} \quad \mathbb{Z}^\infty.$$

We have thus obtained the result from page 310 in a more systematic fashion.

25.3. **The direct limit in the topological category.** We now turn to the direct limit in the topological category. In this setting we will only be interested in the case of a sequence of embeddings.

Proposition 25.4. (Topological-Direct Limit Proposition) Let

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$$

be a sequence of topological spaces such that each $\varphi_i \colon X_i \to X_{i+1}$ is an embedding.

(0) We consider the set

$$\varinjlim X_i \;:=\; \Big(\bigsqcup_{i\in\mathbb{N}_0} X_i\Big)/\sim \quad \text{where } x \sim \varphi_i(x) \text{ for all } i \in \mathbb{N}_0 \text{ and } x \in X_i.$$

Given $i \in \mathbb{N}_0$ we denote by $g_i \colon X_i \to \varinjlim X_i$ the natural map.

(a) We say $U \subset \varinjlim X_i$ is open if and only if for each $i \in \mathbb{N}_0$ the preimage $g_i^{-1}(U)$ is an open subset of X_i. This defines a topology on $\varinjlim X_i$. In the remainder of this proposition we equip $\varinjlim X_i$ with this topology.

(b) A subset $U \subset \varinjlim X_i$ is closed with respect to the above topology if and only if for each $i \in \mathbb{N}_0$ the preimage $g_i^{-1}(U)$ is a closed subset of X_i.

(1) (a) For each $i \in \mathbb{N}_0$ the natural map $g_i \colon X_i \to \varinjlim X_i$ is an embedding.

(b) Let Y' be a topological space. Given a family of continuous maps map $g_i' \colon X_i \to Y'$ with the property that for any $i \in \mathbb{N}_0$ we have $g_{i+1}' \circ \varphi_i = g_i'$ the map

$$\begin{aligned} G' \colon \varinjlim X_i &\to Y' \\ x &\mapsto g_i'(x_i) \qquad \text{if } x = g_i(x_i) \text{ for some } x_i \in X_i \end{aligned}$$

is well-defined and it is continuous.

(∗) A map $h \colon \varinjlim X_i \to Y'$ to some topological space is continuous if and only if each map $h \circ g_i \colon X_i \to Y'$ is continuous.

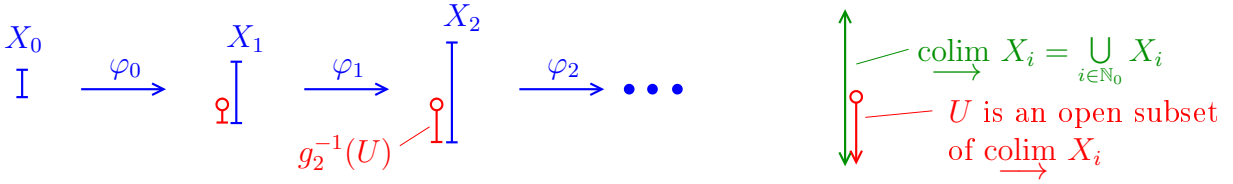
(2) The topological space $\operatorname{colim}\limits_{\longrightarrow} X_i$, together with the natural inclusions $g_i\colon X_i \to \operatorname{colim}\limits_{\longrightarrow} X_i$, is the direct limit of the above sequence in the category of topological spaces.

If all the embeddings $\varphi_i\colon X_i \to X_{i+1}$ are in fact inclusion maps, i.e. if for each $i \in \mathbb{N}_0$ we have $X_i \subset X_{i+1}$ and $\varphi_i\colon X_i \to X_{i+1}$ is the inclusion, then we can replace (0) by

(0') We consider the set

$$\operatorname*{colim}_{\longrightarrow} X_i \;:=\; \bigcup_{i\in\mathbb{N}_0} X_i.$$

We say $U \subset X$ is open if and only if for each $i \in \mathbb{N}_0$ the intersection $U \cap X_i$ is an open subset of X_i. This defines a topology on X which again satisfies (1a), (1b) and (2).



Proof. Let

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$$

be a sequence of topological spaces such that each $\varphi_i\colon X_i \to X_{i+1}$ is an embedding. We set

$$\operatorname*{colim}_{\longrightarrow} X_i \;:=\; \Big(\bigsqcup_{i\in\mathbb{N}_0} X_i\Big)/\sim \quad \text{where } x \sim \varphi_i(x) \text{ for all } i \in \mathbb{N}_0 \text{ and } x \in X_i.$$

For each $i \in \mathbb{N}_0$ we consider the natural inclusion $g_i\colon X_i \to \operatorname{colim}\limits_{\longrightarrow} X_i$.

(0) (a) We set

$$\mathcal{T} := \big\{U \subset \operatorname*{colim}_{\longrightarrow} X_i \,\big|\, \text{for each } i \in \mathbb{N} \text{ the preimage } g_i^{-1}(U) \text{ is open in } X_i\big\}.$$

One can easily verify, say using the Image-Preimage Lemma 0.3, that $\mathcal{T}$ defines a topology on $\operatorname{colim}\limits_{\longrightarrow} X_i$.

(b) This statement follows immediately from the definition of being closed and the observation that by the Image-Preimage Lemma 0.3 (8) we always have the equality $g_i^{-1}(\operatorname{colim}\limits_{\longrightarrow} X_i \setminus U) = g_i^{-1}(\operatorname{colim}\limits_{\longrightarrow} X_i) \setminus g_i^{-1}(U) = X_i \setminus g_i^{-1}(U)$.

(1) (a) Let $j \in \mathbb{N}_0$. We need to show that the natural inclusion $g_j\colon X_j \to \operatorname{colim}\limits_{\longrightarrow} X_i$ is an embedding. The map is evidently injective. It follows immediately from the definition of $\mathcal{T}$ that $g_j\colon X_j \to \operatorname{colim}\limits_{\longrightarrow} X_i$ is continuous. To show that $g_j\colon X_j \to X$ is actually an embedding it remains to prove the following claim.

Claim. Given any open subset $U \subset X_j$ there exists an open subset $V \subset \operatorname{colim}\limits_{\longrightarrow} X_i$ with $V \cap g_j(X_j) = g_j(U)$.[216]

Proof. Let $U_j \subset X_j$ be an open subset. Since $\varphi_j\colon X_j \to X_{j+1}$ is an embedding there exists an open subset $U_{j+1} \subset X_{j+1}$ such that $\varphi_j(X_j) \cap U_{j+1} = \varphi_j(U_j)$. We iterate this procedure to obtain subsets $U_j, U_{j+1}, U_{j+2}, \dots$. We set $V := \bigcup\limits_{k\geq j} g_k(U_k)$. Evidently we have $V \cap g_j(X_j) = g_j(U_j)$. It remains to show that V is open in $\operatorname{colim}\limits_{\longrightarrow} X_i$. This means that we need to show that for every $i \in \mathbb{N}_0$ the preimage

[216] In the proof of this claim it is rather convenient to work with the directed set $(\mathbb{N}_0, \leq)$ instead of a general directed set $(I, \leq)$.

$g_i^{-1}(V)$ is an open subset of X_i. For $i \geq j$ this follows immediately from the construction of V. It remains to consider the case $i < j$. First note that we have $g_{j-1}^{-1}(V) = (g_j \circ \varphi_{j-1})^{-1}(V) = \varphi_{j-1}^{-1}(g_j^{-1}(V)) = \varphi_{j-1}^{-1}(U_j)$. Since $U_j \subset X_j$ is open and since $\varphi_{j-1}\colon X_{j-1} \to X_j$ is continuous we see that $g_{j-1}^{-1}(U_j)$ is indeed an open subset of X_{j-1}. Iterating this argument we see that for all $i \in \{0, \ldots, j-1\}$ the preimage $g_i^{-1}(V)$ is an open subset of X_i. ⊞

(b) As in the proof of the Sets–Direct Limit Existence Proposition 25.2 we see that the map $G'\colon \varinjlim X_i \to Y'$ is well-defined. By construction we have $G' \circ g_i = g_i'$ for each $i \in \mathbb{N}_0$. It remains to show that the map G' is continuous. Thus let V be an open subset of Y'. We need to show that $(G')^{-1}(V)$ is an open subset of $\varinjlim X_i$. By definition of the topology on $\varinjlim X_i$ this means that we have to show that for each $i \in \mathbb{N}_0$ the preimage $g_i^{-1}((G')^{-1}(V))$ is an open subset of X_i. But by construction of G' we have $G' \circ g_i = g_i'$. Thus we see that $g_i^{-1}((G')^{-1}(V)) = (g_i')^{-1}(V)$, which is of course an open subset of X_i since g_i' is continuous.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) This statement follows from (1a) and (1b) together with the observation, which we proved in the Sets–Direct Limit Existence Proposition 25.2, that the map G in (1b) is the unique map which satisfies $G \circ g_i = g_i'$ for all $i \in \mathbb{N}_0$.

Finally note that if all the maps $f_{ji}\colon X_i \to X_j$ are in fact inclusion maps, then it follows by the same argument as in the Sets–Direct Limit Existence Proposition 25.2 (2) that we can work with $\varinjlim X_i := \bigcup_{i \in \mathbb{N}_0} X_i$. We leave it to the reader to make the few adjustments to finish off this proof. ■

Lemma 25.5. (Direct Limit–Open Subset Lemma) Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset \ldots$ be subspaces of X with $\varinjlim X_i = X$. Suppose that we are given $w \in \mathbb{N}_0$ and suppose that for each $i \geq w$ we are given an open subset $U_i \subset X_i$. If for each $i \geq w$ we have $U_i \subset U_{i+1}$, then $\bigcup_{i \geq w} U_i$ is an open subset of X.

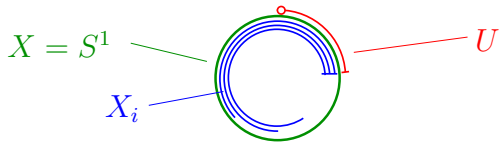
Proof. We set $U := \bigcup_{i \geq w} U_i$. Since $X = \varinjlim X_i$ it suffices to show that for each $j \in \mathbb{N}_0$ the intersection $X_j \cap U$ is open in X_j. Thus let $j \in \mathbb{N}_0$. We set $m := \max\{j, w\}$. Note that

$$X_j \cap U = X_j \cap \bigcup_{i \geq w} U_i \underset{\substack{\uparrow \\ \text{since for each } i \geq w \text{ we have } U_i \subset U_{i+1}}}{=} X_j \cap \bigcup_{i \geq m} U_i = \bigcup_{i \geq m} (X_j \cap U_i).$$

Next note that for $i \geq m$ it follows from the fact that U_i is open in X_i that $X_j \cap U$ is open in X_j. In summary we have shown that $X_j \cap U$ is the union of open subsets of X_j, thus $X_j \cap U$ is an open subset of X_j. ■

The Topological-Direct Limit Proposition 25.4 raises the following question: if we are given a topological space X and subspaces $X_1 \subset X_2 \subset \ldots$ with $\bigcup_{i \in \mathbb{N}} X_i = X$, does the direct limit topology $\varinjlim X_i$ necessarily agree with the original topology on X? It turns out that in general this is *not* the case. For example we consider the circle $X = S^1$ and the subspaces

$X_i := \{\exp(2\pi \mathrm{i} t) \,|\, t \in [0, 1 - \frac{1}{n})\}$. For $U = \{\exp(2\pi \mathrm{i} t) \,|\, t \in [0, \frac{1}{4})\}$ we see that each $U \cap X_i$ is an open subset of X_i, but $U \subset S^1$ is of course not open.



In contrast to the above example, the following lemma gives a useful criterion for when we have $X = \operatorname{colim}_{\longrightarrow} X_i$.

Criterion 25.6. (Union-equals-Direct Limit Criterion) Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset \ldots$ be subspaces of X. We assume that the following two conditions are satisfied:
(1) We have $\bigcup_{i \in \mathbb{N}_0} X_i = X$.
(2) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.
Then $X = \operatorname{colim}_{\longrightarrow} X_i$ as topological spaces.

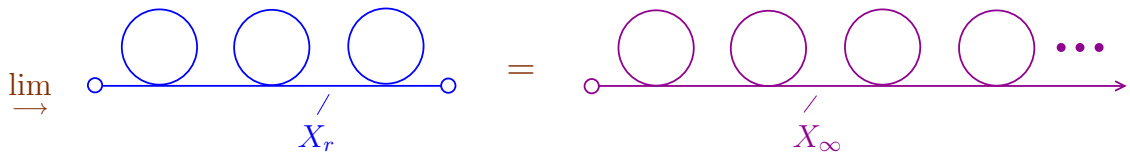
Example. Given $r \in \mathbb{N}_0$ we set

$$X_r \;:=\; \big(0, r + \tfrac{1}{2}\big) \;\cup\; \bigcup_{j=1}^{r} \underbrace{\big\{z \in \mathbb{C} \,\big|\, \big|z - (j + \tfrac{1}{4}\mathrm{i})\big| = \tfrac{1}{4}\big\}}_{\text{circle of radius } \frac{1}{4} \text{ around } j + \frac{1}{4}\mathrm{i} \,\in\, \mathbb{C}} \;\subset\; \mathbb{C}.$$

In other words, for $r \in \mathbb{N}_0$ the set X_r is the subspace of $\mathbb{C} = \mathbb{R}^2$ that is given by the interval $(0, r + \frac{1}{2})$ with r circles attached. We obtain a sequence of continuous maps by considering the inclusions $X_r \to X_{r+1}$. It follows fairly easily from the Union-equals-Direct Limit Criterion 25.6 that the direct limit is the interval $(0, \infty)$ with infinitely many circles attached, i.e.

$$\operatorname*{colim}_{\longrightarrow} X_r \;=\; X_\infty \;:=\; \big(0, \infty\big) \;\cup\; \bigcup_{j=1}^{\infty} \big\{z \in \mathbb{C} \,\big|\, \big|z - (j + \tfrac{1}{4}\mathrm{i})\big| = \tfrac{1}{4}\big\} \;\subset\; \mathbb{C},$$

where the maps $X_s \to \operatorname{colim}_{\longrightarrow} X_r = X_\infty$, $s \in \mathbb{N}_0$ are the obvious inclusion maps.



Proof. If $U \subset X$ is open, then, by definition of the subspace topology, each $U \cap X_i$ is open in X_i. This implies that $U \subset \operatorname{colim}_{\longrightarrow} X_i$ is open.

Now assume that $U \subset \operatorname{colim}_{\longrightarrow} X_i$ is open. We need to show that $U \subset X$ is open. Since U is open in $\operatorname{colim}_{\longrightarrow} X_i$ we know that each $U \cap X_i$ is open in X_i. By definition of the subspace topology this means that for each $i \in \mathbb{N}_0$ there exists an open subset $V_i \subset X$ such that

$V_i \cap X_i = U \cap X_i$. Now we see that

$$\begin{array}{ccccccc} \bigcup_{i\in\mathbb{N}_0}(V_i\cap \mathring{X}_i) & \subset & \bigcup_{i\in\mathbb{N}_0}(V_i\cap X_i) & \overset{\text{since } V_i\cap X_i = U\cap X_i}{=} & \bigcup_{i\in\mathbb{N}_0}(U\cap X_i) & = & U\cap\bigcup_{i\in\mathbb{N}_0} X_i \overset{\text{follows from (1)}}{=} U \\ & = & U\cap\bigcup_{i\in\mathbb{N}_0}\mathring{X}_i & = & \bigcup_{i\in\mathbb{N}_0}(U\cap\mathring{X}_i) & = & \bigcup_{i\in\mathbb{N}_0}(V_i\cap\mathring{X}_i). \\ & & \uparrow & & & & \uparrow \\ & & \text{since it follows from (1) and (2) that } X=\bigcup_{i\in\mathbb{N}_0}\mathring{X}_i & & & & \text{since } V_i\cap X_i = U\cap X_i \text{ and since } \mathring{X}_i\subset X_i \end{array}$$

We see that all inclusions are equalities. This implies that $U = \bigcup_{i\in I}(V_i \cap \mathring{X}_i)$. Thus we have shown that U is the union of open subsets, which implies that U itself is open. ∎

25.4. **Examples and properties of the direct limit of topological spaces.** In this section we study several direct limits of natural sequences of embeddings of topological spaces which lead to new and interesting topological spaces. Along the way we also develop the theory of direct limits of sequences of topological spaces a little further.

Definition.

(1) We consider the *set*

$$\mathbb{R}^\infty := \mathbb{R}^{(\mathbb{N})} = \{(x_1, x_2, \dots) \mid x_i \in \mathbb{R} \text{ but only finitely many } x_i\text{'s are non-zero}\}$$

(2) Given a subset $X \subset \mathbb{R}^k$ we write

$$X \times \{0\} := \{(x, 0, 0, \dots) \mid x \in X\} \subset \mathbb{R}^\infty.$$

We have an obvious bijection of $X \times \{0\}$ to X and we use it to equip $X \times \{0\}$ with a topology.

(3) Unless we explicitly say something else we equip the set $\mathbb{R}^\infty$ with the topology given by $\mathbb{R}^\infty = \operatorname{colim}_{\longrightarrow}(\mathbb{R}^i \times \{0\})$. In other words, $U \subset \mathbb{R}^\infty$ is open if and only if for each $i \in \mathbb{N}_0$ the intersection $U \cap (\mathbb{R}^i \times \{0\})$ is open in $\mathbb{R}^i \times \{0\}$.

(4) Similarly we define $\mathbb{C}^\infty := \operatorname{colim}_{\longrightarrow}(\mathbb{C}^i \times \{0\})$ as a topological space.

Example.

(1) To practice the above definition we will show that any linear map $\mathbb{R}^\infty \to \mathbb{R}^\infty$ is continuous with respect to the above topology. The proof proceeds in two steps:

(a) By the Topological-Direct Limit Proposition 25.4 (1b) it suffices to that show for each $n \in \mathbb{N}_0$ the map

$$\begin{array}{rccc} f_n\colon & \mathbb{R}^n \times \{0\} & \to & \mathbb{R}^\infty \\ & x & \mapsto & f(x,0) \end{array}$$

is continuous.

(b) Let $n \in \mathbb{N}_0$. Note that there exists a $k \in \mathbb{N}$ such that $f(e_1), \dots, f(e_n) \in \mathbb{R}^k \times \{0\}$. Thus we obtain the following commutative diagram

$$\begin{array}{ccc} \mathbb{R}^n \times \{0\} & \xrightarrow{x\mapsto f(x,0)} & \mathbb{R}^\infty \\ \downarrow{\scriptstyle (x,0)\mapsto x} & & \uparrow{\scriptstyle y\mapsto(y,0)} \\ \mathbb{R}^n & \xrightarrow{\widetilde{f}} & \mathbb{R}^k \end{array}$$

for a suitable linear map $\widetilde{f}\colon \mathbb{R}^n \to \mathbb{R}^k$. We make three observations:

- The left vertical map is a homeomorphism.
- Each linear map $\mathbb{R}^n \to \mathbb{R}^k$ is continuous.
- By the Topological-Direct Limit Proposition 25.4 (1a) we know that the right vertical map is continuous.

It follows that the top horizontal map is continuous.

(2) We consider the map

$$\begin{array}{rcl} \mathbb{C}^\infty & \to & \mathbb{R}^\infty \\ (x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2, \dots) & \mapsto & (x_1, y_1, x_2, y_2, \dots). \end{array}$$

This map has an obvious inverse. Using the approach from (1) one can easily show that both maps are continuous, thus they are both homeomorphisms. At times we will use these homeomorphisms to make the identification $\mathbb{C}^\infty =_i \mathbb{R}^\infty$

The above definition motivates the following related definition.

Definition. We consider the topological space

$$S^\infty := \left\{(x_1, x_2, \dots) \in \mathbb{R}^\infty \,\middle|\, \sum_{j=1}^{\infty} x_j^2 = 1\right\} = \varinjlim (S^{i-1} \times \{0\}).$$

Thus $U \subset S^\infty$ is open if and only if for each $i \in \mathbb{N}_0$ the intersection $U \cap (S^{i-1} \times \{0\})$ is an open subset of $S^{i-1} \times \{0\}$.

Before we can formulate our next lemma it is helpful to introduce the following definition.

Definition. Let G be a group that acts on two sets X and Y. A map $f\colon X \to Y$ is called G-equivariant if given any $x \in X$ and $g \in G$ we have $f(g \cdot x) = g \cdot f(x)$.

Lemma 25.7. (Direct Limit–Action Lemma) Let

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$$

be a sequence of topological spaces such that each $\varphi_i\colon X_i \to X_{i+1}$ is an embedding. Let G be a group. Suppose that for each $i \in \mathbb{N}_0$ we are given a continuous action of G on X_i such that each φ_i is G-equivariant. It follows almost immediately from this data that G also acts continuously on $\varinjlim X_i$.

(1) The natural $g_i\colon X_i \to \varinjlim X_i$ maps are G-equivariant.

(2) The resulting natural map $\varinjlim (X_i/G) \to \left(\varinjlim X_i\right)/G$ is a homeomorphism.

Sketch of proof. For $g \in G$ let $\theta_i^g\colon X_i \to X_i$ be the continuous map that is given by multiplication by g. We consider the continuous maps $X_i \xrightarrow{\theta_i^g} X_i \to \varinjlim X_i$. By our hypothesis on the actions and the universal property of direct limits we get a unique induced continuous map

$$\theta^g\colon \varinjlim X_i \to \varinjlim X_i.$$

Using the uniqueness of the universal property one can easily verify that these maps define a continuous G-action on $\varinjlim X_i$ and that the natural inclusions $X_i \to \varinjlim X_i$ are G-equivariant. Thus we get induced maps

$$X_i/G \to (\varinjlim X_i)/G.$$

It follows from the Twice Quotient Lemma 6.3 that these maps are continuous. By the universal property of direct limits we obtain a continuous map

$$\Xi\colon \operatorname*{colim}_{\longrightarrow} (X_i/G) \;\to\; \big(\operatorname*{colim}_{\longrightarrow} X_i\big)/G.$$

It follows from the explicit description of the direct limit in the category of topological spaces that we gave in the Topological-Direct Limit Proposition 25.4 that this map Ξ is a bijection. We want to show that Ξ is a homeomorphism. It remains to prove the following claim.

Claim. The map $\Xi\colon \operatorname*{colim}_{\longrightarrow} (X_i/G) \to \big(\operatorname*{colim}_{\longrightarrow} X_i\big)/G$ is an open map.

Proof. For $i \in \mathbb{N}_0$ we denote by $p_i\colon X_i \to X_i/G$ the natural projection. Furthermore we denote by $q\colon \operatorname*{colim}_{\longrightarrow} X_i \to \big(\operatorname*{colim}_{\longrightarrow} X_i\big)/G$ the natural projection. Let $U \subset \operatorname*{colim}_{\longrightarrow} (X_i/G)$ be an open subset. By definition of the direct limit topology this means that for each $i \in \mathbb{N}_0$ the intersection $U_i := (X_i/G) \cap U$ is an open subset of X_i/G. Note that by definition of the quotient topology this means that $\widetilde{U}_i := p_i^{-1}(U_i)$ is an open subset of X_i. We set $\widetilde{U} := \bigcup_{i\in\mathbb{N}_0} \widetilde{U}_i \subset \operatorname*{colim}_{\longrightarrow} X_i$. It follows easily from the definition of the direct limit topology that $\widetilde{U}$ is an open subset of $\operatorname*{colim}_{\longrightarrow} X_i$. Next note that it follows from an elementary argument that $q(\widetilde{U})$ is an open subset of $\big(\operatorname*{colim}_{\longrightarrow} X_i\big)/G$. Finally note that it follows easily from the various definitions that $\Xi(U) = q(\widetilde{U})$. We have thus shown that Ξ is indeed an open map. ∎

We consider one last example of an interesting sequence of topological spaces.

Definition.

(1) We consider the sequence

$$\mathbb{R}\mathrm{P}^0 \xrightarrow{[x]\mapsto[x:0]} \mathbb{R}\mathrm{P}^1 \xrightarrow{[x]\mapsto[x:0]} \mathbb{R}\mathrm{P}^2 \xrightarrow{[x]\mapsto[x:0]} \dots$$

of embeddings. We refer to

$$\mathbb{R}\mathrm{P}^\infty := \operatorname*{colim}_{\longrightarrow} \mathbb{R}\mathrm{P}^i$$

as the infinite-dimensional real projective space.

(2) The same way we also define the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$.

At this point there is not much to say about the infinite-dimensional projective spaces, except that we can show that there exist several different ways to describe these topological spaces:

Lemma 25.8. (Infinite Projective Space-Sphere Quotient Lemma)

(1) The obvious action of the group $\{\pm 1\}$ on S^∞ is continuous and discrete.

(2) There exists a homeomorphism $S^\infty/\{\pm 1\} \to \mathbb{R}\mathrm{P}^\infty$.

Proof.

(1) We leave the verification of this statement to the reader.

(2) We consider the following commutative diagram:

$$\begin{array}{ccc} \varinjlim\,(S^n/\{\pm 1\}) & \longrightarrow & \varinjlim\,\mathbb{R}\mathrm{P}^n \\ \downarrow & & \downarrow \\ S^\infty/\{\pm 1\} & \longrightarrow & \mathbb{R}\mathrm{P}^\infty. \end{array}$$

It follows from the Projective Space-Sphere Quotient Lemma 6.9 that the top horizontal map is a homeomorphism. The right vertical map is a homeomorphism by definition. Finally the left vertical map is a homeomorphism by the Direct Limit–Action Lemma 25.7. It follows that the bottom map is a homeomorphism. ■

26. Fundamental groups and direct limits

We already showed that fundamental groups behave well with respect to several operations:[217]

(1) In the Product-π_1-Proposition 13.7 we showed that fundamental groups preserve products.
(2) In the Wedge-π_1-Proposition 20.6 we showed that, under slight technical hypotheses, fundamental groups preserve coproducts.
(3) In the Seifert–van Kampen Theorem 20.1 we showed that, again under fairly mild technical hypotheses, fundamental groups preserve pushouts.

This discussion, and the definitions from the last chapter, raise the question, whether fundamental groups preserve direct limits. In this chapter we will see that, under suitable hypotheses, this is indeed the case.

In the second half of this chapter we will use this result to calculate the fundamental group of the surface of infinite genus and to prove several results on fundamental groups of finite genus.

The following proposition gives the promised statement about the relationship between fundamental groups and direct limits.

Proposition 26.1. (Direct Limit-π_1-Proposition) Let

$$(X_0, w_0) \xrightarrow{\varphi_0} (X_1, w_1) \xrightarrow{\varphi_1} (X_2, w_2) \xrightarrow{\varphi_2} \dots$$

be a sequence of pointed topological spaces such that each $\varphi_i \colon X_i \to X_{i+1}$ is an embedding with $\varphi_i(w_i) = w_{i+1}$. For each $j \in \mathbb{N}_0$ we denote by $g_j \colon X_j \to \operatorname{colim} X_i$ the natural map. If for each $i \in \mathbb{N}_0$ we have $g_i(X_i) \subset \operatorname{int}(g_{i+1}(X_{i+1}))$, then we have a natural isomorphism

$$\operatorname{colim} \pi_1(X_i, w_i) \xrightarrow{\cong} \pi_1\big(\operatorname{colim} X_i, g_0(w_0)\big).$$

Before we turn to the proof of the Direct Limit-π_1-Proposition 26.1 we state a corollary, which is the result which in practice will be more relevant.

Proposition 26.2. (Exhaustion-π_1-Proposition) Let X be a topological space and let

$$X_0 \subset X_1 \subset X_2 \subset X_3 \subset \dots$$

be a sequence of subspaces of X. We assume that the following two conditions are satisfied:

(1) We have $\bigcup_{i \in I} X_i = X$.
(2) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.

Then for any $w \in X_0$ we have a natural isomorphism

$$\operatorname{colim} \pi_1(X_i, w) \xrightarrow{\cong} \pi_1(X, w).$$

[217] To smoothen the exposition we are a little ambiguous about whether we are dealing with the category of topological spaces or the category of pointed topological spaces.

Example. We assume that we are in the setting of the Exhaustion-π_1-Proposition 26.2. If all the X_i are simply connected, then we see that

$$\begin{array}{ccccccc} \pi_1(X,w) & \xleftarrow{\cong} & \varinjlim \pi_1(X_i,w) & \cong & \varinjlim \{e\} & = & \{e\}. \\ \uparrow & & \uparrow & & \uparrow & & \end{array}$$

Exhaustion-π_1-Proposition 26.2 — since we assume that the X_i are simply connected — by the Direct Limit Stabilization Lemma 25.1

Thus we recovered the statement of the Exhaustion–Simply Connected Proposition 20.10.

Proof of Proposition 26.2 assuming Proposition 26.1. We consider the following natural commutative diagram:

$$\begin{array}{ccc} \varinjlim \pi_1(X_i,w) & \longrightarrow & \pi_1\big(\varinjlim X_i, w\big) \\ & \searrow \quad \swarrow & \\ & \pi_1(X,w). & \end{array}$$

It follows from our hypothesis that we can apply the Direct Limit-π_1-Proposition 26.1, which then tells us, that the horizontal map is an isomorphism. Since we know that for each $i \in \mathbb{N}_0$ we have $X_i \subset \overset{\circ}{X}_{i+1}$ we obtain from the Union-equals-Direct Limit Criterion 25.6 that we have a natural homeomorphism $\varinjlim X_i \to X$. Thus the right diagonal map is an isomorphism. It follows from this discussion that the left diagonal map is an isomorphism. ■

The proof of the Direct Limit-π_1-Proposition 26.1 rests on the following elementary lemma from earlier on.

Lemma 20.11. (Compact Subspace–Exhaustion Lemma) Let X be a topological space and let

$$X_0 \subset X_1 \subset X_2 \subset X_3 \subset \ldots$$

be a sequence of subspaces of X such the following two conditions are satisfied:

(1) We have $\bigcup\limits_{i\in\mathbb{N}_0} X_i = X$.

(2) For each $i \in \mathbb{N}_0$ we have $X_i \subset \overset{\circ}{X}_{i+1}$.

Given any compact subset K of X there exists an $i \in \mathbb{N}_0$ with $K \subset X_i$.

Now we can provide the proof of the Direct Limit-π_1-Proposition 26.1.

Proof of the Direct Limit-π_1-Proposition 26.1. Let

$$(X_0,w_0) \xrightarrow{\varphi_0} (X_1,w_1) \xrightarrow{\varphi_1} (X_2,w_2) \xrightarrow{\varphi_2} \ldots$$

be a sequence of topological spaces such that each $\varphi_i\colon X_i \to X_{i+1}$ is an embedding. For each $i \in \mathbb{N}_0$ we denote by $g_i\colon X_i \to \varinjlim X_i$ the natural map. We assume that for each $i \in \mathbb{N}_0$ we have $g_i(X_i) \subset \operatorname{int}(g_{i+1}(X_{i+1}))$. We set $X := \varinjlim X_i$. It follows from the Topological-Direct Limit Proposition 25.4 that the natural maps $g_i\colon X_i \to X$ are embeddings and that the union of the images of the X_i is all of X. Thus we can and will view each X_i as a subspace of X. In summary we can now assume that we have a sequence

$$X_0 \subset X_1 \subset X_2 \subset X_3 \subset \ldots$$

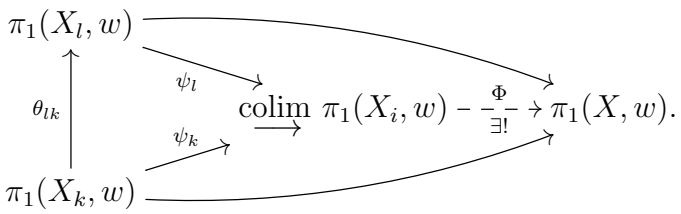
of subspaces of X such the following two conditions are satisfied:

(a) We have $\bigcup_{i\in\mathbb{N}_0} X_i = X$.

(b) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.

Let $w \in X_0$. Given $k < l$ we denote by $\theta_{lk}\colon \pi_1(X_k, w) \to \pi_1(X_l, w)$ the inclusion induced homomorphism. Given $k \in \mathbb{N}_0$ we denote by $\psi_k\colon \pi_1(X_k, w) \to \operatorname{colim} \pi_1(X_i, w)$ the natural homomorphism.

Note that by the universal property of a direct limit there exists a unique homomorphism $\Phi\colon \operatorname{colim} \pi_1(X_i, w) \to \pi_1(X, w)$ that makes the following diagram commute for any $k < l$:



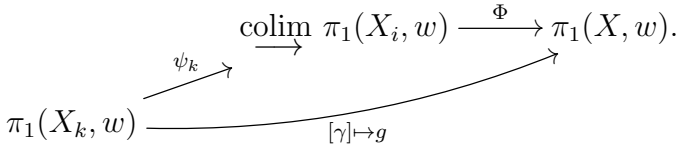
We need to show that Φ is an isomorphism.

(1) First we show that the map Φ is surjective. Let $g \in \pi_1(X, w)$.

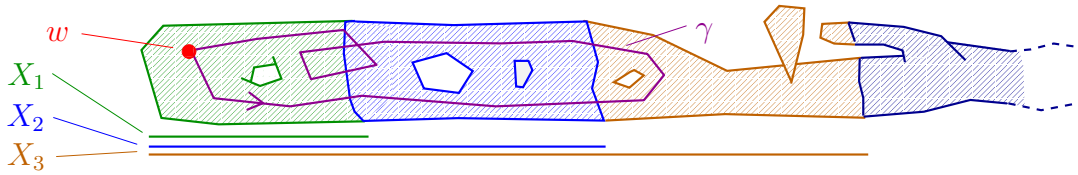
Claim. There exists a compact subset K of X with $w \in K$ such that g lies in the image of the inclusion induced map $\pi_1(K, w) \to \pi_1(X, w)$.

Proof. We pick a loop $\gamma\colon [0, 1] \to X$ that represents $g \in \pi_1(X, w)$. By the Compact Image Lemma 2.8 the image $\gamma([0, 1])$ is compact. Thus $K := \gamma([0, 1])$ has the desired property. ⊞

Note that it follows from the claim together with the Compact Subspace–Exhaustion Lemma 20.11 that there exists a $k \in \mathbb{N}_0$ such that g lies in the image of the inclusion induced map $\pi_1(X_k, w) \to \pi_1(X, w)$. We consider the following commutative diagram



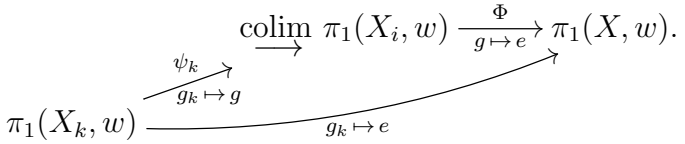
By our choice of k we see that g lies in the image of the curved map, hence it also lies in the image of Φ. This concludes the proof that Φ is surjective.



(2) Now we show that the map Φ is injective. So let $g \in \ker\big(\Phi\colon \operatorname{colim} \pi_1(X_i, w) \to \pi_1(X, w)\big)$. By the Algebraic–Direct Limit Existence Proposition 25.3 we can write

$$\operatorname{colim} \pi_1(X_i, w) \;=\; \Big(\bigsqcup_{i\in\mathbb{N}_0} \pi_1(X_i, w)\Big)/\sim \quad \text{where } g \sim \theta_{lk}(g) \text{ for all } k \le l,\ g \in \pi_1(X_k, w).$$

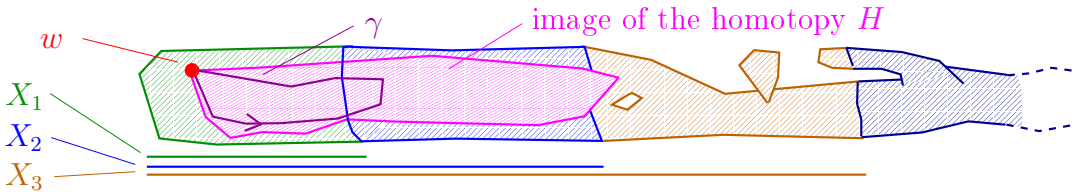
It follows in particular from this explicit description that there exists a $k \in \mathbb{N}_0$ and a $g_k \in \pi_1(X_k, w)$ with $g = \psi_k(g_k)$. In summary, we have the following situation:



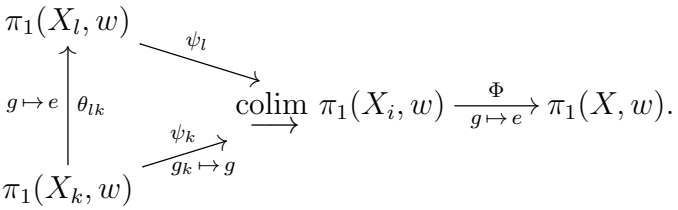
Claim. There exists a compact subset K of X such that g_k lies in the kernel of the inclusion induced map $\pi_1(X_k, w) \to \pi_1(X_k \cup K, w)$.

Proof. We pick a loop $\gamma\colon [0,1] \to X_k$ that represents g_k. By the above we know that $[\gamma] \in \ker(\pi_1(X_k, w) \to \pi_1(X, w))$. By definition this means that there exists a path-homotopy $H\colon [0,1]\times[0,1] \to X$ from the path γ to the constant path. We appeal again to the Compact Image Lemma 2.8 to deduce that $K := H([0,1] \times [0,1])$ is a compact subset of X. Evidently K has the desired property. ⊞

It follows from the claim, the Compact Subspace–Exhaustion Lemma 20.11 and the fact that we always have $X_i \subset X_{i+1}$ that there exists an $l \geq k$ such that $\theta_{lk}([\gamma])$ is trivial in $\pi_1(X_l, w)$.



We obtain the following diagram



It follows immediately that g itself is trivial. This concludes the proof that Φ is injective. ■

Example.

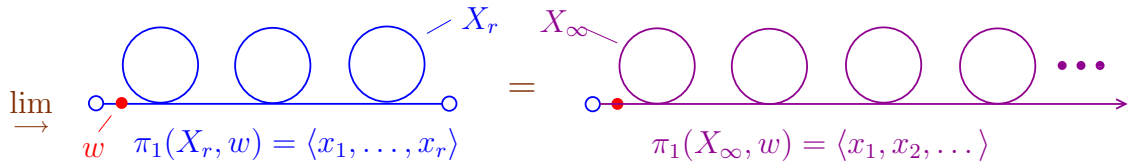
(1) As on page 316 we denote by X_∞ the interval $(0,\infty)$ with infinitely many circles attached. We pick the base point $w = \frac{1}{4}$. We obtain that [218]

$$\pi_1\big(X_\infty, w\big) \underset{\uparrow}{=} \pi_1\big(\varinjlim X_r, w\big) \underset{\uparrow}{=} \varinjlim \pi_1\big(X_r, w\big) \overset{\downarrow}{=} \varinjlim \langle x_1, \dots, x_r\rangle \underset{\uparrow}{=} \langle x_1, x_2, \dots\rangle.$$

$$\begin{array}{ccccc} \text{see page 310} & & \text{Direct Limit-}\pi_1\text{-Proposition 26.1} & & \text{see page 310} \end{array}$$

$$\text{Graph-}\pi_1\text{-Proposition 20.9}$$

Furthermore the proof of the Graph-π_1-Proposition 20.9 and the above argument show that the generators $x_1, x_2, \dots$ of $\pi_1(X_\infty)$ can be chosen such that x_n, $n \in \mathbb{N}$, corresponds to a loop which goes "once around the n-th circle".



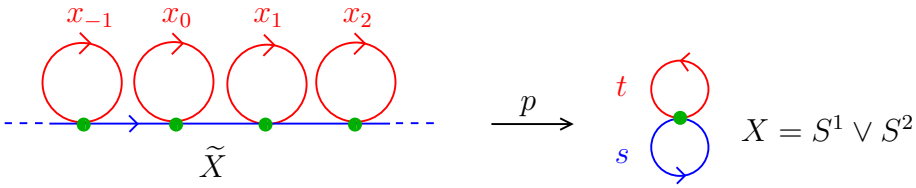
(2) We consider

$$\widetilde{X} := \mathbb{R} \cup \bigcup_{j\in\mathbb{Z}} \underbrace{\big\{z \in \mathbb{C} \,\big|\, |z - (j + \tfrac{1}{4}\mathrm{i})| = \tfrac{1}{4}\big\}}_{\text{circle of radius } \frac{1}{4} \text{ around } j+\frac{1}{4}\mathrm{i} \in \mathbb{C}} \subset \mathbb{C}.$$

Put differently, $\widetilde{X}$ is the real line with infinitely many circles attached. The same argument as in (1) shows that $\pi_1(\widetilde{X}, 0)$ is the free group on infinitely many generators $\{x_n\}_{n\in\mathbb{Z}}$. In the figure below we illustrate a covering map $p\colon \widetilde{X} \to X = S^1 \vee S^1$. By the Covering-$\pi_1$-Monomorphism Proposition 14.11 the induced homomorphism $p_*\colon \pi_1(\widetilde{X}, 0) \to \pi_1(X, *)$ is a monomorphism. We had just seen that $\pi_1(\widetilde{X})$ is a free group on infinitely generators and by the Wedge-of-Circles-π_1-Lemma 20.7 we know that $\pi_1(X)$ is a free group on two generators. In fact, with appropriate orientations and conventions this shows that the homomorphism

$$\begin{array}{ccccccc} \pi_1(\widetilde{X}, 0) & \xleftarrow{\cong} & \langle \dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\rangle & \to & \langle s, t\rangle & \xrightarrow{\cong} & \pi_1(X, *) \\ & & x_n & \mapsto & s^n \cdot t \cdot s^{-n} & & \end{array}$$

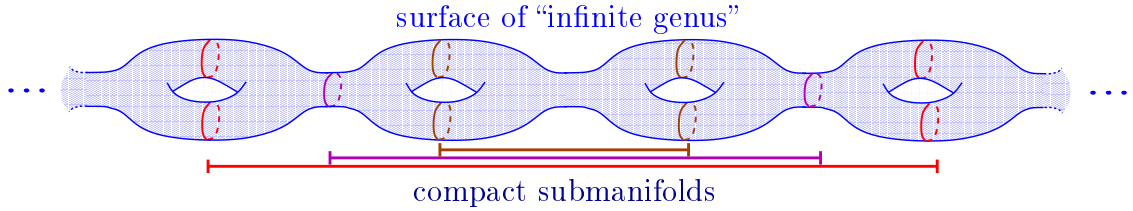
is a monomorphism. In particular we see that the free group $\langle s, t\rangle$ on two generators contains a free group on infinitely many generators.



[218] The third equality follows from the Graph-π_1-Proposition 20.9, to be more precise, it follows from the fact that for any $r \le s$ the inclusion map $\iota_{sr}\colon X_r \to X_s$ and the isomorphisms of the Graph-π_1-Proposition 20.9 give rise to a commutative diagram

$$\begin{array}{ccc} \pi_1(X_r, w) & \xrightarrow{\cong} & \langle x_1, \dots, x_r\rangle \\ {\scriptstyle (\iota_{sr})_*}\downarrow & & \downarrow{\scriptstyle x_i \mapsto x_i} \\ \pi_1(X_s, w) & \xrightarrow{\cong} & \langle x_1, \dots, x_s\rangle. \end{array}$$

Example. In [Fri23] we will use the Direct Limit-π_1-Proposition 26.1 to show that the fundamental group of the surface of infinite genus is the free group on infinitely many generators.



27. Universal coverings

After spending a lot of time on determining the fundamental groups of topological spaces we now want to return to the study of coverings. For the reader's convenience we recall the definition from page 187.

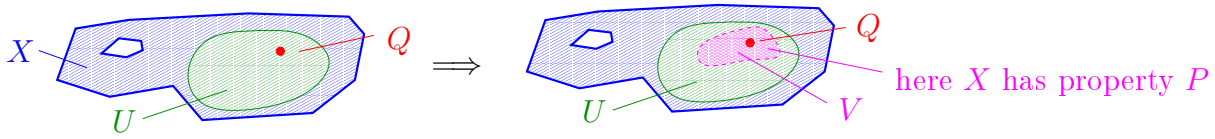
Definition. Let $p\colon X \to B$ be a continuous map between topological spaces.

(1) An open subset $U \subset B$ is called uniformly covered, if $p^{-1}(U) \subset X$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ of X with the property that for each $i \in I$ the map $p|_{V_i}\colon V_i \to U$ is a homeomorphism.
(2) We say the map $p\colon X \to B$ is a covering, if p is surjective and if for every $b \in B$ there exists an open neighborhood U of b that is uniformly covered.
(3) A covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces is a covering $p\colon X \to B$ with $p(x_0) = b_0$.
(4) We say a covering $p\colon X \to B$ is path-connected, if X is path-connected. We adopt obvious variations on this language.

We would like answer the following question: Does every path-connected non-empty topological space B admit a path-connected covering $p\colon X \to B$ such that X is simply connected? We know of course that the answer is yes for the torus, the Klein bottle and the Möbius band. But does such a covering exist for the surface of genus 2? How about the wedge $S^1 \vee S^1$ of two circles?

27.1. **Local properties of topological spaces.** Before we can address the above question we need to recall the definition from Section 2.9 of a local property of topological spaces.

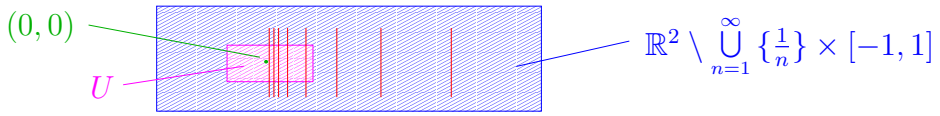
Definition. Let P be a property of topological spaces. We say a topological space X is locally P if given any $Q \in X$ and any neighborhood U of Q there exists an open neighborhood V of Q that is contained in U and that has the property P.



Examples.

(1) Note that every topological manifold (in particular every open subset of $\mathbb{R}^n$) is locally contractible.
(2) One can easily show that every topological graph is locally contractible.
(3) One can easily show that any wedge of topological manifolds (e.g. any wedge of spheres) is locally contractible.
(4) By the Homotopy-π_0-Lemma 9.6 and the Homotopy-π_1-Proposition 13.4 we know that every contractible topological space is path-connected and simply connected. Thus every locally contractible topological space is also locally path-connected and locally simply connected.
(5) We consider
$$X := \mathbb{R}^2 \setminus \bigcup_{n=1}^{\infty} \left\{\tfrac{1}{n}\right\} \times [-1, 1],$$
with the usual subspace topology coming from $\mathbb{R}^2$. Then X is path-connected but not locally path-connected. Indeed, we consider the point $(0, 0) \in X$ and the neighborhood

$U = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2}) \cap X$. It is straightforward to show that there does not exist a path-connected neighborhood V of $(0,0)$ which is contained in U.



(6) On numerous occasions we will use the following elementary fact: if $p\colon X \to B$ is a covering and if B is locally P, then so is X.

27.2. **Lifting maps to coverings.** The following definition is a mild variation on the definition on page 193.

Definition. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Furthermore let $f\colon (Z, z_0) \to (B, b_0)$ be a continuous map. A lift of f is a continuous map $\widetilde{f}\colon (Z, z_0) \to (X, x_0)$ that makes the following diagram commute

$$\begin{array}{ccc} & & (X, x_0) \\ & \overset{\widetilde{f}}{\nearrow} & \downarrow p \\ (Z, z_0) & \xrightarrow[f]{} & (B, b_0). \end{array}$$

It is worth recalling some earlier results on lifts of maps. First of all we have the following proposition that deals with any type of covering.

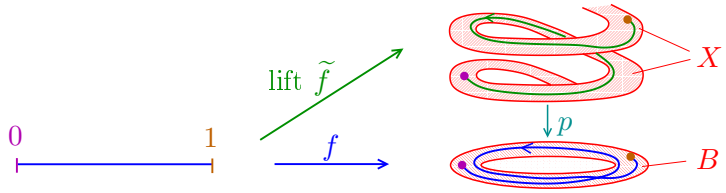
Proposition 14.7. (Lift Uniqueness Proposition) Let $p\colon X \to B$ be a covering and let Z be a connected topological space. Furthermore let $f\colon Z \to B$ be a continuous map and let $\widetilde{f}, \widehat{f}\colon Z \to X$ be two lifts of f. If $\widetilde{f}$ and $\widehat{f}$ agree at some point $z_0 \in Z$, then they agree everywhere.

We are already experts on lifts of paths, i.e. of continuous maps $[0,1] \to B$. We now recall two of the main results.

Proposition 14.10. (Path-Lifting Proposition) Let $p\colon X \to B$ be a covering.

(1) Let $f\colon [0,1] \to B$ be a path. Given any $\widetilde{b} \in X$ with $p(\widetilde{b}) = f(0)$ there exists a unique lift $\widetilde{f}\colon [0,1] \to X$ of f to the starting point $\widetilde{b}$, i.e. with $\widetilde{f}(0) = \widetilde{b}$.

(2) Let $f, g\colon [0,1] \to B$ be two paths and let $\widetilde{f}, \widetilde{g}\colon [0,1] \to X$ be two lifts with the same starting point. If f and g are path-homotopic, then the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree and the paths $\widetilde{f}$ and $\widetilde{g}$ are path-homotopic.



Proposition 14.12 (Loop–Lifting Proposition) Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Let $f\colon [0,1] \to B$ be a loop in (B, b_0). We denote by $\widetilde{f}\colon [0,1] \to \widetilde{B}$ the lift of f to the starting point x_0. The following holds:

$$\widetilde{f} \text{ is a loop in } (X, x_0) \quad \Longleftrightarrow \quad [f] \text{ lies in } p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0).$$

Given a covering $p\colon (X,x_0)\to(B,b_0)$ we now want to study the existence of lifts of arbitrary continuous maps $(Z,z_0)\to(B,b_0)$. We start out with a very instructive example.

Example.

(1) We consider the maps:

$$\begin{array}{ccc} & & (S^1,1) \\ & \overset{???\ \text{lift}\ \widetilde{f}\ ???}{\nearrow} & \big\downarrow{\scriptstyle p}\ z\mapsto z^3 \\ (S^1,1) & \xrightarrow[f]{z\mapsto z^2} & (S^1,1). \end{array}$$

Here the map p is a 3-fold covering. Does the map f lift to a map $\widetilde{f}$?

(2) To answer the question we just posed in (1) it is helpful to consider a more general setup. Thus let $p\colon (X,x_0)\to(B,b_0)$ be a covering of pointed topological spaces and let $f\colon (Z,z_0)\to(B,b_0)$ be a continuous map. Let us suppose that there exists a lift $\widetilde{f}\colon (Z,z_0)\to(X,x_0)$. We obtain the commutative diagram

$$\begin{array}{ccc} & & (X,x_0) \\ & \overset{\widetilde{f}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ (Z,z_0) & \xrightarrow[f]{} & (B,b_0) \end{array}$$

from the functoriality of the fundamental group it follows that $f_* = \big(p\circ\widetilde{f}\big)_* = p_*\circ\widetilde{f}_*$, i.e. we get the commutative diagram

$$\begin{array}{ccc} & & \pi_1(X,x_0) \\ & \overset{\widetilde{f}_*}{\nearrow} & \big\downarrow{\scriptstyle p_*} \\ \pi_1(Z,z_0) & \xrightarrow[f_*]{} & \pi_1(B,b_0). \end{array}$$

Thus it follows that if f lifts, then

$$\operatorname{im}(f_*) \;=\; \operatorname{im}((p\circ\widetilde{f})_*) \;=\; \operatorname{im}(p_*\circ\widetilde{f}_*) \;\subset\; \operatorname{im}(p_*).$$

(3) In the above example (1) we have $\pi_1(S^1,1)=\mathbb{Z}$. By the discussion on page 174 we have $\operatorname{im}(f_*)=2\mathbb{Z}$ and we have $\operatorname{im}(p_*)=3\mathbb{Z}$. Since $2\mathbb{Z}\not\subset 3\mathbb{Z}$ we see that f does *not* lift.

The following proposition says, that the above conclusion has a converse for "reasonably" topological spaces.

Proposition 27.1. (Map Lifting Criterion) Let $p\colon (X,x_0)\to(B,b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected topological space and let $f\colon (Z,z_0)\to(B,b_0)$ be a continuous map. If Z is locally path-connected, then we have the following equivalence of statements:[219]

$$\begin{array}{ccc} & & (X,x_0) \\ & \overset{\text{there exists a lift}\ \widetilde{f}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ (Z,z_0) & \xrightarrow[f]{} & (B,b_0) \end{array} \qquad\Longleftrightarrow\qquad \operatorname{im}(f_*)\subset\operatorname{im}(p_*).$$

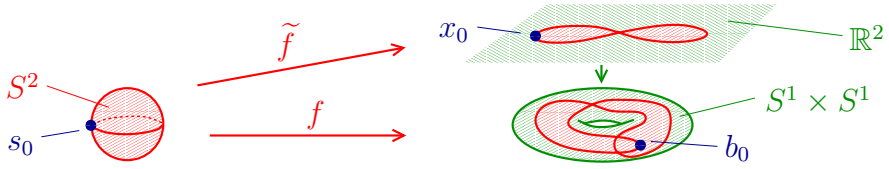
Example. Let $p\colon (X,x_0)\to(B,b_0)$ be a covering of pointed topological spaces.

(1) The condition on the fundamental groups in the Map Lifting Criterion 27.1 is automatically satisfied if Z is simply connected.

(2) Let $f\colon (S^n,s_0)\to(B,b_0)$ be a continuous map. The topological manifold S^n is path-connected and locally path-connected. By the Sphere-π_1-Proposition 11.14 the sphere

[219] It is worth recalling that since Z is in particular connected, it follows from the Lift Uniqueness Lemma 11.10 that a lift $(Z,z_0)\to(X,x_0)$, if it exists, is unique.

S^n is simply connected if $n \geq 2$. Thus it follows from (1) that there exists a uniquely determined lift $\widetilde{f}\colon (S^n, s_0) \to (X, x_0)$ of the map $f\colon (S^n, s_0) \to (B, b_0)$.



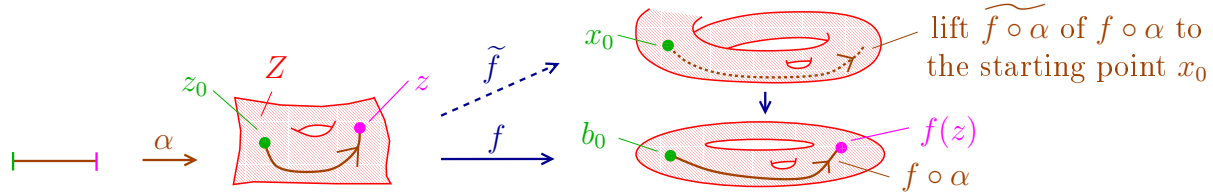
Proof of the Map Lifting Criterion 27.1. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected and locally path-connected topological space and let $f\colon (Z, z_0) \to (B, b_0)$ be a continuous map.

By the discussion on page 329 we only have to prove the "$\Longleftarrow$"-direction of the proposition. So suppose that we have

$$\operatorname{im}(f_*\colon \pi_1(Z, z_0) \to \pi_1(B, b_0)) \;\subset\; \operatorname{im}(p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)).$$

We have to construct a lift $\widetilde{f}\colon (Z, z_0) \to (X, x_0)$. We will do so as follows. Let $z \in Z$ be a point. Since Z is path-connected we can find a path $\alpha\colon [0,1] \to Z$ with $\alpha(0) = z_0$ and $\alpha(1) = z$. According to the Path-Lifting Proposition 14.10 (1) we can lift the path $f \circ \alpha\colon [0,1] \to B$ to a path $\widetilde{(f \circ \alpha)}\colon [0,1] \to X$ with starting point x_0. We set

$$\widetilde{f}(z) \;:=\; \widetilde{(f \circ \alpha)}(1).$$



It remains to prove the following claim.

Claim.
(1) The map $\widetilde{f}\colon Z \to X$ is well-defined.
(2) The map $\widetilde{f}$ is continuous.
(3) The map $\widetilde{f}$ is a lift.

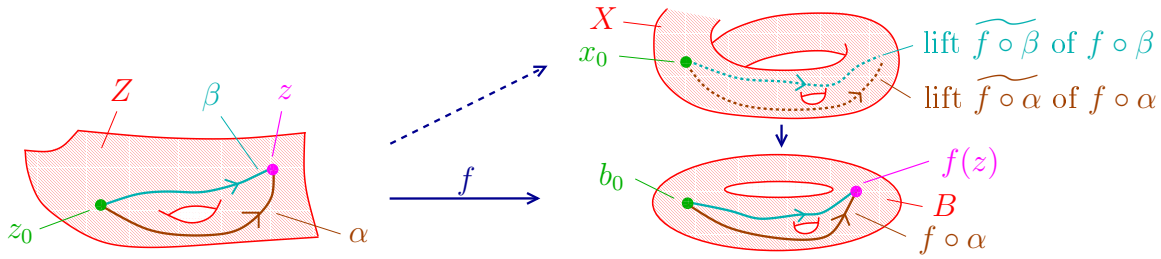
Proof.

(1) We have to show that for $z \in Z$ the point $\widetilde{f}(z) \in X$ is well-defined, i.e. independent of the choice of the path α. Thus let $\beta\colon [0,1] \to Z$ be another path with $\beta(0) = z_0$ and

$\beta(1) = z$. It follows that

$$\begin{aligned}
\widetilde{(f\circ\alpha)}(t=1) &= (\text{lift of } f\circ\alpha \text{ to the starting point } x_0)(t=1)\\
&= (\text{lift of } (f\circ\alpha) * \overline{(f\circ\beta)} * (f\circ\beta) \text{ to the starting point } x_0)(t=1)\\
&\quad \uparrow \\
&\quad \text{by the Path-Lifting Proposition 14.10 (2) since } f\circ\alpha \text{ and } (f\circ\alpha)*\overline{(f\circ\beta)}*(f\circ\beta) \text{ are path-homotopic}\\
&= (\text{lift of } (f\circ(\underbrace{\alpha * \overline{\beta}}_{=:\gamma,\ \text{a loop in } (Z,z_0)}))*(f\circ\beta) \text{ to the starting point } x_0)(t=1)\\
&= (\text{lift of } f\circ\beta \text{ to the starting point } \widetilde{(f\circ\gamma)}(1))(t=1)\\
&= (\text{lift of } f\circ\beta \text{ to starting point } x_0)(t=1) \;=\; \widetilde{(f\circ\beta)}(t=1).\\
&\quad \uparrow \\
&\quad \text{by our hypothesis we have } [f\circ\gamma] = f_*([\gamma]) \in \operatorname{im}(f_*) \subset p_*(\pi_1(X,x_0)),\\
&\quad \text{hence } \widetilde{f\circ\gamma} \text{ is a loop in } x_0 \text{ by the Loop–Lifting Proposition 14.12}
\end{aligned}$$

Thus we have shown that the map $\widetilde{f}$ is well-defined.



(2) We need to show that $\widetilde{f}$ is continuous. By the Pasting Proposition 2.4 it suffices to show that $\widetilde{f}$ is locally continuous. So let $z \in Z$. We pick a path α from z_0 to z. Since p is a covering there exists an open neighborhood V of $\widetilde{f}(z)$ such that the restriction of p to V is an embedding. We denote by $q\colon p(V) \to V$ the inverse map. Since Z is locally path-connected there exists an open neighborhood U of z that is contained in $f^{-1}(p(V))$ and that is path-connected. Since f and q are continuous it suffices to prove the following claim.

Claim. On the open neighborhood U of z the map $\widetilde{f}$ agrees with $q\circ f$.

Proof. So let $w \in U$. Since U is path-connected there exists a path δ from z to w that lies in U. Then

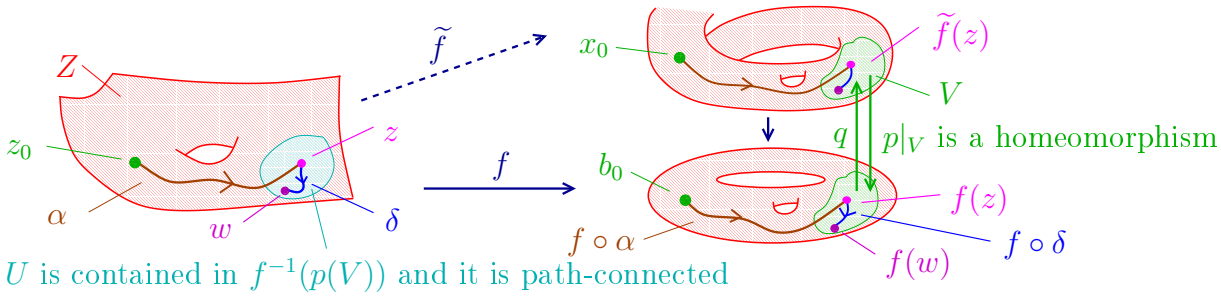
$$\begin{aligned}
\widetilde{f}(w) &= \text{endpoint of the lift of } f\circ(\alpha*\delta) = (f\circ\alpha)*(f\circ\delta) \text{ to the starting point } x_0\\
&= \text{endpoint of the lift of } f\circ\delta \text{ to the starting point } \widetilde{f}(z)\\
&= \text{endpoint of } q\circ(f\circ\delta) \;=\; (q\circ f)(\delta(1)) \;=\; (q\circ f)(w).\\
&\quad \uparrow \\
&\quad \text{since } q = p^{-1} \text{ we know that } q\circ(f\circ\delta) \text{ is the lift of } f\circ\delta \text{ to the starting point } f(z)
\end{aligned}$$

(3) We need to show that the map $\widetilde{f}\colon Z \to X$ is a lift of $f\colon Z \to B$, i.e. we need to show that $p\circ\widetilde{f} = f$. Given any $z \in Z$ we have indeed the following equalities:

$$p\big(\widetilde{f}(z)\big) \;\underset{\uparrow}{=}\; p\big(\widetilde{(f\circ\alpha)}(1)\big) \;=\; (f\circ\alpha)(1) \;=\; f(\alpha(1)) \;=\; f(z).$$

we pick a path α from z_0 to z and we denote by $\widetilde{f\circ\alpha}$ the lift of $f\circ\alpha$

■



27.3. **Existence of covering spaces.** Our next goal will be to show that every "reasonable" path-connected non-empty topological spaces admits a covering that is simply connected.

Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. In the Covering-π_1-Monomorphism Proposition 14.11 we saw that $p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)$ is an injective homomorphism. Thus we can view $\pi_1(X, x_0)$ as a subgroup of $\pi_1(B, b_0)$. One can now ask, whether to each subgroup Γ of $\pi_1(B, b_0)$ there exists a corresponding covering with $p_*(\pi_1(X, x_0)) = \Gamma$. We will see that this is indeed the case, at least under the rather technical assumption that the topological space B is locally simply connected.

Theorem 27.2. (Covering Existence Theorem) Let Y be a topological space that is path-connected and locally simply connected, let $y_0 \in Y$ and let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup.

(1) There exists a path-connected covering $p\colon (X, x_0) \to (Y, y_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$.[220]

(2) For the covering p from (1) we have

$$\underbrace{[X : Y]}_{\substack{\text{degree as defined} \\ \text{on page 189}}} = [\pi_1(Y, y_0) : \Gamma].$$

We will provide the proof of the Covering Existence Theorem 27.2 in the next two sections. In the following we discuss a few examples.

Example. We consider $(Y, y_0) = (S^1, 1)$. We pick an identification[221] $\pi_1(S^1, 1) = \mathbb{Z}$. Given $n \in \mathbb{N}$ we consider $\Gamma = n\mathbb{Z} \subset \mathbb{Z} = \pi_1(S^1, 1)$. It follows from the discussion on page 174 that the covering

$$\begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

has the property that $p_*(\pi_1(S^1, 1)) = n\mathbb{Z}$. Another example is given by the covering

$$\begin{array}{rcl} q\colon \mathbb{R}/n\mathbb{Z} & \to & S^1 \\ t + n\mathbb{Z} & \mapsto & \exp(2\pi i t). \end{array}$$

Here we also have $q_*(\pi_1(\mathbb{R}/n\mathbb{Z}, [0])) = n\mathbb{Z}$.

[220] Note that it follows from the Covering-π_1-Monomorphism Proposition 14.11 that $\pi_1(X, x_0) \cong \Gamma$.

[221] This means that we pick an isomorphism $\pi_1(S^1, 1) \xrightarrow{\cong} \mathbb{Z}$ and then ignore it in the notation.

27.4. **Proof of the Covering Existence Theorem 27.2 I.** We turn to the proof of the Covering Existence Theorem 27.2. Note that we only need to show the existence of such a covering, the statement about the degree of the covering is an immediate consequence of the Covering-Degree Proposition 14.13.

We start out with a general discussion of the construction of the covering. Only afterwards we will turn to the nitty-gritty of the proof. In particular for the time being let Y be any path-connected topological space, without any other adjectives. Furthermore let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup.

To get an idea for the construction, let us first consider the case that Γ is the trivial group. In this case we want to construct a covering $p\colon X \to Y$ such that X is simply connected. Let us assume for a second that we have such an X. We pick $x_0 \in X$ with $p(x_0) = y_0$. Then, somewhat similar to the statement of the Action–π_1-Theorem 15.1, we have a map

$$\begin{array}{rcl} \Phi\colon X & \to & \{\text{path-homotopy classes of paths } [0,1] \to Y \text{ starting in } y_0\} \\ x & \mapsto & p \circ (\text{any path from } x_0 \text{ to } x). \end{array}$$

In fact the proof of the Action–π_1-Theorem 15.1 comes close to showing that the map Φ is a bijection. Hence Φ is in fact a homeomorphism, provided we equip the right-hand side with an appropriate topology.

Now that we are not given X, the idea is, that given (Y, y_0) we can define the right-hand side of the above bijection, and with a suitable topology we will have found our X.

This discussion leads us to the following definition.

Definition. Let Y be a topological space, let $y_0 \in Y$ and let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup.

(1) We denote by $Y^{[0,1]}$ the set of all continuous maps $[0,1] \to Y$.

(2) We consider the path space

$$P_{y_0} := \{\gamma \in Y^{[0,1]} \mid \gamma(0) = y_0\} = \{\text{all paths in } Y \text{ with starting point } y_0\}.$$

(3) For $u, v \in P_{y_0}$ we write

$$u \sim v \quad :\Longleftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [\ \underbrace{u * \overline{v}}_{\text{loop in } (Y, y_0)}\ ] \in \Gamma.$$

Using the fact that $\Gamma \subset \pi_1(Y, y_0)$ is a subgroup one can show easily that $\sim$ is indeed an equivalence relation on P_{y_0}.

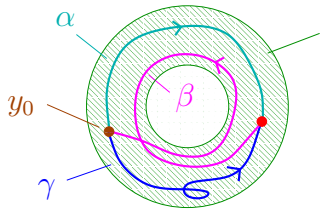
(4) We write $X_{y_0,\Gamma} := P_{y_0}/\sim$ and $x_0 := [e_{y_0}]$, i.e. x_0 is the equivalence class of the constant path.

(5) We consider the evaluation map

$$\begin{array}{rcl} p\colon X_{y_0,\Gamma} = P_{y_0}/\sim & \to & Y \\ [v] & \mapsto & v(1). \end{array}$$

Note that this map is well-defined, i.e. it is independent of the choice of the representative v.

It remains to equip $X_{y_0,\Gamma} = P_{y_0}/\sim$ with a sensible topology and to show that the map $p\colon X_{y_0,\Gamma} = P_{y_0}/\sim \to Y$ has all the desired properties.



Y is an annulus
we consider $\Gamma = 2\mathbb{Z} \subset \pi_1(Y, y_0) = \mathbb{Z}$,
then α and β are equivalent,
but α and γ are inequivalent

The following lemma defines such a topology and thus completes the proof of the Covering Existence Theorem 27.2.

Lemma 27.3. (Whisker Topology Lemma) Let Y be a path-connected topological space, let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup. We write $P := P_{y_0}$ and we write $X := X_{y_0,\Gamma}$. Given $x = [f] \in X = P/\sim$ and given an open neighborhood V of $p(x) = f(1) \in Y$ we consider

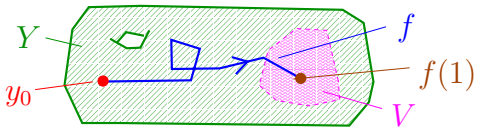
$$U(x, V) \;:=\; \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \subset X = P/\sim .$$

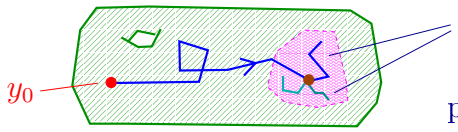
With this notation the following statement holds:

(0) The family of all such subsets $U(x, V)$ of X satisfy the "basis property" that we introduced on page 47.

The topology that is generated on $X = X_{Y_0,\Gamma}$ by the sets $U(x, V)$ is called the whisker topology. If Y is locally simply connected, and if we equip $X = X_{Y_0,\Gamma}$ with the whisker topology, then the following statements hold:

(1) The above evaluation map $p\colon X \to Y$ is continuous and open.
(2) The map $p\colon X \to Y$ is a covering.
(3) The topological space X is path-connected.
(4) We have $p_*(\pi_1(X, x_0)) = \Gamma$.





all these paths
starting at y_0 are
points in $U([f], V)$

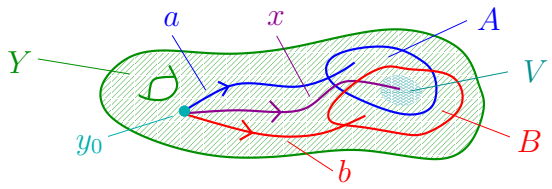
Proof. Let Y be a path-connected topological space, let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup. We write $P := P_{y_0}$ and we write $X := X_{y_0,\Gamma}$. Given $x = [f] \in X = P/\sim$ and given an open path-connected neighborhood V of $p(x) = f(1) \in Y$ we consider

$$U(x, V) \;:=\; \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \subset X = P/\sim .$$

We denote by $\mathcal{B}$ the family of all such subsets of X.

(0) We need to show that $\mathcal{B}$ has the basis property.
  (B1) For each $x \in X$ we have $x \in U(x, X)$. This shows that $\mathcal{B}$ satisfies (B1).
  (B2) Now let $x = [f]$ be a point in the intersection of $U(a, A)$ and $U(b, B)$. Let V be any open neighborhood V of $f(1)$ with $V \subset A \cap B$, e.g. we could take $V = A \cap B$. It is straightforward to see that $U(x, V) \subset U(a, A) \cap U(b, B)$.

In the following we assume that Y is locally path-connected and that Y is locally simply connected. We equip $X = X_{Y_0,\Gamma}$ with the *whisker topology*, i.e. we equip it with the topology generated by $\mathcal{B}$. Since Y is locally path-connected it follows easily that the whisker topology



is also generated by

$$\mathcal{C} := \{U(x,V) \in \mathcal{B} \mid x \in X \text{ and } V \text{ is a } \textit{path-connected} \text{ open neighborhood of } p(x)\}.$$

(1) We need to show that the map $p\colon X = P/\sim \to Y$ is continuous and open. We start with the following observation: for every $x \in X$ and every open path-connected neighborhood V of $p(x)$ we have $p(U(x,V)) = V$.[222]

Now we turn to the actual proof of continuity. So let $A \subset Y$ be an open subset and let $x = [f] \in p^{-1}(A)$. It suffices to show that there exists an open subset B of Y such that $U(x,B) \subset p^{-1}(A)$. Since Y is locally path-connected, there exists an open neighborhood B of x that is contained in A and that is path-connected. By the above observation we have $p(U(x,B)) = B \subset A$, i.e. $U(x,B) \subset p^{-1}(A)$.

Finally we show that p is also open. It suffices to verify that images of the basis $\mathcal{C}$ of the topology of X are open. But this statement we had just verified in the above observation.

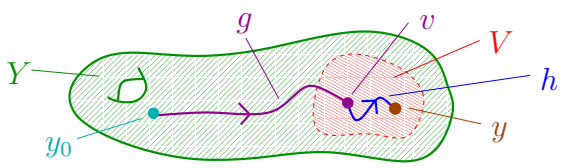
(2) Now we want to prove that $p\colon X = P/\sim \to Y$ is a covering map. First note that it follows from the hypothesis that Y is path-connected that p is surjective. Next let $y \in Y$ be a point. We need to show that there exists a connected open neighborhood V of y such that the restriction of p to each component of $p^{-1}(V)$ is a homeomorphism.

Since Y is locally simply connected, there exists an open neighborhood V of y such that the inclusion induced map $\pi_1(V,x) \to \pi_1(Y,y)$ is the trivial map. We claim that V already has the desired property.

Claim.
$$p^{-1}(V) = \bigsqcup_{\widetilde{y} \in p^{-1}(y)} U(\widetilde{y}, V).$$

Proof. It is clear that the right-hand side is contained in the left-hand side. Next we want to show that the left-hand side is contained in the right-hand side. So let $[g] \in p^{-1}(V)$. Here g is a path in Y from y_0 to a point $v \in V$. Since V is in particular path-connected there exists a path h in V from v to y. We then have

$$[g] = [g * h * \overline{h}] \in U(\underbrace{[g*h]}_{\in p^{-1}(y)}, V).$$



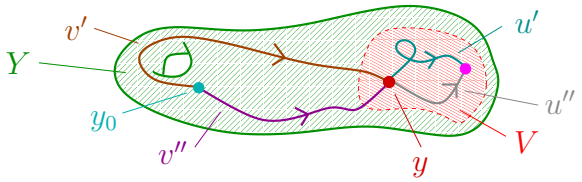
[222] By definition we have $p(U(x,V)) \subset V$. But since V is path-connected we also have $V \subset p(U(x,V))$.

It remains to show that the sets on the right-hand side of the claim are disjoint. So let $[v'], [v''] \in p^{-1}(y)$ and let $[g] \in U([v'], V) \cap U([v''], V)$. Then there exist paths u' and u'' in V with $[v' * u'] = [v'' * u'']$. It follows that

$$[v'] \;=\; [v' * u' * \overline{u}'] \;=\; [v'' * u'' * \overline{u}'] \;\underset{\uparrow}{=}\; [v''].$$

$u'' * \overline{u}'$ is null-homotopic since the map $\pi_1(V, y) \to \pi_1(Y, y)$ is the trivial map

Thus $U([v'], V) \cap U([v''], V)$ intersect only if $[v'] = [v'']$. ⊞



The sets $U(\widetilde{y}, V)$ are by definition open. By the above claim the sets are also disjoint. Thus it suffices to prove the following claim.

Claim. For each $\widetilde{y} \in p^{-1}(y)$ the restriction of the projection $p\colon X = P/\!\sim \to Y$ to the map $p\colon U(\widetilde{y}, V) \to V$ is a homeomorphism.

Proof. So let $\widetilde{y} \in p^{-1}(y)$. As we had just pointed out in (1), the map $p\colon U(\widetilde{y}, V) \to V$ is surjective. From the fact that the inclusion induced map $\pi_1(V, y) \to \pi_1(Y, y)$ is trivial it also follows immediately that p is injective. Furthermore we had just verified in (1) that p is continuous and open. Therefore $p\colon U(\widetilde{y}, V) \to V$ is a homeomorphism. ⊞

(3) Now we want to show that $X = P/\!\sim$ is path-connected. We start out with the following claim.

Claim. For any path $\alpha\colon [0,1] \to Y$ the following map is continuous:

$$\begin{array}{rcl} \widetilde{\alpha}\colon [0,1] & \to & X = P/\!\sim \\ t & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & Y \\ s & \mapsto & \alpha(t \cdot s) \end{array}\right]. \end{array}$$

Proof. By the Subbasis-Continuity Proposition 2.2 it suffices to show that for each $U(x, V)$ in the basis $\mathcal{C}$ of the topology on X the preimage $\widetilde{\alpha}^{-1}(U(x, V))$ is open in $[0,1]$. So suppose we are given $U(x, V)$. We denote by $\{I_j\}_{j \in J}$ the path-components of $\alpha^{-1}(V)$. Since α is continuous each I_j is an open subset of $[0,1]$. Note that for each $j \in J$ we either have[223] $\widetilde{\alpha}(I_j) \subset U(x, V)$ or $\widetilde{\alpha}(I_j) \cap U(x, V) = \varnothing$. It follows that

$$\widetilde{\alpha}^{-1}(U(x, V)) \;=\; \text{union of all the } I_j \text{ with the property that } \widetilde{\alpha}(I_j) \subset U(x, V).$$

Thus we see that $\widetilde{\alpha}^{-1}(U(x, V))$ is a union of open subsets of $[0,1]$, hence the preimage itself is open. ⊞

To show that X is path-connected it suffices to show that for every point $x \in X$ there exists a path in X from $x_0 = [e_{y_0}]$ to x. So let $x = [\alpha] \in X$. Then such a path is given by $\widetilde{\alpha}\colon [0,1] \to X$, where $\widetilde{\alpha}$ is defined as in the claim. By the claim $\widetilde{\alpha}$ is indeed continuous.

[223] This might require a few minutes of thought.

(4) It remains to show that $p_*(\pi_1(X,x_0)) = \Gamma$. Let $g \in \pi_1(Y,y_0)$. We choose a representatives $\alpha\colon [0,1] \to Y$ of g. We consider the following path[224] in $X = P/\sim$

$$\begin{array}{rcl} \widetilde{\alpha}\colon [0,1] & \to & X = P/\sim \\ t & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & Y \\ s & \mapsto & \alpha(t\cdot s) \end{array}\right]. \end{array}$$

Note that $\widetilde{\alpha}$ is evidently the lift of α to the starting point x_0. Then we have

$$g \in p_*(\pi_1(X,x_0)) \underset{\uparrow}{\Leftrightarrow} \widetilde{\alpha} \text{ is a loop} \Leftrightarrow \text{in } X = P/\sim \text{ we have } \widetilde{\alpha}(1) = x_0 = [e_{y_0}] \underset{\uparrow}{\Leftrightarrow} \alpha \in \Gamma.$$

the Loop–Lifting Proposition 14.12 definition of $\sim$ on P ∎

27.5. **Uniqueness of coverings.** Let (B,b_0) be a path connected topological space. We just saw, that under a few mild technical hypothesis, any subgroup $\Gamma \subset \pi_1(B,b_0)$ gives rise to a covering. The following question arises: to what degree is the covering corresponding to the subgroup Γ unique? To answer this question it is helpful to first prove the following universal property of coverings corresponding to a subgroup.

Theorem 27.4. (Covering–Covering Theorem) Let B be a locally path-connected topological space, let $b_0 \in B$ and let $p\colon (X,x_0) \to (B,b_0)$ be a path-connected covering. For every other path-connected covering $q\colon (Y,y_0) \to (B,b_0)$ with $p_*(\pi_1(X,x_0)) \subset q_*(\pi_1(Y,y_0))$ there exists a *unique* covering $r\colon (X,x_0) \to (Y,y_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} (X,x_0) & \overset{\exists!\, r}{\dashrightarrow} & (Y,y_0) \\ {\scriptstyle p}\downarrow & & \swarrow{\scriptstyle q} \\ (B,b_0). & & \end{array}$$

Proof. Let B be a locally path-connected topological space and let $b_0 \in B$ and let $p\colon (X,x_0) \to (B,b_0)$ and $q\colon (Y,y_0) \to (B,b_0)$ be two path-connected coverings such that $p_*(\pi_1(X,x_0)) \subset q_*(\pi_1(Y,y_0))$. Note that X is also locally path-connected. Thus it follows from the Map Lifting Criterion 27.1 together with the Lift Uniqueness Lemma 11.10 that there exists a unique map $r\colon (X,x_0) \to (Y,y_0)$ that makes the following diagram commute

$$\begin{array}{ccc} & & (Y,y_0) \\ & \overset{r}{\nearrow} & \downarrow{\scriptstyle q} \\ (X,x_0) & \xrightarrow[p]{} & (B,b_0). \end{array}$$

It remains to prove that $r\colon X \to Y$ is a covering. Recall that this means that we need to prove the following two statements:

(a) Each $y_1 \in Y$ admits a uniformly covered open neighborhood.
(b) The map $r\colon X \to Y$ is surjective.

We turn to the proofs of these two statements:

(a) Let $y_1 \in Y$. We set $b_1 := q(y_1)$. Since B is locally path-connected and since $p\colon X \to B$ and $q\colon Y \to B$ are coverings there exists a path-connected open neighborhood W of b that is uniformly covered. Since $p\colon X \to B$ and $r\colon Y \to B$ are coverings we can

[224] Note that this map is continuous by the claim in (3).

write $p^{-1}(W) = \bigsqcup_{x\in p^{-1}(b_1)} U_x$ where each $U_x \subset X$ is open and each $p\colon U_x \to W$ is a homeomorphism and we can write $q^{-1}(W) = \bigsqcup_{y\in q^{-1}(b_1)} V_y$ where each $V_y \subset Y$ is open and each $q\colon V_y \to W$ is a homeomorphism.

Claim. Let $x \in p^{-1}(b_1)$ and let $y \in q^{-1}(b_1)$. If $r(U_x) \cap V_y \neq \varnothing$, then $r(x) = y$ and $r(U_x) = V_y$.

Proof. We make the following observations:

(i) Since W is path-connected it is in particular connected. Since U_x is homeomorphic to W we see that U_x is connected.

(ii) Since $p = q \circ r$ we see that $r(U_x) \subset q^{-1}(q(r(U_x)) = q^{-1}(p(U_x)) = q^{-1}(W)$.

(iii) It follows from (i), (ii), the fact $q^{-1}(W) = \bigsqcup_{y\in q^{-1}(b_1)} V_y$ and the fact that the V_y are open that there exists a $y \in q^{-1}(b_1)$ with $r(U_x) \subset V_y$.

(iv) It follows from (iii) together with $p(x) = b_1 = q(y)$, $p^{-1}(b_1) \cap U_x = \{x\}$, $q^{-1}(b_1) \cap V_y = \{y\}$ and $p = q \circ r$ that $r(x) = y$.

(v) Since $p\colon U_x \to W$ and $q\colon V_y \to W$ are homeomorphisms and since $p = q \circ r$ we obtain from (iii) that $r\colon U_x \to V_y$ is a homeomorphism. ⊞

Next note that

$$\bigsqcup_{x\in r^{-1}(y)} \bigsqcup_{y\in q^{-1}(b)} U_x = \bigsqcup_{x\in (q\circ r)^{-1}(b)} U_x = \bigsqcup_{x\in p^{-1}(b)} U_x = p^{-1}(W) = (q\circ r)^{-1}(W)$$
$$= r^{-1}(q^{-1}(W)) = r^{-1}\Big(\bigsqcup_{y\in q^{-1}(b)} V_y\Big) \underset{\uparrow}{=} \bigsqcup_{y\in q^{-1}(b)} r^{-1}(V_y).$$

Image-Preimage Lemma 0.3 (6)

It follows from this equality and the above claim that

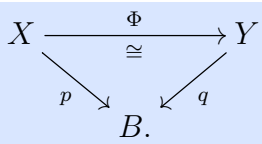
$$\bigsqcup_{x\in r^{-1}(y_1)} U_x = r^{-1}(V_{y_1}).$$

Next let $x \in r^{-1}(y_1)$. We already know that U_x is an open subset of X. Note that the restriction of r to U_x is given by $(q|_{V_y\to W})^{-1} \circ (p\colon U_x \to W)$. This shows that $r\colon U_x \to V_{y_1}$ is a homeomorphism. In summary, we have shown that V_{y_1} is uniformly covered.

(b) We want to show that $r\colon X \to Y$ is surjective. Let $y_1 \in Y$. We just showed in (a) that there exists an open neighborhood V_{y_1} that is uniformly covered with respect to the map $r\colon X \to Y$. But this means that either $V_{y_1} \subset r(X)$ or $V_{y_1} \cap r(X) = \varnothing$. It follows from the JH-Openness Criterion 1.1 that $r(X)$ and $Y \setminus r(X)$ are both open. Since Y is by hypothesis path-connected (and thus connected) and since $X \neq \varnothing$ we see that $r(X) = Y$. ■

Now we want to formulate the promised uniqueness statement for coverings. This statement builds on the following natural definition.

Definition. Let $p\colon X \to B$ and $q\colon Y \to B$ be two coverings of a topological space B. We say p and q are equivalent, if there exists a homeomorphism $\Phi\colon X \to Y$ such that the following diagram commutes:



Similarly we define the equivalence of coverings of pointed topological spaces.

Example. Given $n \in \mathbb{N}$ the two coverings

$$\begin{array}{rcl} q\colon \mathbb{R}/n\mathbb{Z} & \to & S^1 \\ t + n\mathbb{Z} & \mapsto & \exp(2\pi \mathrm{i} t) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

are equivalent. In fact the desired map $\Phi\colon \mathbb{R}/n\mathbb{Z} \to S^1$ is given by $[t] \mapsto \exp(2\pi \mathrm{i} \frac{t}{n})$.

Now we can formulate the promised uniqueness theorem:

Theorem 27.5. (Covering Unique Existence Theorem) Let B be a path-connected topological space that is locally simply connected. Furthermore let $b_0 \in B$ and let $\Gamma \subset \pi_1(B, b_0)$ be a subgroup. There exists, up to equivalence, a *unique* path-connected covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$.

Proof. Let $b_0 \in B$ and let $\Gamma \subset \pi_1(B, b_0)$ be a subgroup. The existence of such a covering is just the statement of the Covering Existence Theorem 27.2.

Now suppose that we are given two path-connected coverings $p\colon (X, x_0) \to (B, b_0)$ and $q\colon (Y, Y_0) \to (B, b_0)$ with $p_*(\pi_1(X, x_0)) = q_*(\pi_1(Y, y_0)) = \Gamma$. There are two approaches to showing that p and q are equivalent:

(1) The uniqueness follows immediately from the Covering–Covering Theorem 27.4 and the usual proof of uniqueness for an object satisfying a universal property. We leave it to the reader to fill in the details.

(2) By the Covering–Covering Theorem 27.4 there exists a covering $f\colon (X, x_0) \to (Y, y_0)$ with $p = q \circ f$. Next note that it follows from our hypotheses and the Covering-π_1-Monomorphism Proposition 14.11 that $p_*\colon \pi_1(X, x_0) \to \Gamma$ and $q_*\colon \pi_1(X, x_0) \to \Gamma$ are isomorphisms. Since $p_* = q_* \circ f_*$ we see that f_* is an isomorphism. It follows from the Covering-Degree Proposition 14.13 that $f\colon X \to Y$ is a covering of degree 1. This implies that f is a homeomorphism. ∎

27.6. **The universal covering.** In the Covering Unique Existence Theorem 27.5 we saw that subgroups of fundamental groups give rise to coverings. An obvious choice of a subgroup is the trivial group and we immediately obtain the following corollary to the Covering Unique Existence Theorem 27.5.

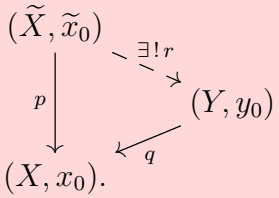
Proposition 27.6. (Universal Covering Proposition) Let X be a topological space that is path-connected and locally simply connected. Furthermore let $x_0 \in X$. The following statements hold:

(1) There exists, up to equivalence, a unique path-connected covering

$$p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$$

such that $\widetilde{X}$ is simply connected.

(2) Let $q\colon (Y, y_0) \to (X, x_0)$ be another path-connected covering. There exists a unique covering $r\colon (\widetilde{X}, \widetilde{x}_0) \to (Y, y_0)$ such that the following diagram commutes:



Proof. Let X be a topological space that is path-connected and locally simply connected. Furthermore let $x_0 \in X$.

(1) Note that if we apply the Covering Unique Existence Theorem 27.5 to the trivial subgroup of $\pi_1(X, x_0)$ and we obtain, up to equivalence, a unique path-connected covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ is the trivial group. It follows from the Covering-π_1-Monomorphism Proposition 14.11 that $\widetilde{X}$ is simply connected.

(2) The second statement is an immediate consequence of the Covering–Covering Theorem 27.4. ■

Definition. Let X be a topological space that is path-connected, locally simply connected[225] and let $x_0 \in X$. We refer to any covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $\widetilde{X}$ is simply connected as a universal covering[226] of (X, x_0).

Example.

(1) For any $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & (S^1)^n \\ (t_1, \ldots, t_n) & \mapsto & \big(\exp(2\pi \mathrm{i} t_1), \ldots, \exp(2\pi \mathrm{i} t_n)\big) \end{array}$$

is a universal covering of the n-dimensional torus $(S^1)^n$.

(2) The projection $p\colon S^3 \to S^3/\mathbb{Z}_p = L(p, q)$ is a universal covering of the lens space $L(p, q)$.

Convention. In practice one often suppresses the actual map p from the language. For example, often one says that "$\mathbb{R}^n$ is a universal covering of $(S^1)^n$".

Example.

(1) Let $n \geq 2$. It follows from the Projective Space-Sphere Quotient Lemma 6.9 together with the Sphere-π_1-Proposition 11.14 that the universal covering of $\mathbb{R}\mathrm{P}^n$ is given by S^n.

(2) It follows from the Projective Space-Avatar Lemma 6.11 that a universal covering of $\mathrm{SO}(3)$ is given by $\mathrm{SU}(2)$.

(3) By the Klein Bottle-as-Quotient Lemma 14.1 we know that a universal covering of the Klein bottle is given by $\mathbb{R}^2$.

---

[225] Recall that most "reasonable" topological spaces (e.g. topological manifolds) are locally simply connected.

[226] The name "universal covering" stems from the fact, proved in the Universal Covering Proposition 27.6 (2), that it has a universal property, namely the universal covering of a topological space X "covers" every other path-connected covering of X.

27.7. **Explicit descriptions of the universal covering.** We have seen in the Universal Covering Proposition 27.6 that every "reasonable" topological space X admits a universal covering $p\colon \widetilde{X} \to X$. In the proof we gave abstract construction. In the previous section we also gave an explicit description of the universal covering of some popular topological spaces, namely the n-dimensional torus $(S^1)^n$, the lens spaces, the projective spaces $\mathbb{R}\mathrm{P}^n$ with $n \geq 2$, $\mathrm{SO}(3)$ and the Klein bottle. In this section we will give explicit descriptions of the universal coverings of more complicated examples, namely of $X = S^1 \vee S^2$ and of $X = S^1 \vee S^1$.

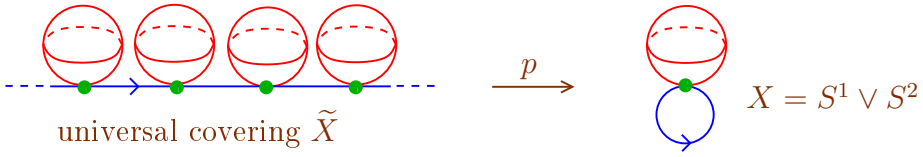
Example. We consider $X = S^1 \vee S^2$ where we identify $\mathrm{i} \in S^1$ with $(0,0,-1) \in S^2$. We also consider the topological space

$$\widetilde{X} \;=\; \big(\mathbb{R} \cup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n,(0,0,-1)), \quad n \in \mathbb{Z}.$$

It is pretty straightforward to see that the map

$$\begin{array}{rcl} p\colon \big(\mathbb{R} \cup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n,(0,0,-1)) & \to & S^1 \vee S^2 \\ [P] & \mapsto & \begin{cases} \exp(\mathrm{i}(2\pi P + \frac{\pi}{2})), & \text{if } P \in \mathbb{R} \\ Q, & \text{if } P = (n,Q) \in \mathbb{Z} \times S^2 \end{cases} \end{array}$$

is well-defined and in fact a covering. On page 272 we showed that $\widetilde{X}$ is simply connected. Therefore $p\colon \widetilde{X} \to X$ is a universal covering of X.



Now we turn to the universal covering of the wedge $X = S^1 \vee S^1$. By the discussion on page 221 we know that X is homeomorphic to the topological realization of the unique abstract graph G that is given by the single vertex $*$ and the edge set $\{x,y\}$.[227] It is convenient to work in the following in the setup of abstract graphs.

Convention. Let V be a set and let $E \subset V \times V$ be a subset. The pair (V,E) defines an abstract graph (V,E,i,t) by setting $i(g,h) = g$ and $t(g,h) = h$. We use this observation to view such (V,E) as an abstract graph.

Proposition 27.7. (Wedge-of-Circles–Universal Covering Proposition) We consider the abstract graph $G = (\{*\},\{x,y\})$. We also consider the abstract graph

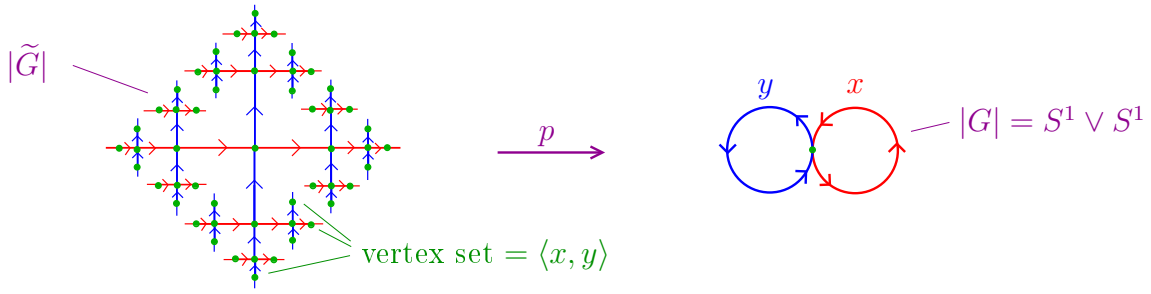
$$\widetilde{G} \;=\; (\underbrace{\widetilde{V} = \text{ free group } \langle x,y\rangle}_{\text{vertex set}}, \underbrace{\widetilde{E} = \{(g,h) \in \widetilde{V} \times \widetilde{V} \,|\, h = g \cdot x \text{ or } h = g \cdot y\}}_{\text{edge set}}).$$

The maps

$$\begin{array}{rcl} \text{vertex set of } \widetilde{G} & \to & \text{vertex set of } G \\ g & \mapsto & * \end{array} \quad \text{and} \quad \begin{array}{rcl} \text{edge set of } \widetilde{G} & \to & \text{edge set of } G \\ (g,h) & \mapsto & g^{-1} \cdot h \end{array}$$

[227] Since we have a unique vertex we do not need to specify the maps from the edge set to the vertex set.

define a morphism of graphs in the sense of the definition on page 125. The corresponding map $p\colon |\widetilde{G}| \to |G|$ is a universal covering of $|G| \cong S^1 \vee S^1$.



Proof. We leave it to the reader to verify that the map $p\colon |\widetilde{G}| \to |G|$ is a covering. It remains to show that $|\widetilde{G}|$ is simply connected. This requires some preparations.

Given $g \in \widetilde{V} = \langle x, y \rangle$ we define its length to be

$$l(g) \;:=\; \min\{k \in \mathbb{N}_0 \,|\, g = a_1 \cdot \ldots \cdot a_k \text{ for some } a_1, \ldots, a_k \in \{x^{\pm 1}, y^{\pm 1}\}\}.$$

Given $k \in \mathbb{N}_0$ we consider the subgraph $\widetilde{G}_k = (\widetilde{V}_k, \widetilde{E}_k)$ of $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ that is given by $\widetilde{V}_k := \{g \in \widetilde{V} = \langle x, y \rangle \,|\, l(g) \leq k\}$ and $\widetilde{E}_k := \widetilde{E} \cap \mathcal{P}(\widetilde{V}_k)$.

Claim. For each $k \in \mathbb{N}_0$ we have $\chi(\widetilde{G}_k) = 1$.[228]

Proof. We prove the claim by induction on $k \in \mathbb{N}_0$. First note that $\widetilde{G}_0$ has a single vertex and no edge. Thus we see that $\chi(\widetilde{G}_0) = 1$. Now suppose that we know that $\chi(\widetilde{G}_k) = 1$. We make the following two observations:

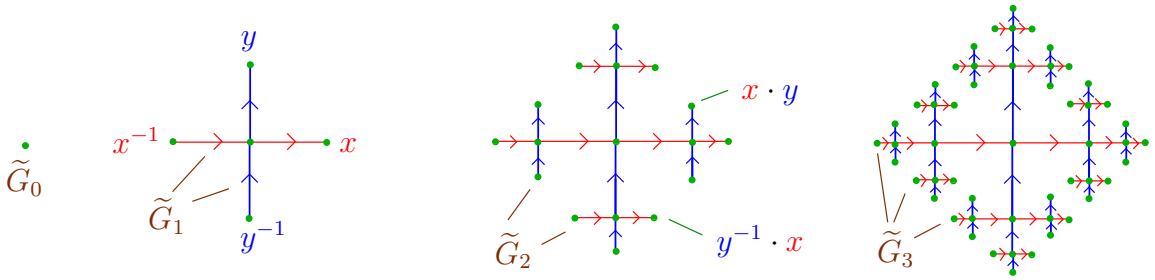
(1) It follows easily from the definition of the free product $\langle x, y \rangle = \langle x \rangle * \langle y \rangle$ as a set of reduced words, see page 238, that each $g \in \widetilde{V}_{k+1} \setminus \widetilde{V}_k$ is uniquely of the form $a_1 \cdot \ldots \cdot a_{k+1}$ with $a_1, \ldots, a_{k+1} \in \{x^{\pm 1}, y^{\pm 1}\}$ and with $g := a_1 \cdot \ldots \cdot a_k \in \widetilde{V}_k$. Note that *precisely one* of $(g, g \cdot a_{k+1})$ or $(g \cdot a_{k+1}, g)$ lies in $\widetilde{E}_{k+1}$.

(2) Let $(g, h) \in \widetilde{E}_{k+1}$. It follows again easily from the definition of $\langle x, y \rangle = \langle x \rangle * \langle y \rangle$ as a set of reduced words that precisely one of g and h lies in $\widetilde{V}_k$ and $\widetilde{V}_{k+1}$.

Thus we see that

$$\chi(\widetilde{G}_{k+1}) \;=\; \chi(\widetilde{V}_k) + \underbrace{\#(\widetilde{V}_{k+1} \setminus \widetilde{V}_k) - \#(\widetilde{E}_{k+1} \setminus \widetilde{E}_k)}_{\substack{= 0, \text{ since by the above two observations} \\ \text{we have a bijection between the} \\ \text{sets } \widetilde{V}_{k+1} \setminus \widetilde{V}_k \text{ and } \widetilde{E}_{k+1} \setminus \widetilde{E}_k}} \;=\; \chi(\widetilde{G}_k) \;=\; 1.$$

⊞



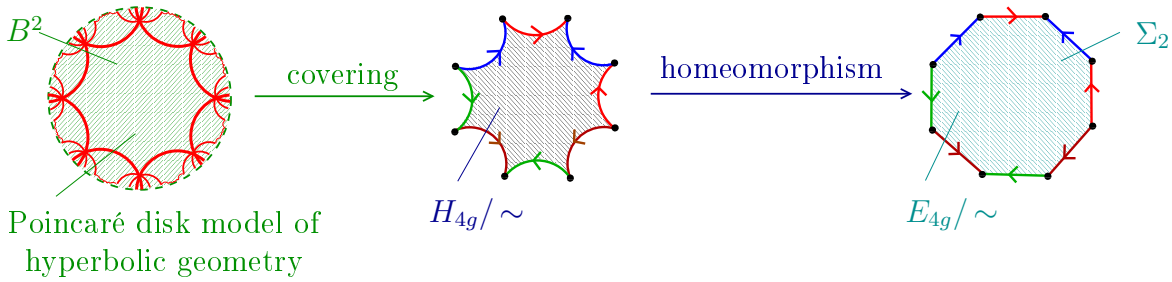
[228] Here χ denotes the Euler characteristic, i.e. the number of vertices minus number of edges.

It follows from the claim together with the Tree Characterization Proposition 7.7 that each $\widetilde{G}_k$ is a tree. The Finite Tree–Contractible Proposition 17.4 together with the Homotopy-π_1-Proposition 13.4 now implies that each $|\widetilde{G}_k|$ is simply connected. It follows easily from the Exhaustion–Simply Connected Proposition 20.10 (or alternatively the Exhaustion-π_1-Proposition 26.2) that $|\widetilde{G}| = \bigcup_{k\in\mathbb{N}} |\widetilde{G}_k|$ itself is simply connected. ■

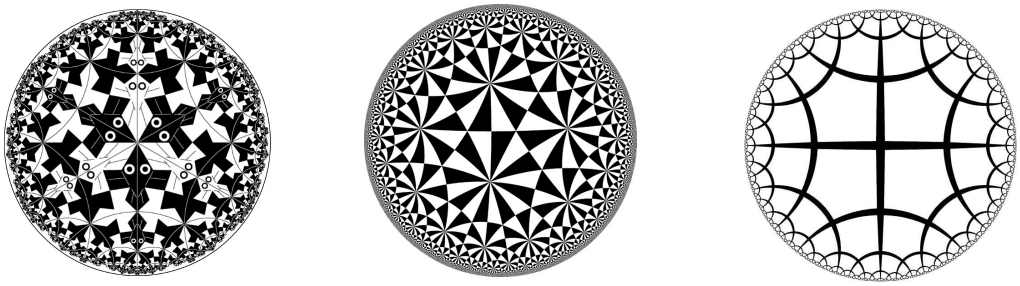
The surfaces of genus $g \geq 2$ are arguably one of the most interesting types of topological manifolds. Evidently we also would like to understand their universal coverings.

Theorem 27.8. (Hadamard's Theorem) Given any $g \geq 2$ there exists a covering map $B^2 \to \Sigma_g = E_{4g}/\sim$.

Proof. The key idea of the proof is to replace the "Euclidean" octagon E_{4g} by a "hyperbolic" octagon H_{4g} and to equip the open disk B^2 with a "hyperbolic structure" via the Poincaré disk model of hyperbolic geometry. The long and difficult details are worked out in [Fri23]. ■



Example. We do not have the time and space to explain the Poincaré disk model of hyperbolic geometry. In the figure below we see three different decompositions of $\mathbb{D}$ into subsets (the pictures are of course due to M.C. Escher). In each of the three cases these subsets have the same hyperbolic size, even though they differ in the usual Euclidean sense.



28. Higher homotopy groups: Definition and basics

28.1. **Definition of the higher homotopy groups.** We start out with the following variation on earlier definitions.

Definition. Let W and X be topological spaces and let $A \subset W$ and $B \subset X$ be subsets.

(1) A continuous map $(W, A) \to (X, B)$ of pairs of topological spaces is a continuous map $f\colon W \to X$ with $f(A) \subset B$.

(2) Let $f, g\colon (W, A) \to (X, B)$ be two continuous maps of pairs of topological spaces.

(a) A homotopy between f and g is a continuous map

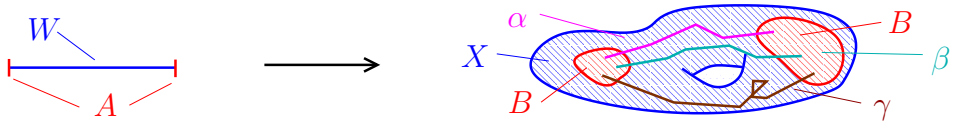
$$H\colon W \times [0,1] \to X,$$

such that $H_0 = f$ and $H_1 = g$ and such that each H_t is a map $(W, A) \to (X, B)$.

(b) If there exists a homotopy, then we say f and g are homotopic and we write $f \simeq g$.

(3) We set

$$[(W, A), (X, B)] := \{\text{continuous maps } (W, A) \to (X, B)\}/\simeq .$$



$\alpha, \beta\colon (W, A) \to (X, B)$ are homotopic but $\alpha, \gamma\colon (W, A) \to (X, B)$ are not homotopic

Notation. Throughout this chapter we write $I = [0, 1]$ for the unit interval. Given $n \in \mathbb{N}$ we denote by $I^n \subset \mathbb{R}^n$ the cube in $\mathbb{R}^n$ and we denote by ∂I^n its boundary as a subspace of $\mathbb{R}^n$.

Let (X, x_0) be a pointed topological space. With the above definitions we can write[229]

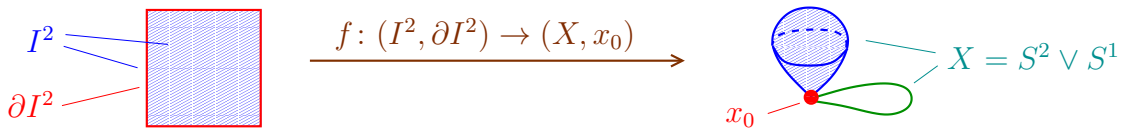
$$\pi_1(X, x_0) \;=\; [(I, \partial I), (X, x_0)].$$

This definition can be easily generalized to higher dimensions:

Definition. Given a pointed topological space (X, x_0) and given $n \in \mathbb{N}_0$ we define

$$\begin{aligned} \pi_n(X, x_0) \;&:=\; [(I^n, \partial I^n), (X, x_0)] \\ &=\; \text{homotopy classes of continuous maps } (I^n, \partial I^n) \to (X, x_0). \end{aligned}$$

For $n \geq 1$ we refer to $\pi_n(X, x_0)$ as the n-th homotopy group. We refer to any $\pi_n(X, x_0)$ with $n \geq 2$ as a higher homotopy group.



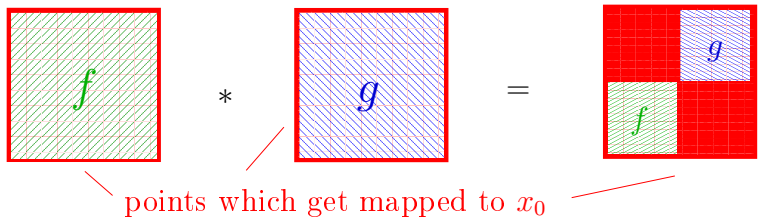
Note that $\pi_n(X, x_0)$ is a priori only a set. So to justify the name "homotopy group" we need to define a group structure on $\pi_n(X, x_0)$ which for $n = 1$ agrees with our previous definition of the group structure. Here is what we do:

[229] For a point $x_0 \in X$ we write (X, x_0) instead of the more correct $(X, \{x_0\})$.

Definition. Let (X, x_0) be a pointed topological space and let $n \geq 1$. Furthermore let $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ be two continuous maps. We define

$$\begin{array}{rcl} f * g\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (x_1, \ldots, x_n) & \mapsto & \begin{cases} f(2x_1, \ldots, 2x_n), & \text{if } (x_1, \ldots, x_n) \in [0, \frac{1}{2}]^n, \\ g(2x_1 - 1, \ldots, 2x_n - 1), & \text{if } (x_1, \ldots, x_n) \in [\frac{1}{2}, 1]^n, \\ x_0, & \text{otherwise.} \end{cases} \end{array}$$

Using the Pasting Proposition 2.4 (2) one can easily show that this map is indeed continuous.



It is straightforward to see that the homotopy class of $f * g$ only depends on the homotopy classes of f and g. This leads us to the following definition:

Definition. Let (X, x_0) be a pointed topological space and let $n \geq 1$. We define

$$\begin{array}{rcl} \pi_n(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([f], [g]) & \mapsto & [f] \cdot [g] := [f * g]. \end{array}$$

Note that for $n = 1$ this is precisely the same definition of the product structure that we already introduced in Chapter 11.2.

Now we can formulate the following proposition:

Proposition 28.1. (π_n-Group Proposition) Let (X, x_0) be a pointed topological space and let $n \geq 1$. The following two statements hold:

(1) The map

$$\begin{array}{rcl} \pi_n(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([f], [g]) & \mapsto & [f] \cdot [g] := [f * g] \end{array}$$

defines a group structure on $\pi_n(X, x_0)$.

(2) Given a continuous map $f\colon (I^n, \partial I^n) \to (X, x_0)$ the inverse of $[f] \in \pi_n(X, x_0)$ is represented by the map

$$\begin{array}{rcl} \overline{f}\colon I^n & \to & X \\ (x_1, x_2, \ldots, x_n) & \mapsto & f(1 - x_1, x_2, \ldots, x_n). \end{array}$$

Example. There is one non-empty topological space for which we can immediately compute the higher homotopy groups, namely the topological space consisting of a single point P. In this case the only map $I^n \to \{P\}$ is the constant map, hence we see that $\pi_n(\{P\}, P) = 0$ for any $n \geq 1$.

The proof of the π_n-Group Proposition 28.1 requires the remainder of this section. As we will see, the proof of the π_n-Group Proposition 28.1 is quite similar to the proof of the Path-Homotopy Product Proposition 11.4 and the proof of the π_1–Group Proposition 11.5.

The technicalities are evidently more difficult than before. We will provide (most) of the gory details. But most readers will be best served by just glimpsing at the pictures in the proof and by ignoring the technical details.

After this preamble let us turn to the actual proof of the π_n-Group Proposition 28.1. To deal with the technicalities we need a few more definitions and one lemma.

Definition.

(1) A hyperrectangle is any subset of $\mathbb{R}^n$ of the form $[c_1, d_1] \times \cdots \times [c_n, d_n]$.

(2) We say a hyperrectangle $[c_1, d_1] \times \cdots \times [c_n, d_n]$ is non-degenerate if for all $i = 1, \ldots, n$ we have $d_i > c_i$.

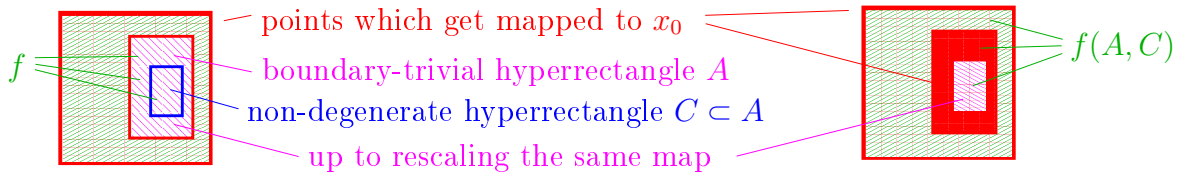
Let X be a topological space, let $x_0 \in X$ and let $f\colon I^n \to X$ be a continuous map.

(3) We say a hyperrectangle $A \subset I^n$ is boundary-trivial if f sends all points on the boundary ∂A of A to x_0.

(4) Let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset I^n$ be a boundary-trivial hyperrectangle. Furthermore let $C = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be a non-degenerate hyperrectangle that is contained in A. We define

$$\begin{array}{rcl} f(A,C)\colon I^n & \to & X \\ x = (x_1, \ldots, x_n) & \mapsto & \begin{cases} f(x), & \text{if } x \notin A, \\ x_0, & \text{if } x \in A \setminus C, \\ f\left(a_1 + (x_1 - c_1) \cdot \frac{b_1 - a_1}{d_1 - c_1}, \cdots, a_n + (x_n - c_n) \cdot \frac{b_n - a_n}{d_n - c_n}\right) & \text{if } x \in C. \end{cases} \end{array}$$

Using the Pasting Proposition 2.4 (2) one can verify that this map is continuous.



Lemma 28.2. (Shrinking Lemma) Let (X, x_0) be a pointed topological space. Furthermore let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a continuous map. Furthermore let $A \subset I^n$ be a boundary-trivial hyperrectangle and let C be a non-degenerate hyperrectangle that is contained in A. Then the maps f and $f(A, C)$ from $(I^n, \partial I^n)$ to (X, x_0) are homotopic.

Proof. Let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a continuous map, let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset I^n$ be a boundary-trivial hyperrectangle and let $C = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be a non-degenerate hyperrectangle that is contained in A. For $t \in [0, 1]$ we write

$$C_t = [a_1 \cdot (1-t) + c_1 \cdot t, b_1(1-t) + d_1 \cdot t] \times \cdots \times [a_n \cdot (1-t) + c_n \cdot t, b_n \cdot (1-t) + d_n \cdot t].$$

Note that $C_0 = A$ and $C_1 = C$. Using the Pasting Proposition 2.4 (2) one can show fairly easily that the map

$$\begin{array}{rcl} F\colon I^n \times [0,1] & \to & X \\ (x, t) & \mapsto & f(A, C_t)(x) \end{array}$$

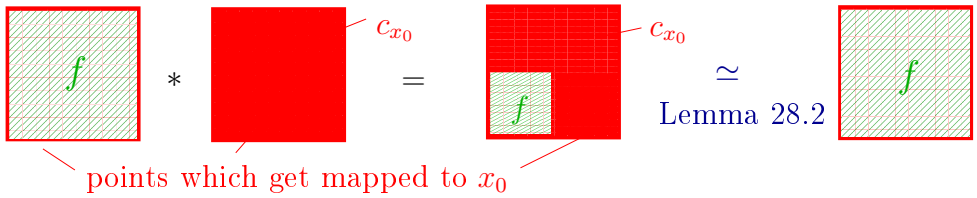
is continuous. By construction we have $F_0 = F$ and $F_1 = f(A, C)$, thus F is a homotopy between f and $f(A, C)$. ∎

Proof of the π_n-Group Proposition 28.1.

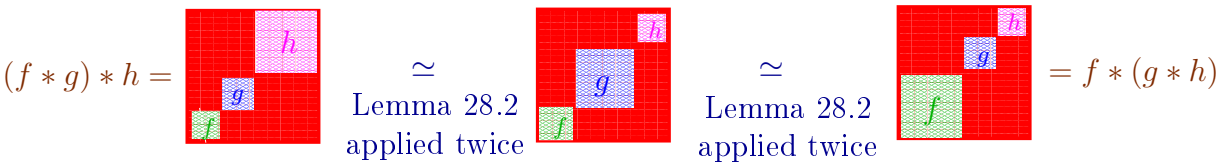
- The neutral element of the product structure is given by the equivalence class of the constant map

$$\begin{array}{rcl} c_{x_0}\colon I^n & \to & X \\ x & \mapsto & x_0. \end{array}$$

Indeed, it follows immediately from the Shrinking Lemma 28.2, applied to $A = [0,1]^n$ and $C = [0, \frac{1}{2}]^n$, that given any continuous map $f\colon (I^n, \partial I^n) \to (X, x_0)$ we have $f * c_{x_0} \simeq f$. Basically the same argument shows also that $c_{x_0} * f \simeq f$.



- We consider three maps $f, g, h\colon (I^n, \partial I^n) \to (X, x_0)$. By applying the Shrinking Lemma 28.2 several times one can easily show that $(f * g) * h \simeq f * (g * h)$, see e.g. the figure below. This proves that the multiplication on $\pi_n(X, x_0)$ is associative.



- Now we turn to the most interesting question, namely finding the inverse of a given element in $\pi_n(X, x_0)$. Thus suppose we are given a continuous map $f\colon (I^n, \partial I^n) \to (X, x_0)$. We consider the map

$$\begin{array}{rcl} \overline{f}\colon I^n & \to & X \\ (x_1, \dots, x_n) & \mapsto & f(1 - x_1, x_2, \dots, x_n). \end{array}$$

We claim that $[f] \cdot [\overline{f}] = [f * \overline{f}] = [c_{x_0}]$. Thus we need to show that $f * \overline{f}$ is homotopic to c_{x_0}. It follows from the Shrinking Lemma 28.2 that $f * \overline{f}$ is homotopic to the map

$$\begin{array}{rcl} h\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (x_1, \dots, x_n) & \mapsto & \begin{cases} f(2x_1, x_2, \dots, x_n), & \text{if } x_1 \in [0, \frac{1}{2}], \\ \underbrace{\overline{f}(2x_1 - 1, x_2, \dots, x_n)}_{= f(2 - 2x_1, x_2, \dots, x_n)}, & \text{if } x_1 \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

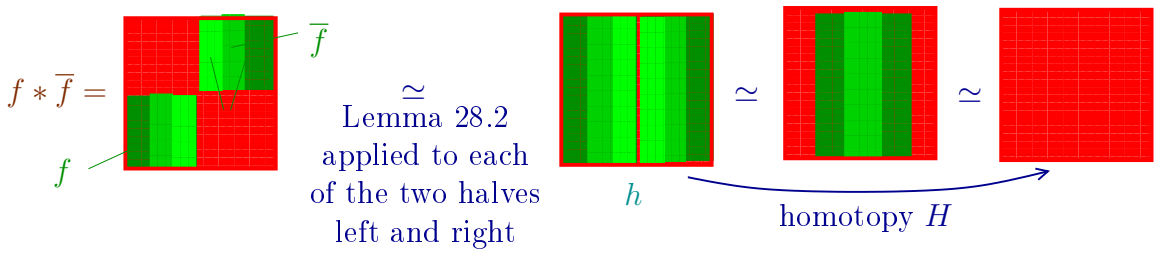
Thus it suffices to show that h is homotopic to c_{x_0}. It is in fact straightforward to see that the map[230] [231]

$$\begin{array}{rcl} H\colon I^n \times [0,1] & \to & X \\ (x_1, \dots, x_n, t) & \mapsto & \begin{cases} f(2(x_1 - \frac{t}{2}), x_2, \dots, x_n), & \text{if } x_1 \in [\frac{t}{2}, \frac{1}{2}], \\ f(2 - 2(x_1 + \frac{t}{2}), x_2, \dots, x_n), & \text{if } x_1 \in [\frac{1}{2}, 1 - \frac{t}{2}], \\ x_0, & \text{otherwise} \end{cases} \end{array}$$

is a homotopy from h to c_{x_0}. We refer to the figure below for an illustration of the argument. Basically the same argument also shows that $[\overline{f}] \cdot [f] = [\overline{f} * f] = [c_{x_0}]$. ■

[230] It follows from the Pasting Proposition 2.4 (2) that this map is continuous.

[231] Even for $n = 1$ this homotopy is different from the homotopy used in the Path-Homotopy Product Proposition 11.4 (3). Why is that?



28.2. **Alternative descriptions of homotopy groups.** In Section 11.6 we showed that there are two different but equivalent ways to define fundamental groups, namely we can either consider the set $[(I, \partial I), (X, x_0)]$ or we can consider the set $[(S^1, 1), (X, x_0)]$. Similarly, in this section we will see that there are several alternative points of view regarding elements in homotopy groups.

Identification. Let $n \in \mathbb{N}$.

(1) We pick and fix a homeomorphism $\Phi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$.
(2) We equip S^n throughout with the base point $* = (0, \ldots, 0, 1) \in S^n$.[232]
(3) We denote by $\Psi\colon (\overline{B}^n/S^{n-1}, [S^{n-1}]) \to (S^n, *)$ the explicit homeomorphism from the Ball Quotient-Sphere Lemma 6.5.
(4) Let X be a topological space and let $x_0 \in X$. It follows easily from the Topological-Quotient Proposition 6.1 (1) together with the Homotopy Quotient Theorem 8.5 that the above homeomorphisms Φ and Ψ define bijections

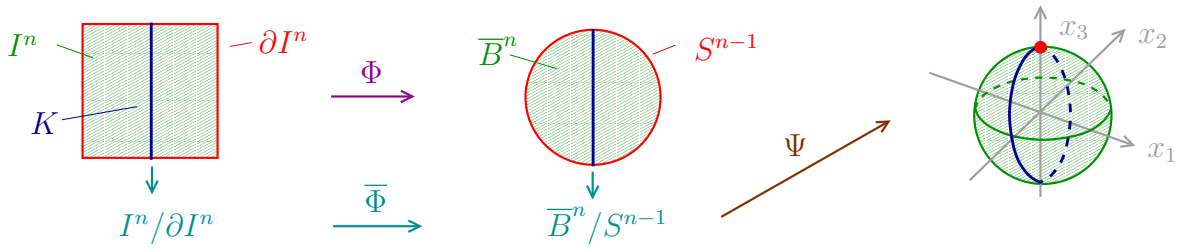
$$\begin{aligned} & \text{(homotopy classes) of continuous maps } (I^n, \partial I^n) \to (X, x_0) \\ \iff\ & \text{(homotopy classes) of continuous maps } (\overline{B}^n, S^{n-1}) \to (X, x_0) \\ \iff\ & \text{(homotopy classes) of continuous maps } (\overline{B}^n/S^{n-1}, [S^{n-1}]) \to (X, x_0) \\ \iff\ & \text{(homotopy classes) of continuous maps } (S^n, *) \to (X, x_0). \end{aligned}$$

The four equivalences combined say that we can view elements in $\pi_n(X, x_0)$ as homotopy classes of continuous maps $(S^n, *) \to (X, x_0)$. This point of view is perhaps closest to our visual approach to thinking about topological spaces. But as we have seen above, for technical purposes it is often much easier to work with the maps $(I^n, \partial I^n) \to (X, x_0)$.

Remark. It is visually most natural to work with the last point of view, namely to think of elements in $\pi_n(X, x_0)$ as homotopy classes of continuous maps $(S^n, *) \to (X, x_0)$. Unfortunately with this definition it is a trifle less clear how the group structure on $\pi_n(X, x_0)$ is defined. To visualize the group structure we need to work a little:

(1) Note that the homeomorphism $\Phi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ descends to a homeomorphism $\overline{\Phi}\colon I/\partial I^n \to \overline{B}^n/S^{n-1}$.
(2) We consider the homeomorphism $\Theta := \overline{\Phi}^{-1} \circ \Psi^{-1}\colon S^n \to I^n/\partial I^n$. Note that it identifies the North Pole $(0, \ldots, 0, 1)$ with the point $[\partial I^n]$.
(3) We write $K := \{(x_1, \ldots, x_n) \in I^n \,|\, x_1 = \frac{1}{2}\}$.

[232] Here is a slight nuisance: For S^1, viewed as a subspace of $\mathbb{C}$, it seems most reasonable to pick the base point $1 = (1, 0) \in \mathbb{C} = \mathbb{R}^2$. On the other hand for $S^n \subset \mathbb{R}^{n+1}$ it seems most reasonable to pick the North Pole $(0, \ldots, 0, 1)$ as a base point. Of course mathematically there is no difference and we are fully confident that the reader can handle the fact that for S^1 we use at times two different base points.



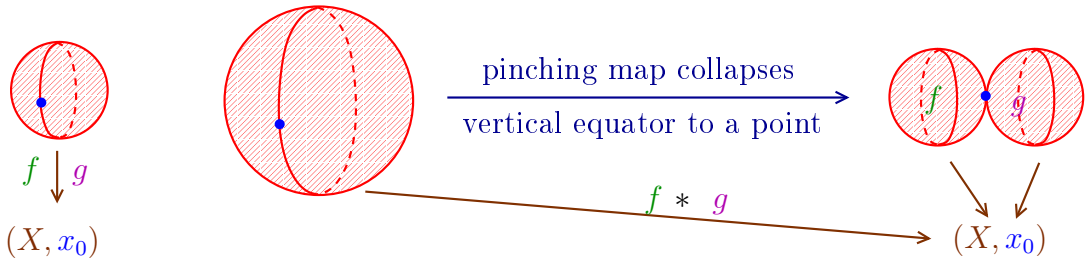
(4) We denote by $S^n_1 \vee S^n_2$ the wedge of two copies of S^n along the point $* \in S^n$.
(5) We refer to the map[233]

$$S^n \xrightarrow{\Theta} I^n/\partial I^n \to I^n/(\partial I^n \cup K) \to S^n_1 \vee S^n_2$$
$$[x_1, \ldots, x_n] \mapsto \begin{cases} \Theta^{-1}([2x_1, x_2, \ldots, x_n]) \in S^n_1, & \text{if } x_1 \in [0, \frac{1}{2}], \\ \Theta^{-1}([2x_1 - 1, x_2, \ldots, x_n]) \in S^n_2, & \text{if } x_1 \in [\frac{1}{2}, 1] \end{cases}$$

as the *pinching map*.
(6) It follows basically immediately from the definitions that the group structure on $\pi_n(X, x_0)$ is given by

$$\pi_n(X, x_0) \times \pi_n(X, x_0) \to \pi_n(X, x_0)$$
$$([f\colon S^n \to X], [g\colon S^n \to X]) \mapsto \left[S^n \to S^n_1 \vee S^n_2 \xrightarrow{f \vee g} X\right].$$

In the figure below we sketch the pinching map and we sketch the corresponding definition of the group structure on $\pi_n(X, x_0)$.[234]



28.3. **Properties and basic calculations of higher homotopy groups.** Now we want to understand some of the basic properties of higher homotopy groups. The higher homotopy groups $\pi_n(X, x_0)$ seem even more complicated than the fundamental group. Surprisingly at least their algebraic structure is much simpler namely, with a great sight of relief, we will now see that higher homotopy groups are abelian.

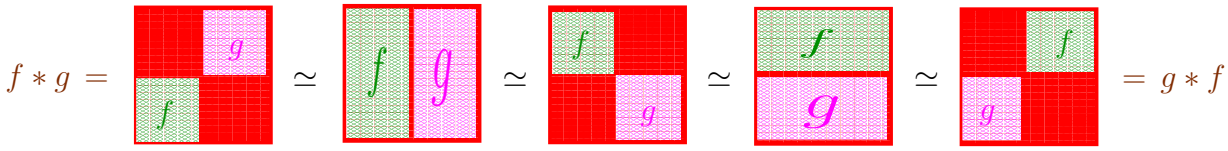
Proposition 28.3. (π_n-Abelian Proposition) For any pointed topological space (X, x_0) and any $n \geq 2$ the homotopy group $\pi_n(X, x_0)$ is abelian.

Proof. Suppose we are given two continuous maps $f, g\colon (I^n, \partial I^n) \to (X, x_0)$. We have to show that $f * g$ is homotopic to $g * f$. The homotopy is described in the figure below for $n = 2$. It is straightforward to write down an explicit homotopy between $f * g$ and $g * f$

[233] It follows easily from the Pasting Proposition 2.4 (2′), the Topological-Quotient Proposition 6.1 (1b) and the Wedge Proposition 17.1 (1a) that this map is continuous.
[234] The meticulous reader will notice that it is perhaps not completely clear how the maps f and g are defined on the two smaller spheres.

by composing four instances of the homotopy provided by the Shrinking Lemma 28.2. We leave the details to the reader. ■

all homotopies are provided by Lemma 28.2



The following proposition can be viewed as a generalization of the key statements of the Change-of-Base Point Proposition 11.6.

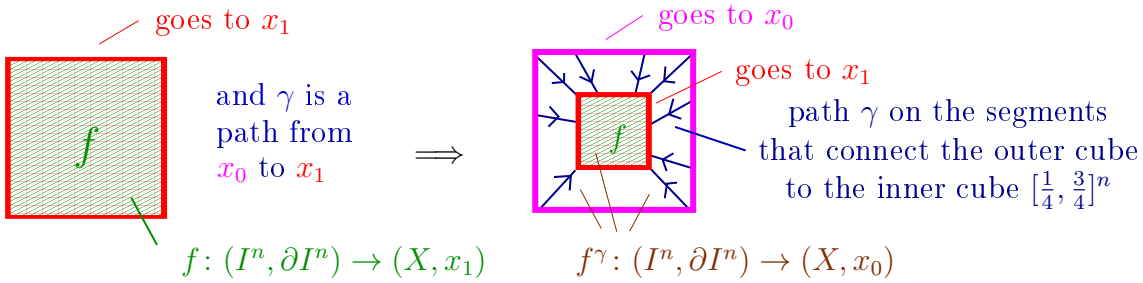
Proposition 28.4. (Change-of-Base Point Proposition) Let X be a topological space, let x_0 and x_1 be two points in X and let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1. Given a continuous map $f\colon (I^n, \partial I^n) \to (X, x_1)$ we consider the map[235]

$$\begin{array}{rcl} f^\gamma\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (t_1, \dots, t_n) & \mapsto & \begin{cases} f\big(2t_1 - \frac{1}{2}, \dots, 2t_n - \frac{1}{2}\big), & \text{if } (t_1, \dots, t_n) \in \big[\frac{1}{4}, \frac{3}{4}\big]^n, \\ \gamma(2-4t), & \text{if } t := \max\{|t_1 - \frac{1}{2}|, \dots, |t_n - \frac{1}{2}|\} \in [\frac{1}{4}, \frac{1}{2}]. \end{cases} \end{array}$$

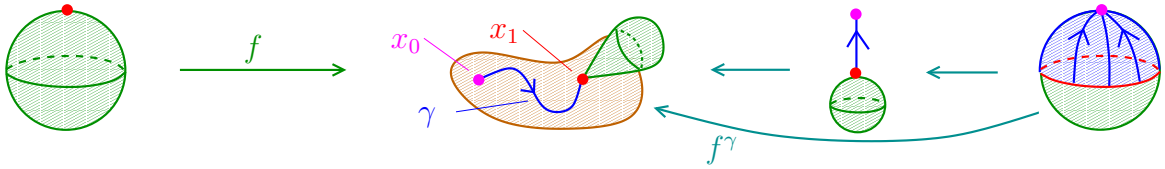
The map

$$\begin{array}{rcl} \gamma_*\colon \pi_n(X, x_1) & \to & \pi_n(X, x_0) \\ [f] & \mapsto & [f^\gamma] \end{array}$$

is well-defined and it is a group isomorphism.



Remark. In the figure below we show the induced map $\gamma_*\colon \pi_n(X, x_1) \to \pi_n(X, x_0)$ given by $f \mapsto f^\gamma$, if in the definition of homotopy groups we work with $(S^n, *)$ instead of $(I^n, \partial I^n)$.



Proof of the Change-of-Base Point Proposition 28.4. Let X be a topological space, let x_0 and x_1 be two points in X and let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1. We need to prove the following claim.

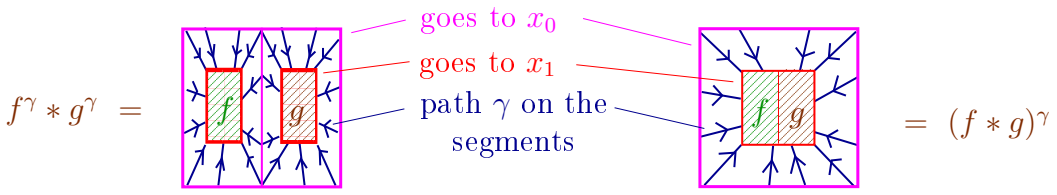
Claim.

[235] Using the Pasting Proposition 2.4 (2) one can easily verify that the map is continuous.

(1) The map $\Phi\colon \pi_n(X, x_1) \to \pi_n(X, x_0)$ given by $[f] \mapsto [f^\gamma]$ is well-defined.
(2) The map Φ is a bijection.
(3) The map Φ is a group homomorphism.

Proof.

(1) This statement can be verified easily by hand.
(2) As usual we denote by $\overline{\gamma}$ the path given by $\overline{\gamma}(t) = \gamma(1-t)$. It is straightforward to verify that the map $\Psi\colon \pi_n(X, x_0) \to \pi_n(X, x_1)$ given by $[g] \mapsto [f^{\overline{\gamma}}]$ is an inverse to Φ.
(3) Let $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ be continuous maps. We need to show that $f^\gamma * g^\gamma$ is homotopic to $(f * g)^\gamma$.



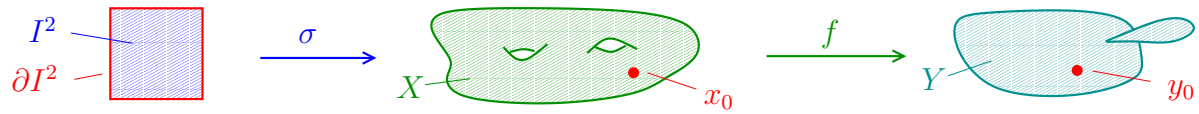
It should be clear that verifying this statement is elementary, in the sense that one can write down an explicit homotopy. On the other hand it should also be clear that writing down an explicit homotopy is supremely painful and highly non-instructive. Thus we will skip this step and leave it to the eager reader to fill in the details. ■

28.4. **Functoriality of homotopy groups.** Next we will see that higher homotopy groups are covariantly functorial.

Definition. Let $f\colon X \to Y$ be a continuous map between two topological spaces, let $n \in \mathbb{N}$ and let $x_0 \in X$. We write $y_0 = f(x_0)$. One can easily verify that the map

$$\begin{array}{rcl} f_*\colon \pi_n(X, x_0) & \to & \pi_n(Y, y_0) \\ [\sigma\colon (I^n, \partial I^n) \to (X, x_0)] & \mapsto & [f \circ \sigma\colon (I^n, \partial I^n) \xrightarrow{\sigma} (X, x_0) \xrightarrow{f} (Y, y_0)] \end{array}$$

is well-defined. As usual we say that f_* is the map induced by f.



The next proposition is a straightforward generalization of the π_1-Functor Proposition 13.1.

Proposition 28.5. (π_n-Functor Proposition) Let $n \geq 1$. Recall that we denote by $\mathcal{PTop}$ the category of pointed topological spaces and that we denote by $\mathcal{G}roup$ be the category of groups. The map[236]

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PTop}) & \to & \mathrm{Ob}(\mathcal{G}roup) \\ (X, x_0) & \mapsto & \pi_n(X, x_0) \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}(\mathcal{PTop}) & \to & \mathrm{Mor}(\mathcal{G}roup) \\ (f\colon (X, x_0) \to (Y, y_0)) & \mapsto & (f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)) \end{array}$$

form a covariant functor.

[236]It follows from the π_n-Abelian Proposition 28.3 that for $n \geq 2$ this is also a covariant functor to the category $\mathcal{A}b\mathcal{G}roup$ of abelian groups.

Proof. First note that it follows immediately from the definitions that given a continuous map $f\colon (X,x_0)\to(Y,y_0)$ the induced map $f_*\colon \pi_n(X,x_0)\to\pi_n(Y,y_0)$ is actually a group homomorphism. Furthermore note that it follows immediately from the definitions that

$$\begin{array}{rcll} \mathrm{id}_* & = & \mathrm{id} & \text{for all pointed pairs } (X,x_0) \\ (g\circ f)_* & = & g_*\circ f_* & \text{for all continuous maps } f\colon (X,x_0)\to(Y,y_0) \text{ and } g\colon (Y,y_0)\to(Z,z_0). \end{array}$$

But this is exactly what we needed to show. ■

We continue with the following proposition, which generalizes the essentials of the Homotopy-π_1-Proposition 13.4.

Proposition 28.6. (Homotopy-π_n-Proposition) Let $n\geq 1$.

(1) Let $f,g\colon X\to Y$ be two continuous maps between topological spaces and let $x_0\in X$. If f and g are homotopic, then

$$\underset{\text{is an isomorphism}}{f_*\colon \pi_n(X,x_0)\to\pi_n(Y,f(x_0))} \iff \underset{\text{is an isomorphism.}}{g_*\colon \pi_n(X,x_0)\to\pi_n(Y,g(x_0))}$$

The analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism".

(2) (a) Let $f\colon X\to Y$ be a continuous map between topological spaces and let $x_0\in X$ be a point. If f is a homotopy equivalence, then

$$f_*\colon \pi_n(X,x_0) \;\to\; \pi_n(Y,f(x_0))$$

is an isomorphism.

(b) Let X be a topological space and let $A\subset X$ be a subspace. We denote by $i\colon A\to X$ the inclusion. If A is a deformation retract of X, then for any $a_0\in A$ the induced map

$$i_*\colon \pi_n(A,a_0) \;\to\; \pi_n(X,a_0)$$

is an isomorphism.

(3) If X is a contractible topological space, then for any $x_0\in X$ we have $\pi_n(X,x_0)=0$.

Proof. The proof of Statement (1) is quite similar to the proof of Statement (1) of the Homotopy-π_1-Proposition 13.4. The only problem is that the notation gets a little more messy. As in the Homotopy-π_1-Proposition 13.4 one can easily deduce Statements (2) and (3) from Statement (1). ■

Example. The topological space $\mathbb{R}^k$ is homotopy equivalent to a point $\{P\}$. Thus it follows from the Homotopy-π_n-Proposition 28.6 (2) and the discussion on page 345 that $\pi_n(\mathbb{R}^k)\cong\pi_n(\{P\})\cong 0$. Exactly the same way, using the discussion on page 149, one can show that all the higher homotopy groups of convex subsets of $\mathbb{R}^k$ vanish, e.g. we have $\pi_n\big(\overline{B}^k\big)=\pi_n(B^k)=0$.

The next proposition is the high-dimensional analogue of the Product-π_1-Proposition 13.7.

Proposition 28.7. (Product-π_n-Proposition) Let $n\in\mathbb{N}$. Let $(X_i,x_i)_{i\in I}$ be a family of pointed topological spaces. For each $j\in I$ we denote by $p_j\colon \prod_{i\in I} X_i\to X_j$ the natural

projection. The induced homomorphism[237]

$$\begin{array}{rccc} \Phi\colon \pi_n\Big(\prod\limits_{i\in I} X_i, (x_i)_{i\in I}\Big) & \to & \prod\limits_{i\in I} \pi_n(X_i, x_i) \\ [f] & \mapsto & (p_{i*}([f]))_{i\in I} \end{array}$$

is an isomorphism.

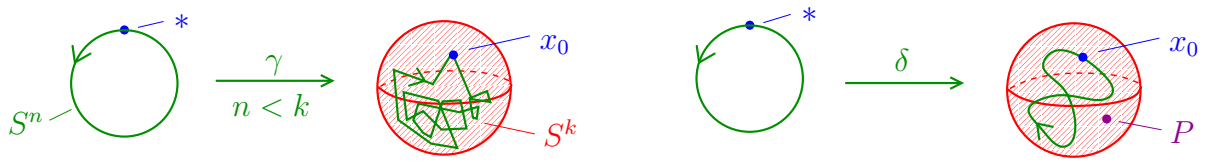
Proof. The proof is basically identical to the proof of the Product-π_1-Proposition 13.7. ■

In the Sphere-π_1-Proposition 11.14 we saw that $\pi_1(S^n) = 0$ for $n \geq 2$. The following proposition generalizes this results to suitable higher dimensions.

Proposition 28.8. (Sphere–Low-π_n-Proposition) Let $k \in \mathbb{N}$. For any $n < k$ we have

$$\pi_n(S^k) \;=\; 0.$$

Proof. Let $k \geq 2$, let $x_0 \in S^k$ and let $\gamma\colon (S^n, *) \to (S^k, x_0)$ be a continuous map. It follows immediately from the Whitney Approximation Theorem [Fri23], applied to the closed subset $\{*\} \subset S^n$, that there exists a homotopy rel $\{*\}$ from $\gamma\colon S^n \to S^k$ to a smooth map $\delta\colon S^n \to S^k$ with $\delta(*) = x_0$. Since δ is smooth and since $k > n$ we see that the image of δ has measure zero, which implies that the smooth map $\delta\colon S^n \to S^k$ is not surjective. We pick $P \in S^n \setminus \delta(S^n)$.



Note that $S^k \setminus \{P\}$ is homeomorphic to $\mathbb{R}^k$, in particular contractible, which implies that $[\delta] = 0 \in \pi_n(S^k \setminus \{P\}, *)$, which implies that $[\delta] = 0 \in \pi_n(S^k, *)$. ■

The following lemma is the obvious extension of the Exhaustion–Simply Connected Proposition 20.10.

Proposition 28.9. (Exhaustion-π_n-Zero-Proposition) Let $n \in \mathbb{N}$, let X be a path-connected topological space and let

$$X_0 \;\subset\; X_1 \;\subset\; X_2 \;\subset\; X_3 \;\subset\; \ldots$$

be a sequence of subspaces of X such the following two conditions are satisfied:

(1) We have $\bigcup\limits_{i\in\mathbb{N}_0} X_i = X$.

(2) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.

(3) For each $i \in \mathbb{N}_0$ the following holds: X_i is path-connected and we have $\pi_n(X_i) = 0$.

Then for each $x_0 \in X$ we have $\pi_n(X, x_0) = 0$.

Proof. Let $x_0 \in X$ and let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a continuous map. By the Compact Image Lemma 2.8 we know that $f([0,1]^n)$ is a compact subset of X. By the Compact Subspace–Exhaustion Lemma 20.11 we know that $f([0,1]^n)$ is contained in a single X_i.

[237] The projections p_i induce homomorphisms on fundamental groups which in turn, by the Product-of-Sets-and-Groups Lemma 4.2, induce the given homomorphism.

But then it follows from Hypothesis (1) that $[f] = 0 \in \pi_n(X_i, x_0)$, which implies that $[f] = 0 \in \pi_n(X, x_0)$. ■

28.5. **Covering spaces and higher homotopy groups.** Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. In the Covering-π_1-Monomorphism Proposition 14.11 we saw that the induced map $p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)$ is a monomorphism. We saw in many examples that the map is in general not an isomorphism. Perhaps somewhat surprisingly, for the homotopy groups π_n with $n \geq 2$ the induced map is in fact always an isomorphism:

Proposition 28.10. (Covering-π_n-Isomorphism Proposition) Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Given any $n \geq 2$ the following two statements hold:

(1) The map $p_*\colon \pi_n(X, x_0) \to \pi_n(B, b_0)$ is an isomorphism.
(2) The map[238] [239]

$$\begin{array}{rcl} \pi_n(B, b_0) & \to & \pi_n(X, x_0) \\ [f\colon (S^n, *) \to (B, b_0)] & \mapsto & [\text{unique lift of } f \text{ to a map } (S^n, *) \to (X, x_0)] \end{array}$$

is well-defined and it is the inverse to p_*.

Before we provide the proof of the Covering-π_n-Isomorphism Proposition 28.10 let us first give a few examples. We will consider a few more examples after the proof of the Covering-π_n-Isomorphism Proposition 28.10.

Examples.

(1) For $n \geq 2$ we have

$$\pi_n((S^1)^k) \quad \cong \quad \pi_n(\mathbb{R}^k/\mathbb{Z}^k) \quad \xleftarrow[\uparrow]{\cong} \quad \pi_n(\mathbb{R}^k) \quad \underset{\uparrow}{=} \quad 0.$$

Covering-π_n-Isomorphism Proposition 28.10 — page 352

In particular we see that all of the higher homotopy groups of S^1 and the torus $S^1 \times S^1$ vanish.

(2) Let Σ be a surface of genus ≥ 2. For any $n \geq 2$ we see that

$$\pi_n(\Sigma) \quad \underset{\uparrow}{\cong} \quad \pi_n(\text{universal covering of } \Sigma) \quad \underset{\uparrow}{\cong} \quad \pi_n(B^2) \quad \underset{\uparrow}{=} \quad 0.$$

Covering-π_n-Isomorphism Proposition 28.10 — Hadamard's Theorem 27.8 — see page 352

Together with the example on page 354 we have now shown that the higher homotopy groups of every surface of genus ≥ 1 vanish.

Proof of the Covering-π_n-Isomorphism Proposition 28.10. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces and let $n \geq 2$. Our goal is to construct an inverse to the map $p_*\colon \pi_n(X, x_0) \to \pi_n(B, b_0)$.

We saw on page 348 that we can think of elements in $\pi_n(B, b_0)$ as homotopy classes of continuous maps $(S^n, *) \to (B, b_0)$. Thus let $f\colon (S^n, *) \to (B, b_0)$ be a continuous map.

[238] Here we take the point of view from page 348 where we saw that we view the elements of $\pi_n(B, b_0)$ as homotopy classes of continuous maps $(S^n, *) \to (B, b_0)$.
[239] We will discuss this map $\pi_n(B, b_0) \to \pi_n(X, x_0)$ in greater detail in the proof.

We are in the following setting[240] [241]

$$\begin{array}{ccc} & & (X,x_0) \\ & & \downarrow p \\ (S^n,*) & \xrightarrow[f]{} & (B,b_0). \end{array} \qquad \begin{array}{c} \text{by the Map Lifting Criterion 27.1} \\ \text{since by } n\geq 2 \text{ we have } \pi_1(S^n)=0 \end{array} \qquad \begin{array}{ccc} & \exists!\,\widetilde{f} \nearrow & (X,x_0) \\ & \nearrow & \downarrow p \\ (S^n,*) & \xrightarrow[f]{} & (B,b_0). \end{array}$$

Furthermore it follows again from the Map Lifting Criterion 27.1, applied to the pointed topological space $(Z,z_0)=(S^n\times[0,1],(*,0))$, that the lifts of two homotopic maps $(S^n,*)\to(B,b_0)$ are two homotopic maps $(S^n,*)\to(X,x_0)$.[242] Summarizing we just constructed a map

$$\begin{array}{rccc} \lambda\colon & \pi_n(B,b_0) & \to & \pi_n(X,x_0) \\ & [f] & \mapsto & [\widetilde{f}]. \end{array}$$

We have to show that λ is an inverse to p_*. First of all, given any $[f]\in\pi_n(B,b_0)$ we have

$$(p_*\circ\lambda)([f]) \;=\; p_*\big([\widetilde{f}]\big) \;=\; [p\circ\widetilde{f}] \;=\; [f].$$

Now let $[g]\in\pi_n(X,x_0)$. We then have

$$(\lambda\circ p_*)([g]) \;=\; \lambda([p\circ g]) \;\underset{\uparrow}{=}\; [g].$$

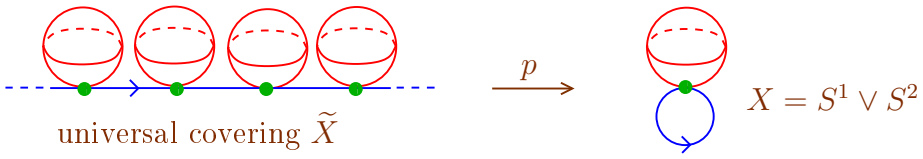
since g is the unique lift of $p\circ g$ to x_0.

■

Now we give a few more applications of the Covering-π_n-Isomorphism Proposition 28.10.

Example.

(1) We consider $X=S^1\vee S^2$. On page 341 we determined its universal covering $\widetilde{X}$, which is given by the wedge of $\mathbb{R}$ with infinitely many spheres. It follows from the Covering-π_n-Isomorphism Proposition 28.10 that $\pi_2(S^1\vee S^2)$ is isomorphic to $\pi_2(\widetilde{X})$. At the moment we cannot determine $\pi_2(\widetilde{X})$, but since $\widetilde{X}$ has "infinitely many spheres" one might expect $\pi_2(\widetilde{X})$ to be pretty big.



[240] In the Map Lifting Criterion 27.1 we need that the sphere S^n is path-connected and locally path-connected, but there are no hypotheses on B or X.

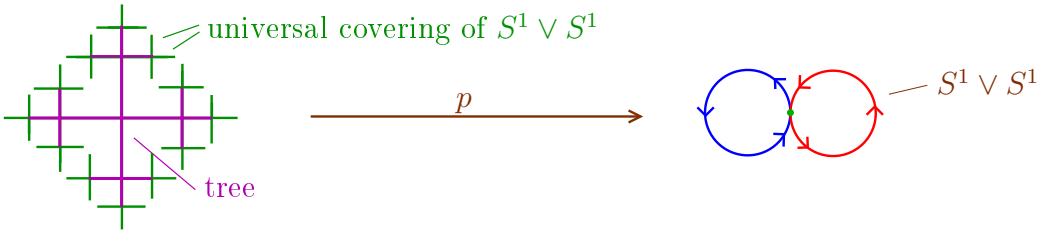
[241] We also use throughout the proof that the Lift Uniqueness Lemma 11.10 guarantees that the lifts, if they exist, are unique.

[242] Note that there is a little hidden subtlety: if $F\colon S^n\times[0,1]\to B$ is a homotopy between two maps $f,g\colon(S^n,*)\to(B,b_0)$, then we can lift f,g to maps $\widetilde{f},\widetilde{g}\colon(S^n,*)\to(X,x_0)$. Furthermore we can apply the Map Lifting Criterion 27.1 to lift F to a map $\widetilde{F}\colon S^n\times[0,1]\to X$ such that $\widetilde{F}(*,0)=x_0$. We still need to show that $\widetilde{F}(*,t)=x_0$ for all $t\in[0,1]$. We know that for every $t\in[0,1]$ we have $\widetilde{F}(*,t)\in p^{-1}(b_0)$. By the Covering–Basics Lemma 14.6 (2) the preimage $p^{-1}(b_0)$ is discrete. Since $[0,1]$ is connected it follows that the map $p\colon[0,1]\to p^{-1}(b_0)$ is constant by the Connected-to-Discrete Lemma 2.18. But since it takes the value x_0 for $t=0$ it takes the value x_0 for all $t\in[0,1]$.

(2) We have

$$\pi_n(S^1 \vee S^1) \underset{\text{Covering-}\pi_n\text{-Isomorphism Proposition 28.10}}{\cong} \pi_n(\text{universal covering of } S^1 \vee S^1) \underset{\text{Wedge-of-Circles–Universal Covering Proposition 27.7}}{\cong} \pi_n\Big(\bigcup_{k\in\mathbb{N}} \text{trees}\Big) \underset{\text{by the Exhaustion-}\pi_n\text{-Zero-Proposition 28.9, since trees are contractible}}{=} 0.$$

We will generalize this argument in the 1-Complex-π_n-Lemma 32.2.



28.6. **Are there any higher homotopy groups that are non-trivial?** We have now computed lots of higher homotopy groups, but so far the results have been rather dull, all the groups we have computed so far are trivial.
So are there are any higher homotopy groups that are non-trivial? A natural candidate for a non-trivial group is surely $\pi_n(S^n)$. The following question naturally arises:

Question 28.11. Let $n \geq 2$.
(1) Does the identity map $\mathrm{id}\colon S^n \to S^n$ define a non-trivial element in the homotopy group $\pi_n(S^n, *)$?
(2) Do we have an isomorphism $\pi_n(S^n, *) \cong \mathbb{Z}$?

We can also ask, what are the higher homotopy groups of spheres, i.e. what can we say about $\pi_k(S^n)$ for $k > n$? On page 354 we saw that all these groups are trivial for $n = 1$. But what happens for $n > 1$? Can the higher homotopy groups of spheres be non-trivial? We will answer this question in Algebraic Topology II using singular homology.

To construct our first example of a potentially interesting map from a higher-dimensional to a lower-dimensional sphere we recall that on page 116 we gave an identification $\mathbb{C}\mathrm{P}^1 =_i S^2$. which identifies $[0:1]$ with the North Pole $N = (0,0,1)$.

Definition. We refer to

$$\begin{array}{rcl} H\colon S^3 = \{(z_1,z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 =_i \mathbb{C}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1 : z_2] \end{array}$$

as the Hopf map[243]. The homotopy class of H defines an element in $\pi_3(S^2, N)$.

Question 28.12. Does the Hopf map $H\colon S^3 \to S^2$ represent a non-trivial element in the homotopy group $\pi_3(S^2, N)$?

We will answer this question in Algebraic topology III

Remark. The definition of the Hopf map is very elegant, but unfortunately we pay for the elegance by loosing any intuition for what the map looks like. So how are supposed to think about the Hopf map? A straightforward calculation shows that for each $P \in S^2$ the

[243] Heinz Hopf (1894-1971) was a German mathematician.

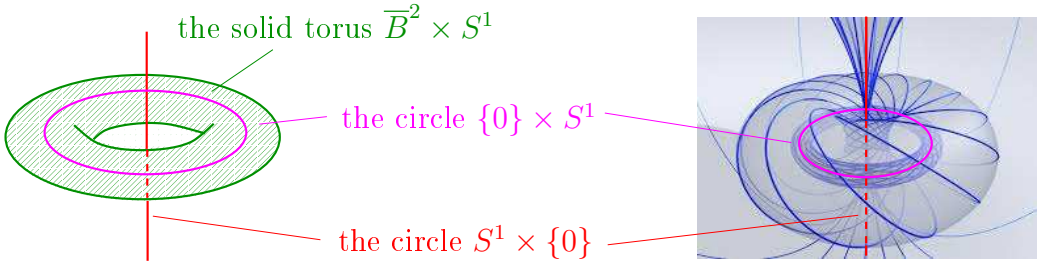
preimage $H^{-1}(P)$ is a circle. Furthermore, if we use the explicit homeomorphism from the Stereographic Projection Lemma 2.11 to make the identification of S^3 with $\mathbb{R}^3 \cup \{\infty\}$ then we can make the following observations:

(1) The circle $S^1 \times \{0\}$ corresponds to the points $(z, 0) \in S^3$, in particular the circle gets sent to the point $[1:0] \in \mathbb{C}\mathrm{P}^1$.
(2) The circle that is given by the z-axis together with ∞ corresponds to the points $(0, z) \in S^3$, in particular the circle gets sent to the point $[0:1] \in \mathbb{C}\mathrm{P}^1$.

One can show that there exists a homeomorphism

$$\left(S^1 \times \overline{B}^2\right) \cup_{S^1 \times S^1} \left(\overline{B}^2 \times S^1\right) \ \overset{\cong}{\to} \ S^3.$$

It follows from the definition of this identification that on the torus $S^1 \times S^1$ the "diagonal circle" $\{(z, z) \,|\, z \in S^1\}$ is the preimage of the point $[1:1] \in \mathbb{C}\mathrm{P}^1$. In the figure below we show the preimages of H for some points in $\mathbb{C}\mathrm{P}^2 = S^2$.



We introduce one more construction of continuous maps that represent potentially non-trivial elements in homotopy groups of spheres.

Definition.

(1) Let X be a non-empty topological space. We define the suspension of X to be the topological space

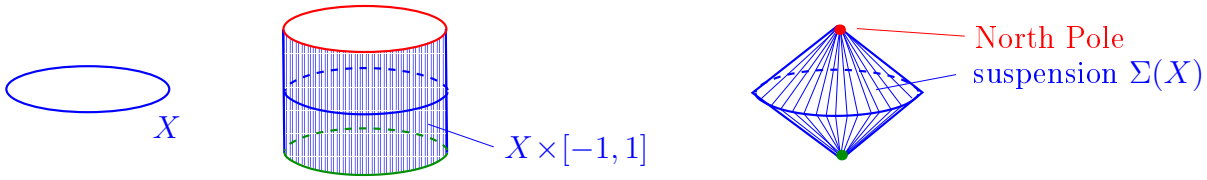
$$\Sigma(X) \ := \ (X \times [-1, 1])/(X \times \{-1\})/\sim$$

where we identify all points in $X \times \{-1\}$ to a single point and we identify all points in $X \times \{1\}$ to a single point. We refer to the point $[X \times \{1\}]$ as the North Pole of $\Sigma(X)$.

(2) Given a continuous map $f\colon X \to Y$ between topological spaces we denote by

$$\begin{array}{rccc} \Sigma(f)\colon & \Sigma(X) & \to & \Sigma(Y) \\ & [(x,t)] & \mapsto & [(f(x),t)] \end{array}$$

the suspension of f.



Remark. Let $n \in \mathbb{N}_0$. One can easily write down an explicit homeomorphism $\Sigma(S^n) \to S^{n+1}$ that sends the North Pole of $\Sigma(S^n)$ to the North Pole $(0, \ldots, 0, 1)$ of S^{n+1}. We will use this homeomorphism to identify $\Sigma(S^n)$ with S^{n+1}.

Now we have the following elementary lemma.

Lemma 28.13. (Suspension Homomorphism Lemma) Let $k \in \mathbb{N}_0$.

(1) Let (X, x_0) be a pointed topological space. We denote by $*$ the North Pole of $\Sigma(X)$. The map

$$\begin{array}{rcl} \pi_k(X, x_0) & \to & \pi_{k+1}(\Sigma(X), *) \\ [f\colon S^k \to X] & \mapsto & \big[\Sigma(f)\colon \Sigma(S^k) = S^{k+1} \to \Sigma(X)\big] \end{array}$$

is a well-defined group homomorphism. We refer to it as the suspension homomorphism.

(2) Let $n \in \mathbb{N}_0$. We denote by $*$ the North Pole of S^n and S^{n+1}. The map

$$\begin{array}{rcl} \pi_k(S^n, *) & \to & \pi_{k+1}(S^{n+1}, *) \\ [f\colon S^k \to S^n] & \mapsto & \big[\Sigma(f)\colon \Sigma(S^k) =_i S^{k+1} \to \Sigma(S^n) =_i S^{n+1}\big] \end{array}$$

is a well-defined group homomorphism. We refer to it also as the suspension homomorphism.

Proof.

(1) We leave the task of verifying that the map is in fact a group homomorphism as a somewhat challenging exercise to the reader. Alternatively we refer to [Hil53, Theorem 2.1] for a proof.

(2) This statement follows immediately from (1). ■

We conclude this chapter with the following question:

Question 28.14. Let $H\colon S^3 \to S^2$ be the Hopf map. Is the image of $[H] \in \pi_3(S^2)$ under the iterated suspension homomorphisms $\pi_3(S^2, x_0) \to \pi_{3+k}(S^{2+k}, x_1)$ non-trivial?

Unfortunately at the moment we do not have many tools to address the above questions. In general it is very hard to compute higher homotopy groups as there are few computational techniques that can be used to show that higher homotopy groups are non-zero. One of the key problems is that there is no analogue to the Seifert–van Kampen-Theorem 20.1. We will get to know a few techniques to compute higher homotopy groups of spheres, but we will cover these only in much later chapters, since the calculations will require the use of singular homology groups. But even then we will only be able to determine some homotopy groups of spheres. But note that the full computation of homotopy groups of spheres is still an open problem.

This sorry state of affairs means that as of right now we cannot use higher homotopy groups to distinguish any of the topological spaces that we could not distinguish already using the fundamental group. In particular we still cannot show that for $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic. We will only be able to do so after developing a new tool, namely the singular homology groups, in the next semester.

Part III

CW-complexes

29. Cell attachments

In this chapter we introduce the concept of attaching a cell to a topological space. We will see that many topological spaces arise that way and we will see that attaching cells changes the fundamental group in a controlled way. This allows us to give the overdue calculation of the fundamental groups of the complex projective spaces $\mathbb{C}\mathrm{P}^n$. More importantly though the study of cell attachments will be essential in the follow-up chapter where will graduate to the more general concept of a CW-complex.

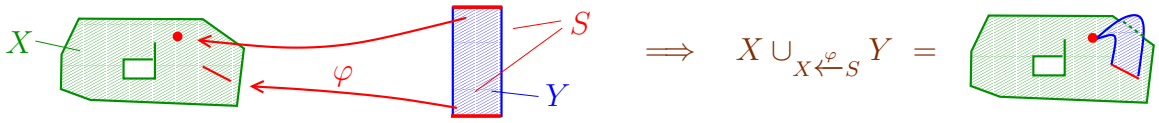
29.1. **Attaching a cell a to topological space.** We start out with a very general definition.

Definition. Let X and Y be topological spaces, let $S \subset Y$ be a subspace and let $\varphi\colon S \to X$ be a continuous map. (Note that φ does not need to be injective.) We write

$$X \cup_\varphi Y \;:=\; X \cup_{X \xleftarrow{\varphi} S} Y \;:=\; (X \sqcup Y)/\sim$$

where $\sim$ is the equivalence relation generated by $\varphi(s) \sim s$ for $s \in S$. We call $X \cup_\varphi Y$ the topological space obtained by gluing Y to X via the map attaching φ. Note that according to the Topological Pushout Proposition 6.12 we obtain a pushout diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \cup_\varphi Y. \end{array}$$



In this chapter we will mostly be interested in the following special case.

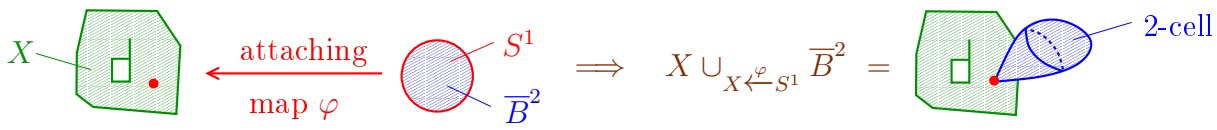
Definition. Let X be a topological space and let $\varphi\colon S^{n-1} \to X$ be a continuous map. We consider the topological space

$$X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n.$$

We introduce the following language:

(1) We refer to the image of B^n as an open n-cell.
(2) We refer to the image of $\overline{B}^n$ as an n-cell.
(3) We refer to φ as the attaching map of the n-cell.

We say that $X \cup_\varphi \overline{B}^n$ is obtained from X by attaching an n-cell along the attaching map φ. It follows from the fact that the inclusion $S^{n-1} \to \overline{B}^n$ is a closed embedding together with the Topological Pushout-Maps Lemma 6.14 (4) that the natural map $X \to X \cup_\varphi \overline{B}^n$ is a closed embedding. Often we will use this fact as an excuse to view X as a closed subset of $X \cup_\varphi \overline{B}^n$.



In the following we provide a very long list of examples of cell attachments. These examples train our geometric intuition and they will come in super handy in the next chapter, when we study CW-complexes.

Examples.

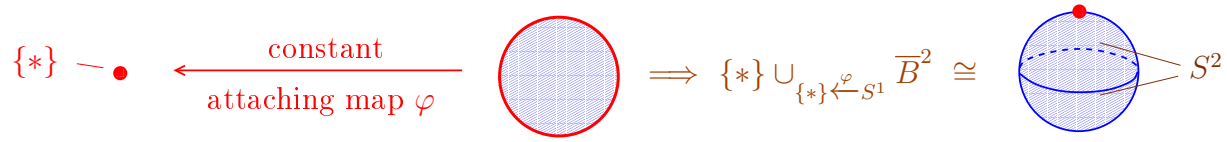
(1) We consider the effect of adding a 0-cell to a topological space X. Since $S^{-1} = \varnothing$ and since $\overline{B}^0 = \{0\}$ we see that attaching a 0-cell is the same as taking the disjoint union of X with a point.

(2) Let $X = \{*\}$ and let $\varphi\colon S^{n-1} \to \{*\}$ be the constant map. We have homeomorphisms

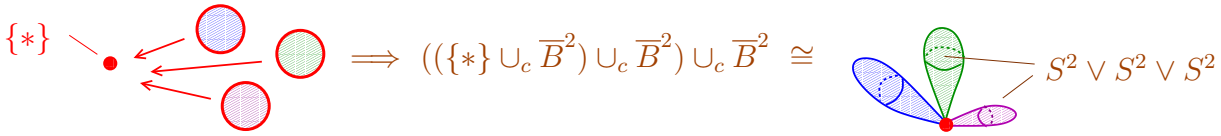
$$\{*\} \cup_{\{*\} \overset{\varphi}{\leftarrow} S^{n-1}} \overline{B}^n \quad \overset{\cong}{\longrightarrow} \quad \overline{B}^n / S^{n-1} \quad \overset{\cong}{\longrightarrow} \quad S^n.$$

see the homeomorphism from page 120 — identification from the Ball Quotient-Sphere Lemma 6.5

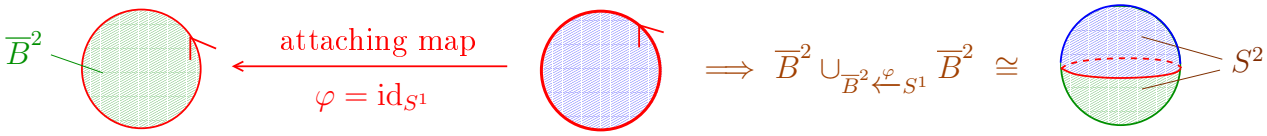
In fact, looking at these explicit homeomorphisms we see that these homeomorphisms send $\{*\}$ to the North Pole $(0, \dots, 0, 1)$.



(3) Again let $X = \{*\}$. Iteratively we attach k n-cells according to the constant attaching map $c\colon S^{n-1} \to \{*\}$. It follows easily from the Topological Pushout Proposition 6.12 (1∗), the Compact-Hausdorff Proposition 2.10 (3), and the Wedge-Properties Lemma 17.2 (3) that the resulting topological space is homeomorphic to the wedge of k copies of S^n.

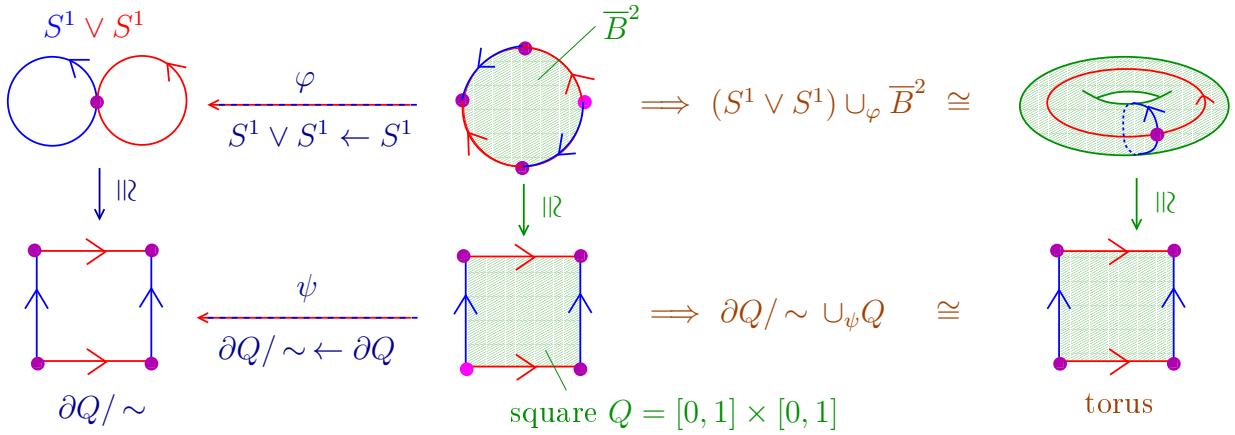


(4) Let $n \in \mathbb{N}_0$. We consider the n-ball $\overline{B}^n$ and we attach an n-cell to it along the attaching map $\varphi\colon S^{n-1} \to \overline{B}^n$ that is given by the identity. On page 122 we already saw that $\overline{B}^n \cup_{\overline{B}^n \overset{\varphi}{\leftarrow} S^{n-1}} \overline{B}^n$ is homeomorphic to the n-sphere S^n.

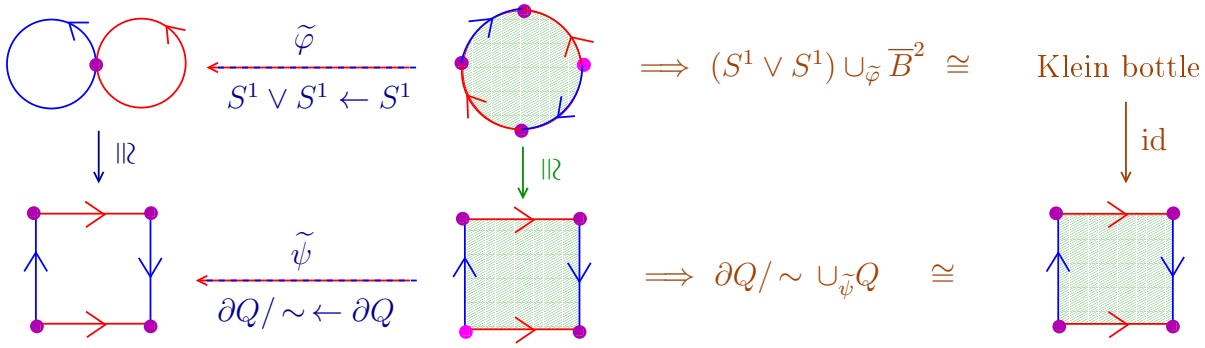


(5) We consider $X = S^1 \vee S^1$.

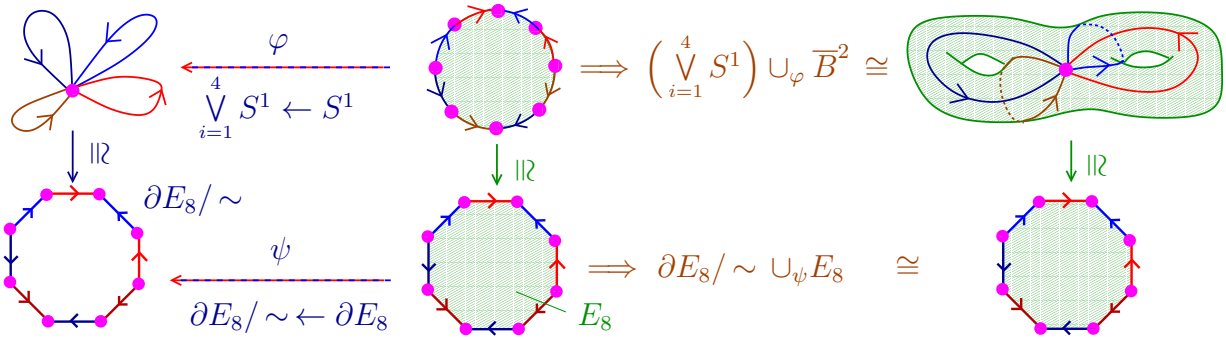
(a) We attach a 2-cell along the attaching map $\varphi\colon S^1 \to S^1 \vee S^1$ that is illustrated below. The Convex-to-Ball Proposition 2.12 gives us an explicit homeomorphism $\overline{B}^2 \to Q := [0,1] \times [0,1]$. This homeomorphism, together with the Topological Pushout Proposition 6.12 (1∗) and the Compact-Hausdorff Proposition 2.10 (3) can be used to show that $(S^1 \vee S^1) \cup_{S^1 \vee S^1 \overset{\varphi}{\leftarrow} S^1} \overline{B}^2$ is homeomorphic to the 2-torus $Q/\sim \cong ([0,1] \times [0,1])/\sim$.



(b) If instead we consider the attaching map $\widetilde{\varphi}\colon S^1 \to S^1 \vee S^1$ that is shown in the figure below, then one easily sees that $(S^1 \vee S^1) \cup_{S^1 \vee S^1 \overset{\widetilde{\varphi}}{\leftarrow} S^1} \overline{B}^2$ is homeomorphic to the Klein bottle.



(c) In the figure below we show that if we attach a 2-cell in a suitable way to the wedge of four circles, then we obtain a topological space that is homeomorphic to the surface of genus 2.



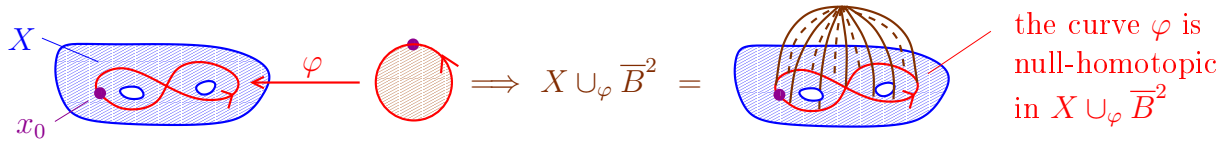
29.2. **Attaching cells and fundamental groups.** The following proposition tells us how attaching a cell affects the fundamental group of a given topological space.

Proposition 29.1. (Cell Attachment-π_1-Proposition) Let X be a topological space, let $n \geq 1$ and let $\varphi\colon S^{n-1} \to X$ be a continuous map. We set $x_0 := \varphi(\underbrace{(0,\ldots,0,1)}_{\in S^{n-1}})$. The following statements hold:

($n > 2$) The inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^n, x_0)$ is an isomorphism.

($n = 2$) The map $[0,1] \to X$ given by $t \mapsto \varphi(\exp(2\pi i t))$ defines an element in $\pi_1(X, x_0)$ that we denote by $[\varphi]$. The inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^n, x_0)$ descends to an isomorphism

$$\pi_1(X, x_0)/\langle\langle [\varphi] \rangle\rangle \xrightarrow{\cong} \pi_1\big(X \cup_\varphi \overline{B}^2, x_0\big).$$



Proposition 29.1. (Cell Attachment-π_1-Proposition) We continue with the above notation.

($n = 1$) We set $x_0 := \varphi(1)$ and $y_0 := \varphi(-1)$.

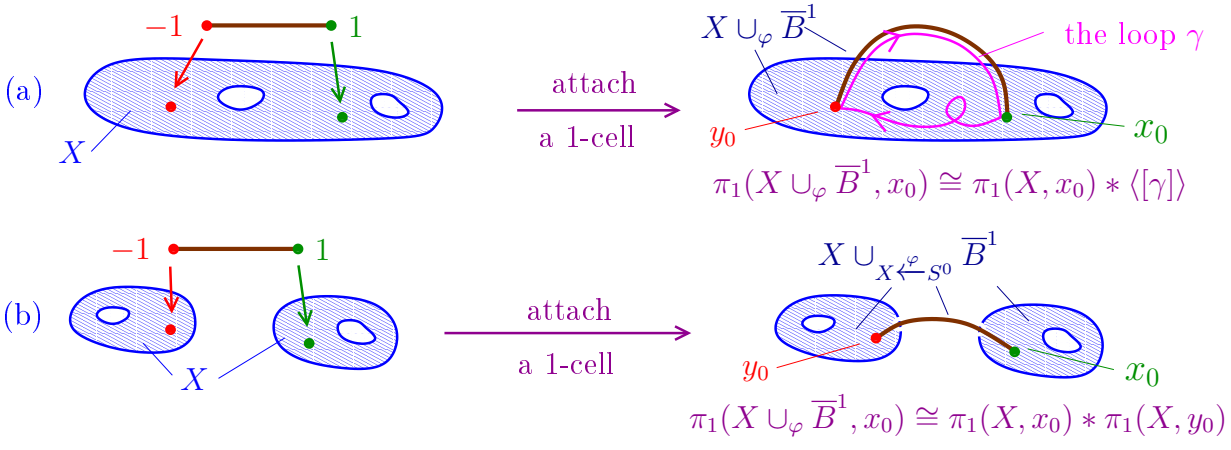
(a) Suppose that x_0 and y_0 lie in the same path-component of X. Let γ be a loop in $X \cup_\varphi \overline{B}^1$ that is given by the concatenation of the obvious path in the 1-cell from $x_0 = \varphi(1)$ to $y_0 = \varphi(-1)$ and some path in X from $y_0 = \varphi(-1)$ to $x_0 = \varphi(1)$. With this notation the map

$$\pi_1(X, x_0) * \langle t \rangle \;\to\; \pi_1(X \cup_\varphi \overline{B}^1, x_0)$$

that is given by the inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^1, x_0)$ and given by $t \mapsto [\gamma]$ is an isomorphism.

(b) Suppose that x_0 and y_0 lie in two different path components of X. Then we get an isomorphism

$$\pi_1(X, x_0) * \pi_1(X, y_0) \;\to\; \pi_1(X \cup_\varphi \overline{B}^1, x_0).$$



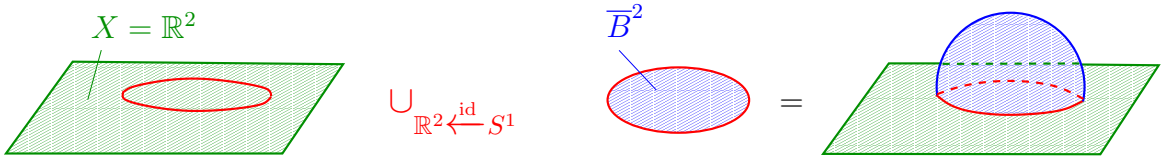
Examples.

(1) Let $X = S^1$ and let $n \in \mathbb{N}$. We attach to S^1 a 2-cell via the attaching map $\varphi\colon S^1 \to S^1$ that is given by $\varphi(z) = z^n$. Then

$$\pi_1\big(S^1 \cup_\varphi \overline{B}^2, 1\big) \underset{\uparrow}{=} \pi_1(S^1,1)/\langle\langle[\varphi]\rangle\rangle \underset{\uparrow}{=} \langle x\rangle/\langle\langle x^n\rangle\rangle \cong \mathbb{Z}_n.$$

Cell Attachment-π_1-Proposition 29.1 — see the Circle-π_1-Theorem 11.13

(2) We attach a 2-cell to $X = \mathbb{R}^2$ via the attaching map $\varphi := \mathrm{id}_{S^1}\colon S^1 \to S^1$ that is given by the identity. By the Cell Attachment-π_1-Proposition 29.1 the new topological space $\mathbb{R}^2 \cup_\varphi \overline{B}^2$ is again simply connected

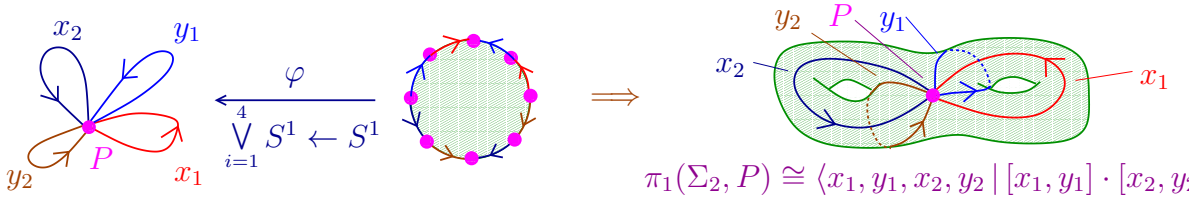


(3) In the previous section we already saw that many interesting topological spaces can be built out of easier topological spaces by attaching cells. In many cases we can use the Cell Attachment-π_1-Proposition 29.1 to calculate the fundamental group. For example, on page 362 we showed that we can obtain the surface of genus 2 from the wedge of four circles by attaching a 2-cell. With the notation from the figure below we now see that

by the Cell Attachment-π_1-Proposition 29.1

$$\pi_1(\Sigma_2, P) \cong \pi_1\Big(\Big(\bigvee_{i=1}^4 S^1\Big) \cup_\varphi \overline{B}^2, P\Big) \underset{\downarrow}{\cong} \pi_1\Big(\bigvee_{i=1}^4 S^1, P\Big)/\langle\langle[\varphi]\rangle\rangle \underset{\uparrow}{\cong} \langle x_1, y_1, x_2, y_2\rangle/\langle\langle[x_1,y_1]\cdot[x_2,y_2]\rangle\rangle.$$

by the Wedge-π_1-Proposition 20.6 and since $[\varphi] = [x_1,y_1]\cdot[x_2,y_2]$



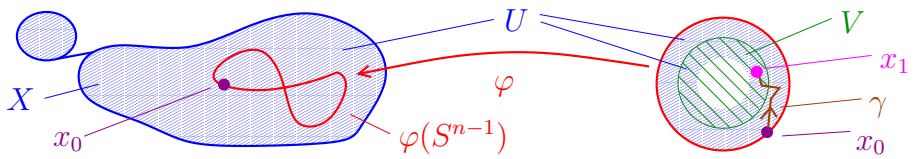
Of course we obtained the same result much earlier. The advantage of the current approach is that the Cell Attachment-π_1-Proposition 29.1 absorbs many of the technical arguments we had to deal with in the proof of the Surface-π_1-Proposition 22.1, i.e. in our initial calculation of $\pi_1(\Sigma_2, P)$.

Proof of the Cell Attachment-π_1-Proposition 29.1 for $n \geq 2$. Let X be a topological space, let $n \geq 2$ and let $\varphi\colon S^{n-1} \to X$ be a continuous map. We set $x_0 := \varphi((0,\ldots,0,1))$. The key idea of the proof is to apply the Seifert–van Kampen-Theorem 20.1 to the following two subsets of $X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n$

$$U = X \cup_{X \xleftarrow{\varphi} S^{n-1}} \big\{z \in \overline{B}^n \,\big|\, \|z\| > \tfrac{1}{4}\big\} \quad \text{and} \quad V = \big\{z \in \overline{B}^n \,\big|\, \|z\| < \tfrac{3}{4}\big\}.$$

The remainder of the proof consists in showing that the resulting isomorphism implies the desired, a priori slightly different, isomorphism.

We pick a point $x_1 \in U \cap V$ and we pick a path γ in U from x_0 to x_1.



Claim. The following diagram of maps

$$\begin{array}{ccc}
\pi_1(X,x_0) & \longrightarrow & \pi_1(X\cup_\varphi \overline{B}^n,x_0) \\
\cong\downarrow & & \downarrow\mathrm{id} \\
\pi_1(U,x_0) & \longrightarrow & \pi_1(X\cup_\varphi \overline{B}^n,x_0) \\
\cong\uparrow\gamma_* & & \cong\uparrow\gamma_* \\
\pi_1(U,x_1) & \longrightarrow & \pi_1(X\cup_\varphi \overline{B}^n,x_1) \\
\downarrow & & \downarrow\mathrm{id} \\
\pi_1(U,x_1)/\langle\langle\pi_1(U\cap V,x_1)\rangle\rangle & \longrightarrow & \pi_1(X\cup_\varphi \overline{B}^n,x_1) \\
\cong\uparrow & \text{Amalgamated Product–Quotient Lemma 19.13} & \downarrow\mathrm{id} \\
\pi_1(U,x_1)*_{\pi_1(U\cap V,x_1)} \underbrace{\pi_1(V,x_1)}_{=\{e\}} & \xrightarrow[\text{Seifert–van Kampen Theorem 20.1}]{\cong} & \pi_1(X\cup_\varphi \overline{B}^n,x_1)
\end{array}$$

has the following three properties:

(i) The diagram commutes.
(ii) All vertical maps except possibly the third vertical map on the left are isomorphisms.
(iii) The bottom horizontal map is an isomorphism.

Proof.

(i) The two vertical maps from the third to the second level are the homomorphisms induced by the choice of base point coming from the path γ from x_0 to x_1. It follows from the Change-of-Base Point Proposition 13.2 that the second square commutes. All the other vertical and horizontal maps are induced by inclusions or they are the obvious algebraic map. It follows easily from functoriality that the diagram commutes.

(ii) First note that it follows from the Change-of-Base Point Proposition 11.6 (1) that the vertical maps from the third to the second level are isomorphisms. Using the Homotopy Pushout Lemma from [Fri23] one can easily show that X is a deformation retract of U. Thus it follows from the Homotopy-π_1-Proposition 13.4 (2b) that the top left vertical map is an isomorphism.

The natural map $\{z\in B^n \,|\, \|z\|<\frac{3}{4}\}\to X\cup_{X\xleftarrow{\varphi}S^{n-1}}\overline{B}^n$ is easily seen to be an embedding. It follows that V is homeomorphic to an open ball. Hence $\pi_1(V,x_1)$ is the trivial group. It follows from this observation and the Amalgamated Product–Quotient Lemma 19.13 that the bottom vertical map on the left is an isomorphism.

(iii) It follows easily from the Topological-Pushout Proposition 6.12 (0) and the Topological-Coproduct Proposition 5.10 (0) that U and V are indeed open subsets of $X\cup_{X\xleftarrow{\varphi}S^{n-1}}\overline{B}^n$. We can therefore apply the Seifert–van Kampen-Theorem 20.1 and we obtain that the bottom horizontal map is an isomorphism. ⊞

Now we consider the case $n \geq 3$. In this case $\pi_1(U \cap V, x_1)$ is trivial since $U \cap V$ is homotopy equivalent to S^{n-1}. It follows from the Homotopy-π_1-Proposition 13.4 and the Sphere-π_1-Proposition 11.14 that $\pi_1(U \cap V, x_1) \cong \pi_1(S^{n-1}) = 0$. It is now clear that the one missing vertical map on the left is also an isomorphism. Since all vertical maps are isomorphisms, since the diagram commutes and since the bottom horizontal map is an isomorphism it follows that the top horizontal map is an isomorphism.

Finally we consider the case $n = 2$. In this case $U \cap V$ is homotopy equivalent to S^1 which implies, by the explicit isomorphism from the Circle-π_1-Theorem 11.13 and the Homotopy-π_1-Proposition 13.4, that $\pi_1(U \cap V, x_1) \cong \mathbb{Z}$. Note that under the explicit isomorphism $\Phi\colon \pi_1(X, x_0) \to \pi_1(U, x_1)$ this infinite cyclic subgroup corresponds to $\langle [\varphi] \rangle$. It is straightforward to see that we get a commutative diagram

$$\begin{array}{ccc}
\pi_1(X, x_0) & \longrightarrow & \pi_1\big(X \cup_\varphi \overline{B}^n, x_0\big) \\
{\scriptstyle \gamma_*}\downarrow{\scriptstyle \cong} & \nearrow_{\cong} & \downarrow \\
\pi_1(U, x_1)/\langle\langle \pi_1(U \cap V, x_1) \rangle\rangle & \xrightarrow[\cong]{} & \pi_1\big(X \cup_\varphi \overline{B}^n, x_1\big).
\end{array}$$

This concludes the proof of the proposition. ∎

Proof of the Cell Attachment-π_1-Proposition 29.1 for $n = 1$.

(a) We consider the case $\varphi(-1)$ and $\varphi(1)$ lie in the same path-component of X. This case is treated in [Fri23]. We leave it to the reader to fill in the details.

(b) We consider the case $\varphi(-1)$ and $\varphi(1)$ lie in the same path-component of X. This case follows once again fairly easily from the Seifert–van Kampen-Theorem 20.1. Again we leave it to the reader to fill in the details. ∎

29.3. **Projective spaces.** We just saw that the Cell Attachment-π_1-Proposition 29.1 can be very useful for calculating fundamental groups. Before we had already been quite successful with calculating fundamental groups, but it turns out that there is one interesting example of topological spaces that we could not handle yet, namely the complex projective spaces $\mathbb{C}\mathrm{P}^n$.

For the reader's convenience we recall the definition of real and complex projective spaces which we defined on page 113.

Definition. Let $\mathbb{A} = \mathbb{R}$ or $\mathbb{A} = \mathbb{C}$.

(1) For $n \in \mathbb{N}_0$ we set $\quad \mathbb{A}\mathrm{P}^n := (\mathbb{A}^{n+1} \setminus \{0\})/\sim \quad$ where $v \sim \lambda \cdot w$ for $\lambda \in \mathbb{A} \setminus \{0\}$.

Given a point $(v_0, \ldots, v_n) \in \mathbb{A}^{n+1} \setminus \{0\}$ we denote by $[v_0 : \ldots : v_n]$ its equivalence class.

(2) For $k \leq n$ we refer to the map

$$\begin{array}{rcl} \iota\colon \mathbb{A}\mathrm{P}^k & \to & \mathbb{A}\mathrm{P}^n \\ [v_0 : \ldots : v_k] & \mapsto & [v_0 : \ldots : v_k : 0 : \ldots : 0] \end{array}$$

as the standard embedding. We saw on page 115 that this map is indeed an embedding.

The following lemma shows that we can obtain $\mathbb{R}\mathrm{P}^n$ from $\mathbb{R}\mathrm{P}^{n-1}$ by attaching a single n-cell and that we can obtain $\mathbb{C}\mathrm{P}^n$ from $\mathbb{C}\mathrm{P}^{n-1}$ by attaching a single $2n$-cell.

Lemma 29.2. (Projective Space–Cell Attachment Lemma) Let $n \in \mathbb{N}$.

(1) We consider the continuous map

$$\begin{array}{rcl} \varphi\colon S^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1} \\ (x_1, \ldots, x_n) & \mapsto & [x_1 : \ldots : x_n]. \end{array}$$

There exists a homeomorphism $\mathbb{R}\mathrm{P}^{n-1} \cup_{\mathbb{R}\mathrm{P}^{n-1} \xleftarrow{\varphi} S^{n-1}} \overline{B}^n \to \mathbb{R}\mathrm{P}^n$ such that the restriction to $\mathbb{R}\mathrm{P}^{n-1}$ is given by the standard embedding $\mathbb{R}\mathrm{P}^{n-1} \to \mathbb{R}\mathrm{P}^n$.

(2) We think of $\overline{B}^{2n}$ and S^{2n-1} as subspaces of $\mathbb{C}^n =_i \mathbb{R}^{2n}$. We consider the attaching map

$$\begin{array}{rcl} \varphi\colon S^{2n-1} & \to & \mathbb{C}\mathrm{P}^{n-1} \\ (z_1, \ldots, z_n) & \mapsto & [z_1 : \ldots : z_n]. \end{array}$$

There exists a homeomorphism $\mathbb{C}\mathrm{P}^{n-1} \cup_{\mathbb{C}\mathrm{P}^{n-1} \xleftarrow{\varphi} S^{2n-1}} \overline{B}^{2n} \to \mathbb{C}\mathrm{P}^n$ such that the restriction to $\mathbb{C}\mathrm{P}^{n-1}$ is given by the standard embedding $\mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^n$.

Proof.

(1) We consider the map

$$\begin{array}{rcl} \Theta\colon \mathbb{R}\mathrm{P}^{n-1} \cup_\varphi \overline{B}^n & \to & \mathbb{R}\mathrm{P}^n \\ [P] & \mapsto & \begin{cases} [x_1 : \cdots : x_n : 0], & \text{if } P = [x_1 : \cdots : x_n] \in \mathbb{R}\mathrm{P}^{n-1}, \\ [P : \sqrt{1 - \|P\|^2}], & \text{if } P \in \overline{B}^n. \end{cases} \end{array}$$

It follows immediately from the definition of φ that the map Θ is well-defined. Furthermore it follows easily from the Topological Pushout Proposition 6.12 (1$*$) that the map Θ is continuous. It is basically clear that the map Θ is a surjection and it is straightforward to verify that the map Θ is an injection. Using the Topological-Quotient Proposition 6.1 (1b), the Projective Spaces-Compact+Hausdorff Proposition 6.10 and the Compact-Hausdorff Proposition 2.10 (3) one can easily verify that the map is a homeomorphism. Finally it is clear that the restriction of the map Θ to $\mathbb{R}\mathrm{P}^{n-1}$ is the standard embedding.

(2) Similar to (1) we see that the following map is a homeomorphism:

$$\begin{array}{rcl} \Theta\colon \mathbb{C}\mathrm{P}^{n-1} \cup_\varphi \overline{B}^{2n} & \to & \mathbb{C}\mathrm{P}^n \\ [P] & \mapsto & \begin{cases} [z_1 : \cdots : z_n : 0], & \text{if } P = [z_1 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^{n-1}, \\ [P : \underbrace{\sqrt{1 - \|P\|^2}}_{\in \mathbb{R} \subset \mathbb{C}}], & \text{if } P \in \overline{B}^{2n}. \end{cases} \end{array}$$

■

The combination of the Cell Attachment-π_1-Proposition 29.1 together with the Projective Space–Cell Attachment Lemma 29.2 allows us to (re-) calculate the fundamental groups of projective spaces.

Proposition 29.3. (Projective Space–π_1-Proposition)

(1) We have $\pi_1(\mathbb{R}\mathrm{P}^1) \cong \mathbb{Z}$ and for $n \geq 2$ we have $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}_2$.

(2) For any $n \in \mathbb{N}_0$ we have $\pi_1(\mathbb{C}\mathrm{P}^n) = 0$.

Proof.

(1) Since we already calculated the fundamental groups of $\mathbb{R}\mathrm{P}^n$ in the Real Projective Space–π_1-Proposition 15.3 we outsource this part as an exercise to the reader.
(2) Note that $\mathbb{C}\mathrm{P}^0$ consists of a single point. It follows from the Projective Space–Cell Attachment Lemma 29.2 that we obtain $\mathbb{C}\mathrm{P}^n$ from $\mathbb{C}\mathrm{P}^0$ by iteratively attaching cells of dimension $2, 4, \ldots, 2n$. It follows from the Cell Attachment-π_1-Proposition 29.1 that $\pi_1(\mathbb{C}\mathrm{P}^n) \cong \pi_1(\mathbb{C}\mathrm{P}^0) = 0$. ■

29.4. **Attaching arbitrarily many cells.** In the following we slightly generalize the concepts from page 360 to the following notationally less pleasant definition.

Definition. Let X be a topological space and let $\{\varphi\colon S^{n_i} \to X\}_{i\in I}$ be a family of continuous maps.

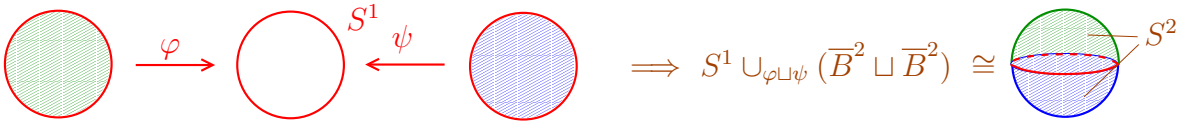
(1) We consider the topological space

$$X \cup_{X \leftarrow \bigsqcup_{i\in I} S^{n_i-1} \,:\, \bigsqcup_{i\in I} \varphi_i} \bigsqcup_{i\in I} \overline{B}^{n_i}.$$

As before we introduce the notion of an (open) cell and of an attaching map of a cell. We say that the above topological space is obtained from X by attaching cells along the attaching maps φ_i.
(2) It follows from the fact that the inclusions $S^{n-1} \to \overline{B}^n$ are closed embeddings together with the Topological Pushout-Maps Lemma 6.14 (4) that the natural map of X into the above topological space is a closed embedding. Often we will use this fact as an excuse to view X as a closed subset of the above topological space.

Example.

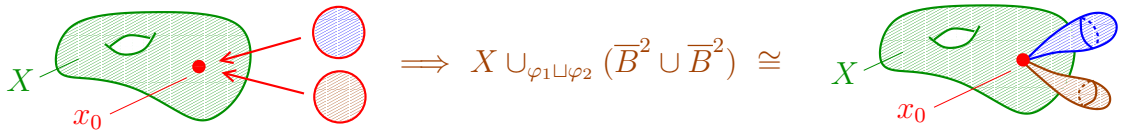
(1) Let $n \in \mathbb{N}_0$. We attach two n-cells to S^{n-1} where in both cases the attaching map is given by the identity map $\varphi := \psi := \mathrm{id}\colon S^{n-1} \to S^{n-1}$. Using the Compact-Hausdorff Proposition 2.10 (3) one can easily show that there exists a homeomorphism $S^{n-1} \cup_{\varphi\sqcup\psi} (\overline{B}^n \sqcup \overline{B}^n) \to S^n$ which on S^{n-1} is given by $x \mapsto (x, 0)$.



(2) Let (X, x_0) be a pointed topological space. Furthermore let I be a set and for each $i \in I$ let $n_i \in \mathbb{N}_0$. Given $i \in I$ we denote by $\varphi_i\colon S^{n_i-1} \to X$ the constant map that sends all points to x_0. Using the Ball Quotient-Sphere Lemma 6.5 one can show that there exists a homeomorphism

$$X \cup_{X \leftarrow \bigsqcup_{i\in I} S^{n_i-1} \,:\, \bigsqcup_{i\in I} \varphi_i} \bigsqcup_{i\in I} \overline{B}^{n_i} \xrightarrow{\cong} \underbrace{(X, x_0) \vee \bigvee_{i\in I} (S^{n_i}, N)}_{\substack{\text{wedge of pointed topological} \\ \text{spaces where } N \in S^{n_i} \text{ is the} \\ \text{North Pole}}}$$

that is the identity on X. [244]



The following proposition shows that the Hausdorff property is preserved under cell attachments.

Proposition 29.4. (Cell Attachment–Hausdorff Proposition) Let X be a topological space and let $\{\varphi_i\colon S^{n_i} \to X\}_{i\in I}$ be a family of continuous maps. We consider the topological space

$$\widehat{X} \;:=\; X \cup_{X \leftarrow \bigsqcup_{i\in I} S^{n_i-1} \,:\, \bigsqcup_{i\in I}\varphi_i} \bigsqcup_{i\in I} \overline{B}^{n_i}$$

that we obtain from X by attaching cells along the attaching maps φ_i.

(1) If U, V are disjoint open subset of X, then there exist disjoint open subsets $\widehat{U}$ and $\widehat{V}$ of $\widehat{X}$ with $X \cap \widehat{U} = U$ and $X \cap \widehat{V} = V$.
(2) If X is Hausdorff, then $\widehat{X}$ is also Hausdorff.

Proof. For each $i \in I$ we consider the natural map $\Phi_i\colon \overline{B}^n_i \to \widehat{X}$. It follows immediately from the Topological-Pushout Proposition 6.12 (0) and the Topological-Coproduct Proposition 5.10 (0) that given any subset $U \subset \widehat{X}$ the following statement holds:

$$U \text{ is open in } \widehat{X} \iff \begin{array}{l}\text{(a) the intersection } U \cap X \text{ is open in } X \text{ and}\\ \text{(b) for all } i \in I \text{ the preimage } \Phi_i^{-1}(U) \text{ is open in } \overline{B}^n.\end{array}$$

After this preparation we turn to the actual proof of the two statements:

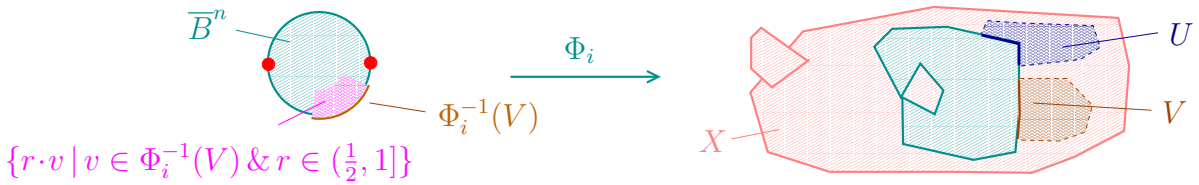
(1) Let U and V be two disjoint open subsets of X. We set

$$\begin{aligned}\widehat{U} &:= U \cup \bigcup_{i\in I} \Phi_i(\{r\cdot u \,|\, u \in \Phi_i^{-1}(U) \text{ and } r \in (\tfrac{1}{2}, 1]\}).\\ \widehat{V} &:= V \cup \bigcup_{i\in I} \Phi_i(\{r\cdot v \,|\, v \in \Phi_i^{-1}(V) \text{ and } r \in (\tfrac{1}{2}, 1]\}).\end{aligned}$$

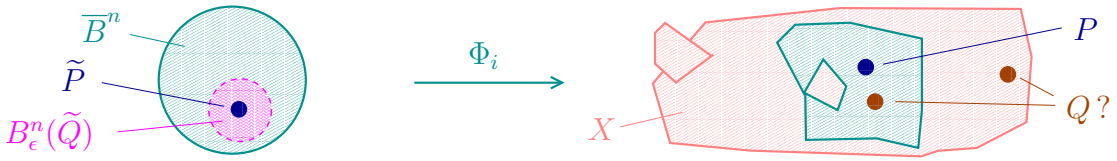
It follows immediately from the above discussion that $\widehat{U}$ and $\widehat{V}$ are open subsets of $\widehat{X}$. Furthermore it is clear that they are disjoint.

(2) Let $P, Q \in \widehat{X}$ be two distinct points. If P and Q lie in X, then it follows from the hypothesis that X is Hausdorff and (1) that we can find disjoint open neighborhoods $\widehat{U}$ of P and $\widehat{V}$ of Q in $\widehat{X}$.

[244] For fans of checking continuity let us elaborate a little bit. Let $p_n\colon \overline{B}^n \to \overline{B}^n/S^{n-1}$ be the projection and let $\Theta_n\colon \overline{B}^n/S^{n-1} \to S^n$ be the homeomorphism from the Ball Quotient-Sphere Lemma 6.5. The maps $\Theta_{n_i}\circ p_{n_i}\colon \overline{B}^{n_i} \to S^{n_i}$ together with the identity on X define, by the Topological-Coproduct Proposition 5.10 (1b), Topological Pushout Proposition 6.12 (1b) and the Wedge Proposition 17.1 (1a), a continuous map $\alpha\colon X \cup \bigcup_{i\in I} \overline{B}^{n_i} \to X \vee \bigvee_{i\in I} S^{n_i}$. Conversely note that the natural map $\overline{B}^{n_i} \to X \cup \bigsqcup_{i\in I} \overline{B}^{n_i}$ descends to a continuous map $f_i\colon \overline{B}^{n_i}/S^{n_i-1} \to X \cup \bigsqcup_{i\in I} \overline{B}^{n_i}$. The maps $f_i \circ \Theta_{n_i}^{-1}\colon S^n \to X \cup \bigsqcup_{i\in I} \overline{B}^{n_i}$ and the identity on X define, by the Wedge Proposition 17.1 (1a), a continuous map $\beta\colon X \vee \bigvee_{i\in I} S^{n_i} \to X \cup \bigsqcup_{i\in I} \overline{B}^{n_i}$. One can easily verify that these two maps α and β are inverses of one another.



It remains to consider the case that at least one of P and Q lies in $\widehat{X}\setminus X$. Let us assume that $P\in\widehat{X}\setminus X$. This means that there exists an $i\in I$ and a $\widetilde{P}\in B^n$ such that $P=\Phi_i(\widetilde{P})$. Since $Q\neq P$ there exists an $\epsilon>0$ such that $B^n_\epsilon(\widetilde{P})\subset B^n$ and such that $Q\notin\Phi_i(B^n_\epsilon(\widetilde{P}))$. We set $U:=\Phi_i(B^n_{\frac{\epsilon}{2}}(\widetilde{Q}))$ and $V:=X\setminus\Phi_i(\overline{B}^n_{\frac{\epsilon}{2}}(\widetilde{Q}))$. It is clear that U and V are disjoint. It follows from the discussion at the beginning of the proof that U and V are open subsets of $\widehat{X}$. ■



30. CW-complexes: Definitions and basic properties

30.1. **Definition of CW-complexes and examples.** After the long discussion of cell attachments in the last chapter we are now mentally ready for the following all-important notion of a CW-complex.

Definition. A CW-complex X[245][246] is a topological space X together with a filtration

$$\varnothing = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X$$

by subspaces which has the following properties:

(CW1) We have $X = \operatorname{colim} X^n$, as described in the Topological-Direct Limit Proposition 25.4. In other words, as a set we have $X = \bigcup_{n\in\mathbb{N}_0} X^n$ and $U \subset X$ is open if and only if for each $n \in \mathbb{N}_0$ the intersection $U \cap X^n$ is open in X^n.

(CW2) For each $n \in \mathbb{N}_0$ there exists a set I, a family $\{\varphi_i \colon S^{n-1} \to X^{n-1}\}_{i\in I}$ of continuous maps and a homeomorphism[247][248]

$$\Xi_n \colon X^{n-1} \cup_{X^{n-1} \leftarrow \bigsqcup_{i\in I} S^{n-1} \,:\, \bigsqcup_{i\in I} \varphi_i} \bigsqcup_{i\in I} \overline{B}^n \;\to\; X^n$$

which is the inclusion on X^{n-1}.

Before we discuss examples of CW-complexes it is useful to already introduce some extra language.

Definition. Using the notation from above we introduce the following language:

(1) Given $n \in \mathbb{N}_{\geq 0}$ we refer to X^n as the n-skeleton of X.

(2) The components of $X^n \setminus X^{n-1}$ are called the open n-cells of X.[249][250]

(3) For each open n-cell σ of X we refer to its closure $\overline{\sigma}$ in X as an n-cell of X.

(4) We set

$$\dim(X) \;:=\; \sup\{n \in \mathbb{Z}_{\geq -1} \,|\, X^n \neq X^{n+1}\} \;\in\; \mathbb{Z}_{\geq -1} \cup \{\infty\}.$$

(5) We say the CW-complex is finite if it has only finitely many cells and we say it is countable if it has only countably many cells.

(6) A CW-structure on a topological space X is filtration as in (1).

---

[245] CW-complexes were introduced by John H. C. Whitehead[246]. The name comes from "C" for "closure-finite", and the "W" for "weak topology":

(1) "Closure finite" means that the closure of each open cell is contained in the union of finitely many open cells. It follows from Lemma 30.12 (3) and the Cell-Compact Proposition 30.4 (2) that this is in fact the case.

(2) "Weak topology" means that a subset is open if and only if each intersection $U \cap X^n$ is open.

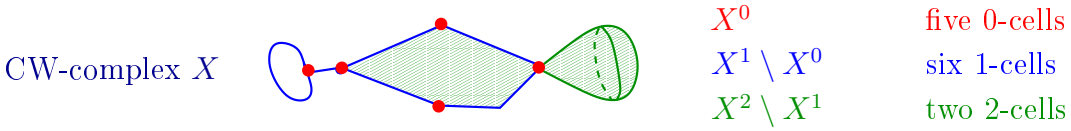
Some people also say that the name CW goes back to the initials C. W. of Whitehead.

[246] At first glance the definitions of CW-complexes that appear in the literature look widely different. Some differences just stem from different notation. Some authors, see e.g. [Lee00, p. 132] and [Mun84, p. 214] require "closure-finiteness" as part of the definition, but as we will see in Lemma 30.12 (3) and the Cell-Compact Proposition 30.4 (2), this property follows from the definition as given. One could also make the case that the definitions of CW-complexes in some references are a little vague.

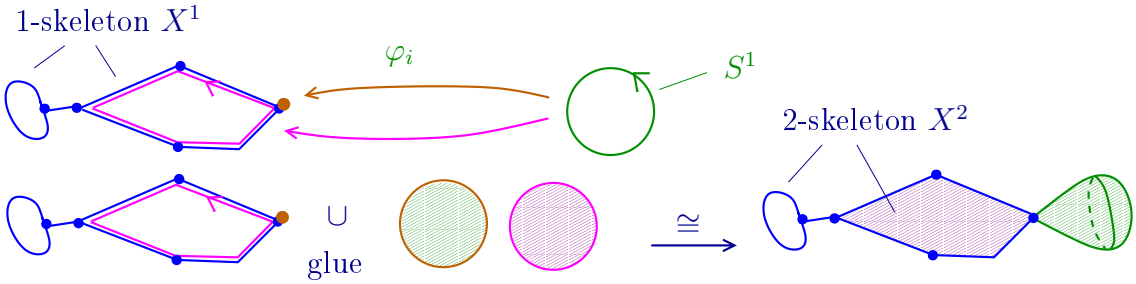
[247] In other words, X^n is obtained from attaching n-cells to X^{n-1} along the maps φ_c.

[248] Note that the φ_c and the homeomorphism are *not* part of the data of a CW-complex.

Illustration. In the figure below we show a 2-dimensional CW-complex X with its filtration $X^0 \subset X^1 \subset X^2$.



In the next figure we indicate how we obtain the 2-skeleton from the 1-skeleton by gluing on 2-disks.



Our long list of examples of cell attachments from page 361 provides us, almost for free, with a long list of CW-complexes.

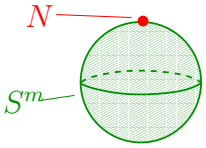
Examples.

(1) The one type of CW-complexes that is easy to understand are 0-dimensional CW-complexes. Similar to the discussion on page 361 it follows from the observation that $\overline{B}^0$ consists of a point and that $S^{-1} = \varnothing$ that a CW-complex of dimension ≤ 0 is essentially the same as a set of points with the discrete topology.

(2) Let $m \in \mathbb{N}_0$. As usual we denote by $N := (0, \ldots, 0, 1)$ the North Pole of S^m. We consider the filtration $\varnothing \subset \{N\} \subset S^m$. That this is a CW-structure can be seen as follows:

(CW1) This condition is vacuous since the m-skeleton equals S^m.

(CW2) The case $n = 0$ is clear since $\{N\}$ has the discrete topology. For $n = 1, \ldots, m-1$ there is nothing to show. Finally note that on page 361 we showed that we obtain S^m from $\{N\}$ by attaching an m-cell.



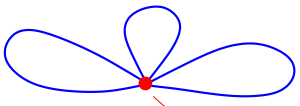
the filtration $\varnothing \subset \{N\} \subset S^m$ is a CW-structure with
one 0-cell
one m-cell

[249] It follows from the Disjoint Union Topology-Properties Lemma 5.11 (2b) and the Topological Pushout Proposition 6.12 (0) that, in the notation of (1), the open cells are precisely the images of the B^n_c and that each open n-cell is an open subset of X^n.

[250] It follows almost immediately from the definitions that, *as a set*, any CW-complex is the disjoint union of all the open cells.

(3) Let $n \in \mathbb{N}_0$ and let $X = \bigvee_{i=1}^{n} S^1$ be the wedge of n circles with wedge point $*$. The discussion on page 361 implies that the filtration $\varnothing \subset \{*\} \subset \bigvee_{i=1}^{n} S^1$ is a CW-complex with one 0-cell and n 1-cells.

$X = S^1 \vee S^1 \vee S^1$



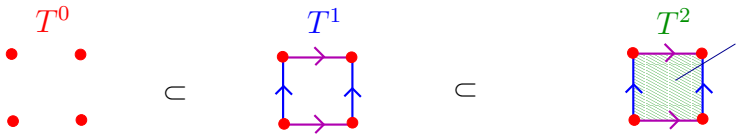
wedge point $\{*\}$

the filtration $\varnothing \subset \{*\} \subset S^1 \vee S^1 \vee S^1$ is a CW-complex with
one 0-cell
three 1-cells

(4) The figure below shows a filtration of the torus $T = ([0,1] \times [0,1]) \sim$ that is a CW-structure with one 0-cell, two 1-cells and one 2-cell. That this is a CW-structure can be seen as follows:

(CW1) This condition is vacuous since $T^2 = T$.

(CW2) The case $n = 0$ is clear since T^0 has the discrete topology. For $n = 1$ note that T^1 is clearly homeomorphic to the wedge of two circles along the unique point of T^0. Thus we obtain from the previous example that (CW2) is satisfied for $n = 1$. Finally note that on page 361 we showed that we obtain T from attaching a 2-cell to $T^1 \cong S^1 \vee S^1$.

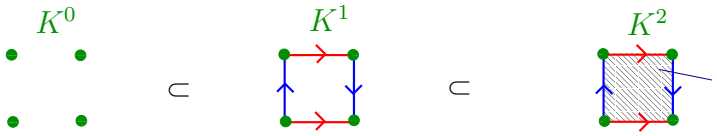


torus $T = ([0,1] \times [0,1])/\sim$ with
one 0-cell
two 1-cells
one 2-cell

For fun we also show the filtration if we view the torus as a subspace of $\mathbb{R}^3$:

torus as subspace of $\mathbb{R}^3$

(5) The figure below shows a filtration of the Klein bottle that is a CW-structure with one 0-cell, two 1-cells and one 2-cell. We see that the two CW-structures of the torus



Klein bottle $[0,1] \times [0,1]/\sim$

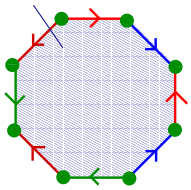
and the Klein bottle have the same 1-skeleton, but the 2-cells have different attaching maps.

(6) In the figure below we sketch CW-structures for the surface of genus 2, the projective plane $\mathbb{R}\mathrm{P}^2 =_i \overline{B}^2/\sim$, the non-orientable surface of genus 3 and the Möbius band as CW-complexes. Generalizing these examples one can easily show, using the Surface Classification Theorem 22.8, that every compact 2-dimensional topological manifold admits a CW-structure.

So far we only considered CW-complexes with finitely many cells and for the most part we just recycled material from the previous chapter. We now move onto CW-complexes with infinitely many cells:

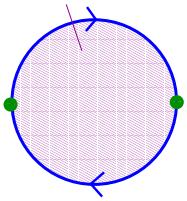
Examples.

surface of genus 2



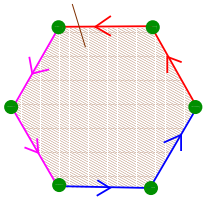
one 0-cell
four 1-cells
one 2-cell

$\mathbb{R}\mathrm{P}^2 =_i \overline{B}^2/\sim$



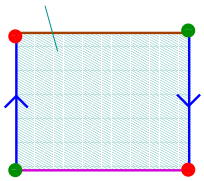
one 0-cell
one 1-cell
one 2-cell

non-orientable surface of genus 3



one 0-cell
three 1-cells
one 2-cell

Möbius band



two 0-cells
three 1-cells
one 2-cell

(1) (a) Let us consider the topological space $\mathbb{R}$. We equip it with the filtration $\varnothing \subset \mathbb{Z} \subset \mathbb{R}$. We consider the family $\{\varphi_i \colon S^0 \to \mathbb{Z}\}_{i\in\mathbb{Z}}$ of maps given by $\varphi_i(\epsilon) = i + \frac{1+\epsilon}{2}$. Using the Cover-Open Subset Lemma 1.5 (2) one can verify that the map

$$\begin{aligned} \mathbb{Z} \cup_{\mathbb{Z}\hookleftarrow \bigsqcup_{i\in\mathbb{Z}} S_i^0 \,:\, \bigsqcup_{i\in\mathbb{Z}} \varphi_i} \bigsqcup_{i\in\mathbb{Z}} \overline{B}_i^1 &\to \mathbb{R} \\ P &\mapsto \begin{cases} n, & \text{if } P = [n] \text{ for some } n \in \mathbb{Z} \\ i + \frac{1+t}{2}, & \text{if } P = [(t,i)] \text{ for some } t \in \overline{B}^1 = [-1,1] \text{ and } i \in \mathbb{Z} \end{cases} \end{aligned}$$

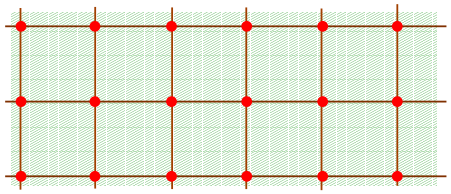
is a homeomorphism. Thus the filtration $\varnothing \subset \mathbb{Z} \subset \mathbb{R}$ is a CW-structure on $\mathbb{R}$.

(b) Using the Convex-to-Ball Proposition 2.12 one can easily generalize the above example and one can show that the subspaces

$$X^i := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, \text{at least } n-i \text{ coordinates lie in } \mathbb{Z}\} \text{ with } i = 0, \ldots, n$$

form a CW-structure on $\mathbb{R}^n$.

CW-structure on $\mathbb{R}$

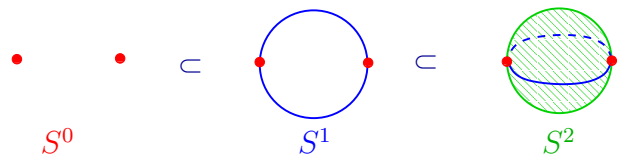


CW-structure on $\mathbb{R}^2$

(2) Let I be a set and for each $i \in I$ let $n_i \in \mathbb{N}$. We consider the wedge $\bigvee_{i\in I} S_i^{n_i}$. Given $n \in \mathbb{N}_0$ we denote by X_n the wedge of all spheres of dimension $\leq n$. It follows from the discussion on page 368 that the filtration $\varnothing \subset X_0 \subset X_1 \subset \ldots$ is a CW-structure on $\bigvee_{i\in I} S_i^{n_i}$.

Next we want to consider the infinite families S^n, $\mathbb{R}\mathrm{P}^n := (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\})$ and $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$. To study these family we introduce, with great reluctance, the following convention.

Convention. For $k < n$ we use in the following discussion the embedding $S^k \to S^n$ given by $x \mapsto (x, 0)$ to think of S^k as a subspace of S^n. Similarly we view $\mathbb{R}\mathrm{P}^k$ as a subspace of $\mathbb{R}\mathrm{P}^n$ and we view $\mathbb{C}\mathrm{P}^k$ as a subspace of $\mathbb{C}\mathrm{P}^n$.



It is natural to try to find CW-structures which are "compatible" with these inclusions. For example we one could try to find CW-complexes such that the "smaller family members" are "subcomplexes". This leads us to the following definition:

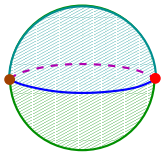
Definition. We say that a CW-complex A is a subcomplex of a CW-complex X if $A \subset X$ and if the following two conditions are satisfied:
(1) For each $n \in \mathbb{N}_0$ we have $A^n = A \cap X^n$.
(2) Each open n-cell of A is also an open n-cell of X.
(For example, it follows immediately from the definitions that any skeleton is a subcomplex.)

Examples.

(1) We consider the spheres S^n.
  (a) On page 372 we found a "tiny" CW-structure for S^n with just one 0-cell and one n-cell. But this CW-structure does not have any interesting subcomplexes. To fix this, recall that on page 368 we saw that we obtain S^{k+1} from S^k by attaching a single $(k+1)$-cell. It follows that a CW-structure for S^n is given by

$$\varnothing \subset S^0 \subset S^1 \subset S^2 \subset \ldots \subset S^n.$$

  With this CW-structure we can view each S^k with $k < n$ as a subcomplex of S^n.



  the filtration $\varnothing \subset S^0 \subset S^1 \subset S^2$ is a CW-structure with
  two 0-cells
  two 1-cells
  two 2-cells

  (b) The new CW-structure on spheres gives us an immediate bonus: As on page 318 we consider the topological space

$$S^\infty := \left\{(x_1, x_2, \ldots) \in \mathbb{R}^\infty \,\middle|\, \sum_{i=1}^{\infty} x_i^2 = 1\right\} = \operatorname*{colim}_{\longrightarrow} S^{i-1}.$$

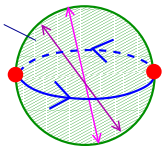
  It follows immediately from (a) and the definitions that a CW-structure for S^∞ is given by

$$\varnothing \subset S^0 \subset S^1 \subset S^2 \subset \ldots \subset S^\infty.$$

(2) We consider the real projective spaces $\mathbb{R}\mathrm{P}^n$.
  (a) In the Projective Space–Cell Attachment Lemma 29.2 (1) we showed that we can obtain $\mathbb{R}\mathrm{P}^{k+1}$ from $\mathbb{R}\mathrm{P}^k$ by attaching a single $(k+1)$-cell. It follows that a CW-structure for $\mathbb{R}\mathrm{P}^n$ is given by

$$\varnothing \subset \mathbb{R}\mathrm{P}^0 \subset \mathbb{R}\mathrm{P}^1 \subset \mathbb{R}\mathrm{P}^2 \subset \ldots \subset \mathbb{R}\mathrm{P}^n$$

  and that this CW-structure has precisely one cell in dimensions $0, 1, 2, \ldots, n$.

$\mathbb{R}\mathrm{P}^2 =_i S^2/\sim$ 

the filtration $\varnothing \subset \mathbb{R}\mathrm{P}^0 \subset \mathbb{R}\mathrm{P}^1 \subset \mathbb{R}\mathrm{P}^2$ is a CW-structure with
one 0-cell
one 1-cell
one 2-cell

(b) As on page 319 we consider $\mathbb{R}\mathrm{P}^\infty = \operatorname{colim} \mathbb{R}\mathrm{P}^i$. As in (1b) we see that

$$\varnothing \subset \mathbb{R}\mathrm{P}^0 \subset \mathbb{R}\mathrm{P}^1 \subset \mathbb{R}\mathrm{P}^2 \subset \ldots \subset \mathbb{R}\mathrm{P}^\infty$$

is a CW-structure which has precisely one cell in each dimension.

(3) Finally we consider the complex projective spaces $\mathbb{C}\mathrm{P}^n$.

(a) As in (2a) we see, this time using the Projective Space–Cell Attachment Lemma 29.2 (2), that a CW-structure for $\mathbb{C}\mathrm{P}^n$ is given by

$$\varnothing \subset \mathbb{C}\mathrm{P}^0 \subset \mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^2 \subset \ldots \subset \mathbb{C}\mathrm{P}^n$$

and that this CW-structure has precisely one cell in dimensions $0, 2, 4, \ldots, 2n$.

(b) As on page 319 we consider $\mathbb{C}\mathrm{P}^\infty = \operatorname{colim} \mathbb{C}\mathrm{P}^i$ and we see that

$$\varnothing \subset \mathbb{C}\mathrm{P}^0 \subset \mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^2 \subset \ldots \subset \mathbb{C}\mathrm{P}^\infty$$

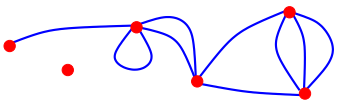
is a CW-structure which has precisely one cell in each even dimension.

The following lemma shows that topological graphs and CW-complexes of dimension ≤ 1 are essentially the same.

Lemma 30.1. (Graph–CW-Complex Lemma)

(1) If $(X, G = (V, E, \theta), \Theta\colon |G| \to X)$ is an undirected topological graph then $\varnothing \subset V \subset X$ is a CW-structure.

(2) Let $\varnothing \subset X^0 \subset X^1 = X$ be a CW-complex of dimension ≤ 1. There exists an undirected abstract graph (V, E, θ) and a homeomorphism $f\colon |G| \to X$ such that $f(V) = X^0$.



vertices form the 0-skeleton
edges correspond to 1-cells

Sketch of proof.

(1) This statement follows basically immediately from the definitions.

(2) By definition of a CW-complex there exists a set C, a family $\{\varphi_c\colon S^0 \to X^0\}_{c\in C}$ of continuous maps and a homeomorphism

$$\Xi\colon X^0 \cup_{X^0 \hookleftarrow \bigsqcup_{c\in C} S^0_c \,:\, \bigsqcup_{c\in C} \varphi_c} \bigsqcup_{c\in C} \overline{B}^1_c \;\to\; X^1 = X$$

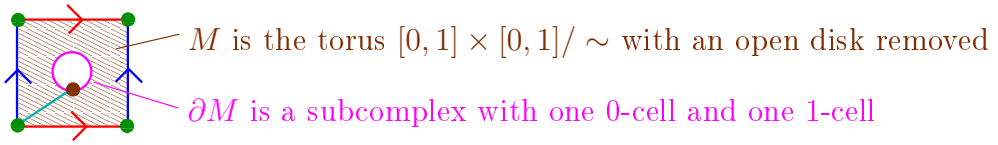
which is the inclusion on X^0. Given $c \in C$ we set $\theta(c) := \{\varphi_c(-1), \varphi_c(1)\} \subset \mathcal{P}(X^0)$. It is clear that $G := (X^0, C, \theta)$ is an undirected abstract graph. The map Ξ gives rise to a homeomorphism $f\colon |G| \to X$ with $f(V) = X^0$. ∎

We conclude our long list of examples of CW-complexes with a sneak preview of a later result.

Theorem 30.2. (Smooth Manifold-CW-Theorem) Every compact connected n-dimensional smooth manifold M admits a finite CW-structure with the following properties:
(1) the maximal dimension of a cell is n,
(2) the boundary ∂M is a subcomplex.

Proof. This theorem is proved in [Fri23]. ■

Example. In the figure below we consider the smooth manifold M that is given by removing an open disk from a torus. We see that M has a CW-structure such that the boundary ∂M is a subcomplex.



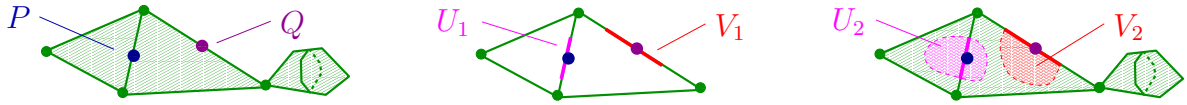
30.2. **Attaching maps and characteristic maps.** Before we continue in developing our language for CW-complexes it is essential to get the following proposition out of the way.

Proposition 30.3. (CW-Complex–Hausdorff Proposition) Every CW-complex is Hausdorff.

Proof. Let X be a CW-complex and let P, Q be two distinct points in X. Recall that $X = \bigcup_{n\in\mathbb{N}_0} X^n$. Thus there exists an $n \in \mathbb{N}_0$ with $P, Q \in X^n$.

Since the empty topological space is Hausdorff we obtain from iteratively applying the Cell Attachment–Hausdorff Proposition 29.4 (2) that $X^0, \dots, X^n$ are Hausdorff. In particular we see that there exist disjoint open neighborhoods U_n of $P \in X^n$ and V_n of $Q \in X^n$.

It follows from iteratively applying the Attachment–Hausdorff Proposition 29.4 (1) that for each $d \geq n$ we can find disjoint open neighborhoods U_d of $P \in X^d$ and V_d of $Q \in X^d$ such that for $d \geq n$ we have $U_d \subset U_{d+1}$ and $V_d \subset V_{d+1}$.



It remains to prove the following claim.

Claim. The sets $U := \bigcup_{d\geq n} U_d$ and $V := \bigcup_{d\geq n} V_d$ are disjoint open neighborhoods of P and Q in X.

Proof. Note that for each $d \geq n$ we have $(U \cap X^d) \cap (V \cap X^d) = U^d \cap V^d = \varnothing$. This shows that U and V are disjoint. It follows immediately from the Direct Limit–Open Subset Lemma 25.5 that U and V are open subsets of X. ■

Before we can continue with the theory of CW-complexes we need to introduce the following definition.

Definition. Let X be a CW-complex

(1) Given $n \in \mathbb{N}_0$ we denote by $\mathrm{Cell}_n(X)$ the set of n-cells of X. Furthermore we denote by $\mathrm{Cell}(X) := \bigcup_{n\in\mathbb{N}_0} \mathrm{Cell}_n(X)$ the set of all cells of X.

(2) Let $c \in \mathrm{Cell}_n(X)$ be an n-cell of X.

(a) An attaching map for c is a continuous map $\varphi\colon S^{n-1} \to X^{n-1}$ such that there exists a homeomorphism

$$X^{n-1} \cup_{X^{n-1} \hookleftarrow S^{n-1} :\varphi} \overline{B}^n \;\to\; \underbrace{X^{n-1} \cup c}_{\text{subspace of } X^n}$$

which is the identity on X^{n-1}.

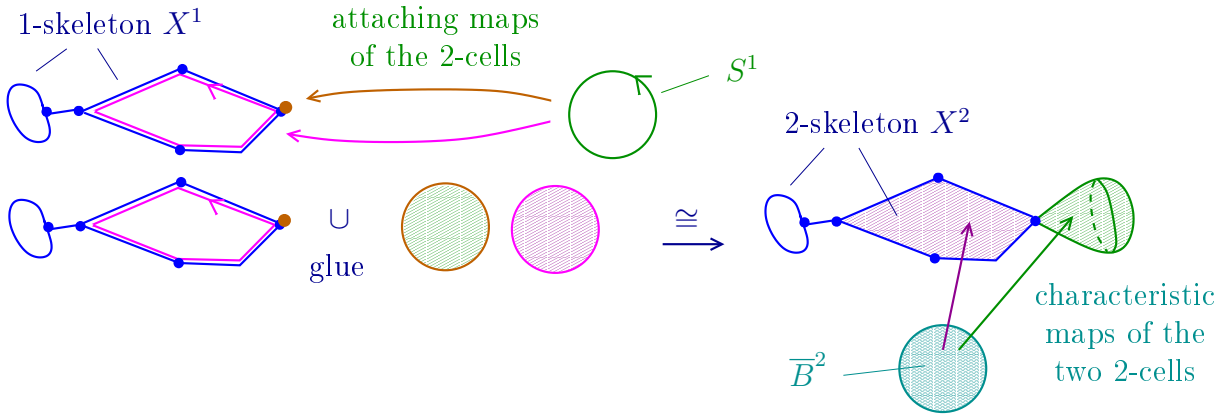
(b) A characteristic map for c is a continuous map $\Phi\colon \overline{B}^n \to X$ such that the map

$$X^{n-1} \cup_{X^{n-1} \hookleftarrow S^{n-1} :\Phi|_{S^{n-1}}} \overline{B}^n \;\to\; X^{n-1} \cup c$$

induced by the identity on X^{n-1} and by the map Φ on $\overline{B}^n$ is well-defined and it is a homeomorphism.

It follows fairly easily from the definitions that a characteristic map, and thus also an attaching map, always exists.

(3) A full set of attaching maps for the cells of the CW-complex X is defined as a family of continuous maps $\{\varphi_c\colon S^{\dim(c)-1} \to X^{\dim(c)-1}\}_{c\in\mathrm{Cell}(X)}$ such that each φ_c is an attaching map for the cell c. Similarly we define a full set of characteristic maps for the n-cells of X.



Using the CW-Complex–Hausdorff Proposition) 30.3 we can prove the following essential proposition.

Proposition 30.4. (Cell-Compact Proposition) Let X be a CW-complex.

(1) Let c be an n-cell of X. If $\Phi\colon \overline{B}^n \to X$ is a characteristic map for c, then $\Phi(\overline{B}^n) = c$.

(2) Every cell is compact.

Proof. Let c be an n-cell of X.

(1) Let $\Phi\colon \overline{B}^n \to X$ be a characteristic map for c. We start out with the following claim.

Claim. The image $\Phi(\overline{B}^n)$ is a closed subset of X.

Proof. Since characteristic maps are continuous and since $\overline{B}^n$ is compact we obtain from the Compact Image Lemma 2.8 that $\Phi(\overline{B}^n)$ is compact. By the CW-Complex–Hausdorff Proposition 30.3 we know that X is Hausdorff. Thus it follows from the Compact-Closed Lemma 1.9 (2) that $\Phi(\overline{B}^n)$ is a closed subset of X. ⊞

Note that we have the following inclusions

$$\Phi(\overline{B}^n) \quad \underset{\substack{\uparrow \\ \text{by an elementary argument}}}{\subset} \quad \underbrace{\overline{\Phi(B^n)}}_{=c=\overline{f}} \quad \underset{\substack{\uparrow \\ \text{definition of the closure}}}{=} \quad \bigcap \text{closed subsets containing } \Phi(B^n) \quad \underset{\substack{\uparrow \\ \text{by the claim}}}{\subset} \quad \Phi(\overline{B}^n).$$

It follows that $\overline{\Phi(B^n)} = \Phi(\overline{B}^n)$. Next recall that by definition of an n-cell there exists an open n-cell $f \subset X$ with $\overline{f} = c$. It follows easily from the definition of a characteristic map that the restriction of Φ to B^n defines a homeomorphism $\Phi\colon B^n \to f$. It follows from the above that we have $c = \overline{f} = \overline{\Phi(B^n)} = \Phi(\overline{B}^n)$.

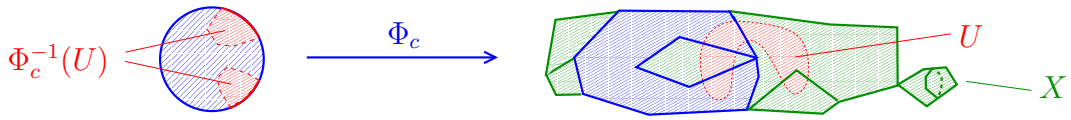
(2) This statement follows from (1) and the Compact Image Lemma 2.8. ■

Proposition 30.5. (Characteristic Maps Proposition) Let X be a CW-complex. Furthermore let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X^{\dim(c)}\}_{c\in \mathrm{Cell}(X)}$ be a full set of characteristic maps for the cells of X.

(1) Given any subset $U \subset X$ the following statements are equivalent:
   (a) U is open in X.
   (b) For every $c \in \mathrm{Cell}(X)$ the intersection $c \cap U$ is open in c.
   (c) For every $c \in \mathrm{Cell}(X)$ the preimage $\Phi_c^{-1}(U)$ is open in $\overline{B}^{\dim(c)}$.

   The same statements also hold if we replace "open" by "closed".

(2) Let $f\colon X \to Y$ be a map to a topological space Y. The following statements are equivalent:
   (a) f is continuous.
   (b) For every $c \in \mathrm{Cell}(X)$ the restriction of f to c is continuous.
   (c) For every $c \in \mathrm{Cell}(X)$ the map $f \circ \Phi_c\colon \overline{B}^{\dim(c)} \to Y$ is continuous.



Proof. Let X be a CW-complex. We write $C := \mathrm{Cell}(X)$ and given $n \in \mathbb{N}_0$ we write $C_n := \mathrm{Cell}_n(X)$. By definition of a CW-complex we know that for each $n \in \mathbb{N}_0$ there exists a family $\{\varphi_c\colon S^{n-1} \to X^{n-1}\}_{c\in C_n}$ of continuous maps and a homeomorphism

$$\Xi_n\colon X^{n-1} \cup_{X \leftarrow \bigsqcup_c S_c^{n-1} \,:\, \bigsqcup_c \varphi_c} \bigsqcup_{c\in C_n} \overline{B}_c^n \;\to\; X^n$$

which is the inclusion on X^{n-1}. We start out with the following technical claim:

Claim. For each $n \in \mathbb{N}_0$ the map

$$\Omega_n\colon X^{n-1} \cup_{X \leftarrow \bigsqcup_{c\in C} S_c^{n-1} \,:\, \bigsqcup_{c\in C} \Phi_c|_{S^{n-1}}} \bigsqcup_{c\in C} \overline{B}_c^n \;\to\; X^n$$

induced by the identity on X^{n-1} and the maps $\Phi_c\colon \overline{B}^n_c \to X^n$ is well-defined and it is a homeomorphism.

Proof. We consider the map

$$\underbrace{X^{n-1} \cup_{X \leftarrow \bigsqcup_{c\in C_n} S^{n-1}_c \,:\, \bigsqcup_{c\in C_n} \Phi_c|_{S^{n-1}}} \bigsqcup_{c\in C_n} \overline{B}^n_c}_{=:\widetilde{S}} \xrightarrow{\Omega_n} X^n \xrightarrow[\cong]{\Xi_n^{-1}} \underbrace{X^{n-1} \cup_{X \leftarrow \bigsqcup_{c\in C_n} S^{n-1}_c \,:\, \bigsqcup_{c\in C_n} \varphi_c} \bigsqcup_{c\in C_n} \overline{B}^n_c}_{=:S}.$$

Since Ξ_n is a homeomorphism it suffices to show that $\Xi_n^{-1}\circ\Omega_n\colon \widetilde{S} \to S$ is a homeomorphism.

First note that one can easily verify that $\Xi_n^{-1} \circ \Omega_n\colon \widetilde{S} \to S$ is a bijection. Since each Φ_c is continuous we obtain from the Topological-Pushout Proposition 6.12 (1b) and the Topological-Coproduct Proposition 5.10 (1b) that $\Xi_n^{-1} \circ \Omega_n\colon \widetilde{S} \to S$ is continuous.

Given $c \in C_n$ we denote by $\widetilde{c} \subset \widetilde{S}$ the image of $\overline{B}^n_c$ on the left hand side. Note that $\Xi_n^{-1} \circ \Omega_n\colon \widetilde{c} \to c$ is a continuous bijection. By the Compact Image Lemma 2.8 we know that $\widetilde{c}$ is compact and by the CW-Complex–Hausdorff Proposition 30.3 we know that c is Hausdorff. Therefore it follows from the Compact-Hausdorff Proposition 2.10 (3) that $\Xi_n^{-1} \circ \Omega_n\colon \widetilde{c} \to c$ is a homeomorphism. We denote by $g_c\colon c \to \widetilde{c}$ its inverse. Furthermore we denote by $p_c\colon \overline{B}^n_c \to S$ the natural map. Since each $g_c \circ p_c$ is continuous we obtain from the Topological-Pushout Proposition 6.12 (1b) and the Topological-Coproduct Proposition 5.10 (1b) that $\Omega_n^{-1} \circ \Xi_n\colon S \to \widetilde{S}$ is continuous. ⊞

Now we turn to the actual proof of the statements of the proposition.

(1) Let $U \subset X$ be a subset. The (a)⇒(b)-direction follows from the definition of the subspace topology and the (b)⇒(c)-direction follows from the continuity of the attaching maps. It remains to deal with the (c)⇒(a)-direction. Thus we assume that U satisfies (c). It follows from the claim and the Topological-Pushout Proposition 6.12 (0) and the Topological-Coproduct Proposition 5.10 (0), applied iteratively for $n = 0, 1, 2, \ldots$, that each $X^n \cap U$ is open. But by definition of a CW-complex this means that U is open in X.

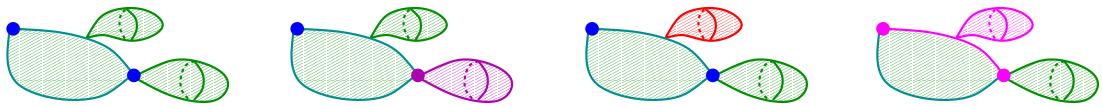
(2) This statement follows immediately from (1). ■

30.3. **Subcomplexes.** We recall the following definition from page 375.

Definition. We say that a CW-complex A is a subcomplex of a CW-complex X if $A \subset X$ and if the following two conditions are satisfied:

(1) For each $n \in \mathbb{N}_0$ we have $A^n = A \cap X^n$.

(2) Each open n-cell of A is also an open n-cell of X.



Example. If X is a CW-complex, then any skeleton X^n, with the obvious filtration, is basically by definition a subcomplex of X.

The above definition of a subcomplex is short and succinct. But in practice it is not always that easy to work with it. In the following lemma we first show that subcomplexes are always closed subsets. Afterwards we will give a convenient characterization of subcomplexes, which in practice is quite helpful.

Lemma 30.6. (Subcomplex–Closed Lemma) Each subcomplex of a CW-complex X is a closed subset of X.

Proof. We provide the somewhat technical proof in [Fri23]. ■

The following proposition gives a very convenient characterisation of subcomplexes.

Proposition 30.7. (Subcomplex–Characterization Proposition) Let X be a CW-complex.

(1) Every subcomplex of X is the union of open cells of X.
(2) Let $\{\sigma_i\}_{i\in I}$ be a family of open cells of X. We set $A := \bigcup_{i\in I} \sigma_i$. The following statements are equivalent:
   (a) A is a subcomplex.
   (b) A is a closed subset of X.
   (c) For every $i \in I$ the closure $\overline{\sigma}_i$ of $\sigma_i \subset X$ is contained in A.

Proof. Also in this case we outsource the rather technical proof to [Fri23]. ■

The Subcomplex–Characterization Proposition 30.7 allows us to prove the following frequently used lemma.

Lemma 30.8. (Subcomplex–Intersection-Union Lemma) Let X be a CW-complex. The union and also the intersection of arbitrarily many subcomplexes of X is again a subcomplex of X.

Proof. Let X be a CW-complex and let $\{A_k\}_{k\in K}$ be a family of subcomplexes of X. For each $k \in K$ let $\{\sigma_i\}_{i\in I_K}$ be the corresponding family of open cells of X.

(1) It follows easily from the Subcomplex–Characterization Proposition 30.7 (1), together with the (a)⇔(b)-direction of the Subcomplex–Characterization Proposition 30.7 (2) that $\bigcup_{k\in K} A_k = \bigcup_{k\in K} \bigcup_{i\in I_K} \sigma_i$ is a subcomplex of X.
(2) It follows from the Subcomplex–Characterization Proposition 30.7 (1), together with the (a)⇔(c)-direction of the Subcomplex–Characterization Proposition 30.7 (2) and the fact that the intersection of closed sets is always closed that $\bigcap_{k\in K} A_k = \bigcup_{i\in \bigcap_{k\in K} I_K} \sigma_i$ is a subcomplex of X. ■

Lemma 30.9. (CW-Complex Pasting Lemma) Let X be a CW-complex and let $\{X_i\}_{i\in I}$ be a family of subcomplexes such that $\bigcup_{i\in I} X_i = X$. The following three statements hold:

(1) (a) A subset $U \subset X$ is open if and only if each $X_i \cap U$ is an open subset of X_i.
   (b) A subset $A \subset X$ is closed if and only if each $X_i \cap A$ is a closed subset of X_i.
(2) A map $f\colon X \to Y$ to a topological space Y is continuous if and only if each map $f|_{X_i}\colon X_i \to Y$ is continuous.

Proof.

(1) (a) Given any subset $U \subset X$ we have

$$U \subset X \text{ is open } \overset{\text{by the Characteristic Maps Proposition 30.5 (1)}}{\Downarrow} \bigwedge_{c \in \mathrm{Cell}(X)} c \cap U \text{ is open in } c \overset{\text{follows from } \cup X_i = X \text{ and the observation that it is a consequence of the Subcomplex–Characterization Proposition 30.7 (a)}\Rightarrow\text{(c) that every cell of some } X_i \text{ is a cell of } X}{\Downarrow} \bigwedge_{i \in I} \bigwedge_{c \in \mathrm{Cell}(X_i)} c \cap U \text{ is open in } c$$
$$\overset{\text{follows from the Characteristic Maps Proposition 30.5 (1) applied to the CW-complex } X_i}{\Uparrow} \bigwedge_{i \in I} X_i \cap U \text{ is open in } X_i.$$

(b) The same logic as in (a) also applies for "closed" instead of "open". This statement is an easy consequence of the Subcomplex–Characterization Proposition 30.7 (a)⇒(c) and the Characteristic Maps Proposition 30.5 (1).

(2) The "only if"-direction is an immediate consequence of the Basics-of-Continuity Lemma 2.1 (3). The "if"-direction follows easily from (1a). ■

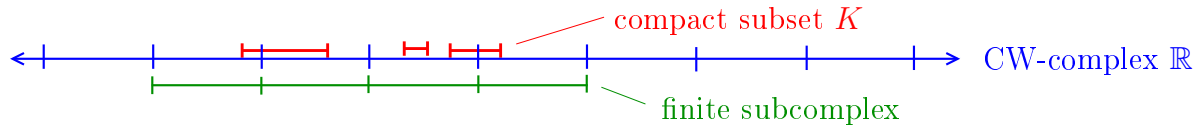
30.4. **The Finiteness Theorem for CW-complexes.** The following proposition is one of the key technical results about infinite CW-complexes.

Theorem 30.10. (CW-Complex-Finiteness Theorem) Every compact (connected) subset of a CW-complex X is contained in a finite (connected) subcomplex.

Example.

(1) We consider the wedge $X := \bigvee_{i \in \mathbb{N}_0} S^1_i$. As discussed on page 374, we can view X as a CW-complex with one 0-cell, given by the wedge point, and 1-cells indexed by $c \in \mathbb{N}_0$. It follows from the CW-Complex-Finiteness Theorem 30.10 that each compact subset of X is contained in the union of finitely many S^1_i.

(2) We consider the real line $\mathbb{R}$ with the CW-structure $\varnothing \subset \mathbb{Z} \subset \mathbb{R}$. In this context the CW-Complex-Finiteness Theorem 30.10 gives us the familiar statement that any compact subset of $\mathbb{R}$ is bounded.



Before we start out with the proof of the CW-Complex-Finiteness Theorem 30.10 we formulate the following pretty corollary.

Corollary 30.11. (CW-Complex Compactness Corollary) A CW-complex is compact if and only if it is finite.

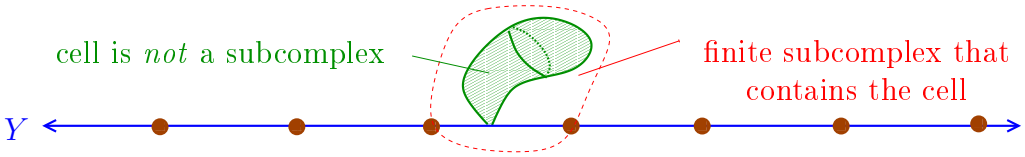
Proof. A finite CW-complex is by definition the union of finitely many cells. Thus it follows from the Cell-Compact Proposition 30.4 (2) that a finite CW-complex is compact.

Conversely let X be a compact CW-complex. It follows from the CW-Complex-Finiteness Theorem 30.10 that X is contained in a finite subcomplex Y of X. But from $X \subset Y \subset X$ it follows that $X = Y$, i.e. X is a finite CW-complex. ■

The proof of the CW-Complex-Finiteness Theorem 30.10 will be a fairly straightforward consequence of the following rather technical lemma.

Lemma 30.12. Let X be a CW-complex.
(1) Let $A \subset X$ be a subset. If every open cell of X contains at most one point of A, then A is closed and discrete.
(2) If $C \subset X$ is a compact subset, then C is contained in a finite union of open cells.
(3) Every cell of X is contained in a finite subcomplex of X.

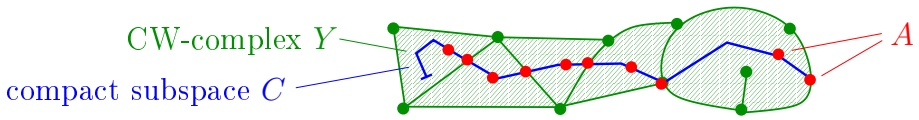
Example. Let X be a CW-complex. The example below shows that a cell itself is not necessarily a subcomplex, but, as predicted by Lemma 30.12, we see in our example that every cell is contained in a finite subcomplex.



Proof of Lemma 30.12. We start out with the following claim.

Claim. For any CW-complex Y the statements (1), (2) and (3) of the lemma are equivalent.

Proof. Let Y be a CW-complex.
(1)⇒(2). Let $C \subset Y$ be a compact subset. For each open cell σ that intersects C we pick a single point in $A \cap \sigma$. We denote the resulting set of points by A. By (1) the set A is closed and discrete. Since A is a closed subset of the compact set C we obtain from the Compact-Closed Lemma 1.9 (1) that A is compact. Therefore A is in fact finite. But that means that C intersects only finitely many cells, i.e. C is contained in a finite union of open cells. This is precisely what we had to show.



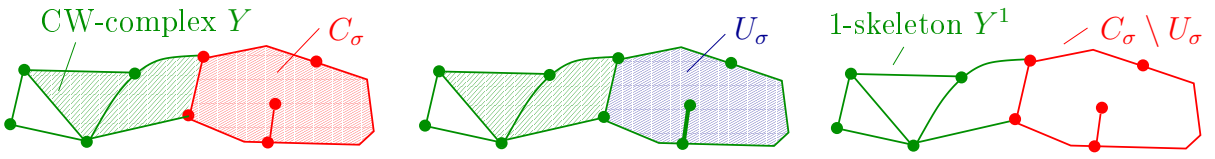
(2)⇒(3). This direction follows from the following stronger subclaim.

Subclaim. For every $n \in \mathbb{N}_0$ the following holds: If Y^{n-1} satisfies (2), then Y^n satisfies (3).

Proof. We prove the claim by induction on $n \in \mathbb{N}_0$. For $n = 0$ there is nothing really to show. Now suppose that the statement holds for some $n \in \mathbb{N}_0$. Let $C_\sigma \subset Y$ be an $(n+1)$-cell. We denote by U_σ the corresponding open cell. Note that by the Cell-Compact Proposition 30.4 (2) we know that C_σ is compact. Furthermore note that it follows from the discussion on page 372 that U_σ is an open subset of Y^{n+1}. In particular we see that U_σ is an open subset of C_σ. Finally note that it follows from these observations and the Compact-Closed Lemma 1.9 that $C_\sigma \setminus U_\sigma$ is a compact subset of the n-skeleton Y^n.

Since we assume that (2) holds for n it follows that the subset $C_\sigma \setminus U_\sigma$ is contained in a finite union of open cells of dimension $\leq n$. By induction each of these open cells is contained in a finite subcomplex of A. By the Subcomplex–Intersection-Union Lemma 30.8 the union of these finitely many finite subcomplexes forms a subcomplex of Y. This subcomplex is clearly finite and it contains C_σ. □

(3)⇒(1). We start out with the following subclaim.



Subclaim. If $B \subset Y$ is a subset such that no two points lie in the same open cell, then B is a closed subset.

Proof. Let $B \subset Y$ be a subset such that no two points lie in the same open cell. We need to show that B is a closed subset of X. By the Characteristic Maps Proposition 30.5 it suffices to show that the intersection of B with each cell c is closed. Thus let c be a cell of X. By (3) we know that c is contained in a finite subcomplex Y'. Since B has at most one point in common with any open cell we see that the intersection $B \cap Y'$ must be finite. Thus the intersection $B \cap c$ is finite. It follows from the CW-Complex–Hausdorff Proposition 30.3 that $B \cap c$ is a closed subset of c. □

Now let $A \subset Y$ be a subset such that no two points lie in the same open cell. By the subclaim we know that A is a closed subset of Y. We still need to show that A is discrete. Note that for any point $x \in A$ the subset set $A \setminus \{x\}$ also satisfies the hypothesis of the subclaim. Thus $A \setminus \{x\}$ is a closed subset of X. Hence $\{x\}$ is open in A, so A is discrete. ⊞

Finally we turn to the actual proof of the lemma. Let X be a CW-complex. We need to show that for X all three statements hold. First note that Statement (1) holds for X^0 by definition of the topology on X^0. Now suppose that for some $n \in \mathbb{N}_0$ we have (1) for the n-skeleton X^n By claim we know that (2) and (3) also hold for X^n. By the subclaim in the proof of (2)⇒(3) we know that (3) actually holds for X^{n+1}. Thus we also have (1) for X^{n+1}. Summarizing we have all three statements for X^n for all $n \in \mathbb{N}_0$. But any cell lies in some X^n, and so we know (3) for X itself, and by the claim this implies that (1) and (2) also hold for X. ■

Proof of the CW-Complex-Finiteness Theorem 30.10. Let X be a CW-complex and let $C \subset X$ be a compact subset. By Lemma 30.12 (2) the subset C is contained in a union of finitely many open cells. By Lemma 30.12 (3) each of these open cells is contained in a finite subcomplex. The union of these finitely many finite subcomplexes is a finite subcomplex Z which contains C.

If C is connected, then we take the component of Z that contains C. ■

30.5. **Fundamental groups of CW-complexes.** Before we head to the fundamental groups of CW-complexes let us quickly study the path-connectedness of CW-complexes. The following lemma says that the path-connectedness depends only on the 1-skeleton.

Lemma 30.13. (CW-Skeleton-π_0-Lemma) Let X be a CW-complex.
(1) The inclusion induced map $\pi_0(X^0) \to \pi_0(X)$ is a surjection.
(2) For every $k \geq 1$ the inclusion induced map $\pi_0(X^1) \to \pi_0(X)$ is a bijection.

Proof. We leave the proof as an exercise to the reader. ■

Example. We equip S^∞, $\mathbb{R}\mathrm{P}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$ with the usual CW-structures. Their 1-skeleta are $S^1, \mathbb{R}\mathrm{P}^1$ and $\mathbb{C}\mathrm{P}^1$. Since these are path-connected we obtain from the CW-Skeleton-π_0-Lemma 30.13 that S^∞, $\mathbb{R}\mathrm{P}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$ are path-connected.

In a similar fashion the following proposition says that for CW-complexes the fundamental group is determined by the fundamental group of its 2-skeleton.

Proposition 30.14. (CW-Skeleton-π_1-Proposition) Let X be a connected CW-complex and let x_0 be a point in the 0-skeleton of X.

(1) The inclusion induced map $\pi_1(X^1, x_0) \to \pi_1(X, x_0)$ is an epimorphism.
(2) For any $k \geq 2$ the inclusion induced map $\pi_1(X^k, x_0) \to \pi_1(X, x_0)$ is an isomorphism.

Example. It follows from the CW-Skeleton-π_1-Proposition 30.14 together with some of our earlier calculations that

$$\begin{array}{lcllcll} \pi_1(S^\infty) & \cong & \pi_1(\text{2-skeleton of } S^\infty) & \cong & \pi_1(S^2) & = & 0, \\ \pi_1(\mathbb{R}\mathrm{P}^\infty) & \cong & \pi_1(\text{2-skeleton of } \mathbb{R}\mathrm{P}^\infty) & \cong & \pi_1(\mathbb{R}\mathrm{P}^2) & = & \mathbb{Z}_2, \\ \pi_1(\mathbb{C}\mathrm{P}^\infty) & \cong & \pi_1(\text{2-skeleton of } \mathbb{C}\mathrm{P}^\infty) & \cong & \pi_1(\mathbb{C}\mathrm{P}^1) & = & 0. \end{array}$$

Proof.

(1) Let $\gamma\colon [0,1] \to X$ be a loop in (X, x_0). It follows from the Compact Image Lemma 2.8 and the CW-Complex-Finiteness Theorem 30.10 that there exists a finite connected subcomplex A of X with $\gamma([0,1]) \subset A$. We consider the following commutative diagram

$$\begin{array}{ccc} \pi_1(A^1, x_0) & \longrightarrow & \pi_1(X^1, x_0) \\ \downarrow & & \downarrow \\ \pi_1(A, x_0) & \longrightarrow & \pi_1(X, x_0). \end{array}$$

Since A is a finite CW-complex we see that A is obtained from its 1-skeleton A^1 by attaching finitely many cells of dimension ≥ 2. We obtain from iteratively applying the Cell Attachment-π_1-Proposition 29.1 that the left vertical map is an epimorphism. By choice of A we see that $[\gamma]$ lies in the image of the bottom horizontal map. Since the diagram commutes we see that $[\gamma]$ lies in the image of the right vertical map. We have thus shown that the right vertical map is indeed an epimorphism.

(2) The proof of this statement is a variation on the argument of (1) and it follows from the same logic as in the proof of the Direct Limit-π_1-Proposition 26.1. We leave it to the reader to fill in the details. ∎

Proposition 30.15. (CW-π_1-Finite Presentation–Proposition) Let X be a connected CW-complex and let $x_0 \in X$.

(1) If there are only finitely many cells in dimensions 0 and 1, then $\pi_1(X, x_0)$ is finitely generated.
(2) If there are only finitely many cells in dimensions 0, 1 and 2, then $\pi_1(X, x_0)$ is finitely presented.

Proof.

(1) By the CW-Skeleton-π_1-Proposition 30.14 it suffices to show that the fundamental group of the 1-skeleton X^1 is finitely generated. Note that by hypothesis and by the CW-Skeleton-π_0-Lemma 30.13 we know that the 1-skeleton is a finite connected

1-dimensional CW-complex. It follows now easily from the Cell Attachment-π_1-Proposition 29.1, or alternatively from the Graph–CW-Complex Lemma 30.1 together with the Graph-π_1-Proposition 20.9, that $\pi_1(X^1)$ is a finitely generated free group.

(2) As in (1) we see that by the CW-Skeleton-π_1-Proposition 30.14 it suffices to show that a finite connected 2-dimensional CW-complex X has a finitely presented fundamental group. We pick $x_0 \in X$. Let $\varphi_1, \dots, \varphi_k \colon S^1 \to X$ be attaching maps of all of the 2-cells. For $i = 1, \dots, k$ we consider the corresponding path $\widetilde{\varphi}_i \colon [0,1] \to X$ that is given by $\widetilde{\varphi}_i(t) := \varphi(\exp(2\pi \mathrm{i} \cdot t))$. Furthermore, for $i = 1, \dots, k$ we pick a path p_i from x_0 to the point $\widetilde{\varphi}_i(0) = \widetilde{\varphi}_i(1)$ and we write $\psi_i = p_i * \widetilde{\varphi}_i * \overline{p}_i$. Now we see that

$$\pi_1(X) \underset{\uparrow}{\cong} \pi_1(X^1)/\langle\langle [\psi_1] \rangle\rangle \dots /\langle\langle [\psi_k] \rangle\rangle = \pi_1(X^1)/\langle\langle [\psi_1], \dots, [\psi_k] \rangle\rangle$$

by the Change-of-Base Point Proposition 11.6 and by the Cell Attachment-π_1-Proposition 29.1 — follows from the Quotient Presentation–Lemma 21.6

$$= \underset{\uparrow}{} \{\text{finitely generated free group}\}/\langle\langle [\psi_1], \dots, [\psi_k] \rangle\rangle.$$

see the argument in (1)

We have thus shown that $\pi_1(X)$ is a finitely presented group. ■

30.6. **CW-complexes and direct limits.** The following propositions give a convenient variation on the Exhaustion-π_1-Proposition 26.2.

Proposition 30.16. (CW-Direct Limit-π_n-Proposition) Let X be a CW-complex and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subcomplexes such that $X = \bigcup_{i \in \mathbb{N}} X_i$.

(1) We have $X = \operatorname{colim}_{\longrightarrow} X_i$.

(2) For any $n \in \mathbb{N}_0$ and any $w \in X_1$ we have a natural isomorphism[251]

$$\operatorname{colim}_{\longrightarrow} \pi_n(X_i, w) \xrightarrow{\cong} \pi_n\big(\underbrace{\operatorname{colim}_{\longrightarrow} X_i}_{=X}, w\big).$$

Sketch of proof.

(1) This statement follows immediately from the characterization of the direct limit $\operatorname{colim}_{\longrightarrow} X_i$ in the Topological-Direct Limit Proposition 25.4 together with the CW-Complex Pasting Lemma 30.9.

(2) The proof of this proposition is basically identical to the proof of the Exhaustion-π_1-Proposition 26.2. We just need to replace the Compact Subspace–Exhaustion Lemma 20.11 by the CW-Complex-Finiteness Theorem 30.10. ■

Examples.

(1) For ever $n \in \mathbb{N}_0$ we have

$$\pi_n(S^\infty) \underset{\uparrow}{\cong} \operatorname{colim}_{\longrightarrow} \pi_n(S^i) \underset{\uparrow}{=} \operatorname{colim}_{\longrightarrow} (\,? \to \dots \to ? \to \underset{\downarrow}{\pi_n(S^{n+1})} 0 \to 0 \to \dots) \underset{\uparrow}{\cong} 0.$$

by the Direct Limit-π_n Proposition 30.16 — since by the Sphere-Low-π_n Proposition 28.8 we have $\pi_n(S^i) = 0$ for $i > n$ — by the CW-Direct Limit Stabilization Lemma 25.1

[251] For $n = 0$ this is an isomorphism of sets, i.e. a bijection. For $n \geq 1$ this is an isomorphism of groups.

(2) Recall that by the Infinite Projective Space-Sphere Quotient Lemma 25.8 we know that the group $\{\pm 1\}$ acts continuously and discretely on S^∞ and that $S^\infty/\{\pm 1\}$ is homeomorphic to $\mathbb{R}\mathrm{P}^\infty$. Therefore it follows from (1) and the Action-π_1–Theorem 15.1 that $\pi_1(\mathbb{R}\mathrm{P}^\infty) \cong \pi_1(S^\infty/\{\pm 1\}) \cong \mathbb{Z}_2$. Furthermore it follows from (1), the Action–Covering Proposition 14.5 and the Covering-π_n-Isomorphism Proposition 28.10 that for every $n \in \mathbb{N}_{\geq 2}$ we have $\pi_n(\mathbb{R}\mathrm{P}^\infty) \cong \pi_n(S^\infty) = 0$.

31. Homotopies on CW-complexes

In this chapter we study homotopies on CW-complexes. The technical tool for doing so is the all important CW×Intervall Proposition 31.1. We will use this result to show that the infinite dimensional sphere S^∞ is contractible and we will use this result to show that CW-complexes are locally contractible.

31.1. **The topology of the product of a CW-complex with an interval.** We want to study homotopies of continuous maps $X \to Y$ between topological spaces. In other words, we want to study *continuous* maps $X \times [0,1] \to Y$. In general it is quite difficult to show that a given map out of a product is continuous. Fortunately the following important proposition simplifies life quite a bit if X is a CW-complex.

Proposition 31.1. (CW×Intervall Proposition) Let X be a CW-complex.

(1) The following two topologies on $X \times [0,1]$ agree:
   (a) The product topology on the product $X \times [0,1]$.
   (b) The topology on $X \times [0,1]$ viewed as the direct limit $\operatorname{colim}_{\longrightarrow} \underbrace{(X^k \times [0,1])}_{\text{product topology}}$.

(2) Let $f\colon X \times [0,1] \to Y$ be a map to a topological space Y. The following statements are equivalent:
   (a) The map f is continuous where $X \times [0,1]$ is equipped with the product topology.
   (b) For every $k \in \mathbb{N}_0$ the restriction of f to $X^k \times [0,1]$ is continuous.

Proof. First we show that any subset of $X \times [0,1]$ that is open with respect to (a) is also open with respect to (b). Thus let $U \subset X \times [0,1]$ be a subset that is open with respect to the product topology on $X \times [0,1]$. We need to show that U is open in $\operatorname{colim}_{\longrightarrow} (X^k \times [0,1])$. By definition of the direct limit topology this means that we need to show that for each $k \in \mathbb{N}_0$ the intersection $U \cap (X^k \times [0,1])$ is open in $X^k \times [0,1]$ where we equip $X^k \times [0,1]$ with the product topology. By the Product Topology-Basics Lemma 5.3 (1a) we know that the subspace topology is compatible with taking products. In our context this implies that the product topology on $X^k \times [0,1]$ conveniently enough agrees with the subspace topology coming from $X \times [0,1]$. It follows from this discussion that $U \cap (X^k \times [0,1])$ is indeed open in $X^k \times [0,1]$.

The converse, that every subset of $X \times [0,1]$ that is open with respect to (a) is also open with respect to (b) is much more difficult and technical. It is not a particularly interesting task to read the proof, but it is good to know that this direction is non-trivial and that it requires some thought. In this spirit we outsource the proof to [Fri23]. ■

We now consider the infinite-dimensional sphere S^∞ which is defined as[252]

$$S^\infty := \left\{(x_1, x_2, \dots) \in \mathbb{R}^\infty \,\middle|\, \sum_{j=1}^{\infty} x_j^2 = 1\right\} = \operatorname*{colim}_{\longrightarrow} S^i.$$

On page 386 we showed that $\pi_n(S^\infty) = 0$ for every $n \in \mathbb{N}$. Thus none of the homotopy groups prevent S^∞ from being contractible. Indeed the next mind-bending lemma shows that S^∞ is, in contrast to the finite dimensional spheres, contractible.

[252] To simplify the notation we think of S^i as a subset of S^∞ via the injective map $x \mapsto (x, 0)$.

Lemma 31.2. (Infinite Sphere–Contractibility Lemma) The infinite-dimensional sphere S^∞ is contractible.

Proof. We want to show that S^∞ is contractible. It suffices to show that there exists a homotopy H from id_{S^∞} to a constant map. Our goal is to find such H. We set

$$X \;:=\; \{(0, x_2, x_3, \dots) \in \mathbb{R}^\infty \,|\, x_2, x_3, \cdots \in \mathbb{R}\}.$$

Next we consider the maps

$$\begin{array}{rcl} F\colon \mathbb{R}^\infty \times [0,1] & \to & \mathbb{R}^\infty \\ (P,t) & \mapsto & \underbrace{P\cdot(1-t) + (0,P)\cdot t}_{\neq 0 \text{ if } P\neq 0} \end{array} \quad \text{and} \quad \begin{array}{rcl} G\colon X \times [0,1] & \to & \mathbb{R}^\infty \\ (Q,t) & \mapsto & \underbrace{Q\cdot(1-t) + (1,0,\dots)\cdot t}_{\neq 0 \text{ if } Q\neq 0}. \end{array}$$

Note that F_1 takes values in X and note that G_1 is a constant map. For non-zero $P \in \mathbb{R}^\infty$ and $P \in X$ we define $\widetilde{F}(P,t) \in S^\infty$ and $\widetilde{G}(P,t) \in S^\infty$ to be the normalizations of $F(P,t)$ and $G(P,t)$ to length one. Next we consider the map[253]

$$\begin{array}{rcl} H\colon S^\infty \times [0,1] & \to & S^\infty \\ (P,t) & \mapsto & \begin{cases} \widetilde{F}(P,2t), & \text{if } t \in [0,\frac{1}{2}], \\ \widetilde{G}(\underbrace{\widetilde{F}(P,1)}_{\in X}, 2t-1), & \text{if } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

Clearly $H_0 = \mathrm{id}$ and it is straightforward to see that H_1 is a constant map. It is tempting to conclude that we are done. But in fact we still need to show the following claim.

Claim. The map H is continuous.

Proof. The proof proceeds in several steps:

(1) On page 375 we saw that we can equip S^∞ with a CW-structure where each i-skeleton is given by $S^i \subset S^\infty$.

(2) By (1) and by the CW×Intervall Proposition 31.1 (2) it suffices to that show for each $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} H\colon S^n \times [0,1] & \to & S^\infty \\ (x,t) & \mapsto & H((x,0),t) \end{array}$$

is continuous.

(3) Let $n \in \mathbb{N}_0$. Note that $H(S^n \times [0,1]) \subset S^{n+1}$. Thus we obtain the following commutative diagram

$$\begin{array}{ccc} S^n \times [0,1] & \xrightarrow{\;H\;} & S^\infty \\ & \searrow^{H} & \uparrow \\ & & S^{n+1} \end{array} .$$

By the Topological-Direct Limit Proposition 25.4 (1a) we know that the vertical map is a continuous. It follows from (2) and this discussion that it suffices to show that for each $n \in \mathbb{N}_0$ the diagonal map is a continuous.

(4) By (3) and by the Homotopy Combination Lemma 8.2 (1) it suffices to show that for each $n \in \mathbb{N}_0$ the restriction of $H\colon S^n \times [0,1] \to S^{n+1}$ to the subspaces $S^n \times [0,\frac{1}{2}]$ and $S^n \times [\frac{1}{2},1]$ are continuous. But now we are in familiar territory and the continuity is a consequence of elementary arguments. ∎

[253] One can easily verify that the map is well-defined for $t = \frac{1}{2}$.

31.2. **Trees.** In this section we want to return to the study of "trees", which we first encountered on page 130. Our main result on trees builds on the following proposition, which is of interest in its own right, and which is an application of the CW×Intervall Proposition 31.1 (2).

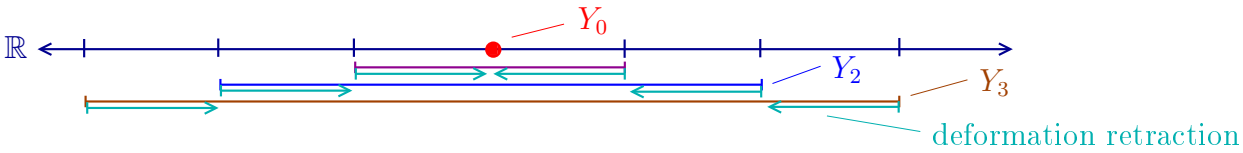
Proposition 31.3. (Iterated Deformation Retract Proposition) Let X be a CW-complex and let $Y_0 \subset Y_1 \subset Y_2 \subset \dots$ be a sequence of subcomplexes of X such that the following two conditions are satisfied:

(1) we have $X = \bigcup_{k\in\mathbb{N}} Y_k$,

(2) each Y_k is a deformation retract of Y_{k+1}.

Then Y_0 is a deformation retract of X.

Example. We consider the CW-complex $\mathbb{R}$ with the usual CW-structure that is given by $\varnothing \subset \mathbb{Z} \subset \mathbb{R}$. For $k \in \mathbb{N}_0$ we consider the subcomplex $Y_k := [-k, k]$. Evidently each Y_k is a deformation retract of Y_{k+1}. Thus we obtain from the Iterated Deformation Retract Proposition 31.3 that $\{0\}$ is a deformation retraction of $\mathbb{R}$. This statement is not very surprising. But it is perhaps somewhat puzzling how one can get out of this infinite family of deformation retractions from Y_{k+1} to Y_k a deformation retraction from $\mathbb{R}$ to $\{0\}$.



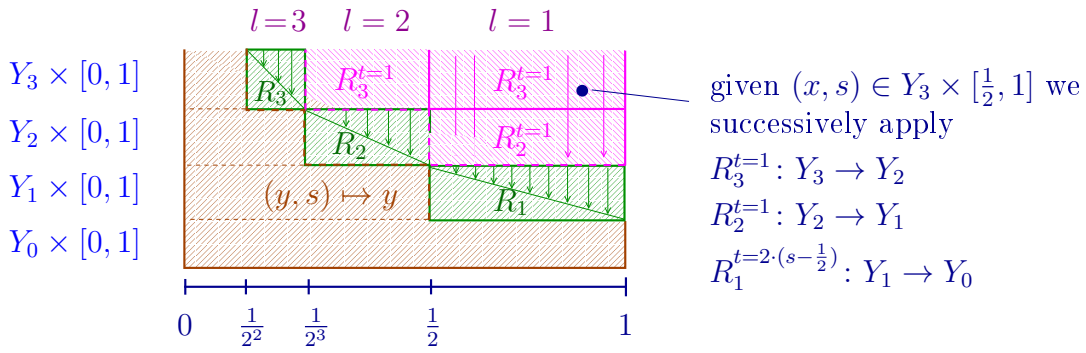
Proof. By hypothesis there exists for each $k \in \mathbb{N}_0$ a deformation retraction

$$R_k \colon Y_k \times [0,1] \;\to\; Y_k$$

from Y_k to Y_{k-1}. We consider the map

$$F \colon X \times [0,1] \;\to\; X$$

$$(x,s) \;\mapsto\; \begin{cases} x, & \text{if } x \in Y_k \text{ and } s \in [0, \frac{1}{2^k}], \\ \underbrace{(R_l^{t=2^l\cdot(s-\frac{1}{2^l})} \circ R_{l+1}^{t=1} \circ \dots \circ R_k^{t=1})(x)}_{\in Y_{l-1}}, & \text{if } x \in Y_k \text{ and } s \in [\frac{1}{2^l}, \frac{1}{2^{l-1}}] \text{ with } 1 \le l \le k. \end{cases}$$



One can easily verify that this map is well-defined. It follows from the Homotopy Combination Lemma 8.2 (1) that the restriction of F to each $Y_k \times [0,1]$ is continuous. Thus it

follows from the CW×Intervall Proposition 31.1 (2) that $F\colon X \times [0,1] \to X$ is continuous. This map is the desired deformation retraction from X to Y_0. ■

Next we turn to the study of trees. On page 130 we defined what it means for an undirected abstract graph to be a tree. We now define what it means for a CW-complex to be a tree:

Definition. A simply connected CW-complex of dimension ≤ 1 is called a tree.

Remark. As we just mentioned, on page 130 we defined what it means for an undirected abstract graph to be a tree. In the Graph–CW-Complex Lemma 30.1 we saw that undirected abstract graphs and CW-complexes of dimension ≤ 1 are essentially the same. It follows from the Tree–π_1-Proposition 20.12 that the two definitions of a tree are compatible.

In the Finite Tree–Contractible Proposition 17.4 we showed that finite trees are contractible. Now we can generalize this statement to arbitrary trees:

Proposition 31.4. (Tree–Contractible Proposition) Let T be a non-empty connected CW-complex of dimension ≤ 1. The following statements are equivalent:
(1) T is a tree.
(2) Each 0-cell is a deformation retract of T.
(3) T is contractible.

Proof. Let T be a non-empty connected CW-complex of dimension ≤ 1. The (2)⇒(3)-direction follows from the Deformation Retract–Homotopy Equivalence Lemma 9.3 and the (3)⇒(1)-direction follows from the Homotopy-π_1-Proposition 13.4.

Thus at this stage it remains to prove the (1)⇒(2)-direction. Let $v \in T$ be a 0-cell. By the Iterated Deformation Retract Proposition 31.3 it suffices to prove the following claim:

Claim. There exists a sequence $Y_0 \subset Y_1 \subset Y_2 \subset \dots$ of subcomplexes of T with the following properties:
(1) We have $\bigcup_{i \in \mathbb{N}_0} Y_i = T$.
(2) Each Y_i is a deformation retract of Y_{i+1}.

Proof. We construct a sequence $\{v\} = Y_0 \subset Y_1 \subset Y_2 \subset \dots$ of subcomplexes of T as follows:[254]

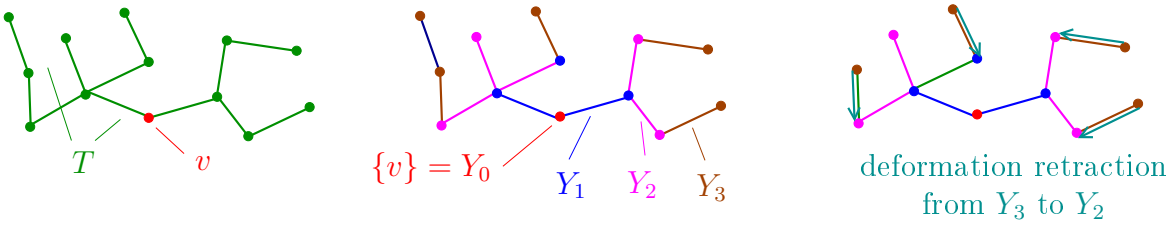
(a) We set $Y_0 := \{v\}$.
(b) Given Y_i we define Y_{i+1} to be the union of Y_i with all 1-cells which contain at least one 0-cell of Y_i.

We make the following observations:

(i) Note that every 1-cell of any CW-complex defines a subcomplex of T.[255] Thus it follows from the Subcomplex–Intersection-Union Lemma 30.8 that Y_{i+1} is a subcomplex of T.

[254] The construction is very similar to a construction that we gave in the proof of the Spanning Tree–Existence Proposition 7.6.
[255] This follows from the observation that any non-empty subset of a 0-cell is all of the 0-cell.

(ii) We set $Y := \bigcup_{i\in\mathbb{N}} Y_i$. We claim that $T = Y$. This can be seen as follows. First note that, basically by construction of Y, we have for each characteristic map[256] $\Phi\colon \overline{B}^n \to T$, $n = 0, 1$ that $\Phi^{-1}(Y) = \varnothing$ or $\Phi^{-1}(Y) = \overline{B}^n$. Since T is 1-dimensional it follows from the Characteristic Maps Proposition 30.5 (1) that Y is a subset of T that is closed and that is open. Since T is connected and since Y is non-empty we see that $Y = T$.



It remains to show that each Y_i is a deformation retract of Y_{i+1}. As we will see, this is almost obvious, except that we need to worry a little bit about continuity issues. Let V be the set of 0-cells in Y_{i+1} that are not contained in Y_i. Note that for each $v \in V$ the corresponding 1-cell contains two 0-cells, namely v and one 0-cell w in Y_i. Finally we consider the map

$$F\colon \overbrace{\Big(\Big(Y_i \sqcup \bigsqcup_{v\in V} [0,1]_v\Big)/v \sim (0,v)\Big)}^{=Y_{i+1}} \times [0,1] \to Y_{i+1} = \Big(Y_i \sqcup \bigsqcup_{v\in V} [0,1]_v\Big)/v \sim (0,v)$$
$$([P],t) \mapsto \begin{cases} P, & \text{if } P \in Y_i, \\ [(x\cdot(1-t),v)], & \text{if } P = (x,v) \in [0,1]_v. \end{cases}$$

It follows almost immediately from the Homotopy Pushout Lemma from [Fri23] that the map F is continuous. It is now basically clear that F defines a deformation retraction from Y_{i+1} to Y_i. ∎

31.3. **The fundamental group of 1-dimensional CW-complexes.** Our next goal is to use the results from the previous section to calculate the fundamental groups of 1-dimensional CW-complexes. To formulate our theorem we need to introduce the following definition.

Definition. Let X be a 1-dimensional CW-complex. A spanning tree is a subcomplex T of X that is a tree and that contains all 0-cells of X.[257]

[256] For $n = 0$ the statement is a tautology. For $n = 1$ note that if $\Phi^{-1}(Y) \neq \varnothing$, then there exists an $i \in \mathbb{N}_0$ with $\Phi^{-1}(Y_i) \neq \varnothing$. Since Y_i is a subcomplex we see that $\Phi^{-1}(Y_i)$ contains at least one 0-cell. But by construction this means that $\Phi^{-1}(Y_{i+1}) = \overline{B}^1$.

[257] On page 131 we introduced the notion of a spanning tree for an undirected abstract graph. It follows immediately from the definitions together with the Tree–π_1-Proposition 20.12 that the two notions of spanning tree match under the correspondence given by the Graph–CW-Complex Lemma 30.1.

The following theorem can be viewed as a generalization of the Graph-π_1-Proposition 20.9 and the Graph-Homotopy Type Corollary 17.6.

Theorem 31.5. (1-Complex-π_1-Theorem) Let X be a connected non-empty 1-dimensional CW-complex X.

(1) X admits a spanning tree.

Let T be a spanning tree for X and let I be the set of 1-cells of X that are not contained in T.

(2) (a) X is homotopy equivalent to $\bigvee_{i\in I} S^1$.

(b) Given any $x_0 \in X$ the group $\pi_1(X, x_0)$ is isomorphic to the free group on the set I.

(3) If the CW-complex X is finite, then we have $\#I = 1 - \chi(X)$, where the Euler characteristic $\chi(X)$ of X is defined as $\chi(X) := \#\{\text{0-cells}\} - \#\{\text{1-cells}\}$.

The proof of the 1-Complex-π_1-Theorem 31.5 rests on the following proposition, which is perhaps of independent interest.

Proposition 31.6. (1-Complex-mod-Tree Proposition) Let X be a 1-dimensional CW-complex. For every subtree T of X the projection $X \to X/T$ is a homotopy equivalence.

Proof of the 1-Complex-mod-Tree Proposition 31.6. The proof of the proposition is basically identical to the proof of the Graph-mod-Tree Proposition 17.5. We just need to replace the Finite Tree–Contractible Proposition 17.4 by the Tree–Contractible Proposition 31.4. ■

Proof of the 1-Complex-π_1-Theorem 31.5. Let X be a connected non-empty 1-dimensional CW-complex X.

(1) It follows almost immediately from the Spanning Tree–Existence Proposition 7.6 together with the correspondence given by the Graph–CW-Complex Lemma 30.1 that X admits a spanning tree.

Let T be a spanning tree for X and let I be the set of 1-cells of X that are not contained in T.

(2) (a) By the 1-Complex-mod-Tree Proposition 31.6 we know that the projection $X \to X/T$ is a homotopy equivalence. It follows from the Abstract Graph–Quotient Lemma 7.9 that X/T admits the structure of a 1-dimensional CW-complex with a single 0-cell and one 1-cell for each $i \in I$. It follows from the discussion on page 221 that X/T is homeomorphic to the wedge $\bigvee_{i\in I} S^1$.

(b) This statement follows from (a) together with with the Homotopy-π_1-Proposition 13.4 and the Wedge-of-Circles–π_1-Lemma 20.7.

(3) This statement follows from the Graph-Homotopy Type Corollary 17.6 together with the Graph–CW-Complex Lemma 30.1. ■

31.4. **Local properties of CW-complexes.** The following, rather technical proposition, says that CW-complexes are locally in many ways very well-behaved.

[257] The two paths exist since a tree is in particular connected.

Proposition 31.7. (CW-Complex–Local Properties Proposition)

(1) Let X be a CW-complex, let $x \in X$ and let U be an open neighborhood of x. There exists an open neighborhood V of x that is contained in U and that admits a deformation retraction to $\{x\}$.

(2) CW-complexes are locally contractible, locally path-connectedand locally simply-connected.

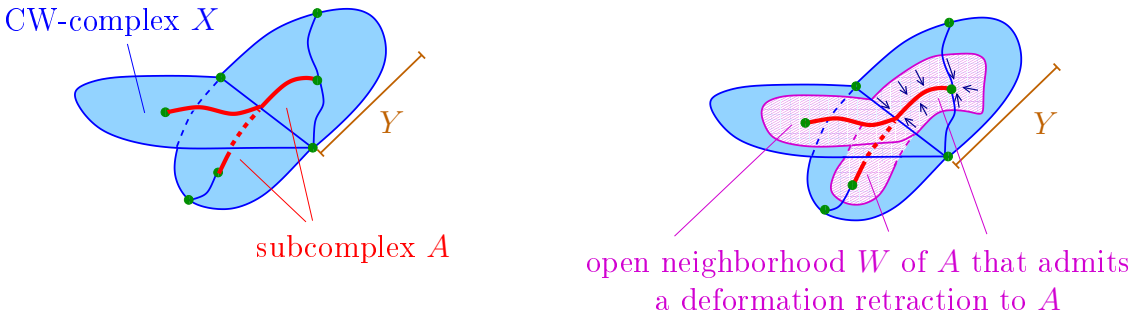
Proof. The proof of this proposition is technically fairly demanding and it also rests on the CW×Intervall Proposition 31.1 (2). Since the proof, despite or because of the difficulty, is not very interesting to read we hide it in [Fri23]. ■

On several occasions we will need the following variation on the CW-Complex–Local Properties Proposition 31.7.

Proposition 31.8. (Subcomplex–Neighborhood Proposition) Let X be a CW-complex.

(1) For each subcomplex A of X there exists an open neighborhood W of A and a deformation retraction $F\colon W \times [0,1] \to W$ from W to A.

(2) One can choose the deformation retraction such that for each subcomplex Y of X the map F restricts to a deformation retraction $F\colon (Y \cap W) \times [0,1] \to Y \cap W$ from $Y \cap W$ to $Y \cap A$.



Proof. The proof of this proposition is similar to the proof of the CW-Complex–Local Properties Proposition 31.7. Again we refer to [Fri23] for details. ■

32. Coverings of CW-complexes

Let X be a connected CW-complex, let $x_0 \in X$ and let $\Gamma \subset \pi_1(X, x_0)$ be a subgroup. By the CW-Complex–Local Properties Proposition 31.7 we know that X is locally simply connected. It follows from the Covering Existence Theorem 27.2 that there exists an (essentially unique) covering $p\colon \widetilde{X} \to X$ with $p_*(\pi_1(\widetilde{X}, \widetilde{x})) = \Gamma$. This discussion shows that CW-complexes tend to have many coverings. In this short chapter we will see that any covering of a CW-complex inherits a natural CW-structure. We will use this topological result to prove several "topology free" statements in group theory.

32.1. **Coverings of CW-complexes.** In this section we will see that, as promised above, a covering of a CW-complex inherits a natural CW-structure.

Proposition 32.1. (CW-Complex Covering Proposition) Let $p\colon \widetilde{X} \to X$ be a connected covering of a non-empty CW-complex X.

(1) The filtration

$$\varnothing = p^{-1}(X^{-1}) \subset p^{-1}(X^0) \subset p^{-1}(X^1) \subset \ldots \subset \widetilde{X}$$

defines a CW-structure for $\widetilde{X}$ which has the property that for any $k \in \mathbb{N}_0$ we have[258][259]

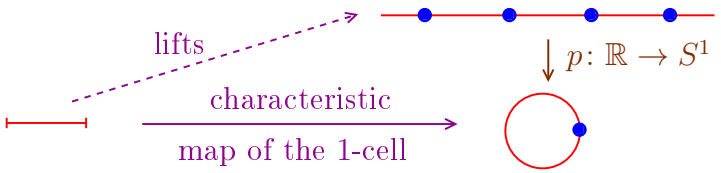
$$\#k\text{-cells of } \widetilde{X} \;=\; \big[\widetilde{X} : X\big] \cdot \#k\text{-cells of } X.$$

(2) (a) Each characteristic map $\Phi\colon \overline{B}^n \to X^n$ admits precisely $[\widetilde{X} : X]$ lifts.

(b) Let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c \in \mathrm{Cell}(X)}$ be a full set of characteristic maps for X. The set of all lifts of all these characteristic maps is a full set of characteristic maps for $\widetilde{X}$.

Examples.

(1) We equip S^1 with the CW-structure $\varnothing \subset \{1\} \subset S^1$ consisting of one 0-cell and one 1-cell. We denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$ the universal covering of S^1. It follows from the CW-Complex Covering Proposition 32.1, that a CW-structure on $\mathbb{R}$ is given by $\varnothing \subset p^{-1}(\{1\}) \subset S^1$, i.e. by $\varnothing \subset \mathbb{Z} \subset \mathbb{R}$.

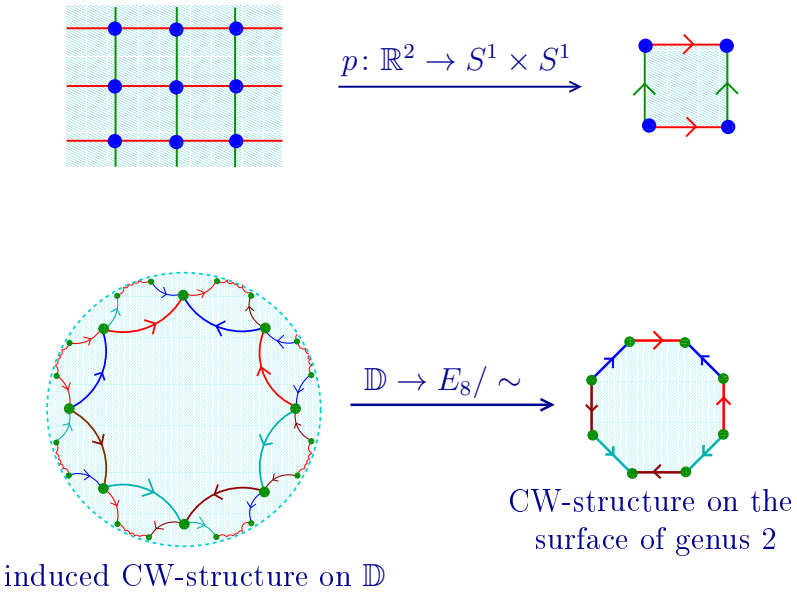


(2) We equip $S^1 \times S^1 =_i ([0,1] \times [0,1])/\sim$ with the usual CW-structure with one 0-cell, two 1-cells and one 2-cell. We consider the universal covering $\mathbb{R}^2 \to S^1 \times S^1$. The resulting CW-structure on $\mathbb{R}^2$ is illustrated in the figure below.

(3) We consider $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$. Recall that Hadamard's Theorem 27.8 says that there exists a covering map $\mathbb{D} \to H_8/\sim \cong E_8/\sim$. If we equip $E_8/\sim$ with the "usual" CW-structure then we obtain from the CW-Complex Covering Proposition 32.1 the CW-structure on $\mathbb{D}$ that is shown in the figure below.

[258] Recall that on page 189 we defined $[\widetilde{X} : X] := \#p^{-1}(x)$ where x is any point in X. Since X is non-empty and connected this definition makes sense.

[259] Note that this is an equality of cardinalities. For example, if $[\widetilde{X} : X]$ is infinite countable and if X has finitely many k-cells, then $\widetilde{X}$ has infinitely many, but countably many, k-cells.



Proof. We start out with an observation. Let $\Phi\colon \overline{B}^n \to X$ be any continuous map. Since each $\overline{B}^n$ is simply connected and locally path-connected we obtain from the Map Lifting Criterion 27.1 and the Lift Uniqueness Proposition 14.7 that for each $y \in p^{-1}(\Phi(0))$ there exists a unique lift $\widetilde{\Phi}_y\colon \overline{B}^n \to \widetilde{X}$ of $\Phi\colon \overline{B}^n \to X$ with $\widetilde{\Phi}_y(0) = y$. Note that this observation already implies (2a).

We turn to the proof of (1) and (2b). We introduce the following notation:

- For $n \in \mathbb{N}_0$ we set $\widetilde{X}^n := p^{-1}(X^n)$.
- Let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c\in \mathrm{Cell}(X)}$ be a full set of characteristic maps for the CW-complex X. By the above observation we obtain for each $c \in \mathrm{Cell}(X)$ the family of continuous maps $\{\widetilde{\Phi}_{c,y}\colon \overline{B}^{\dim(c)} \to \widetilde{X}\}_{y\in p^{-1}(\Phi_c(0))}$.

Statements (1) and (2b) follow from the following two claims.

Claim 1. A set $U \subset \widetilde{X}$ is open if and only if for each $n \in \mathbb{N}_0$ the intersection $\widetilde{X}^n \cap U$ is open in $\widetilde{X}^n$.

Claim 2. Let $n \in \mathbb{N}_0$. We consider the map

$$\Theta\colon \widehat{X}^n := \widetilde{X}^{n-1} \cup \bigcup_{c\in \mathrm{Cell}_n(X)} \bigcup_{y\in p^{-1}(\Phi_c(0))} \overline{B}^n_{c,y} \;\to\; \widetilde{X}^n$$

that is induced by the identity on $\widetilde{X}^{n-1}$ and by the maps $\widetilde{\Phi}_{c,y}\colon \overline{B}^n \to \widetilde{X}^n$. This map is a homeomorphism.

Proof of Claim 1. Since $p\colon \widetilde{X} \to X$ is a covering there exists an open cover $\{\widetilde{W}_i\}_{i\in I}$ of $\widetilde{X}$ such that for each $i \in I$ the image $p(\widetilde{W}_i) =: W_i$ is an open subset of X and that the map

$\widetilde{p}_i := p|_{\widetilde{W}_i} \colon \widetilde{W}_i \to W_i$ is a homeomorphism. Now we see that

$$
\begin{aligned}
& \bigwedge_{n\in\mathbb{N}_0} \widetilde{X}^n \cap U \text{ is open in } \widetilde{X}^n \\
\iff & \bigwedge_{n\in\mathbb{N}_0} \bigwedge_{i\in I} \widetilde{W}_i \cap (\widetilde{X}^n \cap U) \text{ is open in } \widetilde{W}_i \cap \widetilde{X}^n && \text{(Cover-Open Subset Lemma 1.5)} \\
\iff & \bigwedge_{n\in\mathbb{N}_0} \bigwedge_{i\in I} p_i(\widetilde{W}_i \cap (\widetilde{X}^n \cap U)) \text{ is open in } p_i(\widetilde{W}_i \cap \widetilde{X}^n) && \text{(since each } p_i \text{ is a homeomorphism)} \\
\iff & \bigwedge_{i\in I} \bigwedge_{n\in\mathbb{N}_0} (W_i \cap p_i(U)) \cap X^n \text{ is open in } W_i \cap X^n && \text{(since } \widetilde{X}^n = p^{-1}(X^n) \text{ and } p(\widetilde{W}_i) = W_i) \\
\iff & \bigwedge_{i\in I} \bigwedge_{n\in\mathbb{N}_0} (W_i \cap p_i(U)) \cap X^n \text{ is open in } X^n && \text{(since } W_i \text{ is open in } X) \\
\iff & \bigwedge_{i\in I} W_i \cap p_i(U) \text{ is open in } W_i && \text{(since } X \text{ is a CW-complex)} \\
\iff & \bigwedge_{i\in I} \widetilde{W}_i \cap U \text{ is open in } \widetilde{W}_i && \text{(since } p_i \text{ is a homeomorphism)} \\
\iff & U \text{ is open in } \widetilde{X} && \text{(Cover-Open Subset Lemma 1.5).}
\end{aligned}
$$

⊞

Proof of Claim 2. First note that one can easily verify that Θ is a bijection. Also note that it follows from the Topological-Pushout Proposition 6.12 (0) that the map Θ is continuous. To show that it is actually a homeomorphism we have to work a little harder.

To do so we consider the map

$$\widehat{p} \colon \widehat{X}^n = \widetilde{X}^{n-1} \cup \bigcup_{c\in \mathrm{Cell}_n(X)} \bigcup_{y\in p^{-1}(\Phi_c(0))} \overline{B}^n_{c,y} \;\to\; X^n$$

that is induced by the projection $p \colon \widetilde{X}^{n-1} \to X^{n-1}$ and the maps $\overline{B}^n_{c,y} \to \overline{B}^n_c \xrightarrow{\Phi_c} X^n$. It follows again from the Topological-Pushout Proposition 6.12 (0) that $\widehat{p}$ is continuous. Also note that p satisfies $p \circ \Theta = \widehat{p} \colon \widetilde{X}^n \to X^n$, i.e. we have the following little commutative diagram:

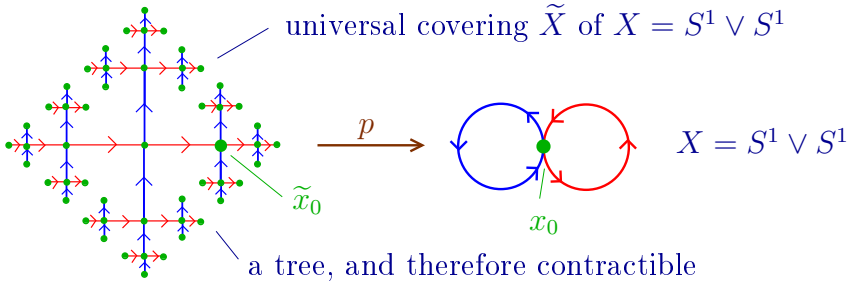
$$\begin{array}{ccc} \widehat{X}^n & \xrightarrow{\quad\Theta\quad} & \widetilde{X}^n \\ & {\scriptstyle\widehat{p}}\searrow \quad \swarrow{\scriptstyle p} & \\ & X^n. & \end{array}$$

It is not hard to show that $\widehat{p} \colon \widehat{X}^n \to \widetilde{X}^n$ is a covering. (We leave it to the reader to write down the details.) We now see that p and $\widehat{p}$ are both local homeomorphisms. It follows from $p \circ \Theta = \widehat{p}$ that Θ is a local homeomorphism. Since Θ is a bijection it follows that Θ is a homeomorphism. ■

The CW-Complex Covering Proposition 32.1, together with the Tree–Contractible Proposition 31.4, allows us to calculate the higher homotopy groups of 1-dimensional CW-complexes.

Lemma 32.2. (1-Complex-π_n-Lemma) Let X be a connected 1-dimensional CW-complex and let $x_0 \in X$ be a base point. Then $\pi_n(X, x_0) = 0$ for every $n \geq 2$.

Proof. Let X be a connected 1-dimensional CW-complex, let $x_0 \in X$ and let $n \geq 2$. By the CW-Complex–Local Properties Proposition 31.7 we know that X is locally simply connected. It follows from the Covering Existence Theorem 27.2 that there exists a universal covering $p \colon \widetilde{X} \to X$, i.e. a covering such that $\widetilde{X}$ is simply connected. We pick $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = x_0$.



We perform the following calculation:

$$\begin{array}{ccccc}
\text{by the Covering-}\pi_n\text{-Isomorphism Proposition 28.10, since } n \geq 2 & & \text{by the CW-Complex Covering Proposition 32.1} & & \\
\downarrow & & \downarrow & & \\
\pi_n(X, x_0) & \xleftarrow{\cong} & \pi_n(\widetilde{X}, \widetilde{x}_0) & = & \pi_n(\text{simply connected 1-dimensional CW-complex}) \\
 & & & = & \pi_n(\text{contractible topological space}) \quad = \quad 0. \\
 & & & \uparrow & \uparrow \\
 & & \text{by the Tree–Contractible Proposition 31.4} & & \text{by the Homotopy-}\pi_n\text{-Proposition 28.6}
\end{array}$$

■

32.2. **Subgroups of finitely presented groups.** We conclude this chapter with two purely group theoretic results, which we will prove using topological methods. In particular the CW-Complex Covering Proposition 32.1 will play a prominent role in both proofs. We start out with the following result.

Proposition 32.3. (Finite-Index Subgroup–Presentation Proposition) Every finite-index subgroup of a finitely presented group is finitely presented.

We will prove the Finite-Index Subgroup–Presentation Proposition 32.3 by switching from the group-theoretic setting to the topological setting. The key to doing so is the following construction, which associates to any finite presentation a 2-dimensional CW-complex.

Construction. Let $\langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle$ be a finite presentation. For $i = 1, \ldots, k$ let S^1_i be a copy of S^1. For $j = 1, \ldots, m$ we do the following:

(1) We write the relation r_j as a reduced word $g_{s_1}^{\epsilon_1} \cdot \ldots \cdot g_{s_n}^{\epsilon_n}$ with $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$.

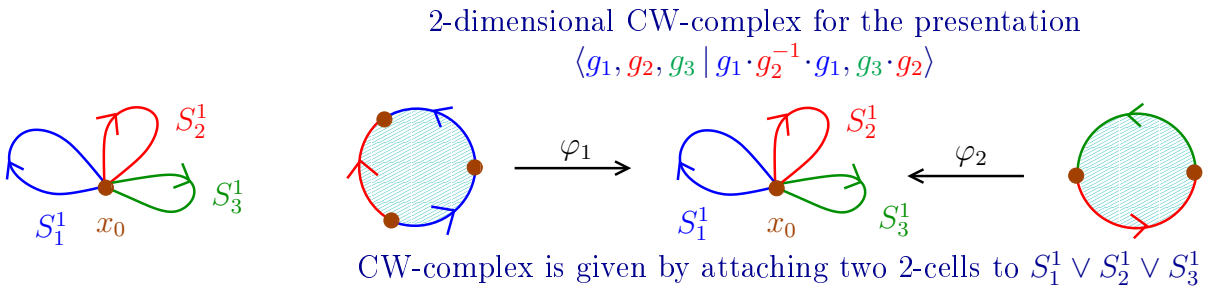
(2) We denote by $\varphi_j \colon S^1 =_i [0,1]/0 \sim 1 \to \bigvee_{i=1}^{k} S^1_i$ the (hopefully obvious) path which on the interval $[\frac{i-1}{n}, \frac{i}{n}]$ travels through $S^1_{s_i}$ at constant speed in the direction determined by ϵ_i.

We denote by Y the CW-complex that is given by attaching m 2-cells to $\bigvee_{i=1}^{k} S^1_i$ along the attaching maps $\varphi_1, \ldots, \varphi_m$. We refer to Y as the CW-complex associated to the presentation.

Now we can state the following unsurprising lemma.

Lemma 32.4. (Presentation–CW Lemma) Let $\langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle$ be a finite presentation for a group π. If Y is the associated CW-complex, then $\pi_1(Y) \cong \pi$.

Proof of the Presentation–CW Lemma 32.4. Let $\pi = \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle$ be a finitely presented group. We use the notation introduced in the above construction of the CW-complex associated to π. We denote by X the wedge of the k circles and we denote by



x_0 the wedge point. By the Wedge-of-Circles–π_1-Lemma 20.7 we can make the identification $\pi_1(X, x_0) = \langle g_1, \dots, g_k \rangle$. This identification has the property that for $j = 1, \dots, m$ the corresponding loop $\varphi_j \colon S^1 \to X$ represents $r_j \in \pi_1(X, x_0) = \langle g_1, \dots, g_k \rangle$. We obtain that

$$\begin{array}{ccccc} & \text{Cell Attachment-}\pi_1\text{-Proposition 29.1, applied } m \text{ times} & & \text{since } [\varphi_j] = r_j & \\ & \downarrow & & \downarrow & \\ \pi_1(Y, x_0) & \cong & \langle g_1, \dots, g_k \rangle / \langle\langle [\varphi_1] \rangle\rangle \dots / \langle\langle [\varphi_m] \rangle\rangle & = & \langle g_1, \dots, g_k \rangle / \langle\langle r_1 \rangle\rangle \dots / \langle\langle r_m \rangle\rangle \\ & = & \langle g_1, \dots, g_k \rangle / \langle\langle r_1, \dots, r_m \rangle\rangle & = & \langle g_1, \dots, g_k \,|\, r_1, \dots, r_m \rangle. \\ & \text{follows from the Quotient Presentation–Lemma 21.6} & & \text{by definition} & \end{array} \qquad \blacksquare$$

Proof of the Finite-Index Subgroup–Presentation Proposition 32.3. Suppose we are given a finitely presented group $\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_m \rangle$. By the Presentation–CW Lemma 32.4 there exists a finite connected 2-dimensional CW-complex Y with $\pi_1(Y) \cong \pi$. Now let Γ be a finite-index subgroup of π. By the CW-Complex–Local Properties Proposition 31.7 (2) together with the Covering Existence Theorem 27.2 there exists a finite covering $p \colon \widetilde{Y} \to Y$ such that $\pi_1(\widetilde{Y}) \cong \Gamma$. By the CW-Complex Covering Proposition 32.1 (1) the 2-complex $\widetilde{Y}$ has finitely many 1-cells and 2-cells. Thus it follows from the CW-π_1-Finite Presentation–Proposition 30.15 that $\Gamma \cong \pi_1(\widetilde{Y})$ is a finitely presented group. $\blacksquare$

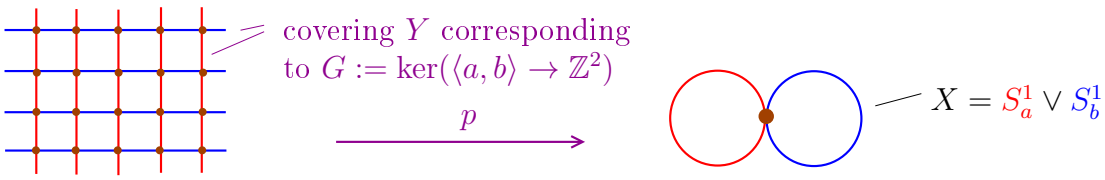
32.3. **Subgroups of free groups.** We conclude this chapter with our second application of our topological methods and results to group theory:

Proposition 32.5. (Free Subgroup Proposition)
(1) Every subgroup of a free group (of any rank) is again a free group.
(2) If F is finitely generated and if G is a finite-index subgroup of F, then

$$\operatorname{rank}(G) - 1 \;=\; [F : G] \cdot (\operatorname{rank}(F) - 1).$$

Proof. Let $F = \langle S \rangle$ be a free group on a set S and let G be a subgroup of F.

(1) We consider the 1-dimensional CW-complex $X = \bigvee_{s \in S} S_s^1$. By the 1-Complex-$\pi_1$-Theorem 31.5 we have $\pi_1(X, P) \cong F$. By the CW-Complex–Local Properties Proposition 31.7 (2) and the Covering Existence Theorem 27.2 there exists a covering $p \colon (Y, Q) \to (X, P)$ of degree $[F : G]$ such that $p_*(\pi_1(Y, Q)) = G$.



Recall that by the Covering-π_1-Monomorphism Proposition 14.11 the induced map $p_*\colon \pi_1(Y,Q) \to \pi_1(X,P)$ is injective. Thus we see that $\pi_1(Y,Q)$ itself is isomorphic to G. By the CW-Complex Covering Proposition 32.1 we can equip Y with the structure of a 1-dimensional CW-complex. We deduce from the 1-Complex-π_1-Theorem 31.5 that $G \cong \pi_1(Y,Q)$ is a free group.

(2) Now we suppose that F is a free group of finite rank and we suppose that G is a finite-index subgroup of F. Then

$$\begin{aligned}
\operatorname{rank}(G) &\overset{\text{by the 1-Complex-}\pi_1\text{-Theorem 31.5}}{\downarrow} \\
&= \#\{\text{1-cells of } Y\} - \#\{\text{0-cells of } Y\} + 1 \\
&= [F:G]\cdot(\underbrace{\#\{\text{1-cells of } X\}}_{=\operatorname{rank}(F)} - \underbrace{\#\{\text{0-cells of } X\}}_{=1}) + 1 \;=\; [F:G]\cdot(\operatorname{rank}(F)-1)+1. \\
&\underset{\text{by the CW-Complex Covering Proposition 32.1}}{\uparrow}
\end{aligned}$$

■

33. Constructions of CW-complexes

In this chapter we will introduce and discuss several ways to construct CW-complexes out of given CW-complexes. In particular we will consider the disjoint union, the pushout, the quotient and the wedge of CW-complexes. We will also consider the somewhat tricky topic of products of CW-complexes.

33.1. **The two categories of CW-complexes.** As fans of category theory it is natural to view CW-complexes as objects of a category. We start out with one obvious way of doing so.

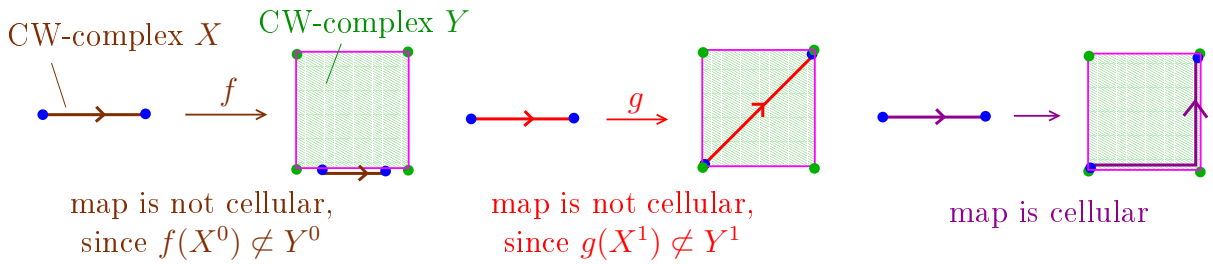
Definition. The full category of CW-complexes is the category $\mathcal{FullCW}$ that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{FullCW}) &:= \text{all CW-complexes,} \\ \mathrm{Mor}(X,Y) &:= \text{all } \textit{continuous} \text{ maps from } X \text{ to } Y. \end{aligned}$$

As we will see, in many settings it is reasonable to consider maps between CW-complexes that respect the CW-structure to a certain degree. This leads us to the following definition.

Definition. A map $f\colon X \to Y$ between two CW-complexes is called cellular if f is continuous and if for each $n \in \mathbb{N}_0$ we have $f(X^n) \subset Y^n$.

Example. In the figure below we consider maps from the CW-complex $X = [0,1]$ to the CW-complex $Y = [0,1] \times [0,1]$, where each is equipped with the "obvious" CW-structure. The first two maps are not cellular, the one to the right is cellular.



It is clear that the composition of two cellular maps is again cellular. This leads us to the following definition.

Definition. The category of CW-complexes is the category $\mathcal{CW}$ that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{CW}) &:= \text{all CW-complexes,} \\ \mathrm{Mor}(X,Y) &:= \text{all } \textit{cellular} \text{ maps from } X \text{ to } Y. \end{aligned}$$

33.2. **Constructions of CW-structures: Pushouts.** In this section we finally turn to constructions of CW-complexes. We start out with the following elementary lemma.

Lemma 33.1. (CW-Disjoint Union Proposition) Let $\{X_i\}_{i\in I}$ be a family of CW-complexes. The disjoint union $\bigsqcup_{i\in I} X_i$ admits a natural CW-structure where each n-skeleton is given by $\bigsqcup_{i\in I} X_i^n$.

(a) The natural maps $X_j \to \bigsqcup_{i\in I} X_i$ are cellular.

(b) If Z is a CW-complex and if $f_i\colon X_i \to Z$, $i \in I$ are continuous (cellular) maps, then the corresponding map $\bigsqcup_{i\in I} f_i\colon \bigsqcup_{i\in I} X_i \to Z$ is also continuous (cellular).

Proof. The proof is elementary, and the more one says, the more confusing it gets. ■

We continue with the following much more interesting proposition.

Proposition 33.2. (CW-Gluing Proposition) Let X and Y be CW-complexes, let A be a subcomplex of X and let $f\colon A \to Y$ be a cellular map. As on page 120 we consider the following topological space:

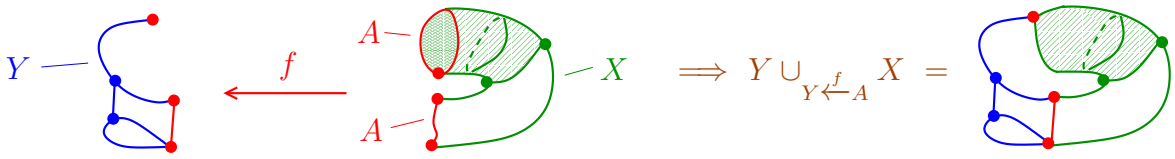
$$Y \cup_{Y \xleftarrow{f} A} X \;:=\; Y \cup_f X \;:=\; (Y \sqcup X)/\!\sim \qquad \text{where } f(a) \sim a \text{ for } a \in A$$

By the Topological-Pushout Proposition 6.12 the following diagram is a pushout diagram in the category $\mathcal{T}op$ of topological spaces:

$$\begin{array}{ccc} A & \hookrightarrow & X \\ \downarrow{\scriptstyle f} & \lrcorner\, p & \downarrow{\scriptstyle x\mapsto [x]} \\ Y & \xrightarrow[q]{y\mapsto [y]} & Y \cup_f X. \end{array}$$

(1) The subspaces $(Y \cup_f X)^n := q(Y^n) \cup p(X^n)$ define a CW-structure on $Y \cup_f X$.

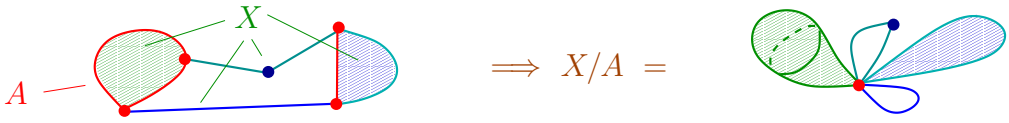
(2) If $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c\in \mathrm{Cell}(X)}$ and $\{\Phi_d\colon \overline{B}^{\dim(d)} \to Y\}_{d\in \mathrm{Cell}(Y)}$ are full sets of characteristic maps of X and Y, then $\{p \circ \Phi_c\colon \overline{B}^{\dim(c)} \to Y \cup_f X\}_{c\in \mathrm{Cell}(X)\setminus \mathrm{Cell}(A)}$ together with $\{q \circ \Phi_d\colon \overline{B}^{\dim(d)} \to Y \cup_f X\}_{d\in \mathrm{Cell}(Y)}$ is a full set of characteristic maps of $Y \cup_f X$.



Proof. The proof is somewhat lengthy, but at the end of the day there are no real technical difficulties. Thus we outsource the proof to [Fri23]. ■

In the following lemmas we will spell out some important special cases of the CW-Gluing Proposition 33.2.

Lemma 33.3. (CW-Quotient Lemma) Let X be a CW-complex and let A be a non-empty subcomplex. We denote by $p\colon X \to X/A$ the natural projection. The subsets $p(X^n)$, $n \in \mathbb{N}_0$, define a CW-structure on X/A.



Proof. On page 120 we pointed out that we have the following pushout diagram:

$$\begin{array}{ccc} A & \hookrightarrow & X \\ \downarrow & x\mapsto [x] & \lrcorner\, \downarrow \\ \{*\} & \longrightarrow & X/A. \end{array}$$

The lemma is thus an almost immediate consequence of the CW-Gluing Proposition 33.2. ■

Lemma 33.4. (CW-Wedge Lemma) Let $\{X_i\}_{i\in I}$ be a family of CW-complexes and suppose that for each $i \in I$ we are given a point $x_i \in X_i^0$. The corresponding wedge $\bigvee_{i\in I}(X_i, x_i)$, admits a natural CW-complex structure.

Proof. It follows from the CW-Disjoint Union Proposition 33.1 that $X := \bigsqcup_{i\in I} X_i$ has a natural CW-structure such that $A := \bigsqcup_{i\in I}\{x_i\}$ is a subcomplex of X. It follows from the CW-Quotient Lemma 33.3 that $X/A = \bigvee_{i\in I}(X_i, x_i)$ inherits a natural CW-structure. ■

Later on we will make good use of the following lemma.

Lemma 33.5. (Skeleton-Quotient Lemma) Let X be a CW-complex and let $n \in \mathbb{N}$. Let $\{\Phi_c\colon \overline{B}^{n+1} \to X^n\}_{c\in \mathrm{Cell}(X)}$ be a full set of characteristic maps of the $(n+1)$-cells of X. The map

$$\begin{aligned} \bigvee_{c\in\mathrm{Cell}(X)} \overbrace{(\overline{B}_c^{n+1}/S_c^n, [S_c^n])}^{\text{the pointed topological space } (\overline{B}^{n+1}/S^n,[S^n]) \text{ is by the Ball Quotient-Sphere Lemma 6.5 homeomorphic to } (S^{n+1},N)} &\to X^{n+1}/X^n \\ [x] &\mapsto [\Phi_c(x)] \quad \text{if } x \in \overline{B}_c^{n+1} \end{aligned}$$

is a homeomorphism.

Example. In the figure below we see the torus together with a CW-structure X that has two 2-cells. It follows from the Skeleton-Quotient Lemma 33.5 that $X/X^1 = X^2/X^1$ is homeomorphic to the wedge of two spheres.

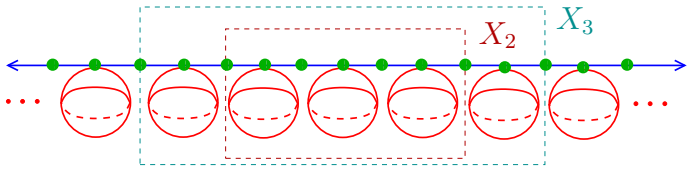
Proof. It is basically clear that the given map $f\colon \bigvee_{c\in\mathrm{Cell}(X)}(\overline{B}_c^{n+1}/S_c^n, [S_c^n]) \to X^{n+1}/X^n$ is a bijection. Next note that it follows immediately from the Twice Quotient Lemma 6.3, the Wedge Proposition 17.1 (1b) and the Topological-Quotient Proposition 6.1 (1) that f is continuous. To upgrade this bijective continuous map to a homeomorphism we have to work a little harder.

To do so note that by the Ball Quotient-Sphere Lemma 6.5 the pointed topological space $(\overline{B}^{n+1}/S^n, [S^n])$ is homeomorphic to (S^{n+1}, N). It follows from the discussion on page 372 that we can view $\overline{B}^{n+1}/S^n$ as a CW-complex where the point $[S^n]$ is the unique 0-cell. Therefore we can use the CW-Wedge Lemma 33.4 and the CW-Quotient Lemma 33.3 to view both sides of the Skeleton-Quotient Lemma 33.5 as CW-complexes. The map f gives a bijection between the cells of the two sides. Using the Compact-Hausdorff Proposition 2.10 (3) one can show that the restriction of f^{-1} to each cell of X^{n+1}/X^n is continuous. It follows from the Characteristic Maps Proposition 30.5 that f^{-1} is indeed continuous. ■

We conclude this section with the following lemma.

Lemma 33.6. (Direct Limit–CW Lemma) Let $\{f_i\colon X_i \to X_{i+1}\}_{i\in\mathbb{N}_0}$ be a sequence of cellular embeddings between CW-complexes. The direct limits $\operatorname{colim}\limits_{\longrightarrow} X_i^n$ of the skeleta of the X_i define a CW-structure on $\operatorname{colim}\limits_{\longrightarrow} X_i$.

Example. The Direct Limit–CW Lemma 33.6 allows us to view the line with infinitely many spheres attached as a CW-complex.

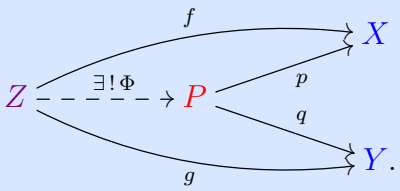


the topological space
$X = \bigcup\limits_{i\in\mathbb{N}_0} X_i = \operatorname{colim}\limits_{\longrightarrow} X_i$

Proof. As so often, it is hard to tell whether a proof is elementary or not. This proof turns out is not that elementary, but it is certainly not very enlightening. Thus we hide the proof in [Fri23]. ■

33.3. **Products of CW-complexes.** The reader will have noticed that a discussion of products is long overdue. As we will see, there was a reason why we were procrastinating. In fact it is perhaps helpful to reflect about what "product" is supposed to mean in the first place. Thus let us recall the following from page 80.

Definition. Let $\mathcal{C}$ be a category and let $X, Y \in \operatorname{Ob}(\mathcal{C})$ be two objects. A product of X and Y in the category $\mathcal{C}$ is some object $P \in \operatorname{Ob}(\mathcal{C})$ together with two morphisms $p\colon P \to X$ and $q\colon P \to Y$ such that the following property is satisfied: Whenever we are given two morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ there exists a unique morphism $\Phi\colon Z \to P$ such that $f = p \circ \Phi$ and $g = q \circ \Phi$. The situation is summarized in the following diagram



By the Product Uniqueness Lemma 4.1 we know that if the product exists, then it is unique in a suitable sense.

We remind the reader of the following simplified version of the Topological-Product Proposition 5.1.

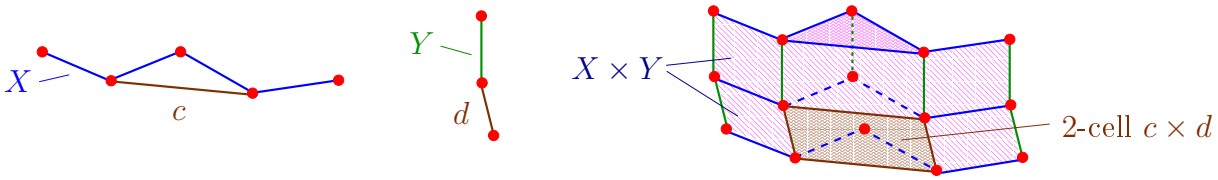
Proposition 5.1. (Topological-Product Proposition) Let X and Y be two topological spaces. We consider the topology given by

$$\mathcal{TP} := \left\{ U \subset X \times Y \;\middle|\; \begin{array}{l} \text{for all } (x,y) \in U \text{ there exist open neighborhoods} \\ A \subset X \text{ of } x \text{ and } B \subset Y \text{ of } Y \text{ with } A \times B \subset U \end{array} \right\}.$$

(1) (a) The natural projections $X \times Y \to X$ and $X \times Y \to Y$ are continuous.
(b) Given a topological space Z and given continuous maps $f\colon Z \to X$ and $g\colon Z \to Y$ the map $(f,g)\colon Z \to (X \times Y, \mathcal{TP})$ that is given by $z \mapsto (f(z), g(z))$ is continuous.

(2) The topological space $X \times Y$ with the natural projections is a product of X and Y in the category $\mathcal{T}op$ of topological spaces.

Now let X and Y be CW-complexes. We want to equip $X \times Y$ with a CW-structure. Perhaps initially it is easier to think in terms of cells instead of filtrations. If c is a p-cell of X and if d is a q-cell of Y, then it seems reasonable that $c \times d$ should be a $(p+q)$-cell of $X \times Y$.



If that is the idea, then it follows from the Characteristic Maps Proposition 30.5 that $X \times Y$ should have the topology that is described in the following definition.

Definition. Let X and Y be two CW-complexes.

(1) Given a p-cell c for X and given a q-cell d for Y we equip c and d with the subspace topology and we equip $c \times d$ with the product topology.

(2) We denote by $\mathcal{CWP}$ the topology on $X \times Y$ that is defined by the condition that $U \subset X \times Y$ is open if and only if each intersection $(c \times d) \cap U \subset c \times d$ is open.

Proposition 33.7. (CW-Category–Product Proposition) Let X and Y be two CW-complexes.

(0) (a) A CW-structure for the topological space $(X \times Y, \mathcal{CWP})$ is given by the filtration

$$(X \times Y)^n \;:=\; \bigcup_{\substack{p,q \in \mathbb{N}_0 \text{ with} \\ p+q=n}} (X^p \times Y^q) \qquad \text{with } n \in \mathbb{N}_0.$$

(b) The n-cells of $X \times Y$ are precisely the sets of the form $c \times d$ where c is a p-cell of X and d is an $(n-p)$-cell of Y.

(1) (a) The natural projections $X \times Y \to X$ and $X \times Y \to Y$ are cellular, in particular they are continuous with respect to $\mathcal{CWP}$.

(b) If Z is a CW-complex and if $f\colon Z \to X$ and $g\colon Z \to Y$ are continuous maps, then the map $Z \to (X \times Y, \mathcal{CWP})$ that is given by $z \mapsto (f(z), g(z))$ is also continuous.[260]

(2) The CW-complex $(X \times Y, \mathcal{CWP})$ with the natural projections is a product of the CW-complexes X and Y in the full category $\mathcal{FullCW}$ of CW-complexes.

In the proof of the CW-Category–Product Proposition 33.7 we will need the following elementary lemma.

Lemma 33.8. (Ball-times-Ball Lemma) Let $p, q \in \mathbb{N}_0$. There exists a homeomorphism $\overline{B}^p \times \overline{B}^q \to \overline{B}^{p+q}$ that restricts to a homeomorphism $(\overline{B}^p \times S^{q-1}) \cup (S^{p-1} \times \overline{B}^q) \to S^{p+q-1}$.

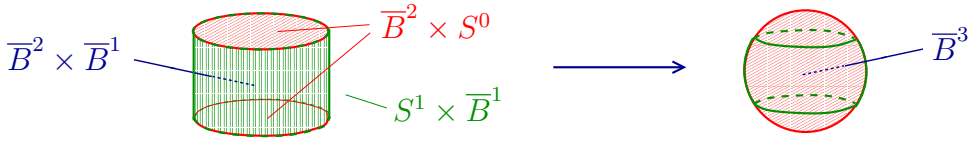
Proof of the Ball-times-Ball Lemma 33.8. This lemma is proved in [Fri23]. ■

Proof of the CW–Product Proposition 33.7. Let X and Y be two CW-complexes.

(0) We start out with the following claim.

[259] It follows easily from the Image-Preimage Lemma 0.3 that this defines a topology on $X \times Y$.

[260] Note that if f and g are cellular, this does *not* mean that the map $(f, g)\colon Z \to X \times Y$ is cellular.



Claim 0. Let $p, q \in \mathbb{N}_0$. We view the skeleta X^p and Y^q as CW-complexes in their own right. The $\mathcal{CWP}$-topology on $X^p \times Y^q$ agrees with the subspace topology coming from $(X \times Y, \mathcal{CWP})$

Proof. We consider the inclusion map $f\colon (X^p \times Y^q, \mathcal{CWP}) \to (X \times Y, \mathcal{CWP})$. Note that it follows from the Subcomplex–Characterization Proposition 30.7 that every cell of X^p is a cell of X and that every cell of Y^q is a cell of Y. It follows basically immediately from this observation and the definition of the CW-product topology that the map $f\colon (X^p \times Y^q, \mathcal{CWP}) \to (X \times Y, \mathcal{CWP})$ is continuous. By the Open-Injective Lemma 2.9 it remains to show that f is a closed map. Let A be a subset that is closed in $(X^p \times Y^q, \mathcal{CWP})$. We need to show that A is closed in $(X \times Y, \mathcal{CWP})$. By definition of $(X \times Y, \mathcal{CWP})$ and by an elementary argument it suffices to show that for each cell c of X and each cell d of Y the intersection $(c \times d) \cap A$ is closed in $c \times d$.

Note that it follows from the Cell-Compact Proposition 30.4, the Subcomplex–Closed Lemma 30.6 and the CW-Complex-Finiteness Theorem 30.10 that there exist finitely many cells $x_1, \dots, x_r$ of X^p such that $c \cap X^p \subset x_1 \cup \dots \cup x_r$ and that there exist finitely many cells $y_1, \dots, y_s$ of X^p such that $d \cap Y^q \subset y_1 \cup \dots \cup y_s$. We now see that

$$\begin{aligned}
(c \times d) \cap A &\overset{\substack{\text{since } A \subset X^p \times Y^q \\ \downarrow}}{=} (X^p \times Y^q) \cap ((c \times d) \cap A) \;=\; ((c \cap X^p) \times (d \cap Y^q)) \cap A \\
&= ((c \cap X^p) \times (d \cap Y^q)) \cap ((x_1 \cup \dots \cup x_r) \times (y_1 \cup \dots \cup y_s)) \cap A \\
&= \underbrace{((c \cap X^p) \times (d \cap Y^q))}_{\substack{\text{closed in } c \times d \text{ since it follows from} \\ \text{the Subcomplex–Closed Lemma 30.6} \\ \text{that } X^p \subset X \text{ and } Y^q \subset Y \text{ are closed}}} \cap \Bigg(\bigcup_{i=1}^{r} \bigcup_{j=1}^{s} \underbrace{((x_i \times y_j) \cap A)}_{\substack{\text{by hypothesis closed} \\ \text{in } x_i \times y_j \text{, thus compact,} \\ \text{thus closed in } c \times d}} \Bigg).
\end{aligned}$$

We have thus shown that $(c \times d) \cap A$ is indeed a closed subset of $c \times d$. ⊞

Now we turn to the actual proof that the filtration $(X \times Y)^n := \bigcup_{p+q=n} (X^p \times Y^q)$ defines a CW-structure on $(X \times Y, \mathcal{CWP})$. The following two claims show that this filtration satisfies the two conditions (CW1) and (CW2) from page 371.

Claim 1. A subset $U \subset (X \times Y, \mathcal{CWP})$ is open if and only if for each $n \in \mathbb{N}_0$ the intersection $(X \times Y)^n \cap U$ is open in $(X \times Y)^n$.

Proof. Let $U \subset X \times Y$. Then:

$$\begin{aligned}
U \text{ is open in } \mathcal{CWP} &\iff \forall_{p \in \mathbb{N}_0} \forall_{q \in \mathbb{N}_0} \forall_{c \in \mathrm{Cell}_p(X)} \forall_{d \in \mathrm{Cell}_q(Y)} (c \times d) \cap U \text{ is open in } c \times d \\
&\underset{\substack{\uparrow \\ \text{follows from Claim 0}}}{\iff} \forall_{p \in \mathbb{N}_0} \forall_{q \in \mathbb{N}_0} (X^p \times Y^q) \cap U \text{ is open in } X^p \times Y^q \\
&\underset{\substack{\uparrow \\ \text{follows from the Pasting Proposition 2.4 (2), since } (X \times Y)^n \text{ is the union} \\ \text{of the finitely many closed subsets } X^p \times Y^q \text{, where } p+q=n}}{\iff} \forall_{n \in \mathbb{N}_0} (X \times Y)^n \cap U \text{ is open in } (X \times Y)^n
\end{aligned}$$

⊞

Claim 2. For each $n \in \mathbb{N}_0$ there exists a set I, a family $\{\varphi_i \colon S^{n-1} \to (X \times Y)^{n-1}\}_{i \in I}$ of continuous maps and a homeomorphism

$$(X \times Y)^{n-1} \cup_{X \times Y \hookleftarrow \bigsqcup_{i \in I} S_i^{n-1} \,:\, \bigsqcup_{i \in I} \varphi_i} \bigsqcup_{i \in I} \overline{B}_i^n \;\to\; (X \times Y)^n$$

which is the inclusion on $(X \times Y)^{n-1}$.

Proof. Let $\{\Phi_c \colon \overline{B}^{\dim(c)} \to X\}_{c \in \mathrm{Cell}(X)}$ and $\{\Psi_d \colon \overline{B}^{\dim(d)} \to Y\}_{d \in \mathrm{Cell}(Y)}$ be full sets of characteristic maps of the CW-complexes X and Y. Note that for each $p, q \in \mathbb{N}_0$ the restriction of the map $\Phi_c \times \Psi_d \colon \overline{B}^p \times \overline{B}^q \to X \times Y$ to $(\overline{B}^p \times S^{q-1}) \cup (S^{p-1} \times \overline{B}^q)$ takes values in $(X^p \times Y^{q-1}) \cup (X^{p-1} \cup Y^q) \subset (X \times Y)^{n-1}$. Thus we can consider the topological space

$$(X \times Y)^{n-1} \;\cup_\sim \bigsqcup_{\substack{p,q \in \mathbb{N}_0 \text{ with} \\ p+q=n}} \bigsqcup_{c \in \mathrm{Cell}_p(X)} \bigsqcup_{d \in \mathrm{Cell}_q(Y)} (\overline{B}^p \times \overline{B}^q)$$

where given c, d we use the restrictions of $\Phi_p \times \Psi_q$ to $(\overline{B}^p \times S^{q-1}) \cup (S^{p-1} \times \overline{B}^q)$ to glue $\overline{B}^p \times \overline{B}^q$ to $(X \times Y)^{n-1}$. Next we consider the map

$$(X \times Y)^{n-1} \;\cup_\sim \bigsqcup_{\substack{p,q \in \mathbb{N}_0 \text{ with} \\ p+q=n}} \bigsqcup_{c \in \mathrm{Cell}_p(X)} \bigsqcup_{d \in \mathrm{Cell}_q(Y)} (\overline{B}^p \times \overline{B}^q) \;\to\; (X \times Y)^n$$

that is given by the inclusion on $(X \times Y)^{n-1}$ and that is given by all the product maps $\Phi_c \times \Psi_d \colon \overline{B}^p \times \overline{B}^q \to X \times Y$. It follows from the Topological Pushout Proposition 6.12 $(0*)$ that this map is continuous. It is clearly a bijection. Using the $\mathcal{CWP}$ topology on $(X \times Y)^n$ it is also not difficult to show that the map is closed. It follows from the Open-Injective Lemma 2.9 that the above map is a homeomorphism.

The claim now follows from the above discussion and the observation that the Ball-times-Ball Lemma 33.8 allows us to replace $\overline{B}^p \times \overline{B}^q$ by $\overline{B}^{p+q} = \overline{B}^n$ and it allows us to replace $(\overline{B}^p \times S^{q-1}) \cup (S^{p-1} \times \overline{B}^q)$ by $S^{p+q-1} = S^{n-1}$. ⊞

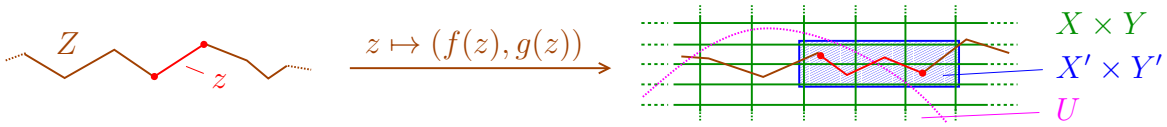
It remains to show that the n-cells of $X \times Y$ are precisely the sets of the form $c \times d$ where c is a p-cell of X and d is an $(n-p)$-cell of Y. But this follows easily from Claim 2.

(1) (a) We consider the natural projection $p \colon (X \times Y, \mathcal{CWP}) \to X$. We need to show that this map is continuous. Note that it suffices to show that for each cell $c \subset X$ and each cell $d \subset Y$ the map $c \times d \to X \times Y \to X$ is continuous. But by the Product Topology-Basics Lemma 5.3 (1a) and the Topological-Product Proposition 5.1 (1a) we know that the maps $c \times d \to (X \times Y, \mathcal{TP})$ and $(X \times Y, \mathcal{TP}) \to X$ are continuous. Thus the composition is also continuous. The same argument of course also shows that the natural projection $p \colon X \times Y \to Y$ is continuous.

(b) Let Z be a CW-complex and let $f \colon Z \to X$ and $g \colon Z \to Y$ be two continuous maps. We need to show that the map $Z \to (X \times Y, \mathcal{CWP})$ that is given by $z \mapsto (f(z), g(z))$ is also continuous.[261] Let $U \subset (X \times Y, \mathcal{CWP})$ be an open subset. We need to show that $(f, g)^{-1}(U)$ is an open subset of the CW-complex Z. By the Characteristic

[261] In the CW-Complex Product Proposition 33.9 we will see that $\mathcal{TP} \subset \mathcal{CWP}$. This implies that it is easier to show that maps *out* of $(X \times Y, \mathcal{CWP})$ are continuous than to show that maps *out* of $(X \times Y, \mathcal{TP})$ are continuous. But this fact also implies that it harder to show that maps *to* $(X \times Y, \mathcal{CWP})$ are continuous

Maps Proposition 30.5 it remains to show that for each cell z of Z the intersection $z \cap (f,g)^{-1}(U)$ is open in z.
It follows from the Cell-Compact Proposition 30.4, the Compact Image Lemma 2.8 and the CW-Complex-Finiteness Theorem 30.10 that there exist finite subcomplexes X' of X and Y' of Y with $f(z) \subset X'$ and $g(z) \subset Y'$.



Claim. The intersection $(X' \times Y') \cap U$ is an open subset of $(X' \times Y', \mathcal{TP})$.

Proof. First note that we have the equality

$$(X' \times Y') \cap U \;=\; \bigcup_{c \in \mathrm{Cell}(X')} \bigcup_{d \in \mathrm{Cell}(Y')} (c \times d) \cap U.$$

Since U is open in $\mathcal{CWP}$ we see that each $(c \times d) \cap U$ is open in $c \times d$. Note that each $c \times d$ is a closed subset of $X' \times Y'$. It follows from these observations, and the fact that the finite CW-complexes X' and Y' have only finitely many cells, together with the Pasting Proposition 2.4 (2) that $(X' \times Y') \cap U$ is indeed an open subset of $X' \times Y'$. ⊞
Next note that

$$z \cap (f,g)^{-1}(U) \;=\; z \cap (f,g)^{-1}((X' \times Y') \cap U).$$

By the elementary Basics-of-Continuity Lemma 2.1 the map $(f,g)\colon Z \to X' \times Y'$ is also continuous. Thus it follows from the above equality and the claim, that $z \cap (f,g)^{-1}(U)$ is indeed an open subset of z.

(2) As in the case of the Topological-Product Proposition 5.1 this statement follows almost immediately from (1). ■

Let X and Y be two CW-complexes. The obvious question that now arises is, under what circumstances does the CW-product topology $\mathcal{CWP}$ that we used in the CW-Category–Product Proposition 33.7 agree with the usual product topology $\mathcal{TP}$. The following proposition shows that in two settings, which are of relevance to us, this is indeed the case.

Proposition 33.9. (CW-Complex Product Proposition) Let X and Y be two CW-complexes.

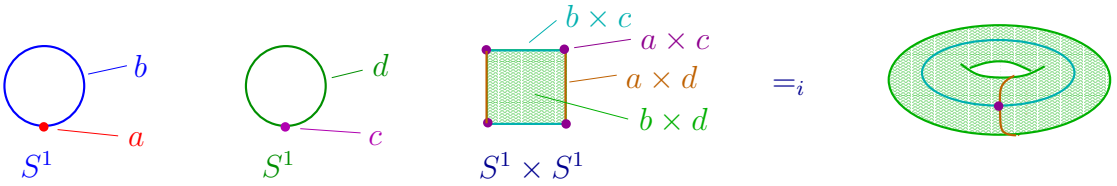
(1) The identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is continuous, in other words, we have the inclusion $\mathcal{TP} \subset \mathcal{CWP}$.
(2) Suppose we are in one of the following two settings:
  (a) X and X are both finite CW-complexes, or
  (b) $Y = [0,1]$ where Y is equipped with the usual CW-structure with two 0-cells and one 1-cell,

  then the identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is a homeomorphism, in other words, we have $\mathcal{CWP} = \mathcal{TP}$. Thus the CW-product structure from the CW-Category–Product Proposition 33.7 is a CW-structure on $(X \times Y, \mathcal{TP})$.

---

than to show that maps *to* $(X \times Y, \mathcal{TP})$ are continuous. This observation explains why the proof of (1b) is trickier than the proof of (1a).

Example. If we equip $S^1 =_i [0,1]/0 \sim 1$ with the usual CW-structure with one 0-cell and one 1-cell, then the product CW-structure on the torus $S^1 \times S^1 =_i ([0,1] \times [0,1])/\sim$ is precisely the CW-structure that we had already encountered on page 373. We refer to the figure below for an illustration.



Proof. Let X and Y be two CW-complexes.

(1) In the Product Topology-Basics Lemma 5.3 (1a) we showed that the product topology is compatible with taking the subspace topology. In our context this means that for cells $c \subset X$ and $d \subset Y$ the product topology on $c \times d$ agrees with the subspace topology on $c \times d \subset (X \times Y, \mathcal{TP})$. It follows immediately from this discussion that the identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is continuous.

(2) (a) Suppose that X and X are both finite CW-complexes. It follows from the CW-Category–Product Proposition 33.7 (0b) that the CW-complex $X \times Y$ has only finitely many cells. It thus follows from the CW-Complex Compactness Corollary 30.11 that $(X \times Y, \mathcal{CWP})$ is compact. Furthermore note that it follows from the CW-Complex–Hausdorff Proposition 30.3 and the Product Topology-Properties Proposition 5.4 that $(X \times Y, \mathcal{TP})$ is Hausdorff. It follows from these facts, the Compact-Hausdorff Proposition 2.10 (3) and Statement (1) that $\mathrm{id}\colon (X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is indeed a homeomorphism.

(b) Suppose we have $Y = [0,1]$ with the obvious CW-structure. By (1) we know that $\mathcal{TP} \subset \mathcal{CWP}$. It remains to show that $\mathcal{CWP} \subset \mathcal{TP}$. Thus let $U \in \mathcal{CWP}$. Let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c \in \mathrm{Cell}(X)}$ be a full set of characteristic maps for the CW-complex X. Since $U \in \mathcal{CWP}$ we know that for each $c \in \mathrm{Cell}(X)$ the intersection $(c \times [0,1]) \cap U$ is open in $c \times [0,1]$. Next note that it follows from the Topological-Product Proposition 5.1 (1) that the map $\Phi_c \times \mathrm{id}_{[0,1]}\colon \overline{B}^{\dim(c)} \times [0,1] \to X \times [0,1]$ is continuous. Thus we see that for each $c \in \mathrm{Cell}(X)$ the preimage $(\Phi_c \times \mathrm{id}_{[0,1]})^{-1}(U)$ is open in $\overline{B}^{\dim(c)} \times [0,1]$. It follows from this discussion together with the (c)⇒(a)-direction of the CW×Intervall Proposition 31.1 (1) that $U \in \mathcal{TP}$. ■

Example. The formulation of the CW-Complex Product Proposition 33.9 already suggests that the conclusion does not hold without some assumptions on X and Y. In [Fri23] we show that if X is the wedge of uncountably many intervals and if Y is the wedge of countably many intervals, then the topologies $\mathcal{CWP}$ and $\mathcal{TP}$ disagree on $X \times Y$.

Part IV

Singular Homology

34. The idea behind singular homology

The definition and the theory of singular homology was developed over several decades in the first half of the 20th century. In this chapter we will try to outline how one might have come up with the idea behind singular homology. The stress in this chapter is on explaining ideas and examples, without getting bogged down with technical discussions. We will not make use of any of the results and definitions of this chapter. We will use this feature as an excuse for taking a few liberties here and there.

So let's start with our story. Historically mathematicians found, and still find it, convenient to work with simplicial complexes. Loosely speaking a simplicial complex is a subset of $\mathbb{R}^n$ that is built out of simplices. Thus we first should tell the reader what we mean by a simplex:

Definition.

(1) We say $k+1$ points in $\mathbb{R}^n$ are in general position if they are not contained in a $(k-1)$-dimensional affine hyperplane of $\mathbb{R}^n$.[262]

(2) Given any finite subset $S \subset \mathbb{R}^n$ we denote by[263]

$$\text{Hull}(S) := \left\{ \sum_{x\in S} \lambda_x \cdot x \;\middle|\; \text{each } \lambda_x \in \mathbb{R}_{\geq 0} \text{ and } \sum_{x\in S} \lambda_x = 1 \right\}$$

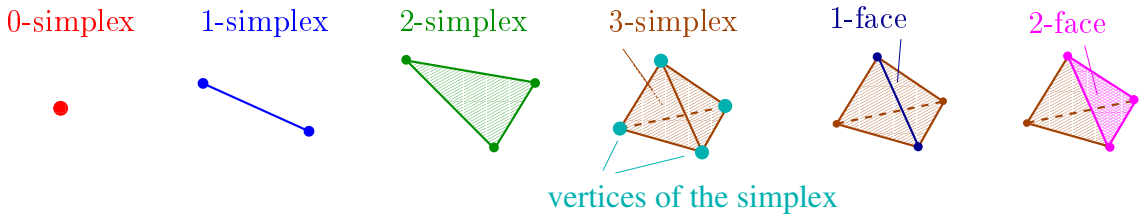
the convex hull of S.

(3) A k-simplex in $\mathbb{R}^n$ is the convex hull of $k+1$ points of $\mathbb{R}^n$ that are in general position.

(4) Let $\sigma \subset \mathbb{R}^n$ be a k-simplex.

(a) There exists a unique[264] subset $V \subset \mathbb{R}^n$ with $k+1$ elements and with $\sigma = \text{Hull}(V)$. We refer to V as the vertex set of σ.

(b) Let V be the vertex set of σ. For any subset $T \subset V$ of with $(l+1)$ elements we refer to $\text{Hull}(T)$ as an l-face of $\sigma = \text{Hull}(V)$.



Simplices can be viewed as convenient building blocks to describe topological spaces. This observation leads us straight to the following definition.

Definition. A simplicial complex in $\mathbb{R}^n$ is a finite family $X = \{\sigma_i\}_{i=1,\dots,k}$ of simplices in $\mathbb{R}^n$ with the following properties:

(1) Any face of a simplex is also contained in the family.

(2) The intersection of any two simplices is either empty or it is a single simplex, which is a face of both simplices.

Given a simplicial complex $X = \{\sigma_i\}_{i=1,\dots,k}$ as above we denote by X also the subspace $\sigma_1 \cup \cdots \cup \sigma_k \subset \mathbb{R}^n$.

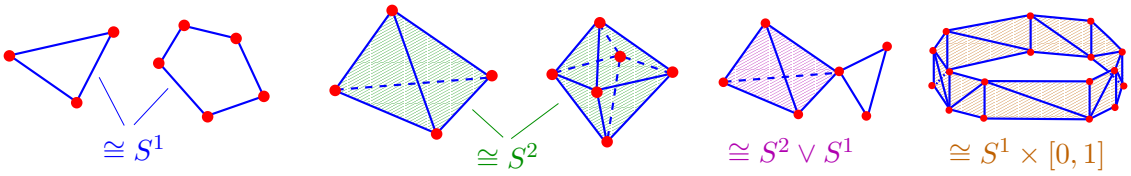
---

[263] In other words, $P_0, \dots, P_k \in \mathbb{R}^n$ are in general position if $P_1 - P_0, \dots, P_k - P_0$ are linearly independent.

[264] It is a nice little exercise to figure out what S is unique.

Example.

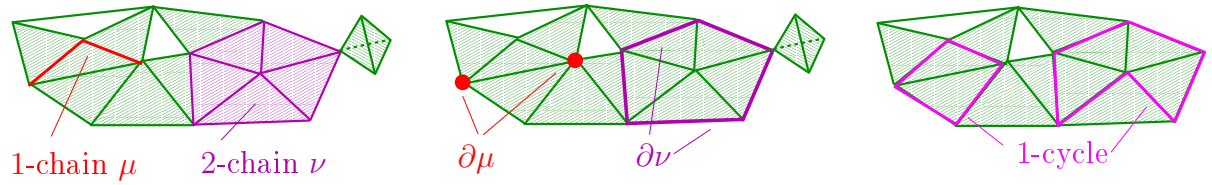
(1) In the figure below we see many examples of simplicial complexes. In particular we see that many topological spaces that we like, are actually homeomorphic to a simplicial complex.



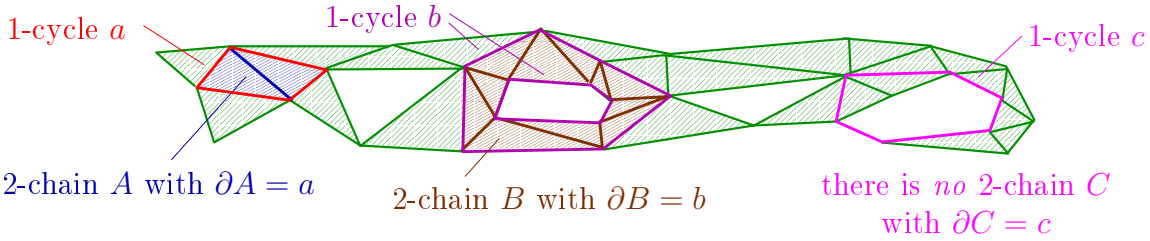
(2) In [Fri23] it is shown that every compact smooth manifold is homeomorphic to a simplicial complex and in [Fri23] it is also shown that every finite CW-complex is homotopy equivalent to a simplicial complex.

Definition. Let X be a simplicial complex.

(1) An n-chain in X is a collection of n-simplices of X, in other words, it is a subset of the set of n-simplices of X.
(2) The boundary $\partial\sigma$ of an n-chain σ is the union of all $(n-1)$-simplices that are the faces of an *odd* number of n-simplices of the n-chain.
(3) We say an n-chain is an n-cycle if its boundary is the empty set of $(n-1)$-simplices.



In the next figure we see that cycles come in two different flavors. Some n-cycles are the boundary of an $(n+1)$-chain and are thus not very interesting. On the other hand there are n-cycles which are not the boundary of an $(n+1)$-chain, and these non-bounding cycles seem to detect something interesting, like the presence of a "hole".



The idea now is to "measure" how many n-cycles are not the boundary of an $(n+1)$-cycle by introducing some algebra. This leads us to the following definition.

Definition. Let X be a simplicial complex.

(1) We define the n-th simplicial chain group of X as

$$\mathrm{C}_n^{\mathrm{simp}}(X) \;:=\; \text{set of all } n\text{-chains} \;=\; \mathbb{Z}_2^{\text{set of } n\text{-simplices of } X}.$$

for any set W we have $\mathcal{P}(W) = \mathbb{Z}_2^W$, we apply this logic to the set of all n-simplices of X

In other words the n-th chain group $\mathrm{C}_n^{\mathrm{simp}}(X)$ of X is defined as the free $\mathbb{Z}_2$-vector space on the set of all n-simplices of X. More informally speaking, $\mathrm{C}_n^{\mathrm{simp}}(X)$ is the set of all finite $\mathbb{Z}_2$-linear combinations of n-simplices of X.

(2) We define the boundary map

$$\begin{aligned} \partial_n \colon \mathrm{C}_n^{\mathrm{simp}}(X) &\to \mathrm{C}_{n-1}^{\mathrm{simp}}(X) \\ \sum_{i=1}^{r} a_i \cdot \sigma_i &\mapsto \sum_{i=1}^{r} a_i \cdot \underbrace{\partial \sigma_i}_{\uparrow} \end{aligned}$$

we view $\partial\sigma_i$ as a linear combination of $(n-1)$-simplices

(3) We refer to the sequence

$$\ldots \xrightarrow{\partial_3} \mathrm{C}_2^{\mathrm{simp}}(X) \xrightarrow{\partial_2} \mathrm{C}_1^{\mathrm{simp}}(X) \xrightarrow{\partial_1} \mathrm{C}_0^{\mathrm{simp}}(X) \to 0$$

as the simplicial chain complex of X.

Remark. Note that it follows easily from the definitions that an n-chain μ is an n-cycle if and only if $\partial_n(\mu) = 0$.[265]

The following lemma is the crucial observation, which makes the whole theory come alive.

Lemma 34.1. (Simplicial–Double Boundary Lemma) Let X be a simplicial complex. For every $n \in \mathbb{N}_0$ the composition $\partial_n \circ \partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathrm{simp}}(X) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(X)$ of two consecutive boundary maps is the zero map.

Sketch of proof. Since $\mathrm{C}_{n+1}^{\mathrm{simp}}(X)$ is a vector space with basis given by the $(n+1)$-simplices of X it suffices to prove that for each $(n+1)$-simplex of X we have $\partial_n(\partial_{n+1}(\sigma)) = 0$. Thus let σ be an $(n+1)$-simplex with vertex set $V = \{x_0, \ldots, x_{n+2}\}$. We compute[266]

$$\begin{aligned} \partial_n(\partial_{n+1}(\sigma)) &= \partial_n\Big(\sum_{i=0}^{n+2} \mathrm{Hull}(\{x_0, \ldots, \widehat{x_i}, \ldots, x_{n+2}\}\Big) \\ &= \sum_{i=0}^{n+2} \sum_{\substack{j=0 \\ i \neq j}}^{n+2} \mathrm{Hull}(\{x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, x_{n+2}\}) \;=\; 0. \end{aligned}$$

since each $(n-1)$-face appears twice and since we work with $\mathbb{Z}_2$-coefficients

Another way of stating this argument is to say that each $(n-1)$-face of σ is the face of precisely two n-faces of σ. ∎

[265] Here the odd "odd" condition in the definition of the boundary of an n-chain on page 413 is absolutely crucial.

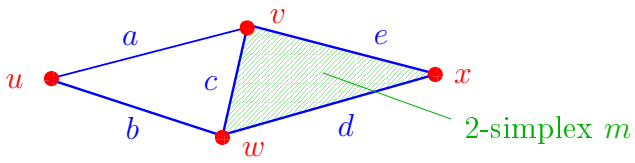
[266] Here $\widehat{x_i}$ and $\widehat{x_j}$ means that we leave out this point.

The Simplicial–Double Boundary Lemma 34.1 allows us to introduce the following definition.

Definition. Let X be a simplicial complex. Furthermore let $n \in \mathbb{N}_0$. It follows immediately from the above Simplicial–Double Boundary Lemma 34.1 that we have the inclusion $\operatorname{im}(\partial_{n+1}\colon \mathrm{C}_{n+1}^{\mathrm{simp}}(X) \to \mathrm{C}_n^{\mathrm{simp}}(X)) \subset \ker(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(X) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(X))$. Thus it makes sense to define the n-th simplicial homology group as follows:

$$\mathrm{H}_n^{\mathrm{simp}}(X) \;:=\; \frac{\ker(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(X) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(X))}{\operatorname{im}(\partial_{n+1}\colon \mathrm{C}_{n+1}^{\mathrm{simp}}(X) \to \mathrm{C}_n^{\mathrm{simp}}(X))}.$$

Example. We consider the simplicial complex X that is shown below. It has four 0-simplices u, v, w, x, five 1-simplices a, b, c, d, e and one 2-simplex m.



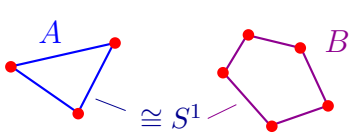
One can easily verify that the corresponding simplicial chain complex is given as follows:

$$0 \;\to\; \underbrace{\mathbb{Z}_2 \cdot m}_{=\mathrm{C}_2^{\mathrm{simp}}(X)} \xrightarrow{\begin{pmatrix}0\\0\\1\\1\\1\end{pmatrix}} \underbrace{\begin{matrix}\mathbb{Z}_2 \cdot a\\ \mathbb{Z}_2 \cdot b\\ \mathbb{Z}_2 \cdot c\\ \mathbb{Z}_2 \cdot d\\ \mathbb{Z}_2 \cdot e\end{matrix}}_{=\mathrm{C}_1^{\mathrm{simp}}(X)} \xrightarrow{\begin{pmatrix}1&1&0&0&0\\1&0&1&0&1\\0&1&1&1&0\\0&0&0&1&1\end{pmatrix}} \underbrace{\begin{matrix}\mathbb{Z}_2 \cdot u\\ \mathbb{Z}_2 \cdot v\\ \mathbb{Z}_2 \cdot w\\ \mathbb{Z}_2 \cdot x\end{matrix}}_{=\mathrm{C}_0^{\mathrm{simp}}(X)} \;\to\; 0.$$

An elementary, albeit rather painful calculation shows that

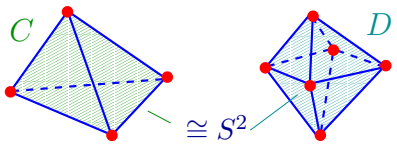
$$\mathrm{H}_k^{\mathrm{simp}}(X) \;\cong\; \begin{cases} \mathbb{Z}_2, & \text{if } k = 0,\\ \mathbb{Z}_2, & \text{if } k = 1,\\ 0, & \text{if } k \geq 2. \end{cases}$$

The previous example should convince the reader that simplicial homology groups can be computed, with enough patience, for any given simplicial complex. This is a good moment to return to some our previous examples. With sufficient energy one obtains the following simplicial homology groups:



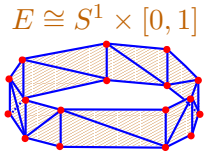
$\mathrm{H}_0^{\mathrm{simp}}(A) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(A) \cong \mathbb{Z}_2$
$\mathrm{H}_2^{\mathrm{simp}}(A) = 0$

$\mathrm{H}_0^{\mathrm{simp}}(B) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(B) \cong \mathbb{Z}_2$
$\mathrm{H}_2^{\mathrm{simp}}(B) = 0$



$\mathrm{H}_0^{\mathrm{simp}}(C) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(C) = 0$
$\mathrm{H}_2^{\mathrm{simp}}(C) \cong \mathbb{Z}_2$
$\mathrm{H}_3^{\mathrm{simp}}(C) = 0$

$\mathrm{H}_0^{\mathrm{simp}}(D) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(D) = 0$
$\mathrm{H}_2^{\mathrm{simp}}(D) \cong \mathbb{Z}_2$
$\mathrm{H}_3^{\mathrm{simp}}(D) = 0$



$\mathrm{H}_0^{\mathrm{simp}}(E) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(E) \cong \mathbb{Z}_2$
$\mathrm{H}_2^{\mathrm{simp}}(E) = 0$

Using these examples we can already make a couple of educated guesses:

- The simplicial complexes A and B are homeomorphic and the corresponding simplicial homology groups $\mathrm{H}_k^{\mathrm{simp}}(A)$ and $\mathrm{H}_k^{\mathrm{simp}}(B)$ are isomorphic. We see the same behavior for the simplicial complexes C and D which are also homeomorphic. This suggests that the isomorphism type of simplicial homology might only depend on the homeomorphism type of the simplicial complex.
- If we look at the simplicial homology groups of the simplicial complexes that are homeomorphic to S^1 and that are homeomorphic to S^2 we might spot a pattern, namely one could make the guess that for any[267] $n \geq 1$ and any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k^{\mathrm{simp}}(\text{simplicial complex homeomorphic to } S^n) \;\cong\; \begin{cases} \mathbb{Z}_2, & \text{if } k = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

- The simplicial complexes $A \cong S^1$ and $E \cong S^1 \times [0,1]$ are homotopy equivalent. We notice that the simplicial homology groups are isomorphic. This suggests, on admittedly rather tenuous evidence, that the isomorphism type of simplicial homology might actually only depend on the homotopy type of the simplicial complex.

All this sounds very promising. But there are serious issues with the concept of simplicial homology:

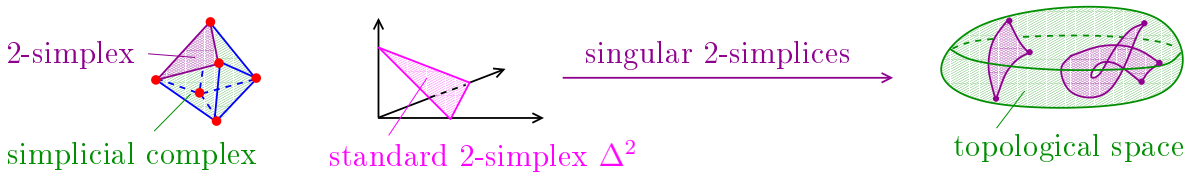
- First of all it seems it seems rather restrictive to work just with simplicial complexes. Even though many topological spaces are homeomorphic to simplicial complexes, already for simple topological spaces, like the torus $S^1 \times S^1$, it is quite painful and unnatural to represent them as a simplicial complex.
- For simplicial homology to be really useful as a topological tool one would need to show that simplicial homology groups depend only on the homeomorphism type (or better homotopy type) of the simplicial complex. The longer one thinks about it, the harder it seems to prove such a statement.
- When we studied fundamental groups we saw that in applications it was essential that fundamental groups are covariantly functorial, i.e. that continuous maps induce maps on fundamental groups. Given a continuous map between two simplicial complexes there is no reasonable way how such a map will induce a map on the corresponding simplicial homology groups.

Because of all these issues we have to go back to the drawing board and come up with a better concept. The key idea is as follows:

- In *simplicial* homology $\mathrm{H}_k^{\mathrm{simp}}(X)$ we consider only simplicial complexes X and we study the simplices contained in them.
- In *singular* homology $\mathrm{H}_k^{\mathrm{simp}}(X)$ we allow any topological space X and this time we take the drastic step of considering "singular simplices", namely we study all continuous maps of the standard simplices Δ^n to X, regardless of whether or not the maps are injective.

This new concept will eventually solve all of the technical problems. For example we will see that, almost by definition, singular homology groups are covariantly functorial under continuous maps and thus homeomorphism invariants. Furthermore it will also not be particularly difficult to show that singular homology groups depend only on the homotopy

[267] What about $n = 0$?



type of the topological space. The problem with singular homology is that in our attempt to solve the technical issues of simplicial homology we threw computability out of the window. We will have to work quite hard to calculate singular homology groups. But eventually we will see that singular homology encompasses simplicial homology and that it is the best of two worlds: it has very nice theoretical properties and highly computable.

35. The singular homology groups of a topological space

In the previous chatty chapter we give a very short introduction to simplicial homology, we calculated a few examples and we discovered serious issues with the concept of simplicial homology. As indicated towards the end of the previous chapter the solution to these problems is to introduce a "homology theory" where we study "singular simplices". In this chapter we will follow up on this idea, we will introduce singular homology, we will see it is covariantly functorial and we will calculate singular homology in a few very basic examples. In the coming chapters we will then develop the theory of singular homology groups in its full glory.

35.1. **Algebraic chain complexes.** In the previous chapter, in the discussion of simplicial homology, we implicitly briefly encountered the purely algebraic notion of a chain complex. This notion will play a prominent role in the discussion of homology and it thus useful to introduce this algebraic notion before we return to topology and before we get to the actual definition of singular homology.

Definition.

(1) An (algebraic) chain complex (C_*, ∂_*) is a sequence

$$\ldots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots C_1 \xrightarrow{\partial_1} C_0 \to 0$$

of homomorphisms between abelian groups such that the composition of any two consecutive boundary maps is the zero map, i.e. for every $i \in \mathbb{N}$ we have $\partial_{i-1} \circ \partial_i = 0$. For $n \in \mathbb{Z}_{<0}$ we always set $C_n := 0$.

(2) Let (C_*, ∂_*) be a chain complex and let $n \in \mathbb{N}_0$. By definition of a chain complex we have $\operatorname{im}(\partial_{n+1} \colon C_{n+1} \to C_n) \subset \ker(\partial_n \colon C_n \to C_{n-1})$. Thus it makes sense to define the n-th homology group of the chain complex as follows:

$$\mathrm{H}_n(C) \ := \ \frac{\ker(\partial_n \colon C_n \to C_{n-1})}{\operatorname{im}(\partial_{n+1} \colon C_{n+1} \to C_n)}.$$

(3) (a) We refer to elements in the kernel of $\partial_n \colon C_n \to C_{n-1}$ as cycles.
(b) We say that two elements $c, d \in C_n$ are homologous if there exists an $e \in C_{n+1}$ with $c - d = \partial_{n+1} e$.
(c) We say that a cycle $z \in C_n$ is null-homologous if z is homologous to the zero element of C_n, i.e. if $[z] = 0 \in \mathrm{H}_n(C)$.

As always we not only care about objects, but we also want to allow reasonable morphisms between the objects. This quest leads us the following definition.

Definition. A chain map $f \colon C_* \to D_*$ between two chain complexes[268] consists of a family $\{f_n \colon C_n \to D_n\}_{n \in \mathbb{N}_0}$ of homomorphisms such that for each $n \in \mathbb{N}_0$ the following equality holds:[269]

$$f_{n-1} \circ \partial_n \ = \ \partial_n \circ f_n.$$

Put differently the homomorphisms f_n have the property that the following diagram commutes:

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{\partial_{n+2}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \ldots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \ldots & \xrightarrow{\partial_{n+2}} & D_{n+1} & \xrightarrow{\partial_{n+1}} & D_n & \xrightarrow{\partial_n} & D_{n-1} & \xrightarrow{\partial_{n-1}} & \ldots \end{array}$$

The above definitions naturally lead to the following definition.

Definition. We refer to category $\mathcal{ChCplx}$ that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{ChCplx}) &:= \text{ all chain complexes,} \\ \text{and with morphisms} \quad \mathrm{Mor}(C_*, D_*) &:= \text{ all chain maps from } C_* \text{ to } D_*. \end{aligned}$$

with the obvious composition of morphisms[270] as the category of chain complexes.

Lemma 35.1. (Chain Complex-H_*-Functor Lemma)

(1) If $f\colon C_* \to D_*$ is a chain map between chain complexes C_* and D_*, then the map

$$\begin{array}{rcl} f_*\colon \mathrm{H}_n(C_*) & \to & \mathrm{H}_n(D_*) \\ [c] & \mapsto & [f_n(c)] \end{array}$$

is a well-defined homomorphism.

(2) Given $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} C_* & \mapsto & \mathrm{H}_n(C_*) \\ (f\colon C_* \to D_*) & \mapsto & (f_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)) \end{array}$$

define a covariant functor from the category $\mathcal{ChCplx}$ of chain complexes to the category $\mathcal{AbGroup}$ of abelian groups.

Definition. In the setting of the Chain Complex-H_*-Functor Lemma 35.1 (1) we refer to the corresponding map $f_*\colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ as the induced map on homology.

Proof.

(1) Let $c \in C_n$ be a cycle. First we have to show that $f_n(c)$ is again a cycle. Indeed we have

$$\partial_n(f_n(c)) \underset{\text{definition of a chain map}}{\overset{\uparrow}{=}} f_{n-1}(\partial_n c) \underset{\text{since } c \text{ is a cycle}}{\overset{\uparrow}{=}} f_n(0) \underset{\substack{\text{since } f_n \text{ is a} \\ \text{homomorphism}}}{\overset{\uparrow}{=}} 0.$$

Now let $c, d \in C_n$ be two cycles which represent the same element in $\mathrm{H}_n(C)$. We have to show that $[f_n(c)] = [f_n(d)]$. Indeed we have

$$[f_n(c)] - [f_n(d)] = [f_n(c-d)] \underset{\substack{\text{since } c,d \text{ are homologous there} \\ \text{exists an } e \in C_{n+1} \text{ with } \partial_{n+1} e = c-d}}{\overset{\uparrow}{=}} [f_n(\partial_{n+1}(e))] \underset{\substack{\text{by the definition} \\ \text{of a chain map}}}{\overset{\uparrow}{=}} [\partial_{n+1}(f_{n+1}(e))] \underset{\substack{\text{by the definition} \\ \text{of homology}}}{\overset{\uparrow}{=}} 0 \in \mathrm{H}_n(D).$$

[268] As usual we suppress bits from the notation which are understood from the context. In this instance we suppress the notation for the boundary maps in the chain complexes. Put differently, instead of saying "let (C_*, ∂_*) be a chain complex" we say simply "let C_* be a chain complex".

[269] More precisely, for any $c \in C$ we have $f_{n-1}(\partial_n(c)) = \partial_n(f_n(c))$ where on the left-hand side we consider the boundary map $C_n \to C_{n-1}$, and on the right-hand side we consider the boundary map $D_n \to D_{n-1}$ which we also denote by ∂_n. Usually it is obvious from the context which boundary map we consider.

[270] Here we use the rather elementary observation that the composition of two chain maps $f\colon C_* \to D_*$ and $g\colon D_* \to E_*$ defines a chain map $g \circ f\colon C_* \to E_*$.

This shows that the map $f_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)$ is well-defined. It follows easily from the definitions that this map is actually a homomorphism.

(2) The statement follows easily from (1) and the definitions. ■

At this stage there is only one little lemma on chain complexes that we want to state. Before we can do so, we recall the following definition from page 420.

Definition. Let $\{X_s\}_{s\in S}$ be a family of abelian groups.

(1) We define the product of the X_s to be the abelian group

$$\prod_{s\in S} X_s \;:=\; \Big\{ f\colon S \to \bigcup_{s\in S} X_s \;\Big|\; \text{for every } s\in S \text{ we have } f(s)\in X_s \Big\},$$

where the group structure is given by the obvious "component wise addition".

(2) We define the direct sum of the X_s to be the group

$$\bigoplus_{s\in S} X_s \;:=\; \Big\{ (g_s)_{s\in S} \in \prod_{s\in S} X_s \;\Big|\; \text{all but finitely many } g_s \text{ are trivial} \Big\}.$$

In the Coproduct-of-Abelian Groups Lemma 18.1 we showed that the direct sum with the above natural homomorphisms is the coproduct of $\{X_s\}_{s\in S}$ in the category of abelian groups.

Basically the same way as we defined the direct sum of a family of abelian groups we can also define the direct sum of a family of chain complexes. Now we can formulate the following lemma.

Lemma 35.2. (Direct Sum-H$_*$-Lemma) Let $\{C_a\}_{a\in A}$ be a family of chain complexes. For every $n\in\mathbb{N}_0$ the natural maps $C_a \to \bigoplus_{a\in A} C_a$ induce an isomorphism

$$\bigoplus_{a\in A} \mathrm{H}_n(C_a) \;\xrightarrow{\cong}\; \mathrm{H}_n\Big(\bigoplus_{a\in A} C_a\Big).$$

Proof. The proof is a standard exercise in homological algebra, which, like every other author, we leave to the reader. ■

We could easily fill the remainder of these notes with the general theory of chain complexes. We will not do so, but instead we will develop the theory of chain complexes along the way, as we develop the theory of the singular chain complexes and singular homology.

35.2. **The singular chain complex.** With this section we return to topology. As a first step towards the definition of singular homology we need to introduce the notion of a (standard) n-simplex:

Definition. Let $n\in\mathbb{N}_0$.

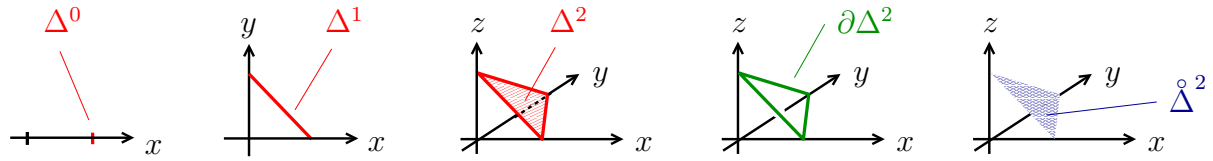
(1) Given $n\in\mathbb{N}_0$ we refer to the convex hull of the standard basis vectors $e_1,\dots,e_{n+1}$ of $\mathbb{R}^{n+1}$, i.e. to

$$\Delta^n \;:=\; \{(x_0,\dots,x_n)\in\mathbb{R}^{n+1} \,|\, x_0+\cdots+x_n = 1 \text{ and } x_i\geq 0 \text{ for all } i=0,\dots,n\}$$

as the standard n-simplex.

(2) Given the standard n-simplex Δ^n we define its boundary as

$$\partial\Delta^n := \bigcup_{i=0}^{n} \{(t_0, \ldots, t_n) \in \Delta^n \mid t_i = 0\}.$$



For the record we state the following elementary lemma.

Lemma 35.3. (Δ^n-Homeomorphism Lemma) Let $n \in \mathbb{N}$.

(1) We set

$$\triangle_n := \left\{(x_1, \ldots, x_n) \in [0,1]^n \,\middle|\, \sum_{i=1}^{n} x_i \in [0,1]\right\}$$

and refer to it as the planar n-simplex. The maps

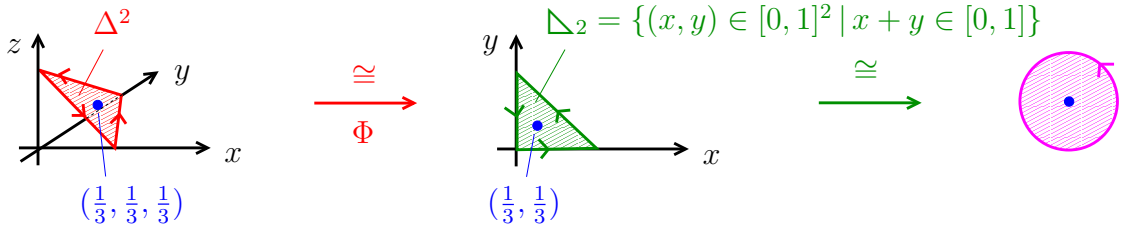
$$\begin{array}{rcl} \Phi\colon \Delta^n & \to & \triangle_n \\ (t_0, \ldots, t_n) & \mapsto & (t_0, \ldots, t_{n-1}) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon \triangle_n & \to & \Delta^n \\ (x_1, \ldots, x_n) & \mapsto & \left(x_1, \ldots, x_n, 1 - \sum_{i=1}^{n} x_i\right) \end{array}$$

are homeomorphisms that are inverses of one another.

(2) The map

$$\Delta^n \xrightarrow{\Phi} \triangle_n \xrightarrow[\text{applied to } \triangle_n \text{ and } Q = (\frac{1}{n+1}, \ldots, \frac{1}{n+1})]{\text{the homeomorphism } \Xi \text{ obtained from the Convex-to-Ball Proposition 2.12 (2)}} \overline{B}^n$$

is a homeomorphism that restricts to a homeomorphism from $\partial\Delta^n$ to the sphere S^{n-1}.



Proof.

(1) Note that Φ extends "by the same formula" to a map $\mathbb{R}^{n+1} \to \mathbb{R}^n$ which is evidently continuous. Hence the restriction to Δ^n is also continuous. Now that this is settled we see that the remaining statements are basically trivial.

(2) The statement follows easily from (1), the observation that $\triangle_n$ is compact (which can be shown easily using the Heine-Borel Theorem 1.11) and the Convex-to-Ball Proposition 2.12 (2). ■

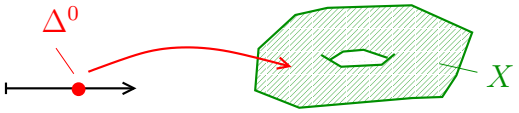
Definition. Let X be a topological space and let $n \in \mathbb{N}_0$. A singular n-simplex is a continuous map $\sigma\colon \Delta^n \to X$.

Example. Let X be a topological space.

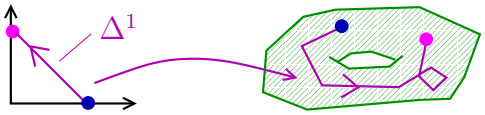
(1) A singular 0-simplex is a continuous map from Δ^0 to X. Note that $\Delta^0 = \{1\} \subset \mathbb{R}$. Thus a map $\Delta^0 = \{1\} \to X$ is uniquely determined by a point in X. Given a point

$P \in X$ we frequently denote by P also the singular 0-simplex $\Delta^0 \to X$ that is given by sending the single point in Δ^0 to P.

(2) A singular 1-simplex is a continuous map $\Delta^1 \to X$. Since the map $[0,1] \to \Delta^1$ given by $t \mapsto (1-t, t)$ is a homeomorphism we see that a singular 1-simplex is essentially the same as a path in X.



a singular 0-simplex in X is essentially the same as a point in X



a singular 1-simplex in X is essentially the same as a path in X

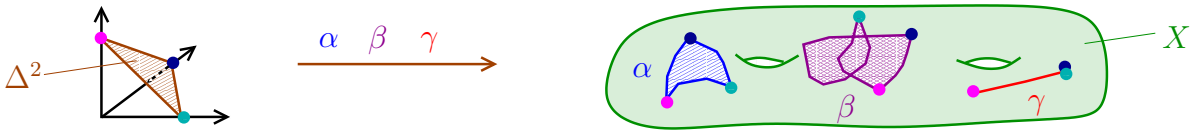
Definition. Let X be a topological space and let $n \in \mathbb{N}_0$.

(1) We define the n-th singular chain group as follows:

$$\mathrm{C}_n(X) \;:=\; \text{free abelian group generated by the set of singular } n\text{-simplices in } X.$$

For $n \in \mathbb{Z}_{<0}$ we set $\mathrm{C}_n(X) := 0$.

(2) We refer to an element in $\mathrm{C}_n(X)$ as a singular n-chain in X.

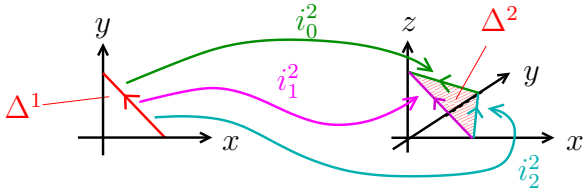


$-2\alpha + \beta + 3\gamma$ is a singular 2-chain

Motivated by the discussion of the last chapter we now want to find suitable boundary maps between the singular chain groups $\mathrm{C}_n(X)$. In the simplicial chain complex our generators where simplices and on page 414 we defined the boundary of a simplex to be a linear combination of codimension-one faces of the simplex. In our current setting generators are singular simplices, i.e. continuous maps $\Delta^n \to X$. To such a singular simplex we now have to associate a singular $(n-1)$-simplex, i.e. a linear combination of continuous maps $\Delta^{n-1} \to X$. To to so it is convenient to introduce the following definition.

Definition. Let $n \in \mathbb{N}_0$. For $j \in \{0, \dots, n\}$ we define the j-th face map as follows:[271272]

$$\begin{aligned} i_j^n \colon \Delta^{n-1} &\to \Delta^n \\ (t_0, \dots, t_{n-1}) &\mapsto (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1}). \end{aligned}$$



Now we can define the promised boundary maps $\mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$.

[272]One can of course easily verify that this map is continuous.

[272]The image of a face map is often called an $(n-1)$-*dimensional face* of the standard simplex. The boundary $\partial\Delta^n$ of the standard n-simplex is the union of all $(n-1)$-dimensional faces of Δ^n.

Definition. Let X be a topological space and let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. The boundary $\partial_n\sigma \in \mathrm{C}_{n-1}(X)$ of σ is defined as[273]

$$\partial_n\sigma \;:=\; \sum_{j=0}^{n} (-1)^j \cdot \underbrace{\sigma \circ i_j^n}_{\text{map } \Delta^{n-1} \to X}.$$

By the Free Abelian Group–Existence Lemma 18.4 the map $\sigma \mapsto \partial_n\sigma$ extends uniquely to a linear map

$$\begin{aligned} \partial_n\colon \mathrm{C}_n(X) &\to \mathrm{C}_{n-1}(X) \\ \sum_{r=1}^{k} a_r \cdot \sigma_r &\mapsto \sum_{r=1}^{k} a_r \cdot \partial\sigma_r, \end{aligned}$$

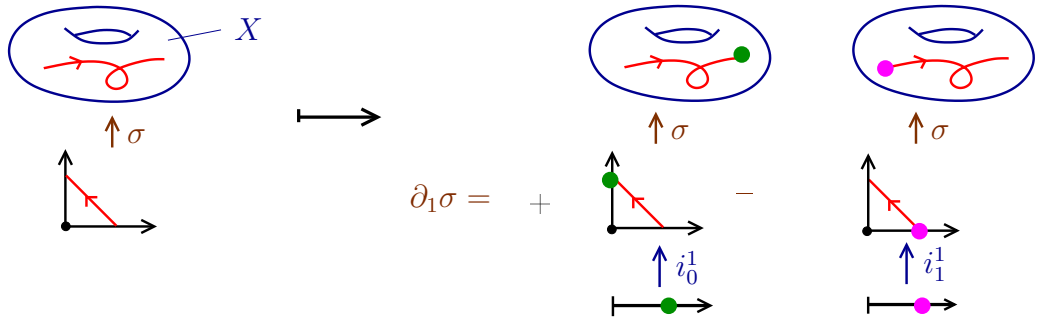
which we refer to as the n-th boundary map. If it is clear with what dimension we are working with, then we just write ∂ instead of ∂_n.

Examples.

(1) For any singular 1-simplex $\sigma\colon \Delta^1 \to X$ we have[274]

$$\partial_1\sigma \;=\; (-1)^0 \cdot \sigma \circ i_0^1 + (-1)^1 \cdot \sigma \circ i_1^1 \;=\; \sigma(0,1) - \sigma(1,0) \;\in\; \mathrm{C}_0(X).$$

Recall that a a singular 1-simplex can be viewed as a map σ from the "interval" Δ^1 to X. The boundary of σ is thus the "endpoint" $\sigma(0,1)$ "minus" the "starting point" $\sigma(1,0)$.



(2) In the figure below we sketch the boundary of a singular 2-simplex.

The following proposition, which is an analogue of the Simplicial–Double Boundary Lemma 34.1, shows that these boundary maps have the fundamental property that the composition of two consecutive boundary maps is the zero map.

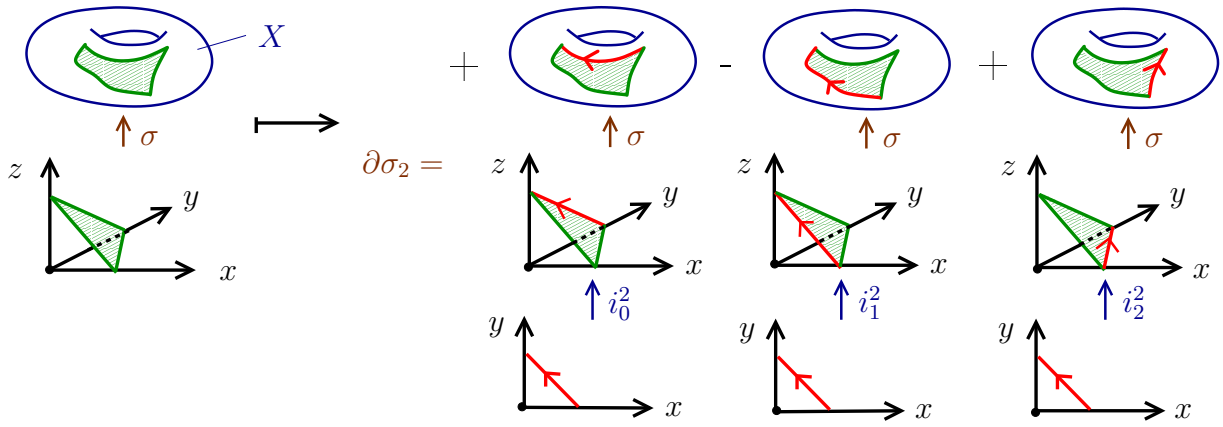
Proposition 35.4. (Double Boundary Proposition) Let X be a topological space. For each $n \in \mathbb{N}_0$ the composition

$$\partial_{n-1} \circ \partial_n\colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_{n-2}(X)$$

is the zero map.

[273] The sign $(-1)^j$ is surely somewhat unexpected. As we will see shortly, it is necessary to ensure that the composition of two consecutive boundary maps is indeed zero.

[274] Here we again use the convention that given a point $x \in X$ we denote by x also the corresponding singular 0-simplex.



Proof. Let $n \in \mathbb{N}_0$. It suffices to show that for every singular n-simplex $\sigma\colon \Delta^n \to X$ we have $\partial_{n-1}(\partial_n(\sigma)) = 0$. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. We calculate that

$$\begin{aligned}\partial_{n-1}(\partial_n(\sigma)) &\overset{\text{definition of } \partial_n}{=} \partial_{n-1}\Big(\sum_{j=0}^{n}(-1)^j \cdot \sigma \circ i_j^n\Big) \overset{\text{definition of } \partial_{n-1}}{=} \sum_{k=0}^{n-1}(-1)^k \cdot \sum_{j=0}^{n}(-1)^j \cdot (\sigma \circ i_j^n) \circ i_k^{n-1} \\ &= \sum_{0=k<j\leq n}(-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) + \sum_{0=j\leq k\leq n-1}(-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) =: (*).\end{aligned}$$

Now we want to show that each expression $\sigma \circ (i_j^n \circ i_k^{n-1})$ appears precisely twice, but with opposite signs. From the definitions it follows easily that for $j \leq k$ we have[275]

$$i_j^n \circ i_k^{n-1} = i_{k+1}^n \circ i_j^{n-1}.$$

Now we continue with the above calculation:

$$\begin{aligned}(*) &\overset{\text{above calculation of } i_j^n \circ i_k^{n-1}}{=} \sum_{0=k<j\leq n}(-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) + \sum_{0=j\leq k\leq n-1}(-1)^{k+j} \cdot \sigma \circ (i_{k+1}^n \circ i_j^{n-1}) \\ &\overset{\text{substitution } l=k+1}{=} \sum_{0=k<j\leq n}(-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) + \sum_{0=j<l\leq n}(-1)^{l+j-1} \cdot \sigma \circ (i_l^n \circ i_j^{n-1}) \overset{\text{all summands left and right cancel}}{=} 0.\end{aligned}$$

■

The Double Boundary Proposition 35.4 says that the map $X \mapsto (\mathrm{C}_*(X), \partial_*)$ assigns to each topological space a chain complex. Of course, as experienced algebraic topologists, we want this assignment to be functorial. This goal leads us to the following definition.

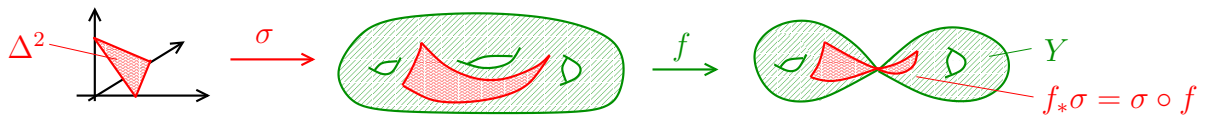
Definition. Let $f\colon X \to Y$ be a continuous map between topological spaces. For each $n \in \mathbb{N}_0$ we refer to

$$\begin{aligned} f_*\colon \mathrm{C}_n(X) &\to \mathrm{C}_n(Y) \\ \sum_{i=1}^{k} a_i \cdot \underbrace{\sigma_i}_{\text{map } \Delta^n \to X} &\mapsto \sum_{i=1}^{k} a_i \cdot (\underbrace{f \circ \sigma_i}_{\text{map } \Delta^n \to Y}).\end{aligned}$$

as the induced chain map.

The above definition leads to the following lemma:

[275] Indeed, if $j \leq k$, then the zero-entries of $(i_j^n \circ i_k^{n-1})(t_0, \ldots, t_{n-1})$ are the entries j and $k+1$.



Lemma 35.5. (C_*–Functor Lemma)

(1) Let $f\colon X \to Y$ be a continuous map between two topological spaces. The induced maps $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ form a chain map, i.e. given any $c \in \mathrm{C}_n(X)$ we have

$$f_*(\partial_n c) \;=\; \partial_n(f_*(c)) \;\in\; \mathrm{C}_n(Y).$$

(2) The maps

$$\begin{array}{rcl} \text{topological space } X & \mapsto & (\mathrm{C}_*(X), \partial_*) \\ (f\colon X \to Y) & \mapsto & (f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)) \end{array}$$

and

define a covariant functor from the category $\mathcal{T}op$ of topological spaces to the category $\mathcal{ChCplx}$ of chain complexes.

Proof.

(1) Let $f\colon X \to Y$ be a continuous map between topological spaces. Given any singular n-simplex $\sigma\colon \Delta^n \to X$ we have

$$f_*(\partial\sigma) \underset{\text{definition of } \partial}{\overset{\uparrow}{=}} f_*\Big(\sum_{j=0}^{n}(-1)^j\cdot\sigma\circ i_j^n\Big) \underset{\text{definition of } f_*}{\overset{\uparrow}{=}} \sum_{j=0}^{n}(-1)^j\cdot f\circ(\sigma\circ i_j^n) = \sum_{j=0}^{n}(-1)^j\cdot\overbrace{(f\circ\sigma)}^{=f_*\sigma}\circ i_j^n \underset{\text{definition of } \partial}{\overset{\uparrow}{=}} \partial(f_*\sigma).$$

Since the singular n-simplices generate $\mathrm{C}_n(X)$ we obtain the desired equality for all singular n-chains. By definition this means that $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ is a chain map.

(2) It follows immediately from the definitions that for two continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$ we have $(g \circ f)_* = g_* \circ f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Z)$. With this observation the second statement is an almost immediate consequence of the first statement. ∎

For any functor it is interesting to know whether or not products, coproducts and more generally limits and colimits, etc. are preserved. It follows easily from the following lemma that the functor $X \mapsto (\mathrm{C}_*(X), \partial_*)$ preserves colimits.

Lemma 35.6. (Path-Component–C_*–Lemma) Let X be a topological space with path-components $\{X_a\}_{a\in A}$. Given $a \in A$ we denote by $i_a\colon X_a \to X$ the inclusion map. The map

$$\bigoplus_{a\in A} i_a\colon \bigoplus_{a\in A} \mathrm{C}_*(X_a) \;\to\; \mathrm{C}_*(X)$$

is an isomorphism of chain complexes.

Proof. Evidently the standard simplex Δ^n is path-connected. Thus the image of a continuous map $\sigma\colon \Delta^n \to X$ is contained in a path-component X_a. This shows that

$$\bigsqcup_{a\in A} \text{set of singular } n\text{-simplices in } X_a \;=\; \text{set of singular } n\text{-simplices in } X.$$

It follows that the map

$$\begin{array}{rcl} \displaystyle\bigoplus_{a\in A} i_a\colon \bigoplus_{a\in A} \mathrm{C}_n(X_a) & \to & \mathrm{C}_n(X) \\ \Big(\displaystyle\sum_{i=1}^{k} a_i\cdot(\sigma\colon \Delta^n \to X_{a_i})\Big) & \mapsto & \Big(\displaystyle\sum_{i=1}^{k} a_i\cdot(\sigma\colon \Delta^n \to X_{a_i})\Big) \end{array}$$

is an isomorphism of free abelian groups. It follows immediately from the definitions that this a chain map. ■

35.3. **Definition of the singular homology groups.** It follows from the Double Boundary Proposition 35.4 that given a topological space X the sequence $(\mathrm{C}_*(X), \partial_*)$ is a chain complex. This allows us to make the following definition, which is arguably one of the most influential and important definitions of algebraic topology.

Definition. Let X be a topological space and let $n \in \mathbb{N}_0$. We define the n-th singular homology group of X as the n-homology group of the singular chain complex. In other words, we set

$$\mathrm{H}_n(X) \;:=\; \mathrm{H}_n(\mathrm{C}_*(X)) \;=\; \frac{\ker(\partial_n \colon \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X))}{\mathrm{im}(\partial_{n+1} \colon \mathrm{C}_{n+1}(X) \to \mathrm{C}_n(X))}.$$

Throughout the lecture notes we will mostly shorten the name "singular homology" to "homology".

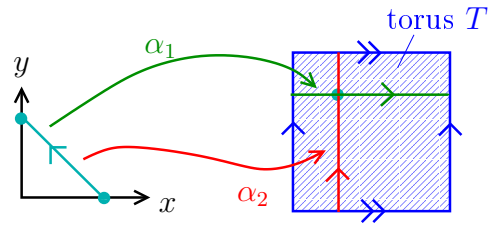
Example. There is only one example that we can do immediately: for the empty topological space $\varnothing$ we have for all $n \in \mathbb{N}_0$ that $\mathrm{C}_n(\varnothing) = 0$. It follows that for all $n \in \mathbb{N}_0$ we have $\mathrm{H}_n(\varnothing) = 0$.

The following definitions are just adaptations of the definitions on page 418.

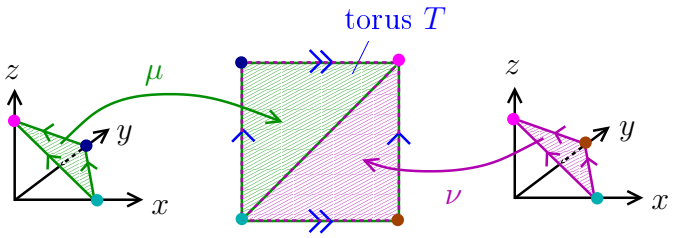
Definition. Let X be a topological space.

(1) We say $c \in \mathrm{C}_n(X)$ is a singular n-cycle, or short n-cycle, if $\partial c = 0$.
(2) We say two singular n-chains c and d are homologous if there exists a singular $(n+1)$-chain $e \in \mathrm{C}_{n+1}(X)$ with $\partial e = c - d$.
(3) An n-cycle c is called null-homologous, if $[c] = 0 \in \mathrm{H}_n(X)$.

Example. In the figure below we show the torus $T = ([0,1] \times [0,1])/\sim$ together with two singular 1-simplices α_1 and α_2 and together with two singular 2-simplices μ and ν. It follows easily from the definitions that α_1 and α_2 are 1-cycles. It is a good exercise to verify that $\mu - \nu$ is a 2-cycle. With a little bit of practice it is fairly easy to write down cycles, but usually it is very hard to show that a cycle represents a non-trivial element in homology. For example, with our present knowledge it is impossible though to determine whether the cycles $\alpha_1, \alpha_2, \mu - \nu$ represent non-trivial homology classes.



α_1 and α_2 are 1-cycles



$\mu - \nu$ is a 2-cycle

The following lemma is so fundamental that we will use it constantly without referring to it.

Lemma 35.7. (H_n–Functor Lemma)
(1) Let $f\colon X \to Y$ be a continuous map between topological spaces. The map

$$\begin{array}{rcl} f_*\colon \mathrm{H}_n(X) & \to & \mathrm{H}_n(Y) \\ \left[\sum_{i=1}^{k} a_i \cdot \sigma_i\right] & \mapsto & \left[\sum_{i=1}^{k} a_i \cdot (f \circ \sigma_i)\right] \end{array}$$

is a well-defined homomorphism.
(2) For each $n \in \mathbb{N}_0$ the map

$$X \mapsto \mathrm{H}_n(X)$$

and the map

$$(f\colon X \to Y) \mapsto \left(\begin{array}{rcl} f_*\colon \mathrm{H}_n(X) & \to & \mathrm{H}_n(Y) \\ \left[\sum_{i=1}^{k} a_i \cdot \sigma_i\right] & \mapsto & \left[\sum_{i=1}^{k} a_i \cdot (f \circ \sigma_i)\right] \end{array} \right)$$

define a covariant functor from the category of topological spaces to the category of abelian groups.

Example. It follows from the H_n–Functor Lemma 35.7 that every homeomorphism induces an isomorphism of homology groups.

Proof.
(1) This statement follows from the C_*–Functor Lemma 35.5 (1) together with the Chain Complex-H_*-Functor Lemma 35.1 (1).
(2) This statement follows from the combination of (1), the C_*–Functor Lemma 35.5 (2) and the Chain Complex-H_*-Functor Lemma 35.1 (2). ■

Again the question is how singular homology groups behave under coproducts, products, etc. At this stage we have the following very long hanging fruit, from which it follows easily that homology groups preserve coproducts.

Lemma 35.8. (Path-Component–H_*–Lemma) Let X be a topological space with path-components $\{X_a\}_{a\in A}$. Given $a \in A$ we denote by $i_a\colon X_a \to X$ the inclusion map. For each $n \in \mathbb{N}_0$ the map

$$\bigoplus_{a\in A} i_{a*}\colon \bigoplus_{a\in A} \mathrm{H}_n(X_a) \to \mathrm{H}_n(X)$$

is an isomorphism.

Proof. This lemma is an immediate consequence of the Path-Component–C_*–Lemma 35.6 and the Direct Sum-H_*-Lemma 35.2. ■

35.4. **First calculations of homology groups.** Given a topological space the corresponding homology groups are defined as the quotient of two abelian groups which in general will be infinitely generated. Somewhat surprisingly we will see in the coming chapters that the homology groups are nonetheless very frequently finitely generated abelian groups.

We start out our computations of homology groups with the 0-th homology of a path-connected topological space. The key to the calculation is the following definition.

Definition. Given a topological space X we refer to the homomorphism

$$\begin{aligned} \epsilon_X \colon \mathrm{C}_0(X) &\to \mathbb{Z} \\ \sum_{i=1}^{k} a_i \cdot \sigma_i &\mapsto \sum_{i=1}^{k} a_i \end{aligned}$$

as the augmentation homomorphism.

The following lemma shows in particular that the augmentation map is a neat addition to the singular chain complex of a topological space.

Lemma 35.9. (Augmentation Lemma) Let X be a topological space.

(1) The map $\epsilon_X \circ \partial_1 \colon \mathrm{C}_1(X) \to \mathbb{Z}$ is the zero map.

(2) The map

$$\begin{aligned} \epsilon_X \colon \mathrm{H}_0(X) &\to \mathbb{Z} \\ [\sigma] &\mapsto \epsilon_X(\sigma) \end{aligned}$$

is well-defined. In the following we refer to this map also as the augmentation map.

(3) Given any continuous map $f \colon X \to Y$ between topological spaces the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}_0(X) & \xrightarrow{\quad f_* \quad} & \mathrm{H}_0(Y). \\ & {\scriptstyle \epsilon_X} \searrow \quad \swarrow {\scriptstyle \epsilon_Y} & \\ & \mathbb{Z}. & \end{array}$$

Proof. Let X be a topological space.

(1) Since the abelian group $\mathrm{C}_1(X)$ is generated by singular 1-simplices it suffices to prove that for each singular 1-simplex $\sigma \colon \Delta^1 \to X$ we have $(\epsilon_X \circ \partial_1)(\sigma) = 0$. But this is the case since

$$\epsilon_X(\partial_1(\sigma)) \underset{\substack{\uparrow \\ \text{see the discussion on page 423}}}{=} \epsilon_X(\sigma(0,1) - \sigma(1,0)) \underset{\substack{\uparrow \\ \text{by definition of } \epsilon_X}}{=} 1 - 1 = 0.$$

(2) If we have $\sigma, \tau \in \mathrm{C}_0(X)$ with $[\sigma] = [\tau] \in \mathrm{H}_0(X)$, then by definition of $\mathrm{H}_0(X)$ there exists a $\mu \in \mathrm{C}_1(X)$ with $\partial\mu = \sigma - \tau$. We now see that

$$\epsilon_X(\sigma) = \epsilon_X(\tau - \partial\mu) = \epsilon_X(\tau) - \underbrace{\epsilon_X(\partial\mu)}_{=0 \text{ by (1)}} = \epsilon_X(\tau).$$

(3) This statement follows easily from the definitions. ∎

Using the augmentation map we can formulate the following proposition.

Proposition 35.10. (H_0-Proposition)

(1) Let X be a path-connected non-empty topological space. The augmentation map

$$\epsilon_X \colon \mathrm{H}_0(X) \to \mathbb{Z}$$

is a natural[276] isomorphism. In fact given any point $P \in X$ the inverse is given by the map

$$\begin{aligned} \iota \colon \mathbb{Z} &\to \mathrm{H}_0(X) \\ n &\mapsto n \cdot [P]. \end{aligned}$$

(2) Let $f \colon X \to Y$ be a continuous map between two topological spaces.

(a) If X and Y are path-connected non-empty topological spaces, then the induced map $f_* \colon \mathrm{H}_0(X) \to \mathrm{H}_0(Y)$ is an isomorphism.

(b) If $X = Y$ is a non-empty path-connected topological space, then the induced $f_*\colon \mathrm{H}_0(X) \to \mathrm{H}_0(X)$ is in fact the identity map.

Example. Let X be a topological space. As on page 64 we denote by $\pi_0(X)$ the set of path-components of X. It follows from the H_0-Proposition 35.10 (1) together with the Direct Sum-H_*-Lemma 35.2 that we have a natural isomorphism

$$\mathrm{H}_0(X) \xrightarrow{\cong} \mathbb{Z}^{(\pi_0(X))}.$$

In particular, if X consists of n path-components with $n \in \mathbb{N}_0$, then we have an isomorphism

$$\mathrm{H}_0(X) \xrightarrow{\cong} \mathbb{Z}^n.$$

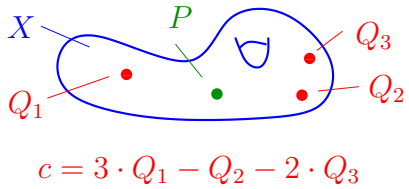
Proof.

(1) Let X be a path-connected non-empty topological space and let P be a point in X (which exists, since X is non-empty). It is clear that $\epsilon_X \circ \iota = \mathrm{id}_{\mathbb{Z}}$, in particular the homomorphism ϵ_X is surjective. It remains to show that ϵ_X is injective.

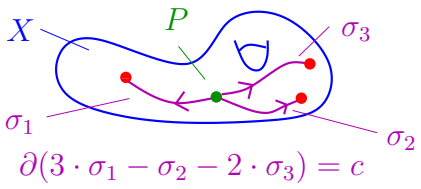
Thus let $c \in \mathrm{C}_0(X)$ with $\epsilon_X(c) = 0$. By definition of $\mathrm{C}_0(X)$ we can find points $Q_1, \dots, Q_k$ in X and $a_1, \dots, a_k \in \mathbb{Z}$ such that $c = \sum_{i=1}^{k} a_i \cdot Q_i$. Since X is path-connected we can find, for $i = 1, \dots, k$, a singular 1-simplex σ_i with $\sigma_i((0,1)) = Q_i$ and $\sigma_i((1,0)) = P_i$. Then we have

$$\partial\Big(\sum_{i=1}^{k} a_i \cdot \sigma_i\Big) = \sum_{i=1}^{k} a_i \cdot \partial\sigma_i \underset{\text{see the discussion on page 423}}{\uparrow} = \sum_{i=1}^{k} a_i \cdot (Q_i - P) = \sum_{i=1}^{k} a_i \cdot Q_i - \underbrace{\Big(\sum_{i=1}^{k} a_i\Big)}_{=\epsilon_X(c)=0} \cdot P = \underbrace{\sum_{i=1}^{k} a_i \cdot Q_i}_{=c}.$$

This implies that $[c] = 0 \in \mathrm{H}_0(X)$. Thus we have shown that ϵ_X is injective.



$c = 3 \cdot Q_1 - Q_2 - 2 \cdot Q_3$



$\partial(3 \cdot \sigma_1 - \sigma_2 - 2 \cdot \sigma_3) = c$

(2) (a) Let $f\colon X \to Y$ be a continuous map between two topological spaces. By the Augmentation Lemma 35.9 (3) we have the commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_0(X) & \xrightarrow{f_*} & \mathrm{H}_0(Y). \\ & {\scriptstyle \epsilon_X} \searrow \quad \swarrow {\scriptstyle \epsilon_Y} & \\ & \mathbb{Z}. & \end{array}$$

If X and Y are path-connected non-empty topological spaces, then we know by (1) that the diagonal maps are isomorphisms. Thus the horizontal map is also an isomorphism.

(b) If $X = Y$ is a path-connected non-empty topological space, then in the above diagram the two diagonal maps are the same isomorphism, thus the horizontal map is the identity. ∎

We continue with another elementary but essential calculation.

[276] What does "natural" mean in this context? What are the two functors?

Lemma 35.11. (Singleton-H_*-Lemma) If X is a singleton, i.e. if X is a topological space which consists of a single point, then

$$\mathrm{H}_n(X) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Proof. Let $X = \{\star\}$ be a topological space consisting of a single point. Given $n \in \mathbb{N}_0$ let $\sigma_n \colon \Delta^n \to \{\star\}$ be the unique map. We calculate that

$$\partial_n(\sigma_n) \underset{\substack{\uparrow \\ \text{see the definition} \\ \text{of } \partial_n \text{ on page 423}}}{=} \sum_{j=0}^{n} \cdot (-1)^j \cdot \underbrace{\sigma_n \circ i_j^n}_{=\sigma_{n-1}} = \sigma_{n-1} \cdot \sum_{j=0}^{n} \cdot (-1)^j = \begin{cases} \sigma_{n-1}, & \text{if } n \text{ even}, \\ 0, & \text{if } n \text{ odd}. \end{cases}$$

We see that the singular chain complex is of the form

$$\ldots \mathbb{Z} \cdot \sigma_4 \xrightarrow{\cong} \mathbb{Z} \cdot \sigma_3 \xrightarrow{=0} \mathbb{Z} \cdot \sigma_2 \xrightarrow{\cong} \mathbb{Z} \cdot \sigma_1 \xrightarrow{=0} \mathbb{Z} \cdot \sigma_0 \to 0.$$

The lemma follows almost immediately from this observation. ■

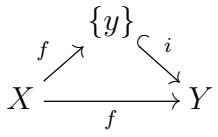
Example.

(1) One of our goals is to calculate the homology groups of the spheres. Now we can at least calculate the homology groups of the 0-dimensional sphere $S^0 = \{\pm 1\}$. More precisely, by the Singleton-H_*-Lemma 35.11 and the Path-Component–H_*–Lemma 35.8 we have

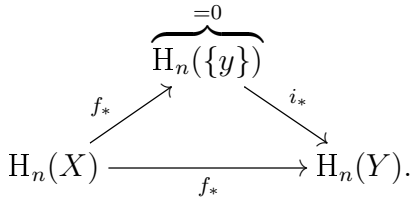
$$\mathrm{H}_n(S^0) \cong \begin{cases} \mathbb{Z}^2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Later we will cleverly reduce the computation of the homology groups of all other spheres to this calculation.

(2) Let X and Y be non-empty topological spaces. Let $f \colon X \to Y$ be a continuous map. We suppose that f is constant, i.e. we suppose that there exists a $y \in Y$ such that $f(X) = \{y\}$. We denote by $i \colon \{y\} \to Y$ the inclusion. Let $n \in \mathbb{N}$. We consider the commutative diagram



which by the functoriality of homology gives rise to the commutative diagram



By the Singleton-H_*-Lemma 35.11 we know that $\mathrm{H}_n(\{y\}) = 0$. Thus we see that $f_* \colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ factors through the trivial group, i.e. the map is the trivial map. On many occasions we will make use of this observation without referring to this discussion.

The calculation of $\mathrm{H}_n(\varnothing)$ on page 426, of $\mathrm{H}_0(X)$ in the H_0-Proposition 35.10 (1) and the calculation of the homology of a point in the Singleton-H_*-Lemma 35.11 are the only calculations of homology groups of path-connected topological spaces that one can do "by hand". For all other topological spaces we will need to work significantly harder, and this task will keep us busy for many chapters.

35.5. **Singular 1-chains.** It follows from the H_0-Proposition 35.10 (1) together with the Direct Sum-H_*-Lemma 35.2 that we have no troubles calculating 0-th homology groups. Unfortunately, going up by one dimension, things become already much harder. At the moment we do not have the tools to calculate the homology groups in dimension 1. But in this last section we will prove one result on singular 1-chains which will come in handy on numerous occasions. To formulate our lemma we need the following notation.

Notation. Let X be a topological space. Given a singular 1-simplex $\sigma\colon \Delta^1 \to X$ we denote by $\overline{\sigma}$ the singular 1-simplex that is given by $\overline{\sigma}(t, 1-t) = \sigma(1-t, t)$. Informally speaking $\overline{\sigma}$ is given by reversing the orientation of σ.

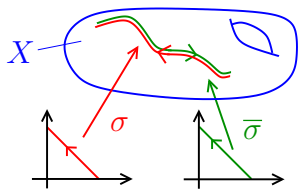
Lemma 35.12. (Singular 1-Simplex Lemma) Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex.

(1) σ is a cycle if and only if $\sigma((0,1)) = \sigma((1,0))$.
(2) The singular 1-chains $-\sigma$ and $\overline{\sigma}$ are homologous.
(3) Let $s \in (0,1)$. We consider the singular 1-simplices[277]

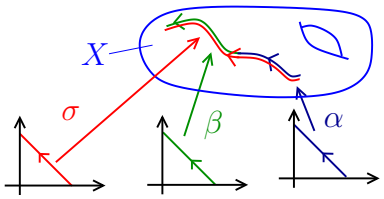
$$\begin{array}{rcl} \alpha\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(st, 1-st) \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(s+t(1-s), 1-(s+t(1-s))). \end{array}$$

The singular 1-chains σ and $\alpha + \beta$ are homologous.

The second part of the lemma says that for a singular 1-simplex the geometric operation of reversing the orientation is (up to the boundary of a simplicial 2-chain) the same as algebraically inverting the sign. Furthermore, the third part of the lemma says that for a singular 1-simplex (again up to the boundary of a simplicial 2-chain) it does not make a difference whether or not we split the singular 1-simplex into two singular 1-simplices.



σ and $-\overline{\sigma}$ are homologous

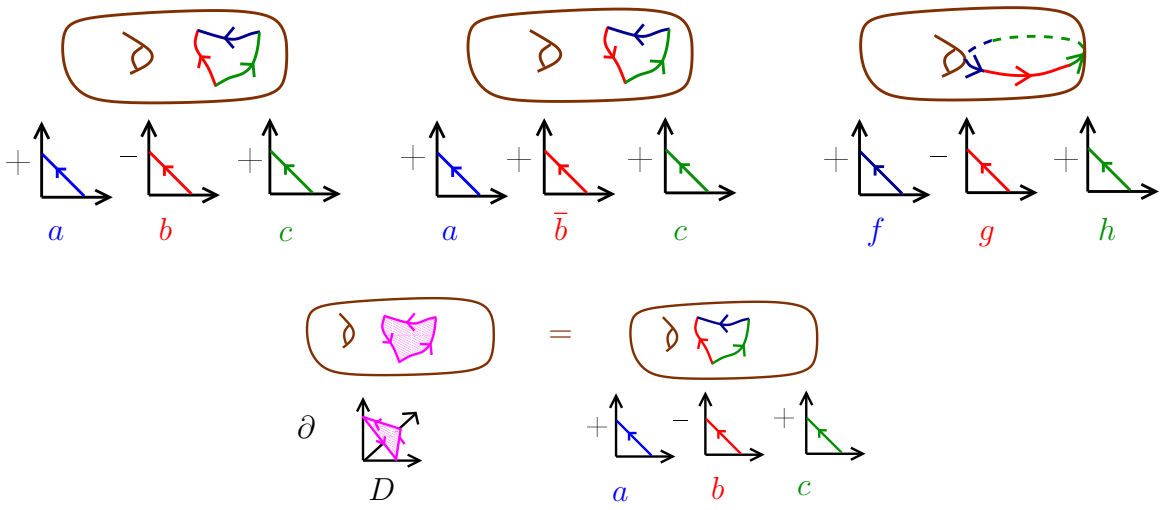


σ and $\alpha + \beta$ are homologous

Proof. The first statement follows from the discussion of the boundary of a singular 1-simplex on page 423. We leave the proof of the remaining two statements as an exercise to the reader. ■

Example. In the figure below we see three singular chains on the torus. It is straightforward to show that all three singular chains are cycles.
Now the question arises, which of these cycles are in fact null-homologous and which represent non-zero classes in homology? In the figure below we see that the first cycle $a - b + c$ is the boundary of a singular 2-simplex D.
One can easily see that the second cycle, $a + \overline{b} + c$ in the above figure is *not* the boundary of a singular 2-simplex. On the other hand by the Singular 1-Simplex Lemma 35.12 we know that $b + \overline{b}$ is a null-homologous cycle, i.e. there exists an $E \in C_2(X)$ such that

[277] Loosely speaking we break the singular 1-simplex σ into two singular 1-simplices α and β.



$b+\overline{b}=\partial E\in \mathrm{C}_1(X)$. Thus it follows that

$$a+\overline{b}+c \;=\; a+\partial E-b+c \;=\; a-b+c+\partial E=\partial D+\partial E \;=\; \partial(D+E),$$

i.e. $a+\overline{b}+c$ is null-homologous.

It is not clear whether the third cycle in the figure below is the boundary of a 2-dimensional singular chain. We will see later that this not the case. But it will take us quite some time to develop the techniques which will allow us to prove this statement.

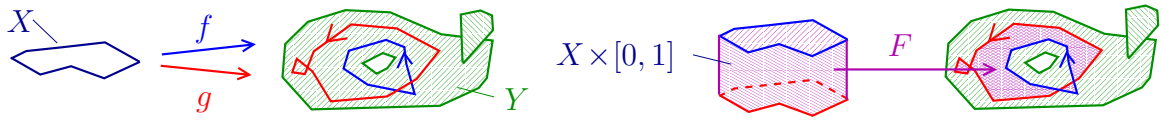
36. Homology and homotopies

We start out with recalling the following definition from page 136.

Definition. Let $f, g\colon X \to Y$ be two continuous maps between topological spaces. A homotopy between the maps f and g is a continuous map

$$F\colon X \times [0,1] \to Y,$$

such that for all $x \in X$ we have $F(x,0) = f(x)$ and $F(x,1) = g(x)$. If there exists a homotopy between f and g, then we say that f and g are homotopic and we write $f \simeq g$.



We also recall the following related two definitions.

Definition. Let X be and Y be topological spaces.

(1) A continuous map $f\colon X \to Y$ is a homotopy equivalence between X and Y, if f admits a homotopy inverse, i.e. a continuous map $g\colon Y \to X$ with $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.
(2) If there exists a homotopy equivalence between X and Y, then we say that X and Y are homotopy equivalent and we write $X \simeq Y$.

Example.

(1) In the Deformation Retract–Homotopy Equivalence Lemma 9.3 we showed that if A is a deformation retract of a topological space X, then the inclusion $A \to X$ is a homotopy equivalence.
(2) It follows easily from (1) that each non-empty convex subset of $\mathbb{R}^n$ is homotopy equivalent to a singleton, i.e. to a topological space consisting of a single point.

Next we recall the following definition from page 146.

Definition. We define the homotopy category $\mathcal{HomTop}$ to be the category given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{HomTop}) &:= \text{all topological spaces,} \\ \mathrm{Mor}(X,Y) &:= \text{set of homotopy equivalence classes of continuous maps } X \to Y \end{aligned}$$

with the composition[278]

$$\begin{array}{rcl} \mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,E_*) & \to & \mathrm{Mor}(X,Y) \\ ([f],[g]) & \mapsto & [g \circ f]. \end{array}$$

It is worth recalling that in the Homotopy Equivalence-Reformulation Lemma 9.4 we proved the elementary fact that a continuous map $f\colon X \to Y$ is a homotopy equivalence if and only if f represents an isomorphism in the homotopy category.

In the Homotopy-π_1–Proposition 13.4 and in the Homotopy-π_n-Proposition 28.6 we showed that homotopic maps induce essentially the same map on homotopy groups. We used this fact to show that homotopy equivalent path-connected topological spaces have isomorphic homotopy groups.

In this chapter we will see that homotopic maps induce the same map on homology. We will use this result to show that homotopy equivalent topological spaces have isomorphic

[278] It follows from the Homotopy Composition Lemma 8.4 (2) that this composition map is well-defined.

homology groups. This result, together with the results from the previous chapter, allows us for example to compute the homology groups of closed and open balls.

In the next section we first make some algebraic preparations. In particular we will introduce algebraic analogues of homotopies, of being "homotopy equivalent" and of the homotopy category.

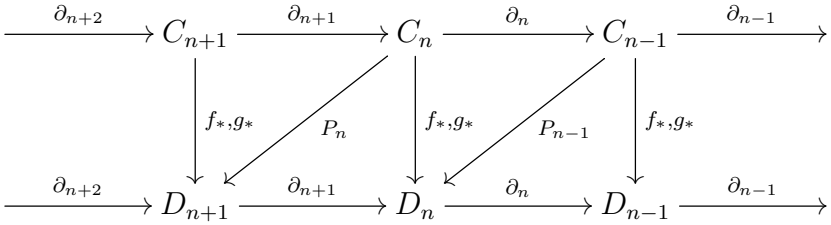
36.1. **Chain homotopies.** In this purely algebraic section we introduce a criterion that ensures that two chain maps $f, g\colon C_* \to D_*$ induce the same maps on the corresponding homology groups. In the next section we will use this to study the effect of homotopic maps on homology groups.

Definition. [279] Let $f, g\colon C_* \to D_*$ be chain maps between two chain complexes (C_*, ∂_*) and (D_*, ∂_*). A chain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between f and g is defined as a family $\{P_n\colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ of homomorphisms such that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; f_n - g_n\colon C_n \;\to\; D_n.$$

If there exists a chain homotopy between chain maps f and g, then we say that f and g are chain homotopic and we write $f \simeq g$.

For the definition of a chain homotopy and the subsequent discussion of chain homotopies it is sometimes helpful to keep in mind the following diagram that combines all objects involved in the definition:



The following lemma explains perhaps why we are interested in chain homotopies.

Lemma 36.1. (Chain Homotopy–H_*-Lemma) Let $f, g\colon C_* \to D_*$ be two chain maps between chain complexes. If f and g are chain homotopic, then $f_* = g_*\colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ for all $n \in \mathbb{N}_0$.

Proof. Let $P = \{P_n\}_{n\in\mathbb{N}_0}$ be a chain homotopy between f and g. Then for any cycle $z \in C_n$ we get the equalities

$$\begin{aligned} f(z) - g(z) \;=\; (f-g)(z) \;&=\; (\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n)(z) \\ &=\; (\partial_{n+1} \circ P_n)(z) + P_{n-1}(\underbrace{\partial_n(z)}_{=0}) \;=\; \partial(P_n(z)), \end{aligned}$$

i.e. $f(z)$ and $g(z)$ are homologous. ∎

Now that we know that chain homotopies are useful it seems reasonable to develop the theory a bit further. The next lemma can be viewed as an algebraic analogue of the Homotopy-Equivalence Relation Lemma 8.3 and the Homotopy Composition Lemma) 8.4.

[279] We apologize for the fact that we present this definition without any motivation. One could spend some time trying to motivate this definition, but we think that this would not really improve matters.

Lemma 36.2. (Chain Homotopy-Lemma) Let C_* and D_* be two chain complexes.

(1) Chain homotopy defines an equivalence relation on the set of all chain maps from C_* to D_*. In particular, if $f, g, h\colon C_* \to D_*$ are chain maps with $f \simeq g$ and $g \simeq h$, then $f \simeq h$.

(2) Let $f, f'\colon C_* \to D_*$ and $g, g'\colon D_* \to E_*$ be chain maps. Then

$$f \simeq f' \text{ and } g \simeq g' \implies g \circ f \simeq g' \circ f'.$$

Proof. We leave the elementary proof of the lemma to the reader. ■

The following definition is modelled on the definition of the homotopy category $\mathcal{HomTop}$ of topological spaces which we recalled in the beginning of the chapter.

Definition. We define the chain homotopy category $\mathcal{HomCplx}$

$$\begin{aligned} \mathrm{Ob}(\mathcal{HomCplx}) &:= \text{all chain complexes,} \\ \mathrm{Mor}(C_*, D_*) &:= \text{set of chain homotopy equivalence classes of chain maps } C_* \to D_* \end{aligned}$$

with the composition[280]

$$\begin{array}{ccc} \mathrm{Mor}(C_*, D_*) \times \mathrm{Mor}(D_*, E_*) & \to & \mathrm{Mor}(C_*, D_*) \\ ([f], [g]) & \mapsto & [g \circ f]. \end{array}$$

The following definition is also modelled on a definition which we recalled in the beginning of the chapter.

Definition. Let C_* and D_* be two chain complexes.

(1) A chain map $f\colon C_* \to D_*$ is called a chain homotopy equivalence between C_* and D_*, if it admits a chain homotopy inverse, i.e. a chain map $g\colon D_* \to C_*$ with $g \circ f \simeq \mathrm{id}_{C_*}$ and $f \circ g \simeq \mathrm{id}_{D_*}$.

(2) If there exists a chain homotopy equivalence between C_* and D_*, then we say that the chain complexes C_* and D_* are chain homotopy equivalent and we write $C_* \simeq D_*$.

The following lemma is the algebraic analogue of the Homotopy Equivalence-Reformulation Lemma 9.4.

Lemma 36.3. (Chain Homotopy Equivalence-Reformulation Lemma) Let C_* and D_* be two chain complexes and let $f\colon C_* \to D_*$ be a chain map. The following statements are equivalent:

(1) The element $[f] \in \mathrm{Mor}_{\mathcal{HomCplx}}(C_*, D_*)$ is an isomorphism in the chain homotopy category $\mathcal{HomCplx}$.

(2) f is a chain homotopy equivalence.

In particular the following statements are equivalent:

(1′) C_* and D_* are isomorphic in the chain homotopy category $\mathcal{HomCplx}$.

(2′) C_* and D_* are chain homotopy equivalent.

Example. It follows from the Chain Homotopy Equivalence-Reformulation Lemma 36.3 and the Isomorphism Lemma 3.3 that the relation "chain homotopy equivalence" of chain complexes is indeed an equivalence relation on the class of all chain complexes.

[280] It follows from the Chain Homotopy-Lemma 36.2 (2) that this composition map is well-defined.

Proof. As in the case of the proof of the Homotopy Equivalence-Reformulation Lemma 9.4 the statements follow almost immediately from the definitions. ■

For later we record the following straightforward corollary to the Chain Homotopy–H_*-Lemma 36.1.

Corollary 36.4. (Chain Homotopy Equivalence-H_*-Corollary) If $f\colon C_* \to D_*$ is a chain homotopy equivalence between two chain complexes, then for each $n \in \mathbb{N}_0$ the induced map $f_*\colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ is an isomorphism.

Proof. In the following we give the same proof twice, but with different language. First the language is a little more high-brow, the second time it is more down-to-earth:

(1) First note that it follows from the Chain Homotopy–H_*-Lemma 36.1 that the homology functor $\mathrm{H}_n\colon \mathcal{ChCplx} \to \mathcal{AbGroup}$ from the category of chain complexes to the category of abelian groups factors through a functor $\mathcal{HomCplx} \to \mathcal{AbGroup}$ from the homotopy category of chain complexes to the category of abelian groups. It follows from this observation, the Chain Homotopy Equivalence-Reformulation Lemma 36.3 and the fact that functors send isomorphisms to isomorphisms that chain homotopies give rise to isomorphisms of abelian groups.

(2) Let $g\colon D_* \to C_*$ be a chain homotopy inverse to f. Then we see that

$$g_* \circ f_* \quad = \quad (g \circ f)_* \quad \underset{\uparrow}{=} \quad (\mathrm{id}_{C_*})_* \quad = \quad \mathrm{id}_{\mathrm{H}_n(C)}\,.$$

by the Chain Homotopy–H_*-Lemma 36.1 since $g \circ f$ is chain homotopic to the identity

The same way we see that $f_* \circ g_* = \mathrm{id}_{\mathrm{H}_n(D)}$. ■

36.2. **Homology and homotopic maps.** The name *chain homotopy* introduced in the previous section suggests that this concept is related to the notion of a homotopy between two topological spaces. The following proposition shows that this is indeed the case.

Proposition 36.5. (Homotopy-H_*-Proposition) Let X and Y be two topological spaces and let $f, g\colon X \to Y$ be two continuous maps. If f and g are homotopic, then the following hold:

(1) The induced maps f_* and g_* from $\mathrm{C}_*(X)$ to $\mathrm{C}_*(Y)$ are chain homotopic.
(2) For each $n \in \mathbb{N}_0$ we have $f_* = g_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$.

Remark. The proposition can be stated more succinctly as follows:

$$f \simeq g\colon X \to Y \quad \Longrightarrow \quad f_* \simeq g_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y) \quad \Longrightarrow \quad f_* = g_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y).$$

We postpone the lengthy proof of the Homotopy-H_*-Proposition 36.5 to Section 36.3. Let us first turn to the fun part, namely let us jot down a few consequences. We start out with the following straightforward consequence.

Corollary 36.6. (Homotopy Category-H_*-Functor Corollary) Each of our homology functors $\mathrm{H}_n\colon \mathcal{Top} \to \mathcal{AbGroup}$ defines a functor $\mathrm{H}_n\colon \mathcal{HomTop} \to \mathcal{AbGroup}$ from the homotopy category of topological spaces to the category of abelian groups.

Proof. This statement follows immediately from the Homotopy-H_*-Proposition 36.5 and the definitions. ■

The following corollary. This corollary mirrors the fact, proved in the Homotopy-π_1-Proposition 13.4 (2) and the Homotopy-π_n-Proposition 28.6 (2a), that homotopy equivalences induce isomorphisms of fundamental groups and higher homotopy groups.

Corollary 36.7. (Homotopy Equivalence-H_*-Corollary) Let X and Y be topological spaces and let $n \in \mathbb{N}_0$.

(1) (a) If $f\colon X \to Y$ is a homotopy equivalence, then the induced map $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism.
(b) If X is a deformation retract of Y, then the inclusion induced map $\mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism.

(2) If X and Y are homotopy equivalent, then $\mathrm{H}_n(X) \cong \mathrm{H}_n(Y)$.

Proof.

(1) (a) This statement follows immediately from the Homotopy Category-H_*-Functor Corollary 36.6 and the fact that functors send isomorphisms to isomorphisms.
(b) This statement follows immediately from (1a) and the Deformation Retract–Homotopy Equivalence Lemma 9.3.

(2) This statement follows immediately from (1a).

■

Corollary 36.8. (Contractible-H_*-Corollary) If X is contractible (i.e. if X is homotopy equivalent to a singleton), then we have $\mathrm{H}_0(X) \cong \mathbb{Z}$ and we have $\mathrm{H}_n(X) = 0$ for all $n \in \mathbb{N}$.

Proof. This statement follows from the Homotopy Equivalence-H_*-Corollary 36.7 (2) and the Singleton-H_*-Lemma 35.11. ■

Example. On page 149 we saw that every convex subset of $\mathbb{R}^n$, e.g. the open disk B^n, the closed disk $\overline{B}^n$, any simplex or $\mathbb{R}^n$, is contractible. Thus it follows from the Contractible-H_*-Corollary 36.8, that for any $i \in \mathbb{N}_0$ we have

$$\mathrm{H}_i(\underbrace{\text{convex subset of } \mathbb{R}^n}_{\text{e.g. } \mathbb{R}^n,\ \overline{B}^n,\ B^n}) \ \cong\ \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$

36.3. **Proof of the Homotopy-H_*-Proposition 36.5.** For the reader's convenience we recall the proposition that we need to prove.

Proposition 36.5. (Homotopy-H_*-Proposition) Let X and Y be two topological spaces and let $f, g\colon X \to Y$ be two continuous maps. If f and g are homotopic, then the following hold:

(1) The induced maps f_* and g_* from $\mathrm{C}_*(X)$ to $\mathrm{C}_*(Y)$ are chain homotopic, i.e. there exists a family $\{P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(Y)\}_{n\in\mathbb{N}_0}$ of homomorphisms such that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \ = \ f_n - g_n\colon \mathrm{C}_n(X) \ \to\ \mathrm{C}_{n+1}(Y).$$

(2) For each $n \in \mathbb{N}_0$ we have $f_* = g_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$.

We start out with a discussion of the key idea behind the proof of the Homotopy-H_*-Proposition 36.5. Afterwards we will fill in the details. So let $f, g\colon X \to Y$ be two continuous maps between topological spaces and let $F\colon X \times [0,1] \to Y$ be a homotopy between the

maps f and g. Our goal is to find a chain homotopy between f_* and g_*. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. We need to find $P(\sigma) \in \mathrm{C}_{n+1}(Y)$ and $P(\partial\sigma) \in \mathrm{C}_n(Y)$ such that

$$\partial P(\sigma) \;=\; -P(\partial\sigma) + f_*(\sigma) - g_*(\sigma).$$

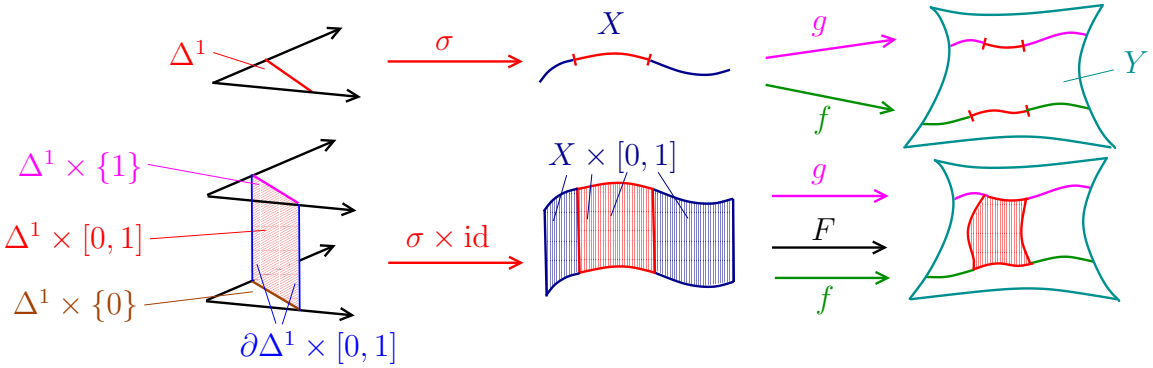
Note that $F \circ (\sigma \times \mathrm{id}_{[0,1]})$ defines a map $\Delta^n \times [0,1] \to Y$. Now we can rewrite the above desired equality as follows:

$$\partial P(\sigma) \;=\; -P(\partial\sigma) + \underset{\text{restricted to } \underbrace{\Delta^n \times \{0\}}_{=_i\Delta^n}}{\text{map } F \circ (\sigma \times \mathrm{id}_{[0,1]})} - \underset{\text{restricted to } \underbrace{\Delta^n \times \{1\}}_{=_i\Delta^n}}{\text{map } F \circ (\sigma \times \mathrm{id}_{[0,1]})}.$$

To get an idea we consider the figure below. Looking at the figure we note that

$$\partial(\Delta^n \times [0,1]) \;=\; (\partial\Delta^n \times [0,1]) \;\cup\; (\Delta^n \times \{0\}) \;\cup\; (\Delta^n \times \{1\}).$$

Comparing the desired equality of maps and the equality of sets suggests that each map $P_k\colon \mathrm{C}_k(X) \to \mathrm{C}_{k+1}(Y)$ should take the role of replacing a singular k-simplex $\mu\colon \Delta^k \to X$ by the map $F \circ (\mu \times \mathrm{id}_{[0,1]})\colon \Delta^k \times [0,1] \to Y$. This is a good idea, but the problem is that $\Delta^k \times [0,1]$ is not a $(k+1)$-simplex. The idea now is to write $\Delta^k \times [0,1]$ as the union of $(k+1)$-simplices. In the figure below we see for example that we can view $\Delta^1 \times [0,1]$ as the union of the triangles with the vertices v_0, w_0, w_1 respectively v_0, v_1, w_1.



To generalize this observation we have to introduce some more notation.

Notation. Let $n \in \mathbb{N}_0$.

(1) For $i = 0, \ldots, n$ we write

$$v_i \;:=\; (0,\ldots,1,0,\ldots,0) \times \{0\} \;\in\; \mathbb{R}^{n+1} \times \{0\},$$
$$\text{and} \qquad w_i \;:=\; (0,\ldots,1,0,\ldots,0) \times \{1\} \;\in\; \mathbb{R}^{n+1} \times \{1\}.$$

The points $v_0, \ldots, v_n$ thus span the simplex $\Delta^n \times 0$ and the points $w_0, \ldots, w_n$ span the simplex $\Delta^n \times 1$.

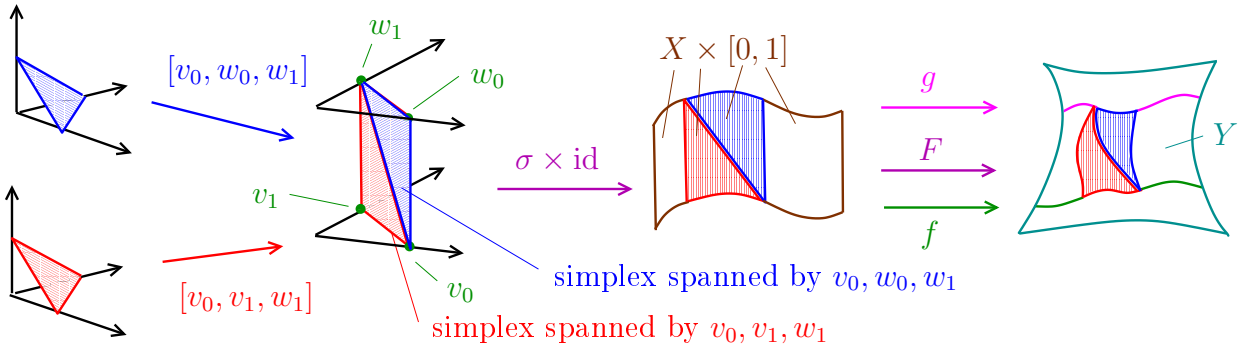
(2) For $a_0, \ldots, a_k \in \Delta^n \times [0,1]$ we consider the map

$$[a_0,\ldots,a_k]\colon \Delta^k \;\to\; \Delta^n \times [0,1]$$
$$(\lambda_0,\ldots,\lambda_k) \;\mapsto\; \sum_{j=0}^{k} \lambda_j \cdot a_j.$$

The map $[a_0, \ldots, a_k]$ thus sends the standard k-simplex Δ^k to the simplex spanned by $a_0, \ldots, a_k$ in $\Delta^n \times [0,1]$.

(3) We set

$$\Omega_n \;:=\; \sum_{j=0}^{n} (-1)^j \cdot \underbrace{[v_0,\dots,v_j,w_j,\dots,w_n]}_{\text{map } \Delta^{n+1}\to\Delta^n\times[0,1]} \;\in\; \mathrm{C}_{n+1}(\Delta^n\times[0,1]).$$



In the following lemma we compute the boundary of $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n\times[0,1])$. The statement might not be particularly visually appealing, but it will turn out to be very useful.

Lemma 36.9. (Prism Boundary Lemma) Let $n\in\mathbb{N}_0$. Given $t\in\{0,1\}$ we write

$$\begin{aligned}\eta_t\colon \Delta^n &\to \Delta^n\times[0,1]\\ x &\mapsto (x,t).\end{aligned}$$

With this notation we have[281]

$$\partial_{n+1}(\Omega_n) + \sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times\mathrm{id}_{[0,1]})_*\Big(\underbrace{\Omega_{n-1}}_{\in \mathrm{C}_n(\Delta^{n-1}\times[0,1])}\Big) \;=\; \eta_1-\eta_0 \;\in\; \mathrm{C}_n(\Delta^n\times[0,1]).$$

Proof. The proof of the lemma is basically an elementary, albeit slightly confusing calculation. First note that

$$\begin{aligned}\partial\Omega_n &= \partial\Big(\sum_{j=0}^{n}(-1)^j\cdot[v_0,\dots,v_j,w_j,\dots,w_n]\Big)\\ &= \sum_{k=0}^{n+1}(-1)^k\cdot\sum_{j=0}^{n}(-1)^j\cdot[v_0,\dots,v_j,w_j,\dots,w_n]\circ i_k^{n+1}.\end{aligned}$$

It is straightforward to see that for any $k\in\{0,\dots,n+1\}$ we have[282][283]

$$[v_0,\dots,v_j,w_j,\dots,w_n]\circ i_k^{n+1} \;=\; \begin{cases}[v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n], & \text{if } k\le j,\\ [v_0,\dots,v_j,w_j,\dots,\widehat{w_{k-1}},\dots,w_n], & \text{if } k>j.\end{cases}$$

[281] Here $i_j^n\colon\Delta^{n-1}\to\Delta^n$ denotes the j-th face map as introduced on page 422.

[282] The first case, $k\le j$ is slightly easier: one has to figure out, where the h-th vertex e_h of the standard n-simplex gets sent to. If $h<k$, then i_k^{n+1} sends the h-th vertex again to the h-th vertex, and $[v_0,\dots,v_j,w_j,\dots,w_n]$ sends this vertex to v_h. If $h\ge k$, then i_k^{n+1} sends the h-th vertex again to the $(h+1)$-th vertex, and $[v_0,\dots,v_j,w_j,\dots,w_n]$ sends this vertex to v_{h+1}. If $k\le j$ this corresponds just to the map $[v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n]$.
The second case, $k>j$ is proved precisely the same way.

[283] Here we use the usual notation that $\widehat{v_k}$ means that we do not plug in $\widehat{v_k}$, the same applies to $\widehat{w_{k-1}}$.

Thus it follows that

$$\begin{aligned}\partial\Omega_n &= \sum_{0\le k\le j\le n} (-1)^k\cdot(-1)^j\cdot[v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n]\\ &\quad+\sum_{n+1\ge k>j\ge 0}(-1)^k\cdot(-1)^j\cdot[v_0,\dots,v_j,w_j,\dots,\widehat{w_{k-1}},\dots,w_n].\end{aligned}$$

Now we split the first sum into the cases $k=j$ and $k<j$ and we split the second sum into the cases $k=j+1$ and $k>j+1$. This leads us to the following equality:

$$\begin{aligned}\partial\Omega_n &= \sum_{k=0}^{n}(-1)^{2k}\cdot[v_0,\dots,v_{k-1},w_k,\dots,w_n]\\ &\quad+\sum_{k=1}^{n+1}(-1)^{2k+1}\cdot[v_0,\dots,v_{k-1},w_k,\dots,w_n]\\ &\quad+\sum_{0\le k<j\le n}(-1)^k\cdot(-1)^j\cdot[v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n]\\ &\quad+\sum_{n+1\ge k>j+1\ge 1}(-1)^k\cdot(-1)^j\cdot[v_0,\dots,v_j,w_j,\dots,\widehat{w_{k-1}},\dots,w_n].\end{aligned}$$

Next we consider the first two sums. The summands cancel each other, except for the summands for $k=0$ and $k=n+1$. Thus we obtain that

$$\begin{aligned}\partial\Omega_n &= [w_0,\dots,w_n]-[v_0,\dots,v_n]\\ &\quad+\sum_{0\le k<j\le n}(-1)^k\cdot(-1)^j\cdot[v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n]\\ &\quad+\sum_{n+1\ge k>j+1\ge 1}(-1)^k\cdot(-1)^j\cdot[v_0,\dots,v_j,w_j,\dots,\widehat{w_k},\dots,w_n].\end{aligned}\tag{$*$}$$

The first summand is just η_1 and the second is $-\eta_0$. It remains to show that the remaining summand corresponds to the remaining term in the formula. Thus we continue our proof with the discussion of the one term in the statement of the lemma that we have not studied yet. We have[284]

$$\sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times\mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \underset{\substack{\uparrow\\ \text{definition of } (i_j^n\times\mathrm{id}_{[0,1]})_* \text{ and substitution } r=j}}{=} \sum_{r=0}^{n}\sum_{s=0}^{n-1}(-1)^r\cdot(-1)^s\cdot(i_r^n\times\mathrm{id}_{[0,1]})\circ[v_0,\dots,v_s,w_s,\dots,w_{n-1}].$$

It follows from an elementary (but arguably confusing) argument, using only the definitions, that we have the following equalities of maps $\Delta^n\to\Delta^n\times[0,1]$:

$$(i_r^n\times\mathrm{id}_{[0,1]})\circ[v_0,\dots,v_s,w_s,\dots,w_{n-1}] = \begin{cases}[v_0,\dots,\widehat{v_r},\dots,v_{s+1},w_{s+1},\dots,w_n], & \text{if } r\le s,\\ [v_0,\dots,v_s,w_s,\dots,\widehat{w_{r+1}},\dots,w_n], & \text{if } r>s.\end{cases}$$

Thus we see that

$$\begin{aligned}\sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times\mathrm{id}_{[0,1]})_*(\Omega_{n-1}) &= \sum_{0\le r\le s\le n-1}(-1)^s\cdot(-1)^r\cdot[v_0,\dots,\widehat{v_r},\dots,v_{s+1},w_{s+1},\dots,w_n]\\ &\quad+\sum_{n\ge r>s\ge 0}(-1)^s\cdot(-1)^r\cdot[v_0,\dots,v_s,w_s,\dots,\widehat{w_{r+1}},\dots,w_n].\end{aligned}$$

[284] Here $v_0,\dots,v_{n-1},w_0,\dots,w_{n-1}$ are now, unfortunately somewhat confusingly, the corresponding vertices of $\Delta^{n-1}\times[0,1]$ instead of $\Delta^n\times[0,1]$.

In the first term we perform the substitutions $r = k$ and $s = j - 1$ and in the second term we perform the substitutions $r = k - 1$, $s = j$. We obtain the following equality:

$$\begin{aligned}\sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \;&=\; \sum_{0\le k\le j-1\le n-1} (-1)^{j-1}\cdot(-1)^k \cdot [v_0,\dots,\widehat{v_k},\dots,v_j,w_j,\dots,w_n] \\ &\quad + \sum_{n\ge k-1>j\ge 0} (-1)^j\cdot(-1)^{k-1}\cdot[v_0,\dots,v_j,w_j,\dots,\widehat{w_k},\dots,w_n].\end{aligned}$$

We see immediately that these terms are just the negatives of the corresponding terms in the above equation $(*)$. ■

Now we can finally provide the proof of the Homotopy-H_*-Proposition 36.5.

Proof of the Homotopy-H_*-Proposition 36.5. Let X be a topological space and let $F\colon X \times [0,1] \to Y$ be a homotopy from a map f to a map g.

(1) This statement follows immediately from the following claim.

Claim. Given any $n \in \mathbb{N}_0$ we define

$$P_n\colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_{n+1}(Y)$$

via

$$(\sigma\colon \Delta^n \to X) \;\mapsto\; (F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\underset{\uparrow}{\Omega_n}).$$

we defined $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0,1])$ on page 439

For any $n \in \mathbb{N}_0$ we have $\quad \partial P_n + P_{n-1}\partial \;=\; g_* - f_*$

as maps from $\mathrm{C}_n(X)$ to $\mathrm{C}_n(Y)$.

Proof. As always it suffices to verify the equality for each singular n-simplex. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. We calculate that

$$\begin{aligned}(\partial P_n + P_{n-1}\partial)(\sigma) \;&=\; \partial(P_n(\sigma)) + P_{n-1}\Big(\sum_{j=0}^{n} (-1)^j \cdot \sigma \circ i_j^n\Big) \\ &=\; \partial(P_n(\sigma)) + \sum_{j=0}^{n} (-1)^j \cdot P_{n-1}(\sigma \circ i_j^n) \\ &=\; \partial((F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n)) + \sum_{j=0}^{n} (-1)^j \cdot (F \circ (\sigma \circ_j^n \times \mathrm{id}_{[0,1]}))_*(\Omega_{n-1}) \\ &=\; \underset{\uparrow}{(F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*}\Big(\partial(\Omega_n)) + \sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*(\Omega_{n-1})\Big)\end{aligned}$$

by the C_*–Functor Lemma 35.5 (1) and since we have $(\sigma \times \mathrm{id}) \circ (i_j^n \times \mathrm{id}) = (\sigma \circ i_j^n) \times \mathrm{id}$ as maps $\Delta^n \times [0,1] \times Y \times [0,1]$

$$\underset{\uparrow}{=}\; (F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\eta_1 - \eta_0) \;\underset{\uparrow}{=}\; f \circ \sigma - g \circ \sigma \;=\; g_*(\sigma) - f_*(\sigma).$$

by the Prism Boundary Lemma 36.9 since F is a homotopy from f to g ⊞

(2) This statement follows immediately from Statement (1) together with the Chain Homotopy–H_*-Lemma 36.1. ■

Exercise 36.1. Let $\varphi\colon A \to B$ be a homomorphism of free abelian groups and let $C \subset A$ and $D \subset B$ be subgroups such that the natural map $C/D \to A/B$ is an isomorphism. Show that the chain complex $0 \to A \xrightarrow{\varphi} B \to 0$ is chain homotopy equivalent to the chain complex $0 \to C \xrightarrow{\varphi} D \to 0$.

37. Exact sequences and long exact sequences

In the next chapter, given a topological space X and a subspace $A \subset X$, we will introduce the relative chain complex given by the groups $\mathrm{C}_n(X, A) := \mathrm{C}_n(X)/\mathrm{C}_n(A)$. To understand the relationship between the homology groups of the chain complexes $\mathrm{C}_*(A)$, $\mathrm{C}_*(X)$ and $\mathrm{C}_*(x, A)$ we will need to develop our algebraic skills.

In this purely algebraic chapter we will therefore introduce the concept of a "(short) exact sequence" and will see that a "short exact sequence" of chain complexes gives rise to a "long exact sequence" of homology groups. These concepts will play an essential role in the theory of homology.

37.1. **Exact sequences.** We start out with the following definition which appears in many parts of mathematics.

Definition. A sequence of homomorphisms of abelian groups

$$\ldots \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \ldots$$

is called exact, if for each n we have

$$\mathrm{im}(f_{n+1}\colon A_{n+1} \to A_n) \;=\; \ker(f_n\colon A_n \to A_{n-1}).$$

In the following lemma we collect a few basic statements about exact sequences which we will use frequently without referring to the lemma.

Lemma 37.1. (Exactness Lemma)

(1) In any exact sequence the composition of two consecutive maps is the zero map.

(2) We have the following equivalences of statements:

$$\begin{aligned}
\text{the sequence } 0 \to A \xrightarrow{f} B \quad \text{is exact} &\iff f \text{ is injective}\\
\text{the sequence } A \xrightarrow{f} B \to 0 \text{ is exact} &\iff f \text{ is surjective}\\
\text{the sequence } 0 \to A \xrightarrow{f} B \to 0 \text{ is exact} &\iff f \text{ is an isomorphism.}
\end{aligned}$$

(3) Given a monomorphism $i\colon A \to B$ and an epimorphisms $p\colon B \to C$ of groups we have

$$\begin{aligned}
0 \to A \xrightarrow{i} B \xrightarrow{f} C \text{ is exact} &\iff i \text{ induces an isomorphism } A \xrightarrow{\cong} \ker(f)\\
A \xrightarrow{f} B \xrightarrow{p} C \to 0 \text{ is exact} &\iff p \text{ descends to an isomorphism } \mathrm{coker}(f) \xrightarrow{\cong} C.
\end{aligned}$$

Proof. The lemma follows easily from the definitions. We leave it to the reader to provide the details. ∎

Example. Let $f\colon A \to B$ be a homomorphism of groups. It follows from the Exactness Lemma 37.1 that the following sequence is exact

$$0 \;\to\; \ker(f) \;\to\; A \;\xrightarrow{f}\; B \;\to\; \mathrm{coker}(f) \;\to\; 0.$$

37.2. **Split exact sequences.** In many future settings we will be given an exact sequence where we know some of the groups and we will wonder, whether we can determine the missing groups. In this section we will study the simplest instant of such a problem, namely the case of a short exact sequence:

[284] Note that we do not assume that the groups are abelian. In most cases the groups will be abelian though, that is why in this section we denote the trivial group by "0".

Definition. A short exact sequence is an exact sequence of the form

$$0 \to A \to B \to C \to 0,$$

which consists of at most three non-trivial abelian groups.

Let

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

be a short exact sequence. If we know $\varphi\colon A \to B$ or if we know $\psi\colon B \to C$ then using the Exactness Lemma 37.1 we can read off the third group. Now suppose we know that the two "outer" groups A and C. Can we read "read off" B from this short exact sequence? In general the answer is no. For example the short exact sequence

$$0 \to \mathbb{Z}_2 \longrightarrow B \longrightarrow \mathbb{Z}_2 \to 0$$

could be of the form

$$0 \to \mathbb{Z}_2 \xrightarrow{x\mapsto(x,0)} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{(x,y)\mapsto y} \mathbb{Z}_2 \to 0$$

but it could also be of the form[285]

$$0 \to \mathbb{Z}_2 \xrightarrow{[x]\mapsto[2x]} \mathbb{Z}_4 \xrightarrow{[x]\mapsto[x]} \mathbb{Z}_2 \to 0.$$

In the remainder of this section we will give a criterion for when the naive guess $B \cong A \oplus C$ is actually the correct answer.

Definition. We say that a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of groups splits, if there exists a splitting $s\colon C \to B$, i.e. a homomorphism $s\colon C \to B$ such that $p \circ s = \mathrm{id}_C$.

Example.

(1) The short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{k\mapsto(k,0)} \mathbb{Z} \oplus \mathbb{Z}_n \xrightarrow{(k,l)\mapsto l} \mathbb{Z}_n \to 0$$

splits. In fact a splitting is given by the homomorphism $b \mapsto (0, b)$. On the other hand the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{k\mapsto n\cdot k} \mathbb{Z} \to \mathbb{Z}_n \to 0.$$

does not split.

(2) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. It follows from the Free Abelian Group–Splitting Lemma 18.5, that if the last group C is a free abelian group, then the short exact sequence splits.

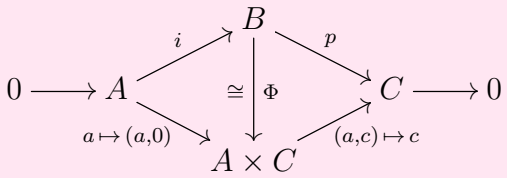
The following lemma will be useful on numerous occasions.

Lemma 37.2. (Splitting Lemma) Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

[285] Are there any other possibilities for the isomorphism type of B?

be a short exact sequence of abelian groups. The following three statements are equivalent:
(1) The short exact sequence splits.
(2) There exists a homomorphism $t\colon B \to A$ such that $t \circ i = \mathrm{id}_A$.
(3) There exists an isomorphism $\Phi\colon B \to A \times C$ such that the following diagram commutes:



where the lower left diagonal map is the obvious inclusion and the lower right diagonal map is the natural projection.

Proof. Let

$$0 \;\to\; A \;\xrightarrow{i}\; B \;\xrightarrow{p}\; C \;\to\; 0$$

be a short exact sequence of abelian groups.
(1)⇒(3). Suppose that $s\colon C \to B$ is a splitting, i.e. suppose that s is a homomorphism such that $p \circ s = \mathrm{id}_C$. We consider the map

$$\begin{array}{rcl} \Psi\colon A \times C & \to & B \\ (a,c) & \mapsto & i(a) + s(c). \end{array}$$

Using the fact that B is abelian one can easily verify that the map is a homomorphism.

Claim. The above homomorphism $\Psi\colon A \times C \to B$ is an isomorphism and the inverse $\Phi := \Psi^{-1}\colon B \to A \times C$ has all desired properties.

Proof. First we show that Ψ is an epimorphism. Let $b \in B$. Note that

$$p(b - s(p(b))) \underset{\substack{\uparrow \\ \text{since } p \text{ is a homomorphism}}}{=} p(b) - \underbrace{(p \circ s)}_{=\mathrm{id}}(p(b)) = p(b) - p(b) = 0.$$

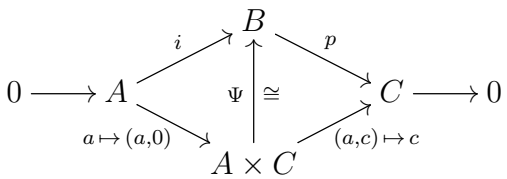
It follows from this calculation and the exactness of the short exact sequence that there exists a unique $a \in A$ with $i(a) = b - s(p(b))$. Then $\Psi(a, p(b)) = i(a) + s(p(b)) = b$.

Next we show that Ψ is a monomorphism. Let $(a,c) \in A \times B$ with $\Psi(a,c) = 0$. Then

$$0 = p(0) = p(\Psi(a,c)) = p(i(a) + s(c)) = \underbrace{(p \circ i)}_{=0}(a) + \underbrace{(p \circ s)}_{=\mathrm{id}_C}(c) = c.$$

Thus we have shown that $c = 0$. It follows that $0 = \Psi(a,c) = i(a) + s(c) = i(a)$. Since i is a monomorphism we see that $a = 0$. In summary we have shown that $(a,c) = 0$.

It remains to show that the following diagram commutes:



It is clear that the left hand triangle commutes. Furthermore it follows from $p \circ i = 0$ and $p \circ s = \mathrm{id}_C$ that the triangle to the right commutes.

(2)⇒(3). Let $t\colon B \to A$ be a homomorphism such that $t \circ i = \mathrm{id}_A$. We define

$$\begin{array}{rcl} \Phi\colon B & \to & A \times C \\ b & \mapsto & (t(b), p(b)). \end{array}$$

As in proof of (1)⇒(2) it follows easily from the exactness of the initial sequence that this map is in fact an isomorphism. It is straightforward to verify that Φ has all the other desired properties.

(3)⇒(1),(2). Now suppose we are given an isomorphism $\Phi\colon B \to A \times C$ such that the given diagram commutes. It follows immediately from the definitions and given properties that the maps

$$B \xrightarrow{\Phi} A \times C \xrightarrow{(a,c)\mapsto a} A \quad \text{and} \quad C \xrightarrow{c\mapsto(0,c)} A \times C \xrightarrow{\Phi^{-1}} B$$

have the required properties. ■

37.3. **Long exact sequence of homology groups.** In the previous two sections we studied (short) exact sequences of abelian groups. In this section we will study short exact sequences of chain complexes and we will see that they give rise to an exact sequence of homology groups.

Definition. A short exact sequence of chain complexes

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$

consists of chain complexes A_*, B_*, C_* and chain maps $i\colon A_* \to B_*$ and $p\colon B_* \to C_*$ such that for each n the maps

$$0 \to A_n \xrightarrow{i} B_n \xrightarrow{p} C_n \to 0$$

form a short exact sequence.

Definition. Let $0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$ be short exact sequence of chain complexes. Let $n \in \mathbb{N}_0$. We define a homomorphism

$$\partial_n\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$$

as follows:

(0) Let $z \in \mathrm{H}_n(C)$.

(1) We pick an n-cycle $c \in C_n$ with $[c] = z$.

(2) Since $0 \to A_n \xrightarrow{i} B_n \xrightarrow{p} C_n \to 0$ is exact we know by the Exactness Lemma 37.1 that $p\colon B_n \to C_n$ is an epimorphism. Thus there exists a $b \in B_n$ with $p(b) = c$.

(3) We have

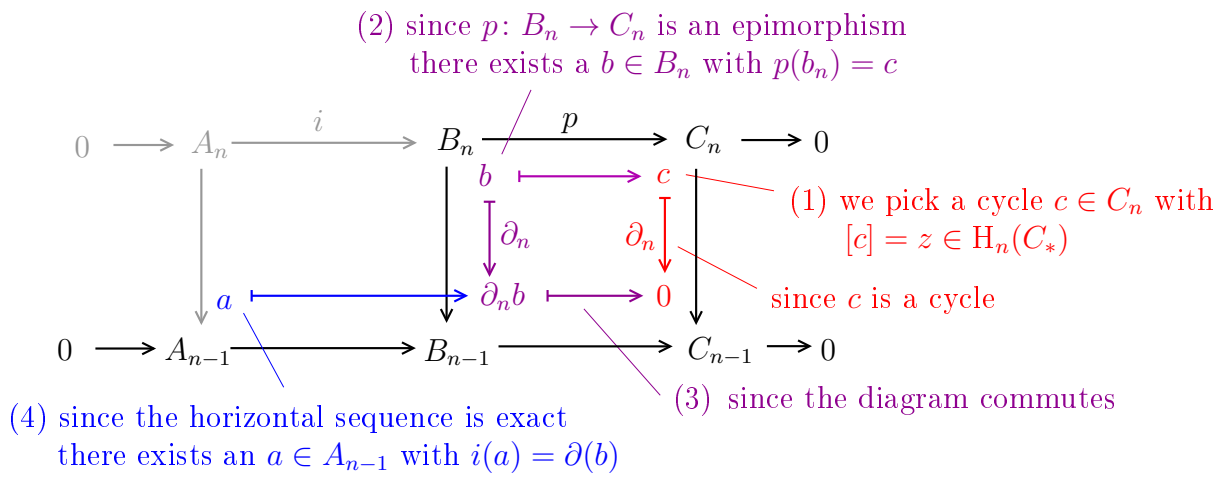
$$p(\partial b) \underset{\substack{\uparrow \\ \text{since } p \text{ is a chain map}}}{=} \partial p(b) = \partial c \underset{\substack{\uparrow \\ \text{since } c \text{ is a cycle}}}{=} 0.$$

(4) Since $0 \to A_{n-1} \xrightarrow{i} B_{n-1} \xrightarrow{p} C_{n-1} \to 0$ is exact it follows from (3) that there exists a unique $a \in A_{n-1}$ with $i(a) = \partial b$.

(5) We set $\partial_n(z) := [a] \in \mathrm{H}_{n-1}(A)$.

We refer to this map $\partial_n\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ as the connecting homomorphism. [286]

[287] The fact that here the symbol "∂_n" denotes the boundary map of the various chain complexes and also the connecting homomorphism $\mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ is somewhat unfortunate.



The following lemma summarizes two of the key properties of the connecting homomorphisms.

Lemma 37.3. (Connecting Homomorphism Lemma)

(1) Let $0 \to A_* \to B_* \to C_* \to 0$ be a short exact sequence of chain complexes. For each $n \in \mathbb{N}_0$ the connecting homomorphism $\partial_n\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ is well-defined and it is indeed a homomorphism.

(2) The connecting homomorphism is natural, i.e. if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A'_* & \xrightarrow{i} & B'_* & \xrightarrow{p} & C'_* & \longrightarrow & 0 \end{array}$$

is a commutative diagram of chain maps where the horizontal sequences are exact, then the following diagram commutes for every $n \in \mathbb{N}$:

$$\begin{array}{ccc} \mathrm{H}_n(C) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A) \\ {\scriptstyle c_*}\downarrow & & \downarrow{\scriptstyle a_*} \\ \mathrm{H}_n(C') & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A') \end{array}$$

Proof. In the proof of the lemma it is helpful to keep in mind the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{n+1} & \xrightarrow{i} & B_{n+1} & \xrightarrow{p} & C_{n+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \\ 0 & \longrightarrow & A_n & \xrightarrow{i} & B_n & \xrightarrow{p} & C_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} & & \\ 0 & \longrightarrow & A_{n-1} & \xrightarrow{i} & B_{n-1} & \xrightarrow{p} & C_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial_{n-1}} & & \downarrow{\scriptstyle \partial_{n-1}} & & \downarrow{\scriptstyle \partial_{n-1}} & & \\ 0 & \longrightarrow & A_{n-2} & \xrightarrow{i} & B_{n-2} & \xrightarrow{p} & C_{n-2} & \longrightarrow & 0. \end{array}$$

The proof of this lemma and the following proposition are done by a "diagram chase". This means that one needs to keep the above commutative diagram in mind and on several occasions one has to use that the vertical maps are chain maps and that the horizontal sequences are exact. In both cases the proofs are quite elementary and utterly unreadable. In both cases it is thus highly recommended to prove the statements on your own and to ignore the proofs provided below.

For the undeterred reader we now give the fully gory details.

(1) We start out with the following claim.

Claim 1. The map $\partial_n \colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ is well-defined.

Proof. Let $z \in \mathrm{H}_n(C)$. We denote by $c \in C_n$, $b \in B_n$ and $a \in A_{n-1}$ the elements that arise in the definition of $\partial_n(z)$. We need to show that the element a is indeed a cycle and that the definition of $\partial_n(z)$ does not depend on the choices of c and b.

The proofs of these two statements are not particularly difficult and they are done via "diagram chasing". First we note that

$$i(\partial a) \underset{\substack{\uparrow \\ \text{since } i \text{ is a chain map}}}{=} \partial i(a) = \partial(\partial b) = 0.$$

Since i is injective we obtain that $\partial a = 0$, i.e. a is indeed a cycle. Now we show that the homology class $[a] \in \mathrm{H}_{n-1}(A)$ depends only on the homology class $z \in \mathrm{H}_n(C)$.

(a) If we chose a different element $\widetilde{b}$ in B_n with $p(\widetilde{b}) = p(b) = c$, then it follows from the fact that the horizontal sequences are exact that this element would be of the form $\widetilde{b} = b + i(a')$ for some $a' \in A_n$. But then we obtain the following equality:

$$\partial\widetilde{b} = \partial(b + i(a')) = \partial b + \partial(i(a')) \overset{\substack{\text{since } i \text{ is a chain map} \\ \downarrow}}{=} \underbrace{\partial b}_{=i(a)} + i(\partial a') = i(a + \partial a').$$

Since $[a] = [a + \partial a'] \in \mathrm{H}_{n-1}(A)$ this does not change the homology class.

(b) If we picked a different representative $\widetilde{c}$ for $z \in \mathrm{H}_n(C)$, then this representative would be of the form $\widetilde{c} = c + \partial c'$ for some $c' \in C_{n+1}$. Since the map $p \colon B_n \to C_n$ is an epimorphism there exists a $b' \in B_{n+1}$ with $p(b') = c'$. It follows that

$$p(b + \partial b') = p(b) + p(\partial b') \underset{\substack{\uparrow \\ \text{since } p \text{ is a chain map}}}{=} c + \partial p(b') = c + \partial c' = \widetilde{c}.$$

Since $\partial(b + \partial b') = \partial b$ we see that we have ended up with the same element in B_{n-1} as before. ⊞

Claim 2. The map $\partial_n \colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ is a homomorphism.

Proof. Let $[c_1], [c_2] \in \mathrm{H}_n(C)$. For $j = 1, 2$ we choose $b_j \in B_n$ as in the definition of the connecting homomorphism $\partial_n \colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ and we denote by $a_j \in A_{n-1}$ the uniquely determined element with $i(a_j) = \partial b_j$. Since i and p are homomorphisms we

also have $p(b_1+b_2)=c_1+c_2$ and $i(a_1+a_2)=i(a_1)+i(a_2)=\partial b_1+\partial b_2=\partial(b_1+b_2)$. It follows that

$$\partial_n([c_1+c_2]) \;=\; [a_1+a_2] \;=\; [a_1]+[a_2] \;=\; \partial_n([c_1])+\partial_n([c_2]).$$

⊞

(2) It follows easily from the definitions that the connecting homomorphisms are natural. We leave the verification of the details to the reader. ■

Now we can formulate the following proposition which explains the name "connecting homomorphism".

Proposition 37.4. (LES-Proposition)

(1) Let

$$0 \;\to\; A_* \;\xrightarrow{i}\; B_* \;\xrightarrow{p}\; C_* \;\to\; 0$$

be a short exact sequence of chain complexes. The sequence

$$\ldots \to \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B) \xrightarrow{p_*} \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \xrightarrow{i_*} \mathrm{H}_{n-1}(B) \xrightarrow{p_*} \ldots$$

is exact. This sequence is called the long exact sequence in homology corresponding to the short exact sequence of chain complexes.

(2) If

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & A'_* & \xrightarrow{i} & B'_* & \xrightarrow{p} & C'_* & \longrightarrow & 0 \end{array}$$

is a commutative diagram of chain maps where the horizontal sequences are exact, then the following diagram commutes:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \mathrm{H}_n(A) & \xrightarrow{i_*} & \mathrm{H}_n(B) & \xrightarrow{p_*} & \mathrm{H}_n(C) & \xrightarrow{\partial} & \mathrm{H}_{n-1}(A) & \longrightarrow & \ldots \\ & & \downarrow a_* & & \downarrow b_* & & \downarrow c_* & & \downarrow a_* & & \\ \ldots & \longrightarrow & \mathrm{H}_n(A') & \xrightarrow{i_*} & \mathrm{H}_n(B') & \xrightarrow{p_*} & \mathrm{H}_n(C') & \xrightarrow{\partial} & \mathrm{H}_{n-1}(A') & \longrightarrow & \ldots \end{array}$$

Proof of the LES-Proposition 37.4. First note that Statement (2) is an immediate consequence of the Chain Complex-H_*-Functor Lemma 35.1 (2) together with the Connecting Homomorphism Lemma 37.3. Thus let us turn to the actual meat, namely the proof of Statement (1).

Let

$$0 \;\to\; A_* \;\xrightarrow{i}\; B_* \;\xrightarrow{p}\; C_* \;\to\; 0$$

be a short exact sequence of chain complexes. The sequence

$$\ldots \to \mathrm{H}_{n+1}(C) \xrightarrow{\partial} \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B) \xrightarrow{p_*} \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \xrightarrow{i_*} \mathrm{H}_{n-1}(B) \xrightarrow{p_*} \ldots$$

is exact. We examine the exactness at the three groups $\mathrm{H}_n(A), \mathrm{H}_n(B)$ and $\mathrm{H}_n(C)$. Put differently, at each of the three groups we want to show the following two statements:

(i) The image of the map to the left is contained in the kernel of the map to the right, or equivalently, the composition of the map to the left and the map to the right is zero,

(ii) The kernel of the map to the right is contained in the image of the map to the left.

Now we perform these steps:

(A) (i) Let $[c] \in \mathrm{H}_{n+1}(C)$. We want to show that $i_*(\partial[c]) = 0$. We choose $b \in B_{n+1}$ with $p(b) = c$ and we denote by $a \in A_n$ the element with $i(a) = \partial b$. Then we have

$$i(\partial[c]) \underset{\substack{\uparrow \\ \text{definition of } \partial\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)}}{=} i([a]) = [\partial b] = 0 \in \mathrm{H}_n(B).$$

(ii) Let $[a] \in \ker(i_*)$. Then $i(a) = \partial b$ for some $b \in B_{n+1}$. It follows immediately from the definitions that $\partial([p(b)]) = [a]$.

(B) (i) From $p \circ i = 0$ it follows that $p_* \circ i_* = 0$, i.e. $\mathrm{im}(i_*) \subset \ker(p_*)$.

(ii) Let $b \in \ker(p_*)$. Then $p(b) = \partial c'$ for some $c' \in C_{n+1}$. Since p is surjective, there exists a $b' \in B_{n+1}$ with $p(b') = c'$. Since p is a chain map we see that $p(\partial b') = \partial p(b') = \partial c' = p(b)$. Thus it follows that $p(b - \partial b') = 0$, i.e. $b - \partial b' = i(a)$ for some $a \in A_n$. But then it follows that $[b] = [b - \partial b'] = [i(a)] \in \mathrm{im}(i_*)$.

(C) (i) Let $[b] \in \mathrm{H}_n(B)$. We have to show that $\partial(p_*([b])) = 0$. In the definition of $\partial p_*([b])$ we need to choose a $\widetilde{b} \in B_n$ with $[p(\widetilde{b})] = p_*([b])$. Of course we take $\widetilde{b} = b$. Since $\partial b = 0$ it follows immediately that $\partial p_*([b]) = 0$.

(ii) Let $[c] \in \ker(\partial\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A))$. We choose $b \in B_n$ with $p(b) = c$. Recall that there exists a uniquely determined $a \in A_{n-1}$ with $i(a) = \partial b$. Since we have $[a] = \partial[c] = 0 \in \mathrm{H}_{n-1}(A)$ there exists an $a' \in A_n$ with $\partial a' = a$. It follows that

$$\partial(b - i(a')) = \partial b - \partial i(a') \underset{\substack{\uparrow \\ \text{since } i \text{ is a chain map}}}{=} \partial b - i(\partial a') = \partial b - i(a) = 0,$$

i.e. $b - i(a') \in B_n$ is a cycle. On the other hand note that we have the equalities $p(b - i(a')) = p(b) - (p \circ i)(a') = p(b)$. Thus we see that $p_*([b - i(a')]) = [c]$. We have thus shown that $\ker(\partial) \subset \mathrm{im}(p_*)$.

Thus we have proved the claim and therefore we have also completed the proof of the proposition. ∎

We conclude this chapter with the following lemma, which will often be useful when we compare two long exact sequences of homology groups.

Lemma 37.5. (Five Lemma) Let

$$\begin{array}{ccccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\
\downarrow{\scriptstyle a} & & \cong\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \cong\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E'
\end{array}$$

be a diagram of groups (here for once we do not suppose that the groups are abelian) such that the horizontal sequences are exact and such that every squares commute (if all the groups are abelian, then it suffices that the squares commute up to a sign). If a is surjective, if b and d are isomorphisms, and if e is injective, then c is an isomorphism.[288]

Proof. One can prove the lemma via a "diagram chase". We leave the pleasure of the chase as an exercise to the reader. ∎

[288] In many applications it suffices to remember the following slightly weaker statement: if the two vertical maps on the left and right are isomorphisms, then the middle map is also an isomorphism.

38. Relative homology

Let X be a topological space and let $A \subset X$ be a subspace. We denote by $i\colon A \to X$ the inclusion. Does there exist a relationship between the homology groups of X and A? The first, very optimistic, guess might be that perhaps the projection $X \to X/A$ induces an isomorphism $\mathrm{H}_k(X)/i_*(\mathrm{H}_k(A)) \to \mathrm{H}_k(X/A)$. But this idea breaks down almost immediately, for example if we have $X = A$, then for any $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X)/i_*(\mathrm{H}_k(A)) = 0$ but by the H_0-Proposition 35.10 we have $\mathrm{H}_0(X/A) \cong \mathrm{H}_0(\{*\}) \cong \mathbb{Z}$.

This simple example shows that the covariant functor $X \to \mathrm{H}_k(X)$ does not preserve quotients. But recall that this functor is really the composition of the two covariant functors $X \mapsto (\mathrm{C}_*(X), \partial_*)$ and $(C_*, \partial_*) \mapsto \mathrm{H}_k((C_*, \partial_*))$. In this and the following chapters we will study the question to which degree the covariant functor $X \mapsto (\mathrm{C}_*(X), \partial_*)$ preserves quotients (at least say up to chain homotopy).

38.1. **Definition of the relative homology groups.** As we just discussed it is worth exploring whether the chain complex functor $X \mapsto (\mathrm{C}_*(X), \partial_*)$ preserves taking quotients. This question motivates the following definition, which will play a major role for the remainder of these notes.

Definition. Let (X, A) be a pair of topological spaces.[289] We view each chain group $\mathrm{C}_n(A)$ as a subgroup of $\mathrm{C}_n(X)$. Thus we can define

$$\mathrm{C}_n(X, A) \;:=\; \mathrm{C}_n(X)/\mathrm{C}_n(A)$$

and we consider the boundary map[290]

$$\begin{array}{rccc} \partial_n\colon & \mathrm{C}_n(X, A) & \to & \mathrm{C}_{n-1}(X, A) \\ & [c] & \mapsto & [\partial_n c]. \end{array}$$

We refer to

$$\ldots \xrightarrow{\partial_{n+2}} \mathrm{C}_{n+1}(X, A) \xrightarrow{\partial_{n+1}} \mathrm{C}_n(X, A) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(X, A) \xrightarrow{\partial_{n-1}} \ldots$$

as the chain complex of the pair (X, A).[291] Given any $n \in \mathbb{N}_0$ we define the n-th relative homology group of (X, A) as follows:

$$\mathrm{H}_n(X, A) \;:=\; \mathrm{H}_n(\mathrm{C}_*(X, A)).$$

Examples.

(1) If $A = \varnothing$, then for all $n \in \mathbb{N}_0$ we have $\mathrm{C}_n(\varnothing) = 0$ and thus we have a natural isomorphism $\mathrm{C}_n(X, \varnothing) = \mathrm{C}_n(X)/\{0\} \xrightarrow{\cong} \mathrm{C}_n(X)$, i.e. we have a natural isomorphism $\mathrm{H}_n(X, \varnothing) \xrightarrow{\cong} \mathrm{H}_n(X)$. In the following we will use these natural isomorphisms to freely go back and forth between the notations $\mathrm{H}_n(X, \varnothing)$ and $\mathrm{H}_n(X)$.

(2) Whereas it is often difficult to write down singular cycles in absolute chain complexes $\mathrm{C}_*(X)$, it can be surprisingly easy to write down cycles in a relative chain complex

[289] This means that X is a topological space and $A \subset X$ is a subspace.

[290] The map ∂_n is well-defined, since for any $c \in \mathrm{C}_n(A) \subset \mathrm{C}_n(X)$ the boundary ∂c obviously lies in $\mathrm{C}_{n-1}(A) \subset \mathrm{C}_{n-1}(X)$.

[291] Since $\partial_{n-1} \circ \partial_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n-2}(X)$ is the zero map it follows immediately that the composition $\partial_{n-1} \circ \partial_n\colon \mathrm{C}_n(X, A) \to \mathrm{C}_{n-2}(X, A)$ is also the zero map.

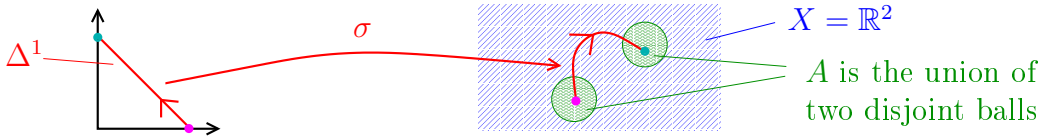
$\mathrm{C}_*(X,A)$: Let $\sigma\colon \Delta^n \to X$ be a continuous map with $\sigma(\partial\Delta^n) \subset A$. Note that

$$\partial_n(\sigma) \;=\; \sum_{j=0}^{n}(-1)^j \cdot \sigma \circ i_j^n \;\underset{\uparrow}{=}\; 0 \;\in\; \mathrm{C}_{n-1}(X,A) = \mathrm{C}_{n-1}(X)/\mathrm{C}_{n-1}(A).$$

since $\sigma \circ i_j^n\colon \Delta^{n-1}$ takes values in A we see that $\sigma \circ i_j^n \in \mathrm{C}_{n-1}(A)$

Thus we have shown that *any continuous map* $\sigma\colon \Delta^n \to X$ *with* $\sigma(\partial\Delta^n) \subset A$ *defines a cycle in the chain complex* $\mathrm{C}_n(X,A)$.

As an example we consider $X = \mathbb{R}^2$ with the subspace $A = B_1 \sqcup B_2$ that is given by the union of two disjoint closed disks in $\mathbb{R}^2$. Let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex with $\sigma(0,1) \in B_1$ and $\sigma(1,0) \in B_2$. It follows from the above discussion that σ is a cycle in $\mathrm{C}_*(X,A)$. Later, on page 455, we will see whether or not $[\sigma] \in \mathrm{H}_1(X,A)$ is non-trivial.

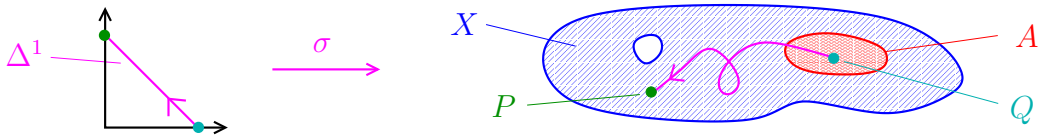


(3) If X is a path-connected topological space and if A is non-empty, then $\mathrm{H}_0(X,A) = 0$. This can be seen as follows: First note that $\mathrm{H}_0(X,A)$ is a quotient of $\mathrm{C}_0(X)$, which is generated by singular 0-simplices, i.e. by points. Thus it suffices to show that for each $P \in X$ we have $[P] = 0 \in \mathrm{H}_0(X,A)$. Thus let $P \in X$ be a point. Since A is non-empty we can pick a point $Q \in A$. Since X is path-connected we can pick a continuous map $\sigma\colon \Delta^1 \to X$ such that $\sigma(0,1) = P$ and $\sigma(1,0) = Q$. We see that

$$\partial_1([\sigma]) \;=\; [\partial_1(\sigma)] \;\underset{\uparrow}{=}\; [P-Q] \;=\; [P]-[Q] \;\underset{\uparrow}{=}\; [P].$$

see page 423 since $Q \in A$ we have $[Q] = 0 \in \mathrm{C}_0(X,A)$

We have thus shown that $\partial_1([\sigma]) = [P]$, which implies that $[P] = 0 \in \mathrm{H}_0(X,A)$.



As always we expect that relative homology groups are functorial in a suitable way. This thought leads us to the following definition.

Definition.

(1) We refer to the category that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{PairTop}) &:= \text{all pairs of topological spaces,} \\ \mathrm{Mor}((X,A),(Y,B)) &:= \text{all continuous maps } f\colon X \to A \text{ with } f(A) \subset B \end{aligned}$$

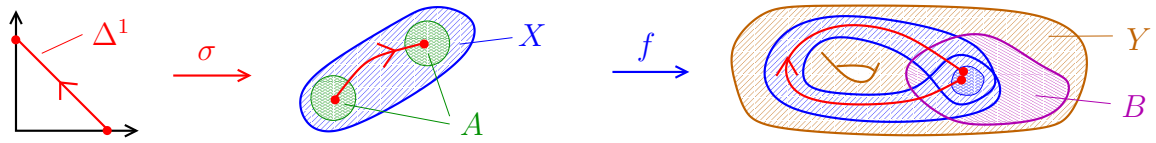
with the usual composition of maps as the category of pairs of topological spaces.

(2) Let $f\colon (X,A) \to (Y,B)$ be a continuous map of pairs of topological spaces, i.e. a continuous map $f\colon X \to Y$ with $f(A) \subset B$. If follows from $f(A) \subset B$ that the induced chain maps $f_*\colon \mathrm{C}_n(X) \to \mathrm{C}_n(Y)$ from the H_n–Functor Lemma 35.7 descends to a chain map

$$f_*\colon \mathrm{C}_n(X,A) \;\to\; \mathrm{C}_n(Y,B).$$

In particular we also get induced homomorphisms

$$f_*\colon \mathrm{H}_n(X,A) \;\to\; \mathrm{H}_n(Y,B).$$



The following lemma is a variation on the C_*–Functor Lemma 35.5 and the H_n–Functor Lemma 35.7. Often we will use the lemma without actually referring to it.

Lemma 38.1. (Relative-H_*–Functor Lemma) The maps

$$\begin{array}{rcl}(X,A) & \to & (\mathrm{C}_*(X,A),\partial_*)\\ (f\colon (X,A)\to(Y,B)) & \mapsto & (f_*\colon \mathrm{C}_*(X,A)\to \mathrm{C}_*(Y,B))\end{array}$$

define a covariant functor from the category *PairTop* of pairs of topological spaces to the category *ChCplx* of chain complexes and for each $n\in\mathbb{N}_0$ the maps

$$\begin{array}{rcl}(X,A) & \to & \mathrm{H}_n(X,A)\\ (f\colon (X,A)\to(Y,B)) & \mapsto & (f_*\colon \mathrm{H}_n(X,A)\to \mathrm{H}_n(Y,B))\end{array}$$

define a covariant functor from the category *PairTop* of pairs of topological spaces to the category *AbGroup* of abelian groups.

Proof. The lemma follows almost immediately from the definitions. ■

38.2. **The long exact sequence of a triple.** Let (X,A) be a pair of topological spaces. In the following we want to consider the relationship between the homology groups of A, X and (X,A). Cleverly we prepared the ground in Chapter 37. Using the tools from the previous chapter we can easily show the existence of interesting long exact sequences of homology groups.

In fact it is convenient to study a slightly more general setup, which leads us to the harmless notion of a triple of topological spaces:

Definition.

(1) A triple of topological spaces is a triple (X,B,A) where X is a topological space and $A\subset B\subset X$ are subspaces.

(2) A map $f\colon (X,B,A)\to(X',B',A')$ between triples of topological spaces is a continuous map $f\colon X\to X'$ such that $f(B)\subset B'$ and $f(A)\subset A'$.

Proposition 38.2. (LES-of-Triple Proposition) Let X be a topological space and let $A\subset B\subset X$ be two subspaces. We denote by $i\colon (B,A)\to(X,A)$ and by $p\colon (X,A)\to(X,B)$ the obvious maps.

(1) For each $n\in\mathbb{N}_0$ the following map is a well-defined homomorphism:[292]

$$\begin{array}{rcl}\partial_n\colon \mathrm{H}_n(X,B) & \to & \mathrm{H}_{n-1}(B,A)\\ \Big[\sum_{j=1}^{k} s_j\cdot(\sigma_j\colon\Delta^n\to X)\Big] & \mapsto & \Big[\sum_{j=1}^{k} s_j\cdot \underbrace{\partial\sigma_j}_{\text{singular } (n-1)\text{-chain in } B}\Big].\end{array}$$

We refer to ∂_n as a connecting homomorphism of the triple (X, B, A).

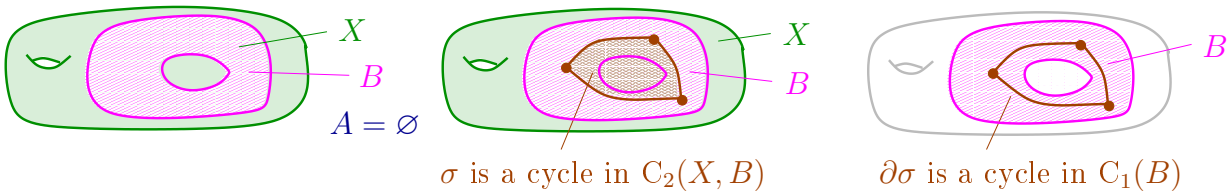
(2) The connecting homomorphisms are natural. In particular for any continuous map $f\colon (X, A, B) \to (\widetilde{X}, \widetilde{A}, \widetilde{B})$ of triples of topological spaces the following diagram commutes:[293]

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \mathrm{H}_n(B, A) & \xrightarrow{i_*} & \mathrm{H}_n(X, A) & \xrightarrow{p_*} & \mathrm{H}_n(X, B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B, A) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\
\ldots & \longrightarrow & \mathrm{H}_n(\widetilde{B}, \widetilde{A}) & \xrightarrow{i_*} & \mathrm{H}_n(\widetilde{X}, \widetilde{A}) & \xrightarrow{p_*} & \mathrm{H}_n(\widetilde{X}, \widetilde{B}) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(\widetilde{B}, \widetilde{A}) & \longrightarrow & \ldots.
\end{array}$$

(3) The maps

$$\ldots \to \mathrm{H}_n(B, A) \xrightarrow{i_*} \mathrm{H}_n(X, A) \xrightarrow{p_*} \mathrm{H}_n(X, B) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(B, A) \to \ldots$$

form an exact sequence. It is called the called the long exact sequence (LES) of the triple (X, B, A).



Proof.

(1) First note that it follows immediately from the definitions and the third isomorphism theorem in group theory[294] that

$$0 \to \mathrm{C}_*(B, A) \xrightarrow{i_*} \mathrm{C}_*(X, A) \xrightarrow{p_*} \mathrm{C}_*(X, B) \to 0$$

is a short exact sequence of chain complexes.

Claim. The maps ∂_n introduced in Statement (1) of the propositions are precisely the connecting homomorphisms, as defined on page 445, of the above short exact sequence of chain complexes. In particular the maps ∂_n are well-defined homomorphisms.

Proof. The claim follows basically immediately from the definitions. But for completeness' sake we give some details. We consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{C}_n(B, A) & \longrightarrow & \mathrm{C}_n(X, A) & \longrightarrow & \mathrm{C}_n(X, B) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
0 & \longrightarrow & \mathrm{C}_{n-1}(B, A) & \longrightarrow & \mathrm{C}_{n-1}(X, A) & \longrightarrow & \mathrm{C}_{n-1}(X, B) & \longrightarrow & 0.
\end{array}$$

[291] Here we use that every homology class in $\mathrm{H}_n(X, B)$ admits by definition a representative in the quotient $\mathrm{C}_n(X, B) = \mathrm{C}_n(X)/\mathrm{C}_n(B)$, so it can be represented by a singular n-chain in X.

[292] We consider the category of triples of topological spaces. The connecting homomorphisms ∂_n define a natural transformation from the functor $(X, B, A) \to \mathrm{H}_n(X, B)$ to the functor $(X, B, A) \to \mathrm{H}_{n-1}(B, A)$.

[293] The fact that ∂_n is a natural transformation means precisely that the third square commutes.

[294] The third isomorphism theorem in group theory implies that we have a natural isomorphism $(\mathrm{C}_*(X)/\mathrm{C}_*(A))/(\mathrm{C}_*(B)/\mathrm{C}_*(A)) \cong \mathrm{C}_*(X)/\mathrm{C}_*(B)$.

Let $z \in \mathrm{H}_n(X, A)$. Now we will determine the image of z under the connecting homomorphism $\mathrm{H}_n(X, B) \to \mathrm{H}_{n-1}(B, A)$ which during the proof of the claim we denote by ∂^c.

We pick a representative $\sum_{j=1}^{k} s_j \cdot (\sigma_j \colon \Delta^n \to X) \in \mathrm{C}_n(X)$ of $z \in \mathrm{H}_n(X, B)$. Note that $\sum_{j=1}^{k} s_j \cdot (\sigma_j \colon \Delta^n \to X)$ also defines an element in $\mathrm{C}_n(X, A)$. We take the boundary of the above singular n-chain under the boundary map $\partial_n \colon \mathrm{C}_n(X, A) \to \mathrm{C}_{n-1}(X, A)$ and we end up with the singular n-chain $\sum_{j=1}^{k} s_j \cdot \partial\sigma_j \in \mathrm{C}_{n-1}(X, A)$. As we have seen in the proof of the Connecting Homomorphism Lemma 37.3 this singular n-chain lies in fact in $\mathrm{C}_{n-1}(B, A) \subset \mathrm{C}_{n-1}(X, A)$. The homology class thereof is by definition precisely $\partial^c(z) \in \mathrm{H}_{n-1}(B, A)$. Summarizing we have shown that

$$\partial^c(z) \;=\; \Big[\sum_{j=1}^{k} s_j \cdot \partial\sigma_j\Big].$$

This shows that the map defined in (1) agrees with ∂^c. ⊞

(2) It follows from the above claim together with the Connecting Homomorphism Lemma 37.3 that the connecting homomorphisms are natural.

(3) It follows from the above claim together with the LES-Proposition 37.4 that the given sequence is exact. ■

In a second we will state an important special case of the LES-of-Triple Proposition 38.2. This will require the following notation.

Notation. Let X be a topological space, let $A \subset X$ be a subspace and let $n \in \mathbb{N}_0$. We introduce the following homomorphisms[295]

$$\mathrm{H}_n(X) \xrightarrow{=} \mathrm{H}_n(X, \varnothing) \longrightarrow \mathrm{H}_n(X, A), \qquad =:\text{я}=:\text{Я}_{X,A}$$

Now we can formulate the promised proposition.

Proposition 38.3. (LES-of-Pair Proposition)

(1) Let X be a topological space and let $B \subset X$ be a subspace. The connecting homomorphism of the LES-of-Triple Proposition 38.2 applied to $A = \varnothing$ gives rise to an exact sequence

$$\begin{array}{ccccccccccc} \ldots & \to & \mathrm{H}_n(B) & \xrightarrow{i_*} & \mathrm{H}_n(X) & \xrightarrow{\text{я}} & \mathrm{H}_n(X, B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B) & \to & \ldots \\ & & & & & & \Big[\sum_{j=1}^{k} s_j \cdot \sigma_j\Big] & \mapsto & \Big[\sum_{j=1}^{k} s_j \cdot \partial\sigma_j\Big] & & \end{array}.$$

This exact sequence is called the long exact sequence of the pair (X, B).

(2) The connecting homomorphisms in (1) are natural. In particular, given a continuous map $f \colon (X, A) \to (Y, B)$ between pairs of topological spaces we obtain a commutative

[295] We apologize to whoever wants to teach out of these notes for using Cyrillic symbols.

diagram

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & \mathrm{H}_n(A) & \xrightarrow{i_*} & \mathrm{H}_n(X) & \xrightarrow{\text{я}} & \mathrm{H}_n(X,A) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\
\dots & \longrightarrow & \mathrm{H}_n(B) & \xrightarrow{i_*} & \mathrm{H}_n(Y) & \xrightarrow{\text{я}} & \mathrm{H}_n(Y,B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B) & \longrightarrow & \dots
\end{array}$$

Proof. This proposition follows immediately from the LES-of-Triple Proposition 38.2 applied to $A = \varnothing$. ∎

Example. As on page 451 we consider topological space let $X = \mathbb{R}^2$ and the subspace $A = B_1 \sqcup B_2$ that is given by the disjoint union of two closed disks in $\mathbb{R}^2$. We want to determine the relative homology groups of (X, A). We denote by $i\colon A \to X$ the inclusion map. By the LES-of-Pair Proposition 38.3 we have a long exact sequence

$$\mathrm{H}_2(X) \to \mathrm{H}_2(X,A) \to \mathrm{H}_1(A) \xrightarrow{i_*} \mathrm{H}_1(X) \to \mathrm{H}_1(X,A) \to \mathrm{H}_0(A) \to \mathrm{H}_0(X) \to \mathrm{H}_0(X,A) \to 0.$$

By the Contractible-H_*-Corollary 36.8 together with the Path-Component–H_*–Lemma 35.8 we have $\mathrm{H}_i(X) = \mathrm{H}_i(\mathbb{R}^2) = 0$ and also $\mathrm{H}_i(A) = \mathrm{H}_i(B_1) \oplus \mathrm{H}_i(B_2) = 0$ for $i \geq 1$. Thus we see from the above long exact sequence that $\mathrm{H}_i(X,A) = 0$ for $i \geq 2$.

Next let $\epsilon\colon \mathrm{H}_0(\mathbb{R}^2) \to \mathbb{Z}$ be the augmentation map. Since $\mathbb{R}^2$ is path-connected we know by the H_0-Proposition 35.10 (1) that this map is an isomorphism. Similarly we denote by $\mu_1\colon \mathrm{H}_0(B_1) \to \mathbb{Z}$ and $\mu_2\colon \mathrm{H}_0(B_2) \to \mathbb{Z}$ the isomorphisms that are given by the augmentation maps. We obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}_1(X,A) & \xrightarrow{\partial} & \mathrm{H}_0(A) & \xrightarrow{i_*} & \mathrm{H}_0(X) & \longrightarrow & \mathrm{H}_0(X,A) & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mu_1\oplus\mu_2}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathrm{H}_1(X,A) & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{H}_0(X,A) & \longrightarrow & 0.
\end{array}$$

It follows easily from the Augmentation Lemma 35.9 that the homomorphism $\mathbb{Z}^2 \to \mathbb{Z}$ is given by the matrix $(1\ 1)$. It follows that

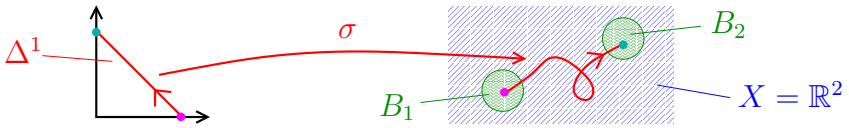
$$\mathrm{H}_1(X,A) \underset{\text{by the Exactness Lemma 37.1 (3)}}{\overset{\uparrow}{\cong}} \ker\left(\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}\right) \cong \mathbb{Z} \qquad \text{and} \qquad \mathrm{H}_0(X,A) \underset{\text{by the Exactness Lemma 37.1 (3)}}{\overset{\uparrow}{\cong}} \operatorname{coker}\left(\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}\right) \cong 0.$$

Finally we want to find an explicit generator of $\mathrm{H}_1(X,A) \cong \mathbb{Z}$. As on page 451 let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex with $\sigma(0,1) \in B_2$ and $\sigma(1,0) \in B_1$. Note that

$$((\mu_1 \oplus \mu_2) \circ \partial_1)[\sigma] \underset{\substack{\text{see the description of the connecting homomorphism } \partial_1\colon \mathrm{H}_1(X,A) \to \mathrm{H}_0(A) \text{ in the LES-of-Triple Proposition 38.2}}}{\overset{\uparrow}{=}} ((\mu_1 \oplus \mu_2)([\partial_1 \sigma]) = (\mu_1 \oplus \mu_2)([\underbrace{\sigma(0,1)}_{\in B_2}] - [\underbrace{\sigma(1,0)}_{\in B_1}]) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a generator of the kernel of $\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}$ we see that $[\sigma]$ is in fact a generator of $\mathrm{H}_1(X,A) \cong \mathbb{Z}$. In particular it is non-zero.

We conclude this section with two more theoretical applications of the LES-of-Pair Proposition 38.3.



Corollary 38.4. (Homotopy Equivalence-Relative-H_*-Corollary) In the following let $f\colon (X,A) \to (Y,B)$ be a continuous map of pairs of topological spaces. If $f\colon X \to Y$ is a homotopy equivalence and if $f\colon A \to B$ is a homotopy equivalence[296], then for every $n \in \mathbb{N}_0$ the induced map $f_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B)$ is an isomorphism.

Example. It follows from the Homotopy Equivalence-Relative-H_*-Corollary 38.4 that for each $k \in \mathbb{N}_0$ the inclusion induced homomorphisms $\mathrm{H}_n(\overline{B}^k, S^{k-1}) \to \mathrm{H}_n(\overline{B}^k, \overline{B}^k \setminus \{0\})$ are isomorphisms.

Proof. We consider the following diagram

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & \mathrm{H}_n(A) & \xrightarrow{i_*} & \mathrm{H}_n(X) & \xrightarrow{\text{я}} & \mathrm{H}_n(X,A) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A) & \xrightarrow{i_*} & \mathrm{H}_{n-1}(X) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\
\ldots & \longrightarrow & \mathrm{H}_n(B) & \xrightarrow{i_*} & \mathrm{H}_n(Y) & \xrightarrow{\text{я}} & \mathrm{H}_n(Y,B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B) & \xrightarrow{i_*} & \mathrm{H}_{n-1}(Y) & \longrightarrow & \ldots
\end{array}$$

By the LES-of-Pair Proposition 38.3 (2) we know that the diagram commutes. Furthermore note that it follows from our hypothesis on f and the Homotopy Equivalence-H_*-Corollary 36.7 that the two vertical maps to the left and the two vertical maps to the right are isomorphisms. It follows from the Five Lemma 37.5 that the middle vertical map is, as promised, an isomorphism. ■

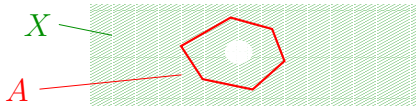
Corollary 38.5. (Deformation Retract-Relative H_*-Corollary) Let X be a topological space. If $A \subset X$ is a deformation retract of X, then $\mathrm{H}_n(X,A) = 0$ for all $n \in \mathbb{N}_0$.

Proof. Let $i\colon A \to X$ be the inclusion map. By the LES-of-Pair Proposition 38.3 we have a long exact sequence

$$\ldots \to \mathrm{H}_n(A) \xrightarrow[\uparrow]{i_*} \mathrm{H}_n(X) \to \mathrm{H}_n(X,A) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A) \xrightarrow[\uparrow]{i_*} \mathrm{H}_{n-1}(X) \to \ldots$$

it follows from the Deformation Retract–Homotopy Equivalence Lemma 9.3 that $i\colon A \to X$ is a homotopy equivalence, thus it follows from the Homotopy Equivalence-H_*-Corollary 36.7 that i_* is an isomorphism

Since the maps $i_*\colon \mathrm{H}_n(A) \to \mathrm{H}_n(X)$ are isomorphisms and since the sequence is exact we see that the relative homology groups $\mathrm{H}_n(X,A)$ are all zero. ■

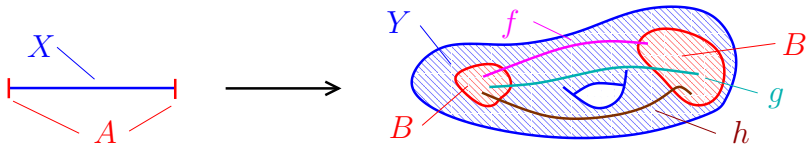


A is a deformation retract of X therefore each $i_*\colon \mathrm{H}_n(A) \to \mathrm{H}_k(X)$ is an isomorphism

[296] In many applications $f\colon (X,A) \to (Y,B)$ will be an inclusion, where X is a deformation retract of Y and A is a deformation retract of B. It then follows from the Deformation Retract–Homotopy Equivalence Lemma 9.3 that $f\colon X \to Y$ and $f\colon A \to B$ are homotopy equivalences.

38.3. Relative homology groups and homotopies.

Definition. We say that two continuous maps $f, g\colon (X, A) \to (Y, B)$ between pairs of topological spaces are homotopic, if there exists a continuous map $F\colon X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$, and such that $F(a,t) \in B$ for all $a \in A$ and all $t \in [0,1]$.



$f, g\colon (X, A) \to (Y, B)$ are homotopic but $f, h\colon (X, A) \to (Y, B)$ are not homotopic

The following proposition extends the Homotopy-H_*-Proposition 36.5 to the relative case.

Proposition 38.6. (Homotopy-H_*-Proposition) Let $f, g\colon (X, A) \to (Y, B)$ be two continuous maps between pairs of topological spaces. If f and g are homotopic, then the chain maps $f_*, g_*\colon \mathrm{C}_*(X, A) \to \mathrm{C}_*(Y, B)$ are chain homotopic, in particular for every $n \in \mathbb{N}_0$ we have

$$f_* \;=\; g_*\colon \mathrm{H}_n(X, A) \;\to\; \mathrm{H}_n(Y, B).$$

Proof. Let $f, g\colon (X, A) \to (Y, B)$ be two continuous maps between pairs of topological spaces and let $F\colon X \times [0,1] \to Y$ be a homotopy between f and g as maps of pairs of topological spaces.[297] Let $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0,1])$ be the singular chain that we defined on page 439. On page 441 we showed that the homomorphisms

$$\begin{array}{rcl} P_n\colon \mathrm{C}_n(X) & \to & \mathrm{C}_{n+1}(Y) \\ \sum_{i=1}^{k} s_i \cdot (\sigma\colon \Delta^n \to X) & \mapsto & \sum_{i=1}^{k} s_i \cdot (F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n) \end{array}$$

define a chain homotopy between the maps $f_*, g_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$, i.e. for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; f_* - g_*\colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_n(Y).$$

Since F restricts to a map $A \times [0,1] \to B$ we see that these homomorphisms P_n have the property that $\mathrm{C}_n(A)$ gets sent to $\mathrm{C}_{n+1}(B)$. Thus the maps $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(Y)$ descend to homomorphisms $\overline{P}_n\colon \mathrm{C}_n(X, A) \to \mathrm{C}_{n+1}(Y, B)$. These satisfy

$$\partial_{n+1} \circ \overline{P}_n + \overline{P}_{n-1} \circ \partial_n \;=\; f_* - g_*\colon \mathrm{C}_n(X, A) \;\to\; \mathrm{C}_n(Y, B),$$

i.e. the maps $\overline{P}_n$ form a chain homotopy between the maps f_* and g_* from $\mathrm{C}_*(X, A)$ to $\mathrm{C}_*(Y, B)$. It follows from this observation together with the Chain Homotopy–H_*-Lemma 36.1 that $f_* = g_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B)$. ∎

[297] In most cases we just want the statement that $f_* = g_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B)$. This statement follows easily from the LES-of-Pair Proposition 38.3 (2), the Homotopy-H_*-Proposition 36.5 and the Five Lemma 37.5.

39. Reduced homology

In this very short chapter we introduce the *reduced* homology groups of topological spaces. These reduced homology groups are almost the same as the by now familiar homology groups. But they will come in handy in the calculations of homology groups in the next chapters, since reduced homology groups are often a little easier to deal with than ordinary homology groups, even though they contain the same information.

39.1. Definition and basic properties of reduced homology.

The Augmentation Lemma 35.9 (1) motivates the following definition of the reduced homology groups of a topological space.

Definition. Let X be a topological space. By the Augmentation Lemma 35.9 (1) we know that the following is a generalized[298] chain complex

$$\begin{array}{ccccccccccc} & & & & & & \text{degree } 0 & & \text{degree } -1 & & \\ & & & & & & \downarrow & & \downarrow & & \\ \ldots & \xrightarrow{\partial_3} & \mathrm{C}_2(X) & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \xrightarrow{\epsilon_X} & \mathbb{Z} & \to & 0. \\ & & & & & & \sum_{i=1}^{k} r_i \cdot \sigma_i & \mapsto & \sum_{i=1}^{k} r_i & & \\ & & & & & & \uparrow & & & & \\ & & & & & & \text{augmentation map from page 428} & & & & \end{array}$$

We introduce the following two definitions.

(1) We refer to this chain complex as the augmented chain complex $\widetilde{\mathrm{C}}_*(X)$ of X.

(2) Given $n \in \mathbb{Z}_{\geq -1}$ we refer to $\widetilde{\mathrm{H}}_n(X) := \mathrm{H}_n(\widetilde{\mathrm{C}}_*(X))$ as the n-th reduced homology group of X.

These reduced homology groups are a minute but at times surprisingly useful variation on the usual homology groups. In the following we will explore the properties of reduced homology groups and we will explore their relationship to the traditional homology groups.

The following lemma is a variation on the by now familiar H_n–Functor Lemma 35.7.

Lemma 39.1. ($\widetilde{\mathbf{H}}_n$–Functor Lemma)

(1) If $f\colon X \to Y$ is a continuous map between two topological spaces, then we denote by $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ the map that is the usual map on $\widetilde{\mathrm{C}}_n(X) = \mathrm{C}_n(X)$ for $n \geq 0$ and that is the identity on the final $\mathbb{Z}$-term. This map is a chain map and thus induces homomorphisms $f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$ on the reduced homology groups.

(2) For any $n \in \mathbb{Z}_{\geq -1}$ the maps

$$\begin{array}{rcl} X & \mapsto & \widetilde{\mathrm{H}}_n(X) \\ (f\colon X \to Y) & \mapsto & (f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)) \end{array}$$

define a covariant functor from the category of topological spaces to the category of abelian groups.

(3) For any $n \in \mathbb{Z}_{\geq -1}$ the maps

$$\begin{array}{rcl} \text{ж}\colon \widetilde{\mathrm{H}}_n(X) & \to & \mathrm{H}_n(X) \\ [\sigma] & \mapsto & [\sigma] \end{array}$$

[298] Recall that a generalized chain complex is like a chain complex, but we also allow non-zero groups in negative degrees.

are well-defined homomorphisms and they define a natural transformation from the functor $X \mapsto \widetilde{\mathrm{H}}_n(X)$ to the functor $X \mapsto \mathrm{H}_n(X)$.

Proof.

(1) Let $f\colon X \to Y$ be a continuous map between two topological spaces. As in the Augmentation Lemma 35.9 (3) it is a straightforward consequence of the definition of the augmentation maps that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}_0(X) & \xrightarrow{f_*} & \mathrm{C}_0(Y) \\ {\scriptstyle \epsilon_X}\downarrow & & \downarrow{\scriptstyle \epsilon_Y} \\ \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z}. \end{array}$$

It follows from this observation together with the C_*–Functor Lemma 35.5 that the induced maps $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ are chain maps. Furthermore we obtain from the Chain Complex-H_*-Functor Lemma 35.1 (1) that we get induced maps on reduced homology.

(2) This statement is basically obvious.

(3) This statement follows easily from the definitions. ■

The following elementary lemma explains the relationship between ordinary and reduced homology: Basically they are the same, except that for non-empty topological spaces they "differ by $\mathbb{Z}$" in dimension 0.

Lemma 39.2. (Reduced Homology Lemma) For every topological space X the following statements hold:

(0) We have[299]
$$\widetilde{\mathrm{H}}_{-1}(X) \;=\; \begin{cases} 0, & \text{if } X \neq \varnothing, \\ \mathbb{Z}, & \text{if } X = \varnothing. \end{cases}$$

(1) (a) If X is non-empty, then the sequence
$$0 \;\to\; \widetilde{\mathrm{H}}_0(X) \;\xrightarrow{\text{ж}}\; \mathrm{H}_0(X) \;\xrightarrow[\text{Augmentation Lemma 35.9}]{\text{augmentation map } \epsilon_X \text{ from the}}\; \mathbb{Z} \;\to\; 0$$
is exact.

(b) Given any point $P \in X$ the map
$$\begin{array}{ccc} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} & \to & \mathrm{H}_0(X) \\ [\sigma] \oplus n & \mapsto & [\sigma] + [n \cdot P] \end{array}$$
is an isomorphism.

(c) If X has $n \in \mathbb{N}$ path-components, then $\widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}^{n-1}$. In particular X is path-connected if and only if $\widetilde{\mathrm{H}}_0(X) = 0$.

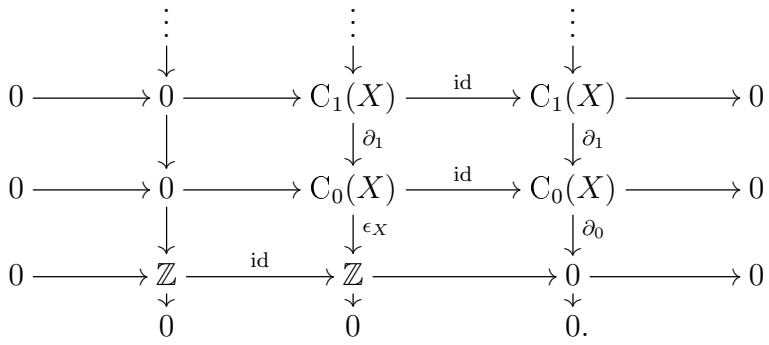
(2) For every $n \geq 1$ the natural map ж$\colon \widetilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X)$ is the identity.

Proof. Let X be a topological space.

(0) First note that clearly we have $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$. Now we assume that X is non-empty. It follows immediately from the definitions that the augmentation map ϵ_X is an epimorphism which entails that $\widetilde{\mathrm{H}}_{-1}(X) = 0$.

[299] This calculation shows that in almost all cases we can ignore $\widetilde{\mathrm{H}}_{-1}(X)$. Nonetheless, some formulas later on become cleaner if we take into account that $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$.

(1) (a) We consider the following short exact sequence of generalized chain complexes:



The desired statement follows easily from the corresponding long exact sequence of homology groups that is given by the LES-Proposition 37.4.

(b) Let $P \in X$ be a point. It is straightforward to show that the maps

$$\begin{array}{rcl} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} & \to & \mathrm{H}_0(X) \\ [\sigma] \oplus n & \mapsto & [\sigma] + [n \cdot P] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{H}_0(X) & \to & \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} \\ [\sigma] & \mapsto & [\sigma - \epsilon_X(\sigma) \cdot P] + \epsilon_X(\sigma) \end{array}$$

are inverses of one another. In particular both maps are isomorphisms.

(c) This statement follows almost immediately from (1a), the H_0-Proposition 35.10 (1) together with the Direct Sum-H_*-Lemma 35.2.

(2) This statement is clear. ■

Many statements for ordinary homology groups carry over to the setting of reduced homology groups. For completeness' sake we spell out the analogues of the Homotopy-H_*-Proposition 36.5 and the Homotopy Equivalence-H_*-Corollary 36.7:

Proposition 39.3. (Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition) Let X and Y be two topological spaces and let $f, g\colon X \to Y$ be two continuous maps. If f and g are homotopic, then the following hold:

(1) The induced maps f_* and g_* from $\widetilde{\mathrm{C}}_*(X)$ to $\widetilde{\mathrm{C}}_*(Y)$ are chain homotopic.

(2) For each $n \in \mathbb{N}_0$ we have $f_* = g_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$.

Sketch of proof.

(1) This statement follows easily from the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 39.3 (2).

(2) This statement follows immediately from (1). ■

Corollary 39.4. (Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary) Let X and Y be topological spaces and let $n \in \mathbb{N}_0$.

(1) (a) If $f\colon X \to Y$ is a homotopy equivalence, then the induced map $f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$ is an isomorphism.

(b) If X is a deformation retract of Y, then the inclusion induced map $\widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$ is an isomorphism.

(2) If X and Y are homotopy equivalent, then $\widetilde{\mathrm{H}}_n(X) \cong \widetilde{\mathrm{H}}_n(Y)$.

Sketch of proof. This corollary can be obtained easily from the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 39.3. ■

The statement of the following corollary differs subtly from the corresponding "unreduced" version.

Corollary 39.5. (Contractible-$\widetilde{\mathrm{H}}_*$-Corollary) If X is contractible (i.e. if X is homotopy equivalent to a singleton), then we have $\widetilde{\mathrm{H}}_n(X) = 0$ for all $n \in \mathbb{Z}_{\geq -1}$.

Example. On page 149 we saw that every convex subset of $\mathbb{R}^n$ is contractible. Thus it follows from the Contractible-H_*-Corollary 36.8, that for any $i \in \mathbb{Z}_{\geq -1}$ we have

$$\widetilde{\mathrm{H}}_i(\underbrace{\text{convex subset of } \mathbb{R}^n}_{\text{e.g. } \mathbb{R}^n,\ \overline{B}^n,\ B^n \text{ or simplex}}) \ = \ 0.$$

Proof. This corollary follows immediately from the Contractible-H_*-Corollary 36.8 together with the Reduced Homology Lemma 39.2. ■

39.2. **The long exact sequence of a pair in reduced homology.** Shortly we will see that a pair of topological spaces gives rise to a long exact sequence that involves reduced homology groups. Before we can formulate this result we need to introduce the following notation:

Notation. Let X be a topological space, let $A \subset X$ and let $n \in \mathbb{N}_0$. We introduce the following map:

$$\widetilde{\mathrm{H}}_n(X) \xrightarrow{\text{ж}} \mathrm{H}_n(X) \xrightarrow{\text{я}} \mathrm{H}_n(X, A), \qquad \widetilde{\mathrm{H}}_n(X) \xrightarrow{=:\text{ж}_{X,A}} \mathrm{H}_n(X, A),$$

where ж: $\widetilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X)$ is the natural homomorphism introduced in the $\widetilde{\mathrm{H}}_n$–Functor Lemma 39.1 and я: $\mathrm{H}_n(X) \to \mathrm{H}_n(X, A)$ is the natural homomorphism that we introduced on page 454.

The following slight variation on the LES-of-Pair Proposition 38.3 will turn out to be surprisingly useful.

Proposition 39.6. (Reduced LES-of-Pair Proposition)

(1) Let X be a topological space and let $A \subset X$ be a subspace. The following sequence is exact[300]

$$\begin{array}{ccccccccccc} \ldots & \to & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{\text{ж}_{X,A}} & \mathrm{H}_n(X, A) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(A) & \to & \ldots \\ & & & & & & \left[\sum_{j=1}^{k} s_j \cdot \sigma_j\right] & \mapsto & \left[\sum_{j=1}^{k} s_j \cdot \partial\sigma_j\right] & & \end{array}$$

This sequence is called the long exact sequence in reduced homology of the pair (X, A).

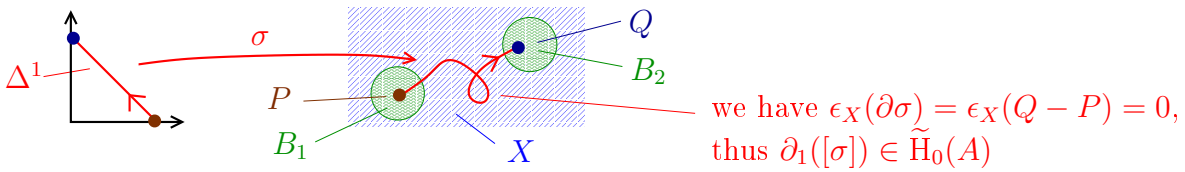
(2) The connecting homomorphisms in (1) are natural. In particular, given a continuous map $f\colon (X, A) \to (Y, B)$ between pairs of topological spaces we obtain a commutative diagram

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{\text{ж}_{X,A}} & \mathrm{H}_n(X, A) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(A) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ \ldots & \longrightarrow & \widetilde{\mathrm{H}}_n(B) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(Y) & \xrightarrow{\text{ж}_{Y,B}} & \mathrm{H}_n(Y, B) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(B) & \longrightarrow & \ldots \end{array}$$

Example. As on pages 451 and 455 we consider the topological space $X = \mathbb{R}^2$ and the subspace $A = B_1 \sqcup B_2$ that is given by the disjoint union of two closed disks in $\mathbb{R}^2$. We want to determine the relative homology groups of (X, A). We denote by $i\colon A \to X$ the inclusion map. By the Reduced LES-of-Pair Proposition 39.6 we have a long exact sequence

$$\widetilde{\mathrm{H}}_2(X) \to \mathrm{H}_2(X, A) \to \widetilde{\mathrm{H}}_1(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_1(X) \to \mathrm{H}_1(X, A) \to \mathrm{H}_0(A) \to \widetilde{\mathrm{H}}_0(X) \to \mathrm{H}_0(X, A) \to 0.$$

By the Contractible-H_*-Corollary 36.8, the Path-Component–H_*–Lemma 35.8 and the Reduced Homology Lemma 39.2 we have $\widetilde{\mathrm{H}}_i(X) = \widetilde{\mathrm{H}}_i(\mathbb{R}^2) = 0$ for all $i \geq 0$, we have $\widetilde{\mathrm{H}}_i(A) = 0$ for $i \geq 1$ and we have $\widetilde{\mathrm{H}}_1(A) \cong \mathbb{Z}$. Thus we see from the above long exact sequence that $\mathrm{H}_i(X, A) = 0$ for $i \neq 1$ and that $\mathrm{H}_1(X, A) \cong \mathbb{Z}$. We get the same result as on page 455, but with much less pain.



Proof.

(1) We consider the following short exact sequence of chain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_1(A) & \longrightarrow & \mathrm{C}_1(X) & \longrightarrow & \mathrm{C}_1(X, A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_1} & & \\
0 & \longrightarrow & \mathrm{C}_0(A) & \longrightarrow & \mathrm{C}_0(X) & \longrightarrow & \mathrm{C}_0(X, A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon_A} & & \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \partial_0} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

From the LES-Proposition 37.4 we obtain a long exact sequence of homology groups, which in this case involve the reduced homology groups of A and X, and the relative homology groups of (X, A). Furthermore, as in the proof of the LES-of-Triple Proposition 38.2 we see that the connecting homomorphism is given by the map

$$\begin{array}{ccc}
\mathrm{H}_n(X, A) & \to & \widetilde{\mathrm{H}}_{n-1}(A) \\
\left[\sum_{j=1}^{k} s_j \cdot \sigma_j\right] & \mapsto & \left[\sum_{j=1}^{k} s_j \cdot \partial\sigma_j\right]
\end{array}$$

Finally note that one can easily easily verify that the maps $\widetilde{\mathrm{H}}_k(X) \to \mathrm{H}_k(X, A)$ are precisely the above maps $\text{ж}_{X,A}$.

[300] Note that if the subspace B is non-empty, then it follows from the Reduced Homology Lemma 39.2 (0) that $\widetilde{\mathrm{H}}_{-1}(B) = 0$. Thus in this case the last possibly non-zero term is $\mathrm{H}_0(X, B)$. On the other hand, if $B = \varnothing$, then we have $\widetilde{\mathrm{H}}_{-1}(B) = \widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$, and if $X = \varnothing$, then we have $\widetilde{\mathrm{H}}_{-1}(X) = \widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$. Therefore, if $B = \varnothing$ the sequence is understood to continue with $\mathrm{H}_0(X, \varnothing) \xrightarrow{\epsilon} \mathrm{H}_{-1}(\varnothing) \to \mathrm{H}_{-1}(X) \to 0$.

(2) This statement follows easily from the naturality of the connecting homomorphism proved in the LES-Proposition 37.4 ■

We conclude this chapter with the following lemma that elucidates the relationship between reduced and relative homology.

Lemma 39.7. (Reduced-to-Relative-H_*-Lemma) Let X be a topological space and let $n \in \mathbb{N}_0$. For each $x_0 \in X$ the map

$$\text{ж}_{X,\{x_0\}} \colon \widetilde{\mathrm{H}}_n(X) \;\to\; \mathrm{H}_n(X, \{x_0\})$$

is a natural isomorphism.

Proof. Let X be a topological space and let $x_0 \in X$. We apply the Reduced LES-of-Pair Proposition 39.6 to the pair $(X, \{x_0\})$ and we obtain a long exact sequence

$$\ldots \to \underbrace{\widetilde{\mathrm{H}}_n(\{x_0\})}_{=0} \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{\text{ж}_{X,\{x_0\}}} \mathrm{H}_n(X, \{x_0\}) \xrightarrow{\partial_n} \underbrace{\widetilde{\mathrm{H}}_{n-1}(\{x_0\})}_{=0} \to \ldots$$

The lemma follows from the exactness of this sequence and the observation that it follows from the Singleton-H_*-Lemma 35.11 and the Reduced Homology Lemma 39.2 that all reduced homology groups of $\{x_0\}$ vanish. ■

40. The Excision Theorem

We introduced singular homology on page 426, many chapters ago. Since then we have developed the theory of singular homology groups and related topics. But the reader will have noticed that up to now we have not really calculated any interesting homology groups.

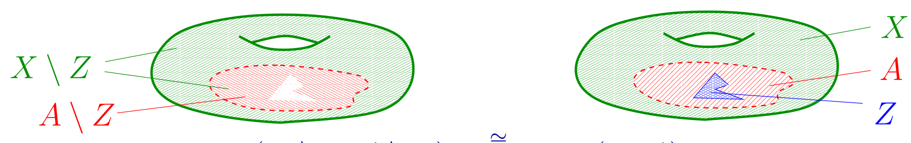
This is about to change. In this chapter we will state and prove the Excision Theorem 40.1, which is easily one of the most essential theorems in algebraic topology. We will use it, and the techniques from the previous chapters, to finally provide the calculation of the homology groups of spheres. In the next chapter we will use the Excision Theorem 40.1 to find, given a suitable pair of topological spaces (X, A), a connection between the groups $\mathrm{H}_n(X, A)$ and $\mathrm{H}_n(X/A)$.

40.1. The Excision Theorem.

Here is the main theorem of this chapter.

Theorem 40.1. (Excision Theorem) Let X be a topological space and let $Z \subset A \subset X$ be subsets. If the closure of Z is contained in the interior of A, then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathrm{H}_n(X, A).$$

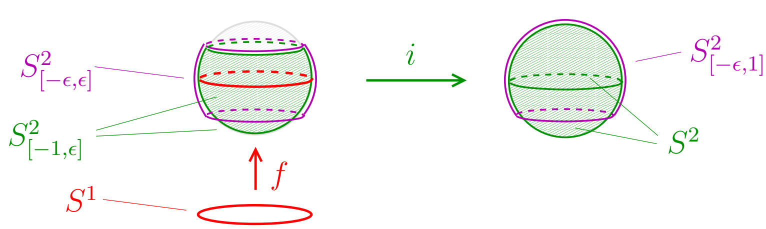


$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathrm{H}_n(X, A)$$

The Excision Theorem 40.1 thus says in particular that if we are given a pair of topological spaces (X, A) and if we excise a closed subset Z from the interior of A, then this does not affect the relative homology groups. In the next chapter we will use the Excision Theorem 40.1 to perform many calculations. As an amuse bouche we will now compute the homology groups of the spheres S^n:

Example. We want to determine the reduced homology groups of S^n. We pick some $\epsilon \in (0, 1)$ and for each $m \in \mathbb{N}_0$ we consider the following homomorphisms:

$$\begin{array}{c}
\longrightarrow \widetilde{\mathrm{H}}_m(S^n_{[-\epsilon,1]}) \longrightarrow \widetilde{\mathrm{H}}_m(S^n) \xrightarrow{\cong} \mathrm{H}_m(S^n, S^n_{[-\epsilon,1]}) \xrightarrow{\partial_m} \widetilde{\mathrm{H}}_{m-1}(S^n_{[-\epsilon,1]}) \longrightarrow \dots \\
i_* \uparrow \cong \\
\dots \to \widetilde{\mathrm{H}}_{m-1}(S^n_{[-\epsilon,\epsilon]}) \to \widetilde{\mathrm{H}}_m(S^n_{[-1,\epsilon]}) \to \mathrm{H}_m(S^n_{[-1,\epsilon]}, S^n_{[-\epsilon,\epsilon]}) \xrightarrow[\cong]{\partial_m} \widetilde{\mathrm{H}}_{m-1}(S^n_{[-\epsilon,\epsilon]}) \to \widetilde{\mathrm{H}}_{m-1}(S^n_{[-1,\epsilon]}) \to \\
f_* \uparrow \cong \\
\widetilde{\mathrm{H}}_{m-1}(S^{n-1})
\end{array}$$



We make the following clarifications and explanations:

(1) For an interval $I \subset [-1,1]$ we write $S^n_I := \{(x_1,\dots,x_{n+1}) \in S^n \,|\, x_{n+1} \in I\}$.
(2) The top horizontal sequence is the long exact sequence in reduced homology of the pair $(S^n, S^n_{[-\epsilon,1]})$ that is given by the Reduced LES-of-Pair Proposition 39.6.
(3) The bottom horizontal sequence is the long exact sequence in reduced homology of the pair $(S^n_{[-1,\epsilon]}, S^n_{[-\epsilon,\epsilon]})$ that is given by the Reduced LES-of-Pair Proposition 39.6.
(4) It follows from the Stereographic Projection Lemma 2.11 that $S^n_{[-\epsilon,1]}$ and $S^n_{[-1,\epsilon]}$ are homeomorphic to the closed n-ball $\overline{B}^n$. It follows from this observation and the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that for all $k \in \mathbb{N}_0$ we have $\widetilde{\mathrm{H}}_k(S^n_{[-\epsilon,1]}) = 0$ and that we have $\widetilde{\mathrm{H}}_k(S^n_{[-1,\epsilon]}) = 0$.
(5) It follows from (4) and the exactness of the horizontal sequences that the horizontal maps shown in red are isomorphisms.
(6) It follows from the Excision Theorem 40.1, applied to $X = S^n$ and the two subspaces $A = S^n_{[-\epsilon,1]}$ and $Z = S^n_{(\epsilon,1]}$, that the vertical map $i_*\colon \mathrm{H}_m(S^n_{[-1,\epsilon]}, S^n_{[-\epsilon,\epsilon]}) \to \mathrm{H}_m(S^n, S^n_{[-\epsilon,1]})$ is an isomorphism.
(7) We denote by $f\colon S^{n-1} \to S^n_{[-\epsilon,\epsilon]}$ the map that is given by $x \mapsto (x,0)$. This map is clearly an embedding. It is not hard to show that $f(S^{n-1})$ is a deformation retract of $S^n_{[-\epsilon,\epsilon]}$. It follows from the Deformation Retract–Homotopy Equivalence Lemma 9.3 that $f\colon S^{n-1} \to S^n_{[-\epsilon,\epsilon]}$ is a homotopy equivalence.
(8) It follows from (7) together with the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 39.4 for reduced homology that the induced map $f_*\colon \widetilde{\mathrm{H}}_{m-1}(S^{n-1}) \to \widetilde{\mathrm{H}}_{m-1}(S^n_{[-\epsilon,\epsilon]})$ is an isomorphism.

We have now shown that all red homomorphisms are isomorphisms. We now see that for each $m \in \mathbb{Z}_{\geq -1}$ we have[301]

$$\begin{array}{ccccccccc} \widetilde{\mathrm{H}}_m(S^n) & \cong & \widetilde{\mathrm{H}}_{m-1}(S^{n-1}) & \cong & \dots & \cong & \widetilde{\mathrm{H}}_{m-n}(S^0) & \cong & \begin{cases} \mathbb{Z}, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases} \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \end{array}$$

since all red homomorphisms are isomorphisms

iterating the argument

follows from the Singleton-H_*-Lemma 35.11, the Path-Component–H_*–Lemma 35.8 and the Reduced Homology Lemma 39.2

We have thus finally calculated the homology groups of spheres. We will redo the calculation in the Sphere-H_n-Proposition 41.2 using a slightly different approach. Furthermore in Section 41.2 we will discuss several applications of this calculation.

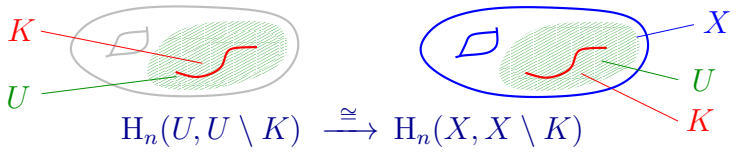
The plan for the remainder of this chapter is as follows: In the next section we will formulate several variations on the Excision Theorem 40.1. We give a short outline of the proof of the Excision Theorem 40.1 in Section 40.3 and we will provide the full daunting details of the proof of the Excision Theorem 40.1 in Section 40.4. Readers who want to see the Excision Theorem 40.1 in action are advised to move to the next chapter.

[301] We could even have gone one step further and ended with $S^{-1} = \varnothing$ and the final calculation would have been provided by the Reduced Homology Lemma 39.2 (0).

40.2. **Variations on the Excision Theorem 40.1.** In this section we will state and prove several variations on the Excision Theorem 40.1. The following theorem is in fact just a convenient reformulation of the Excision Theorem 40.1.

Theorem 40.2. (Excision Theorem) Let X be a topological space, let K be a subset of X and let U be a neighborhood of the closure of K. The inclusion $(U, U \setminus K) \to (X, X \setminus K)$ induces for each $n \in \mathbb{N}_0$ an isomorphism[302]

$$\mathrm{H}_n(U, U \setminus K) \xrightarrow{\cong} \mathrm{H}_n(X, X \setminus K).$$



Proof of Theorem 40.2 assuming Theorem 40.1. We have

$$\mathrm{H}_n(U, U \setminus K) \underset{\uparrow}{=} \mathrm{H}_n(X \setminus Z, A \setminus Z) \underset{\uparrow}{\xrightarrow{\cong}} \mathrm{H}_n(X, A) = \mathrm{H}_n(X, X \setminus K).$$

set $A := X \setminus K$ and $Z := X \setminus U$

since U is a neighborhood of the closure of K we see that the closure of Z is contained in the interior of A, thus we can apply the Excision Theorem 40.1

■

In many occasions we will deal with manifolds and CW-complexes. In these settings the following two theorems will be useful.

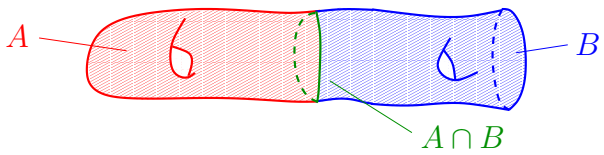
Theorem 40.3. (Manifold–Excision Theorem) Let X be an m-dimensional topological manifold and let $A, B \subset X$ be two codimension-zero submanifolds such that the following statements hold:

(a) $X = A \cup B$.
(b) $A \cap B$ is a union of components of ∂A.
(c) A and B are closed subsets of X.[303]

For every $n \in \mathbb{N}_0$ the inclusion induced map

$$\mathrm{H}_n(B, A \cap B) \to \mathrm{H}_n(X, A)$$

is an isomorphism.



the inclusion induced map $\mathrm{H}_n(B, A \cap B) \to \mathrm{H}_n(X, A)$ is an isomorphism

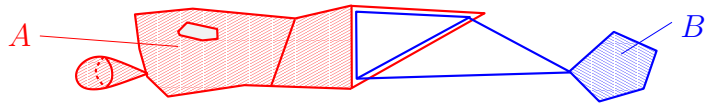
Theorem 40.4. (CW-Excision Theorem) Let X be a CW-complex and let A and B be two subcomplexes of X with $X = A \cup B$. For every $n \in \mathbb{N}_0$ the inclusion induced map

$$\mathrm{H}_n(B, A \cap B) \xrightarrow{\cong} \mathrm{H}_n(X, A)$$

[302] It follows from the Compact-Closed Lemma 1.9 (2) that the hypotheses are for example satisfied if X is Hausdorff (e.g. if X is a topological manifold) and if K is a compact subset that is contained in an open subset U.

[303] Note that by the Compact-Closed Lemma 1.9 (2) this condition is satisfied if A and B are compact.

is an isomorphism.



the inclusion induced map
$\mathrm{H}_n(B, A \cap B) \to \mathrm{H}_n(X, A)$
is an isomorphism

The proofs of the Manifold–Excision Theorem 40.3 and the CW-Excision Theorem 40.4 both rely on the following somewhat technical theorem.

Theorem 40.5. (Excision-via-Deformation Theorem) Let X be a topological space and let $Z \subset A \subset X$ be two subsets. We suppose that there exists a subset $U \subset Z$ such that the closure of U is contained in the interior of A and such that the following statements hold:

(1) $A \setminus Z$ is a deformation retract of $A \setminus U$.
(2) $X \setminus Z$ is a deformation retract of $X \setminus U$

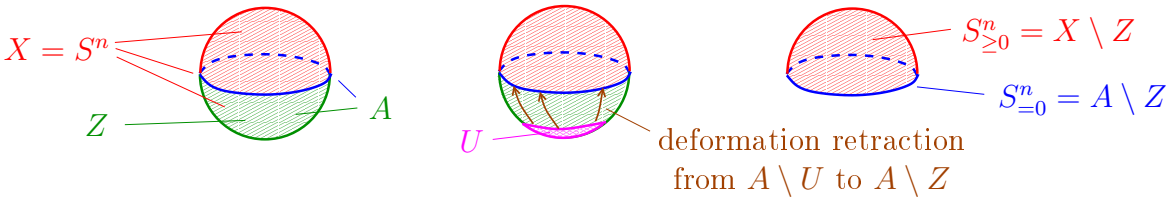
Then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathrm{H}_n(X, A).$$

Example. Let $n \in \mathbb{N}$. As usual we denote by $S^n_{\geq 0}$ and $S^n_{\leq 0}$ the upper and lower hemisphere of S^n and we write $S^n_{=0} = S^n_{\geq 0} \cap S^n_{\leq 0}$. It follows from the Excision-via-Deformation Theorem 40.5 and the choices of X, A, Z and U illustrated in the figure below that for each $k \in \mathbb{N}_0$ the inclusion induced map

$$\mathrm{H}_k(S^n_{\geq 0}, S^n_{n=0}) \;\to\; \mathrm{H}_k(S^n, S^n_{\leq 0})$$

is an isomorphism.



Proof of Theorem 40.5 assuming the Excision Theorem 40.1. We consider the following commutative diagram of inclusion induced maps

$$\begin{array}{ccc} \mathrm{H}_n(X \setminus Z, A \setminus Z) & \longrightarrow & \mathrm{H}_n(X, A) \\ \downarrow & \nearrow_{\cong} & \\ \mathrm{H}_n(X \setminus U, A \setminus U). & & \end{array}$$

Since by hypothesis we know that $\overline{U} \subset \mathring{A}$ we obtain from the Excision Theorem 40.1 that the diagonal map is an isomorphism. Thus it suffices to show that the vertical map is an isomorphism.

To do so we consider the following diagram:

$$\begin{array}{ccccccccc}
\mathrm{H}_n(A\setminus Z) & \longrightarrow & \mathrm{H}_n(X\setminus Z) & \longrightarrow & \mathrm{H}_n(X\setminus Z, A\setminus Z) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A\setminus Z) & \longrightarrow & \mathrm{H}_{n-1}(X\setminus Z) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
\mathrm{H}_n(A\setminus Z) & \longrightarrow & \mathrm{H}_n(X\setminus U) & \longrightarrow & \mathrm{H}_n(X\setminus U, A\setminus Z) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A\setminus Z) & \longrightarrow & \mathrm{H}_{n-1}(X\setminus U) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}_n(A\setminus U) & \longrightarrow & \mathrm{H}_n(X\setminus U) & \longrightarrow & \mathrm{H}_n(X\setminus U, A\setminus U) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A\setminus U) & \longrightarrow & \mathrm{H}_{n-1}(X\setminus U).
\end{array}$$

Here the horizontal maps are pieces of the long exact sequences of the various pairs of topological spaces provided by the LES-of-Pair Proposition 38.3 (1). Note that by the LES-of-Pair Proposition 38.3 (2) we know that the diagram commutes.

Furthermore the vertical maps are induced by the various inclusion maps. On the upper level the first and the fourth vertical maps are the identity. Furthermore it follows from the hypothesis that $X\setminus Z$ is a deformation retract and Homotopy Equivalence-H_*-Corollary 36.7 that the second and the fifth vertical map are also isomorphisms. It follows from the Five Lemma 37.5 that the upper middle vertical map is also an isomorphism.

Using the hypothesis that $A\setminus Z$ is a deformation retract of $A\setminus U$ one can show exactly the same way that the lower middle vertical map is also an isomorphism. Combining the two middle vertical maps we obtain the desired isomorphism. ■

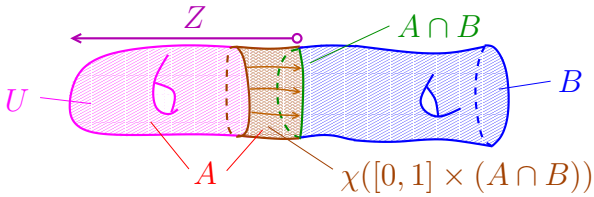
Proof of Theorem 40.3 assuming Theorem 40.5. Let X be an m-dimensional topological manifold and let $A, B \subset X$ be two codimension-zero submanifolds such that the following statements hold:

(a) $X = A \cup B$.
(b) $A \cap B$ is a union of components of ∂A.
(c) A and B are closed subsets of X.

It follows from (b) together with the Collar Neighborhood Theorem 23.2 that there exists a continuous map $\chi\colon [0,1]\times(A\cap B)\to A$ with the following three properties:

(d) The map χ is an embedding.
(e) The image of $[0,1)\times(A\cap B)$ is an open subset of A.
(f) For every $P\in A\cap B$ we have $\chi(0,P)=P$.

We set $Z := X\setminus B$ and we set $U := X\setminus(B\cup\chi([0,1)\times(A\cap B))) = A\setminus\chi([0,1)\times(A\cap B))$.



Claim.
(0) The closure of U is contained in the interior of A.
(1) $A\setminus Z = A\cap B$ is a deformation retract of $A\setminus U = \chi([0,1]\times(A\cap B))$.
(2) $X\setminus Z = B$ is a deformation retract of $X\setminus U = (\chi([0,1]\times(A\cap B))\cup B$.

Proof.

(0) It follows from (b) that $A \setminus (A \cap B) = X \setminus B$ is an open subset of X. Furthermore it follows from (c), (e), (f) and the Cover-Open Subset Lemma 1.5 (2b) that U is a closed subset of X. It follows from this discussion with the fact that $U \subset X \setminus B$ that the closure of U is contained in the interior of A.
(1) We have an obvious deformation retraction from $[0,1] \times (A \cap B)$ to $\{0\} \times (A \cap B)$. It follows from (d) and (f) that this deformation retraction gives rise to a a deformation retraction from $\chi([0,1] \times (A \cap B))$ to $A \cap B$.
(2) It follows from (c) and the Homotopy Combination Lemma 8.2 that we can extend the deformation retraction from (2) in an obvious way to the desired deformation retraction from $X \setminus Z$ to $X \setminus U$. ⊞

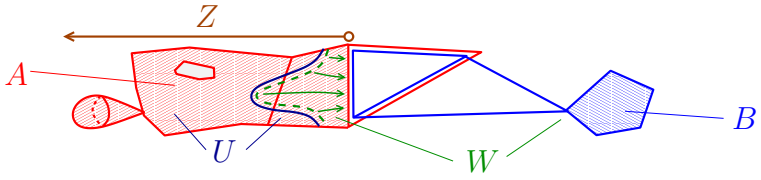
It follows from the claim that we can apply the Excision-via-Deformation Theorem 40.5 which tells us that the inclusion induced map

$$\mathrm{H}_n(\underbrace{X \setminus Z}_{=B}, \underbrace{A \setminus Z}_{=A\cap B}) \xrightarrow{\cong} \mathrm{H}_n(X, A)$$

is an isomorphism. But that is exactly what we needed to show. ■

Proof of the CW-Excision Theorem 40.4 assuming Theorem 40.5. Let X be a CW-complex and let A and B be two subcomplexes of X with $X = A \cup B$. Note that by the Subcomplex-Neighborhood Proposition 31.8, applied to the subcomplex B, there exists an open neighborhood W of B and a deformation retraction $F\colon W \times [0,1] \to W$ from W to B with the extra property that the map F restricts to a deformation retraction $F\colon (A \cap W) \times [0,1] \to A \cap W$ from $A \cap W$ to $A \cap B$.

We set $Z := X \setminus B$ and we set $U := X \setminus W$. Since W is an open neighborhood of B we see that the closure of U is contained in the interior of A.



Recall that F is a deformation retraction from $W = X \setminus U$ to $B = X \setminus Z$ and that F restricts to a deformation retraction from $A \cap W = A \setminus U$ to $A \cap B = A \setminus Z$. Therefore it follows from the Excision-via-Deformation Theorem 40.5 that the inclusion induced map

$$\mathrm{H}_n(\underbrace{X \setminus Z}_{=B}, \underbrace{A \setminus Z}_{=A\cap B}) \xrightarrow{\cong} \mathrm{H}_n(X, A)$$

is an isomorphism. But that is exactly what we needed to show. ■

40.3. **The proof of the Excision Theorem 40.1: the idea.** In this section we want to come up with the idea for the proof of the Excision Theorem 40.1. The full details of the proof will then be given in the following section.

Thus let X be a topological space and let $Z \subset A \subset X$ be two subsets such that the closure of Z is contained in the interior of A. Our goal is to show that the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A).$$

Let us attempt to prove that the map $\mathrm{H}_n(X\setminus Z, A\setminus Z) \to \mathrm{H}_n(X, A)$ is surjective. We take an element in $\mathrm{H}_n(X, A) = \mathrm{H}_n(\mathrm{C}_*)X, A)) = \mathrm{H}_n(\mathrm{C}_*(X)/\mathrm{C}_*(A))$ and we choose a representative $a_1 \cdot \sigma_1 + \cdots + a_r \cdot \sigma_r \in \mathrm{C}_n(X)$ where $\sigma_i \colon \Delta^n \to X$, $i = 1, \ldots, r$ are singular n-simplices in X.

We make the following two basic observations:

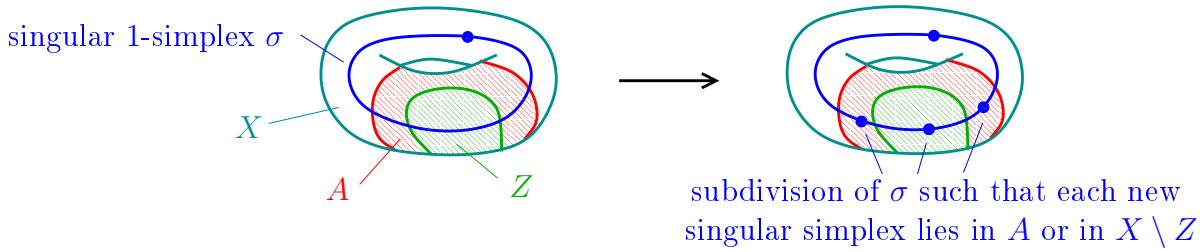
(1) If the singular n-simplices $\sigma_1, \ldots, \sigma_r$ lie in $X \setminus Z$,[304] then $[a_1 \cdot \sigma_1 + \cdots + a_r \cdot \sigma_r]$ evidently lies in the image of the inclusion induced map $\mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A)$.

(2) If a singular n-simplex lies entirely in A, then it represents the zero element in the quotient group $\mathrm{C}_n(X, A) = \mathrm{C}_n(X)/\mathrm{C}_n(A)$.

The combination of these two observations proves the following lemma.

Lemma 40.6. Let X be a topological space and let $Z \subset A$ be two subsets of X. Let $c = a_1 \cdot \sigma_1 + \cdots + a_r \cdot \sigma_r$ be a cycle in X such that each σ_i lies entirely in A or it lies entirely in $X \setminus Z$. Then $[c]$ lies in the image of $\mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A)$.

Unfortunately not every cycle in $\mathrm{C}_n(X)$ satisfies the hypothesis of the lemma. But we will see in the next section that we can subdivide each singular simplex into smaller singular simplices that lie entirely in A or lie entirely in $X \setminus Z$. This then implies that $\mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A)$ is an epimorphism. A very similar argument then also shows that the map is a monomorphism.



40.4. **The proof of the Excision Theorem 40.1: the full details.** In this chapter we will now turn the vague idea of the last section into a rigorous proof of the Excision Theorem 40.1. This will take a surprisingly long time.

First we introduce a few definitions.

Definition. Let again X be a topological space.

(1) On page 41 we defined a cover of X to be a family $\{U_i\}_{i\in I}$ of subsets of X such that $X = \bigcup_{i\in I} U_i$.

(2) We say a cover $\{U_i\}_{i\in I}$ is comfortable if $X = \bigcup_{i\in I} \mathring{U}_i$, i.e. if the interiors of the U_i also form a cover of X.

We move straight to the next definition.

[304] Given a topological space X and a subset A we say that *a singular simplex* $\sigma \colon \Delta^n \to X$ *lies in* A if the image of σ lies in A.

Definition. Let again X be a topological space. Given a comfortable cover $\mathcal{U} = \{U_i\}_{i\in I}$ we consider the following subgroup of $\mathrm{C}_n(X)$:

$$\mathrm{C}_n^{\mathcal{U}}(X) := \left\{ \sum_{j=1}^{k} a_j \cdot \sigma_j \;\middle|\; \text{for each } j \text{ there exists a } U_i \in \mathcal{U} \text{ such that the image of } \sigma_j \text{ lies in } U_i \right\}.$$

It follows quite easily from the definitions that the restriction of the usual boundary map $\partial_n \colon \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$ to $\mathrm{C}_n^{\mathcal{U}}(X)$ restricts to a map $\partial_n \colon \mathrm{C}_n^{\mathcal{U}}(X) \xrightarrow{\partial} \mathrm{C}_{n-1}^{\mathcal{U}}(X)$. Since this map is the restriction of the usual boundary map it follows in particular that the composition $\partial_n \circ \partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathcal{U}}(X) \xrightarrow{\partial} \mathrm{C}_{n-1}^{\mathcal{U}}(X)$ is the zero map. Thus we can define

$$\begin{aligned} \mathrm{H}_n^{\mathcal{U}}(X) &:= \mathrm{H}_n\big(C_*^{\mathcal{U}}(X), \partial_*\big) \\ &= \ker\big(\partial_n \colon \mathrm{C}_n^{\mathcal{U}}(X) \to \mathrm{C}_{n-1}^{\mathcal{U}}(X)\big) \,/\, \mathrm{im}\,\big(\partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathcal{U}}(X) \to \mathrm{C}_n^{\mathcal{U}}(X)\big). \end{aligned}$$

The homology groups $\mathrm{H}_n^{\mathcal{U}}(X)$ are thus defined by singular chains which are "small" in the sense that the image of each singular simplex lies in some U_i.

The following proposition says that we can determine the homology of a topological space through such "small" singular chains. As we will see later on many occasions, this proposition is at the heart of many methods for computing homology groups.

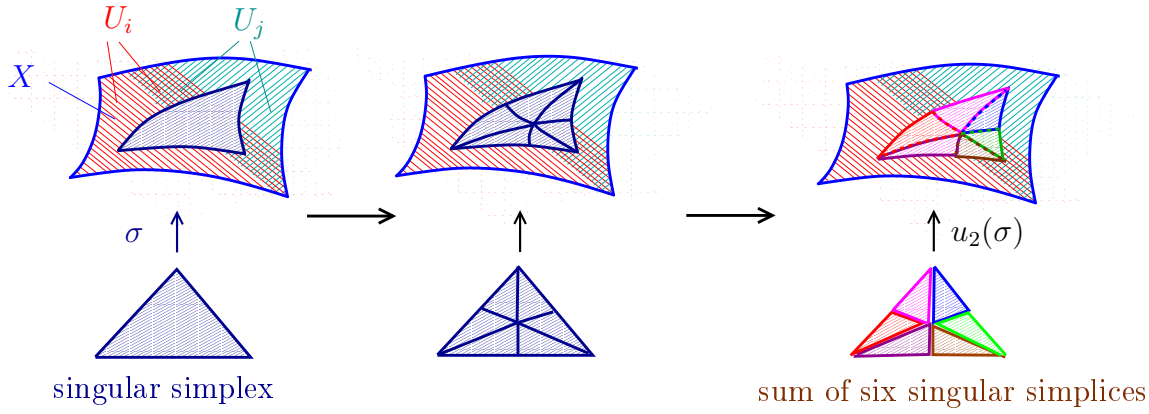
Proposition 40.7. (Comfortable Covering Proposition) Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a comfortable cover of X. The inclusion map

$$\mathrm{C}_*^{\mathcal{U}}(X) \;\to\; \mathrm{C}_*(X)$$

induces for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n^{\mathcal{U}}(X) \;\xrightarrow{\cong}\; \mathrm{H}_n(X).$$

The idea of the proof of the Comfortable Covering Proposition 40.7 is the one we already described in the previous section. We want to replace systematically each singular n-simplex $\sigma \colon \Delta^n \to X$ by a singular n-chain of the form $\sum_{j=1}^{r} a_j \cdot \sigma_j$ such that the image of each singular simplex σ_j lies in some $U_i \in \mathcal{U}$. As we have seen before in the Singular 1-Simplex Lemma 35.12 in the 1-dimensional case, an appropriate subdividing procedure does not change the homology class. The systematic approach we are going to use for the general case is given by "barycentric subdivision". The key idea is sketched in the figure below for the case $n = 2$.



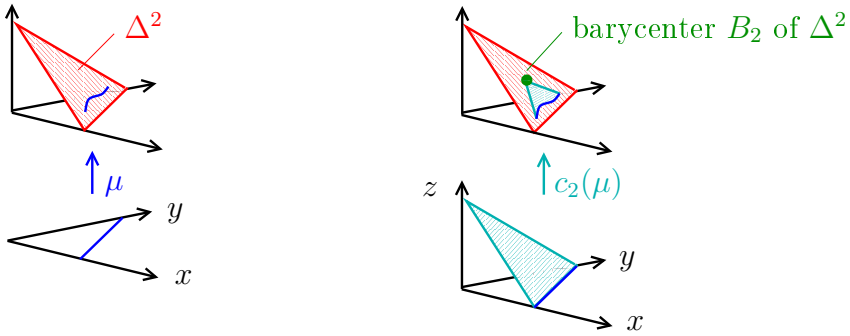
Before we can execute this idea we have to introduce a few more definitions.

(1) Given $n \in \mathbb{N}_0$ we denote by $B_n := (\frac{1}{n+1}, \dots, \frac{1}{n+1}) \in \Delta^n$ the *barycenter*[305] of the standard n-simplex Δ^n.

(2) Given a singular l-simplex $\mu\colon \Delta^l \to \Delta^n$ we consider the singular $(l+1)$-simplex $c_{l+1}(\mu)\colon \Delta^{l+1} \to \Delta^n$ which is defined as the composition of the (inverses) of the following two maps:

$$\begin{array}{rcccl} & & \text{homeomorphism by the Compact-Hausdorff Proposition 2.10 (3)} & & \text{continuous by the Topological-Quotient Proposition 6.1 (1b)} \\ & & \downarrow \cong & & \downarrow \\ c_{l+1}(\mu)\colon \Delta^{l+1} & \xleftarrow{\cong} & (\Delta^l \times [0,1])/\Delta^l \times \{1\} & \to & \Delta^n \\ ((t_0,\dots,t_l)\cdot(1-s),s) & \mapsto & [((t_0,\dots,t_l),s)] & & \\ & & [((t_0,\dots,t_l),s)] & \mapsto & \underbrace{\mu(t_0,\dots,t_l)\cdot(1-s)+B\cdot s.}_{\text{contained in } \Delta^n \text{ since } \Delta^n \text{ is convex}} \end{array}$$

Pictorially speaking the singular $(l+1)$-simplex $c_{l+1}(\mu)$ thus corresponds to the "cone" which is given by the singular l-simplex μ and the barycenter B_n. Note that by the Free Abelian Group–Existence Lemma 18.4 this map $\mu \mapsto c_{l+1}(\mu)$ defines a linear map $\mathrm{C}_l(\Delta^n) \to \mathrm{C}_{l+1}(\Delta^n)$ that we also denote by c_{l+1} and which we call the $(l+1)$-st cone map.



(3) For $n = 0, 1, 2, 3, \dots$ we inductively do the following:
  (a) we define an element $\chi_n \in \mathrm{C}_n(\Delta^n)$, and
  (b) for each topological space X we define a homomorphism $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$.

We do this inductive procedure as follows:
  (i) We set $\chi_0 = \mathrm{id}\colon \Delta^0 \to \Delta^0$ and for each topological space X we define u_0 to be the identity on $\mathrm{C}_0(X)$.
  (ii) Now we suppose that $\chi_{n-1} \in \mathrm{C}_{n-1}(\Delta^{n-1})$ is already defined and we suppose that the homomorphisms $u_{n-1}\colon \mathrm{C}_{n-1}(X) \to \mathrm{C}_{n-1}(X)$ are already defined. Then we set

$$\chi_n := (-1)^n \cdot c_n(u_{n-1}(\underbrace{\partial\,\mathrm{id}_{\Delta^n}}_{\in \mathrm{C}_{n-1}(\Delta^n)})) \in \mathrm{C}_n(\Delta^n).$$

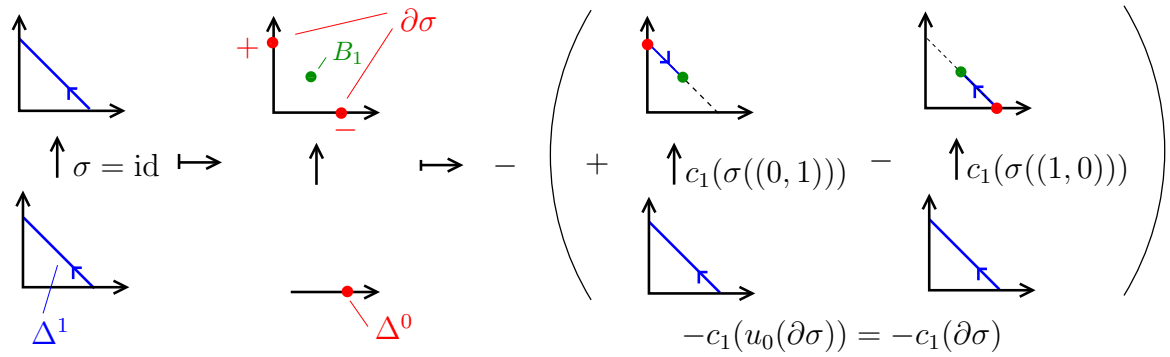
[305] The word "barycenter" comes from the Greek word "barys" which means heavy. In physics the barycenter of an object is the center of mass, in our case the barycenter of the standard n-simplex is the center of mass if we view Δ^k as a physical object of uniform density.

Furthermore, for each topological space X we define

$$\begin{array}{rcl} u_n\colon \mathrm{C}_n(X) & \to & \mathrm{C}_n(X) \\ \sum_{l=1}^{r} a_l \cdot (\sigma_l\colon \Delta^n \to X) & \mapsto & \sum_{l=1}^{r} a_l \cdot (\sigma_l)_*(\underbrace{\chi_n}_{\in \mathrm{C}_n(\Delta^n)}). \end{array}$$

map $\sigma_l\colon \Delta^l \to X$ induces a map $(\sigma_l)_*\colon \mathrm{C}_n(\Delta^l) \to \mathrm{C}_n(X)$

Note that with this notation we have $u_n(\mathrm{id}_{\Delta^n}) = \chi_n$. The map u_n is illustrated for $n = 1$ in the figure below. For $n = 2$ we already gave a more schematic sketch in the figure above without considering signs and orientations.



Before we can continue we need to prove several technical lemmas about the subdivision maps. We start out with a straightforward lemma.

Lemma 40.8. (Subdivision–Natural Lemma) The maps $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$ are natural,[306] in particular for any continuous map $f\colon X \to Y$ between topological spaces and any $n \in \mathbb{N}_0$ we have

$$f_* \circ u_n \;=\; u_n \circ f_*\colon \mathrm{C}_n(X) \to \mathrm{C}_n(Y).$$

Proof. This lemma follows immediately from the definitions. Indeed, let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. Then

$$\begin{array}{ccccc} f_*(u_n(\sigma)) & = & f_*(\sigma_*(\chi_n)) & = & (f \circ \sigma)_*(\chi_n) \\ \uparrow & & \uparrow & & \uparrow \\ \text{by definition of } u_n(\sigma) & & \text{since } (f\circ\sigma)_* = f_* \circ \sigma_* & & \text{by definition of } u_n(f_*(\sigma)) \\ & & \text{as maps } \mathrm{C}_n(\Delta^n) \to \mathrm{C}_n(Y) & & \end{array} \quad = \; u_n(f_*(\sigma)).$$

Since the desired equality holds for a generating set of $\mathrm{C}_n(X)$ it also holds for all of $\mathrm{C}_n(X)$. ■

We continue with a lemma which is much harder to prove.

[306] Let us explain in what sense these maps deserve to be called "natural". Given $n \in \mathbb{N}_0$ we consider the functor C_* from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{ChCplx}$ of chain complexes which is given by

$$\text{topological space } X \;\mapsto\; \mathrm{C}_*(X)$$

and

$$(f\colon X \to Y) \;\mapsto\; (f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)).$$

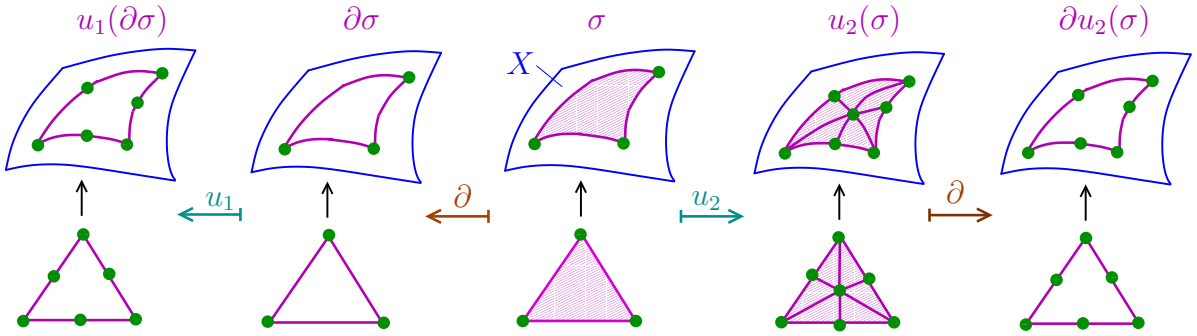
Given a topological space X we denote by $\Phi_X\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X)$ the map that is given by the n-th subdivision map that we defined on page 473. Lemma 40.9 and the Subdivision–Natural Lemma 40.8 say that these maps define a natural transformation from the functor C_* to itself.

Lemma 40.9. (Chain Subdivision Lemma) Let X be a topological space. The maps $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$, $n \in \mathbb{N}_0$, form a chain map from the chain complex $(\mathrm{C}_*(X), \partial)$ to itself.

The statement of the Chain Subdivision Lemma 40.9 is equivalent to the statement that for every singular n-simplex σ in X we have

$$u_{n-1}(\partial\sigma) \;=\; \partial(u_n(\sigma)).$$

Put differently, the Chain Subdivision Lemma 40.9 says that it does not make a difference whether we first take the boundary and then subdivide, or whether we first subdivide and then take the boundary. This statement is sketched in the figure below.



Proof of the Chain Subdivision Lemma 40.9. [307] We adopt the following notation and we make the following remarks:

(1) We denote the barycenter of Δ^n by b.

(2) Given points $x_0, \ldots, x_k \in \Delta^n$ we denote by $[x_0, \ldots, x_k]$ the singular k-chain $\Delta^k \to \Delta^n$ that is given by[308]

$$\begin{array}{rcl} \Delta^k & \to & \Delta^n \\ (t_0, \ldots, t_k) & \mapsto & \sum_{i=0}^{k} t_i \cdot x_i, \end{array}$$

we refer to such a singular k-simplex as a *linear* singular simplex.

(3) We denote by $\mathrm{C}_k^{\mathrm{lin}}(\Delta^n)$ the subgroup of $\mathrm{C}_k(\Delta^n)$ generated by all the linear singular k-simplices. Note that the boundary map $\partial\colon \mathrm{C}_k(\Delta^n) \to \mathrm{C}_{k-1}(\Delta^n)$ restricts to a boundary map $\partial\colon \mathrm{C}_k^{\mathrm{lin}}(\Delta^n) \to \mathrm{C}_{k-1}^{\mathrm{lin}}(\Delta^n)$.

(4) For $x_0, \ldots, x_k \in \Delta^n$ we write

$$[x_0, \ldots, \widehat{x_i}, \ldots, x_k] \;:=\; [x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k].$$

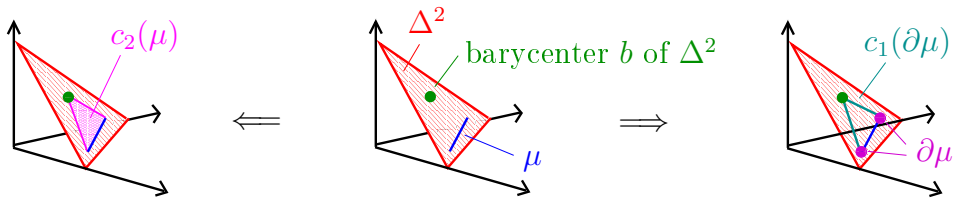
The proof of the lemma will be broken into three claims.

Claim 1. For any $\mu \in \mathrm{C}_k^{\mathrm{lin}}(\Delta^n)$ we have

$$\partial(c_{k+1}(\mu)) \;=\; c_k(\partial\mu) + (-1)^{k+1} \cdot \mu \;\in\; \mathrm{C}_k(\Delta^n).$$

[307] We will not do this proof in the lectures. The proof is somewhat lengthy but straightforward. The cleverness hereby does not lie in the proof, but in the cleverly chosen definitions, which turn the verification into a simple calculation.

[308] It is straightforward to show that this map does indeed take values in Δ^n.



Proof. To prove the claim we first note that for any choice of points $x_0, \dots, x_{l-1} \in \Delta^n$ we have by definition of c_l that

$$c_l([x_0, \dots, x_{l-1}]) \;=\; [x_0, \dots, x_{l-1}, b].$$

It follows that for any $w_0, \dots, w_k \in \Delta^n$ we have the following equality in $\mathrm{C}_k(\Delta^n)$:

$$\begin{aligned}
\partial(c_{k+1}([w_0, \dots, w_k])) &\overset{\text{by the above equality}}{\downarrow} \\
&= \partial([w_0, \dots, w_k, b]) \\
&\underset{\text{definition of } \partial}{\uparrow} \sum_{j=0}^{k} (-1)^j \cdot [w_0, \dots, \widehat{w_j}, \dots, w_k, b] + (-1)^{k+1} \cdot [w_0, \dots, w_k] \\
&\overset{\text{definition of } \partial \text{ and by the above equality}}{\downarrow} \\
&= c_k(\partial([w_0, \dots, w_k])) + (-1)^{k+1} \cdot [w_0, \dots, w_k].
\end{aligned}$$

The claim immediately follows from this equality. ⊞

Claim 2. For any $n \in \mathbb{N}_0$ and any $\nu \in \mathrm{C}_n^{\mathrm{lin}}(\Delta^n)$ we have

$$\partial u_n(\nu) \;=\; u_{n-1}(\partial\nu).$$

Proof. We will prove this claim by induction on n. The case $n = 0$ is trivial. So now suppose we have already shown the case $n - 1$. It suffices to show the claim for all linear singular n-simplices. So let $\nu\colon \Delta^n \to \Delta^n$ be a linear singular n-simplex. We have then

$$\begin{aligned}
\partial u_n(\nu) &\overset{\text{definition of } u_n \text{ and } \chi_n, \text{ see page 472}}{\downarrow} \partial(\nu_*(\chi_n)) \overset{\text{since } \partial \circ \nu_* = \nu_* \circ \partial}{\downarrow} (-1)^n \cdot \partial(\nu_*(c_n(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})))) \overset{\downarrow}{=} (-1)^n \cdot \nu_*\big(\partial(c_n(\underbrace{u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})}_{=:\mu}))\big) \\
&\underset{\text{we can apply the previous claim since } \mu := u_{n-1}(\partial\,\mathrm{id}_{\Delta^n}) \in \mathrm{C}_{n-1}^{\mathrm{lin}}(\Delta^n)}{\uparrow} (-1)^n \cdot \nu_*\big(c_{n-1}(\partial(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) + (-1)^n \cdot u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})\big) \\
&\underset{\uparrow}{=} (-1)^n \cdot \nu_*\big(c_{n-1}(u_{n-2}(\underbrace{\partial\partial\,\mathrm{id}_{\Delta^n}}_{=0}))\big) \;+\; \nu_*(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) \\
&\underset{\text{by induction hypothesis we have } \partial u_{n-1}(\rho) = u_{n-2}(\partial\rho) \text{ for } \rho \in \mathrm{C}_{n-1}^{\mathrm{lin}}(\Delta^n), \text{ applied here to } \rho = \partial\,\mathrm{id}_{\Delta^n}}{\uparrow} \\
&\underset{\text{Subdivision–Natural Lemma 40.8 applied to } f = \nu}{=} u_{n-1}(\nu_*(\partial\,\mathrm{id}_{\Delta^n})) \;\underset{\text{by definition}}{=}\; u_{n-1}(\partial\nu).
\end{aligned}$$

⊞

The following claim is just a reformulation of the statement of the lemma.

Claim 3. For any singular n-simplex σ in X we have $u_{n-1}(\partial\sigma) = \partial(u_n(\sigma))$.

Proof. So let $\sigma\colon \Delta^n \to X$ be a singular n-simplex in X. Then we have

$$\begin{array}{rcccccl} & & \text{by the Subdivision–Natural Lemma 40.8 applied to } f=\sigma & & \text{since } \sigma_* \text{ is a chain map} & & \\ \partial(u_n(\sigma)) & = & \partial(u_n(\sigma_*(\mathrm{id}_{\Delta^n}))) & \overset{\downarrow}{=} & \partial(\sigma_*(u_n(\mathrm{id}_{\Delta^n}))) & \overset{\downarrow}{=} & \sigma_*(\partial(u_n(\mathrm{id}_{\Delta^n}))) \\ & = & \sigma_*(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) & = & u_{n-1}(\sigma_*(\partial\,\mathrm{id}_{\Delta^n})) & = & u_{n-1}(\partial\sigma_*(\mathrm{id}_{\Delta^n})) \;=\; u_{n-1}(\partial\sigma). \\ & \overset{\uparrow}{} & & \overset{\uparrow}{} & & \overset{\uparrow}{} & \\ & \text{by the previous claim} & & \text{by the Subdivision–Natural Lemma 40.8 applied to } f=\sigma & & \text{since } \sigma_* \text{ is a chain map} & \end{array}$$

■

The proof of the next lemma is again rather involved.

Lemma 40.10. (Subdivision–Chain Homotopy-Lemma) Let X be a topological space. The chain maps

$$u_n\colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_n(X)$$

are chain homotopic to the identity.

Proof. Showing that the chain maps $(u_n)_{n\in\mathbb{N}}$ are chain homotopic to the identity requires us to find maps $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X), n \in \mathbb{N}_0$ such that for each $n \in \mathbb{N}_0$ the following equality holds:

$$\partial_{n+1} \circ P_n \;=\; -P_{n-1} \circ \partial_n + u_n - \mathrm{id}\,.$$

First we consider the case that X is a convex subset of some $\mathbb{R}^m$. Recall that we defined $\mathrm{C}_{-1}(X) = 0$, accordingly we define $P_{-1}\colon \mathrm{C}_{-1}(X) \to \mathrm{C}_0(X)$ to be the zero map. Let us suppose that we have already found maps $P_{-1}, \ldots, P_{n-1}$ with the desired properties.

We want to define the next map $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X)$. This map has to have the property that for each singular n-simplex σ we have

$$\partial_{n+1}(P_n(\sigma)) \;=\; -(P_{n-1} \circ \partial_n)(\sigma) \;+\; (u_n - \mathrm{id})(\sigma).$$

Put differently, to define $P_n(\sigma)$ we have to find a singular $(n+1)$-chain $E_\sigma \in \mathrm{C}_{n+1}(X)$ with

$$\partial E_\sigma \;=\; -(P_{n-1} \circ \partial)(\sigma) + (u_n - \mathrm{id})(\sigma),$$

so that we can then define $P_n(\sigma) := E_\sigma$. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex.

(1) For $n = 0$ we have

$$\big(\underbrace{P_{-1}}_{=0} \circ \partial_0 + \underbrace{u_0}_{=\mathrm{id}} - \mathrm{id}\big)(\sigma) \;=\; 0 \in \mathrm{C}_0(X),$$

and we put $E_\sigma := 0 \in \mathrm{C}_1(X)$.

(2) Now suppose that $n > 0$. Since X is convex we know from the discussion on page 437 that $\mathrm{H}_n(X) = 0$. Therefore to show that an element $c \in \mathrm{C}_n(X)$ is the boundary of some element in $\mathrm{C}_{n+1}(X)$ it suffices to show that $\partial c = 0$. In our specific situation we calculate

$$\begin{array}{rcl} & & \text{by the Chain Subdivision Lemma 40.9 the subdivision maps form a chain map, i.e. } \partial \circ u_n = u_{n-1} \circ \partial \\ \partial\big(\,(-P_{n-1}\partial_n + u_n - \mathrm{id})\,(\sigma)\big) & \overset{\downarrow}{=} & -\partial P_{n-1}\partial\sigma + u_{n-1}\partial\sigma - \partial\sigma \\ & = & \underbrace{(-\partial P_{n-1} + u_{n-1} - \mathrm{id})}_{=P_{n-2}\partial}(\partial\sigma) \;=\; P_{n-2}(\underbrace{\partial\partial\sigma}_{=0}) \;=\; 0. \end{array}$$

As we had just remarked, this implies that there exists an $E_\sigma \in \mathrm{C}_{n+1}(X)$ which satisfies $\partial(E_\sigma) = (P_{n-1}\partial_n - u_{n-1} - \mathrm{id})(\sigma)$.

Now we denote by $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X)$ the map which assigns to each singular n-simplex σ the element $E_\sigma \in \mathrm{C}_{n+1}(X)$. It follows from the definition of E_σ that

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; u_n - \mathrm{id}$$

as maps $\mathrm{C}_n(X) \to \mathrm{C}_n(X)$. This concludes the proof of the Subdivision–Chain Homotopy-Lemma 40.10 if X is a convex subset of some $\mathbb{R}^m$.

Now we deal with the general case. So let X be a topological space. We consider

$$\begin{array}{rcl} P_n\colon \mathrm{C}_n(X) & \to & \mathrm{C}_{n+1}(X) \\ \sum_{l=1}^{r} a_l \cdot (\sigma_l\colon \Delta^n \to X) & \mapsto & \sum_{l=1}^{r} a_l \cdot (\sigma_l)_*(\underbrace{P_n(\mathrm{id}_{\Delta^n})}_{\substack{\uparrow \\ \text{defined since } \Delta^n \text{ is convex}}}). \end{array}$$

It is straightforward to see that this is a chain homotopy from the subdivision map to the identity map. ■

Definition. Given a topological space X and $m \in \mathbb{N}_0$ we denote by u_n^m the m-fold composition of $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$.

Lemma 40.11. (Iterated Subdivision Lemma) For each topological space X and every $m \in \mathbb{N}$ the maps $\{u_n^m\}_{n\geq 0}$ form a chain map $\mathrm{C}_*(X) \to \mathrm{C}_*(X)$ that is chain homotopic to the identity.

Proof. Recall that in the Chain Subdivision Lemma 40.9 we already saw that the maps $\{u_n\}_{n\geq 0}$ form a chain map $\mathrm{C}_*(X) \to \mathrm{C}_*(X)$. As we remarked on page 419, the composition of chain maps is again a chain map. Thus for any $m \in \mathbb{N}_0$ the maps $\{u_n^m\}_{n\geq 0}$ are also chain maps.

By the Subdivision–Chain Homotopy-Lemma 40.10 we have $u_* \simeq \mathrm{id}$. By the Chain Homotopy-Lemma 36.2 (2) this implies $u_*^2 \simeq u_*$, and by the Chain Homotopy-Lemma 36.2 (1) this implies $u_*^2 \simeq \mathrm{id}$. An induction argument now shows that $u_*^m \simeq \mathrm{id}$ for all m. ■

Definition. Given a singular chain c we can write it uniquely as $c = a_1 \cdot \sigma_1 + \cdots + a_s \cdot \sigma_s$ with $a_1, \ldots, a_s \in \mathbb{Z} \setminus \{0\}$ and where the σ_i are pairwise different. We call $\sigma_1, \ldots, \sigma_s$ the pieces of c.

If $\sigma\colon \Delta^n \to X$ is a singular n-simplex, then the pieces of $u_n(\sigma)$ are evidently "smaller" than the pieces of σ, and by iteratively applying u_n one can hope to make the resulting pieces "arbitrary small". In the following two lemmas we will make it precise what we mean by "arbitrary small". Before we formulate the first lemma we need to recall and introduce several definitions.

Definition.

(1) Given a bounded subset A of $\mathbb{R}^{n+1}$ the diameter is defined as

$$\mathrm{diam}(A) \;:=\; \sup\big\{\|a-b\| \,\big|\, a, b \in A\big\} \in \mathbb{R}_{\geq 0}.$$

(2) Given a singular simple $\sigma\colon \Delta^n \to X$ we denote by $|\sigma|$ the image of σ.

Definition. For points $z_0, \ldots, z_k \in \mathbb{R}^n$ we consider the map

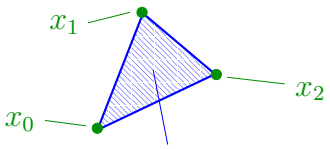
$$\begin{aligned} [z_0, \ldots, z_k]\colon \Delta^k &\to \mathbb{R}^n \\ (t_0, \ldots, t_k) &\mapsto \sum_{j=0}^{k} t_j \cdot z_j. \end{aligned}$$

The map $[z_0, \ldots, z_k]$ thus sends the standard k-simplex Δ^k to the simplex spanned by $z_0, \ldots, z_k$ in $\mathbb{R}^n$. Note that $[z_0, \ldots, z_k]$ is a singular k-simplex in $\mathbb{R}^n$.

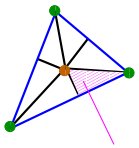
Now we can formulate the following lemma.

Lemma 40.12. (Subdivision–Diameter Lemma) For any $x_0, \ldots, x_k \in \mathbb{R}^n$ and any piece τ of $u_k([x_0, \ldots, x_k])$ we have

$$\operatorname{diam}(|\tau|) \;\leq\; \tfrac{n}{n+1} \cdot \operatorname{diam}(|[x_0, \ldots, x_k]|).$$



the image of $[x_0, x_1, x_2]$



the image of a piece τ of $u_2([x_0, x_1, x_2])$

In the proof of the Subdivision–Diameter Lemma 40.12 we will need the following elementary[309] geometric lemma.

Lemma 40.13. (Simplex–Diameter Lemma) Let $y_0, \ldots, y_k \in \mathbb{R}^n$.

(1) We have the equality

$$\operatorname{diam}(|[y_0, \ldots, y_k]|) \;=\; \max\left\{ \|y_i - y_j\| \,\middle|\, i, j = 1, \ldots, k \right\}.$$

(2) We denote by $b = \frac{1}{k} \cdot \sum_{i=0}^{k} y_i$ the barycenter of $y_0, \ldots, y_k$. For any $i = 0, \ldots, k$ we have

$$\|y_i - b\| \;\leq\; \operatorname{diam}(|[y_0, \ldots, y_k]|).$$

Proof of the Simplex–Diameter Lemma 40.13.

(1) The inequality "$\geq$" follows immediately from the definitions and the observation that each y_i lies in

$$|[y_0, \ldots, y_k]| \;=\; \left\{ \sum_{i=0}^{k} t_i \cdot y_i \,\middle|\, t_i \in [0,1] \text{ and } \sum_{i=0}^{k} t_i = 1 \right\}.$$

We turn to the proof of the reverse inequality "$\leq$". So we pick $a, b \in |[y_0, \ldots, y_k]|$. By definition we can write $a = \sum_{i=0}^{k} s_i \cdot y_i$ and $b = \sum_{i=0}^{k} t_i \cdot y_i$ with $s_i, t_i \in [0,1]$ and such that

[309] Here "elementary" means that the statement and the proof are basically high school level mathematics which does not mean that the proofs are entirely obvious.

$\sum_{i=0}^{k} s_i = \sum_{i=0}^{k} t_i = 1$. We then see that

$$\begin{array}{rcl} \|a-b\| & = & \left\|a - \sum_{i=0}^{k} t_i \cdot y_i\right\| \\ & \leq & \sum_{i=0}^{k} t_i \cdot \max_{i=0,\ldots,k}\{\|a-y_i\|\} \\ & & \uparrow \text{since } \sum_{i=0}^{k} t_i = 1 \end{array} \quad \begin{array}{rcl} & & \overset{\text{since } \sum_{i=0}^{k} t_i = 1}{\downarrow} \\ & = & \left\|\sum_{i=0}^{k} t_i \cdot (a-y_i)\right\| \leq \sum_{i=0}^{k} \|t_i \cdot (a-y_i)\| \\ & = & \max_{i=0,\ldots,k}\{\|a-y_i\|\} \leq \max_{i,j=0,\ldots,k}\{\|y_j-y_i\|\} \\ & & \uparrow \text{same argument applied to } a = \sum_{j=0}^{k} s_j \cdot y_j \end{array}$$

(2) We denote by b_i the barycenter of $y_0, \ldots, \widehat{y_i}, \ldots, y_k$, i.e.

$$b_i = \tfrac{1}{k} \cdot \sum_{j \neq i} y_j.$$

We then see that

$$b = \tfrac{1}{k+1} \cdot y_i + \tfrac{k}{k+1} \cdot b_i.$$

It follows that

$$\begin{array}{rcl} \|b-y_i\| & = & \left\|\frac{1}{k+1} \cdot y_i + \frac{k}{k+1} \cdot b_i - y_i\right\| = \left\|\frac{k}{k+1} \cdot b_i - \frac{k}{k+1} \cdot y_i\right\| \\ & & = \frac{k}{k+1} \cdot \|b_i - y_i\| \leq \frac{k}{k+1} \cdot \operatorname{diam}(|[y_0, \ldots, y_k]|). \end{array}$$

Thus we have obtained the desired inequality. ■

Now we are ready to provide the proof of the Subdivision–Diameter Lemma 40.12.

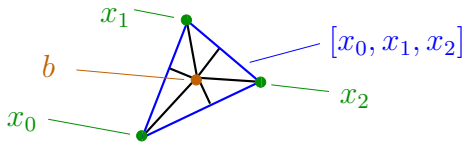
Proof of the Subdivision–Diameter Lemma 40.12. Recall that we need to show that for any $x_0, \ldots, x_k \in \mathbb{R}^n$ and any piece τ of $u_k([x_0, \ldots, x_k])$ we have

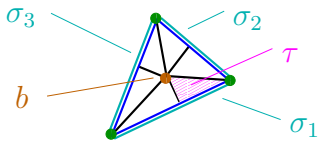
$$\operatorname{diam}(|\tau|) \leq \tfrac{n}{n+1} \cdot \operatorname{diam}(|[x_0, \ldots, x_k]|).$$

We prove the claim by induction on k. For $k = 0$ all the diameters are zero, so the inequality holds trivially. Now suppose that the claim holds for $k-1$. Let $x_0, \ldots, x_k \in \Delta^n$. We denote by $b = \frac{1}{k} \cdot \sum_{i=0}^{k} x_i$ the barycenter of $x_0, \ldots, x_k$. We denote by $\sigma_1, \ldots, \sigma_r$ the pieces of $\partial[x_0, \ldots, x_k]$.[310]

Now let τ be a piece of $u_k([x_0, \ldots, x_k])$. By the definition of the subdivision map given on page 473 we see that $|\tau| = |[w_0, \ldots, w_{k-1}, b]|$ where $[w_0, \ldots, w_{k-1}]$ is a piece of $u_{k-1}(\sigma_s)$ for some $s \in \{1, \ldots, r\}$. By the Simplex–Diameter Lemma 40.13 (1) it now suffices to prove the following claim.

Claim.
(1) For any $i, j = 0, \ldots, k-1$ we have $\|w_i - w_j\| \leq \frac{k}{k+1} \cdot \operatorname{diam}(|[x_0, \ldots, x_k]|)$.
(2) For any $i = 0, \ldots, k-1$ we have $\|w_i - b\| \leq \frac{k}{k+1} \cdot \operatorname{diam}(|[x_0, \ldots, x_k]|)$.





[310] Why does it not say that the number of pieces is $k+1$?

Proof. First let $i, j \in \{0, \dots, k-1\}$. Then

$$\|w_i - w_j\| \;\leq\; \operatorname{diam}([w_0, \dots, w_{k-1}]) \underset{\substack{\uparrow \\ \text{induction hypothesis} \\ \text{applied to } \sigma_s}}{\leq} \frac{k-1}{k} \cdot \operatorname{diam}(\sigma_s) \underset{\substack{\uparrow \\ \text{since } |\sigma_s| \subset |[x_0, \dots, x_k]| \text{ and } \frac{k-1}{k} \leq \frac{k}{k+1}}}{\leq} \frac{k}{k+1} \cdot \operatorname{diam}([x_0, \dots, x_k]).$$

Now let $i \in \{0, \dots, k-1\}$. We then see that

$$\|w_i - b\| \underset{\substack{\uparrow \\ \text{by the Simplex–Diameter Lemma 40.13 (2)}}}{\leq} \frac{k}{k+1} \cdot \operatorname{diam}(|[w_0, \dots, w_{k-1}]|) \underset{\substack{\uparrow \\ \text{since } |[w_0, \dots, w_{k-1}]| \subset |[x_0, \dots, x_k]|}}{\leq} \frac{k}{k+1} \cdot \operatorname{diam}(|[x_0, \dots, x_k]|) \qquad \blacksquare$$

Lemma 40.14. (Simplex–Reduction Lemma) Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. For every singular n-simplex $\sigma\colon \Delta^n \to X$ there exists an $m \in \mathbb{N}_0$ such that $u_n^m(\sigma)$ lies in $\mathrm{C}_n^{\mathcal{U}}(X)$.

Proof. Let X be a topological space, let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X and let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. By the Lebesgue Lemma 11.9 (2) there exists a $\delta > 0$ such that for any subset A of Δ^n with diameter $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ such that $A \subset \sigma^{-1}(\overset{\circ}{U}_i)$.

We pick an $m \in \mathbb{N}$ such that $\left(\frac{n}{n+1}\right)^m < \delta$. We calculate that

$$\begin{aligned} u_n^m(\sigma) \;=\; u_n^m(\sigma_*(\mathrm{id}_{\Delta^n})) &\underset{\substack{\downarrow \\ \text{by the Subdivision–Natural Lemma 40.8}}}{=} \sigma_*(u_n^m(\mathrm{id}_{\Delta^n})) \underset{\substack{\downarrow \\ \text{by the Subdivision–Diameter Lemma 40.12 and } \left(\frac{n}{n+1}\right)^m < \delta}}{=} \sigma_*\begin{pmatrix}\text{simplicial chain such that the image} \\ \text{has diameter less than } \delta\end{pmatrix} \\ &= \sigma_*\begin{pmatrix}\text{simplicial chain such that the image} \\ \text{of each piece lies in some } \sigma^{-1}(\overset{\circ}{U}_i)\end{pmatrix}. \\ &\qquad \underset{\substack{\uparrow \\ \text{by definition of } \delta}}{} \end{aligned}$$

But this means precisely that $u_n^m(\sigma) \in \mathrm{C}_n^{\mathcal{U}}(X)$. $\blacksquare$

Now we can prove the Comfortable Covering Proposition 40.7.

Proof of the Comfortable Covering Proposition 40.7. Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. We want to show that the inclusion map $\iota\colon \mathrm{C}_*^{\mathcal{U}}(X) \to \mathrm{C}_*(X)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \;\xrightarrow{\cong}\; \mathrm{H}_n(X).$$

We first show that the map $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is surjective. Let $z \in \mathrm{H}_n(X)$ be given. We choose a cycle $c = \sum_{j=1}^{r} a_j \cdot \sigma_j \in \mathrm{C}_n(X)$ that represents $z \in \mathrm{H}_n(X)$. It follows from the Simplex–Reduction Lemma 40.14, applied to the singular n-simplices $\sigma_1, \dots, \sigma_r$, that there exists an $m \in \mathbb{N}_0$ such that $u_n^m(c) \in \mathrm{C}_n^{\mathcal{U}}(X)$. By the Iterated Subdivision Lemma 40.11 the maps $\{u_n^m\}_{n \in \mathbb{N}_0}$ form a chain map that is chain homotopic to the identity. By definition this means that there exists a chain homotopy $P_k\colon \mathrm{C}_k(X) \to \mathrm{C}_{k+1}(X)$, $k \in \mathbb{N}_0$. Thus we obtain that

$$z \;=\; [c] \underset{\substack{\uparrow \\ \text{since } \partial_{n+1}P_n + P_{n-1}\partial_n \;=\; u_n^m - \mathrm{id}}}{=} \left[u_n^m(c) - P_{n-1}\overbrace{\partial_n(c)}^{=0} - \partial_{n+1}P_n(c)\right] \;=\; \left[u_n^m(c) - \partial_{n+1}P_n(c)\right] \underset{\substack{\uparrow \\ \text{equality in } \mathrm{H}_n(X)}}{=} \overbrace{[u_n^m(c)]}^{\in \iota_*(\mathrm{H}_n^{\mathcal{U}}(X))}.$$

Thus we have shown that $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is surjective.

The argument that the map $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is injective is very similar. Indeed, let $[c] \in \mathrm{H}_n^{\mathcal{U}}(X)$ be an element such that $[c] = 0 \in \mathrm{H}_n(X)$. This means that there exists an $e \in \mathrm{C}_{n+1}(X)$ with $\partial e = c$. From the Simplex–Reduction Lemma 40.14 it follows, as above, that there exists an $m \in \mathbb{N}_0$ with $u_{n+1}^m(e) \in \mathrm{C}_{n+1}^{\mathcal{U}}(X)$. Again we choose a chain homotopy $\{P_k\}_{k\in\mathbb{N}_0}$ between the chain maps $\{u_k^m\}_{k\in\mathbb{N}_0}$ and id. Then we have

$$c \;=\; \partial e \;\underset{\substack{\uparrow\\ \text{since } \partial P_{n+1} + P_n\partial = u_{n+1}^m - \mathrm{id}}}{=}\; \partial\big(u_{n+1}^m(e) - \partial P_{n+1}(e) - P_n\partial e\big) \;=\; \partial\underbrace{u_{n+1}^m(e)}_{\in \mathrm{C}_{n+1}^{\mathcal{U}}(X)} \;-\; \partial\underbrace{P_n(c)}_{\in \mathrm{C}_{n+1}^{\mathcal{U}}(X)}.$$

Thus have we have shown that c is indeed the boundary of an $(n+1)$-dimensional singular chain in $C_{n+1}^{\mathcal{U}}(X)$ i.e. $[c] = 0 \in \mathrm{H}_n^{\mathcal{U}}(X)$. Put differently, we have now proved that $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is injective. ∎

Using the Comfortable Covering Proposition 40.7 we can now prove the Excision Theorem 40.1.

Proof of the Excision Theorem 40.1. Let X be a topological space. Furthermore let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A. We have to show that the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \;\xrightarrow{\cong}\; \mathrm{H}_n(X, A).$$

We put $B := X \setminus Z$. Note that $A \cap B = A \setminus Z$. It follows from our hypothesis on Z and A that X is the union of the interiors of A and B. Put differently, the two sets $\{A, B\}$ form a comfortable cover of X.

For each $n \in \mathbb{N}_0$ we consider the following commutative diagram of abelian groups:[311]

$$\begin{array}{ccccccccc}
 & & & & & & \mathrm{C}_n(B, A\cap B) & & \\
 & & & & & & \cong\downarrow{\scriptstyle [\sigma]\mapsto[\sigma]} & & \\
0 & \longrightarrow & \mathrm{C}_n(A) & \longrightarrow & \mathrm{C}_n^{\{A,B\}}(X) & \longrightarrow & \mathrm{C}_n^{\{A,B\}}(X)/\mathrm{C}_n(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_n(A) & \longrightarrow & \mathrm{C}_n(X) & \longrightarrow & \mathrm{C}_n(X, A) & \longrightarrow & 0.
\end{array}$$

We make the following two observations:

(1) By definition the horizontal sequences are short exact sequences of chain complexes.

(2) It follows immediately from the definition of the group $\mathrm{C}_n^{\{A,B\}}(X)$ that the inclusion $\mathrm{C}_n(B) \to \mathrm{C}_n^{\{A,B\}}(X)$ induces a chain map $\mathrm{C}_n(B, A\cap B) \to \mathrm{C}_n^{\{A,B\}}(X)/\mathrm{C}_n(A)$ which is in fact an isomorphism.

[311] Here we use the definition from page 471 which in this context says that

$$\mathrm{C}_n^{\{A,B\}}(X) \;:=\; \{\sigma + \tau \,|\, \sigma \in \mathrm{C}_n(A) \text{ and } \tau \in \mathrm{C}_n(B)\}.$$

Now we consider the long exact sequence of homology groups arising from the above short exact sequences of chain complexes. It follows from the LES-Proposition 37.4, together with the above vertical isomorphism, that we obtain the following commutative diagram where the horizontal sequences are exact:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \mathrm{H}_n(A) & \longrightarrow & \mathrm{H}_n^{\{A,B\}}(X) & \longrightarrow & \overbrace{\mathrm{H}_n(B, A\cap B)}^{=\mathrm{H}_n(X\setminus Z, A\setminus Z)} & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A) & \longrightarrow & \mathrm{H}_{n-1}^{\{A,B\}}(X) \longrightarrow \ldots \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
\ldots & \longrightarrow & \mathrm{H}_n(A) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n(X, A) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A) & \longrightarrow & \mathrm{H}_{n-1}(X) \longrightarrow \ldots
\end{array}$$

From the Comfortable Covering Proposition 40.7 it follows that the second and fifth vertical map are isomorphisms. Since the first and fourth map are evidently isomorphisms it now follows from the Five Lemma 37.5 that the middle map is also an isomorphism. ■

41. Applications of the Excision Theorem

In the previous chapter we formulated and proved the Excision Theorem 40.1. The proof turned out to be long and difficult. But in this chapter we will see that this effort really pays off. For example we will use the Excision Theorem 40.1 to compute homology groups of suspensions and wedges of topological spaces.

We will also return to the challenge, posed in Chapter 38, to calculate homology groups of quotient spaces X/A. Using the Excision Theorem 40.1 we will be able, in reasonable settings, to relate the homology groups of A, X and X/A via a long exact sequence.

41.1. **Suspensions and homology groups.** Let us recall the following definitions from pages 252.

Definition. Let X be a non-empty topological space.

(1) We define the suspension of X to be the topological space

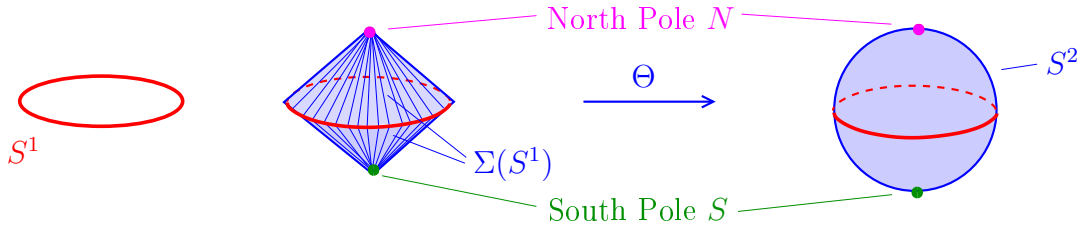
$$\Sigma(X) \;:=\; ((X\times[-1,1])/(X\times\{-1\}))/(X\times\{1\}).$$

(2) We call $S := [X\times\{-1\}]$ the South Pole and we call $N := [X\times\{1\}]$ the North Pole of $\Sigma(X)$.[312]

Example. One can easily verify that the map

$$\begin{array}{rcl}\Theta\colon \Sigma(S^n) & \xrightarrow{\cong} & S^{n+1}\subset\mathbb{R}^{n+1}\times\mathbb{R}\\ [(P,t)] & \mapsto & \big(\underbrace{P\cdot\cos\left(\frac{\pi}{2}t\right)}_{\in\mathbb{R}^{n+1}},\underbrace{\sin\left(\frac{\pi}{2}t\right)}_{\in\mathbb{R}}\big)\end{array}$$

is a homeomorphism.



In the Suspension-π_1-Lemma 20.2 we calculated the fundamental group of the suspension of a non-empty topological space with finitely many path components. In our next proposition we will determine the homology groups of a suspension. To formulate the proposition we need to introduce the following definition.

Definition.

(1) We set

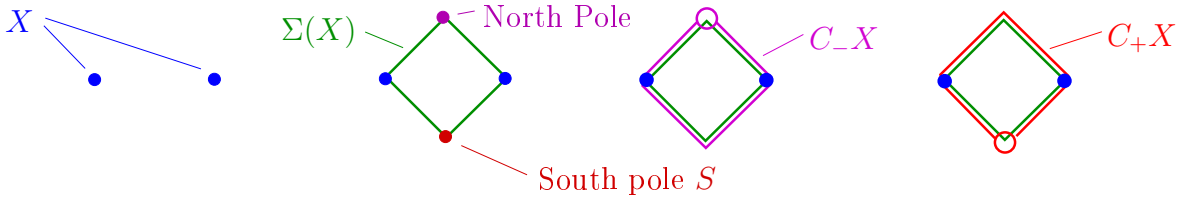
$$C_-X \;:=\; \Sigma(X)\setminus\{N\} \qquad\text{and}\qquad C_+X \;:=\; \setminus\{S\}.$$

(2) Given a continuous map $f\colon X\to Y$ between topological spaces we refer to

$$\begin{array}{rcl}\Sigma(f)\colon \Sigma(X) & \to & \Sigma(Y)\\ [(x,t)] & \mapsto & [(f(x),t)]\end{array}$$

[312] If $X=\varnothing$, then $\Sigma(X)$ consists of precisely two points, which we also call North and South Pole.

as the suspension of f.



Proposition 41.1. (Suspension-H$_*$-Proposition) Let X be a topological space and let $n \in \mathbb{Z}$.[313]

(1) We write $I := (-1,1)$. The following maps are isomorphisms:[314]

$$\widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) \xrightarrow{\cong} \mathrm{H}_{n+1}(\Sigma(X), C_-X) \xleftarrow{\text{induced by the inclusion}} \mathrm{H}_{n+1}(C_+X, X\times I) \xrightarrow{\partial_{n+1}} \widetilde{\mathrm{H}}_n(X\times I) \xleftarrow{\text{induced by the embedding } x\mapsto(x,0)} \widetilde{\mathrm{H}}_n(X).$$

the natural map defined on page 461

connecting homomorphism of the pair $(C_+X, X\times I)$ from the Reduced LES-of-Pair Proposition 39.6

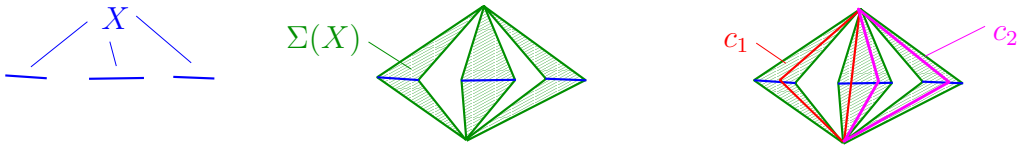
(2) The resulting isomorphism

$$\Sigma_X \colon \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) \xrightarrow{\cong} \widetilde{\mathrm{H}}_n(X)$$

is natural. In other words, given any continuous map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) & \xrightarrow{\Sigma_X} & \widetilde{\mathrm{H}}_n(X) \\ \downarrow{\scriptstyle \Sigma(f)_*} & & {\scriptstyle f_*}\downarrow \\ \widetilde{\mathrm{H}}_{n+1}(\Sigma(Y)) & \xrightarrow{\Sigma_Y} & \widetilde{\mathrm{H}}_n(Y). \end{array}$$

Example. Let $n \in \mathbb{N}$ and let X be the disjoint union of $n \in \mathbb{N}$ intervals. It follows from the Suspension-H$_*$-Proposition 41.1 together with the Reduced Homology Lemma 39.2 that we have isomorphisms $\mathrm{H}_1(\Sigma(X)) \cong \widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}^{n-1}$. In the figure below we show a generating set for $\mathrm{H}_1(\Sigma(X))$ for $n = 3$. We leave it to the reader to deduce from the proof of the Suspension-H$_*$-Proposition 41.1 that the given cycles do indeed form a generating set for $\mathrm{H}_1(\Sigma(X))$.



$\{c_1, c_2\}$ is a basis for $\mathrm{H}_1(\Sigma(X))$

Proof. [315]

[314] One can easily show that the maps $X\times(-1,1) \to C_\pm$ and $X\times(-1,1) \to \Sigma(X)$ given by $(x,t) \mapsto [(x,t)]$ are embeddings. We use this embedding to view $X\times(-1,1)$ as a subspace of $C_\pm X$ and of $\Sigma(X)$.

[315] The proof of the Suspension-H$_*$-Proposition 41.1 is very similar to the calculation of the homology groups of spheres on page 465.

(1) Let X be a topological space. For each $m \in \mathbb{N}_0$ we consider the following homomorphisms:

$$\begin{array}{c}
\longrightarrow \widetilde{\mathrm{H}}_{n+1}(C_-X) \to \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) \xrightarrow{\cong} \mathrm{H}_{n+1}(\Sigma(X), C_-X) \xrightarrow{\partial_{n+1}} \widetilde{\mathrm{H}}_n(C_-X) \longrightarrow \dots \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \cong \\
\dots \to \widetilde{\mathrm{H}}_{n+1}(X \times I) \to \widetilde{\mathrm{H}}_{n+1}(C_+X) \to \mathrm{H}_{n+1}(C_+X, X \times I) \xrightarrow[\cong]{\partial_{n+1}} \widetilde{\mathrm{H}}_n(X \times I) \to \widetilde{\mathrm{H}}_n(C_+X) \to \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad {\scriptstyle i_*}\uparrow \cong \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \widetilde{\mathrm{H}}_n(X)
\end{array}$$

We make the following clarifications and explanations:

(a) The top horizontal sequence is the long exact sequence in reduced homology of the pair $(\Sigma(X), C_-X)$ that is given by the Reduced LES-of-Pair Proposition 39.6.

(b) The bottom horizontal sequence is the long exact sequence in reduced homology of the pair $(C_+X, X \times I)$ that is given by the Reduced LES-of-Pair Proposition 39.6.

(c) As is explained in [Fri23], both C_-X and C_+X admit a deformation retraction to a point. It follows from the Deformation Retract–Homotopy Equivalence Lemma 9.3 together with the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that for all $k \in \mathbb{N}_0$ we have $\widetilde{\mathrm{H}}_k(C_-X) = 0$ and that we have $\widetilde{\mathrm{H}}_k(C_+X) = 0$.

(d) It follows from (c) and the exactness of the horizontal sequences that the horizontal maps shown in red are isomorphisms.

(e) As before let $S = [X \times \{-1\}]$ be the South Pole of the suspension. Note that $\Sigma(X) \setminus \{S\} = C_+X$ and $C_-X \setminus \{S\} = C_-X \cap C_+X = X \times I$. Using this observation we obtain immediately from the Excision Theorem 40.1[316] that the vertical map $\mathrm{H}_{n+1}(C_+X, X \times I) \to \mathrm{H}_{n+1}(\Sigma(X), C_-X)$ is an isomorphism.

(f) We denote by $i\colon X \to X \times I$ the natural embedding given by $x \mapsto (x, 0)$. Since I is an interval one can easily show, using the Deformation Retract–Homotopy Equivalence Lemma 9.3, that i is a homotopy equivalence.

(g) It follows from (f) together with the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 39.4 that the induced map $i_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(X \times I)$ is an isomorphism.

We have now shown that all red homomorphisms are isomorphisms. But that is exactly what he had to do.

(2) Let $f\colon X \to Y$ be a continuous map between two topological spaces. We consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & & \overset{\Sigma_X}{\frown} & & & & \\
\widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) & \xrightarrow{\text{ж}} & \mathrm{H}_{n+1}(\Sigma(X), C_-X) & \longleftarrow & \mathrm{H}_{n+1}(C_+X, X \times I) & \xrightarrow{\partial_{n+1}} & \widetilde{\mathrm{H}}_n(X \times I) & \xleftarrow{i_*} & \widetilde{\mathrm{H}}_n(X) \\
\downarrow{\scriptstyle \Sigma(f)_*} & & \downarrow{\scriptstyle \Sigma(f)_*} & & \downarrow{\scriptstyle (f \times \mathrm{id})_*} & & \downarrow{\scriptstyle (f \times \mathrm{id})_*} & & \downarrow{\scriptstyle f_*} \\
\widetilde{\mathrm{H}}_{n+1}(\Sigma(Y)) & \xrightarrow{\text{ж}} & \mathrm{H}_{n+1}(\Sigma(Y), C_-Y) & \longleftarrow & \mathrm{H}_{n+1}(C_+Y, Y \times I) & \xrightarrow{\partial_{n+1}} & \widetilde{\mathrm{H}}_n(Y \times I) & \xleftarrow{i_*} & \widetilde{\mathrm{H}}_n(Y). \\
 & & & & \underset{\Sigma_Y}{\smile} & & & &
\end{array}$$

[316] One can easily verify that the hypotheses for the Excision Theorem 40.1 are satisfied.

The first, second and fourth square commutes by functoriality of (reduced) homology. The third square commutes since in the LES-of-Triple Proposition 38.2 (2) we showed that the connecting homomorphism is natural. We have thus shown that the diagram commutes, which implies that Σ_X itself is natural. ■

41.2. **The homology groups of spheres.** Using the Suspension-H_*-Proposition 41.1 we can now redo the calculation of the homology groups of spheres which we first did on page 465.

Proposition 41.2. (Sphere-H_n-Proposition) For any $n \in \mathbb{Z}_{\geq -1}$ we have[317]

$$\widetilde{\mathrm{H}}_k(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Proof. For each $m \in \mathbb{Z}_{\geq -1}$ we have

$$\widetilde{\mathrm{H}}_m(S^n) \underset{\text{since } S^n \text{ is homeomorphic to the suspension } \Sigma(S^{n-1})}{\cong} \widetilde{\mathrm{H}}_m(\Sigma(S^{n-1})) \underset{\text{by the Suspension-H}_*\text{-Proposition 41.1}}{\cong} \widetilde{\mathrm{H}}_{m-1}(S^{n-1}) \underset{\text{iterating the argument}}{\cong} \ldots \cong \widetilde{\mathrm{H}}_{m-n}(S^0) \underset{\text{by the Singleton-H}_*\text{-Lemma 35.11, the Path-Component–H}_*\text{–Lemma 35.8 and the Reduced Homology Lemma 39.2}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } m-n=0, \\ 0, & \text{otherwise.} \end{cases}$$

■

Proof. This corollary follows immediately from the calculation of the (reduced) homology groups of spheres in the Sphere-H_n-Proposition 41.2 together with the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 39.4. ■

In the $\mathbb{R}^2$-$\mathbb{R}^3$-Non Homeomorphism–Lemma 13.5 we used the fundamental group to prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \geq 3$. Now we can finally show that for any $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic.

Proposition 41.3. ($\mathbb{R}^k$-$\mathbb{R}^l$-Non Homeomorphism–Proposition) For any $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic.

Proof. Let $k, l \in \mathbb{N}_0$ and suppose there exists a homeomorphism $f\colon \mathbb{R}^k \to \mathbb{R}^l$. Then we obtain that

$$\mathbb{Z} \underset{\text{by the Sphere-H}_n\text{-Proposition 41.2}}{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{k-1}) \underset{\text{this follows from the Homotopy Equivalence-}\widetilde{\mathrm{H}}_*\text{-Corollary 39.4 (1a), since for any } P \in \mathbb{R}^m \text{ the topological space } \mathbb{R}^m \setminus \{P\} \text{ is homotopy equivalent to } S^{m-1}}{\cong} \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^k \setminus \{0\}) \underset{\text{isomorphism since } f \text{ is a homeomorphism}}{\xrightarrow{f_*}} \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^l \setminus \{f(0)\}) \cong \widetilde{\mathrm{H}}_{k-1}(S^{l-1}).$$

So we have $\widetilde{\mathrm{H}}_{k-1}(S^{l-1}) \cong \mathbb{Z}$. But by the Sphere-$\mathrm{H}_n$-Proposition 41.2 that is only possible if $k = l$. ■

We conclude this section with a calculation of a relative homology group, which we will frequently use.

[317] For ordinary homology recall that by the Reduced Homology Lemma 39.2 we know that for $n \geq 0$ we have $\mathrm{H}_0(S^n) \cong \widetilde{\mathrm{H}}_0(S^n) \oplus \mathbb{Z}$ and $\mathrm{H}_k(S^n) \cong \widetilde{\mathrm{H}}_k(S^n)$ for $k > 0$.

Lemma 41.4. (Ball-rel-Sphere-H_*-Lemma) Given any $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\overline{B}^n, S^{n-1}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We consider the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$ which is provided for us by the Reduced LES-of-Pair Proposition 39.6:

$$\ldots \to \underbrace{\widetilde{\mathrm{H}}_k(\overline{B}^n)}_{=0,\text{ see page 461}} \to \mathrm{H}_k(\overline{B}^n, S^{n-1}) \xrightarrow{\partial} \underbrace{\widetilde{\mathrm{H}}_{k-1}(S^{n-1})}_{\substack{\text{calculated in the} \\ \text{Sphere-H}_n\text{-Proposition 41.2}}} \to \underbrace{\widetilde{\mathrm{H}}_{k-1}(\overline{B}^n)}_{=0,\text{ see page 461}} \to \ldots$$

The statement of the lemma is now an immediate consequence of the Sphere-H_n-Proposition 41.2. ∎

41.3. **The Brouwer Fixed Point Theorem.** In this section we discuss a major application of the calculation of the homology groups of spheres, namely the Brouwer Fixed Point Theorem.

For the reader's convenience we recall the following definition from page 142.

Definition. Let X be a topological space. We say a subset $A \subset X$ is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a continuous map with $r(a) = a$ for all $a \in A$.

In the Circle–Non Retract Proposition 13.3 we used the fundamental group to show that S^1 is not a retract of $\overline{B}^2$. Precisely the same argument, together with the Sphere-H_n-Proposition 41.2, now gives us the following high-dimensional generalization.

Proposition 41.5. (Sphere-Non Retract Proposition) For every $n \in \mathbb{N}$ the sphere $S^{n-1} = \partial\overline{B}^n$ is not a retract of the closed ball $\overline{B}^n$.

Remark. I found the following amusing description of the Sphere-Non Retract Proposition 41.5 online:
"Suppose a shark jumps into a shoal of fish (a kind of big ball). The small fishes start escaping in all directions towards the border of the shoal, where the fishes stand still. Yet they escape with a certain disposition to follow a continuous flow, as they usually do, since everybody tends to follow its neighborhoods. But since there is no continuous retraction to the boundary, somebody doesn't know where to go, and stay there for a moment, much to the shark's satisfaction. There is also a 2D version, with a wolf entering into a herd of sheep."

Proof. Suppose there exists a retraction $r\colon \overline{B}^n \to S^{n-1}$. We denote by $i\colon S^{n-1} \to \overline{B}^n$ the inclusion map. We consider the following diagram

$$\begin{array}{ccc} & \widetilde{\mathrm{H}}_{n-1}(\overline{B}^n) & \\ {\scriptstyle i_*}\nearrow & & \searrow{\scriptstyle r_*} \\ \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) & \xrightarrow{\quad (r\circ i)_* \quad} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}). \end{array}$$

This diagram commutes since it follows from the functoriality of the reduced homology groups that $r_* \circ i_* = (r \circ i)_*$. Since r is a retraction it follows that $r \circ i = \mathrm{id}_{S^{n-1}}$, i.e. $(r \circ i)_* = (\mathrm{id}_{S^{n-1}})_*$ is the identity map on $\widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ which, by the Sphere-H_n-Proposition 41.2, is isomorphic to $\mathbb{Z}$. The lower map is thus an isomorphism. But the upper

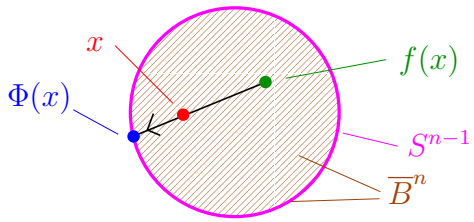
map factorizes through the group $\widetilde{\mathrm{H}}_{n-1}(\overline{B}^n)$, which is the trivial group by the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5. This implies that the composition of i_* and r_* cannot be an isomorphism of $\mathbb{Z}$. Thus we have obtained a contradiction. ■

Now we can formulate and prove the famous Brouwer Fixed Point Theorem 41.6.

Theorem 41.6. (Brouwer Fixed Point Theorem) Every continuous map $f\colon \overline{B}^n \to \overline{B}^n$ admits a fixed point, i.e. for every continuous map $f\colon \overline{B}^n \to \overline{B}^n$ there exists a point $x \in \overline{B}^n$ such that $f(x) = x$.

Proof. Suppose there exists a continuous map $f\colon \overline{B}^n \to \overline{B}^n$ without a fixed point. It follows from the hypothesis that f admits no fixed points that for each $x \in \overline{B}^n$ the open ray $\{f(x) + t \cdot (x - f(x)) \,|\, t > 0\}$ which starts at $f(x)$ and which goes through x contains precisely one point in S^{n-1}. Thus we can consider the following map:

$$\begin{array}{rcl} \Phi\colon \overline{B}^n & \to & \overline{B}^n \\ x & \mapsto & \text{the intersection point of } S^{n-1} \text{ with the} \\ & & \text{uniquely determined open ray from } f(x) \text{ to } x. \end{array}$$



It is elementary to show that Φ is continuous. It is evident that $\Phi(x) = x$ for all $x \in S^{n-1}$ and that $\Phi(x) \in S^{n-1}$ for all $x \in S^{n-1}$. The map Φ is thus a retraction from $\overline{B}^n$ to S^{n-1}. But according to the Sphere-Non Retract Proposition 41.5 this not possible. Thus we have obtained a contradiction. ■

The Brouwer Fixed Point Theorem pops up many times in mathematics. For example we can use it to prove the following theorem in linear algebra.

Theorem 41.7. (Perron-Frobenius Theorem) Let $A = (a_{ij})$ be an $n \times n$-real matrix. If all entries are positive, then A has a positive eigenvalue λ with an eigenvector v such that all entries of v are non-negative.

Proof. We need to introduce two players:

(1) Given $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ we write $|v| := \sum_{i=1}^{n} |v_i|$.

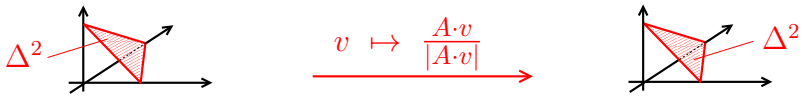
(2) We consider the standard $(n-1)$-simplex

$$\Delta^{n-1} := \{v \in [0,1]^n \,\big|\, |v| = 1\}.$$

We saw in the Δ^n-Homeomorphism Lemma 35.3 (2) that Δ^{n-1} is homeomorphic to $\overline{B}^{n-1}$. It follows immediately from this observation and from the Brouwer Fixed Point Theorem 41.6 that any continuous self-map $f\colon \Delta^{n-1} \to \Delta^{n-1}$ has a fixed point.

Now let $A = (a_{ij})$ be an $n \times n$-real matrix such that all entries are positive. We consider the map

$$\begin{array}{rcl} f\colon \Delta^{n-1} & \to & \Delta^{n-1} \\ v = (v_1, \dots, v_n) & \mapsto & \frac{A\cdot v}{|A\cdot v|}. \end{array}$$



Note that this map is defined since our hypothesis that all a_{ij} are positive, the fact that all entries of v are non-negative and the fact that at least one entry of v is positive imply that $A \cdot v \neq 0$. It is clear that f is continuous. As we had just discussed above, the map $f\colon \Delta^{n-1} \to \Delta^{n-1}$ has a fixed point $v \in \Delta^{n-1}$. But this means that $A \cdot v = |A \cdot v| \cdot v$. Put differently, we have just shown that v is an eigenvector to the positive eigenvalue $|A \cdot v|$. ■

41.4. **Good subsets and relative homology.** A very long time ago, namely in Chapter 38, we started out with our attempt to calculate homology groups of quotients X/A. We developed the concept of relative chain complexes $\mathrm{C}_n(X/A) := \mathrm{C}_n(X)/\mathrm{C}_n(A)$ which, as we saw, turned out to be very interesting in its own right.

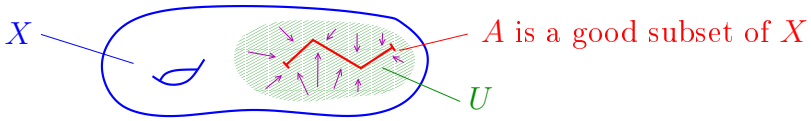
In this section we return to the original question and we will try to determine homology groups of quotients. There is now an obvious idea: we suspect that in many cases the relative homology groups $\mathrm{H}_n(X, A)$ should be isomorphic to the homology groups of the question X/A. In this section we will see that this hunch is basically correct, if A is a "good" subset of X. In the next section we will use this result to determine the homology groups of wedges of topological spaces.

Before we can state the main result of this section we need to introduce the following definition.

Definition. Let X be a topological space and let $A \subset X$ be a[318] subset. We denote by $p\colon (X, A) \to (X/A, A/A)$ the natural projection. For each $n \in \mathbb{N}_0$ we define

$$\mathrm{H}_n(X, A) \xrightarrow{p_*} \mathrm{H}_n(X/A, A/A) \xleftarrow[\cong]{\text{isomorphism ж from the Reduced-to-Relative-H}_*\text{-Lemma 39.7}} \widetilde{\mathrm{H}}_n(X/A)\,.$$
$$\text{=: ю}$$

Definition. Let X be a topological space and let $A \subset X$ be a subset. We say A is good if A is closed and if there exists an open neighborhood U of A in X such that A is a deformation retract of U.



Examples.

(1) Let $x \in X$ be a point. Basically by definition the subset $\{x\} \subset X$ is good in the above sense if and only the point is good in the sense of the definition on page 263.

[318] Note that the definition also makes sense if A is the empty set.

(2) The sphere S^{n-1} is a good subset of the closed ball $\overline{B}^n$. Indeed, it is clear that S^{n-1} is a closed subset of $\overline{B}^n$. Furthermore, if we consider the open neighborhood $U = \overline{B}^n \setminus \{0\}$, then a deformation retraction from U to S^{n-1} is given by the map

$$\begin{array}{rcl} (\overline{B}^n \setminus \{0\}) \times [0,1] & \to & \overline{B}^n \setminus \{0\} \\ (x,t) & \mapsto & \left(1 - t \cdot \frac{1-\|x\|}{\|x\|}\right) \cdot x. \end{array}$$

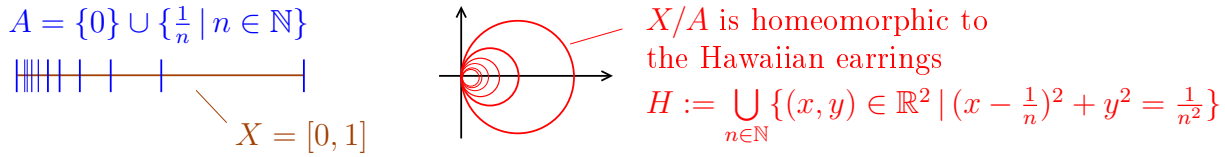
(3) It follows from the Subcomplex–Closed Lemma 30.6 and the Subcomplex–Neighborhood Proposition 31.8 (1) that every subcomplex A of a CW-complex X is good.

Proposition 41.8. (Relative-to-Quotient-H_*-Proposition) Let X be a topological space, let $A \subset X$ be a subset and let $n \in \mathbb{N}_0$. If A is a good subset, then the natural homomorphism $\imath\mathrm{o}\colon \mathrm{H}_n(X,A) \to \widetilde{\mathrm{H}}_n(X/A)$ is an isomorphism.

Example. Let X be a topological space, let $A \subset X$ be a subset and let $n \in \mathbb{N}_0$. For better or worse the natural homomorphism $\mathrm{H}_n(X,A) \to \widetilde{\mathrm{H}}_n(X/A)$ is in general not an isomorphism. In fact in general the two groups $\mathrm{H}_n(X,A)$ and $\widetilde{\mathrm{H}}_n(X/A)$ are not even abstractly isomorphic.

For example let us consider the topological space $X = [0,1]$ together with the subspace $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$. We leave the proof of the following two statements as an exercise to the reader.

(1) The group $\mathrm{H}_1(X,A)$ is countable.
(2) The group $\mathrm{H}_1(X/A)$ is uncountable.



The proof of the Relative-to-Quotient-H_*-Proposition 41.8 relies on the following elementary little lemma.

Auxiliary Lemma 41.9. Let X be a topological space and let A be a closed subset of X.
(1) For every open subset $U \subset X \setminus A$ the obvious map $U \to X/A$ is an open embedding.
(2) For every open neighborhood W of A the obvious map $W \setminus A \to (W/A) \setminus (A/A)$ is a homeomorphism.

Proof of Lemma 41.9. Let X be a topological space and let A be a closed subset of X. We denote by $p\colon X \to X/A$ the natural projection.

(1) Let $U \subset X \setminus A$ be an open subset. We need to show that the map $q\colon U \to X \to X/A$ is an open embedding. It follows from $U \subset X \setminus A$ that q is an injection. Furthermore note that by the Topological-Quotient Proposition 6.1 (1a) we know that q is continuous. By the Open-Injective Lemma 2.9 (2) it remains to show that $q\colon U \to X/A$ is an open

map. Now let $V \subset U$ be an open subset. We need to show that $q(V) \subset X/A$ is open. We break the argument into three easy steps:

(a) It follows from the Open-Closed Inclusion Lemma 2.7 (1) and the fact that A is by hypothesis closed, that V is also an open subset of X.

(b) Since $A \cap V = \varnothing$ we see that $p^{-1}(p(V)) = V$, which by (a) is in fact an open subset of X.

(c) It follows from (b) and the definition of the quotient topology that $p(V)$ is an open subset of X/A.

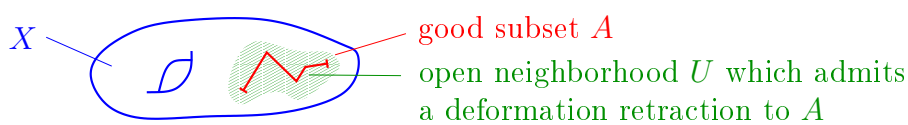
(2) Let W be an open neighborhood of A. We set $U := W \setminus A$. Since A is closed we see that U is an open subset of $W \setminus A$. Recall that we want to show that the obvious map $q\colon W \setminus A \to (W/A) \setminus (A/A)$ is a homeomorphism. It is clear that q is a bijection and it follows from the Twice Quotient Lemma 6.3 (2) that q is continuous. By the Open-Injective Lemma 2.9 (2) it remains to show that $q\colon W \setminus A \to (W/A) \setminus (A/A)$ is an open map. Thus let V be an open subset of $W \setminus A$. We have $q(V) = p(V) \cap (W/A \setminus A/A)$. It follows from (1c) and the definition of the subspace topology that $q(V)$ is an open subset of $(W/A) \setminus (A/A)$. ■

Proof of the Relative-to-Quotient-H$_*$-Proposition 41.8. Let X be a topological space, let $A \subset X$ be a good subset and let $n \in \mathbb{N}_0$. We make the following simple preparations:

(1) Since A is good we can pick an open neighborhood U of A in X such that A is a deformation retract of U.

(2) We denote by $p\colon X \to X/A$ the projection.

(3) We denote by $q\colon X \setminus A \to (X/A) \setminus (A/A)$ the restriction of p to $X \setminus A$. For completeness' sake we point out that it follows from the Basics-of-Continuity Lemma 2.1 that q is continuous.



Now we consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_n(X,A) \arrow[r, "(3)"] \arrow[d, "p_*"] \arrow[dd, bend right=60, "\text{ю}"'] & \mathrm{H}_n(X,U) \arrow[d, "p_*"] & \mathrm{H}_n(X \setminus A, U \setminus A) \arrow[l, "(1)"'] \arrow[d, "q_*"] \\
\mathrm{H}_n(X/A, A/A) \arrow[r, "(4)"'] & \mathrm{H}_n(X/A, U/A) & \mathrm{H}_n((X/A) \setminus (A/A), (U/A) \setminus (A/A)) \arrow[l, "(2)"] \\
\widetilde{\mathrm{H}}_n(X/A), \arrow[u, "\cong", "\text{ж}"'] & &
\end{tikzcd}$$

where all the horizontal maps are induced by inclusions of pairs of topological spaces. Recall that we need to show that $\text{ю}\colon \mathrm{H}_n(X,A) \to \widetilde{\mathrm{H}}_n(X/A)$ is an isomorphism. By the Reduced-to-Relative-H$_*$-Lemma 39.7 we know that $\text{ж}\colon \widetilde{\mathrm{H}}_n(X/A) \to \mathrm{H}_n(X/A, A/A)$ is an isomorphism. Thus it suffices to show that the top left vertical map $p_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(X/A, A/A)$ is an isomorphism.

First note that it follows easily from the functoriality of relative homology groups that the two squares of the diagram commute. Next note that it follows from our hypothesis

that the subset A is closed, together with the above auxiliary Lemma 41.9 (2), that the map $q\colon (X \setminus A, U \setminus A) \to ((X/A) \setminus (A/A), (U/A) \setminus (A/A))$ is a homeomorphism of pairs of topological spaces. We obtain from this observation that the vertical map on the right is an isomorphism. At this stage it remains to prove the following claim:

Claim. All horizontal maps are isomorphisms.

Proof.

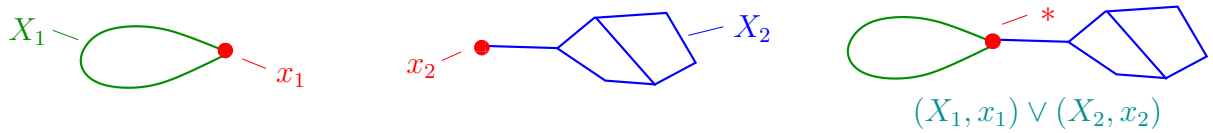
(1) Since $A \subset X$ is closed and since $U \subset X$ is open it follows from the Excision Theorem 40.1 that both the top right horizontal map is an isomorphism.
(2) The hypothesis that $A \subset X$ is closed and that $U \subset X$ is an open neighborhood of $A \subset X$ implies, by definition of the quotient topology, that A/A is a closed subset of X/A and that U/A is an open neighborhood of $A/A \subset X/A$. Thus we can also apply the Excision Theorem 40.1 to the horizontal map on the bottom right, and we see that this map is also an isomorphism.
(3) Since A is deformation retract of U we obtain from the Homotopy Equivalence-Relative-H_*-Corollary 38.4 that the top left horizontal map is an isomorphism.
(4) Recall that A is a deformation retract of U. It follows from the Deformation Retraction-Quotient Theorem 9.2 (2) that A/A is a deformation retract of U/A. As in (3) we now see that the bottom-left horizontal map is an isomorphism. ■

41.5. **The homology groups of the wedge of topological spaces.** We recall the following definition from page 219.

Definition. Let $\{(X_k, x_k)\}_{k\in K}$ be a non-empty family of pointed topological spaces. We refer to

$$\bigvee_{k\in K} (X_k, x_k) \;:=\; \Big(\bigsqcup_{k\in K} X_k\Big)/\sim \qquad \text{where } x_k \sim x_l \text{ for all } k, l \in K$$

as the wedge of the pointed topological spaces (X_k, x_k).. The wedge of the empty family is defined as the singleton $\{*\}$.



Recall that in the Wedge Proposition 17.1 we showed that the wedge $\bigvee_{k\in K} (X_k, x_k)$ is the coproduct in the category $\mathcal{PTop}$ of *pointed* topological spaces. Also recall that we showed in the Coproduct-of-Sets-and-Abelian Groups Lemma 4.4 that in the category $\mathcal{AbGroup}$ of abelian groups the coproduct is given by the direct sum. It is again natural to ask which functors from the category $\mathcal{PTop}$ of pointed topological groups to the category $\mathcal{AbGroup}$ of abelian groups preserve coproducts. One can see in very easy examples that ordinary homology groups do not quite preserve coproducts. But we have alternative functors, namely for each $n \in \mathbb{N}_0$ we can consider the functor

$$\begin{array}{rcl} \mathcal{PTop} & \to & \mathcal{AbGroup} \\ (X, x_0) & \mapsto & \mathrm{H}_n(X, \{x_0\}) \\ (f\colon (X, x_0) \to (Y, y_0)) & \mapsto & (f_*\colon \mathrm{H}_n(X, \{x_0\}) \to \mathrm{H}_n(Y, \{y_0\})). \end{array}$$

The next shows that these relative homology functors preserve coproducts (at least under rather mild hypotheses on the points $x_k \in X_k$).

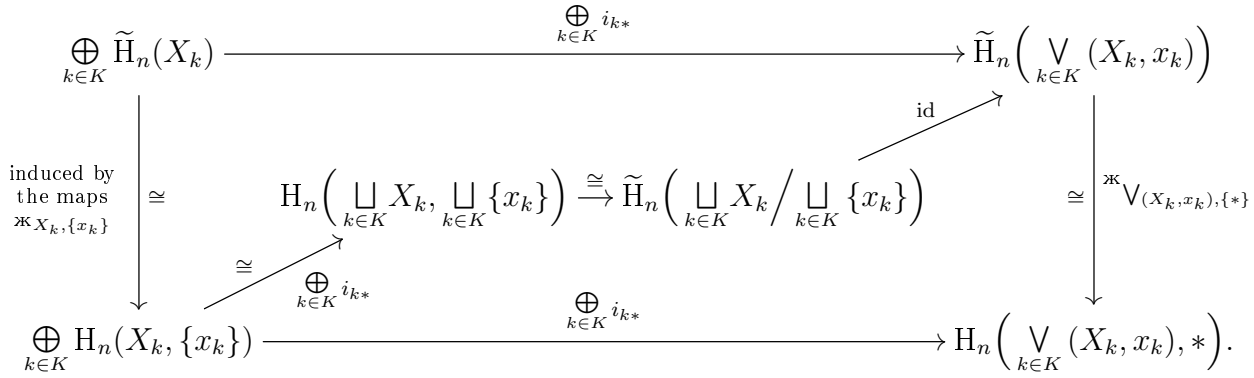
Proposition 41.10. (Wedge-H_*-Proposition) Let $\{(X_k, x_k)\}_{k\in K}$ be a family of pointed topological spaces. We denote the wedge point of $\bigvee_{k\in K}(X_k, x_k)$ by $*$ and given $k \in K$ we denote by $i_k \colon X_k \to \bigvee_{k\in K}(X_k, x_k)$ the natural inclusion that we introduced on page 219. Let $n \in \mathbb{N}_0$. If each $x_k \in X_k$ is good[319], then the map

$$\bigoplus_{k\in K} i_{k*} \colon \bigoplus_{k\in K} \mathrm{H}_n(X_k, \{x_k\}) \to \mathrm{H}_n\Big(\bigvee_{k\in K}(X_k, x_k), \{*\}\Big)$$

is an isomorphism. The same statement also holds for reduced homology instead of relative homology.

Remark. The above proposition can be viewed as an analogue of the Wedge-π_1-Proposition 20.6.

Proof. We consider the following diagram:



The proposition follows from the following clarifications and observations:

(a) It follows from the Reduced-to-Relative-H_*-Lemma 39.7 that the vertical maps to the left and right are isomorphisms.

(b) It follows from the obvious generalization of the Path-Component–H_*–Lemma 35.8 that the bottom left diagonal map is an isomorphism.

(c) By hypothesis each $x_k \in X_k$ is good. It follows easily from the definitions that $\bigsqcup_{k\in K}\{x_k\}$ is a good subset of $\bigsqcup_{k\in K} X_k$. Thus we can apply the Relative-to-Quotient-H_*-Proposition 41.8 to see that the middle horizontal map is an isomorphism.

(d) The top right diagonal map is indeed the identity, we just wrote the wedge of the (X_k, x_k) in two different ways.

(e) One can easily verify that the lower triangle commutes. This implies the desired statement for relative homology groups.

(f) One can also easily verify that the upper triangle commutes. Thus we also get the desired statement on reduced homology groups. ∎

[319] Recall that by the Good Point-Lemma 20.5 (1) we know that any point on a topological manifold is good and that it follows from the CW-Complex-Hausdorff Proposition 30.3 and the CW-Complex–Local Properties Proposition 31.7 (1) that any point on a CW-complex is good.

41.6. **The long exact sequence for quotients.** The following theorem gives us, at least in reasonable settings, the promised connection between the homology groups of a topological space X, a subspace $A \subset X$ and the quotient space X/A.

Theorem 41.11. (Quotient–LES Theorem) Let X be a topological space. Furthermore let $A \subset X$ be a good subset. We denote by $i\colon A \to X$ the inclusion map and we denote by $p\colon X \to X/A$ the projection. For each $n \in \mathbb{N}_0$ there exists a natural[320] homomorphism $\partial_n\colon \widetilde{\mathrm{H}}_n(X/A) \to \widetilde{\mathrm{H}}_{n-1}(A)$ such that the sequence

$$\begin{array}{rrrrrr} & & & \dots & \xrightarrow{p_*} \widetilde{\mathrm{H}}_{n+1}(X/A) & \xrightarrow{\partial_{n+1}} \\ \to & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_n(X/A) & \xrightarrow{\partial_n} \\ \to & \widetilde{\mathrm{H}}_{n-1}(A) & \xrightarrow{i_*} & \dots & & \end{array}$$

is exact.

Proof. Let X be a topological space and let $A \subset X$ be a good subset. We denote by $i\colon A \to X$ the inclusion and we denote by $p\colon X \to X/A$ the projection. We consider the following diagram:

$$\begin{array}{ccccccccccccc} \dots & \longrightarrow & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{\text{ж}_{X,A}} & \mathrm{H}_n(X,A) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_{n-1}(X) & \longrightarrow & \dots \\ & & & & & \searrow^{p_*} & \downarrow{\scriptstyle \text{ю}} & & & & & & \\ & & & & & & \widetilde{\mathrm{H}}_n(X/A). & & & & & & \end{array}$$

We make the following comments and observations:

(1) The horizontal sequence is the long exact sequence coming from the Reduced LES-of-Pair Proposition 39.6.
(2) The vertical map is the natural homomorphism that we defined on page 489. It follows easily from the definitions that the triangle commutes.
(3) Since A is good we obtain from the Relative-to-Quotient-H_*-Proposition 41.8 that the vertical map ю is an isomorphism.

Now we consider the homomorphisms $\widetilde{\partial}_n := \text{ю}^{-1} \circ \partial_n\colon \widetilde{\mathrm{H}}_n(X/A) \to \widetilde{\mathrm{H}}_{n-1}(A)$. Since ю and ∂ are natural we see that the maps $\widetilde{\partial}_n = \text{ю}^{-1} \circ \partial$ are also natural. It follows from the above discussion that the sequence

$$\dots \longrightarrow \widetilde{\mathrm{H}}_n(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{p_*} \widetilde{\mathrm{H}}_n(X/A) \xrightarrow{\widetilde{\partial}_n} \widetilde{\mathrm{H}}_{n-1}(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_{n-1}(X) \longrightarrow \dots$$

is exact. ■

Using the Quotient–LES Theorem 41.11 we can now give a somewhat shorter proof of the Sphere-H_n-Proposition 41.2.

[320] Here "natural" means the following: If $f\colon (X,A) \to (Y,B)$ is a continuous map between pairs of topological spaces and if A and B are good subsets, then the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_n(X/A) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(A) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \widetilde{\mathrm{H}}_n(Y/B) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(B). \end{array}$$

Proposition 41.2. (Sphere-H_n-Proposition) For each $n \in \mathbb{Z}_{\geq -1}$ we have

$$\widetilde{\mathrm{H}}_k(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Proof. We prove the proposition by induction on n. First note that for $n = -1$ we have $S^n = S^{-1} = \varnothing$, thus the statement follows immediately from the Reduced Homology Lemma 39.2 (0).[321]

Now we suppose that the statement of the proposition holds for $n-1$. We want to determine the reduced homology of S^n. The key idea is to use that by the Ball Quotient-Sphere Lemma 6.5 we know that S^n is homeomorphic to $\overline{B}^n/S^{n-1}$. On page 490 we saw that S^{n-1} is a good subset of $\overline{B}^n$. Therefore we can apply the Quotient–LES Theorem 41.11 to $X = \overline{B}^n$ and the closed subset $A = S^{n-1}$ and we obtain the following exact sequence

$$\begin{array}{rcccccc} \dots & \longrightarrow & \widetilde{\mathrm{H}}_{k+1}(\overline{B}^n) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_{k+1}(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial_{k+1}} & \\ \to \quad \widetilde{\mathrm{H}}_k(S^{n-1}) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_k(\overline{B}^n) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_k(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial_k} & \\ \to \quad \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) & \xrightarrow{i_*} & \dots & & & & \end{array}$$

By the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 we know that $\widetilde{\mathrm{H}}_i(\overline{B}^n) = 0$ for all i. Since the sequence is exact we see that all the connecting homomorphisms are isomorphisms. Thus we see that

$$\widetilde{\mathrm{H}}_k(S^n) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_k(\overline{B}^n/S^{n-1}) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } k-1 = n-1 \\ 0, & \text{if } k \neq n. \end{cases}$$

in the Ball Quotient-Sphere Lemma 6.5 we showed that $S^n \cong \overline{B}^n/S^{n-1}$

by the above exact sequence

by induction hypothesis

■

[321] To readers who prefer S^0 as a starting point we point out that S^0 consists of precisely two points. Thus the promised isomorphisms follow from the Singleton-H_*-Lemma 35.11, the Path-Component–H_*–Lemma 35.8 and the Reduced Homology Lemma 39.2.

42. Local homology and explicit generators of homology groups

By the Sphere-H_n-Proposition 41.2 we know that for any $n \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_k(S^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Thus we have determined the isomorphism types of the homology groups of all finite-dimensional spheres. In this chapter we will determine an explicit singular n-cycle in $\mathrm{C}_n(S^n)$ that represents a generator of $\widetilde{\mathrm{H}}_n(S^n)$. Along the way we introduce the concept of the local homology of a topological space X at a point x. This concept will turn out to be quite interesting in its own right.

42.1. **An explicit generator of $\mathbf{H}_n(\Delta^n, \partial\Delta^n)$.** As we mentioned above, in this chapter we eventually want to find explicit generators of $\widetilde{\mathrm{H}}_n(S^n) \cong \mathbb{Z}$ for $n \in \mathbb{N}_0$. We start out with the following lemma, which gives us a closely related statement. In fact the following lemma is the first instance where we can nail down a generator of an interesting homology group.

Lemma 42.1. (Identity-is-Generator Lemma) For every $n \in \mathbb{N}_0$ the following two statements hold:
(1) $\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.
(2) The identity map $\mathrm{id}\colon \Delta^n \to \Delta^n$ represents a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.

Proof. We prove the lemma by induction on $n \in \mathbb{N}_0$. It follows easily from the H_0-Proposition 35.10 (1) that the statement holds for $n = 0$. Now we suppose that the desired statement holds for $n-1$ with $n \in \mathbb{N}$.

Our next goal is to find a connection between the two homology groups $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$ and $\mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$. To do so we introduce the following notation.

(a) We consider

$$\Lambda \;:=\; \bigcup_{i=0}^{n-1} \{(t_0, \ldots, t_n) \in \Delta^n \,|\, t_i = 0\},$$

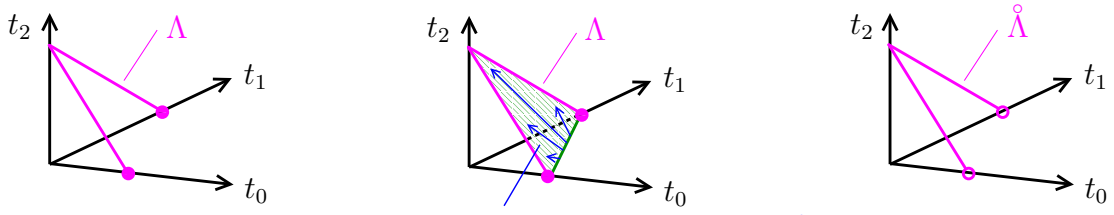
i.e. Λ is the union of all $(n-1)$-dimensional faces of Δ^n except for the face given by $t_n = 0$. Note that Λ is a deformation retract of Δ^n.

(b) We write

$$\mathring{\Lambda} \;:=\; \{(t_0, \ldots, t_n) \in \Lambda \,|\, t_n > 0\}.$$

The set $\mathring{\Lambda}$ can be viewed as the "interior" of Λ.

In the following we view $\mathbb{R}^n$ as the subspace of $\mathbb{R}^{n+1}$ given by setting the last coordinate equal to zero. This point of view allows us to view Δ^{n-1} as a subspace of Δ^n. Note that with this convention we have $\partial\Delta^{n-1} \subset \Lambda$.



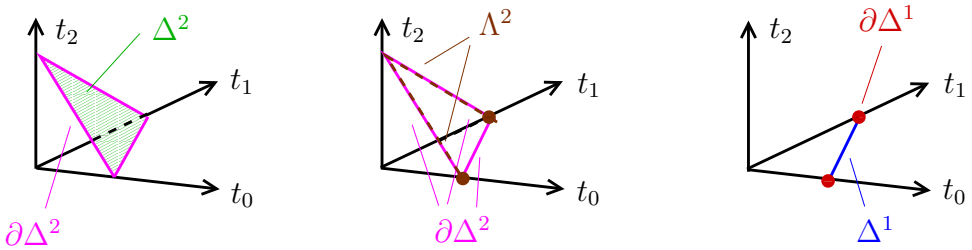
deformation retraction from Δ^2 to Λ

Claim. We consider the maps

$$\mathrm{H}_n(\Delta^n,\partial\Delta^n) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\partial\Delta^n,\Lambda) \xleftarrow{\iota_*} \mathrm{H}_{n-1}(\Delta^{n-1},\partial\Delta^{n-1}).$$

Here the map ∂_n on the left is the connecting homomorphism in the long exact sequence of the triple $(\Delta^n,\partial\Delta^n,\Lambda)$ and the right-hand map is induced by the inclusion. Both maps are isomorphisms and the following equality holds:

$$\partial([\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n\cdot\iota_*([\mathrm{id}_{\Delta^{n-1}}]).$$



It is clear that a proof of the claim together with the induction hypothesis implies that $\mathrm{H}_n(\Delta^n,\partial\Delta^n)\cong\mathbb{Z}$ and that $[\mathrm{id}_{\Delta^n}]$ is a generator of $\mathrm{H}_n(\Delta^n,\partial\Delta^n)$. In particular a proof of the claim implies the Identity-is-Generator Lemma 42.1.

Proof. We consider the two maps separately.

(a) Of the two maps in the claim we first consider the map ∂_n on the left. It follows from the LES-of-Triple Proposition 38.2 applied to the triple $(\Delta^n,\partial\Delta^n,\Lambda)$ of topological spaces that the following sequence is exact:

$$\mathrm{H}_n(\Delta^n,\Lambda) \xrightarrow{p_*} \mathrm{H}_n(\Delta^n,\partial\Delta^n) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\partial\Delta^n,\Lambda) \xrightarrow{\iota_*} \mathrm{H}_{n-1}(\Delta^n,\Lambda)$$

where $p\colon(\Delta^n,\Lambda)\to(\Delta^n,\partial\Delta^n)$ is the obvious inclusion. As we remarked above, one can easily show that Λ is a deformation retract of Δ^n. It follows from the Deformation Retract-Relative H_*-Corollary 38.5 that $\mathrm{H}_n(\Delta^n,\Lambda)=0$, which implies that the connecting homomorphism $\partial_n\colon\mathrm{H}_n(\Delta^n,\partial\Delta^n)\to\mathrm{H}_{n-1}(\partial\Delta^n,\Lambda)$ is an isomorphism. Furthermore note that we have the following equalities in $\mathrm{H}_n(\partial\Delta^n,\Lambda)$:

$$\partial_n([\mathrm{id}_{\Delta^n}]) \underset{\substack{\uparrow\\ \text{by definition of the connecting homomorphism in the LES-of-Triple Proposition 38.2}}}{=} [\partial_n(\mathrm{id}_{\Delta^n})] \overset{\substack{\downarrow\\ \text{see the definition of } \partial_n\colon \mathrm{C}_n(\Delta^n)\to\mathrm{C}_{n-1}(\Delta^n) \text{ on page 423}}}{=} \Big[\underbrace{\sum_{j=0}^{n-1}(-1)^j\cdot\mathrm{id}\circ i_j^n}_{\substack{\text{lies in } \mathrm{C}_{n-1}(\Lambda),\\ \text{thus } =0 \text{ in } \mathrm{C}_n(\partial\Delta^n,\Lambda)}} + (-1)^n\cdot\underbrace{\mathrm{id}\circ i_n^n}_{=\mathrm{id}_{\Delta^{n-1}}}\Big] \underset{\substack{\uparrow\\ \text{since we work in } \mathrm{H}_{n-1}(\partial\Delta^n,\Lambda)}}{=} (-1)^n\cdot[\mathrm{id}_{\Delta^{n-1}}].$$

(b) We turn our attention to the right-hand map $\iota_*\colon\mathrm{H}_{n-1}(\Delta^{n-1},\partial\Delta^{n-1})\to\mathrm{H}_{n-1}(\partial\Delta^n,\Lambda)$ in the statement of the claim. It follows from the Excision-via-Deformation Theorem 40.5, applied to $X=\partial\Delta^n$, $A=\Lambda$, $Z=\mathring{\Lambda}$ and $U=(0,\dots,0,1)$[322] that the inclusion induced map

$$\mathrm{H}_{n-1}(\Delta^{n-1},\partial\Delta^{n-1}) \;=\; \mathrm{H}_{n-1}(\underbrace{\partial\Delta^n\setminus\mathring{\Lambda}}_{=\Delta^{n-1}},\underbrace{\Lambda\setminus\mathring{\Lambda}}_{=\partial\Delta^{n-1}}) \xrightarrow{\iota_*} \mathrm{H}_{n-1}(\partial\Delta^n,\Lambda)$$

[322] One can easily show that U has the property required in the Excision-via-Deformation Theorem 40.5.

is an isomorphism. Finally note that evidently we have $\iota_*([\mathrm{id}_{\Delta^{n-1}}]) = [\mathrm{id}_{\Delta^{n-1}}]$.

Thus we have proved that both maps in the claim are isomorphisms and that

$$\partial([\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n \cdot [\mathrm{id}_{\Delta^{n-1}}] \;=\; (-1)^n \cdot \iota_*([\mathrm{id}_{\Delta^{n-1}}]).$$

■

42.2. **The local homology groups of a topological space.** We start out this section with the following definition which at first glance might look somewhat artificial.

Definition. Given a topological space X and a point $x \in X$ the relative homology group $\mathrm{H}_n(X, X \setminus \{x\})$ is called the n-th local homology group of X at the point x.

As we will see shortly, the concept of local homology is surprisingly useful. In many settings the following proposition allows us to calculate the local homology of a topological space at a point.

Proposition 42.2. (Local-H_*-Proposition) Let X be a topological space and let $x \in X$.

(1) If $\{x\} \subset X$ is a closed subset[323], then for any neighborhood U of $x \in X$ the inclusion induced maps $\mathrm{H}_n(U, U \setminus \{x\}) \to \mathrm{H}_n(X, X \setminus \{x\})$ are isomorphisms.[324]

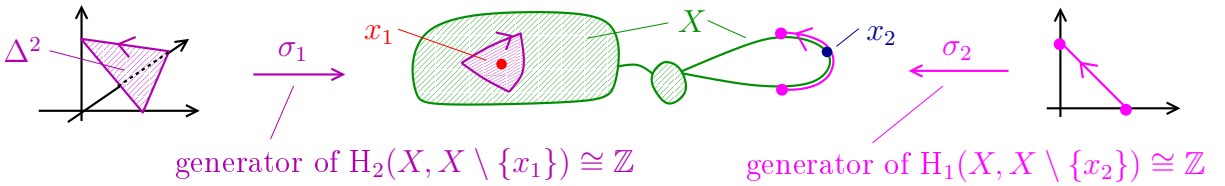
(2) For any homeomorphism $f\colon X \to Y$ the maps $f_*\colon \mathrm{H}_n(X, X \setminus \{x\}) \to \mathrm{H}_n(Y, Y \setminus \{f(x)\})$ are isomorphisms.

(3) (a) If X is an open subset of $\mathbb{R}^n$, then for any $x \in X$ and any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(X, X \setminus \{x\}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

(b) If X is an open subset of $\mathbb{H}^n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \,|\, t_n \geq 0\}$ and if $x \in \mathbb{R}^{n-1} \times \{0\}$, then $\mathrm{H}_k(X, X \setminus \{x\}) = 0$ for all $k \in \mathbb{N}_0$.

(4) If $\sigma\colon \Delta^n \to X$ is a continuous map such that $\sigma\colon \mathring{\Delta}^n \to X$ is an open embedding, then for any $x \in \sigma(\mathring{\Delta}^n)$ we have $\mathrm{H}_n(X, X \setminus \{x\}) \cong \mathbb{Z}$ and the singular n-simplex $\sigma\colon \Delta^n \to X$ represents a generator of $\mathrm{H}_n(X, X \setminus \{x\}) \cong \mathbb{Z}$.



Proof. Let X be a topological space and let $x \in X$.

(1) Let U be a neighborhood of $x \in X$. It follows immediately from the Excision Theorem 40.2 that the inclusion induced maps $\mathrm{H}_n(U, U \setminus \{x\}) \to \mathrm{H}_n(X, X \setminus \{x\})$ are isomorphisms.

(2) This statement follows from the functoriality of relative homology groups.

[323] Recall that by the Points-in-Hausdorff Lemma 1.8 we know that this condition is satisfied if X is Hausdorff, e.g. if X is a topological manifold.

[324] This statement hopefully justifies the name "local" homology.

(3) (a) It follows easily from (1) that it suffices to consider the case that $X = \mathbb{R}^n$. For any $k \in \mathbb{N}_0$ we see that

this is the connecting homomorphism of the long exact sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$ in reduced homology, it is an isomorphism since on page 461 we showed that $\widetilde{\mathrm{H}}_*(\mathbb{R}^n) = 0$

$$\mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \xrightarrow{\partial_k} \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^n \setminus \{x\}) \xleftarrow{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{n-1}(x)) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

isomorphism since the inclusion $S^{n-1}(x) \to \mathbb{R}^n \setminus \{x\}$ is a homotopy equivalence

Sphere-H_n-Proposition 41.2

(b) It follows easily from (1) that it suffices to consider the case that $X = \mathbb{H}^n$ and that $x \in \mathbb{R}^{n-1} \times \{0\}$. By the Reduced LES-of-Pair Proposition 39.6 we have a long exact sequence of the form

$$\ldots \to \widetilde{\mathrm{H}}_k(\mathbb{H}^n) \to \mathrm{H}_k(\mathbb{H}^n, \mathbb{H}^n \setminus \{x\}) \xrightarrow{\cong} \widetilde{\mathrm{H}}_k(\mathbb{H}^n \setminus \{x\}) \to \ldots$$

It follows easily from the fact that $x \in \mathbb{R}^{n-1} \times \{0\}$ that both $\mathbb{H}^n$ and $\mathbb{H}^n \setminus \{x\}$ are contractible. It follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that the reduced homology groups of $\mathbb{H}^n$ and $\mathbb{H}^n \setminus \{x\}$ vanish. But this implies, by exactness, that also the "middle" groups $\mathrm{H}_k(\mathbb{H}^n, \mathbb{H}^n \setminus \{x\})$ have to vanish.

(4) Let X be a topological space and let $\sigma\colon \Delta^n \to X$ be a continuous map such that $\sigma\colon \mathring{\Delta}^n \to X$ is an open embedding. Furthermore let $x \in \sigma(\mathring{\Delta}^n)$. Since $x \in \sigma(\mathring{\Delta}^n)$ and since $\sigma\colon \mathring{\Delta}^n \to X$ is an embedding there exists a unique $a \in \mathring{\Delta}^n$ with $\sigma(a) = x$.

Claim. There exists a continuous map $\tau\colon \Delta^n \to X$ with the following three properties:
(a) $\tau\colon \Delta^n \to X$ is an embedding.
(b) $\tau(\Delta^n)$ is a neighborhood of $x \in X$.
(c) We have $[\sigma] = [\tau] \in \mathrm{H}_n(X, X \setminus \{x\})$.

Proof. We introduce the following notation: We denote by $b := (\frac{1}{n+1}, \ldots, \frac{1}{n+1}) \in \Delta^n$ the barycenter of Δ^n. Furthermore given $\epsilon > 0$ we write

$$\Delta^n_\epsilon = \{(x - b) \cdot (1 - \epsilon) + b \,|\, x \in \Delta^n\}.$$

Note that $\Delta^n_\epsilon \subset \Delta^n$ and note that $\Delta^n_0 = \Delta^n$. It follows from an elementary argument and the hypothesis that $a \in \Delta^n \setminus \partial\Delta^n$ that there exists an $\epsilon > 0$ such that $a \in \Delta^n_\epsilon$. Next we consider the homotopy

$$\begin{aligned} F\colon \Delta^n \times [0,1] &\to \Delta^n \\ (x,t) &\mapsto (x - b) \cdot (1 - t + (1 - \epsilon) \cdot t) + b. \end{aligned}$$

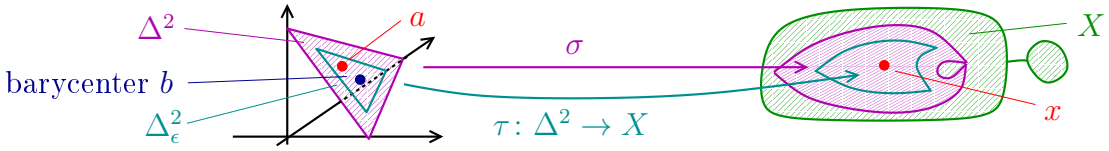
Note that $F_0 = \mathrm{id}_{\Delta^n}$ and note that F_1 defines a homeomorphism $\Delta^n \to \Delta^n_\epsilon$. We set $\tau := \sigma \circ F_1\colon \Delta^n \to X$. We have to verify that τ has the desired properties:
(a) Since $F_1\colon \Delta^n \to \Delta^n$ is an embedding with image Δ^n_ϵ, since $\Delta^n_\epsilon \subset \mathring{\Delta}^n$ and since $\sigma\colon \mathring{\Delta}^n \to X$ is an embedding we see that $\tau = \sigma \circ F_1\colon \Delta^n \to X$ is an embedding.
(b) It is straightforward to verify that Δ^n_ϵ is a neighborhood of $a \in \Delta^n$. By hypothesis $\sigma(\mathring{\Delta}^n)$ is an open subset of X. It follows from the Open-Closed Inclusion Lemma 2.7 that $\tau(\Delta^n) = \sigma(\Delta^n_\epsilon)$ is a neighborhood of $x = \sigma(a) \in X$.

(c) We have the following equalities in $\mathrm{H}_n(X, X\setminus\{x\})$:

$$[\sigma] \;=\; (\sigma\circ F_0)_*([\mathrm{id}_{\Delta^n}]) \;\underset{\uparrow}{=}\; (\sigma\circ F_1)_*([\mathrm{id}_{\Delta^n}]) \;=\; [\sigma\circ F_1\circ \mathrm{id}_{\Delta^n}] \;=\; [\tau].$$

follows from the Homotopy-H_*-Proposition 38.6 since $\sigma\circ F$ defines a homotopy of maps of pairs $(\Delta^n,\partial\Delta^n)\to(X, X\setminus\{x\})$

⊞



Finally we consider the following sequence of homomorphisms:

since $\tau(\Delta^n)$ is a neighborhood of x it follows from the Excision Theorem 40.2 that this inclusion induced map is an isomorphism

isomorphism since τ is an embedding

$\partial\Delta^n$ is a deformation retract of $\Delta^n\setminus\{a\}$, thus it follows from the Homotopy Equivalence–-Relative-H_*-Corollary 38.4 that the map is an isomorphism

$$\begin{array}{ccccccc}
\mathrm{H}_n(X, X\setminus\{x\}) & \leftarrow & \mathrm{H}_n(\tau(\Delta^n), \tau(\Delta^n)\setminus\{x\}) & \xleftarrow{\tau_*} & \mathrm{H}_n(\Delta^n, \Delta^n\setminus\{a\}) & \leftarrow & \mathrm{H}_n(\Delta^n, \partial\Delta^n).\\
[\sigma]=[\tau] & \mapsto & [\tau] & \mapsto & [\mathrm{id}_{\Delta^n}] & \mapsto & [\mathrm{id}_{\Delta^n}].
\end{array}$$

By the Identity-is-Generator Lemma 42.1 we know that id_{Δ^n} represents a generator of $\mathrm{H}_n(\Delta^n,\partial\Delta^n)$. It follows from the above explicit isomorphisms that σ represents a generator of $\mathrm{H}_k(X, X\setminus\{x\})$. ■

We will now see that the Local-H_*-Proposition 42.2 can also be used to show that given cycles represent non-trivial homology classes. To state this result it is convenient to introduce the following definition.

Definition. Let (X, A) be a pair of topological spaces. In the following we say a singular n-chain $c=\sum_{i=1}^m t_i\cdot\sigma_i\in \mathrm{C}_n(X)$ is non-trivial with respect to $x\in X\setminus A$ if the following statements hold:

(a) We have $\partial_n(c)\in \mathrm{C}_{n-1}(A)$, in other words, c represents an n-cycle in $\mathrm{C}_n(X,A)$.

(b) There exists a $j\in\{1,\dots,m\}$ with $t_j=\{\pm 1\}$ such that the map $\sigma_j\colon \mathring{\Delta}^n\to X$ is an open embedding, such that $x\in\sigma_j(\mathring{\Delta}^n)$ and such that for all $i\neq j$ we have $x\notin\sigma_i(\Delta^n)$.

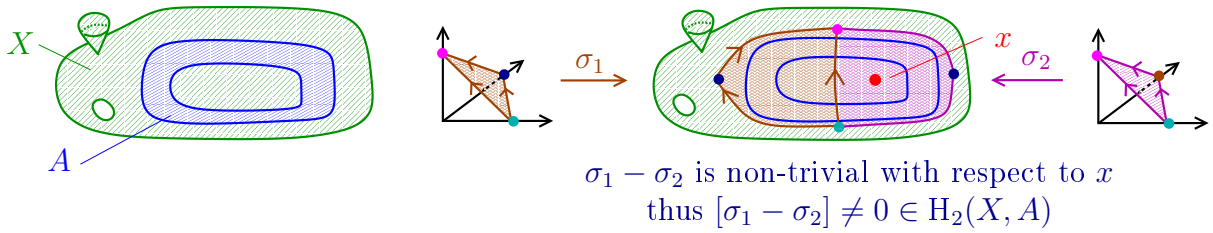
The following proposition gives a convenient criterion for showing that a given cycle represents a non-trivial element in homology.

Proposition 42.3. (Detect-Cycle Proposition) Let (X, A) be a pair of topological spaces and let $c=\sum_{i=1}^m t_i\cdot\sigma_i\in \mathrm{C}_n(X)$ be a singular n-chain that is non-trivial with respect to some $x\in X\setminus A$. The following statements hold:

(1) We have $\mathrm{H}_n(X, X\setminus\{x\})\cong\mathbb{Z}$ and under the map[325] $\mathrm{H}_n(X,A)\to\mathrm{H}_n(X,X\setminus\{x\})$ the singular n-cycle $c\in\mathrm{C}_n(X,A)$ gets sent to a generator of $\mathrm{H}_n(X, X\setminus\{x\})$.

(2) If $\mathrm{H}_n(X,A)\cong\mathbb{Z}$, then c represents a generator of $\mathrm{H}_n(X,A)$.

[325] Note that it follows from $x\in X\setminus A$ that $A\subset X\setminus\{x\}$, in particular this map is indeed defined.



Proof. Let (X, A) be a pair of topological spaces and let $c = \sum_{i=1}^{m} t_i \cdot \sigma_i \in \mathrm{C}_n(X)$ be a singular n-chain with the following two properties:

(a) c represents an n-cycle in $\mathrm{C}_n(X, A)$.

(b) There exists an $x \in X \setminus A$ and a $j \in \{1, \ldots, m\}$ with $t_j = \{\pm 1\}$ such that the map $\sigma_j \colon \mathring{\Delta}^n \to X$ is an open embedding, such that $x \in \sigma_j(\mathring{\Delta}^n)$ and such that for all $i \neq j$ we have $x \notin \sigma_i(\Delta^n)$.

We turn to the actual proofs of the statements:

(1) First note that it follows from the hypothesis that $x \in \varphi_j(\mathring{\Delta}^n)$ and the hypothesis that $\sigma_j \colon \mathring{\Delta}^n \to X$ is an open embedding, together with the the Local-H_*-Proposition 42.2 that $\mathrm{H}_n(X, X \setminus \{x\}) \cong \mathbb{Z}$ and that $[\sigma_j] \in \mathrm{H}_n(X, X \setminus \{x\})$ is a generator. Next we consider the induced homomorphism $\varphi \colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(X, X \setminus \{x\})$. We see that

$$\varphi([c]) \;=\; \varphi\Big(\Big[\sum_{i=1}^{m} t_i \cdot \sigma_i\Big]\Big) \;=\; \underbrace{t_j}_{=\pm 1} \cdot [\sigma_j] + \varphi\Big(\Big[\sum_{i \neq j} \underbrace{t_j \cdot \sigma_j}_{\substack{= 0 \in \mathrm{C}_n(X, X \setminus \{x\}), \\ \text{since } \sigma_j(\Delta^n) \subset X \setminus \{x\}}} \Big]\Big) \;=\; \pm \cdot \underbrace{[\sigma_j]}_{\substack{\uparrow \\ \text{generator of} \\ \mathrm{H}_n(X, X \setminus \{x\})}}.$$

(2) It follows from the hypothesis that $\mathrm{H}_n(X, A) \cong \mathbb{Z}$ and elementary algebra that any epimorphism $\mathrm{H}_n(X, A) \to \mathbb{Z}$ is actually an isomorphism. Statement (2) is an immediate consequence of this observation together with Statement (1). ■

42.3. **The standard generator of $\mathrm{H}_n(S^n)$.** In this section we will use the Local-H_*-Proposition 42.2 to finally exhibit explicit generators of $\widetilde{\mathrm{H}}_n(S^n) \cong \mathbb{Z}$ for any $n \in \mathbb{N}_0$.

We start out by introducing two singular n-simplices on S^n:

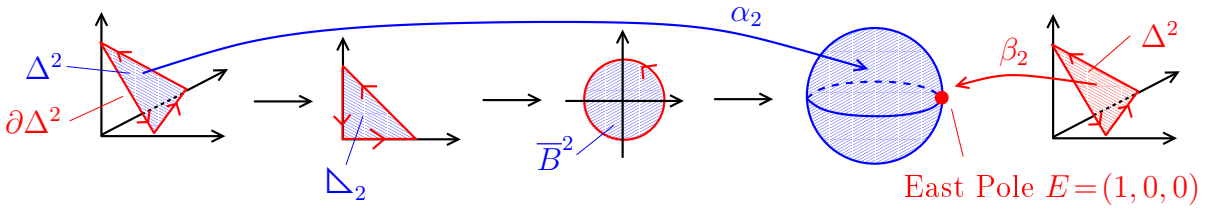
Notation. Let $n \in \mathbb{N}_0$.

(1) We consider the singular n-simplex $\alpha_n \colon \Delta^n \to S^n$ that is given by the concatenation of the following two continuous maps[326]

$$\begin{array}{ccccccc} \Delta^n & \xrightarrow[\Delta^n\text{-Homeomorphism Lemma 35.3 (2)}]{\text{homeomorphism given by the}} & \overline{B}^n & \to & S^n & \subset & \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \\ & & r \cdot v & \mapsto & \underbrace{\begin{pmatrix} -\cos(\pi \cdot r) \\ \sin(\pi \cdot r) \cdot v \end{pmatrix}}_{\in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}}. & & \end{array}$$

Similar to the Ball Quotient-Sphere Lemma 6.5 we see that this map $\Delta^n \to S^n$ sends $\partial\Delta^n$ to the "East Pole" $E := (1, 0, \ldots, 0)$ and that α_n descends to a homeomorphism $\Delta^n / \partial\Delta^n \to S^n$.

(2) We denote by $\beta_n\colon \Delta^n \to S^n$ the constant map that sends all points to the "East Pole" $(1,0,\dots,0)$.



Proposition 42.4. (Standard Generator-$\mathrm{H}_n(S^n)$-Proposition) Let $n \in \mathbb{N}_0$.

(1) The singular n-chain $\alpha_n - \beta_n \in \widetilde{\mathrm{C}}_n(S^n)$ is a cycle and it represents a generator of $\widetilde{\mathrm{H}}_n(S^n)$. (In particular for $n \geq 1$ it represents a generator of $\mathrm{H}_n(S^n)$.)

(2) If n is odd, then $\alpha_n \in \mathrm{C}_n(S^n)$ is also a cycle with $[\alpha_n] = [\alpha_n - \beta_n] \in \widetilde{\mathrm{H}}_n(S^n)$.

Definition. Let $n \in \mathbb{N}_0$. We refer to[327]

$$[S^n] \;:=\; (-1)^{n-1} \cdot [\alpha_n - \beta_n] \quad \in\; \widetilde{\mathrm{H}}_n(S^n)$$

as the standard generator of $\widetilde{\mathrm{H}}_n(S^n)$.

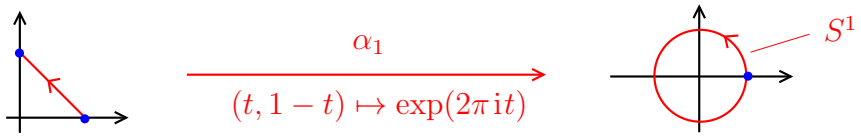
Example.

(1) For $n = 0$ we see that $[S^0] = -[\alpha_0 - \beta_0] = -[(-1) - (1)] = [(1) - (-1)] \in \widetilde{\mathrm{H}}_0(S^0)$.

(2) Next we consider the case $n = 1$. If one goes carefully through the definition of the map $\alpha_1\colon \Delta^1 \to S^1$ one sees that

$$\alpha_1(1-t,t) \underset{\substack{\uparrow \\ \text{since the explicit homeomorphism } \Delta^1 \to \overline{B}^1 \text{ is} \\ \text{given by } (t,1-t) \mapsto 1-2t}}{=} (-\cos(\pi(1-2t)), \sin(\pi(1-2t))) \;=\; (\cos(2t\pi), \sin(2t\pi)) \;=\; \exp(2\pi\,\mathrm{i}t).$$

Thus we see that the singular 1-simplex $(1-t,t) \mapsto \exp(2\pi\,\mathrm{i}t)$ represents the standard generator of $\mathrm{H}_1(S^1)$.



Proof.

(1) We start out with the following claim.

Claim. The singular n-chain $\alpha_n - \beta_n \in \widetilde{\mathrm{C}}_n(S^n)$ is a cycle.

[326] The given map $\overline{B}^n \to S^n$ is a variation on the map given in the Ball Quotient-Sphere Lemma 6.5.

[327] The slightly arbitrary looking factor $(-1)^{n-1}$ will be helpful later to stay consistent with various orientation conventions. In a nutshell it comes from the fact, that if we orient Δ^n with the "outward pointing" normal vector and if we give S^n the standard orientation, then α_n is orientation-preserving if and only if n is odd.

Proof. Let $\gamma\colon \Delta^{n-1} \to S^n$ be the singular $(n-1)$-simplex that sends all points to the "East Pole" $E = (1, 0, \dots, 0)$. For $n \neq 0$ we calculate that

$$\partial_n(\alpha_n - \beta_n) \;=\; \partial_n(\alpha_n) - \partial_n(\beta_n) \;\underset{\substack{\uparrow\\ \text{see the definition}\\ \text{of } \partial_n \text{ on page 423}}}{=} \sum_{j=0}^{n} \cdot (-1)^j \cdot \underbrace{\alpha_n \circ i_j^n}_{\substack{=\gamma, \text{ since}\\ \alpha_n(\partial\Delta^n)=\{E\}}} - \sum_{j=0}^{n} \cdot (-1)^j \cdot \underbrace{\beta_n \circ i_j^n}_{\substack{=\gamma, \text{ since}\\ \beta_n(\partial\Delta^n)=\{E\}}} \;=\; 0.$$

For $n = 0$ the same argument also shows that $\epsilon_{S^0}(\alpha_0 - \beta_0) = 0$ where $\epsilon_{S^0}\colon \mathrm{C}_0(S^0) \to \mathbb{Z}$ denotes the augmentation map. ⊞

We set $x := (-1, 0, \dots, 0)$. We consider the long exact sequence in reduced homology of the pair $(S^n, S^n \setminus \{x\})$:

$$\widetilde{\mathrm{H}}_n(S^n \setminus \{x\}) \;\to\; \widetilde{\mathrm{H}}_n(S^n, S^n \setminus \{x\}) \;\xrightarrow{\text{я}}\; \widetilde{\mathrm{H}}_n(S^n) \;\xrightarrow{\partial_n}\; \widetilde{\mathrm{H}}_{n-1}(S^n \setminus \{x\}).$$

It follows from the Stereographic Projection Lemma 2.11 that $S^n \setminus \{x\}$ is homeomorphic to $\mathbb{R}^n$. This implies by the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that $\widetilde{\mathrm{H}}_n(S^n \setminus \{x\}) = \widetilde{\mathrm{H}}_{n-1}(S^n \setminus \{x\}) = 0$. It follows that $\text{я}\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n \setminus \{x\})$ is an isomorphism.

Next note that it follows easily from the definitions that $\alpha_n - \beta_n$ is non-trivial with respect to the given $x \in S^n$. It follows from the Detect-Cycle Proposition 42.3 (1) that $\alpha_n - \beta_n$ represents a generator of $\widetilde{\mathrm{H}}_n(S^n, S^n \setminus \{x\})$. Thus by the above it also represents a generator of $\widetilde{\mathrm{H}}_n(S^n)$.[328]

(2) If n is odd, then we know by the simple calculation in the proof of the Singleton-H_*-Lemma 35.11 that β_n is a cycle which is the boundary of the singular $(n+1)$-simplex that is given by the constant map $\gamma\colon \Delta^{n+1} \to (1, 0, \dots, 0)$. Thus we see that $[\alpha_n - \beta_n] = [\alpha_n - \partial\gamma] = [\alpha_n] \in \widetilde{\mathrm{H}}_n(S^n)$. ■

[328] We could also have argued using the Detect-Cycle Proposition 42.3 (2). But it seems to us that it is more elegant to give a more direct proof.

43. Homology and self-maps of spheres

To understand the concept of homology we need to be able to compute homology groups of interesting topological spaces *and* we should acquire techniques for calculating induced maps $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$. In the last chapters we got to know the homology groups of spheres quite well. In this and the following chapter we will get to know several techniques for determining induced maps $f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)$.

These results will allow us to prove various interesting results, e.g. the mysterious sounding Hairy Ball Theorem.

43.1. The degree of a self-map of a sphere.

If we want to understand the concept of homology properly, then not only do we need to determine the isomorphism types of homology groups of interesting topological spaces, but we also need to determine induced maps on homology. In this section we will study the maps on homology induced by continuous self-maps of spheres. This topic is interesting in its own right, but the results will also play a major role later on when we will study Mayer-Vietoris sequences and when we try to determine the homology groups of CW-complexes.

Throughout this chapter we will work a lot with the following simple-minded definition.

Definition. Let $(G,+)$ be a group that is isomorphic to $\mathbb{Z}$ and let $\Phi\colon G \to G$ be a homomorphism. Note that Φ is given by multiplication by a unique integer which we refer to as the degree $\deg(\Phi)$.

Next we relate the above definition to topology.

Definition. Let $n \in \mathbb{N}_0$ and let $f\colon S^n \to S^n$ be a continuous map. By the Sphere-H_n-Proposition 41.2 we know that $\widetilde{\mathrm{H}}_n(S^n)$ is isomorphic to $\mathbb{Z}$. Thus we can define the degree of f:

$$\deg(f) \;:=\; \deg\big(f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)\big).$$

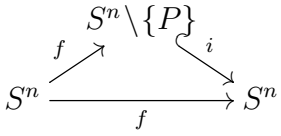
The following lemma summarizes several basic properties of the degree of continuous self-maps of spheres.

Lemma 43.1. (Sphere–Degree Lemma) Let $f, g\colon S^n \to S^n$ be two continuous maps.

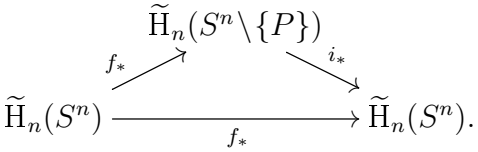
(1) $\deg(\mathrm{id}_{S^n}) = 1$.
(2) If f is not surjective, then $\deg(f) = 0$.
(3) If f is homotopic to g, then $\deg(f) = \deg(g)$.
(4) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
(5) If f is a self-homeomorphism of S^n, then $\deg(f^{-1}) = \deg(f)$.

Proof.

(1) Since $(\mathrm{id}_{S^n})_* = \mathrm{id}_{\widetilde{\mathrm{H}}_n(S^n)}$ we know that $\deg(\mathrm{id}_{S^n}) = 1$.
(2) Suppose that the map $f\colon S^n \to S^n$ misses the point $P \in S^n$. We consider the commutative diagram



which gives rise to the commutative diagram



As we already mentioned on page 503, $S^n \setminus \{P\}$ is homeomorphic to $\mathbb{R}^n$, which implies by the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that $\widetilde{\mathrm{H}}_n(S^n \setminus \{P\}) = 0$. But this means that $f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)$ factors through the zero group, i.e. f_* is the zero map.

(3) If f is homotopic to g, then it follows from the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 39.3 that $\deg(f) = \deg(g)$.

(4) We consider the following diagram

$$\widetilde{\mathrm{H}}_n(S^n) \xrightarrow[\cdot \deg(f)]{f_*} \widetilde{\mathrm{H}}_n(S^n) \xrightarrow[\cdot \deg(g)]{g_*} \widetilde{\mathrm{H}}_n(S^n). \qquad (f\circ g)_*$$

This diagram commutes since $(f \circ g)_* = f_* \circ g_*$. It follows that the curved map is given by multiplication by $\deg(f) \cdot \deg(g)$, i.e. $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

(5) This statement follows immediately from (1) and (4). ■

In the following we want to determine the degrees of interesting self-maps of spheres. We start out with self-maps of circles:

Lemma 43.2. (Degree-of-Power Lemma) Given $k \in \mathbb{Z}$ we have[329]

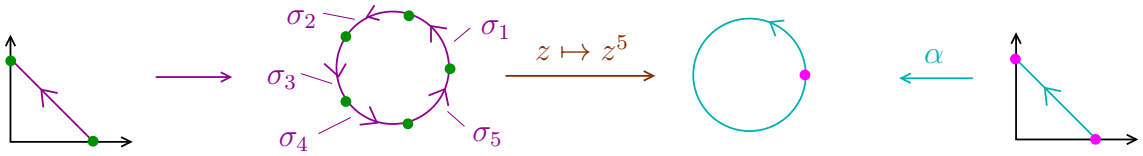
$$\deg \begin{pmatrix} S^1 & \to & S^1 \\ z & \mapsto & z^k \end{pmatrix} = k.$$

Proof. Let $k \in \mathbb{Z}$. We denote by $f\colon S^1 \to S^1$ the map that is given by $z \mapsto z^k$. We need to show that $\deg(f) = k$. Let $\alpha\colon \Delta^1 \to S^1$ be the singular 1-simplex that is given by $\alpha(1-t,t) = \exp(2\pi \mathrm{i} t)$. By the discussion on page 502 we know that $\alpha \in \mathrm{C}_1(S^1)$ is a cycle and $[\alpha] \in \mathrm{H}_1(S^1)$ is a generator.

First we first consider the case $k \geq 1$. For $j = 0, \ldots, k-1$ we denote by $\sigma_j\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $(1-t,t) \mapsto \exp(2\pi \mathrm{i}(\frac{t+j}{k}))$. Then

$$f_*([\alpha]) \underset{\text{by the Singular 1-Simplex Lemma 35.12 (3)}}{=} f_*([\sigma_0 + \cdots + \sigma_{k-1}]) = [\underbrace{f \circ \sigma_0}_{=\alpha} + \cdots + \underbrace{f \circ \sigma_{k-1}}_{=\alpha}] = k \cdot [\alpha].$$

It follows from this calculation that $\deg(f) = k$.



The case $k < 0$ is proved is proved in a similar fashion using the Singular 1-Simplex Lemma 35.12 (2). Finally, if $k = 0$, then $f\colon S^1 \to S^1$ is not surjective, thus it follows from the Sphere–Degree Lemma 43.1 (2) that $\deg(f) = 0$. ■

The combination of the Degree-of-Power Lemma 43.2 and the following lemma shows that every $k \in \mathbb{Z}$ is realized as the degree of any sphere S^n of dimension $n \geq 1$.

[329] This result is analogous to the fact, observed on page 174, that for fundamental groups the induced map $f_*\colon \pi_1(S^1,1) \to \pi_1(S^1,1)$ is given by multiplication by k.

Lemma 43.3. (Degree-Suspension Lemma) Let $n \in \mathbb{N}_0$.

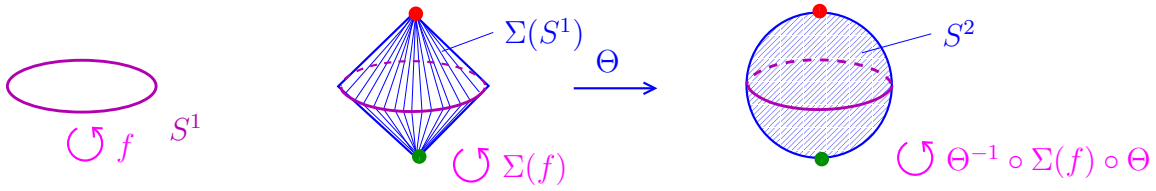
(1) The map

$$\begin{array}{rcl} \Theta\colon \Sigma(S^n) & \to & S^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P,t)] & \mapsto & \big(\underbrace{P \cdot \cos\left(\frac{\pi}{2}t\right)}_{\in \mathbb{R}^{n+1}}, \underbrace{\sin\left(\frac{\pi}{2}t\right)}_{\in \mathbb{R}}\big) \end{array}$$

is a homeomorphism.

(2) Let $f\colon S^n \to S^n$ be a continuous map. Let $\Sigma(f)\colon \Sigma(S^n) \to \Sigma(S^n)$ be the suspension of f that we defined on page 484. Furthermore let $\Theta\colon \Sigma(S^n) \to S^{n+1}$ be any homeomorphism. Then

$$\deg(\Theta \circ \Sigma(f) \circ \Theta^{-1}\colon S^{n+1} \to S^{n+1}) \;=\; \deg(f\colon S^n \to S^n).$$



Proof. Let $n \in \mathbb{N}_0$.

(1) One can easily verify that the given map $\Theta\colon \Sigma(S^n) \to S^{n+1}$ is a homeomorphism.

(2) Let $f\colon S^n \to S^n$ be a continuous map and let $\Theta\colon \Sigma(S^n) \to S^{n+1}$ be a homeomorphism. We consider the following diagram

$$\begin{array}{ccccc} \widetilde{\mathrm{H}}_n(S^n) & \xrightarrow[\cong]{\Sigma_{S^{n+1},*}} & \widetilde{\mathrm{H}}_n(\Sigma(S^n)) & \xrightarrow[\cong]{\Theta_*} & \widetilde{\mathrm{H}}_{n+1}(S^{n+1}) \\ {\scriptstyle \cdot \deg(f)}\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \Sigma(f)_*} & & \downarrow{\scriptstyle (\Theta \circ \Sigma(f) \circ \Theta^{-1})_*} \\ \widetilde{\mathrm{H}}_n(S^n) & \xrightarrow[\cong]{\Sigma_{S^{n+1},*}} & \widetilde{\mathrm{H}}_{n+1}(\Sigma(S^n)) & \xrightarrow[\cong]{\Theta_*} & \widetilde{\mathrm{H}}_{n+1}(S^{n+1}). \end{array}$$

First note that it follows from the Suspension-H_*-Proposition 41.1 (2) that the square to the left commutes and that the horizontal maps to the left are isomorphisms. It is clear that the square to the right commutes. Furthermore, since Θ is a homeomorphism we see that the horizontal maps to right are isomorphism. It follows immediately from the above discussion together with the basic observation that $\deg(\Theta \circ \Sigma(f) \circ \Theta^{-1}) = \deg(f)$. ∎

We move on to our final calculation of a degree:

Lemma 43.4. (Sphere–Degree–Matrix Lemma) Given any $A \in \mathrm{O}(n+1)$ we have

$$\deg\begin{pmatrix} S^n & \to & S^n \\ x & \mapsto & A \cdot x \end{pmatrix} \;=\; \underbrace{\det(A)}_{\in\{\pm 1\}}.$$

Example.

(1) If $f\colon S^n \to S^n$ is the reflection in any hyperplane of $\mathbb{R}^{n+1}$, then it follows from elementary linear algebra and the Sphere–Degree–Matrix Lemma 43.4 that $\deg(f) = -1$,

(2) If $f\colon S^n \to S^n$ is the reflection in the origin i.e. if $f(x) = -x$ for all $x \in S^n$, then it follows from the Sphere–Degree–Matrix Lemma 43.4 that $\deg(f) = \det(-\mathrm{id}_{n+1}) = (-1)^{n+1}$.

Proof. We start out with the following claim.

Claim. Given $n \in \mathbb{N}_0$ we denote by $\rho_n \colon S^n \to S^n$ the reflection in the $(x_1{=}0)$-hyperplane. For any $n \in \mathbb{N}_0$ we have $\deg(\rho_n) = -1$.

Proof. We start out with the case $n = 0$. We denote by $\sigma_+ \colon \Delta^0 \to S^0$ and $\sigma_- \colon \Delta^0 \to S^0$ the singular 0-simplices that are given by $\{1\} \mapsto \{1\}$ and $\{1\} \mapsto \{-1\}$. With this notation let us consider the reduced chain complex of S^0:

$$\ldots \to \mathrm{C}_1(S^1) \xrightarrow[\substack{=0,\text{ see}\\ \text{page 423}}]{\partial_1} \underbrace{\mathrm{C}_0(S^0)}_{=\mathbb{Z}\cdot\sigma_+\oplus\mathbb{Z}\cdot\sigma_-} \xrightarrow[=\epsilon_{S^0}]{\substack{\sigma_+ \mapsto 1\\ \sigma_- \mapsto 1}} \mathbb{Z} \to 0.$$

We see that $\widetilde{\mathrm{H}}_0(S^0) = \ker(\epsilon_{S^0} \colon \mathbb{Z}\cdot\sigma_+ \oplus \mathbb{Z}\cdot\sigma_- \to \mathbb{Z}) = \mathbb{Z}\cdot(\sigma_+ - \sigma_-)$. Thus we see that

$$\rho_{0*}(\sigma_+ - \sigma_-) = \underbrace{\rho_0 \circ \sigma_+}_{=\sigma_-} - \underbrace{\rho_0 \circ \sigma_-}_{=\sigma_+} = \sigma_- - \sigma_+ = (-1)\cdot(\sigma_+ - \sigma_-).$$

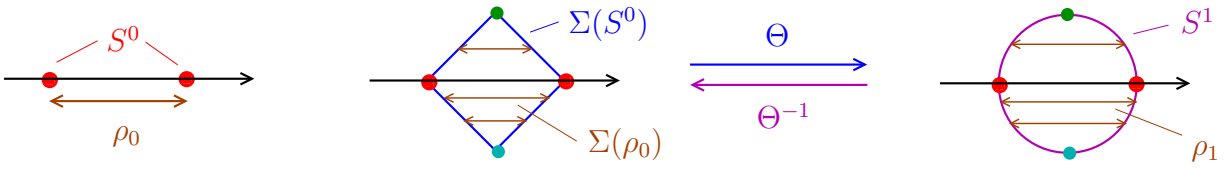
It follows that $\deg(\rho_0) = -1$. We have thus proved the desired statement for $n = 0$.

Now we turn to the proof of the induction step. Thus we assume that we know that the statement holds for some $n \in \mathbb{N}_0$. Let

$$\begin{array}{rcl} \Theta \colon \Sigma(S^n) & \xrightarrow{\cong} & S^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P,t)] & \mapsto & (P \cdot \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t)) \end{array}$$

be the homeomorphism from page 483. It follows immediately from the explicit definitions of the maps that for each $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \Sigma(S^n) & \xrightarrow{\Theta} & S^{n+1} \\ {\scriptstyle \Sigma(\rho_n)}\downarrow & & \downarrow{\scriptstyle \rho_{n+1}} \\ \Sigma(S^n) & \xrightarrow{\Theta} & S^{n+1}. \end{array}$$



Thus it follows from the Degree-Suspension Lemma 43.3 and the induction hypothesis that $\deg(\rho_n) = \deg(\rho_{n-1}) = -1$. ⊞

Before we get to the denouement of the proof, it is helpful to introduce the following notation: given $B \in \mathrm{O}(n+1)$ we denote by $\mu(B) \colon S^n \to S^n$ the map that is given by $x \mapsto B \cdot x$.

Now let $A \in \mathrm{O}(n+1)$. First we consider the case $\det(A) = 1$, i.e. we assume that $A \in \mathrm{SO}(n+1)$. By the Matrix Group-π_0-Proposition 2.23 we know that $\mathrm{SO}(n+1)$ is path-connected, so there exists a continuous map $h \colon [0,1] \to \mathrm{SO}(n+1)$ with $h(0) = \mathrm{id}$ and $h(1) = A$. But then $F(x,t) = h(t) \cdot x$ defines a homotopy from $\mathrm{id}_{S^n} = \mu(\mathrm{id})$ to $\mu(A)$. It follows from the Sphere–Degree Lemma 43.1 (1) and (3) that $\deg(\mu(A)) = \deg(\mathrm{id}_{S^n}) = 1$.

Finally we consider the case that $\det(A) \neq 1$. Let $R \in \mathrm{O}(n+1)$ be the matrix such that $\mu(R) = \rho_n$. Now we see that

$$\deg(\mu(A)) \;=\; \deg(\mu(A \cdot R \cdot R^{-1})) \underset{\substack{\uparrow \\ \text{by the Sphere–Degree} \\ \text{Lemma 43.1 (4)}}}{=} \deg(\mu(A \cdot R)) \cdot \deg(\mu(R)) \underset{\substack{\uparrow \\ \text{since } \det(A \cdot R) = 1 \text{ we have } \deg(\mu(A \cdot R)) = 1 \\ \text{and by the claim we have } \deg(\mu(R)) = -1}}{=} 1 \cdot (-1) \;=\; -1.$$

■

43.2. Application: The Hairy Ball Theorem.

Theorem 43.5. (Sphere-Degree Fixed Point Theorem) Let $f\colon S^n \to S^n$ be a continuous map. If $\deg(f) \neq (-1)^{n+1}$, then f has a fixed point.

Proof. Let $f\colon S^n \to S^n$ be a continuous map that does not admit a fixed point. This means that for all $x \in S^n$ we have $f(x) \neq x$. We need to show that $\deg(f) = (-1)^{n+1}$. We will do this by showing that f is homotopic to $-\mathrm{id}$.

Our hypothesis implies that $(1-t) \cdot f(x) \neq t \cdot x$ for all $t \in [0,1]$ and all $x \in S^n$.[330] Therefore we can consider the map

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (x,t) & \mapsto & \dfrac{(1-t)\cdot f(x) - t \cdot x}{\|(1-t)\cdot f(x) - t \cdot x\|}. \end{array}$$

This map is a homotopy between the maps $x \mapsto f(x)$ and $x \mapsto -x$. Thus it follows that

$$\deg(f) \underset{\substack{\uparrow \\ \text{Sphere–Degree Lemma 43.1 (3)}}}{=} \deg(-\mathrm{id}) \underset{\substack{\uparrow \\ \text{Sphere–Degree–Matrix Lemma 43.4}}}{=} (-1)^{n+1}.$$

■

To formulate our next theorem we need to introduce or recall the following definition.

Definition. Let $M \subset \mathbb{R}^n$ be a submanifold. Given $P \in M \setminus \partial M$ we denote by $\mathrm{T}_P M \subset \mathbb{R}^n$ the tangent vector space. A vector field on M is a continuous map $v\colon M \to \mathbb{R}^n$ such that for any $P \in M$ we have $v(P) \in \mathrm{T}_P M$.

Examples.

(1) Let P be a point on S^n. As is explained in [Fri23], the tangent space is given by

$$\mathrm{T}_P S^n \;=\; \{v \in \mathbb{R}^{n+1} \mid \langle P, v\rangle = 0\}.$$

(2) For an odd-dimensional sphere S^{2k-1} a vector field is given by[331]

$$\begin{array}{rcl} v\colon S^{2k-1} & \to & \mathbb{R}^{2k} \\ (x_1, x_2, \ldots, x_{2k-1}, x_{2k}) & \mapsto & (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1}). \end{array}$$

This vector field has the property that it is nowhere-vanishing. In the case of the 1-sphere S^1 this is just the vector field of constant length 1 that points in the "counterclockwise direction".

---

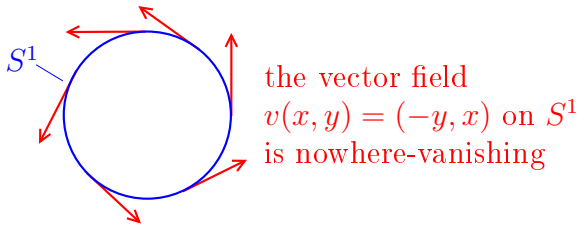
[330] This case be seen as follows: if $t \neq \frac{1}{2}$ we have $\|(1-t)\cdot f(x)\| = 1 - t \neq t = \|t \cdot x\|$. On the other hand, for $t = \frac{1}{2}$ we have $(1-t)\cdot f(x) = \frac{1}{2} f(x) \neq \frac{1}{2} x = t \cdot x$.

[331] Indeed, for any $P \in S^{2k-1}$ we have $\langle P, v(P)\rangle = 0$, i.e. $v(P)$ does indeed lie in the tangent space $\mathrm{T}_P S^{2k-1}$.

(3) For an even-dimensional sphere S^{2k} a vector field is similarly given by

$$\begin{array}{rcl} v\colon S^{2k} & \to & \mathbb{R}^{2k+1} \\ (x_1, x_2, \ldots, x_{2k-1}, x_{2k}, x_{2k+1}) & \mapsto & (-x_2, x_1, \ldots, -x_{2k}, 0). \end{array}$$

This vector field vanishes at the "North Pole" $(0, \ldots, 0, 1)$ and it also vanishes at the "South Pole" $(0, \ldots, 0, -1)$.



The above examples raise the question whether it is possible to find a nowhere vanishing vector field on even-dimensional spheres. The following theorem shows that this is *not* possible.

Theorem 43.6. (Hairy Ball Theorem) Every vector field on an even-dimensional sphere S^{2k} vanishes on at least one point.[332]

Proof. Let v be a vector field on the sphere S^n which is nowhere-vanishing. We have to show that n is odd. Since v is nowhere vanishing we can consider the vector field w on S^n that is given by rescaling v to length one, i.e. that is given by

$$\begin{array}{rcl} S^n & \to & \mathbb{R}^{n+1} \\ x & \mapsto & w(x) := \frac{1}{\|v(x)\|} \cdot v(x). \end{array}$$

Furthermore we consider the map[333]

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (x, t) & \mapsto & \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x). \end{array}$$

The map $F_0 = F(-, 0)$ is the identity and the map $F_1 = F(-, 1)$ equals F is $-\mathrm{id}$. Put differently, the map F is a homotopy between the two maps id and $-\mathrm{id}$. It follows that

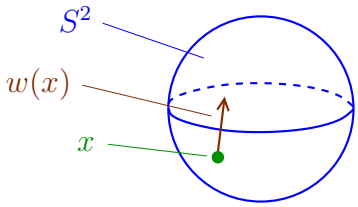
$$1 \underset{\text{by the Sphere–Degree Lemma 43.1 (1)}}{=} \deg(\mathrm{id}) \underset{\text{Lemma 43.1 (3)}}{=} \deg(-\mathrm{id}) \underset{\text{by the Sphere–Degree–Matrix Lemma 43.4}}{=} \det(-\mathrm{id}_{n+1}) = (-1)^{n+1}.$$

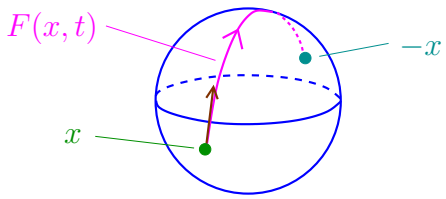
But this means that n is odd. ∎

[332] On S^{2k} we just found a vector field which vanishes at *two* points. Can we find a vector field on S^{2k} which vanishes on only *one* point?

[333] The map does indeed take values in S^n, since

$$\langle \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x), \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x) \rangle \underset{\text{since } \langle x, w(x) \rangle = 0}{=} \cos(\pi t)^2 \cdot \|x\|^2 + \sin(\pi t)^2 \cdot \|w(x)\|^2 \underset{\text{since } \|x\| = \|w(x)\| = 1}{=} 1.$$


S^2
$w(x)$
x


$F(x,t)$
$-x$
x

44. Local degrees of self-maps of spheres

In the previous chapter we introduced the notion of the degree of a continuous self-map of a sphere. We managed to calculate the degree in simple situations, but so far we have no general scheme for calculating degrees. For example in the proof of the Degree-of-Power Lemma 43.2 we had to use a rather ad hoc argument to get the desired result.

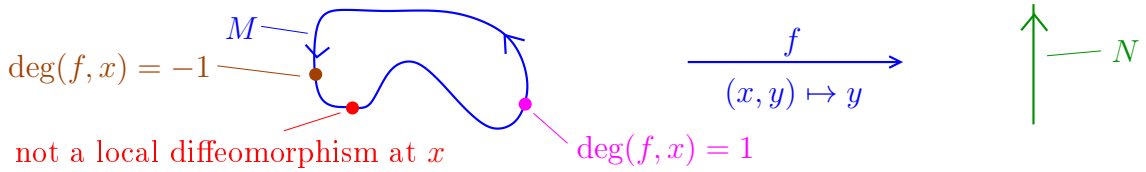
In this chapter we will formulate and prove the Local-Global Degree Proposition 44.2 and the Degree-via-Local Orientations Proposition 44.7. These propositions show how one can determine the degree of a continuous map $f\colon S^n \to S^n$ by considering "local degrees" and "local information" of f.

44.1. Degrees and local orientations.

We introduce the following harmless definition.

Definition. Let $f\colon M \to N$ be a continuous map between two oriented n-dimensional smooth manifolds and let $x \in M \setminus \partial M$. If f is a local diffeomorphism[334] at x, then we define the local degree of f at the point x as[335]

$$\deg(f,x) := \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$



For most computational purposes the following proposition is the only result from this chapter one really needs:

Proposition 44.7. (Degree-via-Local Orientations Proposition) Let $n \in \mathbb{N}$. Let $f\colon S^n \to S^n$ be a continuous map and let $y \in S^n$ such that the following conditions are satisfied:

(1) $f^{-1}(y)$ consists of finitely many points $x_1, \dots, x_m$.
(2) The map f is a local diffeomorphism at each x_i.

Then

$$\deg(f) \;=\; \sum_{i=1}^{m} \deg(f,x_i) \;=\; \sum_{i=1}^{m} \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x_i, \\ -1, & \text{if } f \text{ is orientation-reversing at } x_i. \end{cases}$$

As the number of the proposition suggests, it will take a while to get to the proof. Meanwhile let us consider some examples:

Examples.

(1) Let $m \in \mathbb{Z}$. We consider the map

$$\begin{array}{rccc} f\colon & S^1 & \to & S^1 \\ & z & \mapsto & z^m \end{array}$$

[336]By the Inverse Mapping Theorem, see [Fri23], the map f is a local diffeomorphism at $x \in M$ if there exists an open neighborhood U around x such that $f\colon U \to N$ is smooth and such that $\mathrm{D}f_x\colon \mathrm{T}_xM \to \mathrm{T}_xN$ is an isomorphism.

In the Degree-of-Power Lemma 43.2 we already proved that $\deg(f) = m$. Now we will show, just for fun, that the degree of f can also easily be determined using the Degree-via-Local Orientations Proposition 44.7. First let $m \neq 0$. Then we choose $y = 1$. The preimages of y are given by $x_j = \exp(2\pi \mathrm{i} j/m)$ where $j = 1, \ldots, m$. It follows easily from the definitions that for each $j \in \{1, \ldots, m\}$ we have

$$\deg(f, x_j) = \begin{cases} 1, & \text{if } m > 0, \\ -1, & \text{if } m < 0. \end{cases}$$

It follows from this calculation and the Degree-via-Local Orientations Proposition 44.7 that $\deg(f) = |m| \cdot \operatorname{sign}(m) = m$. If $m = 0$, then we take $y = -1$. Note that for $m = 0$ we have $f^{-1}(y) = \varnothing$, thus we see that $\deg(f) = 0$.

(2) If $f\colon S^n \to S^n$ is a diffeomorphism, then we can apply the Degree-via-Local Orientations Proposition 44.7 with any $y \in S^n$ and we obtain that

$$\deg(f) = \begin{cases} 1, & \text{if } f \text{ is orientation-preserving}, \\ -1, & \text{if } f \text{ is orientation-reversing}. \end{cases}$$

44.2. **The local degree.** In this section we define, for continuous self-maps $f\colon S^n \to S^n$ and suitable points in S^n a "local degree". We will then prove that the degree of a continuous map $S^n \to S^n$ is determined by the local degrees. In the next section we will relate the new notion of local degree to the notion introduced in the beginning of the chapter. This will then allow us to finally prove the Degree-via-Local Orientations Proposition 44.7 which we formulated in the beginning of the chapter.

Definition. We say a map $f\colon X \to Y$ between two topological spaces is nice at a point $x \in X$ if there exists a neighborhood U of x such that for all $x' \neq x$ in U we have $f(x') \neq f(x)$. We refer to such a neighborhood as a nice neighborhood of x.

Examples.

(1) Let $f\colon X \to Y$ be a continuous map between two topological spaces and let $x \in X$. If U is a neighborhood such that $f|_U$ is injective, then U is evidently a nice neighborhood of x.

(2) We consider the maps

$$f\colon \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & t^2 \end{array} \qquad \text{and} \qquad g\colon \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} 0, & \text{if } t = 0, \\ t \cdot \sin(\frac{1}{t}), & \text{if } t \neq 0. \end{cases}$$

For the map f every point on $\mathbb{R}$ is nice, in fact for $x < 0$ we can take $U = (-\infty, 0)$, for $x > 0$ we can take $U = (0, \infty)$ and for $x = 0$ we can take $U = \mathbb{R}$. On the other hand it is straightforward to see that for the map g the point $x = 0$ is not nice.

The following is the key definition of this section.

Definition. Let $n \geq 1$, let $f\colon S^n \to S^n$ be a continuous map and let $x \in S^n$ be a nice point. Let U be a nice neighborhood of x. The local degree of f at the point x is defined

as the degree[337] of the homomorphism

$$\mathrm{H}_n(S^n) \xrightarrow[\cong]{\text{я}} \mathrm{H}_n(S^n, S^n\backslash\{x\}) \xleftarrow[\text{isomorphism by the Excision Theorem 40.2}]{\cong} \mathrm{H}_n(U, U\backslash\{x\}) \xrightarrow[338]{f_*} \mathrm{H}_n(S^n, S^n\backslash\{f(x)\}) \xleftarrow[\cong]{\text{я}} \mathrm{H}_n(S^n).$$

the long exact sequence of the pair $(S^n, S^n \setminus \{y\})$ shows that these maps are isomorphisms

We denote this local degree by $\deg(f, x)$.

Lemma 44.1. (Local Degree–Well-Defined Lemma) The local degree $\deg(f, x)$ of a map $f\colon S^n \to S^n$ at a nice point $x \in S^n$ is well-defined.

Proof. Let $n \in \mathbb{N}$, let $f\colon S^n \to S^n$ be a continuous map and let $x \in S^n$ be a nice point. Let U, U' be nice neighborhoods of x. After replacing U' by $U \cap U'$ we can without loss of generality assume that $U' \subset U$. We consider the following diagram

$$\begin{array}{ccccccc}
 & & \mathrm{H}_n(U', U'\backslash\{x\}) & \xrightarrow{f_*} & \mathrm{H}_n(S^n, S^n\backslash\{f(x)\}) & & \\
 & \overset{\cong}{\swarrow} & \downarrow \cong & & \downarrow \mathrm{id} & \overset{\cong}{\searrow} & \\
\mathrm{H}_n(S^n) \longrightarrow \mathrm{H}_n(S^n, S^n\backslash\{x\}) & & & & & & \mathrm{H}_n(S^n). \\
 & \underset{\cong}{\nwarrow} & & & & \underset{\cong}{\nearrow} & \\
 & & \mathrm{H}_n(U, U\backslash\{x\}) & \xrightarrow{f_*} & \mathrm{H}_n(S^n, S^n\backslash\{f(x)\}). & &
\end{array}$$

All the non-decorated maps are induced by inclusions. It follows from the functoriality of relative homology that the diagram commutes. Also note that the left vertical map is an isomorphism by the Excision Theorem 40.2. It follows immediately from these observations that the local degree defined using U' agrees with the local degree defined using U. ■

Using the following proposition we can determine the degree of a map from local degrees.

Proposition 44.2. (Local-Global Degree Proposition) Let $n \in \mathbb{N}$, let $f\colon S^n \to S^n$ be a continuous map and let $y \in S^n$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$. Then all the points $x_1, \ldots, x_m$ are nice and we obtain the equality

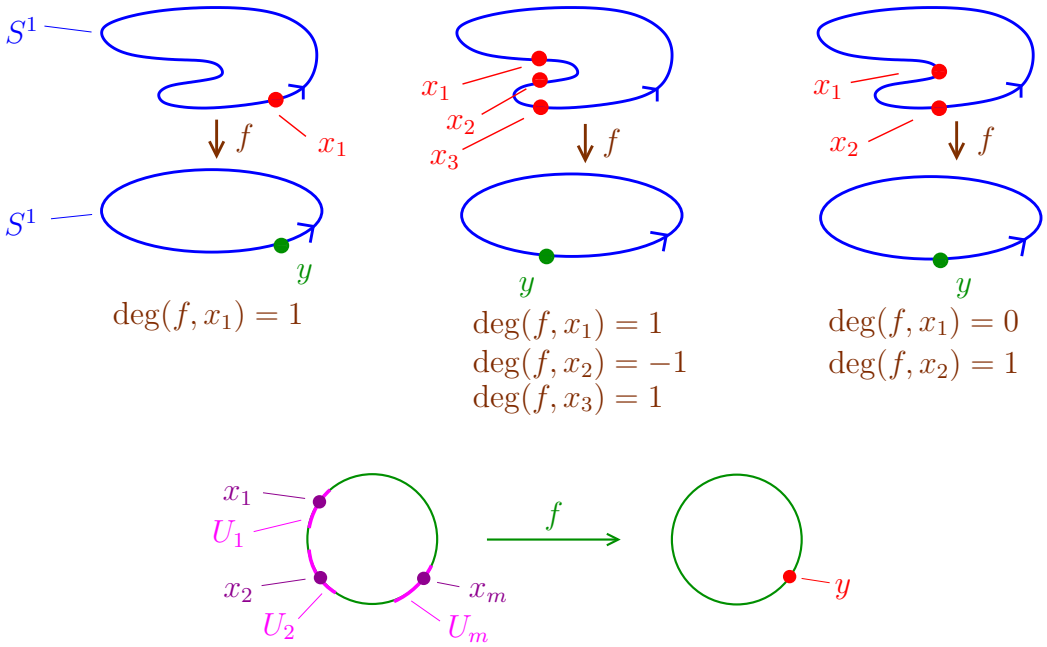
$$\deg(f) \;=\; \sum_{i=1}^{m} \deg(f, x_i).$$

Example. In the figure below we sketch a map $f\colon S^1 \to S^1$ and we choose several points $y \in S^1$. We see that the number of preimages and the local degrees at the points in the preimages vary, but in all three cases the sum of the local degrees at the points in the preimage is the same.

Proof. Let $n \in \mathbb{N}$, let $f\colon S^n \to S^n$ be a continuous map and let $y \in S^n$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$. It follows easily from the fact that S^n is Hausdorff that we can choose pairwise disjoint open neighborhoods $U_1, \ldots, U_m$ for $x_1, \ldots, x_m$. Note that this observation implies already that all the x_i are nice.

[337] By the Sphere-H_n-Proposition 41.2 we know that $\mathrm{H}_n(S^n) \cong \mathbb{Z}$. As on page 504 we note that any homomorphism $\mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is multiplication by an integer, which we call the degree of this homomorphism.

[338] Note that f induces a map of pairs $(U, U \setminus \{x\}) \to (S^n, S^n \setminus \{f(x)\})$ since U is nice.



In the following we adopt the following notation:

(1) Given homomorphisms $\varphi_i \colon A_i \to B$ we write $\bigoplus_{i=1}^{m} \varphi_i \colon \bigoplus_{i=1}^{m} A_i \to B$.

(2) Given homomorphisms $\varphi_i \colon A \to B_i$ we write $(\varphi_1, \ldots, \varphi_m) \colon A \to \bigoplus_{i=1}^{m} B_i$.

(3) Given homomorphisms $\varphi_i \colon A_i \to B_i$ we write $\operatorname{diag}(\varphi_1, \ldots, \varphi_m) \colon \bigoplus_{i=1}^{m} A_i \to \bigoplus_{i=1}^{m} B_i$.

We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}_n(S^n) & & \xrightarrow{\quad f_* \quad} & & \mathrm{H}_n(S^n) \\
{\scriptstyle (\text{я}_1, \ldots, \text{я}_m)} \downarrow & \searrow^{\text{я}} & & & \\
\bigoplus_{i=1}^{m} \mathrm{H}_n(S^n, S^n \setminus \{x_i\}) & \xleftarrow[\cong]{(p_1, \ldots, p_m)} & \mathrm{H}_n(S^n, S^n \setminus f^{-1}(y)) & & \downarrow {\scriptstyle \text{я}} \cong \\
{\scriptstyle \operatorname{diag}(r_{1*}, \ldots, r_{m*})} \uparrow \cong & & {\scriptstyle \iota_*} \uparrow \cong & \searrow^{f_*} & \\
 & \nearrow^{\cong}_{\oplus s_i} & \mathrm{H}_n\Big(\bigcup_{i=1}^{m} U_i, \bigcup_{i=1}^{m} U_i \setminus \{x_i\} \Big) & & \\
\bigoplus_{i=1}^{m} \mathrm{H}_n(U_i, U_i \setminus \{x_i\}) & & \xrightarrow{\quad \oplus f_{i*} \quad} & & \mathrm{H}_n(S^n, S^n \setminus \{y\}).
\end{array}$$

We make the following clarifications and observations:

(1) The maps r_i, s_i, p_i and ι are the maps induced by the obvious inclusions of pairs of topological spaces. Furthermore the maps $f_i \colon (U_i, U_i \setminus \{x_i\}) \to (S^n, S^n \setminus \{y\})$ are given by the restriction of f to U_i.

(2) It follows pretty easily from the functoriality of relative homology groups and elementary considerations that the diagram commutes.

(3) It follows easily from the fact that the U_i are disjoint open subsets of S^n and the Path-Component–H_*–Lemma 35.8 that the diagonal map $\bigoplus_{i=1}^{m} s_i$ to the bottom left is an isomorphism.
(4) It follows from the Excision Theorem 40.2 that the vertical map ι_* and the maps r_{i*} are isomorphisms.
(5) It follows from the fact that the diagram commutes together with (3) and (4) that the horizontal map $(p_1, \dots, p_m)$ is also an isomorphism.
(6) As we pointed out on page 503, it follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that the reduced homology groups of $S^n \setminus \{y\}$ vanish. Since $n \geq 1$ it follows from the LES-of-Pair Proposition 38.3 that $\mathrm{я}\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n, S^n \setminus \{y\})$ is an isomorphism.

By (6) it suffices to prove the following claim.

Claim. We have $$\mathrm{я}(f_*([S^n])) \;=\; \sum_{i=1}^{m} \deg(f, x_i) \cdot \mathrm{я}([S^n]) \;\in\; \mathrm{H}_n(S^n, S^n \setminus \{y\}).$$

Proof. We verify the claim by performing the following calculation:

$$\begin{aligned}
\mathrm{я}(f_*([S^n])) &\overset{\text{since the diagram commutes and since the map } \mathrm{diag}(r_{1*}, \dots, r_{m*}) \text{ is an isomorphism}}{=} \Big(\bigoplus_{i=1}^{m} f_{i*}\Big) \circ \mathrm{diag}(r_{1*}, \dots, r_{m*})^{-1} \circ (\mathrm{я}_1, \dots, \mathrm{я}_m)([S^n]) \overset{\text{since each } r_{i*} \text{ is an isomorphism}}{=} \\
&= \Big(\bigoplus_{i=1}^{m} f_{i*}\Big) \circ (r_{1*}^{-1} \circ \mathrm{я}_1, \dots, r_{m*}^{-1} \circ \mathrm{я}_m)([S^n]) \\
&= \sum_{i=1}^{m} (f_i \circ r_i^{-1} \circ \mathrm{я}_i)([S^n]) \overset{\text{by the definition of the local degree, see page 513}}{=} \sum_{i=1}^{m} \deg(f, x_i) \cdot \mathrm{я}([S^n]).
\end{aligned}$$

since we always have the equality $\Big(\Big(\bigoplus_{i=1}^{m} \varphi_i\Big) \circ (\psi_1, \dots, \psi_m)\Big)(x) = \sum_{i=1}^{m} (\varphi_i \circ \psi_i)(x)$

■

44.3. **The local degree and local diffeomorphisms.** In the previous section we introduced the notion of a local degree of a continuous self-map of a sphere. In this section we will see that for local diffeomorphisms the above notion of a local degree matches the definition that we gave in the beginning of this chapter.

Our proof of the equality of the two local degrees relies on the following proposition, which is interesting in its own right.

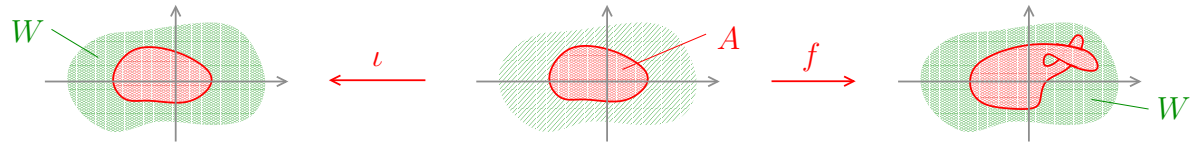
Proposition 44.3. (Local Diffeomorphism-Degree Proposition) Let W be a subset of $\mathbb{R}^n$ and let A be a subset of W such the origin 0 is contained in the interior of W. We denote by $\iota\colon A \to W$ the inclusion. Let $f\colon A \to W$ be an injective continuous map with $f(0) = 0$. We suppose that f is a local diffeomorphism at 0. Then the degree of the homomorphism[339]

$$\mathrm{H}_n(W, W \setminus \{0\}) \xleftarrow[\cong]{\iota_*} \mathrm{H}_n(A, A \setminus \{0\}) \xrightarrow{f_*} \mathrm{H}_n(W, W \setminus \{0\})$$

equals $+1$ if f is orientation-preserving at 0 and -1 if f is orientation-reversing at 0.

In the proof of the Local Diffeomorphism-Degree Proposition 44.3 we will make use of the following two lemmas which we will also use on several other occasions later on.

[339]By the Local-H_*-Proposition 42.2 we know that $\mathrm{H}_n(W, W \setminus \{0\}) \cong \mathbb{Z}$, thus it makes sense to talk of the degree of an automorphism of $\mathrm{H}_n(W, W \setminus \{0\})$.



Lemma 44.4. (Linearization–Homotopy Lemma) Let $f\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. Let $P \in U$. We denote by

$$\begin{array}{rcl} \lambda\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ x & \mapsto & \mathrm{D}f_P \cdot x + f(P) \end{array}$$

the linearization of f at P. There exists an open neighborhood U' of P such that the restrictions of f and λ to maps $(U', U' \setminus \{P\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{f(P)\})$ of pairs of topological spaces are homotopic.

Proof. Let $f\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. To simplify the notation we assume that $P = 0$ and that $f(P) = 0$. Furthermore, by restriction to an open ball we might as well assume that U is convex. We write $D := \mathrm{D}f_0$. Thus the linearization of f at $P = 0$ is given by the map $\lambda(x) = D \cdot x$. Since f is in particular a local diffeomorphism a $P = 0$ we know by the Chain Rule that D is an invertible matrix.

Since $U \subset \mathbb{R}^n$ is convex we obtain from the Map-as-Matrix Multiplication Proposition [Fri23] that there exists a smooth map $A\colon U \to \mathrm{M}(n \times n, \mathbb{R})$ such that $A(0) = \mathrm{D}f_0$ and such that $f(x) = A(x) \cdot x$ for all $x \in U$. Since $\det(A(0)) \neq 0$ we see that there exists an $r > 0$ with $B^n_r(0) \subset U$ such that $\det(A(x)) \neq 0$ for all $x \in B^n_r(0)$. This implies that the map

$$\begin{array}{rcl} B^n_r(0) \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & A(x \cdot (1-t)) \cdot x \end{array}$$

is a homotopy of the maps f and λ as maps $(B^n_r(0), B^n_r(0) \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ of pairs of topological spaces. We now see that $U' = B^n_r(0)$ has the desired properties. ∎

Before we can formulate our next lemma we need to introduce the following notation:

Notation.
(1) Given a matrix $P \in \mathrm{GL}(n,\mathbb{R})$ we denote by $\rho(P)\colon \mathbb{R}^n \to \mathbb{R}^n$ the map given by $x \mapsto P \cdot x$.
(2) Given a non-zero real number r we denote its sign by $\mathrm{sign}(r) \in \{-1, +1\}$.

Lemma 44.5. (Matrix Multiplication–Degree Lemma) Let $n \in \mathbb{N}$. Given any invertible matrix $A \in \mathrm{GL}(n,\mathbb{R})$ we have[340]

$$\deg\left(\rho(A)_*\colon \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})\right) = \mathrm{sign}(\det(A)).$$

Proof of the Matrix Multiplication–Degree Lemma 44.5. Let $A \in \mathrm{GL}(n,\mathbb{R})$. We start out by picking an orthogonal matrix $B \in \mathrm{O}(n)$ such that $\mathrm{sign}(\det(B)) = \mathrm{sign}(\det(A))$.

[340] It follows from the Reduced LES-of-Pair Proposition 39.6, the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5, the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 39.4 and the Sphere-H_n-Proposition 41.2 that we have isomorphisms $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus \{0\}) \cong \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \cong \mathbb{Z}$. Thus the degree of $\rho(A)_*$ is indeed defined.

Next we consider the following diagram

$$\begin{array}{ccccccc}
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow{\mathrm{id}} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow[\cong]{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus \{0\}) & \xleftarrow[\cong]{} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \\
\downarrow{\scriptstyle \rho(A)_*} & & \downarrow{\scriptstyle \rho(B)_*} & & \downarrow{\scriptstyle \rho(B)_*} & & \downarrow{\scriptstyle \rho(B)_*} \\
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow{\mathrm{id}} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow[\cong]{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus \{0\}) & \xleftarrow[\cong]{} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}).
\end{array}$$

We make the following observations:

(1) Since $\mathrm{sign}(\det(B)) = \mathrm{sign}(\det(A))$ we obtain from the Matrix Group-π_0-Proposition 2.23 (1) that there exists a path $[0,1] \to \mathrm{GL}(n,\mathbb{R})$ from A to B. Similar to the discussion on page 137 this path gives rise to a homotopy between $\rho(A)$ and $\rho(B)$ as maps of pairs of topological spaces $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. We obtain from the Homotopy-H_*-Proposition 38.6 that $\rho(A)_* = \rho(B)_*$, in other words, the square to the left commutes.

(2) The square in the middle commutes by the naturality of the connecting homomorphism, see the LES-of-Pair Proposition 38.3.

(3) The horizontal maps to the right are induced by the inclusion $S^{n-1} \to \mathbb{R}^n \setminus \{0\}$. Since S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$ we obtain from the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 39.4 that these horizontal maps are isomorphisms.

(4) Since $B \in \mathrm{O}(n)$ we see that $\rho(B)\colon \mathbb{R}^n \to \mathbb{R}^n$ restricts to a self-map of S^{n-1}. It follows from the functoriality of reduced homology groups that the square to the right commutes.

Now we see that

$$\begin{array}{ccccc}
\deg\left(\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \xrightarrow{\rho(A)_*} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})\right) & \overset{\text{by the above commutative diagram}}{=} & \deg\left(S^{n-1} \xrightarrow{\rho(B)} S^{n-1}\right) & \overset{\text{by the Sphere–Degree–Matrix Lemma 43.4}}{=} & \det(B) \\
= \mathrm{sign}(\det(B)) & & = \mathrm{sign}(\det(A)). & & \\
\uparrow{\scriptstyle \text{since } B \in \mathrm{O}(n) \text{ we have } \det(B) \in \{\pm 1\}} & & \uparrow{\scriptstyle \text{choice of } B} & &
\end{array}$$

■

Now we are finally in a position to prove the Local Diffeomorphism-Degree Proposition 44.3.

Proof of the Local Diffeomorphism-Degree Proposition 44.3. Let W be a subset of $\mathbb{R}^n$ and let A be a subset of W such that the origin 0 is contained in the interior. Let $f\colon A \to W$ be an injective continuous map with $f(0) = 0$. We suppose that f is a local diffeomorphism at 0. Recall that this means that there exists an open neighborhood U of 0 that is contained in A and an open neighborhood V of 0 such that the map $f\colon U \to V$ is a diffeomorphism.

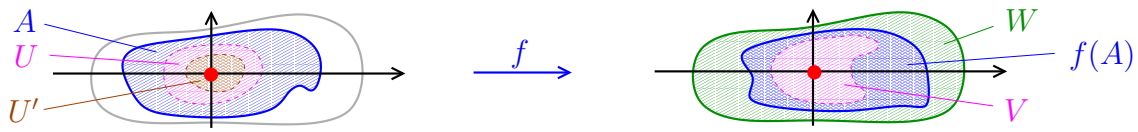
We start out with the following preparations.

(1) We set

$$\epsilon := \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } 0, \\ -1, & \text{if } f \text{ is orientation-reversing at } 0. \end{cases}$$

Note that $\epsilon = \mathrm{sign}(\det(\mathrm{D}f_0))$.

(2) Note that by the Linearization–Homotopy Lemma 44.4 there exists an open neighborhood U' of 0 that is contained in U such that the restrictions of f and $\rho(\mathrm{D}f_0)$ to maps $(U', U' \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ are homotopic.



Next we consider the following diagram

$$\begin{array}{ccccc}
\mathrm{H}_n(W, W\setminus\{0\}) & \xleftarrow{\cong} & \mathrm{H}_n(A, A\setminus\{0\}) & \xrightarrow{f_*} & \mathrm{H}_n(W, W\setminus\{0\}) \\
\downarrow\cong & & \uparrow\cong & & \swarrow\cong \\
 & & \mathrm{H}_n(U', U'\setminus\{0\}) & \xrightarrow{f_*} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \\
\downarrow\cong & & \downarrow\mathrm{id} & & \downarrow\mathrm{id} \qquad \downarrow\cong \\
 & \swarrow\cong & \mathrm{H}_n(U', U'\setminus\{0\}) & \xrightarrow{\rho(\mathrm{D}f_0)_*} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \qquad \swarrow\cong \\
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) & & \xrightarrow{\rho(\mathrm{D}f_0)_*} & & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}).
\end{array}$$

We continue with the following clarifications and observations:

(3) The undecorated maps are induced by inclusions. All of them are isomorphisms by the Excision Theorem 40.2.

(4) It follows from (2) and the Homotopy-H_*-Proposition 38.6 that the central square commutes. All other regions of the diagram commute by the functoriality of relative homology groups.

(5) By the Matrix Multiplication–Degree Lemma 44.5 the bottom horizontal map equals multiplication by $\mathrm{sign}(\det(\mathrm{D}f_0))$.

(6) It follows from all of the above that the top map is given by multiplication by ϵ. But that is precisely what we wanted to show. ■

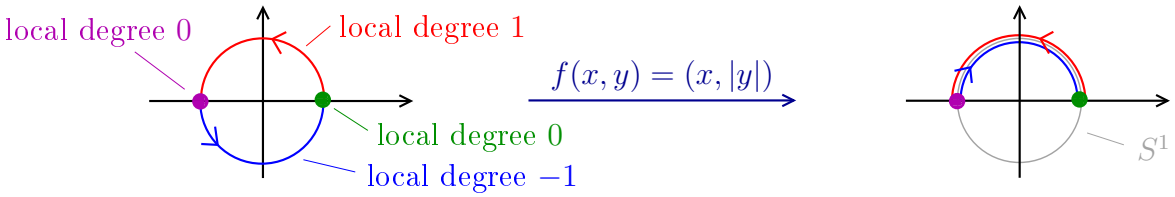
Let $f\colon S^n \to S^n$ be a continuous map with $n \geq 1$ and let $x \in S^n$ be a point such that f is a local diffeomorphism at x. In this setting we now have two different notions of a local degree, namely the notion from page 511 and the notion from page 513. The following proposition shows that the two notions of a local degree agree.

Proposition 44.6. (Local Degree-Calculation Proposition) Let $f\colon S^n \to S^n$ be a continuous map with $n \geq 1$ and let $x \in S^n$. If f is a local diffeomorphism at x, then x is nice and

$$\deg(f, x) \;=\; \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x, \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

Example. We consider the map $f\colon S^1 \to S^1$ that is given by $(x, y) \mapsto (x, |y|)$. It follows from the Local Degree-Calculation Proposition 44.6 that for every point on the open upper semicircle the local degree is $+1$ and that for every point on the open lower semicircle the local degree is -1. One can easily show that the local degree at the points ± 1 is in both cases zero.

Proof. Let $f\colon S^n \to S^n$ be a continuous map with $n \geq 1$ and let $x \in S^n$. We assume that f is a local diffeomorphism at x. By elementary linear algebra we can pick an $A \in \mathrm{SO}(n+1)$ with $A \cdot f(x) = x$. Note that the self-map of S^n given by $z \mapsto A \cdot z$ is orientation-preserving.

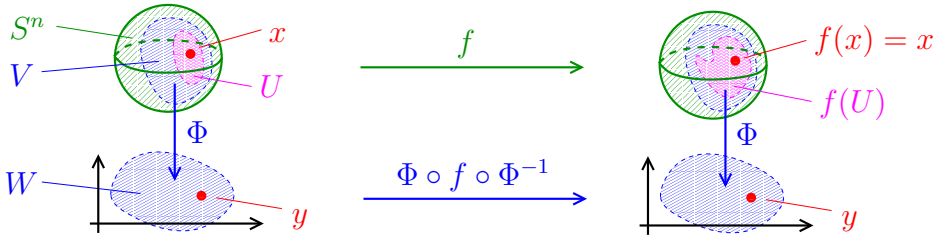


Furthermore by the Sphere–Degree–Matrix Lemma 43.4 we know that $\deg(z \mapsto A \cdot z) = 1$. It follows from these observations, together with the Sphere–Degree Lemma 43.1 (4), that after replacing f by the map $z \mapsto A \cdot f(z)$, we might as well assume that $f(x) = x$.

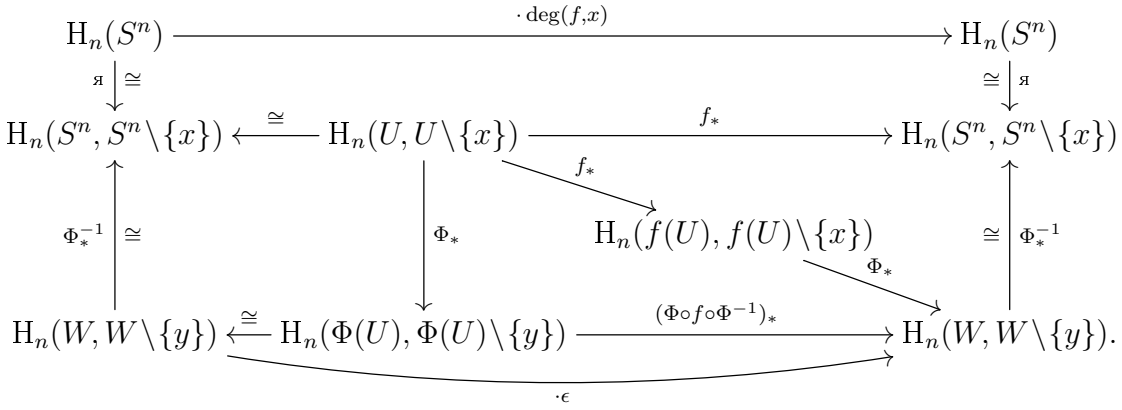
We make a few further preparations.

(1) We pick a chart $\Phi\colon V \to W$, from the smooth atlas, around $x \in S^n$.
(2) Since f is a local diffeomorphism at x there exists an open neighborhood U of x such that $f\colon U \to f(U)$ is a diffeomorphism. Note that after possibly replacing U by the subset $U \cap V \cap f^{-1}(V)$ we can assume that $U \subset V$ and $f(U) \subset V$.
(3) We set $y := \Phi(x)$.
(4) We set

$$\epsilon := \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x, \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$



Next we consider the following diagram



We make the following observations and clarifications.

(1) As usual the undecorated maps are induced by inclusions.
(2) The top rectangle commutes by the definition of $\deg(f, x)$. The middle regions of the diagram commute by the functoriality of relative homology. Finally, note that $\Phi \circ f \circ \Phi^{-1}\colon \Phi^{-1}(U) \to W$ is orientation-preserving at y if and only if f is orientation-preserving at x. Thus we see that the bottom part of the diagram commutes by the Local Diffeomorphism-Degree Proposition 44.3.

(3) By the Excision Theorem 40.2 the map $\Phi_*\colon \mathrm{H}_n(W, W \setminus \{y\}) \to \mathrm{H}_n(S^n, S^n \setminus \{x\})$ is an isomorphism.
(4) It follows from the above that $\deg(f, x) = \epsilon$. ■

Now we can finally prove the following proposition, which we already mentioned on page 511, at the very beginning of this chapter.

Proposition 44.7. (Degree-via-Local Orientations Proposition) Let $n \in \mathbb{N}_0$. Let $f\colon S^n \to S^n$ be a continuous map and let $y \in S^n$ such that the following conditions are satisfied:[341]

(1) $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$.
(2) The map f is a local diffeomorphism at each x_i.

Then[342]

$$\deg(f) \;=\; \sum_{i=1}^{m} \deg(f, x_i) \;=\; \sum_{i=1}^{m} \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x_i, \\ -1, & \text{if } f \text{ is orientation-reversing at } x_i. \end{cases}$$

Proof. The proposition is an immediate consequence of the Local-Global Degree Proposition 44.2 together with the Local Degree-Calculation Proposition 44.6. ■

45. The Mayer–Vietoris sequence and its applications

Recall that in the Seifert–van Kampen-Theorem 20.1 we saw how, given a suitable decomposition $X = U \cup V$ of a topological space, one can determine the fundamental group $\pi_1(X, x_0)$ in terms of the two inclusion induced homomorphisms $\pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$.

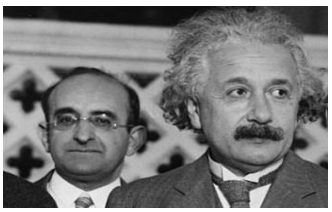
In this chapter we will state and prove the Mayer–Vietoris Theorem 45.1 which gives us a similar result for the homology groups of $X = U \cup V$. As we will see shortly, the Mayer–Vietoris Theorem 45.1 is just a user friendly variation on the Excision Theorem 40.1. But we will also see that it is a very useful tool for computing homology groups. In particular, together with results of the previous chapters, we will be able to determine the homology groups of the torusand the Klein bottle.

45.1. The Mayer–Vietoris sequence.

In the following theorem and throughout the subsequent discussion we will use the following notation:

Notation. Given a topological space D and a subspace $C \subset D$ we denote the corresponding inclusion map $C \to D$ by i_C^D. To avoid pandemonium with subscripts we often drop the "$*$" when we deal with inclusion induced maps on homology, e.g. we will write $i_C^D \colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ instead of the more accurate $(i_C^D)_* \colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$. We use similar notation for inclusions of pairs of topological spaces.

The following theorem, which is the main result of this chapter, has its origin in the work of Walther Mayer [**MayeW1929**] and Leopold Vietoris [**Vie1930**].



Walther Mayer (1887-1948) with some other person

Leopold Vietoris (1891-2002) at 110

Theorem 45.1. (Mayer–Vietoris Theorem) Let X be a topological space and let $A, B \subset X$ be subspaces such that such that one of the following three conditions holds:

(1) X is the union of the interiors of A and B, i.e. $X = \mathring{A} \cup \mathring{B}$.
(2) X is a CW-complex and A and B are subcomplexes with $X = A \cup B$.
(3) X is a topological manifold and $A, B \subset X$ are two codimension-zero submanifolds such that the following conditions are satisfied:
   - $X = A \cup B$.
   - $A \cap B$ is a union of components of ∂A and it is a union of components of ∂B.
   - A and B are closed subsets of X.[343]

By (1) the Excision Theorem 40.1, (2) the CW-Excision Theorem 40.4 or (3) the Manifold–Excision Theorem 40.3 each inclusion induced map $i_{(A, A\cap B)}^{(X,B)} \colon \mathrm{H}_n(A, A \cap B) \to \mathrm{H}_n(X, B)$ is

an isomorphism. We refer to the homomorphism

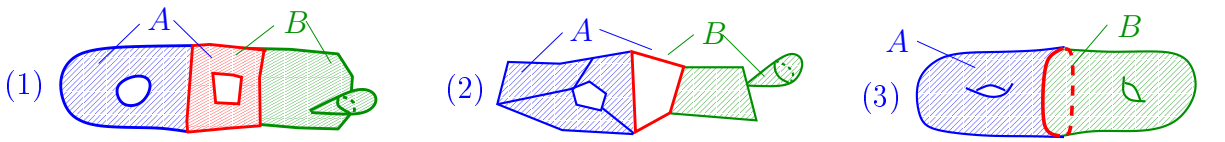
$$\mathrm{H}_n(X) \xrightarrow{\text{я}} \mathrm{H}_n(X,B) \xrightarrow{\left(i^{(X,B)}_{(A,A\cap B)}\right)^{-1}} \mathrm{H}_n(A,A\cap B) \xrightarrow[\uparrow]{\partial_n} \mathrm{H}_{n-1}(A\cap B)$$

connecting homomorphism of the long exact sequence corresponding to the pair $(A, A\cap B)$, provided by the LES-of-Pair Proposition 38.3

as the connecting homomorphism $\partial_n\colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A\cap B)$ corresponding to (X,A,B).[344] With this notation the following sequence is exact:

$$\begin{array}{ccccccc} \ldots \to \mathrm{H}_n(A\cap B) & \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} & \mathrm{H}_n(A)\oplus\mathrm{H}_n(B) & \xrightarrow{i^X_A \oplus i^X_B} & \mathrm{H}_n(X) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A\cap B)\to\ldots \\ s & \longmapsto & (i^A_{A\cap B}(s), -i^B_{A\cap B}(s)) & & & & \\ & & & & s & \longmapsto & i^X_A(s)+i^X_B(s) \end{array}$$

Totally analogous statements also hold if we replace the homology groups of $A\cap B, A, B$ and X by the corresponding reduced homology groups.[345]



A and B are open — A and B are subcomplexes — A and B are submanifolds

$$\cdots \to \mathrm{H}_n(A\cap B) \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{H}_n(A)\oplus\mathrm{H}_n(B) \xrightarrow{i^X_A\oplus i^X_B} \mathrm{H}_n(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A\cap B)\to\ldots$$

Definition. The long exact sequence of the Mayer–Vietoris Theorem 45.1 is called the Mayer–Vietoris sequence of the decomposition $X = A\cup B$.

Proof. There are two approaches to proving the Mayer–Vietoris Theorem 45.1:

(1) One can deduce the Mayer–Vietoris Theorem 45.1 directly from the Excision Theorem 40.1 using an algebraic manipulation. We will carry out this approach in Section 45.2.

(2) One can prove the Mayer–Vietoris Theorem 45.1 using the Comfortable Covering Proposition 40.7. We will carry out this approach in Section 45.3.

The first proof is slicker and it has the advantage that it applies to other functors, besides singular homology, that also satisfy a suitable version of the Excision Theorem 40.1. On the other hand the second proof makes it arguably clearer how the long exact sequence arises. ■

45.2. **First proof of the Mayer–Vietoris Theorem 45.1.** Our first proof of the Mayer–Vietoris Theorem 45.1 rests on the shoulders of the following purely algebraic lemma.

[343] Note that by the Compact-Closed Lemma 1.9 (2) this condition is satisfied if A and B are compact.

[344] It follows immediately from the definitions and the LES-of-Pair Proposition 38.3 (2) that this connecting homomorphism $\mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A\cap B)$ is natural in a suitable sense.

[345] Note though that for reduced homology groups we might need to use the fact, obtained in the Reduced Homology Lemma 39.2 (0), that $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$.

Lemma 45.2. (Barratt-Whitehead Lemma) We consider the following commutative diagram of homomorphisms between abelian groups:

$$\begin{array}{ccccccccccc}
\dots & \xrightarrow{\partial_{n+1}} & C'_n & \xrightarrow{i_n} & C_n & \xrightarrow{p_n} & C''_n & \xrightarrow{\partial_n} & C'_{n-1} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle f'_n} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f''_n} & & \downarrow{\scriptstyle f'_{n-1}} & & \\
\dots & \xrightarrow{\partial_{n+1}} & D'_n & \xrightarrow{j_n} & D_n & \xrightarrow{q_n} & D''_n & \xrightarrow{\partial_n} & D'_{n-1} & \longrightarrow & \dots
\end{array}$$

We suppose that the horizontal sequences are exact and we suppose that each $f''_n \colon C''_n \to D''_n$ is an isomorphism. For each $n \in \mathbb{N}_0$ we set $\Delta_n := \partial_n \circ (f''_n)^{-1} \circ q_n \colon D_n \to C'_{n-1}$. The following sequence is exact:

$$\dots \longrightarrow C'_n \xrightarrow{(i_n, -f'_n)} C_n \oplus D'_n \xrightarrow{f_n \oplus j_n} D_n \xrightarrow{\Delta_n} C'_{n-1} \longrightarrow \dots$$

Proof of the Barratt-Whitehead Lemma 45.2. Let us prove exactness at D_n. First we show that $\Delta_n \circ (f_n \oplus j_n) = 0$. Thus let $(c, d) \in C_n \oplus D'_n$. Then

$$\begin{array}{rcl@{\qquad}rcl}
 & & \text{definition of } \Delta_n & & & \text{since } q_n \circ j_n = 0 \\
 & & \downarrow & & & \downarrow \\
(\Delta_n \circ (f_n \oplus j_n))((c,d)) & = & (\partial_n \circ (f''_n)^{-1} \circ q_n)(f_n(c) + j_n(d)) & = & (\partial_n \circ (f''_n)^{-1} \circ q_n \circ f_n)(c) \\
 & = & (\partial_n \circ (f''_n)^{-1} \circ f''_n \circ p_n)(c) & = & (\partial_n \circ p_n)(c) \quad = \quad 0. \\
 & & \uparrow & & & \uparrow \\
 & & \text{since the diagram commutes we have } q_n \circ f_n = f''_n \circ p_n & & & \text{since } \partial_n \circ p_n = 0
\end{array}$$

Now suppose we have $d \in D_n$ with $\Delta_n(d) = 0$. Since $0 = \Delta_n(d) = \partial_n((f''_n)^{-1} \circ q_n)(d))$ and since the top horizontal sequence is exact we see that there exists a $c \in C_n$ with $p_n(c) = ((f''_n)^{-1} \circ q_n)(d)$. Next we see that

$$\begin{array}{rcl@{\qquad}rcl}
q_n(\underbrace{f_n(c) - d}_{\in D_n}) & = & (q_n \circ f_n)(c) - q_n(d) & = & (f''_n \circ p_n)(c) - q_n(d) & = & 0. \\
 & & & & \uparrow & & \uparrow \\
 & & & & \text{since the diagram commutes} & & \text{choice of } c
\end{array}$$

It follows from this calculation and the hypothesis that the bottom horizontal sequence is exact that there exists a $d' \in D'_n$ with $j_n(d') = f_n(c) - d$. Finally we calculate that

$$(f_n \oplus j_n)((c, -d')) \;=\; f_n(-c) - j_n(d') \;=\; f_n(c) - (f_n(c) - d) \;=\; d.$$

We have thus shown that the given sequence is exact at D_n. The proof of the exactness at C'_n and $C_n \oplus D'_n$ is done in a similar fashion. We leave them to the reader. ∎

With the Barratt-Whitehead Lemma 45.2 at our disposal we can easily prove the Mayer–Vietoris Theorem 45.1:

Proof of the Mayer–Vietoris Theorem 45.1. Let X be a topological space. Furthermore let $A, B \subset X$ be two subspaces. We assume that one of the following three conditions holds:

(1) $X = \mathring{A} \cup \mathring{B}$.
(2) X is a CW-complex and A and B are subcomplexes with $X = A \cup B$.
(3) X is a topological manifold and $A, B \subset X$ are two codimension-zero submanifolds such that the following conditions are satisfied:
   - $X = A \cup B$.
   - $A \cap B$ is a union of components of ∂A and it is a union of components of ∂B.
   - A and B are closed subsets of X.[346]

We consider the following commutative diagram of continuous maps between pairs of topological spaces:

$$\begin{array}{ccccc}
(A\cap B,\varnothing) & \xrightarrow{i^A_{A\cap B}} & (A,\varnothing) & \longrightarrow & (A,A\cap B)\\
\downarrow{\scriptstyle i^B_{A\cap B}} & & \downarrow{\scriptstyle i^X_A} & & \downarrow{\scriptstyle i^{(X,B)}_{(A,A\cap B)}}\\
(B,\varnothing) & \xrightarrow{i^X_B} & (X,\varnothing) & \longrightarrow & (X,B).
\end{array}$$

From the LES-of-Pair Proposition 38.3 we get two long exact sequences which, together with the vertical maps, form the following commutative diagram:

$$\begin{array}{ccccccccc}
\dots\longrightarrow & \mathrm{H}_n(A\cap B) & \xrightarrow{i^A_{A\cap B}} & \mathrm{H}_n(A) & \xrightarrow{\text{я}} & \mathrm{H}_n(A,A\cap B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A\cap B) & \longrightarrow\dots\\
& \downarrow{\scriptstyle i^B_{A\cap B}} & & \downarrow{\scriptstyle i^X_A} & & \downarrow{\scriptstyle i^{(X,B)}_{(A,A\cap B)}} & & \downarrow{\scriptstyle i^B_{A\cap B}} & \\
\dots\longrightarrow & \mathrm{H}_n(B) & \xrightarrow{i^X_B} & \mathrm{H}_n(X) & \xrightarrow{\text{я}} & \mathrm{H}_n(X,B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B) & \longrightarrow\dots.
\end{array}$$

Depending on the setting we are in we obtain from

(1) the Excision Theorem 40.1 (applied with $Z = X \setminus A$), or
(2) the Manifold–Excision Theorem 40.3, or
(3) the CW-Excision Theorem 40.4

that the inclusion induced map $i^{(X,B)}_{(A,A\cap B)}\colon \mathrm{H}_n(A,A\cap B)\to \mathrm{H}_n(X,B)$ is an isomorphism. We apply the Barratt-Whitehead Lemma 45.2 to the above setup and we obtain that the following sequence is exact:

$$\dots\to \mathrm{H}_n(A\cap B)\xrightarrow{(i^A_{A\cap B},-i^B_{A\cap B})} \mathrm{H}_n(A)\oplus\mathrm{H}_n(B)\xrightarrow{i^X_A\oplus i^X_B} \mathrm{H}_n(X)\xrightarrow{\partial_n\circ(i^{(X,B)}_{(A,A\cap B)})^{-1}\circ\text{я}} \mathrm{H}_{n-1}(A\cap B)\to\dots$$

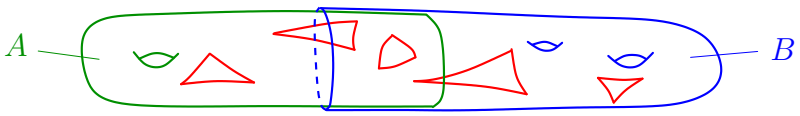
But that was exactly our goal. Finally note that by replacing the LES-of-Pair Proposition 38.3 by the Reduced LES-of-Pair Proposition 39.6 we get the analogous result for reduced homology groups. ■

45.3. **Second proof of the Mayer–Vietoris Theorem 45.1.** In this section we give our second proof of the Mayer–Vietoris Theorem 45.1. Here it is:

Proof of the Mayer–Vietoris Theorem 45.1. Let X be a topological space.

(1) First we suppose we are given two subsets $A, B \subset X$ such that $X = \mathring{A}\cup\mathring{B}$. We recall the following notation from page 471:

$$\mathrm{C}_n^{\{A,B\}}(X) \;:=\; \{a+b\,|\,a\in\mathrm{C}_n(A) \text{ and } b\in\mathrm{C}_n(B)\}.$$



singular simplices in $\mathrm{C}_2^{\{A,B\}}(X)$

The inclusion maps induce the following sequence of chain maps:

$$0\to \mathrm{C}_*(A\cap B)\xrightarrow{(i^A_{A\cap B},-i^B_{A\cap B})} \mathrm{C}_*(A)\oplus\mathrm{C}_*(B)\xrightarrow{i^X_A\oplus i^X_B} \mathrm{C}_*^{\{A,B\}}(X)\to 0.$$

Claim. The above sequence of chain maps is exact.

Proof. We prove the claim in the following four easy steps:

(a) It is clear that each of the maps $\mathrm{C}_*(A\cap B)\to\mathrm{C}_*(A)$ and $\mathrm{C}_*(A\cap B)\to\mathrm{C}_*(B)$ is already injective, in particular the map $(i^A_{A\cap B},-i^B_{A\cap B})$ is injective.

(b) It is an immediate consequence of the definition of $\mathrm{C}_*^{\{A,B\}}(X)$ that the map $i^X_A\oplus i^X_B$ is surjective.

(c) It follows immediately from the definitions that the composition of both maps is the zero map,[347] i.e. $\mathrm{im}((i^A_{A\cap B},-i^B_{A\cap B}))\subset\ker(i^X_A\oplus i^X_B)$.

(d) It remains to show that $\ker(i^X_A\oplus i^X_B)\subset\mathrm{im}((i^A_{A\cap B},-i^B_{A\cap B}))$. Thus let $x\in\mathrm{C}_n(A)$ and $y\in\mathrm{C}_n(B)$ such that $(i^X_A\oplus i^X_B)((x,y))=0$. By definition this means that x and y, viewed as elements in $\mathrm{C}_n(X)$, satisfy $x=-y$. This means that x lies in $\mathrm{C}_n(A)$ and also in $\mathrm{C}_n(B)$. Put differently, $x\in\mathrm{C}_n(A)\cap\mathrm{C}_n(B)=\mathrm{C}_n(A\cap B)$. But then we have $((i^A_{A\cap B},-i^B_{A\cap B}))(x)=x+y$. ⊞

Next we consider the following maps:

$$\begin{array}{ccccccccc}
\ldots\longrightarrow \mathrm{H}_n(A\cap B) & \xrightarrow{(i^A_{A\cap B},-i^B_{A\cap B})} & \mathrm{H}_n(A)\oplus\mathrm{H}_n(B) & \xrightarrow{i^X_A\oplus i^X_B} & \mathrm{H}_n^{\{A,B\}}(X) & \xrightarrow{p_n} & \mathrm{H}_{n-1}(A\cap B)\longrightarrow\ldots \\
 & & & \searrow^{i^X_A\oplus i^X_B} & \Phi_n\downarrow\cong & \nearrow_{:=\partial_n} & \\
 & & & & \mathrm{H}_n(X) & &
\end{array}$$

We make the following clarifications and observations:

(a) Given $n\in\mathbb{N}_0$ we denote by $\mathrm{H}_n^{\{A,B\}}(X)$ the n-th homology group of the chain complex $\mathrm{C}_*^{\{A,B\}}(X)$ and we denote by $p_n\colon\mathrm{H}_n^{\{A,B\}}(X)\to\mathrm{H}_{n-1}(A\cap B)$ the connecting homomorphism of the above short exact sequence of chain complexes, as defined on page 445.

(b) By the LES-Proposition 37.4 the top horizontal sequence is exact.

(c) It follows from the Comfortable Covering Proposition 40.7, together with our hypothesis that $X=\mathring{A}\cup\mathring{B}$, that for every $n\in\mathbb{N}_0$ the inclusion map

$$\begin{array}{rcl}\mathrm{C}_*^{\{A,B\}}(X) & \to & \mathrm{C}_*(X)\\ \Phi_n\colon\mathrm{H}_n^{\{A,B\}}(X) & \xrightarrow{\cong} & \mathrm{H}_n(X).\end{array}$$

induces an isomorphism

(d) For each $n\in\mathbb{N}_0$ we write $\partial_n:=p_n\circ\Phi_n^{-1}\colon\mathrm{H}_n(X)\to\mathrm{H}_{n-1}(A\cap B)$. Using the explicit description of the connecting homomorphisms on page 445 and in the LES-of-Pair Proposition 38.3 one can show easily that $\partial_n:=p_n\circ\Phi_n^{-1}\colon\mathrm{H}_n(X)\to\mathrm{H}_{n-1}(A\cap B)$ equals the connecting homomorphism given in the theorem.

(2) Now we suppose that we are dealing with a CW-complex X and two subcomplexes A and B with $X=A\cup B$. One can use the Subcomplex–Neighborhood Proposition 31.8 together with the Homotopy Equivalence-H_*-Corollary 36.7 to reduce the case of CW-complexes and subcomplexes to the case of an open cover. We will not carry out the technical details for CW-complexes, since we already provided a complete proof of the Mayer–Vietoris Theorem 45.1 (2) in Section 45.2.

[347] Note that here we use the minus sign that appears in the first map.

(3) Similarly to (2), as in the proof of the Manifold-Seifert–van Kampen-Theorem 23.1, we can use the Topological Collar Neighborhood Theorem [Fri23] to reduce the manifold case to the case of an open cover. As in (2) we leave the details to the reader.

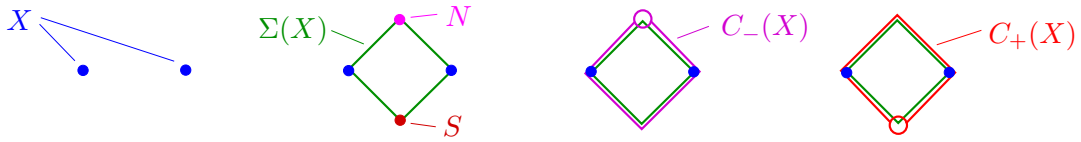
Finally note that the statements for reduced homology can be proved as above, we just need to replace $\mathrm{C}_*^{\{A,B\}}(X)$ by its reduced analogue. Alternatively one can deduce the reduced long exact sequence from the above unreduced long exact sequence and the Reduced Homology Lemma 39.2. Again there is no point in providing the details. ■

45.4. **The Mayer–Vietoris sequence applied to suspensions.** To get used to dealing with the Mayer–Vietoris sequence we will give an alternative proof of a slightly simplified version of the Suspension-H_*-Proposition 41.1:

Proposition 41.1. (Suspension-H_*-Proposition) Given any topological space X and given any $k \in \mathbb{Z}$ we have a natural isomorphism

$$\Sigma_X \colon \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \xrightarrow{\cong} \widetilde{\mathrm{H}}_k(X).$$

Proof. Let X be a topological space and let $k \in \mathbb{Z}$. As on page 483 we consider the subspaces $C_+ := C_+(X) := \Sigma(X) \setminus \{S\}$ and $C_- := C_-(X) := \Sigma(X) \setminus \{N\}$, where S and N denote the South and North Pole of the suspension $\Sigma(X)$. Note that $C_+(X)$ and $C_-(X)$ are both open subsets of $\Sigma(X)$ and that $\Sigma(X) = C_+(X) \cup C_-(X)$.



We consider the corresponding long exact sequence in reduced homology from the Mayer–Vietoris Theorem 45.1 (1):

$$\underbrace{\widetilde{\mathrm{H}}_{k+1}(C_+)}_{=0} \oplus \underbrace{\widetilde{\mathrm{H}}_{k+1}(C_-)}_{=0} \longrightarrow \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \xrightarrow{\partial_{k+1}} \widetilde{\mathrm{H}}_k(C_+ \cap C_-) \longrightarrow \underbrace{\widetilde{\mathrm{H}}_{k+1}(C_+)}_{=0} \oplus \underbrace{\widetilde{\mathrm{H}}_{k+1}(C_-)}_{=0}$$

$$\widetilde{\mathrm{H}}_k(X) \xrightarrow[\cong]{i_*} \widetilde{\mathrm{H}}_k(C_+ \cap C_-) \xrightarrow{\Sigma_X} \widetilde{\mathrm{H}}_k(X).$$

We make the following clarifications and observations:

(1) Note that both $C_+ = C_+(X)$ and $C_- = C_-(X)$ are contractible. It follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that the reduced homology groups of $C_+(X)$ and $C_-(X)$ vanish.

(2) Next note that it follows immediately from (1) that the connecting homomorphism $\partial_{k+1} \colon \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \to \widetilde{\mathrm{H}}_k(C_+(X) \cap C_-(X))$ is in fact an isomorphism.

(3) We consider the following little diagram:

$$X \xrightarrow{x \mapsto (x,0)} X \times (-1,1) \xrightarrow{(x,t) \mapsto [(x,t)]} C_+(X) \cap C_-(X).$$
$$\underset{=:i}{\curvearrowright}$$

The map to the left is evidently a homotopy equivalence. Furthermore one can easily show, either by hand, or using Lemma 41.9 (1), that the map to the right is a homeomorphism. We obtain that the map $i \colon X \to C_+(X) \cap C_-(X)$ is a homotopy equivalence.

Thus we obtain from the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 39.4 (2) that the induced map $i_*\colon \widetilde{\mathrm{H}}_k(X) \to \widetilde{\mathrm{H}}_k(C_+(X) \cap C_-(X))$ is an isomorphism.

It follows from the above that we get an isomorphism $\Sigma_X\colon \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \xrightarrow{\cong} \widetilde{\mathrm{H}}_k(X)$. It follows easily from the definition of Σ_X and the naturality of the connecting homomorphisms of the Mayer–Vietoris sequence (which is a consequence of the LES-of-Pair Proposition 38.3 (2)), that the isomorphism Σ_X is natural. ■

45.5. **The homology groups of a mapping torus.** For convenience we recall from page 209 the definition of the mapping torus:

Definition. Let X be a topological space and let $f\colon X \to X$ be a continuous map. We refer to

$$\mathrm{Tor}(X, f) \;:=\; (X \times [0,1])/(x,0) \sim (f(x),1)$$

as the mapping torus of (X, f).

Examples.

(1) Let X be a topological space. In [Fri23] it is shown that the map

$$\begin{array}{rccc} \Phi\colon & \mathrm{Tor}(X, \mathrm{id}) & \to & X \times S^1 \\ & [(x,t)] & \mapsto & (x, \exp(2\pi\,\mathrm{i}t)) \end{array}$$

is a homeomorphism.

(2) Let $X = [-1,1]$ and let $f\colon [-1,1] \to [-1,1]$ be the map that is given by $x \mapsto -x$. The corresponding mapping torus $\mathrm{Tor}([-1,1], x \mapsto -x)$ is clearly homeomorphic to the Möbius band.

$\mathrm{Tor}([-1,1], x \mapsto -x)$ $\cong$

In the Mapping Torus-π_1-Proposition 15.7 we determined the fundamental group of a given mapping torus. In many cases the following proposition allows us to determine the homology groups of a given mapping torus.

Proposition 45.3. (Mapping Torus-H_*-Proposition) Let X be a topological space and let $f\colon X \to X$ be a continuous map. We denote by $i\colon X \to \mathrm{Tor}(X, f)$ the natural map that is given by $x \mapsto [(x,0)]$. For every $n \in \mathbb{N}_0$ there exists a natural homomorphism

$$\partial_n\colon \mathrm{H}_n(\mathrm{Tor}(X, f)) \;\to\; \mathrm{H}_{n-1}(X)$$

(which is described in greater detail in the proof of the proposition) such that the following sequence is exact

$$\cdots \to \mathrm{H}_n(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_n(X) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{Tor}(X,f)) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_{n-1}(X) \to \ldots$$

Proof. Let X be a topological space and let $f\colon X \to X$ be a continuous map. For notational convenience we work in this proof with the following topological space which is clearly naturally homeomorphic to the mapping torus:

$$M \;:=\; (X \times [0,10])/(x,0) \sim (f(x),10).$$

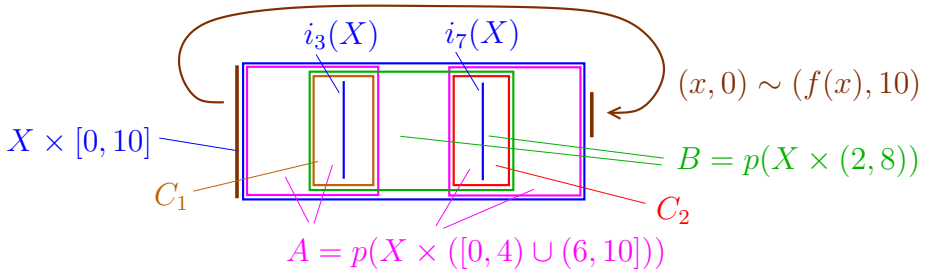
We introduce the following notation:

(1) We denote by $p\colon X \times [0,10] \to M = (X \times [0,1])/\!\sim$ the natural projection.

(2) We consider the subsets $A := p(X \times ([0,4) \cup (6,10])) \subset M$ and $B := p(X \times (2,8)) \subset M$.
(3) We consider the two subsets $C_1 = p(X \times (2,4))$ and $C_2 = p(X \times (6,8))$. Note that $A \cap B = C_1 \cup C_2$.
(4) Given $t \in [0,10]$ we consider the map

$$\begin{array}{rcl} i_t \colon X & \to & p(X \times \{t\}) \\ x & \mapsto & p(x,t). \end{array}$$

(5) We write $i_3^A \colon X \to A$ for the composition of $i_3 \colon A \to p(X \times \{3\})$ with the inclusion map $p(X \times \{3\}) \to A$. We denote analogous compositions by the obvious variations in the notation.
(6) To simplify the notation we drop the "$*$" when we write down the corresponding induced maps on homology.



We can now consider the following diagram:[348]

$$\begin{array}{ccccccccccc}
\to & \begin{matrix} \mathrm{H}_n(C_1) \\ \mathrm{H}_n(C_2) \end{matrix} & \xrightarrow{\begin{pmatrix} i^A_{C_1} & i^A_{C_2} \\ -i^B_{C_1} & -i^B_{C_2} \end{pmatrix}} & \begin{matrix} \mathrm{H}_n(A) \\ \mathrm{H}_n(B) \end{matrix} & \xrightarrow{(i^M_A \ i^M_B)} & \mathrm{H}_n(M) & \xrightarrow{\partial_n} & \begin{matrix} \mathrm{H}_{n-1}(C_1) \\ \mathrm{H}_{n-1}(C_2) \end{matrix} & \xrightarrow{\begin{pmatrix} i^A_{C_1} & i^A_{C_2} \\ -i^B_{C_1} & -i^B_{C_2} \end{pmatrix}} & \begin{matrix} \mathrm{H}_{n-1}(A) \\ \mathrm{H}_{n-1}(B) \end{matrix} & \to \\
 & \uparrow{\scriptstyle \begin{pmatrix} i_3 & 0 \\ 0 & i_7 \end{pmatrix}} \cong & & \cong \uparrow{\scriptstyle \begin{pmatrix} i_7 & 0 \\ 0 & i_7 \end{pmatrix}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \begin{pmatrix} i_3 & 0 \\ 0 & i_7 \end{pmatrix}} \cong & & \cong \uparrow{\scriptstyle \begin{pmatrix} i_7 & 0 \\ 0 & i_7 \end{pmatrix}} & \\
\to & \begin{matrix} \mathrm{H}_n(X) \\ \mathrm{H}_n(X) \end{matrix} & \xrightarrow{\begin{pmatrix} f_* & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}} & \begin{matrix} \mathrm{H}_n(X) \\ \mathrm{H}_n(X) \end{matrix} & \xrightarrow{(i_7 \ i_7)} & \mathrm{H}_n(M) & \xrightarrow{=:\partial_n} & \begin{matrix} \mathrm{H}_{n-1}(X) \\ \mathrm{H}_{n-1}(X) \end{matrix} & \xrightarrow{\begin{pmatrix} f_* & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}} & \begin{matrix} \mathrm{H}_{n-1}(X) \\ \mathrm{H}_{n-1}(X) \end{matrix} & \to \\
 & \uparrow{\scriptstyle \begin{pmatrix} -\mathrm{id} \\ \mathrm{id} \end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \begin{pmatrix} -\mathrm{id} \\ \mathrm{id} \end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}} & \\
\to & \mathrm{H}_n(X) & \xrightarrow{\mathrm{id} - f_*} & \mathrm{H}_n(X) & \xrightarrow{i_7} & \mathrm{H}_n(M) & \xrightarrow{=:\partial_n} & \mathrm{H}_{n-1}(X) & \xrightarrow{\mathrm{id} - f_*} & \mathrm{H}_{n-1}(X) & \to
\end{array}$$

We make the following clarifications and observations:

(1) Note that $A = p(X \times ([0,4) \cup (6,10])) \subset M$ and $B = p(X \times (2,8)) \subset M$ are open subsets of M that cover all of M. Furthermore note that we have $A \cap B = C_1 \sqcup C_2$.[349] On the

[348] Here we write

$$\begin{matrix} \mathrm{H}_i(C_1) \\ \mathrm{H}_i(C_2) \end{matrix} \text{ instead of } \mathrm{H}_i(C_1) \oplus \mathrm{H}_i(C_2) \quad \text{and} \quad \begin{matrix} \mathrm{H}_i(A) \\ \mathrm{H}_i(B) \end{matrix} \text{ instead of } \mathrm{H}_i(A) \oplus \mathrm{H}_i(B).$$

The reason is that we think of direct sums as columns vectors so that we can use the usual matrix notation for describing the maps between the direct sums.

[349] It follows from the Path-Component-H_*-Lemma 35.8 that we have a natural isomorphism $\mathrm{H}_n(A \cap B) = \mathrm{H}_n(C_1 \sqcup C_2) \xleftarrow{\cong} \mathrm{H}_n(C_1) \oplus \mathrm{H}_n(C_2)$.

top of the diagram we wrote down the corresponding exact Mayer–Vietoris sequence from the Mayer–Vietoris Theorem 45.1 (1).

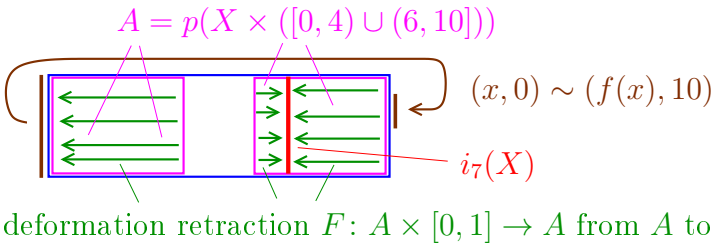
(2) We want to show that the vertical maps from the second to the first row are isomorphisms. In fact this statement follows immediately from the following claim together with the Homotopy Equivalence-H_*-Corollary 36.7 (1a).

Claim. The maps $i_3\colon X \to C_1$, $i_7\colon X \to C_2$, $i_7\colon X \to B$ and $i_7\colon X \to A$ are homotopy equivalences.

Proof. One can easily verify that the maps $i_t\colon X \to p(X \times \{t\})$, $p\colon X \times (2,4) \to C_1$, $p\colon X \times (6,8) \to C_2$ and $p\colon X \times (2,8) \to B$ are homeomorphisms. Using this observation one can easily verify that the first three maps are homotopy equivalences.

If f is not a homeomorphism, then A is not necessarily homeomorphic to a product. Thus we need to work somewhat harder. Let us consider the following map:

$$F\colon A \times [0,1] \to A$$
$$([(x,t)],s) \mapsto \begin{cases} [(x,t)], & \text{if } s \in [0,\tfrac{1}{2}] \text{ and } t \in (6,10], \\ [(x,t\cdot(1-2s))], & \text{if } s \in [0,\tfrac{1}{2}] \text{ and } t \in [0,4), \\ [(x,t\cdot(2-2s)+7\cdot(2s-1))], & \text{if } s \in [\tfrac{1}{2},1] \text{ and } t \in (6,10], \\ [(f(x),10\cdot(2-2s)+7\cdot(2s-1))], & \text{if } s \in [\tfrac{1}{2},1] \text{ and } t \in [0,4). \end{cases}$$



First note that it follows from $(x,0) \sim (f(x),10)$ that this map F is well-defined. One can now easily verify, say using the Pasting Proposition 2.4, that the map F is continuous. It is now clear that the above map F is a deformation retraction from A to $i_7(X)$. This implies, by the Deformation Retract–Homotopy Equivalence Lemma 9.3, that $i_7\colon X \to A$ is a homotopy equivalence. ⊞

(3) Next we want to show that the top part of the diagram commutes.

(a) We start with the square to the left. We need to show that the following equality holds:

$$\underbrace{\begin{pmatrix} i^A_{C_1} & i^A_{C_2} \\ -i^B_{C_1} & -i^B_{C_2} \end{pmatrix}\begin{pmatrix} i_3^{C_1} & 0 \\ 0 & i_7^{C_2} \end{pmatrix}}_{=\begin{pmatrix} i_3^A & i_7^A \\ -i_3^B & -i_7^B \end{pmatrix}} = \underbrace{\begin{pmatrix} i_7^A & 0 \\ 0 & i_7^B \end{pmatrix}\begin{pmatrix} f_* & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}}_{=\begin{pmatrix} i_7^A \circ f_* & i_7^A \circ \mathrm{id} \\ i_7^B \circ (-\mathrm{id}) & i_7^B \circ (-\mathrm{id}) \end{pmatrix}}.$$

If we compare the two matrices we see that it remains to prove the following claim.

Claim. For each $n \in \mathbb{N}_0$ we have $i_3^A = i_7^A \circ f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(A)$ and we also have $i_3^B = i_7^B\colon \mathrm{H}_n(X) \to \mathrm{H}_n(B)$.

Proof. We consider the maps[350]

$$X\times[0,1] \to A \qquad \& \qquad X\times[0,1] \to B$$
$$(x,t) \mapsto \begin{cases} p(x,3-6t), & \text{if } t\in[0,\tfrac12],\\ p(f(x),13-6t) & \text{if } t\in[\tfrac12,1]\end{cases} \qquad (x,t)\mapsto p(x,3\cdot(1-t)+7\cdot t).$$

These two maps are homotopies between the maps $i_3^A, i_7^A\circ f\colon X\to A$ and the maps $i_3^B, i_7^B\colon X\to B$. The claim now follows from the Homotopy-H_*-Proposition 36.5. ⊞

(b) The second square commutes by the functoriality of homology groups.

(c) Recall that in (1) we just showed that the vertical maps from the second to the first row are isomorphisms. Therefore we can define the connecting homomorphism $\partial_n\colon \mathrm{H}_n(M)\to \begin{smallmatrix}\mathrm{H}_{n-1}(X)\\ \mathrm{H}_{n-1}(X)\end{smallmatrix}$ in the second row in such a way that the third square commutes.

(4) It is straightforward to see that the bottom squares in the above diagram commute. We know that the middle sequence is exact. The following claim implies that the bottom sequence is also exact:

Claim. Suppose we are given the following commutative diagram of homomorphisms of abelian groups:

$$\begin{array}{ccccccccccc}
\longrightarrow & \begin{smallmatrix}A_n\\A_n\end{smallmatrix} & \xrightarrow{\begin{pmatrix}\varphi_n & \mathrm{id}\\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}} & \begin{smallmatrix}A_n\\A_n\end{smallmatrix} & \xrightarrow{(\alpha_n\ \alpha_n)} & B_n & \xrightarrow{\delta_n} & \begin{smallmatrix}A_{n-1}\\A_{n-1}\end{smallmatrix} & \xrightarrow{\begin{pmatrix}\varphi_{n-1} & \mathrm{id}\\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}} & \begin{smallmatrix}A_{n-1}\\A_{n-1}\end{smallmatrix} & \longrightarrow\\
 & \uparrow{\scriptstyle \begin{pmatrix}-\mathrm{id}\\ \mathrm{id}\end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix}\mathrm{id}\\0\end{pmatrix}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \begin{pmatrix}-\mathrm{id}\\ \mathrm{id}\end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix}\mathrm{id}\\0\end{pmatrix}} & \\
\longrightarrow & A_n & \xrightarrow{\mathrm{id}-\varphi_n} & A_n & \xrightarrow{\alpha_n} & B_n & \xrightarrow{\delta_n} & A_{n-1} & \xrightarrow{\mathrm{id}-\varphi_{n-1}} & A_{n-1} & \longrightarrow
\end{array}$$

If the top sequence is exact, then the bottom sequence is also exact.

Proof. We consider the following vertically expanded diagram:

$$\begin{array}{ccccccccccc}
\longrightarrow & A_n & \xrightarrow{-\mathrm{id}} & A_n & \longrightarrow & 0 & \longrightarrow & A_{n-1} & \xrightarrow{-\mathrm{id}} & A_{n-1} & \longrightarrow\\
 & \uparrow{\scriptstyle (\mathrm{id}\ \ \mathrm{id})} & & \uparrow{\scriptstyle (0\ \ \mathrm{id})} & & \uparrow & & \uparrow{\scriptstyle (\mathrm{id}\ \ \mathrm{id})} & & \uparrow{\scriptstyle (0\ \ \mathrm{id})} & \\
\longrightarrow & \begin{smallmatrix}A_n\\A_n\end{smallmatrix} & \xrightarrow{\begin{pmatrix}\varphi_n & \mathrm{id}\\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}} & \begin{smallmatrix}A_n\\A_n\end{smallmatrix} & \xrightarrow{(\alpha_n\ \alpha_n)} & B_n & \xrightarrow{\delta_n} & \begin{smallmatrix}A_{n-1}\\A_{n-1}\end{smallmatrix} & \xrightarrow{\begin{pmatrix}\varphi_{n-1} & \mathrm{id}\\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}} & \begin{smallmatrix}A_{n-1}\\A_{n-1}\end{smallmatrix} & \longrightarrow\\
 & \uparrow{\scriptstyle \begin{pmatrix}-\mathrm{id}\\ \mathrm{id}\end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix}\mathrm{id}\\0\end{pmatrix}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \begin{pmatrix}-\mathrm{id}\\ \mathrm{id}\end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix}\mathrm{id}\\0\end{pmatrix}} & \\
\longrightarrow & A_n & \xrightarrow{\mathrm{id}-\varphi_n} & A_n & \xrightarrow{\alpha_n} & B_n & \xrightarrow{\delta_n} & A_{n-1} & \xrightarrow{\mathrm{id}-\varphi_{n-1}} & A_{n-1} & \longrightarrow
\end{array}$$

By hypothesis the bottom part of the diagram commutes. It follows easily from the definitions that the top part also commutes. Furthermore note that the vertical sequences, with 0 added on top and bottom, are exact.

Next note that evidently the top sequence is exact. Furthermore by hypothesis the middle sequence is exact. Since the bottom vertical maps are monomorphisms we see that the bottom sequence is a chain complex.

[350]Note that it follows from $(x,0)\sim(f(x),10)$ that the map to the left is well-defined. It now follows from the Homotopy Combination Lemma 8.2 that the map to the left is indeed continuous.

It follows from the above discussion that all these maps form a short exact sequence of chain complexes. Clearly the homology groups of the top chain complex are zero. By hypothesis the homology groups of the middle chain complex are also zero. It follows from the LES-Proposition 37.4 that the homology groups of the bottom chain complex are also zero. But this means that the bottom sequence is exact.

We have thus completed the proof that the bottom horizontal sequence of the diagram on page 528 is exact. If one goes carefully through the definitions one sees that the connecting homomorphism $\partial_n\colon \mathrm{H}_n(\mathrm{Tor}(X,f)) \to \mathrm{H}_{n-1}(X)$ is given by the composition of the homomorphisms in the following unreadable sequence of homomorphisms:[351]

$$\begin{aligned}\mathrm{H}_n(\mathrm{Tor}(X,f)) &\xrightarrow{\text{я}} \mathrm{H}_n(\mathrm{Tor}(X,f),A) \xleftarrow{\cong} \mathrm{H}_n(B, C_1\cup C_2)\\ &\xrightarrow{\partial_n} \mathrm{H}_{n-1}(C_1\cup C_2) \quad \xleftarrow{\cong} \mathrm{H}_{n-1}(C_1)\oplus\mathrm{H}_{n-1}(C_2) \to \mathrm{H}_{n-1}(C_2) \underset{i_7^{C_2}}{\overset{\cong}{\leftarrow}} \mathrm{H}_{n-1}(X).\end{aligned}$$

■

Now we can easily prove the following proposition.

Proposition 45.4. (Product-with-S^1-H_*-Proposition) Let X be a topological space. We denote by $p\colon X\times S^1 \to X$ the natural projection. For every $n\in\mathbb{N}_0$ there exists a natural[352] homomorphism $\psi_X\colon \mathrm{H}_n(X\times S^1)\to \mathrm{H}_{n-1}(X)$ (which will be described in greater detail in the proof) such that the map

$$\Xi_X := p_* \oplus \psi_X\colon \mathrm{H}_n(X\times S^1) \xrightarrow{\cong} \mathrm{H}_n(X)\oplus\mathrm{H}_{n-1}(X)$$

is an isomorphism.[353]

Proof. Let $n\in\mathbb{N}_0$. It is pretty elementary to show that the map

$$\begin{aligned}\Theta\colon \mathrm{Tor}(X,\mathrm{id}) &\to X\times S^1\\ [(x,t)] &\mapsto (x,\exp(2\pi\,\mathrm{i}t))\end{aligned}$$

is a natural homeomorphism. We denote by $i\colon X\to \mathrm{Tor}(X,\mathrm{id})$ the natural map that is given by $x\mapsto[(x,0)]$ and we denote by $j\colon X\to X\times S^1$ the natural embedding that is given by $x\mapsto(x,1)$. Next we consider the following diagram:

$$\begin{array}{ccccccccc}
\mathrm{H}_n(X) & \xrightarrow{\mathrm{id}_*-\mathrm{id}_*} & \mathrm{H}_n(X) & \xrightarrow{i_*} & \mathrm{H}_n(\mathrm{Tor}(X,\mathrm{id})) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(X) & \xrightarrow{\mathrm{id}_*-\mathrm{id}_*} & \mathrm{H}_{n-1}(X)\\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \overset{\cong}{\downarrow}{\scriptstyle \Theta_*} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathrm{H}_n(X) & \underset{p_*}{\overset{j_*}{\rightleftarrows}} & \mathrm{H}_n(X\times S^1) & \xrightarrow{\partial_n\circ\Theta_*^{-1}} & \mathrm{H}_{n-1}(X) & \longrightarrow & 0.
\end{array}$$

[351] The second homomorphism is an isomorphism by the Excision Theorem 40.1, the third homomorphism is the connecting homomorphism of the given pair of topological spaces, the fourth homomorphism is the isomorphism given by the Path-Component-H_*-Lemma 35.8, the fifth homomorphism is the natural projection and the last isomorphism is induced by the map $i_7^{C_2}\colon X\to C_2 = p(X\times(\frac{6}{10},\frac{8}{10}))$, which is easily seen to be a homotopy equivalence.

[352] Here natural means that for any continuous map $f\colon X\to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}_n(X\times S^1) & \xrightarrow[\cong]{\psi_X} & \mathrm{H}_{n-1}(X)\\
\downarrow{\scriptstyle (f\times\mathrm{id})_*} & & {\scriptstyle f_*}\downarrow\\
\mathrm{H}_n(Y\times S^1) & \xrightarrow[\cong]{\psi_Y} & \mathrm{H}_{n-1}(Y).
\end{array}$$

[353] The direct sum of natural homomorphisms is of course also natural.

We make the following clarifications and observations:

(1) The top sequence is the short exact sequence that we obtain from the Mapping Torus-H_*-Proposition 45.3.
(2) Since $\mathrm{id}_* - \mathrm{id}_* = 0$ we see that the homomorphisms to the top left and top right are the zero map.
(3) Since Θ is a homeomorphism and since $\Theta \circ i = j$ we obtain from (1) and (2) that the bottom sequence is also exact.
(4) The map $p\colon X \times S^1 \to X$ is the natural map that is given by projection onto the first factor. Note that $p_* \circ j_* = (p \circ j)_* = \mathrm{id}$.
(5) We define

$$\begin{array}{rcl} \psi_X\colon \mathrm{H}_n(X \times S^1) & \to & \mathrm{H}_{n-1}(X) \\ \sigma & \mapsto & (\partial_n \circ \Theta_*^{-1})(\sigma - (j_* \circ p_*)(\sigma)). \end{array}$$

It is straightforward to verify that this homomorphism is natural.[354]

Finally note that as in the proof of the Splitting Lemma 37.2 (3)⇒(2) one can show that the map

$$\begin{array}{rcl} \Xi_X\colon \mathrm{H}_n(X \times S^1) & \to & \mathrm{H}_n(X) \oplus \mathrm{H}_{n-1}(X) \\ \sigma & \mapsto & p_*(\sigma) \oplus \underbrace{(\partial_n \circ \Theta_*^{-1})(\sigma - (j_* \circ p_*)(\sigma))}_{=\psi_X(\sigma)} \end{array}$$

is an isomorphism. ■

Using the Product-with-S^1-H_*-Proposition 45.4 we can now finally compute the homology groups of tori $(S^1)^n$:

Proposition 45.5. (Torus-H_*-Proposition) Let $n \in \mathbb{N}_0$.

(1) For $k = 0, \dots, n$ we have

$$\mathrm{H}_k(n\text{-dimensional torus } (S^1)^n) \;=\; \text{free abelian group of rank } \binom{n}{k}$$

and all other homology groups of $(S^1)^n$ vanish.

(2) For $i = 1, \dots, n$ we consider the singular 1-cycles

$$\begin{array}{rcl} \alpha_i\colon \Delta^1 & \to & (S^1)^n \\ (t, 1-t) & \mapsto & (1, \dots, 1, \underbrace{\exp(2\pi i t)}_{\substack{\uparrow \\ i\text{-th coordinate}}}, 1, \dots, 1). \end{array}$$

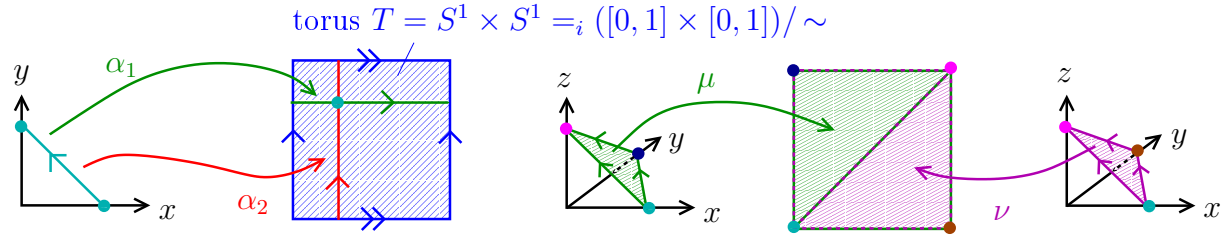
These singular 1-cycles represent a basis for $\mathrm{H}_1((S^1)^n) \cong \mathbb{Z}^n$.

Example. Let $T = S^1 \times S^1 = ([0,1] \times [0,1])/\sim$ be the 2-dimensional torus. It follows from the Torus-H_*-Proposition 45.5 and the Detect-Cycle Proposition 42.3 that

$$\mathrm{H}_n(T) \;=\; \begin{cases} \mathbb{Z} \cdot [P], & \text{if } n = 0, \\ \mathbb{Z} \cdot [\alpha_1] \oplus \mathbb{Z} \cdot [\alpha_2], & \text{if } n = 1, \\ \mathbb{Z} \cdot [\mu - \nu], & \text{if } n = 2, \\ 0, & \text{otherwise,} \end{cases}$$

[354] By the discussion on page 531 one can also describe ψ_X even more explicitly.

where $P \in T$ is any point and where $\alpha_1, \alpha_2 \colon \Delta^1 \to T$ and $\mu, \nu \colon \Delta^2 \to T$ are indicated in the figure below.[355]



Proof.

(1) We prove this statement by induction on n. For $n = 0$ this statement follows from the Singleton-H_*-Lemma 35.11. The induction step is a straightforward consequence of the Product-with-S^1-H_*-Proposition 45.4. We leave it to the reader to fill in the details.

(2) For $i = 1, \dots, n$ let $p_i \colon (S^1)^n \to S^1$ be the projection onto the i-th factor. We consider the homomorphism

$$\begin{array}{rcl} \Psi \colon \mathrm{H}_1((S^1)^n) & \to & \mathrm{H}_1(S^1) \times \cdots \times \mathrm{H}_1(S^1) \\ \alpha & \mapsto & (p_{1*}(\alpha_1), \dots, p_{n*}(\alpha_n)) \end{array}$$

We make the following observations:

(a) By Statement (1) we know that Ψ is a homomorphism between abelian groups that are both isomorphic to $\mathbb{Z}^n$.

(b) On page 502 we showed that the singular 1-cycle $\alpha \colon \Delta^1 \to S^1$ that is given by $(t, 1-t) \mapsto \exp(2\pi i t)$ represents a generator of $\mathrm{H}_1(S^1)$.

(c) It follows easily from (b) and the discussion on page 430 that Ψ is an epimorphism that sends $[\alpha_1], \dots, [\alpha_n]$ to a basis of $\mathrm{H}_1(S^1) \times \cdots \times \mathrm{H}_1(S^1)$.

(d) It follows from (a) and (c)and some elementary algebra that Ψ is an isomorphism.

(e) It follows from (c) and (d) that $[\alpha_1], \dots, [\alpha_n]$ is a basis of $\mathrm{H}_1((S^1)^n)$. ■

The Product-with-S^1-H_*-Proposition 45.4 also provides us with the tools for computing the homology groups of the Klein bottle:

Lemma 45.6. (Klein Bottle-H_*-Lemma) Let

$$K \;=\; ([0,1] \times [0,1])/\sim \qquad \text{where } (x,0) \sim (x,1) \text{ and } (0,y) \sim (1, 1-y)$$

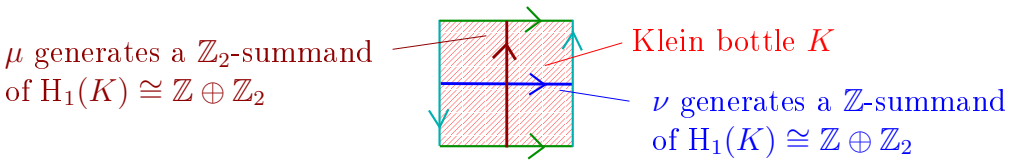
be the Klein bottle as defined on page 108. We consider the two singular 1-cycles

$$\begin{array}{rcl} \mu \colon \Delta^1 & \to & K \\ (t, 1-t) & \mapsto & [(\frac{1}{2}, t)] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \nu \colon \Delta^1 & \to & K \\ (t, 1-t) & \mapsto & [(\frac{1}{2}, t)] \end{array}$$

For any $P \in K$ we have

$$\mathrm{H}_n(K) \;=\; \begin{cases} \mathbb{Z} \cdot [P], & \text{if } n = 0, \\ \mathbb{Z}_2 \cdot [\mu] \oplus \mathbb{Z} \cdot [\nu], & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

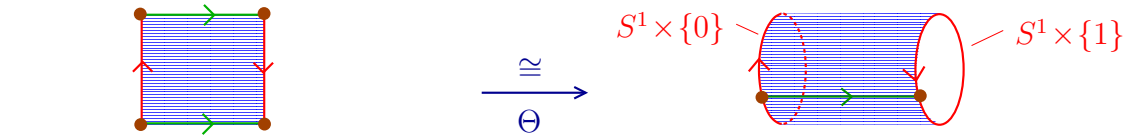
[355] In particular this answers our question from page 426, the cycle $\alpha - \beta$ does indeed represent a non-trivial element in $\mathrm{H}_2(T)$.



Proof. We consider $X = S^1$ and we consider the self-homeomorphism $f\colon S^1 \to S^1$ that is given by $f(z) = \overline{z} = z^{-1}$. It is not difficult to show that the map

$$\begin{array}{rcl} \Theta\colon ([0,1]\times[0,1])/\sim & \to & \mathrm{Tor}(S^1, z \mapsto \overline{z}) \\ [(x,y)] & \mapsto & [(\exp(2\pi\,\mathrm{i}x), y)] \end{array}$$

is a homeomorphism. We will use this homeomorphism as an excuse to study the homology groups of the mapping torus $\mathrm{Tor}(S^1, z \mapsto \overline{z})$.



Klein bottle $([0,1]\times[0,1])/\sim$ mapping torus $\mathrm{Tor}(S^1, z \mapsto \overline{z})$

First note that it follows from the Mapping Torus-H_*-Proposition 45.3 that we get the long exact sequence

$$\begin{array}{ccccccccc} & & & & 0 & \longrightarrow & \mathrm{H}_2(\mathrm{Tor}(S^1,f)) & \xrightarrow{\partial_2} & \\ \to & \mathrm{H}_1(S^1) & \xrightarrow{f_*-\mathrm{id}} & \mathrm{H}_1(S^1) & \xrightarrow{i_*} & \mathrm{H}_1(\mathrm{Tor}(S^1,f)) & \xrightarrow{\partial_1} & & \\ \to & \mathrm{H}_0(S^1) & \xrightarrow{f_*-\mathrm{id}} & \mathrm{H}_0(S^1) & \xrightarrow{i_*} & \mathrm{H}_0(\mathrm{Tor}(S^1,f)) & \longrightarrow & 0. & \end{array}$$

To get our hands on the homology groups of $K \cong \mathrm{Tor}(S^1, f)$ we make the following observations:

(1) By the Sphere-H_n-Proposition 41.2 we know that $\mathrm{H}_0(S^1) \cong \mathbb{Z}$, $\mathrm{H}_1(S^1) \cong \mathbb{Z} \cdot [\alpha]$ where $\alpha(1-t,t) = \exp(2\pi\,\mathrm{i}t)$, and that $\mathrm{H}_k(S^1) = 0$ for $k \geq 2$.

(2) It follows from the H_0-Proposition 35.10 (2) that the induced map $f_*\colon \mathrm{H}_0(S^1) \to \mathrm{H}_0(S^1)$ is the identity map.

(3) It follows from the Sphere–Degree–Matrix Lemma 43.4 that the induced homomorphism $f_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1)$ is given by multiplication by -1.

It follows easily from these observations that $\mathrm{H}_2(K) \cong \mathrm{H}_2(\mathrm{Tor}(S^1,f)) = 0$ and that $\mathrm{H}_0(K) \cong \mathrm{H}_0(\mathrm{Tor}(S^1,f)) \cong \mathbb{Z}$. Furthermore it follows that we have a short exact sequence

$$0 \to \underbrace{\mathrm{coker}(\mathrm{H}_1(S^1) \xrightarrow{\cdot(-2)} \mathrm{H}_1(S^1))}_{\cong \mathbb{Z}_2} \xrightarrow{i_*} \mathrm{H}_1(\mathrm{Tor}(S^1,f)) \xrightarrow{\partial_1} \underbrace{\mathrm{H}_0(S^1)}_{\cong \mathbb{Z}} \to 0.$$

Note that $i_*([\alpha]) = [\Theta \circ \mu]$. Furthermore, using the explicit description of the connecting homomorphisms ∂_1 on page 531 and using the Singular 1-Simplex Lemma 35.12 one can show that $\partial_1([\Theta \circ \nu])$ is a generator of $\mathrm{H}_0(S^1)$. Using the Free Abelian Group–Splitting Lemma 18.5 together with the Splitting Lemma 37.2 (1)⇒(3) we now deduce that

$$\mathrm{H}_1(\mathrm{Tor}(S^1,f)) \;=\; \mathbb{Z}_2 \cdot \underbrace{[\Theta \circ \nu]}_{=\Theta_*([\mu])} \oplus \mathbb{Z} \cdot \underbrace{[\Theta \circ \nu]}_{=\Theta_*([\nu])}.$$

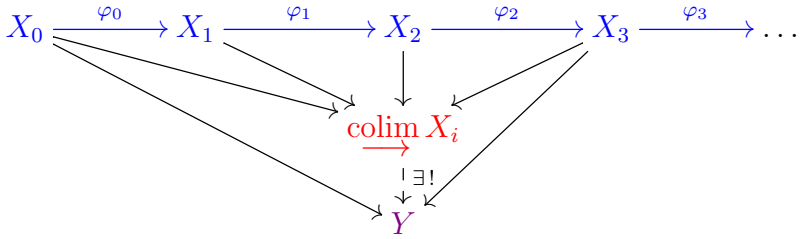
Finally note that by the isomorphism $\Theta_*\colon \mathrm{H}_1(K) \to \mathrm{H}_1(\mathrm{Tor}(S^1, f))$ this implies that $\mathrm{H}_1(K) = \mathbb{Z}_2 \cdot [\mu] \oplus \mathbb{Z} \cdot [\nu]$. ■

46. Homology groups and direct limits

On several occasions we were already wondering whether various functors, e.g. π_1, π_n and H_n, preserve (co-) limits. In particular in the Exhaustion-π_1-Proposition 26.2 and in the CW-Direct Limit-π_n-Proposition 30.16 we studied the question whether (higher) homotopy groups preserve direct limits.

In this chapter we will discuss the question to what degree homology groups preserve direct limits. The results of this chapter will allow us in particular to determine the homology groups of suitable non-compact topological spaces. On a more theoretical level, the results of this chapter will be important in the next chapter when we study cellular homology of CW-complexes.

46.1. **Direct limits of topological spaces and homology.** First let us remind the reader of the notion of a direct limit of a sequence of morphisms, as defined on page 308: Given a sequence $X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$ of morphisms the notion of a direct limit can be summarized in the following diagram:



More precisely, a direct limit $\operatorname{colim} X_i$, if it exists, comes with morphisms $X_i \to \operatorname{colim} X_i$ which form a commutative diagram; furthermore given morphisms $X_i \to Y$ which form commutative diagrams, we get a unique morphism $\operatorname{colim} X_i \to Y$ which makes everything commute.

Next let us look at two of our favorite categories and types of sequences:

- First we consider the category *Top* of topological spaces. Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset \dots$ be a sequence of subspaces of X with $X = \bigcup_{i \in \mathbb{N}_0} X_i$. We suppose that one of the following two conditions is satisfied:
  (1) Each X_i is contained in the interior of X_{i+1}.
  (2) X is a CW-complex and each X_i is a subcomplex of X.
  It follows from (1) the Union-equals-Direct Limit Criterion 25.6 and (2) the Direct Limit–CW Lemma 33.6 that in these two settings we have $X = \operatorname{colim} X_i$.
- Next we consider the category *AbGroup* of abelian groups. If we are given a sequence $A_0 \xrightarrow{\varphi_0} A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \dots$ of homomorphisms between abelian groups, then it follows easily from the Algebraic–Direct Limit Existence Proposition 25.3 (1b), that[356] [357]

$$\operatorname{colim} A_i \;=\; \Big(\bigoplus_{i \in \mathbb{N}_0} A_i \Big) / \langle \varphi_i(a) - a \mid \; i \in \mathbb{N}_0 \text{ and } a \in A_i \rangle.$$

[356] Given a group G and a subset $S \subset G$ we denote by $\langle S \rangle \subset G$ the subgroup generated by S, see page 247 for details.

[357] We use the natural monomorphism $A_j \to \bigoplus_{i \in \mathbb{N}_0}, A_i$ to view A_j as a subgroup of $\bigoplus_{i \in \mathbb{N}_0}, A_i$.

The following lemma, together with the above discussion, shows that in reasonable circumstances for each $n \in \mathbb{N}_0$ the singular chain group functor $X \mapsto \mathrm{C}_n(X)$ from the category *Top* of topological spaces to the category *AbGroup* of abelian groups preserves direct limits.

Lemma 46.1. (Exhaustion-C_*-Lemma) Let X be a topological space. Furthermore let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of subspaces of X with $\bigcup\limits_{i\in\mathbb{N}_0} X_i = X$. Let $n \in \mathbb{N}_0$. By the universal property of direct limits the inclusion induced maps $\mathrm{C}_n(X_i) \to \mathrm{C}_n(X)$ induce a natural homomorphism

$$\varinjlim_i \mathrm{C}_n(X_i) \;\to\; \mathrm{C}_n(X).$$

If one of the following conditions is satisfied

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X,

then the above map $\varinjlim\limits_i \mathrm{C}_n(X_i) \to \mathrm{C}_n(X)$ is an isomorphism.

Proof. Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of subspaces with $\bigcup\limits_{i\in\mathbb{N}_0} X_i = X$ such that one of the following conditions is satisfied:

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X.

We need to show that for each $n \in \mathbb{N}_0$ the natural map $\varinjlim\limits_i \mathrm{C}_n(X_i) \to \mathrm{C}_n(X)$ is an isomorphism. We start out with the following claim:

Claim. For each compact subset $K \subset X$ there exists an $i \in \mathbb{N}_0$ with $K \subset X_i$.

Proof. Under the hypothesis (1) the claim follows immediately from the elementary Compact Subspace–Exhaustion Lemma 20.11, whereas under the hypothesis (2) the claim follows immediately from the CW-Complex-Finiteness Theorem 30.10. ⊞

In the following, given a topological space Y we denote by $\mathrm{S}_n(Y)$ the set of singular n-simplices in Y. We consider the following diagram:

$$\begin{array}{ccccc}
\varinjlim\limits_i \mathrm{C}_n(X_i) & & \longrightarrow & & \mathrm{C}_n(X) \\
\big\downarrow{\scriptstyle \mathrm{id}} & & & & \big\downarrow{\scriptstyle \mathrm{id}} \\
\varinjlim\limits_i \mathbb{Z}^{(\mathrm{S}_n(X_i))} & \xrightarrow{\;\cong\;} & \mathbb{Z}^{\left(\bigcup\limits_{i\in\mathbb{N}_0} \mathrm{S}_n(X_i)\right)} & \xrightarrow{\;\mathrm{id}\;} & \mathbb{Z}^{(\mathrm{S}_n(X))}.
\end{array}$$

The lemma follows from the following observations:

(a) The top map and the bottom left map are induced by the inclusions $X_i \to X$.
(b) It follows easily from (a) and the definitions that the diagram commutes.
(c) On page 309 we saw that the bottom left homomorphism is an isomorphism.
(d) It follows immediately from the fact that Δ^n is compact, the Compact Image Lemma 2.8 and the above claim that the image of any continuous map $\Delta^n \to X$ is contained in some X_i. It follows from this observation and the hypothesis that $X = \bigcup\limits_{i\in\mathbb{N}_0} X_i$ that $\mathrm{S}_n(X) = \bigcup\limits_{i\in\mathbb{N}_0} \mathrm{S}_n(X_i)$. In other words, the bottom right map is indeed the identity.

(e) It follows from (b), (c) and (d) that the top homomorphism is also an isomorphism. ■

Next we would like to understand the relationship between direct limits and homology groups. The key to doing so is the following purely algebraic lemma.

Lemma 46.5. (Direct Limit-Homology Lemma) Suppose that for each $i \in \mathbb{N}_0$ we are given a chain complex $(C_{i*}, \partial_*) = (\{C_{in}\}_{n\in\mathbb{N}_0}, \{\partial_n\}_{n\in\mathbb{N}_0})$. Furthermore suppose that we are given a sequence $C_{0*} \to C_{1*} \to C_{2*} \to \dots$ of chain maps.

(1) The groups $\operatorname{colim}_{\longrightarrow i} C_{in}$ with $n \in \mathbb{N}_0$ naturally form a chain complex such that the natural maps $C_{j*} \to \operatorname{colim}_{\longrightarrow i} C_{i*}$ are chain maps.

(2) For each $n \in \mathbb{N}_0$ the homomorphisms $\mathrm{H}_n(C_{j*}) \to \mathrm{H}_n\big(\operatorname{colim}_{\longrightarrow i} C_{i*}\big)$ from (1) induce an isomorphism

$$\operatorname*{colim}_{\longrightarrow j} \mathrm{H}_n(C_{j*}) \xrightarrow{\cong} \mathrm{H}_n\big(\operatorname*{colim}_{\longrightarrow i} C_{i*}\big).$$

Proof. To keep the conversation going we outsource the proof of this algebraic lemma to Section 46.2. In fact in that section we will formulate and prove a slightly more general statement which we will need later on in a different setting. ■

By the discussion on page 536 the following proposition shows in particular that homology groups preserve many direct limits.

Proposition 46.2. (Exhaustions-H_*-Proposition) Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset \dots$ be a sequence of subspaces such that $X = \bigcup_{i\in\mathbb{N}_0} X_i$ and such that one of the following conditions holds:

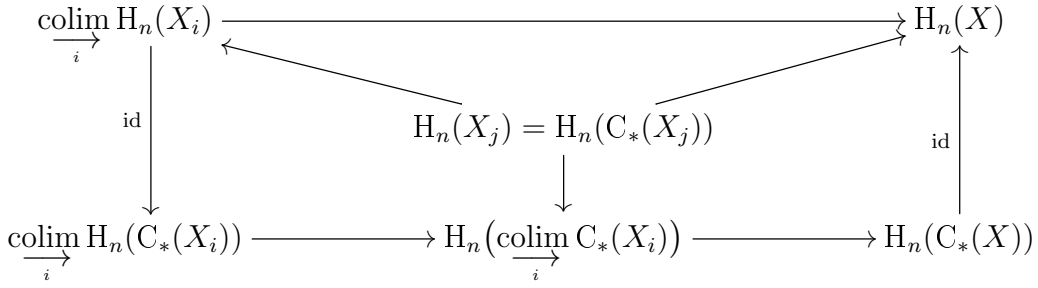
(1) each X_i is contained in the interior of X_{i+1}, or

(2) X is a CW-complex and each X_i is a subcomplex of X,

then for any $n \in \mathbb{N}_0$ the inclusion induced maps $\mathrm{H}_n(X_i) \to \mathrm{H}_n(X)$ induce an isomorphism[358]

$$\operatorname*{colim}_{\longrightarrow} \mathrm{H}_n(X_i) \xrightarrow{\cong} \mathrm{H}_n(X).$$

Proof. For each $j \in \mathbb{N}_0$ we consider the following diagram:



One can easily verify that each segment of the diagram involving $\mathrm{H}_n(X_j)$ commutes. It follows from the universal property of direct limits that the total diagram commutes.

[358] This statement is very similar to the statements of the Exhaustion-π_1-Proposition 26.2 and the CW-Direct Limit-π_n-Proposition 30.16. In fact, as we will see, the proof of our current proposition is at least philosophically also very similar to the earlier proofs.

Next note that the bottom left map is an isomorphism by the purely algebraic Direct Limit-Homology Lemma 46.5 and that the bottom right map is an isomorphism by the Exhaustion-C_*-Lemma 46.1 (here we use our hypothesis on the sequence of subspaces). It follows that the top map is also an isomorphism. ■

We conclude this section with an example:

Example. For any $n \in \mathbb{N}_0$ we have

$$\begin{array}{ccccccc} \mathrm{H}_n(S^\infty) & = & \varinjlim_k \mathrm{H}_n(S^k) & = & \varinjlim_k \begin{cases} \mathbb{Z}, & \text{if } n = k, 0, \\ 0, & \text{else} \end{cases} & = & \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{else.} \end{cases} \\ \uparrow & & & & \uparrow & & \uparrow \end{array}$$

by the Homology-via-Exhaustions Proposition 46.2, where we view S^∞ as a CW-complex as on page 375

by the Sphere-H_n-Proposition 41.2

by the Direct Limit Stabilization Lemma 25.1

46.2. **Direct limits of chain complexes.** In this section we will provide a proof of the purely algebraic Direct Limit-Homology Lemma 46.5. In fact we will consider a slightly more general statement than before, namely we will now consider general direct systems instead of just considering sequences.

First we recall the following definition from pages 307 and 308.

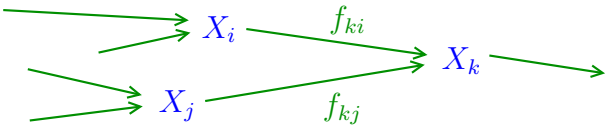
Definition.

(1) A directed set is a set I together with a partial order $\leq$ that has the extra property that for any $i, j \in I$ there exists a $k \in I$ with $i \leq k$ and $j \leq k$. (In most cases the directed set will just be $(\mathbb{N}_0, \leq)$.)

(2) Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. A direct system in the category $\mathcal{C}$ over I is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ji} \colon X_i \to X_j\}_{i \leq j}$ such that the following two conditions are satisfied:

(a) $f_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$,

(b) $f_{ki} = f_{kj} \circ f_{ji}$ for all $i, j, k \in I$ with $i \leq j \leq k$.



Example. Let $\mathcal{C}$ be a category. Any sequence $X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$ of morphisms in $\mathcal{C}$ gives rise to a direct system over the directed set $(\mathbb{N}_0, \leq)$ by considering for $i \leq j$ the morphism $f_{ji} := \varphi_{j-1} \circ \dots \circ \varphi_{i+1} \circ \varphi_i \colon X_i \to X_j$.

We continue with a new definition.

Definition. Let $(I, \leq)$ be a directed set and let $(\{K_i\}_{i\in I}, \{\kappa_{ji}\}_{i\leq j})$ and $(\{L_i\}_{i\in I}, \{\lambda_{ji}\}_{i\leq j})$ be direct systems in some category $\mathcal{C}$.

(1) A morphism between the above two direct systems is a family $\{f_i\colon K_i \to L_i\}_{i\in I}$ of morphisms such that for any $i \leq j$ the following diagram commutes:

$$\begin{array}{ccc} K_i & \xrightarrow{f_i} & L_i \\ {\scriptstyle \kappa_{ji}}\downarrow & & \downarrow{\scriptstyle \lambda_{ji}} \\ K_j & \xrightarrow{f_j} & L_j. \end{array}$$

(2) By the universal property of direct limits the morphisms $K_i \to L_i \to \varinjlim L_i$ give rise to a natural morphism

$$\varinjlim f_i\colon \varinjlim K_i \to \varinjlim L_i.$$

Now we can formulate our first result of this section.

Proposition 46.3. (Direct Limit-Exactness Proposition) Let $(I, \leq)$ be a directed set. Also let $(\{K_i\}_{i\in I}, \{\kappa_{ji}\}_{i\leq j})$, $(\{L_i\}_{i\in I}, \{\lambda_{ji}\}_{i\leq j})$ and $(\{M_i\}_{i\in I}, \{\mu_{ji}\}_{i\leq j})$ be direct systems of abelian groups. Finally suppose that we are given morphisms $f_i\colon K_i \to L_i$ and $g_i\colon L_i \to M_i$ of the direct systems.

(1) If for all $i \in I$ we have $g_i \circ f_i = 0$, then $\varinjlim g_i \circ \varinjlim f_i = 0$.

(2) If for all $i \in I$ the sequence

$$K_i \xrightarrow{f_i} L_i \xrightarrow{g_i} M_i$$

is exact, then the induced sequence

$$\varinjlim K_i \xrightarrow{\varinjlim f_i} \varinjlim L_i \xrightarrow{\varinjlim g_i} \varinjlim M_i$$

is also exact.

Proof. Let $(I, \leq)$ be a directed set. We start out with two general observations regarding direct limits:

(a) Let $(\{A_i\}_{i\in I}, \{\alpha_{ji}\}_{i\leq j})$ be a direct system of abelian groups. By the Algebraic–Direct Limit Existence Proposition 25.3 (1b) we know that

$$\varinjlim A_i = \Big(\bigoplus_{i\in I} A_i\Big) / \langle \alpha_{ji}(x) - x \mid \ i \leq j \text{ and } x \in A_i\rangle.$$

We adopt the following notation: Given $a_j \in A_j \subset \bigoplus_{i\in I} A_i$ we denote by $[a_j] \in \varinjlim A_j$ the element represented by a_j.

Claim.

(i) Any element in $\varinjlim A_i$ is of the form $[a_i]$ for some $a_i \in A_i$.

(ii) If we are given $a_j \in A_j$ with $[a_j] = 0 \in \varinjlim A_i$, then there exists a $k \geq j$ such that $\alpha_{kj}(a) = 0 \in A_k$.

Proof.

(i) By the above construction any element in $\varinjlim A_i$ is of the form $[a_{i_1}] + \cdots + [a_{i_m}]$ where $a_{i_j} \in A_{i_j}$. Since $(I, \leq)$ is a directed set we see easily, by considering an $i \in I$

with $i \geq i_1, \ldots, i_m$, that any element of $\varinjlim A_i$ is in fact contained in the image of a *single* A_i, i.e. is of the form $[a_i]$ for some $a_i \in A_i$.

(ii) If we are given $a_j \in A_j$ with $[a_j] = 0 \in \varinjlim A_i$, then it follows from the above description of $\varinjlim A_i$ that in $\bigoplus_{i\in I} A_i$ we can write $a_j = \sum_{r=1}^{m} c_r \cdot (\alpha_{t_r,s_r}(x_r) - x_r) \in \bigoplus_{i\in I} A_i$ where $c_r \in \mathbb{Z}$, $x_r \in A_{s_r}$ and $t_r \geq s_r$. Since we are dealing with a directed set there exists a $k \in I$ with $k \geq j, t_{r_1}, \ldots, t_{r_m}$. Then we obtain the following equality in A_k:

$$\alpha_{kj}(a_j) = \Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)(a_j) = \Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)\Big(\sum_{r=1}^{m} c_r \cdot (\alpha_{t_r,s_r}(x_r) - x_r)\Big)$$
$$= \sum_{r=1}^{m} c_r \cdot \Big(\underbrace{\Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)(\alpha_{t_r,s_r}(x_r))}_{=\alpha_{k,t_r}(\alpha_{t_r,s_r}(x_r))=\alpha_{k,s_r}(x_r)} - \underbrace{\Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)(x_r)}_{=\alpha_{k,s_r}(x_r)} \Big) = 0. \qquad \boxplus$$

(b) If $(\{A_i\}_{i\in I}, \{\alpha_{ji}\}_{i\leq j})$ and $(\{B_i\}_{i\in I}, \{\beta_{ji}\}_{i\leq j})$ are two direct systems of abelian groups and if $\{f_i \colon A_i \to B_i\}_{i\in I}$ is a morphism between these two direct systems, then it follows easily from the definitions and (a) that the induced map of direct systems is given by

$$\begin{array}{ccc} \varinjlim A_i & \to & \varinjlim B_i \\ [a_i] & \mapsto & [f_i(a_i)] \\ \uparrow & & \\ \text{lies in } A_i & & \end{array}$$

Now we turn to the actual proof of the two statements of the proposition. We write $f = \varinjlim f_i$ and $g = \varinjlim g_i$.

(1) We assume that for all $i \in I$ we have $g_i \circ f_i = 0$. We need to show that $g \circ f = 0$. Suppose we are given an element of $\varinjlim K_i$. By (a-i) we know that it is of the form $[k_i]$ for some $i \in I$ and some $k_i \in K_i$. We now see that

$$(g \circ f)([k_i]) = g(f([k_i])) \underset{\text{by (b)}}{=} g_i([f_i(k_i)]) \underset{\text{by (b)}}{=} [g_i(f_i(k_i))] = [(g_i \circ f_i)(k_i)] \underset{\text{since } g_i \circ f_i = 0}{=} 0.$$

(2) Now we assume that each $K_j \xrightarrow{f_j} L_j \xrightarrow{g_j} M_j$ is exact. Note that by (1) it remains to prove that $\ker(g) \subset \operatorname{im}(f)$. Thus suppose we are given an element of $\ker(g) \in \varinjlim L_i$. By (a-i) we know that it is of the form $[l_i]$ for some $i \in I$ and some $l_i \in L_i$. Since $g([l_i]) = 0$ it follows from (a-ii) and (b) that there exists a $j \geq i$ such that $\mu_{ji}(g_i(l_i)) = 0$. Next we consider the following diagram

$$\begin{array}{ccccc} K_i & \xrightarrow{f_i} & L_i & \xrightarrow{g_i} & M_i \\ \downarrow{\scriptstyle \kappa_{ji}} & & \downarrow{\scriptstyle \lambda_{ji}} & & \downarrow{\scriptstyle \mu_{ji}} \\ K_j & \xrightarrow{f_j} & L_j & \xrightarrow{g_j} & M_j. \end{array}$$

By hypothesis the diagram commutes, thus we see that $g_j(\lambda_{ji}(l_i)) = \mu_{ji}(g_i(l_i)) = 0$. Since $K_j \xrightarrow{f_j} L_j \xrightarrow{g_j} M_j$ is exact we know that there exists a $k_j \in K_j$ with $f_j(k_j) = \lambda_{ji}(l_i)$. It follows from (b) that $f([k_j]) = [f_j(k_j)] = [l_j] = [l_i] \in \varinjlim L_i$. We have thus shown that $[l_i] \in \operatorname{im}(f)$. ■

The following corollary says that direct limits commute with kernels and cokernels.

Corollary 46.4. (Direct Limit-(Co-)-Kernel Corollary) Let $(I, \leq)$ be a directed set and let $(\{K_i\}_{i\in I}, \{\kappa_{ji}\}_{i\leq j})$ and $(\{L_i\}_{i\in I}, \{\lambda_{ji}\}_{i\leq j})$ be direct systems of abelian groups. Furthermore suppose that we are given a morphism $f_i\colon K_i \to L_i$ between the direct systems. There exist natural isomorphisms

$$\operatorname*{colim}_{\longrightarrow} \ker(f_i\colon K_i \to L_i) \xrightarrow{\cong} \ker\left(\operatorname*{colim}_{\longrightarrow} f_i\colon \operatorname*{colim}_{\longrightarrow} K_i \to \operatorname*{colim}_{\longrightarrow} L_i\right)$$

and

$$\operatorname*{colim}_{\longrightarrow} \operatorname{coker}(f_i\colon K_i \to L_i) \xrightarrow{\cong} \operatorname{coker}\left(\operatorname*{colim}_{\longrightarrow} f_i\colon \operatorname*{colim}_{\longrightarrow} K_i \to \operatorname*{colim}_{\longrightarrow} L_i\right).$$

Proof. By the elementary Exactness Lemma 37.1 we know that for each $i \in I$ the sequence

$$0 \to \ker(f_i) \to K_i \xrightarrow{f_i} L_i \to \operatorname{coker}(f_i) \to 0.$$

is exact. One can easily verify that these homomorphisms form morphisms between the various direct systems. It follows from the Direct Limit-Exactness Proposition 46.3 (2) (applied four times!) that the sequence

$$0 \to \operatorname*{colim}_{\longrightarrow} \ker(f_i) \to \operatorname*{colim}_{\longrightarrow} K_i \xrightarrow{\operatorname*{colim} f_i} \operatorname*{colim}_{\longrightarrow} L_i \to \operatorname*{colim}_{\longrightarrow} \operatorname{coker}(f_i) \to 0$$

is also exact. By the Exactness Lemma 37.1 this observation implies the desired statements. ■

The following lemma is a slightly more general version of the lemma stated on page 538 in the first section of this chapter.

Lemma 46.5. (Direct Limit-Homology Lemma) Let $(I, \leq)$ be a directed set and let $(\{C_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ be a direct system of chain complexes.

(1) The groups $\operatorname*{colim}_{\longrightarrow i} C_{i*}$ naturally form a chain complex such that the natural maps $C_{j*} \to \operatorname*{colim}_{\longrightarrow i} C_{i*}$ are chain maps.

(2) For each $n \in \mathbb{N}_0$ the homomorphisms $\mathrm{H}_n(C_{j*}) \to \mathrm{H}_n\left(\operatorname*{colim}_{\longrightarrow i} C_{j*}\right)$ from (1) induce an isomorphism

$$\operatorname*{colim}_{\longrightarrow j} \mathrm{H}_n(C_{j*}) \xrightarrow{\cong} \mathrm{H}_n\left(\operatorname*{colim}_{\longrightarrow i} C_{i*}\right).$$

Proof.

(1) It follows immediately from the Direct Limit-Exactness Proposition 46.3 (1), applied to the maps $C_{i,n+1} \xrightarrow{\partial_{n+1}} C_{i,n} \xrightarrow{\partial_n} C_{i,n-1}$, that the direct limits $\operatorname*{colim}_{\longrightarrow i} C_{i*}$ form a chain complex. Furthermore note that it follows immediately from the universal properties of direct limits that the natural maps $C_{j*} \to \operatorname*{colim}_{\longrightarrow i} C_{i*}$ are chain maps.

(2) Given any $n \in \mathbb{N}_0$ we see that

$$\begin{aligned}
\operatorname*{colim}_i \mathrm{H}_n(C_{i*}) &= \operatorname*{colim}_i \overbrace{\operatorname{coker}\left(C_{i,n+1} \xrightarrow{\partial_{n+1}} \ker(C_{i,n} \xrightarrow{\partial_n} C_{i-1,n})\right)}^{=\mathrm{H}_n(C_{i*})} \\
&\overset{\cong}{\uparrow} \operatorname{coker}\left(\operatorname*{colim}_i C_{i,n+1} \xrightarrow{\operatorname{colim}\partial_{n+1}} \ker\left(\operatorname*{colim}_i C_{i,n} \xrightarrow{\operatorname{colim}\partial_n} \operatorname*{colim}_i C_{i,n-1}\right)\right) \\
&\qquad \text{follows from applying the Direct Limit-(Co-)-Kernel Corollary 46.4 twice} \\
&= \mathrm{H}_n\big(\operatorname*{colim}_i C_{i*}\big). \\
&\qquad \text{by definition of the } n\text{-th homology group}
\end{aligned}$$

The fact that this isomorphism is induced by the maps $\mathrm{H}_n(C_{i*}) \to \mathrm{H}_n(\operatorname{colim} C_{i*})$ follows easily from the various definitions. We leave the details to the reader. ■

It follows from the 1-Complex-π_1-Theorem 31.5 and the Cell Attachment-π_1-Proposition 29.1 that it is relatively easy to compute the fundamental group of a finite CW-complex.

In this chapter we will develop an algorithm which reduces the calculation of homology groups of CW-complexes to determining the degrees of suitable self-maps of spheres. We will use this method to finally determine the homology groups of the (non-orientable) surfaces of genus ≥ 2, of all the projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ and even of the infinite-dimensional projective spaces $\mathbb{R}\mathrm{P}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$.

47.1. **Relative homology groups of CW-complexes.** In the following section we will introduce and study the cellular homology groups of a CW-complex X. Throughout this chapter we will use the following familiar notation:

Notation. Let X be a CW-complex
(1) As usual, we denote the filtration of X by $\varnothing = X^{-1} \subset X^0 \subset X^1 \subset X^2 \subset \dots$ and we refer to X^n as the n-skeleton of X.
(2) We denote by $\mathrm{Cell}_n(X)$ the set of n-cells of X.

In preparation for the next section we now prove several basic statements about homology groups of CW-complexes:

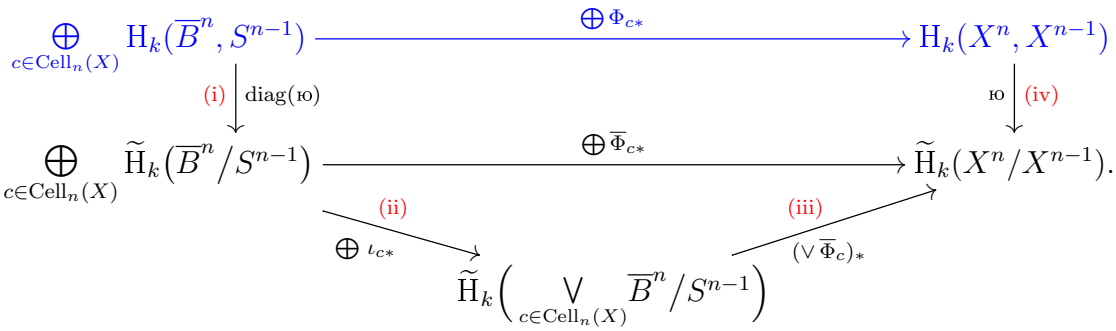
Lemma 47.1. (Skeleton-H_*-Lemma) Let X be a CW-complex and let $n \in \mathbb{N}_0$.
(1) Let $\{\Phi_c \colon \overline{B}^n \to X^n\}_{c \in \mathrm{Cell}_n(X)}$ be a full set of characteristic maps for the n-cells of X.[359] For every $k \in \mathbb{N}_0$ the following map is an isomorphism:
$$\bigoplus_{c \in \mathrm{Cell}_n(X)} \Phi_{c*} \colon \bigoplus_{c \in \mathrm{Cell}_n(X)} \mathrm{H}_k(\overline{B}^n, S^{n-1}) \;\to\; \mathrm{H}_k(X^n, X^{n-1}).$$
In particular the following statements hold:
(a) For $k \neq n$ we have $\mathrm{H}_k(X^n, X^{n-1}) = 0$.
(b) The group $\mathrm{H}_n(X^n, X^{n-1})$ is a free abelian group where the rank equals the cardinality of the set $\mathrm{Cell}_n(X)$ of n-cells in the CW-structure.
(2) For $k > n$ we have $\mathrm{H}_k(X^n) = 0$.
(3) For $k < n$ the inclusion $i \colon X^n \to X$ induces an isomorphism $i_* \colon \mathrm{H}_k(X^n) \xrightarrow{\cong} \mathrm{H}_k(X)$.
(4) The inclusion $i \colon X^n \to X$ induces an epimorphism $i_* \colon \mathrm{H}_n(X^n) \twoheadrightarrow \mathrm{H}_n(X)$.

Remark. The statements (3) and (4) of the Skeleton-H_*-Lemma 47.1 can be viewed as "homological analogues" of the CW-Skeleton-π_1-Proposition 30.14.

[359] We refer to page 378 for the definition of a full set of characteristic maps and a justification why such a set always exists.

Proof. Let X be a CW-complex and let $\{\Phi_c\colon \overline{B}^n \to X^n\}_{c\in \mathrm{Cell}_n(X)}$ be a full set of characteristic maps for the n-cells of X.

(1) We consider the following diagram:



We make the following clarifications and observations:

- Given a pair (Y, A) of topological space we consider the natural homomorphism ю$\colon \mathrm{H}_n(Y, A) \to \widetilde{\mathrm{H}}_n(Y/A)$ from page 489.
- We use (obvious generalizations of) the notation from page 514 regarding maps out of direct sums of abelian groups.
- Given $c \in \mathrm{Cell}_n(X)$ we denote by $\overline{\Phi}_c\colon \overline{B}^n/S^{n-1} \to X^n/X^{n-1}$ the map that is induced by the characteristic map $\Phi_c\colon \overline{B}^n \to X^n$.
- It follows almost immediately from the naturality of the maps ю that the rectangle commutes.
- We use the points $[S^{n-1}] \in \overline{B}^n/S^{n-1}$ to form the wedge $\bigvee_{c\in\mathrm{Cell}_n(X)} \overline{B}^n/S^{n-1}$.
- For $j \in \mathrm{Cell}_n(X)$ let $\iota_j\colon \overline{B}^n/S^{n-1} \to \bigvee_{c\in\mathrm{Cell}_n(X)} \overline{B}^n/S^{n-1}$ be the natural inclusion.
- For each $j \in \mathrm{Cell}_n(X)$ we have $\overline{\Phi}_j = \big(\bigvee_{c\in\mathrm{Cell}_n(X)} \overline{\Phi}_c\big) \circ \iota_j$. It follows from this observation and the definition of $\bigoplus_{c\in\mathrm{Cell}_n(X)} \overline{\Phi}_{c*}$ that the triangle commutes.

We want to show that the top horizontal map is an isomorphism. By the above it remains to show that the maps (i)–(iv) are isomorphisms.

(i) This map is an isomorphism by the Relative-to-Quotient-H_*-Proposition 41.8.[360]
(ii) This map is an isomorphism by the Wedge-H_*-Proposition 41.10.
(iii) This map is an isomorphism by the Skeleton-Quotient Lemma 33.5.
(iv) It follows from the Relative-to-Quotient-H_*-Proposition 41.8 together with the Subcomplex–Neighborhood Proposition 31.8 that this map is an isomorphism.

The remaining statements of (1) are now an immediate consequence of the Ball-rel-Sphere-H_*-Lemma 41.4.

(2) We fix $k \in \mathbb{N}$. We need to prove the following claim.

Claim. For $n = 0, \dots, k-1$ we have $\mathrm{H}_k(X^n) = 0$.

Proof. We prove the claim by induction on $n \in \mathbb{N}_0$. First we consider the case $n = 0$. By the discussion on page 372 we know that the 0-skeleton X^0 has the discrete topology.

[360] Here we use that $S^{n-1} \subset \overline{B}^n$ is clearly a good subset.

It follows from the Singleton-H_*-Lemma 35.11, the Path-Component–H_*–Lemma 35.8 and the hypothesis that $k > 0$ that $\mathrm{H}_k(X^0) = 0$.

Now assume that the statement holds for some $n \in \{0, \dots, k-2\}$. We consider the long exact sequence of the pair (X^{n+1}, X^n):

$$\dots \to \mathrm{H}_{k+1}(X^{n+1}, X^n) \to \underbrace{\mathrm{H}_k(X^n)}_{=0 \text{ by induction}} \to \mathrm{H}_k(X^{n+1}) \to \underbrace{\mathrm{H}_k(X^{n+1}, X^n)}_{\substack{=0 \text{ by (1) since } n < k-1 \\ \text{which implies that } k \neq n+1}} \to \dots$$

It follows that we have an epimorphism $\mathrm{H}_k(X^n) \to \mathrm{H}_k(X^{n+1})$. Since $\mathrm{H}_k(X^n) = 0$ we conclude that $\mathrm{H}_k(X^{n+1}) = 0$. ⊞

(3) Let $k \in \mathbb{N}_0$ and $n > k$. Recall that we need to show that the inclusion induced map $\mathrm{H}_k(X^n) \to \mathrm{H}_k(X)$ is an isomorphism. We consider the following diagram of inclusion induced homomorphisms:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathrm{H}_k(X^{n-1}) & \longrightarrow & \mathrm{H}_k(X^n) & \xrightarrow{\cong} & \mathrm{H}_k(X^{n+1}) & \xrightarrow{\cong} & \mathrm{H}_k(X^{n+2}) \xrightarrow{\cong} \dots \\ & & & \searrow & \downarrow & \swarrow & \swarrow & \nearrow & \\ & & & & \mathrm{H}_k(X). & & & & \end{array}$$

We make the following observations:

(a) The diagram commutes by the functoriality of homology.

(b) We just want to prove the following claim:

Claim. For any $m \geq n$ the inclusion induced homomorphism $\mathrm{H}_k(X^m) \to \mathrm{H}_k(X^{m+1})$ is an isomorphism.

Proof. This claim follows by considering the following excerpt from the long exact sequence of the pair (X^{m+1}, X^m):

$$\dots \to \underbrace{\mathrm{H}_{k+1}(X^{m+1}, X^m)}_{=0 \text{ by (1) since } k < n \leq m} \to \mathrm{H}_k(X^m) \to \mathrm{H}_k(X^{m+1}) \to \underbrace{\mathrm{H}_k(X^{m+1}, X^m)}_{=0 \text{ by (1) since } k < n \leq m} \to \dots$$

⊞

(c) By the Exhaustions-H_*-Proposition 46.2 (2) we know that $\mathrm{H}_k(X) = \operatorname{colim} \mathrm{H}_k(X^n)$.

(d) It follows from (a)–(c) and the Direct Limit–Stabilization Lemma 25.1 that the vertical inclusion induced homomorphism $\mathrm{H}_k(X^n) \to \mathrm{H}_k(X)$ is an isomorphism.

(4) We consider the following excerpt from the long exact sequence of the pair (X^{n+1}, X^n):

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathrm{H}_{n+1}(X^{n+1}, X^n) & \longrightarrow & \mathrm{H}_n(X^n) & \longrightarrow & \mathrm{H}_n(X^{n+1}) & \longrightarrow & \underbrace{\mathrm{H}_n(X^{n+1}, X^n)}_{=0 \text{ by (1)}} \longrightarrow \dots \\ & & & & & \searrow & \downarrow \cong \text{ by (3)} & & \\ & & & & & & \mathrm{H}_n(X) & & \end{array}$$

We see that the inclusion induced map $\mathrm{H}_n(X^n) \to \mathrm{H}_n(X)$ is an epimorphism. ■

47.2. **The definition of cellular homology.** In this section we will introduce the main protagonists of this chapter, namely the cellular homology groups of a CW-complex. We start out with the definition of the cellular boundary map.

Definition. Given a CW-complex X and given $n \in \mathbb{N}_0$ we denote by d_n the homomorphism

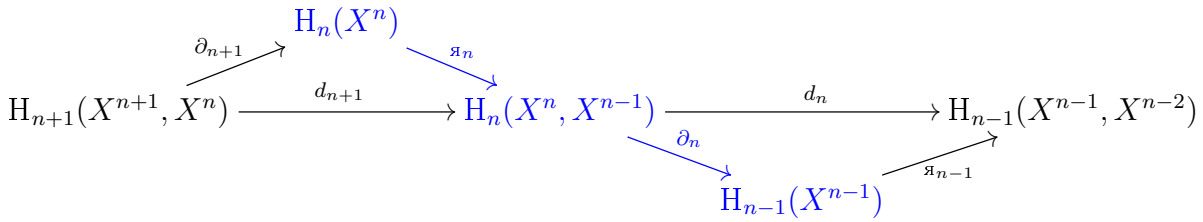
$$\mathrm{H}_n(X^n, X^{n-1}) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X^{n-1}) \xrightarrow{\text{я}_{n-1}} \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$$
$$=: d_n$$

where ∂_n is the connecting map in the long exact sequence of the pair (X^n, X^{n-1}) and where $\text{я}_{n-1} \colon \mathrm{H}_{n-1}(X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$ is the natural map introduced on page 454. We refer to d_n as the cellular boundary map.

The name "cellular boundary map" already suggests that the maps d_n have the following important property:

Lemma 47.2. (Cellular Boundary Map Lemma) For all $n \in \mathbb{N}_0$ we have $d_n \circ d_{n+1} = 0$.

Proof. We consider the following diagram of homomorphisms:



Note that the diagram commutes by the definition of the cellular boundary maps. The map $\partial_n \circ \text{я}_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_{n-1}(X^{n-1})$ is the composition of two successive maps in the long exact sequence of the pair (X^n, X^{n-1}). This implies that the map $\partial_n \circ \text{я}_n$ is the zero map. Since the diagram commutes we obtain that $d_n \circ d_{n+1} = 0$. ■

By the Cellular Boundary Map Lemma 47.2 it now makes sense to introduce the following definition.

Definition. Let X be a CW-complex.

(1) Given $n \in \mathbb{N}_0$ we write $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$.[361]
(2) We refer to $(\mathrm{C}_*^{\mathrm{CW}}(X), d_*)$ as the cellular chain complex of the CW-complex X.[362]
(3) We denote the homology groups of the cellular chain complex by $\mathrm{H}_n^{\mathrm{CW}}(X)$ and we refer to these groups as the cellular homology groups of the CW-complex X.

The following lemma shows that cellular chain complexes tend to be "small".

Example. For $n \geq 2$ we consider the n-dimensional sphere S^n. On page 372 we showed that we can view S^n as a CW-complex with one 0-cell and one n-cell. It follows from the Skeleton-H_*-Lemma 47.1 (1) that the cellular chain complex $(\mathrm{C}_*^{\mathrm{CW}}(S^n), d_*)$ is of the form

$$0 \to \underbrace{\mathrm{C}_n^{\mathrm{CW}}(S^n)}_{\cong \mathbb{Z}} \to 0 \to \ldots \to 0 \to \underbrace{\mathrm{C}_0^{\mathrm{CW}}(S^n)}_{\cong \mathbb{Z}} \to 0.$$

[361] Note that the Skeleton-H_*-Lemma 47.1 (1) says that the n-th cellular chain group $\mathrm{C}_n^{\mathrm{CW}}(X)$ is a free abelian group where the rank is given by the cardinality of the set of n-cells.
[362] It follows from the Cellular Boundary Map Lemma 47.2 that $(\mathrm{C}_*^{\mathrm{CW}}(X), d_*)$ is indeed a chain complex.

Since $n \geq 2$ the two $\mathbb{Z}$'s are separated by at least one zero group. The cellular boundary maps are therefore necessarily the zero maps. This implies that[363]

$$\mathrm{H}_i^{\mathrm{CW}}(S^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

These cellular homology groups are of course isomorphic to the "usual" homology groups of S^n. In the next section we will see that this is not a coincidence.

47.3. **The relationship between cellular and singular homology.** Given a CW-complex X we now have two "competing" homology groups, namely the singular homology groups $\mathrm{H}_n(X)$ and the cellular homology group $\mathrm{H}_n^{\mathrm{CW}}(X)$. The following proposition explains the relationship between these two homology groups.

Proposition 47.3. (Singular-Cellular-H_*-Proposition) Let $n \in \mathbb{N}_0$.

(1) For each CW-complex X there exists a uniquely determined *isomorphism*

$$\Theta_X \colon \mathrm{H}_n(X) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{CW}}(X)$$

with the property that the following diagram commutes:[364]

$$\begin{array}{ccc} \mathrm{H}_n(X^n) & \twoheadrightarrow & \mathrm{H}_n(X) \\ {\scriptstyle я_n}\downarrow & & \downarrow{\scriptstyle \Theta_X} \\ \ker(d_n) & \xrightarrow{a \mapsto [a]} & \mathrm{H}_n^{\mathrm{CW}}(X). \end{array}$$

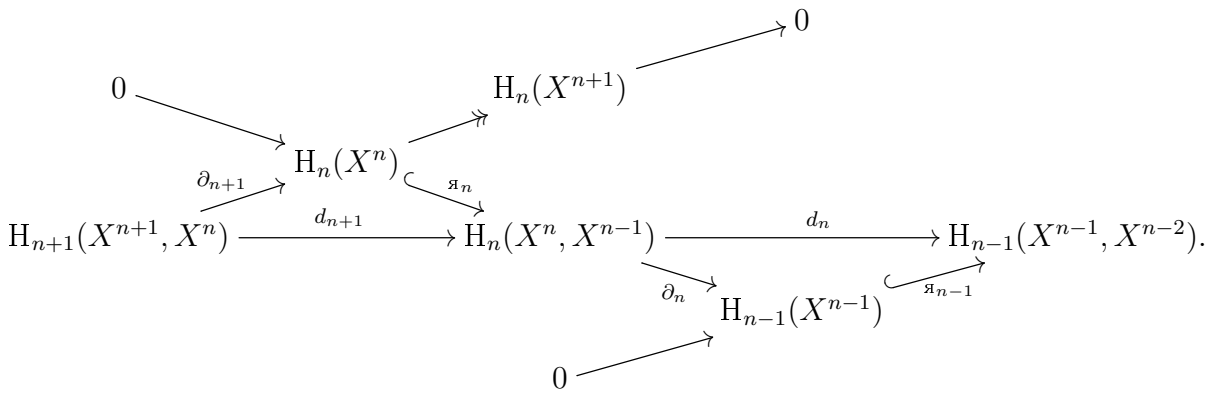
(2) The homomorphisms from (1) define a natural isomorphism between the functors $X \mapsto \mathrm{H}_n(X)$ and the functor $X \mapsto \mathrm{H}_n^{\mathrm{CW}}(X)$. In other words, for every cellular map $f \colon X \to Y$ between CW-complexes the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}_n(X) & \xrightarrow{f_*} & \mathrm{H}_n(Y) \\ {\scriptstyle \Theta_X}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Theta_Y} \\ \mathrm{H}_n^{\mathrm{CW}}(X) & \xrightarrow{f_*} & \mathrm{H}_n^{\mathrm{CW}}(Y). \end{array}$$

[363] It is a fun exercise to calculate the cellular boundary maps of S^1.

[364] Here we again denote by $я_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ the obvious map. In the long exact sequence of the pair (X^n, X^{n-1}) we see that the image of $\mathrm{H}_n(X^n)$ in $\mathrm{H}_n(X^n, X^{n-1})$ lies in the kernel of the connecting homomorphism ∂_n. Since $d_n = я_n \circ \partial_n$ this implies that the map $я_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ takes values in $\ker(d_n)$.

Proof of Proposition 47.3 (1). We add a couple of maps to the commutative diagram that we had already encountered in the proof of the Cellular Boundary Map Lemma 47.2:



Claim 1. All the diagonal sequences in the above commutative diagram are exact.

Proof. By the Skeleton-H_*-Lemma 47.1 (1) and (2) we know that

$$\mathrm{H}_n(X^{n+1}, X^n) = 0, \quad \mathrm{H}_n(X^{n-1}) = 0 \quad \text{and} \quad \mathrm{H}_{n-1}(X^{n-2}) = 0.$$

It thus follows that the diagonal sequences in the above commutative diagram are just excerpts from the long exact sequences of the corresponding pairs of topological spaces. In particular all the diagonal sequences are exact. ⊞

Claim 2. The map $\mathrm{я}_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ induces an isomorphism

$$\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \;\to\; \underbrace{\ker(d_n)/\operatorname{im}(d_{n+1})}_{=\mathrm{H}_n^{\mathrm{CW}}(X)}.$$

Proof.

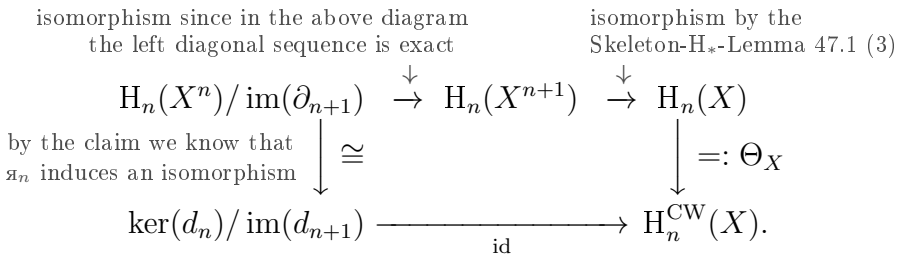
(1) The given map is well-defined since it follows immediately from $\partial_n \circ \mathrm{я}_n = 0$ that $\mathrm{я}_n(\mathrm{H}_n(X^n)) \subset \ker(d_n)$.

(2) [365] The given map is an epimorphism. This can be seen as follows:
   (a) It follows from Claim 1 that the map $\mathrm{я}_{n-1}$ is injective.
   (b) It follows from (a) and Claim 1 that $\ker(d_n) = \ker(\partial_n) = \operatorname{im}(\mathrm{я}_n)$.
   (c) It follows from (b) that the map $\mathrm{я}_n$ induces an epimorphism $\mathrm{H}_n(X^n) \to \ker(d_n)$.
   (d) Since the left triangle commutes we obtain from (c) that $\mathrm{я}_n$ also induces an epimorphism $\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \to \ker(d_n)/\operatorname{im}(d_{n+1})$.

(3) The given map is a monomorphism: We need to show that if we are given $c \in \mathrm{H}_n(X^n)$ such that $\mathrm{я}_n(c) \in \mathrm{H}_n(X^n, X^{n-1})$ lies in the image of d_{n+1}, then $c \in \operatorname{im}(\partial_{n+1})$. Thus suppose that $\mathrm{я}_n(c) \in \mathrm{H}_n(X^n, X^{n-1})$ lies in the image of d_{n+1}. This means that there exists an $e \in \mathrm{H}_{n+1}(X^{n+1}, X^n)$ with $\mathrm{я}_n(c) = d_{n+1}(e) = \mathrm{я}_n(\partial_{n+1}(e))$. Since by Claim 1 we know that $\mathrm{я}_n$ is injective we obtain that $c = \partial_{n+1}(e)$. ⊞

[365] It remains to show that the given map is an epimorphism and a monomorphism. This is shown by a diagram chase. It is arguably best not to read the remainder of the proof of the claim but to go on the chase on one's own.

Next we consider the following diagram:



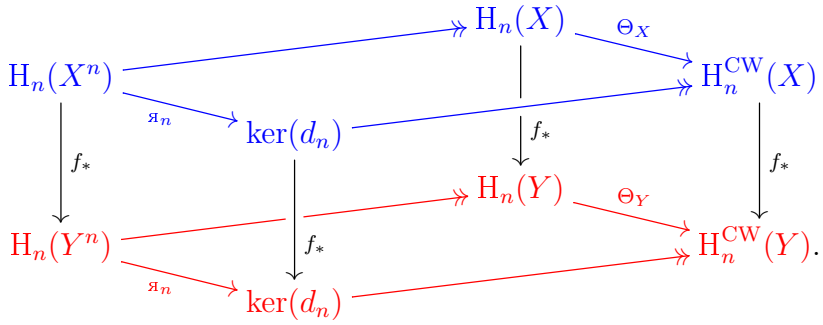
Since all the given maps are isomorphisms we can now define the right vertical map Θ_X as the unique map which makes the diagram commute. Evidently Θ_X is also an isomorphism.

Finally we note that by construction the isomorphism Θ_X has the property that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(X^n) & \twoheadrightarrow & \mathrm{H}_n(X) \\ {\scriptstyle \mathfrak{я}_n}\downarrow & & \downarrow{\scriptstyle \Theta_X} \\ \ker(d_n) & \twoheadrightarrow & \mathrm{H}_n^{\mathrm{CW}}(X). \end{array}$$

Note that by the Skeleton-H$_*$-Lemma 47.1 (4) the top horizontal map is an epimorphism. This implies immediately that Θ_X is the unique homomorphism that makes the diagram commute. ■

Proof of Proposition 47.3 (2). We need to show that the isomorphisms Θ_X that we just constructed are natural. Thus suppose that we are given a cellular map $f\colon X \to Y$ and let $n \in \mathbb{N}_0$. We consider the following diagram:



We make the following observations:

(a) The sides of the hyperrectangle that do not involve Θ_X and Θ_Y commute basically by definition.

(b) The top and bottom parallelogram commute by definition of Θ_X and Θ_Y.

(c) By the Skeleton-H$_*$-Lemma 47.1 (4) the map $\mathrm{H}_n(X^n) \to \mathrm{H}_n(X)$ is an epimorphism.

It follows from these observation that also the parallelogram involving Θ_X and Θ_Y commutes. But that is exactly, what we had to show. ■

Even though the cellular boundary maps in the cellular chain complex are still somewhat mysterious, we can already use the above proposition to carry out an interesting calculation, namely we can finally calculate the homology groups of the complex projective spaces $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$ and of $\mathbb{C}\mathrm{P}^\infty = \operatorname{colim}_{\longrightarrow} \mathbb{C}\mathrm{P}^n$:

Proposition 47.4. ($\mathbb{C}\mathrm{P}^n$-H_*-Proposition)

(1) Given $n \in \mathbb{N}_0$ we have[366]

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \dots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

(2) We have

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty) \cong \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is } k \text{ is odd.} \end{cases}$$

Proof.

(1) Let $n \in \mathbb{N}_0$. On page 376 we showed that $\mathbb{C}\mathrm{P}^n$ admits a CW-structure with exactly one cell in dimensions $0, 2, \dots, 2n$ and no other cells. It follows from the Skeleton-H_*-Lemma 47.1 (1) that the corresponding cellular chain complex is of the form

$$0 \to \underbrace{\mathrm{C}^{\mathrm{CW}}_{2n}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to \underbrace{\mathrm{C}^{\mathrm{CW}}_{2n-1}(\mathbb{C}\mathrm{P}^n)}_{=0} \to \dots \to \underbrace{\mathrm{C}^{\mathrm{CW}}_{2}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to \underbrace{\mathrm{C}^{\mathrm{CW}}_{1}(\mathbb{C}\mathrm{P}^n)}_{=0} \to \underbrace{\mathrm{C}^{\mathrm{CW}}_{0}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to 0.$$

Since every other group is trivial it follows that all boundary maps are zero. In particular the cellular homology groups equal the cellular chain groups. It follows that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \underset{\substack{\uparrow \\ \text{by the Singular-Cellular} \\ \text{Homology Proposition 47.3}}}{\cong} \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^n) \underset{\substack{\uparrow \\ \text{follows from the above discussion}}}{\cong} \mathrm{C}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^n) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \dots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Just for fun we provide two different ways to calculate $\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty)$:

(a) On page 376 we saw that we can equip the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$ with a CW-structure that has precisely one cell in each even dimension and no cells in odd dimensions. From the same arguments as in (1) it follows that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty) \cong \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^\infty) \cong \mathrm{C}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^\infty) \cong \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

(b) We start out with the following claim.

Claim. Let $j \geq i$. Let $\iota\colon \mathbb{C}\mathrm{P}^i \to \mathbb{C}\mathrm{P}^j$ be the usual embedding that is given by $[x] \mapsto [(x, 0)]$. The induced map $\iota_*\colon \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^i) \to \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^j)$ on singular homology is an isomorphism for $k = 0, \dots, 2i$.

Proof. Note that $\iota\colon \mathbb{C}\mathrm{P}^i \to \mathbb{C}\mathrm{P}^j$ is clearly cellular. Thus we obtain from the Singular-Cellular-H_*-Proposition 47.3 (2) the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_k(\mathbb{C}\mathrm{P}^i) & \xrightarrow{\iota_*} & \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^j) \\ {\scriptstyle \Theta_{\mathbb{C}\mathrm{P}^i}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Theta_{\mathbb{C}\mathrm{P}^j}} \\ \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^i) & \xrightarrow{\iota_*} & \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^j) \end{array}.$$

Next note that it follows from inspecting the cellular chain complexes that for any $k \leq 2i$ the bottom horizontal map is an isomorphism. The claim follows immediately from these two observations.[367] ⊞

[367] Alternatively this claim follows from the Skeleton-H_*-Lemma 47.1 (3).

It follows from the claim, the Exhaustions-H_*-Proposition 46.2 and the Direct Limit–Stabilization Lemma 25.1 that for any $k \leq 2i$ the inclusion induced map $H_k(\mathbb{C}P^i) \to H_k(\mathbb{C}P^\infty)$ is an isomorphism. Together with (1) we thus get the promised result. ■

47.4. **The cellular boundary maps.** In this section we want to describe an approach to determine the cellular boundary homomorphisms for a given CW-complex. Note that in general an explicit description of a homomorphism $\varphi\colon C \to D$ between two free abelian groups requires the following:

(1) We need bases $\mathcal{C}$ and $\mathcal{D}$ for the free abelian groups.
(2) With respect to these bases we need to express φ by a matrix, i.e. we need to determine the integral matrix $\{a_{dc}\}_{d\in\mathcal{D},c\in\mathcal{C}}$ such that for any $c \in \mathcal{C}$ we have $\varphi(c) = \sum_{d\in\mathcal{D}} a_{dc} \cdot d$.

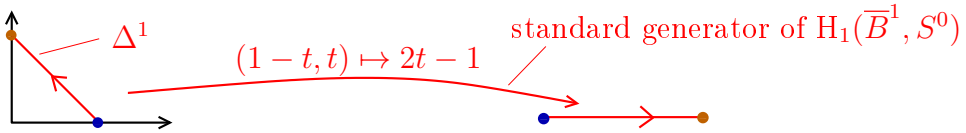
Now we want to apply this scheme to the cellular boundary maps $d_n\colon C_n^{CW}(X) \to C_{n-1}^{CW}(X)$. Let us start with Step (1): By the Singular-Cellular-H_*-Proposition 47.3 (1) the key to describing such a basis is to fix a basis for $H_n(\overline{B}^n, S^{n-1}) \cong \mathbb{Z}$, i.e. we need to fix a preferred generator of $H_n(\overline{B}^n, S^{n-1})$. We do so in the following definition.

Definition. Let $n \in \mathbb{N}$.

(1) We denote by $[S^{n-1}] \in \widetilde{H}_{n-1}(S^{n-1})$ the standard generator of $\widetilde{H}_n(S^{n-1})$, as defined on page 502.
(2) The standard generator $[\overline{B}^n]$ of $H_n(\overline{B}^n, S^{n-1})$ is defined to be the unique element with $\partial_n([\overline{B}^n]) = [S^{n-1}] \in \widetilde{H}_{n-1}(S^{n-1})$ where $\partial_n\colon H_n(\overline{B}^n, S^{n-1}) \to \widetilde{H}_{n-1}(S^{n-1})$ is the connecting homomorphism of the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$.

Example. Recall that on page 502 we saw that the standard generator of $\widetilde{H}_0(S^0)$ is given by $[S^0] = [(+1) - (-1)] \in \widetilde{H}_0(S^0)$. It follows easily from this observation, the explicit description of the connecting homomorphism in the LES-of-Triple Proposition 38.2, and the explicit description of the boundary of a singular 1-simplex on page 423, that the standard generator of $H_1(\overline{B}^1, S^0)$ is represented by the following cycle in $C_1(\overline{B}^1, S^0)$:

$$\begin{array}{rcl} \Delta^1 & \to & \overline{B}^1 \\ (1-t,t) & \mapsto & 2t-1. \end{array}$$



Now we can provide a scheme for equipping the cellular chain groups with bases.

Definition.

(1) A charted CW-complex is a pair consisting of a CW-complex X together with a full set $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X^{\dim(c)}\}_{c\in \mathrm{Cell}(X)}$ of characteristic maps of the cells of X.

(2) Let X be a charted CW-complex as in (1). For an n-cell α we write[368]

$$\Xi_\alpha \;:=\; \Phi_{\alpha *}([\overline{B}^n]) \;\in\; \mathrm{H}_n(X^n, X^{n-1}) \;=\; \mathrm{C}_n^{\mathrm{CW}}(X).$$

With this notation the Skeleton-H_*-Lemma 47.1 (1) says that $\{\Xi_c\}_{c\in \mathrm{Cell}(X)}$ is a basis of the free abelian group $\mathrm{C}_n^{\mathrm{CW}}(X)$.[369]

Our goal now is, given a charted CW-complex X, to determine the matrix of the cellular boundary map

$$d_n\colon \quad \underbrace{\mathrm{C}_n^{\mathrm{CW}}(X)}_{\substack{\text{free abelian group with}\\ \text{basis } \{\Xi_c\}_{c\in \mathrm{Cell}_n(X)}}} \quad \to \quad \underbrace{\mathrm{C}_{n-1}^{\mathrm{CW}}(X)}_{\substack{\text{free abelian group with}\\ \text{basis } \{\Xi_c\}_{c\in \mathrm{Cell}_{n-1}(X)}}}$$

As a warm-up with start out with the case $n = 1$. The following lemma shows that it is straightforward to determine the first cellular boundary map:

Lemma 47.5. (Cellular-First Boundary Map Lemma) Let X be a charted CW-complex. For a 1-cell α with characteristic map $\Phi_\alpha\colon \overline{B}^1 \to X^1$ we have

$$\underset{\substack{\uparrow\\ \text{element of } \mathrm{C}_1^{\mathrm{CW}}(X)}}{d_1(\Xi_\alpha)} \;=\; \Xi_{\Phi_\alpha(1)} - \Xi_{\Phi_\alpha(-1)} \quad \in\; \mathrm{C}_0^{\mathrm{CW}}(X).$$

In particular, if X has precisely one 0-cell, then the cellular boundary map d_1 is the zero map.

Proof. Recall that the cellular boundary map d_n is defined as follows:

$$\begin{array}{ccc} \mathrm{H}_n(X^n, X^{n-1}) & \xrightarrow{\quad =:d_n \quad} & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}). \\ & {\scriptstyle \partial_n}\searrow \quad \nearrow{\scriptstyle \text{я}} & \\ & \mathrm{H}_{n-1}(X^{n-1}) & \end{array}$$

In our special case $n = 1$ we see that $X^{n-2} = \varnothing$, thus d_1 equals the connecting homomorphism ∂_1 of the pair (X^1, X^0) as defined in the LES-of-Pair Proposition 38.3. Now we see that

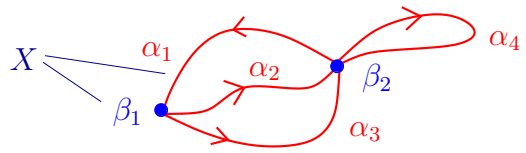
$$\begin{array}{lcll} & \overset{\text{by definition of } \Xi_\alpha \in \mathrm{C}_1^{\mathrm{CW}}(X)}{\downarrow} & \overset{\text{by the above discussion}}{\downarrow} & \overset{\text{see the description of } [\overline{B}^1] \text{ on page 552}}{\downarrow} \\ d_1(\Xi_\alpha) \;=\; d_1(\Phi_{\alpha *}([\overline{B}^1])) & = & \partial_1(\Phi_{\alpha *}([\overline{B}^1])) & = \quad \partial_1([(t, 1-t) \mapsto \Phi_\alpha(2t-1)]) \\ & & & = \quad \Xi_{\Phi_\alpha(1)} - \Xi_{\Phi_\alpha(-1)}. \\ & & & \quad \uparrow \\ & & & \text{see the definition of the connecting homomorphism } \partial_1 \text{ in} \\ & & & \text{the LES-of-Pair Proposition 38.3 and the discussion of } \partial_1 \text{ on page 423} \end{array}$$

∎

Example. We consider the charted 1-dimensional CW-complex X with two 0-cells β_1, β_2 and four 1-cells $\alpha_1, \ldots, \alpha_4$. that is given in the figure below:
By the Cellular-First Boundary Map Lemma 47.5 the cellular chain complex of X is of the form

[368] Note that Φ_α defines a map $(\overline{B}^n, S^{n-1}) \to (X^n, X^{n-1})$ of pairs of topological spaces. Thus we get an induced map $\Phi_{\alpha *}\colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \mathrm{H}_n(X^n, X^{n-1})$, which we can apply to the standard generator $[\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$.

[369] It is quite common to just write α instead of Ξ_α, i.e. it is common to view the cells themselves as elements of the cellular chain complex. With this risque language one can then view $\mathrm{Cell}_n(X)$ as a basis of $\mathrm{C}_n^{\mathrm{CW}}(X)$. For clarity's sake we will not adopt this language.



$$0 \to \underbrace{\begin{matrix}\mathbb{Z}\cdot\Xi_{\alpha_1}\\ \mathbb{Z}\cdot\Xi_{\alpha_2}\\ \mathbb{Z}\cdot\Xi_{\alpha_3}\\ \mathbb{Z}\cdot\Xi_{\alpha_4}\end{matrix}}_{=\mathrm{C}_1^{\mathrm{CW}}(X)} \xrightarrow{\begin{matrix}\ & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ \beta_1 & 1 & -1 & -1 & 0\\ \beta_2 & -1 & 1 & 1 & 0\end{matrix}} \underbrace{\begin{matrix}\mathbb{Z}\cdot\Xi_{\beta_1}\\ \mathbb{Z}\cdot\Xi_{\beta_2}\end{matrix}}_{=\mathrm{C}_0^{\mathrm{CW}}(X)} \to 0.$$

It follows that

$$\begin{array}{ccccc} \mathrm{H}_n(X) & \cong & \mathrm{H}_n^{\mathrm{CW}}(X) & = & \begin{cases} 0, & \text{if } n \geq 2,\\ \mathbb{Z}^3, & \text{if } n = 1,\\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \\ & \uparrow & & \uparrow & \\ & \text{by the Singular-Cellular-H}_*\text{-Proposition 47.3} & & \text{by the above calculation and elementary linear algebra} & \end{array}$$

The next proposition shows that we can calculate the cellular boundary maps d_n in terms of degrees of suitable continuous self-maps of S^{n-1}.

Proposition 47.6. (Cellular Boundary–Degree Proposition) Let X be a charted CW-complex and let $n \in \mathbb{N}$.[370] Given an n-cell α and an $(n-1)$-cell β we consider the following continuous self-map of S^{n-1}:

$$\begin{array}{ccccccc} S^{n-1} & = & S^{n-1} & \xrightarrow{\Phi_\alpha|_{S^{n-1}}} & X^{n-1} & \xrightarrow{q} & X^{n-1}/X^{n-2} \\ & & & & & & \cong \uparrow \bigvee_{j\in\mathrm{Cell}_{n-1}(X)} \overline{\Phi}_j \\ & & & & & & \bigvee_{j\in\mathrm{Cell}_{n-1}(X)} \overline{B}^{n-1}/S^{n-2} \xrightarrow{p_\beta} \overline{B}^{n-1}/S^{n-2} \xrightarrow[\cong]{g} S^{n-1} \end{array}$$

where we adopt the following notation:

- For $c \in \mathrm{Cell}(X)$ we denote by Φ_c the corresponding characteristic map.
- $q\colon X^{n-1} \to X^{n-1}/X^{n-2}$ is the natural projection.
- Let $j \in \mathrm{Cell}_{n-1}(X)$. Since $\Phi_j(S^{n-2}) \subset X^{n-2}$ we see that the characteristic map $\Phi_j\colon \overline{B}^{n-1} \to X^{n-1}$ descends to a continuous map $\overline{\Phi}_j\colon \overline{B}^{n-1}/S^{n-2} \to X^{n-1}/X^{n-2}$.
- $\bigvee_{j\in\mathrm{Cell}_{n-1}(X)} \overline{\Phi}_j\colon \bigvee_{j\in\mathrm{Cell}_{n-1}(X)} \overline{B}^{n-1}/S^{n-2} \to X^{n-1}/X^{n-2}$ is the natural map provided by the Wedge Proposition 17.1. Note that it follows from the Skeleton-Quotient Lemma 33.5 that this map is a homeomorphism.
- $p_\beta\colon \bigvee_{j\in\mathrm{Cell}_n(X)} \overline{B}^{n-1}/S^{n-2} \to \overline{B}^{n-1}/S^{n-2}$ is the natural map that is induced by the identity on the β-copy of $\overline{B}^{n-1}/S^{n-2}$ and the constant map on all other copies of $\overline{B}^{n-1}/S^{n-2}$.
- $g\colon \overline{B}^{n-1}/S^{n-2} \xrightarrow{\cong} S^{n-1}$ denotes the homeomorphism from the Ball Quotient-Sphere Lemma 6.5.

For any n-cell α we have[371]
$$d_n(\Xi_\alpha) \;=\; \sum_{\beta\in \mathrm{Cell}_{n-1}(X)} \deg(\Omega_{\beta,\alpha})\cdot \Xi_\beta \;\in\; \mathrm{C}^{\mathrm{CW}}_{n-1}(X)$$
where[372]
$$\Omega_{\beta,\alpha} \;:=\; g\circ p_\beta \circ \Big(\bigvee_{j\in \mathrm{Cell}_{n-1}(X)} \overline{\Phi}_j\Big)^{-1} \circ q \circ \Phi_\alpha|_{S^{n-1}} \colon S^{n-1} \;\to\; S^{n-1}.$$

The proof of the Cellular Boundary–Degree Proposition 47.6 requires the following lemma.

Lemma 47.7. (Standard Generator-of-$\overline{B}^n$-Lemma) Let $n\in\mathbb{N}_0$. Furthermore let $ю\colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \widetilde{\mathrm{H}}_n(\overline{B}^n/S^{n-1})$ be the natural map and let $g\colon \overline{B}^n/S^{n-1}\to S^n$ be the homeomorphism from the Ball Quotient-Sphere Lemma 6.5. Then
$$(g_*\circ ю)([\overline{B}^n]) \;=\; [S^n] \quad \in \widetilde{\mathrm{H}}_n(S^n).$$

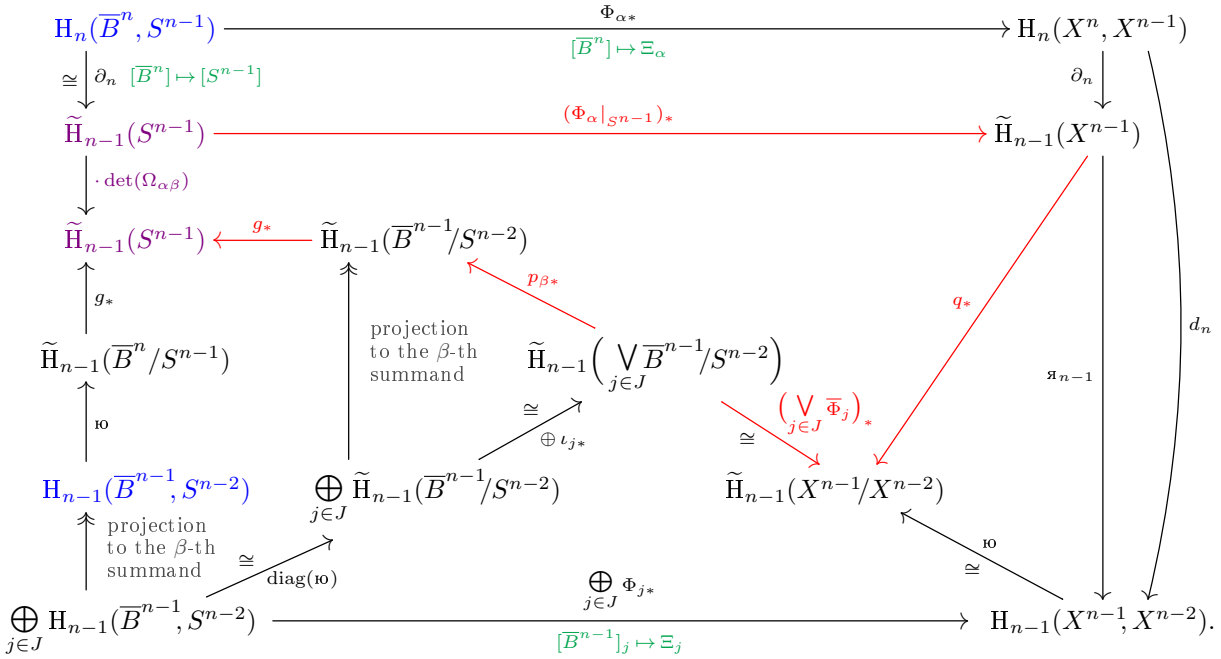
Proof of the Standard Generator-of-$\overline{B}^n$-Lemma 47.7. First note that it follows from the Relative-to-Quotient-H_*-Proposition refprop:homology-of-x-mod-a that the map $g_*\circ ю\colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \widetilde{\mathrm{H}}_n(S^n)$ is an isomorphism. Since $[\overline{B}^n]\in \mathrm{H}_n(\overline{B}^n, S^{n-1})$ and $[S^n]\in \widetilde{\mathrm{H}}_n(S^n)$ are generators we thus see that $(g_*\circ ю)([\overline{B}^n]) = \pm[S^n]\in \widetilde{\mathrm{H}}_n(S^n)$. Therefore it remains to determine the precise sign. This is a little tricky and we outsource the proof to [Fri23]. For most purposes the precise sign is not really relevant though. For example in the statement of the Cellular Boundary–Degree Proposition 47.6 a different sign in the equality $(g_*\circ ю)([\overline{B}^n]) = \pm[S^n]\in \widetilde{\mathrm{H}}_n(S^n)$ changes at worst all $\deg(\Omega_{\beta,\alpha})$ by the *same* sign, which has no effect later on when we need to determine (co-) kernels. ■

Proof of the Cellular Boundary–Degree Proposition 47.6. We adopt the following notation:

(1) We write $I := \mathrm{Cell}_n(X)$ and $J := \mathrm{Cell}_{n-1}(X)$.
(2) Given $k\in J$ we denote by $\iota_k\colon \overline{B}^{n-1}/S^{n-2} \to \bigvee_{j\in J} \overline{B}^{n-1}/S^{n-2}$ the natural injection.

[370] It takes a while to see that the statement for $n=1$ is equivalent to the statement of the Cellular-First Boundary Map Lemma 47.5.
[371] We refer to page 504 for the definition of the degree of a continuous self-map of a sphere.
[372] It follows easily from the CW-Complex-Finiteness Theorem 30.10 that for all but finitely many β the given map $\Omega_{\beta,\alpha}\colon S^{n-1}\to S^{n-1}$ is the constant map, which implies that for all but finitely many β we have $\deg(\alpha_{\beta,\alpha}) = 0$.

For any n-cell α and any $(n-1)$-cell β we consider the following diagram:



We make the following claim.

Claim. The diagram commutes and all the maps decorated with "$\cong$" are indeed isomorphisms.

Proof. We consider the various parts of the diagram.

(1) It follows from the naturality of the connecting homomorphism, see the Reduced LES-of-Pair Proposition 39.6 (2), that the upper rectangle in the diagram commutes.

(2) The red arrows trace out the map $\Omega_{\beta,\alpha}$ of which we take the degree. In other words, the oddly shaped hexagon in the center of the diagram commutes by definition.

(3) We consider the hexagon to the lower left of the diagram. Using the functoriality of relative homology one can easily verify that the hexagon commutes on each summand, hence it commutes for the direct sum.

(4) (a) Note that the map $p_\beta \circ \iota_j$ equals $\mathrm{id}_{\overline{B}^{n-1}/S^{n-2}}$ if $j = \beta$ and that it is constant otherwise. It follows easily from this observation that the triangle towards the center left, involving the map $p_{\beta *}$ commutes.

(b) It follows from the Wedge-H_*-Proposition 41.10 that the map decorated with "$\cong$" in the triangle is indeed an isomorphism.

(5) In the proof of the Skeleton-H_*-Lemma 47.1 (1) we showed that the large triangle at the bottom commutes and we showed that all the maps involved in the large triangle are indeed isomorphisms.

(6) (a) It follows easily from the definitions that the triangle to the bottom right commutes.

(b) It follows from the Relative-to-Quotient-H_*-Proposition 41.8 that the diagonal map at the very bottom right is an isomorphism. ⊞

Next note that by the definition of the standard generator of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ on page 552 we have $\partial_n([\overline{B}^n]) = [S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ and note that the Standard Generator-of-$\overline{B}^n$-Lemma 47.7 says that $g_*(\mathfrak{to}([\overline{B}^{n-1}])) = [S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$. The reader who is still awake will notice that the proposition follows immediately from the claim and this observation. ■

47.5. **The homology groups of surfaces.** In this section we will use the Cellular Boundary–Degree Proposition 47.6 to calculate the isomorphism types of the homology groups of compact 2-dimensional topological and smooth manifolds. First let us recall the following standard notation.

Notation. Given $g \in \mathbb{N}_0$ and given $k \in \mathbb{N}$ we write as usual

$$\begin{aligned} \Sigma_g &:= \text{surface of genus } g \\ N_k &:= \text{non-orientable surface of genus } k. \end{aligned}$$

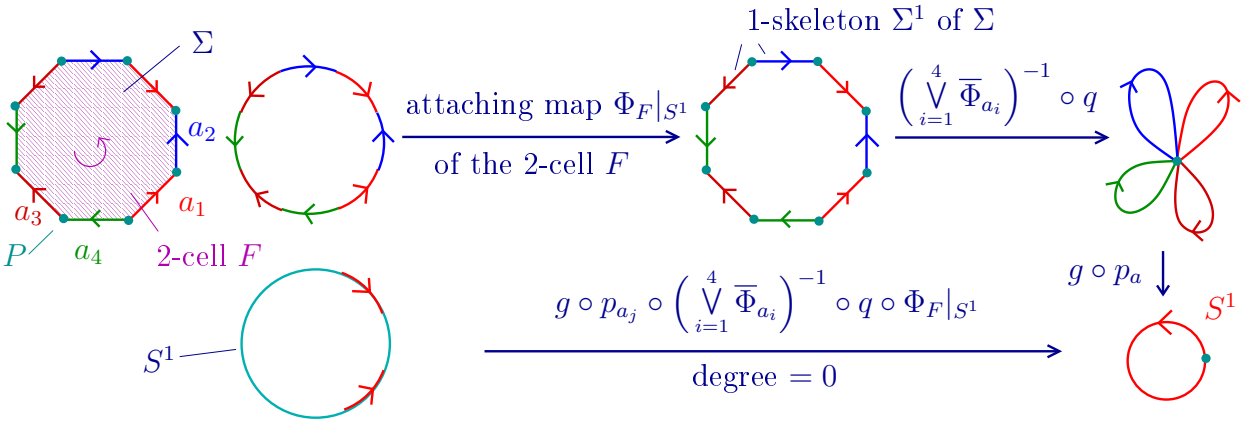
Furthermore given $m \in \mathbb{N}_0$ we write [373] [374]

$$\begin{aligned} \Sigma_{g,m} &:= \text{the surface of genus } g \text{ minus } m \text{ open disks} \\ N_{k,m} &:= \text{the non-orientable surface of genus } k \text{ minus } m \text{ open disks.} \end{aligned}$$

In the following we will first consider some explicit examples before stating a proposition that gives us the calculation of the isomorphism types of the homology groups for all the above surfaces.

Example.

(1) We consider the surface Σ_2 of genus 2. As usual we view Σ_2 as a charted CW-complex with one 0-cell P, four 1-cells a_1, a_2, a_3, a_4 and one 2-cell F.



Note that it follows easily from the definitions, or alternatively from the Degree-via-Local Orientations Proposition 44.7, that for each $j \in \{1, \dots, 4\}$ the degree of the map $\Omega_{F,a_j} = g \circ p_a \circ \left(\bigvee_{i=1}^4 \overline{\Phi}_{a_i}\right)^{-1} \circ q \circ \Phi_F|_{S^1} \colon S^1 \to S^1$ is zero. It follows from the Cellular Boundary–Degree Proposition 47.6 that the cellular boundary map d_2 in the cellular

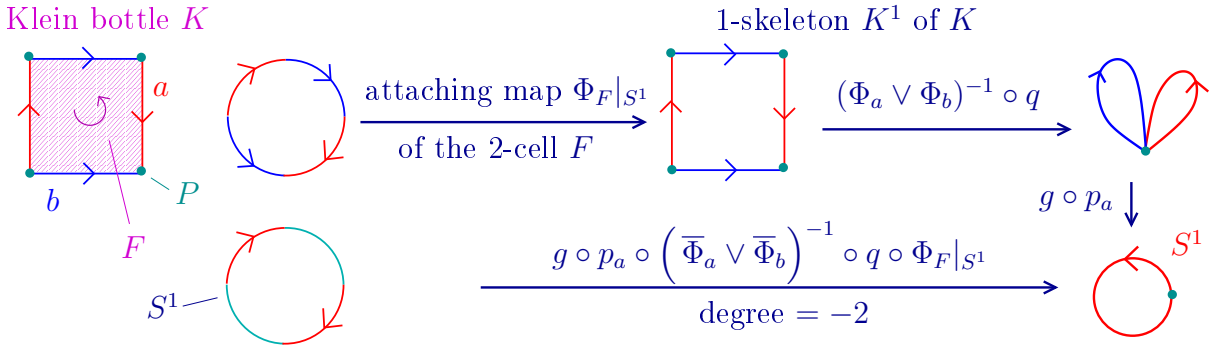
[373] Recall the definition on page 289 of what we mean by a "surface minus m open disks".

[374] Recall that in the Surface Classification Theorem 22.8 we saw that every connected, compact 2-dimensional topological manifold is homeomorphic to either $\Sigma_{g,m}$ for unique $g, m \in \mathbb{N}_0$ or to $N_{k,m}$ for unique $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$.

chain complex of Σ_2 is zero. Since Σ_2 has only one 0-cell we know from the Cellular-First Boundary Map Lemma 47.5 that the cellular boundary map d_1 is also zero. Thus we see that

$$\underset{\text{by the Singular-Cellular-H}_*\text{-Proposition 47.3}}{\mathrm{H}_n(\Sigma_2) \cong \mathrm{H}_n^{\mathrm{CW}}(\Sigma_2)} = \underset{\text{since all the cellular boundary maps are zero}}{\mathrm{C}_n^{\mathrm{CW}}(\Sigma_2)} \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

(2) As a reality check we consider once again the Klein bottle $K = N_2$. We view K as a charted CW-complex with one 0-cell P, two 1-cells a, b and one 2-cell F, as shown in the figure below.



It follows from the Cellular Boundary–Degree Proposition 47.6, the Degree-via-Local Orientations Proposition 44.7 and the Cellular-First Boundary Map Lemma 47.5 that the cellular chain complex is given by

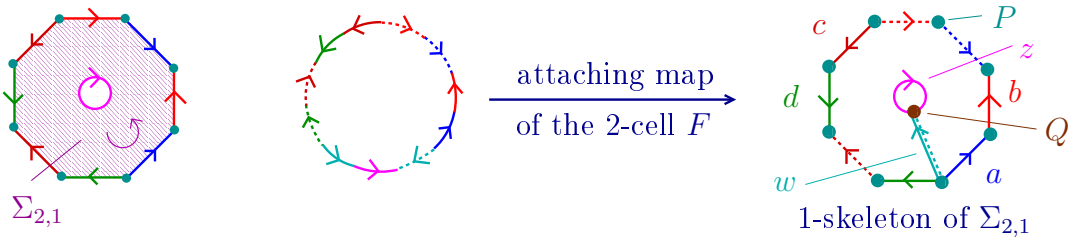
$$0 \to \mathbb{Z} \cdot \Xi_F \xrightarrow{\begin{pmatrix} -2 \\ 0 \end{pmatrix}} \begin{matrix} \mathbb{Z} \cdot \Xi_a \\ \mathbb{Z} \cdot \Xi_b \end{matrix} \xrightarrow{\begin{pmatrix} 0 & 0 \end{pmatrix}} \mathbb{Z} \cdot \Xi_P \to 0.$$

Therefore we see that

$$\underset{\text{by the Singular-Cellular-H}_*\text{-Proposition 47.3}}{\mathrm{H}_n(K) \cong \mathrm{H}_n^{\mathrm{CW}}(K)} = \underset{\text{by the above calculation and elementary linear algebra}}{} \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z} \oplus \mathbb{Z}_2, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

This is of course the same result that we had already obtained in the Klein Bottle-H$_*$-Lemma 45.6.

(3) We consider $\Sigma_{2,1}$ and we equip it with the charted CW-structure that is illustrated in the figure below.



This CW-structure has two 0-cells P, Q, six 1-cells a, b, c, d, w, z and one 2-cell F. Using the Cellular Boundary–Degree Proposition 47.6 and using the Cellular-First Boundary 47.5 one can easily show that the cellular chain complex is of the following form:

$$0 \to \mathbb{Z}\cdot\Xi_F \xrightarrow{\begin{matrix} F \\ a \\ b \\ c \\ d \\ w \\ z \end{matrix}\!\!\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \begin{matrix} \mathbb{Z}\cdot\Xi_a \\ \mathbb{Z}\cdot\Xi_b \\ \mathbb{Z}\cdot\Xi_c \\ \mathbb{Z}\cdot\Xi_d \\ \mathbb{Z}\cdot\Xi_w \\ \mathbb{Z}\cdot\Xi_z \end{matrix} \xrightarrow{\begin{matrix} \quad a & b & c & d & w & z \\ P \\ Q \end{matrix}\!\!\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.} \begin{matrix} \mathbb{Z}\cdot\Xi_P \\ \mathbb{Z}\cdot\Xi_Q \end{matrix} \to 0.$$

Now we see that

$$\mathrm{H}_n(\Sigma_{2,1}) \cong \mathrm{H}_n^{\mathrm{CW}}(\Sigma_{2,1}) \cong \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

by the Singular-Cellular-H_*-Proposition 47.3

by the above chain complex and some elementary linear algebra

In the following proposition we give the calculation for the isomorphism types of the homology groups of all surfaces $\Sigma_{g,n}$ and $N_{k,m}$:

Proposition 47.8. (Surface-H_*-Proposition) For any $g \in \mathbb{N}_0$ we have

$$\mathrm{H}_n(\Sigma_g) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^{2g}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Furthermore for $m \geq 1$ we have

$$\mathrm{H}_n(\Sigma_{g,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{2g+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

For any $k \in \mathbb{N}$ we have

$$\mathrm{H}_n(N_k) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{k-1} \oplus \mathbb{Z}_2, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Furthermore for $m \geq 1$ we have

$$\mathrm{H}_n(N_{k,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{k+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof. The calculation of the homology groups Σ_g is a slight generalization of the arguments that we provided on page 557. Furthermore the case N_k is dealt with the same way as we dealt with the Klein bottle $K = N_2$ on page 558. Finally note that for $m \in \mathbb{N}$ the calculation of the homology groups of $\Sigma_{g,m}$ and $N_{k,m}$ is a slight generalization of the argument preceding the proposition. We leave the details to the reader. ■

47.6. **The homology groups of real projective spaces.** Recall that in the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 47.4 we calculated the homology groups of the complex projective spaces $\mathbb{C}\mathrm{P}^n$. In this section we will finally compute the homology groups of the real projective spaces $\mathbb{R}\mathrm{P}^n$. This calculation is significantly trickier than the calculation for $\mathbb{C}\mathrm{P}^n$, since this time we will need the theory of local degrees, which we developed in Section 44.2, to determine cellular boundary maps.

Proposition 47.9. ($\mathbb{R}\mathbf{P}^n$-$\mathbf{H}_*$-Proposition)
(1) For any $n \in \mathbb{N}_0$ we have[375]

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd and } k < n, \\ 0, & \text{if } k \text{ is even and } 0 < k \neq 0, \\ 0, & \text{if } k > n, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd.} \end{cases}$$

(2) We have

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^\infty) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even and } k > 0. \end{cases}$$

Proof.

(1) Let $n \in \mathbb{N}_0$. On page 115 we showed that for any $k \leq n$ the map $\mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^n$ that is given by $[x] \mapsto [(x, 0)]$ is an embedding. For the duration of the proof we use this embedding as an excuse to view $\mathbb{R}\mathrm{P}^k$ as a subspace of $\mathbb{R}\mathrm{P}^n$.

Next recall we recall the CW-structure on $\mathbb{R}\mathrm{P}^n$ which we introduced on page 375: For each $k = 0, \ldots, n$ we define

$$\begin{aligned} \Phi_k \colon \overline{B}^k &\to \mathbb{R}\mathrm{P}^k \\ x &\mapsto [(x, \sqrt{1 - \|x\|^2})]. \end{aligned}$$

We saw in the proof of the Projective Space–Cell Attachment Lemma 29.2 that the map

$$\begin{aligned} \mathbb{R}\mathrm{P}^{k-1} \cup_{\mathbb{R}\mathrm{P}^{k-1} \xleftarrow{\Phi_k|_{S^{k-1}}} S^{k-1}} \overline{B}^k &\xrightarrow{\cong} \mathbb{R}\mathrm{P}^k \\ [x] &\mapsto \begin{cases} [(x, 0)], & \text{if } x \in \mathbb{R}\mathrm{P}^{k-1} \\ \Phi_k(x), & \text{if } x \in \overline{B}^k \end{cases} \end{aligned}$$

is a homeomorphism. Thus we see inductively that we can view $\mathbb{R}\mathrm{P}^n$ as a charted CW-complex with a single cell in the dimensions $0, \ldots, n$ where the characteristic maps are given precisely by the maps $\Phi_0, \ldots, \Phi_n$.

Claim. For each $k = 1, \ldots, n$ the cellular boundary map

$$d_k \colon \mathbb{Z} \cong \mathrm{C}_k^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^n) \to \mathrm{C}_{k-1}^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}$$

is given by multiplication by $\pm(1 + (-1)^k)$.

Proof. Let $k \in \{1, \ldots, n\}$. We denote by $q \colon \mathbb{R}\mathrm{P}^{k-1} \to \mathbb{R}\mathrm{P}^{k-1}/\mathbb{R}\mathrm{P}^{k-2}$ the projection and we denote by $g \colon \overline{B}^{k-1}/S^{k-2} \to S^{k-1}$ the homeomorphism from the Ball Quotient-Sphere Lemma 6.5. By the Cellular Boundary–Degree Proposition 47.6 it suffices to show that the degree of the map

$$S^{k-1} \xrightarrow{\Phi_k|_{S^{k-1}}} \mathbb{R}\mathrm{P}^{k-1} \xrightarrow{q} \mathbb{R}\mathrm{P}^{k-1}/\mathbb{R}\mathrm{P}^{k-2} \xrightarrow{(\overline{\Phi}_{k-1})^{-1}} \overline{B}^{k-1}/S^{k-2} \xrightarrow{g} S^{k-1}$$

$$=: \Omega$$

[375] The calculation looks more complicated than it is, in fact the most efficient way to remember the calculation is to remember the description of the cellular chain complex given in the proof.

equals $\pm(1+(-1)^k)$. We will do so using the Degree-via-Local Orientations Proposition 44.7. To apply this proposition we recall a few definitions and we make a few preparations:

(a) If $\Omega\colon S^{k-1}\to S^{k-1}$ is a local diffeomorphism at a point x, then as on page 511 we write

$$\deg(\Omega,x) \;:=\; \begin{cases} +1, & \text{if } \Omega \text{ is orientation-preserving at } x \\ -1, & \text{if } \Omega \text{ is orientation-reversing at } x. \end{cases}$$

(b) If $f\colon S^{k-1}\to S^{k-1}$ is a diffeomorphism, then we write

$$\deg(f) \;:=\; \begin{cases} +1, & \text{if } f \text{ is orientation-preserving} \\ -1, & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

(c) If $\Omega\colon S^{k-1}\to S^{k-1}$ is a local diffeomorphism at a point x and if $f\colon S^{k-1}\to S^{k-1}$ is a diffeomorphism, then it follows immediately from $\mathrm{D}(\Omega\circ f)_{f^{-1}(x)} = \mathrm{D}\Omega_x \circ \mathrm{D}\Omega_{f^{-1}(x)}$ that $\deg(\Omega\circ f, f^{-1}(x)) = \deg(\Omega,x)\cdot\deg(f)$.

(d) We pick the point $y=(0,\dots,0,1)\in S^{k-1}$. The preimages of y under the map Ω are $x_+=(0,\dots,0,1)$ and $x_-=(0,\dots,0,-1)$. One can easily verify that the restrictions of Ω to the open hemispheres $B^n_{>0}$ and $B^n_{<0}$ are smooth embeddings. Since $x_+\in B^n_{>0}$ and $x_-\in B^n_{<0}$ we see that Ω is a local diffeomorphism around x_+ and x_-.[376]

(e) Let $\rho\colon S^{k-1}\to S^{k-1}$ be the reflection in the origin, i.e. ρ is given by $z\mapsto -z$.

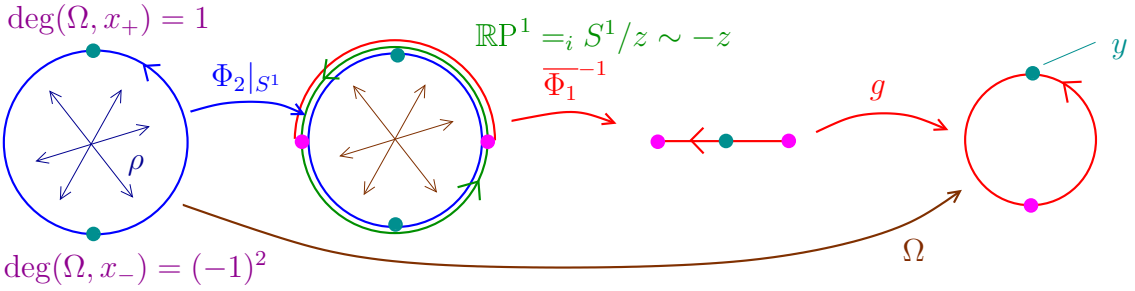
(f) One can easily show that $\deg(\rho)=\det(-\mathrm{id}_k)=(-1)^k$.

(g) Note that $(\Phi_k|_{S^{k-1}})\circ\rho = \Phi_k|_{S^{k-1}}\colon S^{k-1}\to \mathbb{R}\mathrm{P}^{k-1}$. This implies that $\Omega\circ\rho=\Omega$.

We now see that

$$\begin{array}{rcll} \deg(\Omega) & \overset{\text{by the Degree-via-Local Orientations Proposition 44.7 and by (d)}}{\downarrow} & & \overset{\text{since by (g) we know that } \Omega\circ\rho=\Omega}{\downarrow} \\ & \deg(\Omega,x_+)+\deg(\Omega,x_-) & & \deg(\Omega,x_+)+\deg(\Omega\circ\rho,x_-) \\ & \uparrow \text{follows from (c), since } \rho(x_+)=x_- & & \\ & \deg(\Omega,x_+)+\deg(\Omega,x_+)\cdot\deg(\rho) & = & \underbrace{\deg(\Omega,x_+)}_{=\pm 1}\cdot(1+\underbrace{\deg(\rho)}_{=(-1)^k \text{ by (f)}}). \end{array}$$

⊞



It follows from the claim that the cellular chain complex of $\mathbb{R}\mathrm{P}^n$ is of the form

$$0 \to \underset{\substack{\uparrow\\ \text{degree } n}}{\mathbb{Z}} \xrightarrow{\pm(1+(-1)^n)} \mathbb{Z} \to \dots \xrightarrow{\cdot(\pm 2)} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot(\pm 2)} \mathbb{Z} \xrightarrow{0} \underset{\substack{\uparrow\\ \text{degree } 0}}{\mathbb{Z}} \to 0.$$

[376] If one struggles through all the orientation conventions and all the maps, then one can show that $\deg(\Omega,x_+)=1$ and that $\deg(\Omega,x_-)=(-1)^k$. We will not go down that route, instead we will use a more indirect, but hopefully more robust and more convincing argument.

It is now straightforward to determine the cellular homology groups which, by the Singular-Cellular-H_*-Proposition 47.3, are isomorphic to the singular homology groups of $\mathbb{R}P^n$.

(2) On page 376 we equipped $\mathbb{R}P^\infty = \operatorname{colim}_{\longrightarrow} \mathbb{R}P^n$ with a CW-structure with precisely one cell in each dimension. Basically the same calculation as in (1) gives the desired result. Alternatively, as in the proof of the $\mathbb{C}P^n$-H_*-Proposition 47.4 (2) one can use the calculation of (1) and the Exhaustions-H_*-Proposition 46.2 to obtain the desired result. We leave it to the reader to work out the details for this second approach. ■

Remark. We have now seen that the homology groups of CW-complexes can be computed fairly easily using cellular chain complexes. One might ask, why did we introduce singular chain complexes, if it is so much easier to do calculations with cellular chain complexes? There are two related answers:

(1) We also want to study topological spaces, e.g. topological manifolds, that are not (known to be) CW-complexes.

(2) It is much harder to prove some statements using cellular chain complexes, for example it is very difficult to show (without making recourse to singular homology) that homeomorphic CW-complexes have isomorphic homology groups.

A general slogan in topology (and many other fields of mathematics) is that for calculations it is usually best to work with "small objects" (in our case the cellular chain complex), whereas for proving general properties, it is often best to work with "big objects" (in our case the singular chain complex).

48. The Euler characteristic

In this chapter we introduce the Euler characteristic $\chi(X) \in \mathbb{N}_0$ of a finite CW-complex X. This invariant has two interesting features:

(1) The Euler characteristic is easy to define and it is fun to calculate Euler characteristics.
(2) Using singular homology we will see that the Euler characteristic depends only on the homotopy type of X.

In this and the next chapter we will see that the Euler characteristic can be used to prove many statements and answer many classical problems (e.g. why does a football have 12 pentagons?).

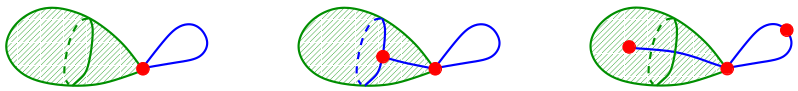
48.1. **The Euler characteristic of a CW-complex.** Let us start out with the straightforward definition of the Euler characteristic of a finite CW-complex.

Definition. Given a finite CW-complex X we define its Euler characteristic $\chi(X)$ as follows:

$$\chi(X) := \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \text{number of } n\text{-cells}.$$

Examples.

(1) If G is a finite topological graph, then, according to the Graph–CW-Complex Lemma 30.1 we can view G as a 1-dimensional CW-complex where the number of 0-cells equals the number of vertices and the number of 1-cells equals the number of edges of G. Thus it follows immediately from the definitions that in this case the above Euler characteristic agrees with the definition of the Euler characteristic of a finite topological graph that we gave on page 132.
(2) In the figure below we show the topological space $S^2 \vee S^1$ with three different CW-structures. We notice that even though the numbers of cells in the various dimensions differ, the Euler characteristic is always the same.



number of 0-cells	1	2	3
number of 1-cells	1	3	3
number of 2-cells	1	2	1
Euler characteristic	1	1	1

The above example suggests that the Euler characteristic depends perhaps only on the homeomorphism type of the CW-complex. The following results shows that this is indeed the case.

Proposition 48.1. (Homology-χ-Proposition) Given any finite CW-complex X we have

$$\chi(X) = \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \operatorname{rank}(\mathrm{H}_n(X)).$$

Proof. We will provide the proof of Homology-χ-Proposition 48.1 in the next section. ■

Before we do anything else let us record the following pretty obvious corollary.

Corollary 48.2. (Homotopy Type-χ-Corollary) Let X and Y be finite CW-complexes. If X and Y are homotopy equivalent, then $\chi(X) = \chi(Y)$.[377]

Proof of the Homotopy Type-χ-Corollary 48.2. The corollary is an immediate consequence of the Homology-χ-Proposition 48.1 together with the Homotopy Equivalence-H_*-Corollary 36.7 (2). ■

Now that we know that the Euler characteristic is a homotopy invariant (in particular a homeomorphism invariant) it is worth determining it for some of our favorite topological spaces.

Examples.

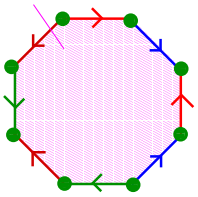
(1) Let $n \in \mathbb{N}_0$. On page 372 we showed that the n-dimensional sphere S^n has a CW-structure of S^n with one 0-cell and one extra cell in dimension n. It follows that[378]

$$\chi(S^n) \;=\; 1+(-1)^n \;=\; \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

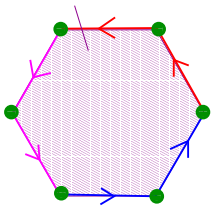
(2) Given $g \in \mathbb{N}_0$ we consider the surface Σ_g of genus g and given $k \in \mathbb{N}$ we consider the non-orientable surface N_k of genus k. On page 564 we equipped Σ_g and N_k with CW-structures that are also shown (for $g=2$ and $k=3$) in the figure below. One can now easily calculate that[379]

$$\chi(\Sigma_g) \;=\; 2-2g \qquad \text{and} \qquad \chi(N_k) \;=\; 2-k.$$

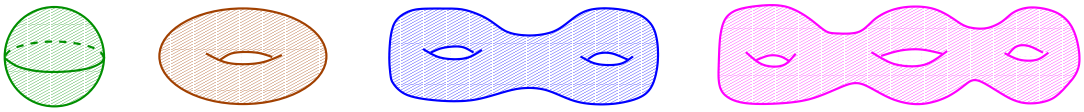
surface Σ_g of genus $g=2$ | non-orientable surface N_k of genus $k=3$



1	number of 0-cells	1
$2\cdot g$	number of 1-cells	k
1	number of 2-cells	1
$2-2\cdot g$	Euler characteristic	$2-k$



In the following figure we show the surfaces with increasing genus. The surface of genus 0, i.e. the sphere S^2, has Euler characteristic $+2$. When we increase the genus by one, the Euler characteristic drops by two.

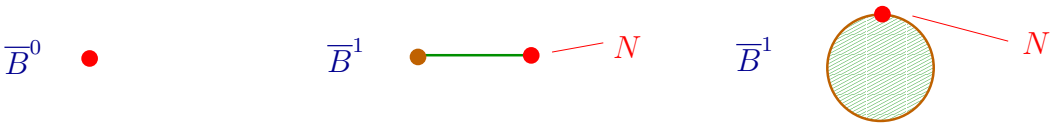


$\chi =$ $+2$ | 0 | -2 | -4

(3) Let $n \in \mathbb{N}_0$. For $n=0$ we have $\overline{B}^0 = \{0\}$, thus $\chi(\overline{B}^0) = 1$. For $n \geq 1$ a CW-structure for the ball $\overline{B}^n$ is given by the filtration $\varnothing \subset \{N\} \subset S^{n-1} \subset \overline{B}^n$, thus we have a

[377] This corollary implies in particular, that the Euler characteristic of a CW-complex does not depend on the choice of CW-structure.
[378] One can also calculate $\chi(S^n)$ using the Sphere-H_n-Proposition 41.2 and Homology-χ-Proposition 48.1.
[379] One can also calculate the Euler characteristics using the Surface-H_*-Proposition 47.8 together with the Homology-χ-Proposition 48.1.

CW-structure with one 0-cell $\{N\}$, one extra cell in dimension $n-1$ and one extra cell of dimension n. Thus again we obtain that $\chi(\overline{B}^n) = 1 + (-1)^{n-1} + (-1)^n = 1$.[380]



(4) On page 375 we showed that $\mathbb{R}\mathrm{P}^n$ admits a CW-structure with precisely one cell of dimensions $0, 1, \ldots, n$ and no other cell. It follows that[381]

$$\chi(\mathbb{R}\mathrm{P}^n) \;=\; \sum_{i=0}^{n} (-1)^n \;=\; \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

(5) On page 376 we showed that $\mathbb{C}\mathrm{P}^n$ admits a CW-structure with precisely one cell of dimensions $0, 2, \ldots, 2n$ and no other cell. It follows that[382] $\chi(\mathbb{C}\mathrm{P}^n) = n + 1$.

48.2. **Proof of the Homology-χ-Proposition 48.1.** The following purely algebraic lemma is the key ingredient in the proof of the Homology-χ-Proposition 48.1.

Lemma 48.3. (Chain Complex–H_*–χ-Lemma) Given any chain complex

$$0 \;\to\; C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \ldots\; C_1 \xrightarrow{\partial_1} C_0 \;\to\; 0$$

of finitely generated abelian groups the following equality holds:

$$\sum_{n=0}^{k} (-1)^n \cdot \mathrm{rank}(C_n) \;=\; \sum_{n=0}^{k} (-1)^n \cdot \mathrm{rank}(\mathrm{H}_n(C_*)).$$

Proof. For the above chain complex (C_*, ∂_*) we write for each $n \in \mathbb{N}_0$[383]

$$Z_n := \ker(\partial_n), \quad B_n := \mathrm{im}(\partial_{n+1}) \quad \text{and} \quad H_n = Z_n/B_n.$$

Note that these groups form the following two types of short exact sequences

$$\begin{array}{lccccccccc} \text{(a)} & 0 & \hookrightarrow & Z_n & \to & C_n & \xrightarrow{\partial_n} & B_{n-1} & \to & 0, \\ \text{(b)} & 0 & \hookrightarrow & B_n & \to & Z_n & \to & H_n & \to & 0. \end{array}$$

We see that

[380] Here is another approach to calculating the Euler characteristic: Note that $\overline{B}^n$ is contractible, i.e. $\overline{B}^n$ is homotopy equivalent to a singleton $\{*\}$. Thus it follows from the Homotopy Type-χ-Corollary 48.2 that $\chi(\overline{B}^n) = \chi(\{*\}) = 1$.

[381] We can also calculate $\chi(\mathbb{R}\mathrm{P}^n)$ using the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 47.9 and the Homology-χ-Proposition 48.1.

[382] We can also calculate $\chi(\mathbb{C}\mathrm{P}^n)$ using the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 47.4 and the Homology-χ-Proposition 48.1.

[383] It is a common notation to call these groups Z_n and B_n, the names go back to the German words "Zykel" and "Bild".

$$\begin{aligned}
&\text{by (a) we have } \operatorname{rank} C_n = \operatorname{rank} Z_n + \operatorname{rank} B_{n-1}\\
\sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank} C_n &\overset{\downarrow}{=} \sum_{n=0}^{k}(-1)^n\cdot(\operatorname{rank} Z_n+\operatorname{rank} B_{n-1})\\
&\overset{\uparrow}{=} \sum_{n=0}^{k}(-1)^n\cdot(\operatorname{rank} B_n+\operatorname{rank} H_n+\operatorname{rank} B_{n-1})\\
&\text{by (b) we have } \operatorname{rank} Z_n = \operatorname{rank} B_n + \operatorname{rank} H_n\\
&= \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank} H_n + \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank} B_n + \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank} B_{n-1}\\
&\overset{\uparrow}{=} \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank} H_n + \sum_{n=0}^{k-1}(-1)^n\cdot\operatorname{rank} B_n - \sum_{m=0}^{k-1}(-1)^m\cdot\operatorname{rank} B_m\\
&\text{we do the substitution } m=n-1 \text{ and we use that } B_k=0 \text{ and } B_{-1}=0\\
&= \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank} H_n. \qquad \blacksquare
\end{aligned}$$

Now we can easily provide the proof of the Homology-χ-Proposition 48.1.

Proof of the Homology-χ-Proposition 48.1. Let X be a finite k-dimensional CW-complex. We perform the following calculation:

$$\begin{array}{lll}
& \text{by definition of } \chi(X) & \text{by the Skeleton-H}_*\text{-Lemma 47.1 (1)}\\
\chi(X) & \overset{\downarrow}{=} \sum_{n=0}^{k}(-1)^n\cdot\text{number of } n\text{-cells} & \overset{\downarrow}{=} \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank}(\mathrm{C}_n^{\mathrm{CW}}(X))\\
& \overset{\uparrow}{=} \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank}(\mathrm{H}_n^{\mathrm{CW}}(X)) & \overset{\uparrow}{=} \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank}(\mathrm{H}_n(X)).\\
\text{by the Chain Complex–H}_*\text{–}\chi\text{-Lemma 48.3} & & \text{by the Singular-Cellular-H}_*\text{-Proposition 47.3}
\end{array}$$

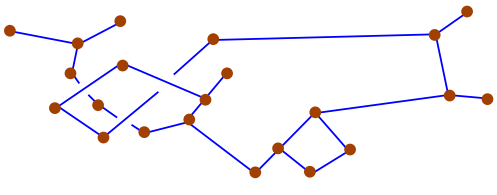
$\blacksquare$

48.3. **Homology groups of topological graphs.** Using the Homology-χ-Proposition 48.1 we can now easily prove the following lemma:

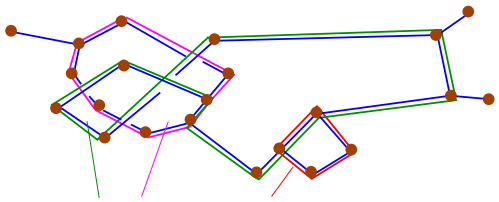
Lemma 48.4. (Graph-H$_*$-Lemma) Let G be a connected non-empty topological graph with v vertices and e edges. Then

$$\mathrm{H}_0(G)\cong\mathbb{Z},\quad \mathrm{H}_1(G)\cong\mathbb{Z}^{e-v+1}\quad\text{and}\quad \mathrm{H}_i(G)=0 \text{ for } i\geq 2.$$

Example. In the figure below we see a connected topological graph G with 22 vertices and 24 edges. It follows from the Graph-H$_*$-Lemma 48.4 that its first homology group is isomorphic to $\mathbb{Z}^3$. Using the observation that an epimorphism of $\mathbb{Z}^3$ is an isomorphism one can show fairly easily that the three cycles which are shown in the figure below represent a basis for $\mathrm{H}_1(G)\cong\mathbb{Z}^3$.



topological graph G with 22 vertices and 24 edges



basis for $\mathrm{H}_1(G)\cong\mathbb{Z}^3$

Proof. Let G be a connected topological graph with v vertices and e edges. Recall that by the Graph–CW-Complex Lemma 30.1 we can view G as a 1-dimensional CW-complex with

v 0-cells and e 1-cells. From the H_0-Proposition 35.10 (1) together with the CW-Complex–Local Properties Proposition 31.7 (2) and the Locally Path-Connected Lemma 2.24 it follows immediately that $\mathrm{H}_0(G) \cong \mathbb{Z}$. Furthermorefrom considering the cellular chain complex it follows that $\mathrm{H}_i(G) = 0$ for $i \geq 2$ and that $\mathrm{H}_1(G)$ is torsion-free. We now see that

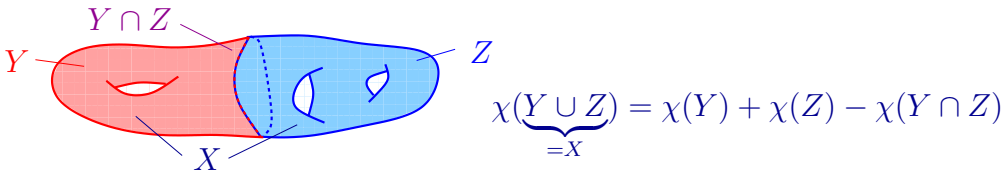
$$1 - \mathrm{rank}(\mathrm{H}_1(G)) \underset{\text{by the above}}{\uparrow} = \sum_i (-1)^i \cdot \mathrm{rank}(\mathrm{H}_i(X)) \underset{\text{by by the Homology-}\chi\text{-Proposition 48.1}}{\uparrow} = \chi(X) = \#0\text{-cells} - \#1\text{-cells} = v - e.$$

Since $\mathrm{H}_1(G)$ is torsion-free we obtain that $\mathrm{H}_1(G) \cong \mathbb{Z}^{e-k+1}$. ■

48.4. **Properties of the Euler characteristic.** The Euler characteristic is an invariant which "sees less" than the homology groups. On the other hand we will see in this section that it behaves better than the homology groups under several natural operations. This which makes the Euler characteristic an eminently computable invariant.

Proposition 48.5. (Union-χ-Proposition) Let $X = Y \cup Z$ be a decomposition of a finite CW-complex X into two subcomplexes Y and Z. The following equality holds [384]

$$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$



Proof. Given $n \in \mathbb{N}_0$ and given a subcomplex W of X we denote by $n(W)$ the number of n-cells that are contained in W. In our setting we have

$$n(X) \underset{\text{since } X = Y \cup Z \text{ and since } Y \text{ and } Z \text{ are subcomplexes we obtain from the Subcomplex–Characterization Proposition 30.7 together with the Subcomplex–Closed Lemma 30.6 that every } n\text{-cell of } X \text{ lies in } Y \text{ or } Z, \text{ the } n\text{-cells that lie in the intersection } Y \cap Z \text{ get counted twice in } n(Y) + n(Z), \text{ so we have to subtract } n(Y \cap Z) \text{ to obtain } n(X)}{\uparrow} = n(Y) + n(Z) - n(Y \cap Z).$$

By performing the alternating sum over all $n \in \mathbb{N}_0$ we obtain immediately that

$$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

■

Proposition 48.6. (Quotient-χ-Proposition) Let X be a finite CW-complex and let A be a subcomplex of X. We equip the quotient X/A with the CW-structure given by the CW-Quotient Lemma 33.3. Then

$$\chi(X/A) = \chi(X) - \chi(A) + 1.$$

Example. Let $\Sigma_{g,n}$ be the surface of genus g minus n open disks. We equip $\Sigma_{g,n}$ with a CW-structure. Let $C_1, \ldots, C_n$ be the boundary components of $\Sigma_{g,n}$. We denote by $\widehat{\Sigma}_{g,n}$ the topological space that is obtained from $\Sigma_{g,n}$ by collapsing each boundary component

[384] It follows from the Subcomplex–Intersection-Union Lemma 30.8 that $Y \cap Z$ is also a CW-complex.

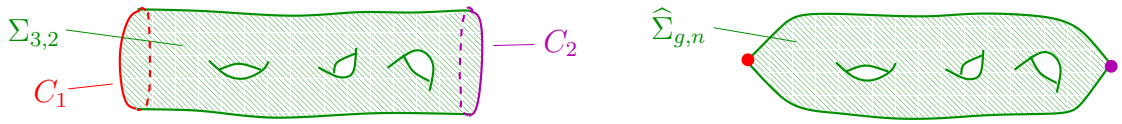
to a point. By the Surface Recovery Lemma 22.6 we know that $\widehat{\Sigma}_{g,n}$ is homeomorphic to the surface of genus g. We calculate that

$$2-2g \underset{\uparrow}{=} \chi(\widehat{\Sigma}_{g,n}) = \chi(\overbrace{(\ldots(\Sigma_{g,n}/C_1)/\ldots/C_k)}^{=\widehat{\Sigma}_{g,n}}) \underset{\uparrow}{=} \chi(\Sigma_{g,n}) - \sum_{i=1}^{n} \underbrace{\chi(C_i)}_{=0} + n = \chi(\Sigma_{g,n}) + n.$$

see page 564

by the Quotient-χ-Proposition 48.6, applied n times

Thus it follows that $\chi(\Sigma_{g,n}) = n = 2-2g-n$.



Proof. Recall that in the CW-Quotient Lemma 33.3 we equipped X/A with a CW-structure such that for each $k \in \mathbb{N}_0$ we have the following equalities:

$$\#k\text{-cells of } X/A = \begin{cases} \#k\text{-cells of } X \text{ not in } A, & \text{if } k > 0, \\ \#k\text{-cells of } X \text{ not in } A \ + \ 1, & \text{if } k = 0. \end{cases}$$

Evidently these equalities imply the equality $\chi(X/A) = \chi(X) - \chi(A) + 1$. ■

Proposition 48.7. (Product-χ-Proposition) Let X and Y be two finite CW-complexes. If we equip $X \times Y$ with the CW-structure from the CW-Category–Product Proposition 33.7 and the CW-Complex Product Proposition 33.9, then

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y).$$

Example. Let X be a finite CW-complex.

(1) It follows from the Product-χ-Proposition 48.7 that $\chi(S^1 \times X) = \chi(S^1) \cdot \chi(X) = 0$.

(2) We consider the suspension $\Sigma(X) = ((X \times [-1,1])/(X \times \{-1\}))/(X \times \{1\})$. We see that

$$\chi(\Sigma(X)) \underset{\downarrow}{=} \chi((X \times [-1,1]) - \chi(X \times \{-1\}) + 1 - \chi(X \times \{1\}) + 1$$
$$= 2 + \chi(X) - \chi(X) - \chi(X) = 2 - \chi(X).$$

follows from applying the Quotient-χ-Proposition 48.6 twice

follows from the Product-χ-Proposition 48.7 and the observation that $\chi([-1,1]) = 1$

Proof. Let X and Y be two finite CW-complexes. Given $k \in \mathbb{N}_0$ we denote by c_k the number of k-cells of X and we denote by d_k the number of k-cells of Y. We equip $X \times Y$ with the product CW-structure from the CW-Category–Product Proposition 33.7 and the CW-Complex Product Proposition 33.9. By construction we have

$$\#n\text{-cells of } X \times Y = \sum_{k+l=n} \#k\text{-cells of } X \cdot \#l\text{-cells of } Y = \sum_{k+l=n} c_k \cdot d_l.$$

It follows that

$$\chi(X) \cdot \chi(Y) = \Big(\sum_{k \in \mathbb{N}_0} (-1)^k \cdot c_k\Big) \cdot \Big(\sum_{l \in \mathbb{N}_0} (-1)^l \cdot d_l\Big) \underset{\downarrow}{=} \sum_{n \in \mathbb{N}_0} \sum_{k+l=n} (-1)^{k+l} \cdot c_k \cdot d_l$$
$$= \sum_{n \in \mathbb{N}_0} (-1)^n \cdot \sum_{k+l=n} c_k \cdot d_l \underset{\uparrow}{=} \chi(X \times Y).$$

by multiplying out

by the above discussion

We have thus proved the desired equality $\chi(X) \cdot \chi(Y) = \chi(X \times Y)$. ■

Next let $p\colon \widetilde{X} \to X$ be a covering of path-connected non-empty topological spaces. We know that the fundamental groups and the higher homotopy groups behave well under coverings:

(1) In the Covering-π_1-Monomorphism Proposition 14.11 we showed that the induced map $p_*\colon \pi_1(\widetilde{X}) \to \pi_1(X)$ is a monomorphism and we saw in the Covering-Degree Proposition 14.13 that $[\pi_1(X) : p_*(\pi_1(\widetilde{X}))] = \deg(p)$.
(2) For $n \geq 2$ we obtain from the Covering-π_n-Isomorphism Proposition 28.10 that the induced map $p_*\colon \pi_n(\widetilde{X}) \to \pi_n(X)$ is an isomorphism.

There are no analogous statements for homology groups. For example we saw on page 189 that there exists a 2-fold covering $p\colon S^1 \times S^1 \to K$ from the torus to the Klein bottle K. In this case we have by the Torus-H_*-Proposition 45.5 and the Klein Bottle-H_*-Lemma 45.6 that

$$\begin{array}{rcl} \mathrm{H}_0(S^1 \times S^1) & \cong & \mathbb{Z}, \\ \mathrm{H}_1(S^1 \times S^1) & \cong & \mathbb{Z}^2, \\ \mathrm{H}_2(S^1 \times S^1) & \cong & \mathbb{Z}, \\ \mathrm{H}_i(S^1 \times S^1) & \cong & 0, \text{ for } i \geq 3, \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{H}_0(K) & \cong & \mathbb{Z}, \\ \mathrm{H}_1(K) & \cong & \mathbb{Z} \oplus \mathbb{Z}_2, \\ \mathrm{H}_2(K) & \cong & 0, \\ \mathrm{H}_i(K) & \cong & 0, \text{ for } i \geq 3. \end{array}$$

Thus we see that the homology groups are quite different and there is no clear connection between the n-th homology group of a covering and the n-th homology group of the base.

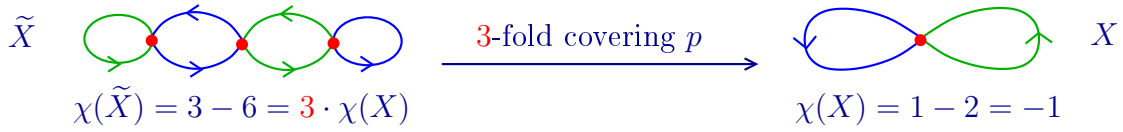
It is therefore perhaps surprising that we have the following proposition:

Proposition 48.8. (Covering-χ-Proposition) Let $p\colon \widetilde{X} \to X$ be a finite covering of a finite CW-complex X. We equip $\widetilde{X}$ with the structure of a CW-complex given by the CW-Complex Covering Proposition 32.1. Then

$$\chi(\widetilde{X}) \;=\; [\widetilde{X} : X] \cdot \chi(X).$$

Example.

(1) We consider again the 2-fold covering $p\colon S^1 \times S^1 \to K$. On page 564 we saw that $\chi(S^1 \times S^1) = 0$ and $\chi(K) = 0$. This is consistent with the Covering-χ-Proposition 48.8.
(2) We consider the 2-fold covering $p\colon S^2 \to \mathbb{R}\mathrm{P}^2$. In this case $\chi(S^2) = 2$ is indeed twice $\chi(\mathbb{R}\mathrm{P}^2) = 1$.
(3) In the Graph-π_1-Proposition 20.9 we showed that if X is a connected finite non-empty 1-dimensional CW-complex, then $\pi_1(X)$ is a free group of rank $1 - \chi(X)$. As we implicitly already argued in the Free Subgroup Proposition 32.5, it follows fairly easily from this observation and the Covering-χ-Proposition 48.8 that if F is a free group of finite rank and if G is a finite-index subgroup of F, then G is a free group with $\mathrm{rank}(G) - 1 = [F : G] \cdot \mathrm{rank}(F)$.



Proof. As we mentioned in the CW-Complex Covering Proposition 32.1 (1), it follows immediately from the construction of the CW-structure on $\widetilde{X}$ that for each $k \in \mathbb{N}_0$ we have

$$\#k\text{-cells of } \widetilde{X} \;=\; \big[\widetilde{X} : X\big] \cdot \#k\text{-cells of } X.$$

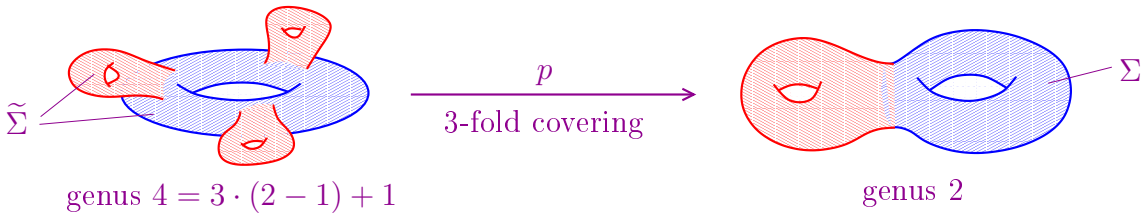
The proposition is an immediate consequence of this equality. ■

48.5. **Applications of Euler characteristics.** The following lemma changes how the genus changes under taking finite coverings.

Lemma 48.9. (Genus-of-Covering Lemma) Let Σ be the surface of genus g. For any connected finite covering $p\colon \widetilde{\Sigma} \to \Sigma$ we have[385]

$$\text{genus}(\widetilde{\Sigma}) \;=\; \deg(p) \cdot (g-1) + 1.$$

Example. In the figure below we illustrate a 3-fold covering of the surface of genus 2 by the surface of genus $4 = 3 \cdot (2-1) + 1$.



Proof. We denote by $\widetilde{g}$ the genus of $\widetilde{\Sigma}$. The lemma follows immediately from the following two facts:

(1) By the discussion on page 564 we have $\chi(\Sigma) = 2 - 2g$ and $\chi(\widetilde{\Sigma}) = 2 - 2\widetilde{g}$.

(2) By the Covering-χ-Proposition 48.8 we have $\chi(\widetilde{\Sigma}) = \deg(p) \cdot \chi(\Sigma)$. ■

[385] Recall that by the Surface Classification Theorem 22.8 there exists precisely one $\widetilde{g}$ such that the closed, oriented 2-dimensional smooth manifold $\widetilde{\Sigma}$ is diffeomorphic to the surface of genus $\widetilde{g}$.

49. Applications of the Euler characteristic

In this chapter we show that the fairly simple concept of the Euler characteristic, which we introduced in the previous chapter, can be used to address the following questions about 3-dimensional objects:

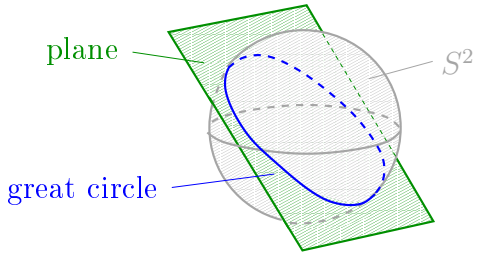
(1) Why is a classical football made using precisely 12 pentagons?
(2) What is Euler's famous formula?
(3) Why are there precisely five types of platonic solids?
(4) Can we embed all finite topological graphs into the plane?

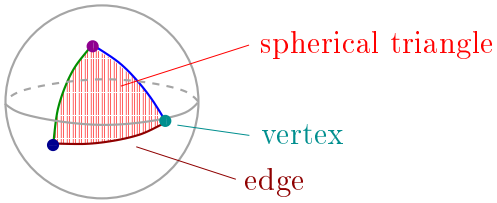
The reader is warned that in this chapter at times we will trade notational and mathematical rigor for readability.

49.1. Building a leather football.

In this section we want to study the classic leather football from a mathematical point of view. This discussion will require the following definitions:

Definition.
(1) A great circle on the 2-dimensional sphere S^2 is the intersection of S^2 with a 2-dimensional vector subspace of $\mathbb{R}^3$.
(2) The angle between two great circles is defined as the smaller of the two angles given by the corresponding 2-dimensional vector subspaces.
(3) A spherical n-gon is a subset of S^2 such that the boundary is the union of n segments of n different great circles. These segments are called edges. The points where two edges intersect are called vertices.





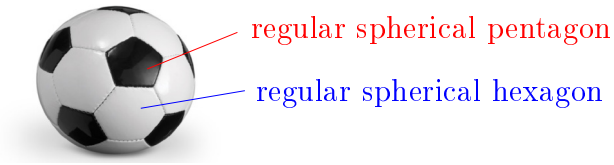
In the following we are mostly interested in highly symmetric spherical n-gons. To define what we mean by "highly symmetric" we need to introduce the following general concept.

Definition. Let P be a subset of $\mathbb{R}^k$.
(1) A symmetry of P in $\mathbb{R}^k$ is a matrix $A \in \mathrm{O}(k)$ such that $A \cdot P = P$.
(2) The symmetry group $\mathrm{Sym}(P) \subset \mathrm{O}(k)$ is defined as the group of all symmetries of P.
(3) We set $\mathrm{Sym}^+(P) := \mathrm{Sym}(P) \cap \mathrm{SO}(k)$.

Using this general definition we can now introduce the main players of this section:

Definition. A spherical polygon P is called regular if the symmetry group $\mathrm{Sym}(P)$ acts transitively on the set of pairs (v, e) where v is a vertex of P and E is an edge of P with $v \in e$.

Example. The classical leather football is made out of 12 regular spherical pentagons and 20 regular spherical hexagons.



We want to determine how many regular spherical pentagons and hexagons are required to make a football. So suppose a football is made out of p regular spherical pentagons and h regular spherical hexagons such that any edge of a polygon is the edge of another polygon. This endows S^2 with a CW-structure such that the following holds:

- The 0-cells correspond to the vertices.
- The 1-cells correspond to the edges.
- The 2-cells correspond to the polygons.

We make the following observations:

(1) We first note that each 1-cell bounds two 2-cells.
(2) The interior angles of a regular n-gon are all the same.
(3) One can show, using elementary spherical geometry (see e.g. [**Bon2009**] or [**Fen2001**]), that the interior angle of a regular spherical n-gon is larger than the angle of the corresponding Euclidean regular n-gon.
(4) It follows from (2) and (3) that in our setting all the interior angles are larger than[386] $\pi - \frac{2\pi}{5} = \frac{3}{5}\pi$. Since the total angle at a vertex is 2π we see that at any vertex there are not more than three regular spherical hexagons and pentagons that meet. Of course this implies that at any vertex precisely three regular spherical hexagons and pentagons meet.

It follows that

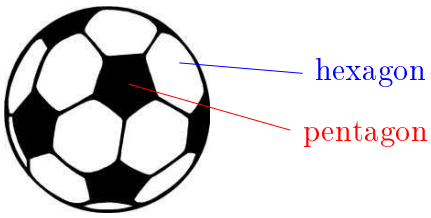
$$\begin{aligned} \#\text{2-cells} &= p+h \\ \#\text{1-cells} &= \tfrac{1}{2}(5p+6h) \quad \text{here we use Observation (1)} \\ \#\text{0-cells} &= \tfrac{1}{3}(5p+6h) \quad \text{here we use Observation (4).} \end{aligned}$$

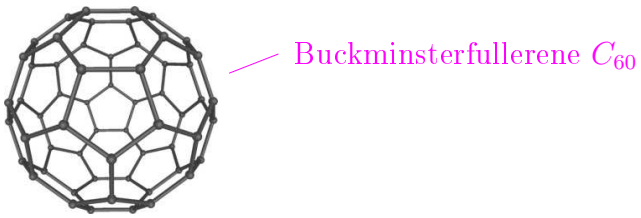
We now see that

$$2 \underset{\uparrow}{=} \chi(S^2) = \sum_{n=0}^{2}(-1)^n \cdot \#n\text{-cells} \underset{\uparrow}{=} \tfrac{1}{3}(5p+6h) - \tfrac{1}{2}(5p+6h) + (p+h) = \tfrac{1}{6}p.$$

by the Homology-χ-Proposition 48.1 — by the above three equalities

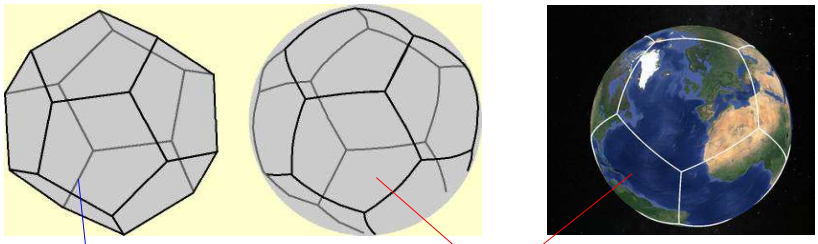
Thus we see that a football has to have precisely 12 regular spherical pentagons. On the other hand, the number of regular spherical hexagons is not fixed. For example, as we mentioned above, the standard football has 20 hexagons.





[386] Where does that formula come from?

But in principle one could also build a football with zero hexagons, see the figure below on the right. This way one obtains a spherical dodecahedron. One could also build footballs with more hexagons, but they would be much less round since they would have bigger flat parts.



Finally note that the 1-skeleton of this CW-structure was in fact already "invented by nature", more precisely, there exists a molecule, called the Buckminsterfullerene C_{60}, built out of 60 carbon atoms, which has exactly the same shape as a football, except that it is a little smaller.

49.2. **Euler's Formula.** We start out by recalling the following definition from page 56:

Definition. Given a subset S of $\mathbb{R}^k$ the convex hull of S is defined as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. Since the intersection of convex sets is again convex we see that the convex hull of S is a convex subset of $\mathbb{R}^n$.

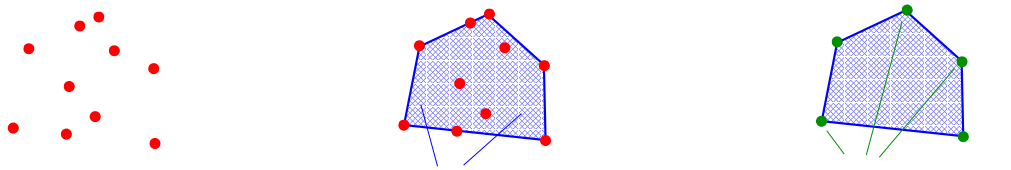
We move onto a new definition.

Definition.

(1) A convex polyhedron is the convex hull of finitely many points in $\mathbb{R}^k$.
(2) The dimension of a convex polyhedron is defined as the minimal dimension of an affine subspace of $\mathbb{R}^k$ that contains the convex polyhedron.

Lemma 49.1. (Vertices-of-Polyhedron Lemma) If P is a convex polyhedron in $\mathbb{R}^k$, then there exists a unique finite set $V \subset \mathbb{R}^k$ such that P is the convex hull of V, but such that for any proper subset W of V, P is not the convex hull of W.

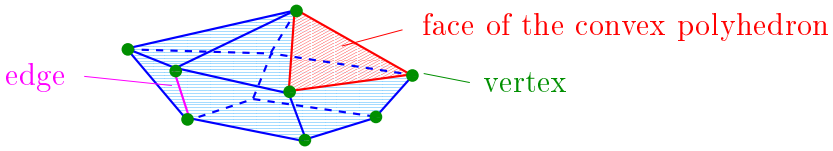
Proof. We leave the proof of this lemma to the reader. ■



Definition. Let P be a convex polyhedron in $\mathbb{R}^k$.

(1) We refer to the finite set of the Vertices-of-Polyhedron Lemma 49.1 as the vertices of the convex polyhedron P.

(2) (a) A closed affine half-space in $\mathbb{R}^k$ is a set of the form $\{v \in \mathbb{R}^k \mid \langle v, w\rangle \geq a\}$ for some fixed non-zero vector $w \in \mathbb{R}^k$ and some $a \in \mathbb{R}$.
(b) A subset $F \subset P$ is called a face of P if there exists a closed affine half-space H of $\mathbb{R}^k$ with $P \subset H$ such that $F = P \cap \partial H$.
Note that a face is a convex polyhedron in its own right, which implies in particular that it makes sense to talk of the dimension of a face.
(3) An edge is a 1-dimensional face. Sometimes we refer to a 2-dimensional face just as a face, without specifying the dimension.



Proposition 49.2. (Euler's Formula) Let P be a convex k-dimensional polyhedron in $\mathbb{R}^k$. The following statements hold:
(1) The interior $\mathring{P}$ of P is non-empty and all faces of P are of dimension $\leq k-1$.
(2) The boundary ∂P admits a CW-structure such that for each $j \in \mathbb{N}_0$ the j-cells are precisely the j-dimensional faces of P.
(3) The following equality hold:

$$\sum_{j=0}^{k-1}(-1)^j \cdot \#j\text{-dimensional faces} \;=\; 1+(-1)^{k-1}.$$

Remark. The 3-dimensional version of Proposition 49.2 gives us that for the boundary of a convex 3-dimensional polyhedron in $\mathbb{R}^3$ we have

$$\text{number of vertices} - \text{number of edges} + \text{number of faces} \;=\; 2.$$

This equality is known as Euler's Formula and was formulated by Leonhard Euler [**Eul1758**] in 1758. This equality was already noted before by Francesco Maurolico [**Friedm2018**] in 1537 and René Descartes [**Rabo2010**] in 1628. Arguably Euler's Formula is historically the first result in topology. A famous book by Imre Lakatos [**Lak1976**] is dedicated to the challenge of finding the correct statement of Euler's formula and of providing a rigorous proof of Euler's Formula without employing the technical machinery, namely singular homology, that we are about to use:

Sketch of proof. We leave the pleasure of proving the first two statements to the reader. Now we calculate that

$$\sum_{j=0}^{k-1}(-1)^j \cdot \#j\text{-dimensional faces} \underset{\text{by (1) and (2)}}{=} \chi(\partial P) \underset{\text{by (1) together with the Convex-to-Ball Proposition 2.12(2)}}{=} \chi(S^{k-1}) \underset{\text{see page 564}}{=} 1+(-1)^{k-1}.$$

■

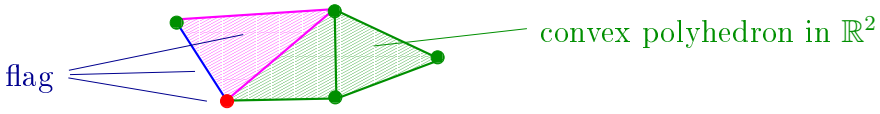
49.3. **Platonic solids.** In this section we will use Euler's Formula 49.2 to address the question: Why are there only five types of Platonic solids. Evidently we first need to introduce the notion of a Platonic solid. This requires some preparations:

[386] Here ∂P is the boundary of P viewed as a subset of $\mathbb{R}^k$.

Definition.

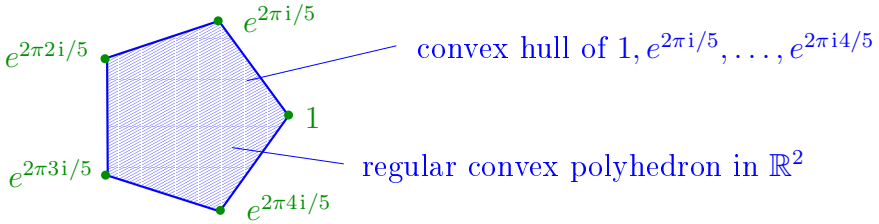
(1) Let P be a convex polyhedron in $\mathbb{R}^k$.
   (a) A flag of P is a sequence $F_0 \subset F_1 \subset \cdots \subset F_{m-1}$ of faces of P such that each F_i is i-dimensional.
   (b) We say P is regular if the symmetry group $\mathrm{Sym}(P)$ acts transitively on the set of flags of P.

(2) Two convex polyhedra P and Q in $\mathbb{R}^k$ are called equivalent if there exists an $r > 0$, a $v \in \mathbb{R}^k$ and an orthogonal matrix $A \in \mathrm{O}(k)$ with $P = A \cdot (r \cdot Q) + v$.



Examples.

(1) Let $n \in \mathbb{N}_{\geq 3}$ and $r > 0$. The convex hull of the n points $r, r \cdot e^{\pi \mathrm{i}/n}, \ldots, r \cdot e^{2\pi \mathrm{i}(n-1)/n}$ in $\mathbb{C} = \mathbb{R}^2$ is a regular convex polyhedron with vertices $r, r \cdot e^{2\pi \mathrm{i}/n}, \ldots, r \cdot e^{2\pi \mathrm{i}(n-1)/n}$. We leave it to the reader to show that, up to equivalence, these are all regular convex 2-dimensional polyhedra in $\mathbb{R}^2$.



(2) The standard n-simplex Δ^n is a convex n-dimensional polyhedron in $\mathbb{R}^{n+1}$ with vertex set $\{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$. It is straightforward to see that Δ^n is regular.

We now specialize to the "classical" setting of 3-dimensional objects. The next definition introduces one of the oldest concepts in the history of mathematics.

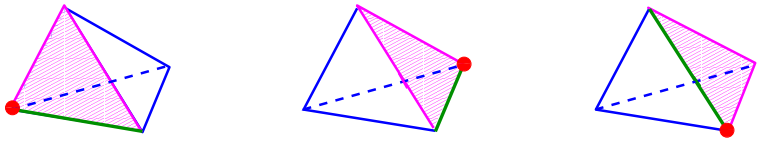
Definition. A Platonic solid is a regular convex 3-dimensional polyhedron in $\mathbb{R}^3$.

Before we discuss the five classical examples of Platonic solids let us introduce the following definition:

Definition. The valence of a vertex of a convex polyhedron is defined as the number of edges that contain the vertex.

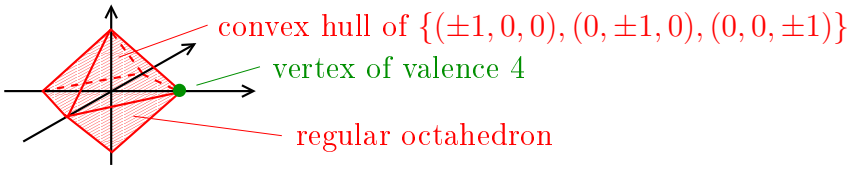
Examples.

(1) Let T be the 3-dimensional polyhedron in $\mathbb{R}^3$ that has four 2-dimensional faces, each of which is an equilateral triangle. In the figure below we show T together with three flags. Some classical geometry shows that T is, up to equivalence, uniquely defined and it is regular. We refer to T as *the regular tetrahedron*. Finally note that each vertex has valence 3.

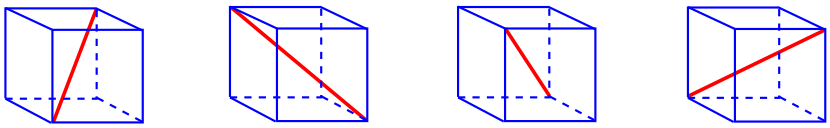


three flags in the regular tetrahedron T

(2) The convex polyhedron in $\mathbb{R}^3$ spanned by $\{(\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1)\}$ is a regular convex 3-dimensional polyhedron where each vertex has valence 4. This convex polyhedron is usually called *the regular octahedron*.



(3) We consider the cube $[-1,1]^3 \subset \mathbb{R}^3$. In [**Arm1988**] it is shown that the group $\mathrm{Sym}^+([-1,1]^3)$ is isomorphic to S_4. More precisely, we denote by D the set of the diagonals in the cube, i.e. the segments in $\mathbb{R}^3$ spanned by opposite vertices of the cube. Note that D has precisely four elements.



In [**Arm1988**] it is shown that the map

$$\begin{array}{rcl} \mathrm{Sym}^+([-1,1]^3) & \to & S_4 \cong \text{permutation group of } D \\ A & \mapsto & \begin{pmatrix} D & \to & D \\ d & \mapsto & A\cdot d \end{pmatrix} \end{array}$$

is an isomorphism. It is clear that the cube is a regular convex polyhedron. Since it has six sides it is in our context also referred to as *the regular hexahedron*.

(4) Let $z := 1+\frac{\sqrt{5}}{2}$ be the golden ratio. We denote by D the polyhedron that is spanned by the 20 vertices

$$(\pm 1,\pm 1,\pm 1),\quad (0,\pm z,\pm \tfrac{1}{z}),\quad (\pm \tfrac{1}{z},0,\pm z),\quad \text{and}\quad (\pm z,\pm \tfrac{1}{z},0).$$

By [**Cox1948**] this is a regular polyhedron. We refer to is as *the regular dodecahedron*. Note that it has 12 faces and 20 vertices. We can inscribe five cubes $C_1,\ldots,C_5$ into the regular dodecahedron. In [**Arm1988**] it is shown that the map

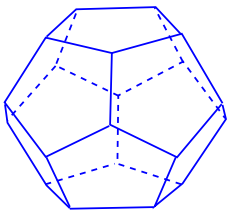
$$\begin{array}{rcl} \mathrm{Sym}^+(D) & \to & A_5 \cong \text{positive permutations of } \{C_1,\ldots,C_5\} \\ A & \mapsto & (C_i \mapsto A\cdot C_i) \end{array}$$

is an isomorphism.

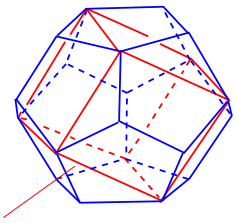
(5) Let $z := 1+\frac{\sqrt{5}}{2}$ be the golden ratio. We denote by I the polyhedron that is spanned by the 12 vertices

$$(0,\pm 1,\pm z),\quad (\pm z,0,\pm 1)\quad \text{and}\quad (\pm 1,\pm z,0).$$

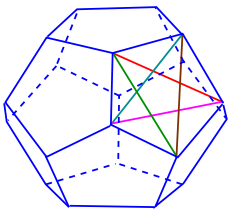
By [**Cox1948**] this is a regular polyhedron. We refer to is as the *regular icosahedron*.



regular dodecahedron

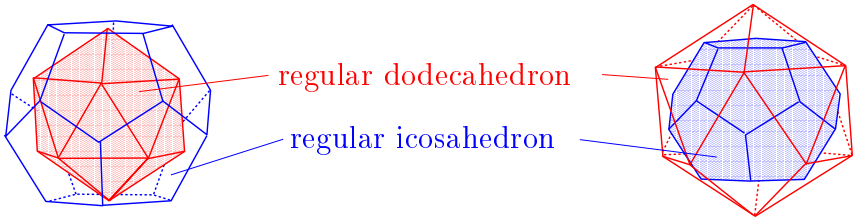


cube inscribed into the regular dodecahedron



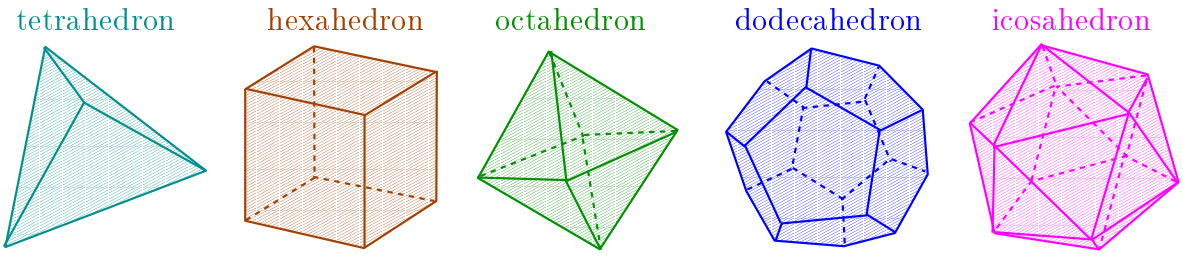
the five diagonals of a face correspond precisely to one inscribed cube

Note that the regular dodecahedron has 20 vertices and 12 faces whereas the regular icosahedron has 12 vertices and 20 faces. In fact the regular icosahedron is dual to the regular dodecahedron in the following sense: up to a scaling factor and a rotation, the regular icosahedron is spanned by the centers of the faces of the regular dodecahedron and vice-versa. We refer to the figure below for an illustration of this fact. It follows that the symmetry group of the regular icosahedron I is in a suitable sense equal to the symmetry group of the regular dodecahedron D. In particular the group $\mathrm{Sym}^+(I)$ is isomorphic to A_5.



Theorem 49.3. (Platonic Solid Theorem) Up to equivalence the only regular convex 3-dimensional polyhedra in $\mathbb{R}^3$ are the platonic solids discussed above and illustrated in the figure below. They have the following types of faces and valences and they have the following symmetry groups:

	tetrahedron	hexahedron	octahedron	dodecahedron	icosahedron
number of faces	4	6	8	12	20
shape of face	3-gon	4-gon	3-gon	5-gon	3-gon
valence of vertex	4	3	4	3	5
$\mathrm{Sym}^+(P)$	A_4	S_4	S_4	A_5	A_5



Sketch of proof. We will not provide a full proof for the theorem. Instead we refer to [**Cox1948**] for a full proof that the list of five platonic solids is indeed complete. Furthermore we refer to [**Arm1988**] or alternatively to [**Ae2008**] for a proof of the statement regarding the symmetry groups.[387]

In the following we will show the weaker statement that any platonic solid fits into the table of the theorem. Put differently, we will show that given any platonic solid the number of faces, the shape of faces and the valence of a vertex corresponds to one of the five columns in the above table.

Thus let P be a regular convex 3-dimensional polyhedron in $\mathbb{R}^3$. We make the following observations:

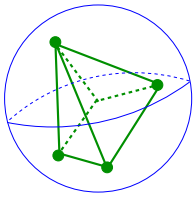
(1) It follows easily from the definition of a regular polyhedron that there exists an $m \in \mathbb{N}$ such that each face of P is a regular m-gon.
(2) The number of faces that meet at a vertex equals the valence of a vertex.
(3) The valences of the vertices are the same and it is at least three.

The following claim tells us the possible values for m and the number f of faces that meet at a vertex.

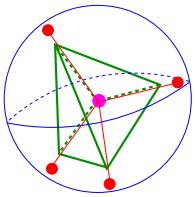
Claim.

possible values for m	3	4	5
f = number of faces that meet at a vertex	$3, 4, 5$	4	3

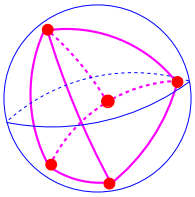
Proof. After a translation we can assume that the sum of all vertices is in the origin. Next we shrink the convex polyhedron so that it fits into the sphere S^2. Then we consider the projection of the vertices and edges of the convex polyhedron onto the sphere S^2.



embed polyhedron in the 3-ball



projection of vertices onto the sphere



projection of edges onto the sphere

Recall that on page 572 we noted that the interior angle of a regular spherical m-gon is greater than the angle of a regular Euclidean m-gon. So it follows that in our case the interior angle is greater than the angle of a regular m-gon in $\mathbb{R}^3$, i.e. it is greater than $\pi - \frac{2\pi}{m}$. Since at any vertex at least three faces meet we obtain the restriction that $f \geq 3$ and that $f \cdot (\pi - \frac{2\pi}{m}) < 2\pi$. It is now straightforward to see that we obtain the stated table. ⊞

Now we treat the cases $m = 3, 4, 5$ separately. We start out with the case $m = 3$, i.e. we assume that all the faces of the convex polyhedron are equilateral triangles. Let us first consider the case that three triangles meet at a vertex. Then we have

$$\#\text{faces} =: k, \qquad \#\text{edges} \underset{\uparrow}{=} \frac{1}{2} \cdot 3k \qquad \text{and} \qquad \#\text{vertices} \underset{\uparrow}{=} \frac{1}{3} \cdot 3k.$$

since $m = 3$ and since each edge cobounds two faces

since $m = 3$ and since each vertex cobounds three faces

[387] We sketched the proofs for the regular hexahedron, the regular dodecahedron and the regular icosahedron.

Now we see that

$$2 \underset{\text{Euler's Formula 49.2}}{=} \#\text{faces} - \#\text{edges} + \#\text{vertices} \underset{\text{by the above}}{=} k - \tfrac{3}{2}k + \tfrac{3}{k}k = \tfrac{1}{2}k.$$

It follows that $k = 4$. Note that this case is realized by the regular tetrahedron. Almost the same calculation shows that if four triangles meet at a vertex, then $2 = k - \frac{3}{2}k + \frac{3}{4}k$, i.e. $k = 8$. Note that this case is realized by the regular octahedron. Finally by the above claim it remains to consider the case that five triangles meet at a vertex. In this case we see that $2 = k - \frac{3}{2}k + \frac{3}{5}k$, i.e. $k = 20$. Note that this case is realized by the regular icosahedron.

Next we consider the case $m = 4$, i.e. each face is a square. The same argument as above shows that the valence of a vertex can only be three and the same calculation shows that there have to be precisely six faces. Note that this case is realized by the regular hexahedron (aka cube).

Finally if $m = 5$, i.e. if each face is a regular pentagon, then as above we see that the valence of a vertex can only be three and that there have to be precisely twelve faces. Note that this case is realized by the regular dodecahedron. ■

Remark. On page 575 we saw that there exist infinitely many equivalence classes of regular convex polyhedra in $\mathbb{R}^2$. On the other hand we have just seen in the Platonic Solid Theorem 49.3 that in $\mathbb{R}^3$ there are only finitely many, namely precisely five equivalence classes of regular convex polyhedra in $\mathbb{R}^3$. This begs the question, what is the number in higher dimensions? It turns out that in $\mathbb{R}^4$ there are six equivalence classes of regular convex polyhedra, whereas in all higher dimensions there are only three equivalence classes of convex polyhedra. A nice exposition of these facts is given on the following website:

`http://math.ucr.edu/home/baez/platonic.html`

49.4. **Planar graphs.** We move on to a completely different subject, namely the study of graphs and their embeddings into the plane.

Definition.

(1) An (undirected abstract) graph G is a triple (V, E, φ) where V is a set, E is a set and φ is a map

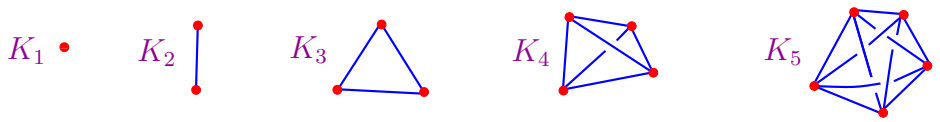
$$\varphi\colon E \to \{\text{subsets of } V \text{ with one or two elements}\}.$$

The elements of V are called vertices of G and the elements of E are called the edges of G.

(2) The topological realization $|G|$ of G is defined as the topological space that is obtained from the vertex set V (equipped with the discrete topology) by attaching an interval for each edge $e \in E$ to the vertices given by $\varphi(e) \subset V$.

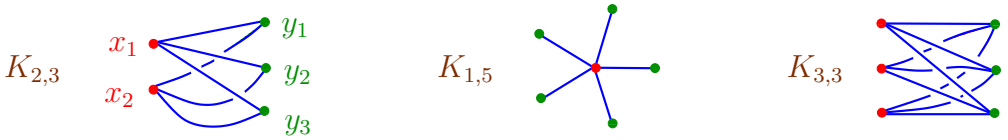
Examples.

(1) For any $n \in \mathbb{N}$ we define the complete graph K_n as the abstract graph with vertex set $V = \{v_1, \dots, v_n\}$, edge set $E = \{\{v_i, v_j\} \mid i, j \in \{1, \dots, n\} \text{ with } i \neq j\}$ and where we define $\varphi\colon E \to \mathcal{P}(V)$ via $\varphi(e) = e$. Put differently, the complete graph K_n has n vertices and each vertex is connected to any other vertex by precisely one edge.



complete graphs K_n

(2) For every choice of $m, n \in \mathbb{N}$ we define the complete bipartite graph $K_{m,n}$ as the abstract graph which is defined by the vertex set $V = \{x_1, \dots, x_m\} \cup \{y_1, \dots, y_n\}$, the edge set $E = \{\{x_i, y_j\} \mid i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\}\}$ and the map $\varphi\colon E \to \mathcal{P}(V)$ that is given by $\varphi(e) = e$. Put differently, the complete bipartite graph $K_{m,n}$ consists of m "red-colored" vertices, n "green-colored" vertices and each "red-colored" vertex is connected to any "green-colored" vertex by precisely one edge.



complete bipartite graphs $K_{m,n}$

Definition. We say a finite graph G is planar if its topological realization $|G|$ admits an embedding $|G| \to \mathbb{R}^2$.

Examples.

(1) The above figures show that the graphs K_1, K_2, K_3 and $K_{1,5}$ are planar. On the other hand the above topological realizations of K_4 and $K_{2,3}$ are not subspaces of $\mathbb{R}^2$. But as we see in the figure below both graphs admit topological realizations that are subspaces of $\mathbb{R}^2$.

topological realizations of K_4 topological realizations of $K_{2,3}$

(2) One can easily verify that given an $m \in \mathbb{N}$ the bipartite graph $K_{2,m}$ is planar.

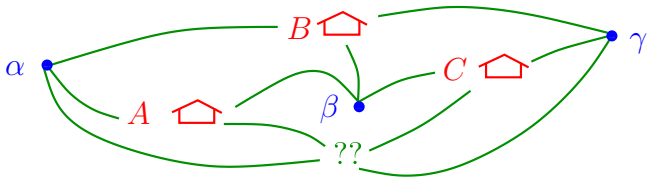
Now the following question arises.

Question 49.4. Are the graphs K_5 and $K_{3,3}$ also planar?

If this question sounds to abstract and bipartite graphs sound too otherworldly let us reformulate the question regarding $K_{3,3}$:

Problem 49.5. (Utility Problem) Suppose that three new houses A, B and C are built and they need to be connected to three utilities, e.g. a water source α, an electricity provider β and an internet provider γ. Can we perform the connections so that no two lines/pipes intersect, except at the endpoints?

To answer that question we need a few more definitions.



Definition. Let G be a graph.

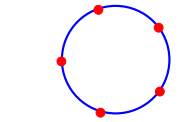
(1) Given $n \in \mathbb{N}$ we denote by X_n the "circular graph with n vertices and n edges", more precisely, X_n is the graph with

$$V(X_n) = \{1, \dots, n\} \quad \text{and} \quad E(X_n) = \{\{1,2\}, \{2,3\}, \dots, \{n-1,n\}, \{n,1\}\}$$

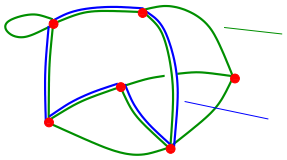
and where $\varphi\colon E(X_n) \to \mathcal{P}(V(X_n))$ is the obvious map.

(2) A cycle of length n in G is an injective morphism $\varphi\colon X_n \to G$ of graphs.

(3) The girth of G is the shortest length of a cycle in G. If G has no cycles, then we define its girth as ∞.



circular graph X_5



Examples.

(1) The girth of the graph shown in the figure above to the right is one, since it has an edge where the two endpoints agree.

(2) One can easily show that for any $n \geq 3$ the girth of the complete graphs K_n equals three.

(3) Similarly one can easily show that for any $n, m \geq 2$ the girth of the complete bipartite graphs $K_{n,m}$ equals four.

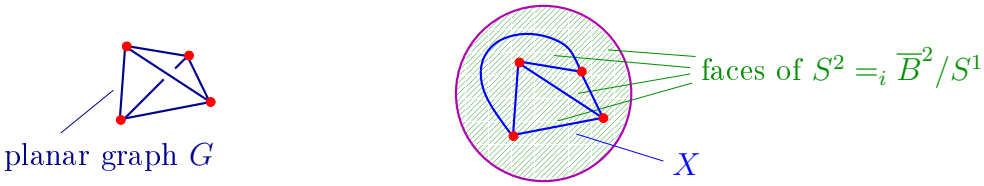
The following proposition gives an obstruction to a graph being planar.

Proposition 49.6. (Planar Graph–Girth Proposition) Let G be a graph with v vertices and e edges. We denote by g the girth of G. If G is planar and if $g < \infty$, then

$$e \leq \frac{g}{g-2} \cdot (v-2).$$

Sketch of proof. Let G be a graph with v vertices and e edges. We suppose that the girth g of G is finite. Furthermore we suppose that G is planar, i.e. we suppose that there exists an embedding $\varphi\colon |G| \to \mathbb{R}^2$. We write $X = \varphi(|G|)$. Without loss of generality we can assume that X lies in the open disk $B^2 \subset \mathbb{R}^2$.

Now we view X as a subspace of the sphere $S^2 =_i \overline{B}^2/S^1$. We refer to the closures of the components of $S^2 \setminus X$ as *faces* and we denote by f the number of faces.



We make the following observation:

(1) The vertices, edges and faces define a CW-structure for $S^2 =_i \overline{B}^2/S^1$.[388]
(2) We make the following two observations:
  (a) The boundary of each face corresponds to a cycle in the graph. By the definition of the girth each face is bounded by at least g edges.
  (b) Each edge lies in the boundary of at most two faces.[389]

  Now we deduce that

$$\begin{aligned} g \cdot f &\overset{\text{follows from (a)}}{\leq} \sum_{\text{faces } f} \text{number of edges in the boundary of } f \\ &= \text{number of pairs } (f,e) \text{ where } f \text{ is a face and } e \text{ an edge in the boundary of } f \\ &= \sum_{\text{edges } e} \text{number of faces that contain } e \text{ in the boundary} \underset{\text{follows from (b)}}{\leq} 2e. \end{aligned}$$

  Summarizing we see that $g \cdot f \leq 2e$.

Now we see that

$$2 \underset{\text{see page 564}}{=} \chi(S^2) \underset{\text{by (1)}}{=} v - e + f \underset{\text{since by (2) we have } g \cdot f \leq 2e}{\leq} v - e + \frac{2}{g} \cdot e = v - \frac{g-2}{g} \cdot e.$$

Solving for e we get the desired inequality $e \leq \frac{g}{g-2} \cdot (v-2)$. ■

The following proposition gives in particular a negative answer to Question 49.4.

Proposition 49.7. (Non-Planar Graph Proposition) If a graph G contains the complete graph K_5 or the complete bipartite graph $K_{3,3}$ as a subgraph, then G itself cannot be planar.

Proof.

(1) Let $m \in \mathbb{N}_{\geq 3}$. It is straightforward to see that the complete graph K_m has m vertices and $\frac{1}{2}m(m-1)$ edges. Furthermore, as we pointed out on page 581, the girth is three. It is straightforward to verify that the inequality of the Planar Graph–Girth Proposition 49.6 is only satisfied for $m = 3$ and $m = 4$.
(2) Let $m, n \in \mathbb{N}_{\geq 2}$. It is easy to verify that the complete bipartite graph $K_{m,n}$ has $m+n$ vertices and $m \cdot n$ edges. Furthermore, as we pointed out above, the girth is four. It is

[388] Here is the reason why it is the sketch of a proof: we do not prove that each face is in fact homeomorphic to a closed disk. This statement is "clear from the picture" and it can be proved using the Schönflies Theorem. But we will not attempt to give a rigorous proof.
[389] This statement also requires a little bit of thought.

straightforward to verify that the inequality of the Planar Graph–Girth Proposition 49.6 is only satisfied if $m = 2$ or $n = 2$.

(3) The proposition now follows from (1) and (2) and the observation that if a graph is planar, then evidently any subgraph is also planar. ■